Congruences of Saito-Kurokawa lifts and divisibility of degree-8 L-values

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Abstract

In this thesis, we study the arithmeticity of critical values of degree-8 tensor product L-functions attached to Siegel modular forms of genus 1 and 2. We show that the congruence between the Hecke eigenvalues of two cuspidal Siegel Hecke eigenforms of genus 2 implies a similar congruence between certain suitably normalised critical values of the associated degree-8 L-functions. This phenomenon is predicted by the Bloch-Kato conjecture, for which we therefore provide further evidence in this particular setting.

We prove this by employing integral representation formulae, due to Saha, and Böcherer and Heim, linking critical L-values to iterated Petersson inner products against diagonally restricted non-holomorphic Eisenstein series.

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Chapter 1

Introduction

§ 1.1 Why L-functions

Some of the most fascinating results in mathematics involve connecting seemingly unrelated objects in a surprising way. This is even more striking when one can apply techniques from different fields to study a common object: L-functions are an example of this phenomenon. If $\{a_n\}_n$ is a sequence of numbers encoding information about some object X, then one can consider the Dirichlet series $L_X(s) = \sum_n a_n n^{-s}$. If $\{a_n\}_n$ has *nice* properties (heavily dependent on the context), the function $L_X(s)$ is *nice* as well (for instance, meromorphic on $\mathbb{C}$, admits functional equation or an Euler product); hence one may use methods from analysis to study it, and maybe get new insights on X via the associated $\{a_n\}_n$.

This approach is quite common in number theory, where sometimes the object we wish to study is the sequence of numbers itself. For instance, let E be an elliptic curve over $\mathbb{Q}$: we know its rational points form a group (with the induced chord/tangent sum) whose structure is well understood, as the Mordell-Weil theorem states being isomorphic to a finite group plus r_E copies of $\mathbb{Z}$. The number r_E is called the rank of E, and (a version of) the Birch and Swinnerton-Dyer conjecture says that r_E equals the order of vanishing of the L-function $L_E(s)$ at $s = 1$, where $L_E(s)$ is constructed in terms of the number of points of the mod p reduction of E for all primes p. Therefore, $L_E(s)$ links analytic and algebraic aspects of E and one can obtain results about one from information about the other.

Surprisingly, there is a third point of view on $L_E(s)$: Galois representation. One can consider what happens to the $\overline{\mathbb{Q}}$-points on E when applying the Galois group $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, since the group operation on E acts in an algebraic way on the coordinates. It turns out that there exists a suitable Galois representation associated to a modular form f which connects it to E, namely via $L_E(s) = L_f(s)$, where $L_f(s)$ is constructed in terms of the Fourier coefficients of f (or, equivalently, in terms of the image of the Frobenius element via the Galois representation attached to f - see section (1.3)). Once again, one can see modular forms as analytic, algebraic or arithmetic objects and attack related problems with radically different tools.

In this thesis, we study arithmetic properties of a degree 8 L-function associated to a pair of Siegel modular forms. The Bloch-Kato conjecture (a generalisation of Birch and Swinnerton-Dyer) predicts that a particular value of this L-function should be divisible by a congruence prime because of *Galois representation reasons* (namely because it divides the order of some Selmer group), and we prove this by applying a combination of algebraic and analytic tools within the context of Siegel modular forms. The next sections first introduce the objects at hand and a naive explanation for the expected result, and then explain why the Bloch-Kato conjecture actually predicts it.

§ 1.2 Goal and motivations

Our goal

Let $h \in \mathcal{S}^1_{2l}$ be an elliptic normalised Hecke cuspidal eigenform of weight $2l$ and genus 1, with l odd; its completed L-function $\Lambda_h(s) := (2\pi)^{-s}\Gamma(s)L_h(s)$ satisfies the functional equation

$$\Lambda_h(s) = (-1)^l \Lambda_h(2l - s)$$

hence L_h has a zero at the central critical value $s = l$.

Let $F \in \breve{\mathcal{S}}^2_{2k}$ be the Saito-Kurokawa lift of an elliptic Hecke cuspidal eigenform $f \in \mathcal{S}^1_{4k-2}$, with $2k \geq 2l$; further, its spinor L-function satisfies

$$Z_F(s) = \zeta(s - 2k + 1)\zeta(s - 2k + 2)L_f(s)$$

(see proposition (2.5.1)). Because of this, the degree 8 tensor product L-function $Z_{F\otimes h}$ defined in (2.23) factorises as

$$Z_{F\otimes h}(s) = L_h(s - 2k + 1)L_h(s - 2k + 2)L_{f\otimes h}(s)$$

and in particular $Z_{F\otimes h}$ has a zero at $s = l + 2k - 1$, the first critical value to the right of the centre, because of the vanishing of $L_h(l)$.

Let $\mathbb{Q}(f)$ be the smallest algebraic extension of $\mathbb{Q}$ containing all the Fourier coefficients of f. One can fix periods $\omega_f^\pm \in \mathbb{C}$ so that

$$L_f^{alg}(j) := \frac{L_f(j)}{\pi^j \omega_f^{(-1)^j}} \in \mathbb{Q}(f)$$

for any integer $1 \leq j \leq 4k-3$.

Fix a *large* prime ideal $\mathfrak{p}$ of $\mathbb{Q}(f)$ dividing $L_f^{alg}(2k)$, for a *suitable* choice of periods $\omega_f^\pm$; see theorems (4.1.3) and (6.1.3) for details. Then a theorem of Katsurada [Kat08, theorem 6.1] ensures that there exists a Hecke eigenform $G \in (\breve{\mathcal{S}}_{2k}^2)^\perp$ whose Hecke eigenvalues $\lambda_T(G)$ are congruent mod $\mathfrak{p}$ to those of F for every Hecke operator $T \in H_{2k}^2$.

The coefficients of the Dirichlet series of $Z_{G\otimes h}$ are defined in terms of the Hecke eigenvalues of G and h. Since $\lambda_T(G) \equiv \lambda_T(F) \mod \mathfrak{p}$ by construction, we roughly expect to see a similar congruence between critical values of $Z_{G\otimes h}$ and $Z_{F\otimes h}$: in particular we would like to observe whether

$$Z_{G\otimes h}(2k+l-1) \equiv Z_{F\otimes h}(2k+l-1) = 0 \mod \mathfrak{p}$$

This is quite imprecise, since $Z_{G\otimes h}(2k+l-1)$ is a complex number so the congruence does not make sense as is. Nonetheless, one can fix periods $\omega_{G\otimes h}^\pm$ so that $Z_{G\otimes h}/\omega_{G\otimes h}^\pm$ takes values in $\overline{\mathbb{Q}}$ up to a power of π, and so what we really expect to see is that $\mathfrak{p}$ divides $Z_{G\otimes h}/\omega_{G\otimes h}^\pm$.

To avoid having to make a particular choice for the periods $\omega_{G\otimes h}^\pm$, we can instead study the ratio

$$\frac{Z_{G\otimes h}(2k+l-1)}{Z_{G\otimes h}(2k+l-1+2m)} \in \pi^{8m}\overline{\mathbb{Q}} \tag{1.1}$$

As there is no particular reason for $\mathfrak{p}$ to appear in the critical values $Z_{G\otimes h}(2k+l-1+2m)$ for $m \geq 1$, we expect to detect a factor of $\mathfrak{p}$ in this ratio for all values of $m \geq 1$ such that $2k+l-1+2m$ is critical; observe that $Z_{G\otimes h}$ does not vanish at these points, as the value there is defined by a convergent Euler product (see section (2.4) for a more detailed discussion on the analytic and functional properties of Z functions).

The congruence mod $\mathfrak{p}$ between F and G is indeed a naive explanation for the expected divisibility of $Z_{G\otimes h}(2k+l-1)$ by $\mathfrak{p}$. However, we explain in the next section how this fact is actually predicted by a deep conjecture in modern number theory, namely the Bloch-Kato conjecture.

Similar results in literature

As explained, L_f^{alg} take values in $\mathbb{Q}(f)$ at its critical points: it is therefore natural to investigate on the nature of the prime factors of this values; in particular, whether congruences between modular forms correspond to congruences between L-values.

A century ago, Ramanujan noted that the Fourier coefficients of the weight 12 Δ cusp form and of the Eisenstein series E_{12} are congruent modulo the large prime 691, which divides the numerator of $\zeta(12)/\pi^{12}$. But 691 also divides the denominator of the rightmost critical value $L_\Delta^{alg}(11)$. In terms of the Bloch-Kato conjecture, the 691 in the numerator is the order of an element of some Selmer group, whilst the one in the denominator is the order of an element in a global torsion group.

Similar phenomena of congruences of critical L-values have been observed multiple times in literature, both experimentally and theoretically. For instance, Dummigan [Dum01, theorem 14.2] proved that, if g is a normalised cuspidal Hecke eigenform of weight k with coefficients congruent to those of E_k modulo a large prime p, then the degree-4 tensor product L-value

$$\frac{Z_{g\otimes g'}(k'/2+k-1)}{\langle g', g'\rangle}$$

is divisible by (a suitable prime ideal above) p, where g' is any cusp form of weight $k' > k$.

Leaving the domain of elliptic modular forms, one can study spinor L-functions attached to Siegel modular forms: in this thesis we focus only on scalar valued ones, but there are analogous conjectures about vector valued forms. For instance, as explained in [BDFK19, section 3.2], we have numerically observed congruences between the Dirichlet coefficients of degree-8 tensor product L-functions, and Bloch-Kato does indeed predict a congruence between particular certain L-values.

Using the method presented in this thesis, we have proved in a recent paper [DHR19] that the *large prime* $\mathfrak{q}$ appears in the rightmost critical value of the tensor product L function $Z_{f\otimes F}(s)$, where $\mathfrak{q}$ divides the rightmost critical value of the symmetric square L-function of $f \in \mathcal{S}_k^1$ and $F \in \mathcal{S}_k^2$ is congruent modulo $\mathfrak{q}$ to the Klingen-Eisenstein series $E_k^{2,1}(f)$ associated to f.

The goal of this thesis is to prove this kind of expectation, at least in the simplest case of scalar valued Siegel modular forms.

§ 1.3 Bloch-Kato conjecture

Statement of Bloch-Kato

Let X be a non-singular projective variety of dimension d, defined over $\mathbb{Q}$. For $0 \leq i \leq 2d$, define the l-adic cohomology V_l by

$$T_l := \varprojlim_n H^i_{\acute{e}t}(X_{\overline{\mathbb{Q}}}, \mathbb{Z}/l^n\mathbb{Z}) \qquad V_l = H^i_l(X_{\overline{\mathbb{Q}}}) := T_l \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

where $H_{\acute{e}t}$ denotes the étale cohomology. Finally, put $A_l := V_l/T_l$ and $A := \oplus_l A_l$, and let $A(m)$ be the m-th Tate twist of A.

For a prime p, define the polynomial $P_p(t) := \det(I - \mathrm{Frob}_p^{-1} t|_{H^i_l(X_{\overline{\mathbb{Q}}})^{I_p}})$, where I_p is the p-inertia subgroup, and let

$$L^i_X(s) := \prod_p P_p(p^{-s})^{-1} \tag{1.2}$$

for $\Re(s) > \frac{i}{2} + 1$. Conjecturally, this admits meromorphic continuation to $\mathbb{C}$ and a functional equation $L^i_X(s) \sim L^i_X(i+1-s)$, where $\sim$ denotes equality up to some explicit factors.

Conjecture 1.3.1 (Bloch-Kato, [BK07])**.** For critical $j \neq \frac{i+1}{2}$, or $j = \frac{i+1}{2}$ with $L^i_X(j) \neq 0$,

$$L^i_X(j) = \frac{\Omega(j)\ \#H^1_f(\mathbb{Q}, A(i+1-j))}{\#H^0(G_{\mathbb{Q}}, A(j))\ \#H^0(G_{\mathbb{Q}}, A(i+1-j))} \prod_{p \leq \infty} c_p(j) \tag{1.3}$$

where $\Omega(j)$ is the Deligne period, $H^1_f(\mathbb{Q}, A(i+1-j))$ a Bloch-Kato Selmer group and $c_p(j)$ the j-th Tamagawa factor (which equals 1 for almost all p).

When X is an elliptic curve, (1.3) reduces to the Birch and Swinnerton-Dyer conjecture.

Galois representations of modular forms

Conjecturally, to a Siegel Hecke eigenform $\varphi \in \mathcal{M}^n_k$ one can attach a degree $2n$ Galois representation ρ_φ so that properties of the Frobenius element are linked to arithmetic information of φ.

For $n = 1$, this is well known: by a theorem of Deligne [Del71], for any normalised eigenform $\varphi \in \mathcal{M}^1_k(\Gamma_1(N), \chi)$, one can construct a semisimple continuous Galois representation $\rho_\varphi : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}_p})$ which is unramified at all primes $l \nmid pN$ and

$$\mathrm{ch}(\rho_\varphi(\mathrm{Frob}_l)) = X^2 - a(l)X + \chi(l)l^{k-1}$$

where $\mathrm{ch}(\cdot)$ denotes the characteristic polynomial and $a(l)$ is the l-th Fourier coefficient of φ.

For $n = 2$ we have some results. Let $\varphi \in \mathcal{S}_k^2$ be a simultaneous cuspidal eigenform for all Hecke operators $T(n)$, $n \in \mathbb{N}$, with eigenvalue λ_n. Let E denote the smallest algebraic extension containing all the eigenvalues λ_n, and let $\mathfrak{p}$ be any extension of a prime p to E. Then, by a theorem of Weissauer [Det01, proposition 3.2], there exists a continuous representation $\rho_{\varphi,\mathfrak{p}} : G_{\mathbb{Q}} \to \mathrm{GL}_4(\overline{E_{\mathfrak{p}}})$, unramified outside p, such that for every prime $l \neq p$

$$\mathrm{ch}(\rho_{\varphi,\mathfrak{p}}(\mathrm{Frob}_l)) = Q_{\varphi}^{(l)}(X) \tag{1.4}$$

where $Q_{\varphi}^{(l)}$ is the Euler factor (2.21) of the spinor L-function Z_{φ}.

In what follows, we will assume the existence of a suitable representation $\rho_{\varphi,\mathfrak{p}}$ whenever needed.

Critical values of modular forms

This section roughly follows [BDFK19, section 5]. Recall f, F, G, h and $\mathfrak{p}$ from section (1.2), and let $\rho_{\star,\mathfrak{p}}$ be the associated Galois representations. In [BDFK19, section 5.1] it is explained how to make suitable choices for the objects appearing in conjecture (1.3.1) so that the Tamagawa factors are integral at $\mathfrak{p}$. Now, we follow the argument of [BDFK19, section 5.3].

Because of the congruence between F and G, the mod $\mathfrak{p}$ reduction $\overline{\rho_{G,\mathfrak{p}}}$ has composition factors $\overline{\rho_{f,\mathfrak{p}}}$, $\mathbb{F}_{\mathfrak{p}}(1-2k)$ and $\mathbb{F}_{\mathfrak{p}}(2-2k)$. Assuming the irreducibility of $\rho_{G,\mathfrak{p}}$, one can choose a $\mathcal{O}_{\mathfrak{p}}$-invariant lattice so that $\mathbb{F}_{\mathfrak{p}}(2-2k)$ is a submodule of $\overline{\rho_{G,\mathfrak{p}}}$, hence $\mathbb{F}_{\mathfrak{p}}$ embeds into $\overline{\rho_{G,\mathfrak{p}}}(2k-2)$.

Since the vanishing of $L_h(l)$ is due to the sign of its functional equation, one has from [Nek13, theorem B] that $H^1_f(\mathbb{Q}, V_{h,\mathfrak{p}}(l))$ is non-empty, and hence there exists a non-zero element $c \in H^1_f(\mathbb{Q}, \overline{\rho_{h,\mathfrak{p}}}(l))$. After tensoring the embedding $\mathbb{F}_{\mathfrak{p}} \to \overline{\rho_{G,\mathfrak{p}}}(2k-2)$ by $\overline{\rho_{h,\mathfrak{p}}}(l)$, one can see $\overline{\rho_{h,\mathfrak{p}}}(l)$ as a submodule of $\overline{\rho_{h,\mathfrak{p}}} \otimes \overline{\rho_{G,\mathfrak{p}}}(l+2k-2)$, hence c maps to a non-zero $c' \in H^1_f(\mathbb{Q}, \overline{\rho_{h,\mathfrak{p}}} \otimes \overline{\rho_{G,\mathfrak{p}}}(l+2k-2))$.

In fact, this produces a non-zero element of $\mathfrak{p}$-torsion in a Selmer group whose order appears in the right-hand side of (1.3), while the left-hand side equals $Z_{G\otimes h}(l+2k-2)$ because of (1.2) and (1.4).

It follows that we should expect $\mathfrak{p}$ to divide $Z_{G\otimes h}(l+2k-2)$, after suitable normalisation which essentially corresponds to a choice of Deligne period $\Omega(l+2k-2)$. By the functional equation of $Z_{G\otimes h}$, the same should hold for the paired critical point $s = l+2k-1$, and hence we expect to detect a factor of $\mathfrak{p}$ in the ratio (1.1).

§ 1.4 Overview of this thesis

The method

For forms φ, ψ of weight l and genus n define the Petersson inner product as

$$\langle \varphi, \psi \rangle := \int_{\mathcal{F}_n} \varphi(Z)\overline{\psi(Z)} \det(\Im(Z))^{l-n-1} dZ$$

whenever convergent, where $\mathcal{F}_n$ is a particular subset of the Siegel upper-half space $\mathcal{H}_n$, namely its Klingen fundamental domain ([Kli90, definition 3.1]). Recall f, h, F, G from section (1.2): we employ Saha's integral representation formula [Sah10, theorem 6.9]

$$\langle\langle E^3_{2k}(Z \times \tau, 2 - k + 2m), \star(Z)h(\tau)\rangle\rangle = \xi_\star \left(\frac{1 + 4m}{6} \right) Z_{\star\otimes h}(3k - 1 + 2m) \quad (1.5)$$

where $\star = F, G$ and $E^3_{2k}(Z \times \tau, \cdot)$ is a Hermitian Eisenstein series of genus 3 diagonally restricted into $\mathcal{H}_2 \times \mathcal{H}_1$, and $\xi_\star(x)$ is an explicit factor depending only on $\star$ and m. This formula requires the weights of F, G and h be equal, so we have to make this assumption even though the claimed result should be true for unequal weights as well.

Using a combination of analytic tools (namely Shimura-Maass differential operator, diagonal restriction and holomorphic projection, see chapter 3) we can replace the complicated Eisenstein series in the integral with the holomorphic function $\Psi_m(Z, \tau) = B^{(m)}_F h(\tau)F(Z) + B^{(m)}_G h(\tau)G(Z)$ for some constants $B^{(m)}_\star$ so that (1.5) yields

$$\xi_\star \left(\frac{1 + 4m}{6} \right) Z_{\star\otimes h}(3k - 1 + 2m) = B^{(m)}_\star \langle \star, \star\rangle\langle h, h\rangle \quad (1.6)$$

since F and G are orthogonal, hence the arithmetic information we are looking for is encoded in $B^{(m)}_G$.

These constants are obtained via the aforementioned process from the Fourier coefficients of the Hermitian Eisenstein series $\mathcal{E}^3_{4+4m}$. For $m \geq 1$, the defining series converges absolutely and we know the Fourier coefficients to be rational numbers whose denominator can be divided only by a finite set of primes (namely the prime factors of some Dirichlet L-values [NT18, theorem 3.1]). For $m = 0$, we can regardless define $\mathcal{E}^3_4$ by analytic continuation of the non-holomorphic Hermitian Eisenstein series [Nag96, theorem 2.2] and then employ a generalisation of the Siegel-Weil formula proven by Ichino [Ich07, theorem 1.1] which expresses it as a rational linear combination of Hermitian theta series, so that the Fourier coefficients of $\mathcal{E}_4^3$ are again rational and non-integral only at some explicit finite set of primes. Assuming that our prime $\mathfrak{p}$ does not divide all of these Fourier coefficients, then the final function Ψ_m has (double) Fourier expansion whose coefficients are integral at $\mathfrak{p}$ and not all divisible by $\mathfrak{p}$. Additionally, we assume that the same property holds for F, G, h.

We use this arithmeticity to compare the constants $B_F^{(m)}$ and $B_G^{(m)}$ in the equality

$$\Psi_m = B_F^{(m)} hF + B_G^{(m)} hG \tag{1.7}$$

Recall that $Z_{F\otimes h}(3k-1) = 0$ because of the assumption on the weight of l and its functional equation: then (1.6) shows $B_F^{(0)} = 0$, hence $B_G^{(0)}$ is integral at $\mathfrak{p}$ since so are the Fourier coefficients of Ψ_0. From (1.6) we also compute $B_F^{(m)} = \xi_F(\frac{1+4m}{6}) Z_{F\otimes h}(3k-1+2m)\langle F, F\rangle^{-1}\langle h, h\rangle^{-1}$, and use the fact that $\langle F, F\rangle$ can be expressed in terms of $\langle f, f\rangle$ and $L_f(2k)$ (since F is the Saito-Kurokawa lift of f, [Bro07, theorem 1]) to deduce that $\mathfrak{p}$ actually divides the denominator of $B_F^{(m)}$; again (1.7) implies that $\mathfrak{p}$ must divide the denominator of $B_G^{(m)}$ as well. Now thanks to formula (1.6) we have

$$\pi^{-6m} \frac{\xi_G(\frac{1}{6})}{\xi_G(\frac{1+4m}{6})} \frac{Z_{G\otimes h}(3k-1)}{Z_{G\otimes h}(3k-1+2m)} = \frac{B_G^{(0)}}{B_G^{(m)}}$$

and, since the right-hand side is divisible by $\mathfrak{p}$, the whole ratio is a multiple of $\mathfrak{p}$ for any suitable choice of $m > 0$ which does not introduce some unrelated factor $\mathfrak{p}$ via $\xi_G(\frac{1+4m}{6})$.

Guided content

In chapter 2 we recall some basic definitions about Siegel modular forms, the Siegel upper-half space and modular group, Hecke operators, L-functions associated to modular forms (via the Satake parameters and related Euler products) and a brief overview of the Saito-Kurokawa lift.

In chapter 3 we define nearly holomorphic modular forms as a generalisation of Siegel modular forms, where we allow the Fourier coefficients to be polynomials rather than constants. We extend some known facts about Siegel forms to this generalisation, such as finite dimensionality of the space of forms of fixed weight and diagonal restriction, and introduce the Shimura-Maass and holomorphic projection operators. The former acts on nearly holomorphic forms by raising the weight at the expense of the complexity of the Fourier coefficients (namely, by raising the degree of the polynomials), notably in the case of Siegel modular forms (where the coefficients are constant); in particular, the Shimura-Maass operator acts on Eisenstein series roughly as $E_k(Z,s) \mapsto E_{k+2}(Z,s-1)$. This allows to study the arithmeticity of the non-holomorphic $E_k(Z,-v)$ as it is obtained by applying this differential operator to the holomorphic $E_{k-2v}(Z,0)$, whose Fourier coefficients are rational with bounded denominator by a classic result of Siegel. Finally, the holomorphic projection maps nearly holomorphic forms to Siegel cusp forms in such a way that $\langle \varphi, G\rangle = \langle \mathrm{Hol}\varphi, G\rangle$ for any cusp form G of suitable weight; again, this operator respects the arithmeticity of Fourier coefficients.

In chapter 4 we employ the integral representation by Böcherer and Heim

$$\langle E^5_{2k}(\tau \times Z \times W, s), \partial^{k-l}_{1,2l} h(\tau) G(Z) F(W)\rangle = \\ = C_{\triangleright}(s)\langle \Phi_F, \Phi_G\rangle L_f(2s+4k-4) Z_{G\otimes h}(s+3k+l-3)$$

and, by employing the tools introduced in chapter 3, we replace $E^5_{2k}(\tau \times Z \times W, s)$ with a suitable linear combination of cusp forms without changing the overall value of the inner product. As explained in the previous section, this linear combination encodes the arithmetic properties of the critical values of $Z_{\star\otimes h}$, which upon further examination reveals that $Z_{\star\otimes h}(s)$ is holomorphic at $s = 2k+l-1+2m$ for $m \geq 0$, and at least integral at $\mathfrak{p}$ at $s = 2k+l-1$. This is close to what we want, but not quite enough: the reason why we cannot prove the claimed divisibility by $\mathfrak{p}$ is that the integral formula employed involves both F and G at the same time, and applying a certain family of Hecke operators to obtain the mentioned linear combination of cusp forms seems to introduce an unwanted factor of $\mathfrak{p}$, due to the congruence between F and G. We conjecture this to be balanced by the effect of these Hecke operators on the Fourier expansions, but we could not prove it.

As this appears to be an issue with the integral formula rather than the method, we have to use a different representation for the critical values: formula (1.5) works indeed, even though we have to restrict F, G, h to have the same weight. As $\mathcal{E}^3_{2k}$ is a Hermitian modular form, we need to obtain arithmetic information about its Fourier coefficients: we do this in chapter 5, where we employ Ichino's unitary version of the Siegel-Weil formula expressing $\mathcal{E}^3_{4+4m}(Z,0)$ as a linear combination of Hermitian theta series (whose coefficients are rational with bounded denominators), which then gets mapped to $\mathcal{E}^3_{2k}(Z,-v)$ by applying the Hermitian version of the Shimura-Maass differential operator. While there are results about the arithmeticity of Hermitian Eisenstein series of high weight (for instance [NT18, theorem 3.1]), to our knowledge this was not known for low weight (i.e. when the series does not converge absolutely); hence corollary (5.5.4) is a new result, interesting in its own right. We want to thank Nagaoka who, in a personal communication, highlighted the connection between some low weight Eisenstein series and theta series, and pointed us to Ichino's version of the Siegel-Weil formula.

At this point, restriction of $\mathcal{E}_{2k}^3(Z,-v)$ to the Siegel upper-half space yields a nearly holomorphic Siegel modular form, and the theory developed in chapter 3 applies. Hence, we detail in chapter 6 the method explained in the previous section, explaining how the wanted divisibility by $\mathfrak{p}$ essentially comes from $\mathfrak{p}$ dividing $L_f(2k)$, in line with what Bloch-Kato predicts.

Chapter 2

Siegel modular forms

In this chapter we present some classical background material about Siegel modular forms. We introduce the Siegel fundamental domain, the vector space of modular and cuspidal forms and the Petersson inner product. Further, we recall the definition of Hecke operators, spinor L-functions associated to Siegel modular forms and some basic facts about the Saito-Kurokawa lift.

§ 2.1 Symplectic group and Siegel half space

Let $n \in \mathbb{N}^+$ be a positive integer and $\mathcal{H}_n$ be the *Siegel upper half space* of genus n

$$\mathcal{H}_n := \left\{ Z \in \mathbb{C}^{n,n} : Z^t = Z, i(\overline{Z} - Z) > 0 \right\} \tag{2.1}$$

where $M \geq 0$ for a symmetric matrix M means positive semi-definite (or definite, according to the inequality strictness); we systematically write $\mathcal{H}_n \ni Z = X + iY$ by decomposing into real and imaginary part, and put $\delta(Z) := \det(Y)$.

Let

$$S_n := \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix} \tag{2.2}$$

and define the *Siegel modular group* of genus n

$$\Gamma_n := \mathrm{Sp}_{2n}(\mathbb{Z}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \gamma \in \mathrm{GL}_{2n}(\mathbb{Z}) : \gamma^t S_n \gamma = S_n \right\} \tag{2.3}$$

where $A, B, C, D \in \mathbb{Z}^{n,n}$ are the matrix blocks of γ.

Proposition 2.1.1 ([Kli90, proposition 1.1])**.** *The action of Γ_n on $\mathcal{H}_n$ given by fractional linear transformations*

$$\gamma(Z) := (AZ + B)(CZ + D)^{-1} \tag{2.4}$$

is well defined. Further, we have

$$\delta(\gamma(Z)) = |\det(CZ+D)|^{-2}\delta(Z) \tag{2.5}$$

for every $\gamma \in \Gamma_n$ and $Z \in \mathcal{H}_n$.

By defining the *automorphy factor*

$$j(\gamma, Z) := \det(CZ+D) \tag{2.6}$$

we can rewrite (2.5) as $\delta(\gamma(Z)) = |j(\gamma, Z)|^{-2}\delta(Z)$.

Following Klingen [Kli90, defin. 3.1], we define the *Siegel fundamental domain*

$$\mathcal{F}_n := \{Z \in \mathcal{H}_n : Z \text{ satisfies (i)-(iii)}\} \tag{2.7}$$

(i) $|j(\gamma, Z)| \geq 1$ for every $\gamma \in \Gamma_n$;

(ii) Y is reduced in the sense of Minkowski [Kli90, defin. 2.1];

(iii) $|X_{lk}| \leq {}^1\!/_2$ for every entry of X.

Then $\mathcal{F}_n$ is a standard representative for $\Gamma_n \backslash \mathcal{H}_n$, in the sense that every $Z \in \mathcal{H}_n$ is a Γ_n-translate of points of $\mathcal{F}_n$ and every identification inside $\mathcal{F}_n$ happens on its boundary. As a special case, when $n = 1$ we recover the classic fundamental domain for $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$ since condition (ii) is vacuous and (i) demands $|z| \geq 1$ by taking $\gamma = S_1$.

§ 2.2 Holomorphic modular forms

A *Siegel modular form* of genus n and weight $k \in \mathbb{Z}$ is a function

$$F : \mathcal{H}_n \to \mathbb{C} \text{ satisfying (i)-(iii)} \tag{2.8}$$

(i) F is holomorphic (in each entry of Z);

(ii) $F(\gamma(Z)) = j(\gamma, Z)^k F(Z)$ for every $Z \in \mathcal{H}_n$ and $\gamma \in \Gamma_n$;

(iii) F is bounded on vertical strips $\{Z \in \mathcal{H}_n : Y > \epsilon I_n\}$ for every $\epsilon > 0$.

Hence a Siegel modular form of genus 1 is an elliptic modular form. More generally, we call *weakly modular* any function transforming like (2.8-ii); by taking $\gamma = -I_{2n}$ we see $F(Z) = F(\gamma(Z)) = (-1)^{nk}F(Z)$, hence nk must be even.

Since $F(Z+B) = F(Z)$ for any symmetric $B \in \mathbb{Z}^{n,n}$, F admits absolutely convergent Fourier series of the form

$$F(Z) = \sum_A F_A e^{2\pi i \mathrm{Tr}(AZ)} \tag{2.9}$$

on $\mathcal{H}_n$ for complex *Fourier coefficients* $F_A \in \mathbb{C}$ and symmetric *half-integral* matrices $A \in \mathbb{Q}^{n,n}$, i.e. $2A_{ij} \in \mathbb{Z}$ and $A_{ii} \in \mathbb{Z}$; as is standard, we will often write $e^{2\pi i \mathrm{Tr}(AZ)} =: q^A$.

