Ramsey classes and partial orders

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The candidate confirms that the work submitted is their own and that appropriate credit has been given where reference has been made to the work of others.

To Toby and Scruffy, for being the fluffiest, most supportive baes.

Just like moons and like suns,
With the certainty of tides,
Just like hopes springing high,
Still I'll rise.
- *Maya Angelou*

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Abstract

We consider the classes of finite coloured partial orders, *i.e.*, partial orders together with unary relations determining the colour of their points. These classes are the ages of the countable homogeneous coloured partial orders, classified by Torrezão de Souza and Truss in 2008. We prove that certain classes can be expanded with an order to become Ramsey classes with the ordering property. The motivation for finding such classes is the 2005 paper of Kechris, Pestov and Todorčević, showing that these concepts are important in topological dynamics for calculating universal minimal flow of automorphism groups of homogeneous structures and finding new examples of extremely amenable groups.

We introduce the *elementary skeletons* to enumerate the classes of ordered shaped partial orders and show that classes are Ramsey using three main approaches. With the Blowup Lemma we use the known results about the Ramsey classes of ordered partial orders, to prove results about shaped classes. We use the Structural Product Ramsey Lemma to show that a class $\mathcal{K}$ is Ramsey when structures in classes known to be Ramsey determine each structure in $\mathcal{K}$ uniquely. Finally, we use the Two Pass Lemma when each structure in the considered class has two dimensions that can be built separately and the classes corresponding to both dimensions of the structure are known to be Ramsey. We then show that the classes of unordered reducts of the structures in the classes enumerated by elementary skeletons are the Fraïssé limits of the countable homogeneous coloured partial orders.

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Chapter 1

Introduction

Stemming from the Classical Ramsey Theorem (Theorem 2.2.2), Structural Ramsey Theory considers classes of structures and is mainly concerned with the question *Does a certain class of structures have a Ramsey property?*

We need to introduce some notation to define the Ramsey property. Given structures $\mathbf{A}$ and $\mathbf{B}$, denote the set of all substructures of $\mathbf{B}$, isomorphic to $\mathbf{A}$, as $\binom{\mathbf{B}}{\mathbf{A}}$. Further, given a positive integer k, we denote the set $\{1, 2, \ldots, k\}$ by $[k]$. Then a class $\mathcal{K}$ of structures has a *Ramsey property* if given any structures $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ there exists a structure $\mathbf{C} \in \mathcal{K}$ such that given any finite colouring

$$c : \binom{\mathbf{C}}{\mathbf{A}} \to [k],$$

there exists a $\mathbf{B}' \in \binom{\mathbf{C}}{\mathbf{B}}$ such that $\binom{\mathbf{B}'}{\mathbf{A}}$ is monochromatic. A class with the Ramsey property is also referred to as a Ramsey class.

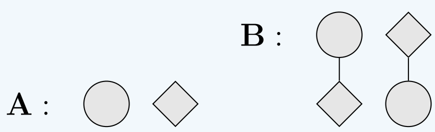
It is particularly intriguing to consider the Structural Ramsey Theory of a classes of structures corresponding to homogeneous structures, as these have been classified in many cases. In this introduction, we will explore why the aim of this thesis was to find the classes of ordered shaped partial orders, that have the Ramsey property, but are also Fraïssé and have the ordering property, and how that is connected to the known classification of all Fraïssé classes of shaped partial orders. We will also provide a summary of the proof of the main result of the thesis, Theorem 2.5.31.

1.1 Motivation

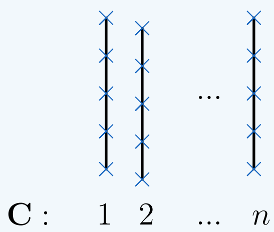
A *shaped partial order* $\mathbf{P}$ is a partial order $(P, <)$, together with a set $\mathfrak{S}$ of shapes and a map $\mathfrak{s} : P \to \mathfrak{S}$, assigning each point in the set P a shape. In the literature, they are referred to as *coloured* partial orders, for example in the Countable homogeneous coloured partial orders by Torrezão de Sousa & Truss (2008). The paper contains the classification of all countable homogeneous shaped partial orders, and thus, by the Fraïssé correspondence, the classification of all Fraïssé classes of shaped partial orders. The reason we refer to the partial orders as *shaped* rather than *coloured* is to avoid the confusion when we consider the colourings in proving the Ramsey property of the classes. We summarise the classification in Section 2.4.

It is perhaps tempting to classify the Fraïssé classes of shaped partial orders with the Ramsey Property, but most of the classes turn out not to have the Ramsey Property for the reason best illustrated by the following example.

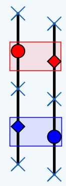
Example 1.1.1. Consider class $\boldsymbol{\mathcal{K}}(AC)$ of shaped antichains of chains and the structures $\mathbf{A}$ and $\mathbf{B}$ in $\boldsymbol{\mathcal{K}}(AC)$.



Let $\mathbf{C}$ be any shaped antichain of chains. Order its chains.

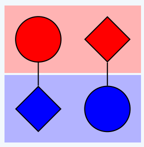
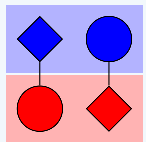


Colour $\binom{\mathbf{C}}{\mathbf{A}}$ as follows. Colour $\mathbf{A}$ red if its circle lies in a chain before its diamond, and blue otherwise.

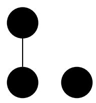
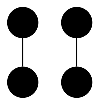


i< j

Then regardless of where the substructure of **C**, isomorphic to **B**, lies, it will contain a red and a blue copy of **A**, as we can see in the picture below.

$\mathbf{B}_1$:  $\mathbf{B}_2$: 

This is not a phenomenon unique to shaped partial orders. The problem is that unless all the structures in the class $\mathcal{K}$ concerned are either highly symmetric (for example, an antichain, or a complete graph) or rigid (these are the structures with only the trivial automorphism; for example, a chain), the class $\mathcal{K}$ fails to have the Ramsey property. For example, considering the antichains of chains **A** and **B** below, a proof like the one in Example 1.1.1 shows that the class $\mathcal{K}(AC_{\aleph_0})$ of all finiter antichains of chains, a Fraïssé class of partial orders, is not a Ramsey class.

A :  **B** : 

On the other hand, considering the classes of rigid structures often leads to discovering new Ramsey classes. For example, Böttcher & Foniok (2011) considers the Fraïssé classes of permutations, based on their classification in Cameron (2002), and shows that they all have the Ramsey property. Thinking of a permutation as a set together with two total orders, it is clear that all structures in those classes are rigid.

It is often interesting to consider classes of structures that are closed under substructures and have the joint embedding property (defined in 2.1.3). The early scholars of Structural Ramsey Theory showed in Nešetřil & Rödl (1977) that amongst such classes, the ones that have the Ramsey property also have the amalgamation property (also defined in 2.1.3), thus making them Fraïssé classes. This indicates that perhaps we should consider classifying Fraïssé classes of rigid structures with the Ramsey property that are somehow related to the Fraïssé classes of shaped partial orders.

An easy way to create a class $\mathcal{K}$ of rigid structures from a class $\mathcal{K}_0$ of structures is to consider an *order class*. That is, we expand the language L_0 of $\mathcal{K}_0$ by a binary relation symbol $\prec$ to get $L = L_0 \cup \{\prec\}$. Then the class $\mathcal{K}$ is an order class in the language L with respect to $\mathcal{K}_0$, if each structure $\mathbf{A} \in \mathcal{K}$ is of the form $\langle \mathbf{A}_0, \prec \rangle$, where $\mathbf{A}_0 \in \mathcal{K}_0$ and $\prec$ defines a total order relation. Due to the total order $\prec$, such structure $\mathbf{A}$ is rigid. The class $\mathcal{K}$ is *reasonable with respect to* $\mathcal{K}_0$, if given any $\mathbf{A} = \langle \mathbf{A}_0, \prec \rangle \in \mathcal{K}$ and any $\mathbf{B}_0 \in \mathcal{K}_0$, then if $\mathbf{A}_0$ is a substructure of $\mathbf{B}_0$, there exists a $\mathbf{B} = \langle \mathbf{B}_0, \prec \rangle \in \mathcal{K}$, such that $\mathbf{A}$ is a substructure of $\mathbf{B}$. This, of course, implies that it contains at least one ordered structure $\mathbf{A}$ of each structure $\mathbf{A}_0$ in $\mathcal{K}_0$. We can find multiple examples of Ramsey classes arising from this construction in Nešetril (2005).

Proposition 2.3.3, proven in Kechris *et al.* (2005), spells out the connection between $\mathcal{K}$ and $\mathcal{K}_0$ in a specific setting. In short, if $\mathcal{K}$ is a Fraïssé class, then $\mathcal{K}$ is a reasonable class with respect to $\mathcal{K}_0$ if and only if $\mathcal{K}_0$ is a Fraïssé class as well. Thus the classification of Fraïssé classes of shaped partial orders provides the basis of classification of the Fraïssé classes of their ordered expansions.

But given a Fraïssé class $\mathcal{K}_0$, there are often several order classes $\mathcal{K}$ that we might want to consider. If we consider the class $\mathcal{K}(AC_{\aleph_0})$ of antichains of chains as the class $\mathcal{K}_0$, the class $\mathcal{K}$ could be formed in at least three ways. We know already that for each $(A, <, \prec) \in \mathcal{K}$, the structure $(A, <)$ will be an antichain of chains. But additionally, there is

(i) $\mathcal{K} = \mathcal{K}(AC_{\aleph_0}, o)$, where $(A, \prec)$ is any total order;

(ii) $\mathcal{K} = \mathcal{K}(AC_{\aleph_0}, e)$, where $(A, \prec)$ is any total order extending the partial order $(A, <)$; and

(iii) $\mathcal{K} = \mathcal{K}(AC_{\aleph_0}, ce)$, where $(A, \prec)$ is obtained by starting with any ordering of the chains of $(A, <)$, say calling the chains A_1, $A_2, \ldots A_n$, and then extending the partial order $(A, <)$ by placing all the points of the chain A_i before all the points of the chain A_j in the total order $(A, \prec)$ whenever $i < j$.

The classes $\mathcal{K}(AC_{\aleph_0}, o)$, $\mathcal{K}(AC_{\aleph_0}, e)$ and $\mathcal{K}(AC_{\aleph_0}, ce)$ are all reasonable classes with respect to $\mathcal{K}(AC_{\aleph_0})$.

Thus an additional property, the *ordering property,* is worth considering. The class $\mathcal{K}$ has the ordering property with respect to $\mathcal{K}_0$, if for each $\mathbf{A}_0 \in \mathcal{K}_0$ there exists a $\mathbf{B}_0 \in \mathcal{K}$, such that for each ordered $\mathbf{A}, \mathbf{B} \in \mathcal{K}$, there exists a substructure of $\mathbf{B}$ isomorphic to $\mathbf{A}$. In Nešetřil & Rödl (1978), the authors show that the classes of ordered sparse graphs (graphs containing no cycles shorter than a specified size) have the ordering property and remark that it is related to the Ramsey property of the class and that the connection provided the original motivation for writing the paper. The property is also considered in Nešetřil & Rödl (1990).

To see how the Ramsey property, the ordering property, Fraïssé classes and their corresponding homogeneous structures fit together consider the following setup. Suppose that $\mathcal{H}$ is a homogeneous structure that is a totally ordered structure for $\prec$ in language $L \supseteq \{\prec\}$. Let $\mathcal{H}_0$ be a reduct of $\mathcal{H}$ to $L \setminus \{\prec\}$. If $\mathcal{H}_0$ is also homogeneous, we have the following correspondence.

$$\begin{tikzcd}
\boxed{\text{homogeneous } \mathcal{H}_0} \arrow[r, leftrightarrow, "\text{Fraïssé}"] \arrow[d, leftrightarrow, "+/\text{- a total order}"'] & \text{class } \mathcal{K}_0 \arrow[d, leftrightarrow, "+/\text{- total orders}"] \\
\boxed{\text{ordered homogeneous } \mathcal{H}} \arrow[r, leftrightarrow, "\text{Fraïssé}"] & \text{order class } \mathcal{K}
\end{tikzcd}$$

The horizontal arrows represent Fraïssé correspondence and the vertical arrows represent adding total orders to get an order class from a Fraïssé class of structures or, as in the setup, taking a reduct of an ordered homogeneous structure to get a homogeneous structure without the total order.

We have already mentioned the Proposition 2.3.3, which shows that in this setup the class $\mathcal{K}$ is a reasonable class with respect to $\mathcal{K}_0$. But Kechris *et al.* (2005) shows that there are further connections. We include the definitions of the topological dynamics concepts mentioned here in Section 2.3.

(i) If $\boldsymbol{\mathcal{K}}$ is a Ramsey class then the automorphism group $Aut(\boldsymbol{\mathcal{H}})$ is *extremely amenable*. This result was significant, as it meant that the authors used the known results about Ramsey classes to find new extremely amenable groups.

(ii) If $\boldsymbol{\mathcal{K}}$ is Ramsey and has the *ordering property*, then $\boldsymbol{\mathcal{K}}$ provides a way to calculate the universal minimal flow of $Aut(\boldsymbol{\mathcal{H}}_0)$. The authors also used the known results about Ramsey classes to calculate the universal minimal flow of various groups.

(iii) If $\boldsymbol{\mathcal{K}}$ is Ramsey, then there exists a class $\boldsymbol{\mathcal{K}}'$, such that

- $\boldsymbol{\mathcal{K}}'$ is a sub class of $\boldsymbol{\mathcal{K}}$, and
- $\boldsymbol{\mathcal{K}}'$ is reasonable, Ramsey and has the ordering property w.r.t. $\boldsymbol{\mathcal{K}}_0$.

(iv) Suppose $\boldsymbol{\mathcal{K}}'$ and $\boldsymbol{\mathcal{K}}''$ are both reasonable Ramsey order classes and have the ordering property with respect to class $\boldsymbol{\mathcal{K}}_0$. Then the classes $\boldsymbol{\mathcal{K}}'$ and $\boldsymbol{\mathcal{K}}''$ are simply bi-definable.

Result (i) provides additional motivation for finding Ramsey classes, while result (ii) ties the classes with the Ramsey and ordering properties to a result about the homogeneous structure, for example the homogeneous shaped partial orders mentioned in the beginning of this section. Further, result (iii) tells us that if we can find an ordered class that is Ramsey, we can also find one that also has the ordering property, while result (iv) tells us that such a class will be unique up to simple bidefinability.

Thus the aim of this thesis was to consider the classes $\boldsymbol{\mathcal{K}}$ of ordered shaped partial orders, and find, up to simple bidefinability, all of them that satisfy the following.

(i) The class $\boldsymbol{\mathcal{K}}$ is Ramsey.

(ii) The class $\boldsymbol{\mathcal{K}}$ is Fraïssé .

(iii) The class $\boldsymbol{\mathcal{K}}$ has the ordering property.

In the thesis we use different *skeletons* Σ to enumerate different classes of shaped partial orders. We denote a class of ordered shaped partial orders by $\boldsymbol{\mathcal{K}}(\Sigma, o)$ and the class of its reducts without the total orders by $\boldsymbol{\mathcal{K}}(\Sigma)$ and vice versa.

We use the good skeletons, defined in Torrezão de Sousa & Truss (2008), to enumerate Fraïssé classes of shaped partial orders $\boldsymbol{\mathcal{K}}(\Sigma)$. In Lemma 2.5.6 we find a criterion that specifies that in some cases there is no order class $\boldsymbol{\mathcal{K}}(\Sigma, o)$ that is Ramsey. In the main theorem in this thesis, Theorem 2.5.31, we show that for the rest of the classes $\boldsymbol{\mathcal{K}}(\Sigma)$, there exists a class $\boldsymbol{\mathcal{K}}(\Sigma', o)$ of ordered shaped partial orders, such that

(i) $\boldsymbol{\mathcal{K}}(\Sigma)$ and $\boldsymbol{\mathcal{K}}(\Sigma')$ are simply bi-definable,

(ii) $\boldsymbol{\mathcal{K}}(\Sigma', o)$ is a Ramsey class and is a reasonable class with respect to $\boldsymbol{\mathcal{K}}(\Sigma')$, and

(iii) in many cases, $\boldsymbol{\mathcal{K}}(\Sigma', o)$ has the ordering property.

The theorem is weaker than the result this thesis aimed for. To achieve the aim and complete the classification, a proof that specific classes of ordered shaped partial orders have the ordering property is needed.

1.2 Summary of the proof

Following this introduction, the thesis contains five chapters.

- Chapter 2 **Preliminaries** formally introduces the concepts and definitions mentioned in the previous section.

- Chapter 3 **Key Technical Lemmas** contains techniques used to show classes of shaped partial orders are Ramsey.

- Chapter 4 **Ramsey Results** proves that some of the classes of shaped partial orders are Ramsey.

- Chapter 5 **Correspondence** is a translation between the language in the classification of homogeneous shaped partial orders and language in the thesis, and contains various simple bi-definability results and results about the ordering property

- Chapter 6 **Conclusion** remarks on the gap between the aim of this thesis and the main result, as well as considers future topics to research.

We have included 'Links' throughout the thesis that look as follows.

Link. Definition 2.1.2

These signpost the reader to related concepts in the thesis, drawing parallels between related concepts and results that couldn't appear near each other in the text. The reader may want to read various chapters in parallel, in which case the links may be helpful. The reader may prefer to read the work linearly and ignore the links.

Before discussing the structure of the proof further, consider a similar pursuit of Fraïssé Ramsey classes with the ordering property in the case of ordered partial orders. Schmerl (1979), as stated in 2.1.10, classified homogeneous partial orders, which are

(i) an antichain A_n of any countable size n,

(ii) a countable chain C (isomorphic to $\mathbb{Q}$ with the natural order),

(iii) an antichain of chains AC_n containing any countable number $n > 1$ of countable chains,

(iv) a countable chain of antichains CA_n of any countable size $n > 1$, or

(v) a generic homogeneous partial order.

Schmerl's classification forms a basis for Sokić (2012a) and Sokić (2012b), which consider various order classes, that are reasonable with respect to Fraïssé classes of partial orders, and determines whether they are Fraïssé , Ramsey or have the ordering property. The subsection starting on page 31 lists all the classes of ordered partial orders considered. In summary, the Fraïssé classes of ordered partial orders that are Ramsey and have the ordering property, up to simple bidefinability, are the following.

(i) The class $\mathcal{K}(A_1, o)$ containing only the antichain with one point and the empty order, reasonable with respect to the class $\mathcal{K}(A_1)$.

(ii) The class $\mathcal{K}(A_{\aleph_0}, o)$ of all finite ordered antichains, reasonable with respect to the class $\mathcal{K}(A_{\aleph_0})$.

(iii) The class $\mathcal{K}(C, e)$ of all ordered chains, where $(P, <)$ and $(P, \prec)$ define the same total order for each $(P, <, \prec) \in \mathcal{K}(C, e)$, reasonable with respect to the class $\mathcal{K}(C)$.

(iv) The class $\mathcal{K}(AC_{\aleph_0}, ce)$ of all ordered antichains of chains with total orders convex on the chains of each structure and extending the partial order, reasonable with respect to the class $\mathcal{K}(AC_{\aleph_0})$.

(v) The class $\mathcal{K}(CA_{\aleph_0}, e)$ of all chains of antichains with total order extensions, reasonable with respect to the class $\mathcal{K}(CA_{\aleph_0})$.

(vi) The class $\mathcal{K}(G, e)$ of all partial orders with total order extensions, reasonable with respect to the class $\mathcal{K}(G)$.

As described in the previous section, the aim of this thesis was to obtain an analogous result about Fraïssé classes of shaped partial orders.

The classification of homogeneous shaped partial orders in Torrezão de Sousa & Truss (2008) is much lengthier than the classification of homogeneous partial orders. It introduces a *skeleton*, which is a partial order, together with the labels for points (G, AC or CA) and the labels for each partial order relation ($<_c$, $<_g$, $<_{sh}$, $<_{pm}$ or $<_{cpm}$). The authors then define a *good skeleton*, with conditions about the labels of the points and relations between them. They show that any homogeneous shaped partial order corresponds to a unique good skeleton, and that any good skeleton, together with a set of shapes and multiplicities for each point in the skeleton, defines a homogeneous shaped partial order. Thus the good skeletons enumerate the shaped homogeneous partial orders. More details about the classification are considered in Section 2.4.

The labels G, AC and CA refer to interdensely shaped components of a homogeneous shaped partial order. They are essentially shaped versions of the structures in the Schmerl classification, but the AC refers to both, antichains of chains and antichains, while CA encompasses chains of antichains as well as a chain. However, to facilitate the Ramsey property proofs, we use a different set of skeletons, introduced in the last part of Section 2.5. Chapter 3 contains the proofs of the lemmas used to show the Ramsey property of classes of ordered shaped partial orders in Chapter 4. And finally, in Chapter 5 we show that the classes discussed so far correspond to the Fraïssé classes precisely to the ordered classes of shaped partial orders, enumerated by the good skeletons. Section 5.2 also contains the ordering property proofs.

We finish this section by introducing the core ideas of the Chapter 3, as these are the methods used in this thesis to show that classes of ordered shaped partial orders are Ramsey.

Bi-definability

If the homogeneous stuctures $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{H}}'$ are simply bi-definable, they are essentially the same structure in different languages. Similarly, simply bi-definable classes $\boldsymbol{\mathcal{K}}$ and $\boldsymbol{\mathcal{K}}'$ represent essentially the same class of structures. More precisely, if $\boldsymbol{\mathcal{K}}$ is a class in language L and $\boldsymbol{\mathcal{K}}'$ is a class in language L', the relations in L' can be defined by simple formulas in language L (and the the ones in L by simple formulas in L'). Formally, this is defined in 3.1.2.

Simple bi-definability is important because it preserves the Ramsey property. Namely, if the classes $\boldsymbol{\mathcal{K}}$ and $\boldsymbol{\mathcal{K}}'$ are simply bi-definable, then $\boldsymbol{\mathcal{K}}$ has the Ramsey property if and only if $\boldsymbol{\mathcal{K}}'$ does, which we show in Lemma 3.1.6.

We consider an informal example, showing that the class of ordered antichains of chains and the class of ordered chains of antichains are simply bi-definable.

Example 1.2.1. Let $\boldsymbol{\mathcal{K}}$ in the language $L = \{<, \prec\}$ be the class $\boldsymbol{\mathcal{K}}(AC_{\aleph_0}, ce)$ of ordered antichains of chains, let $\boldsymbol{\mathcal{K}}'$ in the language $L =$

$\{<', \prec\}$ be the class $\boldsymbol{\mathcal{K}}(CA_{\aleph_0}, ce)$ of ordered chains of antichains, and let $\boldsymbol{\mathcal{K}}_0$ in the language $L = \{\prec\}$ be the class $\boldsymbol{\mathcal{K}}(C)$ of chains.

(i) Let $P = \langle P_0, < \rangle$ be an ordered antichain of chains with the total order P_0. Consider the structure $P' = \langle P_0, <' \rangle$, where $p_1 <' p_2$ if and only if

- $p_1 \prec p_2$, and
- we don't have $p_1 < p_2$.

Then P' is a chain of antichains; denote it by $\Phi(P)$.

(ii) Conversely, let $P' = \langle P_0, <' \rangle$ be an ordered chain of antichains with the total order P_0. Consider the structure $P = \langle P_0, < \rangle$, where $p_1 < p_2$ if and only if

- $p_1 \prec p_2$, and
- we don't have $p_1 <' p_2$.

Then P is an antichain of chains; denote it by $\Phi'(P')$.

Clearly, $\Phi'(\Phi(P)) = P$, and Φ creates a bijections between the expansions of P_0 in $\boldsymbol{\mathcal{K}}$ and those in $\boldsymbol{\mathcal{K}}'$, and similarly for Φ'.
The formula

$$\varphi\big(p_1, p_2\big) = (p_1 \prec p_2) \wedge \neg(p_1 < p_2)$$

then defines $<'$, and similarly

$$\varphi'\big(p_1, p_2\big) = (p_1 \prec p_2) \wedge \neg(p_1 <' p_2)$$

defines $<$.
A different way of reasoning about the situation could be to consider any chain P_0. If we partition it into convex pieces, then we can define $<$ and obtain an ordered antichain of chains P by forgetting the relations between different convex pieces. To define $<'$, forget the relations within each convex piece, and obtain a chain of antichains P'. Since the total order is still present in P and P', it is really the partition into convex pieces that defines

the obtained structure, so P and P', in a sense, represent the same structure in different languages.
This shows that the classes $\mathcal{K}(AC_{\aleph_0}, ce)$ and $\mathcal{K}(CA_{\aleph_0}, ce)$ are simply bi-definable, so the list of the Fraïssé Ramsey classes of ordered partial orders with the ordering property could be even shorter.

Structural Product Ramsey Lemma

Consider a class $\mathcal{K}$ of shaped antichains, where each point is shaped as either a circle or a diamond. If class $\mathcal{K}_1$ contains all circle-shaped antichains and $\mathcal{K}_2$ all diamond-shaped ones, the structures in $\mathcal{K}$ are precisely

(i) circle-shaped antichains $\mathcal{K}_1$,

(ii) diamond-shaped antichains $\mathcal{K}_2$, and

(iii) antichains containing circle-shaped and diamond-shaped points.

To form the class of structures in (iii), we take a structure in $\mathcal{K}_1$, one in $\mathcal{K}_2$, and combine their points to get a unique structure. So by understanding the structures in $\mathcal{K}_1$ and $\mathcal{K}_2$ we can understand the structures in $\mathcal{K}$. In Section 3.2 we consider the cases when that happens even when the classes $\mathcal{K}_1$ and $\mathcal{K}_2$ are more complicated, or when class $\mathcal{K}$ is defined by more than two classes.

We define a *product* of classes in 3.2.3 - in the example above, that would be a subclass of $\mathcal{K}$, containing precisely the structures in (iii), but a class can be a product of many classes. The product of classes, however, is not closed under substructures. Thus we introduce a *full product* of classes, defined in 3.2.7. It formalises the notion of a class $\mathcal{K}$ being defined by classes $\mathcal{K}_i$, $i \in [n]$.

Lemma 3.2.6 shows that if the classes $\mathcal{K}_1, \mathcal{K}_2, \dots \mathcal{K}_n$ are Ramsey classes, the product $\mathcal{K}$ is a Ramsey class as well. But, as in the example above, a product of classes is not necessarily closed under the substructures. For that reason Lemma 3.2.9 shows that if the classes $\mathcal{K}_1, \mathcal{K}_2, \dots \mathcal{K}_n$ are Ramsey classes, the full product $\mathcal{K}'$ is a Ramsey class as well.

The languages we use in definitions 3.2.3 and 3.2.7 are useful for the formal definitions of the concepts. Lemma 3.2.10 and its corollaries translate the Ramsey property results about full products to the results about *merge* classes, which are classes in the language of the ordered shaped partial orders used throughout the thesis. The main difference between a full product of classes and specific merge classes is that the languages L_i of classes $\boldsymbol{\mathcal{K}}_i$ are disjoint and all classes of ordered shaped partial orders contain relation symbols $<$ and $\prec$.

Blowup Lemma

The Blowup Lemma links the results about Ramsey classes of shaped ordered partial orders to the results about the Ramsey properties of their reducts. A *blowup* $\overline{\mathbf{P}}$ of a partial order P (or a shaped partial order $\mathbf{P}$) is a structure, obtained by replacing each point of a partial order P by a partial order, a *block*, containing one point of each shape at least. We denote its unshaped reduct by $\overline{P}$. Given structures $\mathbf{A}$ and $\mathbf{B}$, we consider the unshaped A and $\overline{B}$. When they lie in the class with Ramsey property, we can find a C, such that $C \to (\overline{B})^A_k$. The hard part of the Blowup Lemma is to show that $\overline{\mathbf{C}} \to (\mathbf{B})^{\mathbf{A}}_k$. We use the colouring of $\binom{\overline{\mathbf{C}}}{\mathbf{A}}$ to define a colouring of $\binom{\overline{C}}{A}$. We focus on the set $(\binom{\overline{C}}{A})$, of substructures of $\overline{C}$ with at most one point in each block of $\overline{C}$. Finding a monochromatic $\binom{\overline{B}}{A}$ then translates to finding a monochromatic $\binom{\mathbf{B}}{\mathbf{A}}$ under specific conditions.

Two Pass Lemma

Start with an antichain (Figure 2.2). If we replace each point of the antichain with a chain, we get an antichain of chains (Figure 2.4).

So to build an antichain of chains, we could first decide on the size of the antichain and then on the size of each of the chains. The Two Pass Lemma is useful in the cases that generalise this situation. If, in class $\boldsymbol{\mathcal{K}}$, each structure $\mathbf{A}$ can be viewed as a *quotient structure* $\mathbf{A}_q$ (for example, an antichain) together with *levels* (for example, chains), we can build a structure in the class $\boldsymbol{\mathcal{K}}$ by first choosing the quotient structure and then the levels. In the cases considered, the quotient structure contains an *index set* $\mathcal{I}^{\mathbf{A}}$, and for each $i \in \mathcal{I}^{\mathbf{A}}$ there is one level $\mathbf{A}_i$ of the structure.

Given structures $\mathbf{A}$ and $\mathbf{B}$, we first find a quotient structure $\mathbf{C}_q$ such that $\mathbf{C}_q \to (\mathbf{B}_q)^{\mathbf{A}_q}_k$. The $\mathbf{C}_q$ will be the quotient structure of our $\mathbf{C}$. So next we use the Ramsey properties of various less complex classes with the same levels as the structures in $\boldsymbol{\mathcal{K}}$ (*e.g.* the class of chains, or the class of antichains of a fixed number of chains in our example) to build the levels of $\mathbf{C}$.

We fix a colouring

$$c : \binom{\mathbf{C}}{\mathbf{A}} \to [k]$$

and consider nested substructures $\mathbf{C}^{(1)}, \ldots, \mathbf{C}^{(t)}$; one per each $\mathbf{A}'_q \in \binom{\mathbf{C}_q}{\mathbf{A}_q}$. We use the colours of some $\mathbf{A}' \in \binom{\mathbf{C}^{(i)}}{\mathbf{A}}$ for $i \in [t]$ to define a colouring

$$c' : \binom{\mathbf{C}_q}{\mathbf{A}_q} \to [k].$$

We finally show that the monochromatic $\binom{\mathbf{B}'_q}{\mathbf{A}_q}$ under c' yields a monochromatic $\binom{\mathbf{B}'}{\mathbf{A}}$ under c.

Chapter 2

Preliminaries

In this chapter we present the definitions, concepts and results from the literature that are relevant to the Ramsey classes of shaped partial orders. We present the results using notation compatible with the rest of this thesis. We also introduce classes of ordered shaped partial orders that will be considered in the thesis.

2.1 Homogeneous structures

Basic model theory definitions

We start with the relevant formal model theoretic definitions.

A *language* is a collection $L = \{R_i\}_{i \in I} \cup \{f_j\}_{j \in J} \cup \{c_k\}_{k \in K}$ of distinct *relation*, *function* and *constant* symbols. Each function and relation symbol has an associated number, called its *arity*. The arity $n(i)$ of each relation symbol R_i and the arity $m(j)$ of each function symbol f_j are positive integers. A *structure* for L is an object of the form

$$\mathbf{A} = \langle A, \{R_i^{\mathbf{A}}\}_{i \in I}, \{f_j^{\mathbf{A}}\}_{j \in J}, \{c_k^{\mathbf{A}}\}_{k \in K} \rangle,$$

where A is a non-empty set, called the *universe* of $\mathbf{A}$, $R_i^{\mathbf{A}}$ is a $n(i)$-ary relation on A, *i.e.*, $R_i^{\mathbf{A}} \subset A^{n(i)}$, $f_j^{\mathbf{A}}$ is an $m(j)$-ary function on A, *i.e.*, $f_j^{\mathbf{A}} : A^{m(j)} \to A$, and $c_k^{\mathbf{A}} \in A$.

Given two structures $\mathbf{A}$ and $\mathbf{B}$ in the same language L, an *isomorphism* of $\mathbf{A}$ to $\mathbf{B}$ is a bijective map $\theta : A \to B$, such that

$$R_i^{\mathbf{A}}(a_1, \ldots, a_{n(i)}) \iff R_i^{\mathbf{B}}(\theta(a_1), \ldots, \theta(a_{n(i)}))$$

and

$$\theta(f_j^{\mathbf{A}}(a_1, \ldots, a_{m(j)})) = f_j^{\mathbf{B}}(\theta(a_1), \ldots, \theta(a_{m(j)}))$$

and

$$\theta\left(c_k^{\mathbf{A}}\right) = c_k^{\mathbf{B}}$$

for all $i \in I, j \in J$ and $k \in K$. We write $\theta : \mathbf{A} \to \mathbf{B}$ if that is the case.

A substructure $\mathbf{B}$ of $\mathbf{A}$ has as a universe a non-empty subset $B \subset A$ closed under each $f_j^{\mathbf{A}}$, containing all the $c_k^{\mathbf{A}}$ and satisfying $R_i^{\mathbf{B}} = R_i^{\mathbf{A}} \cap B^{n(i)}$, $f_j^{\mathbf{B}} = f_j^{\mathbf{A}} | B^{m(j)}$. We write $\mathbf{B} \trianglelefteq \mathbf{A}$ to denote that $\mathbf{B}$ is a substructure of $\mathbf{A}$. Note that this is not standard notation.

Suppose that $\mathbf{B}' \trianglelefteq \mathbf{A}$ and $\theta : B \to A$, $\theta(B) = B'$ is a map, that defines an isomorphism when its range is restricted to its image. Then we say that θ is an *embedding* of $\mathbf{B}$ into $\mathbf{A}$ and we write $\theta : \mathbf{B} \to \mathbf{A}$.

Example 2.1.1. A partial order is a structure in a language $L = \{<\}$, with a relation $<$ of arity 2. We will denote a partial order with a universe P as $P = (P, <)$, abbreviating the formal notation $\mathbf{P} = \langle P, <^{\mathbf{P}} \rangle$. The language only contains one relation symbol and no function or constant symbols. A structure P is a partial order if $<$ is irreflexive, antisymmetric and transitive.
Given any subset R of P, the partial order on R induced from the partial order on P defines a partial order; a substructure of P on R.

Note. Let $(P, <)$ be a partial order. If $p \in P$, and neither $p < q$ nor $q < p$ holds, we write $||$.
Let P_1, P_2 be disjoint subsets of P. We write $P_1 < P_2$ if for all $p_1 \in P_1, p_2 \in P_2$ we have $p_1 < p_2$.

In the thesis we will consider structures in different languages. Of particular interest are the structures related in the following way.

Definition 2.1.2. Let L be a language and $\mathbf{A}$ a structure in language L. Let L' be a subset of L and $\mathbf{A}' = \mathbf{A}|_{L'}$, that is, a structure in language L' with universe A that agrees with $\mathbf{A}$ on all relations, functions and constants in L'. Then we call $\mathbf{A}'$ a *reduct* of $\mathbf{A}$. Conversely, we say that $\mathbf{A}$ is an *expansion* of $\mathbf{A}'$.

Fraïssé correspondence

The focus of this thesis is on the classes of finite structures. But there is a correspondence between specific classes of structures and specific countable structures. In this section we present how the two are related.

Definition 2.1.3. A class $\mathcal{K}$ of finite structures:

- is *hereditary* if it is closed under substructures (*i.e.*, for any $\mathbf{A} \in \mathcal{K}$ and $\mathbf{B} \trianglelefteq \mathbf{A}$ we have $\mathbf{B} \in \mathcal{K}$),
- satisfies the *joint embedding property* if for any $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ there is a $\mathbf{C} \in \mathcal{K}$ with $\mathbf{A} \trianglelefteq C$ and $\mathbf{B} \trianglelefteq C$, and
- satisfies the *amalgamation property* if for any embeddings $f_1 : \mathbf{A} \to \mathbf{B}_1$ and $f_2 : \mathbf{A} \to \mathbf{B}_2$, where $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{K}$, there is a $\mathbf{C} \in \mathcal{K}$ and embeddings $g_1 : \mathbf{B}_1 \to \mathbf{C}$ and $g_2 : \mathbf{B}_2 \to \mathbf{C}$, such that $g_1 \circ f_1 = g_2 \circ f_2$.

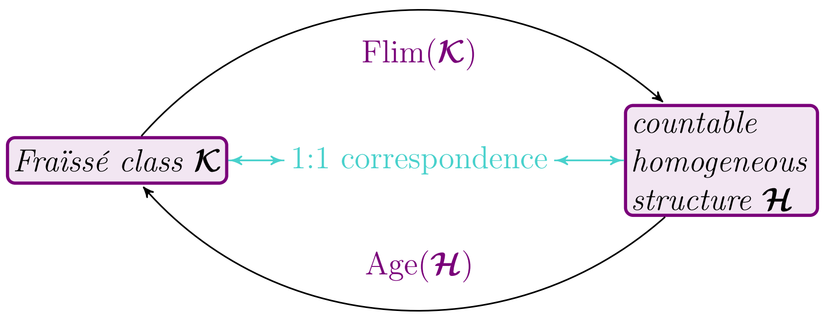
A class $\mathcal{K}$ is a *Fraïssé class* if it is hereditary and satisfies the joint embedding and amalgamation properties.

Note that since $\mathcal{K}$ contains only finite structures, it contains only countably many structures up to isomorphism.

On the side of the structures, we consider the following.

Definition 2.1.4. A countable structure $\mathcal{H}$ is *homogeneous* if every isomorphism between finite substructures of $\mathcal{H}$ extends to an automorphism of $\mathcal{H}$.

Figure 2.1: Fraïssé correspondence



Homogeneous structures have been studied extensively, and an overview of the results can be found in the survey MacPherson (2011).

To link homogeneous structures to classes of finite structures we introduce the following.

Definition 2.1.5. The *age* of a structure $\mathcal{H}$ is the class $\mathcal{K}$ of all finitely generated structures that can be embedded in $\mathcal{H}$. We write $\mathcal{K} = \mathrm{Age}(\mathcal{H})$.

It can be shown that the age of any countably infinite homogeneous structure is a Fraïssé class, giving the correspondence. This was proved in Fraïssé (1954). It can commonly be found in model theory books, including Hodges (1997).

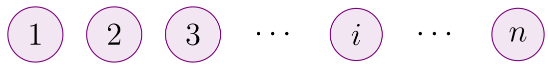
Theorem 2.1.6 (Fraïssé's Theorem)**.** *A class $\mathcal{K}$ is an age of a homogeneous structure $\mathcal{H}$ if and only if the class $\mathcal{K}$ is a Fraïssé class. Furthermore, any two countably infinite homogeneous structures with the same age are isomorphic.*

This theorem provides another link between the Fraïssé classes of finite structures and homogeneous structures.

Definition 2.1.7. Given a Fraïssé class $\mathcal{K}$, a countable homogeneous structure $\mathcal{H}$, such that $\mathcal{K}$ is the age of $\mathcal{H}$, is called a *Fraïssé limit of* $\mathcal{K}$.

By Theorem 2.1.6 the structure $\mathcal{H}$ exists and is unique up to isomorphism.

Figure 2.2: Antichain of size n



Homogeneous partial orders

Classification of homogeneous structures presents an intriguing challenge for mathematicians. An example, particularly relevant to this thesis, is Schmerl's classification of the homogeneous partial orders in Schmerl (1979). It is, perhaps, more intuitive to start by considering their ages, that is, Fraïssé classes of partial orders.

While we usually use the bold letters $\boldsymbol{\mathcal{K}}$ and $\boldsymbol{\mathcal{H}}$ to denote a class and a homogeneous structure, we will use the light letters $\mathcal{K}$ and $\mathcal{H}$ when referring to partial orders in particular. This will help us distinguish between partial orders and *shaped partial orders*, the focus of this thesis.

Throughout this thesis, we will be using the following notation for a disjoint union of sets.

Definition 2.1.8. Given a set X and a collection of sets $\{Y_x : x \in X\}$, the *disjoint union* $X \rtimes Y$ *of sets* $\{Y_x : x \in X\}$ is the set

$$X \rtimes Y = \bigcup_{x \in X} \{x\} \times Y_x.$$

Defined below are certain kinds of partial orders $(P, <)$, Fraïssé classes of partial orders and the corresponding Fraïssé limits.

Definition 2.1.9. Denote by $[n]$ the set $\{1, 2, \ldots, n\}$ of the first n positive integers.

(i) (a) An *antichain* of size n is a partial order on n points, in which all the pairs of distinct points are incomparable. That is,

$$P = \{p_i : i \in [n]\}$$

Figure 2.3: Chain of size n

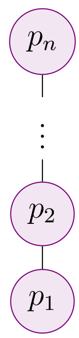


Figure 2.4: Antichain of n chains

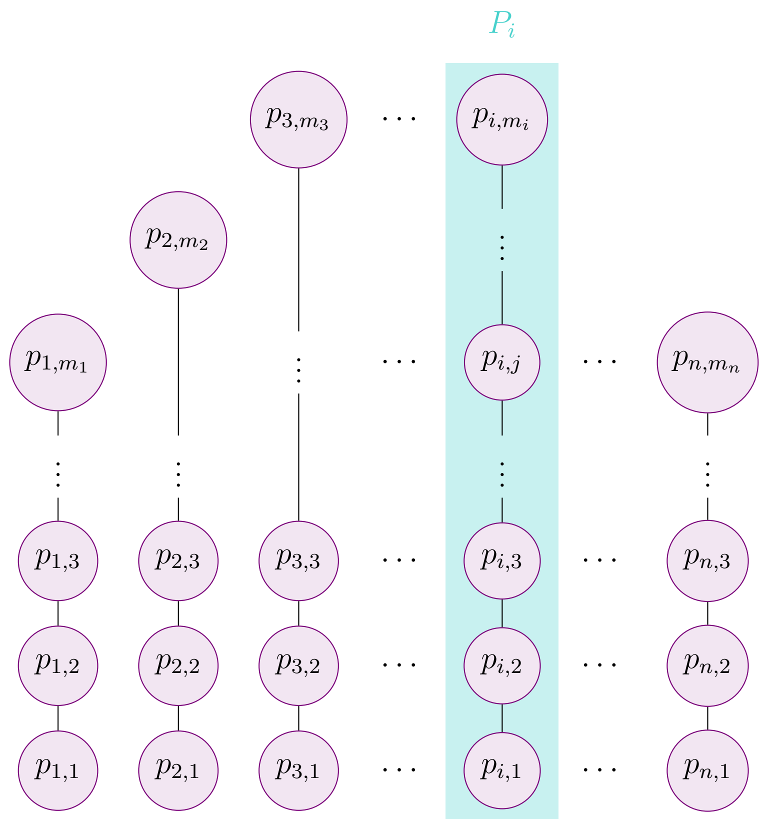
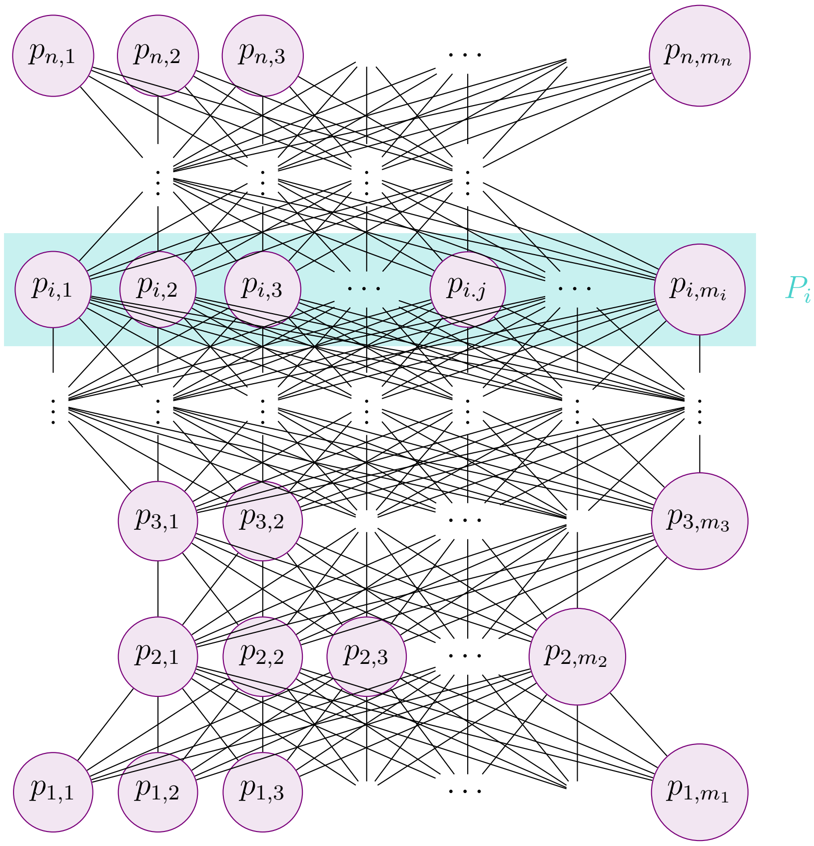


Figure 2.5: Chain of antichains



and $p_i || p_{i'}$ for all distinct $i, i' \in [n]$. See Figure 2.2.

(b) For $1 \leq n < \aleph_0$, let $\mathcal{K}(A_n)$ denote the class of all antichains of size at most n, and $\mathcal{K}(A_{\aleph_0})$ denote the class of all finite antichains.

(c) The corresponding Fraïssé limits $\mathcal{H}(A_n)$, for all $1 \leq n \leq \aleph_0$, are antichains of cardinality n.

(ii) (a) A *chain* of size n is a total or linear order of n points. That is, we can label its points as $P = \{p_i : i \in [n]\}$ such that $p_i < p_{i'}$ if and only if $i < i'$. So P is isomorphic to $[n]$ with its natural order. See Figure 2.3.

(b) The class $\mathcal{K}(C)$ consists of all finite chains.

(c) The Fraïssé limit of $\mathcal{K}(C)$ is denoted by $\mathcal{H}(C)$, a countably infinite homogeneous chain, isomorphic to $(\mathbb{Q}, <)$ with the natural order.

(iii) (a) An *antichain of n chains* consists of n disjoint incomparable chains. That is, there is a partition $\{P_i : i \in [n]\}$ of the set of points P, such that each $P_i = \{p_{i,j} : j \in [m_i]\}$ is a chain. We have $p_{i,j} < p_{i',j'}$ if and only if $i = i'$ and $j < j'$.
Let $M = \{[m_i] : i \in [n]\}$. We write

$$P = \{p_{i,j} : (i, j) \in [n] \rtimes M\}.$$

See Figure 2.4.

(b) The class $\mathcal{K}(AC_n)$ is the class of all finite structures which are antichains of at most n chains, and the class $\mathcal{K}(AC_{\aleph_0})$ is the class of all finite antichains of chains.

(c) The Fraïssé limit of $\mathcal{K}(AC_n)$ is denoted by $\mathcal{H}(AC_n)$. It consists of an antichain of n chains, for $2 \leq n \leq \aleph_0$, with each chain isomorphic to $(\mathbb{Q}, <)$.

(iv) (a) A *chain of antichains* of size at most m consists of a linearly ordered set of disjoint antichains, each of size at most m. That is, there is a partition $\{P_i : i \in [n]\}$ of the set of points P, such that each $P_i = \{p_{i,j} : j \in [m_i]\}$ is an antichain and $1 \leq m_i \leq m$. We have $p_{i,j} < p_{i',j'}$ if and only if $i < i'$. An infinite chain of antichains is defined similarly, but we might have $\{P_i : i \in \aleph_0\}$ or $P_i = \{p_{i,j} : j \in \aleph_0\}$ for some i. Again we write

$$P = \{p_{i,j} : (i, j) \in [n] \rtimes [m]\}.$$

See Figure 2.5.

(b) The class $\mathcal{K}(CA_m)$ is the class of all finite structures which are chains of antichains of size at most m, and the class $\mathcal{K}(CA_{\aleph_0})$ is the class of all finite chains of antichains.

(c) The Fraïssé limit of $\mathcal{K}(CA_m)$ is $\mathcal{H}(CA_m)$, the chain of antichains, with each antichain of size m, for $2 \leq m \leq \aleph_0$.

(v) (a) We denote by $\mathcal{K}(G)$ the class of all finite partial orders.

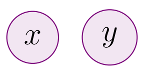
(b) Its Fraïssé limit is $\mathcal{H}(G)$, the generic partial order.

Note. (i) By definition of an antichain of chains, a structure containing only one chain is an antichain of chains. However, we did not list $\mathcal{K}(AC_1)$ as a class of antichains of chains to avoid the clash with $\mathcal{K}(C)$. The same applies to a chain of antichains of size 1 and $\mathcal{K}(CA_1)$.

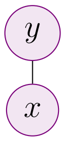
(ii) We can build the structure $\mathcal{K}(CA_m)$ by replacing each point of $\mathbb{Q}$ with an antichain of size m.

Schmerl, by classifying countable homogeneous partial orders, shows that the classes $\mathcal{K}(A_n), \mathcal{K}(A_{\aleph_0}), \mathcal{K}(C), \mathcal{K}(AC_n), \mathcal{K}(AC_{\aleph_0}), \mathcal{K}(CA_n), \mathcal{K}(CA_{\aleph_0})$ and $\mathcal{K}(G)$ above are indeed Fraïssé classes. He classified them by considering the following structures.

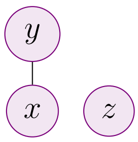
(i) An incomparable pair, $x||y$.



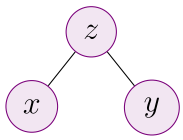
(ii) A comparable pair, $x < y$.



(iii) An L-shape, namely a triplet of points with $x < y$, $x||z$ and $y||z$.



(iv) A Λ-shape, namely a triplet of points with $x||y$, $x < z$ and $y < z$.



He then considers the following classes.

(i) Partial orders that do not contain a comparable pair as a substructure.

(ii) Partial orders that do not contain an incomparable pair as a substructure.

(iii) Partial orders that contain an L-shape but do not contain a Λ-shape.

(iv) Partial orders that contain a Λ-shape but do not contain an L-shape.

(v) Partial orders that contain both, an L-shape and a Λ-shape.

He shows that if $\mathcal{H}$ is a homogeneous partial order, then it satisfies one of the conditions (i)-(v), and that the list of conditions is exhaustive. It is clear that any partial order satisfying the condition (i) is an antichain, and Schmerl shows that any countable antichain is homogeneous. Similarly, a partial order satisfying (ii) must be a chain, and the only countable homogeneous chains are either the trivial chain with one element or a linear order isomorphic to $(\mathbb{Q}, <)$. Schmerl then shows that a homogeneous structure not satisfying conditions (i) and (ii) must contain an L-shape or a Λ-shape as a substructure, and thus satisfy one of the conditions (iii)-(v). The case (iii) yields the homogeneous antichains of chains, case (iv) chains of antichains, and case (v) the generic homogeneous partial order. He proves the following.

Theorem 2.1.10. *The homogeneous partial orders are:*

(i) *an antichain* $\mathcal{H}(A_n)$ *of cardinality* n*, for all* $1 \leq n \leq \aleph_0$*,*

(ii) *a countably infinite chain* $\mathcal{H}(C)$*,*

(iii) *an antichain of* n *chains* $\mathcal{H}(AC_n)$*, for all* $2 \leq n \leq \aleph_0$*,*

(iv) *a chain of antichains of size* n $\mathcal{H}(CA_n)$*, for all* $2 \leq n \leq \aleph_0$*,*

(v) *and a generic partial order* $\mathcal{H}(G)$*.*

Note. The structure $\mathcal{H}(C)$ could be viewed as both, $\mathcal{H}(AC_1)$ and $\mathcal{H}(CA_1)$. In different contexts, it might be classed as one or the other rather than considered separately.

2.2 Introduction to structural Ramsey theory

Structural Ramsey theory stems from the classical Ramsey theorem and extends the concept to whole classes of structures. We will introduce the necessary notation first.

We denote by $\binom{\mathbf{B}}{\mathbf{A}}$ the set of all substructures of $\mathbf{B}$ isomorphic to $\mathbf{A}$.

Ramsey arguments focus on finding, for structures $\mathbf{A}$ and $\mathbf{B}$ and an integer $k > 1$, a structure $\mathbf{C}$, such that given any colouring

$$c : \binom{\mathbf{C}}{\mathbf{A}} \to [k]$$

there is a substructure $\mathbf{B}' \in \binom{\mathbf{C}}{\mathbf{B}}$, such that $\binom{\mathbf{B}'}{\mathbf{A}}$ is *monochromatic*; that is, there is an $l \in [k]$, such that

$$c : \binom{\mathbf{B}'}{\mathbf{A}} \to \{l\},$$

or in other words, the colouring c is constant on $\binom{\mathbf{B}'}{\mathbf{A}}$.

Definition 2.2.1 (Erdős-Rado notation)**.** We write

$$\mathbf{C} \to (\mathbf{B})^{\mathbf{A}}_k$$

if for any k-colouring of substructures of $\mathbf{C}$ isomorphic to $\mathbf{A}$ there is a substructure $\mathbf{B}'$ of $\mathbf{C}$ isomorphic to $\mathbf{B}$, such that all substructures of $\mathbf{B}'$ isomorphic to $\mathbf{A}$ are of the same colour.

Classical Ramsey Theorem

Ramsey theory stems from the Classical Ramsey Theorem, a theorem proven by Frank Plumpton Ramsey in Ramsey (1930).

Theorem 2.2.2 (Classical Ramsey Theorem)**.** *For any three positive integers q, r and k, there is a number p, such that for sets A, B and C of sizes q, r, and p respectively and any colouring*

$$c : \binom{C}{A} \to [k]$$

there is a subset $B' \subset C$ of size r, such that $\binom{B'}{A}$ is monochromatic. In terms of sizes we write

$$p \to (r)^q_k.$$

In other words, Theorem 2.2.2 states that the class of all finite sets is Ramsey.

As well as mentioning the Classical Ramsey Theorem due to its historical importance, we will see in the next section that it can be used to show that certain classes of partial orders are Ramsey.

Definition of a Ramsey class

Structural Ramsey theory focuses on whole classes of structures, rather than randomly picking $\mathbf{A}$ and $\mathbf{B}$ and looking for a $\mathbf{C}$ such that $\mathbf{C} \to (\mathbf{B})^{\mathbf{A}}_k$. It is a study of classes $\mathcal{K}$, in which a $\mathbf{C}$ exists for any choice of $\mathbf{A}$ and $\mathbf{B}$.

Definition 2.2.3. Given a class $\mathcal{K}$ and a structure $\mathbf{A} \in \mathcal{K}$, we say that a class $\mathcal{K}$ is $\mathbf{A}$-*Ramsey*, if given any $\mathbf{B} \in \mathcal{K}$ there is a $\mathbf{C} \in \mathcal{K}$ such that $\mathbf{C} \to (\mathbf{B})^{\mathbf{A}}_k$.
We say that a class $\mathcal{K}$ is *Ramsey*, if it is $\mathbf{A}$-Ramsey for all $\mathbf{A} \in \mathcal{K}$.

Structural Ramsey theory and Fraïssé classes

Structural Ramsey theory was initially studied by Nešetřil and Rödl. In the paper Nešetřil & Rödl (1977) they prove that there is a link between Ramsey classes and Fraïssé classes. In particular, there is a link between Ramsey classes of *rigid* structures, *i.e.*, ones that have no non-trivial automorphisms, and Fraïssé classes. The following result was first proved in Lemma 1 on page 294 of Nešetřil & Rödl (1977), but is also mentioned on page 20 of Kechris *et al.* (2005) in a language more compatible with this thesis.

Theorem 2.2.4. *Let $\mathcal{K}$ be a class of finite rigid structures. If $\mathcal{K}$ is a Ramsey class, hereditary, and has the joint embedding property, then $\mathcal{K}$ has the amalgamation property.*

So any Ramsey class of finite rigid structures, which is hereditary, has the joint embedding property and contains only countably many structures up to isomorphism, is a Fraïssé class.

We have seen a classification of homogeneous partial orders in the first section of this introduction and will see an overview of other classified homogeneous structures in the third section. Suppose now that we are given a classification of specific homogeneous structures and that the corresponding Fraïssé classes consist of rigid structures. Then one can classify the corresponding Ramsey classes by checking whether the Fraïssé classes are Ramsey or not.

However, often, the corresponding Fraïssé classes contain structures that are not rigid. For example, even the antichain $(P, <)$ of size 2 is not rigid. If it consists of points p_1 and p_2, then aside from the trivial automorphism, the map sending p_1 to p_2 and vice versa is an automorphism as well, so $(P, <)$ is indeed not rigid. We will consider what happens when a class contains structures that are not rigid and how one can expand a class to a class of rigid structures by adding a total order relation to the language.

Only some classes of partial orders are Ramsey

What about classes of partial orders encountered in Definition 2.1.9? Consider first the classes $\mathcal{K}(A_n)$. First, any element of $\mathcal{K}(A_1)$ is the partial order with only one element, unique up to isomorphism. So $\mathcal{K}(A_1)$ is trivially Ramsey.

Let $(P, <)$ be an antichain of size p and $(R, <)$ an antichain of size r. Since the partial order relation on any antichain is empty, the substructures of $(P, <)$ isomorphic to $(R, <)$ correspond precisely to subsets of P of size r. So to consider the rest of $\mathcal{K}(A_n)$, we will turn to the Classical Ramsey Theorem.

Lemma 2.2.5. *In the Classical Ramsey Theorem, we have:*

(i) $q \to (q)^q_k$

(ii) If $p \to (r)^q_k$, $k > 1$ *and* $r > q \geq 1$, *then* $p > r$.

Proof. First, given sets A and B of sizes q and r respectively, if there exists a set C of size p such that $C \to (B)^A_k$, we must have that $p \geq r$ for the set C to even contain a subset B' of size r.

If we have $q = r$, we can see that a set C of size $p = q = r$ satisfies $C \to (B)^A_k$, and hence that we have $q \to (q)^q_k$. Indeed, if A, B and C are of the same size, then $\binom{C}{A}$, $\binom{C}{B}$ and $\binom{B}{A}$ are each of size 1 and hence trivially monochromatic, which concludes the proof of (i).

However, as soon as we have $r > q \geq 1$, we have $|\binom{B}{A}| > 1$, so taking a C of size r would not work. Indeed, given a C of size r, we have $|\binom{C}{A}| > 1$, so we can colour one of the structures in $\binom{C}{A}$ with colour 1, and the rest with colour

2. Then the unique subset B' of C of size r, namely C itself, has substructures of two different colours and hence we have $C \nrightarrow (B)^A_k$. Since we have $p \geq r$ and $p \neq r$, we have $p > r$. This yields (ii).

□

Example 2.2.6. To show that $\mathcal{K}(A_n)$ is not a Ramsey class for any $1 < n < \aleph_0$, take $(R, <)$ to be an antichain of size n and $(Q, <)$ one of size q with $n > q \geq 1$. Clearly $(R, <), (Q, <) \in \mathcal{K}(A_n)$. Then by the Classical Ramsey Theorem and Lemma 2.2.5, if $p \rightarrow (n)^q_k$ and $k > 1$, then $p > n$. So for any $(P, <) \in \mathcal{K}(A_n)$, we must have $P \nrightarrow (R)^Q_k$, as P is an antichain of size at most n.

On the other hand, the Classical Ramsey Theorem shows that $\mathcal{K}(A_{\aleph_0})$ is a Ramsey class.

Example 2.2.7. Given antichains $(Q, <)$ and $(R, <)$ of sizes q and r respectively, find p such that $p \rightarrow (r)^q_k$ using Classical Ramsey Theorem.
Let $(P, <)$ be an antichain of size p. Then for any subset Q' of P of size q, the substructure of P on the set Q' is isomorphic to Q, and similarly for any subset R' of P of size r.
The fact that $P \rightarrow (R)^Q_k$ follows trivially.

So far we have shown that $\mathcal{K}(A_n)$ is a Ramsey class if and only if $n = 1$ or $n = \aleph_0$. What about the class $\mathcal{K}(G)$ of all finite partial orders? We will show that $\mathcal{K}(G)$ is not a Ramsey class.

Example 2.2.8. Let Q be a partial order with three points, q_1, q_2, q_3, where $q_2 < q_3$, and R be a partial on four points r_1, r_2, r_3, r_4, where $r_1 < r_2$ and $r_3 < r_4$, and there are no non-specified comparable pairs. That is, Q looks like $. \;\; |$ and R like $| \;\; |$. Let P be any finite partial order, with points $\{p_i : i \in I\}$. Then $P, Q, R \in \mathcal{K}(G)$.
We will find a colouring of $\binom{P}{Q}$ that yields no monochromatic $\binom{R}{Q}$, showing there is no P such that

$$P \rightarrow (R)^Q_k.$$

Indeed, pick a total order on the points of P, for example, $p_i \prec p_j$ if $i < j$. Let $e : Q \to P$ be an embedding. Colour $e(Q)$:

(i) red, if $e(q_1) \prec e(q_2)$, and

(ii) blue, if $e(q_2) \prec e(q_1)$.

Then given any $R' \in \binom{P}{R}$, let $e'(R) = R'$ for some embedding e'. Suppose, without loss of generality, that $e'(r_1) \prec e'(r_3)$. We have

(i) $e_1 : Q \to P, \quad q_1 \mapsto e'(r_1), q_2 \mapsto e'(r_3), q_3 \mapsto e'(r_4)$

(ii) $e_2 : Q \to P, \quad q_1 \mapsto e'(r_3), q_2 \mapsto e'(r_1), q_3 \mapsto e'(r_2)$

So $e_1(Q), e_2(Q) \in \binom{R'}{Q}$, but $e_1(Q)$ is red, and $e_2(Q)$ is blue.
As this is true for any $P \in \mathcal{K}(G)$, the class $\mathcal{K}(G)$ is not Q-Ramsey, and hence not Ramsey.

Order classes

Suppose that $\boldsymbol{\mathcal{K}}_0$ is a class of structures in a language L_0. We can extend the language L_0 to a new language L and consider expansions of members of $\boldsymbol{\mathcal{K}}_0$ to L to obtain a new class $\boldsymbol{\mathcal{K}}$. We have $\boldsymbol{\mathcal{K}}_0 = \boldsymbol{\mathcal{K}}|L_0$, *i.e.*, $\boldsymbol{\mathcal{K}}_0$ consists of reducts of the structures in $\boldsymbol{\mathcal{K}}$. Consider in particular the case where we extend L_0 to L by adding a binary relation symbol $\prec$. We will consider classes in which $\prec$ is a linear order on each of the structures in the class.

Definition 2.2.9. Suppose that L is a language containing a binary relation symbol $\prec$. An *order structure* $\mathbf{A}$ *for* $\prec$ is a structure $\mathbf{A}$ in language L for which $\prec^{\mathbf{A}}$ is a linear ordering. An *order class* $\boldsymbol{\mathcal{K}}$ *for* $\prec$ is one for which all $\mathbf{A} \in \boldsymbol{\mathcal{K}}$ are order structures for $\prec$.

Suppose $\boldsymbol{\mathcal{K}}$ is an order class of finite structures. As $\prec$ is a linear ordering on any $\mathbf{A} \in \boldsymbol{\mathcal{K}}$, the structure $\mathbf{A}$ is rigid, as any automorphism of $\mathbf{A}$ has to respect the linear order on $\mathbf{A}$ and thus must be trivial.

When expanding a class to an order class, we will also insist that it contain enough ordered structures to preserve information about the substructures. We define that notion precisely.

Definition 2.2.10. Let L be a language containing relation $\prec$ and let $L_0 = L \setminus \{\prec\}$. Let $\mathcal{K}$ be an order class for $\prec$ in language L and let $\mathcal{K}_0 = \mathcal{K}|L_0$ be a class of reducts of structures in the class $\mathcal{K}$ in language L_0. The class $\mathcal{K}$ is *reasonable with respect to* $\mathcal{K}_0$ if for every $\mathbf{A}_0, \mathbf{B}_0 \in \mathcal{K}_0$, every embedding $e : \mathbf{A}_0 \to \mathbf{B}_0$, and linear ordering $\prec$ on $\mathbf{A}_0$ with $\mathbf{A} = \langle \mathbf{A}_0, \prec \rangle \in \mathcal{K}$, there is a linear ordering $\prec'$ on $\mathbf{B}_0$, so that $\mathbf{B} = \langle \mathbf{B}_0, \prec' \rangle \in \mathcal{K}$ and $e : \mathbf{A} \to \mathbf{B}$ is also an embedding (*i.e.*, $a \prec b \iff e(a) \prec' e(b)$).

Note. Given a class $\mathcal{K}_0$, we can obtain a reasonable class $\mathcal{K}$ with respect to $\mathcal{K}_0$. Indeed, given $\mathbf{A}_0 \in \mathcal{K}_0$ with universe A, add $\langle \mathbf{A}_0, \prec \rangle$ to $\mathcal{K}$ for each total order $(A, \prec)$. Then given an embedding $e : \mathbf{A}_0 \to \mathbf{B}_0$ and $\mathbf{A} = \langle \mathbf{A}_0, \prec \rangle \in \mathcal{K}$, extend the total order on a part of B, induced from the total order on A and the embedding e, to a total order on B. The structure obtained lies in $\mathcal{K}$ by its definition. Further, $e : \mathbf{A} \to \mathbf{B}$ is an embedding by the definition of the total order on $\mathbf{B}$. So $\mathcal{K}$ is a reasonable class with respect to $\mathcal{K}_0$.

We will outline how the results about order classes relate to the results about the classes of their reducts and corresponding homogeneous structures in Section 1.4. We first state the results about Ramsey classes of ordered partial orders.

Classes of ordered partial orders

In Sokić (2012a) and Sokić (2012b), Sokić considers classes of finite partial orders with arbitrary linear orderings, linear orderings that are linear extensions of the partial ordering, and linear orderings of antichains of chains that are convex on each of the chains. In his papers, $\mathcal{K}^{A_n}$ corresponds to our $\mathcal{K}(A_n)$, $\mathcal{K}^{B_n}$ to $\mathcal{K}(C)$ and $\mathcal{K}(AC_n)$, $\mathcal{K}^{C_n}$ to $\mathcal{K}(CA_n)$ and $\mathcal{K}^D$ to $\mathcal{K}(G)$. The classes of partial orders are classes of structures in the language $L_0 = \{<\}$, but we extend them to classes in language $L = \{<, \prec\}$.

Definition 2.2.11. Given a set P, denote the collection of all linear orderings on the set P by $lo(P)$. Then define the classes of *partial orders with arbitrary linear orderings* as follows.

$$\mathcal{K}(\sigma, o) = \{(P, <, \prec) : (P, <) \in \mathcal{K}(\sigma), \prec \in lo(P)\},$$

for

$$\sigma \in \{A_n\}_{1 \leq n \leq \aleph_0} \cup \{C\} \cup \{AC_n\}_{1 < n \leq \aleph_0} \cup \{CA_n\}_{1 < n \leq \aleph_0} \cup \{G\}.$$

While the classes in Definition 2.2.11 are clearly reasonable order classes, they do not take into account the partial order structure. The classes in the following definition do.

Definition 2.2.12. Given a partial order $(P, <)$, a linear order $\prec$ is an extension of $<$ if for all $p, q \in P$ we have

$$p < q \Rightarrow p \prec q.$$

Denote the collection of all linear orderings on the set P that are extensions of the partial order $(P, <)$ by $le(P, <)$. Then define the classes of *partial orders with linear extensions* as follows.

$$\mathcal{K}(\sigma, e) = \{(P, <, \prec) : (P, <) \in \mathcal{K}(\sigma), \prec \in le(P, <)\},$$

for

$$\sigma \in \{A_n\}_{1 \leq n \leq \aleph_0} \cup \{C\} \cup \{AC_n\}_{1 < n \leq \aleph_0} \cup \{CA_n\}_{1 < n \leq \aleph_0} \cup \{G\}.$$

Note. $\mathcal{K}(A_n, e) = \mathcal{K}(A_n, o)$ for all $1 \leq n \leq \aleph_0$.
Besides, recall that the points of a chain of antichains $P \in \mathcal{K}(CA_n)$ can be labelled $P = \{p_{i,j} : (i, j) \in [n] \rtimes [m]\}$, as in Definition 2.1.9. Then one can easily show that picking a linear extension $(P, \prec)$ of the partial order $(P, <)$ corresponds precisely to picking a total order on each of the maximal

antichains P_i of P.

The classes of chains of antichains in Definition 2.2.12 are also convex on the maximal antichains. Namely, if $p, q \in P$ lie in one of the maximal antichains P_i of P, and there is a point $q \in P$, such that

$$p \prec q \prec r,$$

then the point q also lies in P_i. In fact, given any two maximal antichains in a chain of antichains, one of them is completely below the other in the partial order, and thus in the total order as well. So to ensure the same for maximal chains in the classes of antichains of chains, the following total order is considered.

Definition 2.2.13. Define the classes of *partial orders with convex linear extensions* as follows.

(i) $\mathcal{K}(AC_n, co)$ is the class of structures $(P, <, \prec) \in \mathcal{K}(AC_n, o)$ such that for all $p, q, r \in P$ we have

$$(p < q \text{ or } p > q),\, p \prec r \prec q \Rightarrow (p < r < q \text{ or } p > r > q)$$

(ii) $\mathcal{K}(AC_n, ce)$ is the class of structures $(P, <, \prec) \in \mathcal{K}(AC_n, e)$ such that for all $p, q, r \in P$ we have

$$p < q, p \prec r \prec q \Rightarrow p < r < q$$

The convex extensions induce a total order on the set of maximal chains of an antichain of chains. That is, given any two maximal chains of an antichain of chains, one of them is completely below the other in the total order $\prec$.

Note. We have

(i) $\mathcal{K}(\sigma, o)|L_0 = \mathcal{K}(\sigma, e)|L_0 = \mathcal{K}(\sigma)$ for
$\sigma \in \{A_n\}_{1\leq n\leq\aleph_0} \cup \{C\} \cup \{AC_n\}_{1<n\leq\aleph_0} \cup \{CA_n\}_{1<n\leq\aleph_0} \cup \{G\}$, and

(ii) $\mathcal{K}(\sigma, co)|L_0 = \mathcal{K}(\sigma, ce)|L_0 = \mathcal{K}(\sigma)$ for $\sigma \in \{AC_n\}_{1<n\leq\aleph_0}$.

Besides, recall that the points of an antichain of chains $P \in \boldsymbol{\mathcal{K}}(AC_n)$ can be labelled $P = \{p_{i,j} : (i, j) \in [n] \rtimes M\}$, as in Definition 2.1.9. Then one can easily show that picking a convex linear extension $(P, \prec)$ of the partial order $(P, <)$ corresponds precisely to picking a total order of the set of maximal chains P_i of P.

We introduce the *lexicographic order*, as it will be mentioned in various places in the thesis.

Definition 2.2.14. Suppose that X is a total order and for each $x \in X$ there is a total order Y_x. The lexicographic order on $X \rtimes Y$ is a total order on $X \rtimes Y$, with

$$(x, y) < (x', y')$$

if $x < x'$, or $x = x'$ and $y < y'$.
In general, if $X_1, X_2, \ldots, X_k$ are total orders and

$$Y \subset X_1 \times X_2 \times \ldots \times X_k,$$

then the lexicographic order on Y is defined as

$$(x_1, x_2, \ldots, x_k) < (x'_1, x'_2, \ldots, x'_k)$$

if $x_1 < x'_1$, or $x_1 = x'_1$ and $x_2 < x'_2$ and so on, until we get to $(x_1, x_2, \ldots, x_{k-1}) = (x'_1, x'_2, \ldots, x'_{k-1})$ and $x_k < x'_k$.

Note. In case of $\mathcal{K}(CA_n, e)$ (or $\mathcal{K}(AC_n, ce)$) above, if the indices $i \in [n]$ reflect the total order on the set of maximal antichains (or chains) of a structure P and for each $i \in [n]$ the total order $[m_i]$ reflects the total order of the points in a maximal antichain (or chain) P_i of P, then the total order on P is precisely the lexicographic order.

Example 2.2.15. Consider, for example, the partial order Q from example 2.2.8, with three points q_1, q_2, q_3 and $q_2 < q_3$. Then Q is an antichain of two chains, one of size 1 and one of size 2, so $Q \in \mathcal{K}(AC_2)$. Consider three linear orders on Q:

(i) $\prec_1$: $q_3 \prec_1 q_2 \prec_1 q_1$, let $Q_1 = (Q, <, \prec_1)$,

(ii) $\prec_2$: $q_2 \prec_2 q_1 \prec_2 q_3$, let $Q_2 = (Q, <, \prec_2)$, and

(iii) $\prec_3$: $q_1 \prec_3 q_2 \prec q_3$, let $Q_3 = (Q, <, \prec_3)$.

