UNIVERSITY OF SHEFFIELD

THREE DIMENSIONAL ULTIMATE STRENGTH ANALYSIS OF BEAM – COLUMNS

BY

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DEDICATED
TO MY FAMILY

Summary

A rigorous formulation for the structural response of thin-walled members of arbitrary open cross-section acted upon by a general system of loads is developed based on energy principles and virtual work concepts. Full account is taken of three dimensional behaviour, including sectorial warping effects. The analysis incorporates the effect of initial geometrical deflections. Different patterns of residual stress, non-coincidence of the shear centre and centroid, a complete absence of symmetry in the section and the influence of higher order terms in the strain-displacement relationships including products of the derivatives of axial displacements are also incorporated.

A computer program based on finite element analysis suitable for application in both the elastic and inelastic ranges is developed. This is used to solve the differential equations governing the ultimate strength of beam-columns in space.

The program is written in the Fortran 77 Language. The main function of the program is to follow the loss of stiffness due to spread of yield and hence to trace the full load-deflection response up to collapse. It may be used in a wide variety of ways.

Three types of analysis have been conducted in this study. These are: Linear, Partial Non-linear and Full Non-linear. The Linear involves only the small deflection theory. Partial Non-linear analysis uses non-linear strains while the Full Non-linear analysis incorporates both non-linear strains and nonlinear stiffness matrices. Several illustrative examples, previously investigated either theoretically or experimentally, have been chosen to check the validity of both the

analytical approach and the computer program. These examples cover flexural, flexural-torsional, biaxial bending, and bending and torsional behaviour in the elastic and inelastic ranges. They contain a wide range of parameters e.g. different cross-section shapes, loading, boundary conditions and initial imperfections. Finally the program has been used to study the ultimate strength of steel members subjected to compression, bending and torsion in a more rigorous fashion than has previously been possible.

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List of Publications

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2. El-Khenfas, M.A., and Nethercot, D.A., (1987a), " A General Formulation for Three Dimensional Analysis of Beam-columns", International Journal of Mechanical Sciences, (submitted for publication).
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5. Wang, Y.C., El-Khenfas, M.A., And Nethercot, D.A. (1987), "Lateral- Torsional Buckling of Ben-Restrained Beams", Journal of Construction Steel Research, (in press).
6. El-Khenfas, M.A., (1987a), "TDFE-Computer Program for Analysis of Beams in Space in the Elastic and Inelastic ranges", Report, No. 1CE, Civil and Structural Engineering Department, Sheffield University.
7. El-Khenfas, M.A., (1987b), "Elastic Analysis of Beam-Columns in Space", Report, No. 2CE, Civil and Structural engineering department, Sheffield University.
8. El-Khenfas, M.A., (1987c), "Inelastic Analysis of Beam-Column in 3-Dimensions", Report, No. 3CE, Civil and Structural Engineering Department, Sheffield University.

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Notation

A	Cross-sectional area
A_{fy}	$M_y/M_{yy}+M_z/M_{yz}+M_t/M_{y\omega}$
A_{fp}	$M_y/M_{py}+M_z/M_{pz}+M_t/M_{p\omega}$
AA	$P/P_{pl}+M_y/M_{py}+M_z/M_{pz}+M_t/M_{p\omega}$
a_y, a_z	Distance of transverse load (concentrated or distributed) below or above shear centre
B	Flange width (I-section)
C	centroid of the section
C_1	$\sqrt{\left(\frac{M_y}{M_{yy}}\right)^2 + \left(\frac{M_z}{M_{yz}}\right)^2 + \left(\frac{M_t}{M_{y\omega}}\right)^2}$
C_2	$\sqrt{\left(\frac{M_y}{M_{py}}\right)^2 + \left(\frac{M_z}{M_{pz}}\right)^2 + \left(\frac{M_t}{M_{p\omega}}\right)^2}$
D	Beam depth
E	Elastic modulus.
E_{sh}	Strain hardening modulus
E_t	Tangent modulus
e_y, e_z	Eccentricities of axial load

F_i	Concentrated force
F_x	Internal axial load
f^s	Surface traction force
f^b	Body force
G	Elastic shear modulus
G_{sh}	Strain hardening shear modulus
G_t	Tangent shear modulus
H	$D - T_f$
I_o	Polar moment of inertia
I_y, I_z	Moment of inertia about Y and Z axes
I_{yz}	Product moment of inertia about Y and Z axes
$I_{y\omega}$, $I_{z\omega}$	Warping product moment of inertia about Y and Z axes
I_ω	Warping moment of inertia.
K	Torsional constant
L	Beam length
l	Beam segment length
M_{cr}	$\frac{\Pi}{L}\sqrt{EI_y GK}\sqrt{1-\frac{\Pi^2 EI_\omega}{L^2 GK}}$
M_p	Plastic moment
M_ρ	$\int_A \sigma\rho^2 dA$
$\overline{M}_y$	$\sigma_y \overline{Z}$

M_t	Applied constant torsional moment
M^L_{xi}, M^L_{yi}, M^L_{zi}	Internal moments to the left of node i about X, Y, and Z axes
M^R_{xi}, M^R_{yi}, M^R_{zi}	Internal moments to the right of node i about X, Y, and Z axes
m_{xi}, m_{yi}, m_{xi}	Applied moments to node i about X, Y, and Z axes
M^M_{wi}, M^R_{wi}	Internal moments to the right of node to the right of node i
NEL	Number of elements
NELS	Number of elements in the cross-section
P	Internal axial load
P_t	Total force array
$\overline{P}_y$	$\sigma_y A$
Q_y, Q_z	Shear resultant
q_x, q_y, q_z	Transverse loads applied about X, Y, and Z axes
R	Total displacements array
r_E	Element displacement
r, r_ρ	Projection of ρ on the tangent and perpendicular to the tangent at point A on the contour, Fig. 3.2a
r_y, r_z	Radius of gyration about Y and Z axes
S_c	shear centre coordinate

S_y, S_Z	Plastic section modulii about Y and Z axes
T_f	Flange thickness
t_ω	Web thickness
U, V, W	Displacements in the X, Y, and Z directions
U_o, V_o, Φ_o	Initial displacements
$U_{,yz} = \frac{\partial^2 U}{\partial y \partial z}$	Partial differentiation of U with respect to y and z
$\overline{U}$	Strain energy
$\overline{V}$	Potential energy due to applied load
$\overline{Y}, \overline{Z}$	First moment of area about Y and Z axes
Y_s, Z_s	Coordinates of point s
Z_y, Z_z	ELastic section modulii about Y and Z axes
α	Angle between the tangent at the contour and vertical axis, Fig. 3.2a
$\beta_y, \beta_z, \beta_\omega$	Properties of cross-section defined below

$$\beta_y \qquad \int_A Y(Y^2 + Z^2)dA - 2Z_s$$

$$\beta_z \qquad \int_A Z(Y^2 + Z^2)dA - 2Y_s$$

$$\beta_\omega \qquad \int_A \omega(Y^2 + Z^2)dA$$

$\overline{\beta}_y, \overline{\beta}_z, \overline{\beta}_\omega$	Properties of cross-section defined below
$\overline{\beta}_y$	$\frac{1}{I_y}\int_A Y(Y^2 + Z^2)dA$
$\overline{\beta}_z$	$\frac{1}{I_z}\int_A Z(Y^2 + Z^2)dA$
$\overline{\beta}_\omega$	$\frac{1}{I_\omega}\int_A \omega(Y^2 + Z^2)dA$
γ	Tolérance error
ε_o	Axial strain
ε_r	Residual strain
ε_y	Yield strain
Σ_o^L	Linear strain
Σ_o^L	Nonlinear strain
θ_u, θ_v	Slope of U and V with respect X axis
σ_{by}, σ_{bz}	Bending stress about Y and Z axes
σ_f	Flange tip residual stress
σ_{fw}	Flange-web junction residual stress
σ_r	Residual stress
σ_t	Total stress

σ_y	Yield stress
σ_ω	Warping residual stress
$\overline{\sigma}_\omega$	Web residual stress
ρ	Distance between a point on the cross -section and shear centre, Fig. 3.2a
τ_{xz}, τ_{yz}	Shear stresses.
Φ_Y, Φ_Z	Major and minor curvatures.
Φ_ω	Warping curvature.
Φ_c	Twisting calculated according to equation 8.7
$\Phi_{f.e.}$	Twisting calculated according to finite element analysis
ϖ_n	Normalized warping fuction.
Π	Total potential energy
δU_i	Virtual displacements
δW	Virtual work
< >	Row
{ }	Column
[ ]	Matrix
[ ]	Transpose matrix
$[K_D]$	Diagonal matrix
$[K_E^L]$, $[K_E^{NL}]$	Linear and nonlinear tangential matrix
$[K_G^L]$, $[K_G^{NL}]$	Linear and nonlinear gemetrical matrix

$[K_o]$	Initial geometrical stiffness matrix
$[m_{MNJ}^{mnj}]$	Coefficient matrices as defined in equation 4.7
$[K_t]$	Total stiffness matrix
$[K_{ww}]$	Axial stiffness matrix.
$[K_{uu}]$	Transverse stiffness matrix about Z-Z
$[K_{vv}]$	Transverse stiffness matrix about Y-Y
$[K_{uv}]$	Coupling stiffness matrix about Z-Y
$[K_{wv}]$	Coupling stiffness matrix about W-Y
$[K_{wu}]$	Coupling stiffness matrix about W-Z

Chapter 1

Introduction

1.1 General

The order of complexity of the response of a structural member in three-dimensions depends on a number of factors. These include the nature of the loading, material properties, and the kind of assumptions made in deriving the governing equations. The most general type of member, which combines both axial and flexural loading, is generally termed a 'beam-column'.

The behaviour of beam-columns depends on their slenderness and the load conditions as shown in Fig. 1.1. Failure can occur in either the elastic or inelastic range, in the form of flexural or flexural-torsional buckling, or, more generally, biaxial bending. When a member is bent about its weaker principal axis, or when it is prevented from deflecting laterally while being bent about its stronger principal axis, then only an in-plane flexural response is possible. Flexural-torsional buckling occurs when a member is bent about its stronger axis but is not restrained laterally so that it may buckle out of the plane of bending by deflecting laterally and twisting. If the member is bent about both axes and twisted it will respond in a full three dimensional manner.

1.2 Aim of the study

The purpose of this study is to provide a general formulation suitable for many kinds of cross section such as channel, tee, L, Z, U, I, (mono or doubly symmetric), etc. for structural members acted upon by any form of loading and provided with very general support conditions. The validity and accuracy of this approach is demonstrated by several illustrative examples, the results of which are compared with those obtained previously from either theoretical or experimental investigations.

The domain of this study is summarized in the following:

1- Developing a theoretical analysis for a beam-column having an arbitrary open cross section, which is applicable to both elastic and inelastic analysis.

2- Derivation of linear and nonlinear tangential and geometrical stiffness matrices and strain-displacement matrices for a beam-column in space suitable for many kinds of cross-section subjected to a wide range of loading.

3- Developing a general computer program based on finite element analysis. This program is capable of implementing the formulation to provide numerical solutions.

4- Checking the validity and accuracy of both the derived equations and the computer program by comparing the results against those previously obtained by experimental and theoretical considerations.

5- Investigating problems of bending and torsion not previously fully solved in both the elastic and the inelastic ranges.

1.3 Out-line of the thesis

This thesis contains nine chapters setting out the formulation and implementation of an ultimate strength study of the behaviour of steel beam-columns.

Chapter 1 provides a general introduction to the problem to be investigated. This is followed in Chapter 2 by a selective review of the previous work (theoretical and experimental), within the general area of the structural response of beam-columns.

Chapter 3 presents a pair of general three dimensional formulations, based on the concept of virtual work or the use of energy principles, in which the influence of higher order terms, the effect of initial imperfections (such as residual stresses, initial crockedness, etc.) have been included. Comparison between previous more restricted formulations and this general one are explained in detail. Chapter 4 presents the full stiffness matrices (linear/ nonlinear tangent and geometric stiffnesses and linear/ nonlinear strain matrix together with the interpolation functions and the transformation matrix) required for the implementation of this approach.

Chapter 5 describes both the analytical procedure and the computer program structure. The analytical process is used to generate the section and sectorial properties, internal forces, curvatures, tracing spread of yield through the entire cross-section, etc. The program TDCP (Three Dimensional Computer Program) is based on finite element computer concepts, is written in the Fortran-77 Language, contains a wide variety of options, (in-plane ,out of plane, uniform loads ,distributed loads, initial geometrical imperfections, different boundary conditions, etc.) and may be used to investigate the elastic and inelastic behaviour of members of thin-walled open cross-section, under different load and support arrangements.

Chapters 6 and 7 contain comparisons between the results of this program and those derived previously by other techniques. These illustrate the advantage of both the modified formulation and the more advanced computer program which is capable of correctly accounting for factors such as, absence of symmetry, any form of loading, any pattern of residual stress, any set of initial deformations as well as varying degrees of sophistication in the assumed strain-displacement relations and general geometrical aspects of the problem.

In chapter 8, new problems involving the determination of ultimate strength under bending and torsion are presented in both the elastic and inelastic ranges for beam-columns of I-section.

Chapter 9, presents general conclusions and makes some suggestions for further work.

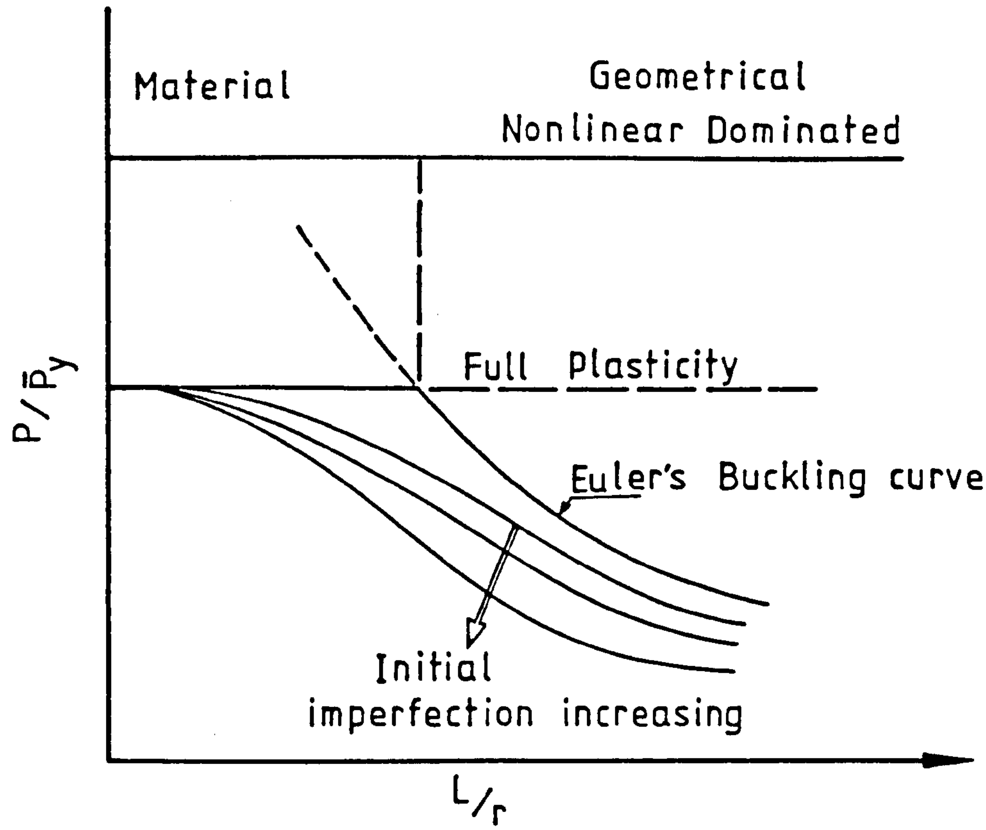


FIG. 1.1 RELATIONS BETWEEN SQUASH LOAD ($\bar{P}_y$), EULER'S LOAD AND FAILURE LOAD OF IMPERFCT COLUMNS

Chapter 2

Review of Previous Work

2.1 Introduction

The behaviour of thin-walled beams and beam-columns of open cross-section is a subject of importance to those concerned with the design of metallic structures. This was initially due to the growth of their use in aircraft followed by increases in their use as members in civil engineering structures. A great deal of research -both theoretical and experimental- has been carried out to provide comprehensive data on which safe and economical designs can be based.

In this thesis a general three-dimensional formulation for the structural response of steel beam-columns having an arbitrary open cross-section is derived taking into account almost all the known factors affecting their behaviour, such as initial crockedness, different patterns of residual stresses, and a wide variety of loads and boundary conditions. The ultimate strength behaviour of beam-columns in both the elastic and inelastic ranges has been investigated by using a computer program based on the resulting finite element analysis. Before undertaking this contribution a review of previous work was conducted.

2.2 Review of Previous Work

2.2.1 Historical & General

The behaviour of a beam-column depends principally on its slenderness ratio. Failure of slender members is often governed by buckling in the elastic range, for which the effects of small geometric imperfections are comparatively unimportant, and the presence of residual stress irrelevant. Thus elastic critical loads may be used to approximate the ultimate strength. However, members of intermediate slenderness normally fail by inelastic buckling, for which the residual stresses are important because they cause early yielding within the cross-section, resulting in a reduction in the effective stiffness of the member, and a lowering of the resistance to buckling. Stocky beam-columns may attain loads determined largely from material strength considerations. Fig. 2.1 represents a typical moment-slenderness relationship for beams. It relates to both idealised members with no initial deflections or residual strains, and a real member for which the effects of these imperfections have been incorporated; failure may be either elastic or inelastic.

A great deal of research work (experimental and theoretical) has been conducted into the behaviour of steel members of thin-walled cross-section subjected to a variety of loading conditions. Some of these studies have included the effects of initial geometrical imperfections. Investigations have covered both the elastic and the inelastic ranges and have considered both two and three dimensional response.

When a laterally unsupported member is subjected to biaxial bending, it will usually deflect in both principal planes and twist at any load level as illustrated in Fig. 2.2. The importance of twisting lies in the fact that the ultimate load carrying capacity of an open cross-section, for which the torsional rigidity is small, will be crucially affected by the torsional aspect of the deformations.

The review given below covers only a selection of contributions to the general subject area, concentrating on some of the more significant developments.

Timoshenko (1910) developed the fundamental differential equations for flexure and torsion of doubly symmetric simply supported I-beams. He solved them for the elastic critical loads by using energy theorems. A particular study was made of the effect of the point of load application when it was remote from the shear centre axis. Wagner (1929) studied the torsional buckling of a thin-walled column; Kappus (1937) modified Wagner's equation and generalized it to deal with any thin-walled open cross section. Bleich (1933,1936) derived an equilibrium equation for a member subjected to axial compression and equal end moments having an I-cross section; in his study the critical loads for flexural-torsional buckling were determined.

A number of investigations have been carried out to determine the critical loads for I-beams subjected to a variety of different load cases. Tabulated results are given by Clark and Hill (1960), Timoshenko and Gere (1961), Vlasov (1961), Galambos (1968), Nethercot and Rockey (1971), and Nethercot (1972). Winter (1941) derived an approximate formula to determine the buckling loads of monosymmetric I-sections under equal end moments. Other load cases have been considered by Petterson (1952), Vlasov (1961), and Anderson and Trahair (1972).

General design methods based on the extensive research on the elastic flexural-torsional buckling of beams have been proposed by Clark and Hill (1960), Trahair (1966), Nethercot and Rockey (1971), and SSRC (1976).

Pekoz and Winter (1966) have noted that the twisting of a beam-column subjected to axial load with eccentricities e_y and e_z, can be explained by considering the physical model illustrated in Fig. 2.3, in which the applied load can be decomposed into four components viz:

1- equivalent to pure axial load P.

2- equivalent to bending moment about weak axis.

3- equivalent to bending moment about strong axis.

4- equivalent to bimoment which causes the bar to warp. (this system produces zero axial load and bending moments on the section.)

A survey of the elastic-plastic behaviour of columns of thin-walled section under biaxial loading has been prepared by Chen and Santathadaporn (1968). A complete account of the developments in this area must necessarily deal with two important aspects: solutions emerging from analytical studies and experimental results obtained by laboratory testing.

Experimental and theoretical investigations have been conducted by Black (1967a) who studied the non-linear elastic behaviour of an unsymmetrical thin-walled beam of open section. The theoretical relationships developed include the effect of torque components arising from the displacements of the beam axes and the point of application of the loads. Experiments were carried out to test the validity of this formulation and he reported that good agreement was obtained

SSRC (1976) reported research results obtained from several theoretical and experimental investigations on beam-columns, including members under uniaxial or biaxial bending, or transverse loads, or combinations of loads. These covered:

1- the evaluation of critical loads in perfect members.

2- the behaviour of beams with initial imperfections.

3- the importance of residual stresses in steel members.

4- developments relating the strain hardening modulus of steel to inelastic buckling behaviour.

5- studies of the combination of the effects of a)initial residual stress, b) geometrical shape imperfections and/ or uncertainties of load location.

