Buildings: An Exposition Detailing Construction and Theorems

An Exposition of the Mathematical Construct of Buildings

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Abstract

In the mid to late twentieth century, Jacques Tits' work in the area of Lie groups and Lie algebras caused him to develop the construct of a building. Since then the topic has expanded to be viewed from a range of different perspectives and has proven useful in a range of other fields of mathematical research. The topic of buildings brings together several areas of mathematics, including combinatorics, incidence geometry, and Coxeter groups, to name but a few. Buildings are used by many as a vehicle to understanding properties of some of the more complex and unworkable groups that one may wish to understand. This is done through having a building upon which a group can act. In this thesis we will see that the building itself can be considered the fundamental object, and motivate ideas that by taking buildings in different geometries we are able to find new examples of groups. There are a variety of ways from which one may approach the study of buildings, each with its own benefits and shortcomings. This thesis provides an introduction to the topic of buildings showing a geometric based construction with recurring examples.

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Declaration

I declare that this thesis is a presentation of original work and I am the sole author. This work has not previously been presented for an award at this, or any other, University. All sources are acknowledged as references.

Introduction

The theory of buildings was originally developed in the late twentieth century by Tits in order to aid his understanding of Lie Groups. Since then buildings have found another use in the wider topic of groups, as they allow a mathematician to obtain data about a group that would be far too complex to analyse alone. As well as groups, the topic of buildings brings together multiple areas of mathematics into a single point of study and as such there are multiple approaches available to reach buildings.

This thesis aims to explore and prepare the reader for the many different pathways that lead into the desired construction of a building. We begin by looking at incidence geometry through chamber systems and more generally wider geometries, before moving on to reflection groups and Coxeter groups and then finally combining these two areas to give us a building. Additionally, concepts such as apartments, panels and isometries are presented. In this thesis we take a predominately example-driven approach, supported by key definitions and theorems. Many of our examples make multiple reappearances throughout, a testament to the multiple topics that buildings cover.

Firstly, we begin in chapter 1 by introducing chamber systems. A chamber system is a geometric structure created from simplices. As well as an introduction to chamber systems we also provide examples of how they can be realised from geometric solids. After more generalised definitions relating to the properties and structure of chamber systems, we close chapter 1 by detailing some of the tools that one has in order to work with such structures. This first section largely follows the motivations used in [**6**] and [**15**].

As we move into chapter 2, we turn away from the group theoretic aspects which we start to see in the language of chamber systems and instead focus on geometry. This chapter provides a solid grounding in Euclidean, spherical and hyperbolic geometries, as required for our purpose (and with possibly a little extra). We begin by defining the different geometries through looking at how to define inner products, norms and metrics on spaces equipped with each geometry. We then move to look specifically at triangles in each of the three geometries, exploring how properties of the triangle are impacted by the geometry of the space they lie in. We end this chapter by looking at how triangles in the plane can be generated by the reflection of regular geometric objects. This gives the reader a stronger appreciation for the close ties between geometry and the material that is to follow. This chapter takes inspiration from [**13**] and [**12**]. Neither of these texts lead the author to the study of buildings, and often in some of the major texts in the area of buildings such as [**1**], [**15**] or [**11**], a comprehensive cover of material related to geometries is lacking.

Chapter 3 focuses on reflections and reflection groups, turning back towards group theory from geometry. We begin this chapter by examining, through examples, what we mean by a reflection. With this understanding, we are able to define a reflection group, before taking some time to point out the links with the previous chapter. We then look at root systems. We provide a running example throughout this section to illustrate some of the key points regarding to root systems. We then look at the classification for irreducible reflection groups. This work is taken almost directly from [**1**], but is worthy of inclusion to illustrate the diversity of reflection groups possible and the importance of the root system. We end this chapter by formalising the notion of a word, a definition which becomes ever more important as we continue.

Moving forwards to chapter 4, we first define what a Coxeter system is before showing by our canonical example, how a chamber system can be realised from a Coxeter system. This section starts to tie much of the previous material into something that starts to feel more purposed. Also in this chapter we explore properties of Coxeter systems and discuss the classification of Coxeter systems. We end this chapter with a range of examples to aid the understanding of the reader and to draw back to the geometric examples seen in chapter 2.

Our final chapter d the introduction of the building, using information taken from each of the previous chapters. We provide some key definitions and accompanying examples, before launching into a much more theoretic approach to both illustrate some key points and also to highlight the range that the construct of the building has across group theory. We conclude this chapter, and indeed the thesis, by providing a brief discussion of alternative approaches to reaching buildings.

CHAPTER 1

Chamber Systems

We begin this exposition leading to the construction of buildings by introducing chamber systems. Our approach to study chamber systems will be to introduce a recurring example, which will be used to illustrate key points throughout both the chapter and the thesis as a whole. Much of this first section may remind the reader of graph theory. This can be a useful view to take in these earlier stages and key results in later chapters will allow us to transfer these simple graphical ideas to the language of buildings. This first chapter follows a similar approach as seen in [**6**], rephrasing and expanding on key points and with addition of some further material.

1. Constructing a Chamber System

1.1. The Flag System. Our first example shows how it is possible to construct a chamber system using a vector space. This example is our canonical example throughout this thesis, indeed we will see it as a motivating example in several places as we continue forwards.

We start by taking an $n+1$-dimensional vector space V over a field k. Let X be the set of all proper, non-zero subspaces of V, and let there be a set $I = \{1, 2, \dots n\}$ which we will call the index set. We now take maximum flags from X.

A flag $f \in X$ is a run of subspaces $x_1 \subset x_2 \subset x_3 \dots \subset x_{m-1} \subset x_m \subset V$ such that $m \leq n$ and for all $1 \leq j \leq m$ we have that $x_j \in X$. A maximal flag on an n-dimensional subspace is a run $x_1 \subset x_2 \subset \dots \subset x_{n-1} \subset V$ where each x_i has dimension i. Essentially we need the spaces to have consecutive dimensions, so that for any x_{i-1} and x_i in a flag, there cannot exist some x_j where $x_{i-1} \subset x_j \subset x_i$. We will call a maximal flag a chamber. We may denote chambers $x_1 \subset x_2 \dots \subset x_n$ as $(x_1, x_2, \dots, x_n)$. For now we can think of these chambers as being denoted graphically, as will soon be shown.

Next we must introduce the notion of being i-adjacent. This relation is an equivalence relation between two chambers. For our example, we will say that two chambers $(x_1, x_2, \dots, x_n)$ and $(y_1, y_2, \dots, y_n)$ are i-adjacent if $x_j = y_j$ for all $j \in \{1, \dots, i-1, i+1, \dots, n\}$. Essentially we are saying here two chambers are i-adjacent if they differ only in the ith position. Considering chambers as vertices, we connect two vertices by an edge whenever the chambers are i-adjacent for some $i \in I$, and the edge is labelled i for the relevant $i \in I$.

This is our chamber system: a collection of chambers and a relation of adjacency between them. We can take this example a little further by inserting some figures to make the example more concrete.

Example 1.1. Set $n = 3$ for the dimension of our vector space V, and set k to be a field of order $q = p^n$ for some p prime. We need to count how many 1-dimensional and 2-dimensional subspaces there are. For the 1-dimensional case each subspace is spanned by a non-zero vector, of which there are $q^3 - 1$. These are all the possible vectors without the zero vector. But then by taking any vector v, it will span the same line as any non-zero scalar multiple. There are $q-1$ scalar multiples of $v \in V$. We are left with $\frac{q^3-1}{q-1} = 1 + q + q^2$ 1-dimensional subspaces of V. A similar argument gives us that there are also $1 + q + q^2$ 2-dimensional subspaces of V.

Now we need to know how many 1-dimensional subspaces are contained in each 2-dimensional subspace and, vice versa, how many 2-dimensional subspaces each 1-dimensional subspace is contained in. We can see using the argument used above that each 2-dimensional subspace must contain $\frac{q^2-1}{q-1} = q + 1$ of the 1-dimensional subspaces. Since we know that there are $(q^2 + q + 1) \times (q + 1)$ distinct inclusions of subspaces $x_1 \subset x_2$ and we also know that there are $q^2 + q + 1$ subspaces of V which are 1-dimensional, we must also have that each 1-dimensional subspace is contained in $q + 1$ subspaces of V which are 2-dimensional .

Setting the subspaces in the flags as vertices (with 1-dimensional subspaces shown in white and 2-dimensional subspaces shown in black) and distinct inclusions $x_1 \subset x_2$ as an edge connecting two vertices, we get the following flag systems when $q = 2$ and $q = 3$. These figures are based on those used in [**6**].

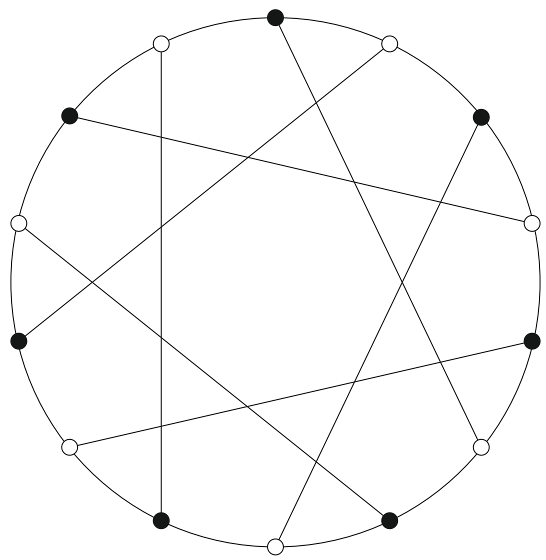


FIGURE 1. The flag complex Δ of the 3-dimensional vector space over the field of order 2

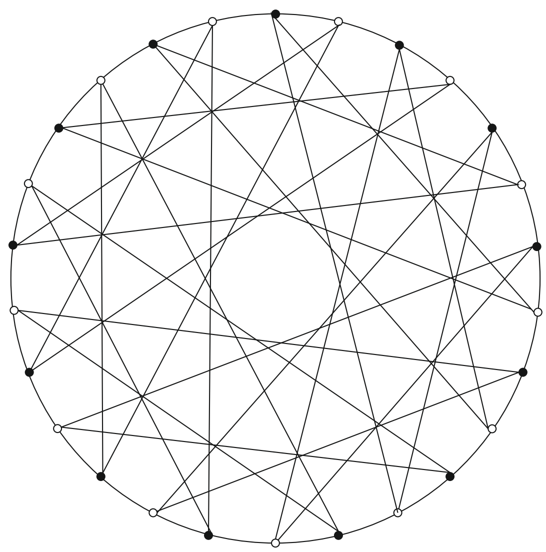


FIGURE 2. The flag complex Δ of the 3-dimensional vector space over the field of order 3

As is clear from the above two figures, the system becomes vastly more complicated with only a small increase in the order of the field.

1.2. Motivating the Chamber System. We can formalise the setup above to give the definition of a chamber system, a construct central to our study of buildings. The following definitions are based on those found in [**7**]. We begin by introducing simplicial complexes: a construct analogous to the flag system introduced in our earlier example. In this exposition, we are only going to consider simplices of finite dimension, although infinite simplices do exist. We then move into the definition for chamber systems which we will focus on in this thesis, using edge coloured graphs.

Definition 1. For a set V, a combinatorial **simplicial complex** with vertices V is a set X such that:

- X is a set of finite subsets of V
- If $x \in X$ and $y \subset x$ then $y \in X$
- Every singleton subset of V is contained in X

Call the elements $x \in X$ **simplices**.

Essentially here, we can think of simplicial complexes as a kind of partially ordered set, indeed this is how we have already seen them presented in our canonical example. Geometrically, simplices are often described using geometric objects to aid visualisation. In such a description, one could describe the zero-simplex as a point, the one-simplex as a line, the two-simplex as a triangle, the three-simplex as a tetrahedron, and so on. Each is contained within the next simplex up. The simplicial complex is then formed through 'sticking' the faces of different tetrahedra together in some reasonable way.

We now look at how to define a maximum simplex, as was taken in our canonical example.

Definition 2. A simplex $x \in X$ is **maximal** if there exists no simplex $z \in X$ such that $x \subset z \in X$.

We have already seen this idea in our canonical example where we required maximum flags of the set of subsets of the vector space. For the sake of this thesis, we are only interested in flag complexes or simplicial complexes where every flag (simplex) is contained in a maximal flag (simplex).

We motivate the chamber system by considering simplicial complexes displaying two qualities, namely that for a simplicial complex X:

- every simplex is contained in a maximal simplex, and
- for maximal simplices $x, y \in X$ there exists a sequence $x_0, x_1, \dots, x_n$ of maximal simplices $x_0 = x$ and $x_n = y$ and x_i is adjacent to x_{i+1} for all indices i. We will refer to the maximal simplicies as chambers and say that two chambers sharing the same face are i-adjacent.

[**7**]

The purpose of starting with simplicial complexes to motivate the concept of a chamber system is to help illustrate the strong links of this topic with geometry, which is further expanded on in the examples provided in section 1.4. Whilst some texts in the literature surrounding this topic focus purely on the simplicial complex approach, we will now take a different route. This approach follow that of [**15**], and begins by defining an edge coloured graph.

Definition 3. For a graph $\Delta = (V, E)$, with V a set of vertices and $E \subset V \times V$ a set of edges between pairs of vertices, say that Δ is an **edge coloured graph** if there exists some set I of 'colours' with a surjective map from E to I.

Indeed, a useful way of thinking about chamber systems is as edge coloured graphs. The set I of colours acts such that it literally colours in the edges of the graph Δ. Denote $x \sim_i y$ to mean the edge connecting x and y is colour i.

Another useful definition, offering a way of isolating important parts of the edge coloured graph, is that of a residue. When we use language such as "connected" or "connected component" in the context of edge coloured graphs and chamber systems, we are talking about the standard notion of connectedness for the underlying graph. So a connected component is a part of the graph where there is a route through a sequence of edges from every vertex in the component to any other vertex in the component. This leads to the definition of a residue.

Definition 4. For some subset $J \subset I$, a **J-residue** or **residue** of an edge coloured graph Δ is a connected component of the subgraph of Δ made by deleting all i-labelled edges for all $i \notin J$. [**15**]

We now have enough terminology to define what we mean by a chamber system.

Definition 5. A **chamber system** is an edge coloured graph Δ with an index set I such that each i-residue of Δ is a complete graph with two or more chambers, for each $i \in I$. [**15**]

Notice that any J-residue of a chamber system is itself a chamber system, as the natural chamber system over the set J. We now consider how one would describe routes through the edge coloured graph.

Definition 6. A **gallery** γ of length n from chambers c to c' in Δ is any sequence of $n+1$ chambers $\gamma = (c_0, c_1, \ldots, c_n)$ in Δ such that $c = c_0$, $c' = c_n$ and for all $1 \leq j \leq n$ we have that $c_{j-1} \sim_{i_j} c_j$ for a given $i_j \in I$. [**15**]

A gallery is effectively a possible route between two chambers in the chamber system. The **type** of a gallery γ with respect to the chamber system is the word $i_1 i_2 i_3 \ldots i_n \in M_I$. Here M_I denotes the free monoid on the set I, giving all possible combinations of elements of I of any length. The definition and use of words is formalised in chapter 3. We will use the definition of a gallery significantly in the study of buildings, primarily as it provides a notion of length.

Clearly we have that our initial example of the flag complexes over vector spaces fits the definition of a chamber system. If we consider the example given in figure 1, that is the flag complex of the 3-dimensional vector space over the field of order 2, we can take the edges of this graph to be chambers. We can label the vertices as 1 or 2 (shown in figure 1 as black or white vertices) to refer to which subspace the flags differ in. Two chambers are then i-adjacent if they share an i labelled vertex.

The notion of i-adjacency gives an equivalence relation. Taking the example above again, it is clear to see this. Given any three adjacent chambers, c_1, c_2 and c_3, for all $i \in I$ we have that $c_1 \sim_i c_1$. This is clear as we can always go from the edge c_1 through an i-labelled vertex back to c_1, hence $\sim$ is reflexive. We also have that if $c_1 \sim_i c_2$ that $c_2 \sim_i c_1$ as going from one edge through an i-labelled vertex to an adjacent edge can always be done in reverse, hence $\sim$ is symmetric. Finally it is clear to see that if $c_1 \sim_i c_2$ and $c_2 \sim_i c_3$, then $c_1 \sim_i c_3$. This can be seen by considering any three edges sharing a vertex on figure 1, and shows transitivity. We can therefore see that $\sim$ is an equivalence relation.

Weiss introduces the reader to chamber systems at the very start of his text on buildings [**15**] and over the course of his first chapter spends time looking at the chamber system as little more than a specialised graph. Conversely, in Brown's book on buildings [**1**], we do not meet the chamber system until almost chapter 2, since Brown takes a completely different approach to arrive at chamber systems later on. We will see both of these alternative definitions as we progress.

1.3. Further Examples. Returning to our example above, we could have taken things further by using a higher dimensional vector space. Even working one dimension higher where V is 4-dimensional increases the complexity significantly. With the same notation as before, in a 4-dimensional vector space we would have $q^3 + q^2 + q + 1$ of the 3-dimensional subspaces, $q^4 + q^3 + 2q^2 + q + 1$ of the 2-dimensional subspaces and $q^3 + q^2 + q + 1$ of the 1-dimensional subspaces. There are now three places in the flag that we can have a different subspace, so the resulting graph would look much more complicated

Alternatively we could have used an entirely different chamber system. Our canonical example at the start was chosen as it has the depths to take us smoothly all the way into the language of buildings. Many chamber systems do not get that far. Indeed, we can consider a chamber system made purely geometrically by labelling a cube as shown below, or even a tiling of the Euclidean plane. The following examples are based on those found in [**12**].

Example 1.2. An abstract chamber system can be taken from examples in basic shapes in geometry. Take a tetrahedron; the edges can be labelled by the set $I = \{1, 2, 3\}$ as shown. Each face, or chamber, then contains exactly one edge labelled by each $i \in I$. This gives us a chamber system comprised of 4 chambers. We could also choose to take the vertices as the chambers in this example, which would give an equivalent chamber system. As can be seen on figure 3 below, we would have that

$$c_1 \sim_3 c_2$$

since c_1 and c_2 share a common edge labelled 3.

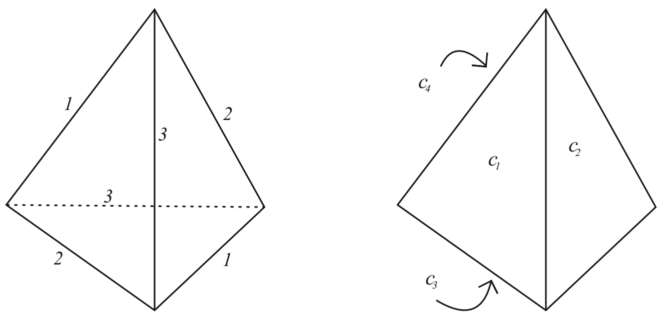


Figure 3. A tetrahedron labelled by the set $I = \{1, 2, 3\}$, giving a chamber system with 4 chambers

Example 1.3. Continuing on the same logic, we could choose to label a cube. In this case we have a choice for our labelling of the edges depending upon whether we view the faces or vertices as the chambers. In this example, each gives rise to different chamber systems.

First consider labelling the edges of the cube using the set $I = \{1, 2, 3\}$ as shown in figure 4 below. In this labelling we view the vertices as the chambers, so each of the vertices of the cube contain exactly one edge labelled for each $i \in I$. This gives a chamber system comprised of 8 chambers for the 8 vertices, as shown below.

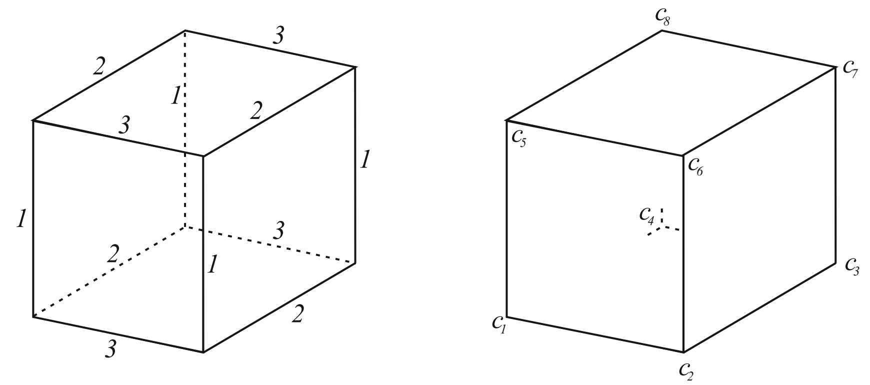


FIGURE 4. A cube with edges labelled using the set $I = \{1, 2, 3\}$ to give a chamber system with 8 chambers

Alternatively we could label the edges of the cube using the set $I = \{1, 2, 3, 4\}$ and take the chambers as the faces of the cube, such that each face contains exactly one edge with label i for each $i \in I$. This gives a chamber system with 6 chambers.

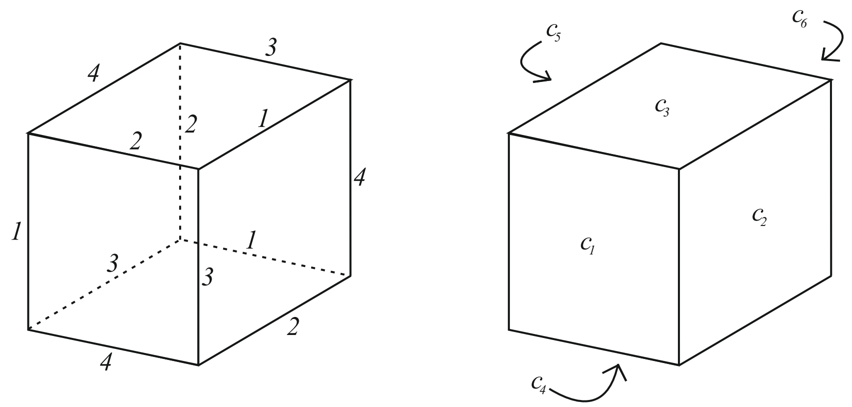


FIGURE 5. A cube with edges labelled using the set $I = \{1, 2, 3, 4\}$ to give a chamber system with 6 chambers

EXAMPLE 1.4. Our final example here involves tiling the 2-dimensional Euclidean plane with triangles. By labelling the edges using the set $I = \{1, 2, 3\}$, we get a chamber system with an infinite number of chambers. The chambers here are the triangles, with each triangle containing exactly one i-labelled edge for each $i \in I$

For further examples of chamber systems, [**12**] provides a very thorough review of chamber systems and a wider exposition of incidence geometry.