The Fourier coefficients satisfy the relation $F_{U^tAU} = F_A$ for any $U \in \mathrm{GL}_n(\mathbb{Z})$ [Kli90, formula 4.2], and using this one can show that $F_A = 0$ whenever A is not positive semi-definite [Kli90, proof 4.1]. For $n \geq 2$, this implies that F is bounded on every vertical strip $\{Z \in \mathcal{H}_n : Y > \epsilon I_n\}$ [Kli90, theorem 4.1], showing that condition (2.8-iii) is implied by (2.8-i) and (2.8-ii). This does not happen for $n = 1$, so we require condition (2.8-iii) anyway to include elliptic modular forms as a special case of Siegel modular forms.

We also observe the following fact about the Fourier indices.

Lemma 2.2.1. *Let A be a symmetric half-integral positive semi-definite matrix. If any of the diagonal entries vanishes, then so does that entire row and column.*

Proof. Assume $A_{mm} = 0$. Fix any $l \neq m$ and consider the vector $x \in \mathbb{Q}^n$ with $x_i = 0$ for $i \neq m, l$ and $x_l = 1$; hence $x^tAx = A_{ll} + 2A_{lm}x_m$.

If $A_{lm} = A_{ml}$ were non-zero, then we could find some $x_m \in \mathbb{Q}$ such that $x^tAx < 0$, against the assumption of $A \geq 0$. □

As in the classical case, *cusp forms* play an important role. We say that a Siegel modular form F is cuspidal if

$$\lim_{t\to\infty} F\left(\begin{pmatrix} Z^* & 0 \\ 0 & it \end{pmatrix}\right) = 0 \tag{2.10}$$

for any $Z^* \in \mathcal{H}_{n-1}$. Looking at the Fourier expansion, by lemma (2.2.1) this is equivalent to saying that $F_A = 0$ whenever A is non-definite; from this follows

$$|F(Z)| \ll_{\epsilon,c} e^{-\mathrm{Tr}(cY)} \tag{2.11}$$

on every vertical strip $\{Z \in \mathcal{H}_n : Y > \epsilon I_n\}$ [Kli90, proposition 5.3] for $\epsilon, c > 0$. We remark that the fundamental domain $\mathcal{F}_n$ is contained in every *large enough* vertical strip, i.e. for ϵ small enough [Kli90, lemma 3.2]; hence every cusp form decreases exponentially fast on $\mathcal{F}_n$ *going up* i.e. for $\mathrm{Tr}(Y) \to \infty$.

We denote

$$\begin{aligned}\mathcal{M}_k^n &:= \{F : \mathcal{H}_n \to \mathbb{C} \text{ modular form of weight } k\} \\ \mathcal{S}_k^n &:= \{F : \mathcal{H}_n \to \mathbb{C} \text{ cusp form of weight } k\}\end{aligned} \tag{2.12}$$

These are obviously complex vector spaces, and $\mathcal{S}_k^n \subseteq \mathcal{M}_k^n$. It is well known they are finite dimensional: this will follow from a more general result in the next chapter - proposition (3.1.2).

Finally, put

$$d\mu(Z) := \frac{dX dY}{\delta(Z)^{n+1}} \tag{2.13}$$

where $dX := \prod_{i\leq k} dX_{ij}$ and $dY := \prod_{i\leq k} dY_{ij}$; then $d\mu$ is a Γ_n left-invariant measure on $\mathcal{H}_n$. The *Petersson inner product* of modular forms F, G of weight k and genus n is defined as

$$\langle F, G\rangle := \int_{\Gamma_n\backslash\mathcal{H}_n} F(Z)\overline{G(Z)}\delta(Z)^k d\mu(Z) \tag{2.14}$$

whenever convergent. Observe that the integrand is Γ_n left-invariant thanks to (2.5), hence the integration domain is well defined: we will often read $\mathcal{F}_n$ for $\Gamma_n\backslash\mathcal{H}_n$, but any other set of representatives can be used as well. Due to (2.11), the integral does converge if any of the forms is cuspidal; in particular the Petersson inner product gives $\mathcal{S}_k^n$ the structure of Hilbert space.

§ 2.3 Hecke operators

Following the normalisation of [AS01], define the *slash operator* $[\star]_k$ by

$$F[\gamma]_k(Z) := \det(\gamma)^{k/2} j(\gamma, Z)^{-k} F(\gamma(Z)) \tag{2.15}$$

where F is any function $F : \mathcal{H}_n \to \mathbb{C}$ and $\gamma \in \mathrm{Sp}_{2n}(\mathbb{R})$, so that $j(\gamma, Z) \neq 0$ - see proof of [Kli90, proposition 1.1]; in particular, we can rewrite (2.8-ii) as

$$F[\gamma]_k = F \quad \forall\gamma \in \Gamma_n$$

i.e. weakly modular forms are slash-invariant functions on $\mathcal{H}_n$. Further, we observe

$$F[\gamma_1\gamma_2]_k = (F[\gamma_1]_k)\,[\gamma_2]_k \tag{2.16}$$

Fix a prime p and let $M \in \mathrm{GSp}_{2n}(\mathbb{Z}[p^{-1}])^+$, i.e. $M^t S_n M = \mu_n(M) S_n$ for some $\mu_n(M) \in \mathbb{Q}^\times$ and $M \in \mathrm{GL}_{2n}(\mathbb{Z}[p^{-1}])$ with positive determinant. Decompose the double coset

$$\Gamma_n M \Gamma_n = \bigsqcup_i \Gamma_n M_i$$

and define the weight k *Hecke operator* $T_M^{(k)}$ by

$$T_M^{(k)}F = F[\Gamma_n M \Gamma_n]_k := \sum_i F[M_i]_k \tag{2.17}$$

If F is weakly modular for weight k, then $T_M^{(k)}F$ is well defined, i.e. it does not depend on the choice of representatives for the double coset decomposition: if $\{N_i\}_i$ is another such choice, then $N_i = \gamma_i M_i$ for some $\gamma_i \in \Gamma_n$ and

$$\sum_i F[N_i]_k = \sum_i F[\gamma_i M_i]_k = \sum_i F[\gamma_i]_k[M_i]_k = \sum_i F[M_i]_k$$

where in the last step we have used $F[\gamma]_k = F$ for every $\gamma \in \Gamma_n$. Further, $T_M^{(k)}F$ is weakly modular itself:

$$\sum_i F[M_i]_k[\gamma]_k = \sum_i F[M_i\gamma]_k = \sum_i F[M_{\sigma(i)}]_k$$

since right-multiplication by $\gamma \in \Gamma_n$ induces a permutation of representatives $\{M_i\}_i$, and hence $(T_M^{(k)}F)[\gamma]_k = T_M^{(k)}F$. If $F \in \mathcal{M}_k^n$, then $T_M^{(k)}F \in \mathcal{M}_k^n$.

The *local Hecke algebra* is the set

$$H_k^n(p) := \{T_M^{(k)} : M \in \mathrm{GSp}_{2n}(\mathbb{Z}[p^{-1}])^+\} \tag{2.18}$$

which is indeed a $\mathbb{Q}$-algebra since the slash operator (2.15) is $\mathbb{Q}$-linear and satisfies (2.16). Define a particular set of (local) Hecke operators by

$$\begin{aligned} T_p^{(k)} &\sim \Gamma_n \ \mathrm{diag}(\underbrace{1,\dots,1}_{n},\underbrace{p,\dots,p}_{n})\Gamma_n \\ T_{p^2,i}^{(k)} &\sim \Gamma_n \ \mathrm{diag}(\underbrace{1,\dots,1}_{n-i},\underbrace{p,\dots,p}_{i},\underbrace{p^2,\dots,p^2}_{n-i},\underbrace{p,\dots,p}_{i})\Gamma_n \end{aligned} \tag{2.19}$$

for $i = 0, \dots, n$; then $H_k^n(p)$ is generated by these operators.

More generally, if $M \in \mathrm{GSp}_{2n}(\mathbb{Q})^+$ then we can write

$$M = \prod_p M_p$$

where $M_p = I_{2n}$ for almost all primes p and $M_p \in \mathrm{GSp}_{2n}(\mathbb{Z}[p^{-1}])^+$, and put $T_M^{(k)} := \prod_p T_{M_p}^{(k)}$. Define the *Hecke algebra* by

$$H_k^n := \{T_M^{(k)} : M \in \mathrm{GSp}_{2n}(\mathbb{Q})^+\} \tag{2.20}$$

then $H_k^n \cong \otimes_p H_k^n(p)$.

§ 2.4 L-functions

A modular form $F \in \mathcal{M}_k^n$ is called *Hecke eigenform* if

$$TF = \lambda_F(T)F$$

for every $T \in H_k^n$, where $\lambda_F(T) \in \mathbb{C}$ is the T-th eigenvalue of F. For the rest of this section, F is assumed to be an eigenform.

Every local Hecke algebra $H_k^n(p)$ is isomorphic via the Satake isomorphism to $\mathbb{Q}[x_0^{\pm}, \ldots, x_n^{\pm}]^{W_n}$ where W_n, is the Weyl group of Sp_{2n}: see [Gro98] for an expository treatment. Then, for every prime p,

$$\lambda_F : H_k^n(p) \cong \mathbb{Q}[x_0^{\pm}, \ldots, x_n^{\pm}]^{W_n} \to \mathbb{C}$$

is an algebra homomorphism, completely determined by a tuple

$$(\alpha_{p,0}, \ldots, \alpha_{p,n}) \in (\mathbb{C}^\times)^{n+1}$$

unique up to the action of the Weyl group. We call these numbers *Satake parameters* of F.

Put

$$Q_F^{(p)}(X) := (1 - \alpha_{p,0}X) \prod_{r=1}^{n} \prod_{1 \leq i_1 < \cdots < i_r \leq n} (1 - \alpha_{p,0}\alpha_{p,i_1} \cdots \alpha_{p,i_r} X) \tag{2.21}$$

and define the *spinor L-function* associated to F by

$$Z_F(s) := \prod_p \left\{ Q_F^{(p)}(p^{-s}) \right\}^{-1} \tag{2.22}$$

If $h \in \mathcal{M}_l^1$ is a *normalised* Hecke eigenform (i.e. the first Fourier coefficient is 1), define the *tensor product L-function* associated to F and h by

$$Z_{F\otimes h}(s) := \prod_p \left\{ Q_F^{(p)}(\beta_{p,0}p^{-s}) Q_F^{(p)}(\beta_{p,0}\beta_{p,1}p^{-s}) \right\}^{-1} \tag{2.23}$$

where $(\beta_{p,0}, \beta_{p,0}\beta_{p,1})$ are the GL_2-Satake parameters of h. Finally, the spinor L-function associated to h is exactly the classical (elliptic) L-function

$$Z_h(s) = L_h(s) := \prod_p \left\{ 1 - h_p p^{-s} + p^{l-1} p^{-2s} \right\}^{-1} \tag{2.24}$$

where h_p is the p-th Fourier coefficient of h. This is because $\beta_{p,0} + \beta_{p,1} = h_p$ and $\beta_{p,0}\beta_{p,1} = p^{k-1}$ for any choice of Satake parameters (up to the action of the Weil group), so the two definitions coincide.

All the object that we have defined as Euler products converge for $\Re(s)$ *large enough*, and the abscissa of convergence varies greatly depending on the nature of the forms in question. Further, these L-functions are conjectured to have a functional equation, but we currently know this only for a small subset of all Siegel modular forms. In our particular case, we only deal with spinor L-functions associated to forms of genus 1 and 2, of which we know more than the general case: we summarise this in the next propositions.

Proposition 2.4.1. *[Bum97, proposition 1.3.6] Let $h \in \mathcal{S}_l^1$ be a normalised Hecke eigenform, with $l \geq 2$ even. Then the Euler product (2.24) converges absolutely for $\Re(s) > l/2$. Further, the completed function*

$$\Lambda_h(s) := (2\pi)^{-s}\Gamma(s)L_h(s)$$

extends to an entire function on $\mathbb{C}$ and satisfies the functional equation

$$\Lambda_h(s) = (-1)^{l/2}\Lambda_h(l-s)$$

Proposition 2.4.2. *[And74, theorem 3.1.1] Let $F \in \mathcal{S}_k^2$ be a Hecke eigenform. Then the Euler product (2.22) converges absolutely for $\Re(s)$ large enough, and the completed function*

$$\Lambda_F(s) := (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)Z_F(s)$$

extends to a meromorphic function on $\mathbb{C}$, with at most simple poles at $k-2$ and k, and satisfies the functional equation

$$\Lambda_F(s) = (-1)^k\Lambda_h(2k-2-s)$$

Proposition 2.4.3. *[BH00, theorem 3.8] Let $h \in \mathcal{S}_l^1$ (normalised) and $F \in \mathcal{S}_k^2$ be Hecke eigenforms, with $k \geq l \geq 2$ even, and assume that the first Fourier-Jacobi coefficient of F does not vanish. Then the Euler product (2.23) converges absolutely at least for $\Re(s) > 3 - k/2$. Further, the completed function*

$$\Lambda_{F\otimes h}(s) := (2\pi)^{-4s}\tilde{\Gamma}_\Delta(s)Z_{F\otimes h}(s)$$

(where $\tilde{\Gamma}_\Delta$ is defined in [BH00, formula 24]) extends to a meromorphic function on $\mathbb{C}$ and satisfies the functional equation

$$\Lambda_{F\otimes h}(s) = \Lambda_{F\otimes h}(2k+l-3-s)$$

Although we do not use it in this thesis, we report for completeness that a similar result holds in the case $k < l$ by [BH00, theorem 6.1].

§ 2.5 Saito-Kurokawa lift

Let $F \in \mathcal{M}_k^2$ with Fourier expansion

$$F(Z) = \sum_{\substack{a,b,c \in \mathbb{Z} \\ a,c \geq 0 \\ b^2 \leq 4ac}} F_{[a,b,c]} q^{[a,b,c]} = \sum_{\substack{a,b,c \in \mathbb{Z} \\ a,c \geq 0 \\ b^2 \leq 4ac}} F_{[a,b,c]} e^{2\pi i a z} e^{2\pi i b w} e^{2\pi i c \tau} \tag{2.25}$$

where

$$[a,b,c] := \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \qquad Z := \begin{pmatrix} z & w \\ w & \tau \end{pmatrix}$$

Define the *c-th Fourier-Jacobi coefficient* $\Phi_F^{(c)}$ associated to F by

$$F(Z) =: \sum_{c \geq 0} \Phi_F^{(c)}(z,w) e^{2\pi i c \tau} = \sum_{c \geq 0} \left(\sum_{\substack{a,b \in \mathbb{Z} \\ a \geq 0 \\ b^2 \leq 4ac}} F_{[a,b,c]} e^{2\pi i a z} e^{2\pi i b w} \right) e^{2\pi i c \tau} \tag{2.26}$$

which is a special case of Jacobi forms, functions on $\mathcal{H}_1 \times \mathbb{C}$ transforming in a particular way under the action of the semi-direct product $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$. We will only need the first Fourier-Jacobi coefficient $\Phi_F = \Phi_F^{(1)}$ in the following chapters; see [Bro05, section 2.4] for more details about Jacobi forms.

Similarly, we refer to [Bro05, section 2.3] for an overview on half-integral weight modular forms; all we need to know now is that these function (like the elliptic forms in $\mathcal{M}_k^1$) admit Fourier expansion

$$g(z) = \sum_{n \geq 0} g_n q^n$$

where g_n is the n-th Fourier coefficient of the half-integral weight form g.

Finally, let $\breve{\mathcal{S}}_k^2$ be the *Maass subspace* of $\mathcal{S}_k^2$, consisting of cusp forms F whose Fourier coefficients as in (2.25) satisfy the relation

$$F_{[a,b,c]} = \sum_{d \mid \gcd(a,b,c)} d^{k-1} F_{[ac/d^2, b/d, 1]} \tag{2.27}$$

Proposition 2.5.1 (Saito-Kurokawa lift). *For every $k \in \mathbb{N}$, there exists a Hecke-equivariant isomorphism of vector spaces*

$$\mathcal{S}_{4k-2}^1 \to \breve{\mathcal{S}}_{2k}^2$$

The isomorphism is the map

$$f \mapsto g_f \mapsto \Phi_F \mapsto F \tag{2.28}$$

where g_f is a half-integral weight modular form (associated to f via the Shintani lift [Bro05, section 3.1]), Φ_F is a Jacobi form ([Bro05, section 3.2]) which turns out to be exactly the first Fourier-Jacobi coefficient of $F \in \breve{\mathcal{S}}^2_{2k}$ [Bro05, section 3.3]: we call F the *Saito-Kurokawa lift* of $f \in \mathcal{S}^1_{4k-2}$. This lift is Hecke-equivariant, in the sense that it maps Hecke eigenforms to Hecke eigenforms; further we have the following relation for the eigenvalues:

$$\begin{aligned} \lambda_F(T_p^{(2k)}) &= p^{2k-1} + p^{2k-2} + \lambda_f(T_p^{(4k-2)}) \\ &= p^{2k-1} + p^{2k-2} + f_p \end{aligned} \tag{2.29}$$

assuming that f is normalised, so that the T_p-th eigenvalue of f is exactly its p-th Fourier coefficient f_p. Because of this relation between eigenvalues, the L-functions of f and its Saito-Kurokawa lift F are connected by

$$Z_F(s) = \zeta(s-2k+1)\zeta(s-2k-2)L_f(s) \tag{2.30}$$

where ζ is the Riemann zeta function.

Remark 2.5.2. While $f \in \mathcal{S}^1_{4k-2}$ is assumed normalised so that its first Fourier coefficient is 1, there is no standard scaling for Siegel forms of higher genus (since the spinor L-function depends only on the eigenvalues), and in particular for the Saito-Kurokawa lift F of f. However, if the coefficients of f are in some ring $\mathcal{O} \subseteq \mathbb{C}$, then there is a suitable scaling of (2.28) such that also F has Fourier coefficients in $\mathcal{O}$ ([Bro05, theorem III.2], [Bro05, theorem III.3] and [Bro05, theorem III.7]).

Chapter 3

Non-holomorphic Siegel Eisenstein series

In this chapter, we introduce nearly holomorphic modular forms, together with some operators that we will need to use in the following chapters, namely diagonal restriction, Shimura-Maass differential operator and holomorphic projection. We also prove some technical lemmas and propositions regarding the arithmeticity of the Fourier coefficients of certain Eisenstein series, which turn out to be examples of nearly holomorphic modular forms. Further, we show how the arithmetic information of the Fourier coefficients is preserved when applying the operators above.

§ 3.1 Nearly holomorphic forms

Nearly holomorphic modular forms were introducted in the 70's to prove algebraicity results for special L-values [Shi78]. Here we follow Mizumoto [Miz97] and give a more restrictive set of conditions that will simplify working with these forms.

A *nearly holomorphic* modular form of genus n, weight k and degree d is a function

$$F : \mathcal{H}_n \to \mathbb{C} \text{ satisfying (i)-(v)} \tag{3.1}$$

(i) $F \in \mathcal{C}^\infty(\mathcal{H}_n)$ in each entry of Z;

(ii) $F(\gamma(Z)) = j(\gamma, Z)^k F(Z)$ for every $Z \in \mathcal{H}_n$ and $\gamma \in \Gamma_n$;

(iii) $\delta(Z)^d F(Z)$ is a polynomial in the entries of Y with bounded holomorphic functions in Z as coefficients.

For ease of notation, we write $\mathbb{C}[Y] := \mathbb{C}[\{Y_{ij}\}_{i,j}]$. Any function F satisfying (3.1-i)-(3.1-iii) admits an absolutely convergent Fourier expansion on $\mathcal{H}_n$ of the form

$$F(Z) = \delta(Z)^{-d} \sum_{A \geq 0} F_A(Y) q^A \tag{3.2}$$

where $F_A(Y) \in \mathbb{C}[Y]$ is a polynomial in the entries of Y. By the same argument of [Kli90, formula 4.2] we have

$$F_{U^t A U}(Y) = F_A(UYU^t) \tag{3.3}$$

for any $U \in \mathrm{GL}_n(\mathbb{Z})$. Similarly to the holomorphic case, F is bounded on vertical strips [Miz97, theorem 1.4(i)]. In addition we require

(iv) $\delta(Z)^d F_A(Y^{-1})$ is also in $\mathbb{C}[Y]$ for every $A \geq 0$;

(v) if $A = \left(\begin{smallmatrix} A^{(r)} & 0 \\ 0 & 0 \end{smallmatrix}\right)$ for a $r \times r$ matrix $A^{(r)}$ and $r = 0, \ldots, n-1$ then $F_A(Y)$ is a polynomial in $\det(Y)$ and the entries of the upper left $r \times r$ block of Y.

Shimura's original definition requires only (i), (ii) and (iv). The latter in particular, demanding the coefficients of F to be polynomials in the inverse of Y, is quite unwieldy; condition (iii) and hence formula (3.2) are instead more practical for what we will discuss later in this chapter.

We now introduce the *multi-index notation* to simplify working with the Fourier expansion (3.2). For any matrices $\alpha \in \mathbb{Q}^{n,n}$ and $Y \in \mathbb{C}^{n,n}$, let

$$Y^\alpha := \prod_{i,j=1}^{n} Y_{ij}^{\alpha_{ij}} \in \mathbb{C} \tag{3.4}$$

and also put $|\alpha| := \sum_{i,j} \alpha_{ij}$. Then Y^α is a (rational) monomial in the entries of Y of total degree $|\alpha|$; conversely every such monomial can be written as Y^α for a suitable multi-index α. If Y is symmetric then α is not uniquely determined: we stipulate to always take the unique multi-index α which is symmetric and half-integral so that

$$Y^\alpha = \prod_{i=1}^{n} Y_{ii}^{\alpha_{ii}} \prod_{i<j} Y_{ij}^{\alpha_{ij}+\alpha_{ji}}$$

is a monomial in $\mathbb{C}[Y]$. We will write any polynomial $p(Y) \in \mathbb{C}[Y]$ as $p(Y) = \sum_\alpha p_\alpha Y^\alpha$ for suitable constants $p_\alpha \in \mathbb{C}$.

Lemma 3.1.1. *Let $F_A(Y)$ be as in (3.2). Then the total degree of $F_A(Y)$ is $\leq nd$, and the degree of each of its monomials satisfies $\alpha_{mm} + 2\sum_{i=1}^n \alpha_{im} \leq d$ for any $m \leq n$.*

Viceversa, any polynomial $p(Y) \in \mathbb{C}[Y]$ with these properties satisfies (3.1-iv).

Proof. Recall that $Y^{-1} = \det(Y)^{-1}(q_{ij}(Y))_{i,j}$ for polynomials $q_{ij}(Y) \in \mathbb{C}[Y]$ (namely its cofactors) so that the degree of Y_{ml} in $q_{ij}(Y)$ is

$$\deg_{Y_{mj}}(q_{ij}(Y)) = \begin{cases} 1 & i \neq m \text{ and } j \neq l \\ 0 & \text{otherwise} \end{cases}$$

hence the degree of Y_{ml} in $\prod_{i,j} q_{ij}(Y)^{\alpha_{ij}}$ equals

$$\sum_{i,j} \deg_{Y_{ml}}(q_{ij}(Y)^{\alpha_{ij}}) = \sum_{i,j}^{i\neq m, j\neq l} \alpha_{ij} = |\alpha| - \sum_i \alpha_{il} - \sum_j \alpha_{mj} + \alpha_{ml}$$

Now, by (3.1-iv)

$$\delta(Z)^d F_A(Y^{-1}) = \sum_\alpha F_{A,\alpha} \det(Y)^{d-|\alpha|} \prod_{i,j} q_{ij}(Y)^{\alpha_{ij}}$$

is in $\mathbb{C}[Y]$, and since the degree of Y_{ml} in $\det(Y)$ is exactly 1,

$$0 \leq \deg_{Y_{ml}}(\delta(Z)^d F_A(Y^{-1})) \leq d - \sum_i \alpha_{il} - \sum_j \alpha_{mj} + \alpha_{ml}$$

By letting $m = l$ we deduce (by the symmetry of α)

$$0 \leq d - 2\sum_{i\neq m} \alpha_{im} - \alpha_{mm} \tag{3.5}$$

which proves the second statement. Summing over $l = 1, \ldots, n$ instead we see

$$0 \leq nd - |\alpha| - \sum_j \alpha_{mj} + \sum_l \alpha_{ml} = nd - |\alpha|$$

as claimed.

For the additional statement, write $p(Y) = \sum_{|\alpha|\leq nd} p_\alpha Y^\alpha$. Then

$$\det(Y)^d p(Y^{-1}) = \sum_{|\alpha|\leq nd} p_\alpha \det(Y)^{d-|\alpha|} \prod_{i,j} q_{ij}(Y)^{\alpha_{ij}}$$

and the degree of Y_{ml} in each of its summands is

$$\deg_{Y_{ml}}(\det(Y)^d p(Y^{-1})) = d - \sum_i \alpha_{il} - \sum_j \alpha_{mj} + \alpha_{ml}$$

which is non-negative by (3.5). □

Let

$$\mathcal{N}^n_{k,d} := \{F : \mathcal{H}_n \to \mathbb{C} \text{ nearly holomorphic of weight } k \text{ and degree } \leq d\} \tag{3.6}$$

Clearly $\mathcal{N}^n_{k,d}$ is a complex vector space and $\mathcal{N}^n_{k,d} \subseteq \mathcal{N}^n_{k,d+1}$ for every $d \geq 0$. By lemma (3.1.1) we see that $\mathcal{N}^n_{k,0} = \mathcal{M}^n_k$, since the Fourier coefficients of $F \in \mathcal{N}^n_{k,0}$ must be polynomials of degree 0 hence constant, making F holomorphic and modular. Finally, it is known that every $\mathcal{N}^n_{k,d}$ is finite dimensional:

Proposition 3.1.2 ([Miz97, proposition 3.2(1)]). *There exist a positive constant c_n depending only on n and a positive constant $b_{n,d}$ depending only on n and d such that*

$$\dim(\mathcal{N}^n_{k,d}) \leq c_n(d+1)(k+b_{n,d})^{\frac{n(n+1)}{2}}$$

for every $k \geq 0$.

§ 3.2 Diagonal restriction

For $Z \in \mathcal{H}_n$ and $W \in \mathcal{H}_m$, let

$$Z \times W := \begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix} \tag{3.7}$$

where 0 represents zero matrices of suitable dimension; then $Z \times W \in \mathcal{H}_{n+m}$. Similarly define $\otimes_i Z_i \in \mathcal{H}_{(\sum n_i)}$ as the block diagonal matrix with $Z_i \in \mathcal{H}_{n_i}$ as blocks.

Vice versa, for $Z \in \mathcal{H}_{n+m}$, let Z^* be the top left $n \times n$ block of Z and Z_* the bottom right $m \times m$ block of Z, so that

$$Z = \begin{pmatrix} Z^* & \star \\ \star & Z_* \end{pmatrix} \tag{3.8}$$

Then $Z^* \in \mathcal{H}_n$ and $Z_* \in \mathcal{H}_m$. More generally, every $n_i \times n_i$ block Z_i on the diagonal of Z is in $\mathcal{H}_{n_i}$; we refer to the map $\mathcal{H}_{\sum n_i} \to \prod_i \mathcal{H}_{n_i}$ given by $Z \mapsto (Z_i)_i$ as *diagonal restriction*, and suppress the dimensions n_i from the notation when clear from the context.

Define

$$\begin{pmatrix} A_\gamma & B_\gamma \\ C_\gamma & D_\gamma \end{pmatrix} \times \begin{pmatrix} A_\eta & B_\eta \\ C_\eta & D_\eta \end{pmatrix} := \begin{pmatrix} A_\gamma & 0 & B_\gamma & 0 \\ 0 & A_\eta & 0 & B_\eta \\ C_\gamma & 0 & D_\gamma & 0 \\ 0 & C_\eta & 0 & D_\eta \end{pmatrix} \tag{3.9}$$

Then $\gamma \times \eta \in \Gamma_{n+m}$ for $\gamma \in \Gamma_n$ and $\eta \in \Gamma_m$. It is immediate to show that the action of the symplectic group on the Siegel space commutes with the diagonal restriction, i.e.

$$\begin{aligned}(\gamma \times \eta)(Z \times W) &= \gamma(Z) \times \eta(W)\\ j(\gamma \times \eta, Z \times W) &= j(\gamma, Z) j(\eta, W)\\ \delta(Z \times W) &= \delta(Z)\delta(W)\end{aligned} \tag{3.10}$$

hence, if F is weakly modular of weight k, then

$$F(\gamma(Z) \times \eta(W)) = j(\gamma, Z)^k j(\eta, W)^k F(Z \times W)$$

This suggests that diagonally restricting a nearly holomorphic modular form should produce something of the same kind.

Lemma 3.2.1. *Let $F \in \mathcal{N}_{k,d}^{n+m}$ and*

$$G_W(Z) := F(Z \times W) =: H_Z(W)$$

for $Z \in \mathcal{H}_n$ and $W \in \mathcal{H}_m$. Then $G_W \in \mathcal{N}_{k,d}^n$ and $H_Z \in \mathcal{N}_{k,d}^m$ for every fixed Z, W.

Proof. We show that $G_W \in \mathcal{N}_{k,d}^n$; the proof for the other statement is analogous. Write $Z = X + iY$ and $W = U + iV$ by decomposing in real and imaginary part, so that by (3.2)

$$G_W(Z) = \delta(Z)^{-d}\delta(W)^{-d} \sum_{A \geq 0} \sum_{|\alpha| \leq (n+m)d} F_{A,\alpha}(Y \times V)^\alpha e^{2\pi i \mathrm{Tr}(A(Z \times W))}$$

Observe that $\mathrm{Tr}(A(Z \times W)) = \mathrm{Tr}(A^* Z) + \mathrm{Tr}(A_* W)$ where A^* and A_* are the top left $n \times n$ and bottom right $m \times m$ blocks of A. With similar notation $(Y \times V)^\alpha = Y^{\alpha^*} V^{\alpha_*}$, hence

$$G_W(Z) = \delta(Z)^{-d} \sum_{B_{(n)} \geq 0} \sum_\beta \left(\delta(W)^{-d} \sum_{A \geq 0}^{A^* = B} \sum_{|\alpha| \leq (n+m)d}^{\alpha^* = \beta} F_{A,\alpha} V^{\alpha_*} e^{2\pi i \mathrm{Tr}(A_*)W} \right) Y^\beta q^B$$

The expression in brackets is constant for fixed W; further the multi-indices β have degree $\leq nd$ and follow (3.5), hence the polynomials in the Fourier expansion of G_W satisfy (3.1-i)-(3.1-iv) by lemma (3.1.1).