Then all three ordered partial orders Q_i are elements of the class $\mathcal{K}(AC_2, o)$. The total order Q_1 is not an extension of the partial order on Q, but $Q_2, Q_3 \in \mathcal{K}(AC_2, e)$. Now both Q_1 and Q_3 are convex on the chains of Q, whilst Q_2 is not, with q_1 between the points q_1 and q_3 of the other chain of Q. So $Q_1, Q_3 \in \mathcal{K}(AC_2, co)$. Requiring that the total order be an extension as well as convex, we get that only $Q_3 \in \mathcal{K}(AC_2, ce)$.

Sokić classifies which of these classes are Fraïssé. We combine Lemma 2 in Sokić (2012a) and Lemma 1 and Lemma 3 in Sokić (2012b).

Theorem 2.2.16. *The following classes are Fraïssé:*

(i) $\mathcal{K}(A_1, o) = \mathcal{K}(A_1, e)$ *and* $\mathcal{K}(A_{\aleph_0}, o) = \mathcal{K}(A_{\aleph_0}, e)$,

(ii) $\mathcal{K}(C, o)$ *and* $\mathcal{K}(C, e)$,

(iii) $\mathcal{K}(AC_n, o)$ *and* $\mathcal{K}(AC_n, e)$ *for all* $1 < n \leq \aleph_0$,

(iv) $\mathcal{K}(AC_{\aleph_0}, co)$ *and* $\mathcal{K}(AC_{\aleph_0}, ce)$,

(v) $\mathcal{K}(CA_{\aleph_0}, o)$ *and* $\mathcal{K}(CA_{\aleph_0}, e)$, *and*

(vi) $\mathcal{K}(G, o)$ *and* $\mathcal{K}(G, e)$.

The following classes are not Fraïssé:

(i) $\mathcal{K}(A_n, o) = \mathcal{K}(A_n, e)$, *for all* $1 < n < \aleph_0$,

(ii) $\mathcal{K}(AC_n, co)$ *and* $\mathcal{K}(AC_n, ce)$ *for* $1 < n < \aleph_0$, *and*

(iii) $\mathcal{K}(CA_n, o)$ *and* $\mathcal{K}(CA_n, e)$ *for all* $1 < n < \aleph_0$.

Definition 2.2.17. Homogeneous ordered partial orders corresponding to the classes in Theorem 2.2.16 are:

(i) $\mathcal{H}(A_1, e) = Flim(\mathcal{K}(A_1, e))$ and $\mathcal{H}(A_{\aleph_0}, e) = Flim(\mathcal{K}(A_{\aleph_0}, e))$;

(ii) $\mathcal{H}(C, o) = Flim(\mathcal{K}(C, o))$ and $\mathcal{H}(C, e) = Flim(\mathcal{K}(C, e))$;

(iii) $\mathcal{H}(AC_n, o) = Flim(\mathcal{K}(AC_n, o))$ and $\mathcal{H}(AC_n, e) = Flim(\mathcal{K}(AC_n, e))$ for all $1 < n \leq \aleph_0$;

(iv) $\mathcal{H}(AC_{\aleph_0}, co) = Flim(\mathcal{K}(AC_{\aleph_0}, co))$ and
$\mathcal{H}(AC_{\aleph_0}, ce) = Flim(\mathcal{K}(AC_{\aleph_0}, ce))$;

(v) $\mathcal{H}(CA_{\aleph_0}, o) = Flim(\mathcal{K}(CA_{\aleph_0}, o))$ and
$\mathcal{H}(CA_{\aleph_0}, e) = Flim(\mathcal{K}(CA_{\aleph_0}, e))$; and

(vi) $\mathcal{H}(G, o) = Flim(\mathcal{K}(G, o))$ and $\mathcal{H}(G, e) = Flim(\mathcal{K}(G, e))$.

Sokić proves which classes are Ramsey as well. We combine Theorem 7 in Sokić (2012a) and Lemma 1 and Lemma 3 in Sokić (2012b).

Theorem 2.2.18. *The following classes are Ramsey:*

(i) $\mathcal{K}(A_1, o)$, $\mathcal{K}(A_{\aleph_0}, o)$, $\mathcal{K}(A_1, e)$ *and* $\mathcal{K}(A_{\aleph_0}, e)$,

(ii) $\mathcal{K}(C, o)$ *and* $\mathcal{K}(C, e)$,

(iii) $\mathcal{K}(AC_{\aleph_0}, co)$ *and* $\mathcal{K}(AC_{\aleph_0}, ce)$,

(iv) $\mathcal{K}(CA_{\aleph_0}, o)$ *and* $\mathcal{K}(CA_{\aleph_0}, e)$, *and*

(v) $\mathcal{K}(G, e)$.

The following classes are not Ramsey:

(i) $\mathcal{K}(A_n, o) = \mathcal{K}(A_n, e)$ *for all* $1 < n < \aleph_0$*,*

(ii) $\mathcal{K}(AC_n, o)$ *and* $\mathcal{K}(AC_n, e)$ *for all* $1 < n \leq \aleph_0$*,*

(iii) $\mathcal{K}(AC_n, co)$ *and* $\mathcal{K}(AC_n, ce)$ *for* $1 < n < \aleph_0$*,*

(iv) $\mathcal{K}(CA_n, o)$ *and* $\mathcal{K}(CA_n, e)$ *for all* $1 < n < \aleph_0$*, and*

(v) $\mathcal{K}(G, o)$*.*

Ordering Property

As above, let $L = \{<, \prec\}$, $L_0 = \{<\}$, let $\boldsymbol{\mathcal{K}}$ be a class in language L and $\boldsymbol{\mathcal{K}}_0 = \boldsymbol{\mathcal{K}}|L_0$.

Definition 2.2.19. A class $\boldsymbol{\mathcal{K}}$ satisfies the *ordering property* with respect to $\boldsymbol{\mathcal{K}}_0$ if for every $\mathbf{A}_0 \in \boldsymbol{\mathcal{K}}_0$ there is a $\mathbf{B}_0 \in \boldsymbol{\mathcal{K}}_0$, such that for every linear ordering $\prec^{\mathbf{A}}$ on $\mathbf{A}_0$ and every linear ordering $\prec^{\mathbf{B}}$ on $\mathbf{B}_0$ with

$$\mathbf{A} = \langle \mathbf{A}_0, \prec^{\mathbf{A}} \rangle \in \boldsymbol{\mathcal{K}} \text{ and } \mathbf{B} = \langle \mathbf{B}_0, \prec^{\mathbf{B}} \rangle \in \boldsymbol{\mathcal{K}},$$

there is an embedding $e : \mathbf{A} \to \mathbf{B}$.

Sokić proves the following. We combine Lemma 3 in Sokić (2012a) and Lemma 2 in Sokić (2012b).

Theorem 2.2.20. *The following classes satisfy OP:*

(i) $\mathcal{K}(A_n, o) = \mathcal{K}(A_n, e)$ *with respect to* $\mathcal{K}(A_n)$ *for all* $1 \leq n \leq \aleph_0$*,*

(ii) $\mathcal{K}(C, e)$ *with respect to* $\mathcal{K}(C)$*,*

(iii) $\mathcal{K}(AC_n, ce)$ *with respect to* $\mathcal{K}(AC_n)$ *for all* $1 < n \leq \aleph_0$*,*

(iv) $\mathcal{K}(CA_n, e)$ *with respect to* $\mathcal{K}(CA_n)$ *for all* $1 < n \leq \aleph_0$*, and*

(v) $\mathcal{K}(G, e)$ *with respect to* $\mathcal{K}(G)$*.*

The following classes do not satisfy OP:

(i) $\mathcal{K}(C, o)$ *with respect to* $\mathcal{K}(C)$*,*

(ii) $\mathcal{K}(AC_n, o)$ *and* $\mathcal{K}(AC_n, e)$ *with respect to* $\mathcal{K}(AC_n)$ *for all* $1 < n \leq \aleph_0$*,*

(iii) $\mathcal{K}(AC_n, co)$ *with respect to* $\mathcal{K}(AC_n)$ *for all* $1 < n \leq \aleph_0$*,*

(iv) $\mathcal{K}(CA_n, o)$ *with respect to* $\mathcal{K}(CA_n)$ *for all* $1 < n \leq \aleph_0$*, and*

(v) $\mathcal{K}(G, o)$ *with respect to* $\mathcal{K}(G)$*.*

2.3 Links to topological dynamics

Kechris, Pestov, Todorčević

The paper Kechris *et al.* (2005) explores the connections between structural Ramsey theory, Fraïssé theory and topological dynamics. In particular, it provides a new way of finding extremely amenable groups and calculating the universal minimal flow.

A *T-flow* is a continuous action of a topological group T on a compact space X. A T-flow is *minimal* if all of its orbits are dense. The *universal minimal flow* $M(T)$ is a minimal T-flow that can be homomorphically mapped onto any other minimal T-flow. A general topological dynamics result states that every topological group T has a universal minimal flow $M(T)$. If $M(T)$ is a singleton, the group T is *extremely amenable*. Equivalently, T is extremely amenable if and only if every T-flow has a fixed point.

Consider S_∞, the group of all permutations of $\mathbb{N}$, with the pointwise convergence topology. That is, the elements of S_∞ are bijections $\mathbb{N} \to \mathbb{N}$, so S_∞ is a subgroup of the group of all functions $\mathbb{N} \to \mathbb{N}$, namely $\mathbb{N}^\mathbb{N}$. Taking the discrete topology on $\mathbb{N}$ (with all subsets of $\mathbb{N}$ being open), and product topology on $\mathbb{N}^\mathbb{N}$ (with a basis made of preimages of any collection of subsets of $\mathbb{N}$ in the product under projection maps), we get precisely the pointwise convergence topology on $\mathbb{N}^\mathbb{N}$. The subspace topology on S_∞ is precisely the pointwise convergence topology as well.

The group S_∞ is an interesting group to be considered because any countable group is a subgroup of S_∞. The paper Kechris *et al.* (2005) in Theorem 4.7 shows that there is a link between extremely amenable subgroups of S_∞ and Fraïssé order classes with the Ramsey property.

Theorem 2.3.1. *Let $T \trianglelefteq S_\infty$ be a closed subgroup. Then the following are equivalent:*

(i) T is extremely amenable.

(ii) $T = Aut(\boldsymbol{\mathcal{H}})$, where $\boldsymbol{\mathcal{H}}$ is the Fraïssé limit of a Fraïssé order class with the Ramsey property.

A standard result states that the closed subgroups of S_∞ under the pointwise convergence topology are precisely the automorphism groups of the homogeneous first order structures. It is proven in Cameron (1990), for example. This leads to the next theorem, Theorem 4.8 from Kechris *et al.* (2005), stating more explicitly the connection between Ramsey classes and extremely amenable groups.

Theorem 2.3.2. *Let $\boldsymbol{\mathcal{K}}$ be a Fraïssé order class and $\boldsymbol{\mathcal{H}} = Flim(\boldsymbol{\mathcal{K}})$. Then the following are equivalent:*

(i) the automorphism group $Aut(\boldsymbol{\mathcal{H}})$ is extremely amenable, and

(ii) $\boldsymbol{\mathcal{K}}$ satisfies the Ramsey property.

Thus finding a new Ramsey class might lead to finding a new extremely amenable group if the automorphism group $Aut(\boldsymbol{\mathcal{H}})$ is not known to be extremely amenable.

A common way to find new order classes to be studied is by expanding a Fraïssé class $\boldsymbol{\mathcal{K}}_0$ in language L_0 to a new Fraïssé class $\boldsymbol{\mathcal{K}}$ in language $L = L_0 \cup \{\prec\}$, such that the new class $\boldsymbol{\mathcal{K}}$ is an order class with respect to $\prec$ and a reasonable class with respect to $\boldsymbol{\mathcal{K}}_0$, as discussed on page 30.

The paper Kechris *et al.* (2005) specifies and proves that in certain circumstances considering $\boldsymbol{\mathcal{K}}$ provides a way to calculate a universal minimal flow of the automorphism group of $Flim(\boldsymbol{\mathcal{K}}_0)$, providing a way to calculate universal minimal flows of a wider range of automorphism groups of homogeneous structures that are not extremely amenable.

Instead of starting with the class $\mathcal{K}_0$, we will start by considering the class $\mathcal{K}$ and work our way back to results related to $\mathcal{K}_0$. The paper Kechris *et al.* (2005) in Proposition 5.2 shows the following.

Proposition 2.3.3. *Let $L \supset \{\prec\}$ be a language and $\mathcal{K}$ a Fraïssé order class for $\prec$ in L. Let $L_0 = L \setminus \{\prec\}$, $\mathcal{K}|_{L_0}$, $\mathcal{H} = Flim(\mathcal{K})$ and $\mathcal{H}_0 = \mathcal{H}|_{L_0}$. Then the following are equivalent:*

(i) $\mathcal{K}_0$ is a Fraïssé class and $\mathcal{H}_0 = Flim(\mathcal{K}_0)$, and

(ii) $\mathcal{K}$ is reasonable with respect to $\mathcal{K}_0$.

In the rest of this section we will consider the case with $\mathcal{K}$, $\mathcal{K}_0$, $\mathcal{H}$ and $\mathcal{H}_0$ as in Proposition 2.3.3, with $\mathcal{K}$ a Fraïssé order class for $\prec$ in L, that is reasonable with respect to $\mathcal{K}_0$.

The homogeneous structures $\mathcal{H}$ and $\mathcal{H}_0$ are both structures with a countable universe $H = \{h_i : i \in \mathbb{N}\}$.

We could view the total order $\prec$ on $\mathcal{H}$ as an element of $2^{\mathbb{N}^2}$: that is, a map

$$\mathbb{1}_\prec : \mathbb{N}^2 \to \{0, 1\}$$

with

$$\mathbb{1}_\prec(i, i') = 1 \iff h_i \prec h_{i'}.$$

Let LO be a subset of $2^{\mathbb{N}^2}$ of all maps corresponding to total orders of H. It is clear that any $\mathbb{1}_{\prec'} \in 2^{\mathbb{N}^2}$ lies in LO precisely when it satisfies the following four conditions:

(i) $\forall i \in \mathbb{N}, \mathbb{1}_{\prec'}(i, i) = 0$,

(ii) $\forall i, i' \in \mathbb{N}, i \neq i' \Rightarrow (\mathbb{1}_{\prec'}(i, i') = 1 \vee \mathbb{1}_{\prec'}(i', i) = 1)$,

(iii) $\forall i, i' \in \mathbb{N}, \neg(\mathbb{1}_{\prec'}(i, i') = 1 \wedge \mathbb{1}_{\prec'}(i', i) = 1)$, and

(iv) $\forall i, i', i'' \in \mathbb{N}, (\mathbb{1}_{\prec'}(i, i') = 1 \wedge \mathbb{1}_{\prec'}(i', i'') = 1) \Rightarrow \mathbb{1}_{\prec'}(i, i'') = 1$.

Taking a discrete topology on $\{0,1\}$ and a product topology on $2^{\mathbb{N}^2}$, we get a compact space $2^{\mathbb{N}^2}$ by Tychonoff's theorem. A quick check shows that LO is a closed subspace of $2^{\mathbb{N}^2}$, and thus compact itself.

Then taking any permutation $g \in S_\infty$ of $\mathbb{N}$, g acts on LO as follows:

$$\forall i, i' \in \mathbb{N}, g \cdot \mathbb{1}_{\prec'}(i,i') = \mathbb{1}_{g\cdot\prec'}(i,i') = 1 \iff \mathbb{1}_{\prec'}(g^{-1}(i), g^{-1}(i')) = 1$$

Let T_0 be the automorphism group of $\boldsymbol{\mathcal{H}}_0$. Then we can view T_0 as isomorphic to a subgroup of S_∞ in a natural way, identifying $t \in T_0$ with $g \in S_\infty$ if

$$\forall i, i' \in \mathbb{N}, t(h_i) = h_{i'} \iff g(i) = i'.$$

Thus we can consider the action of T_0 on LO,

$$T_0 \cdot LO \to LO.$$

Definition 2.3.4. Let $\boldsymbol{\mathcal{K}}$ be a Fraïssé class, that is a reasonable order class with respect to $\boldsymbol{\mathcal{K}}_0$, let $\boldsymbol{\mathcal{H}}_0 = Flim(\boldsymbol{\mathcal{K}}_0)$ and $\boldsymbol{\mathcal{H}} = \langle \boldsymbol{\mathcal{H}}_0, \prec \rangle = Flim(\boldsymbol{\mathcal{H}}_0)$. Denote by $X_{\boldsymbol{\mathcal{K}}}$ the closure of the orbit $\prec$ under the action of the group $T_0 = Aut(H_0)$ on the space of all linear orders of the universe H of $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{H}}_0$;

$$X_{\boldsymbol{\mathcal{K}}} = \overline{T_0 \cdot \prec}.$$

We call any ordering in $X_{\boldsymbol{\mathcal{K}}}$ a $\boldsymbol{\mathcal{K}}$-*admissible ordering.*

Aside from the connection between $X_{\boldsymbol{\mathcal{K}}}$ and the automorphism group of $\boldsymbol{\mathcal{H}} = Flim(\boldsymbol{\mathcal{K}})$, the paper Kechris *et al.* (2005) in Proposition 7.1 shows that there is a more explicit connection between $X_{\boldsymbol{\mathcal{K}}}$ and $\boldsymbol{\mathcal{K}}$.

Proposition 2.3.5. *A linear ordering $\prec'$ is in $X_{\boldsymbol{\mathcal{K}}}$ if and only if for every finite substructure $\mathbf{A}_0$ of $\boldsymbol{\mathcal{H}}_0$, the structure*

$$\mathbf{A} = \langle \mathbf{A}_0, \prec |_{\prec'} \rangle$$

lies in the class $\boldsymbol{\mathcal{K}}$.

It is perhaps more intuitive to think about the total orderings on the finite structures in $\mathcal{K}_0$ that yield the order class $\mathcal{K}$. Similarly to orderings of H in $X_{\mathcal{K}}$ being $\mathcal{K}$-admissible, we will also say that an ordering $\prec$, where for some $\mathbf{A}_0 \in \mathcal{K}_0$ we have $\langle \mathbf{A}_0, \prec \rangle \in \mathcal{K}$, is $\mathcal{K}$-admissible. However, we will be careful to clarify whether we are referring to a $\mathcal{K}$-admissible ordering of a homogeneous structure or a $\mathcal{K}$-admissible ordering of a finite structure in a Fraïssé class.

More importantly, the paper Kechris *et al.* (2005) in Theorem 10.8 shows how to calculate the universal minimal flow of T_0, where T_0 is the automorphism group of a homogeneous structure, in case that there exists a reasonable Fraïssé expansion of its age with specific properties.

Theorem 2.3.6. *Let $L \supset \{\prec\}$ be a language, $L_0 = L \setminus \{\prec\}$, and $\mathcal{K}$ a Fraïssé order class in L, reasonable with respect to the class $\mathcal{K}_0 = \mathcal{K}|L_0$. Let then $\mathcal{H} = Flim(\mathcal{K})$ and $\mathcal{H}_0 = Flim(\mathcal{K}_0) = \mathcal{H}|L_0$.*
Let further $T_0 = Aut(\mathcal{H}_0)$, $T = Aut(\mathcal{H})$ and let $X_{\mathcal{K}}$ be the set of linear orderings on $\mathcal{H}$ which are $\mathcal{K}$-admissible. Then the following are equivalent:

(i) $\mathcal{K}$ has the Ramsey and ordering properties.

(ii) $X_{\mathcal{K}}$ is the universal minimal flow of T_0.

Further, combining part (i) of Theorem 7.5 with the Theorem 10.8 from Kechris *et al.* (2005), we get the following result.

Theorem 2.3.7. *The universal minimal flow $X_{\mathcal{K}}$ in Theorem 2.3.6 is metrizable.*

Existence and uniqueness

Given a Fraïssé class $\mathcal{K}_0$ that does not have Ramsey property, finding a reasonable Fraïssé order class $\mathcal{K}$ with respect to $\mathcal{K}_0$ that has Ramsey property may be difficult or even impossible at times. So one might predict that finding a class $\mathcal{K}$ with the Ramsey and ordering properties, and thus yielding interesting topological results discussed in the previous section, would be even more difficult. While that might be the case, Theorem 10.7 from Kechris *et al.* (2005) assures us that the search is not in vain.

Theorem 2.3.8. *Let $\mathcal{K}_0$ be a Fraïssé class in the language L_0, and assume that $\mathcal{K}$ is a reasonable Fraïssé order class with respect to $\mathcal{K}_0$ in the language $L = L_0 \cup \{\prec\}$ which satisfies the Ramsey property. Then there is a reasonable Fraïssé order class $\mathcal{K}' \subset \mathcal{K}$ with respect to $\mathcal{K}_0$, such that $\mathcal{K}'$ satisfies both the Ramsey and ordering properties.*

However, once the class with the ordering and Ramsey properties is found, it is essentially unique in a specific sense. That is, the class is unique up to *simple bi-definability.*

Definition 2.3.9. A *first order simple formula* is a quantifier-free finite formula in the first order language.
Let $\mathcal{K}$ and $\mathcal{K}'$ both be reasonable order classes in language $L = L_0 \cup \{\prec\}$ with respect to $\mathcal{K}_0$ in language L_0. The classes $\mathcal{K}$ and $\mathcal{K}'$ are *simply bi-definable* if there are simple formulas $\varphi(x, y)$ and $\varphi'(x, y)$ in L, such that for each $\mathbf{A}_0 \in \mathcal{K}_0$, the formulas φ and φ' define (uniformly) a bijection between the expansions of $\mathbf{A}_0$ in L that are in $\mathcal{K}$ with those that are in $\mathcal{K}'$.

The general definition of bi-definability is not restricted only to reasonable order classes. We will explore it further in Section 3.1. Now we consider the definition of simple bi-definability in this specific context.

The paper Kechris *et al.* (2005) in Theorem 9.1 first shows that bi-definability preserves Ramsey and ordering properties.

Theorem 2.3.10. *Suppose that the classes $\mathcal{K}$ and $\mathcal{K}'$ are reasonable Fraïssé order classes in L with respect to $\mathcal{K}_0$. If $\mathcal{K}$ and $\mathcal{K}'$ are simply bi-definable, then the following hold.*

(i) The class $\mathcal{K}$ satisfies the Ramsey property if and only if the class $\mathcal{K}'$ does.

(ii) The class $\mathcal{K}$ satisfies the ordering property if and only if the class $\mathcal{K}'$ does.

The following theorem, Theorem 9.2 in Kechris *et al.* (2005), is the uniqueness result for classes with Ramsey and ordering properties.

Theorem 2.3.11. *Suppose that the classes $\mathcal{K}$ and $\mathcal{K}'$ are reasonable Fraïssé order classes in L with respect to $\mathcal{K}_0$. If $\mathcal{K}$ and $\mathcal{K}'$ satisfy the ordering and Ramsey properties, then they are simply bi-definable.*

Thus, up to simple bi-definability, given a class $\mathcal{K}_0$, if there exists a class $\mathcal{K}$, that is a reasonable Fraïssé order class in L with respect to $\mathcal{K}_0$ and satisfies the ordering and Ramsey properties, then the class $\mathcal{K}$ is unique up to simple bi-definability.

Further, by Theorem 9.5 in Kechris *et al.* (2005), if $\mathcal{K}$ satisfies the ordering and Ramsey properties

Theorem 2.3.12. *Suppose that the classes $\mathcal{K}$ and $\mathcal{K}'$ are reasonable Fraïssé order classes in L with respect to $\mathcal{K}_0$. If class $\mathcal{K}$ has the Ramsey property and class $\mathcal{K}'$ has the Ramsey and ordering properties, then $\mathcal{K}' \subset \mathcal{K}$ up to simple bidefinability.*

Topological dynamics of partial orders

Combining the results from Theorem 2.2.18 and Theorem 2.3.2 Sokić proves the following. We combine Theorem 11 in Sokić (2012a) and Theorem 3 in Sokić (2012b).

Theorem 2.3.13. *The following groups are extremely amenable.*

(i) $Aut(\mathcal{H}(A_{\aleph_0}, e))$,

(ii) $Aut(\mathcal{H}(C, e))$ *and* $Aut(\mathcal{H}(C, o))$,

(iii) $Aut(\mathcal{H}(AC_{\aleph_0}, co))$ *and* $Aut(\mathcal{H}(AC_{\aleph_0}, ce))$,

(iv) $Aut(\mathcal{H}(CA_{\aleph_0}, o))$ *and* $Aut(\mathcal{H}(CA_{\aleph_0}, e))$,

(v) $Aut(\mathcal{H}(G, o))$ *and* $Aut(\mathcal{H}(G, e))$.

Similarly, combining Theorems 2.2.16, 2.2.18, 2.3.6, and 2.2.20 Sokić proves the following. We combine Theorem 13 and 14 in Sokić (2012a) and Theorem 3 in Sokić (2012b).

Theorem 2.3.14. *$X_{\mathcal{K}}$ is a universal minimal T_0-flow for the following.*

(i) $\mathcal{K} = \mathcal{K}(A_{\aleph_0}, e)$ *and* $T_0 = Aut(\mathcal{H}(A_{\aleph_0}))$

(ii) $\mathcal{K} = \mathcal{K}(C, e)$ *and* $T_0 = Aut(\mathcal{H}(C))$

(iii) $\mathcal{K} = \mathcal{K}(AC_{\aleph_0}, ce)$ *and* $T_0 = Aut(\mathcal{H}(AC_{\aleph_0}))$

(iv) $\mathcal{K} = \mathcal{K}(CA_{\aleph_0}, e)$ *and* $T_0 = Aut(\mathcal{H}(CA_{\aleph_0}))$

(v) $\mathcal{K} = \mathcal{K}(G, e)$ *and* $T_0 = Aut(\mathcal{H}(G))$

2.4 Shaped homogeneous partial orders

Countable homogeneous shaped partial orders (or, as referred to in the paper, countable homogeneous coloured partial orders) have been classified in Torrezão de Sousa & Truss (2008). The paper shows that there is an equivalence relation $\sim$, partitioning $\boldsymbol{\mathcal{H}}$ into *interdensely shaped components*. The structure of a homogeneous structure in terms of the equivalence classes of $\sim$ and the connections between them can be described by a labelled partial order Σ, in which the labels of vertices carry information about the equivalence classes of $\sim$ and the labels of comparable pairs carry information about the connections between them. It states and proves a classification theorem for countable homogeneous shaped partial orders in terms of the associated labelled partial orders Σ. This section revisits the results from the paper Torrezão de Sousa & Truss (2008), adapting notation to one that will be used throughout this thesis.

Model theory of shaped partial orders and notation

We begin this section with a formal definition.

Definition 2.4.1. Let $<$ be binary relation and let $\mathbf{s}^a$ for $a \in \mathcal{A}$ be unary relations. Let $\mathfrak{S} = \{\mathbf{s}^a : a \in \mathcal{A}\}$ and let P be a set. An $\mathfrak{S}$-*shaped partial order* $\mathbf{P}$ is a structure

$$\mathbf{P} = \langle P, <, \{\mathbf{s}^a\}_{a \in \mathcal{A}} \rangle,$$

satisfying the following conditions.

(i) Relation $<$ is a partial order relations.

(ii) For all $p \in P$ the following holds:

$$\mathbf{s}^1(p) \vee \mathbf{s}^2(p) \vee \ldots \vee \mathbf{s}^{|\mathcal{A}|}(p).$$

(iii) For all pairs of distinct $a, a' \in \mathcal{A}$ and for all $p \in P$, the we have:

$$\neg(\mathbf{s}^a(p) \wedge \mathbf{s}^{a'}(p)).$$

Note. Part (iii) of the definition requires that each point of $\mathbf{P}$ has a shape and part (iv) ensures that each point has exactly one shape.

However, informally, we replace the unary relations and the related axioms with a map $\mathfrak{s}$, sending the set of points of P to the set of shapes. That is,

$$\mathfrak{s} : P \to \mathfrak{S}, \quad \mathfrak{s}(p) = \mathbf{s}^a \text{ if } \mathbf{s}^a(p) \text{ in the model } \mathbf{P}.$$

Throughout the rest of the thesis we will be using the following notation.

Definition 2.4.2. Let $<$ be a binary relation and let $\mathfrak{S} = \{\mathbf{s}^a : a \in \mathcal{A}\}$ be a finite set. A $\mathfrak{S}$-*shaped partial order* $\mathbf{P}$ is a structure

$$(P, <, \mathfrak{s})$$

satisfying the following conditions.

(i) $(P, <)$ is a partial order, and

(ii) $\mathfrak{s} : P \to \mathfrak{S}$ is a map.

For each $p \in P$, the element $\mathfrak{s}(p) \in \mathfrak{S}$ is the *shape of* P.
The *class* $\mathcal{K}$ *of all finite* $\mathfrak{S}$*-shaped partial orders* $\mathcal{K}(G, \mathfrak{S})$ is the class of all finite $\mathfrak{S}$-shaped partial orders.

Note. The formal definition of a substructure or an isomorphism then translates to the shape maps agreeing on the shape of points. Namely, if $\mathbf{P}, \mathbf{R}$ are shaped partial orders with $\mathbf{P} = (P, <^{\mathbf{P}}, \mathfrak{s}^{\mathbf{P}})$, $\mathbf{R} = (R, <^{\mathbf{R}}, \mathfrak{s}^{\mathbf{R}})$ and

$$e : \mathbf{P} \to \mathbf{R}$$

is an embedding, then for all $p \in P$ we have

$$\mathfrak{s}^{\mathbf{P}}(p) = \mathfrak{s}^{\mathbf{R}}(e(p)).$$

Suppose that $\mathfrak{S}$ is an countably infinite set of shapes. Then for each finite subset $\mathfrak{S}'$ of $\mathfrak{S}$ there is a class $\mathcal{K}(G, \mathfrak{S}')$ of all finite ordered $\mathfrak{S}'$-shaped partial orders. Torrezão de Sousa & Truss (2008) shows that each of these classes is a Fraïssé class, as it shows that a generic countable homogeneous shaped partial order $\mathcal{H}(G, \mathfrak{S}')$ exists. But then by compactness, the structure $\mathcal{H}(G, \mathfrak{S})$ exists as well. Thus the class $\mathcal{K}(G, \mathfrak{S})$ is a Fraïssé class.

In this section we consider the countable homogeneous shaped partial orders.

Definition 2.4.3. A *countable homogeneous shaped partial order* $\mathcal{H}$ is a countable partial order, together with an expansion by unary predicates, that is homogeneous. We will say that $\mathcal{H}$ is an $\mathfrak{S}$*-shaped partial order*, if $\mathcal{H} = (H, <, \mathfrak{s})$, where $(H, <)$ is a partial order and $\mathfrak{s} : H \to \mathfrak{S}$ is a map from the set H of points of $\mathcal{H}$ to the set $\mathfrak{S}$ of shapes of $\mathcal{H}$.

The Interdense Relation

The formal definition of the $\sim$ relation is as follows.

Definition 2.4.4. Let $\mathcal{H} = (H, <, \mathfrak{s})$ be a countable homogeneous shaped partial order. Let $\sim$ be the transitive closure of the following relation:

$$x \sim' y \iff$$

- $\mathfrak{s}(x) = \mathfrak{s}(y)$ or
- $x < y$ and $\mathfrak{s}(x), \mathfrak{s}(y)$ occur interdensely between x and y (that is, for any x', y' such that $x < x' < y' < y$ there are z, z' such that $x' < z < y'$ and $x' < z' < y'$ and $\mathfrak{s}(z) = \mathfrak{s}(x), \mathfrak{s}(z') = \mathfrak{s}(y)$), or
- analogous condition for $y < x$.

According to Torrezão de Sousa & Truss (2008), the relation $\sim$ tells us a lot about the homogeneous shaped partial order $\mathcal{H}$. Remark 2.4.5 part (ii) refers to the equivalence relation on the shapes, denoted by $\approx$, that $\sim$ induces (considered in Section 2 of the paper). In Section 3, the paper states the results summarised in parts (iii)-(v).

Remark 2.4.5. (i) The relation $\sim$ is an equivalence relation, and hence partitions H into equivalence classes H_σ, where $\sigma \in \Sigma$ for some set Σ. We refer to the substructure of $\mathcal{H}$ restricted to an equivalence class H_σ of $\sim$ as the *component* $\mathcal{H}_\sigma$ of $\mathcal{H}$.

(ii) The relation $\sim$ induces a partition of $\mathfrak{S}$ into sets $\mathfrak{S}_\sigma$ for $\sigma \in \Sigma$, such that $\mathfrak{s}(H_\sigma) = \mathfrak{S}_\sigma$ for $\sigma \in \Sigma$. So each $\mathcal{H}_\sigma$ is an $\mathfrak{S}_\sigma$-shaped partial order.

(iii) The next section of this thesis considers the components of a shaped homogeneous partial order. It summarises that for each $\sigma \in \Sigma$, the substructure of $\mathcal{H}$ on the points H_σ is one of the following:

- an $\mathfrak{S}_\sigma$-shaped antichain of chains AC,
- an $\mathfrak{S}_\sigma$-shaped chain of antichains CA, or
- an $\mathfrak{S}_\sigma$-shaped generic partial order G.

(iv) The partial order $(H, <)$ induces a partial order on Σ. For $\sigma, \sigma' \in \Sigma$, set $\sigma < \sigma'$ if there are $x \in H_\sigma$ and $y \in H_{\sigma'}$ such that $x < y$ in $(H, <)$. Then $<$ is a partial order on Σ.

In particular, if $\sigma < \sigma'$, then for all $p \in H_\sigma, q \in H_{\sigma'}$, either $p < q$ or $p||q$, but never $q < p$.

(v) Further, given any distinct σ, σ', the restriction of $<$ on the pairs (x, y) with $x \in H_\sigma, y \in H_{\sigma'}$ is one of the six types of relations we will denote by $||$, $<_c$, $<_g$, $<_{pm}$, $<_{cpm}$ or $<_{sh}$. These are considered in the section starting on page 52.

(vi) The set Σ together with a partial order $(\Sigma, <)$ mentioned in (iv), a map l_1 on points of Σ and a map l_2 on comparable points of $(\Sigma, <)$ will be discussed informally in the following two sections and defined formally in 2.4.14.

Countable homogeneous interdensely shaped partial orders

An *interdensely shaped* homogeneous partial order is one for which the relation $\sim$ has only one equivalence class. That is, $\mathcal{H}$ is a one component homogeneous shaped partial order.

Recall that the set $\mathbb{Q}$ with the natural order is, up to isomorphism, the unique countably infinite homogeneous linear order. Similarly, given a set of shapes $\mathfrak{S}$, there is, up to isomorphism, a unique countably infinite interdensely $\mathfrak{S}$-shaped homogeneous linear order, that we will denote by $\mathbb{Q}_{\mathfrak{S}}$.

The interdensely shaped homogeneous partial orders are classified in Theorem 8.1 of Torrezão de Sousa & Truss (2008). It later combines the shaped partial orders with the underlying partial order an antichain of size n (A_n) or an antichain of n chains (AC_n) under the label AC, and similarly the shaped partial orders with the underlying partial order a chain (C) or a chain of antichains of size at most n (CA_n) under the label CA, so

Definition 2.4.6. Suppose that $\boldsymbol{\mathcal{H}} = (H, <)$ is a countable homogeneous interdensely shaped partial order.

(i) $\boldsymbol{\mathcal{H}}$ is an antichain of chains, denoted by $\boldsymbol{\mathcal{H}}(AC)$, if

  (a) either $(H, <)$ is an antichain of size n, where $1 \leq n \leq \aleph_0$ and $\mathfrak{S} = \{\mathbf{s}^1\}$,

  (b) or $(H, <)$ is an antichain of n chains, where $2 \leq n \leq \aleph_0$ and each chain $\boldsymbol{\mathcal{H}}_i$ of $\boldsymbol{\mathcal{H}}$ is isomorphic to $\mathbb{Q}_{\mathfrak{S}}$.

(ii) $\boldsymbol{\mathcal{H}}$ is a chain of antichains, denoted by $\boldsymbol{\mathcal{H}}(CA)$, where

  (a) there are sets $\mathcal{I}$ and $\mathcal{A}$ and a map $\mathfrak{s}_1 : \mathcal{I} \to \mathcal{A}$, such that $(\mathcal{I}, <, \mathfrak{s}_1)$ is isomorphic to $\mathbb{Q}_{\mathcal{A}}$

  (b) there is a partition $\{H_i : i \in \mathcal{I}\}$ of the set H into maximal antichains of $(H, <)$, with $x < y$ in $(H, <)$ for all $x \in H_i, y \in H_{i'}$ and $i < i'$ in $(\mathcal{I}, <)$,

  (c) the set of shapes $\mathfrak{S}$ partitions as $\{\mathfrak{S}_a : a \in \mathcal{A}\}$, and

  (d) for $i \in \mathcal{I}$ with $\mathfrak{s}_1(i) = a$, H_i is an antichain with $n_{a,b}$ points of shape $\mathbf{s}^{a,b}$ for each $\mathbf{s}^{a,b} \in \mathfrak{S}_a$, where $1 \leq n_{a,b} \leq \aleph_0$.

(iii) $\boldsymbol{\mathcal{H}}$ is a generic $\mathfrak{S}$-shaped partial order, denoted by $\boldsymbol{\mathcal{H}}(G)$, if

  (a) $(H, <)$ is a generic countable homogeneous partial order, and

  (b) $\boldsymbol{\mathcal{H}}$ is interdensely $\mathfrak{S}$-shaped.

Note. (i) We view an antichain as an antichain of chains of length 1. Also, a shaped interdense linear order could be viewed as an antichain of one chain. But in this classification it is viewed as a chain of antichains, each of size 1.

(ii) We will see that for a Σ with more than one point, each component $\boldsymbol{\mathcal{H}}_\sigma$ is isomorphic to an interdensely-shaped homogeneous shaped partial order.

(iii) We introduce the map

$$l_1 : \Sigma \to \{AC, CA, G\},$$

with the label $l_1(\sigma)$ denoting whether the structure is isomorphic to an $\boldsymbol{\mathcal{H}}(AC), \boldsymbol{\mathcal{H}}(CA)$ or $\boldsymbol{\mathcal{H}}(G)$. In light of remark (ii), l_1 can be defined on Σ with more than one component as well and will be formally introduced in Definition 2.4.14.

(iv) The notation $\boldsymbol{\mathcal{H}}(AC)$ introduced in this definition is a shorthand for

$$\boldsymbol{\mathcal{H}}(\Sigma), \quad \Sigma = \{\sigma\}, \quad l_1(\sigma) = AC$$

and similarly for $\boldsymbol{\mathcal{H}}(CA)$ and $\boldsymbol{\mathcal{H}}(G)$. Whilst $\boldsymbol{\mathcal{H}}(AC), \boldsymbol{\mathcal{H}}(CA)$ or $\boldsymbol{\mathcal{H}}(G)$ is shorter, the notation $\boldsymbol{\mathcal{H}}(\Sigma)$ is consistent with Definition 2.4.14.

The paper Torrezão de Sousa & Truss (2008) shows the following in Theorem 8.1 on page 29.

Lemma 2.4.7. *Any countable homogeneous interdensely shaped partial order $\boldsymbol{\mathcal{H}}$ is either an $\boldsymbol{\mathcal{H}}(AC)$, an $\boldsymbol{\mathcal{H}}(CA)$ or an $\boldsymbol{\mathcal{H}}(G)$.*

While it is easy to define countable homogeneous interdensely shaped partial orders, the countable homogeneous shaped partial orders with more than one component have a more complex structure. Thus considering their ages, or even viewing a countable homogeneous shaped partial order primarily as a Fraïssé limit of a particular Fraïssé class of shaped partial orders, is essential. Since the structures $\boldsymbol{\mathcal{H}}$ are one-component structures, so are the corresponding classes.

Definition 2.4.8. (i) The homogeneous structure $\boldsymbol{\mathcal{H}}(AC)$ is a Fraïssé limit of the class $\boldsymbol{\mathcal{K}}(AC)$ of antichains of chains. That is, either,

(i:a) for some $1 \leq n \leq \aleph_0$, $\boldsymbol{\mathcal{K}}(AC)$ is a class of finite $\mathfrak{S}$-shaped antichains of size at most n, where $\mathfrak{S} = \{\mathbf{s}^1\}$, or (i:b) for some $2 \leq n \leq \aleph_0$, $\mathcal{K}(AC)$ is a class of finite $\mathfrak{S}$-shaped antichains of at most n chains.

(ii) The homogeneous structure $\mathcal{H}(CA)$ is a Fraïssé limit of the class $\mathcal{K}(CA)$ of finite $\mathfrak{S}$-shaped antichains of chains. That is,

(a) the set of shapes $\mathfrak{S}$ partitions as $\{\mathfrak{S}_a : a \in \mathcal{A}\}$,

(b) for each $\mathbf{s}^{a,b} \in \mathfrak{S}_a$, there is an $n_{a,b}$, with $1 \leq n_{a,b} \leq \aleph_0$, and

(c) each antichain of a $P \in \mathcal{K}(CA)$ is $\mathfrak{S}_a$-shaped for some $a \in \mathcal{A}$, and has at most $n_{a,b}$ points of shape $\mathbf{s}^{a,b}$ for each $\mathbf{s}^{a,b} \in \mathfrak{S}_a$.

(iii) The homogeneous structure $\mathcal{H}(G)$ is a Fraïssé limit of the class $\mathcal{K}(G)$ of all finite $\mathfrak{S}$-shaped partial orders.

Remark 2.4.9. Notation $\mathcal{K}(AC)$ omits information about the set of shapes $\mathfrak{S}$, whether the class is a class of antichains or proper antichains of chain, and how many chains there could be in any structure. To include that information in the notation we write $\mathcal{K}(AC, \mathfrak{S}, \{n_1, n_2\})$, where n_1 corresponds to n in parts (i:a) and (i:b) of the definition above, and the classes in (i:a) have $n_2 = 1$, while the classes in (i:b) have $n_2 = \aleph_0$.
Similarly, notation $\mathcal{K}(CA)$ omits information about the set of shapes $\mathfrak{S}$ and the related $n_{a,b}$. We let $\mathfrak{S}_a = \{\mathbf{s}^{a,b} : b \in \mathcal{B}_a\}$ for some set $\mathcal{B}_a$ and denote by N the set $\{n_{a,b} : (a, b) \in \mathcal{A} \rtimes \mathcal{B}\}$. We then write $\mathcal{K}(CA, \mathfrak{S}, N)$ to denote the class.
Finally, we write $\mathcal{K}(G, \mathfrak{S})$ to label the class defined in part (iii) of the definition above.

Relations between interdense components

Consider now homogeneous shaped partial orders with two equivalence classes of relation $\sim$, namely H_σ and $H_{\sigma'}$. These homogeneous shaped partial orders are *two-component homogeneous shaped partial orders.*

We consider the following relations:

(i) incomparable label $||$,

(ii) complete label $<_c$,

(iii) perfect matching label $<_{pm}$,

(iv) complement of the perfect matching label $<_{cpm}$,

(v) generic label $<_g$, and

(vi) shuffle label $<_{sh}$.

Despite the symbol $<$, these labels are not partial order relations, but merely labels of relation pairs of a partial order.

Definition 2.4.10. Suppose that $\boldsymbol{\mathcal{H}} = (H, <, \mathfrak{s})$ is a homogeneous shaped partial order, where H is a disjoint union of sets H_σ and $H_{\sigma'}$. We have the following notation and the corresponding conditions.

(i) $\boldsymbol{\mathcal{H}}_\sigma || \boldsymbol{\mathcal{H}}_{\sigma'}$, if for all $x \in H_\sigma$ and $y \in H_{\sigma'}$ we have $x||y$.

(ii) $\boldsymbol{\mathcal{H}}_\sigma <_c \boldsymbol{\mathcal{H}}_{\sigma'}$, if for all $x \in H_\sigma$ and $y \in H_{\sigma'}$ we have $x < y$.

(iii) $\boldsymbol{\mathcal{H}}_\sigma <_{pm} \boldsymbol{\mathcal{H}}_{\sigma'}$, if for some $2 \leq n \leq \aleph_0$,

(a) $\boldsymbol{\mathcal{H}}_\sigma$ is an AC with n chains $\boldsymbol{\mathcal{H}}_{\sigma,i}$, and $\boldsymbol{\mathcal{H}}_{\sigma'}$ is an AC with n chains $\boldsymbol{\mathcal{H}}_{\sigma',i}$, for $1 \leq i \leq n$, where

(b) for all $x \in H_{\sigma,i}$ and $y \in H_{\sigma',i'}$ we have $x < y$ if and only if $i = i'$.

(iv) $\boldsymbol{\mathcal{H}}_\sigma <_{cpm} \boldsymbol{\mathcal{H}}_{\sigma'}$, if, for some $2 \leq n \leq \aleph_0$,

(a) $\boldsymbol{\mathcal{H}}_\sigma$ is an AC with n chains $\boldsymbol{\mathcal{H}}_{\sigma,i}$, and $\boldsymbol{\mathcal{H}}_{\sigma'}$ is an AC with n chains $\boldsymbol{\mathcal{H}}_{\sigma',i}$, for $1 \leq i \leq n$, where

(b) for all $x \in H_{\sigma,i}$ and $y \in H_{\sigma',i'}$ we have $x < y$ if and only if $i \neq i'$.

(v) $\boldsymbol{\mathcal{H}}_\sigma <_g \boldsymbol{\mathcal{H}}_{\sigma'}$, if

(a) $\boldsymbol{\mathcal{H}}_\sigma$ and $\boldsymbol{\mathcal{H}}_{\sigma'}$ are each either an AC with $\aleph_0$ chains or a G,

(b) it is not the case that $\boldsymbol{\mathcal{H}}_\sigma <_{pm} \boldsymbol{\mathcal{H}}_{\sigma'}$ or $\boldsymbol{\mathcal{H}}_\sigma <_{cpm} \boldsymbol{\mathcal{H}}_{\sigma'}$, and

(c) there are $x, x' \in H_\sigma$ and $y, y' \in H_{\sigma'}$ such that $x||y$ and $x' < y'$.

(vi) $\mathcal{H}_\sigma <_{sh} \mathcal{H}_{\sigma'}$, if

(a) $\mathcal{H}_\sigma$ and $\mathcal{H}_{\sigma'}$ are both CA, and

(b) there are $x, x' \in H_\sigma$ and $y, y' \in H_{\sigma'}$ such that $x||y$ and $x' < y'$.

Note. We will refer to conditions (ii)-(vi) as the $<_l$-condition for the corresponding $<_l \in \{<_c, <_g, <_{sh}, <_{pm}, <_{cpm}\}$, *i.e.*, condition (ii) is the $<_c$ condition.

The paper Torrezão de Sousa & Truss (2008) classifies two-component homogeneous shaped partial orders in Theorem 8.2.

Lemma 2.4.11. *If the countable homogeneous shaped partial order $\mathcal{H}$ has two equivalence classes H_σ and $H_{\sigma'}$ under the relation $\sim$, the substructure of $\mathcal{H}$ on each of the classes is a countable homogeneous interdensely shaped partial order. The pair of equivalence classes satisfies one of the conditions in Definition 2.4.10 above. Further, given a pair of countable homogeneous interdensely shaped partial orders $\mathcal{H}_\sigma$, $\mathcal{H}_{\sigma'}$ with disjoint sets of shapes and a compatible condition from the Definition 2.4.10, there is, up to an isomorphism, a unique countable homogeneous shaped partial order $\mathcal{H}$ with $\mathcal{H}_\sigma$, $\mathcal{H}_{\sigma'}$ as substructures and $(H_\sigma \cup H_{\sigma'}, <, \mathfrak{s})$ satisfying the condition.*

Note. (i) Label the unique homogeneous structure on $H = H_\sigma \cup H_{\sigma'}$ satisfying a $<_l$ condition as the structure $\mathcal{H}_{\sigma <_l \sigma'}$.

(ii) Further, for structures satisfying conditions (ii)-(vi), let

$$l_2 : \{(\sigma, \sigma')\} \to \{<_c, <_{pm}, <_{cpm}, <_g, <_{sh}\}$$

be a map with $l_2(\sigma, \sigma') = <_l$ if $\sigma <_l \sigma'$ in Σ. Then each $<_l$ can be thought of as a binary relation on Σ. But the relation is not a partial order relation - not all of the relations are transitive.

(iii) The equivalence relation $\sim$ on $\mathfrak{S}$-shaped $\mathcal{H}_{\sigma <_l \sigma'}$ partitions the set of shapes $\mathfrak{S}$ into sets $\mathfrak{S}_\sigma$ and $\mathfrak{S}_{\sigma'}$, such that $\boldsymbol{\mathcal{H}}_\sigma$ is $\mathfrak{S}_\sigma$-shaped and $\boldsymbol{\mathcal{H}}_{\sigma'}$ is $\mathfrak{S}_{\sigma'}$-shaped.

(iv) Again, in line with Definition 2.4.14, label $\boldsymbol{\mathcal{H}}_{\sigma<_l\sigma'}$ as $\boldsymbol{\mathcal{H}}(\Sigma)$, where Σ has points σ and σ' and either $\sigma||\sigma'$ or $\sigma < \sigma'$ and $l_2(\sigma, \sigma') =<_l$. The skeleton Σ is associated with the structure $\boldsymbol{\mathcal{H}}$, $\boldsymbol{\mathcal{H}}_\sigma$ and $\boldsymbol{\mathcal{H}}_{\sigma'}$ are its components, and l_2 will be the map assigning labels to the comparable pairs of components.

(v) The structure $\boldsymbol{\mathcal{H}}_\sigma$ is isomorphic to one of the structures $\boldsymbol{\mathcal{H}}(AC)$, $\boldsymbol{\mathcal{H}}(CA)$ or $\boldsymbol{\mathcal{H}}(G)$, similarly for σ'. Denote the age of $\boldsymbol{\mathcal{H}}_\sigma$ by $\boldsymbol{\mathcal{K}}(\sigma)$, and similarly for σ'.

Rather than Definition 2.4.10 and Lemma 2.4.11, the paper Torrezão de Sousa & Truss (2008) defines the two-component homogeneous shaped partial orders as a Fraïssé limit of certain Fraïssé classes of partial orders and then shows the list is exhaustive.

Suppose we are given disjoint sets of shapes $\mathfrak{S}_{\sigma_1}$ and $\mathfrak{S}_{\sigma_2}$, and one-component Fraïssé classes of $\mathfrak{S}_{\sigma_i}$-shaped partial orders for $i = 1, 2$. Let $\boldsymbol{\mathcal{H}}$ be a two-component homogeneous $\mathfrak{S}$-shaped partial order, with $\mathfrak{S} = \mathfrak{S}_{\sigma_1} \cup \mathfrak{S}_{\sigma_2}$ and with $\boldsymbol{\mathcal{K}}$ as its age. Namely, $\boldsymbol{\mathcal{H}}$ has two components, the $\mathfrak{S}_{\sigma_i}$-shaped component of $\boldsymbol{\mathcal{H}}$ being $\boldsymbol{\mathcal{H}}_{\sigma_i}$ with age $\boldsymbol{\mathcal{K}}(\sigma_i)$. Then the shaping $\mathfrak{s}$ of any $\mathbf{P} \in \boldsymbol{\mathcal{K}}$ induces a partition on the universe P of $\mathbf{P}$, namely if $P_i = \mathfrak{s}^{-1}(\mathfrak{S}_{\sigma_i})$, then $P = P_1 \cup P_2$. Let then $\mathbf{P}_i$ be the substructure of $\mathbf{P}$ on the points P_i, and refer to $\mathbf{P}_i$ as the components of $\mathbf{P}$. Then we have

$$\boldsymbol{\mathcal{K}} = \{\mathbf{P} : \mathbf{P}_1 \in \boldsymbol{\mathcal{K}}(\sigma_1), \mathbf{P}_2 \in \boldsymbol{\mathcal{K}}(\sigma_2)\},$$

with the shapes and the partial order structure on the sets P_i, described already in the one-component case. To understand what $\boldsymbol{\mathcal{K}}$, and consequently $\boldsymbol{\mathcal{H}}$, looks like, we will specify the partial order on the pairs (p_1, p_2) with $p_i \in P_{\sigma_i}$.

Further, if $\mathbf{P}_i$ is an antichain of chains, then P_i partitions into maximal chains on the sets $\{P_{i,j} : j \in \mathcal{I}_i\}$ for some set $\mathcal{I}_i$, and if $\mathbf{P}_i$ is a chain of antichains, then P_i partitions into maximal antichains on the sets $\{P_{i,j} : j \in \mathcal{I}_i\}$ for some set $\mathcal{I}_i$.

(i) The structure $\boldsymbol{\mathcal{H}}_{\sigma_1||\sigma_2}$ is a Fraïssé limit of the class $\boldsymbol{\mathcal{K}}(\sigma_1||\sigma_2)$, where

$$\forall \mathbf{P} \in \boldsymbol{\mathcal{K}}(\sigma_1||\sigma_2), \forall p_1 \in \mathbf{P}_1, \forall p_2 \in \mathbf{P}_2 \quad p_1||p_2.$$

So the two components $\mathbf{P}_i$ of any shaped partial order $\mathbf{P}$ in $\boldsymbol{\mathcal{K}}(\sigma_1||\sigma_2)$ are *incomparable*, and so are the components $\boldsymbol{\mathcal{H}}_{\sigma_i}$ of $\boldsymbol{\mathcal{H}}_{\sigma_1||\sigma_2}$.

(ii) The structure $\boldsymbol{\mathcal{H}}_{\sigma_1<_c\sigma_2}$ is a Fraïssé limit of the class $\boldsymbol{\mathcal{K}}(\sigma_1 <_c \sigma_2)$, where

$$\forall \mathbf{P} \in \boldsymbol{\mathcal{K}}(\sigma_1 <_c \sigma_2), \forall p_1 \in \mathbf{P}_1, \forall p_2 \in \mathbf{P}_2 \quad p_1 < p_2.$$

In this case then, the component $\mathbf{P}_1$ of any shaped partial order $\mathbf{P}$ in $\boldsymbol{\mathcal{K}}(\sigma_1 <_c \sigma_2)$ is completely below the other component $\mathbf{P}_2$. Hence in $\boldsymbol{\mathcal{H}}_{\sigma_1<_c\sigma_2}$, the component $\boldsymbol{\mathcal{H}}_{\sigma_1}$ is *completely* below the component $\boldsymbol{\mathcal{H}}_{\sigma_2}$.

(iii) The structure $\boldsymbol{\mathcal{H}}_{\sigma_1<_{pm}\sigma_2}$ is a Fraïssé limit of the class $\boldsymbol{\mathcal{K}}(\sigma_1 <_{pm} \sigma_2)$, if for each $\mathbf{P} \in \boldsymbol{\mathcal{K}}(\sigma_1 <_{pm} \sigma_2)$, each chain of $\mathbf{P}_1$ is below at most one chain in $\mathbf{P}_2$.

There is also an n, with $2 \leq n \leq \aleph_0$, such that $\boldsymbol{\mathcal{H}}_{\sigma_i}$ consists of n incomparable chains on the sets of points $H_{i,j}$, each isomorphic to an $\mathfrak{S}_{\sigma_i}$-shaped copy of $\mathbb{Q}$. Then for any $j, j' \in [n]$ the chain $H_{1,j}$ is completely below the chain $H_{2,j}$ and incomparable with all other chains $H_{2,j'}$. So there is a *perfect matching* between the chains of $\boldsymbol{\mathcal{H}}_{\sigma_1}$ and $\boldsymbol{\mathcal{H}}_{\sigma_2}$.

(iv) The structure $\boldsymbol{\mathcal{H}}_{\sigma_1<_{cpm}\sigma_2}$ is a Fraïssé limit of the class $\boldsymbol{\mathcal{K}}(\sigma_1 <_{cpm} \sigma_2)$, if for each $\mathbf{P} \in \boldsymbol{\mathcal{K}}(\sigma_1 <_{cpm} \sigma_2)$, each chain of $\mathbf{P}_1$ is incomparable with at most one chain in $\mathbf{P}_2$, and below all the others.

There is also an n, with $2 \leq n \leq \aleph_0$, such that $\boldsymbol{\mathcal{H}}_{\sigma_i}$ consists of n incomparable chains on the sets of points $H_{i,j}$, each isomorphic to an $\mathfrak{S}_{\sigma_i}$-shaped copy of $\mathbb{Q}$. Then the chain $H_{1,j}$ is incomparable with the chain $H_{2,j}$ and completely below all other chains $H_{2,j'}$. So the relationship between the chains of $\boldsymbol{\mathcal{H}}_1$ and $\boldsymbol{\mathcal{H}}_2$ is a *complement of a perfect matching.*

(v) The structure $\boldsymbol{\mathcal{H}}_{\sigma_1<_g\sigma_2}$ is a Fraïssé limit of the class $\boldsymbol{\mathcal{K}}(\sigma_1 <_g \sigma_2)$, if

(a) the classes $\boldsymbol{\mathcal{K}}(\sigma_i)$ are each a $\boldsymbol{\mathcal{K}}(AC)$ or a $\boldsymbol{\mathcal{K}}(G)$,

(b) if $\mathbf{P}_1 \in \boldsymbol{\mathcal{K}}(\sigma_1)$ and $\boldsymbol{\mathcal{K}}(\sigma_1)$ is a class $\boldsymbol{\mathcal{K}}(AC)$, then P_1 partitions into maximal chains on the sets $\{P_{1,i} : i \in \mathcal{I}\}$, and we have

$$\forall i \in \mathcal{I}, \forall p, p' \in P_{1,i}, \forall q \in P_2 \quad (p < q) \iff (p' < q),$$

and similarly if $\boldsymbol{\mathcal{K}}(\sigma_2)$ is a class $\boldsymbol{\mathcal{K}}(AC)$,

(c) the partial order on P is any partial order extending the partial orders on P_i and satisfying the condition (b).

Condition (b) considers the case when $\mathbf{P}_1$ (or $\mathbf{P}_2$) is in the class of antichains of chains. It says that for any point $q \in \mathbf{P}_2$ (or $q \in \mathbf{P}_1$), any maximal chain $\mathbf{P}_{1,i}$ of $\mathbf{P}_1$ (or $\mathbf{P}_{2,i}$ of $\mathbf{P}_2$) is either all incomparable with q or completely below q (completely above q). Thus in case that $\mathbf{P}_1$ (or $\mathbf{P}_2$) is an antichain, the condition (b) is trivially true.

Consider the case where for $i \in \{1, 2\}$ either $l_1(\sigma_i) = G$ or $l_1(\sigma_i) = AC$ and $\boldsymbol{\mathcal{K}}(\sigma_i)$ is the class of antichains. The class $\boldsymbol{\mathcal{K}}(\sigma_1 <_g \sigma_2)$ consists of all shaped partial orders $\mathbf{P}$ with $\mathbf{P}_i \in \boldsymbol{\mathcal{K}}(\sigma_i)$. So the homogeneous structure $\boldsymbol{\mathcal{H}}_{\sigma_1 <_g \sigma_2}$, the Fraïssé limit of $\boldsymbol{\mathcal{K}}(\sigma_1 <_g \sigma_2)$, consists of $\boldsymbol{\mathcal{H}}_{\sigma_1}$ *generically* below $\boldsymbol{\mathcal{H}}_{\sigma_2}$.

Consider now the case where either $l_1(\sigma_1) = AC$ and $\boldsymbol{\mathcal{H}}(\sigma_1)$ is a homogeneous shaped antichain of chains, and either $l_1(\sigma_2) = G$ or $l_1(\sigma_2) = AC$ and $\boldsymbol{\mathcal{H}}(\sigma_2)$ is a homogeneous shaped antichain. Let σ_1' be such that $\boldsymbol{\mathcal{H}}_{\sigma_1' <_g \sigma_2}$ and $\boldsymbol{\mathcal{H}}(\sigma_1')$ is an antichain. To construct $\boldsymbol{\mathcal{H}}_{\sigma_1 <_g \sigma_2}$, replace each point of $\boldsymbol{\mathcal{H}}(\sigma_1')$ by an $\mathfrak{S}_{\sigma_1}$-shaped copy of $\mathbb{Q}$. Construct $\boldsymbol{\mathcal{H}}_{\sigma_1 <_g \sigma_2}$ similarly if $\boldsymbol{\mathcal{H}}(\sigma_2)$ or both $\boldsymbol{\mathcal{H}}(\sigma_i)$ are homogeneous shaped antichains of chains.

(vi) The structure $\boldsymbol{\mathcal{H}}_{\sigma_1 <_{sh} \sigma_2}$ is a Fraïssé limit of the class $\boldsymbol{\mathcal{K}}(\sigma_1 <_{sh} \sigma_2)$, if

(a) the classes $\boldsymbol{\mathcal{K}}(\sigma_i)$ are both of form $\boldsymbol{\mathcal{K}}(CA)$,

(b) if $\mathbf{P}_i \in \boldsymbol{\mathcal{K}}(\sigma_i)$, then P_i partitions into maximal antichains on the sets $\{P_{i,j} : j \in \mathcal{I}_i\}$, such that

$$\forall p \in P_{i,j}, \forall p' \in P_{i,j'} \quad (p < p') \iff j < j',$$

and

(c) either the structure $\mathbf{P} \in \boldsymbol{\mathcal{K}}(\sigma_1 <_{sh} \sigma_2)$ satisfies the following: given any $a_1 \in \mathcal{A}_1, a_2 \in \mathcal{A}_2$,

i. if $\mathbf{P}_{1,j}$ and $\mathbf{P}_{1,j'}$ are both $\mathfrak{S}_{a_1}$-shaped maximal antichains and $j < j'$, then there exists a $\mathfrak{S}_{a_2}$-shaped maximal antichain $\mathbf{P}_{2,j''}$, such that
$$\mathbf{P}_{1,j} < \mathbf{P}_{2,j''} \quad \text{and} \quad \mathbf{P}_{1,j'} || \mathbf{P}_{2,j''}$$

ii. and if $\mathbf{P}_{2,j}$ and $\mathbf{P}_{2,j'}$ are both $\mathfrak{S}_{a_2}$-shaped maximal antichains and $j < j'$, then there exists a $\mathfrak{S}_{a_1}$-shaped maximal antichain $\mathbf{P}_{1,j''}$, such that
$$\mathbf{P}_{1,j''} || \mathbf{P}_{2,j} \quad \text{and} \quad \mathbf{P}_{1,j''} < \mathbf{P}_{2,j'},$$

(d) or the structure $\mathbf{P} \in \boldsymbol{\mathcal{K}}(\sigma_1 <_{sh} \sigma_2)$ is a substructure of some structure satisfying (c).

According to the paper Torrezão de Sousa & Truss (2008), each point p of $\boldsymbol{\mathcal{H}}_{\sigma_1}$ splits the points of $\boldsymbol{\mathcal{H}}_{\sigma_2}$ into sets $H_{p,l}$ and $H_{p,u}$, the lower and the upper part of $\boldsymbol{\mathcal{H}}_{\sigma_2}$ such that p is incomparable with $H_{p,l}$ and p is completely below $H_{p,u}$.

More precisely, suppose that $\{H_{i,j} : j \in \mathcal{I}_i\}$ is the partition of the set H_{σ_1} into maximal antichains of $\boldsymbol{\mathcal{H}}_{\sigma_1}$. Then for each $j \in \mathcal{I}_1$ there is a $j' \in \mathcal{I}_2$, such that for any $p \in H_{1,j}$ and $q \in H_{1,j''}$ we have $p < q$ precisely if $j' < j''$.

Since $\boldsymbol{\mathcal{H}}_{\sigma_i}$ are both chains of antichains, there are sets $\mathcal{I}_i$ and $\mathcal{A}_i$ and maps $\mathfrak{s}_{\sigma_i,1} : \mathcal{I}_i \to \mathcal{A}_i$, such that $(\mathcal{I}_i, <, \mathfrak{s}_{\sigma_i,1})$ is isomorphic to $\mathbb{Q}_{\mathcal{A}_i}$. Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are disjoint and consider the structure $\mathbb{Q}_{\mathcal{A}_1 \cup \mathcal{A}_2}$, which is unique up to isomorphism. Then its substructure on the points of shapes in $\mathcal{A}_i$ is isomorphic to $\mathbb{Q}_{\mathcal{A}_i}$. We obtain $\boldsymbol{\mathcal{H}}_{\sigma_1 <_{sh} \sigma_2}$ by replacing each a-shaped point of $\mathbb{Q}_{\mathcal{A}_1 \cup \mathcal{A}_2}$ with the appropriate $\mathfrak{S}_a$-shaped antichain of $\boldsymbol{\mathcal{H}}_{\sigma_1}$ or $\boldsymbol{\mathcal{H}}_{\sigma_2}$. The total order on $\mathbb{Q}_{\mathcal{A}_1 \cup \mathcal{A}_2}$ defines the partial order on $\boldsymbol{\mathcal{H}}_{\sigma_1 <_{sh} \sigma_2}$. Suppose that $i, i' \in \mathbb{Q}_{\mathcal{A}_1 \cup \mathcal{A}_2}$ and $i < i'$. If the antichain replacing i is an antichain of $\boldsymbol{\mathcal{H}}_{\sigma_2}$ and the antichain replacing i' is an antichain of $\boldsymbol{\mathcal{H}}_{\sigma_1}$, the two antichains are incomparable. Otherwise, the antichain replacing i lies completely below antichain replacing i'.

We say that the maximal antichains of $\mathcal{H}_{\sigma_1}$ *shuffle* between the maximal antichains of $\mathcal{H}_{\sigma_2}$.

Note. The only difference between conditions (iii) and (iv) in Definition 2.4.10 is that in (iii) the components are either an AC or a G, and in (iv) the components are both a CA. The different notation will, however, be useful later.

Skeleton

The concept of a *skeleton* codes how a homogeneous shaped partial order is built from its components.

Definition 2.4.12. A *skeleton* $\Sigma = (\Sigma, <, l_1, l_2)$ is a partial order $(\Sigma, <)$ together with maps

- $l_1 : \Sigma \to \{AC, CA, G\}$ and
- $l_2 : \{(\sigma, \tau) \in \Sigma^2 : \sigma < \tau\} \to \{<_g, <_c, <_{cpm}, <_{pm}, <_{sh}\}$.

For each $\sigma, \tau \in \Sigma$, with $\sigma < \tau$ we call $l_1(\sigma)$ and $l_2(\sigma, \tau)$ the *label of a point* and the *label of a relation* between two points respectively.

Note. Throughout the thesis we will abuse notation and write

$$l_2 : \{<\} \to \{<_g, <_c, <_{cpm}, <_{pm}, <_{sh}\}.$$

Only $<_c$ and $<_{sh}$ are partial order relations; the rest need not be transitive.

We will restrict our attention to a class of skeletons that additionally satisfy certain conditions. The following correspond to the *abstract skeletons* defined in the paper Torrezão de Sousa & Truss (2008).

Definition 2.4.13. A *good skeleton* is a skeleton that obeys the rules described below. For any two components σ_1, σ_2 we have:
2-chain lemmas: For two components with $\sigma_1 < \sigma_2$ we have the following

options for two components and the relation between them.

	$l_1(\sigma_1)$	$l_1(\sigma_2)$	$l_2(\sigma_1,\sigma_2)$
1.)	CA	CA	$<_c$, $<_{sh}$
2.)	CA	AC	$<_c$
3.)	CA	G	$<_c$
4.)	AC	AC	$<_c$, $<_g$, $<_{pm}$, $<_{cpm}$
5.)	AC	G	$<_c$, $<_g$
6.)	G	G	$<_c$, $<_g$

Moreover, in cases of two components with different labels, the analogous option applies for $\sigma_1 > \sigma_2$.

V-shape lemmas For any three components σ_1, σ_2, σ_3, with $\sigma_1 < \sigma_2$, $\sigma_1 < \sigma_3$ and $\sigma_2 || \sigma_3$ we have:

	$l_1(\sigma_1)$	$l_1(\sigma_2)$	$l_1(\sigma_3)$	$l_2(\sigma_1,\sigma_2)$	$l_2(\sigma_1,\sigma_3)$
1.)	any	any	any	$<_c$	any allowed
2.)	CA	CA	CA	$<_{sh}$	$<_{sh}$
3.)	AC or G	AC or G	AC or G	$<_g$	$<_g$

Λ-shape lemmas

Conditions analogous to conditions in V-shape lemmas, but with $\sigma_1 > \sigma_2$, $\sigma_1 > \sigma_3$ and $\sigma_2 || \sigma_3$.

3-chain lemmas: For three components with $\sigma_1 < \sigma_2 < \sigma_3$ we have the following options for the components and relations between them.

	$l_1(\sigma_1)$	$l_1(\sigma_2)$	$l_1(\sigma_3)$	$l_2(\sigma_1,\sigma_2)$	$l_2(\sigma_2,\sigma_3)$	$l_2(\sigma_1,\sigma_3)$
1.)	any	any	any	$<_c$	any allowed	$<_c$
1.)*	any	any	any	any allowed	$<_c$	$<_c$
2.)	CA	CA	CA	$<_{sh}$	$<_{sh}$	$<_{sh}$
3.)	AC	AC	AC	$<_{pm}$	$<_{pm}$	$<_{pm}$
4.)	AC	AC	AC	$<_{pm}$	$<_{cpm}$	$<_{cpm}$
4.)*	AC	AC	AC	$<_{cpm}$	$<_{pm}$	$<_{cpm}$
5.)	AC	AC	AC/G	$<_{pm}$	$<_g$	$<_g$
5.)*	AC/G	AC	AC	$<_g$	$<_{pm}$	$<_g$
6.)	AC	AC/G	AC	$<_g$	$<_g$	$<_{cpm}$
7.)	AC/G	AC/G	AC/G	$<_g$	$<_g$	$<_c, <_g$

Note that the Case 1 and Case 1* are dual, and similarly for the other paired cases.

To define a shaped homogeneous partial order, we need a good skeleton and additional information.

Definition 2.4.14. Fix a good skeleton Σ, a set of shapes $\mathfrak{S}$, together with a partition $\{\mathfrak{S}_\sigma : \sigma \in \Sigma\}$ of $\mathfrak{S}$, and

- for each $\sigma \in \Sigma$ with $l_1(\sigma) = AC$, numbers $n_{\sigma,1}$ and $n_{\sigma,2}$, with
  - $2 \leq n_{\sigma,1} \leq \aleph_0$ and $n_{\sigma,2} \in \{1, \aleph_0\}$, or $n_{\sigma,1} = n_{\sigma,2} = 1$, and
  - $|\mathfrak{S}_\sigma| = 1$ if $n_{\sigma,2} = 1$;
- for each $\sigma \in \Sigma$ with $l_1(\sigma) = CA$,
  - a partition $\{\mathfrak{S}_{\sigma,a} : a \in \mathcal{A}_\sigma\}$ of $\mathfrak{S}_\sigma$, and
  - for each $\mathbf{s}_\sigma^{a,b} \in \mathfrak{S}_{\sigma,a}$, a number $n_{\sigma,a,b}$ with $1 \leq n_{\sigma,a,b} \leq \aleph_0$.

Then the *shaped partial order* $\boldsymbol{\mathcal{H}}(\Sigma)$ is a structure $\boldsymbol{\mathcal{H}}(\Sigma) = (H, <, \mathfrak{s})$ satisfying the following. The set H partitions as $\{H_\sigma : \sigma \in \Sigma\}$, such that

(i) (a) if $l_1(\sigma) = AC$, $\boldsymbol{\mathcal{H}}_\sigma$ is a homogeneous interdensely $\mathfrak{S}_\sigma$-shaped antichain of $n_{\sigma,1}$ chains of size $n_{\sigma,2}$,

(b) if $l_1(\sigma) = CA$, $\boldsymbol{\mathcal{H}}_\sigma$ is a homogeneous interdensely $\mathfrak{S}_\sigma$-shaped chain of antichains, where for each maximal antichain $H_{\sigma,i}$ there is an $a \in \mathcal{A}_\sigma$ such that the substructure of $\boldsymbol{\mathcal{H}}$ on $H_{\sigma,i}$ consists of $n_{\sigma,a,b}$ points of shape $\mathbf{s}_\sigma^{a,b}$ for each $\mathbf{s}_\sigma^{a,b} \in \mathfrak{S}_{\sigma,a}$, and

(c) if $l_1(\sigma) = G$, $\boldsymbol{\mathcal{H}}_\sigma$ is a homogeneous interdensely $\mathfrak{S}_\sigma$-shaped generic partial order;

(ii) (a) if $l_2(\sigma, \sigma') = <_l$ for some $<_l \in \{<_c, <_g, <_{sh}, <_{pm}, <_{cpm}\}$, then the substructure of $\boldsymbol{\mathcal{H}}$ on the set of points $H_\sigma \cup H_{\sigma'}$ is isomorphic to $\boldsymbol{\mathcal{H}}_{\sigma <_l \sigma'}$, and

(b) if, for distinct σ, σ', we do not have $\sigma < \sigma'$ or $\sigma' < \sigma$ in Σ, then we have $x || y$ for all $x \in H_\sigma, y \in H_{\sigma'}$;

(iii) $(H, <)$ is a partial order.