2. Structure of the Chamber System

In this section we delve deeper within the structure of the chamber system, and introduce a range of new terminology.

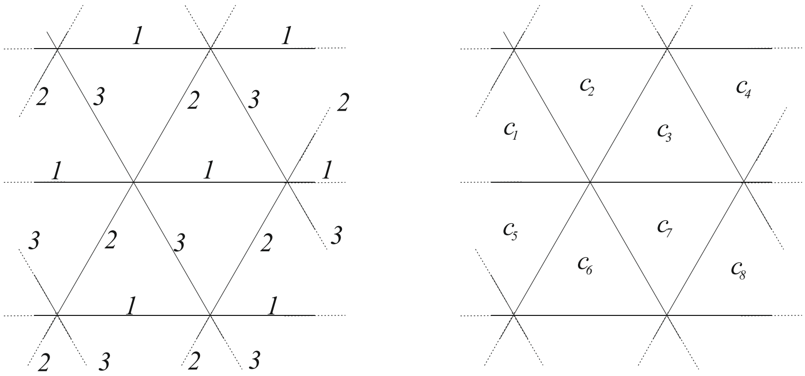


FIGURE 6. Tessellation of the 2-dimensional Euclidean plane by triangles, with edges labelled using $I = \{1, 2, 3\}$ to give an infinite chamber system

2.1. Within Chamber Systems. With an understanding of what a chamber system is, we now look to understand further structure within a chamber system. A natural starting point is the idea of a sub-chamber system, also referred to as a chamber subsystem in [**12**]. The following definitions follow those in [**15**].

DEFINITION 7. For a chamber system Δ, a **subchamber system** is a chamber system Δ' such that:

- $\Delta' \subseteq \Delta$
- For $c_1, c_2 \in \Delta'$, if $c_1 \sim_i c_2$ in Δ then $c_1 \sim_i c_2$ in Δ'.

If we return to the definition of a residue, we now state a further useful piece of terminology, defining a panel in a chamber system.

DEFINITION 8. For a chamber system Δ, a **panel** (of type i) is an i-residue for some $i \in I$. [**15**]

So a panel is one of the connected components that we are left with after deleting every edge between chambers which are not labelled i. The definition of an i-adjacent can now be restated.

DEFINITION 9. For a chamber system Δ and chamber $c, c' \in \Delta$, say that c is ***i*-adjacent** to c' if and only if c and c' lie on the same panel of type i.

Panels have other uses, one of which is to allow us to break chamber systems into different classes.

DEFINITION 10. If every panel in a chamber system contains exactly two chambers, we call the chamber system **thin**. If every panel contains more than two chambers we call the chamber system **thick**. [**15**]

This language of thin and thick will return later. Taking routes through sequences of edges in chamber systems, as is done to get a connected component above, is common when dealing with chamber systems. Recall the definition of a gallery, which we now use to give a notion of distance through chamber systems.

DEFINITION 11. The distance between chambers c and c' is given by the **distance function**, which is defined as:

$$dist(c, c') = min\{n \mid n \text{ is the length of a gallery from } c \text{ to } c'\}.$$

As desired, we now have a distance function. We say that a gallery of length n from c to c' is **minimal** if $dist(c, c') = n$.

We return to our initial example of a flag complex over a vector space. By taking any two chambers on the graph, it is obvious that we have a vast supply of galleries between chambers. Not only are there a variety of direct routes between two chambers but there is also the option to double-back on oneself within the gallery.

This final definition will take us back to subchamber systems, utilising several of the definitions given above.

Definition 12. A subchamber system Δ' of a chamber system Δ is **convex** if for any chambers $c, c' \in \Delta'$, a minimal gallery joining c and c' is the same in Δ as in Δ'.

2.2. Between Chamber Systems. Now that we have looked within the chamber system, we turn our attention to what we can do when we have several chamber systems together. There are a variety of directions in which this subsection could take us, indeed in [**15**] there are a range of different definitions that are covered. For the purposes of this thesis, we will restrict ourselves to the following definitions.

Definition 13. For chamber systems Δ and Δ' with index sets I and I' respectively, a **homomorphism** of chamber systems is two maps $\sigma : I \to I'$ and $\mu : \Delta \to \Delta'$ such that each panel of type i in Δ is mapped to a panel of type $\sigma(i)$ in Δ'.

This leads into the logical extension of an isomorphism.

Definition 14. An **isomorphism** of chamber systems is a homomorphism of chamber systems where σ and μ are bijections.

Isomorphisms are extremely useful as a way of moving between chamber systems as they preserve distance between chambers by mapping galleries to galleries of the same length. Our final definition of this chapter follows from the definition of an isomorphism.

Definition 15. An **automorphism** of a chamber system is an isomorphism such that $\sigma : I \to I$ permutes the index set and $\mu : \Delta \to \Delta$.

Definition 16. A homomorphism (and by extension isomorphism or automorphism) is called a **special homomorphism** (or isomorphism, automorphism) for some σ if $I = I'$ and $\sigma : I \to I$ is the identity map.

For the three definitions of a homomorphism, an isomorphism and an automorphism, we sometimes refer to these maps as a σ-homomorphism (or σ-isomorphism, σ-automorphism respectively) in order to draw attention to the map σ on the index sets. Before finishing this chapter on chamber systems, we will make one final remark about homomorphisms. For any two chambers $c, c' \in \Delta$ and σ-homomorphism μ, we have that $dist(\mu(c), \mu(c')) \leq dist(c, c')$, where $dist$ is defined as above.

CHAPTER 2

Geometries

In this chapter we introduce three fundamental geometries, namely spherical, Euclidean, and hyperbolic geometries. We have already seen examples linking chamber systems and geometry. Whilst a fair portion of the following material should be familiar to the reader, this chapter will provide a short overview of three geometries which can be linked into the study of buildings. We will be showcasing the key differences between each geometry and, for spherical and hyperbolic geometries, models that can be helpful for working with them. It should be noted that the two principle sources of material in this chapter are [**10**] and [**13**].

1. Bilinear forms

We begin our overview of Euclidean, spherical and hyperbolic geometry by introducing some central constructs which exist within each geometry. Our focus to start with will be on the definitions of an inner product, a metric and a norm. Allow us to first recap what each of these means in the Euclidean sense, something the reader will likely be familiar with.

1.1. Euclidean $\mathbb{R}^n$. Our first key construct is that of the inner product, an essential tool for working in a Euclidean vector space. The starting example in Euclidean space of an inner product is the dot product of two vectors.

DEFINITION 17. Given two vectors $x, y \in \mathbb{R}^n$ such that $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, the **dot product** of x and y is defined to be:

$$x \cdot y = x_1 y_1 + \ldots + x_n y_n \in \mathbb{R}.$$

[**10**]

The second key construct that we will focus on is that of a norm, which can be defined as follows:

DEFINITION 18. The **norm** of a vector $v \in \mathbb{R}$ is defined to be the real number

$$||v|| = (v \cdot v)^{\frac{1}{2}}$$

[**10**]

In Euclidean space, the norm is used to assign a strictly positive notion of length to a given vector; we will see that this is similar for the spherical and hyperbolic cases. The Euclidean Norm for a vector $x \in \mathbb{R}^n$ is denoted $|x|$.

DEFINITION 19. The **Euclidean distance** between two vectors x and $y \in \mathbf{R}^n$ is defined as follows:

$$d_E(x, y) = ||x - y||$$

where $|x - y|$ denotes the Euclidean norm of $(x - y)$.

As we move through spherical and hyperbolic geometry, we will also be concerned with viewing hyperplanes in each.

DEFINITION 20. A **hyperplane** in $\mathbb{R}^n$ is an $(n-1)$−dimensional linear subspace.

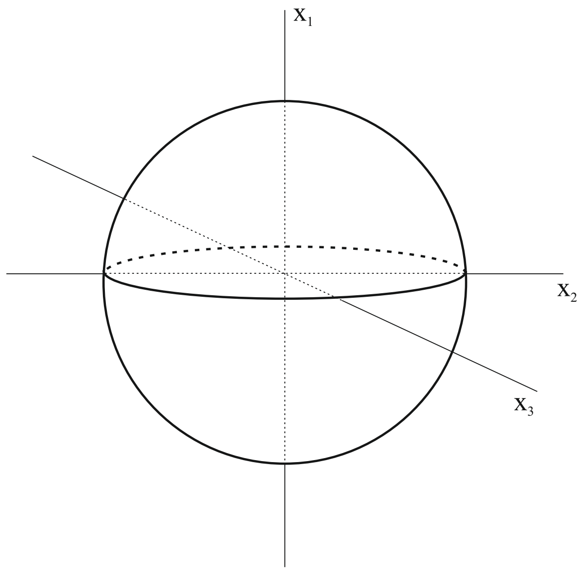


FIGURE 1. Spherical 2-space in $\mathbb{R}^3$

Finally, we discuss models for a geometry. Each geometry which we will look at is defined through a set of axioms. These control how the space behaves. As we move into both spherical and hyperbolic geometries, we shall require a model for each geometry to apply these rules to. These models will be used primarily as an aid to visualising what is going on, just as the space $\mathbb{R}^n$ serves for working with Euclidean geometry.

1.2. Spherical n-Space. Throughout this chapter we will describe and illustrate points using the standard model for spherical geometry, namely the $n-$dimensional unit sphere in $\mathbb{R}^{n+1}$:

$$S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$$

Therefore the picture to keep in mind when dealing with spherical geometry is rather than having 2-dimensional Euclidean space $\mathbb{R}^2$, we have the surface of the unit sphere S^2 which the real Euclidean space has been 'wrapped around'. Throughout this chapter, S^n will be used to denote spherical n-space. The space S^2 in $\mathbb{R}^3$ is shown in the figure 1.

DEFINITION 21. Let x and y be vectors in S^n and let $\theta(x, y)$ denote the Euclidean angle between x and y, such that $0 \leq \theta(x, y) \leq \pi$. The **spherical distance** between x and y is defined to be

$$d_S(x, y) = \theta(x, y).$$

Note that in S^n, we could equally choose to use the Euclidean distance as defined in the previous section. The spherical distance however, is more 'intrinsic' to the space. Despite θ providing a measure of angle, in spherical geometry we are working with radians, so the measure of angle is analogous to the measure of length between two points on the surface of the S^n. Spherical n-space can be defined as the space consisting of S^n together with d_S [**10**].

It is a simple and well known result that our usual inner product has the following relation with the cosine function: for two unit vectors $x, y \in \mathbb{Z}^n$

$$x \cdot y = \cos\theta(x, y).$$

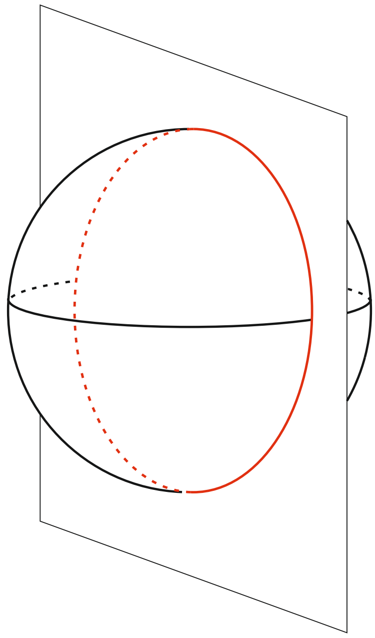


Figure 2. A hyperplane in S^2 (shown in red) is the intersection of S^2 with a hyperplane in $\mathbb{R}^3$

This has an analogous result in S^n using our spherical distance function; for $x, y \in S^n$

$$(x, y) = \cos d_{S^n}(x, y).$$

Thus we are provided with an alternative definition in S^n for using our usual Euclidean distance.

The final aspect that we will look at in spherical geometry is how a hyperplane in S^n will look. [**13**]

Definition 22. A **Hyperplane** in S^n is the intersection of an n-dimensional hyperplane of the ambient $\mathbb{R}^{n+1}$ space with S^n.

This is shown in figure 2.

1.3. Hyperbolic *n*-Space. Finally we look to hyperbolic geometry. As with spherical geometry, we must fist construct our model with which to work with hyperbolic geometry. We do this through first introducing a new product. [**10**]

Definition 23. For x and y vectors in $\mathbb{R}^{n+1}$ the **Lorentzian product**, $(x, y)_L \in \mathbb{C}$, of x and y is defined as:

$$(x, y)_L = x_1y_1 + \ldots + x_ny_n - x_{n+1}y_{n+1}$$

Similarly, we can use the Lorentzian product in order to define the Lorentzian norm function:

Definition 24. The **Lorentzian norm** of a vector $x \in \mathbb{R}^{n+1}$ is given by:

$$\|x\|_L = (x, x)_L^{\frac{1}{2}}$$

Our principle use for this new product is to allows us to dissect the space $\mathbb{R}^n$ into three distinct parts. For any $x \in \mathbb{R}^{n+1}$, exactly one of the following is true:

- $(x, x)_L < 0$
- $(x, x)_L = 0$

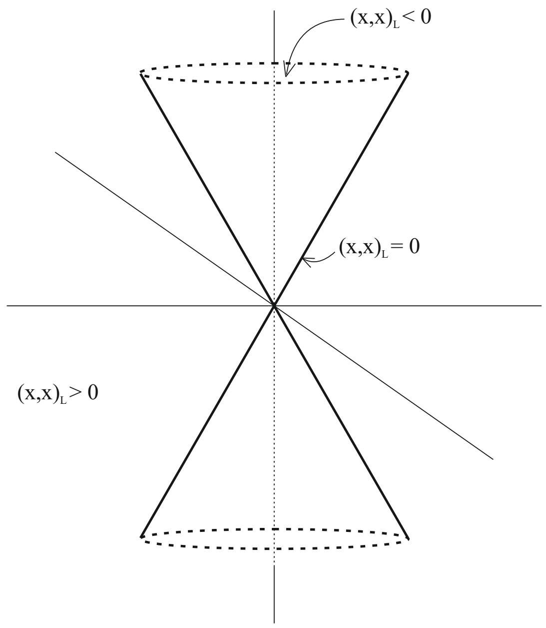


FIGURE 3. The light cone in $\mathbb{R}^3$ defined using inequalities on the Lorentzian inner product

- $(x, x)_L > 0$

In $\mathbb{R}^3$, this splits the space as shown in figure 3.

This diagram also has applications to physics, where the vectors satisfying each option are given the following names:

- A vector $x \in \mathbb{R}^{n+1}$ such that $(x, x)_L < 0$ is called time-like
- A vector $x \in \mathbb{R}^{n+1}$ such that $(x, x)_L = 0$ is called light-like
- A vector $x \in \mathbb{R}^{n+1}$ such that $(x, x)_L > 0$ is called space-like

This is where the cone in space derives its name as a light cone.

We now arrive at our model of hyperbolic n-space.

DEFINITION 25. The hyperboloid model of **Hyperbolic n-space**, denoted $\mathbb{H}^n$, is defined as

$$\mathbb{H}^n = \{x \in \mathbb{R}^{n+1} : ||x||_L^2 = -1, \ x_1 > 0\}$$

Less formally this is described to be the positive sheet of the sphere of radius i, which is the top hyperboloid in figure 4.

With the image of the hyperboloid in mind, it is necessary to consider what we mean when we speak of 'angles' in this space. We now state two theorems from [**10**] without proof, providing comment on the impact in defining time like vectors.

THEOREM 1. *Let $x, y \in \mathbb{R}^n$ both be positive (or both negative) time-like vectors, then $|x \cdot y| \leq |x|_L |y|_L$ with equality if and only if x and y are linearly dependent.*

With x and y as defined above, there exists a unique non-negative $\eta(x, y) \in \mathbb{R}$ such that:

$$x \cdot y = |x|_L |y|_L \cosh \eta(x, y).$$

We call $\eta(x, y)$ the **time-like angle between time-like vectors**.

THEOREM 2. *Let $x, y \in \mathbb{R}^{n+1}$ be linearly independent space-like vectors, then $|x \cdot y| > |x|_L |y|_L$ if and only if the vector subspace V spanned by x and y is time like.*

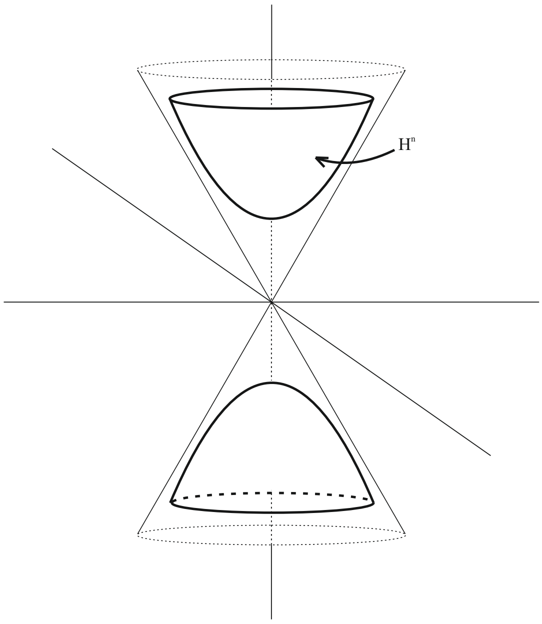


FIGURE 4. A model of H^2 in $\mathbb{R}^3$ inside the light cone, also showing inverse dual hyperboloid

Taking x and y as defined above, we have that there exists a unique positive $\eta(x, y) \in \mathbb{R}$ such that:

$$|x \cdot y| = |x|_L |y|_L \cosh \eta(x, y).$$

We call $\eta(x, y)$ the **time-like angle between space-like vectors**.

We now define distances in $\mathbb{H}^n$, as shown in [**10**].

DEFINITION 26. For vectors $x, y \in \mathbb{H}$ and $\nu(x, y)$ denoting the time-like angle between x and y the **hyperbolic distance**, or **hyperbolic metric** between x and y is defined to be

$$d_{\mathbb{H}}(x, y) = \eta(x, y)$$

and equivalently we have that

$$(x, y)_L = -\cosh d_{\mathbb{H}^n}(x, y).$$

Our next observation to make of hyperbolic geometry is of how a hyperplane in $\mathbb{H}^n$ would look.

DEFINITION 27. A **Hyperplane** in $\mathbb{H}^n$ is the intersection of an n-dimensional hyperplane of the ambient $\mathbb{R}^{n+1}$ space with $\mathbb{H}^n$.

This is illustrated in figure 5.

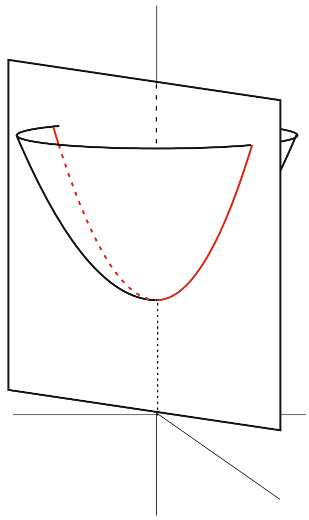


FIGURE 5. A hyperplane in $\mathbb{H}^2$ (depicted in red) is the intersection of the hyperboloid and a hyperplane of $\mathbb{R}^3$

Finally for $\mathbb{H}^2$, we introduce a further model to help our depiction of key points, the Poincaré Disc Model. We will here only provide a short description and illustration of how this model works. As the figure 6 shows, the model essentially involves placing a unit disc in $\mathbb{R}^3$ beneath the hyperboloid, centred at the origin. Any point on the hyperboloid is then joined by a straight line to the top point of the lower hyperboloid, making a stereographic projection through mapping the point on H^n to the unique point that it intersects the disc. This disc has radius 1.

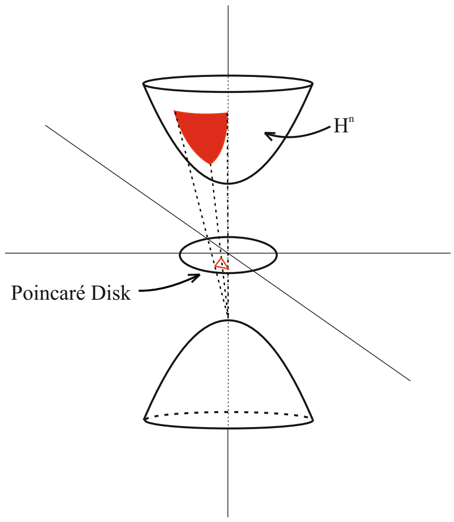


FIGURE 6. The Poincaré disc model

And then a triangle drawn in $\mathbb{H}^n$ would appear on the Poincaré Disk Model shown in figure 7.

As with any models, the Poincaré disc model has its own benefits and limitations. We choose to use this model as it helpfully preserves angles between lines on the hyperbolic plane. In following examples, this allows us to jump from diagrams of the hyperbolic plane and the

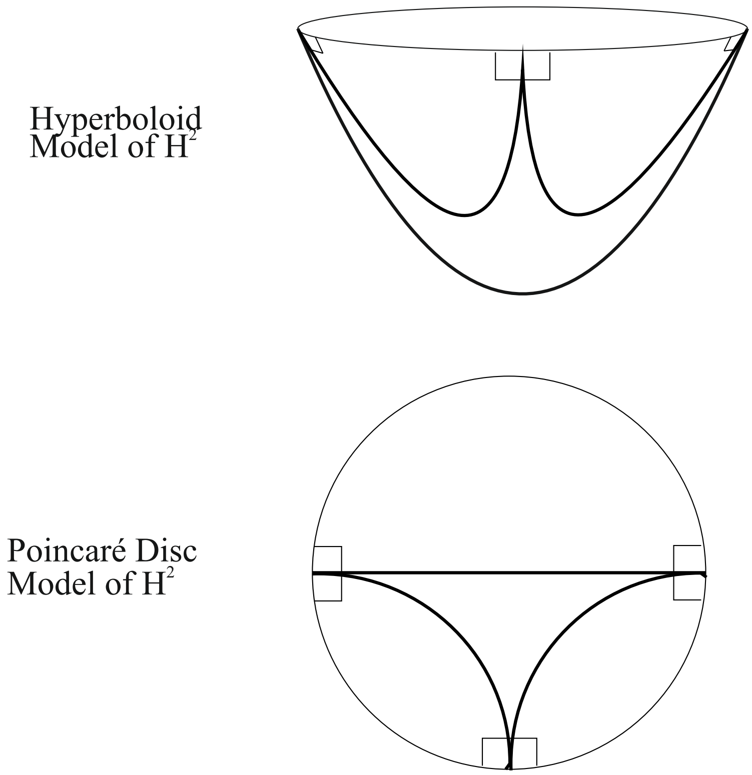


FIGURE 7. A triangle in H^n shown on the Poincaré disc model

disc model when talking about angles, which is much easier to depict. The model also has the benefits that both sits in the two dimensional plane and does not extend to infinity. Other models have benefits such as being able to show geodesics using straight lines (Beltrami-Klein model).