To complete the proof, we need to show that G_W satisfies (3.1-v). Take $B_{(n)} \geq 0$ and assume that $B = B' \times 0_{n-r}$ for its top left $r \times r$ block B'. Fix any $A_{(n+m)} \geq 0$ with $A^* = B$ as above, and take the permutation matrix $U \in \mathrm{GL}_{n+m}(\mathbb{Z})$ such that $U^t A U$ has a zero bottom right $r \times r$ block. By (3.3) we have

$$F_A(Y \times V) = F_{U^t A U}(U^{-1}(Y \times V)U^{-t})$$

which by (3.1-v) is a polynomial in $\det(Y)\det(V)$ and the top left $n+m-r$ entries of $U^{-1}(Y \times V)U^{-t}$, which are exactly the entries of V and Y'. Since the B-th Fourier coefficient of G_W is obtained as a sum of such polynomials, we deduce that (3.1-v) holds for G_W as wanted. □

Proposition 3.2.2. *The diagonal restriction induces a map*

$$\mathcal{N}^{n+m}_{k,d} \to \mathcal{N}^{n}_{k,d} \otimes_{\mathbb{C}} \mathcal{N}^{m}_{k,d}$$

Proof. This proof is taken from [Fre83, pag. 147], but we present it here as we are going to generalise it in the following section. Let $F \in \mathcal{N}^{n+m}_{k,d}$ and observe that $Z \mapsto F(Z \times W)$ is an element of $\mathcal{N}^{n}_{k,d}$ for every choice of $W \in \mathcal{H}_m$ by lemma (3.2.1). This vector space is finite dimensional by proposition (3.1.2), say spanned by a basis $(\varphi_1, \dots, \varphi_R)$, hence

$$F(Z \times W) = \sum_{i=1}^{R} c_i(W)\varphi_i(Z)$$

for some complex coefficients $c_i(W)$. Fix some $Z_1, \dots, Z_R \in \mathcal{H}_n$ and write the system

$$\begin{pmatrix} F(Z_1 \times W) \\ \vdots \\ F(Z_R \times W) \end{pmatrix} = \begin{pmatrix} \varphi_1(Z_1) & \dots & \varphi_R(Z_1) \\ \vdots & \ddots & \vdots \\ \varphi_1(Z_R) & \dots & \varphi_R(Z_R) \end{pmatrix} \begin{pmatrix} c_1(W) \\ \vdots \\ c_R(W) \end{pmatrix} \tag{3.11}$$

If the central matrix is invertible, we can multiple by its inverse and read (say from the q-th line)

$$c_q(W) = \sum_{i=1}^{R} d_i(Z_1, \dots, Z_R) F(Z_i \times W)$$

where the $d_i(Z_1, \dots, Z_R)$ are complex coefficients depending on the choice of Z_i. But the map $W \mapsto F(Z_i \times W)$ is in $\mathcal{N}^{m}_{k,d} = \operatorname{span}(\psi_1, \dots, \psi_S)$ hence

$$F(Z \times W) = \sum_{i,j=1}^{R,S} c_{ij}\varphi_i(Z)\psi_j(W) \tag{3.12}$$

i.e. an element of $\mathcal{N}^{n}_{k,d} \otimes_{\mathbb{C}} \mathcal{N}^{m}_{k,d}$.

To complete the proof we need to show that there exist $Z_1, \dots, Z_R$ such that the central matrix in (3.11) is invertible. We proceed by induction: the matrix $(\varphi_1(Z_1))$ is invertible if Z_1 is not a zero of φ_1 (which is not vanishing everywhere, hence such a Z_1 exists). Assume now we found $Z_1, \dots, Z_p$ (with $1 \le p \le R-1$) such that

$$\begin{pmatrix} \varphi_1(Z_1) & \dots & \varphi_p(Z_1) \\ \vdots & \ddots & \vdots \\ \varphi_1(Z_p) & \dots & \varphi_p(Z_p) \end{pmatrix}$$

is invertible, hence its columns form a basis for $\mathbb{C}^p$ and

$$\begin{pmatrix} \varphi_{p+1}(Z_1) \\ \vdots \\ \varphi_{p+1}(Z_p) \end{pmatrix} = \sum_{j=1}^{p} a_j \begin{pmatrix} \varphi_j(Z_1) \\ \vdots \\ \varphi_j(Z_p) \end{pmatrix}$$

But $\varphi_1, \dots, \varphi_{p+1}$ are linearly independent, so we can find a $Z_{p+1} \in \mathcal{H}_n$ such that $\varphi_{p+1}(Z_{p+1}) \neq \sum_j a_j \varphi_j(Z_{p+1})$, constructing the wanted $p+1$-squared invertible matrix. □

Remark 3.2.3. For fixed $Z_0 \in \mathcal{H}_n$ and $W_0 \in \mathcal{H}_m$, we call *sections* of F the maps $Z \mapsto F(Z \times W_0)$ and $W \mapsto F(Z_0 \times W)$ defined respectively on $\mathcal{H}_n$ and $\mathcal{H}_m$.

In (3.12), the nearly holomorphic forms φ_i and ψ_j are themselves (linear combinations of) sections of F. In fact, by choosing suitable points $W_1, \dots, W_r \in \mathcal{H}_m$ we can write the system

$$\begin{pmatrix} F(Z \times W_1) \\ \vdots \\ F(Z \times W_R) \end{pmatrix} = \begin{pmatrix} \mu_1(W_1) & \dots & \mu_R(W_1) \\ \vdots & \ddots & \vdots \\ \mu_R(W_1) & \dots & \mu_R(W_R) \end{pmatrix} \begin{pmatrix} \varphi_1(Z) \\ \vdots \\ \varphi_R(Z) \end{pmatrix}$$

where $\mu_i = \sum_{j=1}^{S} c_{ij}\psi_j \in \mathcal{N}_{k,d}^m$, with the notation of (3.12). By the same argument of the proof, the square matrix is invertible and (after multiplying by its inverse) we read

$$\varphi_i(Z) = \sum_{j=1}^{S} \eta_i(W_j) F(Z \times W_j)$$

i.e $\varphi(Z)$ is a linear combination of $F(Z \times \star)$. The same argument applies in the variable W.

§ 3.3 Shimura-Maass differential operator

For $n, k, r \in \mathbb{N}^+$ define the *Shimura-Maass operator* $\partial_{n,k}^r$ by

$$\begin{aligned} \partial_{n,k} &:= (2\pi i)^{-n} \delta(Z)^{\frac{n-1}{2}-k} \det(\Delta) \delta(Z)^{k-\frac{n-1}{2}} \\ \partial_{n,k}^r &:= \partial_{n,k+2r-2} \circ \dots \circ \partial_{n,k+2} \circ \partial_{n,k} \end{aligned} \tag{3.13}$$

where Δ is the $n \times n$ matrix of differential operators

$$\Delta := \left(\frac{1+\delta_i^j}{2} \frac{\partial}{\partial Z_{ij}} \right)_{ij} \tag{3.14}$$

and δ_i^j is Kronecker's delta.

Proposition 3.3.1 ([Miz97, proposition 2.1]). *The operator $\partial_{n,k}^r$ acts on nearly holomorphic modular forms as*

$$\partial_{n,k}^r \mathcal{N}_{k,d}^n \subseteq \mathcal{N}_{k+2r,d+r}^n$$

In particular we can construct examples of nearly holomorphic modular forms by applying $\partial_{n,k}^r$ to elements of $\mathcal{M}_k^n$:

Proposition 3.3.2 ([CP91, proposition 3.11]). *Let $F \in \mathcal{M}_k^n$ with Fourier expansion $\sum_A F_A q^A$. Then, for any $r \geq 0$,*

$$\partial_{n,k}^r F = \delta(4\pi Z)^{-r} \sum_{A \geq 0} R_n(4\pi AY; r, \frac{n-1}{2} - k - r) F_A q^A$$

where $R_n(T; r, \beta)$ is the polynomial in $\mathbb{C}[T]$ defined by [CP91, formula 3.7]: its total degree is nr and its term of highest degree is $\det(T)^r$.

The elliptic case

In the special case $n = 1$, we introduce the adjoint of $\partial_{1,k}^r$ with respect to the Petersson inner product given by

$$\begin{aligned} \hat{\partial}_{1,k} &:= -(2\pi i)^{-1} y^2 \frac{\partial}{\partial \bar{z}} \\ \hat{\partial}_{1,k}^r &:= \hat{\partial}_{1,k-2r+2} \circ \ldots \hat{\partial}_{1,k-2} \circ \hat{\partial}_{1,k} \end{aligned} \tag{3.15}$$

which can be directly computed to satisfy

$$\langle f, \partial_{1,k}^r g \rangle = \langle \hat{\partial}_{1,k+2r}^r f, g \rangle \tag{3.16}$$

for any $f \in \mathcal{N}_{k+2r,d}^1$ and $g \in \mathcal{N}_{k,d'}^1$ for which the inner products converge.

In the elliptic case it is easy (e.g. [Hid93, pp. 311-312]) to explicitly compute the effect of the Shimura-Maass operator and its adjoint on the Fourier expansion.

Proposition 3.3.3. *If $f = \sum_A f_A q^A \in \mathcal{M}_k^1$ then*

$$\partial_{1,k}^r f = \sum_{A=0}^{\infty} \left(\sum_{j=0}^{r} \binom{r}{j} (-1)^j \frac{(k+r-1)!}{(k+r-1-j)!} \frac{A^r}{(4\pi A y)^j} \right) f_A q^A$$

Proof. We begin by computing

$$\partial_{1,k}[y^s q^A] = (2\pi i)^{-1} y^{-k} \frac{d}{dz}[y^{s+k} q^A] = \left(-\frac{s+k}{4\pi y} + A \right) y^s q^A$$

for any $s \in \mathbb{C}$. Hence by linearity

$$\partial_{1,k} f = \sum_{A=0}^{\infty} \left(-\frac{k}{4\pi y} + A \right) q^A$$

proving the claim for $r = 1$. We now proceed by induction on $r \geq 1$:

$$\begin{aligned} \partial_{1,k}^{r+1} f &= \partial_{1,k+2r}[\partial_{1,k}^r f] \\ &= \sum_{A=0}^{\infty} f_A \sum_{j=0}^{r} \binom{r}{j} (-1)^j \frac{(k+r-1)!}{(k+r-1-j)!} \frac{A^r}{(4\pi A)^j} \partial_{1,k+2r}[y^{-j} q^A] \\ &= \sum_{A=0}^{\infty} \left[\sum_{j=0}^{r} \binom{r}{j} (-1)^j \frac{(k+r-1)!}{(k+r-1-j)!} \frac{A^r}{(4\pi A y)^j} \left(A - \frac{-j+k+2r}{4\pi y} \right) \right] f_A q^A \end{aligned}$$

Rearranging the content of the square brackets as a polynomial in $(4\pi y A)^{-1}$ yields the wanted expression. □

Proposition 3.3.4. *Let $f \in \mathcal{N}_{k,d}^1$ with expansion*

$$f = \sum_{A=0}^{\infty} \sum_{j=0}^{d} \frac{f_{A,j}}{(4\pi y)^j} q^A$$

Then

$$\hat{\partial}_{1,k}^r f = (4\pi)^{-2r} \sum_{A=0}^{\infty} \sum_{j=r}^{d} \frac{(j+r-1)!}{(j-1)!} \frac{f_{A,j}}{(4\pi y)^{j-r}} q^A \tag{3.17}$$

where we mean $\hat{\partial}_{1,k}^r f = 0$ if $r > d$.

Proof. We begin by computing

$$\hat{\partial}_{1,k}[y^s q^A] = -(2\pi i)^{-1} y^2 \frac{d}{d\bar{z}}[y^s q^A] = -\frac{s}{4\pi} y^{s+1} q^A$$

for any $s \in \mathbb{C}$ hence

$$\hat{\partial}_{1,k} f = (4\pi)^{-2} \sum_{A=0}^{\infty} \sum_{j=1}^{d} j \frac{f_{A,j}}{(4\pi y)^{j-1}} q^A$$

proving the claim for $r = 1$. Now by induction on $r \geq 1$

$$\begin{aligned}
\hat{\partial}_{1,k}^{r+1} f &= \hat{\partial}_{1,k-2r}[\hat{\partial}_{1,k}^{r} f] \\
&= (4\pi)^{-2r} \sum_{A=0}^{\infty} \sum_{j=r}^{d} \frac{(j+r-1)!}{(j-1)!} \hat{\partial}_{1,k-2r} \left[\frac{f_{A,j}}{(4\pi y)^{j-r}} q^A \right] \\
&= (4\pi)^{-2(r+1)} \sum_{A=0}^{\infty} \sum_{j=r+1}^{d} \left[(j-r) \frac{(j+r-1)!}{(j-1)!} \right] \frac{f_{A,j}}{(4\pi y)^{j-r-1}} q^A
\end{aligned}$$

which we see immediately to be of the claimed form. □

Arithmeticy of the Shimura-Maass operator

From proposition (3.3.2) we see that it is more convenient to write the Fourier expansion (3.2) of $F \in \mathcal{N}_{k,d}^n$ as

$$F(Z) = \delta(4\pi Z)^{-d} \sum_{A \geq 0} F_A(4\pi Y) q^A \tag{3.18}$$

for polynomials $F_A(T) \in \mathbb{C}[T]$. More generally, let R be any ring and assume $F_A(T) \in R[T]$ for every $A \geq 0$: denote $\mathcal{N}_{k,d}^n(R)$ the R-module of all nearly holomorphic modular forms with this property.

Proposition 3.3.5. *For any ring $\mathbb{Z}[{}^1/_2] \subseteq R \subseteq \mathbb{C}$,*

$$\partial_{n,k}^r [\mathcal{N}_{k,d}^n(R)] \subseteq \mathcal{N}_{k+2r,d+r}^n(R)$$

Proof. A direct computation shows that, for any $s \in \mathbb{Z}[{}^1/_2]$, half-integral A and $p(T) \in R[T]$

$$\frac{\partial}{\partial Z_{ab}} \left[\delta(4\pi Z)^s p(4\pi Y) q^A \right] = (2\pi i) \cdot \delta(4\pi Z)^{s-1} p'(4\pi Y) q^A$$

for some $p'(T) \in R[T]$ hence

$$\det(\Delta) \left[\delta(4\pi Z)^s p(4\pi Y) q^A \right] = (2\pi i)^n \cdot \delta(4\pi Z)^{s-n} p''(4\pi Y) q^A$$

for some $p''(T) \in R[T]$.

Let $F \in \mathcal{N}^n_{k,d}(R)$ with Fourier coefficients $F_A(T) \in R[T]$ as in (3.18): by (3.13) we then have

$$\partial_{n,k}\left[\delta(4\pi Z)^{-d}F_A(4\pi Y)q^A\right] = \delta(4\pi Z)^{-d-n}F'_A(4\pi Y)q^A$$

with $F'_A(T) \in R[T]$ for every $A \geq 0$, which together with proposition (3.3.1) finishes the proof. $\square$

Proposition 3.3.6. *For any ring $\mathbb{Z}[^1/_2] \subseteq R \subseteq \mathbb{C}$,*

$$\hat{\partial}^r_{1,k}[\mathcal{N}^1_{k,d}(R)] \subseteq \pi^{-2r} \cdot \mathcal{N}^1_{k-2r,d-r}(R)$$

Proof. This follows immediately from the explicit Fourier expansion given in proposition (3.3.4). $\square$

The following proposition shows how arithmetic information is preserved by diagonal restriction. To keep the same notation as proposition (3.2.2), we write $X \otimes_R Y$ for the set of R-linear combinations of products xy with $x \in X$ and $y \in Y$: if X and Y are R-modules, then $X \otimes_R Y$ is indeed (isomorphic to) the standard tensor product. Nonetheless, in the rest of this thesis we will use this notation simply as a shorthand for linear combinations of products, even when this would not make sense algebraically.

Proposition 3.3.7. *For any ring $\mathbb{Z}[^1/_2] \subseteq R \subseteq \mathbb{C}$, the diagonal restriction induces a map*

$$\mathcal{N}^{n+m}_{k,d}(R) \to \mathcal{N}^n_{k,d}(R) \otimes_{R^*} \mathcal{N}^m_{k,d}(R)$$

where R^* is the field of fractions of R.

Proof. Take $F \in \mathcal{N}^{n+m}_{k,d}(R)$, $X + iY = Z \in \mathcal{H}_n$ and $U + iV = W \in \mathcal{H}_m$. By proposition (3.2.2) we have

$$F(Z \times W) = \sum_{i,j=1}^{S_n,S_m} c_{ij}\varphi_i(Z)\psi_j(W)$$

for some constants $c_{ij} \in \mathbb{C}$ and bases $\{\varphi_1, \ldots, \varphi_{S_n}\}$ and $\{\psi_1, \ldots, \psi_{S_m}\}$ of $\mathcal{N}^n_{k,d}$ and $\mathcal{N}^m_{k,d}$ respectively. We write explicitly

$$\varphi_i(Z) = \delta(4\pi Z)^{-d}\sum_{A\geq 0}\sum_{\alpha}\varphi^{(i)}_{A,\alpha}(4\pi Y)^{\alpha}q^A$$
$$\psi_j(W) = \delta(4\pi W)^{-d}\sum_{B\geq 0}\sum_{\beta}\psi^{(j)}_{B,\beta}(4\pi V)^{\beta}q^B$$

with $\varphi^{(i)}_{A,\alpha}, \psi^{(j)}_{B,\beta} \in \mathbb{C}$. By comparing these with

$$F(Z \times W) = \delta(4\pi Z)^{-d}\delta(4\pi W)^{-d} \sum_{C_{(n+m)}\geq 0} \sum_{\gamma} F_{C,\gamma}(4\pi Y)^{\gamma^*}(4\pi V)^{\gamma_*} q_Z^{C^*} q_W^{C_*}$$

we see that

$$\xi^{B,\beta}_{A,\alpha} := \sum_{i,j} c_{ij}\varphi^{(i)}_{A,\alpha}\psi^{(j)}_{B,\beta} = \sum_{C\geq 0}^{C^*=A, C_*=B} \sum_{\gamma}^{\gamma^*=\alpha,\gamma_*=\beta} F_{C,\gamma} \in R$$

for every choice of A, B, α, β, since so are all the $F_{C,\gamma}$; we write this as

$$\begin{pmatrix} \varphi^{(1)}_{A,\alpha} & \dots & \varphi^{(S_n)}_{A,\alpha} \end{pmatrix} \begin{pmatrix} c_{1,1} & \dots & c_{1,S_m} \\ \vdots & \ddots & \vdots \\ c_{S_n,1} & \dots & c_{S_n,S_m} \end{pmatrix} \begin{pmatrix} \psi^{(1)}_{B,\beta} \\ \vdots \\ \psi^{(S_m)}_{B,\beta} \end{pmatrix} = \xi^{B,\beta}_{A,\alpha}$$

Define $H_i(W) := \sum_{j=1}^{S_m} c_{ij}\psi_j(W) = \delta(4\pi W)^{-d} \sum_B \sum_\beta H^{(i)}_{B,\beta}(4\pi V)^\beta q^B$ for $H^{(i)}_{B,\beta} \in \mathbb{C}$, hence

$$\begin{pmatrix} \varphi^{(1)}_{A,\alpha} & \dots & \varphi^{(S_n)}_{A,\alpha} \end{pmatrix} \begin{pmatrix} H^{(1)}_{B,\beta} \\ \vdots \\ H^{(S_n)}_{B,\beta} \end{pmatrix} = \xi^{B,\beta}_{A,\alpha} \tag{3.19}$$

Now, since $\varphi_1, \dots, \varphi_{S_n}$ are linearly independent, we can find S_n choices of A, α such that

$$\begin{pmatrix} \varphi^{(1)}_{A_1,\alpha_1} & \dots & \varphi^{(S_n)}_{A_1,\alpha_1} \\ \vdots & \ddots & \vdots \\ \varphi^{(1)}_{A_{S_n},\alpha_{S_n}} & \dots & \varphi^{(S_n)}_{A_{S_n},\alpha_{S_n}} \end{pmatrix} \begin{pmatrix} H^{(1)}_{B,\beta} \\ \vdots \\ H^{(S_n)}_{B,\beta} \end{pmatrix} = \begin{pmatrix} \xi^{B,\beta}_{A_1,\alpha_1} \\ \vdots \\ \xi^{B,\beta}_{A_{S_n},\alpha_{S_n}} \end{pmatrix} \in R^{S_n} \tag{3.20}$$

where the matrix on the left is invertible (otherwise reduction by rows would show a linear dependency between $\varphi_1, \dots, \varphi_{S_n}$).

Put

$$\begin{aligned} \Psi_j(W) &:= \delta(4\pi W)^{-d} \sum_B \sum_\beta \xi^{B,\beta}_{A_j,\alpha_j}(4\pi V)^\beta q^B \\ &=: \delta(4\pi W)^{-d} \sum_B \sum_\beta \Psi^{(j)}_{B,\beta}(4\pi V)^\beta q^B \end{aligned}$$

then (3.20) says that $\square := \mathrm{span}_{\mathbb{C}}(\Psi_1, \dots, \Psi_{S_n}) = \mathrm{span}_{\mathbb{C}}(H_1, \dots, H_{S_n}) \subseteq \mathcal{N}^m_{k,d}$, which shows at once that $\Psi_j \in \mathcal{N}^m_{k,d}(R)$. Eventually relabelling, assume that

$\Psi_1, \ldots, \Psi_\nu$ form a basis for $\square$ (with $\nu \leq S_n$), so we can find some constants $c'_{i,j} \in \mathbb{C}$ such that (3.19) reads

$$\begin{pmatrix} \varphi^{(1)}_{A,\alpha} & \ldots & \varphi^{(S_n)}_{A,\alpha} \end{pmatrix} \begin{pmatrix} c'_{1,1} & \ldots & c'_{1,\nu} \\ \vdots & \ddots & \vdots \\ c'_{S_n,1} & \ldots & c'_{S_n,\nu} \end{pmatrix} \begin{pmatrix} \Psi^{(1)}_{B,\beta} \\ \vdots \\ \Psi^{(\nu)}_{B,\beta} \end{pmatrix} = \xi^{B,\beta}_{A,\alpha}$$

Repeating the same process for the variable Z, we find $\Phi_1, \ldots, \Phi_\eta \in \mathcal{N}^n_{k,d}(R)$ which are linearly independent and whose coefficients satisfy

$$\begin{pmatrix} \Phi^{(1)}_{A,\alpha} & \ldots & \Phi^{(\eta)}_{A,\alpha} \end{pmatrix} \begin{pmatrix} d_{1,1} & \ldots & d_{1,\nu} \\ \vdots & \ddots & \vdots \\ d_{\eta,1} & \ldots & d_{\eta,\nu} \end{pmatrix} \begin{pmatrix} \Psi^{(1)}_{B,\beta} \\ \vdots \\ \Psi^{(\nu)}_{B,\beta} \end{pmatrix} = \xi^{B,\beta}_{A,\alpha} \in R$$

with constants $d_{i,j} \in \mathbb{C}$ and

$$F(Z \times W) = \sum_{i,j=1}^{\eta,\nu} d_{ij} \Phi_i(Z) \Psi_j(W)$$

Notice that the row/column matrices have entries in R: completing them to R^*-invertible square matrices (it is possible since $\{\Phi_i(Z)\}_i$ and $\{\Psi_j(W)\}_j$ are linearly independent) and multiplying by their inverse shows that the matrix $(d_{i,j})_{i,j}$ is in R^* as claimed. $\square$

§ 3.4 Holomorphic projection

We give a brief review of the holomorphic projection operator as introduced by Sturm in [Stu81]. Let

$$\Gamma_n(s) := \pi^{\frac{n(n-1)}{4}} \prod_{j=0}^{n-1} \Gamma\left(s - \frac{j}{2}\right) \tag{3.21}$$

be the *generalised* Γ *function*, which admits the integral representation [Kli90, lemma 6.2]

$$\Gamma_n(k) = \det(M)^k \int_{Y>0} e^{-\mathrm{Tr}(MY)} \det(Y)^{k-\frac{n+1}{2}} dY \tag{3.22}$$

for any integer $k > (n-1)/2$ and complex symmetric matrix M with positive definite real part: notice the independence of the left-hand side of M, which plays an important role in the following facts, especially (3.26).

Let $\varphi(Z) = \sum_{A\geq 0} \varphi_A(Y)q^A$ be weakly modular of weight k and genus n, where every Fourier coefficient φ_A is a $\mathcal{C}^\infty$ function on $\Im(\mathcal{H}_n)$. Define numbers

$$\tilde{\varphi}_A := \frac{\pi^{n(k-\frac{n+1}{2})}}{\Gamma_n(k-\frac{n+1}{2})} \det(4A)^{k-\frac{n+1}{2}} \int_{Y>0} \varphi_A(Y) e^{-4\pi \operatorname{Tr}(AY)} \det(Y)^{k-n-1} dY \quad (3.23)$$

for every symmetric half-integral $A > 0$, and let *formally*

$$\operatorname{Hol}_{n,k}\varphi := \sum_{A>0} \tilde{\varphi}_A q^A \quad (3.24)$$

Proposition 3.4.1 ([Stu81, theorem 1])**.** *Assume that the integral in (3.23) is convergent for every $A > 0$. If φ is of bounded growth, i.e. for every $\varepsilon > 0$*

$$\int_{\Gamma_{n,0}\backslash\mathcal{H}_n} |\varphi(Z)| e^{-\varepsilon \operatorname{Tr}(Y)} \delta(Z)^k d\mu_Z < \infty \quad (3.25)$$

then $\operatorname{Hol}_{n,k}\varphi$ is a well defined element of $\mathcal{S}_k^n$ and

$$\langle \varphi, G\rangle = \langle \operatorname{Hol}_{n,k}\varphi, G\rangle \quad (3.26)$$

for every $G \in \mathcal{S}_k^n$.

Bounded growth

We present in this section some sufficient conditions for (3.25). These results are a generalisation of [Stu81, corollary 1(C)]: Sturm proves there that a particular bound holds for non-holomorphic Eisenstein series, and we show how the same is true for nearly holomorphic modular forms (of which some Eisenstein series are an example). We begin with a technical lemma, namely a generalisation of the min-max theorem.

Lemma 3.4.2. *Let Y be a symmetric positive definite matrix in $\mathbb{R}^{n,n}$ with eigenvalues $0 < \lambda_1 \leq \cdots \leq \lambda_n$. Then*

$$\min_Q \frac{\det(Q^tYQ)}{\det(Q^tQ)} = \prod_{j=1}^r \lambda_j \quad \textit{and} \quad \max_Q \frac{\det(Q^tYQ)}{\det(Q^tQ)} = \prod_{j=n-r}^n \lambda_j$$

where Q ranges over the set of matrices in $\mathbb{R}^{n,r}$ of rank $r \leq n$.

Proof. We prove the first equality, as the second one is analogous. Since Q is of maximal rank, we can apply the *thin QR decomposition* and write it as $Q = P\binom{R}{0}$, where $P \in O_n(\mathbb{R})$ and R is an upper triangular matrix in $GL_r(\mathbb{R})$. Then

$$\frac{\det(Q^tYQ)}{\det(Q^tQ)} = \frac{\det(R^t(P^tYP)^*R)}{\det(R^tR)} = \det((P^tYP)^*)$$

where A^* denotes the top left $r \times r$ block of a matrix A. If P is the orthogonal matrix that diagonalises Y as $\operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ then

$$\min_Q \frac{\det(Q^t Y Q)}{\det(Q^t Q)} \leq \det((P^t Y P)^*) = \prod_{j=1}^{r} \lambda_j$$

For the opposite inequality, denote $\{e_1, \ldots, e_n\}$ the standard orthonormal basis of $\mathbb{R}^n$ endowed with the inner product $\langle e_i, e_j \rangle = \delta_i^j$ and extended by linearity. The space $\bigwedge^r \mathbb{R}^n$ inherits the standard basis and inner product

$$\langle \wedge_{l=1}^r e_{i_l}, \wedge_{l=1}^r e_{j_l} \rangle := \prod_{l=1}^{r} \delta_{i_l}^{j_l}$$

for $i_1 < \cdots < i_r$ and $j_1 < \cdots < j_r$. Now the bilinear form on $\bigwedge^r \mathbb{R}^n$ given by

$$(\wedge_{l=1}^r u_l, \wedge_{l=1}^r v_l) \mapsto \det(U^t V)$$

(where U and V are the $n \times r$ matrices with $(u_l)_l$ and $(v_l)_l$ as columns) is well defined and coincides with the inner product on the basis $\{\wedge_{l=1}^r e_{i_l}\}$, so they are actually equal.

Let $w_1, \ldots, w_n$ be the eigenvectors of Y corresponding to $\lambda_1, \ldots, \lambda_n$, and since Y is symmetric and positive definite they form an orthonormal basis for $\mathbb{R}^n$, and they induce an orthonormal basis on $\bigwedge^r \mathbb{R}^n$ with respect to the standard inner product too. We construct the square root of Y as follows: if

$$Y = P \operatorname{diag}(\lambda_1, \ldots, \lambda_n) P^t$$

for $P \in O_n(\mathbb{R})$, then

$$\sqrt{Y} := P \operatorname{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}) P^t$$

is a positive definite symmetric matrix and indeed $\sqrt{Y}\sqrt{Y} = Y$. A $n \times n$ matrix defines a linear map on $\bigwedge^r \mathbb{R}^n$ by component-wise multiplication; if we let $v = \sum q_{i_1,\ldots,i_r} \wedge_{l=1}^r w_{i_l}$ then

$$\begin{aligned}
\|\sqrt{Y} v\|^2 &= \| \sum q_{i_1,\ldots,i_r} \left(\sqrt{Y} \wedge_{l=1}^r w_{i_l} \right) \|^2 \\
&= \| \sum \sqrt{\lambda_{i_1} \cdots \lambda_{i_r}} \, (q_{i_1,\ldots,i_r} \wedge_{l=1}^r w_{i_l}) \|^2 \\
&\geq \|v\|^2 \prod_{l=1}^{r} \lambda_l
\end{aligned}$$

We observed that $\langle \wedge_{l=1}^r u_l, \wedge_{l=1}^r v_l\rangle = \det(U^t V)$. In particular, if $v_1, \ldots, v_r$ are the columns of the matrix Q, we let $v = \wedge_{l=1}^r v_l$ and

$$\min_Q \frac{\det(Q^t Y Q)}{\det(Q^t Q)} = \min_Q \frac{\|\sqrt{Y} v\|^2}{\|v\|^2} \geq \min_Q \prod_{l=1}^r \lambda_l = \prod_{l=1}^r \lambda_l$$

□

We immediately use this result to obtain a bound for weakly modular forms. We use from now on the notation

$$|F(Z)| \ll_F \vartheta(Z) \quad \text{on } \Omega$$

to mean

$$|F(Z)| \leq c_F \vartheta(Z) \quad \forall Z \in \Omega$$

where ϑ has image in $\mathbb{R}^+$, Ω is a subset of $\mathcal{H}_n$ and $c_F \in \mathbb{R}^+$ is a suitable constant (which we do not need explicitly) depending only on F.

Lemma 3.4.3. *Let $F : \mathcal{H}_n \to \mathbb{C}$ be weakly modular of weight k and assume that $|F(Z)| \ll_F \delta(Z)^{-\alpha}$ on the fundamental domain $\mathcal{F}_n$ for some $\alpha \in \mathbb{R}$.*

If $0 \leq \alpha \leq k/2$, then

$$|F(Z)| \ll_F \prod_{j=1}^n (\lambda_j^{-\alpha} + \lambda_j^{-k+\alpha})$$

on $\mathcal{H}_n$, where $\{\lambda_i\}_i$ are the eigenvalues of Y.