Remarks 2.4.15. (i) We refer to integers $n_{\sigma,1}, n_{\sigma,2}$ and $n_{\sigma,a,b}$ as *multiplicities*. The notation $\boldsymbol{\mathcal{H}}(\Sigma)$ emphasises the role of Σ in constructing a homogeneous structure, but hides the fact that one has to specify the set of shapes $\mathfrak{S}$ and the multiplicities to define a structure $\boldsymbol{\mathcal{H}}(\Sigma)$.

(ii) The 2-chain lemmas correspond exactly to the $<_l$-conditions in Definition 2.4.10. Thus by Lemma 2.4.11, the condition (ii) (a) in Definition 2.4.14 is compatible with the conditions in (i).

(iii) The 3-chain, V-shape and Λ-shape lemmas ensure that a partial order, with pairs of components isomorphic to $\boldsymbol{\mathcal{H}}_{\sigma<_l\sigma'}$, exists and is homogeneous. For example, if we have $H_{\sigma_1} <_c H_{\sigma_2}$ and $H_{\sigma_2} <_c H_{\sigma_3}$, then by transitivity we must have $H_{\sigma_1} <_c H_{\sigma_3}$. This is confirmed by the 3-chain lemma 1.) or 1.)*.

The main result in Torrezão de Sousa & Truss (2008) is the following.

Theorem 2.4.16. *Any structure $\boldsymbol{\mathcal{H}}(\Sigma)$ defined in Definition 2.4.14 is a countable homogeneous shaped partial order. Further, any countable homogeneous shaped partial order is isomorphic to an $\boldsymbol{\mathcal{H}}(\Sigma)$ for some choice of Σ, with corresponding data as in Definition 2.4.14.*

Note. If $\boldsymbol{\mathcal{H}}(\Sigma)$ has only one component, *i.e.*, $\Sigma = \{\sigma\}$, we know that $l_1(\sigma) \in \{AC, CA, G\}$. Depending on the label of σ, we will refer to one component homogeneous shaped partial orders as $\boldsymbol{\mathcal{H}}(AC)$, $\boldsymbol{\mathcal{H}}(CA)$ or $\boldsymbol{\mathcal{H}}(G)$.

2.5 Ramsey classes of ordered shaped partial orders

Model theory of shaped ordered partial orders

In 2.4.1 we defined a shaped partial order. We add a total order.

Definition 2.5.1. Let $<$ and $\prec$ be binary relations and let $\mathbf{s}^a$ for $a \in \mathcal{A}$ be unary relations. Let $\mathfrak{S} = \{\mathbf{s}^a : a \in \mathcal{A}\}$ and let P be a set. An *ordered $\mathfrak{S}$-shaped partial order* $\mathbf{P}$ is a structure

$$\mathbf{P} = \langle P, <, \prec, \{\mathbf{s}^a\}_{a \in \mathcal{A}} \rangle,$$

where

(i) $\mathbf{P} = \langle P, <, \{\mathbf{s}^a\}_{a \in \mathcal{A}} \rangle$ is a shaped partial order, and

(ii) the structure $(P, \prec)$ is a chain.

Similar to 2.4.2, we use notation $\mathbf{P} = (P, <, \prec, \mathfrak{s})$ to denote an ordered $\mathfrak{S}$-shaped partial order $\mathbf{P}$.

Further, we introduced the interdense relation on the points of any homogeneous shaped partial order $\boldsymbol{\mathcal{H}}(\Sigma)$ in Definition 2.4.4. We observed that it partitions the set $\mathfrak{S}$ of shapes into subsets $\mathfrak{S}_\sigma$ and the similarly the universe H of $\boldsymbol{\mathcal{H}}$ into components H_σ, for some skeleton Σ and for $\sigma \in \Sigma$. So when $\mathfrak{S}_\sigma$ is finite, $\mathfrak{S}_\sigma = \{\mathbf{s}^1_\sigma(p), \mathbf{s}^2_\sigma(p), \ldots, \mathbf{s}^n_\sigma(p)\}$, we could introduce unary relations F_σ to denote membership in the $\mathfrak{S}_\sigma$-shaped component $\boldsymbol{\mathcal{H}}_\sigma(\Sigma)$ of $\boldsymbol{\mathcal{H}}(\Sigma)$. We have

$$\forall p,\ (F_\sigma(p) \iff p \in H_\sigma(\Sigma) \iff \mathfrak{s}(p) \in \mathfrak{S}_\sigma).$$

Then if $\boldsymbol{\mathcal{K}}(\Sigma)$ is the age of $\boldsymbol{\mathcal{H}}(\Sigma)$ and $\mathbf{P} \in \boldsymbol{\mathcal{K}}(\Sigma)$, the relations F_σ denote membership in $\mathfrak{S}_\sigma$-shaped component of $\mathbf{P}$ as well.

We will regularly use shapes to partition a class being studied. For example, we might study a class with a skeleton consisting of two points, σ_1, σ_2, and thus of structures that are $\mathfrak{S}_\sigma$-shaped, or $\mathfrak{S}_{\sigma'}$-shaped, or have two parts, one $\mathfrak{S}_\sigma$-shaped and one $\mathfrak{S}_{\sigma'}$-shaped. At a later point, we might focus on a $\mathfrak{S}_\sigma$-shaped part of a structure in a class $\mathcal{K}(\Sigma)$ for some Σ and combine the other shape relations to get a part that is not $\mathfrak{S}_\sigma$-shaped. Formally, we will consider classes of the following form.

Definition 2.5.2. Let L be a relational language. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be classes in language L, closed under substructures. A class $\mathcal{K}$ is a *merge of classes* $\mathcal{K}_1$ *and* $\mathcal{K}_2$, if $\mathcal{K}$ is a class in language L, that contains relations F_1, F_2 of arity 1, such that the following hold.

(i) For any $\mathbf{A} \in \mathcal{K}$ and $a \in \mathbf{A}$, the following statement is true:

$$(F_1(a) \vee F_2(a)) \wedge \neg(F_1(a) \wedge F_2(a)).$$

(ii) For each $\mathbf{A} \in \mathcal{K}$ the relations F_1, F_2 partition the universe of $\mathbf{A}$ into

$$A = A_1 \cup A_2,$$

such that for all $a \in A_1$ we have $F_1(a)$ and for all $a \in A_2$ we have $F_2(a)$. If A_1 is non-empty, the substructure $\mathbf{A}_1$ of $\mathbf{A}$ with universe A_1 lies in the class $\mathcal{K}_1$; analogous for A_2.

If A_1 and A_2 are both non-empty, the structure $\mathbf{A}$ is a *merge of structures* $\mathbf{A}_1$ *and* $\mathbf{A}_2$.

(iii) For each $\mathbf{A}_1 \in \mathcal{K}_1$, $\mathbf{A}_2 \in \mathcal{K}_2$, the class $\mathcal{K}$ contains $\mathbf{A}_1$, $\mathbf{A}_2$ and at least one structure $\mathbf{A}$, that is a merge of structures $\mathbf{A}_1$ and $\mathbf{A}_2$.

The Definition 2.5.2 extends naturally to more than two classes.

Definition 2.5.3. Let L be a relational language. Let $\mathcal{K}_i$, for $i \in [n]$, be classes in language L, closed under substructures. A class $\mathcal{K}$ is a *merge of classes* $\mathcal{K}_i$*, for* $i \in [n]$, if $\mathcal{K}$ is a class in language L, that contains relations $F_1, F_2, \ldots, F_n$ of arity 1, such that the following hold.

(i) For any $\mathbf{A} \in \mathcal{K}$ and $a \in \mathbf{A}$, the following statements are true:

$$F_1(a) \vee F_2(a) \vee \ldots \vee F_n, \text{ and}$$

$$\text{for all } i \neq j; i,j \in [n] : \neg(F_i(a) \wedge F_j(a)).$$

(ii) For each $\mathbf{A} \in \mathcal{K}$ the relations F_i partition the universe of $\mathbf{A}$ into

$$A = \bigcup_{i \in [n]} A_i$$

such that for all $a \in A_i$ we have $F_i(a)$. If A_i is non-empty, the substructure $\mathbf{A}_i$ of $\mathbf{A}$ with universe A_i lies in the class $\mathcal{K}_i$.

If A_i are all non-empty, the structure $\mathbf{A}$ is a *merge of structures* $\mathbf{A}_i$, *for* $i \in [n]$.

(iii) For each non-empty subset N of $[n]$ and a selection of structures $\mathbf{A}_i \in \mathcal{K}_i$, for $i \in N$, the class $\mathcal{K}$ contains $\mathbf{A}_i$ for each $i \in N$, and at least one structure $\mathbf{A}$, that is a merge of structures $\mathbf{A}_i$.

Link. Theorem 4.3.5, Theorem 4.2.4

While the Definition 2.5.3 is technical, it formalises a very common notion.

Lemma 2.5.4. *Let Σ be a good skeleton, and let $\mathfrak{S}$ be a set of shapes, partitioning into disjoint sets $\mathfrak{S} = \{\mathfrak{S}_\sigma : \sigma \in \Sigma\}$. For each $\sigma \in \Sigma$, let $\mathcal{K}_\sigma$ be a class of $\mathfrak{S}_\sigma$-shaped partial orders ($\mathcal{K}(G), \mathcal{K}(AC)$ or $\mathcal{K}(GA)$), based on the label $l_1(\sigma)$. Then the class $\mathcal{K}$ is a merge of classes $\mathcal{K}_\sigma$ for $\sigma \in \Sigma$.*

The proof of this lemma consists only of unravelling of definitions. The labels of relations between points in Σ further specify which structures the merge $\mathcal{K}$ contains.

Finite restriction

Suppose that $\mathfrak{S} = \{\mathfrak{s}\}$ and that $\mathcal{K}$ is a class of all ordered $\mathfrak{S}$-shaped partial orders. Take any two ordered $\mathfrak{S}$-shaped partial orders $\mathbf{P}, \mathbf{R} \in \mathcal{K}$, with

$$\mathbf{P} = (P, <^{\mathbf{P}}, \prec^{\mathbf{P}}, \mathfrak{s}^{\mathbf{P}}) \text{ and } \mathbf{R} = (R, <^{\mathbf{R}}, \prec^{\mathbf{R}}, \mathfrak{s}^{\mathbf{R}}).$$

Then for each $p \in P, r \in R$ we have

$$\mathfrak{s}^{\mathbf{P}}(p) = \mathfrak{s}^{\mathbf{R}}(r) = \mathfrak{s}.$$

So $\mathbf{P}$ and $\mathbf{R}$ are isomorphic precisely when their unshaped reducts, the structures $(P, <^{\mathbf{P}}, \prec^{\mathbf{P}})$ and $(R, <^{\mathbf{R}}, \prec^{\mathbf{R}})$, are isomorphic.

Lemma 2.5.5. *Let $\mathcal{K}$ be a Fraïssé class of ordered $\mathfrak{S}$-shaped partial orders that is a Ramsey class and let $\mathfrak{s} \in \mathfrak{S}$. Let $\mathcal{K}_{\mathfrak{s}}$ of all $\mathfrak{s}$-shaped structures in $\mathcal{K}$. The following hold.*

(i) If the class $\mathcal{K}$ is Ramsey, then so is $\mathcal{K}_{\mathfrak{s}}$.

(ii) The class $\mathcal{K}_{\mathfrak{s}}$ is Fraïssé.

(iii) For some Fraïssé class $\mathcal{K}_{\mathfrak{s}}$ of ordered partial orders, the class $\mathcal{K}_{\mathfrak{s}}$ consists precisely of structures $\mathbf{P} = (P, <^{\mathbf{P}}, \prec^{\mathbf{P}}, \mathfrak{s}^{\mathbf{P}})$, such that

$$(P, <^{\mathbf{P}}, \prec^{\mathbf{P}}) \in \mathcal{K}_{\mathfrak{s}}.$$

(iv) Let $\mathcal{K}'_{\mathfrak{s}}$ be the class of all reducts $(P, <^{\mathbf{P}})$ of structures $(P, <^{\mathbf{P}}, \prec^{\mathbf{P}})$ in $\mathcal{K}_{\mathfrak{s}}$. Then $\mathcal{K}'_{\mathfrak{s}}$ is a Fraïssé class.

Proof. To prove part (i), take any $\mathbf{Q}, \mathbf{R} \in \mathcal{K}_{\mathfrak{s}}$. Since $\mathbf{Q}, \mathbf{R} \in \mathcal{K}$, there exists $\mathbf{P} \in \mathcal{K}$ such that $\mathbf{P} \to (\mathbf{R})^{\mathbf{Q}}_{k}$. Let $\mathbf{P}_{\mathfrak{s}}$ be the $\mathfrak{s}$-shaped substructure of $\mathbf{P}$. Then for any $\mathbf{Q}' \in \binom{\mathbf{P}}{\mathbf{Q}}$ and $\mathbf{R}' \in \binom{\mathbf{P}}{\mathbf{R}}$, $\mathbf{Q}'$ and $\mathbf{R}'$ are both substructures of $\mathbf{P}_{\mathfrak{s}}$. Thus given any colouring of $\binom{\mathbf{P}'}{\mathbf{Q}}$, we can define the corresponding colouring of $\binom{\mathbf{P}}{\mathbf{Q}}$. The $\mathbf{R}' \in \binom{\mathbf{P}}{\mathbf{R}}$ yielding the monochromatic $\binom{\mathbf{R}'}{\mathbf{Q}}$, that exists as $\mathcal{K}$ is Ramsey, then corresponds to the monochromatic $\mathbf{R}' \in \binom{\mathbf{P}'}{\mathbf{R}}$, showing that $\mathcal{K}_{\mathfrak{s}}$ is Ramsey as well.

The arguments needed to prove part (ii) are similar. Recall the definition of a Fraïssé class, Definition 2.1.3. The class $\boldsymbol{\mathcal{K}}_{\mathbf{s}}$ trivially has hereditary property, as any substructure of an $\mathbf{s}$-shaped structure is $\mathbf{s}$-shaped. To show that $\boldsymbol{\mathcal{K}}_{\mathbf{s}}$ has joint embedding property and amalgamation property proceed similarly. Given a structure $\mathbf{P} \in \boldsymbol{\mathcal{K}}$, let $\mathbf{P}_{\mathbf{s}}$ be the $\mathbf{s}$-shaped substructure of $\mathbf{P}$.

(i) Given $\mathbf{Q}, \mathbf{R} \in \boldsymbol{\mathcal{K}}_{\mathbf{s}}$, there exists $\mathbf{P} \in \boldsymbol{\mathcal{K}}$ showing that $\boldsymbol{\mathcal{K}}$ has joint embedding property. Then $\mathbf{P}_{\mathbf{s}}$ shows that $\boldsymbol{\mathcal{K}}_{\mathbf{s}}$ has joint embedding property.

(ii) Given $\mathbf{Q}, \mathbf{R}_1, \mathbf{R}_2 \in \boldsymbol{\mathcal{K}}_{\mathbf{s}}$ and embeddings $e_1 : \mathbf{Q} \to \mathbf{R}_1, e_2 : \mathbf{Q} \to \mathbf{R}_2$, there exists $\mathbf{P} \in \boldsymbol{\mathcal{K}}$ showing that $\boldsymbol{\mathcal{K}}$ has amalgamation property. Then $\mathbf{P}_{\mathbf{s}}$ shows that $\boldsymbol{\mathcal{K}}_{\mathbf{s}}$ has amalgamation property.

To prove part (iii), observe that for any $\mathbf{P} \in \boldsymbol{\mathcal{K}}_{\mathbf{s}}$ and any $p \in \mathbf{P}$, we have $\mathfrak{s}^{\mathbf{P}}(p) = \mathbf{s}$. So class $\mathcal{K}_{\mathbf{s}}$ of unshaped reducts of structures in $\boldsymbol{\mathcal{K}}_s$ has hereditary property, joint embedding property and amalgamation property because the class $\boldsymbol{\mathcal{K}}_{\mathbf{s}}$ does.

First notice that $\mathcal{K}_{\mathbf{s}}$ plays the role of $\boldsymbol{\mathcal{K}}$ in Proposition 2.3.3 and $\mathcal{K}'_{\mathbf{s}}$ plays the role of $\boldsymbol{\mathcal{K}}_0$. By definition of $\mathcal{K}'_{\mathbf{s}}$, the class $\mathcal{K}_{\mathbf{s}}$ is reasonable with respect to $\mathcal{K}'_{\mathbf{s}}$. Thus the class $\mathcal{K}'_{\mathbf{s}}$ is Fraïssé. $\square$

Consider again the Theorem 2.2.18, the summary of results showing that certain classes of ordered partial orders are Ramsey and that others are not, from Sokić (2012a) and Sokić (2012b). The index n in the notation of A_n, AC_n and CA_n in each case denotes the width of a maximal antichain contained in a class of ordered partial orders. We can see that if $1 < n < \aleph_0$, a class with index n is never Ramsey. In this section we show that a similar result is true for classes of ordered shaped partial orders. The following is the result.

Lemma 2.5.6. *Let $\boldsymbol{\mathcal{K}}$ be a Fraïssé class of ordered shaped partial orders that is a Ramsey class. Suppose that for some shape $\mathbf{s}$, the class $\boldsymbol{\mathcal{K}}$ contains an $\mathbf{s}$-shaped antichain with two points, $\mathbf{A}_2$. Then the class $\boldsymbol{\mathcal{K}}$ contains an $\mathbf{s}$-shaped antichain of any finite size.*

Proof. Let $\boldsymbol{\mathcal{K}}_{\mathbf{s}}$ be the class of all $\mathbf{s}$-shaped structures in $\boldsymbol{\mathcal{K}}$. Then by Lemma 2.5.5, the class $\boldsymbol{\mathcal{K}}_{\mathbf{s}}$ is Fraïssé and so are the class $\mathcal{K}_{\mathbf{s}}$ of unshaped reducts of structures in $\boldsymbol{\mathcal{K}}_{\mathbf{s}}$ and the class $\mathcal{K}'_{\mathbf{s}}$ of unordered reducts of structures in $\boldsymbol{\mathcal{K}}_{\mathbf{s}}$. So the class $\mathcal{K}'_{\mathbf{s}}$ is a Fraïssé class of partial orders. Namely, the class $\mathcal{K}'_{\mathbf{s}}$ is one of the following:

(i) $\mathcal{K}(A_n)$ for $1 \leq n \leq \aleph_0$,

(ii) $\mathcal{K}(C)$,

(iii) $\mathcal{K}(AC_n)$ for $1 < n \leq \aleph_0$,

(iv) $\mathcal{K}(CA_n)$ for $1 < n \leq \aleph_0$, or

(v) $\mathcal{K}(G)$.

These classes are defined in Definition 2.1.9.
Additionally, since $\boldsymbol{\mathcal{K}}$ is a Ramsey class, by Lemma 2.5.5, the class $\boldsymbol{\mathcal{K}}_{\mathbf{s}}$ is a Ramsey class as well.
Now, since $\boldsymbol{\mathcal{K}}$ is a Fraïssé class, it also contains an $\mathbf{s}$-shaped antichain with one point, $\mathbf{A}_1$, as $\mathbf{A}_1$ is a substructure of $\mathbf{A}_2$.
Suppose first that $\mathbf{A}_n$ is the largest $\mathbf{s}$-shaped antichain in the class $\boldsymbol{\mathcal{K}}$ and that $\mathbf{A}_n$ is of size n. Since $A_2 \in \boldsymbol{\mathcal{K}}$, we know that $n \geq 2$. Then the class $\mathcal{K}'_{\mathbf{s}}$ must be one of the following three: $\mathcal{K}(A_n)$, $\mathcal{K}(AC_n)$ or $\mathcal{K}(CA_n)$. We consider two different cases.

(i) The class $\mathcal{K}'_{\mathbf{s}}$ is $\mathcal{K}(AC_n)$.

Take any $\mathbf{P} \in \boldsymbol{\mathcal{K}}_{\mathbf{s}}$. Take any chain $\mathbf{P}_i$ of $\mathbf{P}$ and define a colouring of $\binom{\mathbf{P}}{\mathbf{A}_1}$ as follows:

(a) If $\mathbf{A}'_1 \in \binom{\mathbf{P}_i}{\mathbf{A}_1}$, colour $\mathbf{A}'_1$ with colour 1.
(b) If $\mathbf{A}'_1 \notin \binom{\mathbf{P}_i}{\mathbf{A}_1}$, colour $\mathbf{A}'_1$ with colour 2.

Since $\mathbf{P}$ is an antichain of at most n chains, any antichain of $\mathbf{P}$ of size n will contain a point in the chain $\mathbf{P}_i$. So none of them are monochromatic under the colouring above, and thus

$$\mathbf{P} \nrightarrow (\mathbf{A}_n)^{\mathbf{A}_1}_k.$$

(ii) The class $\mathcal{K}'_{\mathbf{s}}$ is $\mathcal{K}(A_n)$ or $\mathcal{K}(CA_n)$. Take any $\mathbf{P} \in \boldsymbol{\mathcal{K}}_{\mathbf{s}}$. Then $\mathbf{P}$ consists of disjoint maximal antichains of size at most n. Colouring one point in each antichain with colour 1 and the rest with colour 2 again shows that $\mathbf{P} \nrightarrow (\mathbf{A}_n)_k^{\mathbf{A}_1}$.

In both cases we get a contradiction, as $\boldsymbol{\mathcal{K}}_{\mathbf{s}}$ is a Ramsey class. Thus the largest $\mathbf{s}$-shaped antichain in the class $\boldsymbol{\mathcal{K}}$ does not exist and $\boldsymbol{\mathcal{K}}$ contains $\mathbf{s}$-shaped antichains of arbitrarily large size. It contains an $\mathbf{s}$-shaped antichain of any finite size because it has the hereditary property. □

This tells us that many classes of shaped partial orders do not have an ordered expansion that is Ramsey. We can further narrow down the classes to be considered.

Lemma 2.5.7. *Let $\mathfrak{S}$ be a countably infinite set of shapes and let $\boldsymbol{\mathcal{K}}$ be any class of $\mathfrak{S}$-shaped partial orders. If, for each finite subset $\mathfrak{S}'$ of $\mathfrak{S}$ the subclass $\boldsymbol{\mathcal{K}}'(\mathfrak{S})$ of all $\mathfrak{S}'$-shaped partial orders in $\boldsymbol{\mathcal{K}}$ is Ramsey, then the class $\boldsymbol{\mathcal{K}}$ is Ramsey.*

Proof. Take any $\mathbf{Q}, \mathbf{R} \in \boldsymbol{\mathcal{K}}$. Since they are finite, there must be a finite subset $\Sigma' \subset \Sigma$, such that $\mathbf{Q}$ and $\mathbf{R}$ are Σ'-shaped - namely

$$\Sigma' = \{\mathfrak{s}^{\mathbf{Q}}(q) : q \in Q\} \cup \{\mathfrak{s}^{\mathbf{R}}(r) : r \in R\}.$$

Then as $\boldsymbol{\mathcal{K}}'(\mathfrak{S})$ is Ramsey, there exists a $\mathbf{P} \in \boldsymbol{\mathcal{K}}'(\mathfrak{S})$, such that

$$\mathbf{P} \to (\mathbf{R})_k^{\mathbf{Q}}.$$

But $\mathbf{P} \in \boldsymbol{\mathcal{K}}$, so $\boldsymbol{\mathcal{K}}$ is Ramsey. □

Classification

As stated in Theorem 2.4.16, the homogeneous shaped partial orders are defined by a good skeleton Σ (defined in 2.4.13), and for each $\sigma \in \Sigma$ a set $\mathfrak{S}_\sigma$ of shapes and, if necessary, multiplicities $n_{\sigma,1}$ and $n_{\sigma,2}$, or $n_{\sigma,a,b}$ (defined in 2.4.14). By the Fraïssé correspondence, this gives a classification of all Fraïssé classes of shaped partial orders.

Suppose that $\mathcal{K}(\Sigma)$ is a Fraïssé class of shaped partial orders. On one hand, we will use a *good skeleton with a total order* to define a class $\mathcal{K}(\Sigma, o)$, a Fraïssé order class of shaped partial orders that is reasonable with respect to $\mathcal{K}(\Sigma)$.

Definition 2.5.8. A *good skeleton with a total order* is a structure

$$\Sigma = (\Sigma, <, \prec, l_1, l_2),$$

such that

(i) $(\Sigma, <, l_1, l_2)$ is a good skeleton, and

(ii) $(\Sigma, \prec)$ is a total order.

By Lemma 2.5.6, if any of the multiplicities $n_{\sigma,1}$ or $n_{\sigma,a,b}$ is not equal to 1 or $\aleph_0$, the class $\mathcal{K}(\Sigma, o)$ is not Ramsey. We will show that if all of the multiplicities $n_{\sigma,1}$ and $n_{\sigma,a,b}$ are equal to 1 or $\aleph_0$, there exists some Fraïssé order class of shaped partial orders $\mathcal{K}(\Sigma, o)$ that is reasonable with respect to $\mathcal{K}(\Sigma)$, and is a Ramsey class. We introduce *elementary skeletons* to enumerate classes of shaped ordered partial orders in a way that works better for Ramsey proofs.

We will start by considering an *ordered skeleton*, with more labels l_1 than a skeleton (defined in 2.4.12), but with fewer l_2 labels.

Definition 2.5.9. An *ordered skeleton* $\Sigma = (\Sigma, <, \prec, l_1, l_2)$ is structure as follows.

(i) $(\Sigma, <)$ is a partial order.

(ii) $(\Sigma, \prec)$ is a total order.

(iii) $l_1 : \Sigma \to \{A_1, GA, GC, AC, CA, GAC, G\}$

(iv) $l_2 : \{<\} \to \{<_g, <_c\}$

We consider specific ordered skeletons and the classes of ordered shaped partial orders they are enumerating in the following subsections.

Trivial antichain

The simplest class of ordered shaped partial orders is the class containing a single structure on a set of size 1. The corresponding homogeneous ordered shaped partial orders is also a structure on a set of size 1. In the classification of homogeneous shaped partial orders, its unordered reduct corresponds to a good skeleton containing one point labelled AC, as in Definition 2.4.14.

While the label AC is used for all antichains, it is used for for antichains of at least two chains. The 'antichain of one chain', *i.e.*, a chain, is labelled CA.

Since we are only considering classes of all finite antichains and all finite antichains of chains, we introduce a new label, A_1, for the class of ordered shaped partial orders containing a single structure on a set of size 1 instead of considering it as a special case of a structure labelled AC.

Definition 2.5.10. Let Σ be a skeleton with one point labelled A_1, and let $\mathfrak{S} = \{\mathbf{s}\}$ be a set with one shape. Then $\boldsymbol{\mathcal{K}}(A_1, \mathfrak{S})$ is the class containing the $\mathbf{s}$-shaped stucture $\mathbf{P}$ with a universe of size 1 with an empty partial order. The class $\boldsymbol{\mathcal{K}}(A_1, \mathfrak{S}, o)$ additionally contains an empty total order on $\mathbf{P}$.

Link. Lemma 5.2.5

Generic

We first consider a generic homogeneous structure $\boldsymbol{\mathcal{H}}(G)$ and the corresponding Fraïssé class of all shaped partial orders, $\boldsymbol{\mathcal{K}}(G)$. We defined the class of partial orders in 2.1.9. We defined linear extensions in 2.2.12. Finally $\boldsymbol{\mathcal{H}}(G)$ is defined in 2.4.6 and $\boldsymbol{\mathcal{K}}(G)$ in 2.4.8. We pull the definitions together with the aim to define a class of ordered shaped partial orders. We also include the set $\mathfrak{S}$ of shapes in the definitions, as different sets of shapes define different classes of structures.

Definition 2.5.11. Let Σ be a skeleton with one point labelled G, and let $\mathfrak{S}$ be a set of shapes. For any structure $\mathbf{P} \in \boldsymbol{\mathcal{K}}(G, \mathfrak{S}, o)$, there exist

(i) a structure $(P, <, \mathfrak{s}) \in \boldsymbol{\mathcal{K}}(G, \mathfrak{S})$ and

(ii) a chain $\mathcal{I}$,

such that the following hold.

(iii) $\mathbf{P} = (P, <, \prec, \mathfrak{s})$

(iv) $P = \{p_i : i \in \mathcal{I}\}$

(v) If $p_i < p_{i'}$ then $i < i'$ in $\mathcal{I}$.

(vi) $p_i \prec p_{i'}$ precisely when $i < i'$ in $\mathcal{I}$.

Further, given any $(P', <', \mathfrak{s}') \in \boldsymbol{\mathcal{K}}(G, \mathfrak{S})$ and a chain $\mathcal{I}'$ satisfying the conditions (iv), (v) and (vi), the structure $\mathbf{P}' = (P', <', \prec', \mathfrak{s}')$ from condition (iii) is also a structure in $\boldsymbol{\mathcal{K}}(G, \mathfrak{S}, o)$.

The class $\boldsymbol{\mathcal{K}}(G, \mathfrak{S}, o)$ of *ordered shaped partial orders* is the class of $\mathfrak{S}$-shaped partial orders together with *linear extensions* of the partial orders.

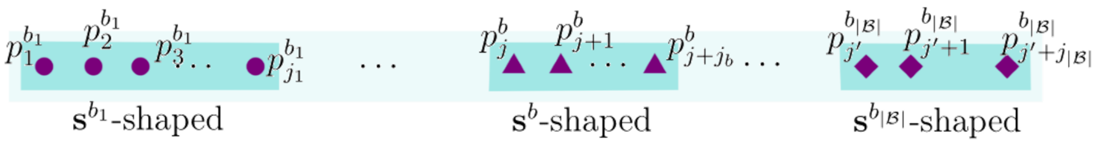
Link. Lemma 5.2.1

Remark 2.5.12. At times we omit the set $\mathfrak{S}$ of shapes and consider the classes $\boldsymbol{\mathcal{K}}(G)$ and $\boldsymbol{\mathcal{K}}(G, o)$ of shaped partial orders and ordered shaped partial orders respectively.

Chains of antichains

We can see that in the case of the class of all shaped partial orders, for any $P \in \mathcal{K}(G)$, we can add any map $\mathfrak{s} : P \to \mathfrak{S}$ to P to get a shaped partial order $\mathbf{P} \in \boldsymbol{\mathcal{K}}(G, \mathfrak{S})$. But the structure of homogeneous shaped chains of antichains induces the structure on the set of shapes as well, and restricts the shapings to the ones that respect the structure of the set of shapes. We defined the class of chains of antichains in 2.1.9. We defined linear extensions in 2.2.12, but we will define total orders on chains of antichains that also respect the structure of the set of shapes. Finally $\boldsymbol{\mathcal{H}}(CA)$ is defined in 2.4.6 and $\boldsymbol{\mathcal{K}}(CA)$ in 2.4.8, and we will combine all to define a class of ordered shaped chains of antichains. But we first introduce a class of *ordered glorified antichains.* A picture of an ordered glorified antichain is in Figure 2.6.

Figure 2.6: Glorified antichain $\mathbf{P}$



Definition 2.5.13. Let Σ be a skeleton with one point labelled GA and consider the following.

(i) A total order $\mathcal{B}$.

(ii) For each $b \in \mathcal{B}$ a number $n_b \in \{1, \aleph_0\}$, and $N = \{n_b : b \in \mathcal{B}\}$.

(iii) A set $\mathfrak{S}$ of shapes, such that $\mathfrak{S} = \{\mathbf{s}^b : b \in \mathcal{B}\}$.

An *ordered glorified antichain* $\mathbf{P} \in \mathcal{K}(GA, \mathfrak{S}, N, o)$ satisfies the following.

(iv) $\mathbf{P} = (P, <, \prec, \mathfrak{s})$ and $(P, <, \mathfrak{s}) \in \mathcal{K}(G, \mathfrak{S})$.

(v) For some chain $\mathcal{J}$, the universe of $\mathbf{P}$ is

$$P = \{p_j^b : j \in \mathcal{J}, \mathfrak{s}(p_j^b) = \mathbf{s}^b\}.$$

(vi) The shapes in $\mathfrak{S}$ induce a partition $\{\mathcal{J}^b : b \in \mathcal{B}\}$ of $\mathcal{J}$ and a partition $\{P^b : b \in \mathcal{B}\}$ of P, such that

$$P^b = \{p_j^b : j \in \mathcal{J}^b\}.$$

(vii) If $n_b = 1$ then $|P^b| \leq 1$.

(viii) $p_j^b || p_{j'}^{b'}$ for all $j \neq j'$.

(ix) $p_j^b \prec p_{j'}^{b'}$ when $j \prec j'$ in $\mathcal{J}$.

(x) For all pairs $p^b_j, p^{b'}_{j'} \in P$, if $j < j'$ then $b \le b'$.

Further, given any $(P', <', \mathfrak{s}') \in \mathcal{K}(G, \mathfrak{S})$ and a chain $\mathcal{J}'$ that define a structure $\mathbf{P}' = (P', <', \prec', \mathfrak{s}')$ satisfying conditions (iv)-(x), the structure $\mathbf{P}'$ lies in $\mathcal{K}(GA, \mathfrak{S}, N, o)$.

If P^b is non-empty, we denote the substructure of $\mathbf{P}$ on P^b by $\mathbf{P}^b$.
The class $\mathcal{K}(GA, \mathfrak{S}, N, o)$ is a class of ordered shaped glorified antichains.

Remark 2.5.14. By this definition, all the points of the glorified antichain $\mathbf{P}$ in Figure 2.6 are incomparable in the partial order $<$. The points shown in the picture are the points of $\mathbf{s}^{b_1}$-shaped antichain P^{b_1}, $\mathbf{s}^b$-shaped antichain P^b and $\mathbf{s}^{b_{|\mathcal{B}|}}$-shaped antichain $P^{b_{|\mathcal{B}|}}$, with the shapes inducing this partition as in part (vi) of this definition.
A point in $\mathbf{P}$ has the label

$$p^b_j,$$

where b tells us that the point is $\mathbf{s}^b$-shaped and the label j tells defines the total order $\prec$. Namely, in the total order $\prec$ we have

$$p^{b_1}_1 \prec p^{b_1}_2 \prec p^{b_1}_3 \prec \ldots \prec p^{b_1}_{j_1} \prec \ldots \prec p^b_j \prec p^b_{j+1} \prec \ldots \prec p^b_{j+j_b} \prec \ldots$$

$$\ldots \prec p^{b_{|\mathcal{B}|}}_{j'} \prec p^{b_{|\mathcal{B}|}}_{j'+1} \prec \ldots \prec p^{b_{|\mathcal{B}|}}_{j'+j_{|\mathcal{B}|}}.$$

By part (ix), this arises from $\mathcal{J}$ being a chain in which

$$1 < \ldots < j_1 < \ldots < j < \ldots < j + j_b < \ldots < j' < j' + 1 < \ldots < j' + j_{|\mathcal{B}|}.$$

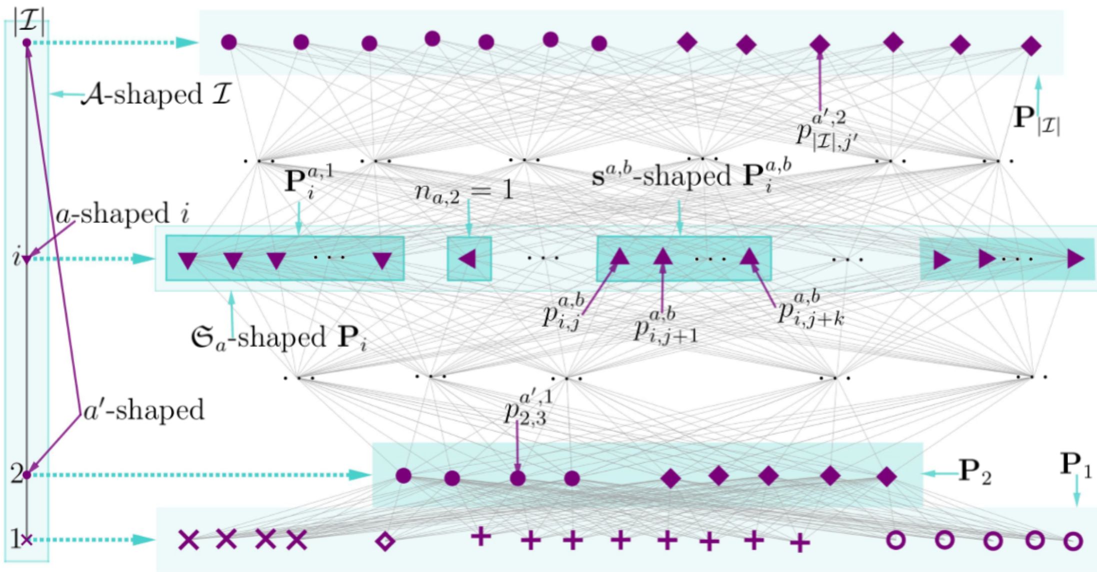
By part (x) we have $b_1 < \ldots b < \ldots < b_{|\mathcal{B}|}$ in the chain $\mathcal{B}$. Thus $\prec$ is convex on each of the $\mathbf{s}^b$-shaped parts $\mathbf{P}^b$ of $\mathbf{P}$ and we have

$$\mathbf{P}^{b_1} \prec \ldots \prec \mathbf{P}^b \prec \ldots \mathbf{P}^{b_{|\mathcal{B}|}}.$$

Also, by part (vii) we have $j_1 \le n_1$, $j_b < n_b$ and $j_{|\mathcal{B}|} < n_{|\mathcal{B}|}$, as $n_1, n_b, n_{|\mathcal{B}|} \in \{1, \aleph_0\}$ by part (iii). This means that in general the part $\mathbf{P}^b$ is finite and has at most one point in the case when $n_b = 1$.

By part (viii) of the definition above, any structure in the class $\mathcal{K}(GA, \mathfrak{S}, N, o)$ is an antichain. We swiftly move on to *chains of antichains*. See Figure 2.7 for a sketch of a chain of antichains.

Figure 2.7: Chain of antichains $\mathbf{P}$



Definition 2.5.15. We define a class $\mathcal{K}(CA, \mathfrak{S}, N, o)$ of *ordered chains of antichains* using the following.

(i) A total order $\mathcal{A}$ and, for each $a \in \mathcal{A}$, a total order $\mathcal{B}_a$.

(ii) A set $\mathfrak{S}$ of shapes, such that

$$\mathfrak{S} = \{\mathbf{s}^{a,b} : (a, b) \in \mathcal{A} \rtimes \mathcal{B}\}.$$

Let also

$$\mathfrak{S}_a = \{\mathbf{s}^{a,b} : b \in \mathcal{B}_a\}.$$

(iii) For each $(a, b) \in \mathcal{A} \rtimes \mathcal{B}$ a number $n_{a,b} \in \{1, \aleph_0\}$, $N_a = \{n_{a,b} : b \in \mathcal{B}_a\}$, and $N = \{n_{a,b} : (a, b) \in \mathcal{A} \rtimes \mathcal{B}\}$.

(iv) For each $a \in \mathcal{A}$, a class $\boldsymbol{\mathcal{K}}(GA, \mathfrak{S}_a, N_a, o)$ of glorified chains.

(v) A class $\boldsymbol{\mathcal{K}}(C, \mathcal{A})$ of $\mathcal{A}$-shaped chains.

For each structure $\mathbf{P} \in \boldsymbol{\mathcal{K}}(CA, \mathfrak{S}, N, o)$ there is an $\mathcal{A}$-shaped chain

$$\mathcal{I} = (\mathcal{I}, <, \mathfrak{s}^{\mathcal{I}}) \in \boldsymbol{\mathcal{K}}(C, \mathcal{A}),$$

and for each $i \in \mathcal{I}$, with $a = \mathfrak{s}^{\mathcal{I}}(i)$, a glorified antichain

$$\mathbf{P}_i \in \boldsymbol{\mathcal{K}}(GA, \mathfrak{S}_a, N_a, o),$$

such that the following hold

(vi) $\mathbf{P} = (P, <, \prec, \mathfrak{s})$.

(vii) $P_i = \{p_{i,j}^{a,b} : j \in \mathcal{J}_i, \mathfrak{s}(p_{i,j}^{a,b}) = \mathbf{s}^{a,b}\}$.

(viii) $P = \{p_{i,j}^{a,b} : (i,j) \in \mathcal{I} \rtimes \mathcal{J}, \mathfrak{s}(p_{i,j}^{a,b}) = \mathbf{s}^{a,b}\}$.

(ix) $p_{i,j}^{a,b} < p_{i',j'}^{a',b'}$ if $i < i'$.

(x) $p_{i,j}^{a,b} \prec p_{i',j'}^{a',b'}$ if

(a) if $i < i'$, or

(b) $i = i'$ and $j < j'$.

Further, given any $\mathcal{I}' \in \boldsymbol{\mathcal{K}}(C, \mathcal{A})$ and for each $i \in \mathcal{I}'$ a structure $\mathbf{P}'_i \in \boldsymbol{\mathcal{K}}(GA, \mathfrak{S}_a, N_a, o)$, where $a = \mathfrak{s}^{\mathcal{I}'}(i)$, the structure $\mathbf{P}' = (P', <', \prec', \mathfrak{s}')$ satisfying conditions (vii)-(x) lies in $\boldsymbol{\mathcal{K}}(CA, \mathfrak{S}, N, o)$.

Link. Lemma 5.2.3

Remarks 2.5.16. (i) Take any $\mathbf{P} \in \boldsymbol{\mathcal{K}}(CA, \mathfrak{S}, N, o)$. Essentially, the partial and total order on $\mathbf{P}$ are reflected in the total orders $\mathcal{I}$ and, for each $i \in \mathcal{I}$, $\mathcal{J}_i$, and thus in indices i and j of the points $p_{i,j}^{a,b}$ of $\mathbf{P}$.

The shaping of $\mathbf{P}$ is reflected in the indices a and b of the points $p_{i,j}^{a,b}$ of $\mathbf{P}$. Thus for any $p_{i,j}^{a,b}, p_{i,j'}^{a',b'} \in \mathbf{P}$ we in fact have $a = a'$.

(ii) Take any $\mathbf{P} \in \mathcal{K}(CA, \mathfrak{S}, N, o)$ and $p_{i,j}^{a,b}, p_{i',j'}^{a',b'} \in \mathbf{P}$. Then the conditions (ix) and (x) show that the total order $\prec$ extends the partial order $<$.

(iii) In Figure 2.7 we can see that to construct a chain of antichains, we start with an $\mathcal{A}$-shaped chain $\mathcal{I}$ and replace each of its a-shaped points i with an $\mathfrak{S}_a$-shaped glorified antichain $\mathbf{P}_i$. This induces the partial order $<$ on the glorified antichains, so we have

$$\mathbf{P}_1 < \mathbf{P}_2 < \ldots < \mathbf{P}_i < \ldots < \mathbf{P}_{|\mathcal{I}|}.$$

The total order $\prec$ is convex on the glorified antichains $\mathbf{P}_i$ and extends the total orders on them, as well as extending $<$, and inducing the total order

$$\mathbf{P}_1 \prec \mathbf{P}_2 \prec \ldots \prec \mathbf{P}_i \prec \ldots \prec \mathbf{P}_{|\mathcal{I}|}.$$

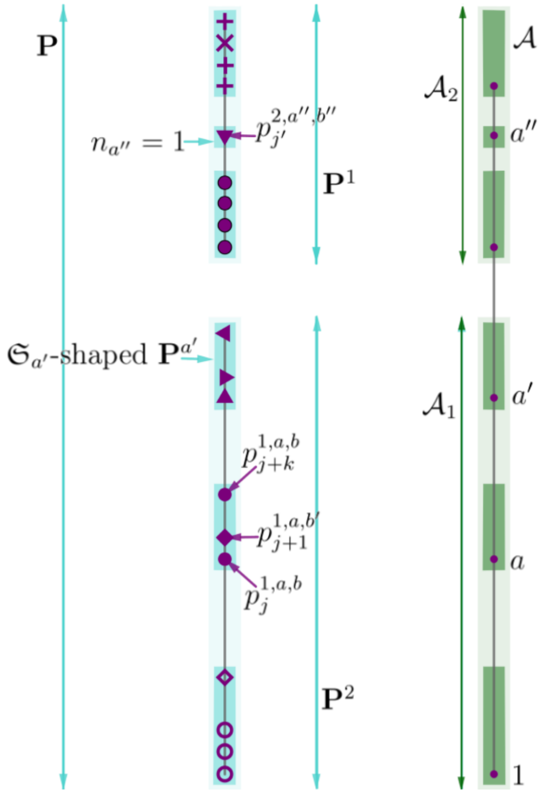
Finally, we consider a particular ordered shaped chain of antichains, a chain.

Definition 2.5.17. Let Σ be a skeleton with one point labelled CA, of $\mathfrak{S}$-shaped chains of antichains with $|\mathcal{B}_a| = 1$ and $n_{a,b_1} = 1$ for all $a \in \mathcal{A}$. Then $\mathcal{K}(\Sigma)$ is the set of all $\mathfrak{S}$-shaped chains, and $\mathcal{K}(\Sigma, o)$ the class of all ordered $\mathfrak{S}$-shaped chains. We denote the class $\mathcal{K}(\Sigma)$ in this case by $\mathcal{K}(C, \mathfrak{S})$. The structures in $\mathcal{K}(C, \mathfrak{S}, o)$ are chains with the partial order $<$ and the total order $\prec$ agreeing on any structure in $\mathcal{K}(C, \mathfrak{S})$, as $\prec$ extends the order $<$, as in any class $\mathcal{K}(CA, \mathfrak{S}, N, o)$.

Glorified antichains of chains

This section will be longer than the previous two on generic partial orders and chains of antichains. Instead of establishing notation for the class of shaped partial orders corresponding to a homogeneous shaped antichain of chains we will introduce a different building block of a shaped partial order, *glorified antichains* *of chains*, built of *glorified chains*. A picture of an ordered glorified chain is in Figure 2.8.

Figure 2.8: Glorified chain **P**



Definition 2.5.18. Let Σ be a skeleton with one point labelled GC and consider the following.

(i) A total order $\mathcal{A}$, with a partition $\{\mathcal{A}_1, \mathcal{A}_2\}$, where $\mathcal{A}_2$ is possibly an empty set, and for all $a_1 \in \mathcal{A}_1$ and $a_2 \in \mathcal{A}_2$ we have $a_1 < a_2$.

(ii) For each $a \in \mathcal{A}$ a number $n_a \in \{1, \aleph_0\}$, and $N = \{n_a : a \in \mathcal{A}\}$.

(iii) A set $\mathfrak{S}$ of shapes with a partition $\{\mathfrak{S}_a : a \in \mathcal{A}\}$, where $|\mathfrak{S}_a| = 1$ when $n_a = 1$ and for each $a \in \mathcal{A}$ there exists a total order $\mathcal{B}_a$, such that $\mathfrak{S}_a = \{\mathbf{s}^{a,b} : b \in \mathcal{B}_a\}$.

An *ordered glorified chain* $\mathbf{P} \in \boldsymbol{\mathcal{K}}(GC, \mathfrak{S}, N, o)$ satisfies the following.

(iv) $\mathbf{P} = (P, <, \prec, \mathfrak{s})$ and $(P, <, \mathfrak{s}) \in \boldsymbol{\mathcal{K}}(G, \mathfrak{S})$.

(v) For some chain $\mathcal{J}$, the universe of $\mathbf{P}$ is

$$P = \{p_j^{h,a,b} : j \in \mathcal{J}, \mathfrak{s}(p_j^{h,a,b}) = \mathbf{s}^{a,b}, a \in \mathcal{A}_h\}.$$

(vi) The partition of shapes from part (iii) induces a partition $\{\mathcal{J}^a : a \in \mathcal{A}\}$ of $\mathcal{J}$ and $\{P^a : a \in \mathcal{A}\}$ of P, so that

$$P^a = \{p_j^{h,a,b} : j \in \mathcal{J}^a\}.$$

(vii) If $n_a = 1$ then $|P_i^a| = 1$.

(viii) $p_j^{h,a,b} < p_{j'}^{h',a',b'}$ when $h = h'$ and $j < j'$.

(ix) $p_j^{h,a,b} \prec p_{j'}^{h',a',b'}$ when $j < j'$.

(x) For all pairs $p_j^{h,a,b}, p_{j'}^{h',a',b'} \in P$, if $j < j'$ then $a \leq a'$.

Further, given any $(P', <', \mathfrak{s}') \in \boldsymbol{\mathcal{K}}(G, \mathfrak{S})$ and a chain $\mathcal{J}'$ that define a structure $\mathbf{P}' = (P', <', \prec', \mathfrak{s}')$ satisfying conditions (iv)-(x), the structure $\mathbf{P}'$ lies in $\boldsymbol{\mathcal{K}}(GA, \mathfrak{S}, N, o)$.

If P^a is non-empty, we denote by $\mathbf{P}^a$ the substructure of $\mathbf{P}$ on P^a.
For $h \in [2]$, let $P^h = \bigcup_{a \in \mathcal{A}_h} P^a$. If P^h is non-empty, we denote by $\mathbf{P}^h$ the substructure of $\mathbf{P}$ on P^h.

Remarks 2.5.19. (i) Essentially, the part (x) says the total order on $\mathbf{P}$ induces the total order on the set of substructures $\mathbf{P}^a$, with

$$\mathbf{P}^a < \mathbf{P}^{a'} \quad \Rightarrow \quad a < a'.$$

The part (viii) then says that $\mathbf{P}^1$ and $\mathbf{P}^2$ are both chains for $<$, incomparable with each other. The part (ix) says that the total order $\prec$ extends the partial order $<$ and places $\mathbf{P}^1$ below $\mathbf{P}^2$.

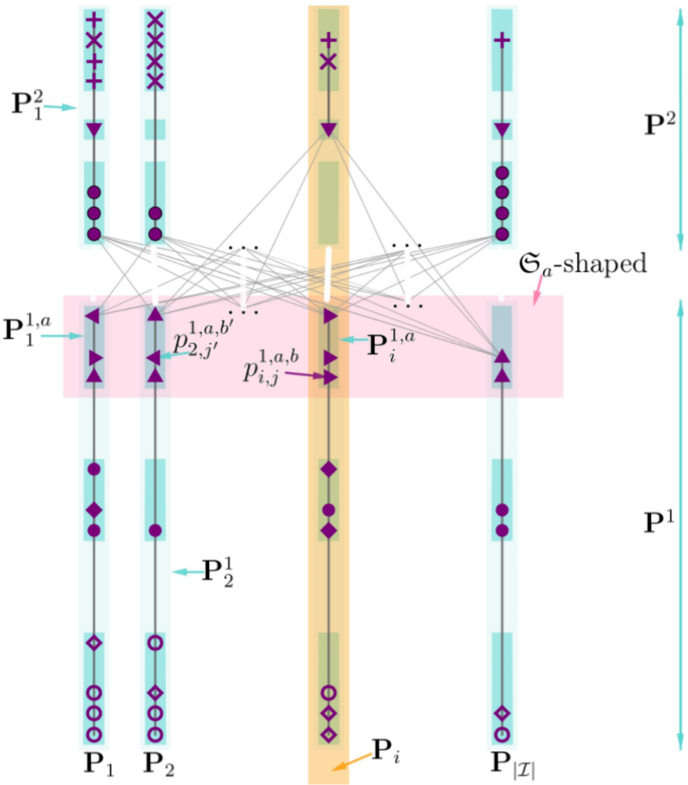
(ii) In the notation for a point in $\mathbf{P}$,

$$p_j^{h,a,b},$$

the label h tells us whether the point lies in $\mathbf{P}_1$ or $\mathbf{P}_2$. The label j tells us where in $\prec$ the point lies by (ix). Finally, by (v), the labels a and b denote the shape $\mathbf{s}^{a,b}$ of the point. For each a the substructure $\mathbf{P}^a$ of $\mathbf{P}$ is $\mathfrak{S}_a$-shaped.

We can see that the class $\mathcal{K}(GC, \mathfrak{S}, N, o)$ is almost a class of chains. Similarly, $\mathcal{K}(GAC, \mathfrak{S}, N, o)$ will resemble a class of antichains of chains. See Figure 2.9 for a sketch of a glorified antichain of chains.

Figure 2.9: Glorified antichain of chains $\mathbf{P}$



Definition 2.5.20. We define a class $\boldsymbol{\mathcal{K}}(GAC, \mathfrak{S}, N, o)$ of *ordered glorified antichains of chains* using the following.

(i) A total order $\mathcal{A}$, with a partition $\{\mathcal{A}_1, \mathcal{A}_2\}$, where $\mathcal{A}_2$ is possibly an empty set, and for all $a_1 \in \mathcal{A}_1$ and $a_2 \in \mathcal{A}_2$ we have $a_1 < a_2$.

(ii) For each $a \in \mathcal{A}$ a number $n_a \in \{1, \aleph_0\}$, and $N = \{n_a : a \in \mathcal{A}\}$.

(iii) A set $\mathfrak{S}$ of shapes with a partition $\{\mathfrak{S}_a : a \in \mathcal{A}\}$, where $|\mathfrak{S}_a| = 1$ when $n_a = 1$ and for each $a \in \mathcal{A}$ there exists a total order $\mathcal{B}_a$, such that $\mathfrak{S}_a = \{\mathbf{s}^{a,b} : b \in \mathcal{B}_a\}$.

(iv) A class $\boldsymbol{\mathcal{K}}(GC, \mathfrak{S}, N, o)$ of glorified chains.

For each structure $\mathbf{P} \in \boldsymbol{\mathcal{K}}(GAC, \mathfrak{S}, N, o)$ there is a chain $\mathcal{I}$, and for each $i \in \mathcal{I}$, a glorified chain

$$\mathbf{P}_i \in \boldsymbol{\mathcal{K}}(GC, \mathfrak{S}, N, o),$$

such that the following hold.

(v) $\mathbf{P} = (P, <, \prec, \mathfrak{s})$.

(vi) $P_i = \{p_{i,j}^{h,a,b} : j \in \mathcal{J}_i\}$.

(vii) $P = \{p_{i,j}^{h,a,b} : (i,j) \in \mathcal{I} \rtimes \mathcal{J}, \mathfrak{s}(p_{i,j}^{h,a,b}) = \mathbf{s}^{a,b}, a \in \mathcal{A}_h\}$.

(viii) $p_{i,j}^{h,a,b} < p_{i',j'}^{h',a',b'}$ if

  (a) if $h = h'$, $i = i'$ and $j < j'$, or

  (b) $h = 1$, $h' = 2$ and $i \neq i'$.

(ix) $p_{i,j}^{h,a,b} \prec p_{i',j'}^{h',a',b'}$ if

  (a) if $i < i'$, or

  (b) $i = i'$ and $j < j'$.

Further, given any chain $\mathcal{I}'$ and for each $i \in \mathcal{I}'$ a structure $\mathbf{P}_i' \in \boldsymbol{\mathcal{K}}(GC, \mathfrak{S}, N, o)$, the structure $\mathbf{P}' = (P', <', \prec', \mathfrak{s}')$ satisfying conditions (vii)-(x) lies in $\boldsymbol{\mathcal{K}}(GAC, \mathfrak{S}, N, o)$.

Link. Subsection of 5.2 (Matching skeletons), Definition 5.2.9, Definition 5.2.11, Lemma 5.2.12

Remark 2.5.21. We defined a glorified antichain of chains by taking a total order $\mathcal{I}$ and for each $i \in \mathcal{I}$ a glorified chain $\mathbf{P}_i$, consisting of chains $\mathbf{P}_i^1$ and $\mathbf{P}_i^2$ for $<$.
The label $p_{i,j}^{h,a,b}$ introduces the label i in addition to labels of points in a glorified chain, denoting that the point lies in the glorified chain $\mathbf{P}_i$.
Take any $i, i' \in \mathcal{I}$ and $h \in \{1, 2\}$. The chains $\mathbf{P}_i^h$ and $\mathbf{P}_{i'}^h$ are incomparable in $<$. But for any $i \in \mathcal{I}$, if P_i^1 and P_i^2 are non-empty, $\mathbf{P}_i^1$ is incomparable with $\mathbf{P}_i^2$, but is below all the other chains in $\mathbf{P}^2$, which creates a complement of a perfect matchin between chains in $\mathbf{P}^1$ and $\mathbf{P}^2$.
The total order $\prec$ extends the total order on the glorified chains, but does not extend the partial order $<$. Instead it is convex on the glorified chains, setting

$$\mathbf{P}_1 \prec \mathbf{P}_2 \prec \ldots \prec \mathbf{P}_i \prec \ldots \prec \mathbf{P}_{|\mathcal{I}|}.$$

Finally, we define particular cases of $\boldsymbol{\mathcal{K}}(GAC, \mathfrak{S}, N, o)$.

Definition 2.5.22. When $\mathcal{A}$ is of size 1, $\mathcal{A} = \{a\}$, we consider two cases.

(i) If $n_a = \aleph_0$, the classes $\boldsymbol{\mathcal{K}}(AC, \mathfrak{S})$ and $\boldsymbol{\mathcal{K}}(AC, \mathfrak{S}, o)$, of shaped antichains of chains and ordered shaped antichains of chains.

(ii) If $n_a = 1$, then $|\mathfrak{S}| = 1$. The classes $\boldsymbol{\mathcal{K}}(A, \{\mathbf{s}^a\})$ and $\boldsymbol{\mathcal{K}}(A, \{\mathbf{s}^a\}, o)$ are the classes of shaped antichains and ordered shaped antichains respectively.

Link. Lemma 5.2.5

Note. Let $\mathcal{K}$ be a class of unshaped reducts of structures in $\boldsymbol{\mathcal{K}}(AC, \mathfrak{S}, o)$. Then $\mathcal{K}$ is precisely the class $\mathcal{K}(AC_{\aleph_0}, ce)$, as defined in 2.2.11.
The class $\boldsymbol{\mathcal{K}}(AC, \mathfrak{S}, o)$ is precisely the class of shaped antichains of chains

with convex extensions, defined in 2.2.13 for the class of antichains of chains, and $\mathcal{K}(A, \{\mathbf{s}\}, o)$ is a class of antichains with arbitrary linear orders, as defined for the class of antichains

Simple skeleton

Simple skeletons are needed when proving the Ramsey property of classes of ordered shaped partial orders.

Definition 2.5.23. Consider an ordered skeleton $\Sigma = (\Sigma, <, \prec, l_1, l_2)$. The Σ is a *simple skeleton*, labelled Σ_{sp}, if

(i) $l_1(\Sigma) \subset \{G, GAC, A\}$,

(ii) $l_2(<) \subset \{<_g, <_c\}$,

(iii) $(\Sigma, \prec)$ extends the partial order $(\Sigma, <)$,

(iv) Σ satisfies the c-condition, and

(v) for each distinct $\sigma, \sigma' \in \Sigma$ there exist σ_i for $i \in [n]$ such that

$$\sigma = \sigma_0 -_g \sigma_1 -_g \ldots -_g \sigma_n -_g \sigma_{n+1} = \sigma'.$$

Suppose that Σ is a simple skeleton.

a) If $l_1(\Sigma) \subset \{G, GAC\}$, then Σ is a *glorified skeleton.*

b) If $l_1(\Sigma) \subset \{G, A\}$, then Σ is a *antichained skeleton.*

c) If $l_1(\Sigma) \subset \{G\}$, then Σ is a *generic skeleton.*

In Definition 2.5.22 we introduced labels AC and A and classes $\mathcal{K}(AC, \mathfrak{S}, o)$ and $\mathcal{K}(A, \{\mathbf{s}\}, o)$ as specific cases of the class $\mathcal{K}(GAC, \mathfrak{S}, N, o)$, namely when

$$\mathcal{A} = \{a\}, \quad \mathfrak{S} = \mathfrak{S}_a,$$

and either $n_a = \aleph_0$ or $n_a = 1$. So different simple skeletons will define the same class. Indeed, a skeleton Σ with a point σ labelled A and a set $\{\mathbf{s}\}$ of shapes defines the class $\boldsymbol{\mathcal{K}}(A, \{\mathbf{s}\}, o)$. A nearly identical skeleton Σ', but with σ labelled GAC, $\mathfrak{S} = \{\mathbf{s}\}$ and $N = \{1\}$ defines the class $\boldsymbol{\mathcal{K}}(GAC, \mathfrak{S}, N, o)$. While they appear different, both classes define a class of $\mathbf{s}$-shaped antichains.

Thus we only need to consider glorified skeletons when focusing on the classification, but use simple skeletons and antichained skeletons to emphasise that some of their points denote classes of antichains rather than any other glorified antichains of chains.

Definition 2.5.24. Suppose that $\Sigma = (\Sigma, <, \prec, l_1, l_2)$ is a glorified skeleton. The skeleton

$$\Sigma' = (\Sigma, <, \prec, l_1', l_2)$$

is the *antichained skeleton of* Σ if

$$l_1(\sigma) = GAC \iff l_1'(\sigma) = A \text{ and } l_1(\sigma) = G \iff l_1'(\sigma) = G.$$

Conversely, the skeleton Σ is the *glorified skeleton of* Σ'.

We first define a class of structures corresponding to an antichained skeleton. See Figure 2.10 for a sketch of a structure in the class defined by an antichained skeletons.

Definition 2.5.25. Let Σ be an antichained skeleton or a generic skeleton, and for each $\sigma \in \Sigma$,

(i) if $l_1(\sigma) = A$, let $\mathfrak{S}_\sigma = \{\mathbf{s}_\sigma\}$ be a set of shapes, and

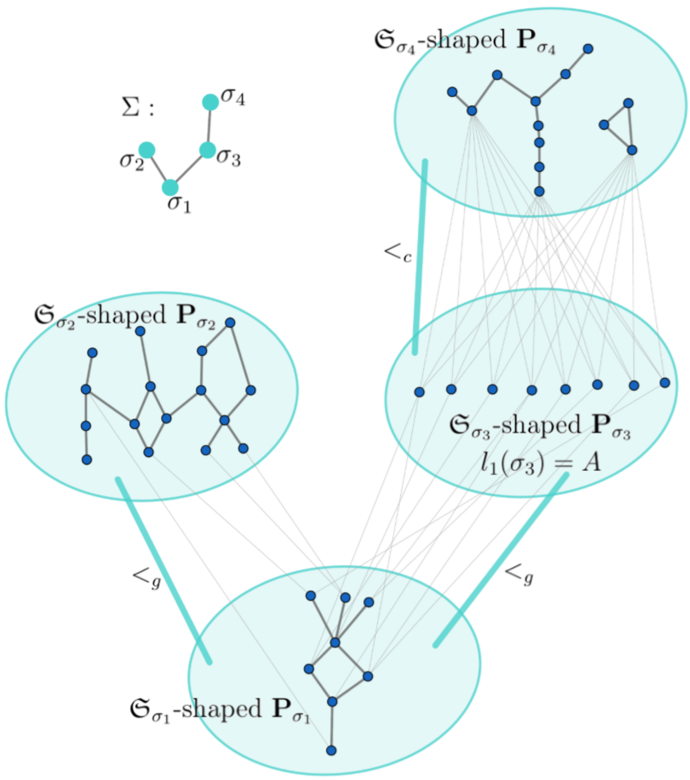
(ii) if $l_1(\sigma) = G$, let $\mathfrak{S}_\sigma$ be a set of shapes.

Let $\mathfrak{S} = \bigcup_{\sigma\in\Sigma} \mathfrak{S}_\sigma$. A structure $\mathbf{P} \in \boldsymbol{\mathcal{K}}(\Sigma, \mathfrak{S}, o)$ is an ordered $\mathfrak{S}$-shaped partial order, such that

(i) there is a partition $\{P_\sigma : \sigma \in \Sigma\}$ of P,

(ii) if P_σ is non-empty, the substructure $\mathbf{P}_\sigma$ of $\mathbf{P}$ on the subset P_σ is $\mathfrak{S}_\sigma$-shaped and $\mathbf{P}_\sigma \in \boldsymbol{\mathcal{K}}(\sigma, \mathfrak{S}_\sigma, o)$

Figure 2.10: A structure $\mathbf{P}$ in a class defined by an antichained skeleton Σ



(iii) if P_σ is non-empty and $l_1(\sigma) = A$, then $\mathbf{P}_\sigma$ is an antichain,

(iv) if $\sigma <_c \sigma'$, then for each $p \in P_\sigma$ and $p' \in P_{\sigma'}$, we have $p < p'$,

(v) if $\sigma \prec \sigma'$, then for each $p \in P_\sigma$ and $p' \in P_{\sigma'}$, we have $p \prec p'$.

The definition of a class of structures corresponding to a glorified skeleton is more complicated. Recall that $\mathcal{K}(GAC, \mathfrak{S}, N, o)$ is defined in 2.5.20. Figure 2.11 explains visually how to build a GAC component of a structure from an A component.

Definition 2.5.26. Let Σ be a glorified skeleton. Consider any $\sigma \in \Sigma$.

(i) If $l_1(\sigma) = G$, let $\mathfrak{S}_\sigma$ be a set of shapes.

(ii) If $l_1(\sigma) = GAC$, consider the following.

(a) A total order $\mathcal{A}_\sigma$, with a partition $\{\mathcal{A}_{\sigma,1}, \mathcal{A}_{\sigma,2}\}$, where $\mathcal{A}_{\sigma,2}$ is possibly an empty set, and for all $a_1 \in \mathcal{A}_{\sigma,1}$ and $a_2 \in \mathcal{A}_{\sigma,2}$ we have $a_1 < a_2$.

(b) For each $a \in \mathcal{A}_{\sigma,1}$ an $n_{\sigma,a} \in \{1, \aleph_0\}$ and $N_\sigma = \{n_{\sigma,a} : a \in \mathcal{A}_\sigma\}$.

(c) A set $\mathfrak{S}$ of shapes with a partition $\{\mathfrak{S}_{\sigma,a} : a \in \mathcal{A}_\sigma\}$, where $|\mathfrak{S}_a| = 1$ when $n_a = 1$ and for each $a \in \mathcal{A}_\sigma$ there exists a total order $\mathcal{B}_{\sigma,a}$, such that $\mathfrak{S}_{\sigma,a} = \{\mathbf{s}_\sigma^{a,b} : b \in \mathcal{B}_{\sigma,a}\}$.

(d) If there exists a $\sigma' \in \Sigma$, such that $\sigma' <_g \sigma$, then $\mathcal{A}_{\sigma,2}$ is empty.

Let $\mathfrak{S} = \bigcup_{\sigma \in \Sigma} \mathfrak{S}_\sigma$.
We define the class $\boldsymbol{\mathcal{K}}(\Sigma, \mathfrak{S})$ of *ordered shaped partial orders with a glorified skeleton* as follows.

Consider an antichained skeleton Σ' of Σ.

(iii) If $l_1(\sigma) = GAC$, let $\mathfrak{S}'_\sigma = \{\mathbf{s}_\sigma\}$.

Let $\mathfrak{S} = \left(\bigcup_{\sigma \in \Sigma, l'_1(\sigma) = G} \mathfrak{S}_\sigma\right) \cup \left(\bigcup_{\sigma \in \Sigma, l'_1(\sigma) = A} \mathfrak{S}'_\sigma\right)$.

A structure $\mathbf{P} \in \boldsymbol{\mathcal{K}}(\Sigma, \mathfrak{S})$ is an $\mathfrak{S}$-shaped partial order, such that the following hold.

(iv) There is a partition $\{P_\sigma : \sigma \in \Sigma\}$ of P.

(v) If P_σ is non-empty, the substructure $\mathbf{P}_\sigma$ of $\mathbf{P}$ on the subset P_σ is $\mathfrak{S}_\sigma$-shaped.

(vi) If P_σ is non-empty and $l_1(\sigma) = GAC$, then $\mathbf{P}_\sigma$ is a glorified antichain of chains, $\mathbf{P}_\sigma \in \boldsymbol{\mathcal{K}}(\Sigma_\sigma, \mathfrak{S}_\sigma, N_\sigma, o)$, with universe

$$P_\sigma = \{p_{\sigma,i,j}^{h,a,b} : (i,j) \in \mathcal{I}_\sigma \rtimes \mathcal{J}_\sigma, \mathfrak{s}(p_{\sigma,i,j}^{h,a,b}) = \mathbf{s}_\sigma^{a,b} \in \mathcal{A}_{\sigma,h}\}.$$

(vii) There is a structure $\mathbf{P}' \in \boldsymbol{\mathcal{K}}(\Sigma', \mathfrak{S}')$ with partition $\{P'_\sigma : \sigma \in \Sigma\}$ of P' and the following.

(a) If $l'_1(\sigma) = G$ and P_σ is non-empty, then $\mathbf{P}'_\sigma = \mathbf{P}_\sigma$.

(b) If $l'_1(\sigma) = A$ and P_σ is non-empty, then $\mathbf{P}'_\sigma$ in an antichain with

$$P_\sigma = \{p_{\sigma,i} : i \in \mathcal{I}_\sigma\}.$$

Let $f : P \to P'$ be a map, such that for $p \in P_\sigma$

(viii) if $l_1(\sigma) = G$, then $f(p) = p$, and

(ix) if $l_1(\sigma) = GAC$, then $p = p^{h,a,b}_{\sigma,i,j}$ and $f(p) = p_{\sigma,i}$.

Suppose that $\sigma \neq \varsigma$, $p \in P_\sigma$, $q \in P_\varsigma$ and that

$$p < q.$$

Then one of the following is true.

(x) $\sigma <_c \varsigma$

(xi) $\sigma <_g \varsigma$, $f(p) < f(q)$ in $\mathbf{P}'$ and either $l_1(\sigma) = G$ or $l_1(\sigma) = GAC$ and $p = p^{1,a,b}_{\sigma,i,j}$, or

(xii) $\varsigma <_g \sigma$, $f(p)||f(q)$ in $\mathbf{P}'$, $l_1(\varsigma) = GAC$ and $q = q^{2,a,b}_{\sigma,i,j}$.

Finally, the structure $\mathbf{P}$ still satisfies this condition for total order $\prec$.

(xiii) If $\sigma \prec \sigma'$, then for each $p \in P_\sigma$ and $p' \in P_{\sigma'}$, we have $p \prec p'$.

Link. Lemma 4.3.6, Theorem 4.3.7, Lemma 5.2.18

Remark 2.5.27. We saw an example of a structure $\mathbf{P}$ in a class of ordered shaped partial orders defined by an antichained skeleton in Figure 2.10. In Remark 2.5.21 we also discussed that we form a glorified antichain of chains from a total order $\mathcal{I}$ and for each $i \in \mathcal{I}$ a glorified chain $\mathbf{P}_i$. Take a glorified skeleton Σ and its antichained skeleton Σ'. We will build a $\mathbf{P} \in \mathcal{K}(\Sigma, \mathfrak{S})$ from a $\mathbf{P}' \in \mathcal{K}(\Sigma', \mathfrak{S}')$ by similarly building a glorified antichain of chains

$\mathbf{P}_\sigma$ from each of the antichains $\mathbf{P}'_\sigma$ for any $\sigma \in \Sigma$ with $l_1(\sigma) = GAC$.
So given a $\mathbf{P}'$, to build $\mathbf{P}$ keep all the components with label G and in each component labelled GAC replace each point in the antichain with a glorified chain to form a glorified antichain of chains. We need to specify the partial order between different components of $\mathbf{P}$. Namely, for any $\sigma \in \Sigma$ with $l_1(\sigma) = GAC$ and any $\sigma' \in \Sigma$, we need to define $<$ between the points in P_σ and $P_{\sigma'}$, whenever the two sets are not empty.

- When $\sigma <_c \sigma'$ we simply place the component $\mathbf{P}_\sigma$ completely below the component $\mathbf{P}_{\sigma'}$ in the partial order $<$ or vice versa when $\sigma' <_c \sigma$.
- When $\sigma' <_g \sigma$, by part (ii)(d) of the definition, the $\mathcal{A}_{\sigma,2}$ defining the glorified antichain of chains $\mathbf{P}_\sigma$ must then be empty, so $\mathbf{P}_\sigma$ only consists of incomparable chains in $\mathbf{P}_{\sigma,1}$. Take any $p \in \mathbf{P}_\sigma$, $q \in \mathbf{P}_{\sigma'}$. Then there are only two options:
$$f(q) < f(p) \quad \text{or} \quad f(q)||f(p).$$
When $f(q) < f(p)$, we place the entire chain $\mathbf{P}^1_{\sigma,i}$ above q, and otherwise the entire chain $\mathbf{P}^1_{\sigma,i}$ is incomparable with q.
- The final case, $\sigma <_g \sigma'$, behaves similarly, but with an additional twist. When $f(q) < f(p)$, we place the entire chain $\mathbf{P}^1_{\sigma,i}$ below q, and the chain $\mathbf{P}^2_{\sigma,i}$ is incomparable with q. The opposite happens when $f(q)||f(p)$, we place the entire chain $\mathbf{P}^2_{\sigma,i}$ above q, and the chain $\mathbf{P}^1_{\sigma,i}$ is incomparable with q.