2. Triangles

In each of Euclidean, spherical and hyperbolic geometries, it is interesting to look at the properties and behaviours of triangles. We open this section looking at the areas of triangles in our three geometries, before shifting our focus to the angles of triangles and how we are able to draw triangles using familiar geometric shapes. We will state many results below without proof.

2.1. Triangles and their Areas. In Euclidean $\mathbb{R}^2$, denoted $\mathbb{E}^2$, it is obvious that a triangle could be drawn as shown in figure 8.

The following is well known regarding the sum of internal angles, which we will denote as $\sum \sphericalangle$, that in $\mathbb{E}^2$ (2-dimensional Euclidean space)

$$\sum \sphericalangle = \pi.$$

Indeed for any n-gon in $\mathbb{E}^2$, we have a well known result that

$$\sum \sphericalangle = (n-2)\pi.$$

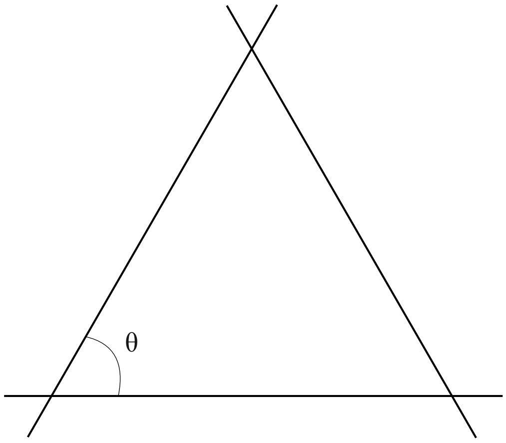


FIGURE 8. A triangle in $\mathbb{E}^2$

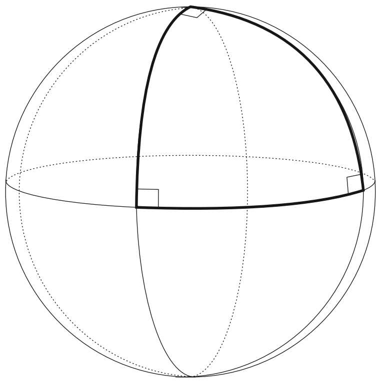


FIGURE 9. A triangle in S^2

We can now consider a triangle in S^2 as shown in figure 9.

In S^2 we have no such requirement that $\sum \sphericalangle = \pi$, as illustrated by figure 9 showing a triangle such that

$$\sum \sphericalangle = \frac{3\pi}{2}.$$

However we are able to go in a different direction with spherical geometry to state that for any triangle in S^2, the area of the triangle (denoted $\triangle$) must satisfy

$$\triangle = \sum \sphericalangle - \pi.$$

This can be easily seen using the example of figure 9 again. Consider the triangle shown above with each internal angle set to $\frac{\pi}{2}$. By the above formula we have that

$$\triangle = \frac{3\pi}{2} - \pi = \frac{\pi}{2}.$$

Since the sphere has total surface area 4π and the triangle shown covers $\frac{1}{8}$ the surface of the sphere, we can indeed see that the area of the given triangle is $\frac{\pi}{2}$ as expected.

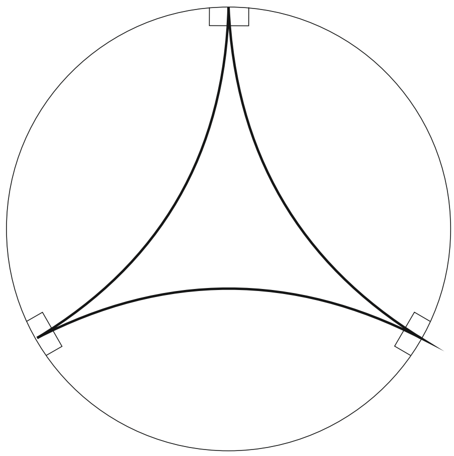


FIGURE 10. A triangle in $\mathbb{H}^2$ with $\sum \sphericalangle = 0$ and area π

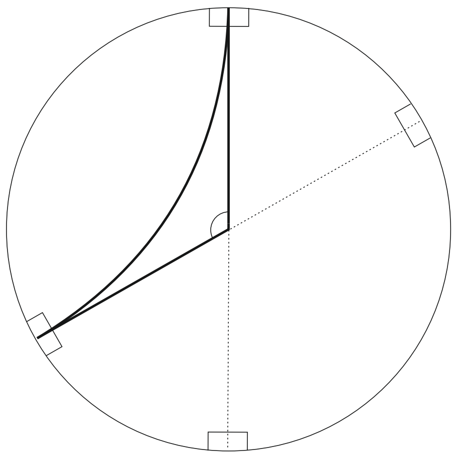


FIGURE 11. A triangle in $\mathbb{H}^2$ with $\sum \sphericalangle = \frac{2\pi}{3}$ and area $\frac{\pi}{3}$

Finally we consider some triangles in $\mathbb{H}^2$, as shown in figure 10 using the Poincaré Disc Model:

For this first triangle, clearly we have that

$$\sum \sphericalangle = 0.$$

We could also take another triangle as shown in the figure 11.

In this example we have that

$$\sum \sphericalangle = \frac{2\pi}{3}.$$

Again however, in hyperbolic geometry we are able to take things a step further and state that the area $\triangle$ of any triangle in $\mathbb{H}^2$ must satisfy the following

$$\Delta = \pi - \sum \sphericalangle.$$

Whist these statements are made without proof, both arise as consequences of the Gauss-Bonnet Theorem. A version of this theorem may be found in [**1**] .

2.2. Triangles in the Plane. Working in not only Euclidean, but also spherical and hyperbolic geometries, opens up new rules on the behaviours of triangles. With these new rules in hand, we are further able to distinguish between different triangles based on their interior angles. Consider a triangle having vertices with interior angles $\frac{\pi}{p}$, $\frac{\pi}{q}$ and $\frac{\pi}{r}$, such that $p, q, r \in \mathbb{Z}^{\geq 2}$ and $p \leq q \leq r$.

We can distinguish between triangles with different values for p, q and r dependant upon what is possible in each geometry. In Euclidean geometry, since

$$\sum \sphericalangle = \pi$$

we get that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1.$$

Considering this in $\mathbb{E}^2$, we must have that $p \leq 3$, as for all $p > 3$

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq \frac{1}{p} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4} < 1.$$

In the case with $p = 2$, we cannot have $q = 2$ as this forces $r = 0$. We must have that $q \leq 4$ since for $q, r > 4$ we have

$$\frac{1}{2} + \frac{1}{q} + \frac{1}{r} \leq \frac{1}{2} + \frac{1}{5} + \frac{1}{5} = \frac{7}{10} < 1.$$

This means that either $(p, q, r) = (2, 4, 4)$ or $(p, q, r) = (2, 3, 6)$.

In the case with $p = 3$ we have that $q \leq 3$ since $q \leq p$. We also have that $q \not> 3$ since for $q, r > 3$ we have

$$\frac{1}{3} + \frac{1}{q} + \frac{1}{r} \leq \frac{1}{3} + \frac{1}{4} + \frac{1}{4} = \frac{5}{6} < 1.$$

This forces that we can only have $(p, q, r) = (3, 3, 3)$. And hence our only options for (p, q, r) in $\mathbb{E}^2$ are:

- (2,4,4)
- (2,3,6)
- (3,3,3)

In spherical geometry our rules are slightly different. Since we have that

$$\sum \sphericalangle - \pi > 0$$

and therefore get that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1.$$

This time we must have that $p = 2$, as for all $p \geq 3$ we have that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1 \not> 1.$$

We must also have that $q \leq 3$ as for $q > 3$ we have that

$$\frac{1}{2} + \frac{1}{q} + \frac{1}{r} \leq \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 \not> 1.$$

For the case where $q = 2$, we have that $r = n$ for all $n \in \mathbb{Z}^{\geq 2}$. For the case where $q = 3$ we have that $r \leq 5$ since for all $r > 5$ we have that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{r} \leq \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = \frac{11}{12} \not> 1.$$

This gives us that in $\mathbb{S}^2$ the possibilities for (p, q, r) are

- (2,2,n) for $n \in \mathbb{Z}^{\geq 2}$
- (2,3,3)
- (2,3,4)
- (2,3,5)

And finally in hyperbolic geometry, we have different rules yet again. Since

$$\pi - \sum \sphericalangle > 0$$

we have that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1.$$

As we have shown the possible values of (p, q, r) for both cases where

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1.$$

and

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1.$$

we must have that all possibilities for (p, q, r) other than those listed above for $\mathbb{E}^2$ and S^2 are possible in $\mathbb{H}^n$.

With the above information in hand, we turn our attention to how we might generate these triangles in each of the geometries listed above. These examples use the vertices, edge centres and face centres of possible tilings of Euclidean, Spherical or Hyperbolic planes. We begin in $\mathbb{E}^2$. What is interesting here is how we are able to draw the triangles using familiar geometric shapes.

First consider the case where $(p, q, r) = (2, 4, 4)$. This triangle is achieved through the tessellation of the plane by squares, as shown in figure 12. Note that we say that a tessellation (often referred to as a tiling) of the Euclidean plane $\mathbb{R}^2$ is said to be **regular** if the tiling is achieved using only one regular polygon of consistent size.

Next we consider the case where $(p, q, r) = (2, 3, 6)$. This is achieved through the tessellation of the plane with hexagons, as shown in figure 13.

And finally we look at the case $(p, q, r) = (3, 3, 3)$. This is achieved through the tessellation of the plane with equilateral triangles, as shown in figure 14.

Moving to our spherical options for (p, q, r), we provide several examples using tilings of S^2.

Figure 15 shows the case with $(p, q, r) = (2, 2, n)$. In this example, we are able to form tessellations of s^2 into triangles with the desired $\frac{\pi}{n}$ angle by using n great circles, by which we mean a line in s^2 dividing the sphere into two equal hemispheres, which all meet a further great circle at an angle of $\frac{\pi}{2}$. This divides S^2 into $4n$ triangles.

Figure 16 shows the case with $(p, q, r) = (2, 3, 3)$. This example is formed using 6 great circles to tessellate S^2 into 24 triangles. Select 4 nodes where great circles intersect to form angles of $\frac{\pi}{3}$ such that the nodes are not joined to another node with angles of $\frac{\pi}{3}$. If we consider these nodes as the centre of faces, we are able to realise the tetrahedron.

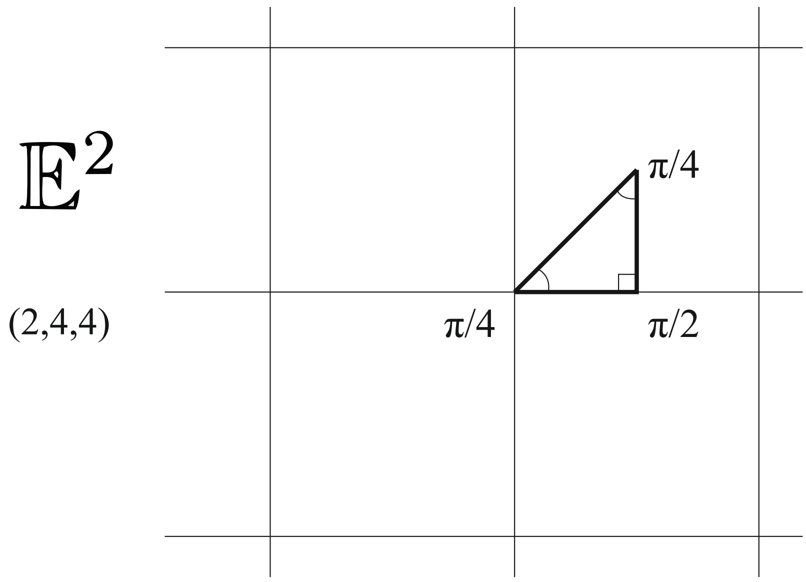


FIGURE 12. Triangle (2,4,4) in $\mathbb{E}^2$ obtained by tessellation of the plane with squares

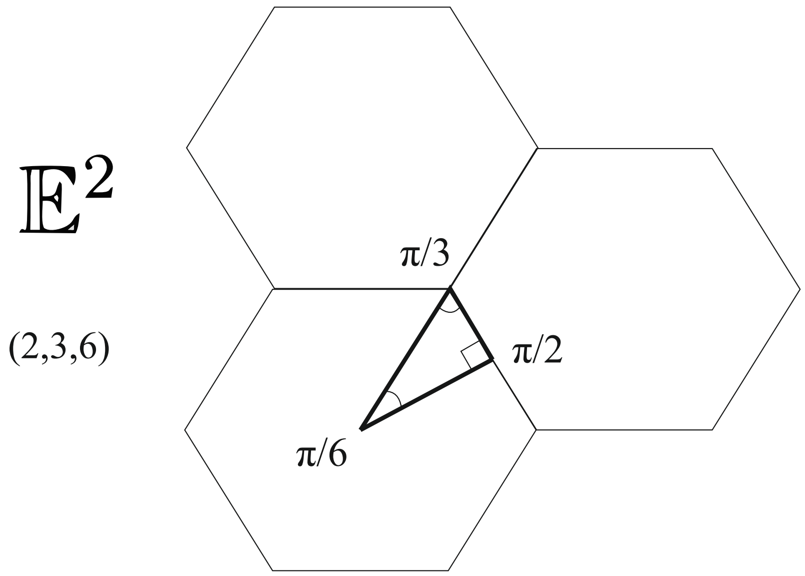


FIGURE 13. Triangle (2,3,6) in $\mathbb{E}^2$ obtained by tessellation of the plane with regular hexagons

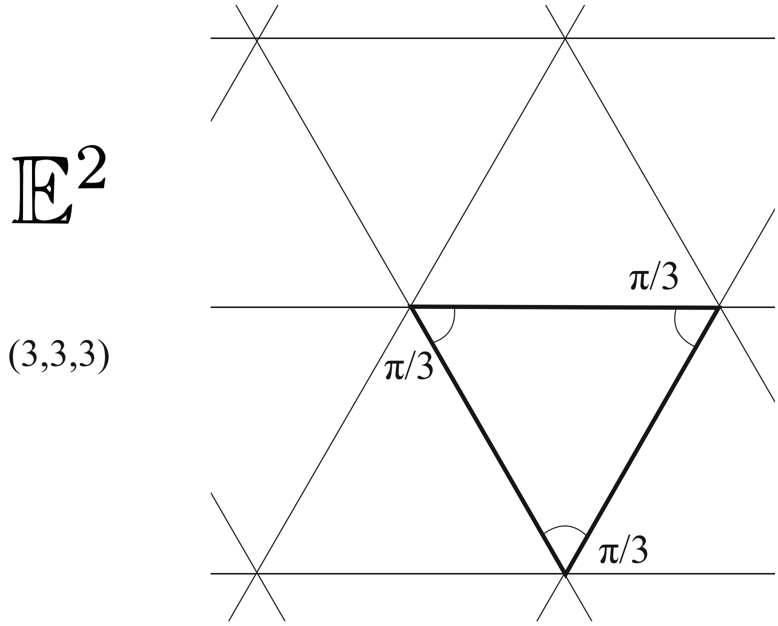


FIGURE 14. Triangle (3,3,3) in $\mathbb{E}^2$ obtained by tessellation of the plane with regular triangles

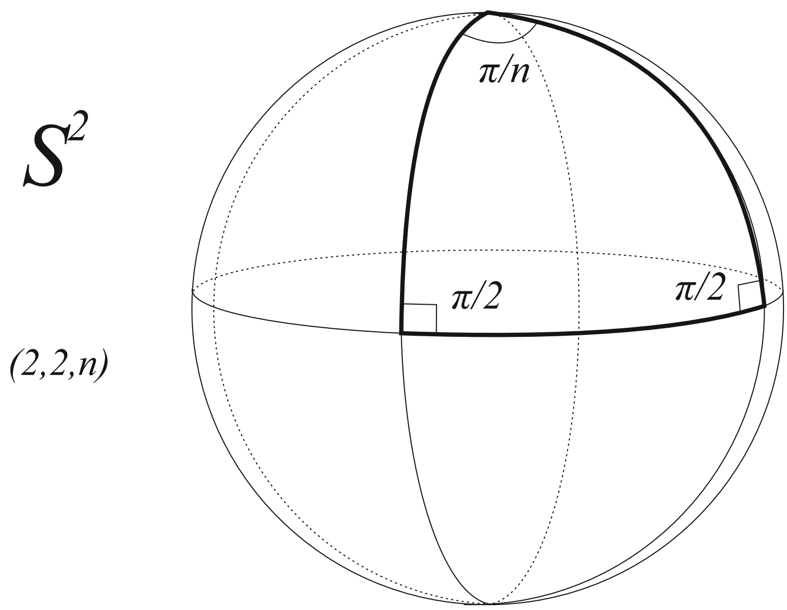


FIGURE 15. Triangle (2,2,n) in S^2

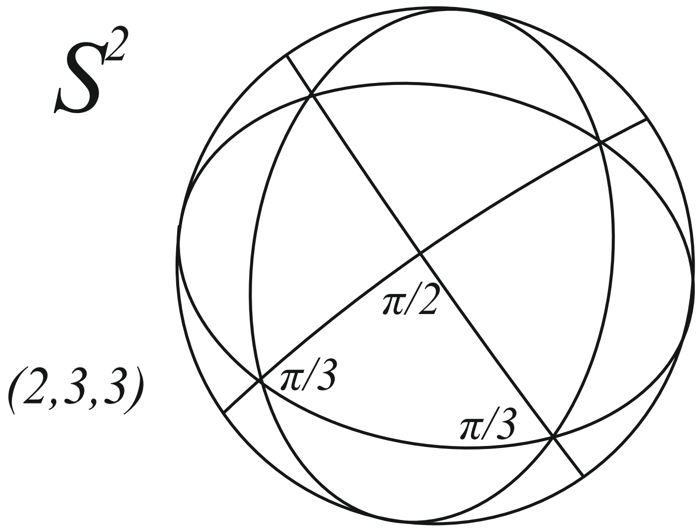


FIGURE 16. Triangle (2,3,3) in S^2, which can be used to realise the tetrahedron

The case where $(p, q, r) = (2, 3, 4)$ tessellates S^2 into 48 triangles. Considering nodes where great circles join at an angle of $\frac{\pi}{4}$ as the centre of each face, we are able to realise the cube. Alternatively if we consider the nodes where great circles join with an angle of $\frac{\pi}{3}$ as the centre of each face, we are able to realise the octahedron.

And finally figure 17 shows the case with $(p, q, r) = (2, 3, 5)$. This example is formed using 14 great circles to tessellate S^2 into 120 triangles. This time if we group the 10 triangles around each node where great circles join at an angle of $\frac{\pi}{5}$ into faces, we realise the dodecahedron.

With this example, we can also group the 6 triangles around each node where great circles intersect at an angle of $\frac{\pi}{3}$ into faces, to realise the icosahedron.

Between the three possible examples of tessellating S^2 by triangles with $(p, q, r) = (2, q, r)$, we are able to realise the 5 platonic solids.

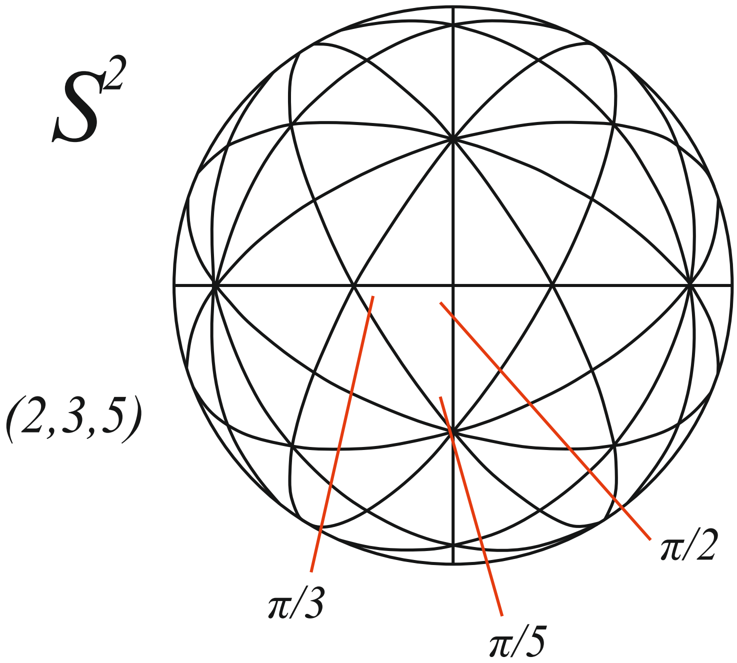


Figure 17. Triangle (2,3,5) in S^2 which can be used to realise both the dodecahedron and the icosahedron

CHAPTER 3

Reflection Groups

In this chapter we look at groups formed from reflections. Coxeter groups, a topic central to the construction of the building, naturally arise as an extension of reflection groups. We start this chapter with some motivating examples, before beginning a more systematic study of reflection groups. The author would like to cite [**9**], as the main source for this chapter with additional material taken from [**6**] and [**1**].

1. Reflections

1.1. Examples of Reflections. To begin this chapter, we take a look at some examples of reflection groups. Linking on nicely from our previous chapter on geometries, we see some of the same shapes that were used for the tessellation of the plane to get triangles emerging again here. We will formalise what we mean by a reflection shortly, however for now the reader should follow these examples with intuition of a reflection.

Example 3.1. The Dihedral Group

In the 2-dimensional Euclidean plane, consider a regular hexagon. We are going to take two reflections, s_1 and s_2, which are depicted in the diagram shown below.

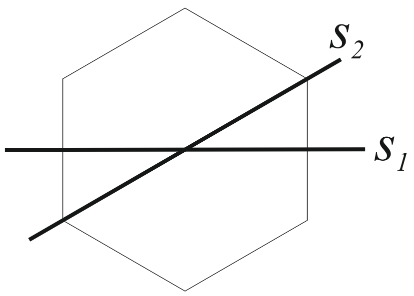


Figure 1. A hexagon with two reflecting lines s_1 and s_2

Using these reflections, it is possible to define a rotation r of the hexagon by $\frac{\pi}{3}$ anticlockwise, such that

$$r = s_1 s_2$$

where $s_1 s_2$ denotes the composite reflection obtained by a reflection in s_1 followed by a reflection in s_2.