Proof. Let $Z \in \mathcal{H}_n$ and take $\gamma \in \Gamma_n$ such that $\gamma(Z) \in \mathcal{F}_n$. Then

$$\begin{aligned} |F(Z)| &= |j(\gamma, Z)^{-k} F(\gamma Z)| \\ &\ll_F |j(\gamma, Z)|^{-k} \delta(\gamma Z)^{-\alpha} \\ &= |j(\gamma, Z)|^{-k+2\alpha} \delta(Z)^{-\alpha} \end{aligned} \tag{3.27}$$

Let C denote the bottom left block of γ and let r be its rank. From the computation in [Maa71, pg. 167], and using the same notation,

$$|j(\gamma, Z)|^2 = \det(C_1)^2 \det(Q^t Y Q)^2 \prod_{v=1}^r (h_v^2 + 1) \geq \det(Q^t Y Q)^2$$

where $Q \in \mathbb{Z}^{n,r}$ is a matrix that can be completed to an element of $\mathrm{GL}_n(\mathbb{Z})$ and $Y = \Im(Z)$. Since the entries of Q are integers, we see

$$|j(\gamma, Z)| \geq \det(Q^t Y Q) \geq \frac{\det(Q^t Y Q)}{\det(Q^t Q)} \geq \prod_{j=1}^r \lambda_j$$

by lemma (3.4.2); hence, if $-k+2\alpha \leq 0$, (3.27) becomes

$$\begin{aligned} |F(Z)| &\ll_F \delta(Z)^{-\alpha} \prod_{j=1}^{r} \lambda_j^{-k+2\alpha} \\ &= \prod_{j=1}^{n} \lambda_j^{-a} \prod_{j=1}^{r} \lambda_j^{-k+2\alpha} \\ &\leq \prod_{j=1}^{n} (\lambda_j^{-\alpha} + \lambda_j^{-k+\alpha}) \end{aligned}$$

uniformly on $\mathcal{H}_n$, provided that $0 \leq \alpha \leq k/2$. □

Lemma 3.4.4. *Let $F \in \mathcal{M}_k^n$. Then*

$$|\partial_{n,k}^r F(Z)| \ll_F \prod_{j=1}^{n} (\lambda_j^{-r} + \lambda_j^{-k-r})$$

uniformly on $\mathcal{H}_n$.

Proof. By proposition (3.3.2), we know that

$$\partial_{n,k}^r F = \delta(4\pi Z)^{-r} \sum_{A \geq 0} R_n(4\pi AY; r, \frac{n-1}{2} - k - r) F_A q^A$$

where $\sum_\beta R_\beta T^\beta := R_n(T; r, \star) \in \mathbb{C}[T]$ and the term of highest degree is $\det(T)^r$. Notice that T is not assumed symmetric, so we have to consider a generic multi-index $\beta \in \mathbb{N}^{n,n}$; on the other hand $\partial_{n,k}^r F \in \mathcal{N}_{k,r}^n$ hence

$$\partial_{n,k}^r F = \delta(4\pi Z)^{-r} \sum_{A \geq 0} \sum_{\alpha} F_{A,\alpha} (4\pi Y)^\alpha q^A$$

By comparing the I-th Fourier coefficient we see $\sum_\alpha F_{I,\alpha} Y^\alpha = F_I \sum_\beta R_\beta Y^\beta$ hence by lemma (3.1.1)

$$\beta_{mm} + \sum_{i \neq m} (\beta_{im} + \beta_{mi}) \leq r \tag{3.28}$$

for every $m \leq n$.

Fix now any β appearing in $R_n(T;r,\star)$. Write explicitly

$$\begin{aligned}(AY)^\beta &= \prod_{i,j}\left(\sum_{l=1}^n A_{il}Y_{lj}\right)^{\beta_{ij}}\\ &= \prod_{i,j}\left[\left(\sum_{l_1^{(ij)}=1}^n A_{il_1^{(ij)}}Y_{l_1^{(ij)}j}\right)\cdots\left(\sum_{l_{\beta_{ij}}^{(ij)}=1}^n A_{il_{\beta_{ij}}^{(ij)}}Y_{l_{\beta_{ij}}^{(ij)}j}\right)\right]\\ &= \prod_{i,j}\sum_{l_1^{(ij)},\cdots,l_{\beta_{ij}}^{(ij)}}\left(A_{il_1^{(ij)}}\cdots A_{il_{\beta_{ij}}^{(ij)}}\right)\left(Y_{l_1^{(ij)}j}\cdots Y_{l_{\beta_{ij}}^{(ij)}j}\right)\end{aligned}$$

Let $Z \in \mathcal{F}_n$, so that [Kli90, proposition 2.1] applies and $|Y_{ij}| \leq Y_{jj}$ for every $i \neq j$, hence

$$\begin{aligned}|(AY)^\beta| &\leq \prod_{i,j}\sum_{l_1^{(ij)},\cdots,l_{\beta_{ij}}^{(ij)}}\left|A_{il_1^{(ij)}}\cdots A_{il_{\beta_{ij}}^{(ij)}}\right|Y_{jj}^{\beta_{ij}^{(ij)}}\\ &= \prod_{i,j}\left(Y_{jj}^{\beta_{ij}^{(ij)}}\sum_{l_1^{(ij)},\cdots,l_{\beta_{ij}}^{(ij)}}\left|A_{il_1^{(ij)}}\cdots A_{il_{\beta_{ij}}^{(ij)}}\right|\right)\\ &= \left(\sum_{\left\{l_1^{(ij)},\cdots,l_{\beta_{ij}}^{(ij)}\right\}_{1\leq i,j\leq n}}\left|A_{il_1^{(ij)}}\cdots A_{il_{\beta_{ij}}^{(ij)}}\right|\right)Y^\beta\\ &=: C(A,\beta)Y^\beta\end{aligned} \tag{3.29}$$

where $C(A,\beta) \in \mathbb{C}[A]$ is a polynomial in the entries of A whose coefficients depend on β, and the last sum is over the mute indices $1 \leq l_{\nu^{(ij)}}^{(ij)} \leq n$, one for each choice of $1 \leq i,j \leq n$ and $1 \leq \nu^{(ij)} \leq \beta_{ij}$.

If $A_{mm} = 0$ for some $m \leq n$, then $A_{\nu m} = A_{m\nu} = 0$ for every $\nu \leq n$ by lemma (2.2.1). In this case

$$(AY)^\beta = \prod_{i,j}\left(\sum_{l=1}^n A_{il}Y_{lj}\right)^{\beta_{ij}} = 0$$

unless $\beta_{m\nu} = 0$ for every $\nu \leq n$; also no $Y_{m\nu}$ appears in $(AY)^\beta$ for any $\nu \leq n$. This means that the variables $Y_{m\nu} = Y_{\nu m}$ do not appear in $(AY)^\beta$ whenever $A_{mm} = 0$, hence we can majorise in (3.29) as

$$|(AY)^\beta| \leq C(A,\beta) \prod_{\substack{m \\ A_{mm}\neq 0}} Y_{mm}^{\beta_{1m}+\cdots+\beta_{nm}} \leq C(A,\beta) \prod_{\substack{m \\ A_{mm}\neq 0}} Y_{mm}^{r} \tag{3.30}$$

where the last inequality follows from (3.28).

We can finally finish the proof. By [Kli90, lemma 2.2] we have $\delta(Z) \gg_n \prod_l Y_l$, so we can write (3.2) as

$$|\partial^r_{n,k} F(Z)| \ll \sum_{A\geq 0} \sum_\beta C(A,\beta) \left(\prod_{\substack{m \\ A_{mm}=0}} Y_{mm}^{-r} \right) |F_A| e^{-2\pi \mathrm{Tr}(AY)}$$

We split the sum over the A whose diagonal entries never vanish, and the remaining ones: we shall show that the first sum decays exponentially in each Y_{ll} while the second one decays as $\delta(Z)^{-r}$ on the fundamental domain $\mathcal{F}_n$, so that lemma (3.4.3) applies.

For the sum over non diagonally vanishing indices, we have

$$\sum_A \sum_\beta C(A,\beta)|F_A|e^{-2\pi\mathrm{Tr}(AY)}$$

Since $Z \in \mathcal{F}_n$, then $Y > \epsilon I$ for some $\epsilon > 0$, and also by [Kli90, lemma 2.2] $\mathrm{Tr}(AY) \gg \sum_l A_{ll}Y_{ll} \geq \mathrm{Tr}(Y)$ hence we majorise by

$$\left(\sum_A \sum_\beta C(A,\beta)|F_A|e^{-\epsilon\pi\mathrm{Tr}(A)} \right) e^{-\pi\mathrm{Tr}(Y)}$$

and the series in brackets converges since $C(A,\beta)$ has polynomial behaviour in the entries of A. The whole expression is then $\ll e^{-\mathrm{Tr}(Y)} \ll \delta(Z)^{-r}$.

For the remaining sum, without loss of generality we assume we are only summing over the matrices A whose only diagonal zero is A_{11} (the general case follows the same idea, but with a more cumbersome notation). Then $\mathrm{Tr}(AY) \geq Y_{22} + \cdots + Y_{nn}$, so that we are dealing with

$$\left(\sum_{\substack{A \\ A_{11}=0}} \sum_\beta C(A,\beta)|F_A|e^{-\epsilon\pi\mathrm{Tr}(A)} \right) Y_{11}^{-r} e^{-\pi(Y_{22}+\cdots+Y_{nn})}$$

Again the series in brackets converges, and we have something decaying polynomially of degree $-r$ in Y_{11} and exponentially in the other Y_{mm}, hence by [Kli90, lemma 2.2] the whole expression is $\ll \delta(Z)^{-r}$. □

Remark 3.4.5. In the proof of lemma (3.4.4) we highlighted how it is the *singular part* (i.e. indexed by diagonally vanishing A) of the Fourier expansion that dictates how fast $|\partial^r_{n,k} F|$ decays. If $n = 1$, the singular part is just the constant term, and by proposition (3.3.3) this equals y^{-r}.

More generally, if $\varphi \in \mathcal{N}^1_{k,d}$, then the asymptotic behaviour for $y \to \infty$ is given by its constant term. Proposition (3.3.3) and (3.3.4) tell us explicitly what happens when applying the operator $\partial_{1,k}$ and its adjoint $\hat{\partial}_{1,k}$: in particular, if $\varphi = y^{-d} + O(y^{-d})$, then $\hat{\partial}^v_{1,k}\varphi = y^{-d+v} + O(y^{-d+v})$ if $v \le d \le k/2$, hence

$$|\hat{\partial}^v_{1,k}\varphi| \ll_\varphi y^{-d+v} + y^{-k+d+v}$$

uniformly on $\mathcal{H}_1$.

Proposition 3.4.6. *If $F : \mathcal{H}_n \to \mathbb{C}$ is weakly modular of weight k and satisfies*

$$|F(Z)| \ll_F \prod_{j=1}^n (\lambda_j^{-\alpha} + \lambda_j^{-\beta})$$

on $\mathcal{H}_n$ for some $\alpha, \beta \in \mathbb{R}$ such that $\max\{\alpha, \beta\} < k - n$, then F is of bounded growth i.e. satisfies (3.25).

Proof. We follow the argument of [Stu81, corollary 2]. Let Ω be the vertical strip $\{Z \in \mathcal{H}_n : |X_{ij}| \le 1/2, Y > 0\}$: we need to prove that

$$\int_\Omega |F(Z)| e^{-\varepsilon \operatorname{Tr}(Y)} \delta(Z)^k d\mu_Z < \infty$$

and, because of the hypothesis, we are left with showing

$$\int_{Y>0} \left(\prod_{j=1}^n (\lambda_j^{-\alpha} + \lambda_j^{-\beta}) \right) e^{-\varepsilon \operatorname{Tr}(Y)} \det(Y)^{k-n-1} dY < \infty$$

The set of matrices $Y > 0$ with distinct eigenvalues is of full measure in the space of all positive definite symmetric matrices, so we can integrate over it instead. Every Y with distinct eigenvalues can be diagonalised as $U\Lambda U^t$, where Λ is diagonal and $\Lambda_{jj} = \lambda_j$, and $U \in O_n(\mathbb{R})$; further U is unique up to multiplication by $\operatorname{diag}(\pm 1, \dots, \pm 1)$, and the Jacobian determinant of the change of variables $Y \to (\Lambda, U)$ is a polynomial $J(\Lambda)$ in the eigenvalues of Λ independent of U. Therefore we need to show that

$$\int_{\{\lambda_i < \lambda_j\}} \int_{O_n(\mathbb{R})} \left(\prod_{j=1}^n (\lambda_j^{-\alpha} + \lambda_j^{-\beta}) \right) e^{-\varepsilon \operatorname{Tr}(\Lambda)} \det(\Lambda)^{k-n-1} J(\Lambda) dU d\lambda_1 \cdots d\lambda_n < \infty$$

The space $O_n(\mathbb{R})$ is of finite dU-measure, and we can majorize the remaining integral as the n-th power of

$$\int_0^\infty (\lambda^{-\alpha} + \lambda^{-\beta}) e^{-\varepsilon\lambda} \lambda^{k-n} \frac{d\lambda}{\lambda} \ll_\varepsilon \Gamma(-\alpha + k - n) + \Gamma(-\beta + k - n)$$

which is indeed convergent provided that $\max\{\alpha, \beta\} < k - n$. $\square$

Arithmeticity of the holomorphic projection

Lemma 3.4.7. *Let* $F \in \mathcal{N}^n_{k,d}$ *with Fourier expansion*

$$F(Z) = \delta(4\pi Z)^{-d} \sum_{A \geq 0} \sum_{\alpha} F_{A,\alpha} (4\pi Y)^\alpha q^A$$

and define

$$M_{A,\alpha} := \frac{\pi^{n(k-\frac{n+1}{2})}}{\Gamma_n(k - \frac{n+1}{2})} \det(4A)^{k-\frac{n+1}{2}} \int_{Y>0} (4\pi Y)^\alpha e^{-4\pi \operatorname{Tr}(AY)} \det(Y)^{k-n-d-1} dY$$

Then, under the conditions of proposition (3.4.1),

$$\operatorname{Hol}_{n,k} F = \sum_{A>0} \left(\sum_{\alpha} M_{A,\alpha} F_{A,\alpha} \right) q^A$$

and $M_{A,\alpha} \in \mathbb{Q}$ *is integral at all primes* $> 2k - n - 4$.

Proof. Let

$$\left(-\frac{d}{dA}\right)^\alpha := \prod_{i \leq j} (-1)^{\alpha_{ij}} \frac{\partial^{\alpha_{ij}}}{\partial A_{ij}^{\alpha_{ij}}}$$

so that

$$\left(-\frac{d}{dA}\right)^\alpha e^{-\operatorname{Tr}(AY)} = Y^\alpha e^{-\operatorname{Tr}(AY)}$$

hence by (3.22)

$$\begin{aligned}
&\int_{Y>0} Y^\alpha e^{-\operatorname{Tr}(AY)} \det(Y)^{k-n-d-1} = \\
&\qquad = \int_{Y>0} \left(-\frac{d}{dA}\right)^\alpha e^{-\operatorname{Tr}(AY)} \det(Y)^{k-n-d-1} \\
&\qquad = \Gamma_n\left(k - n - d - 1 + \frac{n+1}{2}\right) \left[\left(-\frac{d}{dA}\right)^\alpha \left(\det(A)^{k-n-d-1+\frac{n+1}{2}}\right)\right] \\
&\qquad = \Gamma_n\left(k - n - d - 1 + \frac{n+1}{2}\right) c_{A,\alpha}
\end{aligned} \tag{3.31}$$

with $c_{A,\alpha} \in \mathbb{Z}[\frac{1}{2}]$, since so are the entries of A and α. Making a change of variables $Y \mapsto 4\pi Y$ we get

$$M_{A,\alpha} = 4^{-|\alpha|-n(k-d-\frac{n+1}{2})} \det(4A)^{k-\frac{n+1}{2}} \frac{\Gamma_n(k-d-(n+1)/2)}{\Gamma_n(k-(n+1)/2)} c_{A,\alpha}$$

$\square$

Corollary 3.4.8. *Let* $\mathfrak{P}_m^{-1} = \{p^{-1} : p \leq m \text{ prime}\} \subseteq \mathbb{Q}$ *and* $\mathbb{Z}[{}^1/_2] \subseteq R \subseteq \mathbb{C}$ *any ring. Then*

$$\mathcal{N}_{k,d}^n(R)_{\text{bounded}} \xrightarrow{\mathrm{Hol}_{k,n}} \mathcal{S}_k^n(R[\mathfrak{P}_{2k-n-4}^{-1}])$$

where $\mathcal{N}_{k,d}^n(R)_{\text{bounded}}$ *is the subset of* $\mathcal{N}_{k,d}^n(R)$ *whose elements satisfy (3.25).*

Corollary 3.4.9. *Let* $\mathbb{Z}[{}^1/_2] \subseteq R \subseteq \mathbb{C}$ *be any ring, and*

$$\begin{aligned}\varphi(Z_1, \dots, Z_m) &:= \sum_{j_1,\dots,j_m} c_{j_1,\dots,j_m} F_{j_1}(Z_1) \cdots F_{j_m}(Z_m) \\ &\in \bigotimes_{i=1}^m \mathcal{N}_{k_i,d_i}^{n_i}(R)_{\text{bounded}}\end{aligned}$$

for constants $c_{j_1,\dots,j_m} \in \mathbb{C}$. *Then*

$$\begin{aligned}\mathrm{Hol}\varphi(Z_1, \dots, Z_m) &:= \sum_{j_1,\dots,j_m} c_{j_1,\dots,j_m} \mathrm{Hol}_{k_1,n_1} F_{j_1}(Z_1) \cdots \mathrm{Hol}_{k_m,n_m} F_{j_m}(Z_m) \\ &\in \bigotimes_{i=1}^m \mathcal{S}_{k_i}^{n_i}(R[\mathfrak{P}_{2k_i-n_i-4}^{-1}])\end{aligned}$$

obtained by term-wise application of the holomorphic projection operator, is a well defined linear combination of cusp forms and satisfies

$$\langle \varphi(Z_1, \dots, Z_m), G_1(Z_1) \cdots G_m(Z_m) \rangle = \langle \mathrm{Hol}\varphi(Z_1, \dots, Z_m), G_1(Z_1) \cdots G_m(Z_m) \rangle$$

for every $G_i \in \mathcal{S}_{k_1}^{n_1}$.

In addition, if the multivariate Fourier coefficients of φ *are in* R, *then the ones of* $\mathrm{Hol}\varphi$ *are in* $R[\mathfrak{P}_{2K-N-4}^{-1}]$, *where* $K := \max\{k_i\}$ *and* $N := \max\{n_i\}$.

Proof. The first statement follows immediately from corollary (3.4.8) and proposition (3.4.1).

Write the Fourier expansion of F_{j_i} as

$$F_{j_i}(Z) = \delta(4\pi Z)^{-d_i} \sum_{A_i \geq 0} \sum_{\alpha_i} F_{A_i,\alpha_i}^{j_i} (4\pi Y)^{\alpha_i} q^{A_i}$$

and fix any indices $A_1, \dots, A_m$ and monomials $\alpha_i, \dots, \alpha_m$. The corresponding (monomial term of) Fourier coefficient of φ equals therefore

$$\xi^{\alpha_1,\dots,\alpha_m}_{A_1,\dots,A_m} := \sum_{i_1,\dots,i_m} c_{j_1,\dots,j_m} F^{j_1}_{A_1,\alpha_1} \cdots F^{j_m}_{A_m,\alpha_m}$$

which is in R by hypothesis. By lemma (3.4.7), the term-wise application of holomorphic projection to φ acts on its Fourier transform so that the $A_1, \dots, A_m$-th Fourier coefficient of Φ equals

$$\sum_{\alpha_1,\dots,\alpha_m} M_{A_1,\alpha_1} \cdots M_{A_m,\alpha_m} \xi^{\alpha_1,\dots,\alpha_m}_{A_1,\dots,A_m}$$

which is in $R[\mathfrak{P}^{-1}_{2K-N-4}]$ since M_{A_i,α_i} is integral at primes $> 2k_i - n_i - 4$. □

§ 3.5 Eisenstein series and arithmetic properties

Eisenstein series are the main example of (nearly) holomorphic modular forms due to their straightforward definition. For $s \in \mathbb{C}$ and $Z \in \mathcal{H}_n$, define the *Eisenstein series* of weight k and genus n as

$$\begin{aligned} E^n_k(Z, s) &:= \sum_{\gamma \in \Gamma_{n,0} \backslash \Gamma_n} j(\gamma, Z)^{-k} \delta(\gamma Z)^s \\ &= \delta(Z)^s \sum_{\gamma \in \Gamma_{n,0} \backslash \Gamma_n} j(\gamma, Z)^{-k} |j(\gamma, Z)|^{-2s} \end{aligned} \tag{3.32}$$

where

$$\Gamma_{n,0} := \{\gamma \in \Gamma_n : C_\gamma = 0\} \tag{3.33}$$

is the *Siegel parabolic subgroup* of Γ_n of matrices with vanishing bottom left block. The series (3.32) converges absolutely and locally uniformly for $Z \in \mathcal{H}_n$ and $2\Re(s) > n+1-k$, and it admits meromorphic continuation to the whole complex plane in s [Miz93, introduction]. Formally $E^n_k(Z, s)$ is weakly modular: at least when absolutely convergent, $E^n_k(Z, 0)$ defines a Siegel modular form. Other values of s give rise to nearly holomorphic modular forms:

Proposition 3.5.1 ([BH06, proposition 3.1])**.** *Assume $k > n+1$ and let $0 \le v < \frac{k-n-1}{2}$ be an integer. Then*

$$E^n_k(Z, -v) = (-4\pi)^{nv} \prod_{j=1}^{v} \prod_{l=0}^{n-1} \left(k - v - j - \frac{l}{2}\right)^{-1} \partial^v_{n,k-2v} E^n_{k-2v}(Z, 0)$$

and hence $E^n_k(Z, -v) \in \mathcal{N}^n_{k,v}$. If neither $\frac{n+2}{2}$ nor $\frac{n+3}{2}$ is congruent to 2 mod 4, then the statement is valid for $0 \le v < \frac{2k-n+1}{4}$.

Arithmeticity of Siegel Eisenstein series

For any mod N character χ, the *generalised Bernoulli* numbers are defined by

$$\sum_{a=1}^{N} \frac{\chi(a) t e^{at}}{e^{Nt}-1} =: \sum_{k=0}^{\infty} B_{k,\chi} \frac{t^k}{k!} \tag{3.34}$$

and we put $B_k := B_{k,1}$ i.e. associated to the character mod 1.

Proposition 3.5.2 (von Staudt-Clausen). *For $k \in \mathbb{N}$ odd, $B_k = 0$. If $k > 0$ is even, then*

$$B_k + \sum_{(p-1)|k} p^{-1} \in \mathbb{Z}$$

In particular the denominator of B_k is the product of the primes p such that $p-1$ divides k.

Proposition 3.5.3. *Let χ be a primitive quadratic character of order N, and p a prime divisor of the denominator of $B_{k,\chi}/k$ for a fixed $k \geq 1$.*

Then $k \equiv (p-1)/2 \mod p$, and in particular $p \leq 2k-1$.

Proof. Fix a prime p. It is well known [Gui, proposition 3.12] that

$$\frac{B_{\chi,k}}{k} = \frac{1}{1-\chi(g)g^k} \int_{\mathbb{Z}_p^\times} \chi(x) x^{k-1} dE_{1,g}(x)$$

where the equality takes place in the p-completion $\mathbb{Q}_p$ of $\mathbb{Q}$, g is a primitive root mod p, and $dE_{1,g}$ is a certain p-adic measure. The integral is a p-adic integer, hence $B_{k,\chi}/k$ can have a factor of p^{-1} if and only if $1-\chi(g)g^k \equiv 0 \pmod p$.

Since χ is quadratic, $\chi(g) = \pm 1$. If $\chi(g) = 1$, then $\chi(m) = \chi(g^{r_m}) = \chi(g)^{r_m} = 1$ for every $m \in \mathbb{Z}/p\mathbb{Z}^\times = \langle g \rangle$, and χ would be imprimitive against the assumption: hence $\chi(g) = -1$. Therefore a factor of p^{-1} can appear in $B_{k,\chi}/k$ if and only if

$$g^k \equiv -1 \mod p$$

Since g generates $\mathbb{Z}/p\mathbb{Z}^\times$, the smallest value of such k is $(p-1)/2$ and any other value differs from it by a multiple of p. □

In the next chapter we will make use of Eisenstein series of *low weight*, i.e. $k \leq n+1$. For fixed $Z \in \mathcal{H}_n$, the function $s \mapsto E_k^n(Z,s)$ admits meromorphic continuation to the whole of $\mathbb{C}$, so we try to define a Siegel modular form as $Z \mapsto \lim_{s\to 0} E_k^n(Z,s)$: the results of [Shi83, theorem 7.1] and [Har97, theorems 4.4 and 4.9] explain under which conditions this is possible; for our purposes, we only need the following.

Proposition 3.5.4 ([Har97, theorem 4.9(i)]). *Let $4 \leq k = \frac{n+3}{2}$. If $k \equiv 0 \pmod 4$ then $\lim_{s\to 0} E_k^n(Z,s)$ is a well defined element of $\mathcal{M}_k^n$ with Fourier expansion given by*

$$\sum_{\lambda=0}^{n} \sum_{0<h\in\Lambda_\lambda} \sum_{r} a_k(h) q^{rhr^t} \tag{3.35}$$

where r ranges over $\mathbb{Z}_{prim}^{(n,\lambda)}/\mathrm{GL}_\lambda(\mathbb{Z})$, $\mathbb{Z}_{prim}^{(n,\lambda)}$ is the set of matrices in $\mathbb{Z}^{n,\lambda}$ which can be completed to unimodular matrices, Λ_λ is the set of symmetric half-integral matrices (i.e. integral quadratic forms) of size λ and

$$a_k(h) = \begin{cases} (-1)^{\frac{(\lambda+1)(\lambda+3)}{8}} 2^{\frac{3\lambda+1}{2}-k} \dfrac{k!\det(2h)^{k-\frac{\lambda+1}{2}}}{(k-\frac{\lambda+1}{2})!B_k} \times & \\ \qquad \times \displaystyle\prod_{j=1}^{\frac{\lambda-1}{2}} \frac{1}{B_{2k-2j}} P(k,h) & (\lambda \text{ odd}) \\ (-1)^{\frac{\lambda}{2}} 2^{\lambda} \dfrac{k!}{(k-\frac{\lambda}{2})!} \left(\dfrac{\det(2h)}{f}\right)^{k-\frac{\lambda+1}{2}} \dfrac{B_{k-\frac{\lambda}{2},\chi}}{B_k} \times & \\ \qquad \times \displaystyle\prod_{j=1}^{\frac{\lambda}{2}} \frac{1}{B_{2k-2j}} P(k,h) & (\lambda \text{ even}) \end{cases} \tag{3.36}$$

where $d(h) := (-1)^{\lfloor \frac{\lambda}{2} \rfloor} 2^{(\lambda \bmod 2)-1} \det(2h)$, f is the conductor of the quadratic character $\left(\frac{d(h)}{\star}\right)$ and χ is the associated primitive character mod f. The factor $P(k,h)$ is a polynomial in primes which divide $d(h)$, and $B_{k,\chi}$ are the generalised Bernoulli numbers.

For *high weight*, the series $E_k^n(Z,0)$ converges as is, and we also have formulae for its Fourier coefficients:

Proposition 3.5.5 ([Har97, theorem 4.14]). *If $2k > n+4$, then $E_k^n(Z,0)$ has Fourier expansion as (3.35), with coefficients $a_k(h)$ given by (3.36).*

In the next chapter we will deal with the Eisenstein series

$$E_{4+4m}^5(Z) := \lim_{s\to 0} E_{4+4m}^5(Z,0)$$

which is well defined by proposition (3.5.4) for $m = 0$, and by proposition (3.5.5) for $m \geq 1$. In particular, by looking at the explicit formulae given by (3.36) we see:

Corollary 3.5.6. *Let*

$$\begin{aligned} \mathfrak{P}_m^{-1} &:= \{p^{-1} : \textit{prime } p \leq m\} \\ \mathfrak{B}_k^{-1} &:= \{p^{-1} : \textit{prime } p \mid \textit{numerator of } B_k, B_{2k-2} \textit{ or } B_{2k-4}\} \end{aligned} \tag{3.37}$$

Then

$$E^5_{4+4m} \in \mathcal{M}^5_{4+4m}(\mathbb{Z}[\mathfrak{P}^{-1}_{8+8m} \cup \mathfrak{B}^{-1}_{4+4m}])$$

for every $m \in \mathbb{N}$.

Proof. By examining (3.36), we see that the only factors in the denominator of $a_{4+4m}(h)$ arise from the Bernoulli numbers or from $(4+4m-\lambda)!$, for $0 \leq \lambda \leq 5$. By proposition (3.5.3), only primes $< 8+8m-2\lambda$ can divide the denominator of $B_{\chi,4+4m-\lambda}$: hence the only inverse prime factors of $a_k(h)$ are in $\mathfrak{P}^{-1}_{8+8m}$, or in the numerator of the remaining Bernoulli numbers. $\square$

Chapter 4

Integral representation I

In this chapter, we prove a weaker version of the announced result. We employ an integral formula of Böcherer and Heim, expressing L-values as integral against diagonally restricted non-holomorphic Eisenstein series. For the critical values we are interested in, these turn out to be nearly holomorphic modular forms and we can therefore apply the theory developed in the previous chapter to study the arithmetic information of the L-values. At the end of the proof, we remark why we are able to obtain only a partial result.

§ 4.1 Main theorem

An integral formula of Böcherer and Heim

The main tool we use in this chapter is the following result providing an integral representation for the L-functions. Note the change of notation for the weight of modular forms, which is now explicitly doubled as $2k$ as we will often have to use the half weight k.

Proposition 4.1.1 ([BH00, theorem 3.6]). *Let $h \in \mathcal{S}^1_{2l}$ and $F, G \in \mathcal{S}^2_{2k}$ be Hecke eigenform, with F being the Saito-Kurokawa lift of $f \in \mathcal{S}^1_{4k-2}$ and $k \geq l \in \mathbb{N}$. Then*

$$\begin{aligned}\langle E^5_{2k}(\tau \times Z \times W, s), \partial^{k-l}_{1,2l} h(\tau) G(Z) F(W)\rangle &= \\ = C_{\triangleright}(s)\langle \Phi_F, \Phi_G\rangle L_f(2s+4k-4) Z_{G\otimes h}(s+3k+l-3)\end{aligned} \tag{4.1}$$

for $\Re(s) + k > 3$, where $\Phi_\star$ denotes the first Jacobi-Fourier coefficient and

$$\begin{aligned}
C_\triangleright(s) &:= a_\triangleright(s)\zeta_\triangleright(s)\Gamma_\triangleright(s)\\
a_\triangleright(s) &:= 2^{-6s-12k+13}\pi^{6-s-4k}\\
\zeta_\triangleright(s) &:= \zeta(2s+2k)^{-1}\zeta(4s+4k-2)^{-1}\zeta(4s+4k-4)^{-1}\\
\Gamma_\triangleright(s) &:= \frac{\Gamma(2k+s-3/2)\Gamma(2k+s-2)}{\Gamma(2k+s)\Gamma(2k+s-1/2)}\frac{\Gamma(s+k-l)}{\Gamma(s)}\times\\
&\quad\times\frac{\Gamma(s+k+l-1)\Gamma(s+3k+l-3)\Gamma(s+3k-l-2)}{\Gamma(2s+4k-3)}
\end{aligned} \tag{4.2}$$

Our goal

Let $h \in \mathcal{S}^1_{2l}$ be a normalised Hecke eigenform, with l odd; its completed L-function satisfies the functional equation

$$\Lambda_h(s) = (-1)^l\Lambda_h(2l-s)$$

hence L_h has a zero at the central critical value $s = l$.