A full review of research on beam-columns in steel structures conducted during the last forty years was carried out by Massonnet (1976). This covers the behaviour of beams and beam-columns in the elastic and inelastic range subjected to different load patterns. Since that date extensive additional studies (theoretical and experimental) on the behaviour and design of beam-column have been made. In (1976,1977) Chen and Atsuta provided a two volume text on the behaviour of beam-columns in two and three dimensions; Volume 1 helps the reader to develop an understanding of in-plane behaviour, while the second volume provides a comprehensive source of information on biaxially loaded beam-columns as well as an explanation of their space behaviour under various load conditions. The two volumes taken together comprise the first single reference book to discuss the complete theory of beam-columns systematically from the most elementary to the most advanced stage of development. They also covered some publications which provide background and design rules for beam-columns, specifically:

1- "Guide to Stability Design Criteria for Metal Structures", by SSRC,

2- "Stability of Steel Structures" by ECCS,

3- "Handbook of Structural Stability" by CRC of Japan,

Chen (1977) has provided a review of the theory and design rules for beam-columns under different load patterns and boundary conditions. The basic theoretical principles and methods of analysis in two and three dimensions in the elastic and inelastic ranges have been included together with an assessment of the validity of the proposed interaction approach to the design of biaxially loaded members. Vinnakota (1977) has derived governing differential equations for an arbitrary open cross-section, without making use of the notions of centre of gravity, principal axes and shear centre. The finite difference method has been used to solve these equations for a number of problems having different load conditions.

Chen and Cheong-Siat-Moy (1980) have presented a review of the philosophy behind the various interaction formulas for a beam-column that have been proposed and are under consideration by various specification writing bodies such as the American Institute of Steel Constriction. The general validity of these proposed interaction formulas has been demonstrated by comparison of computed loads with test results.

A survey of recent achievements in the analysis (experimental and numerical solutions) and design of steel members in the USA has been produced by Chen (1981), who investigated the behaviour of an isolated beam-column under the influence of the initial deflections, residual stresses, and various loading and boundary conditions. His proposed interaction formulas have been checked against both computed loads and the available test results.

Kennedy and Madugula (1982) made a comprehensive review of both theoretical and experimental work on the buckling of angles, covering single or built-up angles, equal-leg or unequal-leg angles subjected to axial (either concentric or eccentric) load, transverse load, or a combination of loads. Through their study they found that, depending upon the cross-section, effective length and applied load configuration, members comprising of angle shapes can fail by any of the following :

1- Flexural buckling about the minor axis.

2- Torsional buckling about the shear centre.

3- Torsional-flexural buckling.

4- Local plate buckling.

5- Combination of torsional-flexural buckling and local buckling.

Cescotto et al. (1983) developed design rules for determining the buckling strength of beam-columns of monosymmetric sections (Tee and Triangle). The ultimate buckling loads obtained from experiments gave quite satisfactory agreement with numerical simulations for both cases. The suggested design rules appear as a useful complement to the E.C.C.S. Recommendations. Numerical and experimental analyses have been considered by Nakashima, et al.(1983) to investigate the buckling and post buckling behaviour of steel beams having an H-shape, subjected to a constant axial thrust and monotonically increasing end moments.

Interaction equations of beam-columns in the design specification of Western Europe have been investigated by Nethercot (1983). He presented some quantitative evaluation of these proposals. He also provided a tabulated comparison of these interaction formulae. His investigation covered the following aspects:

1- "Uniaxial bending leading to in-plane failure".

2- "Uniaxial bending producing lateral-torsional buckling".

3- "Biaxial bending".

A full review covering the theoretical and experimental analysis in both the elastic and inelastic ranges for columns, beams, and beam-columns covering the years 1744 to 1984 has been presented by Cuk (1984). Nethercot (1986) reviewed comprehensively the lateral buckling of beams dealing with theoretical approaches in both the elastic and inelastic ranges. It was found that the original theoretical developments could be traced through to the most recent ultimate strength approaches.

This introductory review is intended to provide an indication of both the historical development of the subject and the wide range of research already conducted. In the remainder of this chapter attention will be focussed on specific aspects of the subject.

2.2.2 Elastic Behaviour

2.2.2.1 Flexural & Lateral Torsional Buckling

The elastic flexural and lateral-torsional buckling of beams of different cross-section subjected to a wide variety of loads and boundary conditions have been studied by many investigators.

Anderson and Trahair (1972) presented tabulated results for simply supported monosymmetric I-beams and cantilevers with concentrated and distributed loads, and investigated the influence of load height on the elastic buckling moment. Their results compared very well with test data.

Epstein and Murray (1976) developed a three-dimensional large deflection theory for the analysis of thin walled beams. Numerical examples are presented to illustrate the application of their theory to the solution of elastic torsional post buckling behaviour of I-beams. They reported that their solutions compared well with results obtained from experiments.

Kitipornchai and Trahair (1980) developed a simple method for determining section properties for a wide range of monosymmetric

I-beams, including sections with lipped flanges and also they presented a method to calculate the elastic critical loads of monosymmetric beams. Their rule for the calculated elastic critical load for both monosymmetric and doubly symmetric I-beams has been compared with AS 1250, BS 449, and the AISC specification. More accurate and consistent results were obtained.

A second order differential equation has been derived by Warnick and Walston (1980) using a coordinate system whose orientation remains fixed in space for symmetrical members under different loading conditions. Several examples were examined to investigate the lateral buckling of I-beams. Results of their method were comparable with test data.

The behaviour of nonprismatic structural members (simply supported or cantilever) under transverse concentrated loads has been studied by Brown (1981) using the finite difference method to determine the critical loads of simply supported and cantilever beams. He found that the effect of loads placed either below or above the centroid was significant in all types of beams but leads to an increase, with decreasing free end depth for the cantilever.

Cuk (1984) investigated theoretically isolated/ continuous beam-columns subjected to transverse loads and end moments to determine the elastic flexural-torsional buckling. He reported that his results compared favourably with those obtained experimentally.

Kitipornchai et al (1985) proposed an alternative approximation formula to evaluate the elastic lateral buckling of simply supported monosymmetric I-beams under moment gradient. They found three factors affecting the buckling of monosymmetric sections, which were

1- Wagner effect β_x.

2- End moment ratio β.

3- Degree of monosymmetry ρ ($\rho = I_{zt} / (I_{zt}+I_{zb})$).

Their results were compared with those furnished by the design rule, which employed the moment modification factor $m=1.75+1.03\beta+.3\beta^2 \leqslant 2.56$. It was found that the application of their approach gave reasonable results for beams of nearly equal flanges, but for higher degrees of monosymmetric ($\rho<.3$ and $\rho>.7$) unsafe results were obtained when compared with the previous formula.

2.2.2.2 **Biaxial Bending**

Culver (1966a,b) developed an exact numerical method to solve the differential equations governing biaxial bending and torsion established by Timoshenko and Vlasov (1961). His analysis covers two cases viz :

i- Biaxial bending without initial imperfections

ii- Inclusion of the initial imperfections.

The results of both cases were compared with experimental data; satisfactory agreement was obtained.

A governing differential equation for members of thin-walled section subjected to biaxial bending has been derived by Soltis and Christiano (1972). The effects of large deformations and higher order terms were included. Several illustrative examples have been solved by small and large deflection analysis. The results obtained by these analyses yielded similar results up to 80% of the critical load. For higher loads both out-of-plane displacements and the twist were overestimated by the small deformation approach.

2.2.2.3 **Bending and Torsion**

A two-volume treatise on the behaviour of beams and girders subjected to transverse loading causing torsion has been prepared by BCSA (1968,1970). The first of these presents general theory and formulae together with graphs used to display solutions, while the latter presents worked examples (beams and griders subjected to bending and torsion) to calculate stresses (normal stress, bending stresses about Y and Z axes, warping stress, and shearing stresses) and deflections at any point along the member and around the cross-section.

A theoretical approach has been developed by Kitipornchai and Trahair (1975) to study the strength behaviour of tapered monosymmetric I-beams of constant depth subjected to bending and torsion. Also they carried out experiments on small-scale aluminium I-beams to confirm the validity of their theory. They reported that excellent agreement between the two analyses was obtained.

Pastor and DeWolf (1979) investigated theoretically the behaviour of wide-flange beams under equal end moments and a constant torque applied at mid-span. They considered only small deflections and ignored the coupling effects. The results obtained for three beams having sections W12x120, W12x36, and W 12x14 respectively under M_{cr} / 100 at mid-span and monotonically increasing end moments were tabulated. They suggested the design of members subjected to flexural bending and torsion should involve two checks:

1- The total stress should be compared with the yield stress".

2- The applied moments should be compared with critical moments based on lateral-torsional buckling, with safety considerations.

2.2.3 Inelastic

2.2.3.1 **Flexural and Lateral Torsional Buckling**

Kennedy and Murty (1972) have conducted an experimental investigation aimed at verifying the design approaches for angle and tee struts of the AISC (1969) and CSA S16-1969 (1969) standards covering inelastic flexural, torsional-flexural, and plate buckling. As a result they recommended that design be based on the lowest of the calculated values.

For angle struts

$$\sigma = \frac{5625}{1.67\left(\frac{B}{t}\right)^2} \quad \text{if} \quad \frac{B}{t} \geqslant \frac{75}{\sqrt{\sigma}}$$

$$\sigma = 0.6\sigma \quad \text{if} \quad \frac{B}{t} \geqslant \frac{75}{\sqrt{\sigma}}$$

For tee struts

$$\sigma = \frac{16900}{1.6}\left(\frac{D_\omega}{t\omega}\right)^2 \quad \text{if} \quad \frac{D_\omega}{t_\omega} \geqslant \frac{130}{\sqrt{\sigma}}$$

$$\sigma = 0.6\sigma \quad \text{if} \quad \frac{D_\omega}{t_\omega} \geqslant \frac{130}{\sqrt{\sigma}}$$

The inelastic flexural-torsional buckling of simply supported I-beams under uniform moment and different patterns and magnitudes of residual stresses has been studied theoretically by Trahair and Kitipornchai (1972). They deduced that the changes in the residual stress system led to variations in the yielded regions in the cross section, and consequent variation in the section rigidities. These variations cause very significant changes in the inelastic critical moment.

Abdel-Sayed and Aglan (1973) studied the lateral torsional buckling of wide flange beam-columns subjected to axial force and equal end moments about the major axis. Initial imperfections were considered together with strain hardening effects. They come out with these general conclusions; the lateral torsional buckling reduces the strength of beam-column in the inelastic range, while the residual stresses have negligible effect on the buckling in the elastic range but a significant effect in the inelastic range.

Nethercot (1973a) presented a theoretical solution for monosymmetric I-beams loaded by equal end moments acting in the plane of the web to cause inelastic flexural-torsional instability. He employed the expression for the critical moments deduced by Galambos (1968), which was;

$$M_{cr} = \frac{\Pi}{L}\sqrt{EI_z GK\left(1+\frac{M_\rho}{GK}\right)\left[1+\frac{\Pi^2 EI_\omega}{L^2 GK\left(1+\frac{M_\rho}{EI_\omega}\right)}\right.}$$

in which EI_z is the minor flexural rigidity; GK is the torsional rigidity; EI_ω is the warping rigidity; $M_\rho = \int \sigma\rho^2 dA$; and ρ is the distance from the shear centre to the point where the stress, σ, acts. His conclusion was that the Galambos equation was valid for inelastic buckling providing the stiffness terms, EI_z, GK, EI_ω, and M_ρ are correctly reduced to allow for the presence of yielded material.

Yoshida and Maegawa (1984) examined I-beams subjected to the influence of residual stresses, various loading conditions and geometrical imperfections to determine the lateral-torsional buckling strength, the load-deformation behaviour and the spread of yielded portions in the beam. They also examined the relation between the ultimate strength and the buckling strength for the four theoretical models of Fig. 2.4, which are:

Model-I A beam with out-of-plane deflection.

Model-II Straight beam subjected to a concentrated load at the top flange with eccentricity e_y (e_y is the distance from centre line of web to the loading point on the top flange).

Model-III Beam subjected to a vertical concentrated load P_z on the top flange and P_y applied horizontally at the same point, where the ratio $\frac{P_y}{P_z}$ is kept constant.

Model-IV A beam with out-of-plane displacement under eccentric loading.

Matthey (1984) studied the ultimate strength behaviour of beam-columns of I-section subjected to axial force, and bending moments about the X and Y axes. Residual stresses and initial deflections were included. Those variables were arranged in a systematic fashion to form the framework for a study of more than 2500 cases. For each case the compressive load was applied up to a predetermined limit followed by end moment loading to failure. He used these results to calculate the Performance Factor which is defined as the ratio between the ultimate load for every case obtained from his calculations and that given by the design rules of EC3 (1983), SIA (1979), SIAC (1961), and Chen (1979).

Kitipornchai and Lee (1986a) investigated theoretically the inelastic flexural and flexural-torsional buckling of single-angle, tee and double-angle cross-sections used as simply supported columns subjected to axial load. They found that the flexural buckling mode is the dominant failure mode for most shapes, except for single unequal angles, for tees, and for double angles whose radii of gyration give $\frac{r_y}{r_z}$ greater than 1.0. These theoretical results were checked against other analyses and were in reasonable agreement.

2.2.2.3 Biaxial Bending

An approximate formulation has been provided by Syal and Sharma (1971) for the solution of the generalized problem of biaxially loaded columns with equal or unequal load eccentricities. The effects of the residual stresses, cross-section shape B/D, and warping at the ends being either permitted or restrained have been included. They reported that their results match those previously obtained. Linder (1972) conducted a theoretical investigation to determine the ultimate load of columns of bisymmetrical section under biaxial loading. He employed polynomial expressions for the displacements (U, V, and Φ) in order to obtain a general solution. He provided examples to demonstrate his approach, which incorporates different slenderness, eccentricities, and residual stress.

Epstein et al. (1978) extended the work developed by Epstein and Murray (1976) to deal with nontrivial inelastic stability problems for the prediction of the maximum load-carrying capacity of thin-walled beam-columns of open cross-section under biaxial bending. Results for inelastic biaxial bending and for instability of laterally unsupported beams were compared with the experimental results obtained by Birnstiel (1968) and Lee and Galambos (1963) respectively and satisfactory agreement was obtained.

Bending and Torsion

Theoretical studies on biaxially loaded thin-walled beam-columns of open cross-section with and without incorporating torsional effects have been investigated by Razzaq (1974). He undertook some experimental work to verify the validity of the theoretical predictions of his analysis. Twenty beam specimens were tested up to collapse at two different slenderness ratios for the following loading conditions:

1- Subjected to equal end moments about a principal axis.
2- Subjected first to concentrated torque at midspan, and subsequently to equal end-moments about minor axis.
3- Reverse of case 2 .

It was found that good agreement was obtained between theory and experiment.

Kollbrunner, et al. (1978) have examined theoretically the ultimate strength behaviour of a cantilever member of I-section subjected to bending and warping torsion. They reported that the comparison of the analytical results with those obtained from experiments was good. Kollbrunner, et al. (1979) have carried out theoretical investigations on the elastic-plastic behaviour of thin-walled fixed ended I-beams under bending and torsion. The results compared well with those obtained from experiments in terms of ultimate loads, internal forces, and twisting angles.

2.2.3 Experimental Studies

The behaviour of laterally unsupported angles of equal and unequal leg lengths for a variety of $\frac{L}{t}$ have been investigated experimentally by Thomas et al. (1972). Uniform moments were applied about an axis parallel to an angle leg. Their conclusion was that the angle of twist (Φ) causes a reduction in the maximum section stress and has a significant influence on the maximum loads.

Kitipornchai and Trahair (1975) have conducted an experimental investigation of inelastic flexural-torsional buckling of full-scale simply supported I-beams under concentrated loads. The end supports allow for rotation about both minor and major axes, transverse displacements and twisting about the longitudinal axis were restrained, but warping was free. The beams were loaded to failure and all but one failure was inelastic. They found that the effects of residual stresses was not important, and they confirmed that by theoretical predictions, while the geometrical imperfections were more significant in reducing the strength of beams below their theoretical buckling loads. They also reported that their results were consistent when compared against theoretical and test data together with the Australian Code for the elastic and inelastic ranges.

82 experiments have been conduct in Leige (1983) on beam-columns of I-cross-section (HEA200 and steel grade Fe 360), subjected to biaxially eccentric loading. The following variables were accounted for:

1- The slenderness ratio in both planes ($\frac{L}{r_y}$ and $\frac{L}{r_z}$).

2- The bending moments applied in both planes.

3- The axial load applied.

4- The moment gradient in both planes.

An extensive survey of the experimental investigations preformed at various institutions on beams and girders which failed by lateral instability has been prepared by Fukumoto and Kubo (1977). A total of 275 tests have been included in their review; 159 for beams and 119 for welded beams and girders. They employed statistical characteristics (Mean values (m) and mean minus twice standard deviation) in order to have a comparison between test results and those obtained from recommended design formulae.

The variation of inelastic beam capacity with changing moment gradient of simply supported laterally continuous I-beams has been investigated experimentally by Dux and Kitipornchai (1983). Beam slendernesses were chosen such that buckling should occur in the loading range between the first yielding and the attainment of the plastic moment. The points of load application were prevented from moving laterally and twisting. The test results were compared with theoretical predictions. An experimental investigation to determine the inelastic flexural-torsional buckling of continuous beams of I-section in a sub-assemblage of a three dimensional structural framework has been carried out by Cuk (1984). He reported good agreement with the results obtained from his parallel theoretical study.

A series of experiments on tees and angles in compression has been conducted by Wilhoite et al. (1984a,b) to study their behavior and strength. The tested bars were made of high strength, low-alloy steel with improved formability to match the requirements of ASTM-A-718-81, grade 60. They reported that the results obtained for both sections match fairly well the theoretical predictions.

Kitipornchai and Lee (1986b) investigated experimentally the inelastic flexural and flexural-torsional buckling of single-angle, tee and double-angle cross-sections subjected to axial load. They found that flexural buckling is the dominant failure mode for most shapes, except for single unequal angles, for tees, and double angles whose radii of gyration give $\frac{r_y}{r_z}$ greater than 1.0. Their experimental results were carried out on 54 struts with modified slendernesses ranging from

0.33 to 1.08 for simply supported members prevented from twisting about their longitudinal axis. These experimental results are in reasonable agreement with theoretical predictions.

2.2.4 Finite Element Development

Based on the finite element concept much research has been carried out to obtain formulations for beams and beam-columns in three dimensions, using the equilibrium condition, the virtual work principle or energy principles. Some of these have considered only uniform torsion, whilst others have studied both uniform and nonuniform torsion where the loads applied may be axial, flexural (uniaxial & biaxial) or combinations.

The designation 'finite element concepts', as employed by Barsoum and Gallagher (1970) ,was intended to characterize the formulation of a relationship between the forces and displacements of a single member via simplified assumptions as to the behaviour of the element in terms of stress or displacements. Energy theorems were used to develop the governing differential equation to study the torsion and combined flexural-torsional instability of one dimensional members of a constant cross-section in the elastic range. Their stiffness equations gave excellent agreement (as the number of elements was increased) with existing theory when compared with the following cases:

1- Torsional buckling (pure torsion).

2- Lateral buckling of simply supported beam under applied moments.

3- Lateral buckling of a cantilever beam acted upon by concentrated load at the shear centre.

Rajasekaran (1971) presented a finite element analysis, based on the principle of virtual work, for thin-walled members of open section made from material having a trilinear stress-strain curve. The effects of initial imperfections, residual stresses and different patterns of loads have been included . The validity of this formulation has been demonstrated by Rajasekaran and Murray (1973) for several problems in both the elastic and the inelastic ranges by comparison with existing results.

Nethercot (1973b) presented solutions for the inelastic lateral buckling of I-beams loaded with either uniform or nonuniform moments, with or without the inclusion of the effect of residual stresses obtained by the finite element method. His analysis was applied to several illustrative examples and the results compared with existing theoretical and experimental data, good agreement being obtained.

Epstein and Murray (1976) developed a formulation for the analysis of thin -walled beams of arbitrary open cross-section subjected to arbitrary large displacements in three-dimensions based on a set of kinematic assumptions. They reported that numerical solutions obtained for elastic lateral-torsional buckling for several problems employing their model were consistent with experimental results. The approach opened the way for predicting the real behaviour of structural elements in the large deflection range.

Roberts and Azizian (1981) derived expressions for the second order strains in a thin walled bar of open cross section subjected to flexural, torsional and axial displacements based on energy methods. These expressions could be used for nonlinear analysis. Such analysis would be difficult so it was necessary to employ numerical solution techniques. Roberts and Azizian (1983a) used these expressions to derive the equilibrium differential equations by assuming linear elastic material behaviour. They supported their theory by providing several illustrative examples.

Chaudhary (1982) used the differential equation derived by Vlasov (1961) to develop a general stiffness matrix for a structural member (monosymmetric) of thin walled open cross section subjected to a concentric axial force. The analysis built upon the hypothesis that the presence of bimoment leads to coupling of rotational displacements. He found that bimoment has an important effect with a reduction in wall thickness of the cross section of the structure. A three-dimensional formulation for beams of an arbitrary open section based on large deflection assumptions has been derived by Ramm and Osterrieder (1983). Several illustrative examples are compared with previous work.

Opperman (1983) presented a mathematical model to study the spatial behaviour of thin walled open cross-sections using the finite element method. He reported that good agreement had been achieved through comparisons with experimental results for several load cases.

Attard (1985) presented a nonlinear theory of nonuniform torsion for straight prismatic bars having an open section under conservative loads, where the nonlinear effects of changes in the geometry are ignored in the linear elastic theory of nonuniform torsion. Yeong Bin Yang and Mc guire (1985) adopted the equilibrium equations of thin-walled beams based on the principle of virtual displacements and an updated Lagrangian procedure.

Hasegawa, et al. (1985) presented an analysis scheme for the problem of out-of-plane instability of thin-walled beams and frames. Based on the second order strain-displacement relationships, and the theorem of virtual work, they derived the general stiffness equation of linearized finite displacements for a thin-walled member. Numerical examples, which covered the out-of-plane instability for both straight and curved beams, were compared with existing results. Their conclusion was that the results obtained by them were accurate, efficient and versatile for wide applications. Nishida and Fukumoto (1985) derived an exact expression for the fundamental equations of a member with initial imperfections subjected to the action of bending and torsional moments, and also to investigate the strength of a beam under various load and support conditions.