With the complement of a perfect matching between the chains of $\mathbf{P}$ in $\mathbf{P}^1$ and those in $\mathbf{P}^2$ where necessary, the obtained structure is indeed a partial order.
The total order $\prec$, however, is much easier to describe. It places the entire $\mathbf{P}_\sigma$ either completely above or completely below the other components of $\mathbf{P}$ and thus extends the total order on Σ but not the partial order $<$.
Representing an example of what might happen in $\mathbf{P}$ is Figure 2.11. In the picture the chains of $\mathbf{P}^1$ and $\mathbf{P}^2$ are represented by turqouise rectangles. The reason that not all red edges from the complement of the perfect

matching appear in the final combined picture is that there is already a $<$ relationship between some pairs of chains arising from the blue edges.

Elementary skeleton

Finally, consider an *elementary skeleton*.

Definition 2.5.28. An *elementary skeleton* $\Sigma = (\Sigma, <, \prec, l_1, l_2)$ is a structure defined as follows.

(i) $\Sigma = (\Sigma, <, \prec, l_1, l_2)$ is an ordered skeleton,

(ii) $l_1(\Sigma) \subset \{A_1, CA, GAC, G\}$,

(iii) for any distinct $\sigma, \sigma' \in \Sigma$ with $l_1(\sigma) \in \{A_1, CA\}$ we have $\sigma || \sigma'$, and

(iv) any two distinct $\sigma, \sigma' \in \Sigma$ with $\sigma <_c \sigma'$ satisfy the *c-condition*, defined as follows.

  (a) If there is $\tau \in \Sigma$, such that $\tau < \sigma$, then $\tau <_c \sigma'$.

  (b) If there is $\tau \in \Sigma$, such that $\sigma' < \tau$, then $\sigma <_c \tau$.

Remark 2.5.29. So the subset of Σ of the points labelled A_1 or CA is an antichain, incomparable with the rest of the points in the skeleton.

While the definition of an elementary skeleton is short compared to the definition of the good skeleton (2.4.13), it will be accompanied by a longer counterpart to Definition 2.4.14. We aimed to prove the following.

Conjecture 2.5.30. Suppose that $\mathcal{K}$ is a class of ordered shaped partial orders, that is Ramsey and has ordering property. Then $\mathcal{K}$ is simply bi-definable with $\mathcal{K}(\Sigma, o)$, where Σ is an elementary skeleton.

This aim was not achieved, and we proved a weaker result.

Theorem 2.5.31. *Suppose that Σ is a good skeleton. Suppose that $\mathfrak{S}$ is a set of shapes, that there is a partition $\{\mathfrak{S}_\sigma : \sigma \in \Sigma\}$ of $\mathfrak{S}$, and*

- *for each $\sigma \in \Sigma$ with $l_1(\sigma) = AC$, numbers $n_{\sigma,1}$ and $n_{\sigma,2}$, with*
  - *$n_{\sigma,1} = \aleph_0$ and $n_{\sigma,2} \in \{1, \aleph_0\}$, or $n_{\sigma,1} = n_{\sigma,2} = 1$, and*
  - *$|\mathfrak{S}_\sigma| = 1$ if $n_{\sigma,2} = 1$;*
- *for each $\sigma \in \Sigma$ with $l_1(\sigma) = CA$,*
  - *a partition $\{\mathfrak{S}_{\sigma,a} : a \in \mathcal{A}_\sigma\}$ of $\mathfrak{S}_\sigma$, and*
  - *for each $\mathbf{s}_\sigma^{a,b} \in \mathfrak{S}_{\sigma,a}$, a number $n_{\sigma,a,b} \in \{1, \aleph_0\}$.*

Let $\boldsymbol{\mathcal{K}}(\Sigma)$ be the class of shaped partial orders as defined in 2.4.14. Then there exists an elementary skeleton Σ' and a class $\boldsymbol{\mathcal{K}}(\Sigma', o)$ of ordered shaped partial orders, such that

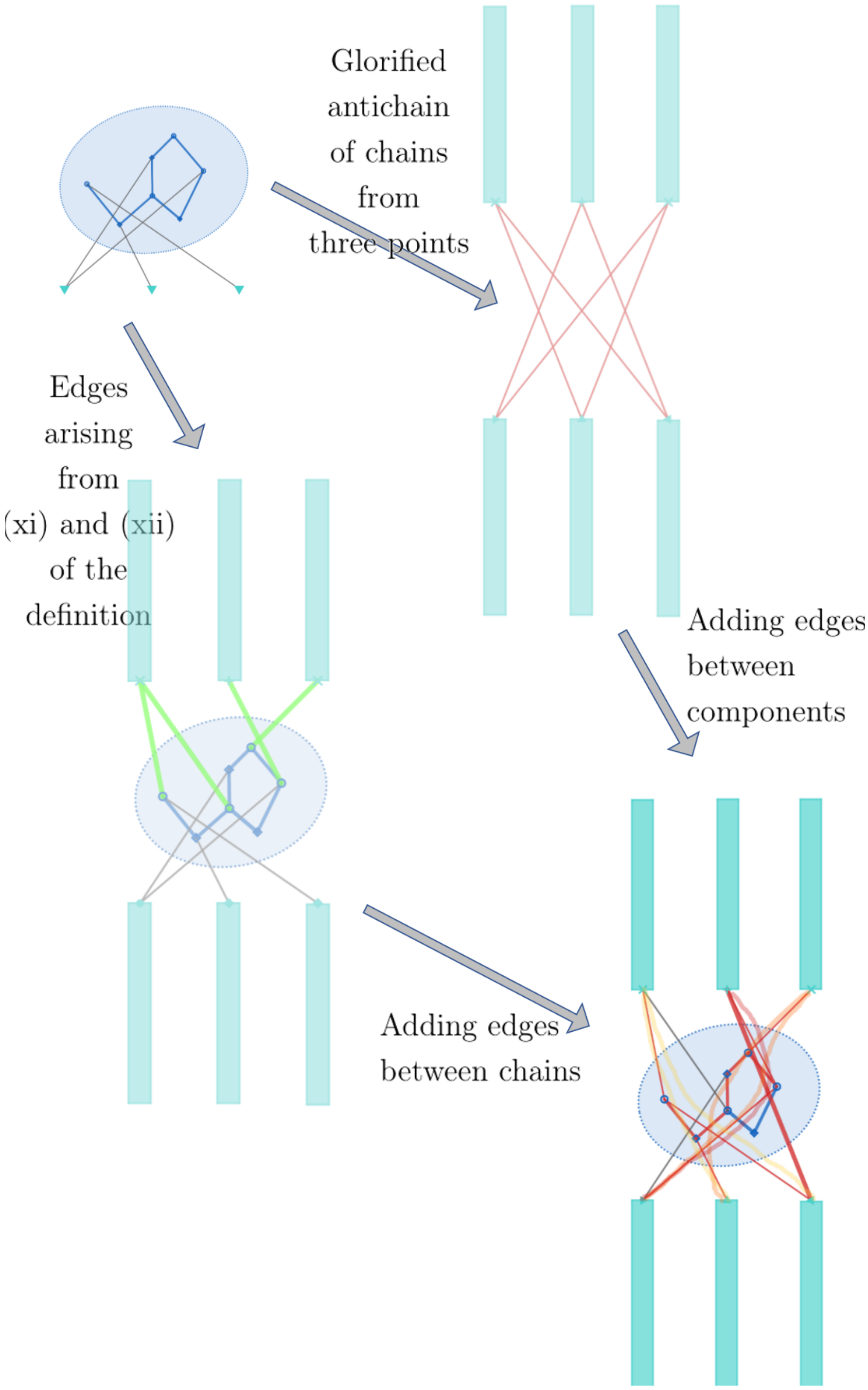
(i) the classes $\boldsymbol{\mathcal{K}}(\Sigma)$ and $\boldsymbol{\mathcal{K}}(\Sigma')$ are simply bi-definable, and

(ii) the class $\boldsymbol{\mathcal{K}}(\Sigma', o)$ is a reasonable class with respect to $\boldsymbol{\mathcal{K}}(\Sigma')$ and is a Ramsey class.

Further, when the elementary skeleton Σ' does not contain edges labelled $<_g$, the class $\boldsymbol{\mathcal{K}}(\Sigma', o)$ has the ordering property.

Note. The author was planning on proving a stronger result, by additionally proving that $\boldsymbol{\mathcal{K}}(\Sigma', o)$ has the ordering property. The author discovered a flaw in the proof and had to amend the statement of the main theorem.

Figure 2.11: Building a structure in a class defined by a glorified skeleton from a structure in a class defined by an antichained skeleton



Chapter 3

Key technical lemmas

In this chapter we introduce technical lemmas that will be used in proving that classes of ordered shaped partial orders are Ramsey. This chapter is heavy on notation, so the results in Chapter 4 should be viewed as examples of structures satisfying the definitions in this chapter. Thus the definitions are often followed by Links, signposting the reader to relevant lemmas in Chapter 4. The reader is advised to read the two chapters in parallel.

3.1 Bi-definability

Recall again the Classical Ramsey Theorem (2.2.2) and related Example 2.2.7. While the Classical Ramsey Theorem is about finite sets, it is clear that it also shows that the class of all finite antichains is Ramsey as well. In both cases, any two structures or substructures of the same size are isomorphic. The same is true in the case of the class of chains, the classes of ordered chains and antichains, and also in the classes of ordered shaped antichains, $\mathcal{K}(A, o)$. In these cases it is easy to see that all classes are Ramsey for essentially the same reason - the size of a structure determines its isomorphism class, so it's not really important what the structures in the class look like. Consider, for example, the following proof.

Example 3.1.1. Given ordered shaped antichains $\mathbf{Q} = (Q, <, \prec, \mathfrak{s})$ and $\mathbf{R} = (R, <, \prec, \mathfrak{s})$ of sizes q and r respectively, find p such that $p \to (r)^q_k$

using Classical Ramsey Theorem.
Let $\mathbf{P} = (P, <, \prec, \mathfrak{s})$ be an ordered shaped antichain of size p. Then for any subset Q' of P of size q, the substructure of $\mathbf{P}$ on the set Q' is isomorphic to $\mathbf{Q}$, and similarly for any subset R' of P of size r.
The fact that $\mathbf{P} \to (\mathbf{R})^{\mathbf{Q}}_k$ follows trivially.
This shows that the class $\mathcal{K}(A, o)$, defined in 2.5.22, is a Ramsey class.

The proof in Example 3.1.1 is essentially identical to the proof in Example 2.2.7. That happens because all the mentioned classes of structures are *simply bi-definable*. We mentioned the simple bi-definability for the specific case of classes of structures with different total orderings in Definition 2.3.9 already. We revisit the definition here, for classes of relational structures. We also show that any two simply bi-definable classes are either both Ramsey, or neither is. That will be one of the techniques of showing that a class of ordered shaped partial orders is Ramsey. This result is mentioned in paper Kechris *et al.* (2005) for the specific case of simply bi-definable classes of two extensions of Fraïssé classes by a total order relation.

We start with a formal definition.

Definition 3.1.2. Let $\mathcal{K}_0$ be a class of structures in a relational language L_0. Let $L_1 = L_0 \cup \{R_{1,j}\}_{j \in J_1}$ and $L_2 = L_0 \cup \{R_{2,j}\}_{j \in J_2}$ be languages of relations of arities $n(1, j)$ and $n(2, j)$ respectively, and let $\mathcal{K}_1$ be a reasonable class of structures in language L_1 and $\mathcal{K}_2$ in L_2. The classes $\mathcal{K}_1$ and $\mathcal{K}_2$ are *simply bi-definable over* $\mathcal{K}_0$ if the following are true.

(i) There are quantifier-free formulas $\{\varphi_{1,j}\}_{j \in J_2}$ with $n(1, j)$ variables in language L_1, and given any $\mathbf{A}_0 \in \mathcal{K}_0$ and $\mathbf{A}_1 \in \mathcal{K}_1$ with $\mathbf{A}_1 = \langle \mathbf{A}_0, \{R^{\mathbf{A}_1}_{1,j}\}_{j \in J_1} \rangle$, there exists a structure $\mathbf{A}_2 \in \mathcal{K}_2$ with $\mathbf{A}_2 = \langle \mathbf{A}_0, \{R^{\mathbf{A}_2}_{2,j}\}_{j \in J_2} \rangle$, with $R^{\mathbf{A}_2}_{2,j}$ on A_0 defined by $\varphi_{1,j}$, *i.e.*,

$$R^{\mathbf{A}_2}_{2,j}(a_1, a_2, \ldots, a_{n(2,j)}) \iff \mathbf{A}_1 \models \varphi_{1,j}[a_1, a_2, \ldots, a_{n(2,j)}].$$

We define Φ_1 by setting $\Phi_1(\mathbf{A}_1) = \langle \mathbf{A}_0, \{R^{\mathbf{A}_2}_{2,j}\}_{j \in J_2} \rangle$.

(ii) Similarly, there are simple formulas $\{\varphi_{2,j}\}_{j \in J_1}$ in L_2, defining for each

$\mathbf{A}_0 \in \mathcal{K}_0$ and $\mathbf{A}_2 \in \mathcal{K}_2$ with $\mathbf{A}_2 = \langle \mathbf{A}_0, \{R_{2,j}^{\mathbf{A}_2}\}_{j\in J_2}\rangle$, a structure

$$\Phi_2(\mathbf{A}_2) = \langle \mathbf{A}_0, \{R_{1,j}^{\mathbf{A}_1}\}_{j\in J_1}\rangle.$$

(iii) For each $\mathbf{A}_0 \in \mathcal{K}_0$, the map Φ_1 is a bijection between the expansions of $\mathbf{A}_0$ in $\mathcal{K}_1$ and expansions of $\mathbf{A}_0$ in $\mathcal{K}_2$, with inverse Φ_2.

Remarks 3.1.3. (i) Technically, we define a Φ_1 and a Φ_2 for each $\mathbf{A}_0 \in \mathcal{K}_0$, so they could be referred to as $\Phi_1^{\mathbf{A}_0}$ and a $\Phi_2^{\mathbf{A}_0}$. But since $\mathcal{K}_1$ and $\mathcal{K}_2$ are both reasonable expansions of the class $\mathcal{K}_0$, if $\mathcal{K}_1$ and $\mathcal{K}_2$ were sets, we could patch together different $\Phi_1^{\mathbf{A}_0}$'s and $\Phi_2^{\mathbf{A}_0}$'s to get a bijection Φ_1 between $\mathcal{K}_1$ and $\mathcal{K}_2$ with an inverse Φ_2.

So, informally, we will say that $\mathcal{K}_1$ and $\mathcal{K}_2$ are simply bi-definable if there exists a unifom bijection $\Phi_1 : \mathcal{K}_1 \to \mathcal{K}_2$, with inverse Φ_2.

We sometimes omit reference to $\mathcal{K}_0$.

(ii) When proving that two classes of structures are simply bi-definable, we will omit formally defining the languages and writing explicit simple formulas φ. Instead, we will explain how the relations in class $\mathcal{K}_2$ can be defined by the relations in $\mathcal{K}_1$ and vice versa.

The link between simple bi-definability and Ramsey classes is the fact that Φ_1 and Φ_2 'preserve' substructures as well.

Lemma 3.1.4. *Suppose that $\mathcal{K}_1$ and $\mathcal{K}_2$ are simply bi-definable over $\mathcal{K}_0$. Then for any structures $\mathbf{A}_1, \mathbf{B}_1 \in \mathcal{K}_1$, $\mathbf{A}_2, \mathbf{B}_2 \in \mathcal{K}_2$ with $\Phi_1(\mathbf{A}_1) = \mathbf{A}_2$ and $\Phi_1(\mathbf{B}_1) = \mathbf{B}_2$, there is a bijection*

$$f^{\mathbf{A}_1,\mathbf{B}_1} : \binom{\mathbf{B}_1}{\mathbf{A}_1} \to \binom{\mathbf{B}_2}{\mathbf{A}_2},$$

with inverse $f^{\mathbf{A}_2,\mathbf{B}_2}$. Moreover, $f^{\mathbf{A}_1,\mathbf{B}_1}$ sends a substructure of $\mathbf{B}_1$ with universe A' to a substructure of $\mathbf{B}_2$ with universe A'.

Proof. Take any $\mathbf{A}'_1, \in \binom{\mathbf{B}_1}{\mathbf{A}_1}$. Then we must have

$$\mathbf{A}'_1 = \langle \mathbf{A}'_0, \{R_{1,j}^{\mathbf{A}'_1}\}_{j \in J_1} \rangle, \quad \mathbf{A}'_0 = \langle A'_0, \{R_{0,j}^{\mathbf{A}'_0}\}_{j \in J_0} \rangle$$

with

$$R_{1,j}^{\mathbf{A}'_1} = R_{1,j}^{\mathbf{B}_1} \cap {A'}_0^{n(1,j)}, \mathbf{A}'_0 \in \binom{\mathbf{B}_0}{\mathbf{A}_0}.$$

Besides, there is a map $\theta : A_0 \to A'_0$, such that

$$R_{1,j}^{\mathbf{A}_1}(a_1, a_2, \ldots, a_{n(1,j)}) \iff R_{1,j}^{\mathbf{A}'_1}(\theta(a_1), \theta(a_2), \ldots, \theta(a_{n(1,j)}))$$

for all relations $R_{1,j}$, with $j \in J_1$.
Consider the structure $\mathbf{A}'_2$, with

$$\mathbf{A}'_2 = \langle \mathbf{A}'_0, \{R_{2,j}^{\mathbf{A}'_2}\}_{j \in J_2} \rangle$$

and

$$R_{2,j}^{\mathbf{A}'_2} = R_{2,j}^{\mathbf{B}_2} \cap {A'}_0^{n(2,j)}.$$

We will show that

$$R_{2,j}^{\mathbf{A}_2}(a_1, a_2, \ldots, a_{n(2,j)}) \iff R_{2,j}^{\mathbf{A}'_2}(\theta(a_1), \theta(a_2), \ldots, \theta(a_{n(2,j)}))$$

for all relations $R_{2,j}$, with $j \in J_2$.
Take any $(a'_1, a'_2, \ldots a'_{n(2,j)}) \in {A'}_0^{n(2,j)}$. Then we have,

$$R_{2,j}^{A'_2}(a'_1, a'_2, \ldots a'_{n(2,j)}) \iff R_{2,j}^{B_2}(a'_1, a'_2, \ldots a'_{n(2,j)})$$

by definition of a substructure.
Now recall that $R_{2,j}^{\mathbf{B}_2}$ is defined as

$$R_{2,j}^{\mathbf{B}_2}(b_1, b_2, \ldots, b_{n(2,j)}) \iff \mathbf{B}_1 \models \varphi_{1,j}[b_1, b_2, \ldots, b_{n(2,j)}].$$

So we have

$$R_{2,j}^{\mathbf{B}_2}(a'_1, a'_2, \ldots a'_{n(2,j)}) \iff \mathbf{B}_1 \models \varphi_{1,j}[a'_1, a'_2, \ldots a'_{n(2,j)}].$$

But since $(a'_1, a'_2, \dots a'_{n(2,j)}) \in A'^{n(2,j)}_0$ and A'_1 is a substructure of B_1, we must also have

$$\mathbf{B}_1 \models \varphi_{1,j}[a'_1, a'_2, \dots a'_{n(2,j)}] \iff \mathbf{A}'_1 \models \varphi_{1,j}[a'_1, a'_2, \dots a'_{n(2,j)}].$$

The map θ defines an isomorphism between A_1 and A'_1, so we have

$$\mathbf{A}'_1 \models \varphi_{1,j}[a'_1, a'_2, \dots a'_{n(2,j)}] \iff \mathbf{A}_1 \models \varphi_{1,j}[\theta^{-1}(a'_1), \theta^{-1}(a'_2), \dots \theta^{-1}(a'_{n(2,j)})].$$

But, again, by definition of $R^{\mathbf{A}_2}_{2,j}$, we have

$$\begin{aligned} &\mathbf{A}_1 \models \varphi_{1,j}[\theta^{-1}(a'_1), \theta^{-1}(a'_2), \dots \theta^{-1}(a'_{n(2,j)})] \\ \iff &R^{\mathbf{A}_2}_{2,j}(\theta^{-1}(a'_1), \theta^{-1}(a'_2), \dots \theta^{-1}(a'_{n(2,j)})). \end{aligned}$$

Which means that we've just shown

$$R^{A'_2}_{2,j}(a'_1, a'_2, \dots a'_{n(2,j)}) \iff R^{\mathbf{A}_2}_{2,j}(\theta^{-1}(a'_1), \theta^{-1}(a'_2), \dots \theta^{-1}(a'_{n(2,j)})),$$

which finishes the proof.

□

Corollary 3.1.5. *Suppose that $\mathcal{K}_1$ and $\mathcal{K}_2$ are simply bi-definable over $\mathcal{K}_0$. Then for any structures $\mathbf{A}_1, \mathbf{B}_1, \mathbf{C}_1 \in \mathcal{K}_1$, $\mathbf{A}_2, \mathbf{B}_2, \mathbf{C}_2 \in \mathcal{K}_2$ with $\Phi_1(\mathbf{A}_1) = \mathbf{A}_2$, $\Phi_1(\mathbf{B}_1) = \mathbf{B}_2$ and $\Phi_1(\mathbf{C}_1) = \mathbf{C}_2$, if*

$$\mathbf{B}'_1 \in \binom{\mathbf{C}_1}{\mathbf{B}_1}, \text{ with } f^{\mathbf{B}_1,\mathbf{C}_1}(\mathbf{B}'_1) = \mathbf{B}'_2,$$

then the maps $f^{\mathbf{A}_1,\mathbf{B}'_1}$ and $f^{\mathbf{A}_1,\mathbf{C}_1}$ agree on the subset $\binom{\mathbf{B}'_1}{\mathbf{A}_1}$ of $\binom{\mathbf{C}_1}{\mathbf{A}_1}$ and we write

$$f^{\mathbf{A}_1,\mathbf{B}'_1} = f^{\mathbf{A}_1,\mathbf{C}_1}|_{\binom{\mathbf{B}'_1}{\mathbf{A}_1}}.$$

Proof. Take any $\mathbf{A}_1 \in \binom{\mathbf{B}'_1}{\mathbf{A}_1}$, and let $\mathbf{A}'_2 = f^{\mathbf{A}_1,\mathbf{B}'_1}(\mathbf{A}'_1)$.
Suppose that the universe of $\mathbf{C}$ is C, of $\mathbf{B}'_1$ is $B \subset C$ and of $\mathbf{A}'_1$ is A. Then

the universe of $\mathbf{B}_2'$ is $B \subset C$ as well, by Lemma 3.1.4. Similarly, the universe of $\mathbf{A}_2' = f^{\mathbf{A}_1,\mathbf{B}_1'}(\mathbf{A}_1')$ is precisely $A \subset B \subset C$. So $f^{\mathbf{A}_1,\mathbf{B}_1'}$ and $f^{\mathbf{A}_1,\mathbf{C}_1}$ both send $\mathbf{A}_1'$ precisely to the substructure of $\mathbf{B}_2' \trianglelefteq \mathbf{C}_2$ with the universe A, *i.e.*,

$$f^{\mathbf{A}_1,\mathbf{B}_1'}(\mathbf{A}_1') = f^{\mathbf{A}_1,\mathbf{C}_1}(\mathbf{A}_1').$$

$\square$

Now we are ready to show that simply bi-definable classes are either both Ramsey, or neither is.

Lemma 3.1.6 (Simply Bi-definable Ramsey Lemma)**.** *Suppose that $\mathcal{K}_1$ and $\mathcal{K}_2$ are simply bi-definable classes and that $\mathcal{K}_1$ is a Ramsey class. Then $\mathcal{K}_2$ is a Ramsey class as well.*

Proof. Take any $\mathbf{A}, \mathbf{B} \in \mathcal{K}_2$. Let $\mathbf{A}_1 = \Phi_1(\mathbf{A})$ and $\mathbf{B}_1 = \Phi_1(\mathbf{B})$. Then there exists $\mathbf{C}_1 \in \mathcal{K}_1$ such that

$$\mathbf{C}_1 \to (\mathbf{B}_1)_k^{\mathbf{A}_1}.$$

Let $\mathbf{C} = \Phi_2(\mathbf{C}_1)$. We will show that

$$\mathbf{C} \to (\mathbf{B})_k^{\mathbf{A}}.$$

Let

$$c : \binom{\mathbf{C}}{\mathbf{A}} \to [k]$$

be any colouring.
Then, using $f^{\mathbf{A}_1,\mathbf{C}_1}$ from Lemma 3.1.4, there is a colouring

$$c \circ f^{\mathbf{A}_1,\mathbf{C}_1} : \binom{\mathbf{C}_1}{\mathbf{A}_1} \to [k].$$

Since $\mathbf{C}_1 \to (\mathbf{B}_1)_k^{\mathbf{A}_1}$, there exists $\mathbf{B}_1' \in \binom{\mathbf{C}_1}{\mathbf{B}_1}$, such that $\binom{\mathbf{B}_1'}{\mathbf{A}_1}$ is monochromatic. Now consider $B' = f^{\mathbf{B}_1,\mathbf{C}_1}(\mathbf{B}_1')$. By Lemma 3.1.5, we have

$$f^{\mathbf{A}_1,\mathbf{C}_1}(\binom{\mathbf{B}_1'}{\mathbf{A}_1}) = \binom{\mathbf{B}'}{\mathbf{A}}.$$

So for any $\mathbf{A}' \in \binom{\mathbf{B}'}{\mathbf{A}}$, we have $\mathbf{A}' = f^{\mathbf{A}_1,\mathbf{C}_1}(\mathbf{A}'_1)$ for some $\mathbf{A}'_1 \in \binom{\mathbf{B}'_1}{\mathbf{A}_1}$ and

$$c(\mathbf{A}') = c\left(f^{\mathbf{A}_1,\mathbf{C}_1}(\mathbf{A}'_1)\right) = c \circ f^{\mathbf{A}_1,\mathbf{C}_1}(\mathbf{A}'_1),$$

so $\binom{\mathbf{B}'}{\mathbf{A}}$ is monochromatic. □

3.2 Structural Product Ramsey Lemma

Aside from the Classical Ramsey theorem, a very useful result is the Product Ramsey Theorem.

Theorem 3.2.1 (Product Ramsey Theorem). *Let $B = B_1 \times \ldots \times B_t$ be a product of non-empty sets of sizes $|B_i| = b_i$ for $1 \leq i \leq t$. Let also $A = A_1 \times \ldots \times A_t$ be a product of sets of sizes $|A_i| = a_i \leq b_i$ for $1 \leq i \leq t$, and let k be a non-negative integer. Then there is a number N such that for any set $C = C_1 \times \ldots \times C_t$, where $|C_i| \geq N$ for $1 \leq i \leq t$, we have*

$$C \rightarrow (B)^A_k.$$

Again, in terms of sizes, we can write

$$N \rightarrow (b_1, \ldots, b_t)^{(a_1,\ldots,a_t)}_k.$$

The proof of this theorem is commonly known, and was also an inspiration for the proof of Theorem 3.2.6.

The proof of the Product Ramsey Theorem can be applied to a result about the class of chains of antichains, although proving that the class is Ramsey requires an argument that inspired the Two Pass Lemma (see 3.6.15).

Example 3.2.2. Take any two structures $Q, R \in \mathcal{K}(CA_{\aleph_0})$, each consisting of $|\mathcal{I}|$ maximal antichains, an antichain Q_i of size q_i and an antichain R_i of size r_i for each $i \in \mathcal{I}$. Then $\binom{R}{Q}$ is non-empty precisely when each antichain R_i is at least as long the antichain Q_i, $1 \leq q_i \leq r_i$.

By Product Ramsey Theorem, there is a number p, such that

$$p \to (r_1, \ldots, r_t)_k^{(q_1,\ldots,q_t)}.$$

Let P be a structure with maximal antichains P_i of size p for each $i \in \mathcal{I}$. Now, importantly, given a subset Q'_i of P_i of size q_i for each $i \in \mathcal{I}$, the substructure of P on the set $\bigcup_{i\in\mathcal{I}} Q'_i$ is isomorphic to Q, and similarly for subsets R'_i of P_i of size r_i. Thus by the Product Ramsey Theorem, we have $P \to (R)_k^Q$.

The underlying reasons making the Product Ramsey Theorem work, that will allow us to state a similar result for a much wider selection of classes of structures, are the following.

(i) There is a set $[t]$ and each structure considered consists of substructures $\mathbf{A}_i$ for each $i \in [t]$.

(ii) If the set $\binom{\mathbf{B}}{\mathbf{A}}$ is non-empty, then for any $\mathbf{A}' \in \binom{\mathbf{B}}{\mathbf{A}}$, the substructure $\mathbf{A}'_i$ of $\mathbf{A}'$ is a substructure of the substructure $\mathbf{B}_i$ of $\mathbf{B}$ for each $i \in [t]$.

(iii) Picking a set of substructures $\mathbf{A}'_i \trianglelefteq \mathbf{B}_i$ for each $i \in [t]$ yields precisely one substructure $\mathbf{A}'$ of $\mathbf{B}$.

We will define a *product* of classes of structures and use it on the way to proving a Structural Product Ramsey Lemma, analogous to the Product Ramsey Theorem, but extending it formally to classes of structures.

Definition 3.2.3. Let $\{L_i\}_{i\in[t]}$ be disjoint relational languages. For each $i \in [t]$ let $L_i = \{R_{i,j}\}_{j\in J_i}$, and let $\mathcal{K}_i$ be a class of structures in language L_i, closed under substructures. A *product* $\mathcal{K}$ *of classes* $\mathcal{K}_i$, $\mathcal{K} = \prod_{i\in[t]} \mathcal{K}_i$ in language $L = \bigcup_{i\in[t]} L_i$ is defined as follows.

(i) Given, for each $i \in [t]$, a structure $\mathbf{A}_i \in \mathcal{K}_i$, there is a structure $\prod_{i\in[t]} \mathbf{A}_i \in \prod_{i\in[t]} \mathcal{K}_i$ such that the following hold.

(a) The universe of $\prod_{i\in[t]} \mathbf{A}_i$ is $[t] \rtimes A$.

(b) For any $i \in I$ and $j \in J_i$, the relation $R_{i,j}$ is defined for $\prod_{i\in[t]} \mathbf{A}_i$

as follows:

$$R^{\mathbf{A}}_{i,j}((i_1, a_1), (i_2, a_2), \ldots, (i_{n(i,j)}, a_{n(i,j)})) \iff$$

$$i_1 = i_2 = \ldots = i_{n(i,j)} = i \text{ and } R^{\mathbf{A}_i}_{i,j}(a_1, a_2, \ldots, a_{n(i,j)})$$

(ii) The product $\prod_{i\in[t]} \mathcal{K}_i$ consists precisely of the structures in (i).

Remarks 3.2.4. (i) Take any $\mathbf{B} \in \mathcal{K}$. For any set of non-empty subsets $\{A_i\}_{i\in[t]}$ with $A_i \subset B_i$, the subset $[t] \rtimes A_i$ of $[t] \rtimes B_i$, with the induced relations in L defines a substructure $\mathbf{A}$ of $\mathbf{B}$. Further, any substructure $\mathbf{A}'$ of $\mathbf{B}$ in $\mathcal{K}$ is defined precisely by a set of non-empty subsets $\{A'_i\}_{i\in[t]}$ with $A'_i \subset B_i$.

(ii) Structures $\mathbf{A} = \prod_{i\in[t]} \mathbf{A}_i$ and $\mathbf{B} = \prod_{i\in[t]} \mathbf{B}_i$ are isomorphic precisely when, for all $i \in [t]$, $\mathbf{A}_i$ is isomorphic to $\mathbf{B}_i$. This follows straightforward from part (i)(b) of the definition above and the definition of an isomorphism.

(iii) Essentially, for any $i \in I$ and $j \in J_i$, we have

$$R^{\mathbf{A}_i}_{i,j} \subset (A_i)^{n(i,j)}$$

and by part (i)(b) of the definition above we have

$$R^{\mathbf{A}}_{i,j} \subset (\{i\} \times A_i)^{n(i,j)} \quad \text{and}$$

$$(a_1, a_2, \ldots, a_{n(i,j)}) \in R^{\mathbf{A}_i}_{i,j} \iff ((i, a_1), (i, a_2), \ldots, (i, a_{n(i,j)})) \in R^{\mathbf{A}}_{i,j}.$$

Lemma 3.2.5. *Suppose that $\mathcal{K}$ is a product of classes $\mathcal{K}_i$ for $i \in [t]$. Then for any $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ there is a bijection*

$$f^{\mathbf{A},\mathbf{B}} : \binom{\mathbf{B}}{\mathbf{A}} \to \prod_{i\in[t]} \binom{\mathbf{B}_i}{\mathbf{A}_i}.$$

Proof. Follows from Remarks 3.2.4. ☐

Lemma 3.2.6. *Suppose that $\mathcal{K}$ is a product of classes, $\mathcal{K} = \prod_{i\in[t]} \mathcal{K}_i$. Suppose that for each $i \in [t]$, $\mathcal{K}_i$ is a Ramsey class. Then $\mathcal{K}$ is a Ramsey class.*

Proof. Take any $\mathbf{A}, \mathbf{B} \in \mathcal{K}$. For $i \in [t]$, define $\mathbf{C}_i \in \mathcal{K}_i$ and a number l_i recursively as follows.

$$\mathbf{C}_1 \to (\mathbf{B}_1)_k^{\mathbf{A}_i}, \quad l_1 = \left|\binom{\mathbf{C}_1}{\mathbf{B}_1}\right|$$

and

$$\mathbf{C}_i \to (\mathbf{B}_i)_{k\cdot\prod_{j=1}^{i-1} l_j}^{\mathbf{A}_i}, \quad l_i = \left|\binom{\mathbf{C}_i}{\mathbf{B}_i}\right|.$$

The $\mathbf{C}_i$ exist because the classes $\mathcal{K}_i$ are Ramsey.
Now let $\mathbf{C} = \prod_{i\in[t]} \mathbf{C}_i$. We will show that $\mathbf{C} \to (\mathbf{B})_k^{\mathbf{A}}$.
To start, set $f_1 = f^{\mathbf{A},\mathbf{B}}$, $f_2 = f^{\mathbf{A},\mathbf{C}}$ and $f_3 = f^{\mathbf{B},\mathbf{C}}$.
Colour $c : \binom{\mathbf{C}}{\mathbf{A}} \to [k]$. Then $c' = c \circ f_2^{-1} : \prod_{i\in[t]} \binom{\mathbf{C}_i}{\mathbf{A}_i} \to [k]$ is a colouring as well, and finding a monochromatic $\prod_{i\in[t]} \binom{\mathbf{B}'_i}{\mathbf{A}_i}$ will yield a monochromatic

$$\binom{\mathbf{B}'}{\mathbf{A}} = f_1^{-1}\left(\prod_{i\in[t]} \binom{\mathbf{B}'_i}{\mathbf{A}_i}\right).$$

Start with $t = 2$. Enumerate $\binom{\mathbf{C}_1}{\mathbf{B}_1} = \{\mathbf{B}_{1,1}, \dots, \mathbf{B}_{1,l_1}\}$ and fix an $\mathbf{A}'_2 \in \binom{\mathbf{C}_2}{\mathbf{A}_2}$.
Then c' induces a colouring

$$c'|_1 : \binom{\mathbf{C}_1}{\mathbf{A}_1} \to [k], \quad \mathbf{A}'_1 \mapsto c'((\mathbf{A}'_1, \mathbf{A}'_2)).$$

Since $\mathbf{C}_1 \to (\mathbf{B}_1)_k^{\mathbf{A}_1}$, there is a $j \in [l_1]$ such that $\binom{\mathbf{B}_{1,j}}{\mathbf{A}_1}$ is monochromatic of colour $k_i \in [k]$. So let

$$c'_2 : \binom{\mathbf{C}_2}{\mathbf{A}_2} \to [k] \times [l_1], \quad \mathbf{A}'_2 \mapsto (k_i, j).$$

Note that $|[k] \times [l_1]| = k \cdot l_1$. So since $\mathbf{C}_2 \to (\mathbf{B}_2)^{\mathbf{A}_2}_{kl_1}$, the colouring c_2' gives us a monochromatic $\binom{\mathbf{B}_2'}{\mathbf{A}_2}$. That is, for each $\mathbf{A}_2' \in \binom{\mathbf{B}_2'}{\mathbf{A}_2}$,

$$c_2'(\mathbf{A}_2') = (k_i, j),$$

so for each $\mathbf{A}_1' \in \mathbf{B}_{1,j}$ we must have

$$c'((\mathbf{A}_1', \mathbf{A}_2')) = k_i.$$

So we must have $\binom{\mathbf{B}_{1,j}}{\mathbf{A}_1} \times \binom{\mathbf{B}_2'}{\mathbf{A}_2}$ monochromatic under c' of colour k_i and f_1^{-1} gives us the monochromatic $\binom{\mathbf{B}'}{\mathbf{A}}$ as explained above.
Now proceed by induction. Suppose the statement is true for $t \leq T$. Set $L = \prod_{j=1}^{T} l_j$. Then

$$c' : \prod_{i \in [T]} \binom{\mathbf{C}_i}{\mathbf{A}_i} \times \binom{\mathbf{C}_{T+1}}{\mathbf{A}_{T+1}} \to [k]$$

and picking an $\mathbf{A}_{T+1}' \in \binom{\mathbf{C}_{T+1}}{\mathbf{A}_{T+1}}$ induces a colouring

$$c'|_1 : \prod_{i \in [T]} \binom{\mathbf{C}_i}{\mathbf{A}_i} \to [k], \quad (\mathbf{A}_1', \ldots, \mathbf{A}_T') \mapsto c'((\mathbf{A}_1', \ldots, \mathbf{A}_{T+1}')).$$

We have that $\left|\prod_{i \in [T]} \binom{\mathbf{C}_i}{\mathbf{B}_i}\right| = L$, so enumerate $\prod_{i \in [T]} \binom{\mathbf{C}_i}{\mathbf{B}_i}$. By induction there is a monochromatic $\prod_{i \in [T]} \binom{\mathbf{B}_i'}{\mathbf{A}_i}$, the j^{th} such in the enumeration, of colour k_i, so define

$$c_2' : \binom{\mathbf{C}_2}{\mathbf{A}_2} \to [k] \times [L], \quad \mathbf{A}_2' \mapsto (k_i, j).$$

Note that $|[k] \times [L]| = k \cdot L$ and we have

$$\mathbf{C}_{T+1} \to (\mathbf{B}_{T+1})^{\mathbf{A}_{T+1}}_{k \cdot L}.$$

Hence the monochromatic $\mathbf{B}'_{T+1} \in \binom{\mathbf{C}_{T+1}}{\mathbf{B}_{T+1}}$ under the colouring c'_2 together with the j^{th} element of $\prod_{i\in[T]} \binom{\mathbf{C}_i}{\mathbf{B}_i}$ give us the monochromatic element of $\prod_{i\in[T+1]} \binom{\mathbf{C}_i}{\mathbf{B}_i}$ of colour k_i, finishing the proof.

□

Now, by definition of the product $\boldsymbol{\mathcal{K}} = \prod_{i\in[t]} \boldsymbol{\mathcal{K}}_i$, given any $\mathbf{A} \in \boldsymbol{\mathcal{K}}$, each of the $\mathbf{A}_i$ is a structure, and thus A_i is non-empty. But taking any non-empty subset A' of $[t] \rtimes A_i$, there exists a relational structure $\mathbf{A}'$, a substructure of $\mathbf{A}$ with universe A'. So unless $t = 1$, the class $\boldsymbol{\mathcal{K}}$ is not closed under substructures. To get a class closed under substructures, we introduce a full product of classes.

Definition 3.2.7. Let $\{L_i\}_{i\in[t]}$ be disjoint relational languages. For each $i \in [t]$ let $L_i = \{R_{i,j}\}_{j\in J_i}$, and let $\boldsymbol{\mathcal{K}}_i$ be a class of structures in language L_i. Let S be the set of non-empty subsets of $[t]$. Let $L = \bigcup_{i\in[t]} L_i$ be a union of languages L_i. The class $\boldsymbol{\mathcal{K}}$ in L is a *full product of classes* $\boldsymbol{\mathcal{K}}_i$ if

$$\boldsymbol{\mathcal{K}} = \bigcup_{T\in S} \left(\prod_{i\in T} \boldsymbol{\mathcal{K}}_i \right).$$

Remarks 3.2.8.

(i) Take $\mathbf{A} = \prod_{i\in T_A} \mathbf{A}_i \in \boldsymbol{\mathcal{K}}$. Then in particular, for each $i \in T_A$, the substructure of $\mathbf{A}$ on the set of points $\{i\} \times A_i$ lies in $\boldsymbol{\mathcal{K}}$. We abuse notation and denote it by $\mathbf{A}_i$, despite the fact $\mathbf{A}_i$ is technically a structure in $\boldsymbol{\mathcal{K}}_i$.

(ii) Technically, again, if $\boldsymbol{\mathcal{K}}_i$ are proper classes, we can't take a union of them. But we abuse the notation to mean that $\mathbf{A} \in \boldsymbol{\mathcal{K}}$ if and only if $\mathbf{A} \in \prod_{i\in T} \boldsymbol{\mathcal{K}}_i$ for some $T \in S$.

Lemma 3.2.9 (Full Structural Product Ramsey Lemma)**.** *Suppose that $\boldsymbol{\mathcal{K}}$ is a full product of classes. Suppose that for each $i \in [t]$, $\boldsymbol{\mathcal{K}}_i$ is a Ramsey class. Then $\boldsymbol{\mathcal{K}}$ is a Ramsey class.*

Proof. First note that by Lemma 3.2.6 the class $\prod_{i \in T} \mathcal{K}_i$ is Ramsey for each $T \in S$.
Take any $\mathbf{A} = \prod_{i \in T_A} \mathbf{A}_i, \mathbf{B} = \prod_{i \in T_B} \mathbf{B}_i \in \mathcal{K}$. Unless $T_A \subset T_B$, the set $\binom{\mathbf{B}}{\mathbf{A}}$ is empty, and trivially $\mathbf{B} \to (\mathbf{B})^{\mathbf{A}}_k$. So consider the case when $T_A \subset T_B$.
Clearly, if $T_A = T_B$, $\mathbf{A}, \mathbf{B} \in \prod_{i \in T_A} \mathcal{K}_i$, and thus there exists a $\mathbf{C} \in \prod_{i \in T_A} \mathcal{K}_i$ (and thus $\mathbf{C} \in \mathcal{K}$), such that $\mathbf{C} \to (\mathbf{B})^{\mathbf{A}}_k$.
Otherwise consider the substructure $\prod_{i \in T_A} \mathbf{B}_i$ of $\mathbf{B}$. Again there exists $\prod_{i \in T_A} \mathbf{D}_i \in \prod_{i \in T_A} \mathcal{K}_i$, such that

$$\prod_{i \in T_A} \mathbf{D}_i \to \left(\prod_{i \in T_A} \mathbf{B}_i \right)^{\mathbf{A}}_k .$$

Define $\mathbf{C}$ as $\prod_{i \in T_B} \mathbf{C}_i$, where

(i) $\mathbf{C}_i = \mathbf{D}_i$ if $i \in T_A$, and

(ii) $\mathbf{C}_i = \mathbf{B}_i$ if $i \in T_B \setminus T_A$.

Then $\mathbf{C} \to (\mathbf{B})^{\mathbf{A}}_k$.
Indeed. Suppose that $\mathbf{B}' \in \binom{\mathbf{C}}{\mathbf{B}}$. Then for $i \in T_B \setminus T_A$, we must have $\mathbf{B}'_i = \mathbf{C}_i$. For $i \in T_A$, B'_i is a substructure of $\mathbf{C}_i = \mathbf{D}_i$, isomorphic to $\mathbf{B}_i$. Thus finding a monochromatic substructure of $\prod_{i \in T_A} \mathbf{D}_i$ isomorphic to $\prod_{i \in T_A} \mathbf{B}_i$ corresponds to finding a a monochromatic substructure of $\mathbf{C}$ isomorphic to $\mathbf{B}$.

□

Recall that we can build new classes by *merging* them, as in Definition 2.5.2. In the case where the merge of any two structures is unique, we can apply the Full Structural Product Ramsey Lemma.

Lemma 3.2.10. *Let $L = \{R_i\}_{i \in I}$ be a relational language and let $\mathcal{K}, \mathcal{K}_1$ and $\mathcal{K}_2$ be classes in language L. Suppose that $\mathcal{K}$ is a merge of classes $\mathcal{K}_1$ and $\mathcal{K}_2$.*
Let $L_1 = \{R_{1,i}\}_{i \in I}$ and $L_1 = \{R_{2,i}\}_{i \in I}$ be disjoint copies of language L, and let $\mathcal{K}'_1$ be a copy of the class $\mathcal{K}_1$ in language L_1, and let $\mathcal{K}'_2$ be a copy of the class $\mathcal{K}_2$ in language L_2.
If, for each $\mathbf{A}_1 \in \mathcal{K}_1$ and $\mathbf{A}_2 \in \mathcal{K}_2$ the merge $\mathbf{A}$ of structures $\mathbf{A}_1$ and $\mathbf{A}_2$

is unique up to isomorphism, then the class $\mathcal{K}$ is simply bi-definable with the full product of classes $\mathcal{K}_1'$ and $\mathcal{K}_2'$.

Note. By 'a copy of the language' and 'a copy of the class' we mean that $\mathcal{K}_1$ and $\mathcal{K}_1'$ are essentially the same class, up to slightly different notation. We could say that they are trivially simply bi-definable. We introduce L_1 and L_2 because we define a product of classes for classes in disjoint languages in 3.2.3.

Proof. Trivially, for any $\mathbf{A}_1 \in \mathcal{K}_1$ there exists a structure $\mathbf{A}_1' \in \mathcal{K}_1'$ such that for all $i \in I$ we have

$$R_{1,i}^{\mathbf{A}_1'}(a_1, a_2, \ldots, a_{n(i)}) \iff \mathbf{A}_1 \models R_i^{\mathbf{A}_1}(a_1, a_2, \ldots, a_{n(i)}).$$

Similarly, there exists a structure $\mathbf{A}_2' \in \mathcal{K}_2'$ for any $\mathbf{A}_2 \in \mathcal{K}_2$.
Conversely, for $h \in [2]$ and any $\mathbf{A}_h' \in \mathcal{K}_h'$, there is a $\mathbf{A}_h \in \mathcal{K}_h$, such that

$$R_{h,i}^{\mathbf{A}_h}(a_1, a_2, \ldots, a_{n(i)}) \iff \mathbf{A}_h' \models R_i^{\mathbf{A}_h'}(a_1, a_2, \ldots, a_{n(i)}).$$

This formalises the assertion that $\mathcal{K}_h'$ is a copy of $\mathcal{K}_h$ for $h \in [2]$.

Take any $\mathbf{A} \in \mathcal{K}$. If A_2 is empty, then $\mathbf{A} = \mathbf{A}_1 \in \mathcal{K}_1$, so

$$R_{1,i}^{\mathbf{A}_1'}(a_1, a_2, \ldots, a_{n(i)}) \iff \mathbf{A} \models R_i^{\mathbf{A}}(a_1, a_2, \ldots, a_{n(i)}).$$

Similarly if A_1 is empty. In an analogous way, we can define $R_i^{\mathbf{A}}$ using $R_{1,i}^{\mathbf{A}_1'}$ for any $\mathbf{A}_1' \in \mathcal{K}_1'$, and using $R_{2,i}^{\mathbf{A}_2'}$ for any $\mathbf{A}_2' \in \mathcal{K}_2'$.

Otherwise $\mathbf{A}$ is a merge of $\mathbf{A}_1$ and $\mathbf{A}_2$. Then we have

$$R_{1,i}^{\mathbf{A}_1'}(a_1, \ldots, a_{n(i)}) \iff \mathbf{A} \models R_i^{\mathbf{A}}(a_1, \ldots, a_{n(i)}) \wedge F_1(a_1) \wedge \ldots \wedge F_1(a_{n(i)})$$

and

$$R_{2,i}^{\mathbf{A}_2'}(a_1, \ldots, a_{n(i)}) \iff \mathbf{A} \models R_i^{\mathbf{A}}(a_1, \ldots, a_{n(i)}) \wedge F_2(a_1) \wedge \ldots \wedge F_2(a_{n(i)}).$$

So using the definition of relations on the product $\mathbf{A}' = \prod_{h' \in [2]} \mathbf{A}_{h'}$, that defines the relations $R_{h,i}^{\mathbf{A}'}$.
Conversely, a relation $R_{h,i}^{\mathbf{A}'}$ defines the relation $R_i^{\mathbf{A}_h}$. Since given any two structures $\mathbf{A}_1 \in \mathcal{K}_1$, $\mathbf{A}_2 \in \mathcal{K}_2$ there is a unique merge $\mathbf{A} \in \mathcal{K}$ of $\mathbf{A}_1$ and $\mathbf{A}_2$, that means that the relations $R_{1,i}^{\mathbf{A}'}$ and $R_{2,i}^{\mathbf{A}'}$ define the relation $R_i^{\mathbf{A}}$. This concludes the proof.

□

Corollary 3.2.11. *Let $L = \{R_i\}_{i \in I}$ be a relational language and let $\mathcal{K}$ and $\mathcal{K}_i$, for $i \in [n]$, be classes in language L. Suppose that $\mathcal{K}$ is a merge of classes $\mathcal{K}_i$.*
Let, for $h \in [n]$, $L_h = \{R_{h,i}\}_{i \in I}$ be copies of language L, and let $\mathcal{K}'_i$ be a copy of the class $\mathcal{K}_i$ in language L_i.
If, for each selection of $\mathbf{A}_i \in \mathcal{K}_i$, the merge $\mathbf{A}$ of structures $\mathbf{A}_i$ is unique up to isomorphism, then the class $\mathcal{K}$ is simply bi-definable with the full product of classes $\mathcal{K}'_i$.

Proof. The proof follows from 3.2.10, by induction on the number n of the classes involved. □

Corollary 3.2.12. *Let $L = \{R_i\}_{i \in I}$ be a relational language and let $\mathcal{K}$ and $\mathcal{K}_i$, for $i \in [n]$, be classes in language L. Suppose that $\mathcal{K}$ is a merge of classes $\mathcal{K}_i$ and suppose that each class $\mathcal{K}_i$ is a Ramsey class.*
If, for each selection of $\mathbf{A}_i \in \mathcal{K}_i$ the merge $\mathbf{A}$ of structures $\mathbf{A}_i$ is unique up to isomorphism, then the class $\mathcal{K}$ is a Ramsey class.

Link. Theorem 4.3.5, Theorem 4.2.4

Proof. By Lemma 3.2.11, the class $\mathcal{K}$ is simply bidefinable with the full product of classes $\mathcal{K}'_i$, defined in the lemma. Now by Lemma 3.1.6, each class $\mathcal{K}'_i$ is a Ramsey class, as it is simply bi-definable with the Ramsey class $\mathcal{K}_i$. The full product of classes $\mathcal{K}'_i$ is Ramsey by Lemma 3.2.9, so $\mathcal{K}$ is indeed also a Ramsey class. □

3.3 Substructures

Throughout this section, let $\boldsymbol{\mathcal{K}}$ be a class of ordered $\mathfrak{S}$-shaped partial orders closed under substructures and isomorphisms, and let $\mathcal{K}$ be a Fraïssé class of ordered partial orders containing the unshaped reducts of the structures in $\boldsymbol{\mathcal{K}}$, *i.e.*,

$$\text{if } (P,<,\prec,\mathfrak{s}) \in \boldsymbol{\mathcal{K}}, \text{ then } (P,<,\prec) \in \mathcal{K}.$$

Note. Given a shaped partial order $\mathbf{P} = (P,<,\prec,\mathfrak{s})$, refer to the reduct $(P,<,\prec)$ as P. This is a slight abuse of notation, as P also refers to the universe of $\mathbf{P}$ (and of $(P,<,\prec)$). It makes the heavy notation in this section somewhat lighter.

Definition 3.3.1. Given a $P \in \mathcal{K}$, define the *set* $\left[\mathfrak{S}^P\right]$ *of shapings of* P to be the set

$$\left[\mathfrak{S}^P\right] = \{\mathfrak{s} : (P,<,\prec,\mathfrak{s}) \in \boldsymbol{\mathcal{K}}\}.$$

Lemma 3.3.2. *Let* $\mathbf{P} \in \boldsymbol{\mathcal{K}}$, $\mathbf{P} = (P,<^{\mathbf{P}},\prec^{\mathbf{P}},\mathfrak{s}^{\mathbf{P}})$ *and let*

$$\mathbf{i}_{P'} : (P,<,\prec) \to (P',<,\prec)$$

be an isomorphism with the underlying bijection

$$i_{P'} : P \to P'.$$

Then $\mathfrak{s}^{\mathbf{P}} \circ i_{P'}^{-1} \in [\mathfrak{S}^{P'}]$.

Proof. We will show that $i_{P'}$ defines an isomorphism

$$\mathbf{i}_{\mathbf{P}'} : (P,<,\prec,\mathfrak{s}^{\mathbf{P}}) \to (P',<,\prec,\mathfrak{s}^{\mathbf{P}} \circ i_{P'}^{-1}).$$

For each $p \in P$,

$$\mathfrak{s}^{\mathbf{P}}(p) = \mathfrak{s}^{\mathbf{P}} \circ (i_{P'}^{-1} \circ i_{P'})(p) = \mathfrak{s}^{\mathbf{P}} \circ i_{P'}^{-1}(i_{P'}(p)).$$

So since $\mathbf{i}_{P'}$ is an isomorphism and $\mathcal{K}$ is closed under isomorphisms, $(P', <, \prec, \mathfrak{s}^{\mathbf{P}} \circ i_{P'}^{-1}) \in \mathcal{K}$ and hence $\mathfrak{s}^{\mathbf{P}} \circ i_{P'}^{-1} \in [\mathfrak{S}^{P'}]$. $\square$

Lemma 3.3.3. *Let $\mathbf{P} \in \mathcal{K}$, $(P, <^{\mathbf{P}}, \prec^{\mathbf{P}}, \mathfrak{s}^{\mathbf{P}})$ and $R' \in \binom{P}{R}$. If $\mathcal{K}$ is closed under substructures and isomorphisms, then for*

$$\mathfrak{s}^{\mathbf{P}}|_{R'} : R' \to \mathfrak{S}, \quad r \mapsto \mathfrak{s}^{\mathbf{P}}(r)$$

and the isomorphism $\mathbf{i} : R \to R'$ of partial orders with the underlying bijection

$$i : R \to R'$$

we have $\mathfrak{s}|_{R'} \circ i \in [\mathfrak{S}^R]$.

Proof. Since $\mathcal{K}$ is a class of ordered $\mathfrak{S}$-shaped partial orders closed under substructures, for any subset Q on P, the structure of $\mathbf{P}$ induces a structure on Q, namely

$$(Q, <^{\mathbf{Q}}, \prec^{\mathbf{Q}}, \mathfrak{s}^{\mathbf{Q}}), \text{ where } <^{\mathbf{Q}} = <^{\mathbf{P}} |_Q, \prec^{\mathbf{Q}} = \prec^{\mathbf{P}} |_Q, \mathfrak{s}^{\mathbf{Q}} = \mathfrak{s}^{\mathbf{P}}|_Q.$$

So the isomorphism $\mathbf{i}$ is a map

$$\mathbf{i} : (R, <^R, \prec^R) \to (R', <^{\mathbf{P}} |_{R'}, \prec^{\mathbf{P}} |_{R'}).$$

Then $(R, <^R, \prec^R, \mathfrak{s}|_{R'} \circ i)$ is isomorphic to $(R', <^{\mathbf{P}} |_{R'}, \prec^{\mathbf{P}} |_{R'}, \mathfrak{s}^{\mathbf{P}}|_{R'})$, as for any $r \in R$ we have

$$\mathfrak{s}|_{R'} \circ i(r) = \mathfrak{s}|_{R'}(i(r)).$$

So, indeed, $\mathfrak{s}|_{R'} \circ i \in [\mathfrak{S}^R]$. $\square$

Corollary 3.3.4. *If $\mathcal{K}$ is closed under substructures and isomorphisms, $P, R \in \mathcal{K}$ and for each $R' \in \binom{P}{R}$, $\mathbf{i}_{R'}$ is the isomorphism*

$$\mathbf{i}_{R'} : R \to R'.$$

Then there is a map

$$\theta^{P,R} : \binom{P}{R} \times \left[\mathfrak{S}^P\right] \to \left[\mathfrak{S}^R\right], \quad (R', \mathfrak{s}) \mapsto \mathfrak{s}|_{R'} \circ i_{R'}.$$

Note. Since $\mathbf{i}_{R'}$ is an isomorphism between two rigid structures, it is unique.

Proof. Straightforward from Lemma 3.3.3. □

Definition 3.3.5. A class $\mathcal{K}$ has a *neat shapings property* if for any $R' \in \binom{P}{R}$ we have

$$\theta\Big(\{R'\} \times \left[\mathfrak{S}^P\right]\Big) = \left[\mathfrak{S}^R\right].$$

Lemma 3.3.6. *Suppose that $\mathcal{K}$ is a class of ordered partial orders and $P, P', R \in \mathcal{K}$. Then an isomorphism*

$$\mathbf{i} : (P, <, \prec) \to (P', <, \prec), \text{ with } i : P \to P'$$

yields a bijection

$$\phi_R : \binom{P}{R} \to \binom{P'}{R}, \quad R' \mapsto i(R').$$

Proof. Since $\mathbf{i}$ is an isomorphism, the map i is a bijection. Clearly $i(R')$ is a subset of P' by definition of i. Also, if $R' \in \binom{P}{R}$ then for any $r, r' \in R'$ we have

$$r < r' \iff i(r) < i(r') \quad \text{and} \quad r \prec r' \iff i(r) \prec i(r').$$

So the map

$$i|_{R'} : R' \to i(R'), \quad r \mapsto i(r)$$

defines an isomorphism $\mathbf{i}' : R' \to i(R')$ and thus $i(R')$ is isomorphic to R and lies in $\binom{P'}{R}$. So $\phi_R(\binom{P}{R}) \subset \binom{P'}{R}$ and ϕ_R is well defined.

Similarly we could use i^{-1} to show that $\binom{P'}{R} \subset \binom{P}{R}$, considering

$$\phi_R^{-1} : \binom{P'}{R} \to \binom{P}{R}, \quad R' \mapsto i^{-1}(R').$$

Finally, given $R' \in \binom{P}{R}$, we have

$$\phi_R^{-1}(\phi_R(R')) = \phi_R^{-1}(i(R')) = i^{-1}(i(R')) = R'$$

and similarly $\phi_R(\phi_R^{-1}(R')) = R'$, so ϕ_R is indeed a bijection. $\square$

3.4 Blowup Lemma

Definition 3.4.1. Let $\boldsymbol{\mathcal{K}}$ be a class of ordered $\mathfrak{S}$-shaped partial orders, and let $\mathcal{K}$ be a Fraïssé class of ordered partial orders containing the unshaped reducts of the structures in $\boldsymbol{\mathcal{K}}$. Let X be a set of size at least $|\mathfrak{S}|$ and let $\alpha : \mathfrak{S} \to X$ be an injective map and $\beta : X \to \mathfrak{S}$ a map such that $\beta(\alpha(\mathbf{s})) = \mathbf{s}$.
A *weak* (X, α, β)*-blowup* $\overline{P}$ of an ordered partial order $P \in \mathcal{K}$ is any ordered partial order in $\mathcal{K}$ on the set of points

$$\overline{P} = P \times X.$$

Given a $p \in P$, a p*-block* of $\overline{P}$ is the substructure of $\overline{P}$ on the set of points $\{p\} \times X$.

Link. Lemma 4.1.1, Lemma 5.2.2

Given a set X, the maps α, β and a partial order $P \in \mathcal{K}$ we can define a map

$$s : \overline{P} \to \mathfrak{S}, \quad (p, x) \mapsto \beta(x).$$

Since $\overline{P} \in \mathcal{K}$ and we did not require $\boldsymbol{\mathcal{K}}$ to have any properties, it could happen that $\left[\mathfrak{S}^{\overline{P}}\right]$ is empty. So in general, the map s need not be a shaping of $\overline{P}$. But we will consider the cases where it is, and where additionally the blowup

$\overline{\mathbf{P}} = (\overline{P}, <, \prec, s)$ contains $(P, <, \prec, \mathfrak{s})$ as a substructure for any $\mathfrak{s} \in \left[\mathfrak{S}^P\right]$.

Note. Given a shaping $\mathfrak{s} \in \left[\mathfrak{S}^P\right]$, there is a shaped partial order $(P, <, \prec, \mathfrak{s})$ in the class $\boldsymbol{\mathcal{K}}$ by definition. But given a partial order $\mathbf{P} \in \boldsymbol{\mathcal{K}}$, we might want to denote the specific shaping on $\mathbf{P}$ by $\mathfrak{s}^{\mathbf{P}}$, reverting to the more formal notation when many shapings are involved.

There is a natural way to find such a substructure of $\overline{\mathbf{P}}$. Indeed, take any shaped partial order $\mathbf{P} = (P, <, \prec, \mathfrak{s}) \in \boldsymbol{\mathcal{K}}$ and let

$$\mathfrak{s} : P \to \mathfrak{S}, \quad p \mapsto \mathfrak{s}(p).$$

Then for a subset $P(\mathfrak{s})$ of $\overline{P}$ defined as

$$P(\mathfrak{s}) = \{(p, x) : p \in P, x = \alpha(\mathfrak{s}(p))\}$$

we have:

$$s((p, x)) = \beta(x) = \beta(\alpha(\mathfrak{s}(p))) = \mathfrak{s}(p).$$

Denote by $P(\mathfrak{s})$ also the substructure of $\overline{P}$ on the set $P(\mathfrak{s})$. Suppose that we have $P(\mathfrak{s}) \in \binom{\overline{P}}{P}$ and that $\overline{\mathbf{P}} = (\overline{P}, <, \prec, s) \in \boldsymbol{\mathcal{K}}$. Let $\mathbf{P}(\mathfrak{s})$ be the substructure of $\overline{\mathbf{P}}$ on the points $P(\mathfrak{s})$. Then we have

$$\mathbf{P}(\mathfrak{s}) \in \binom{\overline{\mathbf{P}}}{\mathbf{P}}.$$

We will additionally require that the blowup and substructures are related in a natural way.

Definition 3.4.2. Let $\boldsymbol{\mathcal{K}}$ be a class of ordered $\mathfrak{S}$-shaped partial orders closed under substructures and isomorphisms, and let $\mathcal{K}$ be a Fraïssé class of ordered partial orders containing the unshaped reducts of the structures in $\boldsymbol{\mathcal{K}}$. Let X be a set of size at least $|\mathfrak{S}|$ and let $\alpha : \mathfrak{S} \to X$ be an injective map and $\beta : X \to \mathfrak{S}$ a map such that $\beta(\alpha(\mathbf{s})) = \mathbf{s}$.
The class $\boldsymbol{\mathcal{K}}$ *admits* (X, α, β)*-blowups* if it satisfies the following two con-

ditions.

(i) For any reduct $P \in \mathcal{K}$ of a structure $\mathbf{P} \in \boldsymbol{\mathcal{K}}$, there is a unique weak (X, α, β)-blowup $\overline{P}$, which, together with a map

$$s : \overline{P} \to \mathfrak{S}, \quad (p, x) \mapsto \beta(x),$$

forms an ordered shaped partial order in $\boldsymbol{\mathcal{K}}$, denoted by $\overline{\mathbf{P}} = (\overline{P}, <, \prec, s)$.

(ii) Given any $\mathbf{P}, \mathbf{R} \in \boldsymbol{\mathcal{K}}$ and their reducts $P, R \in \mathcal{K}$, the following maps are well-defined:

(a) the map

$$\begin{aligned} gg : \binom{P}{R} \times [\mathfrak{S}^R] &\to \binom{\overline{P}}{R}, \\ (R', \mathfrak{s}') &\mapsto R'(\mathfrak{s}') = (R'(\mathfrak{s}'), <, \prec), \end{aligned}$$

where $R'(\mathfrak{s}') = \{(r, x) : r \in R', x = \alpha(\mathfrak{s}'(i_{R'}^{-1}(r)))\}$, and

(b) the map

$$\begin{aligned} ff : \binom{P}{R} &\to \binom{\overline{\mathbf{P}}}{\mathbf{R}}, \\ R' &\mapsto \mathbf{R}' = (R'(\mathfrak{s}^{\mathbf{R}}), <, \prec, \mathfrak{s}^{\mathbf{R}} \circ i_{R'}^{-1}), \end{aligned}$$

where $R'(\mathfrak{s}^{\mathbf{R}}) = \{(r, x) : r \in R', x = \alpha(\mathfrak{s}^{\mathbf{R}}(i_{R'}^{-1}(r)))\}$.

Link. Lemma 4.1.3, Lemma 5.2.2

Note. We defined gg as a map sending a substructure of P, together with a shaping of the substructure R' of P, to substructure on a specific subset of $\overline{P}$. Since $\overline{P}$ is a partial order, a subset of its universe defines a unique substructure of $\overline{P}$. But insisting that the map gg be well-defined requires that the substructure of $\overline{P}$ on the specific subset is isomorphic to the structure R. Similarly with the map ff.
Besides, we should actually write $gg^{P,R}$ and $ff_1^{\mathbf{P},\mathbf{R}}$, since the maps exist for

each pair $\mathbf{P}, \mathbf{R}$ of structures in $\boldsymbol{\mathcal{K}}$ and their reducts $P, R \in \mathcal{K}$. But we will drop the labels of the maps unless they're needed.

Definition 3.4.3. For the maps gg and ff defined in 3.4.2, define the set of *partial transversals* of P isomorphic to R as the set

$$gg\left(\binom{P}{R} \times \left[\mathfrak{S}^R\right]\right) = \left(\binom{\overline{P}}{R}\right) \subset \binom{\overline{P}}{R}.$$

Let the set of *shaped partial transversals* be the set

$$ff\left(\binom{P}{R}\right) = \left(\binom{\overline{\mathbf{P}}}{\mathbf{R}}\right) \subset \binom{\overline{\mathbf{P}}}{\mathbf{R}}$$

Lemma 3.4.4. *For any partial transversal R'' of $\overline{P}$ and any $p \in P$ we have*

$$\left|R'' \cap \overline{P}_p\right| \leq 1.$$

Proof. If $R'' \in \left(\binom{\overline{P}}{R}\right)$, then $R'' = (R'(\mathfrak{s}'), <, \prec)$ for some $R' \in \binom{P}{R}$ and $\mathfrak{s}' \in \left[\mathfrak{S}^R\right]$ and $R'(\mathfrak{s}') = \{(r, x) : r \in R', x = \alpha(\mathfrak{s}'(i_{R'}^{-1}(r)))\}$. We also have $\overline{P}_p = \{p\} \times X$. So for $p \in P$ we have

(i) $R' \cap \overline{P}_p = \{(p, x) : x = \alpha(\mathfrak{s}'(i_{R'}^{-1}(p)))\}$ if $p \in R'$ and

(ii) $R'' \cap \overline{P}_p = \emptyset$ otherwise.

□

Lemma 3.4.5. *The following maps are bijections.*

(i) $g : \binom{P}{R} \times \left[\mathfrak{S}^R\right] \to \left(\binom{\overline{P}}{R}\right), \quad g(R', \mathfrak{s}') = gg(R', \mathfrak{s}')$, *and*

(ii) $f : \binom{P}{R} \to \left(\binom{\overline{\mathbf{P}}}{\mathbf{R}}\right), \quad f(R') = ff(R').$

Note. The maps f amd g are restrictions of the maps ff and gg to their codomains respectively.
Similar to the labels of maps gg and ff, we should actually write $g^{P,R}$ and $f^{\mathbf{P},\mathbf{R}}$, since the maps exist for each pair $\mathbf{P},\mathbf{R}$ of structures in $\boldsymbol{\mathcal{K}}$ and their reducts $P,R \in \mathcal{K}$. But we will drop the labels of the maps again unless they're needed.

Proof. The maps f and g are surjective by definition of $\left(\binom{\overline{P}}{R}\right)$ and $\left(\binom{\mathbf{P}}{\mathbf{R}}\right)$.
Suppose that $gg(R',\mathfrak{s}') = gg(R'',\mathfrak{s}'')$, *i.e.*, $R'(\mathfrak{s}') = R''(\mathfrak{s}'')$ and thus

$$\{(r,x) : r \in R', x = \alpha(\mathfrak{s}'(i_{R'}^{-1}(r)))\} = \{(r,x) : r \in R'', x = \alpha(\mathfrak{s}''(i_{R''}^{-1}(r)))\}.$$

Now, for $r \in R'$, $(r,x) \in R'(\mathfrak{s}') = R''(\mathfrak{s}'')$, so $r \in R''$ and thus $R' \subset R''$. Analogously we can show $R'' \subset R'$ and thus $R' = R''$. Also $i_{R'}^{-1} = i_{R''}^{-1}$. Now take any $r \in R' = R''$. We have $i_{R'}^{-1}(r) \in R$ and $i_{R'}^{-1}(R') = R$. Also $(r,x) \in R'(\mathfrak{s}') = R''(\mathfrak{s}'')$ and so

$$\alpha(\mathfrak{s}^{\mathbf{R}'}(i_{R'}^{-1}(r))) = x = \alpha(\mathfrak{s}^{\mathbf{R}''}(i_{R'}^{-1}(r))).$$

By definition, α is an injective map, so we have $\mathfrak{s}^{\mathbf{R}'}(i_{R'}^{-1}(r)) = \mathfrak{s}^{\mathbf{R}''}(i_{R'}^{-1}(r))$. The shapings $\mathfrak{s}^{\mathbf{R}'}$ and $\mathfrak{s}^{\mathbf{R}''}$ agree on all $i_{R'}^{-1}(r) \in R$ and are thus the same shaping. So in fact we have $(R',\mathfrak{S}') = (R'',\mathfrak{S}'')$ and gg is an injection.
We omit the proof that ff is a bijection. $\square$

Lemma 3.4.6. *Suppose that $P,R,Q \in \mathcal{K}$, $R' \in \binom{P}{R}$ and $Q' \in \binom{R'}{Q}$. Then*

$$g^{R',Q} = g^{P,Q}|_{\binom{R'}{Q}}.$$

Proof. Since $Q' \in \binom{R'}{Q}$ and $R' \in \binom{P}{R}$, we have $Q' \in \binom{P}{Q}$. Pick any $\mathfrak{s}' \in [\mathfrak{S}^Q]$. By definition we have

$$\begin{aligned} g^{P,Q} : \binom{P}{Q} \times [\mathfrak{S}^Q] &\to \left(\binom{\overline{P}}{Q}\right), \\ (Q',\mathfrak{s}') &\mapsto Q'(\mathfrak{s}') = (Q'(\mathfrak{s}'), <, \prec), \end{aligned}$$

where $Q'(\mathfrak{s}') = \{(q,x) : q \in Q', x = \alpha(\mathfrak{s}'(i_{Q'}^{-1}(q)))\}$, and

$$\begin{array}{rccccl} g^{R',Q} : & \binom{R'}{Q} \times [\mathfrak{S}^Q] & \to & \left(\binom{\overline{R'}}{Q}\right), \\ & (Q', \mathfrak{s}') & \mapsto & \overline{Q'}(\mathfrak{s}') = (Q'(\mathfrak{s}'), <, \prec), \end{array}$$

where $Q'(\mathfrak{s}') = \{(q,x) : q \in Q', x = \alpha(\mathfrak{s}'(i_{Q'}^{-1}(q)))\}$.
Then indeed $g^{R',Q}(Q',\mathfrak{s}') = \overline{Q'}(\mathfrak{s}') = g^{P,Q}(Q',\mathfrak{s}')$, so $g^{R',Q} = g^{P,Q}|_{\binom{R'}{Q}\times[\mathfrak{S}^Q]}$. ∎

Lemma 3.4.7. *Suppose that*

$$R' \in \left(\binom{\overline{P}}{R}\right) \text{ and } Q' \in \binom{R'}{Q}.$$

Then the following are true.

(i) Let $R' = g^{P,R}((R'',\mathfrak{s}))$, *where* $R'' \in \binom{P}{R}$ *and* $\mathfrak{s} \in [\mathfrak{S}^R]$. *Then*

(a) for each $Q'' \in \binom{R''}{Q}$ *there is a* $\mathfrak{s}' \in [\mathfrak{S}^Q]$ *such that*

$$g^{P,Q}((Q'',\mathfrak{s}')) \in \binom{R'}{Q}$$

(b) and

$$Q' = g^{P,Q}((Q'',\mathfrak{s}'))$$

for some $Q'' \in \binom{R''}{Q}$ *and* $\mathfrak{s}'$ *from part (i)(a).*

(ii) $Q' \in \left(\binom{\overline{P}}{Q}\right)$.