In fact, all the other reflections of the hexagon can be labelled using only s_1 and s_2, as shown in the following figure. Here we denote $r = s_1 s_2$ and write terms such as r^2 to denote $s_1 s_2 s_1 s_2$ and r^{-2} to denote $s_2 s_1 s_2 s_1$.

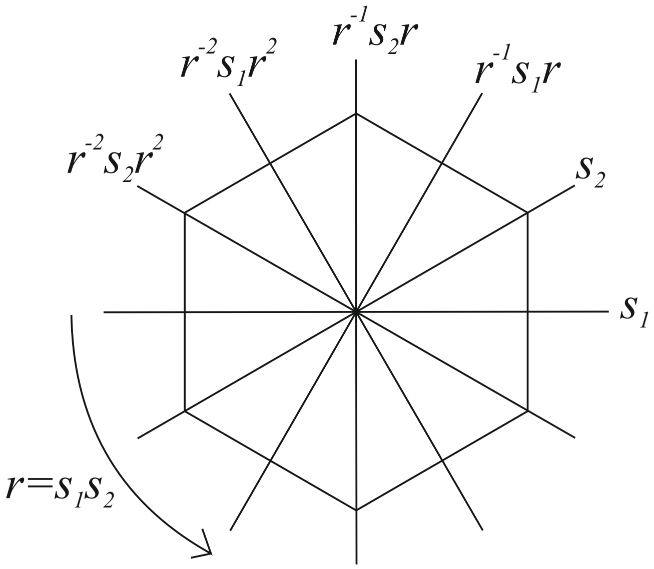


FIGURE 2. The 6 reflections of a hexagon labelled using reflections s_1 and s_2

Thus we have all 6 possible reflections taking the hexagon to itself labelled using only the original 2 reflecting lines. We have already labelled one of the rotations of the hexagon using s_1 and s_2. It turns out that all 6 possible rotations of the hexagon which map the shape to itself can be described using our two reflecting lines. We therefore have that if r denotes a rotation of $\frac{\pi}{3}$, then note that

$$r = s_1 s_2$$

and

$$r^6 = (s_1 s_2)^6 = I = s_1^2 = s_2^2$$

where I denotes the identity map of the hexagon, obtained by mapping each edge of the hexagon to itself. We therefore have that every possible symmetry of the hexagon is contained in the group containing s_1 and s_2 with presentation:

$$D_6 = \langle s_1, s_2 \,|\, s_1^2 = s_2^2 = (s_1 s_2)^6 = 1 \rangle.$$

This is the **dihedral group of order 12**. We can generalise to give the presentation for the dihedral group of order 2m:

$$D_m = \langle s_1, s_2 \,|\, s_1^2 = s_2^2 = (s_1 s_2)^m = 1 \rangle.$$

Dihedral groups are an example of **finite reflection groups**. We will return to this example to illustrate further points as we move through the chapter.

EXAMPLE 3.2. An Affine Weyl Group

In Euclidean $\mathbb{R}^2$, consider the reflecting lines in the figure shown below. Again this diagram is one that we saw at the end of the last chapter, the triangle in E^2 labelled $(2, 4, 4)$.

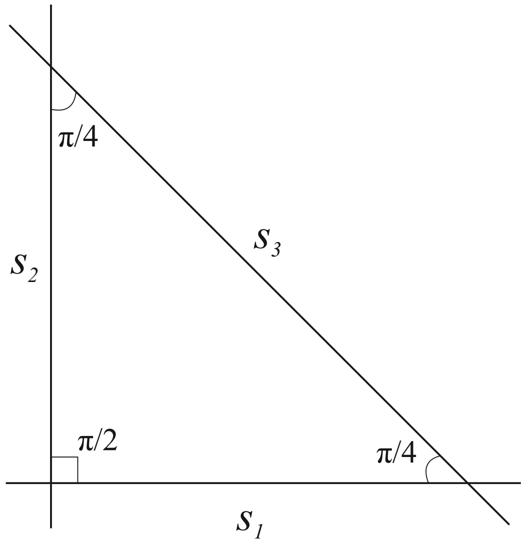


FIGURE 3. A triangle formed using reflections s_1, s_2 and s_3

If we take the three lines forming the edge of the triangle to be reflecting lines, and have that s_1, s_2 and s_3 all denote reflections in these lines mapping $\mathbb{R}^2 \to \mathbb{R}^2$ then there is then a group which is generated by these reflections. This is the group with presentation:

$$G = \langle s_1, s_2, s_3 \,|\, s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^2 = (s_2 s_3)^4 = (s_1 s_3)^4 = 1 \rangle.$$

This is an example of an infinite reflection group; we will come back to these later in the chapter.

1.2. Defining a Reflection. We now move onto the definition of a reflection .

DEFINITION 28. [**9**] Let V be an n-dimensional vector space over $\mathbb{R}$ equipped with the Euclidean distance, (λ, μ) on V. Fix a vector $v \in V$. A **reflection** in V is a map $s_v : V \to V$ such that for all $u \in V$ and some $k \in \mathbb{Z}$

$$s_{v,k}(u) = u - \frac{2\,((u, v) - k)}{(v, v)} v.$$

When $k = 0$ we have a **linear reflection**.

Call the set $\{u \in V : (u, v) = k\}$ the **reflecting hyperplane**, denoted $H_{v,k}$, for the reflection $s_{v,k}$. The example shown in figure 3 is an example of an affine reflection, which is not a linear reflection. When $k = 0$, we denote the reflection as simply s_v.

The figure 4 illustrates the definition of a linear reflection in $\mathbb{R}^2$.

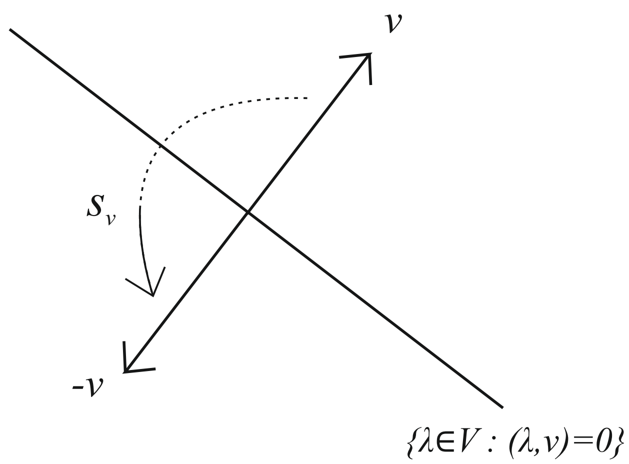


FIGURE 4. A reflection s_v in $\mathbb{R}^2$

Using the idea of a hyperplane to deconstruct the reflection, we can restate the above definition more intuitively as follows.

DEFINITION 29. [6] A **reflection** of a finite-dimensional vector space V is a linear map $s : V \to V$ for which there is a decomposition:

$$V = H_s \oplus L_s$$

where H_s is a codimensional one subspace and L_s is 1-dimensional , such that:

- The restriction of s to H_s is the identity
- The restriction of s to L_s is the map $v \mapsto -v$ for all $v \in V$

This is illustrated in the figure 5.

The hyperplane is, in effect, a fixed mirror of points in V which reflects the rest of the space by multiplication of -1 in a direction determined by the reflecting line L_s. The reflecting line should not lie in the hyperplane, as it would leave the whole of V invariant under s.

Whilst this is all relatively easy to picture in two or three dimensions, we can believe that this definition will also hold in higher dimensions in a similar manner. Thinking back to the previous section on geometries, we can also see that the definition should hold intuitively for reflections in spherical and hyperbolic space.

2. Reflection Groups

We have already seen two examples of reflection groups, namely the dihedral group and an affine Weyl group. It is obvious through our construction of these groups that reflections are central to the groups' behaviour.

2.1. Defining a Reflection Group. We now provide a formal definition for a reflection group.

DEFINITION 30. A **reflection group** W is a group generated by finitely many reflections.

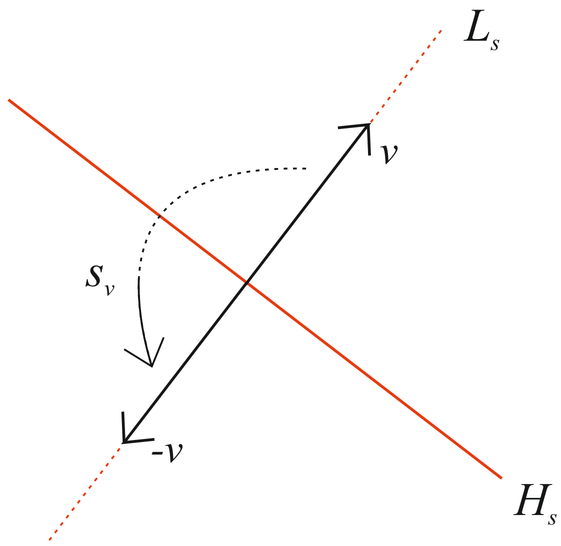


FIGURE 5. A reflection s_v in $\mathbb{R}^2 = H_v \oplus L_s$

By convention, we denote the set of hyperplanes generating a reflection group as $\mathcal{H}$. It is useful to note that in the case where all reflections are linear maps, then the reflection group W generated by these reflections is finite. We will prove this later in the chapter.

EXAMPLE 3.3. The Symmetric Group
For a Euclidean vector space $V = \mathbb{R}^n$ with an orthonormal basis $\{e_1, e_2, \dots e_n\}$, define the **Symmetric group**

$$\mathcal{S}_n = Sym\{1, 2, \dots, n\}$$

as the group of all possible permutations of the set $\{1, 2, \dots, n\}$. For each $\sigma \in \mathcal{S}_n$, we uniquely define a linear map

$$\Sigma : \mathbb{R}^n \to \mathbb{R}^n$$
$$e_i \mapsto e_{\sigma(i)}$$

We now confirm that this is a reflection group. We must first show that in the case where

$$\sigma = (i, j) \in \mathcal{S}_n$$

we have that Σ is the reflection s_v with $v = e_i - e_j$. Indeed, we have that

$$\Sigma(e_i - e_j) = e_j - e_i = -(e_i - e_j)$$

meeting our first criterion for a reflection group. Now for the hyperplane of points which are invariant by the reflection. Notice that for the vector $u = e_i + e_j$ orthogonal to $v = e_i - e_j$, then we have that

$$\Sigma(u) = e_j + e_i = e_i + e_j = u.$$

Hence u is invariant under Σ and indicates the reflecting line. Therefore we have a linear reflection Σ which corresponds to $v = e_i - e_j$. We denote the reflection $\Sigma = s_v$. We can therefore consider $\mathcal{S}_n$ as a reflection group as it is generated by transpositions, which are reflections on $\mathbb{R}^n$. We can define the set of hyperplanes associated to the reflection group as

$$\mathcal{H} = \{H_{i,j} : H_{i,j} = (v_i - v_j)^{\perp}, 1 \le i \neq j \le n + 1\}$$

meaning that $\mathcal{H}$ is the set of hyperplanes with equation $x_i - x_j = 0$.

2.2. Reflection Groups and Geometries. Clearly from the nature of reflection groups depending upon reflections, there is a strong relation between geometry in the previous chapter and our reflection groups here. This becomes more obvious with the following definition.

Definition 31. A reflection group is called a **spherical reflection group** if it has finite order.

Recall the examples at the end of Chapter 2 showing the tessellation of the 2-dimensional Euclidean, spherical and hyperbolic planes. It was clear to see from these examples that, due to the Euclidean and hyperbolic planes extending infinitely, reflecting a triangle could produce a tiling of the plane made of an infinite number of triangles. Considering the spherical example, we observed how a tiling of the 2-dimensional spherical plane by reflecting a triangle could only be achieved using a finite number of triangles. Consider that each triangle is an element of the reflection group, and it is clear to see why we call finite reflection groups spherical.

3. Root Systems

In this section we introduce and study root systems including what they are, how they arise, the group given by a root system, positive and simple systems and how a root system can be generated by reflections. Again the author would like to reference chapter 1 of [**9**] as the key source for content presented.

3.1. Motivation for Root Systems. We begin our introduction to root systems by looking at reflection groups from a different point of view. This begins with defining an orthogonal map.

Definition 32. An **orthogonal map** on a vector space V is a linear map

$$f : V \to V$$

such that

$$(f\mu, fu) = (\mu, u)$$

for all vectors $\mu, u \in V$, where (μ, u) denotes the Euclidean distance.

This means that the map f will preserve the origin, lengths and angles. We can thus describe a reflection as orthogonal, bijective and of order 2 (since $s_v^2 = 1$ for s_v a reflection on V).

We can define a group of orthogonal transformations:

Definition 33. The group $O(V)$ denotes the group of all orthogonal transformations on a vector space V.

Since all reflections are orthogonal, we have that any reflection group G is a subgroup of $O(V)$. We now want to look at reflection groups in a different way. To get to this point, we need the following lemma.

Proposition 1. *For $f \in O(V)$ and any non-zero vector $\alpha \in V$, we have that*

$$s_{f\alpha} = f s_\alpha f^{-1}$$

Proof. We are aiming to show that $f s_\alpha f^{-1}$ sends $f\alpha$ to its negative and also preserves the orthogonal complement of $f\alpha$, denoted $(f\alpha)^\perp$, point-wise. Clearly we have that

$$f s_\alpha f^{-1}(f\alpha) = f s_\alpha(\alpha) = f(-\alpha) = -(f\alpha).$$

Now, for any $u \in V$ we have that

$$(f\alpha, fu) = (\alpha, u),$$

meaning that

$$u \in \alpha^\perp \Leftrightarrow fu \in (f\alpha)^\perp.$$

Therefore we have that any element of $(f\alpha)^{\perp}$ must be of the form fu for some $u \in \alpha^{\perp}$, and that

$$fs_{\alpha}f^{-1}(fu) = fs_{\alpha}(u) = fu$$

as required. $\square$

If we suppose that W is a reflection group, we can take an interesting corollary, which we will state without proof.

Corollary 1. *For W a reflection group and some $w \in W$, then $s_{w\alpha} \in W$ if and only if $s_{\alpha} \in W$.*

Hence, for a vector α corresponding to a reflection $s_{\alpha} \in W$ and some $w \in W$, we have that $s_{w\alpha}$ is also a reflection corresponding to some reflection in W. The key message here is that a reflection group W permutes the vectors $\pm\alpha$, corresponding to the reflections it contains. This is our alternative way of thinking about reflection groups.

3.2. Defining a Root System. We now state the definition of a root system.

Definition 34. Let Φ be a finite set of non-zero vectors in V. The set Φ is called a **root system** if

(1) $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \Phi$, here $\mathbb{R}\alpha$ denotes the $\mathbb{R}$-span of α, meaning the line containing α.
(2) $s_{\alpha}\Phi = \Phi$ for all $\alpha \in \Phi$.

Similarly we can define a root.

Definition 35. For a root system Φ, the elements $\alpha \in \Phi$ are called **roots**.

We will look at further examples of roots and root systems as we continue.

3.3. The Group Given by a Root System. There is a clear link between reflection groups and root systems. The following definition formalises this.

Definition 36. For a root system Φ and any root $\alpha \in \Phi$, define a **linear reflection group with root system** Φ as the group W which is generated by the reflections s_{α}.

This is best illustrated with an example, for which we return to the dihedral group.

Example 3.4. The Dihedral Group
We have already seen that the dihedral group is an example of a reflection group. The diagram below illustrates the root system for the dihedral group of order 12, D_6.

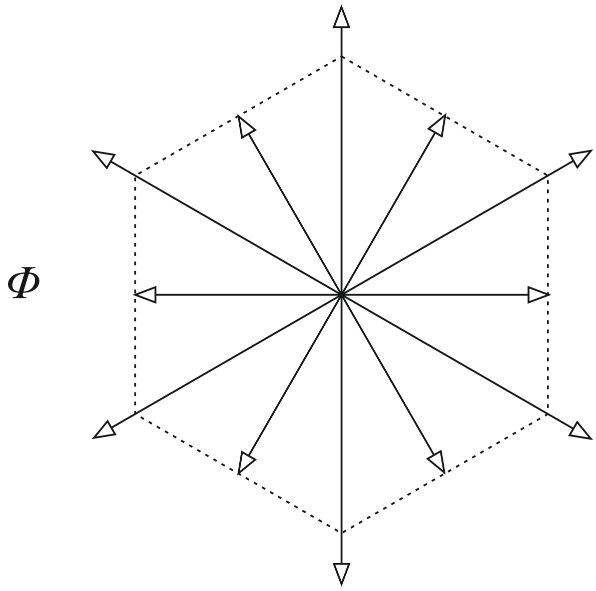


FIGURE 6. A root system Φ for the dihedral group D_6

However, if we were to look for the root system of our affine Weyl group from example 2 in this chapter, we may have some difficulty. This is explained as a consequence of the following theorem.

THEOREM 3. *If W is a linear reflection group with root system Φ, then W is a spherical (finite) reflection group.*

PROOF. If $|\Phi| = n$ then W permutes elements of Φ, meaning there is a homomorphism

$$\theta : W \to \mathcal{S}_n.$$

We can write the vector space V as

$$V = \text{span}\Phi \oplus \text{span}\Phi^{\perp}$$

where w fixes $\text{span}\Phi^{\perp}$ point-wise. If we have that

$$w \in \ker\theta$$

then w must also fix $\text{span}\Phi$ point-wise. Therefore W fixes the whole of the space V so we must have that

$$w = 1.$$

Therefore θ is injective and W is isomorphic to a subgroup of $\mathcal{S}_n$. This gives us that $|W| < \infty$ as required. □

This tells us that if a reflection group has a root system, it must be a spherical reflection group.

3.4. Positive, Negative and Simple Systems. We now continue with the example of the dihedral group D_6. We have just seen that this group has 12 roots; however, when we first introduced the dihedral group at the start of the chapter it was shown with only two generators, not twelve. This leads us to the definition of a positive and a negative vector.

DEFINITION 37. For a reflection group W with root system Φ and a vector $t \in V$ such that for all $\alpha \in \Phi$ we have $(\alpha, t) \neq 0$, a vector $v \in V$ is **positive** if $(v, t) > 0$ or **negative** if $(v, t) < 0$

The concept of positive and negative vectors enables us to define a positive system and a negative system.

DEFINITION 38. For a reflection group W with root system Φ and a vector $t \in V$ such that for all $\alpha \in \Phi$ we have $(\alpha, t) \neq 0$, the set

$$\Pi = \{\alpha \in \Phi : (\alpha, t) > 0\}$$

is called a **positive system**.

DEFINITION 39. For a reflection group W with root system Φ and a vector $t \in V$ such that for all $\alpha \in \Phi$ we have $(\alpha, t) \neq 0$, the set

$$-\Pi = \{\alpha \in \Phi : (\alpha, t) < 0\}$$

is called a **negative system**.

There are two features of importance to note here. Firstly, that Φ is the disjoint union of its positive system, Π, and negative system, $-\Pi$. This is true since

$$(\alpha, t) > 0 \Leftrightarrow -(\alpha, t) < 0 \Leftrightarrow (-\alpha, t) < 0.$$

The second point of note is that it is always the case that positive α lies on the same side of the hyperplane $t^{\perp}$ as t. This is true because we have that $(\alpha, t) > 0$.

We now turn our attention from positive and negative systems, to simple roots and simple systems.

DEFINITION 40. For a positive system Π, a subset $\Delta \subseteq \Pi$ is called a **simple system** if δ is the smallest set such that every element of Π can be written as a linear combination of elements of Δ with non-negative coefficients.

This also provides us with the definition of a simple root.

DEFINITION 41. The elements $\delta \in \Delta$ for a simple system Δ are called **simple roots**.

We will now return to our standing example of the dihedral group to further aid our understanding of what these new definitions provide us.

EXAMPLE 3.5. The Dihedral Group

See the following diagram from our previous example. On this diagram, we can choose any vector t which is not contained within one of the roots.

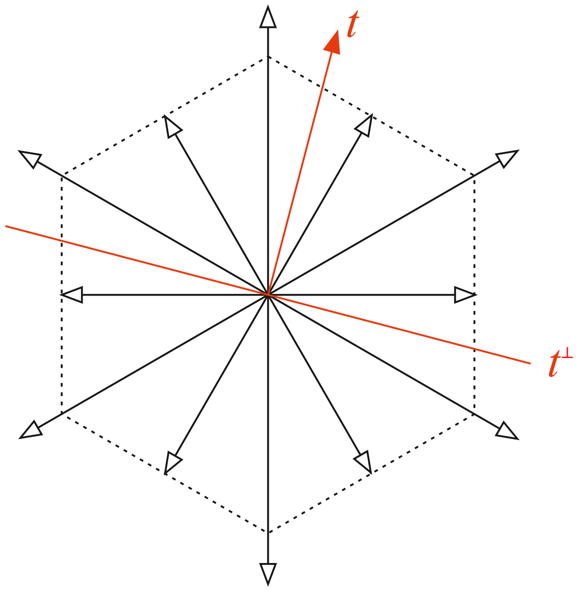


FIGURE 7. A root system Φ for the dihedral group D_6 showing t and $t^{\perp}$

This will give us a positive system (Π) and a negative system ($-\Pi$), as shown in the following figure.

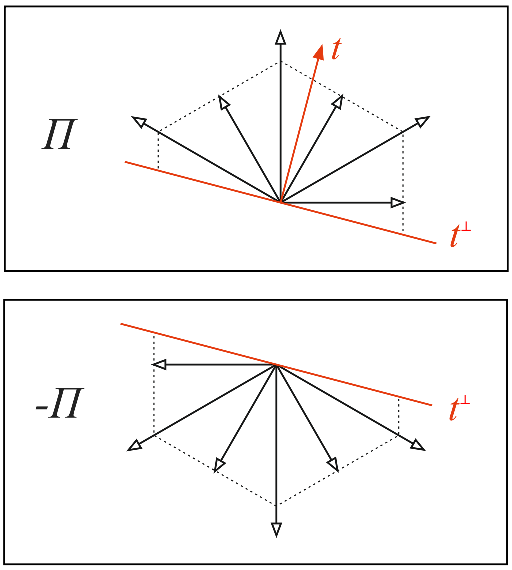


FIGURE 8. A root system Φ for the dihedral group D_6 showing the positive system (Π) and a negative system ($-\Pi$)

Furthermore, we must have that any simple system Δ will have at least two vectors. Since we must have that Δ is the smallest subset of Π such that all $\alpha \in \Pi$ are linear combinations of the elements of Δ with non-negative coefficient, for our choice of t there is only one option for Δ. We must have that $\Delta = \{\alpha_1, \alpha_2\}$ as shown below.