Wekezer (1985) developed an analysis for the nonlinear torsion of bars of variable cross-section. The stiffness matrix is expressed as a function of the coordinates of a discrete set of points selected from the mid-surface of the bar. The geometrical description of the mid-surface of a bar and its strain were considered. Stresses were obtained from the linear stress-strain relations. He reported good agreement with previously obtained results, either theoretical or experimental.

An incremental equilibrium equation has been derived by Sakimoto, et al. (1985) for a beam-column with arbitrary open cross-section. Features provided in the analysis are:

i- Inelastic warping torsion of a member.

ii- Yield of the material is judged as a bi-axial stress problem associated with both normal and shear stresses.

iii- The stress distribution development of the plastic zones in the cross-section can be easily displayed at each incremental load step.

vi- The effect of arbitrarily distributed residual stresses can be considered.

They reported that their approach shows good agreement when compared with available theoretical and experimental results.

Attard (1986) developed two finite element formulations, the first one ignores initial bending curvature and the second takes into consideration the first order initial curvature. The lateral buckling loads for straight elastic prismatic beams of thin-walled section under conservative loads were investigated. Close agreement was obtained with experimental data.

Ohga and Hara (1986) developed a finite element-transfer matrix method that can be applied to linear and nonlinear problems of thin-walled members under various loading conditions. They employed the Newton-Raphson method to achieve convergence of each iteration step. The section was divided into small layers in order to trace the spread of yield. The von-Mises yield criterion was used. The accuracy of their method was demonstrated by the results obtained by experimental evidence.

Kanok-Nukulchai et al. (1986) presented a formulation for the large deflection of members using a Lagrangian mode of description for the structural elements. Their assumptions were based on

1- Appropriate selection of element geometry, nodes as well as nodal variables

2- Implementation of an element shape function which incorporate all the kinematic characteristics of the applied class of structure.

3- Reduction of the three dimensional model into a suitable form. To establish an element model, several problems have been solved which were found to be in close agreement with experimental results.

Total potential energy has been used by Chan and Kitipornchai (1986) to provide a formulation for a general thin walled beam-column incorporating member geometrical nonlinearity. The proposed finite element formulations were demonstrated on a number of buckling problems ,including flexural-torsional buckling of rectangular beams, tee beams under moment gradient and angle beam-columns. Good agreement has been achieved when compared with independent numerical solutions. The efficiency of his method has been compared with the experimental results obtained by Fukumoto and Nishida (1981).

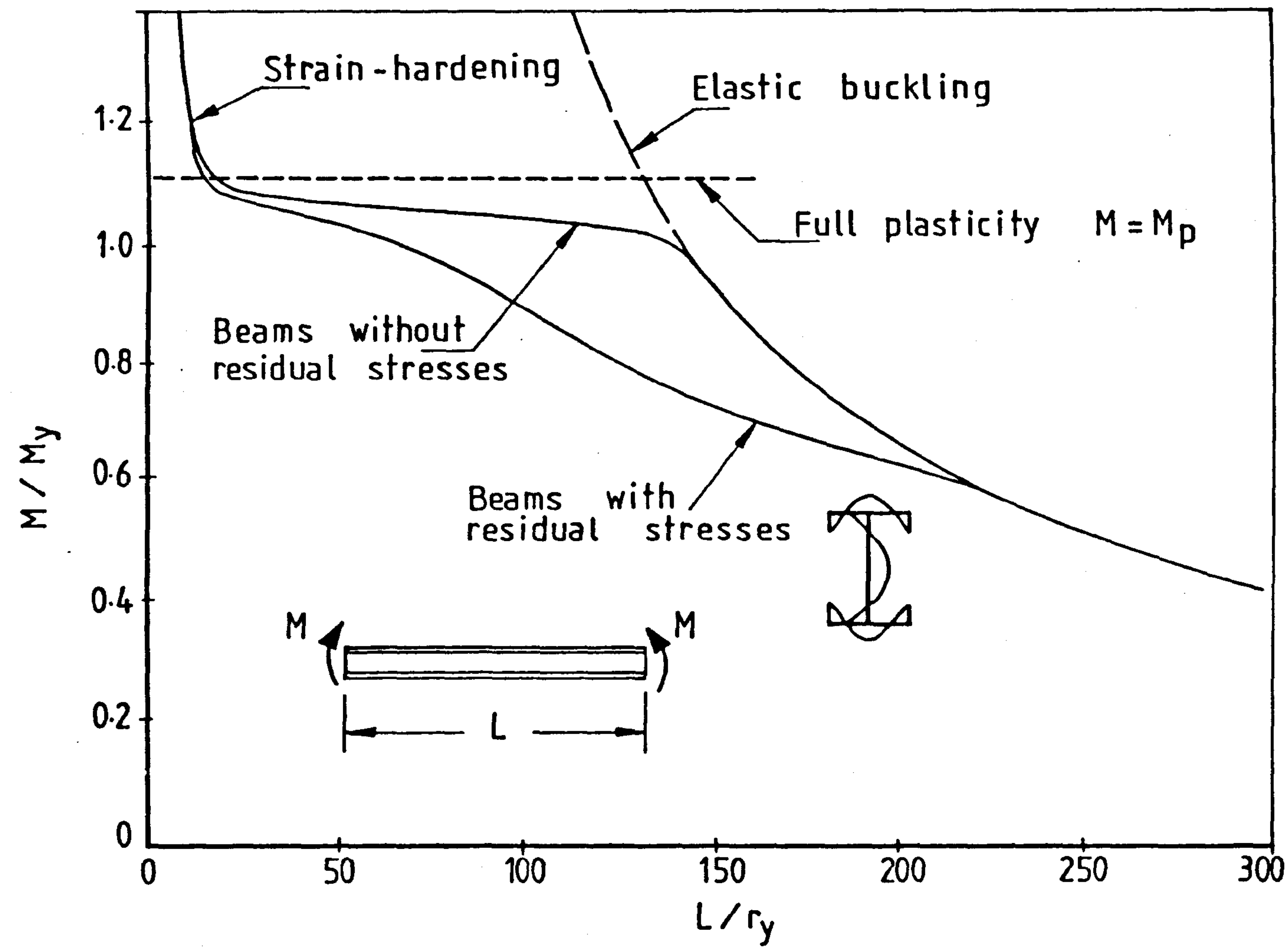
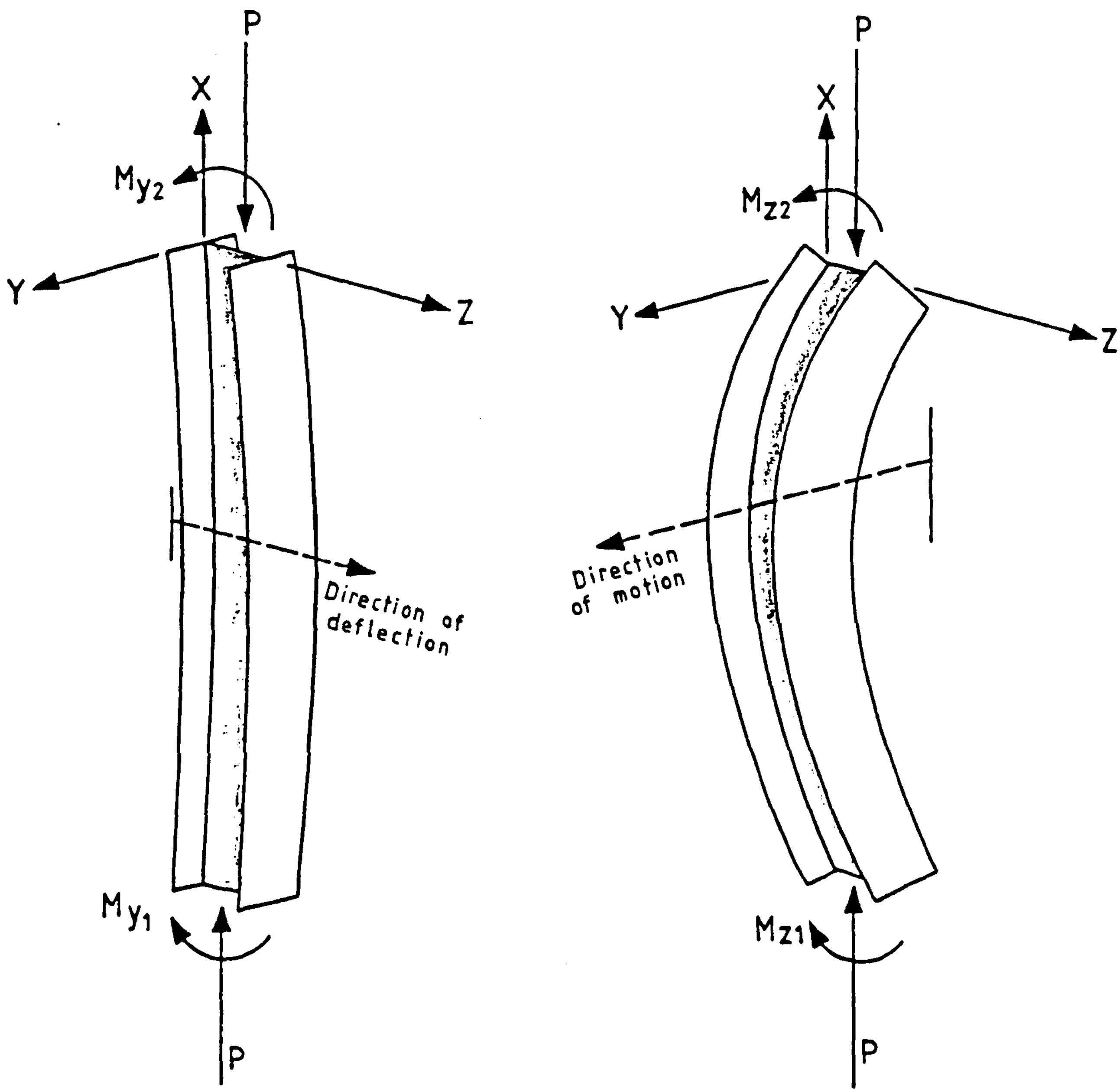


FIG. 2.1 LATERAL BUCKLING STRENGTH OF SIMPLY SUPPORTED I-BEAMS (TRAHAIR 1975)



a - Bending Moment about Y-axis

b - Bending Moment about Z-axis

FIG. 2.2 BEAM-COLUMNS SUBJECTED TO BIAXIAL BENDING ABOUT BOTH AXES

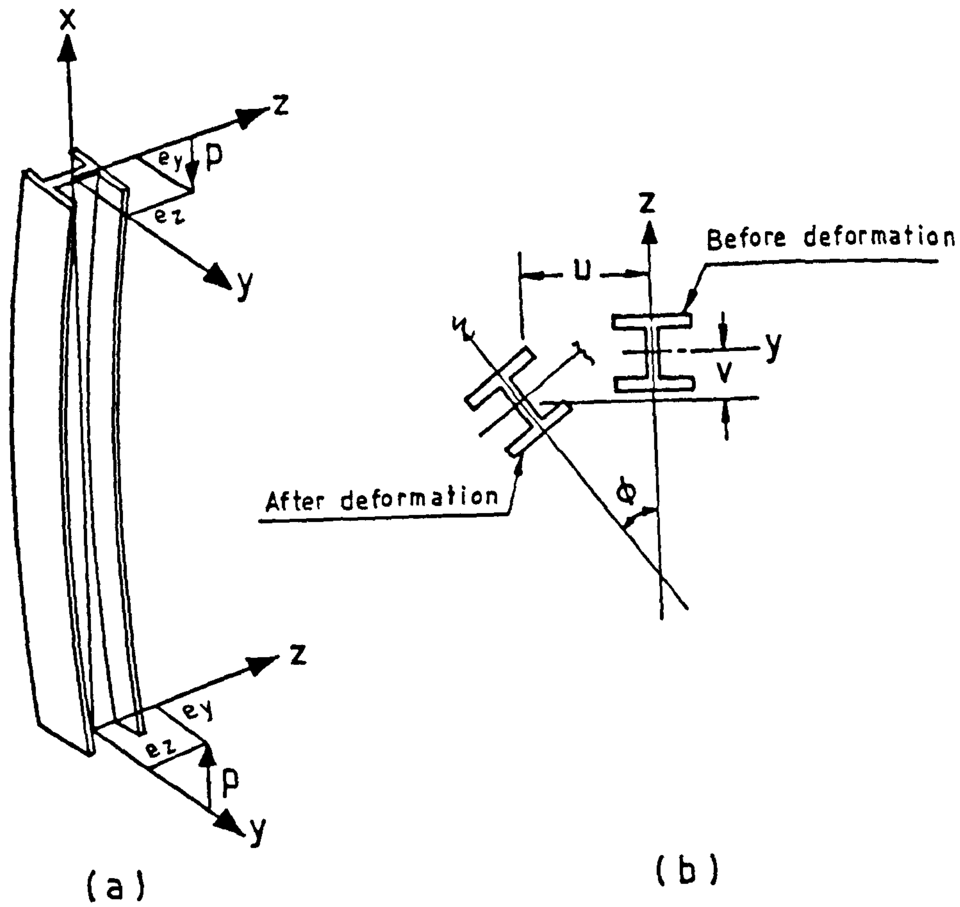


FIG. 2.3 a ISOLATED H-COLUMN UNDER BIAXIAL LOADING.

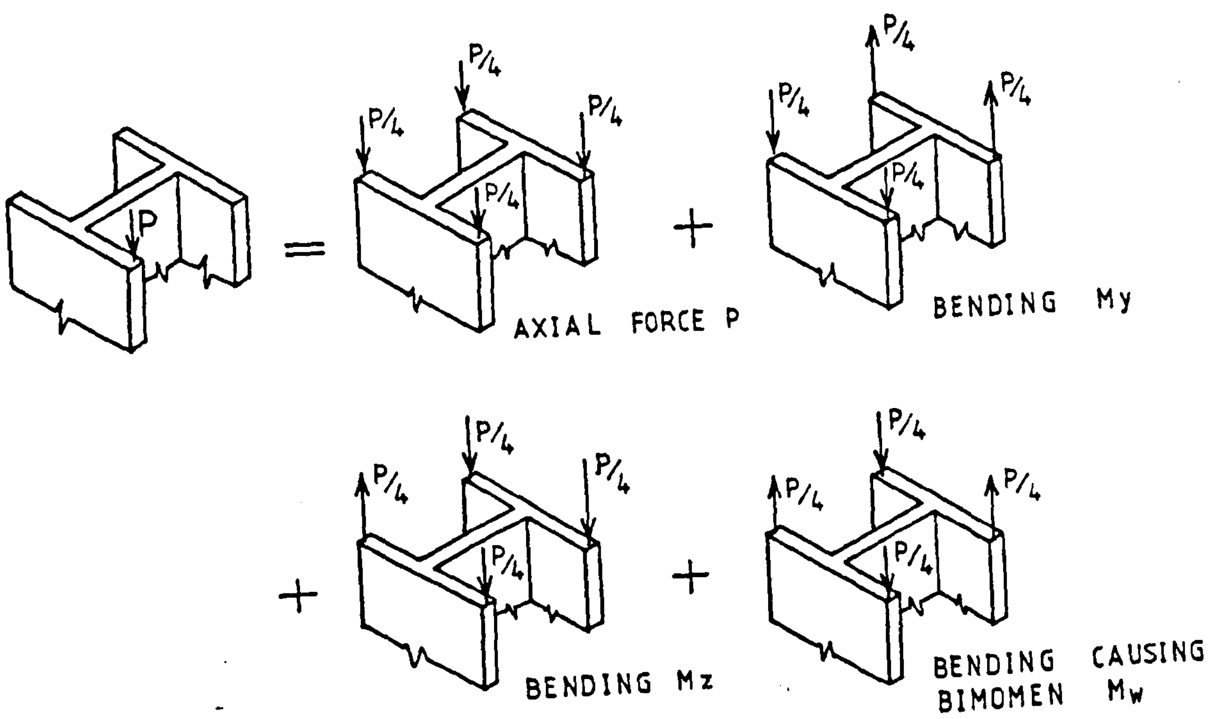


FIG. 2.3 b DECOMPOSITION OF A BIAXIAL LOADING (PERKOZ AND WINTER 1966)

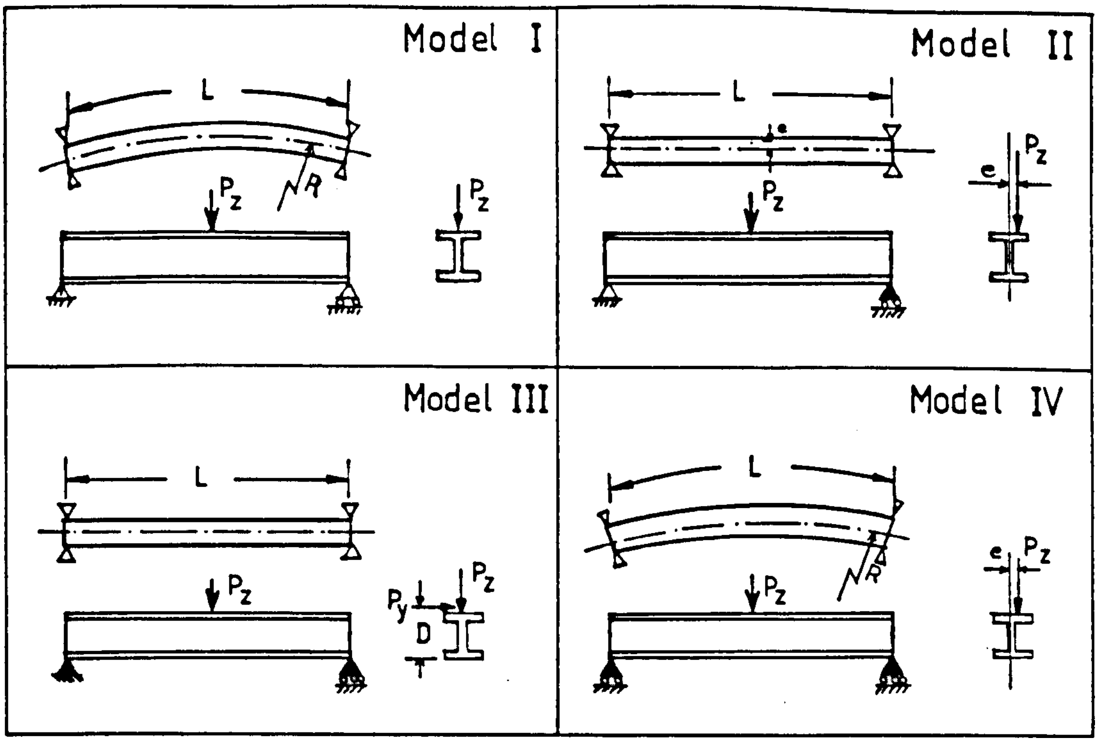


FIG. 2.4a IDEALIZATION OF BEAMS WITH GEOMETRICAL IMPERFECTIONS [YOSHIDA AND MAEGAWA 1984]

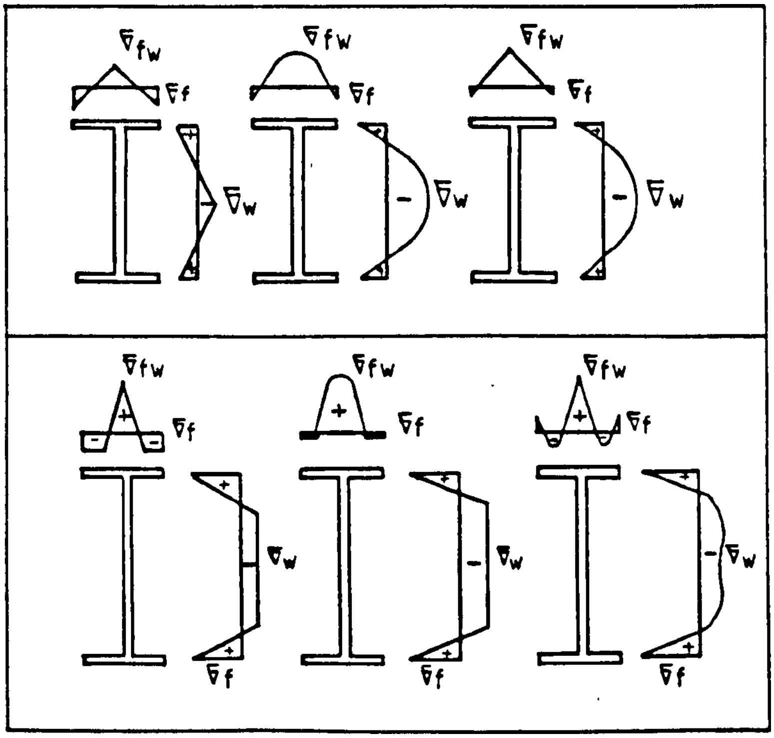


FIG. 2.4b RESIDUAL STRESS DISTRIBUTIONS [YOSHIDA AND MAEGAWA 1984]

Chapter 3

General Formulation of Beam-Column Analysis in Three Dimensions

3.1 Introduction

Much previous research has been conducted to obtain the governing differential equations for beams and beam-columns in three dimensions based on considerations of equilibrium, virtual work, or total potential energy. Some studies considered only uniform torsion, whilst others studied both uniform and non-uniform torsion.