Proof. For $R' \in \left(\binom{\overline{P}}{R}\right)$, there are $R'' \in \binom{P}{R}$ and $\mathfrak{s} \in [\mathfrak{S}^R]$ such that

$$\begin{array}{rccccl} g^{P,R} : & \binom{P}{R} \times [\mathfrak{S}^R] & \to & \left(\binom{\overline{P}}{R}\right), \\ & (R'', \mathfrak{s}) & \mapsto & R' = \overline{R''}(\mathfrak{s}) = (R''(\mathfrak{s}), <, \prec), \end{array}$$

where $R''(\mathfrak{s}) = \{(r,x) : r \in R'', x = \alpha(\mathfrak{s}(i_{R''}^{-1}(r))\}$.

i) a)
Take any $Q'' \in \binom{R''}{Q}$. Then let

$$i_{R'} : R \to R', \quad i_{R''} : R \to R'' \quad \text{and} \quad i_{Q''} : Q \to Q''$$

be isomorphisms, and

$$\mathfrak{s}' = \mathfrak{s} \circ i_{R''}^{-1}\big|_{Q''} \circ i_{Q''}$$

Start by showing that $\mathfrak{s}' \in \left[\mathfrak{S}^Q\right]$. Since $\mathfrak{s} : R \to \mathfrak{S}$ and $i_{R''} : R \to R''$ is an isomorphism, we have $\mathfrak{s} \circ i_{R''}^{-1} \in \left[\mathfrak{S}^{R''}\right]$ by Lemma 3.3.2.
Further, as

$$\mathfrak{s} \circ i_{R''}^{-1}\big|_{Q''} : Q'' \to \mathfrak{S} \quad \text{and} \quad i_{Q''} : Q \to Q'',$$

with $i_{Q''}$ an isomorphism, we have $\mathfrak{s}' = \mathfrak{s} \circ i_{R''}^{-1}\big|_{Q''} \circ i_{Q''} \in \left[\mathfrak{S}^Q\right]$ by Lemma 3.3.3.
Now since $Q'' \in \binom{R''}{Q}$ and $R'' \in \binom{P}{R}$, we have $Q'' \in \binom{P}{Q}$. So by definition,

$$\begin{array}{rccl} g^{P,Q} : & \binom{P}{Q} \times \left[\mathfrak{S}^Q\right] & \to & \left(\!\binom{\overline{P}}{Q}\!\right), \\ & (Q'', \mathfrak{s}') & \mapsto & Q''(\mathfrak{s}') = (Q''(\mathfrak{s}'), <, \prec), \end{array}$$

where $Q''(\mathfrak{s}) = \{(q, x) : q \in Q'', x = \alpha(\mathfrak{s}'(i_{Q''}^{-1}(q))\}$.
Since $q \in Q'' \subset R''$, we have

$$\begin{aligned} x = \alpha(\mathfrak{s}'(i_{Q''}^{-1}(q)) &= \alpha(\mathfrak{s} \circ i_{R''}^{-1}\big|_{Q''} \circ i_{Q''}(i_{Q''}^{-1}(q)) = \\ &= \alpha(\mathfrak{s} \circ i_{R''}^{-1}\big|_{Q''}(i_{Q''}(i_{Q''}^{-1}(q))) = \\ &= \alpha(\mathfrak{s} \circ i_{R''}^{-1}\big|_{Q''}(q)) = \\ &= \alpha(\mathfrak{s} \circ i_{R''}^{-1}(q)) \end{aligned}$$

Therefore

$$Q''(\mathfrak{s}') = \{(q, x) : q \in Q'' \subset R'', x = \alpha(\mathfrak{s}(i_{R''}^{-1}(q))\} \subset R''(\mathfrak{s}).$$

Since $Q''(\mathfrak{s}')$ is isomorphic to Q, we indeed have

$$Q''(\mathfrak{s}') = g^{P,Q}\left((Q'', \mathfrak{s}')\right) \in \binom{R'}{Q}.$$

i) b)

Consider now

$$W = \left\{ (Q'', \mathfrak{s} \circ i_{R''}^{-1}|_{Q''} \circ i_{Q''}) : Q'' \in \binom{R''}{Q} \right\}.$$

We claim that

$$\binom{R'}{Q} = g^{P,Q}(W).$$

We have already shown that for $(Q'', \mathfrak{s}') \in W$, $g^{P,Q}((Q'', \mathfrak{s}')) \in \binom{R'}{Q}$, so $g^{P,Q}(W) \subset \binom{R'}{Q}$. But the size of W is the size of $\binom{R''}{Q}$, and $g^{P,Q}$ is a bijection. Since the sets are finite that means that indeed

$$\binom{R'}{Q} = g^{P,Q}(W).$$

Hence for any $Q' \in \binom{R'}{Q}$,

$$Q' = g^{P,Q}((Q'', \mathfrak{s}'))$$

for some $Q'' \in \binom{R''}{Q}$ and $\mathfrak{s}' \in [\mathfrak{S}^Q]$.

ii) Follows immediately from part i).

$\square$

Corollary 3.4.8. *The following is true.*

$$\left(\!\!\binom{\overline{\mathbf{P}}}{\mathbf{R}}\!\!\right) = \left\{ \mathbf{R}' = (R', <, \prec, \mathfrak{s}) : \mathbf{R}' \in \binom{\overline{\mathbf{P}}}{\mathbf{R}}, R' \in \left(\!\!\binom{\overline{P}}{R}\!\!\right) \right\}.$$

Suppose that

$$\mathbf{R}' \in \left(\!\!\binom{\overline{\mathbf{P}}}{\mathbf{R}}\!\!\right) \text{ and } \mathbf{Q}' \in \binom{\mathbf{R}'}{\mathbf{Q}}.$$

Then the following are also true.

(i) For $\mathbf{R}' = f^{\mathbf{P},\mathbf{R}}(R'')$, we have

$$\mathbf{Q}' = f^{\mathbf{P},\mathbf{Q}}(Q'')$$

for some $Q'' \in \binom{R''}{Q}$.

(ii) $\mathbf{Q}' \in \left(\!\!\binom{\overline{\mathbf{P}}}{\mathbf{Q}}\!\!\right)$.

Proof. Take any $\mathbf{R}' = (R', <, \prec, \mathfrak{s}) \in \left(\!\!\binom{\overline{\mathbf{P}}}{\mathbf{R}}\!\!\right)$. By definition of f we have

$$\mathbf{R}' = f^{\mathbf{P},\mathbf{R}}(R'') = (R''(\mathfrak{s}^{\mathbf{R}}), <, \prec, \mathfrak{s}^{\mathbf{R}} \circ i_{R''}^{-1})$$

for some $R'' \in \binom{P}{R}$. We know by definition of $g^{P,R}$ that the substructure of $\overline{P}$ on the points $R''(\mathfrak{s}^{\mathbf{R}})$ is exactly the structure $g^{P,R}((R'', \mathfrak{s}^{\mathbf{R}}))$.
We already know that $\left(\!\!\binom{\overline{\mathbf{P}}}{\mathbf{R}}\!\!\right) \subset \binom{\overline{\mathbf{P}}}{\mathbf{R}}$, by definition. We have also shown that $R' = R''(\mathfrak{s}^{\mathbf{R}}) = g^{P,R}((R'', \mathfrak{s}^{\mathbf{R}}))$ for some $R'' \in \binom{P}{R}$. Then by definition of $g^{P,R}$ we indeed have $R' \in \left(\!\!\binom{\overline{P}}{R}\!\!\right)$.

(i) Since $\mathbf{Q}' \in \binom{\mathbf{R}'}{\mathbf{Q}}$, we have

$$\mathbf{Q}' = (Q', <, \prec, \mathfrak{s}^{\mathbf{Q}} \circ i_{Q'}^{-1})$$

with $Q' = (Q', <, \prec)$ isomorphic to $(Q, <, \prec)$ as $\mathbf{Q}'$ is isomorphic to $\mathbf{Q}$.
Also we must have $Q' \subset R'$ since $\mathbf{Q}'$ is a substructure of $\mathbf{R}'$.
But then $Q' \in \binom{R'}{Q}$. So by part i) of Lemma 3.4.7 we have

$$Q' = g^{P,Q}((Q'', s'))$$

for some $Q'' \in \binom{R''}{Q}$ and $\mathfrak{s}' \in [\mathfrak{S}^Q]$. So

$$Q' = \left\{(q, x) : q \in Q'', x = \alpha(\mathfrak{s}'(i_{Q''}^{-1}(q)))\right\}.$$

By definition of the shaping s on $\overline{P}$ we have

$$s((q, x)) = \beta(x) = \beta(\alpha(\mathfrak{s}'(i_{Q''}^{-1}(q)))) = \mathfrak{s}'(i_{Q''}^{-1}(q)).$$

But as $\mathbf{Q}'$ is isomorphic to $\mathbf{Q}$ we also have

$$s((q,x)) = \mathfrak{s}^{\mathbf{Q}}(i_{Q'}^{-1}((q,x))).$$

So for $q' \in Q$ with $q' = i_{Q''}^{-1}(q) = i_{Q'}^{-1}((q,x))$ we have

$$\mathfrak{s}'(q') = \mathfrak{s}'(i_{Q''}^{-1}(q)) = s((q,x)) = \mathfrak{s}^{\mathbf{Q}}(i_{Q'}^{-1}((q,x))) = \mathfrak{s}^{\mathbf{Q}}(q')$$

so $\mathfrak{s}' = \mathfrak{s}^{\mathbf{Q}}$, and thus

$$\mathbf{Q}' = (Q''(\mathfrak{s}^{\mathbf{Q}}), <, \prec, \mathfrak{s}^{\mathbf{Q}} \circ i_{Q''}^{-1}) = f^{\mathbf{P},\mathbf{Q}}(Q'').$$

(ii) Follows immediately from ii).

□

Definition 3.4.9. A class $\mathcal{K}$, which admits (X,α,β)-blowups, has the two ways *partial transversal property*, if for any $\overline{R}' \in \binom{\overline{P}}{R}$ we have

$$\left(\binom{\overline{R}'}{R}\right) \cap \left(\binom{\overline{P}}{R}\right) \neq \emptyset.$$

That is, any substructure of $\overline{P}$ isomorphic to an X-blowup of a structure R has a partial transversal isomorphic to R that is also a partial transversal of $\overline{P}$.

Link. Lemma 4.1.4

Theorem 3.4.10. *Let $\mathcal{K}$ be a class of ordered $\mathfrak{S}$-shaped partial orders closed under substructures and isomorphisms, and let $\mathcal{K}$ be a Fraïssé class of ordered partial orders containing the unshaped reducts of the structures in $\mathcal{K}$. Let X be a set of size at least $|\mathfrak{S}|$ and let $\alpha : \mathfrak{S} \to X$ be an injective map and $\beta : X \to \mathfrak{S}$ a map such that $\beta(\alpha(\mathbf{s})) = \mathbf{s}$.*
If the class $\mathcal{K}$ admits (X,α,β)-blowups and has the two way partial

transversal property, $\mathcal{K}$ is Ramsey and for $P, Q, R \in \mathcal{K}$ we have

$$P \to (R)^Q_k \quad \Rightarrow \quad \overline{P} \to (R)^Q_k,$$

then $\boldsymbol{\mathcal{K}}$ is Ramsey.

Link. Theorem 4.1.5

Proof. Take $\mathbf{Q}, \mathbf{R} \in \boldsymbol{\mathcal{K}}$. Then $Q, \overline{R} \in \mathcal{K}$. Since $\mathcal{K}$ is a Ramsey class, there is a $P \in \mathcal{K}$ such that

$$P \to (\overline{R})^Q_k.$$

We will show that

$$\overline{\mathbf{P}} \to (\mathbf{R})^{\mathbf{Q}}_k.$$

Let

$$c : \binom{\overline{\mathbf{P}}}{\mathbf{Q}} \to [k].$$

We will show that there is a shaped partial transversal $\mathbf{R}'$ of $\overline{\mathbf{P}}$, isomorphic to $\mathbf{R}$, such that $\binom{\mathbf{R}'}{\mathbf{Q}}$ is monochromatic. In that case, by Corollary 3.4.8, we know that all of $\mathbf{Q}' \in \binom{\mathbf{R}'}{\mathbf{Q}}$ are partial transversals as well. We will thus focus on the restriction

$$((c)) : \left(\binom{\overline{\mathbf{P}}}{\mathbf{Q}}\right) \to [k], \quad ((c))(\mathbf{Q}) = c(\mathbf{Q}).$$

By Lemma 3.4.5, we have the bijection

$$f : \binom{P}{Q} \to \left(\binom{\overline{\mathbf{P}}}{\mathbf{Q}}\right).$$

Combining it with $((c))$, we get the colouring

$$c' = ((c)) \circ f : \binom{\overline{\mathbf{P}}}{\mathbf{Q}} \to [k].$$

Extend this colouring to all of $\binom{\overline{P}}{Q}$, by defining c'' that is constant on the partial transversals of each block substructure $\overline{Q}'$ of $\overline{P}$. That is, let

$$c'' : \binom{\overline{P}}{Q} \to [k]$$

such that for $Q' \in \binom{\overline{P}}{Q}$

(i) if $Q' = g(Q'', \mathfrak{s}')$ for some $(Q'', \mathfrak{s}) \in \binom{P}{Q} \times [\mathfrak{S}^Q]$,

$$c''(Q') = c''(g(Q'', \mathfrak{s}')) = c'(Q''),$$

(ii) and $c''(Q') = 1$ otherwise.

Since $P \to (\overline{R})^Q_k$ implies $\overline{P} \to (\overline{R})^Q_k$, we can find a $\overline{R}' \in \binom{P}{\overline{R}}$ such that c'' is constant on $\binom{\overline{R}'}{Q}$. Then as $\boldsymbol{\mathcal{K}}$ has the two way partial transversal property, we can find an R^* such that

$$R^* \in \left(\binom{\overline{R}'}{R}\right) \cap \left(\binom{\overline{P}}{R}\right).$$

Since $R^* \in \left(\binom{\overline{R}'}{R}\right)$ and $\binom{\overline{R}'}{Q}$ is monochromatic, $\binom{R^*}{Q}$ is monochromatic as well. Let

$$c''(Q') = l \quad \text{for all} \quad Q' \in \binom{R^*}{Q}.$$

Since $R^* \in \left(\binom{\overline{P}}{R}\right)$, for some $R^{**} \in \binom{P}{R}$ and $\mathfrak{s} \in \left[\mathfrak{S}^R\right]$

$$R^* = g^{P,R}((R^{**}, \mathfrak{s}))$$

We will show that for $\mathbf{R}' = f^{P,R}(R^{**})$ the set $\binom{\mathbf{R}'}{\mathbf{Q}}$ is monochromatic. So take any $\mathbf{Q}' \in \binom{\mathbf{R}'}{\mathbf{Q}}$. Due to Corollary 3.4.8 we have

$$\mathbf{Q}' = f^{\mathbf{P},\mathbf{Q}}(Q'')$$

for some $Q'' \in \binom{R^{**}}{Q}$.

Also by Lemma 3.4.7 part i) a), there is a $\mathfrak{s}' \in \mathfrak{S}$ such that

$$g(Q'', \mathfrak{s}') = Q' \in \binom{R^*}{Q}.$$

But then

$$((c))(\mathbf{Q}') = ((c))(f^{\mathbf{P},\mathbf{Q}}(Q'')) = c'(Q'') = c''(g(Q'', \mathfrak{s}')) = c''(Q') = l.$$

So $\binom{\mathbf{R}'}{\mathbf{Q}}$ is monochromatic with respect to $((c))$. But then $\binom{\mathbf{R}'}{\mathbf{Q}}$ is also monochromatic with respect to c as $\mathbf{R}'$ is a shaped partial transversal. Thus indeed $\overline{\mathbf{P}} \to (\mathbf{R})^{\mathbf{Q}}_k$.

□

Corollary 3.4.11. *Let $\boldsymbol{\mathcal{K}}, \mathcal{K}_u, X, \alpha, \beta$ satisfy the conditions in Theorem 3.4.10. Suppose that the class $\boldsymbol{\mathcal{K}}$ contains a class $\boldsymbol{\mathcal{K}}'$ of structures that is closed under (X, α, β)-blowups and that the class $\mathcal{K}'_u$ of unshaped reducts of $\boldsymbol{\mathcal{K}}'$ is Ramsey. Then the class $\boldsymbol{\mathcal{K}}'$ is Ramsey.*

Proof. Since $\boldsymbol{\mathcal{K}}$ admits (X, α, β)-blowups and $\boldsymbol{\mathcal{K}}'$ is closed under (X, α, β)-blowups, the class $\boldsymbol{\mathcal{K}}'$ admits (X, α, β)-blowups.
The class $\boldsymbol{\mathcal{K}}'$ also has the twofold partial transversal property, as for any $R, P \in \mathcal{K}'$ we have $R, P \in \mathcal{K}$ by definition, so for any $\overline{R}' \in \binom{\overline{P}}{R}$ we have

$$\left(\binom{\overline{R}'}{R}\right) \cap \left(\binom{\overline{P}}{R}\right) \neq \emptyset,$$

since $\boldsymbol{\mathcal{K}}$ has the twofold partial transversal property.
Similarly,

$$P \to (R)^Q_k \Rightarrow \overline{P} \to (R)^Q_k$$

is true in $\mathcal{K}'_u$ since it is true in $\mathcal{K}$.
So $\boldsymbol{\mathcal{K}}'$ and $\mathcal{K}'_u$ satisfy all the conditions in Theorem 3.4.10 and $\boldsymbol{\mathcal{K}}'$ is Ramsey.

□

3.5 Order classes

Consider first the class $\mathcal{K}(C)$ of all finite chains, or all finite total orders. It is well-known and easy to verify that any structure $P \in \mathcal{K}(C)$ is isomorphic to the total order $[n]$ with the natural order inherited from $\mathbb{N}$ and where $n = |P|$.

Let L be a language containing a binary relation symbol $\prec$ and let $\boldsymbol{\mathcal{K}}$ be an order class for $\prec$. Suppose further that $\boldsymbol{\mathcal{K}}$ only contains finite structures. Take a structure $\mathbf{A} \in \boldsymbol{\mathcal{K}}$. Then $\mathbf{A}$ is a finite structure and $\prec^{\mathbf{A}}$ is a total order by definition, so the reduct of $\mathbf{A}$ to language $\{\prec\}$ is a finite total order. It is therefore tempting to account for the total order $\prec^{\mathbf{A}}$ by writing

$$A = \{a_i : i \in [n], n = |A|\}, \quad \text{where} \quad a_i \prec a_{i'} \iff i < i',$$

using $[n]$ as an index structure of the points of $\mathbf{A}$.

But then, taking $\mathbf{B}' \in \binom{\mathbf{A}}{\mathbf{B}}$, we would have

$$B' = \{b'_j : j \in [m], m = |B'|\}, \quad \text{where} \quad b'_j \prec b'_{j'} \iff j < j'$$

and

$$B' = \{b'_j : j \in [m], m = |B'|\} \subset \{a_i : i \in [n], n = |A|\},$$

which does not clearly denote which substructure of $\mathbf{A}$ the structure $\mathbf{B}'$ is. We could denote the points of $\mathbf{B}'$ as the points a_{i_j} for $j \in [m]$. However, we will avoid the double indices wherever possible.

When defining classes of partial orders in Chapter 2, we wrote

$$P = \{p_i : i \in \mathcal{I}\}$$

or

$$P = \{p_{i,j} : (i,j) \in \mathcal{I} \rtimes \mathcal{J}\}$$

and used the total order on $\mathcal{I}$ and the total orders on $\mathcal{J}_i$ for $i \in \mathcal{I}$ to define the total order of a structure $\mathbf{P}$ (see Definition 2.5.11, Definition 2.5.15 and Definition 2.5.20). We will introduce notation in which the substructures of *index structures* will define the total order of the substructures as well.

Denote by $\mathcal{D}^{\mathbf{A}}$ a finite total order in $\mathcal{K}(C)$ isomorphic to $\langle A, \prec^{\mathbf{A}} \rangle$ and write

$$A = \{a_d : d \in \mathcal{D}^{\mathbf{A}}\}$$

to denote

$$a_d \prec a_{d'} \quad \iff d < d' \text{ in } \mathcal{D}^{\mathbf{A}},$$

encoding the total order $\prec^{\mathbf{A}}$ in the notation for the points of the universe A of $\mathbf{A}$.

Example 3.5.1. Let A be a partial order on the set of points $\{a, b, c, d\}$ and B a partial order on the set of points $\{k, l, m\}$, such that

$$a < b, c, d; c < d \quad \text{and} \quad k < l, m.$$

Suppose further that A and B are ordered partial orders, with the total orders extending the partial order, namely such that

$$a \prec b \prec c \prec d \quad \text{and} \quad k \prec l \prec m.$$

Then we can set

$$a = a_1, b = a_2, c = a_3, d = a_4 \quad \text{and} \quad k = b_1, l = b_2, m = b_3.$$

In this case, we set $\mathcal{D}^{\mathbf{A}} = [4]$ and $\mathcal{D}^{\mathbf{B}} = [3]$. Then $\binom{A}{B}$ contains two structures, namely the structures on the points

$$B_1 = \{a_i : i \in \{1, 2, 3\}\} \quad \text{and} \quad B_2 = \{a_i : i \in \{1, 2, 4\}\}.$$

Then for any $\mathbf{B}' \in \binom{\mathbf{A}}{\mathbf{B}}$, we write

$$B' = \{a_d : d \in \mathcal{D}^{\mathbf{B}'}\}$$

where

$$e : \mathbf{B} \to \mathbf{A}$$

is an embedding with $e(\mathbf{B}) = \mathbf{B}'$. Since any embedding has to preserve a total order $\prec^{\mathbf{B}}$ (and the rest of structure on $\mathbf{B}$), e induces a map

$$e_1 : \mathcal{D}^{\mathbf{B}} \to \mathcal{D}^{\mathbf{A}}, \quad \text{with} \quad e_1(\mathcal{D}^{\mathbf{B}}) = \mathcal{D}^{\mathbf{B}'}$$

and

$$d < d' \text{ in } \mathcal{D}^{\mathbf{B}} \iff e_1(d) < e_1(d') \text{ in } \mathcal{D}^{\mathbf{A}}.$$

In this case we then get $\mathcal{D}^{\mathbf{B}'} \in \binom{\mathcal{D}^{\mathbf{A}}}{\mathcal{D}^{\mathbf{B}}}$, with $e_1 : \mathcal{D}^{\mathbf{B}} \to \mathcal{D}^{\mathbf{A}}$ an embedding of total orders.

Instead of requiring that the index structures be total orders, we will require they have a total relation in the class, but permit additional structure. Formally, we capture that in the following definition.

Definition 3.5.2. Suppose that the language L contains the relation symbol $\prec$ and L_D contains a relation $<_D$. Let $\mathcal{K}$ be a class of structures in language L and $\mathcal{K}_D$ in language L_D.
A class $\mathcal{K}$ is *an order class with respect to* $\mathcal{K}_D$ if

(i) $\mathcal{K}$ is an order class with respect to $\prec$,

(ii) $\mathcal{K}_D$ is an order class with respect to $<_D$,

(iii) for each $\mathbf{A} \in \mathcal{K}$ there exists a structure $\Phi(\mathbf{A}) \in \mathcal{K}_D$ and such that for the map
$$\Phi : \mathbf{A} \to \Phi(\mathbf{A})$$
we have
$$a \prec a' \quad \iff \quad \Phi(a) <_D \Phi(a').$$
We denote $\Phi(\mathbf{A})$ by $\mathcal{D}^{\mathbf{A}}$, $\Phi(a) = d$ and write $A = \{a_d : d \in \mathcal{D}^{\mathbf{A}}\}$.

Link. Lemma 4.2.1

3.6 Two Pass Lemma

In this section, we will use letters $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ to refer to shaped partial orders most of the time, and $\mathbf{P}, \mathbf{Q}$ and $\mathbf{R}$ when we consider structures in a specific class of shaped partial orders.

The idea that lies behind the Two Pass Lemma can be illustrated by the proof that the class of chains of antichains, $\mathcal{K}(CA_{\aleph_0})$, is a Ramsey class. A similar proof is also used in Sokić (2012a) to prove that the classes $\mathcal{K}(AC_{\aleph_0}, ce)$ and $\mathcal{K}(CA_{\aleph_0}, e)$ are Ramsey, but we adapt notation to be consistent with the rest of the thesis.

Example 3.6.1. This is a continuation of Example 3.2.2.
Take any two structures $Q, R \in \mathcal{K}(CA_{\aleph_0})$, Q with $|\mathcal{I}^Q|$ maximal antichains and R with $|\mathcal{I}^R|$ maximal antichains. As we mentioned in the beginning of Section 3.1, the class $\mathcal{K}(C)$ of all chains is Ramsey. Thus there exists a chain $\mathcal{I}^P$, such that

$$\mathcal{I}^P \to (\mathcal{I}^R)_k^{\mathcal{I}^Q}.$$

Let

$$\binom{\mathcal{I}^P}{\mathcal{I}^Q} = \left\{(\mathcal{I}^Q)^{(1)}, (\mathcal{I}^Q)^{(2)}, \ldots, (\mathcal{I}^Q)^{(w)}\right\}.$$

Let the maximal length of the maximal antichain of R be r. Let the structure $\tilde{M}^{(0)}$ be a chain of $|\mathcal{I}^Q|$ maximal antichains of size r.
We will alternate between constructing structures $M^{(n)}$ and structures $\tilde{M}^{(n-1)}$, each with $|\mathcal{I}^Q|$ maximal antichains, for $n \in [w]$.

(i) Given $\tilde{M}^{(n-1)}$, let $M^{(n)}$ be a structure satisfying

$$M^{(n)} \to (\tilde{M}^{(n-1)})_k^Q.$$

The structure $M^{(n)}$ exists by Example 3.2.2.

(ii) Then let $m^{(n)}$ be the maximal length of the maximal antichain of $M^{(n)}$ and let $\tilde{M}^{(n)}$ be a chain of $|\mathcal{I}^Q|$ maximal antichains of size $m^{(n)}$.

Now since $M^{(n)} \to (\tilde{M}^{(n-1)})_k^Q$, the structure $\tilde{M}^{(n-1)}$ is isomorphic to a substructure of $M^{(n)}$. Further, the antichains of $\tilde{M}^{(n)}$ are all at least as long as the antichains of $M^{(n)}$. Thus we have

$$\tilde{M}^{(0)} \trianglelefteq M^{(1)} \trianglelefteq \tilde{M}^{(1)} \trianglelefteq \ldots \trianglelefteq \tilde{M}^{(w-1)} \trianglelefteq M^{(w)} \trianglelefteq \tilde{M}^{(w)}$$

Let P be a chain of $|\mathcal{I}^P|$ maximal antichains of size $m^{(w)}$. We will show that $P \to (R)^Q_k$.
Let $c : \binom{P}{Q} \to [k]$ be a colouring.
Set $P^{(0)} = P$ and consider the substructure of $P^{(0)}$ on the subset

$$\bigcup_{i \in (\mathcal{I}^Q)^{(1)}} P_i^{(0)},$$

a union of antichains of $P^{(0)}$, labelled as $(P^{(0)}|(\mathcal{I}^Q)^{(1)})$.
This substructure is a chain of $|\mathcal{I}^Q|$ maximal antichains of size $m^{(w)}$, thus isomorphic to $\tilde{M}^{(w)}$. Thus there exists a substructure $N^{(1)}$ of $(P^{(0)}|(\mathcal{I}^Q)^{(1)})$, such that

(i) $\binom{N^{(1)}}{Q}$ is monochromatic of colour l_1,

(ii) $N^{(1)}$ is isomorphic to $\tilde{M}^{(w-1)}$, and

(iii) $N^{(1)} = \bigcup_{i \in (\mathcal{I}^Q)^{(1)}} N_i^{(1)}$, such that $N_i^{(1)} \subset P_i^{(0)}$.

Define $P^{(1)}$ as follows.

(i) If $i \in (\mathcal{I}^Q)^{(1)}$, let $P_i^{(1)} = N_i^{(1)}$.

(ii) If $i \notin (\mathcal{I}^Q)^{(1)}$, let $P_i^{(1)} = P_i^{(0)}$.

Then $P^{(1)}$ is a chain of $|\mathcal{I}^P|$ maximal antichains of size at least $m^{(w-1)}$.
We define $N^{(n)}$, $P^{(n)}$ and l_i recursively for $n \in [w]$, in an analogous fashion.
We obtain a sequence of structures $P^{(n)}$, with

$$P^{(w)} \subset P^{(w-1)} \subset \ldots \subset P^{(1)} \subset P^{(0)}$$

and with

$$N^{(n)} = (P^{(n)}|(\mathcal{I}^Q)^{(n)}),$$

such that $\binom{N^{(n)}}{Q}$ is monochromatic of colour l_n.
But then, for each $n \in [w]$ and for $N'^{(n)} = (P^{(w)}|(\mathcal{I}^Q)^{(n)})$, the set $\binom{N'^{(n)}}{Q}$ is monochromatic of colour l_n. In addition, the structure $P^{(w)}$ is a chain of $|\mathcal{I}^P|$ maximal antichains of size at least r.

Define the colouring c' on $\binom{\mathcal{I}^P}{\mathcal{I}^Q}$ as $c'((\mathcal{I}^Q)^{(n)}) = l_n$. Since $\mathcal{I}^P \to (\mathcal{I}^R)_k^{\mathcal{I}^Q}$, there exists a $\mathcal{I'}^R \in \binom{\mathcal{I}^P}{\mathcal{I}^R}$ such that $\binom{\mathcal{I'}^R}{\mathcal{I}^Q}$ is monochromatic.
Then $(P^{(w)}|(\mathcal{I'}^R)$ is a chain of $|\mathcal{I}^R|$ maximal antichains of size at least r, and

$$\binom{(P^{(w)}|\mathcal{I'}^R)}{Q}$$

is monochromatic. Since r is the maximal length of the maximal antichain of R, $(P^{(w)}|\mathcal{I'}^R)$ contains a substructure isomorphic to R, ending the proof that $P \to (R)^Q$.

Consider the class $\mathcal{K}(C)$ of chains and the class $\mathcal{K}(A)$ of antichains. In Example 3.6.1 we used the fact that any structure $P \in \mathcal{K}(CA_{\aleph_o})$ can be built from a structure $\mathcal{I} \in \mathcal{K}(C)$ and for each $i \in \mathcal{I}$ a structure $P_i \in \mathcal{K}(A)$. We will refer to $\mathcal{I}$ as the index of P and to P_i as the levels of P. In the configuration for Two Pass Lemma (3.6.2), the class $\mathcal{K}_I$ will play the role of the class $\mathcal{K}(C)$ in the example above and $\mathcal{K}_{\mathbf{z}}$, for each $\mathbf{z} \in \mathfrak{Z}$ will play the role of the class $\mathcal{K}(A)$. The shapes $\mathfrak{Z}$ are added since, unlike in the example above, we will be considering shaped partial orders with differently shaped levels.

We will formalise 'building structures from an index structure and levels' in Definition 3.6.4 of a strongly levelled class. We will build a strongly levelled class $\mathcal{K}_s$ from $\mathcal{K}_I$ and classes $\mathcal{K}_{\mathbf{z}}$.

Now consider a class $\mathcal{K}$ with a glorified skeleton $\sigma <_g \sigma'$, where $l_1(\sigma) = G$ and $l_1(\sigma') = GAC$. As we have seen in Definition 2.5.26, we built the class $\mathcal{K}$ using the class $\mathcal{K}_q$ with an antichained skeleton $\sigma <_g \sigma'$, where $l_1(\sigma) = G$ and $l_1(\sigma') = A$. Given a structure $\mathbf{P}' \in \mathcal{K}_q$, we build a structure $\mathbf{P} \in \mathcal{K}$ by picking a glorified chain for each of the $\mathfrak{S}_2$-shaped antichain of $\mathbf{P}'$. So rather than building a $\mathcal{K}_s$ from $\mathcal{K}_I$ and $\mathcal{K}_2$, we could build $\mathcal{K}$ from $\mathcal{K}_q$ and $\mathcal{K}_{\mathbf{z}}$. This process of building structures is formalised in Definition 3.6.9.

We formalise this setup in Definition 3.6.2. In addition to the classes already mentioned, we add the classes $\mathcal{K}_J$ and $\mathcal{K}_D$ to provide indices for other classes, as in Definition 3.5.2.

The idea in the proof of Two Pass Lemma (3.6.15) is the same as idea in Example 3.6.1, but the structures involved are more complicated. The reader may wish to return to the example above when reading through the proof of the Two Pass Lemma.

Definitions

The notation in this section is heavy. The reader may wish to flip between this section and the lemmas mentioned in the Link boxes below the definitions.

Since the definitions in this section involve a selection of classes, we define the *two pass configuration*, so we can refer to it in the following definitions.

Recall Definition 3.5.2.

Definition 3.6.2. Let $\mathfrak{Z}$ be a set of shapes and let $\mathfrak{S}_1$ and $\{\mathfrak{S}_{\mathbf{z}}\}_{\mathbf{z}\in\mathfrak{Z}}$ be disjoint sets of shapes, with $\mathfrak{S} = \bigcup_{\mathbf{z}\in\mathfrak{Z}} \mathfrak{S}_{\mathbf{z}}$. Consider the languages $L_I = \{<_I, \ldots\}$, $L_J = \{<_J, \ldots\}$, $L_D = \{<_D, \ldots\}$, $L = \{<, \prec, \ldots\}$, $L_{\mathfrak{Z}} = \{\mathbf{z} : \mathbf{z} \in \mathfrak{Z}\}$, $L_{\mathfrak{S}_1} = \{\mathbf{s} : \mathbf{s} \in \mathfrak{S}_1\}$ and $L_{\mathfrak{S}} = \{\mathbf{s} : \mathbf{s} \in \mathfrak{S}\}$. Consider also the following classes.

- A class $\boldsymbol{\mathcal{K}}_I$ of all $\mathfrak{Z}$-shaped chains in language $L_I \cup L_{\mathfrak{Z}}$.
- Classes $\mathcal{K}_J$ and $\mathcal{K}_D$ of all chains in languages L_J and L_D respectively.
- A class $\boldsymbol{\mathcal{K}}_1$ of $\mathfrak{S}_1$-shaped partial orders in language $L \cup L_{\mathfrak{S}_1}$, that is an order class with respect to $\mathcal{K}_D$.
- A class $\boldsymbol{\mathcal{K}}_2$ of ordered $\mathfrak{Z}$-shaped partial orders in language $L \cup L_{\mathfrak{Z}}$, that is bi-definable with $\boldsymbol{\mathcal{K}}_I$, with the relations on any $\mathbf{A}_2 \in \boldsymbol{\mathcal{K}}_2$ defining the structure $\Phi(\mathbf{A}_2) = \mathcal{I}^{\mathbf{A}}$, so that

  (i) $\mathbf{A}_2 = (A_2, <, \prec, \mathfrak{s})$, with $A_2 = \{a_i : i \in \mathcal{I}^{\mathbf{A}}\}$,

  (ii) $\mathcal{I}^{\mathbf{A}} = (\mathcal{I}^{\mathbf{A}}, <_I, \mathfrak{z})$,

  (iii) $a_i \prec a_{i'}$ precisely when $i <_I i'$, and

  (iv) $\mathfrak{s}(a_i) = \mathfrak{z}(i)$.

  As the classes are bidefinable, the relations on any $\mathcal{I}^{\mathbf{B}}$ define a structure $\Phi'(\mathcal{I}^{\mathbf{B}}) = \mathbf{A}_2 \in \boldsymbol{\mathcal{K}}_2$.

- For each $\mathbf{z} \in \mathfrak{Z}$, let $\boldsymbol{\mathcal{K}}_{\mathbf{z}}$ be a class of ordered $\mathfrak{S}_{\mathbf{z}}$-shaped partial orders in language $L \cup L_{\mathfrak{S}}$, that is an order class with respect to $\mathcal{K}_J$, closed under substructures and with the joint embedding property. Let $K_{\mathfrak{Z}} = \{\boldsymbol{\mathcal{K}}_{\mathbf{z}} : \mathbf{z} \in \mathfrak{Z}\}$.

The classes $\boldsymbol{\mathcal{K}}_I$, $\mathcal{K}_J, \mathcal{K}_D, \boldsymbol{\mathcal{K}}_1, \boldsymbol{\mathcal{K}}_2$ and $K_{\mathfrak{Z}}$ are in a *two pass configuration.* The classes $\boldsymbol{\mathcal{K}}_I$, $\mathcal{K}_J$ and $K_{\mathfrak{Z}}$ are in a *strongly levelled configuration*

Link. Lemma 4.2.2, Lemma 4.3.3

Remark 3.6.3. The classes $\mathcal{K}_J$ and $\mathcal{K}_D$ are essentially the class $\mathcal{K}(C)$ of finite chains, but in languages L_J and L_D respectively, instead of in the language containing only the partial order relation $<$. The distinction is important for the clarity throughout this section.

Definition 3.6.4. Let classes $\boldsymbol{\mathcal{K}}_I$, $\mathcal{K}_J$ and $K_{\mathfrak{Z}}$ be in a strongly levelled configuration. A *strongly levelled class* $\boldsymbol{\mathcal{K}}_s$ *defined by* $\boldsymbol{\mathcal{K}}_I, \mathcal{K}_J$ *and* $K_{\mathfrak{Z}}$ is a class in language $L \cup L_{\mathfrak{S}}$ where the following hold.

(i) For each $\mathcal{I} = (\mathcal{I}, <_I, \mathfrak{z}) \in \boldsymbol{\mathcal{K}}_I$ and a selection $\{\mathbf{A}_i : i \in \mathcal{I}\}$ of structures with $\mathbf{A}_i \in \boldsymbol{\mathcal{K}}_{\mathfrak{z}(i)}$, there exists a unique structure $\mathbf{A} \in \boldsymbol{\mathcal{K}}_s$, where the following is true.

  (a) For each $i \in \mathcal{I}$, the universe of $\mathbf{A}_i$ is $A_i = \{a_{i,j} : j \in \mathcal{J}_i\}$ for some $\mathcal{J}_i \in \mathcal{K}_J$.

  (b) The universe of $\mathbf{A}$ is $A = \{a_{i,j} : (i,j) \in \mathcal{I} \rtimes \mathcal{J}\}$.

  (c) The substructure of $\mathbf{A}$ on the set A_i is isomorphic to $\mathbf{A}_i$.

  (d) The total order on $\mathbf{A}$ is defined as

$$a_{i,j} \prec a_{i',j'}$$

  if $i <_I i'$ or $i = i'$ and $j <_J j'$.

The shaped total order $\mathcal{I}$ and the total orders $\mathcal{J}_i$ for $i \in \mathcal{I}$ are also *index sets* of $\mathbf{A}$. When multiple structures are discussed, we write

$$A = \left\{a_{i,j} : (i,j) \in \mathcal{I}^{\mathbf{A}} \rtimes \mathcal{J}^{\mathbf{A}}\right\}.$$

We denote the structure $\mathbf{A}$ by $\bigcup_{i\in\mathcal{I}^{\mathbf{A}}} \mathbf{A}_i$, to denote that the isomorphism type of $\mathbf{A}$ is defined by the isomorphism types of $\mathcal{I}^{\mathbf{A}}$ and $\{\mathbf{A}_i : i \in \mathcal{I}^{\mathbf{A}}\}$.

(ii) Any structure $\mathbf{A} \in \mathcal{K}_s$ is one arising in part (i).

(iii) For all $\mathbf{A}, \mathbf{B} \in \mathcal{K}_s$ and $\mathbf{A}' \in \binom{\mathbf{B}}{\mathbf{A}}$, with an embedding $e : \mathbf{A} \to \mathbf{B}$, $e(\mathbf{A}) = \mathbf{A}'$, there is an embedding

$$e_1 : \mathcal{I}^{\mathbf{A}} \to \mathcal{I}^{\mathbf{B}},$$

such that

$$e(A_i) \subset B_{e_1(i)}.$$

Link. Lemma 4.2.3, Lemma 4.3.4

In the Two Pass Lemma proof we will use the properties of structures in strongly levelled classes. In particular, we will build a structure with an index set $\mathcal{I}$ of a sufficient size, and then build a large enough selection of structures $\{\mathbf{A}_i : i \in \mathcal{I}\}$. We proceed by formally defining how to build new structures from existing ones in a strongly levelled class.

Definition 3.6.5. Let $\mathcal{K}_s$ be a strongly levelled class and let $\mathbf{A} = \bigcup_{i\in\mathcal{I}^{\mathbf{A}}} \mathbf{A}_i \in \mathcal{K}_s$, with the map $\mathfrak{z}^{\mathbf{A}}$ defining the shapes on $\mathbf{A}$. The set

$$\tilde{\mathbf{A}} = \{\tilde{\mathbf{A}}_{\mathbf{z}} : \mathbf{z} \in \mathfrak{z}^{\mathbf{A}}(\mathcal{I}^{\mathbf{A}})\}$$

is a *joint embedding set of* $\mathbf{A}$ if for each $\mathbf{z} \in \mathfrak{z}^{\mathbf{A}}(\mathcal{I}^{\mathbf{A}}) \subset \mathfrak{Z}$ there is a structure $\tilde{\mathbf{A}}_{\mathbf{z}} \in \mathcal{K}_{\mathbf{z}}$, such that

$$\mathbf{A}_i \trianglelefteq \tilde{\mathbf{A}}_{\mathbf{z}} \quad \forall i \in (\mathfrak{z}^{\mathbf{A}})^{-1}(\mathbf{z}).$$

That is, if the structure $\tilde{\mathbf{A}}_{\mathbf{z}}$ contains a substructure isomorphic to $\mathbf{A}_i$ for all i that are $\mathbf{z}$-shaped in $\mathcal{I}^{\mathbf{A}}$.
We denote the index set of $\tilde{\mathbf{A}}_{\mathbf{z}}$ by $\mathcal{J}_1^{\tilde{\mathbf{A}}_{\mathbf{z}}}$.

Remark 3.6.6. This is why we required that, for all $\mathbf{z} \in \mathfrak{z}$, the class $\mathcal{K}_{\mathbf{z}}$ have the joint embedding property. By the joint embedding property of the class $\mathcal{K}_{\mathbf{z}}$, the structure $\tilde{\mathbf{A}}_{\mathbf{z}}$ exists, and thus a joint embedding set exists for any $\mathbf{A} \in \mathcal{K}_s$.

Definition 3.6.7. Suppose that $\mathcal{K}_s$ is a strongly levelled class.
Given structures $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ with $\mathfrak{z}^{\mathbf{A}}(\mathcal{I}^{\mathbf{A}}) \subset \mathfrak{z}^{\mathbf{B}}(\mathcal{I}^{\mathbf{B}})$, the structure

$$\langle \mathbf{A}, \mathbf{B} \rangle = \bigcup_{i \in \mathcal{I}^{\mathbf{A}}} \tilde{\mathbf{B}}_{\mathfrak{z}^{\mathbf{A}}(i)}$$

is the *level product of* $\mathbf{A}$ *and* $\mathbf{B}$.

Remark 3.6.8.
(i) We know that $\langle \mathbf{A}, \mathbf{B} \rangle \in \mathcal{K}$, because $\mathcal{I}^{\mathbf{A}} \in \mathcal{K}_I$ and for each $i \in \mathcal{I}^{\mathbf{A}}$ we have $\tilde{\mathbf{B}}_{\mathfrak{z}^{\mathbf{A}}(i)} \in \mathcal{K}_{\mathfrak{z}^{\mathbf{A}}(i)}$. The levels $\tilde{\mathbf{B}}_{\mathfrak{z}^{\mathbf{A}}(i)}$ for $i \in \mathcal{I}^{\mathbf{A}}$ exist because $\mathfrak{z}^{\mathbf{A}}(\mathcal{I}^{\mathbf{A}}) \subset \mathfrak{z}^{\mathbf{B}}(\mathcal{I}^{\mathbf{B}})$.

(ii) For each $i \in \mathcal{I}^{\mathbf{A}}$ with $\mathfrak{z}^{\mathbf{A}}(i) = \mathbf{z} \in \mathfrak{z}^{\mathbf{B}}(\mathcal{I}^{\mathbf{B}})$, $\langle \mathbf{A}, \mathbf{B} \rangle$ has a level $\langle \mathbf{A}, \mathbf{B} \rangle_i$ on the set of points

$$\langle A, B \rangle_i = \{x_{i,j} : j \in \mathcal{J}_i^{\tilde{\mathbf{B}}_{\mathbf{z}}}\},$$

with $\mathcal{J}_i^{\tilde{\mathbf{B}}_{\mathbf{z}}}$ isomorphic to the index set $\mathcal{J}_1^{\tilde{\mathbf{B}}_{\mathbf{z}}}$ of $\tilde{\mathbf{B}}_{\mathbf{z}}$ and $\langle \mathbf{A}, \mathbf{B} \rangle_i$ isomorphic to the structure $\tilde{B}_{\mathbf{z}}$.

(iii) The structure $\langle \mathbf{A}, \mathbf{B} \rangle \in \mathcal{K}$ has the universe

$$\langle A, B \rangle = \{x_{i,j} : (i, j) \in \mathcal{I}^{\mathbf{A}} \rtimes \mathcal{J}^{\tilde{\mathbf{B}}}\}.$$

Definition 3.6.9. Let classes $\boldsymbol{\mathcal{K}}_I, \mathcal{K}_J, \mathcal{K}_D, \boldsymbol{\mathcal{K}}_1, \boldsymbol{\mathcal{K}}_2$ and K_3 be in a two pass configuration.
Let $\boldsymbol{\mathcal{K}}_q$ be a merge of classes $\boldsymbol{\mathcal{K}}_1$ and $\boldsymbol{\mathcal{K}}_2$ in language $L \cup L_{\mathfrak{S}_1} \cup L_3$.
Let $\boldsymbol{\mathcal{K}}_s$ be a strongly levelled class defined by $\boldsymbol{\mathcal{K}}_I, \boldsymbol{\mathcal{K}}_J$ and K_3.
The class $\boldsymbol{\mathcal{K}}$ in language $L \cup L_{\mathfrak{S}_1} \cup L_{\mathfrak{S}}$ is a *class with a strongly levelled part* $\boldsymbol{\mathcal{K}}_s$ *and a quotient* $\boldsymbol{\mathcal{K}}_q$ if the following hold.

(i) The class $\boldsymbol{\mathcal{K}}$ is a merge of classes $\boldsymbol{\mathcal{K}}_1$ and $\boldsymbol{\mathcal{K}}_s$.

(ii) Given any $\mathbf{A}_1 \in \boldsymbol{\mathcal{K}}_1$ and $\mathbf{A}_s \in \boldsymbol{\mathcal{K}}_s$, there are $\mathbf{A}_2 \in \boldsymbol{\mathcal{K}}_2$ and $\mathbf{A}_q \in \boldsymbol{\mathcal{K}}_q$ satisfying

  (a) $A_1 = \{a_d : d \in \mathcal{D}^{\mathbf{A}_1}\}$,

  (b) $A_2 = \{a_i : i \in \mathcal{I}^{\mathbf{A}_2}\}$,

  (c) $A_q = A_1 \cup A_2$, and $\mathbf{A}_1$ and $\mathbf{A}_2$ are substructures of $\mathbf{A}_q$, and

  (d) $A_s = \{a_{i,j} : (i,j) \in \mathcal{I}^{\mathbf{A}_2} \rtimes \mathcal{J}^{\mathbf{A}_s}\}$.

  For each pair of structures $\mathbf{A}_q$ and $\mathbf{K}_s$ satisfying conditions (a) to (d), there is a unique structure $\mathbf{A}$, labelled $[\mathbf{A}_q, \mathbf{A}_s]$, in $\boldsymbol{\mathcal{K}}$. Its universe is

$$A = \{a_d : d \in \mathcal{D}^{\mathbf{A}_1}\} \cup \{a_{i,j} : (i,j) \in \mathcal{I}^{\mathbf{A}_2} \rtimes \mathcal{J}^{\mathbf{A}_s}\},$$

  and it contains $\mathbf{A}_1$ and $\mathbf{A}_s$ as substructures.

  For convenience, we denote its points by

$$A = \{a_d : d \in \mathcal{D}^{\mathbf{A}}\} \cup \{a_{i,j} : (i,j) \in \mathcal{I}^{\mathbf{A}} \rtimes \mathcal{J}^{\mathbf{A}}\}.$$

(iii) For any $\mathbf{A}_1 \in \boldsymbol{\mathcal{K}}_1$, the class cK contains the structure $\mathbf{A}_1$, denoted by $[\mathbf{A}_1, \emptyset]$.

  For any structure $\mathbf{A}_s \in \boldsymbol{\mathcal{K}}_s$ with $A_s = \{a_{i,j} : (i,j) \in \mathcal{I}^{\mathbf{A}_s} \rtimes \mathcal{J}^{\mathbf{A}_s}\}$, the class cK contains the structure $\mathbf{A}_s$, denoted by $[\Phi'(\mathcal{I}^{\mathbf{A}_s}), \mathbf{A}_s]$.

(iv) Any $\mathbf{A} \in \boldsymbol{\mathcal{K}}$ is of the form described in (ii) or (iii).

(v) Take $\mathbf{A} = [\mathbf{A}_q, \mathbf{A}_s] \in \boldsymbol{\mathcal{K}}$ and any subset A_1' of A_1, a subset $\mathcal{I}^{\mathbf{A}'}$ of $\mathcal{I}^{\mathbf{A}}$, and for each $i \in \mathcal{I}^{\mathbf{A}'}$ a subset A_i' of A_i. If the set

$$A' = A_1' \cup \left(\bigcup_{i \in \mathcal{I}^{\mathbf{A}'}} A_i' \right)$$

is non-empty, the substructure of $\mathbf{A}$ on the subset A' is isomorphic to the structure $[\mathbf{A}_q', \mathbf{A}_s']$, where

(a) $\mathbf{A}_q'$ is a substructure of $\mathbf{A}_q$ on the subset $A_1' \cup \{a_i : i \in \mathcal{I}^{\mathbf{A}'}\}$, and

(b) $\mathbf{A}_s'$ is a substructure of $\mathbf{A}_s$ on the subset $\bigcup_{i \in \mathcal{I}^{\mathbf{A}'}} A_i'$.

Link. Lemma 4.3.6

Remarks 3.6.10. (i) In part (iii), note that by Definition 3.6.2, we have $\Phi'(\mathcal{I}^{\mathbf{A}}) \in \boldsymbol{\mathcal{K}}_2$ and thus $\Phi'(\mathcal{I}^{\mathbf{A}}) \in \boldsymbol{\mathcal{K}}_q$.

(ii) We denote the substructure of $\mathbf{A}$ on the set of points A_1 as $\mathbf{A}_1$; this is the same as the substructure of $\mathbf{A}_q$ on the set of points A_1.

(iii) We denote the substructure of $\mathbf{A}$ on the set of points $A_i = \{a_{i,j} : (i,j) \in \{i\} \times \mathcal{J}_i^{\mathbf{A}}\}$ as $\mathbf{A}_i$; this is the same as the substructure of $\mathbf{A}_s$ on the set of points A_i.

Definition 3.6.11. Suppose that the class $\boldsymbol{\mathcal{K}}$ is a class with a strongly levelled part $\boldsymbol{\mathcal{K}}_s$ and a quotient $\boldsymbol{\mathcal{K}}_q$. Take any $\mathbf{A} = [\mathbf{A}_q, \mathbf{A}_s] \in \boldsymbol{\mathcal{K}}$ and a substructure $\mathbf{A}_q'$ of $\mathbf{A}_q$. Then a *restriction* $(\mathbf{A}|\mathbf{A}_q')$ *of* $\mathbf{A}$ *to* $\mathbf{A}_q'$ is the substructure of $\mathbf{A}$ on the subset

$$A_1' \cup \{a_{i,j} : (i,j) \in \mathcal{I}^{\mathbf{A}'} \rtimes \mathcal{J}^{\mathbf{A}}\}.$$

Lemma 3.6.12. *Suppose that the class $\mathcal{K}$ is a class with a strongly levelled part $\mathcal{K}_s$ and a quotient $\mathcal{K}_q$. Take any structures $\mathbf{A}_s, \mathbf{B}_s \in \mathcal{K}_s$ and any structures $\mathbf{A} = [\mathbf{A}_q, \mathbf{A}_s], \mathbf{B} = [\mathbf{B}_q, \mathbf{B}_s] \in \mathcal{K}$. Then there is a bijection*

$$f_{\mathbf{A},\mathbf{B}} : \binom{\mathbf{B}}{\mathbf{A}} \to \binom{\mathbf{B}_q}{\mathbf{A}_q} \rtimes \binom{(\mathbf{B}|\mathbf{A}_q)_s}{\mathbf{A}_s} = \bigcup_{\mathbf{A}'_q \in \binom{\mathbf{B}_q}{\mathbf{A}_q}} \{\mathbf{A}'_q\} \times \binom{(\mathbf{B}|\mathbf{A}'_q)_s}{\mathbf{A}_s},$$

where $\mathbf{A}' = [\mathbf{A}'_q, \mathbf{A}'_s] \mapsto (\mathbf{A}'_q, \mathbf{A}'_s)$.

Proof. The proof of this is just unravelling of the part (v) of Definition 3.6.9 of a class $\mathcal{K}$ is a class with a strongly levelled part $\mathcal{K}_s$ and a quotient $\mathcal{K}_q$, and Definition 3.6.11 of a restriction $(\mathbf{B}|\mathbf{A}'_q)$ of $\mathbf{B}$ to $\mathbf{A}'_q$. □

The two pass lemma will show that under certain conditions a class $\mathcal{K}$ with a strongly levelled part $\mathcal{K}_s$ and a quotient $\mathcal{K}_q$ is Ramsey. But we first show that under certain conditions a part of a strongly levelled class $\mathcal{K}$ is Ramsey. We define that part first.

Definition 3.6.13. Suppose that $\mathcal{K}$ is a strongly levelled class.
Given a structure $\mathcal{I} \in \mathcal{K}_I$, a structure $\mathbf{A} \in \mathcal{K}$ is $\mathcal{I}$-levelled if $\mathbf{A} = \bigcup_{i \in \mathcal{I}^{\mathbf{A}}} \mathbf{A}_i$ and $\mathcal{I}^{\mathbf{A}}$ is isomorphic to $\mathcal{I}$.
The class $\mathcal{K}$ is *$\mathcal{I}$-level-Ramsey* if given any $\mathcal{I}$-levelled structures $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{K}$, there exists an $\mathcal{I}$-levelled structure $\mathbf{C} \in \mathcal{K}$ such that $\mathbf{C} \to (\mathbf{B})^{\mathbf{A}}_k$.
We label the class of all $\mathcal{I}$-levelled structures in $\mathcal{K}$ by $\mathcal{K}_{\mathcal{I}}$.
A class $\mathcal{K}$ is *level-Ramsey* if $\mathcal{K}$ is $\mathcal{I}$-level-Ramsey for all $\mathcal{I} \in \mathcal{K}_I$.

Lemma 3.6.14. *Suppose that $\mathcal{K}$ is a strongly levelled class, defined by $\mathcal{K}_I, \mathcal{K}_J$ and $K_{\mathfrak{Z}}$ and that for each $\mathbf{z} \in \mathfrak{Z}$ the class $\mathcal{K}_{\mathbf{z}}$ is a Ramsey class. Then $\mathcal{K}$ is level-Ramsey.*

Proof. Take any $\mathcal{I} \in \mathcal{K}_I$. We will show that the class $\mathcal{K}_{\mathcal{I}}$ of all $\mathcal{I}$-levelled structures in $\mathcal{K}$ is Ramsey.
Take any selection of structures $\mathbf{A}_i \in \mathcal{K}_{\mathfrak{z}(i)}$ for $i \in \mathcal{I}$. Then, up to isomorphism, there is a unique structure $\mathbf{A} = \bigcup_{i \in \mathcal{I}} \mathbf{A}_i$ in the class $\mathcal{K}$.

We will show that for any $\mathbf{A}, \mathbf{B} \in \mathcal{K}_{\mathcal{I}}$ there is a bijection

$$f^{\mathbf{A},\mathbf{B}} : \binom{\mathbf{B}}{\mathbf{A}} \to \prod_{i \in \mathcal{I}} \binom{\mathbf{B}_i}{\mathbf{A}_i}.$$

By definition, if $\mathbf{A}' \in \binom{\mathbf{B}}{\mathbf{A}}$, then for each $i \in \mathcal{I}$ there is a subset A_i' of B_i such that the substructure $\mathbf{A}_i'$ of $\mathbf{B}$ on the subset A_i' is isomorphic to $\mathbf{A}_i$. So for each $i \in \mathcal{I}$ we must have $\mathbf{A}_i' \in \binom{\mathbf{B}_i}{\mathbf{A}_i}$, so we can set

$$f^{\mathbf{A},\mathbf{B}} : \mathbf{A}' \mapsto (\mathbf{A}_i)_{i \in \mathcal{I}}.$$

Conversely, taking, for each $i \in \mathcal{I}$, a structure $\mathbf{A}_i' \in \binom{\mathbf{B}_i}{\mathbf{A}_i}$, the structure

$$\mathbf{A}' = \bigcup_{i \in \mathcal{I}} \mathbf{A}_i'$$

is isomorphic to $\mathbf{A}$ and is a substructure of $\mathbf{B}$.
This shows that a bijection $f^{\mathbf{A},\mathbf{B}}$ like one in Lemma 3.2.5 exists for any structures $\mathbf{A}, \mathbf{B} \in \mathcal{K}_{\mathcal{I}}$.
Thus the proof that the class $\mathcal{K}_{\mathcal{I}}$ is Ramsey is the same as proof of Lemma 3.2.6.
Note that alternatively, we could show that the class $\mathcal{K}_{\mathcal{I}}$ is simply bi-definable with a product class.
Since $\mathcal{I}$ was any structure $\mathcal{I} \in \mathcal{K}_{\mathcal{I}}$, this shows that $\mathcal{K}$ is level-Ramsey. $\square$

Statement and proof of Two Pass Ramsey Lemma

Lemma 3.6.15 (Two Pass Ramsey Lemma)**.** *Suppose that $\mathcal{K}_s$ is a strongly levelled class defined by $\mathcal{K}_I, \mathcal{K}_J$ and K_3, and that for each $\mathbf{z} \in \mathfrak{Z}$ the class $\mathcal{K}_{\mathbf{z}}$ is Ramsey. Suppose that $\mathcal{K}$ is a class with a strongly levelled part $\mathcal{K}_s$ and a quotient $\mathcal{K}_q$, and that $\mathcal{K}_q$ is Ramsey. Then $\mathcal{K}$ is Ramsey.*

Link. Theorem 4.3.7

Proof. First note that by Lemma 3.6.14, the class $\mathcal{K}_s$ is level-Ramsey.
Take $\mathbf{A}, \mathbf{B} \in \mathcal{K}$, such that $\left|\binom{\mathbf{B}}{\mathbf{A}}\right| > 0$.

If $\mathbf{A} = [\mathbf{A}_q, \emptyset]$ and $\mathbf{B} = [\mathbf{B}_q, \mathbf{B}_s]$, then as $\mathcal{K}_q$ is Ramsey, there exists $\mathbf{C}_q \in \mathcal{K}_q$ such that $\mathbf{C}_q \to (\mathbf{B}_q)_k^{\mathbf{A}_q}$. Then it is easy to check that

$$[\mathbf{C}_q, \mathbf{B}_s] \to ([\mathbf{B}_q, \mathbf{B}_s])_k^{[\mathbf{A}_q, \emptyset]}.$$

Otherwise there exists a structure $\mathbf{A}_s \in \mathcal{K}_s$, such that $\mathbf{A} = [\mathbf{A}_q, \mathbf{A}_s]$. But since $\left|\binom{\mathbf{B}}{\mathbf{A}}\right| > 0$, there must exist $\mathbf{B}_s \in \mathcal{K}_s$, such that $\mathbf{B} = [\mathbf{B}_q, \mathbf{B}_s]$.
Since $\mathcal{K}$ has a strongly levelled part $\mathcal{K}_s$ and quotient $\mathcal{K}_q$, the bijection

$$f_{\mathbf{A},\mathbf{B}} : \binom{\mathbf{B}}{\mathbf{A}} \to \binom{\mathbf{B}_q}{\mathbf{A}_q} \rtimes \binom{(\mathbf{B}|\mathbf{A}_q)_s}{\mathbf{A}_s} = \bigcup_{\mathbf{A}'_q \in \binom{\mathbf{B}_q}{\mathbf{A}_q}} \{\mathbf{A}'_q\} \times \binom{(\mathbf{B}|\mathbf{A}_q)_s}{\mathbf{A}_s}$$

exists by Lemma 3.6.12. Thus also $\left|\binom{\mathbf{B}_q}{\mathbf{A}_q}\right| > 0$ and $\left|\binom{\mathbf{B}_s}{\mathbf{A}_s}\right| > 0$.
Defining a $\mathbf{C}$, such that $\mathbf{C} \to (\mathbf{B})_k^{\mathbf{A}}$ involves quite a few steps.
Let $\mathbf{T} = \mathbf{A}_q$ and $\mathbf{U} = \mathbf{B}_q$. Since $\mathcal{K}_q$ is a Ramsey class, there exists a $\mathbf{V} \in \mathcal{K}_q$, such that $\mathbf{V} \to (\mathbf{U})_k^{\mathbf{T}}$. Enumerate the substructures of $\mathbf{V}$ isomorphic to $\mathbf{T}$ as

$$\binom{\mathbf{V}}{\mathbf{T}} = \{\mathbf{T}^{(1)}, \mathbf{T}^{(2)}, \dots, \mathbf{T}^{(w)}\}.$$

By Definition 3.6.5, the structure

$$\langle \mathbf{A}_s, \mathbf{B}_s \rangle = \bigcup_{i \in \mathcal{I}^{\mathbf{A}}} \tilde{\mathbf{B}}_{\mathfrak{z}^{\mathbf{A}}(i)}$$

is $\mathcal{I}^{\mathbf{A}}$-levelled. Let $\mathbf{M}^{(0)} = \langle \mathbf{A}_s, \mathbf{B}_s \rangle$.
Since $\mathcal{K}_s$ is level-Ramsey, it is, in particular $\mathcal{I}^{\mathbf{A}}$-level-Ramsey. So we can define an $\mathcal{I}^{\mathbf{A}}$-levelled structure $\mathbf{M}^{(n)}$ recursively for $n \in [w]$ as

$$\mathbf{M}^{(n)} \to \left(\langle \mathbf{A}_s, \mathbf{M}^{(n-1)} \rangle\right)_k^{\mathbf{A}_s}.$$

For each $i \in \mathcal{I}^{\mathbf{A}}$, the level $\langle \mathbf{A}_s, \mathbf{M}^{(n)} \rangle_i$ is isomorphic to $\tilde{\mathbf{M}}^{(n)}_{\mathfrak{z}^{\mathbf{A}}(i)}$ by definition (3.6.7). The structures $\mathbf{M}_i^{(n)}$ and $\langle \mathbf{A}_s, \mathbf{M}^{(n)} \rangle_i$ are both $\mathfrak{z}^{\mathbf{A}}(i)$-shaped. So by definition of $\tilde{\mathbf{M}}^{(n)}_{\mathfrak{z}^{\mathbf{A}}(i)}$, we must have $\mathbf{M}_i^{(n)} \trianglelefteq \tilde{\mathbf{M}}^{(n)}_{\mathfrak{z}^{\mathbf{A}}(i)}$.

Thus $\mathbf{M}^{(n)}$ is isomorphic to a substructure of $\langle \mathbf{A}_s, \mathbf{M}^{(n)} \rangle$, implying that

$$\langle \mathbf{A}_s, \mathbf{M}^{(n)} \rangle \to \left(\langle \mathbf{A}_s, \mathbf{M}^{(n-1)} \rangle \right)_k^{A_s}.$$

Also, since $\mathbf{M}^{(n)}$ contains a substructure $\mathbf{M}'$, isomorphic to $\langle \mathbf{A}_s, \mathbf{M}^{(n-1)} \rangle$, such that $\binom{\mathbf{M}'}{\mathbf{A}_s}$ is monochromatic, $\langle \mathbf{A}_s, \mathbf{M}^{(n-1)} \rangle$ is isomorphic to a substructure of $\mathbf{M}^{(n)}$ and hence of $\langle \mathbf{A}_s, \mathbf{M}^{(n)} \rangle$. Thus we have, for each $\mathbf{z} \in \mathfrak{Z}$,

$$\tilde{\mathbf{M}}_{\mathbf{z}}^{(0)} \triangleleft \tilde{\mathbf{M}}_{\mathbf{z}}^{(1)} \triangleleft \ldots \triangleleft \tilde{\mathbf{M}}_{\mathbf{z}}^{(w)}. \tag{3.1}$$

Let $\mathbf{C} = \mathbf{V}_1 \cup \langle \mathbf{V}, \mathbf{M}^{(w)} \rangle$. We will show that indeed $\mathbf{C} \to (\mathbf{B})_k^{\mathbf{A}}$.
Let $c : \binom{\mathbf{C}}{\mathbf{A}} \to k$ be a colouring. Since $\boldsymbol{\mathcal{K}}$ is a class with a strongly levelled part $\boldsymbol{\mathcal{K}}_s$ and a quotient $\boldsymbol{\mathcal{K}}_q$, we have a bijection

$$f_{\mathbf{A},\mathbf{C}} : \binom{\mathbf{C}}{\mathbf{A}} \to \binom{\mathbf{C}_q}{\mathbf{A}_q} \rtimes \binom{(\mathbf{C}|\mathbf{A}_q)_s}{\mathbf{A}_s}.$$

Given a $\mathbf{T}^{(w')} \in \binom{\mathbf{V}}{\mathbf{T}}$ let

$$c|_{\mathbf{T}^{(w')}} : \binom{(\mathbf{C}|\mathbf{T}^{(w')})_s}{\mathbf{A}_s} \to [k], c|_{\mathbf{T}^{(w')}}(\mathbf{A}'_s) = k' \text{ if } c(f^{-1}_{\mathbf{A},\mathbf{C}}(\mathbf{T}^{(w')}, \mathbf{A}'_s)) = k'.$$

We will construct $\mathbf{C}^{(n)}$ such that it satisfies *construction conditions for* $\mathbf{C}^{(n)}$:

(i) for each $i \in \mathcal{I}^{\mathbf{C}}$, $\mathbf{C}_i^{(n)}$ contains a substructure isomorphic to $\tilde{\mathbf{M}}_{\mathfrak{z}(i)}^{(w-n)}$, and

(ii) the structure $(\mathbf{C}^{(n)}|\mathbf{T}^{(n+1)})_s$ contains a substructure $\mathbf{N}^{(n+1,n+1)}$ isomorphic to $\langle \mathbf{A}_s, \mathbf{M}^{(w-n-1)} \rangle$ such that $\binom{\mathbf{N}^{(n+1,n+1)}}{\mathbf{A}_s}$ is monochromatic under $c|_{\mathbf{T}^{(n+1)}}$ of colour l_{n+1}.

Note that by definition of restriction (3.6.11), for any $w' \in [w]$ we have

$$(\mathbf{C}^{(n)}|\mathbf{T}^{(w')})_s = \bigcup_{i \in \mathcal{I}^{\mathbf{T}^{(w')}}} \mathbf{C}_i^{(n)}.$$

Thus the construction condition (i) for $\mathbf{C}^{(n)}$ implies the following.

(i)* For each $\mathbf{T}^{(w')} \in \binom{\mathbf{V}}{\mathbf{T}}$ the structure $(\mathbf{C}^{(n)}|\mathbf{T}^{(w')})_s$ contains a substructure $\mathbf{N}^{(w',n)}$ isomorphic to $\langle \mathbf{A}_s, \mathbf{M}^{(w-n)} \rangle$.

First let $\mathbf{C}^{(0)} = \mathbf{C}$. Supposing that $\mathbf{C}^{(n-1)}$ exists for a $n \in [w]$ and satisfies the construction conditions, we let $\mathbf{C}_1^{(n)} = \mathbf{C}_1$, and

1. $(\mathbf{C}^{(n)}|\mathbf{T}^{(n)})_s = \mathbf{N}^{(n,n)}$ and
2. $\mathbf{C}_{2,i}^{(n)} = \mathbf{C}_{2,i}^{(n-1)}$ if $i \notin \mathcal{I}^{\mathbf{T}^{(n)}}$.

This ensures that $\binom{(\mathbf{C}^{(n)}|\mathbf{T}^{(n)})_s}{\mathbf{A}_s}$ is monochromatic under $c|_{\mathbf{T}^{(n)}}$ of colour l_n by construction condition (ii). The statements 1. and 2. are parts of *definition of* $\mathbf{C}^{(n)}$.
We first check the construction conditions for $\mathbf{C}^{(0)}$.

(i) We have $\mathbf{C}^{(0)} = \mathbf{C} = \mathbf{V}_1 \cup \langle \mathbf{M}^{(w)}, \mathbf{V}_s\rangle$, so $\mathbf{C}_i^{(0)}$ is isomorphic to $\tilde{\mathbf{M}}^{(w)}_{\mathfrak{z}(i)}$, satisfying construction condition (i).

(ii) Following the check of construction condition (i), we know that $(\mathbf{C}^{(0)}|\mathbf{T}^{(1)})_s$ is isomorphic to $\langle \mathbf{M}^{(w)}, \mathbf{A}_s\rangle$. Since we have

$$\langle \mathbf{M}^{(w)}, \mathbf{A}_s\rangle \to \left(\langle \mathbf{M}^{(w-1)}, \mathbf{A}_s\rangle\right)^{\mathbf{A}_s}_k ,$$

the colouring $c|_{\mathbf{T}^{(1)}}$ yields a substructure of $(\mathbf{C}^{(0)}|\mathbf{T}^{(1)})_s$ isomorphic to $\langle \mathbf{M}^{(w-1)}, \mathbf{A}_s\rangle$ with all substructures isomorphic to $\mathbf{A}_s$ of colour l_1, so let that be the structure $\mathbf{N}^{(1,1)}$.

Suppose that $\mathbf{C}^{(n-1)}$ satisfies the construction conditions for $\mathbf{C}^{(n-1)}$. We check the construction conditions for $\mathbf{C}^{(n)}$.

(i) If $i \in \mathcal{I}^{\mathbf{T}^{(n)}}$, then $\mathbf{C}_i^{(n)}$ is isomorphic to $\tilde{\mathbf{M}}^{(w-n)}_{\mathfrak{z}(i)}$ by part 1. of definition of $\mathbf{C}^{(n)}$. Indeed, $\mathbf{N}^{(n,n)}$ is isomorphic to $\langle \mathbf{A}_s, \mathbf{M}^{(w-n)}\rangle$ by construction condition (ii) for $\mathbf{C}^{(n-1)}$ and $\langle \mathbf{A}_s, \mathbf{M}^{(w-n)}\rangle_i = \tilde{\mathbf{M}}^{(w-n)}_{\mathfrak{z}(i)}$.

If $i \notin \mathcal{I}^{\mathbf{T}^{(n)}}$ then $\mathbf{C}_i^{(n)} = \mathbf{C}_i^{(n-1)}$ by part 2. of definition of $\mathbf{C}^{(n)}$. It contains a structure isomorphic to $\tilde{\mathbf{M}}^{(w-n+1)}_{\mathfrak{z}(i)}$ by construction condition (i) for $\mathbf{C}^{(n-1)}$ and hence one isomorphic to $\tilde{\mathbf{M}}^{(w-n)}_{\mathfrak{z}(i)}$ by (3.1) above.

So $\mathbf{C}^{(n)}$ satisfies construction condition (i).

(ii) The structure $(\mathbf{C}^{(n)}|\mathbf{T}^{(n+1)})_s$ contains a substructure $\mathbf{N}^{(n+1,n)}$ isomorphic to $\langle \mathbf{M}^{(w-n)}, \mathbf{A}_s\rangle$ by a corollary (i)* of the construction condition (i) just shown. Since

$$\langle \mathbf{M}^{(w-n)}, \mathbf{A}_s\rangle \to \left(\langle \mathbf{M}^{(w-n-1)}, \mathbf{A}_s\rangle\right)_k^{\mathbf{A}_s}$$

let $\mathbf{N}^{(n+1,n+1)}$ be the substructure of $\mathbf{N}^{(n+1,n)}$ such that $\binom{\mathbf{N}^{(n+1,n+1)}}{\mathbf{A}_s}$ is monochromatic under $c|_{\mathbf{T}^{(n+1)}}$ of colour l_{n+1} and $\mathbf{N}^{(n+1,n)}$ is isomorphic to $\langle \mathbf{M}^{(w-n-1)}, \mathbf{A}_s\rangle$. Thus $\mathbf{C}^{(n)}$ satisfies construction condition (ii).