Let $F \in \breve{\mathcal{S}}^2_{2k}$ be the Saito-Kurokawa lift of a normalised cuspidal Hecke eigenform $f \in \mathcal{S}^1_{4k-2}$, with $2k \geq 2l$. Because of (2.29), the tensor product L-function $Z_{F\otimes h}$ factorises as

$$Z_{F\otimes h}(s) = L_h(s-2k+1)L_h(s-2k+2)L_{f\otimes h}(s) \tag{4.3}$$

and in particular $Z_{F\otimes h}$ has a zero at $s = l+2k-1$, the first critical value to the right of the centre.

Let $\mathbb{Q}(f)$ be the smallest algebraic extension of $\mathbb{Q}$ containing all the Fourier coefficients of f: it is well known that $\mathbb{Q}(f)$ is a number field, i.e. a finite extension. As a consequence of the Eichler-Shimura isomorphism [Koh85, corollary p. 202], we can fix two constants $\omega_f^\pm \in \mathbb{C}$ such that $\omega_f^+\omega_f^- = \langle f, f\rangle$ and

$$L_f^{alg}(j) := \frac{L_f(j)}{\pi^j\omega_f^{(-1)^j}} \in \mathbb{Q}(f) \tag{4.4}$$

for any integer $1 \leq j \leq 4k-3$. The numbers $\omega_f^\pm$ are not unique, since we can rescale them by an algebraic factor; the quantity

$$L_f^{alg}(j)L_f^{alg}(j') = \frac{L_f(j)L_f(j')}{\pi^{j+j'}\langle f, f\rangle} \in \mathbb{Q}(f) \tag{4.5}$$

is instead independent of such choice, for any $j \not\equiv j' \mod 2$. We say that a prime ideal $\mathfrak{p}$ of $\mathbb{Q}(f)$ is a *congruence prime* of f if there exist another Hecke eigenform

$f' \in \mathcal{S}^1_{4k-2}$ such that $\lambda_q(f) \equiv \lambda_q(f') \mod \mathfrak{p}$ for every integer prime q, where λ_q denotes the q-th Hecke eigenvalue and the congruence takes place in a large enough extension of $\mathbb{Q}$.

In what follows, rather than fixing a particular choice of periods, we will consider ratios of critical L-values. While there are some canonical choices we can make, dealing with ratios is easier and will suffice for our goal.

Proposition 4.1.2 ([Kat08, theorem 6.1])**.** *Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Q}(f, \sqrt{D})$ not dividing $(4k-1)!$ such that*

(i) $\mathfrak{p}_0$ *divides* $\pi D^{-1} L_f(2k)/L_f(\chi_D; 2k-1)$ *for some fundamental discriminant* $D < 0$, *where* χ_D *is the Kronecker character corresponding to the quadratic extension* $\mathbb{Q}(\sqrt{D})$*;*

(ii) $\mathfrak{p}_0$ *is not a congruence prime of* f*;*

(iii) $\mathfrak{p}_0$ *does not divide* $\zeta(1-2m)L_f(2m+2k-2)L_f(2m+2k-1)/\langle f, f\rangle$ *for some integer* $2 \leq m \leq k-2$.

Then there exists a Hecke eigenform $G \in (\breve{\mathcal{S}}^2_{2k})^{\perp}$ *such that* $\lambda_T(F) \equiv \lambda_T(G) \mod \mathfrak{p}$ *for every Hecke operator* $T \in H^2_{2k}$*, for any prime ideal* $\mathfrak{p}$ *of* $\mathbb{Q}(f, G, \sqrt{D})$ *above* $\mathfrak{p}_0$*.*

Note that $\mathfrak{p}_0$ needs not to exist in general. In fact, the first example (by weight) of this phenomenon is with $2k = 20$ [Sko92, tables 1–6].

As noted in remark (2.5.2), it is possible to scale the Fourier coefficients of g (the Shintani lift of f) and F (the Saito-Kurokawa lift of f) so that they lie in $\mathbb{Q}(f)$. Let $\mathfrak{p}_0$ be as in proposition (4.1.2); if the $|D|$-th Fourier coefficient $c_g(|D|)$ of g is non-zero in addition to (i), then the conditions of [DIK11, lemma 6.2(2)] are satisfied: hence there is a scaling of F and G so that their Fourier coefficients are integral at $\mathfrak{p}$ and not all divisible by $\mathfrak{p}$ for any prime ideal $\mathfrak{p}$ of $\mathbb{Q}(f, G, \sqrt{D})$ above $\mathfrak{p}_0$.

Theorem 4.1.3. *Let $\mathfrak{p}_0$ as in proposition (4.1.2), and assume it does not divide $c_g(|D|) \neq 0$.*

Let $\{h_v\}_v$ and $\{F_t\}_t$ be orthogonal Hecke eigenforms bases for $\mathcal{S}^1_{2l}$ and $\mathcal{S}^2_{2k}$ respectively with $h_1 = h$, $F_1 = F$ and $F_2 = G$. Assume that no F_t with $t \geq 3$ is congruent to either F or G $\mod \mathfrak{p}$*, and that there exists a rational prime ℓ such that* $\lambda_\ell(F) \not\equiv \lambda_\ell(G) \mod \mathfrak{p}^2$*. Similarly, assume h not congruent* $\mod \mathfrak{p}$ *to any h_v, for $v \geq 2$. Again, $\mathfrak{p}$ denotes any prime ideal of $\mathbb{K}$ above $\mathfrak{p}_0$, where $\mathbb{K}$ is any large enough number field containing all the fields in this chapter: namely $\mathbb{Q}(\sqrt{D})$, $\mathbb{Q}(f)$, $\mathbb{Q}(F_i)$ and $\mathbb{Q}(h_v)$ for all $i \leq \dim(\mathcal{S}^2_{2k})$ and $v \leq \dim(\mathcal{S}^1_{2l})$.*

Let $1 \leq m < \frac{l}{2} - 2$ be any integer such that

(i) $\mathfrak{p}$ *divides* $\pi^{4m} L_f(2k)/L_f(2k+4m)$*;*

(ii) $\mathfrak{p}$ *is coprime with* B_{4+4m}, B_{6+8m} *and* B_{4+8m}*;*

(iii) $\mathfrak{p}$ *is above a rational prime* $> \max\{3k+l-2+2m, 4k-9\}$*;*

(iv) $\mathfrak{p}$ *does not divide* $\pi^{-2l-4m-1} L_h(l+2m) L_h(l+2m+1)/\langle h, h\rangle$*;*

(v) $\mathfrak{p}$ *does not divide* $\pi^{-4k-4m+1} Z_{f\otimes h}(2k+l-1+2m)/\langle f, f\rangle$*.*

Then

$$\frac{\pi^{8m} Z_{G\otimes h}(l+2k-1)}{Z_{G\otimes h}(l+2k-1+2m)} \tag{4.6}$$

is algebraic, and integral at $\mathfrak{p}$*.*

The rest of the chapter is devoted to the proof.

Remark 4.1.4. We have used the expression $\mathfrak{p}$ is *coprime with* or *divides* an algebraic quantity x: we need to explain precisely what we mean. Since we know a priori where all these quantities lie, we can work in fixed large enough number field $\mathbb{K}$ containing everything we need: namely $\mathbb{Q}(\sqrt{D})$, $\mathbb{Q}(f)$, $\mathbb{Q}(F_i)$ and $\mathbb{Q}(h_v)$ for all $i \leq \dim(\mathcal{S}^2_{2k})$ and $v \leq \dim(\mathcal{S}^1_{2l})$. Then, for any $x \in \mathbb{K}$, the ideal $x\mathcal{O}_{\mathbb{K}}$ factorises uniquely as

$$x\mathcal{O}_{\mathbb{K}} = \prod_{i=1}^{g} \mathfrak{L}_i^{e_i}$$

for some $e_i \in \mathbb{Z}$, $g \in \mathbb{N}$ and prime ideals $\mathfrak{L}_i \subset \mathcal{O}_{\mathbb{K}}$. If $\mathfrak{p}$ is a prime ideal of $\mathbb{K}$, put

$$\mathrm{ord}_{\mathfrak{p}}(x) := \begin{cases} e_i & \text{if any } \mathfrak{L}_i = \mathfrak{p} \\ 0 & \text{otherwise} \end{cases}$$

and say that x is divisible by $\mathfrak{p}$ if $\mathrm{ord}_{\mathfrak{p}}(x) > 0$, or coprime with $\mathfrak{p}$ if $\mathrm{ord}_{\mathfrak{p}}(x) = 0$.

§ 4.2 L-values as inner products

Diagonal restriction

By proposition (4.1.1), we express $Z_{G\otimes h}$ as the inner product of a diagonally restricted Eisenstein series against h, F and G: put $s_m := 2-k+2m$ and (4.1) becomes

$$\begin{aligned} &\langle E^5_{2k}(\tau\times Z\times W, s_m), \partial^{k-l}_{1,2l} h(\tau) \star (Z) F(W)\rangle = \\ &\qquad = C_{\triangleright}(s_m)\langle \Phi_F, \Phi_\star\rangle L_f(2k+4m) Z_{\star\otimes h}(2k+l-1+2m) \end{aligned} \tag{4.7}$$

where $\star$ is either F or G.

By proposition (3.5.1), if $2k > 6$ and $-1 < 2m \le k-2$ then

$$E^5_{2k}(\cdot, s_m) = (-4\pi)^{5(k-2-2m)} \prod_{j=1}^{k-2-2m} \prod_{l=0}^{4} \left(k+2+2m-j-\frac{l}{2}\right)^{-1} \partial^{k-2-2m}_{5,4+4m} E^5_{4+4m}(\cdot, 0)$$

With the notation of (3.37), put

$$R_m := \mathbb{Z}[\mathfrak{B}^{-1}_{4+4m} \cup \mathfrak{P}^{-1}_{8+8m} \cup \mathfrak{P}^{-1}_{k+1+2m}]$$

so that

$$E^5_{2k}(\cdot, s_m) \in \pi^{5(k-2-2m)} \cdot \mathcal{N}^5_{2k,k-2-2m}(R_m)$$

by corollary (3.5.6) and proposition (3.3.5).

After diagonally restricting $E^5_{2k}(\cdot, s_m)$ to $\mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_2$, we see by proposition (3.3.7) that $E^5_{2k}(\tau \times Z \times W, s_m)$ is an element of

$$\pi^{5(k-2-2m)} \cdot \mathcal{N}^1_{2k,k-2-2m}(R_m) \otimes_{\mathbb{Q}} \mathcal{N}^2_{2k,k-2-2m}(R_m) \otimes_{\mathbb{Q}} \mathcal{N}^2_{2k,k-2-2m}(R_m) \tag{4.8}$$

i.e.

$$E^5_{2k}(\tau \times Z \times W, s_m) = \pi^{5(k-2-2m)} \sum_{r,i,j} c^{(m)}_{rij} \varphi^{(m)}_r(\tau) \psi^{(m)}_i(Z) \psi^{(m)}_j(W) \tag{4.9}$$

for constants $c^{(m)}_{rij} \in \mathbb{Q}$, and bases $\{\varphi^{(m)}_r\}_r$ and $\{\psi^{(m)}_i\}_i$ of $\mathcal{N}^1_{2k,k-2-2m}(R_m)$ and $\mathcal{N}^2_{2k,k-2-2m}(R_m)$ respectively.

Plugging (4.9) into the left-hand side of (4.7) we get

$$\begin{aligned}
&\langle E^5_{2k}(\tau \times Z \times W, s_m), \partial^{k-l}_{1,2l} h(\tau) \star (Z) F(W) \rangle = \\
&\qquad = \pi^{5(k-2-2m)} \sum_{r,i,j} c^{(m)}_{rij} \langle \varphi^{(m)}_r, \partial^{k-l}_{1,2l} h \rangle \langle \psi^{(m)}_i, \star \rangle \langle \psi^{(m)}_j, F \rangle \\
&\qquad = \pi^{5(k-2-2m)-2(k-l)} \sum_{r,i,j} c^{(m)}_{rij} \langle \pi^{2(k-l)} \hat{\partial}^{k-l}_{1,2k} \varphi^{(m)}_r, h \rangle \langle \psi^{(m)}_i, \star \rangle \langle \psi^{(m)}_j, F \rangle
\end{aligned} \tag{4.10}$$

where we have used (3.16) in the last equality, and highlighted the factor $\pi^{-2(k-l)}$ of proposition (3.3.6) so that

$$\pi^{2(k-l)} \hat{\partial}^{k-l}_{1,2k} \varphi^{(m)}_r \in \mathcal{N}^1_{2l,l-2-2m}(R_m)$$

By a similar argument as the proof of corollary (3.4.9), the function

$$\begin{aligned}
\mathfrak{E}_m(\tau, Z, W) &:= \sum_{r,i,j} c^{(m)}_{rij} \big(\pi^{2(k-l)} \hat{\partial}^{k-l}_{1,2k} \varphi^{(m)}_r(\tau)\big) \psi^{(m)}_i(Z) \psi^{(m)}_j(W) \\
&\quad \in \mathcal{N}^1_{2l,l-2-2m}(R_m) \otimes_{\mathbb{Q}} \mathcal{N}^2_{2k,k-2-2m}(R_m) \otimes_{\mathbb{Q}} \mathcal{N}^2_{2k,k-2-2m}(R_m)
\end{aligned} \tag{4.11}$$

still has multivariate Fourier coefficients in R_m: the application of $\hat{\partial}_{1,2k}^{k-l}$ acts on the monomials of the Fourier coefficients of $\varphi_r^{(m)}$ by multiplication by a factor depending only on the degree of the monomial itself, as explicitly computed in (3.17). Hence (4.10) yields

$$\begin{aligned}\langle E_{2k}^5(\tau\times Z\times W, s_m), \partial_{1,2l}^{k-l} h(\tau)\star(Z)F(W)\rangle = \\ = \pi^{5(k-2-2m)-2(k-l)}\langle \mathfrak{E}_m(\tau, Z, W), h(\tau)\star(Z)F(W)\rangle\end{aligned} \tag{4.12}$$

Holomorphic projection

If the conditions of proposition (3.4.1) are satisfied, then we can apply holomorphic projection to each nearly holomorphic modular form appearing in (4.10) without changing the value of the inner products. With the notation of theorem (4.1.3), let $\{h_v\}_v$ and $\{F_t\}_t$ be orthogonal cuspidal Hecke eigenform bases of $\mathcal{S}_{2l}^1$ and $\mathcal{S}_{2k}^2$ respectively, with $h_1 = h$, $F_1 = F$ and $F_2 = G$; then (4.12) yields

$$\begin{aligned}\langle E_{2k}^5(\tau\times Z\times W, s_m), \partial_{1,2l}^{k-l} h(\tau)\star(Z)F(W)\rangle = \\ = \pi^{5(k-2-2m)-2(k-l)}\langle \mathfrak{E}_m(\tau, Z, W), h(\tau)\star(Z)F(W)\rangle \\ = \pi^{5(k-2-2m)-2(k-l)}\langle \mathrm{Hol}\mathfrak{E}_m(\tau, Z, W), h(\tau)\star(Z)F(W)\rangle \\ = \pi^{5(k-2-2m)-2(k-l)}\sum_{r,i,j} d_{r,i,j}^{(m)}\langle h_r, h\rangle\langle F_i, \star\rangle\langle F_j, F\rangle \\ = \pi^{5(k-2-2m)-2(k-l)}\langle h, h\rangle\langle F, F\rangle\Big(A^{(m)}\langle F, \star\rangle + B^{(m)}\langle G, \star\rangle\Big)\end{aligned} \tag{4.13}$$

where we have applied term-wise holomorphic projection to $\mathfrak{E}_m$ as explained in corollary (3.4.9), the $d_{r,i,j}^{(m)}$ are suitable constants in $\mathbb{C}$, and the last equality is due to orthogonality after putting $A^{(m)} := d_{1,1,1}^{(m)}$ and $B^{(m)} = d_{1,2,1}^{(m)}$. Again, $\star$ denotes either F or G.

Comparing (4.13) and (4.7) we finally obtain

$$\begin{aligned}\pi^{5(k-2-2m)-2(k-l)}A^{(m)}\langle h, h\rangle\langle F, F\rangle\langle F, F\rangle = \\ = C_\triangleright(s_m)\langle \Phi_F, \Phi_F\rangle L_f(2k+4m)Z_{F\otimes h}(2k+l-1+2m)\end{aligned} \tag{4.14}$$

$$\begin{aligned}\pi^{5(k-2-2m)-2(k-l)}B^{(m)}\langle h, h\rangle\langle F, F\rangle\langle G, G\rangle = \\ = C_\triangleright(s_m)\langle \Phi_F, \Phi_G\rangle L_f(2k+4m)Z_{G\otimes h}(2k+l-1+2m)\end{aligned} \tag{4.15}$$

Hence, the arithmetic information we are looking for is encoded in $B^{(m)}$, which we study in the next section; now we verify that the conditions of proposition (3.4.1) are indeed satisfied by $\varphi_r^{(m)}$ and $\psi_i^{(m)}$.

By remark (3.2.3), $\psi_i^{(m)}$ is a section of $E^5_{2k}(\cdot, s_m)$, and in particular

$$|\psi_i^{(m)}(Z)| \ll_{\tau,W} |E^5_{2k}(\tau \times Z \times W, s_m)|$$

uniformly on $Z \in \mathcal{H}_2$, for any fixed τ and W. Since $E^5_{2k}(\cdot, s_m)$ is obtained by applying $\partial^{k-2-2m}_{5,4+4m}$ to an element of $\mathcal{M}^5_{4+4m}$, lemma (3.4.4) applies and

$$|E^5_{2k}(\cdot, s_m)| \ll \prod_{j=1}^{5} (\lambda_j^{-k+2+2m} + \lambda_j^{-k-2-2m})$$

uniformly on $\mathcal{H}_5$, and in particular (with natural labelling)

$$|\psi_i^{(m)}(Z)| \ll_{\tau,W} \prod_{j=2}^{3} (\lambda_j^{-k+2+2m} + \lambda_j^{-k-2-2m})$$

If $k+2+2m < 2k-2$ (or equivalently $2m < k-4$) then, by proposition (3.4.6), $\psi_i^{(m)}$ is of bounded growth as claimed. By the same argument

$$|\varphi_r^{(m)}(\tau)| \ll_{Z,W} \lambda_1^{-k+2+2m} + \lambda_1^{-k-2-2m}$$

and then, by remark (3.4.5),

$$|\hat{\partial}^{k-l}_{1,2k}\varphi_r^{(m)}(\tau)| \ll_{Z,W} \lambda_1^{-l+2+2m} + \lambda_1^{-l-2-2m}$$

so that $\hat{\partial}^{k-l}_{1,2k}\varphi_r^{(m)}$ is of bounded growth by proposition (3.4.6), provided that $l+2+2m < 2l-1$, i.e. $2m < l-3$. Finally, we need to check that (3.23) is convergent for every $A > 0$: in the case of nearly holomorphic modular forms, this integral was computed in (3.31), and is indeed convergent if

$$\begin{cases} k-2-2m < 2k-2 \\ l-2-2m < 2l-1 \end{cases}$$

which is surely satisfied for any $m \geq 0$.

§ 4.3 Arithmeticity of L-values

Isolating $A^{(m)}$ and $B^{(m)}$

With the notation of (4.11) and (4.13), put

$$\begin{aligned} \Xi_m(\tau, Z, W) &:= \mathrm{Hol}\,\mathfrak{E}_m(\tau, Z, W) \\ &= \sum_{r,i,j} d^{(m)}_{r,i,j} h_r(\tau) F_i(Z) F_j(W) \end{aligned} \tag{4.16}$$

for suitable constants $d^{(m)}_{r,i,j} \in \mathbb{C}$, with $d^{(m)}_{1,1,1} = A^{(m)}$ and $d^{(m)}_{1,2,1} = B^{(m)}$. We observed that $\mathfrak{E}_m$ has Fourier coefficients in R_m, and therefore Ξ_m has coefficients in $R_m[\mathfrak{P}^{-1}_{4k-9}]$ by corollary (3.4.9). Then (4.13) states that

$$\begin{aligned}\langle E^5_{2k}(\tau \times Z \times W, s_m), \partial^{k-l}_{1,2l} h(\tau) \star (Z) F(W)\rangle = \\ = \pi^{5(k-2-2m)-2(k-l)} \langle \Xi_m(\tau, Z, W), h(\tau) \star (Z) F(W)\rangle\end{aligned}$$

and the functions appearing on the right-hand side are holomorphic, hence with numeric Fourier coefficients. We now derive $A^{(m)}$ and $B^{(m)}$ from the coefficients of Ξ_m by applying a certain family of Hecke operators to it.

Recall the basis of orthogonal Hecke eigenforms $\{h_r\}_r \in \mathcal{S}^1_{2l}$, where $h_1 = h$. By hypothesis of theorem (4.1.3), we can find primes $\{q_r\}_r$ such that

$$\lambda_{q_r}(h_r) \not\equiv \lambda_{q_r}(h) \mod \mathfrak{p} \tag{4.17}$$

for every $r \geq 2$. Put

$$T_\tau := \prod_{r=2}^{\dim(\mathcal{S}^1_{2l})} \frac{T^{(2l)}_{q_r} - \lambda_{q_r}(h_r)}{\lambda_{q_r}(h) - \lambda_{q_r}(h_r)}$$

where the subscript $_\tau$ denotes which variable the operator applies to. Since each Hecke operator $T^{(2l)}_{q_r}$ acts as an endomorphism on the space of modular forms with algebraic integral coefficients [DS05, proposition 5.3.1], and the Hecke eigenvalues $\lambda_{q_r}(h_r)$ are algebraic integers themselves, the function $T_\tau \Xi_m$ has algebraic Fourier coefficients which are still integral at $\mathfrak{p}$ by (4.17). On the other hand, observe

$$T_\tau h_u = \begin{cases} h & u = 1 \\ 0 & u \geq 2 \end{cases}$$

hence from (4.16) we get

$$T_\tau \Xi_m(\tau, Z, W) = \sum_{i,j} d^{(m)}_{1,i,j} h(\tau) F_i(Z) F_j(W)$$

and

$$\langle \Xi_m(\tau, Z, W), h(\tau) \star (Z) F(W)\rangle = \langle T_\tau \Xi_m(\tau, Z, W), h(\tau) \star (Z) F(W)\rangle$$

Similarly, we get rid of the redundant forms F_i in the variable Z. By hypothesis, there exist primes $\{q'_i\}_i$ such that

$$\begin{aligned}\lambda_{q'_i}(F_i) \not\equiv \lambda_{q'_i}(F) \mod \mathfrak{p} \\ \lambda_{q'_i}(F_i) \not\equiv \lambda_{q'_i}(G) \mod \mathfrak{p}\end{aligned} \tag{4.18}$$

for $i \geq 3$, where $\{F_i\}_i$ is a basis of orthogonal Hecke eigenforms for $\mathcal{S}^2_{2k}$. Put

$$T_Z := \prod_{i=3}^{\dim(\mathcal{S}^2_{2k})} \frac{T^{(2k)}_{q'_i} - \lambda_{q'_i}(F_i)}{\lambda_{q'_i}(F) - \lambda_{q'_i}(F_i)}$$

and

$$\vartheta := \prod_{i=3}^{\dim(\mathcal{S}^2_{2k})} \frac{\lambda_{q'_i}(G) - \lambda_{q'_i}(F_i)}{\lambda_{q'_i}(F) - \lambda_{q'_i}(F_i)} \in \overline{\mathbb{Q}} \tag{4.19}$$

so that

$$T_Z T_\tau \Xi_m(\tau, Z, W) = \sum_j d^{(m)}_{1,1,j} h(\tau) F(Z) F_j(W) + \vartheta \sum_j d^{(m)}_{1,2,j} h(\tau) G(Z) F_j(W)$$

and

$$\begin{aligned}\langle \Xi_m(\tau, Z, W), h(\tau)F(Z)F(W)\rangle &= \langle T_Z T_\tau \Xi_m(\tau, Z, W), h(\tau)F(Z)F(W)\rangle \\ \langle \Xi_m(\tau, Z, W), h(\tau)G(Z)F(W)\rangle &= \vartheta^{-1}\langle T_Z T_\tau \Xi_m(\tau, Z, W), h(\tau)G(Z)F(W)\rangle\end{aligned}$$

Again, the Hecke operators $T^{(2k)}_{q'_i}$ act as endomorphisms on the space of Siegel modular forms with algebraic integral coefficients [Sko92, theorem section 2]. Also, ϑ is coprime with $\mathfrak{p}$, so $T_Z T_\tau \Xi_m$ still has algebraic Fourier coefficients, integral at $\mathfrak{p}$.

Finally, we kill the redundant forms F_j in the variable W. Put

$$T_W := \prod_{j=2}^{\dim(\mathcal{S}^2_{2k})} \frac{T^{(2k)}_{q'_j} - \lambda_{q'_j}(F_j)}{\lambda_{q'_j}(F) - \lambda_{q'_j}(F_j)} \tag{4.20}$$

where $q'_2 = \ell$ as in the statement of theorem (4.1.3) so that

$$\begin{aligned}\lambda_\ell(F) &\equiv \lambda_\ell(G) \mod \mathfrak{p} \\ \lambda_\ell(F) &\not\equiv \lambda_\ell(G) \mod \mathfrak{p}^2\end{aligned}$$

Then

$$\begin{aligned}\Psi_m(\tau, Z, W) &:= T_W T_Z T_\tau \Xi_m(\tau, Z, W) \\ &= d^{(m)}_{1,1,1} h(\tau) F(Z) F(W) + \vartheta d^{(m)}_{1,2,1} h(\tau) G(Z) F(W) \\ &= A^{(m)} h(\tau) F(Z) F(W) + \vartheta B^{(m)} h(\tau) G(Z) F(W)\end{aligned} \tag{4.21}$$

and

$$\begin{aligned}\langle \Xi_m(\tau, Z, W), h(\tau)F(Z)F(W)\rangle &= \langle \Psi_m(\tau, Z, W), h(\tau)F(Z)F(W)\rangle \\ \langle \Xi_m(\tau, Z, W), h(\tau)G(Z)F(W)\rangle &= \vartheta^{-1}\langle \Psi_m(\tau, Z, W), h(\tau)G(Z)F(W)\rangle\end{aligned} \tag{4.22}$$

Due to the congruence between F and G, by applying T_Z we introduce a factor of $\mathfrak{p}$ in the denominator of Ψ_m; nonetheless, by the assumption on ℓ, the Fourier coefficients of Ψ_m have $\mathrm{ord}_\mathfrak{p} \geq -1$.

Arithmetic properties of $A^{(m)}$ and $B^{(m)}$

We begin by observing that $A^{(0)} = 0$: this follows immediately from (4.14), since $Z_{F\otimes h}(2k+l-1) = 0$ by assumption. Then (4.21) says

$$\Psi_0(\tau, Z, W) = \vartheta B^{(0)} h(\tau) G(Z) F(W)$$

Fix a Fourier term for each variable such that the corresponding Fourier coefficients on the right-hand side are coprime with $\mathfrak{p}$, hence by comparing the order at $\mathfrak{p}$ in each side we deduce $\mathrm{ord}_{\mathfrak{p}}(B^{(0)}) \geq -1$.

Let now $m \geq 1$: to study $A^{(m)}$, we rewrite (4.14) as

$$A^{(m)} = \frac{C_{\triangleright}(2-k+2m)}{\pi^{5(k-2-2m)-2(k-l)}} \frac{\langle \Phi_F, \Phi_F\rangle}{\langle F, F\rangle} \frac{L_f(2k+4m)}{\langle F, F\rangle} \frac{Z_{F\otimes h}(2k+l-1+2m)}{\langle h, h\rangle} \tag{4.23}$$

From (4.2), $C_{\triangleright}(2-k+2m)$ is the product of

$$\begin{aligned}
a_{\triangleright}(2-k+2m) &= 2^{-6k+1-12m}\pi^{4-2m-3k} \\
\zeta_{\triangleright}(2-k+2m) &= 8\frac{(4+4m)!(8m+6)!(8m+4)!}{(2\pi)^{20m+14}B_{4+4m}B_{6+8m}B_{4+8m}} \\
\Gamma_{\triangleright}(2-k+2m) &= \frac{\prod_{j=0}^{k-l-1}(2+2m-k+j)}{(k+2m+1/2)(k+2m)(k+2m+1)}\times \\
&\quad \times \frac{(2m+l)!(2m+3k+l-2)!(2m+2k-l-1)!}{(4m+2k)!}
\end{aligned}$$

hence

$$\frac{C_{\triangleright}(2-k+2m)}{\pi^{5(k-2-2m)-2(k-l)}} \in \pi^{-12m-6k-2l}\mathbb{Q} \tag{4.24}$$

which, under the conditions of theorem (4.1.3), is coprime with $\mathfrak{p}$ (i.e. $\mathfrak{p}$ is not above any of its prime divisors).

Next, recall the formulae [Bro07, theorem 4.1 and corollary 6.3]

$$\langle F, F\rangle = \frac{(2k-1)!L_f(2k)}{3\cdot 2^{4k+1}}\langle \Phi_F, \Phi_F\rangle$$

$$\langle F, F\rangle = \frac{2k-1}{2^4\cdot 3^2}\frac{|c_g(|D|)|^2 L_f(2k)}{\pi|D|^{2k-3/2}L_f(\chi_D; 2k-1)}\langle f, f\rangle$$

where g is the half-integral modular form associated to f via the Saito-Kurokawa correspondence and D is as in proposition (4.1.2). Hence the remaining part of the right-hand side of (4.23) equals

$$
\begin{aligned}
\frac{\langle \Phi_F, \Phi_F \rangle}{\langle F, F \rangle} \frac{L_f(2k+4m)}{\langle F, F \rangle} \frac{Z_{F\otimes h}(2k+l-1+2m)}{\langle h, h \rangle} &= \\
&= \frac{3^3 \cdot 2^{4k+3}\pi |D|^{2k-3/2}}{(2k-1)!(2k-1)|c_g(|D|)|^2} \frac{L_f(2k+4m)}{L_f(2k)} \frac{L_f(\chi_D; 2k-1)}{L_f(2k)} \times \\
&\times \frac{Z_{f\otimes h}(2k+l-1+2m)}{\langle f, f \rangle} \frac{L_h(l+2m)L_h(l+2m+1)}{\langle h, h \rangle}
\end{aligned} \tag{4.25}
$$

where we have factorised $Z_{F\otimes h}$ as in (4.3): we proceed to study each ratio individually.