A total potential energy approach has been used by many researchers. In 1970 Barsoum and Gallagher developed a stiffness equation for torsional and combined flexural-torsional elastic instability of one-dimensional members of uniform cross section of doubly symmetric shape for which the shear centre and centroid coincide. Rajasekaran and Murray (1973) derived the differential equations for an arbitrary cross section including inelastic material behaviour, as well as the effect of initial imperfections such as residual stresses. Roberts and Azizian (1981) derived expressions for the second order strains of thin-walled bars of open cross section subjected to flexural, torsional and axial displacements. Roberts and Azizian (1983a) used these expressions to derive the equilibrium differential equations by assuming linear elastic material behaviour, Attard (1986) presented a nonlinear theory of non-uniform torsion for straight prismatic bars having an open section under the action of conservative loads and, Chan and Kitipornchai (1986) applied it to obtain a general formulation for thin walled beam-columns, incorporating member geometrical nonlinearity. The proposed finite element formulations were applied to a number of buckling problems, including flexural-torsional buckling of rectangular beams, lateral buckling of tee beams under moment gradient and angle section beam-columns.

Nishida and Fukumoto (1985) derived an exact expression for the fundamental equations of a member with initial imperfections subjected to the action of bending and torsional moments using the principle of virtual work. The same approach with the second order strain displacement relationships has been considered by Hasegawa et al. (1985) to develop the equilibrium equation for a member having an open cross-section under flexural loading. Yang and McGuire (1986) employed the equilibrium equations of a thin walled beam based on the principle of virtual displacements and an updated Lagrangian procedure to derive the stiffness matrices.

A complete formulation of the equilibrium of a thin walled member of arbitrary cross section in space is presented herein. The equilibrium equations have been derived by two methods, the first is based on the principle of virtual work, whilst the second is based on the energy theorems. Imperfections are included in both formulations. The validity and accuracy of this formulation has been tested on a number of elastic (chapter 6) and inelastic (chapter 7) applications.

3.2 Assumptions

The following assumptions have been used in the analysis:-

I- The beam-column has a general open cross-section.

II- Transverse displacements are much larger than the longitudinal ones.

III- The member length is assumed very large compared with its cross-sectional dimensions.

IV- No distortion of the cross-section occurs apart from warping, (Vlasov 1961).

V- The shearing strain in the middle surface for open cross-sections and in the planes normal to the individual plate elements may be either neglected or included.

VI- Yielding is governed by normal stresses only.

VII- Applied loads are conservative.

3.3 Theoretical Analysis

The development of the equilibrium equations for a three-dimensional beam-column of thin walled open cross-section requires that attention be given to:

3.3.1- Kinematics of a Section.

3.3.2- Stress-Strain Relationship.

3.3.3- Strain-Displacement Relationship.

3.3.4- Analysis Methods.

3.3.4.1- Virtual work.

3.3.4.2- Total Potential Energy.

3.3.1 Kinematics of the Section

A thin walled open cross-section is shown in Fig. 3.1 having Cartesian coordinates X, Y, and, Z, curvilinear coordinates n and s, where s is measured along the centre line of the section, n is perpendicular to s, and t(s) is the thickness of the cross-section at a distance s. S_c and C are the shear centre and the section's centroid. Let U , V , W ,ζ ,and η denote the displacements in the X,Y ,and Z directions, Φ be the angle of twist and U_s and V_s the displacements of point S from the initial position in which Y_s and Z_s represent its coordinates.

The transverse displacement of any point on the cross section can be described by the transverse displacement of a reference point and the rotation of the cross section in its own plane. U_o, V_o, and Φ_o are the initial deformations from the original position. By taking an arbitrary point A on the cross-section the transverse displacements U_A and V_A can be expressed as:

$$U_A = (U_s+U_o) - (Z - Z_s)SIN(\Phi+\Phi_o) - (Y-Y_s)[1 - COS(\Phi+\Phi_o)] \quad (3.1)$$

$$V_A = (V_s+V_o) + (Y - Y_s)SIN(\Phi+\Phi_o) - (Z-Z_s)[1 - COS(\Phi+\Phi_o)] \quad (3.2)$$

From Fig 3.1 the direction cosines are

$$SIN\alpha = -\frac{dy}{ds} \quad (3.3a)$$

$$COS\alpha = \frac{dz}{ds} \quad (3.3b)$$

where α is the angle between the tangent at any point on the contour to the Z-axis.

If η_A and ζ_A are the displacements of point A in the tangential and normal directions respectively.

$$\eta_A = V_A COS\alpha - U_A SIN\alpha \quad (3.4)$$

$$\zeta_A = V_A SIN\alpha + U_A COS\alpha \quad (3.5)$$

Substituting equations 3.1 and 3.2 into equations 3.4 and 3.5 yields

$$\eta_A = (V_s+V_o)COS\alpha-(U_s+U_o)SIN\alpha-r_n\{1-COS(\phi+\phi_0)\} + rSIN(\phi+\phi_0) \qquad (3.6)$$

$$\zeta_A = (V_s+V_o)SIN\alpha+(U_s+U_o)COS\alpha-r_nSIN(\phi+\phi_0)-r\{1- COS(\phi+\phi_o)\} \qquad (3.7)$$

where

$$r = (Y-Y_s)COS\alpha + (Z - Z_s)SIN\alpha \qquad (3.8a)$$

$$r_n = (Z-Z_s)COS\alpha - (Y - Y_s)SIN\alpha \qquad (3.8b)$$

3.3.2 Stress-Strain Relationships

The stress-strain relationship must either be given in analytic form or may be approximated by one or more polynomials or by piecewise linearization ; the use of both methods is summarized in the following section.

1- Analytical Method

Fig. 3.2a represents a σ - ε relationship. It shows that after initial yielding this relation is not linear and ,in particular, the slope of the stress-strain curve is not constant. The tangent of the slope $d\sigma$ / $d\varepsilon$ is denoted by the tangent modulus E_t.

$$E_t = \frac{d\sigma}{d\varepsilon} < E \quad \text{for} \quad \varepsilon > \varepsilon_y \qquad (3.9)$$

2- **Idealization of Stress-Strain Curve**

An elastic-plastic strain-hardening stress-strain curve is a commonly used idealization of the true $\sigma - \varepsilon$ curve for steel in which three straight lines, as illustrated in Fig. 3.2b, are used with the stress being defined by ;

$$\sigma = \begin{cases} E\varepsilon & |\varepsilon| < \varepsilon_y \\ \sigma_y & \varepsilon_Y < |\varepsilon| < \varepsilon_{st} \\ \sigma_y + E_s(\varepsilon - \varepsilon_{st}) & |\varepsilon| > \varepsilon_{st} \end{cases} \tag{3.10}$$

3.3.3 Strain-Displacement Relationships

The strain configuration is defined by displacements U, V and W which are continuous functions of position within the solid. In general this may involve rigid body displacements and deformation of the body as shown in Fig. 3.3

The general strain-displacement relationships are:

$$\varepsilon_{ij} = \frac{1}{2}[U_{j,i} + U_{i,j} + U_{i,k}U_{j,k}] \tag{3.11}$$

where i,j,and k=1,2,3

In unbridged notation the nine equations are :

$$\varepsilon_{xx} = W_{,x} + \frac{1}{2}\{W^2_{,x} + V^2_{,x} + U^2_{,x}\} \tag{3.12a}$$

$$\varepsilon_{yy} = W_{,y} + \frac{1}{2}\{W^2_{,x} + V^2_{,x} + U^2_{,x}\} \tag{3.12b}$$

$$\varepsilon_{zz} = W_{,z} + \frac{1}{2}\{W^2_{,x} + V^2_{,x} + U^2_{,x}\} \tag{3.12c}$$

$$\varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2}\{W_{,y} + U_{,x} + U_{,x}W_{,y} + U_{,x}U_{,y} + V_{,x}V_{,y}\} \tag{3.12d}$$

$$\varepsilon_{xz} = \varepsilon_{zx} = \frac{1}{2}\{W_{,z} + V_{,x} + W_{,x}W_{,z} + U_{,x}U_{,z} + V_{,x}V_{,z}\} \qquad (3.12e)$$

$$\varepsilon_{zy} = \varepsilon_{yz} = \frac{1}{2}\{U_{,z} + V_{,y} + W_{,y}W_{,z} + U_{,y}U_{,z} + V_{,y}V_{,z}\} \qquad (3.12f)$$

in which subscripts x, y, z denote differentiation with respect to that parameter.

Equation 3.12 allows for shear deformation due to St. Venant torsion based on the hypothesis that the twisting of any cross-section will produce only shear stresses. This shear stress varies from zero at the centre of the section to its maximum value at an external surface. It is assumed that twisting of such sections does not produce any longitudinal stresses (Zbirohowski-Koscia 1967).

Using assumption V leads to

$$\varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_{yz} = 0. \qquad (3.13)$$

so equation 3.12 can be reduced to

$$\varepsilon_{xx} = W_{,x} + \frac{1}{2}\{W^2_{,x} + V^2_{,x} + U^2_{,x}\} \qquad (3.14a)$$

$$\varepsilon_{yx} = \varepsilon_{xy} = \frac{1}{2}\{W_{,y} + U_{,x} + W_{,x}W_{,y} + U_{,x}U_{,y} + V_{,x}V_{,y}\} \qquad (3.14b)$$

$$\varepsilon_{zx} = \varepsilon_{xz} = \frac{1}{2}\{W_{,z} + V_{,x} + W_{,x}W_{,z} + U_{,x}U_{,z} + V_{,x}V_{,z}\} \qquad (3.14c)$$

Equation 3.11 can be written in matrix form by using equations 3.13 and 3.14 as

$$[\varepsilon_{ij}] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{yx} & \varepsilon_{yz} \\ \varepsilon_{zx} & 0 & 0 \\ \varepsilon_{yz} & 0 & 0 \end{bmatrix} \tag{3.15}$$

The curvilinear shear strain can be obtained from equation 3.12 using assumption 2.

$$\gamma_{sx} = 2\varepsilon_{sx} = \frac{d\gamma_A}{dx} + \frac{dW}{ds} + \frac{dU_A}{ds}\frac{dU_A}{dx} + \frac{dV_A}{ds}\frac{dV_A}{dx} \tag{3.16}$$

$$\gamma_{nx} = 2\varepsilon_{nx} = \frac{d\gamma_A}{dx} + \frac{dW}{dn} + \frac{dU_A}{dn}\frac{dU_A}{dx} + \frac{dV_A}{dn}\frac{dV_A}{dx} \tag{3.17}$$

Apply assumption 5 to equation 3.16.

$$\frac{d\gamma_A}{dx} + \frac{dW}{ds} + \frac{dU_A}{ds}\frac{dU_A}{dx} + \frac{dV_A}{ds}\frac{dV_A}{dx} = 0 \tag{3.18}$$

Substituting equations 3.1, 3.2, and 3.7 in equation 3.18 yields

$$\frac{dW}{ds} = -\frac{dy}{ds}[V_{,x}SIN(\Phi+\Phi_o) + U_{,x}COS(\Phi+\Phi_o)]$$

$$-\frac{dz}{ds}[V_{,x}COS(\Phi+\Phi_o) - U_{,x}SIN(\Phi+\Phi_o)] - r\,\Phi_{,x} \tag{3.19}$$

Integrating equation 3.19 as

$$W = W_c - Y[V_{,x}SIN(\Phi+\Phi_o) + U_{,x}COS(\Phi+\Phi_o)]$$

$$- Z[V_{,x}COS(\Phi+\Phi_o) - U_{,x}SIN(\Phi+\Phi_o)] - \int_s \{r\,\Phi_{,x}\}ds \tag{3.20}$$

where

$$W_c = W_A + Y_A[V_{,x}SIN(\Phi+\Phi_o) + U_{,x}COS(\Phi+\Phi_o)]$$

$$- Z_A[V_{,x}COS(\Phi+\Phi_o) - U_{,x}SIN(\Phi+\Phi_o)] \qquad (3.21)$$

where the subscript A is the integral limits e.g. Y at $Y=Y_A$ and $Z= Z_A$.

The last term of equation 3.20 may be determined from the sectional coordinates of a beam having a cross section with a curved outline as given in Fig. 3.4 and is calculated by integration of this equation.

$$\delta\omega_{Ds_1s_2} = \delta S_1 \times r_{s_1s_2} \qquad (3.22a)$$

where Ds_1s_2 is equal to the incremental area, $r_{s_1s_2}$ is equal to the distance of pole D to a line passing through points S_1 and S_2 and dS_{12} is the distance between S_1 and S_2. Integrating the above equation between O and S_n (the length of the centerline of a segment of the section measured from point O) yields.

$$\omega_{DOS} = \int_o^{S_n} r(s)ds \qquad (3.22b)$$

Further details are given in Zbirohowski-Koscia (1967). Denoting the normalized function by $\bar{\omega}_n$

$$\bar{\omega}_n = \int_o^{S_n} rds \qquad (3.23)$$

Substituting equation 3.22b into equation 3.20 yields

$$W = W_c - Y[V_{,x}SIN(\Phi+\Phi_o) + U_{,x}COS(\Phi+\Phi_o)]$$

$$- Z[V_{,x}COS(\Phi+\Phi_o) - U_{,x}SIN(\Phi+\Phi_o)] - \bar{\omega}_n\Phi_{,x} \qquad (3.24)$$

Equation 3.14 in terms of stress-displacements can be rewritten as:

$$\sigma_{xx} = E^{*} [W_{,x} + \frac{1}{2}\{V^2_{,x} + U^2_{,x} + W^2_{,x}\}] + \sigma_r \qquad (3.25a)$$

$$\sigma_{xy} = \frac{E^{*}}{2}[W_{,y} + U_{,x} + W_{,x}W_{,y} + U_{,x}U_{,y} + V_{,x}V_{,y}] \qquad (3.25b)$$

$$\sigma_{xz} = \frac{E^{*}}{2}[W_{,z} + V_{,x} + W_{,x}W_{,z} + U_{,x}U_{,z} + V_{,x}V_{,z}] \qquad (3.25c)$$

Substituting equations 3.1, 3.2 and 3.20 into equation 3.25a yields a longitudinal strain which can be written as .

$$\epsilon_{xx} = [W_{,x} + \frac{1}{2}\{U^2_{,x} + V^2_{,x} + \rho^2\phi^2_{,x}\}]$$

$$-Y \left\{U_{,xx} - V_{,xx}\Phi - U_{,xx}\frac{\Phi^2}{2} + V_{,xx}\Phi_o + U_{,x}\Phi_{,x}\Phi_o\right\}$$

$$-Z \left\{V_{,xx} - V_{,xx}\frac{\Phi^2}{2} - U_{,xx}\Phi - U_{,xx}\Phi_o + V_{,x}\Phi_{,x}\Phi_o\right\}$$

$$- Y_s\Phi_{,x}\{ V_{,x} - U_{,x}\Phi\} + Z_s\Phi_{,x}\{U_{,x} + V_{,x}\Phi\} - \bar{\omega}_n\Phi_{,xx}] + \epsilon_r \qquad (3.26)$$

Equation 3.26 may be written in another form as

$$\varepsilon = \varepsilon_0 - \Phi_z Y - \Phi_y Z + \Phi_\omega \bar{\omega}_n + \varepsilon_r \qquad (3.27)$$

where ε_o , $\Phi_{,x}$, $\Phi_{,y}$ and Φ_ω are the representative axial strain , biaxial curvatures about the Y , and Z axes and warping curvature respectively . They are given by

$$\varepsilon_0 = W_{,x}+\frac{1}{2}\{U^2_{,x} + V^2_{,x} + \rho^2\phi^2_{,x}\}$$

$$- Y_s\Phi_{,x}\{ V_{,x} - U_{,x}\Phi\} + Z_s\Phi_{,x}\{U_{,x} + V_{,x}\Phi\} + \varepsilon_r \qquad (3.28)$$

$$\Phi_y = V_{,xx} - V_{,xx}\frac{\Phi^2}{2} - U_{,xx}\Phi - U_{,xx}\Phi_o - V_{,x}\Phi_{,x}\Phi_o \tag{3.29a}$$

$$\Phi_z = U_{,xx} - U_{,xx}\frac{\Phi^2}{2} - V_{,xx}\Phi + V_{,xx}\Phi_o - U_{,x}\Phi_{,x}\Phi_o \tag{3.29b}$$

$$\Phi_\omega = \Phi_{,xx} \tag{3.29c}$$

Equation 3.27 can be written in matrix form as

$$\sigma = E\,[1 \quad -Z \quad -Y \quad -\overline{\omega}_n]\begin{bmatrix}\varepsilon_0\\ \Phi_y\\ \Phi_z\\ \Phi_w\end{bmatrix} \tag{3.30}$$

The stress resultants for a thin-walled member in space are

$$P = \int_A \sigma_{xx}\, dA \tag{3.31a}$$

$$Q_y = \int_A \sigma_{xy}\, dA \tag{3.31b}$$

$$Q_z = \int_A \sigma_{xz}\, dA \tag{3.31c}$$

$$M_y = \int_A \sigma_{xx}\, Z\, dA \tag{3.31d}$$

$$M_z = \int_A \sigma_{xx}\, Y\, dA \tag{3.31e}$$

$$W_\omega = \int_A \sigma_{xx}\, \overline{\omega}_n\, dA \tag{3.31f}$$

$$M_x = \int_A [\sigma_{xy} (Y-Y_s) - \sigma_{xz} (Z-Z_s)] \, dA \tag{3.31g}$$

$$M_\rho = \int_A \sigma_{xx} [(Y-Y_s)^2 + (Z-Z_s)^2] \, dA \tag{3.31h}$$

The generalized stress and generalized strain relations are then obtained by combining equations 3.30, 3.31, and 3.32.

$$\begin{bmatrix} P \\ M_y \\ M_z \\ M_w \end{bmatrix} = \begin{bmatrix} EA & EA\bar{Z} & EA\bar{Y} & -ES_w \\ & EI_y & -EI_{yz} & -EI_{wz} \\ & & EI_z & -EI_{wy} \\ & & & EI_w \end{bmatrix} \begin{bmatrix} \varepsilon_0 \\ \Phi_y \\ \Phi_z \\ \Phi_w \end{bmatrix} \tag{3.32}$$

3.3.4 **Method of Analysis**

Consider the equilibrium of a general three dimensional body of the type illustrated in Fig. 3.5 The external forces acting on the body are surface traction f^s, body forces f^b and concentrated forces F_i. These forces include all externally applied forces and reactions and have in general three components corresponding to the three coordinate axes.

$$f^b = \begin{bmatrix} f_x^b \\ f_y^b \\ f_z^b \end{bmatrix} , \qquad f^s = \begin{bmatrix} f_x^s \\ f_y^s \\ f_z^s \end{bmatrix} , \qquad F_i = \begin{bmatrix} F_{xi} \\ F_{yi} \\ F_{zi} \end{bmatrix} \tag{3.33}$$

3.3.4.1- Principle of Virtual Work (displacements)

When a particle begins to move from a state of rest, it is gaining kinetic energy. Therefore, according to the law of kinetic energy, the forces F_i which act on the particle are performing net positive work. Hence, the particle does not move unless it can undergo some arbitrary small displacement, say δr_i (virtual displacement), for which the corresponding increment of work δw (virtual work) of forces is positive.

An equivalent approach which may be used to obtain equations that express the equilibrium of the body is through the use of the principle of virtual displacements. This principle states that for the body to be in equilibrium for any compatible set of "small" virtual displacements imposed on the body the total internal virtual work must be equal to the total external virtual work i.e.

$$\int_V \sigma_{ij}\, \delta\, \varepsilon_{ij}\, dv = \int_V f^b\, \delta\, r\, dv$$

$$+ \int_s f^s\, \delta\, r\, ds + \int_s F_i\, \delta\, r\, ds \qquad (3.34)$$

The external work will be given by the product of the actual forces f^b , f^s ,and F_i acting through the virtual displacements δr, where

$$r = [\;U \quad V \quad \Phi \quad W\;] \qquad (3.35)$$

The internal virtual work is equal to the actual stresses σ_{ij} acting through the virtual strains $\delta\varepsilon_{ij}$ corresponding to the imposed virtual displacements.

$$\delta\varepsilon_{ij} = \delta[\;\varepsilon_{xx} \quad \varepsilon_{yy} \quad \varepsilon_{zz} \quad \varepsilon_{xy} \quad \varepsilon_{yz} \quad \varepsilon_{zx}\;] \qquad (3.36)$$

The superscript S denotes that surface displacements are considered and the superscript i denotes the displacements at the point where the concentrated force F_i acts (Bathe and Wilson 1976). If the body forces and traction forces are neglected equation 3.34 can be written as

$$\int_V \sigma_{ij}\ \delta\ \varepsilon_{ij}\ dv = \int_s F_i\ \delta\ r\ ds \qquad (3.37)$$

Substituting equations 3.25a,b, and c into equation 3.37 yields and assuming the initial strains are neglected herein by this approach.

$$\int_V E^{*}\varepsilon_{xx}\delta W_{,x}dV + \frac{1}{2}\int_V \sigma_{,xx}\delta[W^2_{,x} + U^2_{,x} + V^2_{,x}]dV$$

$$+ \int_V G^{*}[\{W_{,y} + U_{,x}\}\ \delta\{W_{,y} + U_{,x}\} + \{W_{,z} + V_{,x}\}\delta\{W_{,z} + V_{,x}\}]dV$$

$$\frac{1}{2}\int_V \sigma_{xz}\delta\{W_{,x}W_{,z} + U_{,x}U_{,z} + V_{,x}V_{,z}\}dV$$

$$\frac{1}{2}\int_V \sigma_{xy}\delta\{W_{,x}W_{,y} + U_{,x}U_{,y} + V_{,x}V_{,y}\}\ dV - \int_s F_i\delta r ds = 0 \qquad (3.38)$$

3.3.4.2 <u>**Total Potential Energy**</u>

Several references such as Bleich (1952), Kerensky et al. (1956), Timoshenko and Gere (1961), Galambos (1968), and Zienkiewicz (1977) have covered full details of the energy theorems, these can be summarized as: the total potential energy is equal to the summation of both strain energy ($\overline{U}$) and potential energy due to applied loads ($\overline{V}$).

$$\Pi = \overline{U} + \overline{V} \qquad (3.39)$$

where

$$\overline{U} = \frac{1}{2}\int_V [\sigma_{ji}][D][\Sigma_{ij} dV] \tag{3.40a}$$

where D is Hookean's constant relating to the material properties. The above equation including the initial strains can be written in an other form as:

$$\overline{U} = \frac{1}{2}\int_V [\Sigma^L + \Sigma^{NL}][D][\Sigma^L + \Sigma^{NL}] \tag{3.40b}$$

where Σ^L and Σ^{NL} are linear and nonlinear strains.

$$\overline{V} = \int_A \overset{=}{P}\delta r dA \tag{3.40c}$$

where

$$P^{=} = [\ F_{xk} \quad Q_{yk} \quad M_{zk} \quad Q_{zk} \quad M_{yk} \quad M_{xk} \quad M_{\omega k}]^T \tag{3.40d}$$

where k is equal to 1 or 2 and the superscipt T is the transpose of a matrix.