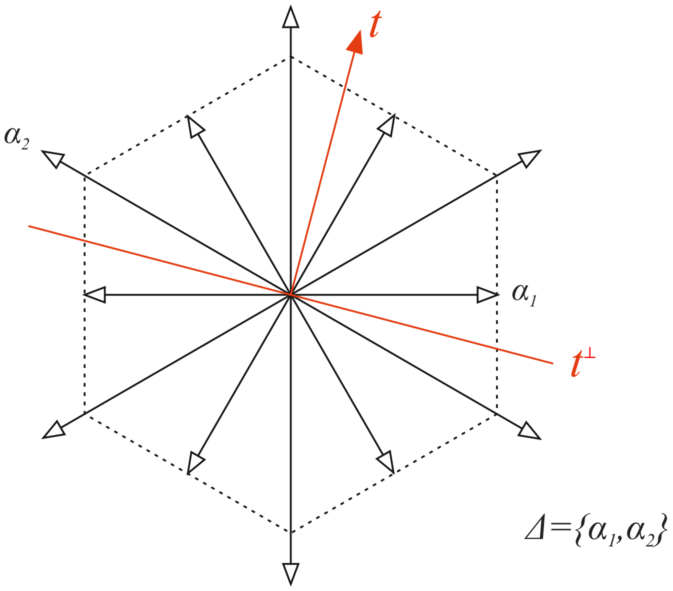


FIGURE 9. The simple system Δ for a root system Φ of the dihedral group D_6

There are several key observation we will make here:

(1) For a root system Φ, the choice of t means that there could be many positive systems $\Pi \subseteq \Phi$ that we could have, however for each Π there is a unique simple system Δ.
(2) $span(\Delta) = span(\Phi)$
(3) For $\Delta = \{\alpha_1, \alpha_2\}$, we have that the reflections $s_{\alpha_1}, s_{\alpha_2}$ generate D_6.

Each of the above observations turn out to be true in general.

4. Classification of Reflection Groups

4.1. Classification. There are many different families of reflection groups. Finite reflection groups have been completely classified up to isomorphism into different families, as can be seen in [**1**][**2**]. We give more details below, however broadly the classification proceeds by observing that each finite refection group is a direct product of certain irreducible factors. We then classify the irriducible finite reflection groups into different '**Types**', each with an associated Coxeter diagram.

Since we also use the word 'type' in the context of galleries in a chamber system, for clarity in this thesis we will distinguish between the two notions by referring to the 'type' of a gallery and the 'Type' of a group.

4.2. Finite Reflection Groups. There are three families of finite reflection groups:

- Type $\mathbf{A}_n$ for $n \geq 1$ is the group of symmetries of a regular n-simplex (or the symmetric group on $n + 1$ elements.
- Type $\mathbf{B}_n$ for $n \geq 2$, sometimes known as Type $\mathbf{C}_n$ is the group of symmetries of the n-cube.
- Type $\mathbf{D}_n$ for $n \geq 4$ is the group that is the Weyl group of a root system. The Weyl group of a root system is a group generated by reflections in hyperplanes that are orthogonal to the roots. This does not correspond to any regular solid.

Then there are seven exceptional groups which are given by the following four Types:

- Type $\mathbf{E}_n$ for $n = 6, 7, 8$ is the Weyl group of the root system of the same name. This group does not correspond to any regular solid. Please see [**1**] for full details.
- Type $\mathbf{F}_4$ is the group of symmetries of a certain self-dual 24-sided regular solid in $\mathbb{R}^4$. The faces of this solid are 3-dimensional octahedra.
- Type $\mathbf{G}_2$ is the group of symmetries of a hexagon, meaning our example D_6 has this Type.
- Type $\mathbf{H}_n$ for $n = 3$ and $n = 4$. When $n = 3$ this is the symmetry group of either the dodecahedron or the icosahedron. For $n = 4$ this is a 120-sided solid in $\mathbb{R}^4$ or alternatively a 600-sided solid in $\mathbb{R}^4$.

4.3. The Dihedral Groups. We have already looked at the dihedral group as a key example thus far. Also notice that for a dihedral group D_n:

- If $n = 2$ then the group is reducible
- If $n = 3$ then the group has Type $\mathbf{A}_2$
- If $n = 4$ then the group has Type $\mathbf{B}_2$
- If $n = 6$ then the group has Type $\mathbf{G}_2$

We denote one final family:

- Type $\mathbf{I}_2(n)$ for $n = 5$ or $n \geq 7$ is the dihedral group of order n.

This concludes the classification of finite reflection groups. For the full classification theorem, the reader should consult [**2**].

5. Words and Reduced Words

As we continue forwards with reflections and begin to turn our mind to Coxeter groups in the following chapter, we want to formalise the concept of a 'word'. We first encountered the terminology of a 'word' back in chapter 1, used to define the type of a gallery. In this chapter we briefly return to the concepts of a word to clarify and expand on the notion. We will focus more deeply on the further related definitions and theorems regarding words once we have covered Coxeter Systems in the following chapter. Our key sources of material for this section are [**7**], [**11**] and [**1**].

5.1. Reduced Words. We begin with some key definitions.

Definition 42. For some set I the **free monoid on** I, denoted M_I, is the set of all finite strings of elements of I, called 'words', with the binary operation of concatenation. The identity element is the empty word, denoted $\emptyset$.

Given a group G with generating set S, an element $s_1, s_2, \ldots s_n \in G$ is set to correspond with the word $s_1, s_2, \ldots s_n \in M_I$, with $1 \in G$ corresponding to $\emptyset$.

Whilst we may not always be so formal, and sometimes for convenience denote the word $1, 2, \ldots n$ as the expression $s_1, s_2, \ldots s_n$. There is a distinction to be made between a word $w \in M_I$ and the element $g = s_1 \ldots s_n \in G$ which it represents. We now define the length of a word.

Definition 43. For a group G with generating set S, the **length** of a word $w \in M_I$ with respect to S, denoted $l(w)$, is the least $n \in \mathbb{Z}$ such that the word w has the expression

$$w = (s_1, \ldots, s_n)$$

such that $s_i \in S$.[**7**]

Using the definition of length, we define what it means for a word to be reduced.

Definition 44. For a group G with generating set S and the word $w = (s_1, \ldots, s_n)$ corresponding to some $g \in G$, say that w is **reduced** with respect to S if and only if $n = l(w)$. If w is reduced, call the expression $(s_1, \ldots, s_n)$ the **reduced decomposition** of g.[**7**]

To put this another way, a word is reduced when there is no shorter word which can represent it.

CHAPTER 4

Coxeter Systems

With an understanding of reflection groups, roots, chambers and words behind us, we move to look at Coxeter groups and Coxeter systems. A Coxeter group is essentially a generalisation of a reflection group. We begin by providing a definition for the Coxeter group before we start to link everything together that we have covered thus far.

1. Coxeter Groups

1.1. Refection Groups to Coxeter Groups. Recall from the previous chapter on reflection groups the example of the dihedral group D_6. As we noted in example 3.1., the group could be generated using only 2 reflecting lines, s_1 and s_2. Of key importance was a remark which we made at the end of example 3.1., that the group could be given with presentation

$$D_6 = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^6 = 1 \rangle.$$

We wish to understand and generalise some key properties of the reflection group to take us to a Coxeter group.

Firstly we must examine the properties given by the reflections s_1 and s_2. We now choose to describe a generating set $S = \{s_1, \ldots, s_n\}$ where the elements are reflections for some reflection group W. Notice that since each s_i is a reflection, we have for all s_i such that $1 \leq i \leq n$ that s_i has order 2 in W. Also notice that there is a relation for $s_1, s_2 \in S$ such that $(s_1 s_2)^6 = 1$. Both of these facts cause the group to have the presentation in the form that we see above.

Secondly, consider the following figure showing the reflections s_1 and s_2 (as previously defined for our example of D_6) and fixing one of the triangular segments labelled I.

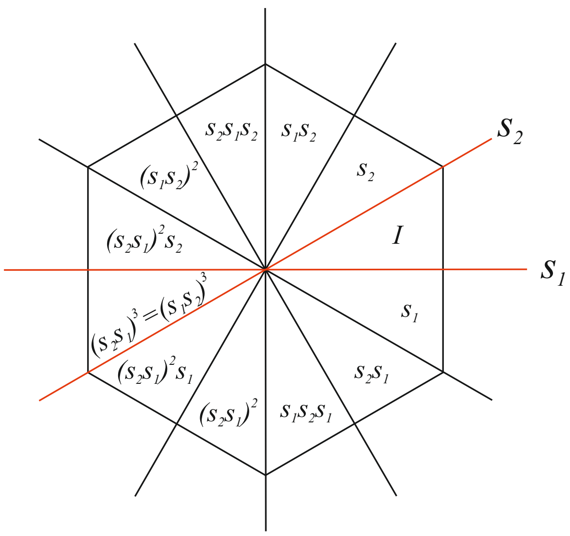


FIGURE 1. A hexagon with two reflecting lines $s_1, s_2 \in S$, a fixed triangle I and all other triangles labelled via the reflections and rotations which move I to that triangle

By fixing our triangle I as the triangle which the reflecting lines in S are the edges, we are able to generate all 12 elements of D_6 on our hexagon.

1.2. Defining a Coxeter System. We will now use the observations made about D_6 to motivate the definition of a Coxeter group. The following definition is based on those found in [**6**] and [**1**].

Definition 45. A **Coxeter System** (W, S) is a group W with generating set S such that every generator $s \in S$ has order 2 with respect to W, and W admits a presentation

$$\langle s \in S \mid (s_i s_j)^{m_{i,j}} = 1 \rangle$$

where $m_{i,j} \in \mathbb{Z}^{\geq 1}$ and is such that $m_{i,j} = m_{j,i}$, and $m_{i,j} = 1$ if and only if $i = j$.

Sometimes we refer to the group W as a Coxeter group and (W, S) as the Coxeter system. We will see the importance of the choice of S later in this chapter. Clearly our dihedral group above satisfies this definition, so is therefore not only a reflection group but also a Coxeter group. We can use the $m_{i,j}$ as given in the definition above to state the definition of a Coxeter matrix.

Definition 46. For a Coxeter group (W, S) with generating set $S = \{s_1, \ldots, s_n\}$, the associated **Coxeter matrix** is an $n \times n$ matrix with $m_{i,j}$ corresponding to the entry (i, j) such that:

- $(i, j) = (j, i)$
- (i, j) is either a positive integer or the symbol ∞ when $(s_i s_j)$ has infinite order in W
- $(i, j) = 1$ if and only if $i = j$.

[**15**]

Clearly this definition gives us a symmetric matrix with 1's on the main diagonal.

Finally we state the following theorem without proof.

Theorem 4. *All reflection groups are Coxeter groups.*

This result was a key contribution of Coxeter. A full proof of this theorem can be found in Coxeter's 1935 paper [**3**].

2. The Associated Chamber System

In this section we return to our material on chamber systems. Indeed we will look at our canonical example (example 1.1.) to observe how we can move from our initial chamber system to a reflection group, and indeed a Coxeter group. This section returns to follow the method used in [**6**] to realise the chamber system.

2.1. Realising a Chamber System. In this section, we are going to start with our chamber system and see that it corresponds to a Coxeter system. This is not true for every chamber system and usually we would look in the opposite direction where we are able to find a chamber system starting from a reflection group.

Recall our canonical example of a flag system of a vector space. When we took the flag system of a 3-dimensional vector space over the field of order 2, we obtained an image of a chamber system Δ as shown in the following figure, which we repeat from a previous chapter. Here the chambers are the edges and are i adjacent for $I = \{1, 2\}$ whenever two edges meet at a node. For $c_1, c_2 \in \Delta$, if $c_1 \sim_1 c_2$ we will denote this with a white node. If $c_1 \sim_2 c_2$ then we will denote this with a black node.

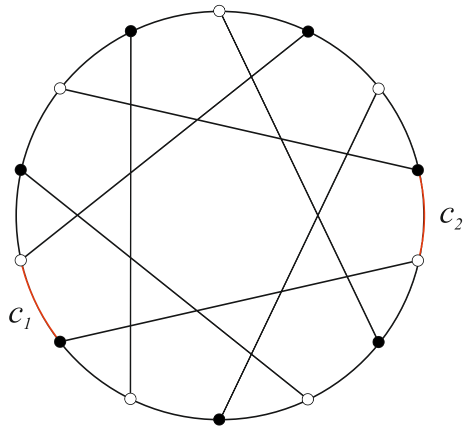


FIGURE 2. The chamber system Δ of a flag system of a 3-dimensional vector space over the field of order 2 (from example 1.1) with chambers c_1 and c_2 labelled

We now make an observation about this chamber system. Notice that we are able to set $c_1, c_2 \in \Delta$ to be any two distinct chambers and always find a closed loop of six chambers $x_1, x_2, x_3, x_4, x_5, x_6 \in \Delta$ with

$$x_1 \sim_1 x_2 \sim_2 x_3 \sim_1 x_4 \sim_2 x_5 \sim_1 x_6 \sim_2 x_1$$

such that $c_1, c_2 \in \{x_1, x_2, x_3, x_4, x_5, x_6\}$. In other words, any two chambers are contained in a common hexagon of chambers. We will call this the local picture of Δ, as shown in the following figure.

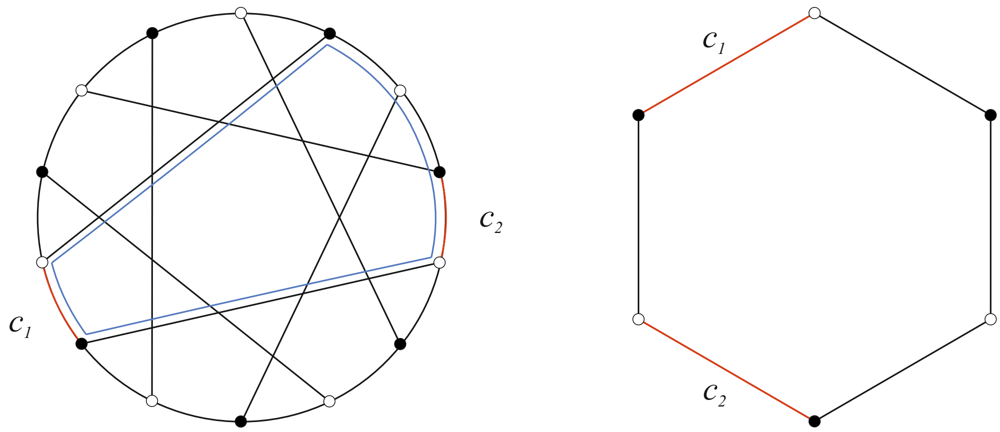


FIGURE 3. The local picture Δ for two chambers c_1 and c_2

We want to get a reflection group from this chamber system. Already we have that there is a set $I = \{1, 2\}$ for the adjacency between chambers. Looking at the example above, we can also see that there are two possible routes between c_1 and c_2. If we were instead to think of the adjacency between chambers as reflections s_1 and s_2, we can say that

$$s_2 s_1 = s_1 s_2 s_1 s_2.$$

We also have that

$$s_1^2 = s_2^2 = 1$$

and by placing the two chambers on opposite sides of the hexagon, that

$$s_1 s_2 s_1 = s_2 s_1 s_2.$$

Let us consider what these reflections are doing. We are taking some chamber c_1 and through reflections we get to another chamber c_2. In fact we are using the reflections as a way of permuting the six chambers. We can think of taking some $g \in W$, where W is our reflection group, such that g sends c_1 to c_2. Clearly if we are looking at a group which is permuting a set of 6 elements then we are talking about the symmetric group $\mathcal{S}_3$. We can think of redrawing our local picture of Δ to show the reflecting lines s_1 and s_2, as shown in the following figure. We show this on the triangle to emphasise that we are talking about $\mathcal{S}_3$, with the reflections permuting the vertices of the triangle.

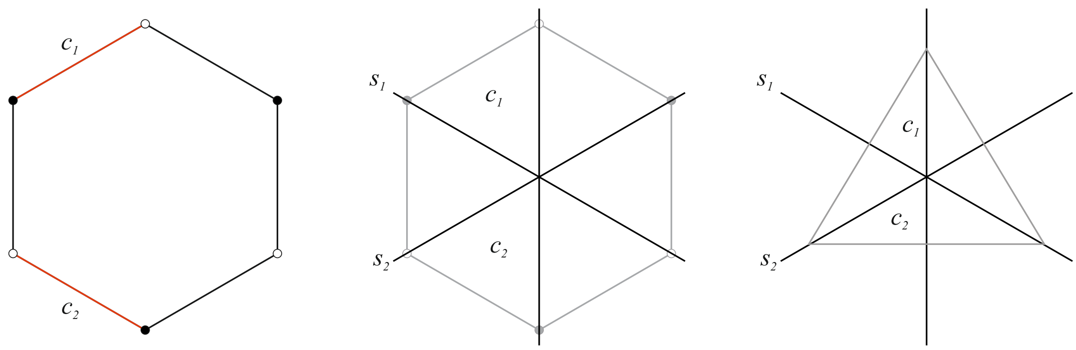


FIGURE 4. The local picture Δ shown with reflections s_1 and s_2 to realise the symmetric group $\mathcal{S}_3$

2.2. Chambers, Panels and Adjacency. We now formalise what is happening in the previous section and reintroduce the language of chamber systems in this new setting. We will continue to follow our example of the flag system in diagrams, but will also speak more generally. This approach uses root systems to follow a similar method in [**6**]

Let V be a Euclidean vector space and W be a reflection group with index set I. Let W be generated by reflections $S = \{s_i \mid i \in I\}$ with a corresponding set of hyperplanes $\mathcal{H} = \{H_i\}_{i \in I}$. Also have that $W\mathcal{H} = \mathcal{H}$ giving that W is finite. We are going to consider the root system Φ of W. The following figure shows a root system for our previous case with 3 hyperplanes.

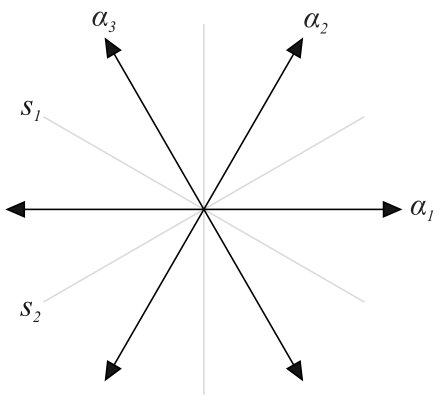


FIGURE 5. A root system Φ of W

We are going to use the roots α_i for $i \in I$ to each define half spaces dividing V. Here (v, α_i) denotes the Euclidean distance of v and α_i. We will use roots to provide partitions of V such that for each hyperplane $H_i \in \mathcal{H}$:

- H_i consist of all v with $(v, \alpha_i) = 0$
- One side of the half space of H_i consists of all v such that $(v, \alpha_i) > 0$
- One side of the half space of H_i consists of all v such that $(v, \alpha_i) < 0$

We now fix an I-tuple, $\epsilon = (\epsilon_i)_{i \in I}$ with $\epsilon_i \in \{\pm 1\}$, and consider a set

$$c = c(\epsilon) = \{v \in V \ : \ \epsilon_i(v, \alpha_i) > 0 \text{ for all } i\}.$$

If this set is non-empty then we say that it is a chamber of W. Furthermore, we say that a panel is a non-empty set of the form

$$a = a(\epsilon) = \{v \in V \ : \ (v, \alpha_{i_0}) = 0 \text{ for some } i_0 \in I \text{ and } \epsilon_i(v, \alpha_i) > 0 \text{ for all } i \neq i_0\}.$$

We return to our running example. We will denote the panels and chambers by their I-tuples. In the following figure, we can see the 6 chambers realised over the group W.

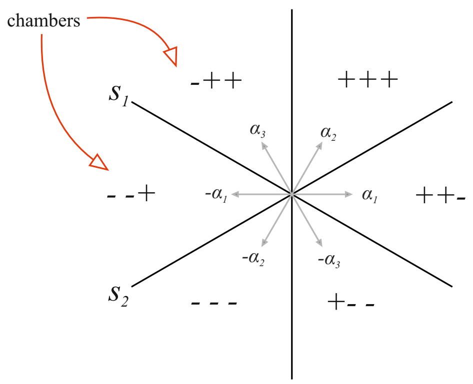


FIGURE 6. Chambers shown on the group W

Furthermore, the following figure shows the 6 panels.

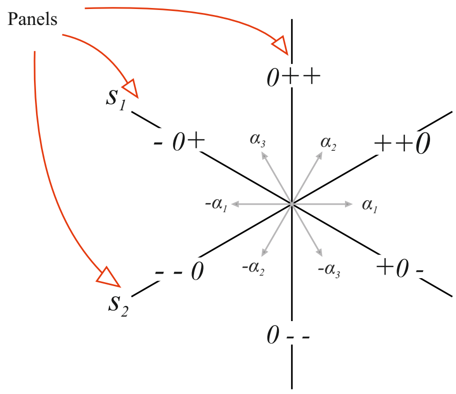


Figure 7. Panels shown on the group W

Note that there are possible I-tuples which are not shown in the two previous figures. This is common, such I-tuples denote empty sets. In our example we have 6 chambers, 6 panels and 8 empty sets.

This construct gives a clear notion of the i-adjacency of chambers. We previously had that two chambers are adjacent if they share a common face. We can specify this definition to say the following.

Definition 47. Two chambers are **adjacent** if they share a common panel.

2.3. Regular Action of W. As we have just seen, we can form a chamber system using reflections. This led to chambers being separated by panels such that two chambers sharing a panel are said to be adjacent. We mentioned earlier that for some $c_1, c_2 \in \Delta$, we can find a $g \in W$ such that we have $c_1 = gc_2$. It turns out that this g is unique for our symmetric group example, and is formalised with the following definition.

Definition 48. For a chamber system Δ and a group W acting on Δ, say that W has a **regular group action** on Δ if for any $c_1, c_2 \in \Delta$ there exists a unique $g \in W$ such that $c_1 = gc_2$.

2.4. The Coxeter Chamber System. A key example of a chamber system which we are now able to state is that of the Coxeter chamber system. For any Coxeter system (W, S) with $S = \{s_i\}_{i \in I}$ the generating set of reflections and index set I, it is possible to construct a chamber system. The chambers in this building are the elements of W. For any two chambers x, y in the chamber system, we define i-adjacency as

$$x \sim_i y \Longleftrightarrow x^{-1}y = s_i.$$

By convention we will denote this chamber system Σ_Π where the Coxeter system is of Type Π. Here, Π refers to the Coxeter diagram of the Coxeter system, which we will define in the following section.