The first ratio on the right-hand side of (4.25) is in $\pi\mathbb{Q}(f)(\sqrt{|D|})$ and coprime with $\mathfrak{p}$ by the assumptions of proposition (4.1.2). By corollary [Koh85, p. 202],

$$
\begin{aligned}
\frac{L_h(l+2m)L_h(l+2m+1)}{\langle h, h \rangle} &= \pi^{2l+4m+1} \frac{L_h(l+2m)}{\pi^{l+2m}\omega_h^-} \frac{L_h(l+2m+1)}{\pi^{l+2m+1}\omega_h^+} \\
&\in \pi^{2l+4m+1}\mathbb{Q}(h)
\end{aligned} \tag{4.26}
$$

and we have assumed $\mathfrak{p}$ coprime with this quantity.

By [Shi78, theorem 3],

$$
\frac{Z_{h\otimes f}(2k+l-1+2m)}{\langle f, f \rangle} \in \pi^{4k+4m-1}\mathbb{Q}(h, f) \tag{4.27}
$$

which is coprime with $\mathfrak{p}$ by hypothesis.

The ratio

$$
\begin{aligned}
\frac{L_f(2k+4m)}{L_f(2k)} &= \pi^{4m} \frac{L_f(2k+4m)}{\pi^{2k+4m}\omega_f^+} \frac{\pi^{2k}\omega_f^+}{L_f(2k)} \\
&\in \pi^{4m}\mathbb{Q}(f)
\end{aligned} \tag{4.28}
$$

has $\mathrm{ord}_{\mathfrak{p}} \leq -1$ by the very definition of $\mathfrak{p}$.

Lastly

$$
\frac{L_f(\chi_D, 2k-1)}{L_f(2k)} \in \pi^{-1}\mathbb{Q}(f)(\sqrt{|D|}) \tag{4.29}
$$

by [Shi77, theorem 1], and we assumed $\mathrm{ord}_{\mathfrak{p}} \leq -1$ for this quantity in proposition (4.1.2).

Combining (4.24), (4.26), (4.27), (4.28) and (4.29) in (4.23), we see that $A^{(m)} \in \overline{\mathbb{Q}}$ with $\mathrm{ord}_{\mathfrak{p}} \leq -2$. Again comparing Fourier expansions in (4.21), we deduce $\mathrm{ord}_{\mathfrak{p}}(B^{(m)}) \leq -2$ as well.

Taking the ratio

By (4.15), we can finally take the ratio

$$\pi^{-10m}\frac{C_{\triangleright}(2-k)}{C_{\triangleright}(2+2m-k)}\frac{L_f(2k)}{L_f(2k+4m)}\frac{Z_{G\otimes h}(2k+l-1)}{Z_{G\otimes h}(2k+l-1+2m)}=\frac{B^{(0)}}{B^{(m)}}\in\overline{\mathbb{Q}}$$

We have established in the previous section that $B^{(0)}/B^{(m)}$ has $\mathrm{ord}_{\mathfrak{p}}\geq 1$. For $m\geq 1$ as in theorem (4.1.3),

$$C_{\triangleright}(2+2m-k)\in\pi^{-10-3k-22m}\overline{\mathbb{Q}}$$

is coprime with $\mathfrak{p}$. Since $\pi^{-4m}L_f(2k)/L_f(2k+4m)\in\overline{\mathbb{Q}}$ has $\mathrm{ord}_{\mathfrak{p}}\geq 1$, we deduce that

$$\frac{\pi^{8m}Z_{G\otimes h}(2k+l-1)}{Z_{G\otimes h}(2k+l-1+2m)}$$

is an algebraic number with $\mathrm{ord}_{\mathfrak{p}}\geq 0$, completing the proof.

Remark 4.3.1. While $\mathrm{ord}_{\mathfrak{p}}(B^{(m)})\leq -2$ for $m\geq 1$ is due to the presence of $L_f(2k)$ twice in (4.25), there really is no reason to expect $\mathrm{ord}_{\mathfrak{p}}(B^{(0)})\geq -1$: in fact, numerical data support $\mathrm{ord}_{\mathfrak{p}}(B^{(0)})\geq 0$ instead. If this is indeed the case, then the ratio $B^{(0)}/B^{(m)}$ has $\mathrm{ord}_{\mathfrak{p}}\geq 2$ and therefore

$$\mathrm{ord}_{\mathfrak{p}}\left(\frac{\pi^{8m}Z_{G\otimes h}(2k+l-1)}{Z_{G\otimes h}(2k+l-1+2m)}\right)\geq 1$$

which is what the Bloch-Kato conjecture actually predicts. We will prove this stronger result in the following chapters by using a different integral representation formula for $Z_{G\otimes h}$, but with the additional restriction that the weights of F,G,h must be equal: this restriction is due to the integral formula itself, as the result is expected to hold regardless of the weights.

The reason we get $\mathrm{ord}_{\mathfrak{p}}(B^{(0)})\geq -1$ is that, when applying the Hecke operator (4.20), we are introducing a factor of $\mathfrak{p}^{-1}$ due to the congruence between $\lambda_{\mathfrak{q}}(F)$ and $\lambda_{\mathfrak{q}}(G)$. We conjecture that this additional factor is balanced by the action of $T_{\mathfrak{q}}-\lambda_{\mathfrak{q}}(G)$, which should act on F by introducing a factor of $\mathfrak{p}$ in its Fourier expansion: we could not prove this, but this would explain the inaccurate inequality for $B^{(0)}$. Again, this issue is due to the nature of integral formula we employ: in proposition (4.1.1), we take the inner product of the Eisenstein series against h,G and F to yield L_f and $Z_{G\otimes h}$, hence F should have no effect on $Z_{G\otimes h}$, but the presence of F apparently forces $\mathfrak{p}^{-1}$ into $B^{(0)}$, when it really should not.

Chapter 5

Non-holomorphic Hermitian Eisenstein series

In this chapter, we study the arithmetic properties of the Fourier coefficients of some Hermitian Eisenstein series. We are particularly interested in those of low weight, i.e. when the defining series does not converge absolutely and we need to consider the meromorphic continuation. For the low weight Eisenstein series we need, we show that the process of meromorphic continuation does indeed produce a holomorphic Hermitian modular form; further, we show that it equals a particular linear combination of theta series, and therefore its Fourier coefficients are rational numbers with bounded denominators. We are grateful to Nagaoka, who pointed out this connection to us in a private communication: this entire chapter is devoted to showing that the different definitions used by Nagaoka, Saha and Ichino describe essentially the same Hermitian Eisenstein series, and Ichino's version of the Siegel-Weil formula equate it with a linear combination of theta series.

§ 5.1 Unitary groups

Let L be an imaginary quadratic extension of $\mathbb{Q}$ with class number 1 and define the *general unitary group*

$$\mathrm{GU}(n,n)(L) := \left\{\gamma \in \mathrm{GL}_{2n}(L) : \gamma^\dagger S_n \gamma = \mu_n(\gamma) S_n, \mu_n(\gamma) \in \mathbb{Q}^\times\right\} \tag{5.1}$$

where $\dagger$ denotes the transpose conjugate, i.e. $\gamma^\dagger = \overline{\gamma}^t$. It is immediate to show that, if $\gamma \in \mathrm{GU}(n,n)(L)$, then so are γ^{-1} and $\gamma^\dagger$. Further define the *unitary group*

$$\mathrm{U}(n,n)(L) := \left\{\gamma \in \mathrm{GL}_{2n}(L) : \gamma^\dagger S_n \gamma = S_n\right\} \tag{5.2}$$

which is a subgroup of $\mathrm{GU}(n,n)(L)$. If $\gamma \in \mathrm{U}(n,n)(L)$, from $\gamma S_n \gamma^\dagger = S_n$ we deduce

$$AB^\dagger = BA^\dagger \qquad CD^\dagger = DC^\dagger \qquad AD^\dagger - BC^\dagger = I_n \tag{5.3}$$

where A, B, C, D are the blocks of γ, thence

$$\gamma^{-1} = \begin{pmatrix} D^\dagger & -B^\dagger \\ -C^\dagger & A^\dagger \end{pmatrix} \tag{5.4}$$

Finally, we have the *Hermitian modular group*

$$\Upsilon_n(L) := \left\{ \gamma \in \mathrm{GL}_{2n}(\mathcal{O}_L) : \gamma^\dagger S_n \gamma = S_n \right\} \tag{5.5}$$

consisting of the elements of $\mathrm{U}(n,n)(L)$ with integral entries, where $\mathcal{O}_L$ denotes the ring of integers of L. Indeed, from (5.4) we see that γ^{-1} belongs to $\mathrm{GL}_{2n}(\mathcal{O}_L)$ and hence to $\Upsilon_n(L)$.

If G is any of the groups (5.1), (5.2) or (5.5), then

$$P(G) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G : C = 0 \right\} \tag{5.6}$$

denotes its *standard parabolic subgroup*. We shall show that $P(G)\backslash G$ defines essentially the same object for any of the groups above. As we are fixing n and L, we often drop them from the notation.

We begin with a characterisation of Υ_n; this is essentially the content of [Maa71, chapter 11] generalised to the Hermitian case. We say that $(C|D) \in \mathcal{O}_L^{n,2n}$ is a *symmetric pair* if $CD^\dagger = DC^\dagger$, and *coprime* if $(MC|MD) \in \mathcal{O}_L^{n,2n}$ implies that M is integral for any $M \in L^{n,n}$.

Lemma 5.1.1. *The matrix $(C|D) \in \mathcal{O}_L^{n,2n}$ is the second row of a $\gamma \in \Upsilon_n$ if and only if it is a symmetric coprime pair.*

Proof. Let $\gamma \in \Upsilon_n$; then by (5.3) its second row $(C|D)$ is a symmetric pair. Now let M be such that $(MC|MD)$ is integral: then by (5.4)

$$(MC|MD)\gamma^{-1} = (MC|MD) \begin{pmatrix} D^\dagger & -B^\dagger \\ -C^\dagger & A^\dagger \end{pmatrix} = (0_n|M)$$

hence M is integral since so are both factors on the left.

Let now $(C|D)$ be any symmetric coprime pair. We first observe that $(C|D)$ is coprime if and only if so is $U_1(C|D)U_2$ for any $U_1 \in \mathrm{GL}_n(\mathcal{O}_L)$ and $U_2 \in \mathrm{GL}_{2n}(\mathcal{O}_L)$. Since $\mathcal{O}_L$ is a PID (as we assume L to have class number 1), by elementary divisor reduction theory we can find unimodular matrices U_1 and U_2 such that

$U_1(C|D)U_2 = (\Lambda|0_n)$ where Λ is a diagonal matrix. Since $(\Lambda|0_n)$ is coprime, we deduce that every diagonal entry of Λ must be in $\mathcal{O}_L^\times$, hence $\Lambda \in \mathrm{GL}_n(\mathcal{O}_L)$. By renaming U_1, we can therefore assume that

$$(C|D)U_2 = (U_1^{-1}|0_n)$$

hence $(C|D)$ is of maximal rank. We let

$$\begin{pmatrix} X \\ Y \end{pmatrix} := U_2 \begin{pmatrix} U_1 \\ 0 \end{pmatrix}$$

and by left-multiplying by $(C|D)$ we see that $CX + DY = I_n$. Also by assumption $CD^\dagger = DC^\dagger$; we let

$$\gamma := \begin{pmatrix} Y^\dagger + X^\dagger Y C & -X^\dagger + X^\dagger Y D \\ C & D \end{pmatrix} \in \mathcal{O}_L^{2n,2n}$$

and an immediate computation shows that (5.3) hold for γ i.e. $\gamma^\dagger S_n \gamma = S_n$. Thence, from (5.4) we observe that $\gamma^{-1} \in \mathcal{O}_L^{2n,2n}$, which proves that γ is in $\mathrm{GL}_{2n}(\mathcal{O}_L)$ and therefore in Υ_n. □

Proposition 5.1.2. *The embedding map*

$$P(\mathrm{U})\backslash\mathrm{U} \to P(\mathrm{GU})\backslash\mathrm{GU}$$

is a group isomorphism.

Proof. The embedding $\mathrm{U} \to P(\mathrm{GU})\backslash\mathrm{GU}$ sending γ to $P(\mathrm{GU})\gamma$ is an obvious group morphism with kernel $P(\mathrm{U})$.

The proof is over once we show it is surjective. Fix any $\gamma \in \mathrm{GU}$ with multiplier $\mu = \mu(\gamma) \in \mathbb{Q}^\times$. Put $\rho := \begin{pmatrix} I_n & 0_n \\ 0_n & \mu^{-1} I_n \end{pmatrix} \in P(\mathrm{GU})$ and $\alpha := \rho\gamma$. It is immediate to see that $\alpha^\dagger S_n \alpha = S_n$, i.e. $\alpha \in \mathrm{U}$, and its image under the embedding is $P(\mathrm{GU})\gamma$. □

Proposition 5.1.3. *The embedding map*

$$P(\Upsilon)\backslash\Upsilon \to P(\mathrm{U})\backslash\mathrm{U}$$

is a group isomorphism.

Proof. Let $V := L^n \oplus L^n$ and $\langle\cdot,\cdot\rangle$ be the standard alternating Hermitian form on V, so that if $x, y \in V$ then $\langle x, y\rangle = x^\dagger S_n y$. In this setting, $\mathrm{U}(n,n)$ is exactly the group of endomorphisms of V fixing this alternating Hermitian form.

A n-dimensional isotropic subspace Ω of V is called a *Lagrangian* of V, where isotropic means that $\langle x, y\rangle = 0$ for any $x, y \in \Omega$. We shall show that both ${}_{P(\mathrm{U})}\backslash^{\mathrm{U}}$ and ${}_{P(\Upsilon)}\backslash^{\Upsilon}$ are in bijection with the set of Lagrangians on V.

We say that a matrix $\omega \in L^{n,2n}$ represents a Lagrangian Ω if its rows form a basis for Ω, and write $\Omega = [\omega]$. We define a right action of $\mathrm{U}(n, n)$ (hence of Υ) on the set of Lagrangians by $[\omega]\gamma := [\omega\gamma]$; we observe in fact that $\omega\gamma$ is of full rank (since γ is invertible) and $(\omega\gamma)S_n(\gamma^\dagger\omega^\dagger) = \omega S_n\omega^\dagger = 0_{2n}$ hence $[\omega\gamma]$ is again isotropic.

We show that this action is transitive. If ω represents a Lagrangian, we write $\omega = (C|D)$ and observe that $(C|D)$ is a symmetric pair by isotropy of $[\omega]$. Eventually rescaling the rows of ω, we may assume that $(C|D) \in \mathcal{O}_L^{n,2n}$. As in the proof of lemma (5.1.1), by elementary divisors theory we can find unimodular matrices U_1, U_2 such that $U_1(C|D)U_2 = (0_n|\Lambda)$ for a diagonal matrix $\Lambda \in \mathcal{O}_L^{n,n}$, which is non-singular as ω has full rank. By letting $\Lambda' := U_1^{-1}\Lambda^{-1}U_1$, we have $(0_n|I_n) = U_1\Lambda'\omega U_2$. Since Λ' is invertible, $\Lambda'\omega$ represents the same Lagrangian as ω; furthermore, $(C'|D') := \Lambda'\omega$ is integral since it equals $U_1^{-1}(0_n|I_n)U_2^{-1}$, and is a coprime pair since so is $(0_n|I_n)$. Since $(C'|D')$ represents a Lagrangian, it is a symmetric pair by isotropy, hence by lemma (5.1.1) $(C'|D')$ is the second row of a matrix $\gamma \in \Upsilon_n$. We have just shown that every Lagrangian $[\omega]$ is realised as $[(0_n|I_n)]\gamma_\omega$ for a suitable $\gamma_\omega \in \Upsilon$, i.e. Υ (hence $\mathrm{U}(n, n)$) acts transitively on the set of Lagrangians of V.

Denote $\omega_0 := (0_n|I_n)$. The stabiliser of $[\omega_0]$ in $\mathrm{U}(n, n)$ is the set of $\gamma \in \mathrm{U}(n, n)$ such that $\omega_0\gamma$ represents the same Lagrangian, i.e. $\omega_0\gamma = P\omega_0$ for some $P \in \mathrm{GL}_n(L)$: if $(C|D)$ is the second row of γ, this means $(C|D) = (0_n|P)$ i.e. γ must lie in the parabolic subgroup $P(U)$. Analogously, $P(\Upsilon)$ is the stabiliser of $[\omega_0]$ in Υ. Therefore the map

$$P(G)\backslash G \to \{\text{Lagrangians of V}\}$$

given by $P(G)\gamma \mapsto [\omega_0]\gamma$ is a bijection for both $G = \mathrm{U}(n, n)$ and $G = \Upsilon_n$. □

Corollary 5.1.4. *The embedding maps*

$$P(\Upsilon)\backslash\Upsilon \to P(\mathrm{U})\backslash\mathrm{U} \to P(\mathrm{GU})\backslash\mathrm{GU}$$

are group isomorphisms, and each quotient admits representatives in Υ.

§ 5.2 Hermitian modular forms

Let $\mathfrak{H}_n$ be the *Hermitian upper half space* of genus n

$$\mathfrak{H}_n := \left\{Z \in \mathbb{C}^{n,n} : i(Z^\dagger - Z) > 0\right\} \tag{5.7}$$

which contains the Siegel upper half space $\mathcal{H}_n$ of (2.1) as a submanifold. We systematically write $Z = X + iY$ where

$$X := \frac{Z + Z^\dagger}{2} \qquad Y := \frac{Z - Z^\dagger}{2i} \tag{5.8}$$

so that X and Y are Hermitian matrices, and $Y > 0$ by definition of $\mathfrak{H}_n$. If $Z \in \mathcal{H}_n$, then X and Y are the standard real and imaginary parts of Z. We put $\delta(Z) := \det(Y)$, which extends the previous definition given in (2.1).

The Hermitian modular group Υ_n (after choosing an embedding of L into $\mathbb{C}$, hence of Υ_n into $\mathbb{C}^{2n,2n}$) acts on $\mathfrak{H}_n$ by fractional linear transformations

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}(Z) := (AZ + B)(CZ + D)^{-1} \tag{5.9}$$

and $j(\gamma, Z) := CZ + D$ is the *automorphy factor*. Again, the Siegel modular group Γ_n of (2.3) is a subgroup of Υ_n and these definitions extend the ones given in (2.4) and (2.6).

We call *Hermitian modular forms* of genus n and weight $k \in \mathbb{Z}$ a function

$$F : \mathfrak{H}_n \to \mathbb{C} \text{ satisfying (i)-(iii)} \tag{5.10}$$

(i) $F(Z)$ is holomorphic (in each entry of Z);

(ii) $F(\gamma(Z)) = j(\gamma, Z)^k F(Z)$ for every $Z \in \mathfrak{H}_n$ and $\gamma \in \Upsilon_n$;

(iii) when $n = 1$, F is bounded on vertical strips $\{z \in \mathcal{H}_1 : y > \epsilon\}$ for every $\epsilon > 0$.

We see from (ii) that the restriction of F to $\mathcal{H}_n$ gives rise to a Siegel (weakly) modular form, since $\mathcal{H}_n \subseteq \mathfrak{H}_n$ and $\Gamma_n \subseteq \Upsilon_n$.

Similarly to (2.9), Hermitian modular forms admit Fourier expansions; let $\mathcal{I}_L$ be the different ideal of L: the Fourier coefficients are indexed by positive semi-definite Hermitian matrices $A \in L^{n,n}$ satisfying $A_{i,i} \in \mathbb{Z}, A_{ij} \in \mathcal{I}_L^{-1}$ and the Fourier expansion is given by

$$\sum_{A \geq 0} F_A q^A$$

where the sum is over such matrices A.

We will not be using the theory of Hermitian modular forms in what follows, as in the next chapter we will immediately pullback a Hermitian Eisenstein series to the space of Siegel modular forms, for which the results of the previous chapters apply. Nonetheless, one can find a recollection of the main results and properties of Hermitian forms in [HK09, section 1.2].

The main example of Hermitian modular forms is given by Eisenstein series (at least, if we relax the holomorphy condition (5.10-i) to smoothness): let $\Upsilon_{n,0} := P(\Upsilon_n)$ be the parabolic subgroup (5.6) of Υ_n, consisting of matrices whose bottom-left block vanishes, and define the *Hermitian Eisenstein series* $\mathcal{E}_n^k : \mathfrak{H}_n \to \mathbb{C}$ of genus n and weight k by

$$\begin{aligned} \mathcal{E}_k^n(Z,s) &:= \sum_{\gamma\in\Upsilon_{n,0}\backslash\Upsilon_n} j(\gamma,Z)^{-k}\delta(\gamma Z)^s \\ &= \delta(Z)^s \sum_{\gamma\in\Upsilon_{n,0}\backslash\Upsilon_n} j(\gamma,Z)^{-k}|j(\gamma,Z)|^{-2s} \end{aligned} \tag{5.11}$$

which converges absolutely and locally uniformly if $2\Re(s)+k>2n$ [Shi83, p. 417]: under these conditions, the series $\mathcal{E}_k^n(Z,0)$ is indeed a Hermitian modular form. Further, for each fixed $Z\in\mathfrak{H}_n$ the function $s\mapsto\mathcal{E}_k^n(Z,s)$ admits meromorphic continuation to the whole complex plane, so it is sometimes possible to get holomorphic Eisenstein series of low weight outside of the original domain of convergence ([Shi83, theorem 7.1], [Shi83, theorem 7.2]).

§ 5.3 Shimura-Maass differential operator

For $n,k,r\in\mathbb{N}^+$ define the *Hermitian Shimura-Maass operator* $\partial_{n,k}^r$ by

$$\begin{aligned} \partial_{n,k} &:= (2\pi i)^{-n}\delta(Z)^{n-1-k}\det(\nabla)\delta(Z)^{k-n+1} \\ \partial_{n,k}^r &:= \partial_{n,k+2r-2}\circ\cdots\circ\partial_{n,k+2}\circ\partial_{n,k} \end{aligned} \tag{5.12}$$

where ∇ is the $n\times n$ matrix of differential operators

$$\nabla := \left(\frac{\partial}{\partial Z_{ij}}\right)_{ij} \tag{5.13}$$

Notice the similarities with the Siegel Shimura-Maass operator of (3.13): in particular the factor of 1/2 in the matrix Δ due to the symmetry of $Z\in\mathcal{H}_n$, which does not appear in ∇ as it applies to $Z\in\mathfrak{H}_n$. We use the same symbol $\partial_{n,k}^r$ for both versions of the operator as it will be clear from the context (i.e. from its operand).

Effect on Eisenstein series

Lemma 5.3.1. *For any* $r \in \mathbb{Z}^+$ *and* $\gamma \in \Upsilon_n$,

$$\partial^r_{n,k}\left[\delta(\gamma(Z))^s j(\gamma, Z)^{-k}\right] = (-4\pi)^{-nr}\left(\prod_{j=1}^{r} c_n(k+j-n+s)\right)\delta(\gamma(Z))^{s-r} j(\gamma, Z)^{-k-2r}$$

where, for $\alpha \in \mathbb{C}$,

$$c_n(\alpha) := \prod_{l=0}^{n-1}(\alpha + l)$$

Proof. Assume $r = 1$. The operator defined in [Shi94, formula 4.2b] is exactly $(2\pi i)^n \partial_{n,k}$ because of our different normalization, hence [Shi94, formula 4.6b] states

$$\partial_{n,k}(f[\gamma]_k) = (\partial_{n,k} f)[\gamma]_{k+2} \tag{5.14}$$

for any function $f : \mathfrak{H} \to \mathbb{C}$. In particular we take $f(Z) := \delta(Z)^s$, so that the left-hand side of (5.14) becomes $\partial_{n,k}\left[j(\gamma, Z)^{-k}\delta(\gamma(Z))^s\right]$. For the right-hand side, we compute

$$\begin{aligned}\partial_{n,k}[\delta(Z)^s] &= (2\pi i)^{-n}\delta(Z)^{n-1-k}\det(\nabla)\delta(Z)^{k+s-n+1} \\ &= (-4\pi)^{-n} c_n(k+s-n+1)\delta(Z)^{s-1}\end{aligned}$$

by [Shi83, lemma 9.1(ii)]. Therefore the right-hand side of (5.14) equals

$$(-4\pi)^{-n} c_n(k+1-n+s)\delta(\gamma(Z))^{s-1} j(\gamma, Z)^{-k-2}$$

as claimed. A repeated application of $\partial_{n,\star}$ proves the result for $r > 1$. □

Hence, at least formally, $\partial_{n,k}$ maps Eisenstein series to Eisenstein series. In the next chapter we are going to use $\mathcal{E}^3_{2k}(Z, s)$, so we state the following proposition for this particular case.

Proposition 5.3.2. *For* $m, k \in \mathbb{N}$ *such that* $k - 2 - 2m \geq 0$,

$$\mathcal{E}^3_{2k}(Z, 2-k+2m) = c^{-1}_{k,m} \cdot \partial^{k-2-2m}_{3,4+4m}[\mathcal{E}^3_{4+4m}(Z, 0)]$$

where

$$c_{k,m} = (-4\pi)^{-3(k-2-2m)}\left(\prod_{j=1}^{k-2-2m}\prod_{l=0}^{2}(4m+1+j+l)\right)$$

Further, $\mathcal{E}^3_{4+4m}(Z, 0)$ *is a well defined Hermitian modular form.*

Proof. Let $f(Z,s)$ be any holomorphic function in $s \in \Omega$ (with some open $\Omega \subseteq \mathbb{C}$) for every fixed $Z \in \mathfrak{H}_n$, and smooth in the entries of Z for every fixed s. Then

$$\oint_{\mathcal{C}} \frac{\partial}{\partial Z_{ab}} f(Z,s)ds = \frac{\partial}{\partial Z_{ab}} \oint_{\mathcal{C}} f(Z,s)ds = \frac{\partial}{\partial Z_{ab}} 0 = 0$$

for any closed curve $\mathcal{C} \subset \Omega$. Since the operator $\partial_{n,k}$ is constructed in terms of ∇, we deduce that $\partial_{n,k}[f(Z,s)]$ is again holomorphic in $s \in \Omega$ for each fixed Z. In particular we take f to be $\mathcal{E}_k^n$ extended beyond its domain of convergence by meromorphic continuation, so $\partial_{n,k}^r \mathcal{E}_k^n$ is still meromorphic (in s for each fixed $\mathcal{Z}$). Lemma (5.3.1) shows that, for $2\Re(s) + k > 2n$,

$$\partial_{n,k}^r[\mathcal{E}_k^n(Z,s)] = c_{n,k,r,s} \cdot \mathcal{E}_{k+2r}^n(Z, s-r)$$

where $c_{n,k,r,s}$ is a constant not depending on Z, by term-wise application of $\partial_{n,k}^r$ to (5.11), since the series converge absolutely. But by the principle of meromorphic continuation both terms must coincide (for every fixed Z) everywhere they are extended to.

In particular, we turn to $\mathcal{E}_{4+4m}^3(Z,0)$. For $m \geq 1$, the series is convergent as is; $\mathcal{E}_4^3(Z,0)$ is defined instead by analytic continuation of $\mathcal{E}_4^3(Z,s)$ beyond its domain of convergence, but [Nag96, theorem 2.2(2)] ensures that $\mathcal{E}_4^3(Z,0)$ is a holomorphic function on $\mathfrak{H}_3$. $\square$

Effect on Fourier expansions

We observe that the work of [CP91, sections 3.3.1-3.3.4] remains valid when replacing $\mathcal{H}_n$ with $\mathfrak{H}_n$ and Δ with ∇ in the definitions (3.13) and (5.12) of $\partial_{n,k}$: both cases are accounted for by [Shi83, lemma 9.1], from which [CP91, formulae 3.10-3.14] follow in the Hermitian case as well and the proofs of [CP91, lemma 3.9] and [CP91, lemma 3.10] remain unchanged. We summarize this in the following:

Proposition 5.3.3. *Let F be a Hermitian modular form with Fourier expansion $\sum_A F_A q^A$, with respect to some imaginary quadratic extension L of $\mathbb{Q}$. Then*

$$\partial_{n,k}^r F = (2i)^{nr}\delta(4\pi Z)^{-r} \sum_{A \geq 0} F_A R_n(4\pi AY; r, n-1-k-r) q^A \tag{5.15}$$

where $R_n(Z; r, \beta)$ is defined in [CP91, formula 3.7] by

$$R_n(Z; r, \beta) := (-1)^{nr} e^{\mathrm{Tr}(Z)} \det(Z)^{r+\beta} \det(\nabla)^r \left(e^{-\mathrm{Tr}(Z)} \det(Z)^{-\beta}\right)$$

and is a polynomial of degree nr in the entries of Z whose highest degree term is $\det(Z)^n$.

Additionally, let $\mathcal{O}_L$ be the ring of integers of L and $R \supseteq \mathcal{O}_L[i, {}^1/_2]$ any ring: if F is R-integral in the sense of (3.18), then so is $\partial_{n,k}^r F$.

Proof. [CP91, lemma 3.10] states

$$\partial^r_{n,k}[q^A] = (2i)^{nr}\delta(4\pi Z)^{-r}R_n(4\pi AY; r, n-1-k-r)q^A$$

Notice the additional factor $(2i)^{nr}$ in the previous formula due to our different normalization for $\partial_{n,r}$ from [CP91, formula 3.21]; this proves the first statement.

Similarly to proposition (3.3.5), when $\beta \in \mathbb{Z}$ the polynomial $R_n(Z; r, \beta)$ has coefficients in $\mathbb{Z}[1/2]$ since it is a repeated application of partial derivatives to $\det(Z)^{-\beta}e^{-\mathrm{Tr}(Z)}$. This observation, combined with (5.15), readily implies R-integrality of $\partial^r_{n,k}F$ whenever F has Fourier coefficients in R. □

§ 5.4 Unitary Siegel-Weil formula and theta series

Adeles

A valuation on a field F is a map $|\cdot| : F \to \mathbb{R}_{\geq 0}$ such that

(a) $|x| = 0$ if and only if $x = 0$

(b) $|xy| = |x||y|$ for every $x, y \in F$

and at least one of

(c) $|x + y| \leq |x| + |y|$ for every $x, y \in F$

(c') $|x + y| \leq \max\{|x|, |y|\}$ for every $x, y \in F$

Observe that (c') implies (c): the valuation is *archimedean* if it satisfies only (c), and *non-archimedean* otherwise.

Two valuations $|\cdot|_1$ and $|\cdot|_2$ are *equivalent* if they induce the same metric on F, i.e. there exist constants $a \leq b \in \mathbb{R}^+$ such that $a|x|_1 \leq |x|_2 \leq b|x|_1$ for every $x \in F$. A *place* v is a class of equivalent valuations on F, and we denote the associated valuation $|\cdot|_v$; further, F_v denotes the completion of F with respect to $|\cdot|_v$.