3.4 Imperfections

The following factors severely weaken the strength of the structure:

i- Initial deflections.

ii- Residual stresses.

i- Initial Deflections

The initial deflection (U_o, V_o, and Φ_o) quantities may be completely arbitrary but for convenience the following basic arrangements are automatically included:

i- Sine wave.

$$U_o = \mathrm{SIN}\,\pi\frac{X}{L} \qquad (X = 0 \text{ to } L) \tag{3.41}$$

ii- Polynomial Function.

$$U_o = a_0 X + a_1 X^2 + a_2 X^3 + \ldots\ldots\ldots\ldots\ldots\ldots + a_n X^n \tag{3.42}$$

Different arrangements are provided for each of the main deflection components i.e. U_o, V_o, Φ_o.

ii- Residual Stresses

The presence of residual stresses, exerts a significant influence on the way in which yield spreads through the cross-section; it must therefore be included in any analysis which attempts to model real beam-column behaviour. These stresses arise from non-uniform temperature distributions during manufacture (Nethercot 1974), so that the parts which cool slowly, such as the web to flange junction of an I-section will normally be in residual tension (Young 1971, Nethercot and Trahair(1983)) with balancing residual compression elsewhere in the section.

These longitudinal residual stresses must produce a system of forces in the cross-section which must itself satisfy the following three equilibrium equations:

$$\int_A \sigma_r \, dA = 0 \tag{3.43}$$

$$\int_A \sigma_r Y dA = 0 \qquad (3.44)$$

$$\int_A \sigma_r Z dA = 0 \qquad (3.45)$$

If the measured residual stress distributions do not satisfy the above conditions, then these stresses must be adjusted until the necessary balance is obtained. For I and H sections this can best be achieved by modifying the maximum web residual stress until equilibrium results.

Several different forms of standard residual stress distribution have been proposed for use in analytical work (O'conner (1955), Young 1975, Nethercot (1974), Nethercot and Trahair (1983), Kitipornchai and Wang-Chung (1985)). Some of these distributions are considered herein although because the present study addresses the full 3-dimensional response of a beam-column some limitations in the proposals not identified in the context of 2-dimensional behaviour have been corrected.

1- Lehigh distribution.

2- Parabolic distribution.

3- Triangle distribution.

1- Lehigh Pattern

Fig. 3.6a illustrates the residual stress distributions used in much of the work conducted at Lehigh University. A linear function is chosen to represent the variations of the stresses along the web and the flanges. It can be written as:

$$\sigma_1 = a_0 X + a_1 \qquad (3.46)$$

Application of the boundary conditions to the above equation yields

i)- For the flange

$$\text{at } X = 0 \qquad \sigma_{fw} = a_1 \qquad (3.47a)$$

$$\text{at } X = B \qquad \sigma_f = a_0 B + \sigma_{fw} \qquad (3.47b)$$

from equation 3.47a and 3.47b, we obtain

$$a_0 = \frac{\sigma_f - \sigma_{fw}}{B} \qquad (3.48)$$

Substituting equations 3.47a and 3.48 into equation 3.46 yields equation 3.49, which can be used to evaluate the residual stress at any point along the flanges.

$$\sigma_1 = \frac{\sigma_f - \sigma_{fw}}{B} X + \sigma_{fw} \qquad (3.49)$$

ii)- for the web

The residual stress is assumed constant. So equation 3.47 after applying the boundary condition becomes

$$\sigma = \sigma_w \qquad (3.50)$$

where

σ_f is the flange tip residual stress.

σ_w is the web residual stress

σ_{fw} is the web to flange junction residual stress

2- Parabolic Distribution

Recent measurements have suggested that the Lehigh model is too simple to accurately model the true patterns found in many shapes. An improvement results if simple parabolic functions are assumed to represent the residual stress distributions for both web and flange at any point along the flange or the web. It can be written as:

$$\sigma_{\backslash} = a_0 X^2 + a_1 \tag{3.51}$$

Where a_0,anda_1 are arbitrary constants to be determined from boundary conditions for the web and flanges.

i)- For the web

$$\text{at } X=0 \quad \sigma_w = a_1 \tag{3.52a}$$

$$\text{at } \frac{X=D}{2} \quad \sigma_{fw} = a_0 \frac{D^2}{2} + \sigma_w \tag{3.52b}$$

$$a_0 = \frac{\sigma_{fw} - \sigma_w}{(\frac{D}{2})^2} \tag{3.53}$$

Substituting equations 3.52a and 3.53 into equation 3.51 yield a general equation which represents distributions of the residual stresses at any point along the web.

$$\sigma = \frac{\sigma_{fw} - \sigma_w}{(\frac{D}{2})^2} X^2 + \sigma_w \tag{3.54}$$

ii)- For the flange

Equation 3.55 can be obtained similar to equation 3.54, by substituting for the boundary conditions in equation 3.51

$$\sigma = \frac{\sigma_f - \sigma_{fw}}{(\frac{B}{2})^2} X^2 + \sigma_f \qquad (3.55)$$

This distribution is illustrated in Fig. 3.6b

3- <u>Triangular Distribution</u>

This is an alternative simple distribution illustrated in Fig. 2c. Equation 3.47 can be implemented to calculate the residual stresses at any point along the web or the flange noting the change in boundary condition required. More sophisticated models (El-Khenfas and Nethercot 1987a), including any arbitrary pattern which satisfies equations 3.43, 3.44, and 3.45, may readily be incorporated.

3.5 Equilibrium Equations

Two methods are employed herein in order to obtain general equilibrium equations for different kinds of open thin-walled sections under a variety of loading and boundary conditions, which are:

1- Virtual Work

2- Total Potential Energy

3.5.1 By Virtual Work

Using Appendix A, assuming the terms containing the products of $W_{,x}$ in the shear strain i.e. $W_{,x}W_{,y}$ and $W_{,x}W_{z}$, to have significant magnitude so that they cannot be assumed equal to zero, in equation 3.38 the general equilibrium equation of an arbitrary thin-walled open cross-section can be written as:

$$\int_0^L \Big[EA[\{W_{,x} + \frac{1}{2}(U^2_{,x} + V^2_{,x} + \rho^2\Phi^2_{,x})$$

$$- \overline{Y}(V_{,xx}\Phi + U_{,xx}) - \overline{Z}(V_{,xx} - U_{,xx}\Phi)$$

$$- Y_s\Phi_{,x}U_{,x} + Z_sV_{,x}\Phi_{,x} - \frac{ES_\omega}{EA}\Phi_{,x}\}\delta W_{,x}]$$

$$- EA\overline{Y}\{W_{,x}\delta(V_{,xx}\Phi + U_{,xx}) + \frac{1}{2}(U^2_{,x} + V^2_{,x})$$

$$+ Y_s\Phi_{,x}V_{,x} - Z_s\Phi_{,x}U_{,x})\}\delta U_{,xx}\Big]$$

$$+ EAW_{,x}\delta(V_{,x}\Phi_{,x}) + \frac{1}{2}EA\overline{Y}(Y_s^2 + Z_s^2)\Phi^2_{,x}\delta U_{,xx}$$

$$- EA\overline{Z}[W_{,x}\delta(V_{,xx} - U_{,xx}\Phi) + \{\frac{1}{2}(U^2_{,x} + V^2_{,x})$$

$$+ Y_s\Phi_{,x}V_{,x} - Z_s\Phi_{,x}U_{,x}\}\delta V_{,xx}]$$

$$- EAW_{,x}\delta(U_{,x}\Phi_{,x}) + \frac{1}{2}EA\{Y_s^2 + Z_s^2\}\Phi^2_{,x}\delta V_{,xx}$$

$$+ EI_\omega\Phi_{,xx}\delta\Phi_{,xx} + EI_y[V_{,xx}\delta(V_{,xx} - U_{,x}\Phi_{,x})$$

$$- \delta(U_{,xx}V_{,xx}\Phi) - Z_s\Phi^2_{,x}\delta V_{,xx}]$$

$$+ EI_z[U_{,xx}\delta(U_{,xx} + V_{,x}\Phi_{,x}) - \delta(U_{,xx}V_{,xx}\Phi) - Y_s\Phi^2_{,x}\delta U_{,xx}]$$

$$+ EI_{yz}[\delta(U_{,xx}V_{,xx} - U_{,xx}\Phi U_{,xx} + V_{,xx}\Phi V_{,xx}) + V_{,xx}\delta(\Phi_{,x}V_{,x})$$

$$- U_{,xx}\delta(\Phi_{,x}U_{,x}) - \Phi^2_{,x}(Z_s\delta U_{,xx} + Y_s\delta V_{,xx})]$$

$$+ EI_{\omega y}[\ \delta\{\Phi_{,xx}(V_{,xx}\Phi + U_{,xx})\} + \Phi_{,xx}\delta(\Phi_{,x}V_{,x}) - Y_s\Phi^2_{,x}\delta\Phi_{,xx}]$$

$$+ EI_{\omega z}[\ \delta\{\Phi_{,xx}(V_{,xx} - U_{,xx}\Phi)\} - \Phi_{,xx}\delta(\Phi_{,x}U_{,x}) - Z_s\Phi^2_{,x}\delta\Phi_{,xx}]$$

$$- ES_\omega[W_{,x} + \frac{1}{2}\Phi_{,x}(U_{,x} + V_{,x}) + Y_s\Phi_{,x}V_{,x} - Z_s\Phi_{,x}U_{,x}$$

$$+ \frac{1}{2}(Y_s^2 + Z_s^2)]\delta\Phi_{,xx}$$

$$+ \beta_\omega \Phi^2_{,x}\delta\Phi_{,xx} + \beta_y \delta(\Phi^2_{,x}U_{,xx}) + \beta_z \delta(\Phi^2_{,x}V_{,xx}) + KG\Phi_{,x}\delta\Phi_{,x}$$

$$+ P[V_{,x}\delta V_{,x} + U_{,x}\delta U_{,x} + Z_s \delta\{\Phi_{,x}(V_{,x}\Phi + U_{,x})\}$$

$$- Y_s \delta\{\Phi_{,x}(V_{,x} - U_{,x}\Phi)\}]$$

$$+ M_\rho \Phi_{,x}\delta\Phi_{,x} + M_z \delta\{\Phi_{,x}(V_{,x} - U_{,x}\Phi)\} - M_y \delta\{\Phi_{,x}(V_{,x}\Phi + U_{,x})\}$$

$$+ Q_y[\delta(V_{,x}\Phi - W_{,x}(V_{,x}\Phi + U_{,x}))$$

$$- Z_s \delta\{V_{,x}(U_{,xx}\Phi + U_{,x}\Phi_{,x}) + V_{,x}(V_{,xx}\Phi_{,x} + U_{,xx}\Phi_{,x})\}]$$

$$- Q_z[\delta(U_{,x}\Phi + W_{,x}(V_{,x} + U_{,x}\Phi))$$

$$+ Y_s \delta\{V_{,x}(V_{,xx}\Phi + V_{,x}\Phi_{,x}) + U_{,x}(U_{,xx}\Phi_{,x} + V_{,x}\Phi_{,x})\}]$$

$$M_x \delta\{V_{,xx}(V_{,x}\Phi + U_{,x}) + U_{,x}(U_{,x}\Phi_{,x} + U_{,xx}\Phi)$$

$$- V_{,x}(V_{,xx}\Phi + V_{,x}\Phi_{,x}) - U_{,xx}(U_{,x}\Phi + V_{,x})\} + \lfloor F_i \rfloor \{r\} \Big] dx \qquad (3.56)$$

If tne higher order terms such as $V_{,x}U_{,xx}\Phi_{,x}$, $V_{,x}V_{,xx}\Phi_{,x}$, $W_{,x}U_{,x}\Phi_{,x}$, etc. are neglected from the above equation, then it can be written as:

$$\int_0^L \Big[EA\{W_{,x} - \bar{Y}U_{,xx} - \bar{Z}V_{,xx} - \frac{ES_\omega}{EA}\Phi_{,xx}\}\delta W_{,x}]$$

$$- EA\bar{Y}W_{,x}\delta U_{,xx} - EA\bar{Z}W_{,x}\delta V_{,xx} + EI_z U_{,xx}\delta U_{,xx} + EI_y V_{,xx}\delta V_{,xx}$$

$$+ EI_\omega \Phi_{,xx}\delta\Phi_{,xx} + EI_{yz}\delta U_{,xx}V_{,xx} + EI_{\omega y}\delta\Phi_{,xx}U_{,xx} + EI_{\omega z}\delta(\Phi_{,xx}V_{,xx})$$

$$- ES_\omega W_{,x}\delta\Phi_{,xx} + KG\Phi_{,x}\delta\Phi_{,x}$$

$$+ P[V_{,x}\delta V_{,x} + U_{,x}\delta U_{,x} + Z_s \delta(\Phi_{,x}U_{,x}) - Y_s \delta(\Phi_{,x}V_{,x})]$$

$$+ M_\rho \Phi_{,x} \delta\Phi_{,x} + M_z \delta(\Phi_{,x} V_{,x}) - M_y \delta(\Phi_{,x} U_{,x}) + Q_y \delta(V_{,x}\Phi)$$

$$+ Q_z \delta(U_{,x}\Phi) + M_x \delta(V_{,xx} U_{,x} - U_{,xx} V_{,x}) + [F_i]\{r\}\Big] dx \qquad (3.57)$$

For a doubly symmetric section, the above equation reduces to

$$\int_0^L \Big[EA(W_{,x} \delta W_{,x}) + EI_z U_{,xx} \delta U_{,xx} + EI_y V_{,xx} \delta V_{,xx} + EI_\omega \Phi_{,xx} \delta\Phi_{,xx}$$

$$+ KG\Phi_{,x} \delta\Phi_{,x} + P[V_{,x} \delta V_{,x} + U_{,x} \delta U_{,x}]$$

$$+ M_\rho \Phi_{,x} \delta\Phi_{,x} + M_z \delta(\Phi_{,x} V_{,x}) - M_y \delta(\Phi_{,x} U_{,x}) + Q_y \delta(V_{,x}\Phi)$$

$$+ Q_z \delta(U_{,x}\Phi) + M_x \delta(V_{,xx} U_{,x} - U_{,xx} V_{,x}) + [F_i]\{r\}\Big] dx \qquad (3.58)$$

The general equation obtained from Appendix B ,where the product terms in the shear strains i.e. ($W_{,x}W_{,y}$ and $W_{,x}W_{,z}$) are **not** included can be written as:-

$$\int_0^L \Big[EA[\{W_{,x} + \frac{1}{2}(U^2_{,x} + V^2_{,x} + \rho^2 \Phi^2_{,x})$$

$$- \overline{Y}(V_{,xx}\Phi + U_{,xx}) - \overline{Z}(V_{,xx} - U_{,xx}\Phi)$$

$$- Y_s V_{,x} \Phi_{,x} + Z_s U_{,x} \Phi_{,x} - \frac{ES_\omega}{EA} \Phi_x\} \delta W_{,x}]$$

$$- EA\overline{Y}[W_{,x} + \{ \frac{1}{2}(U^2_{,x} + V^2_{,x}) - Y_s \Phi_{,x} V_{,x}$$

$$+ Z_s \Phi_{,x} U_{,x} - \frac{1}{2}(Y_s^2 + Z_s^2)\Phi^2_{,x}] \delta U_{,xx}$$

$$- EA\overline{Z}[W_{,x} + \{ \frac{1}{2}(U^2_{,x} + V^2_{,x}) - Y_s \Phi_{,x} V_{,x}$$

$$+ Z_s \Phi_{,x} U_{,x}),x) - \frac{1}{2}\{Y_s^2 + Z_s^2\}\Phi^2_{,x}] \delta V_{,xx}$$

$$+ EI_\omega \Phi_{,xx}\delta\Phi_{,xx} - EI_y\{U_{,xx}\Phi - V_{,xx} - Z_s\frac{\Phi^2_{,x}}{2}\}\ \delta V_{,xx}$$

$$+ EI_z\{V_{,xx}\Phi + U_{,xx} - Y_s\frac{\Phi^2_{,x}}{2}\}\ \delta U_{,xx}$$

$$-EI_{yz}[\{U_{,xx} - V_{,xx} + Z_s\Phi U^2_{,x}\}\delta U_{,xx}$$

$$- \{V_{,xx}\Phi + U_{,xx} + Y_s\Phi^2_{,x}\}\delta V_{,xx}]$$

$$+ EI_{\omega y}[(\Phi_{,xx} - Y_s\Phi^2_{,x})\delta U_{,xx} + (V_{,xx}\Phi + U_{,xx})\delta\Phi_{,xx}]$$

$$+ EI_{\omega z}[(\Phi_{,xx} - Z_s\Phi^2_{,x})\delta V_{,xx} - (U_{,xx}\Phi - V_{,xx})\delta\Phi_{,xx}]$$

$$-ES_\omega[W_{,x} + \frac{1}{2}\{U^2_{,x} + V^2_{,x}\} - Y_s\Phi_{,x}V_{,x} + Z_s\Phi_{,x}U_{,x}$$

$$- \frac{1}{2}(Y_s^2 + Z_s^2)\Phi^2_{,x}]\delta\Phi_{,xx}$$

$$+ \beta_\omega\Phi^2_{,x}\delta\Phi_{,xx} + \beta_y\delta(\Phi^2_{,x}U_{,xx}) + \beta_z\delta(\Phi^2_{,x}V_{,xx}) \quad + GK\Phi_{,x}\delta\Phi_{,x}$$

$$+ P[V_{,x}\delta V_{,x} + U_{,x}\delta U_{,x} + Z_s\delta\{\Phi_{,x}(V_{,x}\Phi + U_{,x})\}$$

$$- Y_s\delta\{\Phi_{,x}(V_{,x} + U_{,x}\Phi)\}] + M_y\delta(U_{,xx}\Phi + V_{,xx}\frac{\Phi^2_{,x}}{2})$$

$$+ M_\rho\Phi_{,x}\delta\Phi_{,x} + M_z\delta\{V_{,xx}\Phi - U_{,xx}\frac{\Phi^2_{,x}}{2}\} + F_i\{r\}\Big]dx \qquad (3.59)$$

Neglecting the higher order terms ($U_{,x}V_{,xx}\Phi$, $\Phi^2_{,x}U_{,x}$, $W_{,x}V^2_{,x}$, etc.) in the above equation, then it can be written as:

$$\int_0^L \Big[EA[W_{,x} - \bar{Y}U_{,xx} - \bar{Z}V_{,xx} - \frac{ES_\omega}{EA}\Phi_{,xx}]\delta W_{,x}$$

$$- EA\bar{Y}W_{,x}\delta U_{,xx} - EA\bar{Z}W_{,x}\delta V_{,xx} + EI_\omega\Phi_{,xx}\delta\Phi_{,xx} - EI_yV_{,xx}\delta V_{,xx}$$

$$+ EI_zU_{,xx}\delta U_{,xx} + EI_{yz}(V_{,xx}\delta U_{,xx} + U_{,xx}\delta V_{,xx})$$

$$+ EI_{\omega y}(\Phi_{,xx}\delta U_{,xx} + U_{,xx})\delta\Phi_{,xx}]$$

$$+ EI_{\omega z}(\Phi_{,xx}\delta V_{,xx} + \quad - V_{,xx}\delta\Phi_{,xx})$$

$$-ES_{\omega}(W_{,x} - Y_s\Phi_{,x}V_{,x} + Z_s\Phi_{,x}U_{,x})\delta\Phi_{,xx}$$

$$+ GK\Phi_{,x}\delta\Phi_{,x} + P[V_{,x}\delta V_{,x} + U_{,x}\delta U_{,x} + Z_s\delta(\Phi_{,x}U_{,x})$$

$$- Y_s\delta(\Phi_{,x}V_{,x}) + M_y\delta(U_{,xx}\Phi) + M_\rho\Phi_{,x}\delta\Phi_{,x} \tag{3.60}$$

$$+ M_z\delta(V_{,xx}\Phi) + F_i\{r\}\Big]dx$$

for doubly symmetric I-sections Y_s , Z_s, $\overline{Y}$, and $\overline{Z}$ are equal to zero, if we substitute these in the above equation it yields equation 3.61

$$\int_0^L \Big[EA[W_{,x} \quad + EI_{\omega}\Phi_{,xx}\delta\Phi_{,xx} - EI_y V_{,xx}\delta V_{,xx} + EI_z U_{,xx}\delta U_{,xx}$$

$$+ GK\Phi_{,x}\delta\Phi_{,x} + P[V_{,x}\delta V_{,x} + U_{,x}\delta U_{,x} + Z_s\delta(\Phi_{,x}U_{,x})$$

$$+ M_y\delta(U_{,xx}\Phi) + M_\rho\Phi_{,x}\delta\Phi_{,x} + M_z\delta(V_{,xx}\Phi) + F_i\{r\}\Big]dx \tag{3.61}$$