3. Coxeter Systems

We now return to our definition of a Coxeter system, which we denoted (W, S). Also recall from the previous chapter on reflection groups that we included the classification of reflection groups (up to isomorphism). In this section we will start by introducing some new terminology regarding Coxeter groups which we will then use when we look at the classification of Coxeter groups.

3.1. Coxeter Diagrams and Classification. The Coxeter diagram is a very visual way of working with Coxeter Groups. The diagram contains information about the Coxeter matrix M of a Coxeter group (W, S), and is defined as follows:

Definition 49. [**1**] For a Coxeter system (W, S) with index set $I = \{1, \ldots, n\}$ and Coxeter Matrix M, the **Coxeter diagram** is a graph vertex set I and an edge connecting vertices $i, j \in I$ if and only if $m_{i,j} \geq 3$. By convention:

- If $m_{i,j} = 3$ the edge (i, j) has no label
- If $m_{i,j} \geq 4$ the edge (i, j) is labelled with the value $m_{i,j}$

Note that the lack of an edge joining two vertices indicates that $m_{i,j} = 2$. We can use Coxeter diagrams to show differences between Coxeter systems. In the final example from the previous section, we used the Coxeter diagram to inndicate the Type of the Coxeter group. As we will see in the next section, each Type of Coxeter group is linked to a Coxeter diagram. Key information about the system is contained in the diagram, as demonstrated by the following definition.

Definition 50. A Coxeter System (W, S) is **irreducible** if and only if the Coxeter diagram for (W, S) is a connected graph.

We will bring up Coxeter diagrams in our examples later in this chapter, however one of the most visual demonstrations of the Coxeter diagram in action is to see our classification for irreducible reflection groups shown by their Coxeter diagram. We will see this in the following subsection.

3.2. Classification of Finitely Generated Coxeter Groups. A great deal can be said about a Coxeter group from its diagram. In Coxeter's 1935 paper [**4**], he gives a complete classification all finite irreducible Coxeter systems according to their Coxeter diagrams. In an earlier paper [**3**], Coxeter also gives a classification for infinite irreducible Coxeter groups according to their symbol.

The following figure shows the Coxeter diagrams for all families of irreducible finite reflection groups, which we last saw in the classification of reflection groups. This figure is based on a similar figure found in [**1**].

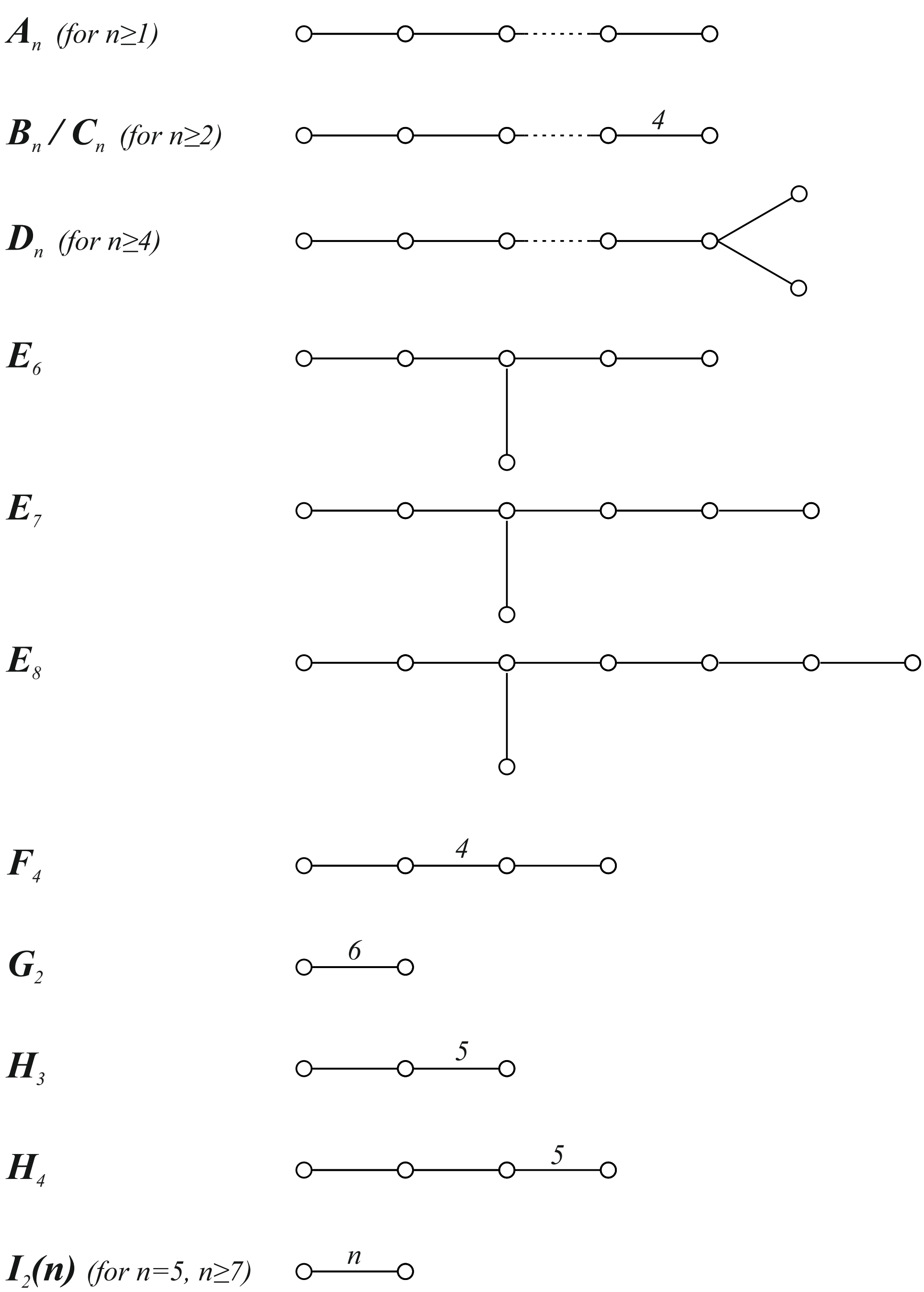


FIGURE 8. Finite reflection group classifications given by their Coxeter diagram [**1**]

3.3. The Choice of S. Earlier we remarked that the purpose of denoting a Coxeter system (W, S) was to emphasise the importance of the choice of the set S. Coxeter diagrams are a simple way to show the consequence of the choice of S. It is possible for two Coxeter groups to have the same group W with different generators S. These groups would then be isomorphic.

A simple illustration of this is to consider our example of the dihedral group D_6. As we have already noted, D_6 is a reflection group of Type $\mathbf{G}_2$ with 2 generators. However, we can also take D_6 to be a group given by the direct product of two groups of Type $\mathbf{A}_1 \times \mathbf{A}_2$ which has 3 generators. This is illustrated by the groups' Coxeter diagrams, as shown in example 4.1. below.

4. Examples of Coxeter Systems

The classification of reflection groups has already raised some examples of reflection, and therefore Coxeter systems. In this section we look at some specific examples to aid clarity and understanding.

4.1. Finite Coxeter Groups. These examples are finite groups formed by reflections. We will formalise the two groups that we have already used as examples in this section.

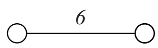
Example 4.1. The dihedral group D_6 with generators $S = \{s_1, s_2\}$ is a Coxeter system (D_6, S) of Type $\mathbf{G}_2$ with presentation

$$(D_6, S) = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^6 = 1 \rangle.$$

This Coxeter system has Coxeter matrix

$$M = \begin{pmatrix} 1 & 6 \\ 6 & 1 \end{pmatrix}$$

and Coxeter diagram as shown below.



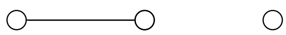
This group is isomorphic to the group D_6 with generators $S' = \{s_1', s_2', s_3'\}$ which is a Coxeter system (D_6, S') of Type $\mathbf{A}_2 \times \mathbf{A}_1$. This Coxeter system admits a presentation

$$(D_6, S') = \langle s_1', s_2', s_3' \mid s_1'^2 = s_2'^2 = s_3'^2 = (s_1' s_2')^3 = (s_1' s_3')^2 = (s_2' s_3')^2 = 1 \rangle,$$

and has Coxeter matrix

$$M = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

and has the following Coxeter diagram.



We note for emphasis that whilst the reflection group D_6 is the same in both examples, the Coxeter systems are different. The two Coxeter systems (D_6, S) and (D_6, S') admit completely different presentations and have different Coxeter diagrams. This is the importance of noting the generating set S when giving a Coxeter system.

It is useful to see how we can relate between (D_6, S) and (D_6, S'). For (D_6, S') notice that the element s'_3 commutes the other generators (such that $(s'_1 s'_3)^2 = 1$ and $(s'_2 s'_3)^2 = 1$). This means that s'_3 must be the centre of the group, and we have that $s'_3 = (s_1 s_2)^3$. Since we have that s'_1 and s'_2 must be reflections with $(s'_1 s'_2)^3 = 1$, we can choose these new generators such that $s'_1 = s_1$ and $s'_2 = s_2 s_1 s_2$.

Now that this is fixed, we can write s_2 in terms of the new generators:

$$s_2 = s_2(s_1 s_1) = (s_2 s_1)s_1 = (s_1 s_2)^5 s_1 = (s_1 s_2)^3 s_1(s_2 s_1 s_2)s_1 = s'_3 s'_1 s'_2 s'_1.$$

This enables us to rewrite all elements of D_6 in terms of generators in S'.

We can imagine a picture of this isomorphism by considering the picture of a six pointed star inside a hexagon, with the points of the star at the vertices of the hexagon. We have that the rotation of s'_3 swaps the two triangles which make up the star and then there is a copy of the symmetric group $\mathcal{S}_3$ generated by s'_1 and s'_2 which permutes the vertices of each triangle amongst themselves. This leads us on to our next example.

Example 4.2. The Symmetric Group

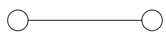
The symmetric group $\mathcal{S}_3$ of order six with generating set $S = \{s_1, s_2\}$ admits a presentation of the form

$$(\mathcal{S}_3, S) = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^3 = 1 \rangle.$$

This Coxeter System, $(\mathcal{S}_3, S)$ has Coxeter matrix

$$M = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

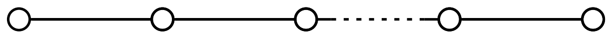
and Coxeter diagram as shown below.



Example 4.3. We can give a generalised presentation for the Coxeter system $(\mathcal{S}_{n+1}, S)$ with generating set S, as

$$(\mathcal{S}_{n+1}, S) = \langle s_1, s_2, \ldots, s_n \mid s_i^2 = (s_i s_{i+1})^3 = (s_i s_j)^2 = 1 \text{ for all } j \neq i+1 \rangle.$$

The Coxeter system $(\mathcal{S}_{n+1}, S)$ is of Type $\mathbf{A}_n$ with the following Coxeter diagram (with n nodes). This clearly holds true for the special case of $\mathcal{S}_3$ shown above which is of Type $\mathbf{A}_2$.



4.2. Infinite Coxeter Groups. At the end of chapter 2 on Geometries, we have already seen several examples of ways that we can tessellate the plane to obtain specific triangles. The motivation for this subsection is to take three lines which intersect each other to form the boundary of a triangle to be generating reflections s_1, s_2, s_3 which tessellate the plane.

Consider a group generated by a finite set of reflecting lines $\mathcal{H}$ in the Euclidean plane $\mathbb{R}^2$. We set a 'fundamental region' of the plane as the geometric shape contained as the interior of the intersections of these reflection lines. For this example we are working with the assumption that there is an interior shape, so the reflections are not linear, and the set of reflection lines are minimal (in the sense that removing any one reflection line will change the fundamental region). We are interested in reflections which form a reflection group giving a complete tiling of the Euclidean plane in copies of the fundamental region.

It turns out that for the 2-dimensional Euclidean plane, there are only four possible tilings which do this. These are denoted $(n; m_1, \ldots m_n)$ where n is the number of reflecting lines generating the tiling and the m_i are such that $\frac{\pi}{m_i}$ are each of the n internal angles of the fundamental shape; we will provide a brief explanation for this shortly. We now state the Coxeter systems coming from regular tilings of the Euclidean plane, with notes on isomorphic groups from [**9**].

- $(3; 2, 4, 4)$, a Coxeter system isomorphic to the semidirect product $\mathbb{Z}^2 \rtimes D_8$
- $(3; 3, 3, 3)$, a Coxeter system isomorphic to the semidirect product $\mathbb{Z}^2 \rtimes D_{12}$
- $(3; 3, 3, 3)$, a Coxeter system isomorphic to the semidirect product $\mathbb{Z}^2 \rtimes S_3$
- $(4; 2, 2, 2, 2)$, a Coxeter system isomorphic to the semidirect product $\mathbb{Z}^2 \rtimes D_4$

Note that the term semidirect product $G = \mathbb{Z}^2 \rtimes D_8$ means that $\mathbb{Z}^2$ is a normal subgroup of the reflection group G and D_8 is a subgroup. For every element $g \in G$ there exists unique $x \in \mathbb{Z}^2$ and $x' \in D_8$ such that $g = xx'$. We return to our claim that these are the only possible Coxeter systems possible from tiling $\mathbb{R}^2$. We will not state an entire proof, however the reader should be motivated that this is true from the following proposition.

Proposition 2. *There are exactly* 3 *regular tilings of Euclidean* $\mathbb{R}^2$*, namely the triangle, square and hexagon tilings.*

Proof. Any regular tiling is achieved using a regular n-gon. Taking any regular n-gon in euclidean $\mathbb{R}^2$, we must have that for the choice of $n \in \mathbb{Z}$, that $n \geq 3$ to give an enclosed area with which to tile $\mathbb{R}^2$. As already noted in section 2.1, in Euclidean $\mathbb{R}^2$ we have that for the sum of the internal angles of an n-gon, denoted $\sum \sphericalangle$, satisfies

$$\sum \sphericalangle = \pi(n-2).$$

Since the n-gon used to tile the plane is a regular n-gon, we must therefore have that each internal angle, $\sphericalangle$, satisfies

$$\sphericalangle = \frac{\pi(n-2)}{n}.$$

We can therefore say that at any point, p, where the regular shapes in the tiling meet, that

$$p\frac{\pi(n-2)}{n} = 2\pi \Rightarrow p = \frac{2n}{n-2}.$$

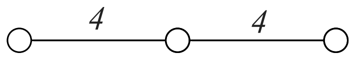
Notice that $\lim_{n\to\infty} \frac{2n}{n-2} = 2$. Taking a brute force approach, we see that

- when $n = 3$ we get $p = 6$, giving the tiling by a regular triangle;
- when $n = 4$ we get $p = 4$, giving the tiling by a square;
- when $n = 5$ we get $p = \frac{10}{3} \notin \mathbb{Z}$, giving no tiling; and
- when $n = 6$ we get $p = 3$, giving the tiling by a regular hexagon.

Since there exists no $p \in \mathbb{Z}$ such that $2 < p < 3$, these are all possible tilings of Euclidean $\mathbb{R}^2$ by regular n-gons. □

Finding appropriate reflection groups for each of these regular shapes yields the Coxeter systems mentioned above.

Example 4.4. We now turn our attention to the tiling $(3; 2, 4, 4)$ of the Euclidean plane as shown in the figure 9. This example uses the square to tile the plane, which we split into non-regular triangles.



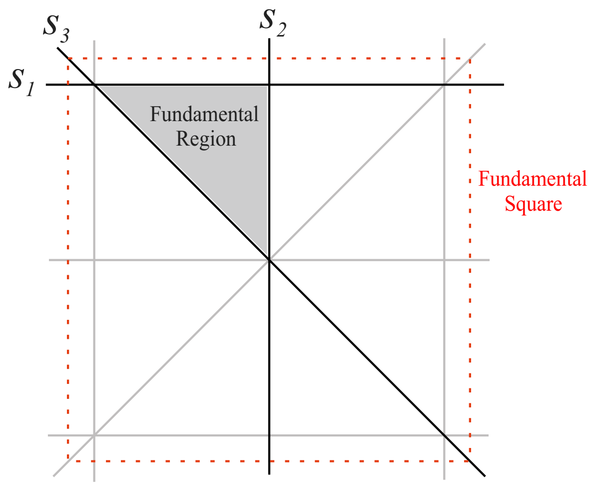


FIGURE 9. Tiling of Euclidean $\mathbb{R}^2$ by reflecting lines s_1, s_2, s_3 showing a fundamental region and fundamental square

We will begin by stating that if we consider the fundamental region enclosed by these reflections as our fixed region 1, we clearly have that $s_1^2 = s_2^2 = s_3^2 = 1$. Also with figure 10 to aid us, we can clearly see that we have $(s_1 s_2)^2 = (s_1 s_3)^4 = (s_2 s_3)^4 = 1$.

We therefore have a Coxeter system (W, S) which admits a presentation

$$(W, S) = \langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^2 = (s_1 s_3)^4 = (s_2 s_3)^4 = 1\rangle$$

with Coxeter matrix

$$M = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 4 \\ 4 & 4 & 1 \end{pmatrix}$$

and Coxeter diagram as follows.

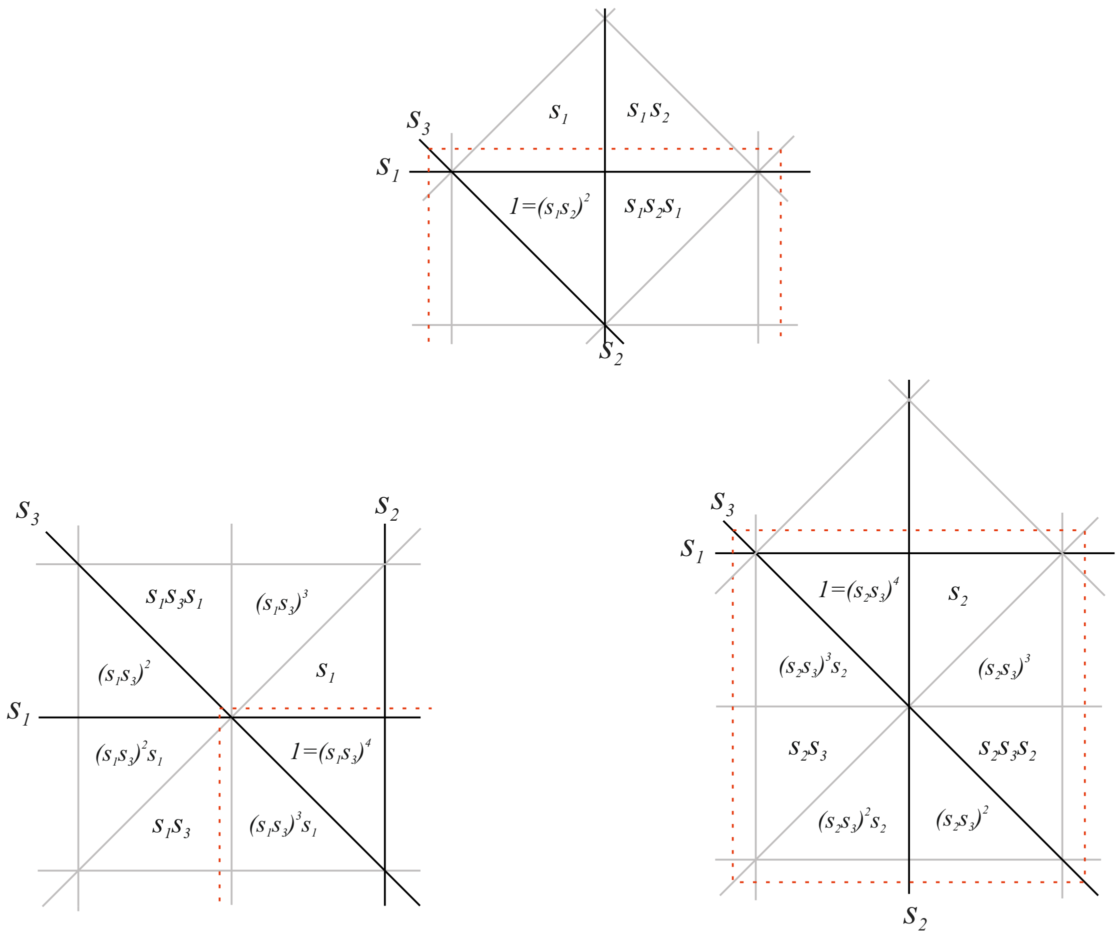


FIGURE 10. Tiling of Euclidean $\mathbb{R}^2$ showing the relations $(s_1s_2)^2 = (s_1s_3)^4 = (s_2s_3)^4 = 1$

Since we are considering translations of the triangle, it is obvious that the triangle must have infinite order so the group must have infinite order. We also stated above that this is a group isomorphic to the semidirect product $\mathbb{Z}^2 \rtimes D_8$, which is infinite. This statement should look a little clearer now as we see that our fundamental square has been divided into 8 segments. Clearly by just using s_1 and s_2 we are able to get the group D_8. The inclusion of a further reflecting line s_1 opens this out to tessellate the plane.

An interesting question to now ask is that given any triangle on the tiling of the Euclidean plane, is it possible to generalise a mapping from the fundamental region to a specified triangle using only our reflections s_1, s_2 and s_3?

This is possible. We can already see from the previous figures that the reflections rotate the fundamental region around the fundamental square. All that remains is to find a way to translate one square to another. We must find an expression for the reflections a, b, c, d, reflecting in the 4 edges of the fundamental square and also e, f, reflecting through the horizontal and vertical symmetry lines of the fundamental square, using only combinations of s_1, s_2 and s_3. The required reflections are shown in the following diagram.