The (classes of) archimedean valuations are called *infinite places*, while the non-archimedean ones are *finite places*. On $\mathbb{Q}$, there is exactly one infinite place, corresponding to the usual Euclidian absolute value $|\cdot|$, and the finite places correspond to the primes $p \in \mathbb{Z}$ via

$$|x|_p := p^{-\mathrm{ord}_p(x)} \tag{5.16}$$

For a finite algebraic extension F of $\mathbb{Q}$, the infinite places are indexed by field embeddings $\tau : F \to \mathbb{C}$ up to conjugation, and

$$|x|_\tau := \begin{cases} |\tau(x)| & \text{if } \tau(F) \subseteq \mathbb{R} \\ \tau(x)\overline{\tau(x)} & \text{if } \tau(F) \not\subseteq \mathbb{R} \end{cases} \tag{5.17}$$

For each rational prime p, the ideal (p) in the ring of integers $\mathcal{O}_F$ of F decomposes as the product of prime ideals of $\mathcal{O}_F$: to each of these corresponds a place v of F, and we write $v|p$.

Proposition 5.4.1 (Product formula). *For each place v of F, one can fix a valuation $|\cdot|_v$ such that*

$$\prod_{v\leq\infty} |x|_v = 1$$

for every $x \in F^\times$.

For any place v of F, $\mathcal{O}_{F_v}$ denotes the ring of integers of the v-completion F_v. Explicitly

$$\mathcal{O}_{F_v} := \{x \in F_v : |x|_v \leq 1\} \tag{5.18}$$

The *adeles ring* $\mathbb{A}_F$ of F is the restricted product

$$\mathbb{A}_F := \prod_v{}' F_v = \left\{ (x_v)_v \in \prod_v F_v : x_v \in \mathcal{O}_{F_v} \text{ for almost all } v \right\} \tag{5.19}$$

and F embeds into $\mathbb{A}_F$ diagonally, i.e. via $x \mapsto (x)_v$: this makes $\mathbb{A}_F$ an F-algebra. If $F = \mathbb{Q}$, we simply write $\mathbb{A} := \mathbb{A}_\mathbb{Q}$.

Let G be an affine algebraic group over F, thus in particular an affine algebraic variety over F, i.e. the elements of G are described as the zeroes of polynomials with coefficients in F. For any F-algebra A, let $G(A)$ be the set of solutions over A of the polynomials defining G. This applies in particular to $G(\mathbb{A}_F)$ and $G(F_v)$, for a place v of F.

We now turn the attention to automorphic forms arising from Hermitian modular forms. Let $L := \mathbb{Q}(\sqrt{-d})$ be a quadratic imaginary number field, and put $G = \mathrm{U}(n, n)$ as defined in (5.2). The entries of $\gamma \in G$ are in L, and can be written as $x_i + y_i\sqrt{-d}$ for $x_i, y_i \in \mathbb{Q}$: the condition $\gamma^\dagger S_n \gamma = S_n$ for $\gamma \in G$ turns into a set of polynomial equations over $\mathbb{Q}$ in these components x_i, y_i, and we can therefore view G as an affine variety over $\mathbb{Q}$.

From [Ich04, p. 244] we have

$$\begin{aligned} M(\mathbb{Q}) &:= \left\{ \begin{pmatrix} a & 0_n \\ 0_n & a^{-\dagger} \end{pmatrix} : a \in \mathrm{GL}_n(L) \right\} \leq G(\mathbb{Q}) \\ N(\mathbb{Q}) &:= \left\{ \begin{pmatrix} I_n & b \\ 0_n & I_n \end{pmatrix} : b^\dagger = b \in L^{n,n} \right\} \leq G(\mathbb{Q}) \\ P(\mathbb{Q}) &:= \left\{ \begin{pmatrix} \star & \star \\ 0_n & \star \end{pmatrix} \in G(\mathbb{Q}) \right\} = M(\mathbb{Q})N(\mathbb{Q}) = N(\mathbb{Q})M(\mathbb{Q}) \end{aligned} \tag{5.20}$$

and the Iwasawa decomposition

$$G(\mathbb{A}) = P(\mathbb{A})K_G \tag{5.21}$$

where $K_G = K_\infty K_{fin} := \prod_{p\leq\infty} K_p \subseteq G(\mathbb{A})$ with

$$\begin{aligned} K_{p<\infty} &:= G(\mathbb{Z}_p) \\ K_\infty &:= G(\mathbb{R}) \cap \mathrm{U}_{2n}(\mathbb{R}) \end{aligned} \tag{5.22}$$

In particular we observe $K_\infty = \{g \in G(\mathbb{R}) : g(iI_n) = iI_n\}$, and by [Ich07, p. 724] any matrix $k_\infty \in K_\infty$ can be written as

$$k_\infty = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \tag{5.23}$$

with $A \pm iB \in \mathrm{U}_n(\mathbb{R})$ and $\det(A - iB) = \overline{\det(A + iB)}$.

By the strong approximation theorem (which holds in this instance because we assumed L of class number 1), any $x \in G(\mathbb{A})$ can be written (not uniquely) as $\gamma x_\infty k \in G(\mathbb{Q})G(\mathbb{R})K_{fin}$. Let $\varphi : G(\mathbb{A}) \to \mathbb{C}$ be any $G(\mathbb{Q})$-left and K_{fin}-right invariant function; put $Z = x_\infty(iI_n) \in \mathfrak{H}_n$, then the function

$$F(Z) := \det(x_\infty)^{-l/2} j(x_\infty, iI_n)^l \varphi(x_\infty) \tag{5.24}$$

is weakly modular of weight l. Viceversa, given such an F, the function

$$\varphi(x) := \det(x_\infty)^{l/2} j(x_\infty, iI_n)^{-l} F(x_\infty(iI_n)) \tag{5.25}$$

on $G(\mathbb{A})$ satifies $\varphi(\gamma x_\infty k) = \varphi(x_\infty)$. A function transforming like this is a particular example of *automorphic form*, together with technical conditions we do not need to worry about here. We call (5.25) and (5.24) the *standard correspondence* between modular and automorphic forms. In (5.24) we will often make the following choice: for $Z = X + iY \in \mathfrak{H}_n$, put

$$g_Z := \begin{pmatrix} I_n & X \\ 0_n & I_n \end{pmatrix} \begin{pmatrix} \sqrt{Y} & 0_n \\ 0_n & \sqrt{Y}^{-1} \end{pmatrix} = \begin{pmatrix} \sqrt{Y} & X\sqrt{Y}^{-1} \\ 0_n & \sqrt{Y}^{-1} \end{pmatrix} \tag{5.26}$$

with $\sqrt{Y} := P^\dagger \mathrm{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})P$, for a suitable $P \in \mathrm{U}_n(\mathbb{R})$ such that $Y = P^\dagger \mathrm{diag}(\lambda_1, \ldots, \lambda_n)P$, since Y is a positive definite Hermitian matrix (hence its eigenvalues are real and positive). Then $g_Z(iI_n) = Z$ and $g_Z \in P(\mathbb{R})$; any other $x_\infty \in G(\mathbb{R})$ such that $x_\infty(iI_n) = Z$ belongs to $g_Z K_\infty$.

Theta series

Let $V := L^m$. For a Hermitian matrix $Q \in L^{m,m}$, let $\langle \cdot, \cdot \rangle_Q : V \times V \to L$ be the Hermitian alternating form

$$\langle x, y \rangle_Q := x^\dagger Q y \tag{5.27}$$

Write $L \cong \mathbb{Q}(\sqrt{-d})$ for square-free integer $d \in \mathbb{N}$; since we assume that L has class number 1,

$$d \in \{1, 2, 3, 7, 11, 19, 43, 67, 163\} \tag{5.28}$$

A *rank* m lattice Λ is a subset of V which is a $\mathcal{O}_L$-module and satisfies $\Lambda \otimes L = V$. Because of our assumption on L, every lattice Λ of rank m is a free $\mathcal{O}_L$-module and there exists a basis $\{b_1, \ldots, b_m\}$ of V such that $\Lambda = \mathrm{span}_{\mathcal{O}_L}(b_1, \ldots, b_m)$. Define the $m \times m$ *Gram matrix* $\Gamma_\Lambda(Q)$ of the lattice Λ with respect to the form $\langle \cdot, \cdot \rangle_Q$ by

$$\Gamma_\Lambda(Q)_{i,j} := \langle b_i, b_j \rangle_Q \tag{5.29}$$

Proposition 5.4.2 ([HK09, theorem 1.19])**.** *Let Λ be a rank m lattice and $\langle \cdot, \cdot \rangle_Q$ a Hermitian form on V such that $\langle x, x \rangle_Q \in 2\mathbb{Z}^+$ for all $x \in \Lambda$ and* $\det(\Gamma_\Lambda(Q)) = 2^m d^{-m/2}$. *Then*

$$\theta_m^n(Z; \Lambda, Q) := \sum_{G \in \mathcal{O}_L^{m,n}} e^{\pi i \mathrm{Tr}(G^\dagger \Gamma_\Lambda(Q) G Z)} \tag{5.30}$$

is a Hermitian modular form of weight m and genus n with Fourier expansion

$$\sum_{A \geq 0} \# \left\{ G \in \mathcal{O}_L^{m,n} : G^\dagger \Gamma_\Lambda(Q) G = 2A \right\} q^A$$

In particular the Fourier coefficients are in $\mathbb{Z}$.

Because of (5.29), we can rewrite (5.30) as

$$\theta_m^n(Z; \Lambda, Q) = \sum_{x \in \Lambda^n} e^{\pi i \mathrm{Tr}(\langle x, x \rangle_Q Z)} \tag{5.31}$$

It is useful to approach theta series from a representation theory perspective. Fix once and for all a Hermitian form Q on V, and let H be the unitary group of (V, Q), i.e. the set of automorphisms of V such that $\langle hx, hy \rangle_Q = \langle x, y \rangle_Q$ for all

$x, y \in V$. H is an algebraic group over $\mathbb{Q}$, so we can consider its adelisation $H(\mathbb{A})$: for a place $v \leq \infty$ of L, the v-th local component of $H(\mathbb{A})$ is the set of endomorphisms of $V_v := V \otimes_L L_v$ fixing the given Hermitian form. Then $H(\mathbb{A})$ acts on the set of lattices λ on V by $(h\lambda) \otimes_L \mathcal{O}_{L_v} = h_v \lambda_v$, where $\lambda_v := \lambda \otimes_L \mathcal{O}_{L_v}$, and we denote $O_\lambda(\mathbb{A})$ the stabiliser of λ.

The $H(\mathbb{A})$-orbit of λ consists of all lattices λ' on V such that λ'_v is Q-isometric to λ_v for every $v < \infty$: this is an equivalence relation on lattices on V, with a finite number of classes. If Λ is a fixed lattice on V, call *genus* its $H(\mathbb{A}_{fin})$-equivalence class: there are finitely many $H(\mathbb{Q})$-orbits (classes) in its genus, which we denote $\{h_i \Lambda\}_i$. Hence we have

$$H(\mathbb{A}) = \bigcup_i H(\mathbb{Q}) h_i O_\Lambda(\mathbb{A}) \tag{5.32}$$

Notice that $H(\mathbb{Q}) h_i O_\Lambda(\mathbb{A})$ does not admit a unique factorisation: put

$$O_{h_i\Lambda} := H(\mathbb{Q}) \cap h_i O_\Lambda(\mathbb{A}) h_i^{-1} \tag{5.33}$$

and then $H(\mathbb{Q}) h_i O_\Lambda(\mathbb{A}) = H(\mathbb{Q}) O_{h_i\Lambda} h_i O_\Lambda(\mathbb{A})$. Observe that $O_{h_i\Lambda}$ is exactly the set of automorphisms of $h_i\Lambda$ with respect to the form $\langle \cdot, \cdot \rangle_Q$.

We fix once and for all the character ψ of $\mathbb{A}/\mathbb{Q}$ given by

$$\begin{aligned} \psi_\infty(x_\infty) &:= e^{\pi i x} \\ \psi_p(x_p) &:= e^{-2\pi i (x_p/2)'} \end{aligned} \tag{5.34}$$

where $'$ denotes the polar part embedding $\mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{R}$, i.e.

$$\left(\sum_{j=-N}^{\infty} \alpha_j p^j \right)' := \sum_{j=-N}^{-1} \alpha_j p^j \in \mathbb{R}$$

For ease of notation put $G := \mathrm{U}(n,n)(L)$, and let $\omega = \omega_\psi$ be the Weil representation of $G(\mathbb{A}) \times H(\mathbb{A})$ on the space $S(V(\mathbb{A})^n)$ of Schwartz-Bruhat functions on $V(\mathbb{A})^n$. Its action is completely determined by [Ich04, p. 246] i.e.

$$\begin{aligned} \omega\left(\begin{pmatrix} a & 0_n \\ 0_n & a^{-\dagger} \end{pmatrix}, 1_H \right) f(x) &= |\det(a)|_{\mathbb{A}_L}^{m/2} f(xa) \\ \omega\left(\begin{pmatrix} I_n & b \\ 0_n & I_n \end{pmatrix}, 1_H \right) f(x) &= \psi(\mathrm{Tr}(\langle x, x \rangle_Q b)) f(x) \\ \omega\left(\begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}, 1_H \right) f(x) &= \int_{V(\mathbb{A}^n)} f(y) \psi(\mathrm{Tr}_{L/\mathbb{Q}} \mathrm{Tr} \langle y, x \rangle_Q) dy \\ \omega(1_G, h) f(x) &= f(h^{-1}x) \end{aligned} \tag{5.35}$$

for any $a \in \mathrm{GL}_n(\mathbb{A}_L)$, $b \in \mathrm{Her}_n(\mathbb{A}_L)$ and $h \in H(\mathbb{A})$, where dy is the self-dual measure associated to the character ψ.

For any $f \in S(V(\mathbb{A})^n)$, define the *theta lift* $\theta_f : G(\mathbb{A}) \to \mathbb{C}$ by

$$\begin{aligned}\Theta_f(g,h) &:= \sum_{x\in V(\mathbb{Q})^n}[\omega(g,h)f](x) = \sum_{x\in V(\mathbb{Q})^n}[\omega(g,1_H)f](h^{-1}x)\\ \theta_f(g) &:= \int_{H(\mathbb{Q})\backslash H(\mathbb{A})}\Theta_f(g,h)dh\end{aligned} \tag{5.36}$$

for $g \in G(\mathbb{A})$ and $h \in H(\mathbb{A})$, where dh is the Haar measure on $H(\mathbb{A})$ normalised so that the volume of $H(\mathbb{Q})\backslash H(\mathbb{A})$ is 1.

Proposition 5.4.3. *Let Λ be a fixed latticed on V of rank m. Let $f(x) \in S(V(\mathbb{A})^n)$ be given by*

$$\begin{aligned}f_\infty(x_\infty) &:= e^{-\pi\mathrm{Tr}(\langle x_\infty,x_\infty\rangle_Q)}\\ f_v(x_v) &:= 1_{\Lambda_v^n}(x_v)\end{aligned}$$

where $1_{\Lambda_v^n}$ is the indicator function of Λ_v^n.

Then its theta lift θ_f is the automorphic form associated via the correspondence (5.25-5.24) to the $\mathbb{Q}$-linear combination of classical theta series

$$\theta_m^n(Z) := \frac{1}{\mathfrak{m}_\Lambda}\sum_i \frac{1}{\#O_{h_i\Lambda}}\theta_m^n(Z;h_i\Lambda,Q) \tag{5.37}$$

where the sum is over all classes in the genus of Λ, $O_{h_i\Lambda}$ is the set (5.33) of automorphisms of $h_i\Lambda$ and

$$\mathfrak{m}_\Lambda := \sum_i \frac{1}{\#O_{h_i\Lambda}} \tag{5.38}$$

is the mass of the lattice Λ.

Proof. Fix $Z \in \mathfrak{H}_n$ and $g_Z \in G(\mathbb{R})$ as in (5.26). Let $g = (g_Z, 1, 1, \dots) \in G(\mathbb{A})$ and compute in (5.36)

$$\begin{aligned}\Theta_f(g,h) &= \sum_{x\in V(\mathbb{Q})^n}\left\{[\omega(g_\infty,1_H)f_\infty](h_\infty^{-1}x_\infty)\prod_v[\omega(g_v,1_H)f_v](h_v^{-1}x_v)\right\}\\ &= \sum_{x\in V(\mathbb{Q})^n}\left\{[\omega(g_\infty,1_H)f_\infty](h_\infty^{-1}x_\infty)\prod_v 1_{\Lambda_v^n}(h_v^{-1}x_v)\right\}\\ &= \sum_{x\in(h\Lambda)^n}[\omega(g_\infty,1_H)f_\infty](x_\infty)\end{aligned}$$

for any $h \in H(\mathbb{A})$. The theta lift $\theta_f(g)$ is then

$$\begin{aligned}
\theta_f(g) &= \int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \Theta_f(g,h)dh \\
&= \sum_i \frac{1}{\# O_{h_i\Lambda}} \int_{O_\Lambda(\mathbb{A})} \Theta_f(g,h_i u)du \\
&= \sum_i \frac{1}{\# O_{h_i\Lambda}} \sum_{x\in(h_i\Lambda)^n} \left(\prod_{v<\infty} \int_{O_\Lambda(\mathbb{Q}_v)} du_v\right) \int_{O_\Lambda(\mathbb{R})} [\omega(g_\infty, 1_H)f_\infty](x_\infty)du_\infty \\
&= \left(\int_{O_\Lambda(\mathbb{A})} du\right) \sum_i \frac{1}{\# O_{h_i\Lambda}} \sum_{x\in(h_i\Lambda)^n} [\omega(g_\infty, 1_H)f_\infty](x_\infty)
\end{aligned}$$

where in the first step we have used the double coset decomposition (5.32) and the observation in (5.33). To compute the factor $\int_{O_\Lambda(\mathbb{A})} du$, we recall that dh is normalised so that $\int_{H(\mathbb{Q})\backslash H(\mathbb{A})} dh = 1$, hence

$$\begin{aligned}
1 &= \int_{H(\mathbb{Q})\backslash H(\mathbb{A})} dh \\
&= \sum_i \frac{1}{\# O_{h_i\Lambda}} \int_{O_\Lambda(\mathbb{A})} du \\
&= \left(\int_{O_\Lambda(\mathbb{A})} du\right) \mathfrak{m}_\Lambda
\end{aligned}$$

Now $g_\infty = g_Z \in P(G(\mathbb{R}))$, for which we have an explicit description via the Weil representation by [Ich04, p. 246], hence

$$\begin{aligned}
\theta_f(g) &= \frac{1}{\mathfrak{m}_\Lambda} \sum_i \frac{1}{\# O_{h_i\Lambda}} \sum_{x\in(h_i\Lambda)^n} \det(Y)^{m/2} e^{\pi i \mathrm{Tr}(\langle x,x\rangle_Q X)} e^{-\pi \mathrm{Tr}(\langle x,x\rangle_Q Y)} \\
&= \det(Y)^{m/2} \frac{1}{\mathfrak{m}_\Lambda} \sum_i \frac{1}{\# O_{h_i\Lambda}} \sum_{x\in(h_i\Lambda)^n} e^{\pi i \mathrm{Tr}(\langle x,x\rangle_Q Z)} \\
&= \det(Y)^{m/2} \frac{1}{\mathfrak{m}_\Lambda} \sum_i \frac{1}{\# O_{h_i\Lambda}} \theta_m^n(Z; h_i\Lambda, Q)
\end{aligned}$$

By (5.24) and proposition (5.4.2), the modular form associated to θ_f is then $\theta_m^n(Z)$ of (5.37) as claimed.

□

Siegel-Weil formula

Recall $V = L^m$, Q and Λ from proposition (5.4.2). In addition to the condition $\langle\lambda,\lambda\rangle_Q \in 2\mathbb{Z}^+$ for all $\lambda \in \Lambda$, we require Q be positive, i.e. $\langle x,x\rangle_Q > 0$ for every non-zero $x \in V$. Hence, the Witt index (i.e. the dimension of a maximal totally isotropic subspace) of (V,Q) is 0; we will use this information in proposition (5.4.4), when specialising [Ich07, theorem 1.1] in our setting.

By the Iwasawa decomposition (5.21), write $g = pk \in G(\mathbb{A})$ with

$$p = \begin{pmatrix} a & \star \\ 0_n & a^{-\dagger} \end{pmatrix} \in P(\mathbb{A}) \tag{5.39}$$

for some $a \in \mathrm{GL}_n(\mathbb{A}_L)$ and $k \in K_G$: with this notation, put $a(g) := N_{L/\mathbb{Q}} \det(a)$. For any $f \in \mathcal{S}(V(\mathbb{A})^n)$, define a holomorphic section of $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(|\det|^s_{\mathbb{A}_L})$ by

$$\Phi_f^{(s)}(g) := |a(g)|^{s-s_0}_{\mathbb{A}_L}[\omega_Q(g,1_H)f](0) \tag{5.40}$$

where $s_0 := (m-n)/2$. As described in [Ich04, section 3], this gives rise to the adelic Eisenstein series

$$E_f(g,s) := \sum_{\gamma\in P(\mathbb{Q})\backslash G(\mathbb{Q})} \Phi_f^{(s)}(\gamma g) \tag{5.41}$$

which converges absolutely for $\Re(s) > n/2$ and has meromorphic continuation to the whole s-plane if f is K_G-finite.

Proposition 5.4.4. *[Ich07, theorem 1.1] If $n < m \le 2n$, then*

$$\lim_{s\to s_0} E_f(g,s) = \theta_f(g)$$

where θ_f is the theta lift of f defined in (5.36).

If $m > 2n$, then both the theta series and the Eisenstein series converge absolutely, and in particular $\lim_{s\to s_0} E_f(g,s) = E_f(g,s_0)$. Weil proved that again they coincide [Wei65]: we summarise this in the following.

Corollary 5.4.5 (Siegel-Weil formula)**.** *Let Λ and $f \in S(V(\mathbb{A})^n)$ as in proposition (5.4.3). If $n < m$, the modular form associated to $\lim_{s\to s_0} E_f(g,s)$ via the standard correspondence is $\theta_m^n(Z)$ of (5.37).*

§ 5.5 Eisenstein series and arithmetic properties

We show that $\mathcal{E}^3_{4+4m}(Z,0)$ equals $\theta^3_{4+4m}(Z)$ by virtue of the Siegel-Weil formula (5.4.5), and has therefore rational Fourier coefficients with bounded denominators. We first prove that $\mathcal{E}^3_{4+4m}(Z,0)$ corresponds to the adelic Eisenstein series defined in [Sah10, section 1D], and then relate the latter to the one appearing in proposition (5.4.4) and (5.4.5).

Lemma 5.5.1. *Put* $G = \mathrm{GU}(n,n)$. *Define a holomorphic section* $\Xi(\cdot,s)$ *of* $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(|\det|^s_{\mathbb{A}_L})$ *by*

$$\begin{aligned}\Xi_\infty(k_\infty,s) &:= \det(k_\infty)^{m/2} j(k_\infty, iI_n)^{-m}\\ \Xi_v(k_v,s) &:= 1\end{aligned} \tag{5.42}$$

for all $k \in K_G$, *and put*

$$E^{\Xi}(g,s) := \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \Xi(\gamma g, s) \tag{5.43}$$

Then E^{Ξ} *is associated via the standard correspondence (5.24) to* $\mathcal{E}^n_m$, *and in particular*

$$\det(g_Z)^{-m/2} j(g_Z, iI_n)^m E^{\Xi}(g_Z,s) = \mathcal{E}^n_m(Z, ns + \frac{n-m}{2}) \tag{5.44}$$

where g_Z *is defined in (5.26).*

Proof. By [Sah10, corollary 6.7], the left hand side of (5.44) depends only on Z, in the sense that replacing g_Z by any other $g \in G(\mathbb{A})$ such that $g(iI_n) = Z$ yields the same value. Further, by corollary (5.1.4), we can take representatives γ of $P(\mathbb{Q})\backslash G(\mathbb{Q})$ in $\Upsilon_n \subseteq \mathcal{O}_L^{2n,2n}$.

We compute the local components of $\Xi(\gamma g_Z, s)$ to finish the proof. As $g_Z \in P(\mathbb{R})$, $(\gamma g_Z)_v = \gamma_v$ for every finite place v hence

$$\Xi_v((\gamma g_Z)_v, s) = \Xi_v(\gamma_v, s) = |N_{L/Q}\det(a_v)|_v^{n(s+1/2)} = |\det(\gamma)|_v^{n(s+1/2)} = 1$$

because of $\Xi \in \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(|\det|^s_{\mathbb{A}_L})$ and (5.42), where $n_v m_v(a_v) k_v = \gamma_v$ is the Iwasawa decomposition. At the infinite place, by [Sah10, lemma 6.6] we have

$$\begin{aligned}\Xi_\infty((\gamma g_Z)_\infty, s) &= \det(\gamma g_Z)^{m/2} j(\gamma g_Z, iI_n)^{-m} \delta(\gamma(Z))^{n(s+1/2)-m/2}\\ &= \det(g_Z)^{m/2} j(g_Z, iI_n)^{-m} j(\gamma, Z)^{-m} \delta(\gamma(Z))^{n(s+1/2)-m/2}\end{aligned}$$

from which (5.44) follows immediately. □

Lemma 5.5.2. *Let $E_f(g,s)$ as in corollary (5.4.5) and $E^{\Xi}(g,s)$ as in lemma (5.5.1). Then*

$$E_f(g_Z, ns+s_0-m/2) = E^{\Xi}(g_Z, s-1/2)$$

for any $H \in \mathfrak{H}_n$ and g_Z as in (5.26).

Proof. Recall from (5.41) and (5.40) that $E_f(g,s)$ is defined in terms of

$$\Phi_f^{(s)}(g) = |a(g)|_{\mathbb{A}_L}^{s-s_0}[\omega_Q(g,1_H)f](0)$$

We will explicitly compute the action of the Weil representation, and compare it with the series defining E^{Ξ}.

First, let $g = k \in K_G$ so that $\Phi_f^{(s)}(k) = [\omega_Q(k,1_H)f](0)$, and we proceed to compute the local components. At each place v, $k_v \in K_v$ is a product of matrices of the form $m(a_v)$, $n(b_v)$ and S as in (5.35).

If v is a finite place, every generator is in $\mathrm{GL}_{2n}(\mathcal{O}_{L,v})$. Let λ denote any lattice on V, then

$$\begin{aligned}
[\omega_Q(n(b_v),1_H)1_{\lambda_v^n}](x) &= \psi_v(\mathrm{Tr}(\langle x,x\rangle_Q b_v)) \cdot 1_{\lambda_v^n}(x) = 1_{\lambda_v^n}(x) \\
[\omega_Q(m(a_v),1_H)1_{\lambda_v^n}](x) &= |\det(a_v)|_v^{m/2} \cdot 1_{\lambda_v^n}(xa_v) = 1_{\lambda_v^n}(x) \\
[\omega_Q(S,1_H)1_{\lambda_v^n}](x) &= \int_{\lambda_v^n} \psi_v(\mathrm{Tr}_{L/\mathbb{Q}}\mathrm{Tr}\langle y,x\rangle_Q)dy = 1_{(\lambda^\perp)_v^n}(x)
\end{aligned}$$

since $\mathrm{Tr}(\langle x,x\rangle_Q b_v) \in \mathcal{O}_{L_v}$; right multiplication by $a_v \in \mathrm{GL}_n(L_v)$ is a change of basis for λ_v^n and $\det(a_v) \in \mathcal{O}_{L_v}^\times$ hence of valuation 1; for the last integral we have studied separately the cases $x \in \lambda_v^n$ and $x \notin \lambda_v^n$, the latter yielding zero as it amounts integrating a non-trivial character over a period. Putting all together, we deduce

$$[\omega_Q(k_v,1_H)f_v](0) = 1_{\Lambda'^n_v}(0) = 1 \tag{5.45}$$

where Λ' is either Λ or $\Lambda^\perp$ according to the number of generators S for k_v.

At the infinite place, we introduce the function

$$w_h(x) := e^{-\pi \mathrm{Tr}(\langle x,x\rangle_Q h)}$$

for $x \in \mathbb{C}^n$ and $h \in \mathbb{C}^{n,n}$ whose eigenvalues have positive real part; observe that $f_\infty(x) = w_I(x)$. We preliminary compute the Fourier transform $\hat{w}_h$ of w_h:

diagonalise $h = UDU^\dagger$ with $U \in \mathrm{U}_n(\mathbb{R})$ and $D = \mathrm{diag}(d_1, \ldots, d_n)$, so that

$$\begin{aligned}
\hat{w_h}(y) &= \int_{\mathbb{C}^{m,n}} w_h(x)\psi_\infty(\mathrm{Tr}_{L/\mathbb{Q}}\mathrm{Tr}\langle x, y\rangle_Q)dx = \\
&= \int_{\mathbb{C}^{m,n}} e^{-\pi \mathrm{Tr}(\langle x,x\rangle_Q h)} e^{2\pi i \Re \mathrm{Tr}\langle x,y\rangle_Q} dx = \Big|_{\{v=yU\}} \\
&= \prod_{l=1}^n \int_{\mathbb{C}^m} e^{\pi\left(-D_l x^\dagger Q x + i x^\dagger Q v_l + i v_l Q x\right)} dx \Big|_{\{z = x - iD_l^{-1} v_l\}} \\
&= \prod_{l=1}^n \int_{\mathbb{C}^m} e^{-\pi D_l \langle z,z\rangle_Q} e^{-\pi D_l^{-1}\langle v_l, v_l\rangle_Q} dz \\
&= \det(h)^{-m} e^{-\pi \mathrm{Tr}(\langle y,y\rangle_Q h^{-1})} \\
&= \det(h)^{-m} w_{h^{-1}}(y)
\end{aligned}$$

where v_l denotes the l-th column of $v = yU$, and the last integral exists because D_l has positive real part. Now

$$\begin{aligned}
[\omega_Q(n(b_\infty), 1_H) w_h](x) &= \psi_\infty(\mathrm{Tr}(\langle x,x\rangle_Q b_\infty)) \cdot w_h(x) = w_{h - ib_\infty}(x) \\
[\omega_Q(m(a_\infty), 1_H) w_h](x) &= |\det(a_\infty)|_\infty^{m/2} \cdot w_h(x a_\infty) = |\det(a_\infty)|^m w_{a_\infty h a_\infty^\dagger}(x) \\
[\omega_Q(S, 1_H) w_h](x) &= \det(h)^{-m} w_{h^{-1}}(x)
\end{aligned}$$

and observe that the eigenvalues of $h - ib_\infty$ and h^{-1} still have positive real part. As observed in (5.23),

$$k_\infty = \begin{pmatrix} D & -C \\ C & D \end{pmatrix}$$

for suitable C, D. Find $U_1, U_2 \in \mathrm{U}_n(\mathbb{R})$ such that

$$U_1 C =: \begin{pmatrix} C_1 & 0 \\ 0 & 0_{n-r} \end{pmatrix} U_2^\dagger \qquad U_1 D =: \begin{pmatrix} D_1 & 0 \\ 0 & I_{n-r} \end{pmatrix} U_2^\dagger$$

where $r = \mathrm{rank}(C)$ and $\det(C_1) \neq 0$. Then one can easily check that $C_1^{-1} D_1$ is Hermitian and

$$\begin{aligned}
k_\infty = m(U_1^\dagger) n \begin{pmatrix} D_1 C_1^{-1} & 0 \\ 0 & 0_{n-r} \end{pmatrix} m \begin{pmatrix} C_1^{-\dagger} & 0 \\ 0 & I_{n-r} \end{pmatrix} \times \\
\times \left\{ n \begin{pmatrix} 0_r & 0 \\ 0 & I_{n-r} \end{pmatrix} S \right\}^3 n \begin{pmatrix} C_1^{-1} D_1 & 0 \\ 0 & 0_{n-r} \end{pmatrix} m(U_2^\dagger)
\end{aligned} \tag{5.46}$$

and use this factorisation to compute

$$[\omega_Q(k_\infty, 1_h) f_\infty](0) = \det(I - iC_1^{-1} D_1)^{-m} |\det(C_1)|^{-m} \tag{5.47}$$

Now that we have an explicit description of $\Phi_f^{(s)}(k)$ for $k \in K_G$ by (5.45) and (5.47), we compare it with $\Xi(k,s)$ of (5.42). As the finite components are identically one, we focus on the infinite place only:

$$\begin{aligned}\Xi_\infty(k_\infty, s) &= \det(k_\infty)^{m/2} j(k_\infty, iI_n)^{-m} \\ &= \det(k'_\infty)^{m/2} j(k'_\infty, iI_n)^{-m} \\ &= \det(C_1)^{m/2} \det(C_1 + D_1 C_1^{-1} D_1)^{m/2} \det(D_1 + iC_1)^{-m}\end{aligned}$$

where $m(U_1)k_\infty m(U_2) =: k'_\infty \times I_{n-r}$. Dividing this quantity by the right-hand side of (5.47) yields 1, showing that

$$\Phi_f^{(s)}(k) = \Xi(k, s')$$

for any $s, s' \in \mathbb{C}$ and $k \in K_G$.