3.5.2 Total Potential Energy

Substituting equations (3.1-3.27) into equation 3.40 yields a general equilibrium equation incorporating the product of the derivatives of longitudinal displacements with respect to the Y and Z directions.

$$\Pi = \frac{1}{2}\int_0^L \Big[[EA\{W^2_{,x} + \frac{1}{2}W_{,x}(U^2_{,x} + V^2_{,x} + \frac{\rho^2}{EA}\Phi^2_{,x})$$

$$- 2\overline{Y}W_{,x}(V_{,xx}\Phi + U_{,xx} + V_{,x}\Phi_{,x})$$

$$- 2\overline{Z}W_{,x}(V_{,xx} - U_{,xx}\Phi - U_{,x}\Phi_{,x})$$

$$- 2W_{,x}Y_s\Phi_{,x}U_{,x} + 2W_{,x}Z_sV_{,x}\Phi_{,x} - 2\frac{ES_\omega}{EA}W_{,x}\Phi_{,xx}$$

$$- 2EAW_{,x}\{V_{,xx}\Phi + U_{,xx} + V_{,x}\Phi_{,x}$$

$$+ V_{,xx} - U_{,xx}\Phi - U_{,x}\Phi_{,x} - \frac{ES_\omega}{EA}\Phi_{,xx}\}$$

$$+ 2EA\overline{Y}\{Y_s\Phi_{,x}V_{,x} - Z_s\Phi_{,x}U_{,x} + \frac{1}{2}(U^2_{,x} + V^2_{,x} + (Y_s^2 + Z_s^2)\Phi^2_{,x})\}U_{,xx}$$

$$+ 2EA\overline{Z}\{Y_s\Phi_{,x}V_{,x} - Z_s\Phi_{,x}U_{,x} + \frac{1}{2}(U^2_{,x} + V^2_{,x} + (Y_s^2 + Z_s^2)\Phi^2_{,x})\}V_{,xx}$$

$$+ 2ES_\omega\{Y_s\Phi_{,x}V_{,x} - Z_s\Phi_{,x}U_{,x} + \frac{1}{2}(U^2_{,x} + V^2_{,x} + (Y_s^2 + Z_s^2)\Phi^2_{,x})\}\Phi_{,xx}$$

$$+ EI_\omega\{\Phi^2_{,xx} + \overline{\beta}_\omega\Phi^2_{,x}\Phi_{,xx}\} + GK\Phi^2_{,x}$$

$$+ EI_y\{(V^2_{,xx} - 2U_{,xx}\Phi V_{,xx}) - 2V_{,xx}(V_{,x}\Phi_{,x} + U_{,x}\Phi_{,x}) + \overline{\beta}_y\Phi^2_{,x}U_{,xx}\}$$

$$+ EI_z\{(2V_{,xx}\Phi U_{,xx} + U^2_{,xx}) + 2U_{,xx}(V_{,x}\Phi_{,x} - U_{,x}\Phi_{,x}) + \overline{\beta}_z\Phi^2_{,x}V_{,xx}\}$$

$$+ 2EI_{yz}\{(U_{,xx}V_{,xx} + (V_{,xx}\Phi + V_{,x}\Phi_{,x} - U_{,x}\Phi_{,x})V_{,xx}$$

$$- (U_{,xx}\Phi + V_{,x}\Phi_{,x} + U_{,x}\Phi_{,x})U_{,xx}\}$$

$$+ 2EI_{y\omega}\{(V_{,xx}\Phi + V_{,x}\Phi_{,x})\Phi_{,xx} - Y_s\Phi^2_{,x}\Phi_{,xx}$$

$$+ (\Phi_{,xx}V_{,x} - U_{,x}\Phi_{,xx})\Phi_{,x} + U_{,xx}\Phi_{,xx}\}$$

$$+ 2EI_{z\omega}\{-(U_{,xx}\Phi + U_{,x}\Phi_{,x})\Phi_{,xx} - Z_s\Phi^2_{,x}\Phi_{,xx}$$

$$- (\Phi_{,xx}U_{,x} + V_{,x}\Phi_{,xx})\Phi_{,x} + V_{,xx}\Phi_{,xx}\}$$

$$+ P\{U^2_{,x} + V^2_{,x} + \rho_o^2\Phi^2_{,x} + 2(-Z_sU_{,x} + Y_sV_{,x})\Phi_{,x}$$

$$+ 2(Z_sV_{,x} + Y_sU_{,x})\Phi_{,x}\Phi \} - 2M_\omega W_{,x}\Phi_{,xx}$$

$$+ Q_y\{W_{,x}(V_{,x} - U_{,x}\Phi) + 2(\Phi U_{,x} + \frac{1}{2}\Phi^2V_{,x})$$

$$+ Y_s(\Phi\Phi_{,x}\Phi + \frac{1}{2}\Phi^2\Phi_{,x}) - Z_s\Phi\Phi_{,x}\}$$

$$- Q_z\{W_{,x}(V_{,x}\Phi + U_{,x}) - 2(\Phi V_{,x} + \frac{1}{2}\Phi^2U_{,x})$$

$$+ Z_s(-\Phi\Phi_{,x}\Phi + \frac{1}{2}\Phi^2\Phi_{,x}) - Y_s\Phi\Phi_{,x}\}$$

$$+ 2M_y\{(U_{,x} - V_{,x}\Phi)\Phi_{,x} - W_{,x}(V_{,xx} - U_{,xx}\Phi - U_{,x}\Phi_{,x})\}$$

$$+ 2M_z\{(V_{,x} - U_{,x}\Phi)\Phi_{,x} - W_{,x}(U_{,xx} + V_{,xx}\Phi + V_{,x}\Phi_{,x})\}$$

$$+ M_x\{\Phi_{,x}\Phi^2 + U_{,x}(U_{,x}\Phi_{,x} + U_{,xx}\Phi) - (U_{,x} - V_{,x}\Phi)V_{,xx}$$

$$- V_{,x}(V_{,x}\Phi_{,x} + V_{,xx}\Phi) + (V_{,x} - U_{,x}\Phi)U_{,xx}\}$$

$$- M_z\Phi_o[V_{,xx} - U_{,x}\Phi_{,x}) + M_z\Phi_o[U_{,xx} + V_{,x}\Phi_{,x})$$

$$+ \frac{1}{2}P[Z_sV_{,x}\Phi_{,x}\Phi_o + Y_sU_{,x}\Phi_{,x}\Phi_o]$$

$$- Q_y\Phi_o[V_{,x} + Y_s\Phi_{,x} + Z_s(\frac{\Phi_o}{2} + 2\Phi]$$

$$+ Q_z\Phi_o[U_{,x} + Z_s\Phi_{,x} - Y_s(\frac{\Phi_o}{2} + 2\Phi]$$

$$+ \langle P\rangle\{r\}]dx + BB + CC \qquad (3.62)$$

where CC is defined in appendix C and BB as

$$BB = [\sigma_{xz}\frac{d\bar{\omega}_n}{dZ} + \sigma_{xy}\frac{d\bar{\omega}_n}{dY}]\bar{X} \qquad (3.63)$$

where

$$\bar{X} = -\{\Phi_{,x} + Y(V_{,x}\Phi_{,x} + V_{,xx}\Phi + U_{,xx}) + Z(U_{,x}\Phi_{,x} + U_{,xx}\Phi - V_{,xx})\}\Phi_{,x} \qquad (3.64)$$

Equation 3.63 is equal to zero and full details are given in appendix A.

The cross section properties $\overline{\beta}_y$, $\overline{\beta}_z$, and $\overline{\beta}_\omega$ are

$$\overline{\beta}_y = \frac{1}{I_y}\int_A Y(Y^2 + Z^2)dA - 2Y_s' \tag{3.65a}$$

$$\overline{\beta}_z = \frac{1}{I_z}\int_A Z(Y^2 + Z^2)dA - 2Z_s \tag{3.65b}$$

$$\overline{\beta}_\omega = \frac{1}{I_\omega}\int_A \omega(Y^2 + Z^2)dA \tag{3.65c}$$

$$\rho_o^2 = \int_A [(Y - Y_s)^2 + (Z - Z_s)^2 dA \tag{3.65d}$$

The variation to the equation 3.62 after substituting for the terms CC from appendix CC yields

$$\delta\Pi = \int_0^l \Big[EA[W_{,x} + \frac{1}{2}(U^2_{,x} + V^2_{,x} + \frac{\rho^2}{EA}\Phi^2_{,x})$$

$$- \overline{Y}(V_{,xx}\Phi + U_{,xx} + V_{,x}\Phi_{,x}) - \overline{Z}(V_{,xx} - U_{,xx}\Phi - U_{,x}\Phi_{,x})$$

$$- Y_s\Phi_{,x}U_{,x} + Z_s\Phi_{,x}V_{,x} - (V_{,xx}\Phi + U_{,xx} + V_{,x}\Phi_{,x}$$

$$+ V_{,xx} - U_{,xx}\Phi - U_{,x}\Phi_{,x} - \frac{ES_\omega}{EA}\Phi_{,xx})]\delta W_{,x}$$

$$+ EAW_{,x}[(U_{,x}\delta U_{,x} + V_{,x}\delta V_{,x} + \frac{\rho_o^2}{EA}\Phi_{,x}\delta\Phi_{,x})$$

$$- \overline{Y}(\Phi\delta V_{,xx} + V_{,xx}\delta\Phi + \delta U_{,xx} + V_{,x}\delta\Phi_{,x} + \Phi_{,x}\delta V_{,x})$$

$$- \overline{Z}(\delta V_{,xx} - U_{,xx}\delta\Phi - \Phi\delta U_{,xx} - U_{,x}\delta\Phi_{,x} - \Phi_{,x}\delta U_{,x})$$

$$- Y_s(\Phi_{,x}\delta U_{,x} + U_{,x}\delta\Phi_{,x}) + Z_s(\Phi_{,x}\delta V_{,x} + V_{,x}\delta\Phi_{,x})$$

$$- (\Phi\delta V_{,xx} + V_{,xx}\delta\Phi + \delta U_{,xx} + V_{,x}\delta\Phi_{,x} + \Phi_{,x}\delta V_{,x}$$

$$+ \delta V_{,xx} - \Phi\delta U_{,xx} - U_{,xx}\delta\Phi - U_{,x}\delta\Phi_{,x} - \Phi_{,x}\delta U_{,x} - \frac{ES_\omega}{EA}\delta\Phi_{,xx})]$$

$$+ EA\overline{Y}[Y_s(V_{,x}\Phi_{,x}\delta U_{,xx} + U_{,xx}V_{,x}\delta\Phi_{,x} + U_{,xx}\Phi_{,x}\delta V_{,x})$$

$$- Z_s(U_{,x}\Phi_{,x}\delta U_{,xx} + U_{,xx}U_{,x}\delta\Phi_{,x} + U_{,xx}\Phi_{,x}\delta U_{,x})$$

$$+ \frac{1}{2}\{U^2_{,x} + V^2_{,x} + (Y_s^2 + Z_s^2)\Phi^2_{,x}\}\delta U_{,xx}$$

$$+ U_{,xx}\{U_{,x}\delta U_{,x} + V_{,x}\delta V_{,x} + (Y_s^2 + Z_s^2)\Phi_{,x}\delta\Phi_{,x}\}]$$

$$+ EA\overline{Z}[Y_s(V_{,x}\Phi_{,x}\delta V_{,xx} + V_{,xx}V_{,x}\delta\Phi_{,x} + V_{,xx}\Phi_{,x}\delta V_{,x})$$

$$- Z_s(U_{,x}\Phi_{,x}\delta V_{,xx} + V_{,xx}U_{,x}\delta\Phi_{,x} + V_{,xx}\Phi_{,x}\delta U_{,x})$$

$$+ \frac{1}{2}\{U^2_{,x} + V^2_{,x} + (Y_s^2 + Z_s^2)\Phi^2_{,x}\}\delta V_{,xx}$$

$$+ V_{,xx}\{U_{,x}\delta U_{,x} + V_{,x}\delta V_{,x} + (Y_s^2 + Z_s^2)\Phi_{,x}\delta\Phi_{,x}\}]$$

$$+ ES_\omega[Y_s(V_{,x}\Phi_{,x}\delta\Phi_{,xx} + \Phi_{,xx}V_{,x}\delta\Phi_{,x} + \Phi_{,xx}\Phi_{,x}\delta V_{,x})$$

$$- Z_s(U_{,x}\Phi_{,x}\delta\Phi_{,xx} + \Phi_{,xx}U_{,x}\delta\Phi_{,x} + \Phi_{,xx}\Phi_{,x}\delta U_{,x})$$

$$+ \frac{1}{2}\{U^2_{,x} + V^2_{,x} + (Y_s^2 + Z_s^2)\Phi^2_{,x}\}\delta\Phi_{,xx}$$

$$+ \Phi_{,xx}\{U_{,x}\delta U_{,x} + V_{,x}\delta V_{,x} + (Y_s^2 + Z_s^2)\Phi_{,x}\delta\Phi_{,x}\}$$

$$+ EI_\omega\{\Phi_{,xx}\delta\Phi_{,xx} + \overline{\beta}_\omega(\Phi^2_{,x}\delta\Phi_{,xx} + 2\Phi_{,xx}\Phi_{,x}\delta\Phi_{,x})\} + GK\Phi_{,x}\delta\Phi_{,x}$$

$$+ EI_y[\{(V_{,xx} - U_{,xx}\Phi) - (V_{,x}\Phi_{,x} + U_{,x}\Phi_{,x})\}\delta V_{,xx}$$

$$- V_{,xx}\{(U_{,xx}\delta\Phi + \Phi\delta U_{,xx})$$

$$- V_{,xx}(\Phi_{,x}\delta V_{,x} + V_{,x}\delta\Phi_{,x} + U_{,x}\delta\Phi_{,x} + \Phi_{,x}\delta U_{,x})\}$$

$$+ \overline{\beta}_y(\Phi^2_{,x}\delta U_{,xx} + 2U_{,xx}\Phi_{,x}\delta\Phi_{,x}\ ]$$

$$+ EI_z[\{(V_{,xx}\Phi + U_{,xx}) + (V_{,x}\Phi_{,x} - U_{,x}\Phi_{,x})\}\delta U_{,xx}$$

$$+ U_{,xx}(\Phi\delta V_{,xx} + V_{,xx}\delta\Phi + V_{,x}\delta\Phi_{,x} + \Phi_{,x}\delta V_{,x}$$

$$- U_{,x}\delta\Phi_{,x} - \Phi_{,x}\delta U_{,x}) + \overline{\beta}_z(\Phi^2_{,x}\delta V_{,xx} + 2V_{,xx}\Phi_{,x}\delta\Phi_{,x}\ )\}]$$

$$+ EI_{yz}(U_{,xx}\delta V_{,xx} + V_{,xx}\delta U_{,xx})$$

$$+ EI_{yz}[(V_{,xx}\Phi - U_{,x}\Phi_{,x} + V_{,x}\Phi_{,x})\delta V_{,xx}$$

$$+ V_{,xx}(V_{,xx}\delta\Phi + \Phi\delta V_{,xx} + V_{,x}\delta\Phi_{,x}$$

$$+ \Phi_{,x}\delta V_{,x} - U_{,x}\delta\Phi_{,x} - \Phi_{,x}\delta U_{,x})$$

$$+ U_{,xx}(U_{,xx}\delta\Phi + \Phi\delta U_{,xx} + V_{,x}\delta\Phi_{,x}$$

$$+ \Phi_{,x}\delta V_{,x} + U_{,x}\delta\Phi_{,x} + \Phi_{,x}\delta U_{,x})]$$

$$- (U_{,xx}\Phi + V_{,x}\Phi_{,x} + U_{,x}\Phi_{,x})\delta U_{,xx}$$

$$+ EI_{\omega y}[(U_{,xx}\delta\Phi_{,xx} + \Phi_{,xx}\delta U_{,xx})$$

$$+ EI_{\omega y}[(V_{,xx}\Phi - V_{,x}\Phi_{,x} - Y_s\Phi^2_{,x})\delta\Phi_{,xx}$$

$$+ \Phi_{,xx}\{V_{,xx}\delta\Phi + \Phi\delta V_{,xx} - V_{,x}\delta\Phi_{,x} - \Phi_{,x}\delta V_{,x} - 2Y_s\Phi_{,x}\delta\Phi_{,x}\}]$$

$$+ (\Phi_{,xx}V_{,x} - U_{,x}\Phi_{,xx})\delta\Phi_{,x} + \Phi_{,x}\{\Phi_{,xx}(\delta V_{,x} - \delta U_{,x}) + (V_{,x} - U_{,x})\delta\Phi_{,xx}\}]$$

$$+ EI_{\omega z}[(V_{,xx}\delta\Phi_{,xx} + \Phi_{,xx}\delta V_{,xx})$$

$$+ EI_{\omega z}[(-U_{,xx}\Phi - U_{,x}\Phi_{,x} - Z_s\Phi^2_{,x})\delta\Phi_{,xx}$$

$$- \Phi_{,xx}\{U_{,xx}\delta\Phi + \Phi\delta U_{,xx}+U_{,x}\delta\Phi_{,x}+\Phi_{,x}\delta U_{,x}-2Z_s\Phi_{,xx}\Phi_{,x}\delta\Phi_{,x}\}]$$

$$- (\Phi_{,xx}U_{,x} + V_{,x}\Phi_{,xx})\delta\Phi_{,x} - \Phi_{,x}\{\Phi_{,xx}(\delta U_{,x}+\delta V_{,x}) + (V_{,x}+U_{,x})\delta\Phi_{,xx}\}]$$

$$+ P[U_{,x}\delta U_{,x} + V_{,x}\delta V_{,x} + \rho_o^2\Phi_{,x}\delta\Phi_{,x}$$

$$+ (Z_sU_{,x} - Y_sV_{,x})\delta\Phi_{,x} + \Phi_{,x}(Z_s\delta U_{,x}-Y_{,s}\delta V_{,x})$$

$$+ \Phi_{,x}\Phi(Z_s\delta V_{,x} + Y_s\delta U_{,x})$$

$$+ (Z_sV_{,x}+Y_{,s}U_{,x})\Phi_{,x}\delta\Phi + (Z_sv_{,x}+y_sU_{,x})\Phi\delta\Phi_{,x}]$$

$$+ \tfrac{1}{2}Q_y[(V_{,x}-\Phi U_{,x})\delta W_{,x}+W_{,x}(\delta V_{,x}-U_{,x}\delta\Phi-\Phi\delta U_{,x})$$

$$+2(\Phi\delta U_{,x}+U_{,x}\delta\Phi+V_{,x}\Phi\delta\Phi+\tfrac{1}{2}\Phi^2\delta V_{,x})$$

$$+ Z_s(\tfrac{1}{2}\Phi^2\delta\Phi_{,x} + \Phi_{,x}\Phi\delta\Phi-2\Phi\Phi_{,x}\delta\Phi-\Phi^2\delta\Phi_{,x})$$

$$- Y_s(\Phi\delta\Phi_{,x} + \Phi_{,x}\delta\Phi)]$$

$$- \tfrac{1}{2}Q_z[(U_{,x}+\Phi V_{,x})\delta W_{,x}+W_{,x}(\delta U_{,x}+V_{,x}\delta\Phi+\Phi\delta V_{,x})$$

$$- 2(\Phi\delta U_{,x}+U_{,x}\delta\Phi+U_{,x}\Phi\delta\Phi+\tfrac{1}{2}\Phi^2\delta U_{,x})$$

$$+ Z_s(\tfrac{1}{2}\Phi^2\delta\Phi_{,x} + \Phi_{,x}\Phi\delta\Phi-2\Phi\Phi_{,x}\delta\Phi-\Phi^2\delta\Phi_{,x})$$

$$- Y_s(\Phi\delta\Phi_{,x} + \Phi_{,x}\delta\Phi)]$$

$$+ M_y[U_{,x}\delta\Phi_{,x}+\Phi_{,x}\delta U_{,x}-V_{,x}\Phi_{,x}\delta\Phi-V_{,x}\Phi\delta\Phi_{,x}-\Phi\Phi_{,x}\delta V_{,x}$$

$$- (V_{,xx} - U_{,xx}\Phi - U_{,x}\Phi_{,x})\delta W_{,x}$$

$$- W_{,x}\{\delta V_{,xx} - U_{,xx}\delta\Phi - \Phi\delta U_{,xx} - U_{,x}\delta\Phi_{,x} - \Phi_{,x}\delta U_{,x}\}]$$

$$+ M_z[V_{,x}\delta\Phi_{,x}+\Phi_{,x}\delta V_{,x}-U_{,x}\Phi_{,x}\delta\Phi-U_{,x}\Phi\delta\Phi_{,x}-\Phi\Phi_{,x}\delta U_{,x}]$$

$$- (V_{,xx}\Phi + U_{,xx} + V_{,x}\Phi_{,x})\delta W_{,x}$$

$$- W_{,x}\{\Phi\delta V_{,xx} + V_{,xx}\delta\Phi + \delta U_{,xx} + V_{,x}\delta\Phi_{,x} + \Phi_{,x}\delta V_{,x}\}]$$

$$- M_\omega[\Phi_{,xx}\delta W_{,x} + W_{,x}\delta\Phi_{,xx}]$$

$$+ \frac{M_x}{2}[V_{,x}\delta U_{,xx}+U_{,xx}\delta V_{,x}-U_{,x}\delta V_{,xx}-V_{,xx}\delta U_{,x}$$

$$+(U^2_{,x}-V^2_{,x}+\Phi^2)\delta\Phi_{,x} + 2\Phi_{,x}(U_{,x}\delta U_{,x}-V_{,x}\delta V_{,x}+\Phi\delta\Phi$$

$$- 2(U_{,x}U_{,xx}\delta\Phi + U_{,x}\Phi\delta U_{,xx} + U_{,xx}\Phi\delta U_{,x})$$

$$- 2(U_{,x}U_{,xx}\delta\Phi + U_{,x}\Phi\delta U_{,xx} + U_{,xx}\Phi\delta U_{,x})$$

$$- M_z\Phi_o[\delta V_{,xx} - U_{,x}\delta\Phi_{,x} - \Phi_{,x}\delta U_{,x}]$$

$$+ M_z\Phi_o[\delta U_{,xx} + V_{,x}\delta\Phi_{,x} + \Phi_{,x}\delta V_{,x}]$$

$$+ \frac{1}{2}P\Phi_o[Z_s(V_{,x}\delta\Phi_{,x} + \Phi_{,x}\delta V_{,x}) + Y_s(U_{,x}\delta\Phi_{,x} + \Phi_{,x}\delta U_{,x})$$

$$- Q_y\Phi_o[\delta V_{,x} +Y_s\delta\Phi_{,x} + 2Z_s\delta\Phi]$$

$$+ Q_z\Phi_o[\delta U_{,x} +Z_s\delta\Phi_{,x} - 2Y_s\Phi]$$

$$+ (D_1U_{,xx} + D_5V_{,xx}+D_6\Phi_{,xx})\delta U_{,xx}$$

$$+ (D_2V_{,xx} + D_4U_{,xx}+D_7\Phi_{,xx})\delta V_{,xx}$$

$$+ (D_3\Phi_{,xx} + D_6U_{,xx}+D_7V_{,xx})\delta\Phi_{,xx}+\langle P\rangle\{\delta r\}\Big]dx \qquad (3.66)$$