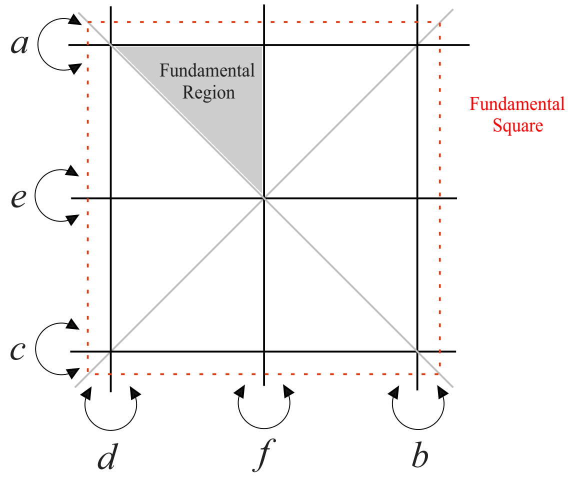


FIGURE 11. Reflections a, b, c, d, e, f required to translate the fundamental square in all directions

It is relatively straightforward to obtain that:

- $a = s_1$
- $b = s_2 s_3 s_1 s_2 s_3$
- $c = (s_3 s_2)^2 s_1 (s_3 s_2)^2$
- $d = s_3 s_1$
- $e = s_3 s_2 s_3$
- $f = s_2$

Combining these reflections, we are able to write a translation by one square up (U), down (D), left (L) or right (R) from the fundamental square.

$$U = s_2 s_1 s_3 s_2 s_3 s_1$$
$$D = s_3 s_2 s_3 s_1$$
$$L = s_3 s_1 s_3 s_2$$
$$R = s_2 s_3 s_1$$

We can now state a formula for a generalised map to any triangle in this tessellation of $\mathbb{R}^2$ from the fundamental region. A triangle which is n_1 squares up, n_2 squares down, n_3 squares left and n_4 squares right of the fundamental region would be obtained by the reflections

$$n_1 U + n_2 D + n_3 L + n_4 R \qquad \text{(+ required rotation within fundamental square)}$$

thus enabling us to map any part of the tiled Euclidean plane to any other part.

4.3. Reflections on a Hyperbolic Plane. We briefly touch on hyperbolic Coxeter systems. In chapter 2, we discussed the different triangles that can be generated by familiar geometric shapes on a plane in each of Euclidean, spherical and hyperbolic geometries. Recall that we were able to specify that a specific set of triangles was possible in each of spherical and Euclidean spaces, owing to the restriction placed from the sum of interior angles on the triangle. We had that most triangles could not be obtained in the spherical or Euclidean spaces and were found instead as tessellations of the hyperbolic plane.

We state without proof that there are an infinite number of hyperbolic Coxeter groups on $\mathbb{H}^2$ and will provide only one example for illustrative purposes.

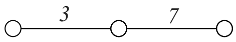
Example 4.5. There exists a Coxeter system (W, S) in $\mathbb{H}^n$ with presentation

$$(W, S) = \langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^2 = (s_1 s_3)^3 = (s_2 s_3)^7 = 1 \rangle$$

with Coxeter matrix

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 7 \\ 3 & 7 & 1 \end{pmatrix}$$

and Coxeter diagram as shown below.



It turns out that we can use the Poincaré disc model to show the reflections of a triangle on the plane which generate this Coxeter system. This image is taken from [**17**].

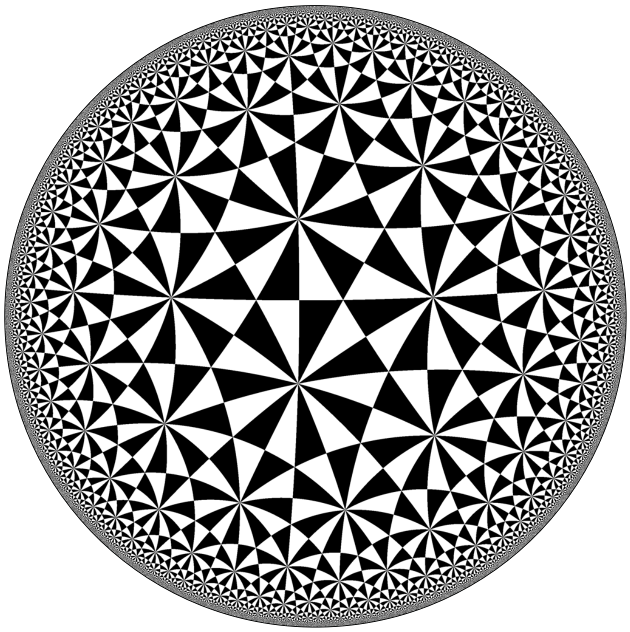


Figure 12. Reflections in $\mathbb{H}^n$ giving the hyperbolic Coxeter group (W, S) [**17**]

5. Words

In this next section we return to look at the concept of words, which we defined in chapter 3. This is our final topic before the grand reveal of the building, and links closely in with the next chapter. This section will use standard definitions shown in [**11**] and [**15**].

Recall definition 42 for which describes the free monoid M_I on a set I and the definition of a word in the M_I. We will use the concept of a word to see how homotopies, expansions and contractions can transform one word into another.

5.1. Homotopy, Expansions and Contractions. For a Coxeter System (W, S), let $[m_{ij}]$ be a Coxeter matrix with an index set I. We define the map $P(i, j)$, for any $i \in I$, as:

$$P(i, j) = \begin{cases} (ij)^{\frac{m_{ij}}{2}} & \text{if } m_{ij} \text{ is even} \\ j(ij)^{\frac{m_{ij}-1}{2}} & \text{if } m_{ij} \text{ is odd} \end{cases}$$

for all ordered pairs with distinct i, j such that $m_{ij} < \infty$. The point of defining $P(i, J)$ is that it always gives a word of length $m_{i,j}$. For example if $m_{i,j} = 3$ we have $P(i, j) = jij$. We now use the definition of $P(i, j)$ to provide the definition of an elementary homotopy of galleries in a building. Recall that in the language of galleries, the type of a gallery was defined in chapter 1 as the word $f \in M_I$ taken by denoting each $i \in I$ which chambers are adjacent in the route of the gallery.

For some word $f \in M_I$ such that $f = i_1 i_2 \ldots i_k$, set $s_f = s_{i_1} s_{i_2} \ldots s_{i_k}$ in the Coxeter system (W, S). Set $r_\emptyset = 1$ where $\emptyset$ denotes the empty word. Finally, we notice that given $c_1, c_2 \in (W, S)$, there is a gallery of type f from x to y if and only if $x^{-1}y = s_f$. We will come back to this fact in the following chapter. We use the above definitions and observation to define an elementary homotopy.

Definition 51. An **elementary homotopy** is an alteration from a word of the form $f_1 p(i, j) f_2$ to the word $f_1 p(j, i) f_2$.

We the use this definition to define what it means for two words to be homotopic.

Definition 52. Two words $f, g \in M_I$ are said to be **homotopic**, denoted $f \simeq g$, if one can be transformed into the other by a finite sequence of elementary homotopies.

Notice that two homotopic words will always have equal length. To transform a word into another word with different length, one will require a contraction or an expansion. First we define a contraction.

Definition 53. An **elementary contraction** is the alteration from a word of the form $f_1 ii f_2 \in M_I$ to the word $f_1 f_2 \in M_I$.

And similarly we can define an expansion.

Definition 54. An **elementary expansion** is the alteration from a word of the form $f_1 f_2 \in M_I$ to the word $f_1 ii f_2 \in M_I$.

Combining all of the above gives us the definition for the equivalence of words.

Definition 55. Two words $f, g \in M_I$ are said to be **equivalent** if f can be transformed to g by a finite sequence of elementary homotopies, contractions or expansions.

This equivalence between words is extremely powerful and enables use to move between words with relative ease. This all brings us to the important definition of a reduced word.

Definition 56. A words $f \in M_I$ is said to be **reduced** if f cannot be transformed into a word of the form $f_1 ii f_2$ by a sequence of elementary homotopies.

Having reduced words enables us to specify a word of minimal possible length with which to work. This will be extremely useful in the following chapter. In the following chapter we will show that if two reduced words are equivalent, then they are homotopic. For now, the best that we can do is the following, which we sketch a proof for.

Proposition 3. *Two words $f, g \in M_I$ are equivalent if and only if $s_f = s_g$.*

Proof. As previously defined in this chapter, in the context of Coxeter groups we have that $s_i^2 = 1$, which enables us to expand and contract words. We also have that $(s_i s_j)^{m_{i,j}} = 1$. This gives us that $s_{P(i,j)} = s_{P(j,i)}$, which enables homotopy of words. Since this holds true for generator elements of (W, S), we must have that $f, g \in M_I$ are equivalent if and only if $s_f = s_g$ as required. $\square$

5.2. The Exchange and Deletion Conditions. We end this chapter by stating three important theorems in this area, without proof. The exchange condition is a theorem regarding words and reflection groups. The deletion condition is then a corollary of this theorem. There is a further interesting theorem that is a consequence of these conditions which links us back into Coxeter systems. These theorems are given as in [**7**], where a worked proof of each can be found. In these theorems, $l(w)$ denotes the length of word w.

Theorem 5 (The Strong Exchange Condition). *For a reflection group W and some $w = s_1 \dots s_n \in W$, if there exists some reflection $t \in W$ such that $l(wt) < l(w)$ then there exists some $i \in \{1, \dots, n\}$ whereby:*

$$wt = s_1 \dots \hat{s}_i \dots s_n$$

where $\hat{s}_i$ denotes omission of s_i. If the word w is reduced then i is unique. [**7**]

Corollary 2 (The Deletion Condition). *If $w = s_1 \dots s_n$ with $n > l(w)$ then there exists $i, j \in \{1, \dots, n\}$ with $i < j$ such that*

$$wt = s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_n$$

with $\hat{s}_i$ again denoting omission. [**7**]

Theorem 6. *Any group W generated by a set S with all $s \in S$ of order 2 and satisfying the deletion condition gives a Coxeter system (W, S).*

CHAPTER 5

Buildings

With the necessary material now covered, we finally arrive at the main topic of interest: the building. We begin by motivating the building from our canonical example of the flag system, before formally defining the building and some further terminology. We conclude this chapter with some examples of buildings and a discussion of where the topic of buildings extends to beyond what we cover in this thesis.

1. The Building

1.1. Our Motivating Example. Recalling what we have seen so far with this example, we started by taking the flag complex of a 3-dimensional vector space over the field of order 2. This was used to motivate the concept of a chamber system. The chambers were maximal flags in the vector space and two chambers were said to be i-adjacent when the flags shared a common face.

In chapter 2 we moved to look more closely at the underlying geometry in action, before carrying forwards into chapter 3 where we encountered reflection groups with a classification, root systems and words. Finally we rejoined our canonical example in chapter 4, when we established that we could realise our chamber system from the symmetric group $\mathcal{S}_3$. We observed that this chamber system over the set $I = \{1, 2\}$ has an associated Coxeter system of Type $\mathbf{A}_2$.

The key picture to think of here is the local picture of our chamber system Δ with the gallery through i labelled nodes between two chambers instead denoted by a series of reflections in corresponding s_i, as shown in the following figure.

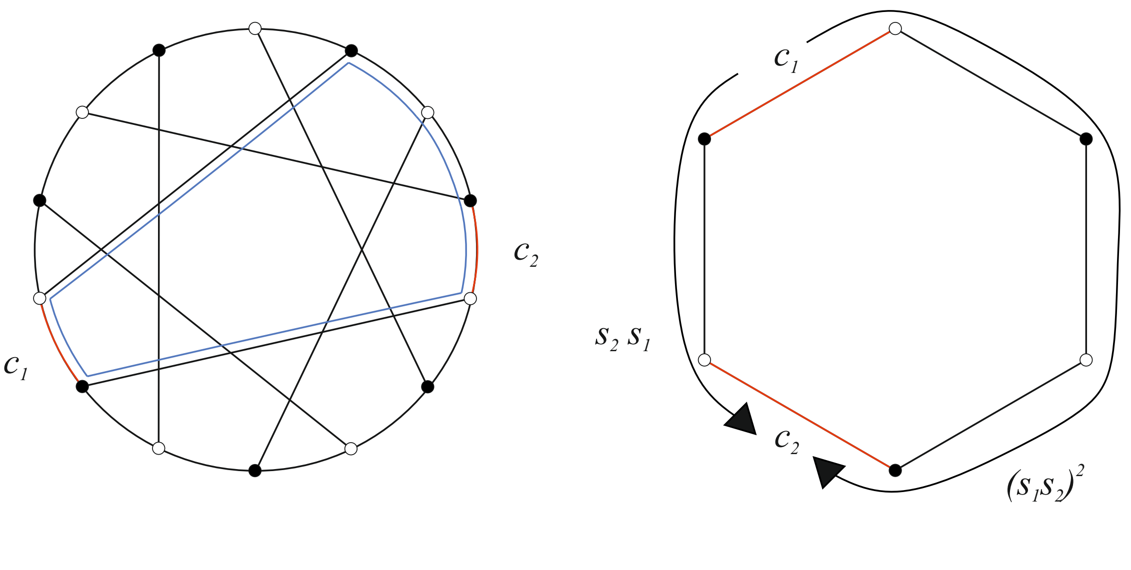


FIGURE 1. The local picture of Δ showing the distance between chambers c_1 and c_2

We can then set $s_1 \in \mathcal{S}_3$ and $s_2 \in \mathcal{S}_3$ such that they satisfy the necessary relations, giving us the symmetric group of order 6 permuting the chambers around. What we have not done, until now, is to use the group $\mathcal{S}_3$ to define a metric on Δ, a way of defining distance between chambers. We then define a map $\delta : \Delta \times \Delta \to \mathcal{S}_3$ such that if c_1, c_2 are joined by a gallery of type $2, 1$ and a gallery of type $1, 2, 1, 2$ as in our example above, then $\delta(c_1, c_2) = s_1 s_2 = s_2 s_1 s_2 s_1$. This is a building, a chamber system together with some metric from a Coxeter system (W, S). We now move to formalise this definition.

1.2. Defining a Building. Having just seen the motivation of a building, we now provide a definition of the building. Here when we say the Type of a building, we refer to the Coxeter diagram of the associated Coxeter system, as we used the definition of Type in the previous chapter.

Definition 57. [**6**] For a Coxeter system (W, S) with $S = \{s_i\}_{i \in I}$, a **building** of Type (W, S) is a chamber system Δ over I such that:

(1) Every panel of Δ contains at least two chambers
(2) Δ has a W-valued metric $\delta : \Delta \times \Delta \to W$ such that if $s_f = s_{i_1} \dots s_{i_k}$ is a reduced word for some expression in W, then

$$\delta(c, c') = s_f \Leftrightarrow \text{ there exists a gallery } c_1 \to_f c_2 \text{ in } \Delta$$

We will sometimes refer to W as the **Weyl group** and δ as the **Weyl distance function** of Δ. Before turning to more building related definitions and theorems, we move to realise a large family of examples of buildings.

1.3. Coxeter systems as buildings. We are going to construct a substantial number of examples of buildings in one go, by realising that every Coxeter system (W, S) of Type Π can be seen as a building (Σ_Π, δ_W). To obtain this building from any Coxeter system, we simply set

$$\delta(x, y) = x^{-1} y.$$

In order for this δ function to serve as our metric, we require the following proposition.

Proposition 4. *For all $x, y \in \Sigma_\Pi$ there exists a gallery of type f from x to y in Σ_Π exactly when $x^{-1} y = s_f$ for some $s_f \in W$.*

Proof. This is true by the construction of Σ_Π. Take chambers $x, y \in \Sigma_\Pi$ and a word $f \in M_I$ such that $x^{-1} y = s_f \in W$, we can write f as $f = ijklm\dots$. Then by definition of Σ_Π there exists a chamber $x_1 = x s_i \in W$ such that $x \sim_i x_1$. Similarly there exists a chamber $x_2 = x s_{ij}$ such that $x_1 \sim_j x_2$, and so on. Therefore there is a unique gallery $(x, x_1, x_2, \dots y)$ of type $ijklm\dots = f$ in Σ_Π from x to y.

In the reverse direction, given a gallery of type f from x to y, by definition of the Coxeter chamber system we will have $x^{-1} y = s_f$ for some $s_f \in W$ as required. □

We will now demonstrate a Coxeter system can be a building using our example the Coxeter group $(D_6, \{s_1, s_2\})$.

Example 5.1. A building from $(D_6, \{s_1, s_2\})$ The 12 elements of this group are shown in the following figure.

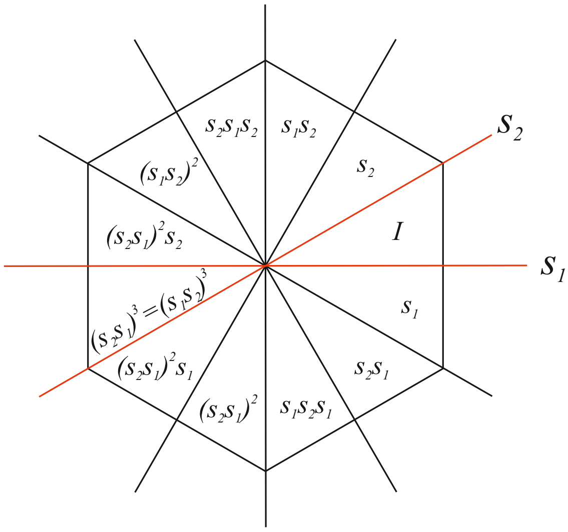


FIGURE 2. The Coxeter system $(D_6, \{s_1, s_2\})$

If we take the intersection of the reflecting lines of the the hexagon with a circle, we are able to realise a chamber system. This is shown in the following figure with white nodes connecting two 1-adjacent chambers and black nodes connecting two 2-adjacent chambers.

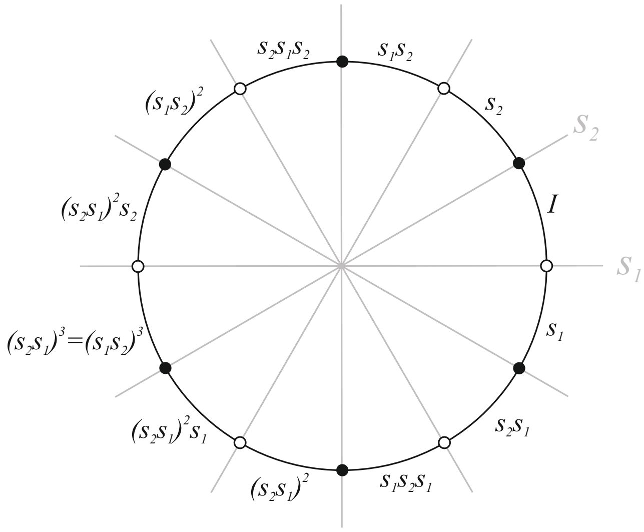


FIGURE 3. The building (Δ, δ) obtained from the Coxeter system $(D_6, \{s_1, s_2\})$

Here, it is clear that the Weyl distance function δ for any two chambers $x, y \in \Delta$ can be defined to be

$$\delta(x, y) = x^{-1}y.$$

This is easy to check, and should be obvious from the fact that the Coxeter group is generated by reflections s_i for $i \in I$ with a chamber system taken over such a set I.

1.4. Thin and Thick Buildings. When describing buildings, a common descriptor for buildings are the definitions of thick and thin. This labelling in fact comes from the underlying chamber system. Recall that a chamber system is called thin if every panel in the chamber system is a face containing exactly 2 chambers. For chamber systems with every panel being a face of at least 3 chambers we say that the chamber system is thick.

The example of Coxeter complexes as buildings are all examples of thin buildings, in fact it turns out that these are our only examples of thin buildings. We now move to look at further definitions, structures and theorems relating to buildings, before returning to state some more example of spherical buildings, by which we mean buildings associate to a finite Coxeter group.

2. Words and Buildings

With the definition of a building behind us, we now move to look at related structures, properties, and maps between buildings. In this next section we return to look at the concept of words, which we have already looked into. We will see that what we have already seen and introduced can now be pulled directly into the language of buildings. Specifically, there are several definitions and results that will prove useful for working with galleries of words. We are able to link these topics because of the W-valued metric on our building, which associates each gallery in a building (Δ, δ) with some s_f in the associated Coxeter system.

2.1. Galleries. We begin by providing another definition of homotopy, with more focus on the building. Recall from Chapter 4 that we defined a map $P(i, j)$.

Definition 58. For a building Δ of Type Π with index set I, consider a gallery γ of the form $(\gamma_{f_1}, \gamma_{P(i,j)}, \gamma_{f_2})$. An **elementary homotopy** of γ is a transformation of γ to a gallery γ' of the form $(\gamma_{f_1}, \gamma_{P(j,i)}, \gamma_{f_2})$.

We the use this definition to define what it means for two galleries to be homotopic.

Definition 59. Two galleries are said to be **homotopic** if one can be transformed, expanded or contracted into the other by a finite sequence of elementary homotopies.

Similarly, the definitions of contraction, expansion and equivalence of words provides analogues to the definitions of contraction, expansion and equivalence of galleries.

We next state an interesting theorem about reduced words. Due to reliance on other proofs omitted from this thesis, this theorem is stated without proof. The reader can find a full proof in [**11**].

Theorem 7. *A gallery of type f is minimal if and only if f is reduced, and moreover any two reduced words $f, g \in M_I$ which are equivalent must be homotopic.*

We can now give a very useful proposition that will ultimately yield a variety of information about the structure and properties of buildings, and will be essential to the major theorem of this thesis that is presented towards the end of this chapter. The proposition is as stated in [**15**], with some further discussion provided in the proof.

Proposition 5. *For a building Δ of Type Π with index set I, chambers $x, y \in \Delta$ and a gallery γ of type f in Δ connecting x and y, the following hold:*

(1) *If $g \in M_I$ is a word homotopic to f, then there exists a gallery from x to y of type g which is homotopic to γ.*
(2) *The gallery γ is minimal if and only if its type f is reduced.*
(3) *If γ is a minimal gallery, then it is the unique gallery of type f from x to y.*

Proof. Let (W, r) denote the Coxeter system of Type Π. As defined earlier in this section, we can take two words $P(i, j)$ and $P(j, i)$ in M_I. Since $P(i, j)$ is either a word of the form $ijij \dots ij$ or $jijij \dots ij$ we have that $P(i, j)$ is reduced for i, j distinct, and similarly for $P(j, i)$.

We also have that the words $P(i, j)$ and $P(j, i)$ have the same image under s, where s denotes the map $M_I \to W$ such that $f \mapsto s_f$ as seen previously.

From the definition of a building, we therefore have that if γ_2 is a gallery in Δ of type $P(i, j)$ for some distinct $i, j \in I$ then there always exists a gallery of type $P(j, i)$ with the same first and last chambers as γ_2. Therefore if we set g to be the word obtained from f by an elementary homotopy, there exists an elementary homotopy transforming γ into a gallery of type g. Thus (1) holds.

For (2), first let us assume that f is not reduced. We want to show that this causes γ to not be a minimal gallery. From part (1) which we have just proven, if f is not reduced then we can assume that f is of the form $f_1 ii f_2$ for some $i \in I$, therefore containing a subgallery (p, w, q) of type (ii). From the definition of a chamber system, we have that this means either $p \sim_i q$ or else $p = q$. Therefore it is possible to delete either one or two chambers from γ giving a gallery of type $f_1 i f_2$ or $f_1 f_2$ from x to y, so γ is not minimal.