By induction, we can indeed construct $\mathbf{C}^{(n)}$ for $n \in [w] \cup \{0\}$ satisfying the construction conditions.

Clearly we have

$$\mathbf{C} = \mathbf{C}^{(0)} \triangleright \mathbf{C}^{(1)} \triangleright \ldots \triangleright \mathbf{C}^{(w)}.$$

This implies, for any $w' \in [w]$,

$$(\mathbf{C}^{(w)}|\mathbf{T}^{(w')})_s \triangleleft (\mathbf{C}^{(w')}|\mathbf{T}^{(w')})_s.$$

Combining that with part 1. of definition of $\mathbf{C}^{(w')}$ we have

$$(\mathbf{C}^{(w)}|\mathbf{T}^{(w')})_s \triangleleft (\mathbf{C}^{(w')}|\mathbf{T}^{(w')})_s = \mathbf{N}^{(w',w')}.$$

Since $\binom{(\mathbf{N}^{(w',w')}|\mathbf{T}^{(w')})_s}{\mathbf{A}_s}$ is monochromatic of colour $l_{w'}$ from construction condition (ii) of $\mathbf{C}^{(w'-1)}$, we see that $\binom{(\mathbf{C}^{(w)}|\mathbf{T}^{(w')})_s}{\mathbf{A}_s}$ is monochromatic of colour $l_{w'}$.

Set

$$\binom{\mathbf{B}}{\mathbf{A}} = \bigcup_{\mathbf{T}' \in \binom{\mathbf{U}}{\mathbf{T}}} K_{\mathbf{T}'}, \qquad \binom{\mathbf{C}}{\mathbf{A}} = \bigcup_{\mathbf{T}' \in \binom{\mathbf{V}}{\mathbf{T}}} L_{\mathbf{T}'},$$

where

$$K_{\mathbf{T}'} = \binom{(\mathbf{B}|\mathbf{T}')}{\mathbf{A}}, \quad L_{\mathbf{T}'} = \binom{(\mathbf{C}|\mathbf{T}')}{\mathbf{A}} \quad \text{and} \quad L'_{\mathbf{T}^{(w')}} = \mathbf{T}_1^{(w')} \cup \mathbf{N}^{(w',w)}.$$

Then let

$$c' : \binom{\mathbf{V}}{\mathbf{T}} \to [k], \quad \mathbf{T}^{(w')} \mapsto l_{w'}.$$

Since we have $\mathbf{V} \to (\mathbf{U})_k^{\mathbf{T}}$, there exists a $\mathbf{U}' \in \binom{\mathbf{V}}{\mathbf{U}}$ such that $\binom{\mathbf{U}'}{\mathbf{T}}$ is monochromatic of colour l.

Consider $(\mathbf{C}^{(w)}|\mathbf{U}')$. By construction condition (i), for all $i \in \mathcal{I}^{\mathbf{C}}$, $\mathbf{C}_i^{(w)}$ contains

a substructure isomorphic to $\tilde{\mathbf{M}}^{(0)}_{\mathfrak{z}(i)}$. But $\mathbf{M}^{(0)} = \langle \mathbf{A}_s, \mathbf{B}_s \rangle$, so $\tilde{\mathbf{M}}^{(0)}_{\mathfrak{z}(i)} = \tilde{\mathbf{B}}_{\mathfrak{z}(i)}$. Thus $(\mathbf{C}^{(w)}|\mathbf{U}')_s$ contains a substructure isomorphic to $\mathbf{B}_s$, call it $\mathbf{B}'_s$.
Since $\mathcal{K}$ has a strongly levelled part $\mathcal{K}_s$ and a quotient $\mathcal{K}_q$, there exists the substructure $\mathbf{B}' = [\mathbf{U}', \mathbf{B}'_s]$ of $\mathbf{C}$, with $\mathbf{B}' = f^{-1}_{\mathbf{B},\mathbf{C}}(\mathbf{U}', \mathbf{B}'_s)$.
Then if $\mathbf{A}' = [\mathbf{A}'_q, \mathbf{A}'_s] \in \binom{\mathbf{B}'}{\mathbf{A}}$ we have a bijection

$$f_{\mathbf{A},\mathbf{B}'} : \binom{\mathbf{B}'}{\mathbf{A}} \to \binom{\mathbf{U}'}{\mathbf{T}} \rtimes \binom{(\mathbf{B}'|\mathbf{T})_s}{\mathbf{A}_s}.$$

Then

$$\mathbf{A}'_s \lhd (\mathbf{B}'|\mathbf{T}')_s$$

for some $\mathbf{T}' \in \binom{\mathbf{U}'}{\mathbf{T}}$. Also, we have $\mathbf{B}' \in \binom{\mathbf{C}^{(w)}}{\mathbf{B}}$. Besides, $\mathbf{U}' \in \binom{\mathbf{V}}{\mathbf{U}}$, so $\binom{\mathbf{U}'}{\mathbf{T}} \subset \binom{\mathbf{V}}{\mathbf{T}}$ and thus $\mathbf{T}' = \mathbf{T}^{(w')}$ for some $\mathbf{T}^{(w')} \in \binom{\mathbf{V}}{\mathbf{T}}$. Combined with a result above we thus get

$$\mathbf{A}'_s \lhd (\mathbf{B}'|\mathbf{T}^{(w')})_s \lhd (\mathbf{C}^{(w)}|\mathbf{T}^{(w')})_s \lhd (\mathbf{C}^{(w')}|\mathbf{T}^{(w')})_s = \mathbf{N}^{(w',w')}.$$

Now, by definition of $\mathbf{U}'$, we have $c'(\mathbf{T}^{(w')}) = l$ and thus $l_{w'} = l$ by definition of c'. By construction condition (ii), $\binom{\mathbf{N}^{(w',w')}}{\mathbf{A}_s}$ is monochromatic under $c|_{\mathbf{T}^{(w')}}$ of colour l. So $c|_{\mathbf{T}^{(w')}}(\mathbf{A}'_w) = l$ and therefore

$$c(f^{-1}_{\mathbf{A},\mathbf{C}}(\mathbf{T}^{(w')}, \mathbf{A}_s)) = l.$$

But $f^{-1}_{\mathbf{A},\mathbf{C}}(\mathbf{T}^{(w')}, \mathbf{A}_s) = \mathbf{A}'$, so $c(\mathbf{A}') = l$. That is true for any $\mathbf{A}' \in \binom{\mathbf{B}'}{\mathbf{A}}$, which finishes the proof. □

Corollary 3.6.16. *Suppose that $\mathcal{K}_s$ is a strongly levelled class defined by $\mathcal{K}_I, \mathcal{K}_J$ and $K_{\mathfrak{Z}}$. Suppose that the class $\mathcal{K}_I$ is Ramsey, and that for each $\mathbf{z} \in \mathfrak{Z}$ the class $\mathcal{K}_{\mathbf{z}}$ is Ramsey. Then $\mathcal{K}_s$ is Ramsey.*

Link. Theorem 4.2.5, Definition 3.6.2, Definition 3.6.9, Definition 3.6.4, Lemma 3.1.6, Lemma 3.6.15

Proof. Suppose that $\mathcal{K}_s$ is a class of ordered $\mathfrak{S}$-shaped partial orders and that $\mathfrak{s}$ is a shape disjoint from $\mathfrak{S}$ and $\mathfrak{Z}$.

Suppose that $\mathcal{K}_1$ is a the class containing only the ordered **s**-shaped antichain $\mathbf{A}_1$ with one point and let $\mathcal{K}_D$ be a class of chains in language L_D, so that the classes $\mathcal{K}_I$, $\mathcal{K}_J, \mathcal{K}_D, \mathcal{K}_1, \mathcal{K}_2$ and K_3 are in a two pass configuration.
Then since $\mathcal{K}_I$ is Ramsey and $\mathcal{K}_I$ and $\mathcal{K}_2$ are simply bi-definable by Definiton 3.6.2 of a two pass configuration, the class $\mathcal{K}_2$ is also Ramsey by Lemma 3.1.6.
Let $\mathcal{K}_q$ be a merge of classes $\mathcal{K}_1$ and $\mathcal{K}_2$, containing

(i) $\mathbf{A}_1$, and

(ii) for each $\mathbf{A}_2 \in \mathcal{K}_2$, the structure $\mathbf{A}_2$ and the unique merge of $\mathbf{A}_1$ and $\mathbf{A}_2$, labelled $(\mathbf{A}_2)^*$.

Then as $\mathcal{K}_2$ is Ramsey, so is $\mathcal{K}_q$.
Similarly let $\mathcal{K}$ be a merge of classes $\mathcal{K}_1$ and $\mathcal{K}_s$, with each structure $\mathbf{A}_s \in \mathcal{K}_s$ defining a unique merge of $\mathbf{A}_1$ and $\mathbf{A}_s$ in $\mathcal{K}$.
Then given $\mathbf{A}_q \in \mathcal{K}_q$ and $\mathbf{A}_s \in \mathcal{K}_s$ as in parts (ii)(a) and (ii)(b) of 3.6.9, the structure $\mathbf{A}_q$ is either the structure $\Phi(\mathcal{I}^{\mathbf{A}}) \in \mathcal{K}_2$ or the unique merge $(\Phi(\mathcal{I}^{\mathbf{A}}))^*$ of $\mathbf{A}_1$ and $\Phi(\mathcal{I}^{\mathbf{A}})$, and the structure $\mathbf{A}$ defined in part (ii) exists in $\mathcal{K}$ and is either the structure $\mathbf{A}_s$ or the unique merge of $\mathbf{A}_1$ and $\mathbf{A}_s$, respectively.
The part (iv) follows by definition of $\mathcal{K}$, and the part (iv) is a consequence of definition of $\mathcal{K}$ and Definition 3.6.4.
So the class $\mathcal{K}$ has a strongly levelled part $\mathcal{K}_s$ and a quotient $\mathcal{K}_q$.
Thus $\mathcal{K}$ is a Ramsey class by Lemma 3.6.15.
Now, $\mathcal{K}$ contains precisely the structures

(i) $[\mathbf{A}_1, \emptyset]$, and

(ii) for each structure $\mathbf{A}_s \in \mathcal{K}_s$, the structure $[\Phi(\mathcal{I}^{\mathbf{A}}), \mathbf{A}_s]$, and

(iii) for each structure $\mathbf{A}_s \in \mathcal{K}_s$, the structure $[(\Phi(\mathcal{I}^{\mathbf{A}}))^*, \mathbf{A}_s]$.

Then given $\mathbf{A}_s, \mathbf{B}_s \in \mathcal{K}_s$, construct $\mathbf{C} \in \mathcal{K}$ to satisfy

$$\mathbf{C} \mapsto ([\Phi(\mathcal{I}^{\mathbf{B}}), \mathbf{B}_s])_k^{[\Phi(\mathcal{I}^{\mathbf{A}}), \mathbf{A}_s]}$$

as in the proof of Lemma 3.6.15. But when defining $\mathbf{V} \in \mathcal{K}_q$ such that

$$\mathbf{V} \mapsto (\Phi(\mathcal{I}^{\mathbf{B}}))_k^{\Phi(\mathcal{I}^{\mathbf{A}})}$$

we know there exists $\mathbf{V} \in \boldsymbol{\mathcal{K}}_2$ satisfying the condition, as $\Phi(\mathcal{I}^{\mathbf{A}}), \Phi(\mathcal{I}^{\mathbf{B}}) \in \boldsymbol{\mathcal{K}}_2$ and $\boldsymbol{\mathcal{K}}_2$ is Ramsey.
But then $\mathbf{C} \in \boldsymbol{\mathcal{K}}_s$, which completes the proof. $\square$

Chapter 4

Ramsey Results

The building blocks of skeletons enumerating the classes of ordered shaped partial orders, A_1, G, CA and GAC, are presented in Figure 4.1.

To define chains of antichains and glorified antichains of chains we introduce glorified antichains (GA, in 2.5.13) and glorified chains (GC, in 2.5.18). Essentially, to construct a chain of antichains, we start with a chain and replace all of its points by glorified antichains, and to construct a glorified antichain of chains, we start with an antichain and replace all of its points by glorified chains. Additionally, we use labels C, AC and A to denote specific classes.

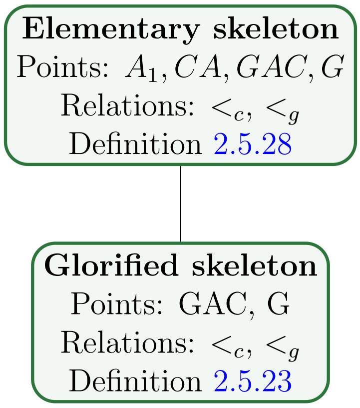
(i) When the class $\boldsymbol{\mathcal{K}}(CA, \mathfrak{S}, N, o)$ consists of shaped ordered chains, we denote it by $\boldsymbol{\mathcal{K}}(C, \mathfrak{S}, o)$, in Definition 2.5.17.

(ii) When the class $\boldsymbol{\mathcal{K}}(GAC, \mathfrak{S}, N, o)$ consists of shaped ordered antichains of chains, we denote it by $\boldsymbol{\mathcal{K}}(AC, \mathfrak{S}, o)$, in Definition 2.5.22.

Figure 4.1: Building blocks of ordered shaped partial orders

A_1 **Singleton** Definition 2.5.10	CA **Chains of antichains** Definition 2.5.15
G **Partial orders** Definition 2.5.11	GAC **Glorified antichains of chains** Definition 2.5.20

Figure 4.2: Elementary skeleton and its subskeleton



(iii) When the class $\mathcal{K}(GAC, \mathfrak{S}, N, o)$ consists of shaped ordered antichains, we denote it by $\mathcal{K}(A, \{\mathbf{s}^a\}, o)$, in Definition 2.5.22.

To enumerate classes of ordered shaped partial orders, we build new skeletons from the building blocks. We define the c-condition in the part (iv) of 2.5.28. Take any partial order, label its points G or GAC, and its relations $<_c$ or $<_g$. If the obtained structure satisfies the c-condition and is $<_g$-connected, it is a glorified skeleton. We can build any elementary skeleton from a selection of glorified skeletons, incomparable with points labelled A_1 and points labelled CA.

We mentioned that the label A denotes a special case of the label GAC. We also mentioned that we build a glorified antichain of chains by replacing points of an antichain with glorified chains. We build the structures in the classes defined by a glorified skeleton in the same way. Let Σ be a glorified skeleton. Replacing each label $'GAC'$ with the label $'A'$, we obtain an antichained skeleton Σ'. The skeleton Σ with any number of the labels $'GAC'$ replaced with labels $'A'$ forms a simple skeleton, and any simple skeleton arises from some glorified skeleton. This is defined formally in 2.5.23 and 2.5.24 . The class of structures enumerated by an antichained skeleton is defined in 2.5.25, and one enumerated by a glorified skeleton in 2.5.26.

In this chapter, we apply the tools from Chapter 3 to prove that the classes enumerated by the skeletons discussed are Ramsey.

The list of Ramsey results in this chapter and the methods used to prove them is in Table 4.1.

Table 4.1

Skeleton	Proof	Method
Generic	4.1.5	Blowup Lemma
G	4.1.6	Corollary of blowup
C	4.1.7	Corollary of blowup
Antichained	4.1.8	Corollary of blowup
GA	4.2.4	Structural Product Ramsey
CA	4.2.5	Two Pass Lemma
GC	4.3.5	Structural Product Ramsey
Glorified	4.3.7	Two Pass Lemma

Structural Product Ramsey Lemma

In Section we proved the Full Structural Product Ramsey Lemma (3.2.9) and link it to the classes of shaped ordered partial orders in Corollary 3.2.12. This formulation of the result makes it very convenient for proving that certain classes of ordered shaped partial orders are Ramsey.

Blowup Lemma

We apply the Blowup Lemma to a class $\mathcal{K}$ of ordered shaped partial orders enumerated by a generic skeleton. We first define maps α and β, and a specific weak blowup of each structure in the class in Definition 3.4.1. We then show that $\mathcal{K}$ admits weak blowups (3.4.2) in Lemma 4.1.3. In 4.1.4 we show that the class $\mathcal{K}$ has the two way partial transversal property, defined in 3.4.9. This allows us to apply Theorem 3.4.10 to show that $\mathcal{K}$ is Ramsey in 4.1.5. Results that other classes of ordered shaped partial orders are Ramsey as well follow easily.

Two Pass Lemma

We apply Two Pass Lemma (3.6.15) in two cases. We apply it directly to a class of shaped ordered partial orders enumerated by a glorified skeleton. Its corollary 3.6.16 allows us to apply the lemma in a simpler context, for example to show that the class of shaped ordered chains of antichains is strongly levelled.

We first show that the classes of glorified antichains and glorified chains are order classes with respect to a class of chains, closed under substructures and satisfy the joint embedding property in lemmas 4.2.1 and 4.3.2. This is needed in order for these classes to play the role of the class $\boldsymbol{\mathcal{K}}_{\mathbf{z}}$ in the definition of the two pass configuration (3.6.2).

Lemma 4.2.2 links the class of chains and classes of glorified antichains to the classes mentioned in Definition 3.6.2 and shows that they are in a two pass configuration. The classes needed for the two pass configuration for the glorified skeleton class are lengthier to define, thus there is Definition 4.3.1, followed by Lemma 4.3.3.

Further, we show that a class of chains of antichains and a class of glorified antichains of chains are strongly levelled (3.6.4) in lemmas 4.2.3 and 4.3.4.

The proof for the case of chains of antichains needs a result that the class of glorified antichains is Ramsey (4.2.4), and combining results in section 4.2 yields Theorem 4.2.5.

The case of a glorified skeleton needs slightly more work. In Lemma 4.3.6 we show that the classes defined so far satisfy Definition 3.6.9. Let Σ be a glorified skeleton. Then we focus on a point $\sigma \in \Sigma$ labelled GAC. We then consider a class of structures defined by a skeleton Σ_q, differing from Σ only in σ having a label A. We show that the class defined by $\boldsymbol{\mathcal{K}}_q$ plays the role of a quotient class, and that we can build any structure in the class defined by Σ by starting with a quotient structure and adding the levels from a class of glorified chains. We show that formally in Lemma 4.3.6. We finish this chapter with Theorem 4.3.7. In the proof, we start with an antichained skeleton of the glorified skeleton given, as a class defined by an antichained skeleton is Ramsey (4.1.8). We proceed by adding GAC labels to the antichained skeleton. Since a class of glorified chains is Ramsey (4.3.5), we obtain Ramsey classes with more and more GAC labels using Lemma 4.3.6, until we get a glorified skeleton.

4.1 Antichained skeleton

Let $\Sigma = (\Sigma, <, \prec, l_1, l_2)$ be a generic skeleton, as defined in Definition 2.5.23. Then $\boldsymbol{\mathcal{K}}(\Sigma, \mathfrak{S}, o)$, defined in 2.5.25, is a class of $\mathfrak{S}$-shaped partial orders, where

$$\mathfrak{S} = \bigcup_{\sigma \in \Sigma} \mathfrak{S}_\sigma, \quad \mathfrak{S}_\sigma = \{\mathbf{s}_\sigma^a : a \in \mathcal{A}_\sigma\}$$

is a disjoint union, and for each $\mathbf{P} \in \boldsymbol{\mathcal{K}}(\Sigma, \mathfrak{S}, o)$, the component $\mathbf{P}_\sigma$ is $\mathfrak{S}_\sigma$-shaped, and $\mathcal{A}_\sigma = [n]$ for some positive integer n. The aim of this section is to show in Theorem 4.1.5 that the class $\boldsymbol{\mathcal{K}}(\Sigma, \mathfrak{S}, o)$ is Ramsey.

Recall the class $\mathcal{K}(G, e)$ of ordered (unshaped) partial orders from Definition 2.2.12. Given $P = (P, <, \prec) \in \mathcal{K}(G, e)$, the set $\left[\mathfrak{S}^P\right]$ of all shapings $\mathfrak{s}$ of P, such that $(P, <, \prec, \mathfrak{s}) \in \boldsymbol{\mathcal{K}}(\Sigma, \mathfrak{S}, o)$, consists precisely of maps

$$\mathfrak{s} \in \left[\mathfrak{S}^P\right], \quad \mathfrak{s} : P \to \mathfrak{S},$$

that satisfy, for all $p, q \in P$,

$$(\sigma \prec \sigma' \wedge \mathfrak{s}(p) \in \mathfrak{S}_\sigma \wedge \mathfrak{s}(q) \in \mathfrak{S}_{\sigma'}) \quad \Rightarrow \quad p \prec q.$$

In other words, if $\sigma \prec \sigma'$, then in the total order $\prec$, the $\mathfrak{S}_\sigma$-shaped component of P is completely below the $\mathfrak{S}_{\sigma'}$-shaped component of P under the shaping $\mathfrak{s}$.

Definition 4.1.1. Let $\boldsymbol{\mathcal{K}}(\Sigma, \mathfrak{S}, o)$ be a class of ordered $\mathfrak{S}$-shaped partial orders corresponding to a generic skeleton Σ.
Let A, playing the role of s in Definition 3.4.1, be the number defined as

$$A = \max_{\sigma \in \Sigma} |\mathfrak{S}_\sigma|$$

and let X be the set

$$X = \Sigma \times [A].$$

For the class $\mathcal{K}(G, e)$, define a weak X-blowup $\overline{P}$ of a partial order

$P \in \mathcal{K}(G, e)$ as an ordered partial order on the set

$$\overline{P} = P \times X = \left\{(p_{\sigma,i}, \varsigma, a) : (\sigma, i, \varsigma, a) \in (\Sigma \rtimes \mathcal{I}^P) \times \Sigma \times [A]\right\}$$

with a partial order defined as

$$(p_{\sigma,i}, \varsigma, a) < (p_{\sigma',i'}, \varsigma', a'), \text{ if}$$

(i) $\varsigma <_c \varsigma'$ in Σ,

(ii) $\varsigma <_g \varsigma'$ in Σ and $p_{\sigma,i} \leq p_{\sigma',i'}$ in P,

(iii) $\varsigma = \varsigma'$ and $p_{\sigma,i} < p_{\sigma',i'}$ in P, or

(iv) $\varsigma = \varsigma'$, $p_{\sigma,i} = p_{\sigma',i'}$ and $a < a'$ in $[A]$.

and the total order defined as

$$(p_{\sigma,i}, \varsigma, a) \prec (p_{\sigma',i'}, \varsigma', a'), \text{ if}$$

(v) $\varsigma \prec \varsigma'$ in Σ,

(vi) $\varsigma = \varsigma'$ and $p_{\sigma,i} \prec p_{\sigma',i'}$ in P, or

(vii) $\varsigma = \varsigma'$, $p_{\sigma,i} = p_{\sigma',i'}$ and $a < a'$ in $[A]$.

Let further α and β be maps as follows

$$\alpha : \mathfrak{S} \to \Sigma \times [A], \mathbf{s}_\sigma^a \mapsto (\sigma, a);$$

$$\beta : \Sigma \times [A] \to \mathfrak{S}, \text{ if } a \leq |\mathfrak{S}_\sigma|, (\sigma, a) \mapsto \mathbf{s}_\sigma^a; \text{ otherwise } (\sigma, a) \mapsto \mathbf{s}_\sigma^1,$$

where for each $\sigma \in \Sigma$, $\mathbf{s}_\sigma^1$ is the least element of $\mathfrak{S}_\sigma$.
Define, for each $\varsigma \in \Sigma$ a substructure $\overline{P}_\varsigma$ of $\overline{P}$ on the set of points

$$\overline{P}_\varsigma = \left\{(p_{\sigma,i}, a) : (\sigma, i, \varsigma, a) \in (\Sigma \rtimes \mathcal{I}^P) \times [A]\right\}.$$

Remark 4.1.2. (i) We know that $\mathcal{K}(G,e)$ is a Fraïssé class of all generic partial orders with total orders that extend the partial orders. It contains all the unshaped reducts of structures in $\boldsymbol{\mathcal{K}}(\Sigma,\mathfrak{S},o)$, since a total order on any $\mathbf{P}\in\boldsymbol{\mathcal{K}}(\Sigma,\mathfrak{S},o)$ extends the partial order on it (the check that this is true is trivial). Similarly, it is easy to see that the total order on $\overline{P}$ indeed extends the partial order on $\overline{P}$, as defined in Definition 4.1.1.

So the weak X-blowup $\overline{P}$ is well-defined.

(ii) In terms of the partial and total order on $\overline{P}$, it would be more intuitive to denote the points of $\overline{P}$ as

$$(\varsigma, p_{\sigma,i}, a)$$

as then the total order $\prec$ on P is the lexicographic order on $\Sigma\times P\times[A]$.

Lemma 4.1.3. *Let Σ be a generic skeleton. The class $\boldsymbol{\mathcal{K}}(\Sigma,\mathfrak{S},o)$ admits $(\Sigma\times[A],\alpha,\beta)$-blowups defined in 4.1.1.*

Proof. By Remark 4.1.2, the class $\boldsymbol{\mathcal{K}}=\boldsymbol{\mathcal{K}}(\Sigma,\mathfrak{S},o)$ is a class of ordered $\mathfrak{S}$-shaped partial orders closed under substructures and isomorphisms, and $\mathcal{K}=\mathcal{K}(G,e)$ is a Fraïssé class (see Theorem 2.2.16) of ordered partial orders containing the unshaped reducts of the structures in $\boldsymbol{\mathcal{K}}$. We verify that these satisfy the assumptions of Definition 3.4.2.

The set $\Sigma\times[A]$ is clearly of size at least $|\mathfrak{S}|$, the map α is injective, and for each $\mathbf{s}^a_\sigma$ we have

$$\beta(\alpha(\mathbf{s}^a_\sigma))=\beta(\sigma,\alpha)=\mathbf{s}^a_\sigma.$$

Now we check the remaining two conditions of Definition 3.4.2.

(i) Take any $\mathbf{P}\in\boldsymbol{\mathcal{K}}(\Sigma,\mathfrak{S},o)$ and a weak $\Sigma\times[A]$-blowup $\overline{P}=(\overline{P},<,\prec)$ of its reduct $P\in\mathcal{K}(G,o)$. Then consider the map

$$s:\overline{P}\to\mathfrak{S},\quad (p_{\sigma,i},\varsigma,a)\mapsto\beta(\varsigma,a).$$

To show that $(\overline{P}, <, \prec, s) \in \boldsymbol{\mathcal{K}}(\Sigma, \mathfrak{S}, o)$, take any $p, q \in \overline{P}$ with

$$s(p) \in \mathfrak{S}_\sigma, s(q) \in \mathfrak{S}_{\sigma'} \text{ and } \sigma \prec \sigma' \text{ in } \Sigma.$$

So if $p = (p', \varsigma, a)$ for some $p' \in P$, then $s(p) = \beta(\varsigma, a) \in \mathfrak{S}_\sigma$. By definition of β that means that in fact $\varsigma = \sigma$. We can reason similarly for q. That means that, for some $p', q' \in P$ and $a, a' \in [A]$ we have

$$p = (p', \sigma, a) \text{ and } q = (q', \sigma', a').$$

But then by definition of the weak $\Sigma \times [A]$-blowup $\overline{P} = (\overline{P}, <, \prec)$, we must have $p \prec q$. So s is indeed a shaping of $\overline{P}$ and

$$\overline{P} = (\overline{P}, <, \prec, s) \in \boldsymbol{\mathcal{K}}(\Sigma, \mathfrak{S}, o).$$

(ii) Now take any $\mathbf{P}, \mathbf{R} \in \boldsymbol{\mathcal{K}}(\Sigma, \mathfrak{S}, o)$ and their reducts $P, R \in \mathcal{K}(G, o)$.

(a) Now consider the map

$$\begin{array}{rccc} gg: & \binom{P}{R} \times \left[\mathfrak{S}^R\right] & \to & \binom{\overline{P}}{R}, \\ & (R', \mathfrak{s}') & \mapsto & R'(\mathfrak{s}') = (R'(\mathfrak{s}'), <, \prec), \end{array}$$

where $R'(\mathfrak{s}') = \{(r, x) : r \in R', x = \alpha(\mathfrak{s}'(i_{R'}^{-1}(r)))\}$.

To check that the map is well-defined, take any $R' \in \binom{P}{R}$ and $\mathfrak{s}' \in \left[\mathfrak{S}^R\right]$. We need to show that the substructure of $\overline{P}$ on the set of points $R'(\mathfrak{s}')$ is isomorphic to R. So fix $R' \in \binom{P}{R}$ and $\mathfrak{s}' \in \left[\mathfrak{S}^R\right]$. Consider the map

$$i : R \to R'(\mathfrak{s}^{\mathbf{R}}), \quad r \mapsto (r', \varsigma, a), \text{ where}$$

$$r' = i_{R'}(r), \text{ and}$$

$$(\varsigma, a) = \alpha\left(\mathfrak{s}'\left(i_{R'}^{-1}(r')\right)\right) = \alpha\left(\mathfrak{s}'\left(i_{R'}^{-1}(i_{R'}(r))\right)\right) = \alpha\left(\mathfrak{s}'(r)\right).$$

So we have $\mathfrak{s}'(r) \in \mathfrak{S}_\varsigma$ by definition of α. Now consider any $r, q \in R$ and let $i(q) = (q', \varsigma', a')$, with $\mathfrak{s}'(q) \in \mathfrak{S}_{\varsigma'}$, as above.

- Suppose that $r < q$ in R. Then as $i_{R'}$ is an isomorphism, we have

$$i_{R'}(r) < i_{R'}(q).$$

By definition of Σ, we must then either have $\varsigma = \varsigma'$ or $\varsigma < \varsigma'$ in Σ.

If $\varsigma = \varsigma'$, then $i(r) < i(q)$ by part (iii) of the definition of the partial order on $\overline{P}$ in Definition 4.1.1. If $\varsigma < \varsigma'$ in Σ, then $i(r) < i(q)$ by part (i) or (ii) of the definition of the partial order on $\overline{P}$ in Definition 4.1.1.

- Suppose now that $r \prec q$ in R. Then as $i_{R'}$ is an isomorphism, we have

$$i_{R'}(r) \prec i_{R'}(q).$$

If we had $\varsigma' \prec \varsigma$ in Σ, then as $\mathfrak{s}'$ is a shaping, $\mathfrak{s}'(r) \in \mathfrak{S}_\varsigma$ and $\mathfrak{s}'(q) \in \mathfrak{S}_{\varsigma'}$, we must have $q \prec r$. That is a contradiction, so we should again have $\varsigma = \varsigma'$ or $\varsigma \prec \varsigma'$ in Σ. So we must indeed have $i(p) \prec i(q)$ by part (i) or (ii) of the definition of the total order $\prec$ on $\overline{P}$ in Definition 4.1.1.

Thus i is an isomorphism and the map gg is well-defined.

(b) Finally, consider the map

$$\begin{array}{rccl} ff: & \binom{P}{R} & \to & \binom{\mathbf{P}}{\mathbf{R}}, \\ & R' & \mapsto & \mathbf{R}' = (R'(\mathfrak{s}^{\mathbf{R}}), <, \prec, \mathfrak{s}^{\mathbf{R}} \circ i_{R'}^{-1}), \end{array}$$

where $R'(\mathfrak{s}^{\mathbf{R}}) = \{(r, x) : r \in R', x = \alpha(\mathfrak{s}^{\mathbf{R}}(i_{R'}^{-1}(r)))\}$.

Since $\mathfrak{s}^{\mathbf{R}} : R \to \mathfrak{S}$ is a shaping we know by part (a) that $R'(\mathfrak{s}^{\mathbf{R}}) \in \binom{\overline{P}}{R}$. So we only need to check that $\mathbf{R}'$ is isomorphic to $\mathbf{R}$. Consider the map

$$i : R \to R'(\mathfrak{s}^{\mathbf{R}}), \quad r \mapsto (r', x) = (i_{R'}(r), \alpha\left(\mathfrak{s}^{\mathbf{R}}(r)\right))$$

again. Then for any $r \in R$, we have

$$s(r', x) = \beta(x) = \beta\left(\alpha\left(\mathfrak{s}^{\mathbf{R}}(r)\right)\right) = \mathfrak{s}^{\mathbf{R}}(r),$$

so i defines an isomorphism $\mathbf{R} \to \mathbf{R}'$ and the map ff is well-defined.

$\square$

Lemma 4.1.4. *Let Σ be a generic skeleton. The class $\boldsymbol{\mathcal{K}}(\Sigma, \mathfrak{S}, o)$ with $(\Sigma \times [A], \alpha, \beta)$-blowups has the two way partial transversal property.*

Proof. The two way partial transversal property makes sense, as we've just shown that $\boldsymbol{\mathcal{K}}(\Sigma, \mathfrak{S}, o)$ admits $(\Sigma \times [A], \alpha, \beta)$-blowups, as in the Definition 3.4.9. So take any $\overline{R}' \in \binom{\overline{P}}{\overline{R}}$. We aim to show that there exists a partial order R'' such that

$$R'' \in \left(\!\binom{\overline{R}'}{R}\!\right) \cap \left(\!\binom{\overline{P}}{R}\!\right).$$

First let

$$i_{\overline{R}'} : \overline{R} \to \overline{R}' \subset \overline{P}$$

be the isomorphism (which is unique, since $\boldsymbol{\mathcal{K}}(\Sigma, \mathfrak{S}, o)$ is a class of ordered shaped partial orders).

Recall that by part (v) of Definition 4.1.1, the total order on Σ induces total orders

$$\overline{R}'_{\sigma_1} \prec \overline{R}'_{\sigma_2} \prec \ldots \prec \overline{R}'_{\sigma_{|\Sigma|}}$$

and

$$\overline{P}_{\sigma_1} \prec \overline{P}_{\sigma_2} \prec \ldots \prec \overline{P}_{\sigma_{|\Sigma|}}.$$

The total order $\prec$ is also convex on each of the $\overline{R}'_{\sigma_i}$, $\overline{P}_{\sigma_j}$, and as $\overline{R}'$ is a substructure of $\overline{P}$, the total order on $\overline{P}$ induces a total order on $\overline{R}'$.

For each $i \in [|\Sigma|]$, let r_i be the least point in $\overline{R}'_{\sigma_i}$, and let $r_{|\Sigma|+1}$ be the greatest point of $\overline{R}'_{\sigma_{|\Sigma|}}$ in the total order $\prec$. By Pigeonhole Principle, two of the r_i must lie in the same $\overline{P}_{\sigma_j}$, say r_i and $r_{i'}$, where $i < i'$. But as $\prec$ is convex on each $\overline{R}'_{\sigma_i}$, the r_{i+1} must lie in $\overline{P}_{\sigma_j}$ as well. But then for any $r \in \overline{R}'_{\sigma_i}$, we have $r \prec r_{i'}$, or $r = r_{i'}$ precisely when $i = |\Sigma|$. Thus, in fact $r \in \overline{P}_{\sigma_j}$.

To summarise, there must be $\sigma, \sigma' \in \Sigma$ such that

$$\overline{R}'_{\sigma} \subset \overline{P}_{\sigma'}.$$

By definition, there is an isomorphism

$$i_{\overline{R}'} : \overline{R} \to \overline{R}' \subset \overline{P}.$$

We will show that the sought R'' is the substructure of $\overline{P}$ on the set of points

$$R'' = \{i_{\overline{R}'}(r, \sigma, 1) : r \in R\}.$$

Define the map

$$\mathfrak{s} : R \to \mathfrak{S}, \quad r \mapsto \mathbf{s}^1_\sigma.$$

Since $\mathfrak{s}(r) \in \mathfrak{S}_\sigma$ for all $r \in R$, the map $\mathfrak{s}$ is a shaping of R, *i.e.*, , $\mathfrak{s} \in \left[\mathfrak{S}^R\right]$. Also $i_R : R \to R$ is the trivial isomorphism, so

$$f^{R,R} : \binom{R}{R} \times \left[\mathfrak{S}^R\right] \to \left(\!\binom{\overline{R}}{R}\!\right), \text{ and}$$

$$(R, \mathfrak{s}) \mapsto R(\mathfrak{s}) = \{(r, x) : r \in R, x = \alpha(\mathfrak{s}(r))\}.$$

But by definition of α, we have $\alpha(\mathfrak{s}(r)) = (\sigma, 1)$, so in fact

$$R'' = \{i_{\overline{R}'}(r, \sigma, 1) : r \in R\} = i_{\overline{R}'}\left(R(\mathfrak{s})\right),$$

and R'' is isomorphic to $R(\mathfrak{s})$ and thus R.
This shows that in fact $R'' \in \left(\!\binom{\overline{R}'}{R}\!\right)$, as $R(\mathfrak{s}) \in \left(\!\binom{\overline{R}}{R}\!\right)$. Since $\overline{R}'$ is a substructure of $\overline{P}$, this further shows that $R'' \in \binom{\overline{P}}{R}$.
It remains to show that $R'' \in \left(\!\binom{\overline{P}}{R}\!\right)$, that is, finding an $R^* \in \binom{P}{R}$ and a shaping $\mathfrak{s}' : R \to \mathfrak{S}$, such that $g^{P,R}(R^*, \mathfrak{s}') = R''$.
For any $r \in R$ there are $p \in P$ and $a \in [A]$, such that the isomorphism $i_{\overline{R}'}$ maps

$$(r, \sigma, 1) \mapsto (p, \sigma', a).$$

So consider the map

$$\mathfrak{s}' : R \to \mathfrak{S}, \quad r \mapsto \mathbf{s}^a_{\sigma'}.$$

Since $\mathfrak{s}'(r) \in \mathfrak{S}_{\sigma'}$ for all $r \in R$, the map $\mathfrak{s}'$ is a shaping of R, *i.e.*, , $\mathfrak{s}' \in \left[\mathfrak{S}^R\right]$. Consider also the map

$$e : R \to P, r \mapsto p.$$

We will show that e is an embedding.

Take any $r, q \in R$ with $r \prec q$. Then for any $a \in [A] \setminus \{1\}$ we have in $\overline{R}$

$$(r, \sigma, 1) \prec (r, \sigma, a) \prec (q, \sigma, 1)$$

But in the total order $\prec$ on $\overline{P}$, there are at most $A - 1$ points above (p, σ', a) of the form

$$(p, \sigma', a')$$

for some $a' \in [A]$. That means that

$$i_{\overline{R}'}(q, \sigma, 1) = (o, \sigma', a')$$

for some $a' \in [A]$ and $o \in P$, such that $p \prec o$.
So if we have $r, q \in R$ with $r < q$, we must also have $r \prec q$ by definition of $\mathcal{K}(G, e)$, as the total order $\prec$ must be an extension of the partial order $<$. So as above

$$i_{\overline{R}'} : (r, \sigma, 1) \mapsto (p, \sigma', a), \quad (q, \sigma, 1) \mapsto (o, \sigma', a')$$

with $p \neq o$. Further, since $R'' \in \binom{\overline{P}}{R}$, we have

$$r < q \iff (r, \sigma, 1) < (q, \sigma, 1) \iff (p, \sigma', a) < (o, \sigma', a').$$

We also know that $p \neq o$ and $\sigma' = \sigma'$, so the fact that $(p, \sigma', a) < (o, \sigma', a')$ must follow from part (ii) of Definition 4.1.1. Thus we must have $p < o$.
This shows that $e : R \to P$ is an embedding and hence $e(R) \in \binom{P}{R}$. So we have

$$g^{P,R} : \binom{P}{R} \to \binom{\overline{P}}{R}, \quad (e(R), \mathfrak{s}') \mapsto R'',$$

showing that $R'' \in \binom{\overline{P}}{R}$, which concludes the proof. $\square$

Theorem 4.1.5. *Let Σ be a generic skeleton. Then the class $\boldsymbol{\mathcal{K}}(\Sigma, \mathfrak{S}, o)$ is Ramsey.*

Proof. This follows trivially from Theorem 3.4.10, Lemma 4.1.3 and Lemma 4.1.4. $\square$

Corollary 4.1.6. *The class $\mathcal{K}(G, o)$ is Ramsey.*

Proof. This follows from Theorem 4.1.5, when Σ only has one point. □

Corollary 4.1.7. *The class $\mathcal{K}(C, o)$ is Ramsey.*

Proof. Recall Definition 2.5.17. Given a set of shapes $\mathfrak{S}$, let $\mathcal{K}(C, o)$ be the class of ordered $\mathfrak{S}$-shaped chains, and $\mathcal{K}(G, o)$ be the class of ordered $\mathfrak{S}$-shaped partial orders. Then $\mathcal{K}(C, o)$ is closed under $(\Sigma \times [A], \alpha, \beta)$-blowups. The class $\mathcal{K}(C, e)$ is a Ramsey class and is also a class of unshaped reducts of structures in $\mathcal{K}(C, o)$. Then by Corollary 3.4.11 the class $\mathcal{K}(C, o)$ is Ramsey.

□

Corollary 4.1.8. *Let Σ' be an antichained skeleton. Then the class $\mathcal{K}(\Sigma', \mathfrak{S}, o)$ is Ramsey.*

Proof. Recall that the antichained skeleton is a simple skeleton with all points labelled either G or A (Definition 2.5.23). For $\Sigma' = (\Sigma, <, \prec, l_1, l_2)$, let $\Sigma = (\Sigma, <, \prec, l_1', l_2)$ be a skeleton with $l_1(\sigma) = G$ for all $\sigma \in \Sigma$. Then by Theorem 4.1.5, the class $\mathcal{K}(\Sigma, o)$ is Ramsey.
Take any $\mathbf{Q}, \mathbf{R} \in \mathcal{K}(\Sigma', o)$. Then $\mathbf{Q}, \mathbf{R} \in \mathcal{K}(\Sigma, o)$, so there exists a $\mathbf{P}' \in \mathcal{K}(\Sigma, o)$, such that $\mathbf{P}' \to (\mathbf{R})^{\mathbf{Q}}_k$.
So for $\mathbf{P}' = (P, <', \prec, \mathfrak{s})$, define the structure $\mathbf{P}$ as follows.

(i) $\mathbf{P} = (P, <, \prec, \mathfrak{s})$.

(ii) $p_{\sigma,i}, < p_{\sigma',i'}$ if $p_{\sigma,i} <' p_{\sigma',i'}$ and either $\sigma \neq \sigma'$, or $\sigma = \sigma'$ and $l(\sigma) \neq A$.

Then for each $\sigma \in \Sigma$ with $l(\sigma) = A$, the substructure $\mathbf{P}_\sigma$ of $\mathbf{P}$ is an antichain. So $\mathbf{P} \in \mathcal{K}(\Sigma', o)$. We claim that $\mathbf{P} \to (\mathbf{R})^{\mathbf{Q}}_k$.
First notice that if $\mathbf{Q}' \in \binom{\mathbf{P}'}{\mathbf{Q}}$ is a substructure of $\mathbf{P}'$ on the subset Q', then the substructure of $\mathbf{P}$ on the subset Q' is isomorphic to $\mathbf{Q}$ as well. This is true because for each $\sigma \in \Sigma$ with $l(\sigma) = A$, the substructure $\mathbf{Q}_\sigma$ of $\mathbf{Q}$ is an antichain, and thus removing any pairs $p_{\sigma,i} <' p_{\sigma,i'}$ in $\mathbf{P}'_\sigma$ to get $\mathbf{P}_\sigma$ doesn't affect the substructures of $\mathbf{P}'$ that were already antichains in the component

σ. The same reasoning applies to any $\mathbf{R}' \in \binom{\mathbf{P}'}{\mathbf{R}}$, yielding injective maps

$$f^{\mathbf{Q}} : \binom{\mathbf{P}'}{\mathbf{Q}} \to \binom{\mathbf{P}}{\mathbf{Q}}, \quad f^{\mathbf{R}} : \binom{\mathbf{P}'}{\mathbf{R}} \to \binom{\mathbf{P}}{\mathbf{R}},$$

sending any structure of $\mathbf{P}'$ to a structure of $\mathbf{P}$ on the same subset of P. Consider any colouring $c : \binom{\mathbf{P}}{\mathbf{Q}} \to [k]$. Then $c \circ f^{\mathbf{Q}} : \binom{\mathbf{P}'}{\mathbf{Q}} \to [k]$. So there exists $\mathbf{R}' \in \binom{\mathbf{P}'}{\mathbf{R}}$ such that $\binom{\mathbf{R}'}{\mathbf{Q}}$ is monochromatic. But then for $\mathbf{R}'' = f^{\mathbf{R}}(\mathbf{R}')$, the set $\binom{\mathbf{R}''}{\mathbf{Q}}$ is monochromatic. So indeed $\mathbf{P} \to (\mathbf{R})^{\mathbf{Q}}_k$.

□

4.2 Chain of antichains

In this section, we will show that a class $\mathcal{K}(CA, \mathfrak{S}, N, o)$ of ordered shaped chains of antichains is a Ramsey class. We will do that using a corollary of the Two Pass Lemma (3.6.16).

Let $\mathcal{K}(CA, \mathfrak{S}, N, o)$ be a class of chains of antichains, as defined in 2.5.15, with a set of shapes

$$\mathfrak{S} = \{\mathbf{s}^{a,b} : (a, b) \in \mathcal{A} \rtimes \mathcal{B}\} = \bigcup_{a \in \mathcal{A}} \mathfrak{S}_a,$$

and for each $(a, b) \in \mathcal{A} \rtimes \mathcal{B}$ a number $n_{a,b} \in \{1, \aleph_0\}$. By Definition 2.5.15, for each $a \in \mathcal{A}$ there is a class $\mathcal{K}(GA, \mathfrak{S}_a, N_a, o)$ of glorified chains.

Lemma 4.2.1. *Let $\mathcal{K}_J$ be a class of all chains in language L_J. The class $\mathcal{K}(GA, \mathfrak{S}_a, N_a, o)$ of glorified antichains is an order class with respect to $\mathcal{K}_J$, closed under substructures and has the joint embedding property.*

Proof. Recall Definition 3.5.2 of a class $\mathcal{K}$ being an order class with respect to $\mathcal{K}_D$. When $\mathcal{K}(GA, \mathfrak{S}_a, N_a, o)$ plays the role of $\mathcal{K}$ and $\mathcal{K}_J$ plays the role of $\mathcal{K}_D$, they clearly satisfy Definition 3.5.2, as for each $\mathbf{P} \in \mathcal{K}(GA, \mathfrak{S}_a, N_a, o)$ there is a chain $\mathcal{J}$, so that

$$P = \{p^{a,b}_j : j \in \mathcal{J}^a, \mathfrak{s}(p^{a,b}_j) = \mathbf{s}^{a,b}\},$$

and by part (ix) of Definition 2.5.15 we have $p^{a,b}_j \prec p^{a,b'}_{j'}$ when $j \prec j'$ in $\mathcal{J}^a$.

This shows that $\mathcal{K}(GA, \mathfrak{S}_a, N_a, o)$ is an order class for $\prec$ and also an order class with respect to $\mathcal{K}_J$.
Given a structure $\mathbf{P} \in \mathcal{K}(GA, \mathfrak{S}_a, N_a, o)$, it is easy to check that for any non-empty subset P' of P, the substructure of $\mathbf{P}$ on the set P' also lies in $\mathcal{K}(GA, \mathfrak{S}_a, N_a, o)$.
Finally, given $\mathbf{P}, \mathbf{R} \in \mathcal{K}(GA, \mathfrak{S}_a, N_a, o)$, for each $b \in \mathcal{B}_a$, let

$$m^b = \max\{|P^{a,b}|, |R^{a,b}|\}.$$

Then if $n_b = 1$, $|P^{a,b}| \leq 1, |R^{a,b}| \leq 1$, so $m^b \leq 1$. If $m^b > 0$, let $\mathbf{Q}^{a,b}$ be an $\mathbf{s}^{a,b}$-shaped antichain of size m^b, and let $\mathbf{Q}$ be an antichain with substructures $\mathbf{Q}^{a,b}$ for $b \in \mathcal{B}_a$ and a total order $\prec$ so that

$$\mathbf{Q}^{a,b} \prec \mathbf{Q}^{a,b'} \iff b < b' \text{ in } \mathcal{B}_a.$$

Then both, $\mathbf{P}$ and $\mathbf{R}$ are substructures of $\mathbf{Q}$ and $\mathbf{Q} \in \mathcal{K}(GA, \mathfrak{S}_a, N_a, o)$, so $\mathcal{K}(GA, \mathfrak{S}_a, N_a, o)$ has joint embedding property. □

We define classes $\mathcal{K}_I$, $\mathcal{K}_J$ and $K_{\mathcal{A}}$ of the two pass configuration (3.6.2).

Lemma 4.2.2. *Let $\mathcal{A}$ be a set of shapes and let $\{\mathfrak{S}_a\}_{a\in\mathcal{A}}$ be disjoint sets of shapes, with $\mathfrak{S} = \bigcup_{a\in\mathcal{A}} \mathfrak{S}_a$. Define the languages:*

- *L_I containing a partial order relation $<_I$,*
- *L_J containing a partial order relation $<_J$,*
- *L containing a partial order relation $<$ and total order relation $\prec$,*
- *$L_{\mathcal{A}}$ containing all the shapes $a \in \mathcal{A}$, and*
- *$L_{\mathfrak{S}}$ containing all the shapes $\mathbf{s} \in \mathfrak{S}$.*

Consider the following classes:

- *Class $\mathcal{K}_I = \mathcal{K}(C, \mathcal{A})$ of all $\mathcal{A}$-shaped chains in language $L_I \cup L_{\mathcal{A}}$.*
- *Class $\mathcal{K}_J$ of all chains in language L_J.*

- *Class* $\mathcal{K}_a = \mathcal{K}(GA, \mathfrak{S}_a, N_a, o)$ *of* $\mathfrak{S}_a$*-shaped glorified antichains, for all* $a \in \mathcal{A}$*, in language* $L \cup L_{\mathfrak{S}}$ *and* $K_{\mathcal{A}} = \{\mathcal{K}_a : a \in \mathcal{A}\}$*.*

The classes $\mathcal{K}_I$*,* $\mathcal{K}_J$ *and* $K_{\mathcal{A}}$ *are in a two pass configuration.*

Proof. This follows straightforward from definition 3.6.2 of a two pass configuration, with the set $\mathfrak{Z}$ of shapes replaced by a set $\mathcal{A}$ of shapes and the classes $\mathcal{K}_D$, $\mathcal{K}_1$ and $\mathcal{K}_2$ omitted. The class $\mathcal{K}_a$ is an order class with respect to $\mathcal{K}_J$, closed under substructures and has the joint embedding property by Lemma 4.2.1. ☐

Lemma 4.2.3. *The class* $\mathcal{K}(CA, \mathfrak{S}, N, o)$ *of ordered shaped chains is a strongly levelled class defined by* $\mathcal{K}_I, \mathcal{K}_J$ *and* $K_{\mathcal{A}}$*.*

Proof. We have already shown that the classes $\mathcal{K}_I$, $\mathcal{K}_J$ and $K_{\mathcal{A}}$ are in a strongly levelled configuration in Lemma 4.2.2.
Parts (i) and (ii) of the definition of a strongly levelled class are trivially true in the class $\mathcal{K}(CA, \mathfrak{S}, N, o)$. Indeed, any $\mathbf{P} \in \mathcal{K}(CA, \mathfrak{S}, N, o)$ is defined by an $\mathcal{A}$-shaped antichain $\mathcal{I}$ and for each $i \in \mathcal{I}$, with $a = \mathfrak{s}^{\mathcal{I}}(i)$, a glorified antichain $\mathbf{P}_i \in \mathcal{K}(GA, \mathfrak{S}_a, N_a, o)$, and the conditions they satisfy in Definition 2.5.15 imply parts (i) and (ii) of Definition 3.6.4.
To show part (iii) of the definition, take any $\mathbf{P}, \mathbf{R} \in \mathcal{K}(CA, \mathfrak{S}, N, o)$. Let $\mathbf{R}' \in \binom{\mathbf{P}}{\mathbf{R}}$, with an embedding $e : \mathbf{R} \to \mathbf{P}$, $e(\mathbf{R}) = \mathbf{R}'$. Then e must send any distinct maximal antichains $\mathbf{R}'_i$, $\mathbf{R}'_{i'}$ to substructures of distinct maximal antichains $\mathbf{P}_{e_1(i)}$, $\mathbf{P}_{e_1(i')}$ of $\mathbf{P}$, with $e(R'_i) \subset P_{e_1(i)}$, $e(R'_{i'}) \subset P_{e_1(i')}$ and

$$\mathbf{R}'_i < \mathbf{R}'_{i'} \iff \mathbf{P}_{e_1(i)} < \mathbf{P}_{e_1(i')},$$

since the partial order on any chain of antichains is defined by the total order on the set of maximal antichains of a chain of antichains. Besides, $\mathbf{R}'_i$ is $\mathfrak{S}_a$-shaped precisely when $\mathbf{P}_{e_1(i)}$ is. This defines the embedding $e_1 : \mathcal{I}^{\mathbf{R}} \to \mathcal{I}^{\mathbf{P}}$ that satisfies condition (iii) of the Definition 3.6.4 and finishes the proof. ☐

Theorem 4.2.4. *The class* $\mathcal{K}(GA, \mathfrak{S}_a, N_a, o)$ *of glorified chains is Ramsey.*

Proof. For each $b \in \mathcal{B}_a$ let $\mathcal{K}_b$ be

(i) the class $\mathcal{K}(A_1, \{\mathbf{s}^{a,b}\}, o)$ if $n_{a,b} = 1$, and

(ii) the class $\mathcal{K}(A, \{\mathbf{s}^{a,b}\}, o)$ if $n_{a,b} = \aleph_0$.

Each class $\mathcal{K}(A_1, \{\mathbf{s}^{a,b}\}, o)$ is trivially Ramsey, as it only contains one structure. Each class $\mathcal{K}(A, \{\mathbf{s}^{a,b}\}, o)$ is Ramsey, as shown in Example 3.1.1.
The class $\mathcal{K}(GA, \mathfrak{S}_a, N_a, o)$ is clearly a merge of classes $\mathcal{K}_b$ for $b \in \mathcal{B}_a$ (see Definition 2.5.3 - the relations $\mathbf{s}^{a,b}$ play the role of F_i). Further, any a non-empty set of antichains $\mathbf{P}^{a,b}$, at most one for each $b \in \mathcal{B}_a$, yields precisely one merge structure $\mathbf{P} \in \mathcal{K}(GA, \mathfrak{S}_a, N_a, o)$.
Thus, by Corollary 3.2.12, the class $\mathcal{K}(GA, \mathfrak{S}_a, N_a, o)$ is Ramsey. □

Theorem 4.2.5. *The class $\mathcal{K}(CA, \mathfrak{S}, N, o)$ of chains of antichains is Ramsey.*

Proof. We have shown in Lemma 4.2.3 that $\mathcal{K}(CA, \mathfrak{S}, N, o)$ is a strongly levelled class defined by $\mathcal{K}_I, \mathcal{K}_J$ and $K_{\mathcal{A}}$.
The class $\mathcal{K}_I$ is a class $\mathcal{K}(C, o)$, with the set $\mathcal{A}$ of shapes. Thus by Corollary 4.1.7, the class $\mathcal{K}_I$ is Ramsey.
For each $a \in \mathcal{A}$, the class $\mathcal{K}(GA, \mathfrak{S}_a, N_a, o)$ is Ramsey by Theorem 4.2.4.
Then by Corollary 3.6.16, the class $\mathcal{K}(CA, \mathfrak{S}, N, o)$ of ordered shaped chains is a Ramsey. □

4.3 Glorified skeleton

We now aim to show that a class $\mathcal{K}(\Sigma_{sp}, o)$ of ordered shaped partial orders with a glorified skeleton is a Ramsey class. We will do so by using the results about antichained skeletons, constructing the glorified antichains of chains from the antichains using the Two Pass Lemma.

In the first subsection, One glorified chain of antichains, we will consider a glorified skeleton and focus on one point of it, labelled GAC. We finish the proof in the following section, Induction.

One glorified antichain of chains

We start with defining skeletons and related classes of structures that will be in use throughout this section.

Definition 4.3.1. Skeletons:

(i) Let $\Sigma = (\Sigma, <, \prec, l_1, l_2)$ be a simple skeleton, and let $\rho \in \Sigma$, such that $l_1(\rho) = GAC$.

(ii) Let $\Sigma_q = (\Sigma, <, \prec, l_1^q, l_2)$ and

(a) $l_1^q(\rho) = A$,

(b) $l_1^q(\sigma) = l_1(\sigma)$ for $\sigma \neq \rho$.

(iii) Let Σ_1 be a substructure of Σ on the subset $\Sigma_1 = \Sigma \setminus \{\rho\}$.

(iv) Let Σ_2 be an ordered skeleton with a single point labelled A.

(v) Let Σ_s be an ordered skeleton with a single point labelled GAC.

Sets of shapes:

(i) Let $\mathfrak{S}$ be a set of shapes with a partition $\{\mathfrak{S}_\sigma : \sigma \in \Sigma\}$.

(ii) For the ρ as in part (i) of Skeletons, consider the following.

(a) A total order $\mathcal{A}_\rho$, with a partition $\{\mathcal{A}_{\rho,1}, \mathcal{A}_{\rho,2}\}$, where $\mathcal{A}_{\rho,2}$ is possibly an empty set, and for all $a_1 \in \mathcal{A}_{\rho,1}$ and $a_2 \in \mathcal{A}_{\rho,2}$ we have $a_1 < a_2$.

(b) For each $a \in \mathcal{A}_\rho$ a number $n_{\rho,a} \in \{1, \aleph_0\}$, and
$N_\rho = \{n_{\rho,a} : a \in \mathcal{A}\}$.

(c) A set $\mathfrak{S}_\rho$ of shapes with a partition $\{\mathfrak{S}_{\rho,a} : a \in \mathcal{A}\}$, where $|\mathfrak{S}_{\rho,a}| = 1$ when $n_{\rho,a} = 1$ and for each $a \in \mathcal{A}$ there exists a total order $\mathcal{B}_{\rho,a}$, such that $\mathfrak{S}_{\rho,a} = \{\mathbf{s}_\rho^{a,b} : b \in \mathcal{B}_{\rho,a}\}$.

(iii) Let $\mathfrak{S}_1 = \mathfrak{S} \setminus \mathfrak{S}_\rho$.

(iv) Let $\mathfrak{S}'_\rho = \{\mathbf{s}_\rho\}$.

Languages:

(i) L_I containing a partial order relation $<_I$,

(ii) L_J containing a partial order relation $<_J$,

(iii) L_D containing a partial order relation $<_D$,

(iv) L containing a partial order relation $<$ and total order relation $\prec$,

(v) L_ρ containing the shape $\mathbf{s}_\rho$,

(vi) $L_{\mathfrak{S}_1}$ containing all the shapes $\mathbf{s} \in \mathfrak{S}_1$, and

(vii) $L_{\mathfrak{S}_\rho}$ containing all the shapes $\mathbf{s} \in \mathfrak{S}_\rho$.

Classes:

(i) $\boldsymbol{\mathcal{K}}_I$ is a class of all $\mathbf{s}_\rho$-shaped chains in language $L_I \cup L_\rho$.

(ii) $\mathcal{K}_J$ and $\mathcal{K}_D$ are classes of all chains in languages L_J and L_D respectively.

(iii) $\boldsymbol{\mathcal{K}}_1$ is a class $\boldsymbol{\mathcal{K}}(\Sigma_1, \mathfrak{S}_1, o)$ in language $L \cup L_{\mathfrak{S}_1}$ of ordered $\mathfrak{S}_1$-shaped partial orders.

(iv) $\boldsymbol{\mathcal{K}}_2$ is a class $\boldsymbol{\mathcal{K}}(\Sigma_2, \{\mathbf{s}_\sigma\}, o)$, in language $L \cup L_\rho$, of ordered $\mathbf{s}_\rho$-shaped antichains.

(v) $\boldsymbol{\mathcal{K}}_{\mathbf{s}_\rho}$ is the class $\boldsymbol{\mathcal{K}}(\Sigma_s, \mathfrak{S}_\rho, N_\rho, o)$. Let $K_\rho = \{\boldsymbol{\mathcal{K}}_{\mathbf{s}_\rho}\}$.

Lemma 4.3.2. *Let $\mathcal{K}_J$ be a class of all chains in language L_J. The class $\boldsymbol{\mathcal{K}}(GC, \mathfrak{S}, N, o)$ of glorified chains is an order class with respect to $\mathcal{K}_J$, closed under substructures and has the joint embedding property.*

Proof. Recall Definition 3.5.2 of a class $\boldsymbol{\mathcal{K}}$ being an order class with respect to $\boldsymbol{\mathcal{K}}_D$. When $\boldsymbol{\mathcal{K}}(GC, \mathfrak{S}, N, o)$ plays the role of $\boldsymbol{\mathcal{K}}$ and $\mathcal{K}_J$ plays the role of $\boldsymbol{\mathcal{K}}_D$, they clearly satisfy Definition 3.5.2, as for each $\mathbf{P} \in \boldsymbol{\mathcal{K}}(GC, \mathfrak{S}, N, o)$ there is a chain $\mathcal{J}$, so that

$$P = \{p_j^{h,a,b} : j \in \mathcal{J}, \mathfrak{s}(p_j^{h,a,b}) = \mathbf{s}^{a,b}, a \in \mathcal{A}_h\},$$

and by part (ix) of Definition 2.5.18 we have $p_j^{h,a,b} \prec p_{j'}^{h',a',b'}$ when $j < j'$. This shows that $\boldsymbol{\mathcal{K}}(GC, \mathfrak{S}, N, o)$ is an order class for $\prec$ and also an order class with respect to $\mathcal{K}_J$.

Given a structure $\mathbf{P} \in \boldsymbol{\mathcal{K}}(GC, \mathfrak{S}, N, o)$, it is easy to check that for any non-empty subset P' of P, the substructure of $\mathbf{P}$ on the set P' also lies in $\boldsymbol{\mathcal{K}}(GC, \mathfrak{S}, N, o)$.

Finally, given $\mathbf{P}, \mathbf{R} \in \boldsymbol{\mathcal{K}}(GC, \mathfrak{S}, N, o)$, for each $a \in \mathcal{A}$ define $\mathbf{Q}^a$ as follows.

(i) If $n_a = 1$, let $\mathbf{Q}^a$ be an antichain of size 1.

(ii) If $n_a = \aleph_0$, and either P^a or R^a is non-empty, let $\mathbf{Q}^a$ be an $\mathfrak{S}_a$-shaped chain, built from $\mathbf{P}^a$ on the bottom and $\mathbf{R}^a$ on top if both structures exist, or from one of the structures if the other is non-empty.

Let $\mathbf{Q}$ be a glorified chain with substructures $\mathbf{Q}^a$ for each $a \in \mathcal{A}$.

Then both, $\mathbf{P}$ and $\mathbf{R}$ are substructures of $\mathbf{Q}$ and $\mathbf{Q} \in \boldsymbol{\mathcal{K}}(GC, \mathfrak{S}, N, o)$, so $\boldsymbol{\mathcal{K}}(GC, \mathfrak{S}, N, o)$ has joint embedding property. □

Lemma 4.3.3. *Classes $\boldsymbol{\mathcal{K}}_I$, $\mathcal{K}_J, \boldsymbol{\mathcal{K}}_D, \boldsymbol{\mathcal{K}}_1, \boldsymbol{\mathcal{K}}_2$ and K_ρ are in a two pass configuration.*

Proof. The language L_ρ plays the role of the language L_3 in Definition 3.6.2, and $L_{\mathfrak{S}_\rho}$ plays the role of $L_{\mathfrak{S}}$.

The class $\boldsymbol{\mathcal{K}}_1$ is an order class with respect to $\prec$, so we can encode the total order on the structures in $\boldsymbol{\mathcal{K}}_1$ using the chains in $\boldsymbol{\mathcal{K}}_D$ to satisfy Definition 3.5.2.

$\boldsymbol{\mathcal{K}}_2$ is bidefinable with $\boldsymbol{\mathcal{K}}_I$ via a bijection $\Phi : \boldsymbol{\mathcal{K}}_I \to \boldsymbol{\mathcal{K}}_2$ that sends each $\mathbf{s}_\rho$-shaped chain to an ordered $\mathbf{s}_\rho$-shaped antichain of the same size, and the partial order on the chain defines the total order on the antichain. That is, for any $\mathcal{I}^{\mathbf{A}} \in \boldsymbol{\mathcal{K}}_I$, with $\Phi(\mathcal{I}^{\mathbf{A}}) = \mathbf{A}_2$ we have:

(i) $\mathbf{A}_2 = (A_2, <, \prec, \mathfrak{s})$, with $A_2 = \{a_i : i \in \mathcal{I}^{\mathbf{A}}\}$,

(ii) $\mathcal{I}^{\mathbf{A}} = (\mathcal{I}^{\mathbf{A}}, <_I, \mathfrak{z})$,

(iii) $a_i \prec a_{i'}$ precisely when $i <_I i'$, and

(iv) $\mathfrak{s}(a_i) = \mathfrak{z}(i) = \mathbf{s}_r ho$.

Finally, the class $\boldsymbol{\mathcal{K}}_{\mathbf{s}_\rho} = \boldsymbol{\mathcal{K}}(GC, \mathfrak{S}_\rho, N_\rho, o)$ of glorified chains is an order class with respect to $\mathcal{K}_J$, closed under substructures and has the joint embedding property by Lemma 4.3.2.

Thus $\boldsymbol{\mathcal{K}}_I$, $\mathcal{K}_J, \mathcal{K}_D, \boldsymbol{\mathcal{K}}_1, \boldsymbol{\mathcal{K}}_2$ and K_ρ are in a two pass configuration. □

Lemma 4.3.4. *The class $\boldsymbol{\mathcal{K}}(GAC, \mathfrak{S}_\rho, N_\rho o)$ of ordered shaped glorified antichains of chains is a strongly levelled class defined by $\boldsymbol{\mathcal{K}}_I, \mathcal{K}_J$ and K_ρ.*

Proof. We have shown that $\boldsymbol{\mathcal{K}}_I, \mathcal{K}_J$ and K_ρ are in a two pass configuration in Lemma 3.6.2. Parts (i) and (ii) of Definition 3.6.4 follow straightforward from the definition of glorified antichains of chains (2.5.20). Indeed, a glorified antichain of chains $\mathbf{P} \in \boldsymbol{\mathcal{K}}(GAC, \mathfrak{S}_\rho, N_\rho o)$ consists of an $\mathbf{s}_\rho$-shaped total order $\mathcal{I}^{\mathbf{A}}$, and for each $i \in \mathcal{I}^{\mathbf{A}}$ a glorified chain $\mathbf{P}_i \in \boldsymbol{\mathcal{K}}(GC, \mathfrak{S}_\rho, N_\rho o)$ by parts (vi) and (vii) of Definition 2.5.20, showing that parts (i)(a), (b) and (c) of Definition 3.6.4 hold. Part (ix) of Definition 2.5.20 shows that part (i)(d) of Definition 3.6.4 holds.

To show part (iii) of the definition, take any $\mathbf{P}, \mathbf{R} \in \boldsymbol{\mathcal{K}}(GAC, \mathfrak{S}_\rho, N_\rho o)$. Let $\mathbf{R}' \in \binom{\mathbf{P}}{\mathbf{R}}$, with an embedding $e : \mathbf{R} \to \mathbf{P}$, $e(\mathbf{R}) = \mathbf{R}'$. Then e must send any distinct maximal glorified chains $\mathbf{R}'_i$, $\mathbf{R}'_{i'}$ to substructures of distinct maximal glorified chains $\mathbf{P}_{e_1(i)}$, $\mathbf{P}_{e_1(i')}$ of $\mathbf{P}$, with $e(R'_i) \subset P_{e_1(i)}$, $e(R'_{i'}) \subset P_{e_1(i')}$ and

$$\mathbf{R}'_i < \mathbf{R}'_{i'} \iff \mathbf{P}_{e_1(i)} < \mathbf{P}_{e_1(i')},$$

since the partial order on any glorified antichain of chains is defined by the total order on the maximal glorified chains of a glorified antichain of chains.

This defines the embedding $e_1 : \mathcal{I}^{\mathbf{R}} \to \mathcal{I}^{\mathbf{P}}$ that satisfies condition (iii) of the Definition 3.6.4 and finishes the proof. □

Glorified chains

Theorem 4.3.5. *The class $\mathcal{K}(GC, \mathfrak{S}, N, o)$ of glorified chains is Ramsey.*

Proof. For $a \in \mathcal{A}$, let $\mathcal{K}_a$ be

(i) The class $\mathcal{K}(A_1, \mathfrak{S}_a, o)$ containing the ordered $\mathbf{s}^{a,1}$-shaped antichain of size 1 if $n_a = 1$.

(ii) The class $\mathcal{K}(C, \mathfrak{S}_a, o)$ of $\mathfrak{S}_a$-shaped chains otherwise.

Then considering relations F_a for $a \in \mathcal{A}$, where for any $\mathbf{P} \in \mathcal{K}(GC, \mathfrak{S}, N, o)$ and $p \in \mathbf{P}$ we have

$$F_a(p) \iff \mathfrak{s}(p) \in \mathfrak{S}_a$$

shows that $\mathcal{K}(GC, \mathfrak{S}, N, o)$ is a merge of classes $\mathcal{K}_a$ for $a \in \mathcal{A}$, as defined in Definition 2.5.3.

In fact, given a non-empty subset A of $\mathcal{A}$ and a structure $\mathbf{P}_a \in \mathcal{K}_a$ for each $a \in A$, the merge of structures $\mathbf{P}_a$ is the unique glorified chain with $\mathbf{P}_a$ below $\mathbf{P}_{a'}$ in the total order for each pair $a, a' \in A$ with $a < a'$.

The class $\mathcal{K}(A_1, \mathfrak{S}_a, o)$ is trivially Ramsey for each $a \in \mathcal{A}$ with $n_a = 1$. By Lemma 4.1.7 the remaining classes $\mathcal{K}(C, \mathfrak{S}_a, o)$ are Ramsey.

Thus, by Corollary 3.2.12, the class $\mathcal{K}(GC, \mathfrak{S}, N, o)$ is Ramsey. □

Ramsey result for glorified antichain of chains

Lemma 4.3.6. *The class $\mathcal{K} = \mathcal{K}(\Sigma, \mathfrak{S}, o)$ is a class with a strongly levelled part $\mathcal{K}_s = \mathcal{K}(GAC, \mathfrak{S}_\rho, o)$ and a quotient $\mathcal{K}_q = \mathcal{K}(\Sigma_q, \mathfrak{S}_1 \cup \{\mathbf{s}_\rho\}, o)$.*

Proof. Recall Definition 3.6.9.

First notice that $\mathcal{K}_q = \mathcal{K}(\Sigma_q, \mathfrak{S}_1 \cup \{\mathbf{s}_\rho\}, o)$ is a merge of classes $\mathcal{K}_1 = \mathcal{K}(\Sigma_1, \mathfrak{S}_1, o)$ and $\mathcal{K}_2 = \mathcal{K}(\Sigma_2, \{\mathbf{s}_\sigma\}, o)$, since $\Sigma_1 = \Sigma \setminus \{\rho\}$, Σ_2 is an ordered skeleton with a single point labelled A, and Σ_q contains Σ_1 as well as the point ρ labelled A. We've shown in Lemma 4.3.4 that the class $\mathcal{K}_s$ is strongly levelled. Clearly also the class $\mathcal{K}$ is a merge of classes $\mathcal{K}_1$ and $\mathcal{K}_s$.

Parts (ii)-(iv) of Definition 3.6.9 hold by the definition of the simple skeleton. Indeed, by parts (vi) and (vii) of Definition 2.5.26, the $\mathfrak{S}_\rho$-shaped part $\mathbf{P}_\rho$ of a structure $\mathbf{P} \in \mathcal{K}$, corresponding to the structure $\mathbf{A}_s$ in Definition 3.6.9, has the corresponding structure $\mathbf{P}' \in \mathcal{K}_q$, corresponding to the structure $\mathbf{A}_q$ in Definition 3.6.9. Any structure $\mathbf{P} \in \mathcal{K}$ is constructed in this manner, and $\mathcal{K}$ contains all structures constructed from structures in $\mathcal{K}_q$ and $\mathcal{K}_s$.