For a general $g \in G$, write the Iwasawa decomposition $g = pk \in P(\mathbb{A})K_G$ as in (5.39) and observe

$$\begin{aligned}\Phi_f^{(s)}(g) &= |N_{L/\mathbb{Q}} \det(a)|_{\mathbb{A}_L}^{s-s_0} \Phi_f^{(s)}(k) \\ \Xi(g, s') &= |N_{L/\mathbb{Q}} \det(a)|_{\mathbb{A}_L}^{n(s'+1/2)} \Xi(k, s')\end{aligned}$$

achieving equality for $s = ns' + n/2 + s_0$ as claimed. □

Corollary 5.5.3. *Under the conditions of corollary (5.4.5),*

$$\mathcal{E}_m^n(\cdot, 0) = \theta_m^n$$

In particular, with the notation of (5.37), put

$$\mathfrak{M}_{m,n}^{-1} := \{p^{-1} : p|\mathfrak{m}_\Lambda\} \cup \bigcup_i \{p^{-1} : p|\#O_{h_i\Lambda} \textit{ prime}\} \tag{5.48}$$

if the genus of Λ is not trivial, or $\emptyset$ otherwise: then $\mathcal{E}_m^n(\cdot, 0)$ has Fourier coefficients in $\mathbb{Z}[\mathfrak{M}_{m,n}^{-1}]$.

Proof. Observe

$$\begin{aligned}\mathcal{E}_m^n(Z, 0) &= \det(g_Z)^{-m/2} j(g_Z, iI_n)^{-m} E^\Xi(g_Z, \tfrac{m}{2n} - \tfrac{1}{2}) \\ &= \det(g_Z)^{-m/2} j(g_Z, iI_n)^{-m} E_f(g_Z, s_0) \\ &= \theta_m^n(Z)\end{aligned}$$

where the first equality is by lemma (5.5.1), the second by lemma (5.5.2) and the last by corollary (5.4.5). □

Corollary 5.5.4. *The Eisenstein series $\mathcal{E}_{4+4m}^3(\cdot, 0)$ is a well defined holomorphic Hermitian modular form with Fourier coefficients in $\mathbb{Z}[\mathfrak{M}_{4+4m,3}^{-1}]$.*

Chapter 6

Integral representation II

In this chapter, we prove the announced result in full. We employ an integral formula of Saha, linking L-values with an integral against a diagonally restricted Hermitian Eisenstein series. After diagonal restriction and Siegel pullback, the theory of chapter 3 applies and we can study the arithmetic properties of the critical L-values, which follow from the arithmeticity of the Fourier coefficients of the Hermitian Eisenstein series, as described in the previous chapter. The method we use in this proof is essentially the same as the one in chapter 4, only applied to a different integral formula.

§ 6.1 Main theorem

An integral formula of Saha

To refine the result of theorem (4.1.3), we appeal to a different integral formula for tensor product L-functions.

Proposition 6.1.1 ([Sah10, theorem 6.9])**.** *Let $h \in \mathcal{S}^1_{2k}$ and $F \in \mathcal{S}^2_{2k}$ be Hecke eigenforms with $2k \geq 6$. Assume that the A-th Fourier coefficient F_A of F is non-zero, for some index A of determinant $4d$ such that $L := \mathbb{Q}(\sqrt{-d})$ has class number 1. Then*

$$\langle \mathcal{E}^3_{2k}(Z \times \tau, -v), F(Z)h(\tau)\rangle = \xi_F\left(\frac{2k-3-2v}{6}\right) Z_{F\otimes h}(4k-3-v) \qquad (6.1)$$

for $v \in \mathbb{N}$, where

$$\xi_F(s) = \frac{(-1)^k 2^{-6s-1}\pi^2(4\pi)^{-3s-3k+3/2} d^{-3s-k}\Gamma(3s+3k-3/2)}{(6s+2k-1)^2\zeta(6s+1)\zeta(6s+3)L(\chi_{-d};6s+2)}\overline{F_\eta}$$

$$\eta = \begin{cases} \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} & 4d \equiv 0 \mod 4 \\ \begin{pmatrix} \frac{4d+1}{4} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} & 4d \equiv 3 \mod 4 \end{cases}$$

Observe that $\mathcal{E}_{2k}^3(\cdot, s)$ is the Hermitian Eisenstein series defined in (5.11), which corresponds via the usual correspondence to the adelic series of [Sah10, section 1D] by lemma (5.5.1); nonetheless, as observed in (5.27), the restriction of $\mathcal{E}_{2k}^3(\cdot, s)$ to the Siegel space $\mathcal{H}_3$ induces a (non-holomorphic) Siegel modular form, so that the left-hand side of (6.1) only involves objects in the Siegel world.

Remark 6.1.2. As explained after (5.11), the Eisenstein series $\mathcal{E}_{2k}^n(Z, s)$ converges absolutely in Z for $\Re(s) + 2k > 2n$, and for each fixed Z it admits meromorphic continuation as a function of s to the complex plane. Formula (6.1) holds unconditionally for $v < k - 3$, but since both $\mathcal{E}_{2k}^3(Z, s)$ and $Z_{F\otimes h}(s)$ can be continued to $\mathbb{C}$, by meromorphic continuation we see that the equality remains true for any value of $v \in \mathbb{N}$.

Our goal

Let h, F, G be as in proposition (4.1.2), with the additional restriction that $l = k$, i.e. all these forms have equal weight $2k \geq 6$: this is due to the conditions of proposition (6.1.1). By putting $v := k - 2 - 2m$, (6.1) becomes

$$\langle \mathcal{E}_{2k}^3(Z \times \tau, 2 - k + 2m), \star(Z)h(\tau)\rangle = \xi_\star\left(\frac{1+4m}{6}\right) Z_{\star\otimes h}(3k - 1 + 2m) \tag{6.2}$$

where $\star$ is either F or G and

$$\xi_\star\left(\frac{1+4m}{6}\right) = \frac{-2^{-4m-2}\pi^2(4\pi)^{-3k+1-2m}d^{-k-2m-1/2}\Gamma(3k-1+2m)}{(2k+4m)^2\zeta(2+4m)\zeta(4+4m)L(\chi_{-d}; 4m+3)}\overline{\star_\eta} \tag{6.3}$$

Observe that, as explained in remark (6.1.2), formula (6.2) holds as is for $m \geq 1$, and for $m \leq 0$ by the principle of meromorphic continuation. This very same phenomenon happened with the Siegel Eisenstein series of theorem (4.1.3): the near-central critical value (which contains the information conjectured by (1.3.1)) corresponds to a non-convergent series, while the *other* critical values are associated with unproblematic objects.

Theorem 6.1.3. *Let $\mathfrak{p}_0$ as in proposition (4.1.2), and assume it does not divide $c_g(|D|) \neq 0$.*

Let $\{h_i\}_i$ and $\{F_j\}_j$ be orthogonal Hecke eigenforms bases for $\mathcal{S}^1_{2k}$ and $\mathcal{S}^2_{2k}$ respectively with $h_1 = h$, $F_1 = F$ and $F_2 = G$. Assume that no F_j with $j \geq 3$ is congruent to either F or $G \mod \mathfrak{p}$, and that there exists a rational prime ℓ such that $\lambda_\ell(F) \not\equiv \lambda_\ell(G) \mod \mathfrak{p}^2$. Similarly, assume h not congruent $\mod \mathfrak{p}$ to any h_i, for $i \geq 2$. Again, $\mathfrak{p}$ denotes any prime ideal of $\mathbb{K}$ above $\mathfrak{p}_0$, where $\mathbb{K}$ is any large enough number field containing all the fields in this chapter: namely $L(\overline{F_\eta})$, $\mathbb{Q}(\sqrt{D})$, $\mathbb{Q}(f)$, $\mathbb{Q}(F_i)$ and $\mathbb{Q}(h_v)$ for all $i \leq \dim(\mathcal{S}^2_{2k})$ and $v \leq \dim(\mathcal{S}^1_{2k})$.

Assume $\mathfrak{p}$ is not above a prime in $\mathfrak{M}^{-1}_{4,3}$, with the notation of (5.48). Further, assume that $\mathfrak{p}$ does not divide $\overline{F_\eta}$.

Let $1 \leq m < \frac{k}{2} - 2$ be any integer such that

(i) $\mathfrak{p}$ divides $\pi^{4m} L_f(2k)/L_f(2k+4m)$;

(ii) $\mathfrak{p}$ is coprime with $B_{4+4m}, B_{3+4m,\chi_{-d}}$ and not in $\mathfrak{M}^{-1}_{4+4m,3}$;

(iii) $\mathfrak{p}$ is above a rational prime $> \max\{3k-2+2m, 2k+4m, 4k-5\}$;

(iv) $\mathfrak{p}$ does not divide $\pi^{-2k-4m-1} L_h(k+2m) L_h(k+2m+1)/\langle h, h\rangle$;

(v) $\mathfrak{p}$ does not divide $\pi^{-4k-4m+1} Z_{f\otimes h}(3k-1+2m)/\langle f, f\rangle$.

Then

$$\frac{\pi^{8m} Z_{G\otimes h}(3k-1)}{Z_{G\otimes h}(3k-1+2m)} \tag{6.4}$$

is algebraic, and divisible by $\mathfrak{p}$.

The rest of the chapter is devoted to the proof.

Remark 6.1.4. The condition $m < \frac{k}{2} - 2$ arises from studying which nearly holomorphic forms can be holomorphicly projected: see section (6.2).

Assume k even. We cannot say anything about $m = \frac{k}{2} - 2$, since our method stops when applying holomorphic projection (as the conditions are not satisfied there). Instead, when $m = \frac{k}{2} - 1$ there is no need to emply holomorphic projection as the Eisenstein series $\mathcal{E}^3_{2k}(\cdot, 0)$ appearing in (6.2) is absolutely convergent hence a well defined holomorphic Hermitian modular forms, and therefore its Siegel-diagonal restriction $\mathcal{E}^3_{2k}(\tau \times Z, 0)$ is an element of $\mathcal{M}^1_{2k} \otimes_{\mathbb{C}} \mathcal{M}^2_{2k}$. Since we are taking its inner product against $h(\tau)F(Z)$ or $h(\tau)G(Z)$ in (6.2), we can decompose every modular form into a sum of Eisenstein series and cusp forms, where only the latters matter. To make sure the remaining cuspidal components are integral at $\mathfrak{p}$ we would need to include additional conditions about the order at $\mathfrak{p}$ of the Eisenstein eigenvalues; on the other hand, we did not want to complicate further the list of conditions in theorem (6.1.3) for this one case only.

Remark 6.1.5. We recall remark (4.1.4) for what we mean by divisibility and coprimality with $\mathfrak{p}$: here the large Galois extension $\mathbb{K}$ also contains $L(\overline{F_\eta})$. Also, note that theorem (6.1.3) is an improvement of theorem (4.1.3): we had previously showed (6.4) to have $\mathrm{ord}_{\mathfrak{p}} \geq 0$, while now we actually deduce that (6.4) has $\mathrm{ord}_{\mathfrak{p}} \geq 1$.

Remark 6.1.6. We are not aware of a way of bounding the primes in $\mathfrak{M}^{-1}_{4+4m,3}$ in general; on the other hand, given a fixed lattice Λ, most CAS applications can compute this data, which is therefore a mild condition for theorem (6.1.3) that can be checked on a case-by-case basis. Further, as we are free to (suitably) choose the field L and the lattice Λ for every m, it could be the case that we can pick a Λ so that its genus has a single class: in this case $\mathfrak{M}^{-1}_{4+4m,3} = \emptyset$ and the condition is therefore vacuous.

§ 6.2 L-values as inner products

Siegel-diagonal restriction

By proposition (5.3.2) and (5.3.3), for $0 \leq m \leq k/2 - 1$

$$\begin{aligned}
&\mathcal{E}^3_{2k}(\mathcal{W}, 2-k+2m) = \\
&\qquad = c_{k,m}^{-1} \cdot \partial^{k-2-2m}_{3,4+4m}[\mathcal{E}^3_{4+4m}(\mathcal{W},0)] \\
&\qquad = c_{k,m}^{-1} \delta(2\pi\mathcal{W})^{2-k+2m} \sum_{\mathcal{A}\geq 0} \mathcal{E}^{(m)}_{\mathcal{A}} R_3(2\pi\mathcal{A}\mathcal{V}; k-2-2m, 4-2k+2m) q^{\mathcal{A}}
\end{aligned}$$

where $\mathcal{W} = \mathcal{U} + i\mathcal{V} \in \mathfrak{H}_3$ and $\mathcal{E}^{(m)}_{\mathcal{A}}$ is the $\mathcal{A}$-th Fourier coefficient of $\mathcal{E}^3_{4+4m}(\mathcal{W},0)$. When restricting to $W \in \mathcal{H}_3$, observe

$$\mathrm{Tr}(\mathcal{A}W) = \sum_{i,j=1}^{n} \mathcal{A}_{ij} W_{ji} = \sum_{i=1}^{n} \mathcal{A}_{ii} W_{ii} + \sum_{i<j} (\mathcal{A}_{ij} + \mathcal{A}_{ji}) W_{ij}$$

and $\mathcal{A}_{ii} \in \mathbb{Z}$ and $\mathcal{A}_{ij} + \mathcal{A}_{ji} \in \frac{1}{2}\mathbb{Z}$. For a symmetric half-integral positive semidefinite $n \times n$ matrix A we say that $A \sim \mathcal{A}$ if $A_{ii} = \mathcal{A}_{ii}$ and $A_{ij} = \mathcal{A}_{ij} + \mathcal{A}_{ji}$, so that $\mathrm{Tr}(\mathcal{A}W) = \mathrm{Tr}(AW)$ whenever $A \sim \mathcal{A}$ (and we observe that for each A there exist finitely many $\mathcal{A}$ with $A \sim \mathcal{A}$). Hence the Siegel pullback $\mathcal{H}_n \to \mathbb{C}$ given by $W \mapsto \mathcal{E}^3_{2k}(W, 2-k+2m)$ is a Siegel (non-holomorphic) form with Fourier expansion

$$\delta(2\pi W)^{2-k+2m} \sum_{A\geq 0} \left(\sum_{\mathcal{A}\sim A} \mathcal{E}^{(m)}_{\mathcal{A}} R_3(2\pi\mathcal{A}V; k-2-2m, 4-2k+2m) \right) q^A \tag{6.5}$$

i.e. an element of $\pi^{3(k-2-2m)} \cdot \mathcal{N}^3_{2k,k-2-2m}$.

Now, by lemma (3.2.2) the diagonal restriction $\mathscr{E}_m(Z,\tau)$ of $\mathcal{E}^3_{2k}(W, 2-k+2m)$ to $\mathcal{H}_2 \times \mathcal{H}_1$ is

$$\begin{aligned} \mathscr{E}_m(Z,\tau) &:= \mathcal{E}^3_{2k}(Z \times \tau, 2-k+2m) \\ &= \pi^{3(k-2-2m)} \sum_{i,j} c^{(m)}_{i,j} \Phi^{(m)}_i(Z) \varphi^{(m)}_j(\tau) \\ &\in \pi^{3(k-2-2m)} \cdot \mathcal{N}^2_{2k,k-2-2m} \otimes_{\mathbb{C}} \mathcal{N}^1_{2k,k-2-2m} \end{aligned} \tag{6.6}$$

for constants $c^{(m)}_{i,j} \in \mathbb{C}$.

Holomorphic projection

By corollary (3.4.9), we can rewrite the left-hand side of (6.2) as

$$\begin{aligned} \langle \mathcal{E}^3_{2k}(Z \times \tau, &2-k+2m), \star(Z)h(\tau)\rangle = \\ &= \pi^{3(k-2-2m)} \sum_{i,j} c^{(m)}_{i,j} \langle \Phi^{(m)}_i, \star\rangle \langle \varphi^{(m)}_j, h\rangle \\ &= \langle \mathscr{E}_m(Z,\tau), \star(Z)h(\tau)\rangle \\ &= \langle \mathrm{Hol}\mathscr{E}_m(Z,\tau), \star(Z)h(\tau)\rangle \\ &= \pi^{3(k-2-2m)} \sum_{i,j} d^{(m)}_{i,j} \langle F_i, \star\rangle \langle h_j, h\rangle \\ &= \pi^{3(k-2-2m)} \langle h, h\rangle \Big(A^{(m)} \langle F, \star\rangle + B^{(m)} \langle G, \star\rangle \Big) \end{aligned} \tag{6.7}$$

where $\{F_i\}_i$ and $\{h_j\}_j$ are orthogonal Hecke eigenforms bases of $\mathcal{S}^2_{2k}$ and $\mathcal{S}^1_{2k}$ respectively, with $h_1 = h$, $F_1 = F$ and $F_2 = G$; the $d^{(m)}_{i,j}$ are suitable constants in $\mathbb{C}$ with $A^{(m)} := d^{(m)}_{1,1}$ and $B^{(m)} := d^{(m)}_{2,1}$, and the last equality follows by orthogonality. Again, recall that $\star$ is either F or G.

Combining (6.7) into (6.2) we get

$$\pi^{3(k-2-2m)} \langle h, h\rangle \langle F, F\rangle A^{(m)} = \xi_F \left(\frac{1+4m}{6} \right) Z_{F\otimes h}(3k-1+2m) \tag{6.8}$$

$$\pi^{3(k-2-2m)} \langle h, h\rangle \langle G, G\rangle B^{(m)} = \xi_G \left(\frac{1+4m}{6} \right) Z_{G\otimes h}(3k-1+2m) \tag{6.9}$$

so that again the arithmetic information we want is encoded in $B^{(m)}$.

To justify the holomorphic projection, we need to check the conditions of proposition (3.4.1). From (6.5), every A-th Fourier coefficient of $\mathcal{E}^3_{2k}(W, 2-k+2m)$

is a finite sum of polynomials in the entries of V with the same structure but different coefficients: therefore the proof of lemma (3.4.4) applies unchanged to deduce

$$|\mathcal{E}^3_{2k}(W, 2-k+2m)| \ll \prod_{j=1}^{3}(\lambda_j^{2+2m-k} + \lambda_j^{-2-2m-k})$$

Since $\Phi_i^{(m)}$ and $\varphi_j^{(m)}$ are sections of $\mathcal{E}^3_{2k}(W, 2-k+2m)$ by remark (3.2.3), they satisfy the same bound and hence are of bounded growth by proposition (3.4.6) provided that $2m < k-4$.

§ 6.3 Arithmeticity of L-values

Isolating $A^{(m)}$ and $B^{(m)}$

Under the assumptions of theorem (6.1.3), the Eisenstein series $\mathcal{E}^3_{4+4m}(\cdot, 0)$ is holomorphic for any $m \geq 0$. By corollary (5.5.4) we have a complete description of the arithmeticity of its Fourier coefficients: with the notation of (5.48), the coefficients of $\mathcal{E}^3_{4+4m}(\cdot, 0)$ are in $\mathbb{Z}[\mathfrak{M}^{-1}_{4+4m,3}]$.

By proposition (5.3.2), $\mathcal{E}^3_{2k}(\cdot, 2+2m-k)$ is obtained by applying $\partial^{k-2-2m}_{3,4+4m}$ to $\mathcal{E}^3_{4+4m}(\cdot, 0)$, and in particular the Fourier coefficients of $\mathcal{E}^3_{2k}(\cdot, 2+2m-k)$ are (polynomials with coefficients) in $\pi^{3(k-2-2m)} \cdot R'_m$ where

$$R'_m := \mathbb{Z}[i, \mathfrak{M}^{-1}_{4+4m,3} \cup \mathfrak{P}^{-1}_{k+1+2m}]$$

with the notation of (3.37). Hence by proposition (3.3.7) the function $\mathscr{E}_m(Z, \tau)$ of (6.6) is an element of

$$\pi^{3(k-2-2m)} \cdot \mathcal{N}^2_{2k,k-2-2m}(R'_m) \otimes_{\overline{\mathbb{Q}}} \mathcal{N}^1_{2k,k-2-2m}(R'_m)$$

and has Fourier coefficients in R'_m.

With the notation of (6.7), put

$$\begin{aligned}\Xi_m(Z, \tau) &:= \mathrm{Hol}\mathscr{E}_m(Z, \tau) \\ &=: \sum_{i,j} d^{(m)}_{i,j} F_i(Z) h_j(\tau)\end{aligned} \tag{6.10}$$

Since Ξ_m is obtained from holomorphic projection of $\mathscr{E}_m(Z, \tau)$, it is an element of

$$\mathcal{S}^2_{2k}(R'_m[\mathfrak{P}^{-1}_{4k-6}]) \otimes_{\overline{\mathbb{Q}}} \mathcal{S}^1_{2k}(R'_m[\mathfrak{P}^{-1}_{4k-5}])$$

with Fourier coefficients in $R'_m[\mathfrak{P}^{-1}_{4k-5}]$ by corollary (3.4.8) and satisfies

$$\langle \mathcal{E}^3_{2k}(Z \times \tau, 2-k+2m), \star(Z)h(\tau)\rangle = \pi^{3(k-2-2m)}\langle \Xi_m(Z, \tau), \star(Z)h(\tau)\rangle$$

for $\star = F$ or $\star = G$.

By the assumptions of theorem (6.1.3), find rational primes $\{q_i\}_i$ and $\{q'_j\}_j$ such that

$$\begin{aligned} \lambda_{q_i}(h) \not\equiv \lambda_{q_i}(h_i) &\quad \bmod \mathfrak{p}, \quad \forall i \geq 2 \\ \lambda_{q'_j}(F) \not\equiv \lambda_{q'_j}(F_j) &\quad \bmod \mathfrak{p}, \quad \forall j \geq 3 \\ \lambda_{q'_j}(G) \not\equiv \lambda_{q'_j}(F_j) &\quad \bmod \mathfrak{p}, \quad \forall j \geq 3 \end{aligned}$$

Define operators

$$T_Z := \prod_{j=3} \frac{T_{q'_j} - \lambda_{q'_j}(F_j)}{\lambda_{q'_j}(F) - \lambda_{q'_j}(F_j)} \qquad T_\tau := \prod_{i=2} \frac{T_{q_i} - \lambda_{q_i}(h_i)}{\lambda_{q_i}(h) - \lambda_{q_i}(h_i)}$$

where the subscript denotes which variable it applies to. Then

$$\begin{aligned} \Psi_m(Z,\tau) &:= T_Z T_\tau \Xi_m(Z,\tau) \\ &= A^{(m)} h(\tau) F(Z) + \vartheta B^{(m)} h(\tau) G(Z) \end{aligned} \tag{6.11}$$

has Fourier coefficients integral at $\mathfrak{p}$, since

$$\vartheta := \prod_{j=3} \frac{\lambda_{q'_j}(G) - \lambda_{q'_j}(F_j)}{\lambda_{q'_j}(F) - \lambda_{q'_j}(F_j)}$$

is coprime with $\mathfrak{p}$.

Arithmetic properties of $A^{(m)}$ and $B^{(m)}$

By (6.8), $A^{(0)} = 0$ since so is $Z_{F\otimes h}(3k-1)$. Then from (6.11)

$$\Psi_0(Z,\tau) = \vartheta B^{(0)} h(\tau) G(Z)$$

Fixing on both sides a choice of Fourier coefficients integral at $\mathfrak{p}$ but not divisible by it, we see that $\mathrm{ord}_{\mathfrak{p}}(B^{0)}) = 0$.

For $m \geq 1$, employ (6.8) to get

$$\begin{aligned} A^{(m)} &= \xi_F\left(\frac{1+4m}{6}\right) \frac{Z_{F\otimes h}(3k-1+2m)}{\pi^{3(k-2-2m)}\langle h,h\rangle\langle F,F\rangle} \\ &= \xi_F\left(\frac{1+4m}{6}\right) \frac{2^4\cdot 3^2 \pi^{-3k+7+6m}|D|^{2k-3/2}}{(2k-1)|c_g(|D|)|^2} \frac{L_h(k+2m)L_h(k+2m+1)}{\langle h,h\rangle} \times \\ &\times \frac{Z_{f\otimes h}(3k-1+2m)}{\langle f,f\rangle} \frac{L_f(\chi_D, 2k-1)}{L_f(2k)} \end{aligned} \tag{6.12}$$

where we have used [Bro07, corollary 6.3] for $\langle F, F\rangle$.

We proceed to study each ratio individually. We have

$$\frac{L_h(k+2m)L_h(k+2m+1)}{\langle h,h\rangle} \in \pi^{2k+4m+1}\mathbb{Q}(h) \tag{6.13}$$

and we have assumed $\mathfrak{p}$ coprime with this quantity.

Also

$$\frac{Z_{h\otimes f}(3k-1+2m)}{\langle f,f\rangle} \in \pi^{4k+4m-1}\mathbb{Q}(h,f) \tag{6.14}$$

is coprime with $\mathfrak{p}$ by hypothesis.

Furthermore

$$\frac{L_f(\chi_D, 2k-1)}{L_f(2k)} \in \pi^{-1}\mathbb{Q}(f)(\sqrt{|D|}) \tag{6.15}$$

by [Shi77, theorem 1], and we assumed $\mathrm{ord}_{\mathfrak{p}} \leq -1$ for this quantity in proposition (4.1.2).

Lastly, from (6.3), $\xi_F(\frac{1+4m}{6})$ equals

$$\pi^{-3-10m-3k}\frac{2^{-12m-4-6k}d^{-k-2m-1/2}(3k-2+2m)!(2+4m)!(4+4m)!}{(2k+4m)^2 B_{2+4m}B_{4+4m}L(\chi_{-d};3+4m)}\overline{F_\eta}$$

and, by [Neu99, p. 443, corollary]

$$L_{\chi_{-d}}(3+4m) = -i\frac{\tau(\chi_{-d})}{2}\left(\frac{2\pi}{d}\right)^{3+4m}\frac{B_{3+4m,\chi_{-d}}}{(3+4m)!}$$

where

$$\tau(\chi_{-d}) := \sum_{j=0}^{d-1}\chi_{-d}(j)e^{\frac{2\pi i}{d}j} = \begin{cases} \pm\sqrt{d} & d\equiv 1 \mod 4 \\ \pm i\sqrt{d} & d \equiv 3 \mod 4\end{cases}$$

hence

$$\xi_F(\tfrac{1+4m}{6}) \in \pi^{-3k-6-14m}\mathbb{Q}(\sqrt{d}) \tag{6.16}$$

is coprime with $\mathfrak{p}$.

Combining (6.13), (6.14), (6.15) and (6.16) into (6.12) we deduce $A^{(m)} \in \overline{\mathbb{Q}}$ and $\mathrm{ord}_{\mathfrak{p}}(A^{(m)}) \leq -1$. Comparing Fourier coefficients on both sides in (6.11), we see that $B^{(m)} \in \overline{\mathbb{Q}}$ with $\mathrm{ord}_{\mathfrak{p}} \leq -1$.

Taking the ratio

Using (6.9) write the ratio

$$\pi^{-6m}\frac{\xi_G(\frac{1}{6})}{\xi_G(\frac{1+4m}{6})}\frac{Z_{G\otimes h}(3k-1)}{Z_{G\otimes h}(3k-1+2m)} = \frac{B^{(0)}}{B^{(m)}} \in \overline{\mathbb{Q}}$$

and the right-hand side has $\mathrm{ord}_{\mathfrak{p}} \geq 1$, for every $m \geq 1$ as in theorem (6.1.3). Since $\xi_G(\frac{1+4m}{6}) \in \pi^{-3k-6-14m}\overline{\mathbb{Q}}$ is coprime with $\mathfrak{p}$, we deduce that

$$\frac{\pi^{8m} Z_{G\otimes h}(3k-1)}{Z_{G\otimes h}(3k-1+2m)} \in \overline{\mathbb{Q}}$$

with $\mathrm{ord}_{\mathfrak{p}} \geq 1$ as claimed.

§ 6.4 Remarks on some conditions

As highlighted in remark (6.1.5), the result of theorem (6.1.3) improves theorem (4.1.3) at the expense of generality: we managed to prove that the congruence suggested by conjecture (1.3.1) holds for $Z_{G\otimes h}(2k+l-1)$ when the weights $2k$ and $2l$ of (respectively) G and h are equal. As this is due to the very nature of Saha's integral formula (6.1), we do not see any immediate way of generalising the result to unequal weights.

On the other hand, under the assumption of equal weights, Saha's work [Sah10] can be employed in the case of modular forms of higher level (whereas we only focused on full level). This is by no means a trivial task, as the Eisenstein series appearing in (6.1) would be a non-converging series with level, whose coefficients' arithmeticity to our knowledge is not clear.

Finally, we could lift the condition that the quadratic extension L be of class number 1. While Saha's result covers any class number, [Sah10, theorem 6.9] expresses the wanted critical values as sum (over the classes) of inner products against various Eisenstein series: again, the main obstacle here is getting arithmetic information about their Fourier coefficients.

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