Instead suppose that f is reduced and that the minimal gallery from x to y is of type h. By the above, h is reduced. From the definition of a building, we have that $s_f = s_h$. All that remains to show is that this means $f \simeq h$. This is true as a consequence of theorem 7. Since we have that $s_f = s_h$, that both f and h are reduced, and that h is a minimal gallery, it follows that $l(f) = l(h)$ therefore f is also a minimal gallery from x to y, therefore we must have that $f \simeq h$ Thus (2) holds.

For the final part of the proposition, suppose that γ is minimal such that the length of γ, denoted as above $|\gamma| = k$ such that $k \geq 2$ (otherwise $x \sim_i y$ and the minimal gallery is obviously unique by the definition of a building). From part (2) of the proposition above, we have that f is reduced.

Next, write γ as $\gamma = (x, \ldots, u, v, y)$ and suppose that there exists another gallery $\gamma' = (x, \ldots u', v', y)$ of type f from x to y. By construction we have that both $v \sim_i y$ and $v' \sim_i y$ (where the last letter of f is i. From the definition of a chamber system, we have that if $v \sim_i y$ and $v' \sim_i y$ then $v \sim_i v'$.

Supposing that we have that v and v' are distinct, then we have that $(x, \ldots u, v)$ is a gallery of length $k - 1$ from x to v and $(x, \ldots, u', v', v)$ is a gallery of length k also from x to v. The second of these galleries is therefore not minimal, however it has type f. This gives us a contradiction since from part (2) we have that all galleries of type f are minimal, so we must have $v = v'$. By induction with respect to k, we have shown that part (3) holds. □

3. Subbuildings

Recalling back to our first chapter on chamber systems, by following our instincts we arrived at the definition sub-chamber system. We now look to find an analogous definition for this special family of chamber systems, and arrive at the definition of a subbuilding.

3.1. Defining a Subbuilding.

Definition 60. Let (Δ, δ) be a building of Type Π with index set I. If Δ_0 is a sub-chamber system of (Δ, δ) with index set $J \subset I$ and (Δ_0, δ_0) is also a building (where δ_0 denotes the restriction of δ_0 to $\Delta_0 \times \Delta_0$), then we say that (Δ_0, δ_0) is a **subbuilding** of Type Π_J of (Δ, δ).

In defining a subbuilding, we are able to come to a fundamental result about buildings which gives a variety of corollaries. This key theorem will allow us to show that we do indeed get subbuildings, and provide us with a way to find concrete examples of such things. To get to this theorem, we will need to come back to the notion of convexity that was introduced in the first chapter on chamber systems. We will then need to give a short proposition.

Recall that a subchamber system is convex if for any two chambers in the sub-chamber system, we have that the minimal gallery joining them is the same gallery in the sub-chamber system as in the wider chamber system.

We can use convexity in the following lemma, as given in [**15**].

Proposition 6. *For a building (Δ, δ) of Type Π with index set I and a sub-chamber system Δ_0 of Δ with index set $J \subset I$, the following are equivalent:*

(1) *(Δ_0, δ_0) is a subbuilding of Type Π_J of (Δ, δ)*
(2) *Δ_0 is convex*

In proving this proposition, we rely heavily on proposition 4.

Proof. We start by showing that (2) implies (1). We have that Δ_0 is a convex subchamber system of Δ and (Δ, δ) is a building. We need to show that (Δ_0, δ_0) is a building.

Firstly, from the definition of a convex chamber system, we have that for any two chambers $x, y \in \Delta_0$ and minimal gallery γ connecting them in Δ then γ is completely contained in Δ_0. Secondly, if we let the type of γ be denoted f. By the proposition at the end of the previous chapter, we have that f is a reduced word (because γ is minimal). By the above two facts, we have that the index set of Δ_0 is J and so we must have that $f \in M_J \subset M_I$. We now bring in the δ function as described in the proposition. Recall from chapter 1 that all residues of chamber systems are themselves sub-chamber systems. Denote the Coxeter system of Type Π (so the Coxeter system associated with our building) as (W, S), and define

$$W_J = \{s_i \,|\, i \in J\}$$

From the definition of a building, we have that $\delta(x, y) = s_f \in W_J$. Hence $\delta(\Delta_0 \times \Delta_0) \subset W_J$.

If we have that $\delta(x, y) = s_g$ for $g \in M_I$ some other reduced word, then again by the definition of a building we must have a gallery γ' from x to y in Δ of type g. Following the same reasoning as above, the previous proposition gives us that γ' must be minimal because g is reduced. Then again, γ' is contained in Δ_0 because Δ_0 is convex. We have shown that (Δ_0, δ_0) is indeed a building, and since it is a sub-chamber system as, we have that indeed (Δ_0, δ_0) is a subbuilding of Type Π_J of (Δ, δ) as required.

In the reverse direction, we start with (Δ_0, δ_0) as a subbuilding of Type Π_J of (Δ, δ). Take any two chambers $x, y \in \Delta_0$ and call the minimal gallery in Δ between them γ. Denote the type of γ as f. Since γ is minimal, by the earlier result we have that f is reduced, hence from the definition of a building we have that $s_f = \delta(x, y)$. Since $\delta(x, y) \in \delta(\Delta_0, \Delta_0)$ and $\delta(\Delta_0, \Delta_0) \subset W_J$ we have that $s_f \in W_J$, and hence $f \in M_J$. Why? If $s_f \in W_J$ that means that $s_f = s_g$ for some reduced $g \in M_J$. □

It is worth noting that in this proof we defined and used a subgroup of the form W_J. There is a fundamental fact that these subgroups are themselves Coxeter groups generated by the corresponding subset of W.

3.2. A Theorem to Find Subbuildings. We will now state and provide a proof for a theorem which will allow us to realise examples of subbuildings from buildings.

Theorem 8. *Let (Δ, δ) be a building of Type Π with index set I. Let J be a subset of I and let Π_J denote the subdiagram of Π spanned by J (the subdiagram obtained from Π by deleting all vertices not contained in J and all edges connected to a vertex in J). Let R be a J-residue of the chamber system Δ. Then R is a subbuilding of Δ of Type Π_J.*

We will prove this theorem using the above proposition giving an alternative definition of a subbuilding involving convexity.

Proof. Again, define

$$W_J = \{s_i \mid i \in J\}$$

and let $u, v \in R$ be chambers in R. Our strategy will be to take a gallery between these chambers and show that it is contained within R. Since this is for a gallery between arbitrary chambers we will get that R is convex.

Let γ be a minimal J-gallery from u to v (a gallery such that the type, f, of γ is contained in M_J). We can assume that f is of reduced. Why? Suppose for a contradiction that f is not a reduced word in M_J, so we have

$$f \cong f_1 i i f_2 = g$$

for some words $f_1, f_2, g \in M_J$. By an earlier result, we have that there then must exists another J-gallery from u to v of type g, which we denote γ_1. Note that we have that

$$|\gamma_1| = |\gamma|$$

where $|\cdot|$ is the length of the gallery with respect to its type.

Because we have the repeated ii in the type of γ_1, we are able to delete at least one chamber from γ_1 to obtain a shorter J-gallery from u to v. Hence γ is not minimal.

Since f is reduced, we have from the definition of a building that $\delta(u, v) = s_f \in S$. We have constructed γ to be a J-residue, so it is unsurprising that γ is contained in R. If we now take an I-gallery from u to v called γ_2 and let the type of this gallery be denoted h, we have from an earlier result that h is also reduced. From the definition of a building therefore, we have that $s_f = s_h \in W_J$ and as such $h \in M_J$. It follows therefore that γ_2 is also contained in R. Therefore R is convex.

By the previous proposition, we have that since R is convex, then (R, δ_R) is a subbuilding of $(\Delta, \delta$ of Type Π_J , where δ_R denotes the map δ restricted to $R \times R$. □

The above result is extremely powerful, since it gives us a way of finding subbuildings using residues. The following corollary formalises a simple consequence of the previous theorem.

Corollary 3. *All residues of buildings are convex.*

Proof. This follows the result of the previous theorem □

4. Maps between buildings

Isometries are essentially maps going between buildings, and play a central role in the study of buildings. We begin by formally defining an isometry of buildings. This section follows the work of [**15**] very closely, with commentary and further expansion in places to aid understanding through the theorems.

4.1. Apartments and Isometries.

Definition 61. [**15**] Let (Δ, δ) and $(\hat{\Delta}, \hat{\delta})$ be two buildings of the same Type, so therefore having the same index set I and same Coxeter group). A map π from a subset $X \subseteq \Delta$ to $\hat{\Delta}$ is called an **isometry** from X to $\hat{\Delta}$ if $\hat{\delta}(\pi(x), \pi(y)) = \delta(x, y)$ for all chambers $x, y \in X$.

Notice here that the isometry need not be between one entire building and another, but rather a subset of one and another. The main notion in this definition is that of isometries preserving the Weyl distance in buildings. With this definition in hand, we now define another central construct in the theory of buildings, apartments.

Definition 62. [**15**] Let (Δ, δ) be a building of Type Π. An **apartment** of Δ is an isometric image of Σ_Π in Δ.

This means that an apartment is the piece of a building that an isometry maps all of a Coxeter chamber system to. In fact, it turns out that apartments are themselves Coxeter chamber systems of Type Π.

4.2. Isometry Theorem. Now we move to the first major theorem we come to, which will allow us to utilise much of our existing knowledge of Coxeter chamber systems into the language of buildings. We will only provide a sketch of the proof for this theorem, with a full worked proof available in [**15**].

THEOREM 9. *For a building Δ of Type Π and a subset $X \subseteq \Sigma_\Pi$, every isometry from X to Δ extends to an isometry from Σ_Π to Δ.*

PROOF. First denote the Coxeter system of type Π by (W, S). Suppose $X \subset \Sigma_\Pi$ and let $\pi : X \to \Delta$ be an isometry. The strategy for this proof is to show that if $X \neq \Sigma_\Pi$, then we can extend π to an isometry from a set $X' \to \Delta$, where $X \subsetneq X' \subseteq \Sigma_\Pi$. Then an application of Zorn's lemma will show that it is possible in the end to extend π to all of Σ_Π.

We may assume that X is non-empty, which means that we can find adjacent chambers $u \in X$ and $v \notin X$, by the connectedness of Σ_Π. Since left multiplication by u^{-1} is an automorphism of Σ_Π which preserves distances and adjacency, we may assume that $u = 1$ and hence, since v is adjacent to u, that $v = s_i$ for some $s_i \in S$, with $i \in I$. Set $X' = X \cup s_i$.

In order to extend the isometry π to X', we need to work out where to send s_i. There are two cases to consider.

The first (easier) case is where for every reduced word g such that $s_g \in X$, the word ig is also reduced. In this case, we can simply choose any chamber $c \in \Delta$ which is i-adjacent to $\pi(1)$ and set $\pi(s_i) = c$. For then, since ig is reduced for all g with $s_g \in X$, we do get that $\pi : X' \to \Delta$ is an isometry.

The second case is much more involved. Here we have at least one reduced word g with $s_g \in X$ and such that ig is not reduced. This means that g is homotopic to a reduced word $f = ih$, say, with h reduced. Now there is a gallery from 1 to $s_f = s_g$ in Σ_Π of type f, and this gallery is minimal by proposition 5. Now, since π is an isometry, there is a gallery from $\pi(1)$ to $\pi(s_g)$ of type f in Δ. Denote this gallery by $(c_0, c_1, \ldots, c_k)$, where $c_0 = \pi(1)$ and $c_k = \pi(s_g)$.

Then we set $\pi(s_i) = c_1$ in this case. It remains to show that this does indeed define an isometry from X' to Δ. It is clear that $\pi(s_i)$ is i-adjacent to $\pi(1)$, since the gallery $(c_0, c_1, \ldots, c_k)$ has type $f = ih$, so c_0 and c_1 are i-adjacent. The last thing to check is that $\delta(c_1, \pi(x)) = s_i^{-1}x$ for all $x \in X$. We skip the details of this check because they involve some technical constructions not included in this thesis.

□

4.3. Consequences of the Isometry Theorem. The above theorem leads onto some very nice results, one of the most important being the following.

COROLLARY 4. *Any two chambers in a building are contained in a common apartment.*

PROOF. If we take two chambers $x, y \in (\Delta, \delta)$ for a building of Type Π; we can the two element subset $\{1, \delta(x, y)\}$ of Σ_Π and an isometry sending $1 \mapsto x$ and $\delta(x, y) \mapsto y$. Using the proof of the previous theorem, we can extend this isometry to an isometry $\Sigma_\Pi \to \delta$, giving us an apartment. □

The following is original work. A sensible question to ask at this point is whether this proof can give us a recipe for finding an apartment containing any two chambers x and y. The way to approach finding a common apartment for a spherical building would be:

(1) Let $\{1, \delta(x, y)\} = X \subseteq \Sigma_\Pi$ and set the isometry π such that $1 \to x$ and $\delta(x, y) \to y$ as in the above proof of corollary 1.
(2) (if repeating process) Use the special auotomorphism obtained by left multiplication on the Coxeter group to shift the most recently added element of the group to be the identity element, $1 \in X$.
(3) Choose an adjacent chamber to 1 in Σ_Π but not in X (so a generating element of the Coxeter group) and call it $v = s_i$ (as in the above proof of theorem).
(4) For all reduced g such that $s_g \in X$, check if ig is a reduced word. If so, continue to (5), if not continue to (6).
(5) Choose any chamber $c \in \Delta$ such that $x \sim_i c$ and set $\pi(v) = c$. Continue to (7)
(6) For the $s_g \in X$ such that ig is not reduced, find a reduced word f homotopic to g beginning with i and the gallery γ of type f from x to $\pi(s_g)$, so that $\gamma = (x, c_1, c_2, \ldots, \pi(s_g))$. Set $\pi(v) = c_1$ from the gallery.
(7) Set a new $X = X \cup s_i$ and go back to (2). If $X = \Sigma_\Pi$, stop as you are done!

There is yet more that we can extract from the powerful theorem that we have just proven, but first we will need a couple of simple propositions that mostly follow as a result of the theorem. See [**15**] for full worked proofs.

Firstly we note that for a building (Δ, δ) of Type Π and a subgraph $\Sigma \subseteq \Delta$, then Σ is an apartment if and only if Σ is a thin subbuilding of Type Π. This provides us with a useful way of identifying apartments.

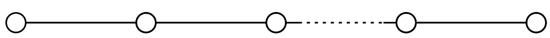
Our second consequence is that all apartments are convex. This again is a useful property which aids in further theorems in the subject area.

Finally, and perhaps of most interest, we have that isomorphisms between buildings map apartments to apartments.

5. A Thick Spherical Building

We have defined a building and considered their structure and properties. So far however, we have only seen a minimal number of explicit examples of buildings. We unfortunately will stop short of showing many thick buildings in this thesis, as the next step in obtaining such a family of thick buildings involves extracting buildings from the structure of reductive algebraic groups. This is a topic that we have not, and will not be covering in this thesis. We finish where we started by stating one final generalised example before entering our closing section . This example follows an example shown in [**6**].

Example 5.2. Spherical buildings of Type $\mathbf{A}_{n-1}$ Let (W, S) be a Coxeter system with Coxeter diagram with n nodes as shown below.



Taking our canonical example of a chamber system Δ formed from a flag complex of a n-dimensional vector space over a field of order k, we put a W-valued metric onto our flag complex. This metric δ is defined as follows. For $1 \leq i \leq n$, let

$$\pi(i) = min\{j \mid V_i' \subset V_{i-1}' + V_j\}$$

We want to show that π is a bijection which permutes the $i \in I$. Whilst this is more involved than it might seem, we will suffice to show that for the case where two flags differ only in the i^{th} position, that π swaps i and $i + 1$.

If we have two flags $V_1 \subset \ldots \subset V_i \subset \ldots \subset V_n$ and $V_1 \subset \ldots \subset V_i' \subset \ldots \subset V_n$ such that $V_i \neq V_i'$, then we must have that

- $\pi(j) = j$ for all $1 \leq j \leq i-1$ and $i+2 \leq j \leq n$;
- $\pi(i) = i+1$; and
- $\pi(i+1) = i$.

This clearly swaps i and $i+1$. Then for $c, c' \in \Delta$ we define

$$\delta(c, c') = \pi$$

This is a spherical building which is an example of a thick building.

6. Alternative Approaches and Further Reading

6.1. Another Approach. In introducing buildings, we have followed a very geometrically based narrative. Our approach came from the perspective of chamber systems, linking them to Coxeter groups and then producing the building. The common alternative approach into the topic of buildings, as is used in [**1**], is to start with Coxeter complexes and show that these are simplicial complexes. The definition of a building then introduces apartments immediately, as is shown in the first definition of a building given by Brown in [**1**]:

Definition 63. A **building** is a simplicial complex Δ which can be expressed as the union of subcomplexes Σ (called apartments) satisfying the following axioms:

- **B0** Each apartment Σ is a Coxeter complex
- **B1** For any two simplices $A, B \in \Delta$, there is an apartment Σ containing both A and B
- **B2** If Σ and Σ' are two apartments containing A and B, then there is an isomorphism $\Sigma \to \Sigma'$ fixing A and B pointwise

As is clear by the above definition, it is the apartment and not the chamber that is seen as the fundamental object in this view of a building.

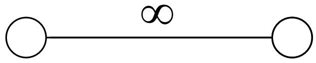
6.2. Beyond Spherical Buildings. The reader may also be wondering why we spent so much time explaining Euclidean and hyperbolic geometries when we only went so far as to show spherical buildings in this thesis. Our reasoning behind understanding all of this material in different geometries is to build the expectation in the reader that spherical buildings are not the end of the story with buildings. For a full introduction and study to Euclidean buildings, the author recommends [**16**] as a well structured narrative. Hyperbolic buildings are also structures which exist, however these are a much more recent development in the topic of buildings and consequently the material is still relatively undeveloped.

The applications of buildings is seen through important results in some areas of mathematics which, without the connection of buildings, can seem fairly disparate. As mentioned in the introduction to the previous section, there are strong links between buildings and reductive algebraic groups. The building provides something for these groups to act on, providing a sometimes unique insight to properties of such group which would go unrealised without buildings.

6.3. The Final Example. We conclude this thesis with a final example, this time illustrating a simple Euclidean building from which the reader can take inspiration to continue exploring this fascinating area further. This example follows work in [**6**].

Example 5.3. A Euclidean Building We motivate this example through showing various figures to illustrate key points, and this example is only to serve as motivation for material in related literature which follows from the conclusion of this thesis.

We start by considering the following Coxeter diagram.



This shows the diagram for the infinite dihedral group W. The question here is what would a building of this Type look like? We can show an example of this building. Take the infinite 3-valent tree and set the edges of the tree to be the chambers. If we colour the vertices black and white, we can get a notion of 1-adjacency and 2-adjacency. Figure 4 shows a portion of this tree labelled as described.

Since there is a unique shortest path between any two chambers in this tree, which will pass through a sequence of 'black' and 'white' nodes, we can associate a sequence of 1's and 2's and hence a corresponding element in the Coxeter group W. It is then straightforward to check the building axioms.

To realise W concretely as an affine Coxeter group, let $\mathbb{V}$ be 2-dimensional and consider two reflections $S = s_1, s_2$ on V, and the group W generated by S. Unlike our previous orthogonal reflections (where the decomposition of V by the reflection s_i was the direct sum of a hyperplane H_{s_i} with an orthogonal reflecting line L_{s_i}), consider s_1, s_2 such that for the decomposition $V = L_{s_i} \oplus H_{s_i}$ we have $H_{s_1} \neq H_{s_2}$ but $L_{s_1} = L_{s_2} = L$. This means that not only is L left invariant by W but that any line parallel to L is also invariant by W. This setup is shown in the figure 5.

However, notice that the reflection $s_0 H_{s_1} \notin \mathcal{H}$. It turns out that we must extend $\mathcal{H}$ to the infinite set shown on the figure 6. If we were to then identify an invariant affine line parallel to L as $\mathbb{R}$, we get that W is isomorphic to the group of affine reflections of $\mathbb{R}$ in $\mathbb{Z}$.

As stated previously, this is not a fully worked example and serves only to further engage the reader in the material of Euclidean Buildings which the author recommends exploring further.

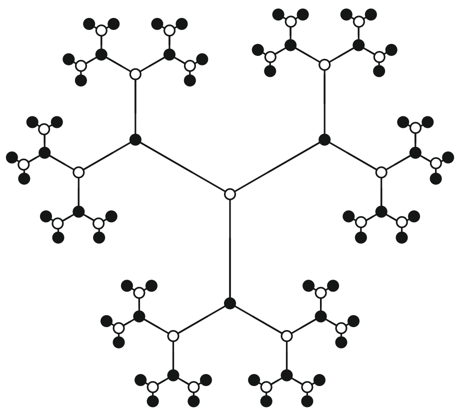


Figure 4. A portion of the 3-valent tree, an example of a building associated with the infinite dihedral group

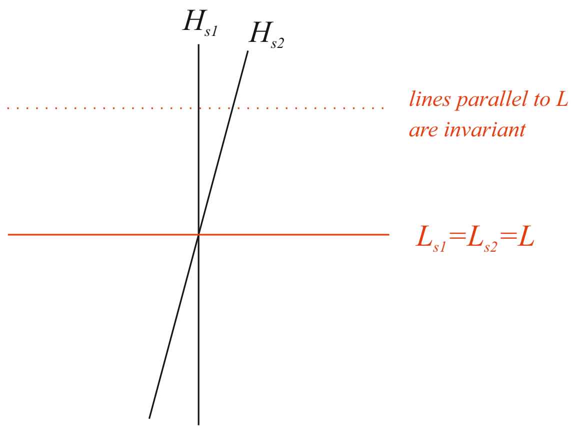


FIGURE 5. Reflecting hyperplanes H_{s_1}, H_{s_2} sharing the same reflecting line $L_{s_1} = L_{s_2} = L$

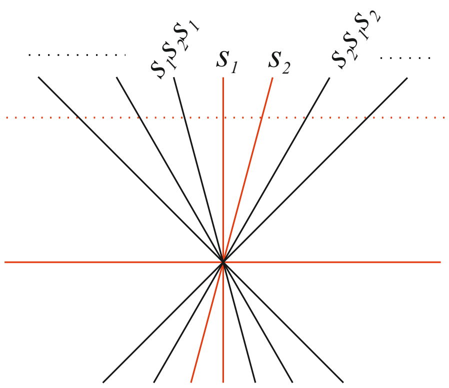


FIGURE 6. Reflecting hyperplanes H_{s_1}, H_{s_2} extend S to an infinite set of reflections

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