Finally, take any structure $\mathbf{P} \in \mathcal{K}$, consisting of a $\mathfrak{S}_1$-shaped substructure $\mathbf{P}_1$ and an $\mathfrak{S}_\rho$-shaped glorified antichain of chains $\mathbf{P}_s$, consisiting of a glorified chain $\mathbf{P}_i$ for each $i \in \mathcal{I}$ for some $\mathbf{s}_\rho$-shaped total order $\mathcal{I}$. Its quotient structure $\mathbf{P}_q$ consists of $\mathbf{P}_1$ and an $\mathbf{s}_\rho$-shaped antichain $\mathbf{P}_2$ with points $P_2 = \{p_i : i \in \mathcal{I}\}$. The classes $\mathcal{K}_q$ and $\mathcal{K}_s$ are both closed under substructures, and taking a subset P_1' of P_1, a subset $\mathcal{I}'$ of $\mathcal{I}$, and for each $i \in \mathcal{I}'$ a subset P_i' of P_i, with at least one of the P_1' or P_i non-empty, indeed defines

(i) a substructure $\mathbf{P}_q'$ of $\mathbf{P}_q$ on the set $P_1' \cup \{p_i : i \in \mathcal{I}'\}$, with $\mathbf{P}_q' \in \mathcal{K}_q$, and

(ii) a substructure $\mathbf{P}_s'$ of $\mathbf{P}_s$ on the set $P_1' \cup \left(\bigcup_{i\in\mathcal{I}'} P_i\right)$, with $\mathbf{P}_s' \in \mathcal{K}_s$.

This finishes the proof. □

Induction argument

Theorem 4.3.7. *Let Σ be a glorified skeleton. Then the class $\mathcal{K}(\Sigma, \mathfrak{S}, o)$ is Ramsey.*

Proof. Let $\Sigma = (\Sigma, <, \prec, l_1, l_2)$ and let $\mathcal{K}(\Sigma, \mathfrak{S}, o)$ be the class defined in 2.5.26. We have the following for any $\sigma \in \Sigma$.

(i) If $l_1(\sigma) = G$, let $\mathfrak{S}_\sigma$ be a set of shapes.

(ii) If $l_1(\sigma) = GAC$, we have.

  (a) A total order $\mathcal{A}_\sigma$, with a partition $\{\mathcal{A}_{\sigma,1}, \mathcal{A}_{\sigma,2}\}$, where $\mathcal{A}_{\sigma,2}$ is possibly an empty set, and for all $a_1 \in \mathcal{A}_{\sigma,1}$ and $a_2 \in \mathcal{A}_{\sigma,2}$ we have $a_1 < a_2$.

  (b) For each $a \in \mathcal{A}_{\sigma,1}$ an $n_{\sigma,a} \in \{1, \aleph_0\}$ and $N_\sigma = \{n_{\sigma,a} : a \in \mathcal{A}_\sigma\}$.

  (c) A set $\mathfrak{S}$ of shapes with a partition $\{\mathfrak{S}_{\sigma,a} : a \in \mathcal{A}_\sigma\}$, where $|\mathfrak{S}_a| = 1$ when $n_a = 1$ and for each $a \in \mathcal{A}_\sigma$ there exists a total order $\mathcal{B}_{\sigma,a}$, such that $\mathfrak{S}_{\sigma,a} = \{\mathbf{s}_\sigma^{a,b} : b \in \mathcal{B}_{\sigma,a}\}$.

(d) If there exists a $\sigma' \in \Sigma$, such that $\sigma' <_g \sigma$, then $\mathcal{A}_{\sigma,2}$ is empty.

We also have $\mathfrak{S} = \bigcup_{\sigma\in\Sigma} \mathfrak{S}_\sigma$.
Let Σ_{GAC} be the subset of Σ of all points in Σ labelled GAC. Enumerate the points in Σ_{GAC} as follows.

$$\Sigma_{GAC} = \{\sigma_1, \sigma_2, \dots, \sigma_w\}.$$

Then for all $v \in [w]$, let $\mathfrak{S}'_v = \{\mathbf{s}_v\}$. Let $\mathfrak{S}_0 = \left(\bigcup_{\sigma\in\Sigma, l'_1(\sigma)=G} \mathfrak{S}_\sigma\right) \cup \left(\bigcup_{v\in[w]} \mathfrak{S}'_v\right)$.
Let $\Sigma_0 = (\Sigma, <, \prec, l_1^0, l_2)$ be the antichained skeleton of the skeleton Σ. Then Σ_0 plays the role of the skeleton Σ' in Definition 2.5.26.
Let, for $v \in [w]$, the skeleton $\Sigma_v = (\Sigma, <, \prec, l_1^v, l_2)$ be a skeleton with points σ_u for $u \in [v]$ labelled GAC, the points σ_u for $u \in [w] \setminus [v]$ labelled A, and agreeing with Σ_0 otherwise. Then Σ_w is precisely the skeleton Σ. Consider also, for $v \in [w]$, the set of shapes

$$\mathfrak{S}_0 = \left(\bigcup_{\sigma\in\Sigma, l'_1(\sigma)=G} \mathfrak{S}_\sigma\right) \cup \left(\bigcup_{u\in[v]} \mathfrak{S}_{\sigma_u}\right) \cup \left(\bigcup_{u\in[w]\setminus[v]} \mathfrak{S}'_u\right).$$

The class $\boldsymbol{\mathcal{K}}(\Sigma_0, \mathfrak{S}_0, o)$ is a Ramsey class by Corollary 4.1.8. Given that the class $\boldsymbol{\mathcal{K}}(\Sigma_{v-1}, \mathfrak{S}_{v-1}, o)$ is Ramsey, we will show that the class $\boldsymbol{\mathcal{K}}(\Sigma_v, \mathfrak{S}_v, o)$ is Ramsey, implying that the class $\boldsymbol{\mathcal{K}}(\Sigma_w, \mathfrak{S}_w, o) = \boldsymbol{\mathcal{K}}(\Sigma, \mathfrak{S}, o)$ is Ramsey and completing the proof.
For $v \in [w]$, the class $\boldsymbol{\mathcal{K}}(\Sigma_{v-1}, \mathfrak{S}_{v-1}, o)$ plays the role of the class $\boldsymbol{\mathcal{K}}_q$ in the Definiton 4.3.1 and $\boldsymbol{\mathcal{K}}(\Sigma_v, \mathfrak{S}_v, o)$ plays the role of the class $\boldsymbol{\mathcal{K}}$. The class $\boldsymbol{\mathcal{K}}(GAC, \mathfrak{S}_{\sigma_v}, N_{\sigma_v}, o)$ plays the role of the class $\boldsymbol{\mathcal{K}}_s$ and the class $\boldsymbol{\mathcal{K}}(GC, \mathfrak{S}_{\sigma_v}, N_{\sigma_v}, o)$ the role of $\boldsymbol{\mathcal{K}}_{\mathbf{s}_\rho}$, with $K_\rho = \{\boldsymbol{\mathcal{K}}_{\mathbf{s}_\rho}\}$.
We have shown that the class $\boldsymbol{\mathcal{K}}(GAC, \mathfrak{S}_{\sigma_v}, N_{\sigma_v}, o)$ is a strongly levelled class defined by $\boldsymbol{\mathcal{K}}_I, \mathcal{K}_J$ and K_ρ. in Lemma 4.3.4. We've shown that the class $\boldsymbol{\mathcal{K}}(GC, \mathfrak{S}, N, o)$ is Ramsey in Lemma 4.3.5. We've also shown that $\boldsymbol{\mathcal{K}}(\Sigma_v, \mathfrak{S}_v, o)$ is a class with a strongly levelled part $\boldsymbol{\mathcal{K}}(GAC, \mathfrak{S}_{\sigma_v}, N_{\sigma_v}, o)$ and a quotient $\boldsymbol{\mathcal{K}}(\Sigma_{v-1}, \mathfrak{S}_{v-1}, o)$ in Lemma 4.3.4. Then since $\boldsymbol{\mathcal{K}}(\Sigma_{v-1}, \mathfrak{S}_{v-1}, o)$ is Ramsey, so is $\boldsymbol{\mathcal{K}}(GAC, \mathfrak{S}_{\sigma_v}, N_{\sigma_v}, o)$ by Two Pass Lemma 3.6.15. $\square$

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Chapter 5

Correspondence

This chapter contains a translation between skeletons from the classification of the shaped homogeneous partial orders in Torrezão de Sousa & Truss (2008) and the skeletons of the ordered shaped partial orders introduced in this thesis, by considering which classes are simply bi-definable and which order classes are reasonable. It also shows that specific classes of ordered shaped partial orders have the ordering property.

In Definition 2.4.8 we introduced classes $\mathcal{K}(AC)$ of shaped antichains of chains, $\mathcal{K}(CA)$ of shaped chains of antichains and $\mathcal{K}(G)$ of shaped partial orders, which are the building blocks of classes $\mathcal{K}(\Sigma)$, defined by a good skeleton Σ; namely the ages of the structures in Definition 2.4.14. In section 5.1 we unravel the conditions in the definition of a good skeleton and introduce some of its subskeletons and the core information about them in Figure 5.1. Breaking down the good skeleton into simpler skeletons provides a way to interpret the long list of conditions in the definition of a good skeleton.

To show this is the case, we consider various equivalence relations, defined in 5.1.2. Denote by Figure 5.2 the statement that an equivalence class of the skeleton $\mathbf{A}$ under the equivalence relation $\sim$ is either a skeleton $\mathbf{B}$ or a skeleton $\mathbf{C}$. Then Figure 5.3 summarises the results proved in Section 5.1. The skeletons denoted by AC, CA and G in the picture are precisely the skeletons containing one point labelled AC, CA and G. We summarise the results in Table 5.1.

Table 5.1

Skeleton	Partition	Proof
Good	Shuffle, Chunk	Lemma 5.1.11
Chunk	G, Pm, Cpm	Lemma 5.1.15
Cpm	Pm	Lemma 5.1.15

Figure 5.1: Good skeleton and its subskeletons

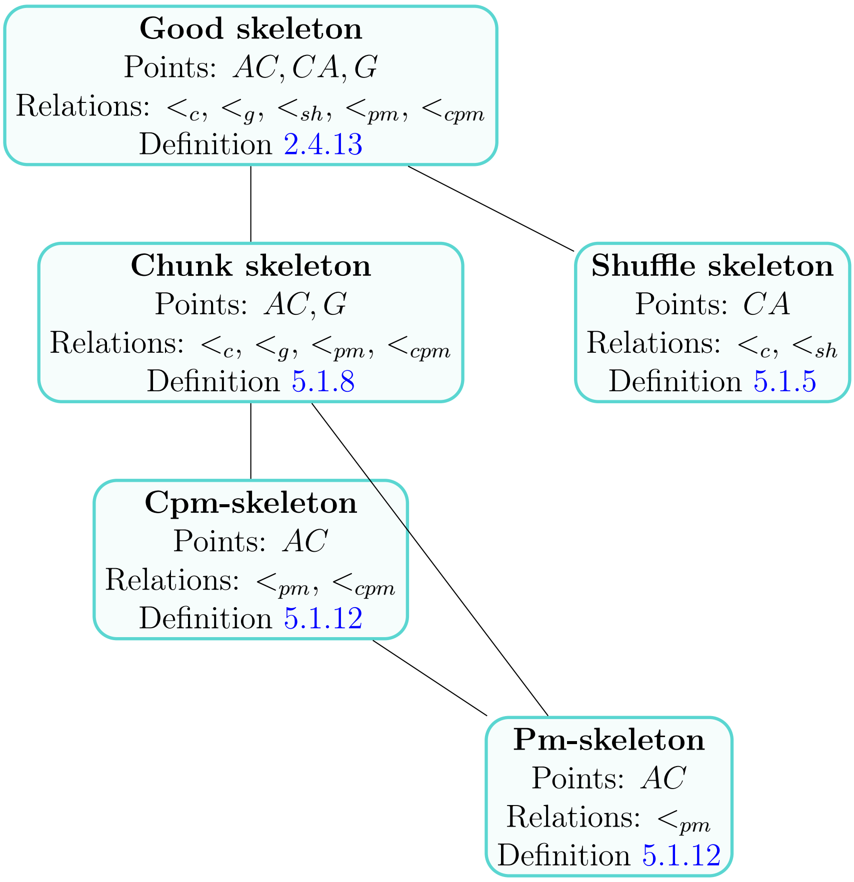


Figure 5.2: Equivalence classes notation

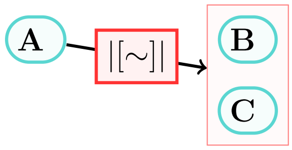
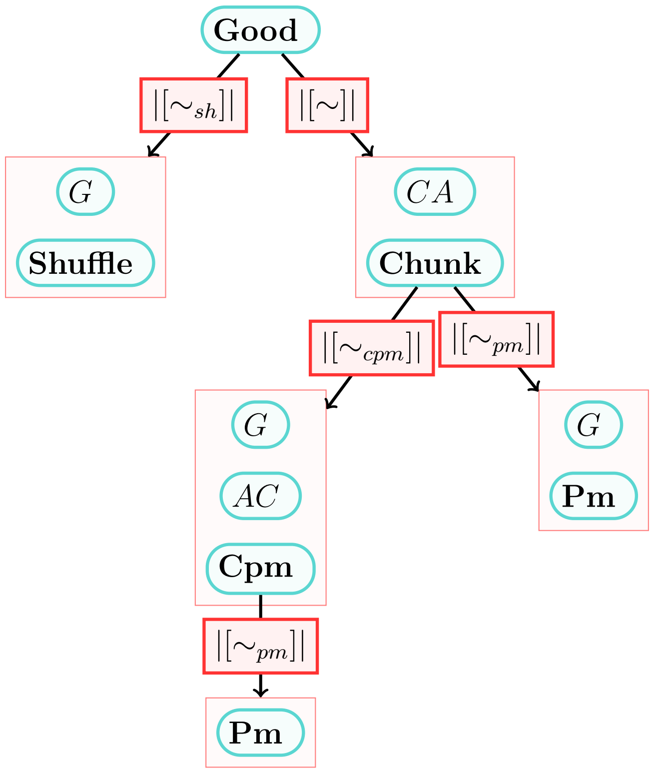


Figure 5.3: Equivalence classes of skeletons



Let $\mathcal{K}(\Sigma)$ be a class of shaped partial orders enumerated by a good skeleton Σ and let $\mathcal{K}'(\Sigma', o)$ be a class of ordered shaped partial orders enumerated by an elementary skeleton Σ'. We summarise the results from Section 5.2 that state that $\mathcal{K}'(\Sigma', o)$ is an order class with respect to $\mathcal{K}(\Sigma)$ and has the ordering property in the tables 5.2 and 5.3. In all cases Chapter 4 contains the proof that $\mathcal{K}'(\Sigma', o)$ is a Ramsey class.

Specific cases of a chunk skeleton are a simple chunk skeleton (Definition 5.2.17), a chunk skeleton (Definition 5.1.18) and a trivial chunk skeleton (Definition 5.2.21).

Table 5.2

$\mathcal{K}(\Sigma)$	$\mathcal{K}'(\Sigma', o)$	**Reasonable**	**OP**
$\mathcal{K}(G, \mathfrak{S})$	$\mathcal{K}(G, \mathfrak{S}, o)$	Lemma 5.2.1	Lemma 5.2.2
$\mathcal{K}(CA, \mathfrak{S}, N)$	$\mathcal{K}(CA, \mathfrak{S}, N, o)$	Lemma 5.2.3	Lemma 5.2.4
$\mathcal{K}(AC, \mathfrak{S}, \{\aleph_0, \aleph_0\})$	$\mathcal{K}(AC, \mathfrak{S}, o)$	Lemma 5.2.5 (i)	Lemma 5.2.14
$\mathcal{K}(AC, \{\mathbf{s}^a\}, \{\aleph_0, 1\})$	$\mathcal{K}(A, \{\mathbf{s}^a\}, o)$	Lemma 5.2.5 (ii)	Lemma 5.2.14
$\mathcal{K}(AC, \{\mathbf{s}\}, \{1, 1\})$	$\mathcal{K}(A_1, \{\mathbf{s}\}, o)$	Lemma 5.2.5 (iii)	trivial
$\mathcal{K}(\Sigma_{(c)pm}, \mathfrak{S}, N)$	$\mathcal{K}(GAC, \mathfrak{S}, N, o)$	Corollary 5.2.13	Lemma 5.2.14
$\mathcal{K}(\Sigma_{(c)pm}, \mathfrak{S}, N)$	$\mathcal{K}(GAC, \mathfrak{S}, N, o)$	Corollary 5.2.13	Lemma 5.2.14
$\mathcal{K}(\Sigma_{(c)pm}, \mathfrak{S}, N)$	$\mathcal{K}(GAC, \mathfrak{S}, N, o)$	Corollary 5.2.13	Lemma 5.2.14

Table 5.3

Σ	Σ'	**Reasonable**	**OP**
(C)pm	GAC	Corollary 5.2.13	Lemma 5.2.14
Simplified chunk	Antichained	Lemma 5.2.16	
Simple chunk	Glorified	Lemma 5.2.18	

Table 5.4

Σ	Σ^*	**Reasonable**
Simplified chunk	Antichained	Lemma 5.2.16
Simple chunk	Glorified	Lemma 5.2.18
Non-trivial chunk	Glorified	Lemma 5.2.22
Shuffle	CA	Lemma 5.2.24
Good	Elementary	Lemma 5.2.26

Definition 5.2.6 defines classes of ordered shaped partial orders enumerated by good skeletons and lemmas 5.2.7 and 5.2.8 provide tools for proving that some classes are simply bi-definable. Using them, the final part of Section 5.2 contains proofs of results of the following kind. Let $\mathcal{K}(\Sigma)$ be a class of shaped partial orders enumerated by the good skeleton Σ. Then there exists an elementary skeleton Σ^* and a class $\mathcal{K}(\Sigma^*, o)$, such that

(i) the classes $\mathcal{K}(\Sigma)$ and $\mathcal{K}(\Sigma^*)$ are simply bi-definable, and

(ii) the class $\mathcal{K}(\Sigma^*, o)$ is a reasonable class with respect to $\mathcal{K}(\Sigma^*)$ and is a Ramsey class.

Table 5.4 summarises the results.

5.1 Substructures of a good skeleton

In this section we analyse the structure of a good skeleton. We divide the good skeleton in smaller components to unravel the extensive list of conditions from Definition 2.4.13.

Relations

Let Σ be an ordered skeleton. To simplify the notation, we will adopt the following. As well as viewing $\{<_g, <_c, <_{cpm}, <_{pm}, <_{sh}\}$ as labels, define relations $<_g$, $<_c$, $<_{cpm}$, $<_{pm}$ and $<_{sh}$ on Σ by $\sigma <_l \tau$ if $\sigma < \tau$ and $l_2(\sigma, \tau) =<_l$, where $l \in \{g, c, cpm, pm, sh\}$. Then by writing $\sigma >_l \tau$ we mean that in the skeleton $\tau < \sigma$ and $l_2(\tau, \sigma) = l$.

Let Σ be a skeleton. We will build up to defining an equivalence relation $\sim$ on Σ as a transitive and symmetric closure of the union of $<_g, <_{sh}, <_{cpm}$ and $<_{pm}$, defining weaker relations on the way.

Recall first that a relation $\sim$ is a reflexive, transitive and symmetric closure of a relation $\sim'$ if it satisfies the following:

- if $\sigma_i \sim' \sigma_j$ then $\sigma_i \sim \sigma_j$ (closure),
- $\sigma_i \sim \sigma_i$ for all σ_i (reflexivity),

- if $\sigma_i \sim \sigma_j$ then $\sigma_j \sim \sigma_i$ (symmetry), and
- if $\sigma_i \sim \sigma_j$ and $\sigma_j \sim \sigma_k$ then $\sigma_i \sim \sigma_k$ (transitivity).

Remark 5.1.1. Suppose that $<_l$ is any relation of arity 2 on a finite structure X. Let $-$ be a symmetric closure of $<_l$ and $\sim$ be a transitive, symmetric and reflexive closure of $<_l$. Then $\sim$ is an equivalence relation. We can view $(X, -)$ as a graph, with $-$ being the edge relation. Then the equivalence classes of $\sim$ correspond exactly to connected components of $(X, -)$. That is, for each x' in the equivalence class $[x]_\sim$ of x, there exist $x_1, x_2, \ldots, x_n \in P$ such that we have

$$x = x_0 - x_1 - \ldots - x_n - x_{n+1} = x'.$$

That, of course, means that for each $i \in [n+1]$ we have $x_{i-1} <_l x_i$ or $x_i <_l x_{i-1}$.

Based on this observation we will define the relevant relations and meta components of a skeleton as follows.

Definition 5.1.2. Define

- $-_{sh}$ as a symmetric closure of $<_{sh}$,
- $-_{pm}$ as a symmetric closure of $<_{pm}$,
- $-_{cpm}$ as a symmetric closure of $<_{cpm}$,
- $-_g$ as a symmetric closure of $<_g$,
- $\sim_{sh}$ as a reflexive, symmetric and transitive closure of $<_{sh}$,
- $\sim_{pm}$ as a reflexive, symmetric and transitive closure of $<_{pm}$,
- $\sim_{cpm}$ as a reflexive, symmetric and transitive closure of $<_{cpm}$,
- $\sigma <' \tau$ if $\sigma < \tau$ and $l_2(\sigma, \tau) \in \{<_g, <_{cpm}, <_{pm}\}$,
- $-$ as a symmetric closure of $<'$, and
- $\sim$ as a reflexive, symmetric and transitive closure of $<'$.

Note. The relations $\sim_{sh}$, $\sim_{pm}$, $\sim_{cpm}$ and $\sim$ are equivalence relations. Let Σ be a skeleton and $\sigma \in \Sigma$. Then denote by $[\![\sigma]\!]_{sh}$, $[\![\sigma]\!]_{pm}$, $[\![\sigma]\!]_{cpm}$ and $[\![\sigma]\!]$ the equivalence classes of σ with respect to $\sim_{sh}$, $\sim_{pm}$, $\sim_{cpm}$ and $\sim$ respectively.

Skeletons

We have already seen the c-condition in the definition of an elementary skeleton (2.5.28). It will apply to good skeletons, alongside the *sh-condition*.

Definition 5.1.3. A skeleton Σ satisfies the *c-condition* if given any $\sigma_1, \sigma_2, \sigma_3 \in \Sigma$ with $\sigma_1 < \sigma_2 < \sigma_3$, then $\sigma_1 <_c \sigma_2$ implies $\sigma_1 <_c \sigma_3$ and $\sigma_2 <_c \sigma_3$ implies $\sigma_1 <_c \sigma_3$.
A skeleton Σ satisfies the *sh-condition* if given any $\sigma_1, \sigma_2, \sigma_3 \in \Sigma$ with $\sigma_1 <_{sh} \sigma_2 <_{sh} \sigma_3$, we have $\sigma_1 <_{sh} \sigma_3$.

Remark 5.1.4. Consider the c-condition. In the 3-chain lemmas part of Definition 2.4.13, we can see that $<_c$ appears in the $l_2(\sigma_1, \sigma_2)$ and $l_2(\sigma_2, \sigma_3)$ columns precisely in parts 1.) and 1.)*. In both cases $l_2(\sigma_1, \sigma_3) =<_c$. So conditions 1.) and 1.)* are equivalent to the c-condition.
Similarly, sh-condition is the part 2.) of the 3-chain lemmas. The sh-condition that shows that $<_{sh}$ is a transitive relation.

Shuffle skeleton

We first define a skeleton with all points being chains of antichains.

Definition 5.1.5. Consider a skeleton $\Sigma = (\Sigma, <, l_1, l_2)$. The skeleton Σ is a *shuffle skeleton*, labelled Σ_{sh}, if

(i) $l_1(\Sigma) \subset \{CA\}$,

(ii) $l_2(<) \subset \{<_{sh}, <_c\}$,

(iii) Σ satisfies the c- and sh-conditions, and

(iv) for each distinct $\sigma, \sigma' \in \Sigma$ there exist σ_i for $i \in [n]$ such that

$$\sigma = \sigma_0 -_{sh} \sigma_1 -_{sh} \dots -_{sh} \sigma_n -_{sh} \sigma_{n+1} = \sigma'$$

Lemma 5.1.6. *Any shuffle skeleton Σ_{sh} is a good skeleton.*

Proof. Since all points of Σ are labelled CA, the relations in Σ are labelled $<_{sh}$ or $<_c$, and Σ satisfies the c- and sh-conditions, we can easily check that Σ satisfies the conditions in definition 2.4.13 of a good skeleton. □

Lemma 5.1.7. *Let Σ be a good skeleton. Then any equivalence class of $\sim_{sh}$ on Σ is one of the following.*

(i) A single point labelled AC or G.

(ii) A shuffle skeleton.

Proof. Take any $\sigma \in \Sigma$. First note that by the Remark 5.1.1, for any $\sigma' \in [\![\sigma]\!]_{sh}$ there exist $\sigma_1, \dots, \sigma_n \in [\![\sigma]\!]_{sh}$ such that

$$\sigma = \sigma_0 -_{sh} \sigma_1 -_{sh} \dots -_{sh} \sigma_n -_{sh} \sigma_{n+1} = \sigma'.$$

So the condition (iv) of the definition of a shuffle skeleton is satisfied by any equivalence class of relation $\sim_{sh}$.
Recall the definition of good skeleton, 2.4.13. By 2-chain lemmas, we see that if $\sigma <_{sh} \sigma'$, we must have $l_1(\sigma) = l_1(\sigma) = CA$. So for any $\sigma \in \Sigma$ with $l_1(\sigma) = AC$ or $l_1(\sigma) = G$, we must have

$$[\![\sigma]\!]_{sh} = \{\sigma\}.$$

This covers case (i) in the lemma 5.1.7.
So consider the case $l_1(\sigma) = CA$. As mentioned above, by 2-chain lemmas all points in $[\![\sigma]\!]_{sh}$ are labelled CA. Also by 2-chain lemmas, a relation between any two points labelled CA is either $<_{sh}$ or $<_c$. So $[\![\sigma]\!]_{sh}$ satisfies conditions (i) and (ii) of the definition of a shuffle skeleton. The parts 1.) and 1.)* of the 3-chain lemmas imply that a good skeleton (and thus $[\![\sigma]\!]_{sh}$) satisfies the c-condition, and the part 2.) implies it satisfies the sh-condition. Thus $[\![\sigma]\!]_{sh}$

is indeed a shuffle skeleton. This completes the proof. □

Chunk skeleton

Definition 5.1.8. Consider a skeleton $\Sigma = (\Sigma, <, l_1, l_2)$. The Σ is a *chunk skeleton*, labelled Σ_{ch}, if

(i) $l_1(\Sigma) \subset \{G, AC\}$

(ii) Σ is a good skeleton, and

(iii) for each distinct $\sigma, \sigma' \in \Sigma$ there exist σ_i for $i \in [n]$ such that

$$\sigma = \sigma_0 - \sigma_1 - \ldots - \sigma_n - \sigma_{n+1} = \sigma'.$$

Remark 5.1.9. Suppose that Σ is a chunk skeleton, and Σ' is a substructure of Σ. Then Σ' satisfies parts (i) and (ii) of Definition 5.1.8 - trivially for (i) and since any substructure of a good skeleton is a good skeleton.

Lemma 5.1.10. *Let Σ be a good skeleton. Then any equivalence class of $\sim$ on Σ is one of the following.*

- *A single point labelled CA.*
- *A chunk skeleton.*

Proof. Since Σ is a good skeleton, any substructure of Σ is a good skeleton, satisfying condition (ii) of Definition 5.1.8.
Take any $\sigma \in \Sigma$. Again, by Remark 5.1.1, the equivalence class $[\![\sigma]\!]$ satisfies part (iii) of Definition 5.1.8.
Now by definition of $-$, if $\sigma < \sigma'$ and $\sigma - \sigma'$, then the label $l_2(\sigma, \sigma')$ is $<_g, <_{pm}$ or $<_{cpm}$. By 2-chain lemmas in Definition 2.4.13 that means that neither of the σ, σ' is a CA.
So if σ is a CA, $[\![\sigma]\!]$ is a single point. Otherwise $[\![\sigma]\!]$ satisfies condition (i) of Definition 5.1.8. It also satisfies the other two conditions, and is thus a chunk skeleton. □

Corollary 5.1.11. *Let Σ be a good skeleton. Then there is a partition of Σ,*

$$\Sigma = \left(\bigcup_{t \in Sh} \Sigma_t \right) \cup \left(\bigcup_{z \in Ch} \Sigma_z \right),$$

such that the following hold.

(i) For each $t \in Sh$, Σ_t is a shuffle skeleton.

(ii) For each $z \in Ch$, Σ_z is a chunk skeleton.

(iii) For any distinct $x, y \in Sh \cup Ch$ and $\sigma \in \Sigma_x$, $\sigma' \in \Sigma_y$, if $\sigma < \sigma'$ in Σ, then $\sigma <_c \sigma'$.

We refer to the partition above as the good partition.

Proof. This is a direct consequence of lemmas 5.1.7 and 5.1.10.
To get $\{\Sigma_t : t \in Sh\}$, take the equivalence classes $[\![\sigma]\!]_{sh}$ of all $\sigma \in \Sigma$ with $l_1(\sigma) = CA$. To get $\{\Sigma_u : u \in Ch\}$, take the equivalence classes $[\![\sigma]\!]$ of all $\sigma \in \Sigma$ with $l_1(\sigma) = AC$ or $l_1(\sigma) = G$.
Using 2-chain conditions of a good skeleton (2.4.13), for any $\sigma < \sigma'$, we have

(i) if $\sigma <_{sh} \sigma'$, then $l_1(\sigma) = l_1(\sigma') = CA$ and $\sigma' \in [\![\sigma]\!]_{sh}$, so for some $t \in Sh$ we have $\sigma, \sigma' \in \Sigma_t$, and

(ii) if $\sigma <_{pm} \sigma'$, $\sigma <_{cpm} \sigma'$ or $\sigma <_g \sigma'$, then $l_1(\sigma), l_1(\sigma') \in \{AC, G\}$ and $\sigma' \in [\![\sigma]\!]$, so for some $z \in Ch$ we have $\sigma, \sigma' \in \Sigma_z$.

So if $\sigma \in \Sigma_x$, $\sigma' \in \Sigma_y$ for some distinct $x, y \in Sh \cup Ch$, we must have $\sigma <_c \sigma'$, proving condition (iii). □

Matching skeletons

We will consider the equivalence classes of relations $\sim_{pm}$ and $\sim_{cpm}$. We start by defining skeletons, and then showing that the structures in the equivalence classes correspond to the skeletons.

Definition 5.1.12. Consider a skeleton $\Sigma = (\Sigma, <, l_1, l_2)$.
The skeleton Σ is a *pm-skeleton*, labelled Σ_{pm}, if

(i) $l_1(\Sigma) \subset \{AC\}$,

(ii) $l_2(<) \subset \{<_{pm}\}$, and

(iii) $(\Sigma, <)$ is a chain.

The skeleton Σ is a *cpm-skeleton*, labelled Σ_{cpm}, if

(i) there are disjoint sets Π_1 and Π_2, and $\Pi = \Pi_1 \cup \Pi_2$,

(ii) $\Sigma = \{\sigma_\pi : \pi \in \Pi\}$,

(iii) the substructures of Σ on the subsets $\Sigma_{cpm,1} = \{\sigma_\pi : \pi \in \Pi_1\}$ and $\Sigma_{cpm,2} = \{\sigma_\pi : \pi \in \Pi_2\}$ are both pm-skeletons, and

(iv) for all $\pi_1 \in \Pi_1, \pi_2 \in \Pi_2$, $\sigma_{\pi_1} <_{cpm} \sigma_{\pi_2}$ (and thus $\sigma_{\pi_1} < \sigma_{\pi_2}$).

The skeleton Σ is a *(c)pm-skeleton* if it is a pm-skeleton or a cpm-skeleton.

Lemma 5.1.13. *Any (c)pm-skeleton is a good skeleton.*

Proof. By definition, a (c)pm skeleton is a chain, so Λ- and V-shape lemmas are irrelevant. A (c)pm-skeleton clearly satisfies part 4.) of the 2-chain lemmas, and parts 3.), 4.) and 4.)* of the 3-chain lemmas. Since $l_2(<) \subset \{<_{pm}, <_{cpm}\}$ for any (c)pm skeleton, those are the only parts of the Definition 2.4.13 that apply. So any (c)pm skeleton is indeed a good skeleton. $\square$

Lemma 5.1.14. *Let Σ be a chunk skeleton. Then any equivalence class of $\sim_{pm}$ on Σ is one of the following.*

- *A single point labelled G.*

- *A pm-skeleton.*

Proof. By part 4.) of the 2-chain lemmas in 2.4.13, if $\sigma <_{pm} \sigma'$ for some $\sigma, \sigma' \in \Sigma$, then $l_1(\sigma) = l_1(\sigma') = AC$, showing that part (i) of the definition of a pm-skeleton (5.1.12). By Remark 5.1.1, if $\sigma' \in [\![\sigma]\!]_{pm}$ and $\sigma' \neq \sigma$, then there exist σ_i for $i \in [n]$ such that

$$\sigma = \sigma_0 -_{pm} \sigma_1 -_{pm} \dots -_{pm} \sigma_n -_{pm} \sigma_{n+1} = \sigma'.$$

We will show that $\sigma -_{pm} \sigma'$. That means that either $\sigma <_{pm} \sigma'$ or $\sigma' <_{pm} \sigma$, showing that indeed parts (ii) and (iii) of the definition of a pm-skeleton hold. Suppose that for some $\sigma_a, \sigma_b, \sigma_c$ in Σ we have

$$\sigma_a -_{pm} \sigma_b -_{pm} \sigma_c.$$

Since no two relations in any part of the V- and Λ-shape lemmas are labelled $<_{pm}$, the 3-chain lemmas apply to $\sigma_a, \sigma_b, \sigma_c$. Two labels $<_{pm}$ only appear in part 3.), so the structure on $\{\sigma_a, \sigma_b, \sigma_c\}$ must be a chain, with all relations labelled $<_{pm}$. Thus we must have $\sigma_a -_{pm} \sigma_c$.
That, of course, means that for the chain $\sigma_0 -_{pm} \sigma_1 -_{pm} \dots -_{pm} \sigma_n -_{pm} \sigma_{n+1}$ above we have $\sigma_0 -_{pm} \sigma_2$, and thus $\sigma_0 -_{pm} \sigma_3$ and eventually $\sigma_0 -_{pm} \sigma_{n+1}$.
So, if $l_1(\sigma) = AC$, the substructure of Σ on the equivalence class $[\![\sigma]\!]_{pm}$ is a pm-skeleton. Otherwise $[\![\sigma]\!]_{pm}$ is a singleton labelled G, concluding the proof. □

Lemma 5.1.15. *Let Σ be a chunk skeleton. Then any equivalence class of $\sim_{cpm}$ on Σ is one of the following.*

- *A single point labelled G or AC.*
- *A cpm-skeleton.*

Further, if $[\![\sigma]\!]_{cpm}$ is a cpm-skeleton and $\sigma' \in [\![\sigma]\!]_{cpm}$, then $[\![\sigma']\!]_{pm} \subset [\![\sigma]\!]_{cpm}$.

Proof. Suppose that there are $\sigma_a, \sigma_b, \sigma_c \in \Sigma$ such that

$$\sigma_a -_{cpm} \sigma_b -_{cpm} \sigma_c.$$

Suppose further that $\sigma_a <_{cpm} \sigma_b$. Then the V- and Λ-shape lemmas again do not apply, and only parts 4.) and 4.)* of 3-chain lemmas apply, showing that

$\sigma_c <_{cpm} \sigma_b$ and $\sigma_a -_{pm} \sigma_c$.
Otherwise we have $\sigma_a >_{cpm} \sigma_b$, which leads to $\sigma_b <_{cpm} \sigma_c$ and again $\sigma_a -_{pm} \sigma_c$.

By Remark 5.1.1, if $\sigma' \in [\![\sigma]\!]_{cpm}$ and $\sigma' \neq \sigma$, then there exist σ_i for $i \in [n]$ such that

$$\sigma = \sigma_0 -_{cpm} \sigma_1 -_{cpm} \ldots -_{cpm} \sigma_n -_{cpm} \sigma_{n+1} = \sigma'.$$

Now by part 4.) of the 2-chain lemmas in 2.4.13, if $\sigma_a <_{cpm} \sigma_b$ for some $\sigma_a, \sigma_b \in \Sigma$, then $l_1(\sigma_a) = l_1(\sigma_b) = AC$, so

$$l_1(\sigma) = l_1(\sigma_1) = \ldots = l_1(\sigma_n) = l_1(\sigma') = AC,$$

showing that part (i) of the definition of a cpm-skeleton (5.1.12) is true for any equivalence class $[\![\sigma]\!]_{cpm}$ with $|[\![\sigma]\!]_{cpm}| > 1$.
Suppose that $\sigma <_{cpm} \sigma_1$. Then by the comments in the beginning of this proof, we must have $\sigma_2 <_{cpm} \sigma_1$ and $\sigma -_{pm} \sigma_2$. Considering σ_1, σ_2 and σ_3, we see that $\sigma_2 <_{cpm} \sigma_3$ and $\sigma_1 -_{pm} \sigma_3$. But then considering σ, σ_2 and σ_3, we see that $\sigma <_{cpm} \sigma_3$. Applying the argument repeatedly we show

$$\sigma <_{cpm} \sigma_{2k-1} \quad \text{and} \quad \sigma -_{pm} \sigma_{2k}$$

for all $k > 0$. So if n is even we have $\sigma <_{cpm} \sigma'$ and if n is odd, we have $\sigma -_{pm} \sigma'$. Further

$$\sigma_{2l} <_{cpm} \sigma_{2k-1}, \quad \sigma_{2l} -_{pm} \sigma_{2k} \quad \text{and} \quad \sigma_{2l+1} -_{pm} \sigma_{2k+1}$$

for all $k, l \geq 0$. Further, if $\sigma -_{cpm} \sigma_1'$, then by considering σ, σ_1 and σ_1' we have $\sigma <_{cpm} \sigma_1'$ and $\sigma_1 -_{pm} \sigma_1'$. So in fact for any $\sigma'' \in [\![\sigma]\!]_{cpm}$ we must have

$$\sigma <_{cpm} \sigma'' \quad \text{or} \quad \sigma -_{pm} \sigma''.$$

Let

$$\Pi_1 = \{\sigma' : \sigma' \in [\![\sigma]\!]_{cpm}, \sigma' -_{pm} \sigma \text{ or } \sigma' = \sigma\} \text{ and}$$

$$\Pi_2 = \{\sigma' : \sigma' \in [\![\sigma]\!]_{cpm}, \sigma <_{cpm} \sigma'\}.$$

Then by the reasoning above, we have $\Pi_1 = [\![\sigma]\!]_{pm}$ and for any $\sigma' \in \Pi_2$ we

have $\Pi_2 = [\![\sigma']\!]_{pm}$, and for any $\sigma'' \in \Pi_1$ and $\sigma' \in \Pi_2$ we have $\sigma'' <_{cpm} \sigma'$. This shows part (iii) of definition of a cpm-skeleton in Definition 5.1.12 (that the substructures of $[\![\sigma]\!]_{cpm}$ on the subsets Π_1 and Π_2 are both pm-skeletons) and part (iv) of definition of a cpm-skeleton.
We proceed analogously if $\sigma_1 <_{cpm} \sigma$, but in that case we set

$$\Pi_1 = \{\sigma' : \sigma' \in [\![\sigma]\!]_{cpm}, \sigma' <_{cpm} \sigma\} \text{ and}$$

$$\Pi_2 = \{\sigma' : \sigma' \in [\![\sigma]\!]_{cpm}, \sigma' -_{pm} \sigma \text{ or } \sigma' = \sigma\}.$$

So if $[\![\sigma]\!]_{cpm}$ with $|[\![\sigma]\!]_{cpm}| > 1$, $[\![\sigma]\!]_{cpm}$ is a cpm skeleton. Otherwise $|[\![\sigma]\!]_{cpm}| = 1$.
Further, if $[\![\sigma]\!]_{cpm}$ is a cpm-skeleton and $\sigma' \in [\![\sigma]\!]_{cpm}$, then $[\![\sigma']\!]_{pm} \subset [\![\sigma]\!]_{cpm}$, as $[\![\sigma']\!]_{pm}$ is either the set Π_1 or the set Π_2. $\square$

Corollary 5.1.16. *Let Σ_{ch} be a chunk skeleton. Then there is a partition of Σ_{ch},*

$$\Sigma_{ch} = \left(\bigcup_{v\in Pm} \Sigma_v\right) \cup \left(\bigcup_{w\in Cpm} \Sigma_w\right) \cup \left(\bigcup_{\gamma\in\Gamma} \sigma_\gamma\right),$$

such that the following hold.

(i) For each $v \in Pm$, Σ_v is a pm-skeleton.

(ii) For each $w \in Cpm$, Σ_w is a cpm-skeleton.

(iii) For each $\gamma \in \Gamma$, σ_γ is labelled G.

(iv) For any distinct $x, y \in Pm \cup Cpm \cup \Gamma$ and

$$\sigma \in \Sigma_x \text{ or } \sigma = \sigma_x, \quad \sigma' \in \Sigma_y \text{ or } \sigma = \sigma_y,$$

if $\sigma < \sigma'$ in Σ, then $\sigma <_g \sigma'$ or $\sigma <_c \sigma'$.

We refer to the partition above as the chunk partition.

Remark 5.1.17. Let $(C)pm = Pm \cup Cpm$. Then

$$\Sigma_{ch} = \left(\bigcup_{u \in (C)pm} \Sigma_u \right) \cup \left(\bigcup_{\gamma \in \Gamma} \sigma_\gamma \right),$$

where for each $u \in (C)pm$, Σ_u is a (c)pm-skeleton.

Proof. To get $\{\Sigma_u : u \in (C)pm\}$, take the equivalence classes $[\![\sigma]\!]_{cpm}$ with $|[\![\sigma]\!]_{cpm}| > 1$, which are cpm-skeletons by the proof of Lemma 5.1.15.
Now consider the equivalence classes $[\![\sigma]\!]_{pm}$ for $\sigma \in \Sigma$. By Lemma 5.1.14, $[\![\sigma]\!]_{pm}$ is either a pm-skeleton or a single point labelled G. We've already shown in Lemma 5.1.15 that if $\sigma' \in [\![\sigma]\!]_{cpm}$ and $|[\![\sigma]\!]_{cpm}| > 1$, then $[\![\sigma']\!]_{pm} \subset [\![\sigma]\!]_{cpm}$. Let

$$\Sigma' = \Sigma \setminus \left(\bigcup_{|[\![\sigma]\!]_{cpm}|>1} [\![\sigma]\!]_{cpm} \right).$$

To get $\{\Sigma_w : w \in Cpm\}$, take $[\![\sigma]\!]_{pm}$ for $\sigma \in \Sigma'$ with $l_1(\sigma) = AC$. Then the remaining points in Σ are labelled G, proving parts (i), (ii) and (iii) of Corollary 5.1.16.
Now if $\sigma, \sigma' \in \Sigma$, then if $\sigma < \sigma'$, we have one of the following.

(i) If $\sigma <_{pm} \sigma'$, then $\sigma' \in [\![\sigma]\!]_{pm}$, so $\sigma, \sigma' \in \Sigma_u$ for some $u \in (C)pm$.

(ii) If $\sigma <_{cpm} \sigma'$, then $\sigma' \in [\![\sigma]\!]_{cpm}$, so $\sigma, \sigma' \in \Sigma_w$ for some $w \in Cpm$.

(iii) Otherwise $\sigma <_g \sigma'$ or $\sigma <_c \sigma'$.

This shows part (iv) of Corollary 5.1.16 and concludes the proof. □

Definition 5.1.18. Let Σ_{ch} be a chunk skeleton with a chunk partition

$$\Sigma_{ch} = \{\Sigma_u : u \in (C)pm\} \cup \{\sigma_\gamma : \gamma \in \Gamma\}.$$

For each $u \in (C)pm$ let σ_u be the least point in the (c)pm-skeleton Σ_u.

The substructure Σ of Σ_{ch} on the set

$$\Sigma = \{\sigma_u : u \in (C)pm\} \cup \{\sigma_\gamma : \gamma \in \Gamma\}$$

is a *simplified skeleton of* Σ_{ch}.

Lemma 5.1.19. *Let* $\Sigma = (\Sigma, <, l_1, l_2)$ *be a simplified skeleton of a chunk skeleton* Σ_{ch}*. Then* $l_2(<) \subset \{<_c, <_g\}$*.*

Proof. This is true because of the part (iv) of Corollary 5.1.16. □

5.2 Good skeleton with a total order

In the introduction we defined a good skeleton (2.4.13) and the related good skeleton with a total order (2.5.8). But then we defined specific ordered skeletons (2.5.9) and considered them in Chapter 3. We started this chapter by considering substructures of a good skeleton, so now we have the vocabulary necessary to link the ordered skeletons to good skeletons with total orders and prove Theorem 2.5.31.

One component

First, recall the following classes, with the numbers of their definitions in brackets in the table below.

label	**class** (2.2.10)	**order class**
G	$\mathcal{K}(G, \mathfrak{S})$	$\mathcal{K}(G, \mathfrak{S}, o)$ (2.5.11)
CA	$\mathcal{K}(CA, \mathfrak{S}, N)$	$\mathcal{K}(CA, \mathfrak{S}, N, o)$ (2.5.15)
AC	$\mathcal{K}(AC, \mathfrak{S}, \{\aleph_0, \aleph_0\})$	$\mathcal{K}(AC, \mathfrak{S}, o)$ (2.5.22)
	$\mathcal{K}(AC, \{\mathrm{s}\}, \{\aleph_0, 1\})$	$\mathcal{K}(A, \{\mathrm{s}\}, o)$ (2.5.22)
	$\mathcal{K}(AC, \{\mathrm{s}\}, \{1, 1\})$	$\mathcal{K}(A_1, \{\mathrm{s}\}, o)$ (2.5.10)

Recall Definition 2.2.10, of a reasonable class $\mathcal{K}$ with respect to a class $\mathcal{K}_0$. We will show that the order classes in the third column of the table above are reasonable with respect to the corresponding class in the second column of the table above.

We also show that the classes $\boldsymbol{\mathcal{K}}(G, \mathfrak{S}, o)$ and $\boldsymbol{\mathcal{K}}(CA, \mathfrak{S}, o)$ have the ordering property. The proof that the classes of antichains of chains have ordering property will be included in the Matching Skeletons sections, in Lemma 5.2.14.

If Σ is a good skeleton of size 1, with $\Sigma = \{\sigma\}$, we will define the class $\boldsymbol{\mathcal{K}}(\sigma, o)$ to be one of the order classes above. When omitting information about the shapes and multiplicities, we will abbreviate the classes as follows.

(i) $\boldsymbol{\mathcal{K}}(G, \mathfrak{S}, o)$ abbreviates as $\boldsymbol{\mathcal{K}}(G, o)$.

(ii) $\boldsymbol{\mathcal{K}}(CA, \mathfrak{S}, N, o)$ abbreviates as $\boldsymbol{\mathcal{K}}(CA, o)$

(iii) $\boldsymbol{\mathcal{K}}(AC, \mathfrak{S}, o)$, $\boldsymbol{\mathcal{K}}(A, \{\mathbf{s}\}, o)$ and $\boldsymbol{\mathcal{K}}(A_1, \{\mathbf{s}\}, o)$ all abbreviate as $\boldsymbol{\mathcal{K}}(AC, o)$.

Generic

The class $\boldsymbol{\mathcal{K}}(G, \mathfrak{S})$ is defined in Definition 2.4.8 and the class $\boldsymbol{\mathcal{K}}(G, \mathfrak{S}, o)$ in 2.5.11.

Lemma 5.2.1. *The class $\boldsymbol{\mathcal{K}}(G, \mathfrak{S}, o)$ is a reasonable class with respect to the class $\boldsymbol{\mathcal{K}}(G, \mathfrak{S})$.*

Proof. By parts (i) and (ii) of the definition of $\boldsymbol{\mathcal{K}} = \boldsymbol{\mathcal{K}}(G, \mathfrak{S}, o)$, we obtain any $(P, <, \prec, \mathfrak{s}) \in \boldsymbol{\mathcal{K}}$ from a structure $(P, <, \mathfrak{s})$ in the class $\boldsymbol{\mathcal{K}}_0 = \boldsymbol{\mathcal{K}}(G, \mathfrak{S})$ and a chain $\mathcal{I}$. By parts (iv) and (vi), the relation $\prec$ is a total order on P, since $\mathcal{I}$ is a chain. So $\boldsymbol{\mathcal{K}}$ is an order class for $\prec$.
Further, the class $\boldsymbol{\mathcal{K}}_0$ is indeed the class of reducts of $\boldsymbol{\mathcal{K}}$. Indeed, taking any $(P, <, \mathfrak{s})$, let $\prec$ be any total order extending the partial order $(P, <)$. Then $(P, \prec)$ is a finite total order. Thus if n is the size of P, there is an isomorphism

$$\iota : P \to [n], \text{ such that } p \prec q \iff \iota(p) < \iota(q).$$

So label any point $q \in P$ as $p_{\iota(q)}$. Then

$$P = \{p_i : i \in [n]\}$$

and the structure $(P, <, \mathfrak{s})$ and the chain $[n]$ with a natural order satisfy conditions (iv)-(vi) of Definition 2.5.11. Thus $(P, <, \prec, \mathfrak{s}) \in \boldsymbol{\mathcal{K}}$ and thus any structure in $\boldsymbol{\mathcal{K}}_0$ is a reduct of a structure in $\boldsymbol{\mathcal{K}}$.

Now take any $\mathbf{P}_0, \mathbf{R}_0 \in \boldsymbol{\mathcal{K}}_0$, such that $\mathbf{P}_0 = (P, <^{\mathbf{P}_0}, \mathfrak{s}^{\mathbf{P}_0})$, $\mathbf{R}_0 = (R, <^{\mathbf{R}_0}, \mathfrak{s}^{\mathbf{R}_0})$ and there exists an embedding

$$e : \mathbf{R}_0 \to \mathbf{P}_0.$$

Further, take any $\mathbf{R} = (R, <^{\mathbf{R}_0}, \prec, \mathfrak{s}^{\mathbf{R}_0}) \in \boldsymbol{\mathcal{K}}$.
For each $p, q \in R$, set

$$e(p) \prec_1 e(q) \iff p \prec q.$$

This defines a total order on a substructure $e(\mathbf{R}_0)$ of $\mathbf{P}_0$. Since e is an embedding, $\prec_1$ extends the partial order $<^{\mathbf{P}_0}$ on the substructure $e(\mathbf{R}_0)$ of $\mathbf{P}_0$ as well. Define $\prec'$ on P so that for $p, q \in P$

(i) if $p <^{\mathbf{P}_0} q$ then $p \prec' q$, and

(ii) if $p \prec_1 q$ then $p \prec' q$.

The partial order exists, as we can start with the relation $<^{\mathbf{P}_0} \cup \prec_1$ and for an incomparable pair in that relation adding the pair and all the comparisons implied by transitivity to the relation, until we've constructed a total order. Then let $\mathbf{P} = (P, <^{\mathbf{P}_0}, \prec', \mathfrak{s}^{\mathbf{P}_0})$. Clearly, the map e defines an embedding

$$e : \mathbf{R} \to \mathbf{P}$$

as well. This finishes the proof. □

Lemma 5.2.2. *The class $\boldsymbol{\mathcal{K}}(G, \mathfrak{S}, o)$ has the ordering property.*

Proof. We have shown in Lemma 5.2.1 that the class $\boldsymbol{\mathcal{K}}(G, \mathfrak{S}, o)$ is a reasonable class with respect to the class $\boldsymbol{\mathcal{K}}(G, \mathfrak{S})$. Also, by part (v) of Definition 2.5.11, for any $(P, <, \prec, \mathfrak{s}) \in \boldsymbol{\mathcal{K}}(G, \mathfrak{S}, o)$, the total order $(P, \prec)$ is an extension of the partial order $(P, <)$.
Recall Definition 4.1.1. When the skeleton Σ in the definition only has one point, we can simplify the notation to the following.
Let $A = |\mathfrak{S}|$ and let $X = [A]$. For the class $\mathcal{K}(G, e)$, define a weak A-blowup

as an ordered partial order on the set

$$\overline{P} = \{(p_i, a) : (i, a) \in \mathcal{I}^{\mathbf{P}} \times [A]\}$$

with a partial order defined as

$$(p_i, a) < (p_{i'}, a'),$$

if

(i) $p_i < p_{i'}$ in P, or

(ii) $i = i'$ and $a < a'$;

and the total order defined as

$$(p_i, a) \prec (p_{i'}, a'),$$

if

(i) $p_i \prec p_{i'}$ in P, or

(ii) $i = i'$ and $a < a'$.

The maps α and β are just the maps $\alpha : \mathbf{s}^a \mapsto a$ and $\beta : a \mapsto \mathbf{s}^a$.
We have shown in Lemma 4.1.3 that $\boldsymbol{\mathcal{K}}(G, \mathfrak{S}, o)$ admits $([A], \alpha, \beta)$-blowups. In this proof we abbreviate this to A-blowup.
Recall the definition of that concept, Definition 3.4.2.
Now, let $\mathfrak{S}'$ be any set of shapes of size $2A - 1$. Similarly, $\boldsymbol{\mathcal{K}}(G, \mathfrak{S}', o)$ admits $(2A - 1)$-blowups.
Take any $\mathbf{P}_0 = (P, <, \mathfrak{s}) \in \boldsymbol{\mathcal{K}}(G, \mathfrak{S})$.
Then for any $\mathbf{P} = (P, <, \prec, \mathfrak{s}) \in \boldsymbol{\mathcal{K}}(G, \mathfrak{S}, o)$, let $\overline{P} = (\overline{P}, <, \prec)$ be the $(2A-1)$-blowup of $\mathbf{P} = (P, <, \prec, \mathfrak{s})$. Then $\overline{P} \in \boldsymbol{\mathcal{K}}(G, e)$.
Let $\overline{P}_0 = (\overline{P}, <)$. Since the class $\mathcal{K}(G, e)$ has the ordering property (see Theorem 2.2.20), there exists a structure $(R, <) \in \mathcal{K}(G)$, such that for any

$$(\overline{P}, <, \prec') \in \mathcal{K}(G, e), (R, <, \prec') \in \mathcal{K}(G, e)$$

there exists an embedding

$$e : (\overline{P}, <, \prec') \to (R, <, \prec').$$

Take $\mathbf{s} \in \mathfrak{S}$ and consider the any ordered $\mathbf{s}$-shaped partial order $\mathbf{R}$ with the universe R and reduct $(R, <)$ - to get it just pick any extension $(R, \prec')$ of $(R, <)$ that is a total order. Then by part (i) of Definition 3.4.2, the weak A-blowup $\overline{R}$, together with a map

$$s : \overline{R} \to \mathfrak{S}, (p, a) \mapsto \mathbf{s}^a$$

is an ordered shaped partial order $\overline{\mathbf{R}}' = (\overline{R}, <, \prec', s) \in \boldsymbol{\mathcal{K}}(G, \mathfrak{S}, o)$.

We will show that for

$$\text{any } \mathbf{P} = (P, <, \prec, \mathfrak{s}) \in \boldsymbol{\mathcal{K}}(G, \mathfrak{S}, o) \text{ and any } \overline{\mathbf{R}} = (\overline{R}, <, \prec, s) \in \boldsymbol{\mathcal{K}}(G, \mathfrak{S}, o),$$

there is an embedding

$$e : \mathbf{P} = (P, <, \prec, \mathfrak{s}) \to \overline{\mathbf{R}} = (\overline{R}, <, \prec, s).$$

Consider first the substructure $\overline{R}_1$ of $\overline{R}$ on the set of points

$$\{(r_i, 1) : i \in \mathcal{I}^{\mathbf{R}}\}.$$

By definition, it is isomorphic to $(R, <)$. Thus for any

$$(\overline{P}, <, \prec') \in \mathcal{K}(G, e), (\overline{R}, <, \prec'') \in \mathcal{K}(G, e)$$

there exists an embedding

$$e : (\overline{P}, <, \prec') \to (\overline{R}, <, \prec'').$$

So take

$$\text{any } \mathbf{P} = (P, <, \prec, \mathfrak{s}) \in \boldsymbol{\mathcal{K}}(G, \mathfrak{S}, o) \text{ and any } \overline{\mathbf{R}} = (\overline{R}, <, \prec, s) \in \boldsymbol{\mathcal{K}}(G, \mathfrak{S}, o).$$

Let $(\overline{P}, <, \prec)$ be the weak A-blowup of $\mathbf{P}$. Then there exists an embedding

$$e : (\overline{P}, <, \prec) \to (\overline{R}, <, \prec).$$

Consider the substructure $\overline{P}_A$ of $\overline{P}$ on the set of points

$$\{(p_i, A) : i \in \mathcal{I}^{\mathbf{P}}\}.$$

Suppose that for each $i \in \mathcal{I}^{\mathbf{R}}$ we have $e(p_i) = (r_{f(i)}, a)$ for some $f(i) \in \mathcal{I}^{\mathbf{R}}, a \in [A]$. Consider any pair $i, i' \in \mathcal{I}^{\mathbf{P}}$, where $i < i'$. Clearly

$$e(p_i, A) = (r_{f(i)}, a) < e(p_{i'}, A) = (r_{f(i')}, a')$$

since e is an embedding. We will show that

$$(r_{f(i)}, A) \prec (r_{f(i')}, 1).$$

By definition of a weak blowup and an embedding we must have

$$e(p_i, A) < e(p_i, A{+}1) < \ldots < e(p_i, 2A{-}1), \quad e(p_{i'}, 1) < e(p_{i'}, 2) < \ldots < e(p_{i'}, A)$$

We must also have

$$e(p_i, A) \prec e(p_i, A{+}1) \prec \ldots \prec e(p_i, 2A{-}1) \prec e(p_{i'}, 1) \prec e(p_{i'}, 2) \prec \ldots < e(p_{i'}, A).$$

We also know that by definition of the weak blowup, while the total order $\prec$ on $\overline{R}$ can be any total order extending the partial order on it, the partial order on $\overline{R}$ tells us that

$$(r_{f(i)}, a) < (r_{f(i)}, a+1) < \ldots < (r_{f(i)}, A)$$

and

$$(r_{f(i')}, 1) < (r_{f(i')}, 2) < \ldots < (r_{f(i')}, a').$$

It also tells us that if $r' \in \overline{R}$, then if

$$(r_{f(i)}, a) < r' < (r_{f(i)}, A)$$

we must have $r' = (r_{f(i)}, a'')$ for some $a < a'' < A$, and if

$$(r_{f(i')}, 1) < r' < (r_{f(i')}, a'),$$

$r' = (r_{f(i')}, a'')$ for some $1 < a'' < a'$. Thus we must have

$$(r_{f(i)}, A) \preceq e(p_i, 2A-1), \quad e(p_{i'}, 1) \preceq (r_{f(i')}, 1).$$

Thus, indeed $(r_{f(i)}, A) \prec (r_{f(i')}, 1)$. Suppose that

$$\mathfrak{s} : P \to \mathfrak{S}, \quad p_i \mapsto \mathbf{s}^{t(i)}.$$

Then an easy check shows that the substructure of $\overline{\mathbf{R}}$ on the subset

$$\{(r_{f(i)}, t(i)) : i \in \mathcal{I}^{\mathbf{P}}\}$$

is indeed isomorphic to $\mathbf{P}$ and

$$e : \mathbf{P} \to \overline{\mathbf{R}}, \quad p_i \mapsto (r_{f(i)}, t(i))$$

is an embedding. □

Chain of antichains

The class $\mathcal{K}(CA, \mathfrak{S}, N)$ is defined in Definition 2.4.8 and the class $\mathcal{K}(CA, \mathfrak{S}, N, o)$ in 2.5.15.

Lemma 5.2.3. *The class $\mathcal{K}(CA, \mathfrak{S}, N, o)$ is a reasonable class with respect to the class $\mathcal{K}(CA, \mathfrak{S}, N)$.*

Proof. Considering the class $\mathcal{K} = \mathcal{K}(CA, \mathfrak{S}, N, o)$ we have the following.

(i) A total order $\mathcal{A}$ and, for each $a \in \mathcal{A}$, a total order $\mathcal{B}_a$.

(ii) A set $\mathfrak{S}$ of shapes, such that

$$\mathfrak{S} = \{\mathbf{s}^{a,b} : (a, b) \in \mathcal{A} \rtimes \mathcal{B}\}.$$

Let also

$$\mathfrak{S}_a = \{\mathbf{s}^{a,b} : b \in \mathcal{B}_a\}.$$

(iii) For each $(a, b) \in \mathcal{A} \rtimes \mathcal{B}$ a number $n_{a,b} \in \{1, \aleph_0\}$, $N_a = \{n_{a,b} : b \in \mathcal{B}_a\}$, and $N = \{n_{a,b} : (a, b) \in \mathcal{A} \rtimes \mathcal{B}\}$.

(iv) For each $a \in \mathcal{A}$, a class $\boldsymbol{\mathcal{K}}(GA, \mathfrak{S}_a, N_a, o)$ of glorified chains.

(v) A class $\boldsymbol{\mathcal{K}}(C, \mathcal{A})$ of $\mathcal{A}$-shaped chains.

Pick an $a \in \mathcal{A}$. The class $\boldsymbol{\mathcal{K}}(GA, \mathfrak{S}_a, N_a, o)$ is defined in 2.5.13. Consider a structure $\mathbf{P} = (P, <, \prec, \mathfrak{s}) \in \boldsymbol{\mathcal{K}}(GA, \mathfrak{S}_a, N_a, o)$. By part (viii) of the definition, $(P, <)$ is an antichain. It is $\mathfrak{S}_a$-shaped, and has at most $n_{a,b}$ points of shape $\mathbf{s}^{a,b}$ for each $\mathbf{s}^{a,b} \in \mathfrak{S}_a$ by part (vii) of the definition. That is, it satisfies part (ii)(c) of Definition 2.4.8.
Now consider any $\mathbf{P} = (P, <, \prec, \mathfrak{s}) \in \boldsymbol{\mathcal{K}}$. By Definition 2.5.15, $\mathbf{P}$ is defined by an $\mathcal{A}$-shaped chain $\mathcal{I}$ and, for each $i \in \mathcal{I}$, a structure $\mathbf{P}_i \in \boldsymbol{\mathcal{K}}(GA, \mathfrak{S}_a, N_a, o)$, where $a = \mathfrak{s}^{\mathcal{I}}(i)$. Part (ix) of the definition then tells us that $\mathbf{P}$ is a chain of antichains $\mathbf{P}_i$, with $\mathbf{P}_i < \mathbf{P}_j$ precisely when $i < j$ in $\mathcal{I}$. Thus the reduct $(P, <, \mathfrak{s})$ satisfies the conditions in part (ii) of Definition 2.4.8, and $(P, <, \mathfrak{s}) \in \boldsymbol{\mathcal{K}}_0 = \boldsymbol{\mathcal{K}}(CA, \mathfrak{S}, N)$.
Consider any chain of antichains $\mathbf{P} = (P, <, \mathfrak{s})$ in the class $\boldsymbol{\mathcal{K}}_0$. We will show that there is a total order $\prec$, so that $(P, <, \prec, \mathfrak{s}) \in \boldsymbol{\mathcal{K}}$.
The structure $\mathbf{P}$ has a finite number of maximal antichains, say n. Since it is a chain of antichains, we can label the maximal antichains as $\mathbf{P}_i$ such that for any $p \in P_i$, $q \in P_j$ we have

$$p < q \quad \Longleftrightarrow \quad i < j.$$

By part (ii)(c) of Definition 2.4.8, each maximal antichain $\mathbf{P}_i$ is $\mathfrak{S}_a$-shaped for some $a \in \mathcal{A}$. So let

$$\mathfrak{s}^{[n]} : [n] \to \mathcal{A}, \quad i \to a.$$

Then $([n], <, \mathfrak{s}^{[n]})$, where $<$ is the natural order on $[n]$, is an $\mathcal{A}$-shaped chain. For any $i \in [n]$, the antichain $\mathbf{P}_i$ has m_i points and is $\mathfrak{S}_a$-shaped, with at most

$n_{a,b}$ points shaped $\mathbf{s}^{a,b}$. Label the points of $\mathbf{P}_i$ as

$$P_i = \{p_{i,j} : j \in [m_i]\},$$

so that

$$\text{if } \mathfrak{s}(p_{i,j}) = \mathbf{s}^{a,b}, \mathfrak{s}(p_{i,j'}) = \mathbf{s}^{a,b'} \text{ and } b < b' \text{ then } j < j'$$

and set $p_{i,j} \prec p_{i,j'}$ if $j < j'$. Then $[m_i]$ with the natural order and the substructure $\mathbf{P}_i$ of $\mathbf{P}$ define a structure $\mathbf{P}'_i = (P_i, <, \prec, \mathfrak{s})$ which satisfies condition (iv)-(x) of Definition 2.5.13. So for any $i \in [n]$, $\mathbf{P}'_i \in \mathcal{K}(GA, \mathfrak{S}_a, N_a, o)$.
An easy check shows that $([n], <, \mathfrak{s}^{[n]})$ and, for each $i \in [n]$, the structure $\mathbf{P}'_i$ define a structure $\mathbf{P}' = (P, <, \prec, \mathfrak{S})$ by setting

$$p_{i,j} \prec p_{i',j'} \text{ if } i < i' \text{ or } i = i' \text{ and } j < j'.$$

The structure $\mathbf{P}'$ satisfies conditions (vii)-(x) of Definition 2.5.15 and therefore lies in $\mathcal{K}$, as claimed. So $\mathcal{K}_0$ is indeed the class of all reducts of structures in $\mathcal{K}$.
Further, for any $(P, <, \mathfrak{s}) \in \mathcal{K}_0$ and a total order $(P, \prec)$, the structure $(P, <, \prec, \mathfrak{s})$ lies in the class $\mathcal{K}$ precisely when $\prec$ is an extension of the partial order $(P, <)$ (and thus convex on the maximal antichains of $(P, <, \mathfrak{s})$) and orders each $\mathfrak{S}_a$-shaped maximal antichain so that for any $b < b'$ in $\mathcal{B}_a$, the $\mathbf{s}^{a,b}$-shaped points are below the $\mathbf{s}^{a,b'}$-shaped points in the total order $\prec$.
Take any $\mathbf{P} = (P, <, \mathfrak{s}) \in \mathcal{K}_0$. Then if $\mathbf{R} = (R, <, \mathfrak{s})$ is a substructure of $\mathbf{P}$ and $(R, <, \prec, \mathfrak{s}) \in \mathcal{K}$, the total order on the substructure of $\mathbf{P}$, which is induced by the total order $(R, \prec)$ can be extended to a total order $\prec'$, so that $(P, <, \prec', \mathfrak{s}) \in \mathcal{K}$, concluding the proof. □

Lemma 5.2.4. *The class $\mathcal{K}(CA, \mathfrak{S}, N, o)$ has the ordering property.*

Proof. Consider a structure $\mathbf{P}_0 = (P, <, \mathfrak{s}) \in \mathcal{K}(CA, \mathfrak{S}, N)$.
Take any structure $\mathbf{P} = (P, <, \prec, \mathfrak{s}) \in \mathcal{K}(CA, \mathfrak{S}, N, o)$. Then

$$P = \{p^{a,b}_{i,j} : (i,j) \in \mathcal{I} \rtimes \mathcal{J}, \mathfrak{s}(p^{a,b}_{i,j}) = \mathbf{s}^{a,b}\},$$

and from part (x) of Definition 2.5.15 $p^{a,b}_{i,j} \prec p^{a',b'}_{i',j'}$ if $i < i'$ or $i = i'$ and $j < j'$.

We know that if $i = i'$ then $a = a'$. Further, from part (x) of Definition 2.5.13, if $i = i'$ and $j < j'$, then $b \leq b'$. This condition induces the total order on the $\mathbf{s}^{a,b}$-shaped substructures $\mathbf{P}_i^{a,b}$ of $\mathbf{P}_i$ for $b \in \mathcal{B}_a$, where for some total order $\mathcal{J}_i^{a,b}$

$$\mathbf{P}_i^{a,b} = \{p_{i,j}^{a,b} : j \in \mathcal{J}_i^{a,b}\}.$$

Then the conditions of Definition 2.5.15 impose the ordering

$$\mathbf{P}_i^{a,b} \prec \mathbf{P}_{i'}^{a',b'}$$

if $i < i'$ or $i = i'$ and $b < b'$.

Suppose that for some other total order $\prec'$, the $\mathbf{P}' = (P, <', \prec, \mathfrak{s})$ lies in $\mathcal{K}(CA, \mathfrak{S}, N, o)$ also.

The substructures $\mathbf{P}_i$ of $\mathbf{P}$ are determined by the partial order $(P, <)$ of $\mathbf{P}_0$. The substructures $\mathbf{P}_i^{a,b}$ of $\mathbf{P}_i$ for $b \in \mathcal{B}_a$ are determined by the shaping $\mathfrak{s}$ of $\mathbf{P}_0$. Thus $\mathbf{P}'$ also consists of maximal antichains $\mathbf{P}_i$, each containing $\mathbf{s}^{a,b}$-shaped substructures $\mathbf{P}_i^{a,b}$ for $b \in \mathcal{B}_a$. Again, we must have

$$\mathbf{P}_i^{a,b} \prec' \mathbf{P}_{i'}^{a',b'}$$

if $i < i'$ or $i = i'$ and $b < b'$.

Thus the total order $(P, \prec')$ permutes the points within each $\mathbf{P}_i^{a,b}$. That means that there must be a total order $\mathcal{J}'^{a,b}_i$ and a bijection

$$f_i^{a,b} : \mathcal{J}_i^{a,b} \to \mathcal{J}'^{a,b}_i$$

such that for $j \in \mathcal{J}_i^{a,b}$

$$p_{i,j}^{a,b} \prec' p_{i,j'}^{a,b} \quad \Longleftrightarrow \quad f_i^{a,b}(j) < f_i^{a,b}(j') \text{ in } \mathcal{J}'^{a,b}_i.$$

For each $i \in \mathcal{I}$, let $\mathcal{J}'_i$ be a total order, such that

$$\mathcal{J}'_i = \bigcup_{b \in \mathcal{B}_a} \mathcal{J}'^{a,b}_i$$

with the total order defined for pairs j, j' with $j \in \mathcal{J}'^{a,b}_i, j' \in \mathcal{J}'^{a,b'}_i$, as

$$j < j' \text{ if } b < b' \text{ in } \mathcal{B}_a \text{ or } b = b' \text{ and } j < j' \text{ in } \mathcal{J}'^{a,b}_i.$$

So consider the map

$$f : \mathcal{I} \rtimes \mathcal{J} \to \mathcal{I} \rtimes \mathcal{J}', \quad f((i,j)) = \left(i, f^{a,b}_i(j)\right) \text{ for } j \in \mathcal{J}^{a,b}_i.$$

The map f allows us to define the map

$$\iota : \mathbf{P} \to \mathbf{P}', \quad p^{a,b}_{i,j} \mapsto p^{a,b}_{f((i,j))}.$$

Since ι maps each antichain $\mathbf{P}_i$ as well as each substructure $\mathbf{P}^{a,b}_i$ to itself, and by definition of $\mathcal{J}'^{a,b}_i$ and $f^{a,b}_i$ also maps the chain $(P^{a,b}_i, \prec)$ to the chain $(P^{a,b}_i, \prec')$, ι is an isomorphism.
Now recall the Definition 2.2.19. We have just shown that for any $\mathbf{P}_0 = (P, <, \mathfrak{s}) \in \mathcal{K}(CA, \mathfrak{S}, N)$ and any total orders $\prec, \prec'$ with

$$\mathbf{P} = (P, <, \prec, \mathfrak{s}), \mathbf{P}' = (P, <, \prec', \mathfrak{s}) \in \mathcal{K}(CA, \mathfrak{S}, N, o)$$

there is an isomorphism $\iota : \mathbf{P} \to \mathbf{P}'$. Thus $\mathbf{P}_0$ can play the role of $\mathbf{A}_0$ and $\mathbf{B}_0$ in Definition 2.2.19 and show that the class $\mathcal{K}(CA, \mathfrak{S}, N, o)$ has ordering property. □

Antichain of chains

The class $\mathcal{K}(AC, \mathfrak{S}, \{n_1, n_2\})$ is defined in Definition 2.4.8.

Lemma 5.2.5. *The following are reasonable classes with respect to $\mathcal{K}(AC, \mathfrak{S}, \{n_1, n_2\})$.*

(i) $\mathcal{K}(AC, \mathfrak{S}, o)$ *with respect to* $\mathcal{K}(AC, \mathfrak{S}, \{\aleph_0, \aleph_0\})$,

(ii) $\mathcal{K}(A, \{\mathbf{s}^a\}, o)$ *with respect to* $\mathcal{K}(AC, \{\mathbf{s}^a\}, \{\aleph_0, 1\})$, *and*

(iii) $\mathcal{K}(A_1, \{\mathbf{s}\}, o)$ *with respect to* $\mathcal{K}(AC, \{\mathbf{s}\}, \{1, 1\})$.