If the higher order terms such as $U_{,x}\Phi_{,x}V_{,xx}$, $\Phi^2U_{,xx}$, $\Phi_{,xx}U_{,xx}\Phi$, etc. are neglected in the above equation it yields

$$\delta\Pi = \int_0^1 \Big[EA[W_{,x} - \bar{Y}U_{,xx} - \bar{Z}V_{,xx} - \frac{Es_\omega}{EA}\Phi_{,xx} + U_{,xx} + V_{,xx}]\delta W_{,x}$$

$$- EAW_{,x}[\bar{Z}\delta V_{,xx} + \bar{Y}\delta U_{,xx} + \delta V_{,xx} + \delta U_{,xx} + \frac{ES_\omega}{EA}\delta\Phi_{,xx})]$$

$$EI_\omega\Phi_{,xx}\delta\Phi_{,xx} + GK\Phi_{,x}\delta\Phi_{,x}$$

$$+ EI_y V_{,xx}\delta V_{,xx} + EI_z U_{,xx}\delta U_{,xx} + EI_{yz}(V_{,xx}\delta U_{,xx} + U_{,xx}\delta V_{,xx})$$

$$+ EI_{\omega y}(V_{,xx}\delta\Phi_{,xx} + \Phi_{,xx}\delta V_{,xx}) + EI_{\omega z}(U_{,xx}\delta\Phi_{,xx} + \Phi_{,xx}\delta U_{,xx})$$

$$+ P[U_{,x}\delta U_{,x} + V_{,x}\delta V_{,x} + \rho_o^2\Phi_{,x}\delta\Phi_{,x}$$

$$+ (Z_s U_{,x} - Y_s V_{,x})\delta\Phi_{,x} + \Phi_{,x}(Z_s\delta U_{,x} - Y_{,s}\delta V_{,x})]$$

$$+ \frac{1}{2}Q_y[V_{,x}\delta W_{,x} + W_{,x}\delta V_{,x}$$

$$+ 2(\Phi\delta V_{,x} + V_{,x}\delta\Phi) - Y_s(\Phi\delta\Phi_{,x} + \Phi_{,x}\delta\Phi)]$$

$$- \frac{1}{2}Q_z[U_{,x}\delta W_{,x} + W_{,x}\delta U_{,x}$$

$$- 2(\Phi\delta U_{,x} + U_{,x}\delta\Phi) - Z_s(\Phi\delta\Phi_{,x} + \Phi_{,x}\delta\Phi)]$$

$$+ M_y[U_{,x}\delta\Phi_{,x} + \Phi_{,x}\delta U_{,x} - (V_{,xx}\delta W_{,x} + W_{,x}\delta V_{,xx})]$$

$$+ M_z[V_{,x}\delta\Phi_{,x} + \Phi_{,x}\delta V_{,x} - (U_{,xx}\delta W_{,x} + W_{,x}\delta U_{,xx})]$$

$$- M_\omega[\Phi_{,xx}\delta W_{,x} + W_{,x}\delta\Phi_{,xx}]$$

$$+ \frac{M_x}{2}[V_{,x}\delta U_{,xx} + U_{,xx}\delta V_{,x} - U_{,x}\delta V_{,xx} - V_{,xx}\delta U_{,x}$$

$$+ (D_1 U_{,xx} + D_5 V_{,xx} + D_6\Phi_{,xx})\delta U_{,xx}$$

$$+ (D_2 V_{,xx} + D_4 U_{,xx} + D_7\Phi_{,xx})\delta V_{,xx}$$

$$- M_z\Phi_o[\delta V_{,xx} - U_{,x}\delta\Phi_{,x} - \Phi_{,x}\delta U_{,x}]$$

$$+ M_z\Phi_o[\delta U_{,xx} + V_{,x}\delta\Phi_{,x} + \Phi_{,x}\delta V_{,x}]$$

$$+ \frac{1}{2}P\Phi_o[Z_s(V_{,x}\delta\Phi_{,x} + \Phi_{,x}\delta V_{,x}) + Y_s(U_{,x}\delta\Phi_{,x} + \Phi_{,x}\delta U_{,x})$$

$$- Q_y\Phi_o[\delta V_{,x} + Y_s\delta\Phi_{,x} + 2Z_s\delta\Phi]$$

$$+ Q_z\Phi_o[\delta U_{,x} + Z_s\delta\Phi_{,x} - 2Y_s\Phi]$$

$$+ (D_3\Phi_{,xx} + D_6U_{,xx} + D_7V_{,xx})\delta\Phi_{,xx} + \langle P\rangle\{\delta r\}\Big]dx \qquad (3.67)$$

For doubly symmetric I-sections equation 3.67 becomes as

$$\delta\Pi = \int_0^l \Big[EA[W_{,x}\delta W_{,x} + EI_\omega\Phi_{,xx}\delta\Phi_{,xx} + GK\Phi_{,x}\delta\Phi_{,x}$$

$$+ EI_yV_{,xx}\delta V_{,xx} + EI_zU_{,xx}\delta U_{,xx} + P[U_{,x}\delta U_{,x} + V_{,x}\delta V_{,x} + \rho_o^2\Phi_{,x}\delta\Phi_{,x}$$

$$+ \frac{1}{2}Q_y[V_{,x}\delta W_{,x} + W_{,x}\delta V_{,x} - 2(\Phi\delta V_{,x} + V_{,x}\delta\Phi]$$

$$- \frac{1}{2}Q_z[U_{,x}\delta W_{,x} + W_{,x}\delta U_{,x} - 2(\Phi\delta U_{,x} + U_{,x}\delta\Phi]$$

$$+ M_y[U_{,x}\delta\Phi_{,x} + \Phi_{,x}\delta U_{,x} - (V_{,xx}\delta W_{,x} + W_{,x}\delta V_{,xx})]$$

$$+ M_z[V_{,x}\delta\Phi_{,x} + \Phi_{,x}\delta V_{,x} - (U_{,xx}\delta W_{,x} + W_{,x}\delta U_{,xx})]$$

$$- M_\omega[\Phi_{,xx}\delta W_{,x} + W_{,x}\delta\Phi_{,xx}]$$

$$+ \frac{M_x}{2}[V_{,x}\delta U_{,xx} + U_{,xx}\delta V_{,x} - U_{,x}\delta V_{,xx} - V_{,xx}\delta U_{,x}$$

$$- M_z\Phi_o[\delta V_{,xx} - U_{,x}\delta\Phi_{,x} - \Phi_{,x}\delta U_{,x}]$$

$$+ M_z\Phi_o[\delta U_{,xx} + V_{,x}\delta\Phi_{,x} + \Phi_{,x}\delta V_{,x}]$$

$$- Q_y\Phi_o[\delta V_{,x}] + Q_z\Phi_o[\delta U_{,x}]$$

$$+ (D_1 U_{,xx} + D_5 V_{,xx} + D_6 \Phi_{,xx}) \delta U_{,xx}$$

$$+ (D_2 V_{,xx} + D_4 U_{,xx} + D_7 \Phi_{,xx}) \delta V_{,xx}$$

$$+ (D_3 \Phi_{,xx} + D_6 U_{,xx} + D_7 V_{,xx}) \delta \Phi_{,xx} + \langle P \rangle \{\delta r\} \Big] dx \qquad (3.68)$$

where

$$\langle P \rangle \{r\} = \begin{bmatrix} \overline{P} \delta W \\ \overline{M}_y \delta V \\ \overline{M}_z \delta U \\ \overline{M}_t \delta \Phi \end{bmatrix}$$

Four equations for longitudinal, in-plane, out-of-plane, and torsional actions have been obtained from equation 3.66. the longitudinal equilibrium equation can be obtained by minimizing equation 3.66 as $\frac{\partial \Pi}{\partial W_{,x}} = 0$, for the in-plane and out-of-plane conditions the equilibrium can be written as $\frac{\partial \Pi}{\partial U_{,x}} = 0$, $\frac{\partial \Pi}{\partial V_{,x}} = 0$, and in the same manner the torsional equation $\frac{\partial \Pi}{\partial \Phi_{,x}} = 0$. Integration by parts and use of the calculus of variations (Erwin (1976)) are incorporated in the analysis; these equations can be represented as:

Longitudinal equation

$$EA[W_{,x} + \tfrac{1}{2}(V^2_{,x} + U^2_{,x} + \rho_o^2 \Phi^2_{,x}) - \overline{Y}(V_{,xx}\Phi + U_{,xx} + V_{,x}\Phi_{,x})$$

$$+ \overline{Z}(V_{,xx} - U_{,xx}\Phi - U_{,x}\Phi_{,x}) - Y_s \Phi_{,x} V_{,x} + Z_s \Phi_{,x} U_{,x}] - ES_\omega \Phi_{,xx}$$

$$- (V_{,xx}\Phi + U_{,xx} + V_{,x}\Phi_{,x} + V_{,xx} - U_{,xx}\Phi - U_{,x}\Phi_{,x})$$

$$- M_\omega \Phi_{,xx} - M_y(V_{,xx} - U_{,xx}\Phi - U_{,x}\Phi_{,x})$$

$$- M_z(V_{,xx}\Phi + U_{,xx} + V_{,x}\Phi_{,x})$$

$$+ \frac{Q_y}{2}(V_{,x}-\Phi U_{,x}) - \frac{Q_y}{2}(U_{,x}+\Phi V_{,x}) + \bar{P}=0 \qquad (3.69)$$

In-plane equation

$$EAW_{,x}\{(V_{,x}+Z_s\Phi_{,x})-\bar{Z}Y_sV_{,x}\Phi_{,xx} + Z_s(U_{,xx}\Phi_{,x}+\Phi_{,xx}U_{,x})$$

$$+ EA\{(U_{,xx}U_{,x}+(Y_s^2+Z_s^2)\Phi_{,x}\Phi_{,xx})+U_{,xx}V_{,x}+\bar{Y}Y_sV_{,xx}\Phi_{,x}\}$$

$$+ ES_\omega(\Phi_{,xx}V_{,x} + Y_s\Phi_{,xx}\Phi)$$

$$- EI_y\{U_{,xxx}\Phi + 2\bar{\beta}_z\Phi_{,x}\Phi_{,xx}\}$$

$$+ EI_y[-V_{,xxx}+U_{,xxx}\Phi+2U_{,xx}\Phi+U_{,x}\Phi_{,xx}+U_{,x}\Phi_{,xx}]$$

$$- EI_{yz}[U_{,xxx}+2\Phi V_{,xxx}+2V_{,xx}\Phi_{,x}+V_{,x}\Phi_{,xx}+U_{,x}\Phi_{,xx}]$$

$$- EI_{z\omega}[\Phi_{,xx}\Phi_{,x} + \Phi_{,xxx}] - EI_{\omega y}[\Phi_{,xxx}\Phi - \Phi_{,x}\Phi_{,xx}]$$

$$+ P[V_{,x} --Y_s\Phi_{,x} + Z_s\Phi_{,x}\Phi]$$

$$- \frac{1}{2}Q_z(W_{,x}\Phi+2W_{,x}) + \frac{1}{2}Q_y(W_{,x}+\Phi^2)$$

$$+ M_z\Phi_{,x} - M_y\Phi\Phi_{,x} - M_{,x}[\Phi_{,x}V_{,x} - U_{,xx}] +\bar{M}_z =0 \qquad (3.70)$$

Out-of-plane equation

$$EAW_{,x}\{(U_{,x}-Y_s\Phi_{,x})-\bar{Y}Y_sU_{,x}\Phi_{,xx}-Y_s(V_{,xx}\Phi_{,x}+\Phi_{,xx}V_{,x})$$

$$- EA\{(V_{,xx}V_{,x}+(Y_s^2+Z_s^2)\Phi_{,x}\Phi_{,xx})+V_{,xx}U_{,x}+\bar{Z}Z_sV_{,xx}\Phi_{,x}\}$$

$$+ ES_\omega(\Phi_{,xx}U_{,x} - Z_s\Phi_{,xx}\Phi)$$

$$+ EI_y\{V_{,xxx}\Phi - 2\bar{\beta}_y\Phi_{,x}\Phi_{,xx}\}$$

$$- EI_z[U_{,xxx}+V_{,xxx}\Phi+2V_{,xx}\Phi+V_{,x}\Phi_{,xx}-U_{,x}\Phi_{,xx}]$$

$$- EI_{yz}[V_{,xxx}-2\Phi U_{,xxx}-2U_{,xx}\Phi_{,x}-V_{,x}\Phi_{,xx}-U_{,x}\Phi_{,xx}]$$

$$- EI_{y\omega}[\Phi_{,xx}\Phi_{,x} + \Phi_{,xxx}] + EI_{\omega z}[\Phi_{,xxx}\Phi - \Phi_{,x}\Phi_{,xx}]$$

$$+ P[U_{,x} + Z_s\Phi_{,x} + Y_s\Phi_{,x}\Phi]$$

$$- \frac{1}{2}Q_y(W_{,x}\Phi+W_{,x}) - \frac{1}{2}Q_z(W_{,x}+\Phi^2)$$

$$+ M_y\Phi_{,x} - M_z\Phi\Phi_{,x} + M_{,x}[\Phi_{,x}U_{,x} - V_{,xx}] + \overline{M}_y = 0 \tag{3.71}$$

Torsional equation

$$EAW_{,x}[\rho_o^2-\overline{Y}V_{,x}+\overline{Z}U_{,x}+Y_sU_{,x}+Z_sV_{,x}-V_{,x}+U_{,x}]$$

$$+ EA\overline{Y}[Y_sU_{,xx}V_{,x}-Z_sU_{,xx}U_{,x}+U_{,xx}\Phi_{,x}(Y_s^2+Z_s^2)]$$

$$+ EA\overline{Z}[Y_sV_{,xx}V_{,x}-Z_sV_{,xx}U_{,x}+V_{,xx}\Phi_{,x}(Y_s^2+Z_s^2)]$$

$$+ ES_\omega[-Y_sV_{,xx}\Phi_{,x}+Z_sU_{,xx}\Phi_{,x}-U_{,xx}U_{,x}-V_{,xx}V_{,x}$$

$$+ \Phi_{,x}\Phi_{,xx}(Y_s^2+Z_s^2)] - EI_\omega\Phi_{,xxx} + GK\Phi_{,x}$$

$$- EI_y[V_{,xxx}U_{,x}+V_{,xx}V_{,x}+V_{,xx}U_{,x}-2\overline{\beta}_yU_{,xx}\Phi_{,x}]$$

$$- EI_z[U_{,xxx}V_{,x}-U_{,xx}V_{,x}+U_{,xx}U_{,x}-2\overline{\beta}_zV_{,xx}\Phi_{,x}]$$

$$- EI_{\omega y}[U_{,xxx}+V_{,xxx}\Phi+\Phi_{,xxx}(V_{,x}-V)+\Phi_{,x}V_{,xx}+U_{,xx}\Phi_{,x}]$$

$$+ EI_{\omega z}[-V_{,xxx}+U_{,xxx}\Phi-\Phi_{,xxx}(U_{,x}-U)+\Phi_{,x}U_{,xx}+V_{,xx}\Phi_{,x}]$$

$$+ P[\rho_o^2\Phi_{,x}-(Y_sV_{,x}-Z_sU_{,x})$$

$$+(Z_s\Phi_{,xx}V+Y_s\Phi_{,xx}U) + (Z_sV_{,x}+Y_sU_{,x})\Phi]$$

$$- Q_y[V+V_{,x}\Phi+Z_s\frac{\Phi^2}{2}] - Q_z[U-U_{,x}\Phi-Y_s\frac{\Phi^2}{2}]$$

$$+ M_y[U_{,x}-V_{,x}\Phi+W_{,x}U_{,x}-V_{,xx}\Phi]$$

$$+ M_z[V_{,x}-U_{,x}\Phi-W_{,x}V_{,x}-U_{,xx}\Phi]$$

$$+ \frac{M_x}{2}[U^2_{,x}+V^2_{,x}-\Phi^2-\Phi_{,x}\Phi] + \overline{M}_t = 0 \qquad (3.72)$$

Neglecting higher order terms $U_{,xx}\Phi_{,x}$, $V_{,xx}V_{,x}$, $U_{,xx}U_{,x}$, etc. from equations 3.70 3.71, 3.72 yields the following equations

Longitudinal equation

$$EAW_{,x} + \overline{P} = 0 \qquad (3.73)$$

In-plane equation

$$+ EI_zU_{,xxx} - EI_yV_{,xxx} - EI_{yz}U_{,xxx} - EI_{z\omega}\Phi_{,xxx}$$

$$+ P[V_{,x} --Y_s\Phi_{,x}] - Q_zW_{,x} + Q_yW_{,x}$$

$$+ M_z\Phi_{,x} + M_{,x}U_{,xx} + \overline{M}_z = 0 \qquad (3.74)$$

Out-of-plane equation

$$+ EI_zU_{,xxx} - EI_yU_{,xxx} - EI_{yz}V_{,xxx} - EI_{y\omega}\Phi_{,xxx}$$

$$+ P[U_{,x} + Z_s\Phi_{,x}] - Q_zW_{,x} + Q_yW_{,x}$$

$$+ M_z\Phi_{,x} + M_{,x}U_{,xx} + \overline{M}_z = 0 \qquad (3.75)$$

Torsional equation

$$- EI_{\omega}\phi_{,xxx} + GK\phi_{,x} - EI_{\omega}yU_{,xxx} - EI_{\omega}zV_{,xxx}$$

$$+ P[\rho_o^2\phi_{,x} - Q_yV - Q_zU$$

$$+ M_yU_{,x} + M_zV_{,x} + \overline{M}_t = 0 \qquad (3.76)$$

Equations 3.73 - 3.76 represent the equilibrium equations of a beam-column in 3-dimensions. Those equations may be shown to reduce to those obtained by Culver (1966), Vlasov (1961), Timoshenko and Gere (1961), etc. if certain limitations are imposed.

3.6 Comparison with Previous Formulations

The result given by equation 3.60 is identical to that obtained by Barsoum and Gallagher (1970) for a doubly symmetric I-beam ,where the effects of the initial imperfections and higher order terms were ignored. If the nonlinear terms and the product of the derivatives of the axial displacements with respect to x, y and x, z ($W_{,x}W_{,y}$ and $W_{,x}W_{,z}$) are neglected and the effect of residual strains is included ,then equation 3.60 gives virtually the same results as that derived previously by Rajasekaran (1971)

The effect of higher order terms, using a principal axes basis ,but neglecting the effect of initial imperfections and the $W_{,x}W_{,y}$ and $W_{,x}W_{,z}$ coupling, has been included in the nonlinear equilibrium equation derived by Attard (1986). His results agree with equation 3.60. Similarly the results of equation 3.63 agree with the solution by Roberts and Azizian (1983), (whose basic assumption were subsequently used by Attard) when the loads are limited to those included by Roberts and Azizian.

The stiffness matrices obtained by Yang and McGuire (1986) for symmetric sections, using the principal axes, neglecting higher order terms and residual stresses and including the products of derivatives of the axial displacements with respect to x, y and x, z respectively i.e ($W_{,x}W_{,y}$ and $W_{,x}W_{,z}$ not equal to zero) are similar to equation 3.57 when the higher order terms such as ($\Phi^2_{,x}U_{,xx}$, $V_{,xx}\Phi\Phi_{,xx}$, etc.) are assumed equal to zero.