Proof. By Definition 2.5.22, we label a class $\mathcal{K}(GAC, \mathfrak{S}, N, o)$ (defined in 2.5.20) with the total order $\mathcal{A}$ only consisting of a total order $\mathcal{A}_1 = \{a\}$ of size 1 and an empty set $\mathcal{A}_2$ as

(i) $\mathcal{K}(AC, \mathfrak{S}, o)$ when $N = \{n_a\} = \{\aleph_0\}$, and

(ii) $\mathcal{K}(A, \{\mathbf{s}^a\}, o)$ when $N = \{n_a\} = \{1\}$.

Then considering Definition 2.5.18, any $\mathbf{P} = (P, <, \prec, \mathfrak{S}) \in \mathcal{K}(GC, \mathfrak{S}, N, o)$ is

(i) an $\mathfrak{S}$-shaped chain, with identical $(P, <)$ and $(P, \prec)$ when $N = \{\aleph_0\}$,

(ii) and an $\mathbf{s}^a$-shaped antichain of size 1 when $N = \{1\}$.

Then Definition 2.5.20 tells us that given a chain $\mathcal{I}$, a structure $\mathbf{P}$ in $\mathcal{K}(AC, \mathfrak{S}, o)$ consists of $|\mathcal{I}|$ $\mathfrak{S}$-shaped chains $\mathbf{P}_i$ for each $i \in \mathcal{I}$, with the total order on $\mathbf{P}$ inducing a total order on the chains $\mathbf{P}_i$ and extending the partial order on $\mathbf{P}$. The total order on the chains is determined by the total order on $\mathcal{I}$, with

$$\mathbf{P}_i \prec \mathbf{P}_j \quad \Longleftrightarrow \quad i < j \text{ in } \mathcal{I}.$$

The situation is analogous in the case of $\mathbf{P}$ in $\mathcal{K}(A, \{\mathbf{s}^a\}, o)$, with each $\mathbf{P}_i$ being an 'chain' of size 1.

Thus any reduct of a structure $\mathbf{P}$ in $\mathcal{K}(AC, \mathfrak{S}, o)$ lies in $\mathcal{K}(AC, \mathfrak{S}, \{\aleph_0, \aleph_0\})$, and any reduct of a structure $\mathbf{P}$ in $\mathcal{K}(A, \{\mathbf{s}^a\}, o)$ lies in $\mathcal{K}(AC, \mathfrak{S}, \{\aleph_0, 1\})$.

Conversely, given any $\mathbf{P} = (P, <, \mathfrak{s})$ in $\mathcal{K}(AC, \mathfrak{S}, \{\aleph_0, \aleph_0\})$ or $\mathbf{P} = (P, <, \mathfrak{s})$ in $\mathcal{K}(AC, \mathfrak{S}, \{\aleph_0, 1\})$, each consisting on n 'chains' $\mathbf{P}_i$, taking a chain $[n]$ with the natural order and extending the partial order $(P, <)$ to a total order $(P, \prec)$ by letting $p \prec q$ if $p \in \mathbf{P}_i$, $q \in \mathbf{P}_j$ and $i < j$, yields an ordered structure $= (P, <, \prec, \mathfrak{s})$ that lies in $\mathcal{K}(AC, \mathfrak{S}, o)$ or $\mathcal{K}(A, \{\mathbf{s}^a\}, o)$, and checking that (i) and (ii) hold is easy. The full proof is omitted.

Finally, the class $\mathcal{K}(A_1, \{\mathbf{s}\}, o)$ is defined in 2.5.10. Clearly $\mathcal{K}(AC, \{\mathbf{s}\}, \{1, 1\})$ contains only an $\mathbf{s}$-shaped antichain of size 1, and $\mathcal{K}(A_1, \{\mathbf{s}\}, o)$ contains only an ordered $\mathbf{s}$-shaped antichain of size 1, so part (iii) of the lemma holds trivially. $\square$

Results about simply bi-definable classes

Definition 5.2.6. Let $\Sigma = (\Sigma, <, \prec, l_1, l_2)$ be a good skeleton with a total order. The class $\mathcal{K}(\Sigma, o)$ consists precisely of all structures $\mathbf{P}$ satisfying the following.

(i) $\mathbf{P} = (P, <^{\mathbf{P}}, \prec^{\mathbf{P}}, \mathfrak{S}^{\mathbf{P}})$.

(ii) For any $\sigma \in \Sigma$, if the subset P_σ of P is non-empty, the substructure $\mathbf{P}_\sigma$ of $\mathbf{P}$ on the subset P_σ lies in the class $\mathcal{K}(\sigma, o)$.

(iii) If $\sigma \prec \sigma'$ and $p \in \mathbf{P}_\sigma$, $p' \in \mathbf{P}_{\sigma'}$, then $p \prec p'$.

Lemma 5.2.7. *Let $\Sigma = (\Sigma, <, \prec, l_1, l_2)$ be a good skeleton with a total order, and let $\Sigma' = (\Sigma, <, \prec', l_1, l_2)$ be a good skeleton with a total order, differing from Σ only in the total order $(\Sigma, \prec')$. Then the classes $\mathcal{K}(\Sigma, o)$ and $\mathcal{K}(\Sigma', o)$ are simply bi-definable.*

Proof. Recall Definition 3.1.2.
Let, for each $\sigma \in \Sigma$ the formula μ_σ be a simple formula

$$\mathfrak{s}(p) \in \mathfrak{S}_\sigma \wedge \mathfrak{s}(q) \in \mathfrak{S}_\sigma \wedge p \prec q.$$

Further, let for each $\sigma, \varsigma \in \Sigma$ with $\sigma \prec' \varsigma$, the formula $\mu_{\sigma,\varsigma}$ be a simple formula

$$\mathfrak{s}(p) \in \mathfrak{S}_\sigma \wedge \mathfrak{s}(q) \in \mathfrak{S}_\varsigma.$$

Finally, let φ be the simple formula

$$\left(\bigvee_{\sigma \in \Sigma} \mu_\sigma\right) \vee \left(\bigvee_{\sigma,\varsigma \in \Sigma, \sigma \prec' \varsigma} \mu_{\sigma,\varsigma}\right).$$

Fix a structure $\mathbf{P}_0 = (P, <^{\mathbf{P}_0}, \mathfrak{s}^{\mathbf{P}_0}) \in \mathcal{K}(\Sigma)$.
Take a structure $\mathbf{P} = (P, <^{\mathbf{P}_0}, \prec^{\mathbf{P}}, \mathfrak{s}^{\mathbf{P}_0}) \in \mathcal{K}(\Sigma, o)$ and define $\prec'^{\mathbf{P}'}$ on P as

$$p \prec'^{\mathbf{P}'} q \iff \mathbf{P} \models \varphi[p, q].$$

The formulae μ_σ tell us that the total orders $\prec^{\mathbf{P}}$ and $\prec'^{\mathbf{P}'}$ agree on each substructure $\mathbf{P}_\sigma$ of $\mathbf{P}$ and $\mathbf{P}'$. The formulae $\mu_{\sigma,\varsigma}$ state that $p{\prec'}^{\mathbf{P}'}q$ if $\sigma \prec' \varsigma$, $\mathfrak{s}^{\mathbf{P}_0}(p) \in \mathfrak{S}_\sigma$, and $\mathfrak{s}^{\mathbf{P}_0}(q) \in \mathfrak{S}_\varsigma$. Thus the structure $\mathbf{P}' = (P, <^{\mathbf{P}_0}, \prec'^{\mathbf{P}}, \mathfrak{s}^{\mathbf{P}_0})$ is indeed the structure in the class $\boldsymbol{\mathcal{K}}(\Sigma', o)$. Set $\Phi_1(\mathbf{P}) = \mathbf{P}'$.
Symmetrically, we can define formulae μ'_σ and $\mu'_{\sigma,\varsigma}$ and φ' that define $\prec^{\mathbf{P}}$ on P from $\prec'^{\mathbf{P}'}$ on $\mathbf{P}' \in \boldsymbol{\mathcal{K}}(\Sigma', o)$, and the map Φ_2 between total order expansions of $\mathbf{P}_0$ in $\boldsymbol{\mathcal{K}}(\Sigma', o)$ and total order expansions of $\mathbf{P}_0$ in $\boldsymbol{\mathcal{K}}(\Sigma, o)$.
We've shown that part (i) and (ii) of Definition 3.1.2 are true. By definition, the structures $\mathbf{P}$ and $\mathbf{P}'$ agree on the total order on each subset P_σ of P, and only differ on the total order for pairs p, q with $p \in P_\sigma, q \in P_\varsigma$ and $\sigma \neq \varsigma$. For those pairs the total order is uniquely determined by Σ for $\mathbf{P}$ and Σ' for $\mathbf{P}'$. Thus Φ_1 is a bijection, which concludes the proof. $\square$

Lemma 5.2.8. *Let $\Sigma = (\Sigma, <, l_1, l_2)$ be a good skeleton and $\Sigma' = (\Sigma, <', l_1, l'_2)$ be a good skeleton, differing from Σ in the partial order $(\Sigma, <')$ and label map l'_2. Let R be a subset of Σ^2, with a partition $\{R_1, R_2, R_3\}$ such that*

(i) If $(\sigma, \varsigma) \in R_1$, then $\sigma <_c \varsigma$ and $\sigma ||' \varsigma$.

(ii) If $(\sigma, \varsigma) \in R_2$, then $\sigma || \varsigma$ and $\sigma <'_c \varsigma$.

(iii) If $(\sigma, \varsigma) \in R_3$, then for some $<_l \in \{<_g, <_c, <_{sh}, <_{pm}, <_{cpm}\}$ we have $\sigma <_l \varsigma$ and $\sigma <'_l \varsigma$.

Suppose further that

(iv) $\sigma < \varsigma$ precisely when $(\sigma, \varsigma) \in R_1 \cup R_3$, and

(v) $\sigma <' \varsigma$ precisely when $(\sigma, \varsigma) \in R_2 \cup R_3$.

Then the classes $\boldsymbol{\mathcal{K}}(\Sigma)$ and $\boldsymbol{\mathcal{K}}(\Sigma')$ are simply bi-definable.

Proof. Recall Definition 3.1.2 again.
Let, for each $\sigma \in \Sigma$ the formula μ_σ be a simple formula

$$\mathfrak{s}(p) \in \mathfrak{S}_\sigma \wedge \mathfrak{s}(q) \in \mathfrak{S}_\sigma \wedge p < q.$$

Further, let for each $(\sigma, \varsigma) \in R_2$ the formula $\mu_{\sigma,\varsigma}$ be a simple formula

$$\mathfrak{s}(p) \in \mathfrak{S}_\sigma \wedge \mathfrak{s}(q) \in \mathfrak{S}_\varsigma.$$

Let also, for each $(\sigma, \varsigma) \in R_3$, the formula $\mu_{\sigma,\varsigma}$ be a simple formula

$$\mathfrak{s}(p) \in \mathfrak{S}_\sigma \wedge \mathfrak{s}(q) \in \mathfrak{S}_\varsigma \wedge p < q.$$

Finally, let φ be the simple formula

$$\left(\bigvee_{\sigma \in \Sigma} \mu_\sigma\right) \vee \left(\bigvee_{(\sigma,\varsigma) \in R_2 \cup R_3} \mu_{\sigma,\varsigma}\right).$$

Fix a structure $\mathbf{P}_0 = (P, \mathfrak{s}^{\mathbf{P}_0})$.
Take a structure $\mathbf{P} = (P, <^{\mathbf{P}}, \mathfrak{s}^{\mathbf{P}_0}) \in \boldsymbol{\mathcal{K}}(\Sigma, o)$ and define $<'^{\mathbf{P}'}$ on P as

$$p <'^{\mathbf{P}'} q \iff \mathbf{P} \models \varphi[p, q].$$

The formulae μ_σ tell us that the partial orders $<^{\mathbf{P}}$ and $<'^{\mathbf{P}'}$ agree on each substructure $\mathbf{P}_\sigma$ of $\mathbf{P}$ and $\mathbf{P}'$. The formulae $\mu_{\sigma,\varsigma}$ define the partial order $<'^{\mathbf{P}'}$ for points p, q in different components, $\mathbf{P}'_\sigma$ and $\mathbf{P}'_\varsigma$ respectively, of $\mathbf{P}'$. The partial orders $<^{\mathbf{P}}$ and $<'^{\mathbf{P}'}$ again agree on pairs $(\sigma, \varsigma) \in R_3$. For $(\sigma, \varsigma) \in R_1$ we have $p ||'^{\mathbf{P}'} q$, as there is no formula $\mu_{\sigma,\varsigma}$. For $(\sigma, \varsigma) \in R_2$ we have $p <'^{\mathbf{P}'} q$, because the formula $\mu_{\sigma,\varsigma}$ applies. Thus the structure $\mathbf{P}' = (P, <^{\mathbf{P}_0}, \mathfrak{s}^{\mathbf{P}_0})$ is indeed the structure in the class $\boldsymbol{\mathcal{K}}(\Sigma', o)$. Set $\Phi_1(\mathbf{P}) = \mathbf{P}'$.
Symmetrically, we can define formulae μ'_σ and $\mu'_{\sigma,\varsigma}$ and φ' that define $<^{\mathbf{P}}$ on P from $<'^{\mathbf{P}'}$ on $\mathbf{P}' \in \boldsymbol{\mathcal{K}}(\Sigma', o)$, and the map Φ_2 between total order expansions of $\mathbf{P}_0$ in $\boldsymbol{\mathcal{K}}(\Sigma', o)$ and total order expansions of $\mathbf{P}_0$ in $\boldsymbol{\mathcal{K}}(\Sigma, o)$.
We've shown that part (i) and (ii) of Definition 3.1.2 are true. By definition, the structures $\mathbf{P}$ and $\mathbf{P}'$ agree on the partial order on each subset P_σ of P, and only differ on the total order for pairs p, q with $p \in P_\sigma, q \in P_\varsigma$ and $(\sigma, \varsigma) \in R_1 \cup R_2$. For those pairs the partial order is uniquely determined by Σ for $\mathbf{P}$ and Σ' for $\mathbf{P}'$. Thus Φ_1 is a bijection, which concludes the proof. $\square$

Matching skeletons

We defined the class $(GAC, \mathfrak{S}, N, o)$ of ordered glorified antichains of chains in Definition 2.5.20. We defined a (c)pm-skeleton in Definition 5.1.12. In this section we explore how the two are related.

Consider the structures $\mathcal{H}_{\sigma<_{pm}\sigma'}$ and $\mathcal{H}_{\sigma<_{cpm}\sigma'}$, satisfying $<_{pm}$- and $<_{cpm}$-condition respectively (defined in 2.4.10).

We know that in $\mathcal{H}_{\sigma<_{pm}\sigma'}$, for any chain $\mathcal{H}_{\sigma,i}$ in $\mathcal{H}_{\sigma}$, there is precisely one chain $\mathcal{H}_{\sigma',i'}$ in $\mathcal{H}_{\sigma'}$, such that

$$\mathcal{H}_{\sigma,i} <_c \mathcal{H}_{\sigma',i'}$$

and vice versa. For any $j \neq i'$ we have

$$\mathcal{H}_{\sigma,i}||\mathcal{H}_{\sigma',j}.$$

In $\mathcal{H}_{\sigma<_{(cpm}\sigma'}$, the two cases are swapped. For any chain $\mathcal{H}_{\sigma,i}$ in $\mathcal{H}_{\sigma}$, there is precisely one chain $\mathcal{H}_{\sigma',i'}$ in $\mathcal{H}_{\sigma'}$, such that

$$\mathcal{H}_{\sigma,i}||\mathcal{H}_{\sigma',i'}$$

and vice versa. For any $j \neq i'$ we have

$$\mathcal{H}_{\sigma,i} <_c \mathcal{H}_{\sigma',j}.$$

In both cases, of course, $\mathcal{H}_{\sigma}$ and $\mathcal{H}_{\sigma'}$ need to have the same number of chains.

In both cases there is also a bijection, or a matching, between the chains in the bottom and the chains on the top; in $\mathcal{H}_{\sigma<_{pm}\sigma'}$, each pair forms a chain, and in $\mathcal{H}_{\sigma<_{cpm}\sigma'}$, each pair is incomparable, and the non-pairs form chains.

Thinking of the chains in the two components as points in the respective components of a bipartite graph and drawing an edge between two points if they represent a pair of chains that form a chain, the graph representing a $\mathcal{H}_{\sigma<_{pm}\sigma'}$ would then be a perfect matching and the one representing $\mathcal{H}_{\sigma<_{cpm}\sigma'}$ its complement.

A different way to describe the similarities between $\boldsymbol{\mathcal{H}}_{\sigma<_{pm}\sigma'}$ and $\boldsymbol{\mathcal{H}}_{\sigma<_{cpm}\sigma'}$ would be to say that they are simply bi-definable. Indeed, suppose that $<$ is the partial order on $\boldsymbol{\mathcal{H}}_{\sigma<_{cpm}\sigma'}$. We can define the partial order $<'$ on the universe of $\boldsymbol{\mathcal{H}}_{\sigma<_{cpm}\sigma'}$ as $p <' q$ if

(i) $p < q$ and $p, q \in \boldsymbol{\mathcal{H}}_\sigma$ or $p, q \in \boldsymbol{\mathcal{H}}_{\sigma'}$, or

(ii) $p \in \boldsymbol{\mathcal{H}}_\sigma$, $q \in \boldsymbol{\mathcal{H}}_{\sigma'}$ and $p||q$.

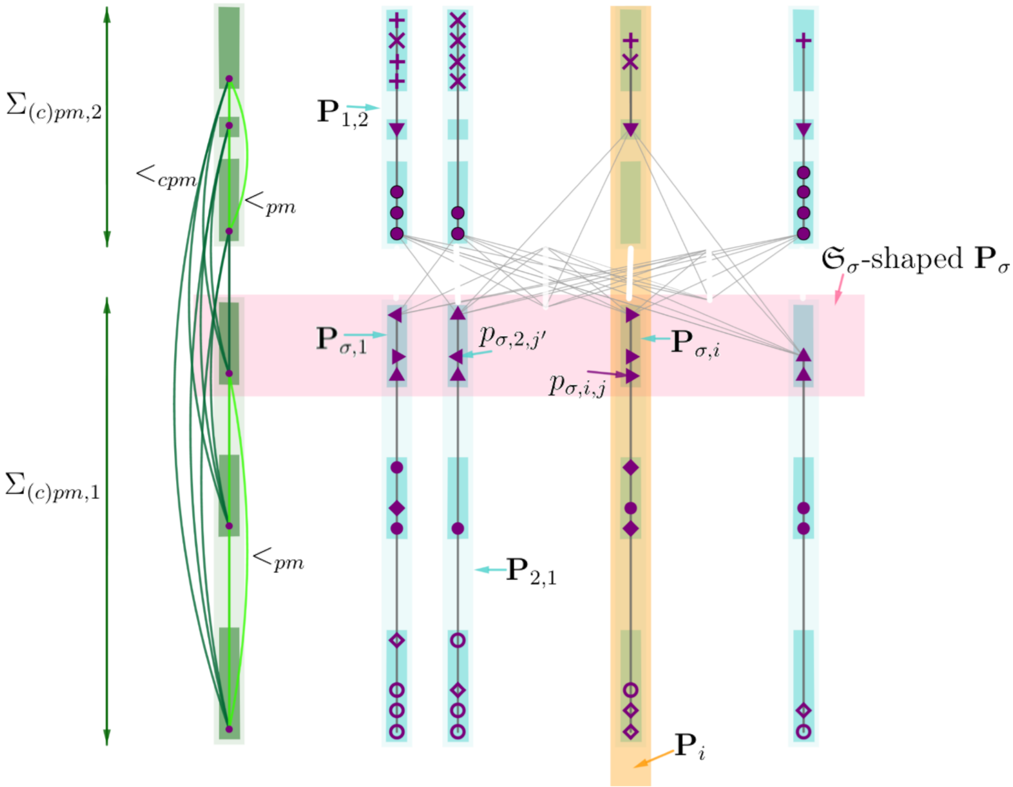
So stacking two (or more) $\boldsymbol{\mathcal{H}}(AC)$ components with the same number of chains on top of each other, with all relations between them being labelled $<_{pm}$ leads to a shaped homogeneous partial order, with an unshaped reduct that is an antichain of chains. Call the skeleton of structure described Σ_1 - it is a chain of points labelled AC, and with relations labelled $<_{pm}$. Suppose Σ_2 is also such a skeleton. If they have the same number of chains, we could 'match' the chains of Σ_1 and Σ_2, by placing the chains of Σ_2 on top of the chains of Σ_1 and making precisely the matched pairs of chains incomparable. Then for any $\sigma \in \Sigma_1$ and $\sigma' \in \Sigma_1$ we have $\boldsymbol{\mathcal{H}}_\sigma <_{cpm} \boldsymbol{\mathcal{H}}_{\sigma'}$. A *glorified chain* then consists of a matched pair of chains, and the constructed structure is a *glorified antichain of chains*. See Figure 5.4 for a sketch of a glorified antichain of chains, as defined here, and compare it to Figure 2.9.

Definition 5.2.9. Let $\Sigma_{(c)pm}$ be a (c)pm-skeleton, with

- a set of shapes $\mathfrak{S}$ with a partition $\bigsqcup_{\sigma\in\Sigma_{(c)pm}} \mathfrak{S}_\sigma$,
- for each $\sigma \in \Sigma_{(c)pm}$ a number $n_\sigma \in \{1, \aleph_0\}$, and
- if $n_\sigma = 1$ then $|\mathfrak{S}_\sigma| = 1$.

Let $N = \{n_\sigma : \sigma \in \Sigma_{(c)pm}\}$.
The homogeneous shaped partial order $\boldsymbol{\mathcal{H}}(\Sigma_{(c)pm}, \mathfrak{S}, N)$ is a *glorified $\mathfrak{S}$-shaped antichain of $\aleph_0$ chains*. For any $\sigma \in \Sigma_{(c)pm}$, $\boldsymbol{\mathcal{H}}_\sigma(\Sigma_{(c)pm}, \mathfrak{S}, N)$ is a $\mathfrak{S}_\sigma$-shaped antichain of $\aleph_0$ chains of size n_σ.
The age $\boldsymbol{\mathcal{K}}(\Sigma_{(c)pm}, \mathfrak{S}, N)$ of $\boldsymbol{\mathcal{H}}(\Sigma_{(c)pm}, \mathfrak{S}, N)$ is a class of *finite shaped glorified antichains of chains*. The universe of any $\mathbf{P} \in \boldsymbol{\mathcal{K}}(\Sigma_{(c)pm}, \mathfrak{S}, N)$, is of

Figure 5.4: Glorified antichain of chains $\mathbf{P}$



the form

$$P = \{p_{\sigma,i,j} : (\sigma, i, j) \in (\Sigma_{(c)pm} \times \mathcal{I}) \rtimes \mathcal{J}\},$$

where $\mathcal{I}$ is a total order of size at most n_1, for each $(\sigma, i) \in \Sigma_{(c)pm} \times \mathcal{I}$, $\mathcal{J}_{\sigma,i}$ is a total order of size at most $n_{\sigma,2}$ or $\mathcal{J}_{\sigma,i}$ is an empty set. The partial order on $\mathbf{P}$ is defined as

$$p_{\sigma,i,j} < p_{\sigma',i',j'}$$

if

(i) $\sigma \in \Sigma_{(c)pm,1}$, $\sigma' \in \Sigma_{(c)pm,2}$ and $i \neq i'$, or

(ii) $\sigma, \sigma' \in \Sigma_{(c)pm,1}$ or $\sigma, \sigma' \in \Sigma_{(c)pm,2}$ and $\sigma < \sigma'$ in $\Sigma_{(c)pm}$ and $i = i'$, or

(iii) $\sigma = \sigma'$, $i = i'$ and $j < j'$ in $\mathcal{J}_{\sigma,i}$.

An $\mathfrak{S}_\sigma$-shaped component $\mathbf{P}_\sigma$ of $\mathbf{P}$ for each $\sigma \in \Sigma_{(c)pm}$, is the substructure of $\mathbf{P}$ on the subset

$$P_\sigma = \{p_{\sigma,i,j} : (\sigma, i, j) \in (\{\sigma\} \times \mathcal{I}) \rtimes \mathcal{J}\}$$

of P. The shaping $\mathfrak{s} : P \to \mathfrak{S}$ sends P_σ to $\mathfrak{S}_\sigma$.
For each $i \in \mathcal{I}$, the *glorified chain* $\mathbf{P}_i$ is the substructure of $\mathbf{P}$ on the subset

$$P_i = \{p_{\sigma,i,j} : (\sigma, i, j) \in (\Sigma_{(c)pm} \times \{i\}) \rtimes \mathcal{J}\}.$$

For each $(\sigma, i) \in \Sigma \times \mathcal{I}$, if the set $\mathcal{J}_{\sigma,i}$ is non-empty, the part $\mathbf{P}_{\sigma,i}$ of $\mathbf{P}$ on

$$P_{\sigma,i} = \{p_{\sigma,i,j} : j \in \mathcal{J}_{\sigma,i}\}.$$

Remarks 5.2.10. (i) For $h \in [2]$, we define

$$\mathfrak{S}_h = \bigcup_{\sigma \in \Sigma_{(c)pm,1}} \mathfrak{S}_\sigma, \quad N_1 = \{n_\sigma : \sigma \in \Sigma_{(c)pm,1}\}$$

and the substructure $\mathbf{P}_{h,i}$ of $\mathbf{P}$ on the set of points

$$P_{h,i} = \{p_{\sigma,i,j} : (\sigma, i, j) \in (\Sigma_{(c)pm,h} \times \{i\}) \rtimes \mathcal{J}\}.$$

Then $\mathbf{P}_i$ consists of a shaped chain $\mathbf{P}_{1,i}$ and/or a shaped chain $\mathbf{P}_{2,i}$. If it consists of two chains, then $\mathbf{P}_{1,i}$ and $\mathbf{P}_{2,i}$ are incomparable.

(ii) One might expect that the universe of a $\mathbf{P} \in \boldsymbol{\mathcal{K}}(\Sigma_{(c)pm}, \mathfrak{S}, N)$ is

$$P = \{p_{\sigma,i,j} : (\sigma, i, j) \in (\Sigma_{(c)pm} \rtimes \mathcal{I}) \rtimes \mathcal{J}\},$$

with each component $\mathbf{P}_\sigma$ of $\mathbf{P}$ having a distinct index set $\mathcal{I}_\sigma$.

But any $\mathbf{P} \in \boldsymbol{\mathcal{K}}(\Sigma_{(c)pm}, \mathfrak{S}, N)$ is a finite substructure of $\boldsymbol{\mathcal{H}}(\Sigma_{(c)pm}, \mathfrak{S}, N)$. As discused in the beginning of this subsection, the partial order reducts of the substructures $\boldsymbol{\mathcal{H}}(\Sigma_{(c)pm,h}, \mathfrak{S}_h, N_h)$ for $h \in [2]$ of $\boldsymbol{\mathcal{H}}(\Sigma_{(c)pm}, \mathfrak{S}, N)$ are antichains of $\aleph_0$ chains, due to the matching between the chains of $\mathfrak{S}_\sigma$-shaped antichains of $\aleph_0$ chains. For any maximal chain $\mathcal{H}_{1,i}$ in $\mathcal{H}(\Sigma_{(c)pm,1}, \mathfrak{S}_1, N_1)$, there is precisely one maximal chain $\mathcal{H}_{2,i}$ in $\mathcal{H}(\Sigma_{(c)pm,2}, \mathfrak{S}_2, N_2)$, such that $\mathcal{H}_{1,i}$ and $\mathcal{H}_{2,i}$ are incomparable. The substructure of $\mathcal{H}(\Sigma_{(c)pm}, \mathfrak{S}, N)$ on $\mathcal{H}_{2,i} \cup \mathcal{H}_{2,i}$ is a glorified chain of $\mathcal{H}(\Sigma_{(c)pm}, \mathfrak{S}, N)$.

So each glorified chain $\mathbf{P}_i$ of $\mathbf{P}$ is a finite substructure of a glorified chain of $\mathcal{H}(\Sigma_{(c)pm}, \mathfrak{S}, N)$. Thus there exists the index set $\mathcal{I}$, which enumerates the glorified chains of $\mathbf{P}$ overall.

Finally, define a class of finite ordered shaped glorified antichains of chains.

Definition 5.2.11. Suppose that $\mathcal{K}(\Sigma_{(c)pm}, \mathfrak{S}, N)$ is a class of finite shaped glorified antichains of chains. The class $\mathcal{K}(\Sigma_{(c)pm}, \mathfrak{S}, N, o)$ of *ordered shaped glorified antichains of chains* is an expansion of the class $\mathcal{K}(\Sigma_{(c)pm}, \mathfrak{S}, N)$ with *glorified convex total orders* of the partial orders. A glorified convex total order $\prec$ of a glorified antichain of chains $\mathbf{P}$ is a total order in which

(i) $\prec$ extends the partial order on each glorified chain, placing the part of the glorified chain defined by $\Sigma_{(c)pm,1}$ below the part defined by $\Sigma_{(c)pm,2}$, and

(ii) $\prec$ is convex on each glorified chain.

Take any $\mathbf{P} = (P, <, \prec, \mathfrak{s}) \in \mathcal{K}(\Sigma_{(c)pm}, \mathfrak{S}, N, o)$. There exists a total order $\mathcal{I}$ and for each $(\sigma, i) \in \Sigma_{(c)pm} \times \mathcal{I}$ a total order $\mathcal{J}_{\sigma,i}$, defining the total order on $\mathbf{P}$, as well as defining the partial order on it. The partial order on $\mathbf{P}$ is defined in 5.2.9. But we have

$$p_{\sigma,i,j} \prec p_{\sigma',i',j'}$$

if

(i) $i < i'$ in $\mathcal{I}$,

(ii) $i = i'$ and $\sigma < \sigma'$ in $\Sigma_{(c)pm}$, or

(iii) $i = i'$, $\sigma = \sigma'$ and $j < j'$ in $\mathcal{J}_{\sigma,i}$.

Note. The total order $\mathcal{I}$ and for each $(\sigma, i) \in \Sigma_{(c)pm} \times \mathcal{I}$ a total order $\mathcal{J}_{\sigma,i}$ exist because the glorified convex total orders essentially define a total order of the glorified chains of a glorified antichain of chains, and then define a total order on a glorified chain as an extension of the partial order, and placing one of the two chains in it below the other based on the skeleton $\Sigma_{(c)pm}$. The total order $\mathcal{I}$ reflects the total order of glorified chains, and the total orders $\mathcal{J}_{\sigma,i}$ together with the chain $\Sigma_{(c)pm}$ reflect the total orders within the glorified chains.

Lemma 5.2.12. *Let $\mathcal{K}(GAC, \mathfrak{S}, N, o)$ be a class of glorified antichains of chains defined by the following.*

(i) A total order $\mathcal{A}$, with a partition $\{\mathcal{A}_1, \mathcal{A}_2\}$, where $\mathcal{A}_2$ is possibly an empty set, and for all $a_1 \in \mathcal{A}_1$ and $a_2 \in \mathcal{A}_2$ we have $a_1 < a_2$.

(ii) For each $a \in \mathcal{A}$ a number $n_a \in \{1, \aleph_0\}$, and $N = \{n_a : a \in \mathcal{A}\}$.

(iii) A set $\mathfrak{S}$ of shapes with a partition $\{\mathfrak{S}_a : a \in \mathcal{A}\}$, where $|\mathfrak{S}_a| = 1$ when $n_a = 1$ and for each $a \in \mathcal{A}$ there exists a total order $\mathcal{B}_a$, such that $\mathfrak{S}_a = \{\mathbf{s}^{a,b} : b \in \mathcal{B}_a\}$.

(iv) A class $\mathcal{K}(GC, \mathfrak{S}, N, o)$ of glorified chains.

Let also Σ be a (c)pm-skeleton with $\mathcal{A}$ playing the role of Π in the definition of the (c)pm-skeleton,

$$\Sigma = \{\sigma_a : a \in \mathcal{A}\}.$$

Then the classes $\mathcal{K}(\Sigma, \mathfrak{S}, N, o)$ and $\mathcal{K}(GAC, \mathfrak{S}, N, o)$ define the same class of structures.

Proof. Recall that $\mathcal{K}(GAC, \mathfrak{S}, N, o)$ is defined in 2.5.20.
Take any $\mathbf{P} \in \mathcal{K}(GAC, \mathfrak{S}, N, o)$. Then

$$P = \{p_{i,j}^{h,a,b} : (i,j) \in \mathcal{I} \rtimes \mathcal{J}, \mathfrak{s}(p_{i,j}^{h,a,b}) = \mathbf{s}^{a,b}, a \in \mathcal{A}_h\}.$$

For each $i \in \mathcal{I}$, the substructure $\mathbf{P}_i$ of $\mathbf{P}$ on the subset $P_i = \{p_{i,j}^{h,a,b} : j \in \mathcal{J}_i\}$ is a glorified chain, defined in 2.5.18. For each $i \in \mathcal{I}$ and $a \in \mathcal{A}$, there is a subset $P_i^a = \{p_j^{h,a,b} : j \in \mathcal{J}_i^a, a \in \mathcal{A}_h\}$ of P_i. If P_i^a is non-empty, $\mathcal{J}_i^a$ is a chain and a substructure of $\mathcal{J}_i$. So we could write

$$P = \{p_{i,j}^{h,a,b} : (a,i,j) \in (\mathcal{A} \times \mathcal{I}) \rtimes \mathcal{J}, \mathfrak{s}(p_{i,j}^{h,a,b}) = \mathbf{s}^{a,b}, a \in \mathcal{A}_h\}.$$

Consider a structure $\mathbf{P}'$ with the universe

$$P' = \{p_{\sigma_a,i,j} : (\sigma_a, i, j) \in (\Sigma \times \mathcal{I}) \rtimes \mathcal{J}\},$$

where $\mathcal{J}_{\sigma_a,i}$ is isomorphic to $\mathcal{J}_i^a$ and

$$\iota_i^a : \mathcal{J}_{\sigma_a,i} \to \mathcal{J}_i^a, j \mapsto \iota(j)$$

is an isomorphism.
Define the shaping

$$\mathfrak{s}' : P' \to \mathfrak{S}, \quad \mathfrak{s}'(p_{\sigma_a,i,j}) = \mathfrak{s}\left(p_{i,\iota(j)}^{h,a,b}\right),$$

and we can check that $\mathfrak{s}'$ indeed satisfes $\mathfrak{s}'(P_{\sigma_a}) \subset \mathfrak{S}_a$ for

$$P_{\sigma_a} = \{p_{\sigma_a,i,j} : (\sigma_a, i, j) \in (\{\sigma_a\} \times \mathcal{I}) \rtimes \mathcal{J}\}.$$

The partial order $<'$ and the total order $\prec'$ are defined in 5.2.9 and 5.2.11 using the total orders $\mathcal{I}$ and $\mathcal{J}_{\sigma_a,i}$.
Let $\mathbf{P}' = (P', <', \prec', \mathfrak{S}')$. Then $\mathbf{P}' \in \boldsymbol{\mathcal{K}}(\Sigma, \mathfrak{S}, N, o)$.
An easy check shows that the map

$$\iota : \mathbf{P}' \to \mathbf{P}, \quad p_{\sigma_a,i,j} \to p_{i,j}^{h,a,b}$$

defines an isomorphism.
Similarly, given a $\mathbf{P}' \in \boldsymbol{\mathcal{K}}(\Sigma, \mathfrak{S}, N, o)$, we can find an isomorphic $\mathbf{P} \in \boldsymbol{\mathcal{K}}(GAC, \mathfrak{S}, N, o)$. Thus $\boldsymbol{\mathcal{K}}(\Sigma, \mathfrak{S}, N, o)$ and $\boldsymbol{\mathcal{K}}(GAC, \mathfrak{S}, N, o)$ both define the same class. $\square$

Corollary 5.2.13. *Consider $\mathcal{K}(GAC, \mathfrak{S}, N, o)$ and $\mathcal{K}(\Sigma, \mathfrak{S}, N)$ defined in Theorem 5.2.12. The class $\mathcal{K}(GAC, \mathfrak{S}, N, o)$ is a reasonable class with respect to the class $\mathcal{K}(\Sigma, \mathfrak{S}, N)$.*

We turn our eyes back to the class $\mathcal{K}(GAC, \mathfrak{S}, N, o)$.

Lemma 5.2.14. *The class $\mathcal{K}(GAC, \mathfrak{S}, N, o)$ has the ordering property.*

Proof. Consider a structure $(P, <, \mathfrak{S}) \in \mathcal{K}(GAC, \mathfrak{S}, N)$.
Take any $\mathbf{P} = (P, <, \prec, \mathfrak{S}) \in \mathcal{K}(GAC, \mathfrak{S}, N, o)$. The total order $\prec$ is convex on the maximal glorified chains of $\mathbf{P}$ and for a chain $\mathcal{I}$ we have

$$\mathbf{P}_i \prec \mathbf{P}_j \quad \Longleftrightarrow \quad i < j \text{ in } \mathcal{I}.$$

Thus $\prec$ for each of the glorified chains is determined already, and any different total order $\prec'$ with $\mathbf{P}' = (P, <, \prec', \mathfrak{S}) \in \mathcal{K}(GAC, \mathfrak{S}, N, o)$ just permutes the maximal glorified chains $\mathbf{P}_i$ of $\mathbf{P}$.
Recall that by Lemma 4.3.2, the class $\mathcal{K}(GC, \mathfrak{S}, N, o)$ has the joint embedding property. Let $\mathbf{P}'$ be a glorified chain with $\mathbf{P}_i$ as a substructure for all $i \in \mathcal{I}$. Then let $\mathbf{P}^* = (P^*, <^*, \mathfrak{S}^*)$ be an antichain of chains $\mathbf{P}_i^*$ for all $i \in \mathcal{I}$, and each $\mathbf{P}_i^*$ isomorphic to $\mathbf{P}'$. Then $\mathbf{P}^* = (P^*, <^*, \mathfrak{S}^*) \in \mathcal{K}(GAC, \mathfrak{S}, N)$ and $(P^*, <^*, \prec^*, \mathfrak{S}^*) \in \mathcal{K}(GAC, \mathfrak{S}, N, o)$ for any total order of the glorified chains $\mathbf{P}_i^*$. Since the glorified chains are isomorphic, the structure $(P^*, <^*, \prec^*, \mathfrak{S}^*)$ is unique up to isomorphism.
Thus for any $\mathbf{P} = (P, <, \prec, \mathfrak{S}) \in \mathcal{K}(GAC, \mathfrak{S}, N, o)$, there is a substructure of $(P^*, <^*, \prec^*, \mathfrak{S}^*)$ isomorphic to $\mathbf{P}$, as, by definition $\mathbf{P}_i$ is a substructure of $\mathbf{P}'$ for all $i \in \mathcal{I}$. This concludes the proof. $\square$

Chunk skeleton

Recall Definition 2.5.23 of a simple skeleton, Definition 5.1.8 of a chunk skeleton and Definition 5.1.18 of a simplified skeleton of a chunk skeleton.

Lemma 5.2.15. *Suppose that $\Sigma = (\Sigma, <, l_1, l_2)$ is a simplified skeleton of some chunk skeleton Σ_{ch}.*
Let $(\Sigma, \prec)$ be a total order extending the partial order $(\Sigma, <)$ and let l'_1 be a map with

$$l'_1(\sigma) = A \quad \Longleftrightarrow \quad l_1(\sigma) = AC.$$

Then $\Sigma' = (\Sigma, <, \prec, l'_1, l_2)$ is an antichained skeleton.

We omit the trivial proof.

Lemma 5.2.15 provides another connection between classes corresponding to good skeletons and classes corresponding to ordered skeletons.

Lemma 5.2.16. *Let Σ and Σ' be a pair of skeletons, as in Lemma 5.2.15. Further, for each $\sigma \in \Sigma$,*

(i) if $l_1(\sigma) = AC$, let $\mathfrak{S}_\sigma = \{\mathbf{s}_\sigma\}$ be a set of shapes, $n_{\sigma,1} = \aleph_0$, $n_{\sigma,2} = 1$,

(ii) and if $l_1(\sigma) = G$, let $\mathfrak{S}_\sigma$ be a set of shapes, such that the sets $\mathfrak{S}_\sigma$ are disjoint for all $\sigma \in \Sigma$.

Let $\mathfrak{S} = \bigsqcup_{\sigma\in\Sigma} \mathfrak{S}_\sigma$.
Then the class $\boldsymbol{\mathcal{K}}(\Sigma', \mathfrak{S}, o)$ is a reasonable class with respect to $\boldsymbol{\mathcal{K}}(\Sigma, \mathfrak{S})$.

Proof. This holds since for each $\sigma \in \Sigma$,

(i) if $l_1(\sigma) = AC$, by part (ii) of Lemma 5.2.5, the class $\boldsymbol{\mathcal{K}}(A, \{\mathbf{s}_\sigma\}, o)$ is a reasonable class with respect to $\boldsymbol{\mathcal{K}}(AC, \{\mathbf{s}_\sigma\}, \{\aleph_0, 1\})$, and

(ii) if $l_1(\sigma) = G$, by Lemma 5.2.1, the class $\boldsymbol{\mathcal{K}}(G, \mathfrak{S}_\sigma, o)$ is a reasonable class with respect to the class $\boldsymbol{\mathcal{K}}(G, \mathfrak{S}_\sigma)$.

The total order on any structure is determined by the total order on each substructure $\mathbf{P}_\sigma$, and $\mathbf{P}_\sigma \prec \mathbf{P}_{\sigma'}$ whenever $\sigma \prec \sigma'$. $\square$

Definition 5.2.17. Let Σ_{ch} be a chunk skeleton with a chunk partition

$$\left(\bigcup_{u\in(C)pm} \Sigma_u\right) \cup \left(\bigcup_{\gamma\in\Gamma} \sigma_\gamma\right).$$

For each $u \in (C)pm$ let σ_u be the least point in the (c)pm-skeleton Σ_u. The skeleton Σ_{ch} is a *simple chunk skeleton* if for each $u \in (C)pm$, $\sigma \in \Sigma_u$ and $\sigma' \notin \Sigma_u$ we have

(i) $\sigma_u <_c \sigma'$ if and only if $\sigma <_c \sigma'$, and

(ii) $\sigma' <_c \sigma_u$ if and only if $\sigma' <_c \sigma$.

Lemma 5.2.18. *Let Σ_{ch} be a simple chunk skeleton with a simplified skeleton Σ. Let Σ' be an antichained skeleton defined in Lemma 5.2.15 and let Σ^* be a glorified skeleton of Σ', defined in 2.5.24.*

(i) For $u \in (C)pm$, with $\Sigma_u = \{\sigma_{u,a} : a \in \mathcal{A}_u\}$ we have the following.

(a) A total order $\mathcal{A}_u$, with a partition $\{\mathcal{A}_{u,1}, \mathcal{A}_{u,2}\}$, where $\mathcal{A}_{u,2}$ is possibly an empty set, and for all $a_1 \in \mathcal{A}_{u,1}$ and $a_2 \in \mathcal{A}_{u,2}$ we have $a_1 < a_2$.

(b) $\Sigma_{u,1} = \{\sigma_{u,a} : a \in \mathcal{A}_{u,1}\}$ and $\Sigma_{u,2} = \{\sigma_{u,a} : a \in \mathcal{A}_{u,2}\}$.

(c) For each $a \in \mathcal{A}_u$, $n_{u,a} \in \{1, \aleph_0\}$, and $N_u = \{n_{u,a} : a \in \mathcal{A}_u\}$.

(d) A set $\mathfrak{S}_u$ of shapes with a partition $\{\mathfrak{S}_{u,a} : a \in \mathcal{A}_u\}$, where $|\mathfrak{S}_{u,a}| = 1$ when $n_{u,a} = 1$ and for each $a \in \mathcal{A}_u$ there exists a total order $\mathcal{B}_{u,a}$, such that $\mathfrak{S}_{u,a} = \{\mathrm{s}_u^{a,b} : b \in \mathcal{B}_{u,a}\}$.

(e) A class $\boldsymbol{\mathcal{K}}(GC, \mathfrak{S}_u, N_u, o)$ of glorified chains.

(ii) For $\gamma \in \Gamma$, a set $\mathfrak{S}_\gamma$ of shapes.

Let $\mathfrak{S} = \left(\bigsqcup_{u\in(C)pm} \mathfrak{S}_u\right) \sqcup \left(\bigsqcup_{\gamma\in\Gamma} \mathfrak{S}_\gamma\right)$.

Then the class $\boldsymbol{\mathcal{K}}(\Sigma^*, \mathfrak{S}, N, o)$ is a reasonable class with respect to the class $\boldsymbol{\mathcal{K}}(\Sigma_{ch}, \mathfrak{S})$.

Proof. By Lemma 5.2.12, for each $u \in (C)pm$, the classes $\mathcal{K}(\Sigma_u, \mathfrak{S}_u, N_u, o)$ and $\mathcal{K}(GAC, \mathfrak{S}_u, N_u, o)$ define the same class of structures, so the class $\mathcal{K}(\Sigma_u, \mathfrak{S}_u, N_u, o)$ is reasonable with respect to $\mathcal{K}(GAC, \mathfrak{S}_u, N_u)$.
For $\gamma \in \Gamma$, by Lemma 5.2.1, the class $\mathcal{K}(G, \mathfrak{S}_\gamma, o)$ is a reasonable class with respect to the class $\mathcal{K}(G, \mathfrak{S}_\gamma)$.
Take any $\mathbf{P} = (P, <, \prec, \mathfrak{S}) \in \mathcal{K}(\Sigma^*, \mathfrak{S}, N, o)$. Then

$$P = \{P_u : u \in (C)pm\} \cup \{P_\gamma : \gamma \in \Gamma\},$$

and for non-empty P_x with $x \in (C)pm \cup \Gamma$, the substructure of $\mathbf{P}$ on P_x is $\mathfrak{S}_x$-shaped.
We know that in the simple chunk skeleton, for each $u \in (C)pm$, $\sigma \in \Sigma_u$ and $\sigma' \notin \Sigma_u$ we have

(i) $\sigma_u <_c \sigma'$ if and only if $\sigma <_c \sigma'$, and

(ii) $\sigma' <_c \sigma_u$ if and only if $\sigma' <_c \sigma$.

The partial order $(P, <)$ satisfies those conditions due to the part (x) of Definition 2.5.26. Thus $(P, <, \mathfrak{S}) \in \mathcal{K}(\Sigma_{ch}, \mathfrak{S})$.
For $x, y \in (C)pm \cup \Gamma$, the total order on any structure is determined by the total order on each substructure $\mathbf{P}_u$ or $\mathbf{P}_\gamma$, and $\mathbf{P}_x \prec \mathbf{P}_y$ whenever $x \prec y$ in the antichained skeleton Σ', which finishes the proof. □

Lemma 5.2.19. *Suppose that $\Sigma_{ch} = (\Sigma, <, l_1, l_2)$ is a chunk skeleton. Then there exists a simple chunk skeleton $\Sigma' = (\Sigma, <', l_1, l_2')$, differing from Σ in the partial order $(\Sigma, <')$ and label map l_2', so that there is a subset R of Σ^2, with a partition $\{R_1, R_2, R_3\}$ such that*

(i) If $(\sigma, \varsigma) \in R_1$, then $\sigma <_c \varsigma$ and $\sigma ||' \varsigma$.

(ii) If $(\sigma, \varsigma) \in R_2$, then $\sigma || \varsigma$ and $\sigma <_c' \varsigma$.

(iii) If $(\sigma, \varsigma) \in R_3$, then for some $<_l \in \{<_g, <_c, <_{sh}, <_{pm}, <_{cpm}\}$ we have $\sigma <_l \varsigma$ and $\sigma <_l' \varsigma$.

(iv) $\sigma < \varsigma$ precisely when $(\sigma, \varsigma) \in R_1 \cup R_3$.

(v) $\sigma <' \varsigma$ *precisely when* $(\sigma, \varsigma) \in R_2 \cup R_3$.

Proof. This is true by the definition of a good skeleton, 2.4.13. For any $u \in (C)pm$, $\sigma \in \Sigma_u$ and $\sigma' \notin \Sigma_u$, we can obtain a simple chunk skeleton from a chunk skeleton by adding a relation $\sigma <_c \sigma'$ or $\sigma' <_c \sigma_u$ or deleting a relation $\sigma_u <_c \sigma'$ or $\sigma' <_c \sigma$ to obtain a skeleton satisfying conditions (i) and (ii) in Definition 5.2.17, that is also a good skeleton. □

Corollary 5.2.20. *Suppose that* Σ_{ch} *is a chunk skeleton. Then there exists a simple chunk skeleton* Σ_{sch} *such that the classes* $\boldsymbol{\mathcal{K}}(\Sigma_{ch})$ *and* $\boldsymbol{\mathcal{K}}(\Sigma_{sch})$ *are simply bi-definable.*

Proof. This follows from Lemma 5.2.8 and Lemma 5.2.19. □

Lemma 5.2.21. *Suppose that* Σ_{ch} *is a chunk skeleton. Suppose that* $\mathfrak{S}$ *is a set of shapes, that there is a partition* $\{\mathfrak{S}_\sigma : \sigma \in \Sigma\}$ *of* $\mathfrak{S}$*, and for each* $\sigma \in \Sigma$ *with* $l_1(\sigma) = AC$*, numbers* $n_{\sigma,1}$ *and* $n_{\sigma,2}$*, with*

(i) $n_{\sigma,1} = \aleph_0$ *and* $n_{\sigma,2} \in \{1, \aleph_0\}$*, or* $n_{\sigma,1} = n_{\sigma,2} = 1$*, and*

(ii) $|\mathfrak{S}_\sigma| = 1$ *if* $n_{\sigma,2} = 1$.

Let $\boldsymbol{\mathcal{K}}(\Sigma)$ *be the class of shaped partial orders as defined in 2.4.14. Then if, for some* $\sigma \in \Sigma$ *we have* $n_{\sigma,1} = n_{\sigma,2} = 1$*, the skeleton* Σ *consists only of the point* σ *labelled* AC*,* $\mathfrak{S}_\sigma = \{\mathbf{s}\}$ *for some shape* $\mathbf{s}$*, and* $\boldsymbol{\mathcal{K}}(\Sigma)$ *contains precisely the* $\mathbf{s}$*-shaped antichain of size 1.*
In this case we call the skeleton Σ_{ch} *a trivial chunk skeleton.*

Proof. If $n_{\sigma,1} = n_{\sigma,2} = 1$, the homogeneous structure $\boldsymbol{\mathcal{H}}_\sigma$ is an an $\mathbf{s}$-shaped antichain of size 1 for the shape $\mathbf{s}$ with $\mathfrak{S}_\sigma = \{\mathbf{s}\}$. Then for any other σ', $\boldsymbol{\mathcal{H}}_\sigma$ can either be incomparable with, completely below, or completely above the structure $\boldsymbol{\mathcal{H}}_{\sigma'}$, as $\boldsymbol{\mathcal{H}}_\sigma$ only contains one point. Thus, by part (iii) of the definition of a chunk skeleton (5.1.8), Σ does not contain any point other than σ, which completes the proof. □

Theorem 5.2.22. *Let Σ_{ch} be a chunk skeleton and $\mathcal{K}(\Sigma_{ch}, \mathfrak{S})$ the class of $\mathfrak{S}$-shaped partial orders, as defined in 2.4.14. If Σ_{ch} is not a trivial chunk skeleton, then there exists a simple skeleton Σ, such that*

(i) the classes $\mathcal{K}(\Sigma_{ch}, \mathfrak{S})$ and $\mathcal{K}(\Sigma, \mathfrak{S})$ are simply bi-definable, and

(ii) the class $\mathcal{K}(\Sigma, \mathfrak{S}, o)$ is a reasonable class with respect to $\mathcal{K}(\Sigma, \mathfrak{S})$ and has the Ramsey property.

Remark 5.2.23. We refer to the skeleton Σ as the simple skeleton of the chunk skeleton Σ_{ch}.

Proof. We have shown in Corollary 5.2.20 that there exists a simple chunk skeleton Σ_{sch} such that the classes $\mathcal{K}(\Sigma_{ch}, \mathfrak{S})$ and $\mathcal{K}(\Sigma_{sch}, \mathfrak{S})$ are simply bi-definable. Further, in Lemma 5.2.18, we have shown that there exists a simple skeleton Σ, such that the class $\mathcal{K}(\Sigma, \mathfrak{S}, o)$ is reasonable with respect to the class $\mathcal{K}(\Sigma_{sch}, \mathfrak{S})$.
Let $\Phi : \mathcal{K}(\Sigma_{sch}, \mathfrak{S}) \to \mathcal{K}(\Sigma_{ch}, \mathfrak{S})$ be a 'map', defined in Remark 3.1.3 after the definition of simple bi-definability.
Define the class $\mathcal{K}(\Sigma_{ch}, \mathfrak{S}, o)$ as follows. Any $\mathbf{P} = (P, <, \prec, \mathfrak{S}) \in \mathcal{K}(\Sigma_{ch}, \mathfrak{S}, o)$ satisfies the following.

(i) $(P, <, \mathfrak{S}) \in \mathcal{K}(\Sigma_{ch}, \mathfrak{S})$

(ii) There is a structure $\mathbf{P}' = (P, <', \prec', \mathfrak{S}) \in \mathcal{K}(\Sigma, \mathfrak{S}, o)$, such that

$$(P, <, \mathfrak{S}) = \Phi\big((P, <', \mathfrak{S})\big)$$

and $p \prec q$ precisely when $p \prec' q$ for all $p, q \in P$.

Thus the classes $\mathcal{K}(\Sigma, \mathfrak{S}, o)$ and $\mathcal{K}(\Sigma_{ch}, \mathfrak{S}, o)$ are simply bi-definable as well and $\mathcal{K}(\Sigma_{ch}, \mathfrak{S}, o)$ is a reasonable class with respect to $\mathcal{K}(\Sigma_{ch}, \mathfrak{S})$.
By Theorem 4.3.7, the class $\mathcal{K}(\Sigma, \mathfrak{S}, o)$ is Ramsey, which finishes the proof. $\square$

Shuffle skeleton

Let Σ be a shuffle skeleton, defined in 5.1.5. In Torrezão de Sousa & Truss (2008), the structures $\boldsymbol{\mathcal{H}}(\Sigma)$, where Σ is a shuffle skeleton, correspond precisely to the SH^*-classes. In the proof of Theorem 5.2, starting on page 20, they show the equivalent of the following.

Lemma 5.2.24. *Suppose that Σ is a shuffle skeleton with the set of shapes*

$$\mathfrak{S} = \bigsqcup_{\sigma \in \Sigma} \mathfrak{S}_\sigma, \quad \mathfrak{S}_\sigma = \bigsqcup_{a \in \mathcal{A}_\sigma} \mathfrak{S}_{\sigma,a}.$$

Then $\boldsymbol{\mathcal{H}}(\Sigma, \mathfrak{S})$ is simply bi-definable with the homogeneous structure $\boldsymbol{\mathcal{H}}(CA, \mathfrak{S})$, where $\mathfrak{S}$ is viewed as

$$\mathfrak{S} = \bigsqcup_{(\sigma,a) \in \Sigma \rtimes \mathcal{A}} \mathfrak{S}_{\sigma,a}.$$

This, of course, implies that given any shuffle skeleton Σ and any compatible set $\mathfrak{S}$ of shapes, the classes $\boldsymbol{\mathcal{K}}(\Sigma, \mathfrak{S})$ and $\boldsymbol{\mathcal{K}}(CA, \mathfrak{S})$ are simply bi-definable.

Good skeleton

Let Σ be a good skeleton. Then by Lemma 5.1.11 there is a partition of Σ,

$$\Sigma = \left(\bigcup_{t \in Sh} \Sigma_t \right) \cup \left(\bigcup_{z \in Ch} \Sigma_z \right),$$

such that the following hold.

(i) For each $t \in Sh$, Σ_t is a shuffle skeleton.

(ii) For each $z \in Ch$, Σ_z is a chunk skeleton.

(iii) For any distinct $x, y \in Sh \cup Ch$ and $\sigma \in \Sigma_x$, $\sigma' \in \Sigma_y$, if $\sigma < \sigma'$ in Σ, then $\sigma <_c \sigma'$.

Lemma 5.2.25. *Let Σ be a good skeleton with a good partition*

$$\Sigma = \left(\bigcup_{t \in Sh} \Sigma_t\right) \cup \left(\bigcup_{z \in Ch} \Sigma_z\right).$$

Then there exists a good skeleton Σ' with the same good partition such that the following hold for Σ'.

(i) For each $t \in Sh$, Σ_t is a shuffle skeleton.

(ii) For each $z \in Ch$, Σ_z is a chunk skeleton.

(iii) For any distinct $x, y \in Sh \cup Ch$ and $\sigma \in \Sigma_x$, $\sigma' \in \Sigma_y$, we have $\sigma || \sigma'$.

(iv) The classes $\boldsymbol{\mathcal{K}}(\Sigma)$ and $\boldsymbol{\mathcal{K}}(\Sigma')$ are simply bi-definable.

We refer to Σ' as a better skeleton.

Proof. This is true by Lemma 5.2.8. □

Theorem 5.2.26. *Suppose that Σ is a good skeleton. Suppose that $\mathfrak{S}$ is a set of shapes, that there is a partition $\{\mathfrak{S}_\sigma : \sigma \in \Sigma\}$ of $\mathfrak{S}$, and*

- *for each $\sigma \in \Sigma$ with $l_1(\sigma) = AC$, numbers $n_{\sigma,1}$ and $n_{\sigma,2}$, with*
  - *$n_{\sigma,1} = \aleph_0$ and $n_{\sigma,2} \in \{1, \aleph_0\}$, or $n_{\sigma,1} = n_{\sigma,2} = 1$, and*
  - *$|\mathfrak{S}_\sigma| = 1$ if $n_{\sigma,2} = 1$;*
- *for each $\sigma \in \Sigma$ with $l_1(\sigma) = CA$,*
  - *a partition $\{\mathfrak{S}_{\sigma,a} : a \in \mathcal{A}_\sigma\}$ of $\mathfrak{S}_\sigma$, and*
  - *for each $\mathrm{s}_\sigma^{a,b} \in \mathfrak{S}_{\sigma,a}$, a number $n_{\sigma,a,b} \in \{1, \aleph_0\}$.*

Let $\boldsymbol{\mathcal{K}}(\Sigma)$ be the class of shaped partial orders as defined in 2.4.14. Then there exists an elementary skeleton Σ^* and a class $\boldsymbol{\mathcal{K}}(\Sigma^*, o)$ of ordered shaped partial orders, such that

(i) the classes $\boldsymbol{\mathcal{K}}(\Sigma)$ and $\boldsymbol{\mathcal{K}}(\Sigma^*)$ are simply bi-definable, and

(ii) the class $\mathcal{K}(\Sigma^*, o)$ is a reasonable class with respect to $\mathcal{K}(\Sigma^*)$ and is a Ramsey class.

Further, when the elementary skeleton Σ^* does not contain edges labelled $<_g$, the class $\mathcal{K}(\Sigma^*, o)$ has the ordering property.

Proof. Let Σ' be the better skeleton defined in 5.2.25. Then $\mathcal{K}(\Sigma, \mathfrak{S})$ and $\mathcal{K}(\Sigma', \mathfrak{S})$ are simply bi-definable.
For each $x \in Sh \cup Ch$, let $\mathfrak{S}_x = \bigcup_{\sigma \in \Sigma_x} \mathfrak{S}_\sigma$.
Then by Lemma 5.2.24, for each $t \in Sh$, the class $\mathcal{K}(\Sigma_t, \mathfrak{S}_t)$ is simply bi-definable with the class $\mathcal{K}(\sigma_t^*, \mathfrak{S}_t, N_t)$, where σ_t^* is labelled CA and

$$N_t = \{n_{\sigma,a,b} : (\sigma, a, b) \in (\Sigma_t \rtimes \mathcal{A}_t) \rtimes \mathcal{B}_t\}.$$

By Theorem 4.2.5, the class $\mathcal{K}(\sigma_t^*, \mathfrak{S}_t, N_t, o)$ is Ramsey and by Lemma 5.2.3 the class $\mathcal{K}(\sigma_t^*, \mathfrak{S}_t, N_t, o)$ is a reasonable class with respect to $\mathcal{K}(\sigma_t^*, \mathfrak{S}_t, N_t)$.
For $z \in Ch$, where Σ_z is a trivial chunk skeleton, defined in 5.2.21, the class $\mathcal{K}(\Sigma_z, \mathfrak{S}_z)$ is simply bi-definable with the class $\mathcal{K}(\sigma_z^*, \mathfrak{S}_z)$, where σ_z^* is labelled A_1. The class $\mathcal{K}(\sigma_z^*, \mathfrak{S}_z, o)$ is trivially Ramsey and the class $\mathcal{K}(\sigma_z^*, \mathfrak{S}_z, o)$ is reasonable with respect to the class $\mathcal{K}(\sigma_z^*, \mathfrak{S}_z)$. Let the set of all such z be Ch_1.
Otherwise, for $z \in Ch \setminus Ch_1$, by Theorem 5.2.22 there exists a simple skeleton Σ_z^*, such that

(i) the classes $\mathcal{K}(\Sigma_z, \mathfrak{S}_z)$ and $\mathcal{K}(\Sigma_z^*, \mathfrak{S}_z)$ are simply bi-definable, and

(ii) the class $\mathcal{K}(\Sigma_z^*, \mathfrak{S}_z, o)$ is a reasonable class with respect to $\mathcal{K}(\Sigma_z^*, \mathfrak{S}_z)$ and has the Ramsey property.

Let

$$\Sigma^* = \left(\bigcup_{t \in Sh} \sigma_t^*\right) \cup \left(\bigcup_{z \in Ch_1} \sigma_z^*\right) \cup \left(\bigcup_{z \in Ch \setminus Ch_1} \Sigma_z^*\right),$$

such that for any distinct $x, y \in Sh \cup Ch$ and $\sigma \in \Sigma_x^*$ or $\sigma = \sigma_x^*$, $\sigma' \in \Sigma_y^*$ or $\sigma' = \sigma_y^*$, we have $\sigma || \sigma'$.
The fact that for distinct x, y we have $\sigma || \sigma'$ implies all of the following.

(i) The classes $\mathcal{K}(\Sigma', \mathfrak{S})$ and $\mathcal{K}(\Sigma^*, \mathfrak{S})$ and thus $\mathcal{K}(\Sigma, \mathfrak{S})$ and $\mathcal{K}(\Sigma^*, \mathfrak{S})$ are simply bi-definable.

(ii) Define $\mathcal{K}(\Sigma^*, \mathfrak{S}, o)$ as consisting of $\mathbf{P} = (P, <, \prec, \mathfrak{s})$, where

  (a) there is a partition $\{P_x : x \in Sh \cup Ch\}$ of P,

  (b) the substructure $\mathbf{P}_x$ of $\mathbf{P}$ for each non-empty P_x lies in the class $\mathcal{K}(\sigma^*_x, \mathfrak{S}_x, o)$ for $x \in Sh \cup Ch_1$ or $\mathcal{K}(\Sigma^*_x, \mathfrak{S}_x, o)$ for $x \in Ch \setminus Ch_1$, and

  (c) there is a total order on $Sh \cup Ch$ inducing a total order on $\mathbf{P}_x$.

Then the class $\mathcal{K}(\Sigma^*, \mathfrak{S}, o)$ is reasonable with respect to the class $\mathcal{K}(\Sigma^*, \mathfrak{S})$.

(iii) The class $\mathcal{K}(\Sigma^*, \mathfrak{S}, o)$ is Ramsey by Corollary 3.2.12. It is a merge of classes $\mathcal{K}(\sigma^*_x, \mathfrak{S}_x, o)$ and $\mathcal{K}(\Sigma^*_x, \mathfrak{S}_x, o)$ for $x \in Sh \cup Ch$, and taking a selection of structures, at most one $\mathbf{P}_x$ from $\mathcal{K}(\sigma^*_x, \mathfrak{S}_x, o)$ per $x \in Sh \cup Ch_1$ or $\mathcal{K}(\Sigma^*_x, \mathfrak{S}_x, o)$ per $x \in Ch \setminus Ch_1$, defines precisely one structure in $\mathcal{K}(\Sigma^*, \mathfrak{S}, o)$; the structure with the total order defined by the total order on $Sh \cup Ch$.

Now Σ^* consists of an antichain of points σ^*_x for $x \in Sh \cup Ch_1$, labelled CA or A_1, as well as the union $\bigcup_{z \in Ch \setminus Ch_1} \Sigma^*_z$ of simple skeletons. Since each of the simple skeletons consists of points labelled GAC or G, relations labelled $<_g$ or $<_c$, and satisfies the c-condition, the skeleton Σ^* is indeed an elementary skeleton, as defined in 2.5.28.
Finally, by condition (v) of the simple skeleton (2.5.23), if Σ^* contains no edges labelled $<_g$, all the simple skeletons consist of a single point, so

$$\Sigma^* = \{\sigma_x : x \in Sh \cup Ch\}.$$

(i) By Lemma 5.2.4, any $\mathcal{K}(\sigma_x, \mathfrak{S}_x, o)$ with $l_1(\sigma_x) = CA$ has the ordering property.

(ii) The class $\mathcal{K}(\sigma_x, \mathfrak{S}_x, o)$ with $l_1(\sigma_x) = A_1$ trivially has the ordering property.

(iii) By Lemma 5.2.2, any $\mathcal{K}(\sigma_x, \mathfrak{S}_x, o)$ with $l_1(\sigma_x) = G$ has the ordering property.

(iv) By Lemma 5.2.14, any $\mathcal{K}(\sigma_x, \mathfrak{S}_x, o)$ with $l_1(\sigma_x) = GAC$ has the ordering property.

So take any structure $\mathbf{P} = (P, <, \mathfrak{s}) \in \mathcal{K}(\Sigma^*, \mathfrak{S})$. Then, for each $x \in Sh \cup Ch$, since $\mathcal{K}(\sigma_x, \mathfrak{S}_x, o)$ has ordering property, there exists a structure $\mathbf{R}_x = (R_x, <, \mathfrak{s}) \in \mathcal{K}(\sigma_x, \mathfrak{S}_x)$, such that for any

$$\mathbf{P} = (P, <, \prec, \mathfrak{s}) \in \mathcal{K}(\Sigma^*, \mathfrak{S}, o)$$

and the corresponding

$$\mathbf{P}_x = (P_x, <, \prec, \mathfrak{s}) \in \mathcal{K}(\sigma_x, \mathfrak{S}_x, o)$$

as well as any

$$\mathbf{R}_x = (R_x, <, \prec, \mathfrak{s}) \in \mathcal{K}(\sigma_x, \mathfrak{S}_x, o)$$

there exists an embedding

$$e_x : \mathbf{P}_x = (P_x, <, \prec, \mathfrak{s}) \to \mathbf{R}_x = (R_x, <, \prec, \mathfrak{s}).$$

Let $\mathbf{R} \in \mathcal{K}(\Sigma^*, \mathfrak{S})$ be the unique merge in of structures

$$\mathbf{R}_x = (R_x, <, \mathfrak{s}) \in \mathcal{K}(\sigma_x, \mathfrak{S}_x, o)$$

for $x \in Sh \cup Ch$. Then any total order $\prec$ on $\mathbf{R}$ induces the total orders $\prec$ on $\mathbf{R}_x = (R_x, <, \prec, \mathfrak{s}) \in \mathcal{K}(\sigma_x, \mathfrak{S}_x, o)$. Since the total order between points in different $\mathbf{P}_x$ (and $\mathbf{R}_x$) is uniquely determined by a total order on $Sh \cup Ch$, we can combine the embeddings e_x to get an embedding

$$e : \mathbf{P} = (P, <, \prec, \mathfrak{s}) \to \mathbf{R} = (R, <, \prec, \mathfrak{s}).$$

Thus the class $\mathcal{K}(\Sigma^*, \mathfrak{S}, o)$ has the ordering property. $\square$

Chapter 6

Conclusion

Consider, again, Definition 2.4.14. It defines a homogeneous shaped partial $\boldsymbol{\mathcal{H}}(\Sigma)$ for any good skeleton Σ, and a set of shapes and multiplicities as follows. Let $\mathfrak{S}$ be a set of shapes, together with a partition $\{\mathfrak{S}_\sigma : \sigma \in \Sigma\}$ of $\mathfrak{S}$, and

- for each $\sigma \in \Sigma$ with $l_1(\sigma) = AC$, numbers $n_{\sigma,1}$ and $n_{\sigma,2}$, with
  - $2 \leq n_{\sigma,1} \leq \aleph_0$ and $n_{\sigma,2} \in \{1, \aleph_0\}$, or $n_{\sigma,1} = n_{\sigma,2} = 1$, and
  - $|\mathfrak{S}_\sigma| = 1$ if $n_{\sigma,2} = 1$;
- for each $\sigma \in \Sigma$ with $l_1(\sigma) = CA$,
  - a partition $\{\mathfrak{S}_{\sigma,a} : a \in \mathcal{A}_\sigma\}$ of $\mathfrak{S}_\sigma$, and
  - for each $\mathbf{s}_\sigma^{a,b} \in \mathfrak{S}_{\sigma,a}$, a number $n_{\sigma,a,b}$ with $1 \leq n_{\sigma,a,b} \leq \aleph_0$.

Suppose that $\boldsymbol{\mathcal{K}}(\Sigma, o)$ is a reasonable order class with respect to $\boldsymbol{\mathcal{K}}(\Sigma)$ and either $1 < n_{\sigma,1} < \aleph_0$ for some $\sigma \in \Sigma$ with $l_1(\sigma) = AC$ (or $1 < n_{\sigma,a,b} < \aleph_0$ for some $\sigma \in \Sigma$ with $l_1(\sigma) = CA$), the class $\boldsymbol{\mathcal{K}}(\Sigma, o)$ contains an $\mathbf{s}_\sigma^i$-shaped $\mathbf{A}_2$ for any $\mathbf{s}_\sigma^i \in \mathfrak{S}_\sigma$ (or an $\mathbf{s}_\sigma^{a,b}$-shaped $\mathbf{A}_2$). By Lemma 2.5.6, $\boldsymbol{\mathcal{K}}(\Sigma, o)$ does not have the Ramsey property. Thus the pair $(n_{\sigma,1}, n_{\sigma,2})$ is restricted to values $(1, 1)$, $(\aleph_0, 1)$ and $(\aleph_0, \aleph_0)$, and $n_{\sigma,a,b}$ can only take values 1 and $\aleph_0$ in Theorem 5.2.26.

The last sentence of Theorem 5.2.26 states that when the elementary skeleton Σ' does not contain edges labelled $<_g$, the class $\boldsymbol{\mathcal{K}}(\Sigma', o)$ has the ordering property. So to achieve the aim of the thesis, the outstanding result needed is that when the elementary skeleton Σ' does contain edges labelled $<_g$, the class $\mathcal{K}(\Sigma', o)$ has the ordering property. The following is needed to complete the aim of the thesis.

Conjecture 6.0.1. Let Σ be a chunk skeleton and Σ^* a glorified skeleton, such that $\mathcal{K}(\Sigma)$ and $\mathcal{K}(\Sigma^*, \mathfrak{S}, N)$ are simply bi-definable. Then $\mathcal{K}(\Sigma^*, \mathfrak{S}, N, o)$ has the ordering property.

The skeleton Σ^* exists by Corollary 5.2.20. In the proof of Theorem 5.2.26 we prove that when all glorified skeleton substructures of an elementary skeleton are singletons, the class of ordered shaped partial orders defined by it has the ordering property. The proof works because we also show in the thesis that the classes of structures defined by a singleton elementary skeleton have the ordering property. So with a proven conjecture, we can similarly show that a class defined by any elementary skeleton has the ordering property. Proving this conjecture is the next step for the author of this thesis.

Next, the Ramsey and ordering properties results lead to topological dynamics results. The classes mentioned in Theorem 5.2.26 can easily be seen to be closed under substructures, so checking that they have the joint embedding property would show that they are Fraïssé by Theorem 2.2.4. This would thus yield the results about the automorphism groups of the corresponding ordered homogeneous shaped partial orders being extremely amenable, and allow us to calculate the universal minimal flow of the automorphism groups of the corresponding homogeneous shaped partial orders.

We have seen in 1.2.1 that the classes $\mathcal{K}(AC_{\aleph_0}, ce)$ and $\mathcal{K}(CA_{\aleph_0}, ce)$ are simply bi-definable. Similarly, there might be classes of ordered shaped partial orders that are simply bi-definable. It might be interesting to find a classification of Ramsey Fraïssé classes of ordered shaped partial orders with the ordering property that contains no simply bi-definable classes.

Another set of questions to consider regards other shaped structures. For example, one might want to classify all shaped homogeneous graphs or shaped homogeneous tournaments. Given the classification, versions of the Two Pass Lemma, the Structural Product Ramsey Lemma or the Blowup Lemma could be relevant to showing that classes of ordered shaped graphs are Ramsey.

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