The present formulation enables the limitations of earlier studies (Barsoum and Gallagher (1970), Rajasekaran (1971), Rajasekaran and Murray (1973), Vinnakota (1977), Nishida and Fukumoto (1985), and Attard (1986)) to be identified. Table 3.1 summarises the findings. Table 3.2 presents a comparison between the assumptions that have employed by the Author and those used previously [Rajasekaran (1971), Rajasekaran and Murray (1973), and Chen and Atsuta (1976)] to develop a formulation for beam-columns in space. Complete derivations of the linear and nonlinear tangential, geometrical, and strain-displacements matrices are presented in Chapter 4. In this context it is particularly important to note that further use of these formulations as the basis for inelastic analysis of biaxial bending leads to effective cross-sections for which several assumptions that might have been valid within the elastic range e.g. doubly symmetric, shear centre and centroid coincide etc., are no longer correct. The importance of not resorting to such simplifications is clearly illustrated by the use of the present formulation in a variety of example problems (El-Khenfas and Nethercot (1987b,c), and chapters 6 and 7 respectively. Specific limitations in previous studies include:

1- Neglect of higher order terms and/ or the effects of residual stress .

2- Using the principal axes for an arbitrary cross-section and neglecting the shift of the shear centre and the centroid even in the elastic range.

3.7- Conclusions

A method has been used to derive the general equilibrium condition for thin walled beams of arbitrary cross-section under general loading. The effects of higher order terms and initial imperfections have been included. The method is based on the principle of virtual work, where the product of the derivatives of axial displacements with respect to x, y and x, z ($W_{,x}W_{,y}$ and $W_{,x}W_{,z}$) are accounted for in equation 3.60 or alternatively are neglected in equation 3.57. An alternative derivation is also possible, based on the use of energy theorems leading to equation 3.63.

Both methods compare very well with the available theories ,permitting their simplifications and limitations to be clearly identified. The validity and accuracy of this theory has been demonstrated in several applications (El-Khenfas and Nethercot (1987b,c)) and Chapters 6/ 7 respectively, ,which cover both elastic and inelastic analysis.

Table 3.1 Comparison between the author's formulation and previous studies

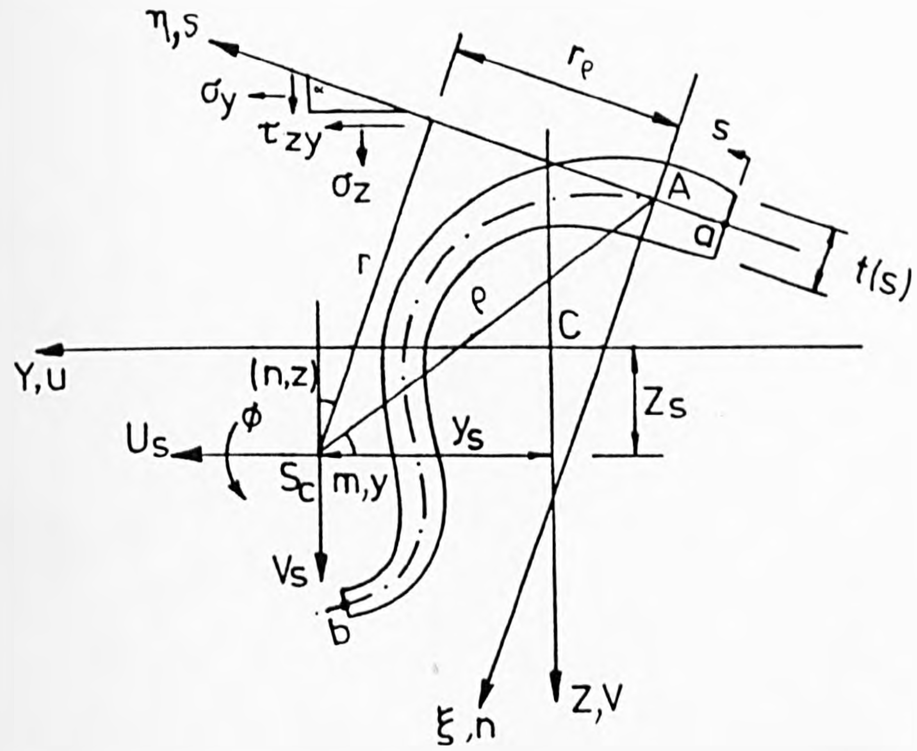
Year	Author	Method of analysis	Analysis type	Imperfections	Axis system	H.O.T. of strain disp. relations	Product of $W_{,x}W_{,y}$ & $W_{,x}W_{,z}$	Non-linear K_G	Non-linear K_Z	Non-linear of strains	Notes
1970	Barsoum and Gallagher	En	E	N	Principal	N	N	N	N	N	1
1971	Rajasckaran	V.W.	E,I	Y	General	N	N	N	N	N	2
1983	Roberts and Azizian	En	E	Y	Principal	Y	N	-	-	-	3
1986	Attard	En	E	-	Principal	Y	Y	N	N	N	4
1977	Vinnokota	Finite of difference	E,I	Y	General	N	N	N	N	N	-
1985	Fukumoto & Nishida	V.W.	E,I	Y	General	N	N	N	N	N	-
1986	Yang and McGuir	V.W.	E,I	Y	Principal	N	Y	N	N	N	-
-	Authors' formulation	V.W.,En	E,I	Y	General	Y	Y	Y	Y	Y	5

1. Barsoum and Gallagher's work is included since it is regarded as the first coverage of torsional-flexural effects by the finite element method.
2. More details are given in Table 3.2
3. This work was performed for general types of cross-sections, but the Wagner effect was not included; it is not clear exactly how non-linear strains were incorporated.
4. Imperfections were ignored (not mentioned).
5. Even for elastic analysis inclusion of higher order terms in the solution has a significant effect on the results, see chapter 6

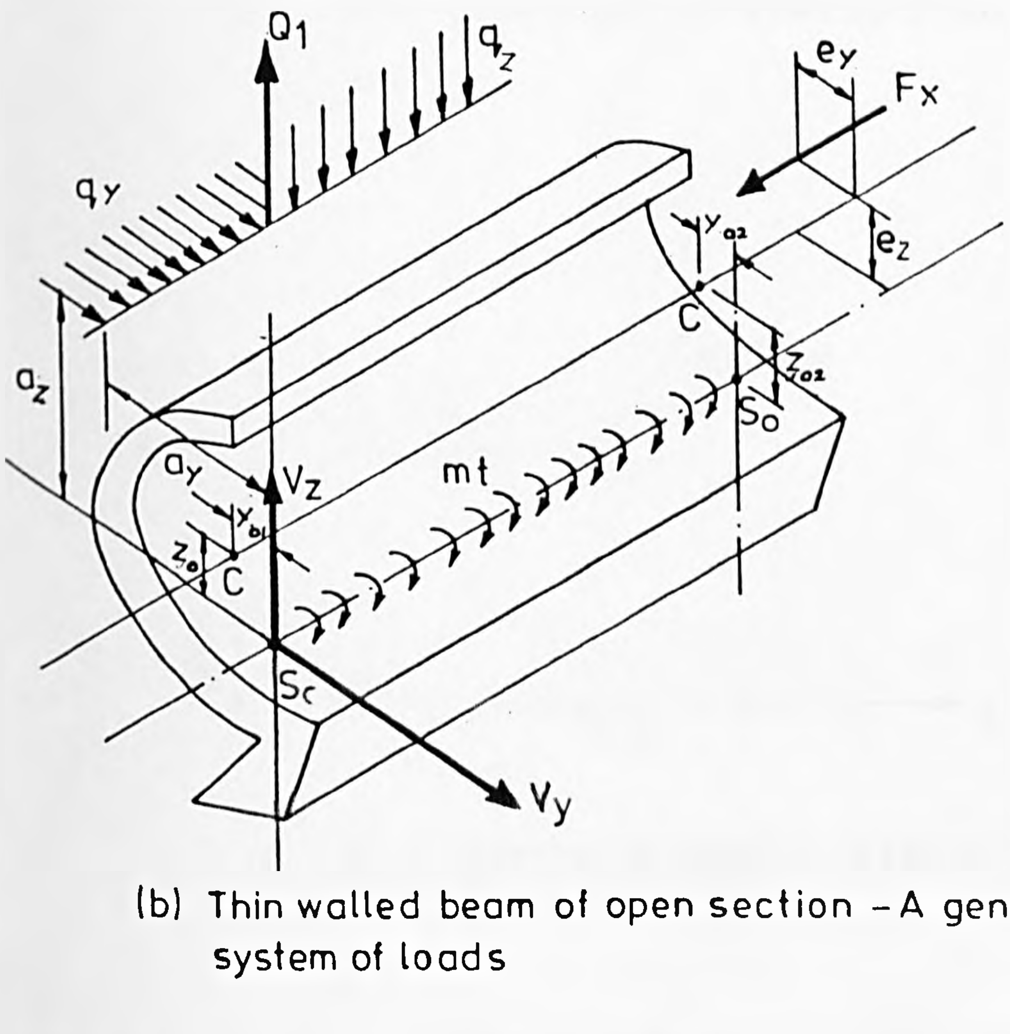
Table 3.2 Comparisons of author's assumptions against refs. Rajasekaran (1970), Rajasekaran and Murray (1973) and Chan & Atsuta (1977) of deriving the general differential equilibrium equation

Item	Descriptions	Refs. (above)		Authors formulations	Note
1		$\text{Sin}\phi = \phi,\ \text{Cos}\phi = 1$		$\text{Sin}\phi = \phi - \frac{\phi^3}{3},\ \text{Cos}\phi = 1 - \frac{\phi^2}{2}$	1
2		$V_s,\ U_s$ shear centre displacements		$V_s,\ U_s$ displacements of point S $\quad V = V_s + V_0$ $V_0,\ U_0$ initial displacements $\quad U = U_s + U_0$ $\phi = \Phi + \phi_0$	
3	In-plane displacement	$V_A = V_s + (Y - Y_0)\phi$		$V_A = V + (Y - Y_s)\text{Sin}\phi + (Z - Z_s)[1 - \text{Cos}\phi]$	
4	Out-of-plane displacement	$U_A = U_s + (Z - Z_s)\phi$		$U_A = U - (Y - Y_s)[1 - \text{Cos}\phi] + (Z - Z_s)\text{Sin}\phi$	
5	Shear strain	$\frac{\partial\eta}{\partial x} + \frac{\partial w}{\partial s} = 0$	2	$\frac{\partial\eta}{\partial x} + \frac{\partial w}{\partial s} + \frac{\partial u}{\partial s}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial s}\frac{\partial v}{\partial x} = 0$	2
6	Longitudinal displacement	$W = W_c - ZV_{,x} - YU_{,x} + \bar{w}_n\Phi_{,x}$	3	$W = W_c - Y[V_{,x}\text{Sin}\phi + U_{,x}\text{Cos}\phi] - Z(V_{,x}\text{Cos}\phi - U_{,x}\text{Sin}\phi) + \bar{w}_n\Phi$	3
7	Axial strain	$\epsilon = w_{,x} - ZV_{,xx} - YU_{,xx} + \bar{w}_n\Phi_{,xx}$	4	$\epsilon = W_{,x} + \frac{1}{2}[U_{,x}^2 + V_{,x}^2 + \rho_0^2{}_{,x}] - Y_s\Phi_{,x}(V_{,x} - U_{,x}\Phi) - Z_s\Phi_{,x}(U_{,x} + V_{,x}\Phi)$ $- Y[U_{,xx} + V_{,xx}\Phi - U_{,xx}\frac{\Phi^2}{2}] - Z[V_{,xx} - V_{,xx}\frac{\Phi^2}{2} - U_{,xx}\Phi] + \bar{w}_n\Phi_{,xx}$	
8	Product of derivatives of axial displacement	$W_{,x}W_{,Y}$ and $W_{,x}W_{,Z} = 0$	5	$W_{,x}W_{,Y}$ and $W_{,x}W_{,Z} \neq 0$	5
9	Linear K_G	–		Torsional moment and shear forces are included	6
10	Non-linear stiffness matrices	Neglected		Considered	7

1. Substitution for $\text{SIN}\phi$ and $\text{COS}\phi$ takes place after item No. 7.
2. Because of the bending and torsional flexibility of thin walled open cross-sections, the effects of shearing strain along the mid surface of the plate segment are extremely small and can be neglected.
3. The longitudinal displacement obtained from integrating the equation presented in item No. 4.
4. The higher order strain terms such as $V_{,xx}\Phi$, $U_{,xx}\Phi$, $Y_s\Phi_{,x}V_{,x}$, $Z_s\Phi_{,x}U_{,x}$, etc. have significant values and are incorporated in the present approach, but are not considered in Refs. (Rajasekaran (1970), Rajasekaran & Murray (1973), and Chen & Atsuta 1977)
5. The product of the derivative of axial displacements with respect to X/Y and X/Z ($W_{,x}W_{,Y}$ and $W_{,x}W_{,Z}$) are accounted for but neglected by other authors as higher order terms, while $W_{,x}^2$ is ignored in all cases.
6. Inclusion of the previous step affects even the linear geometrical stiffness matrix through inclusion of the terms presented in Chapter 4 which are not given by Ref. ' (Rajasekaran (1970), Rajasekaran & Murray (1973), and Chen & Atsuta 1977)
7. The non-linear matrices (tangential, geometrical, strain-displacement) have an important effect because in thin-walled sections the coupling between the displacements and rotations are significant. Neglecting this coupling leads to overestimates of load carrying

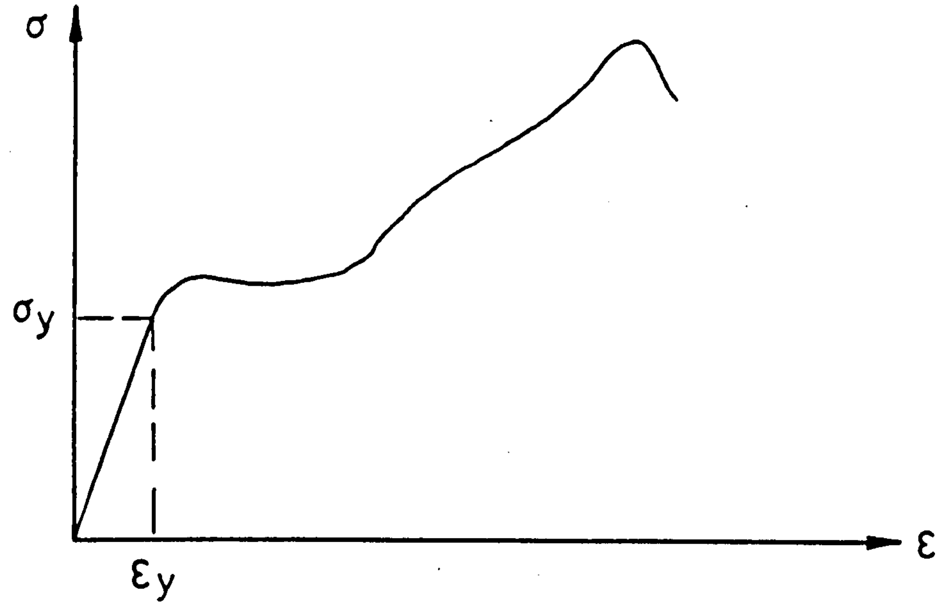


(a) Cross section of thin walled open section

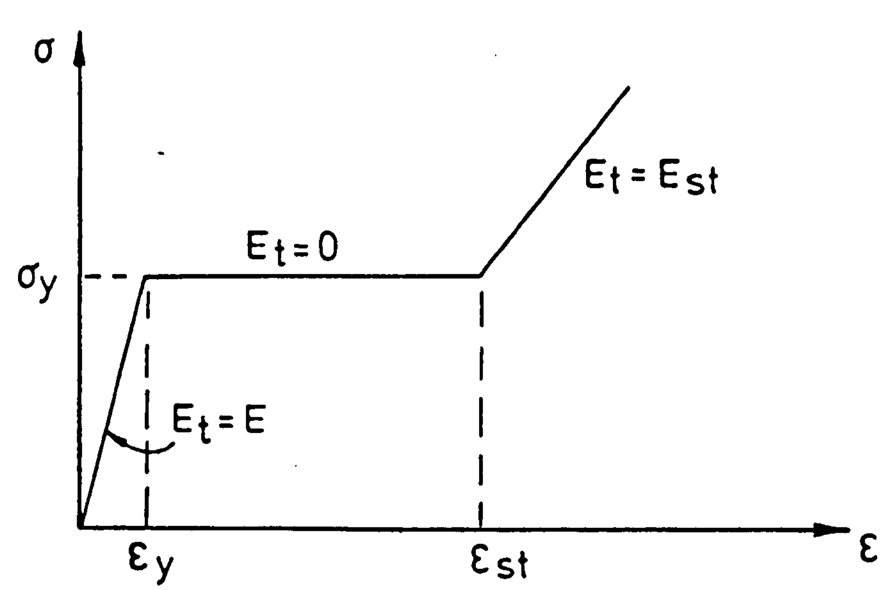


(b) Thin walled beam of open section – A general system of loads

FIG. 3.1 DEFINITION OF PROBLEM



a TRUE BEHAVIOUR OF STRESS-STRAIN CURVE



b IDEALIZATION OF STRESS- STRAIN CURVE

FIG.3.2 STRESS-STRAIN RELATIONSHIP

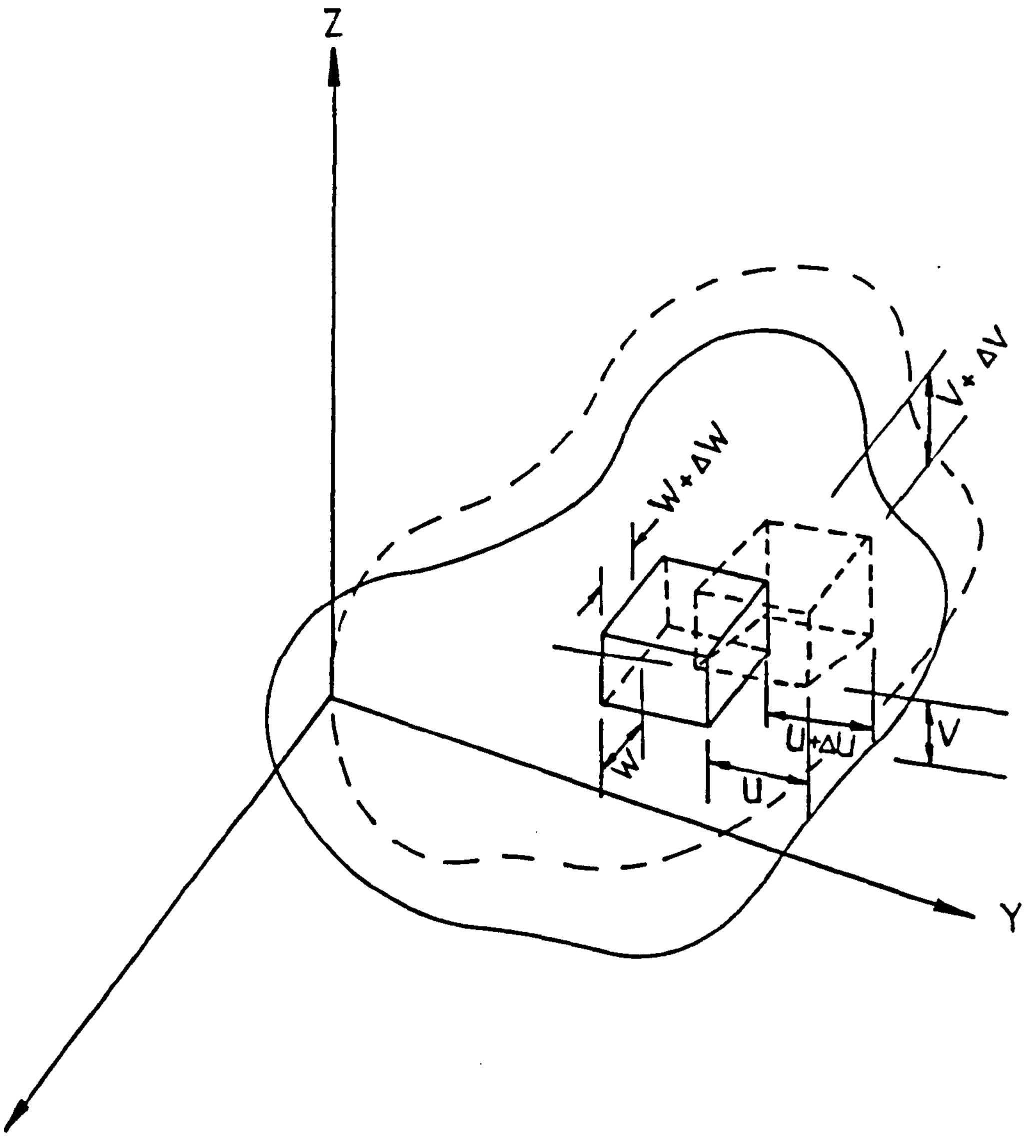


FIG. 3.3 DEFORMED AND UNDEFORMED BODY

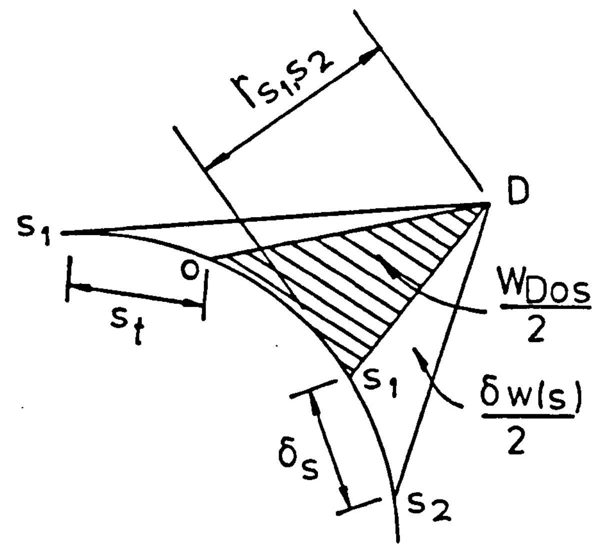


FIG. 3.4 SECTIONAL COORDINATES OF THIN WALLED SECTION

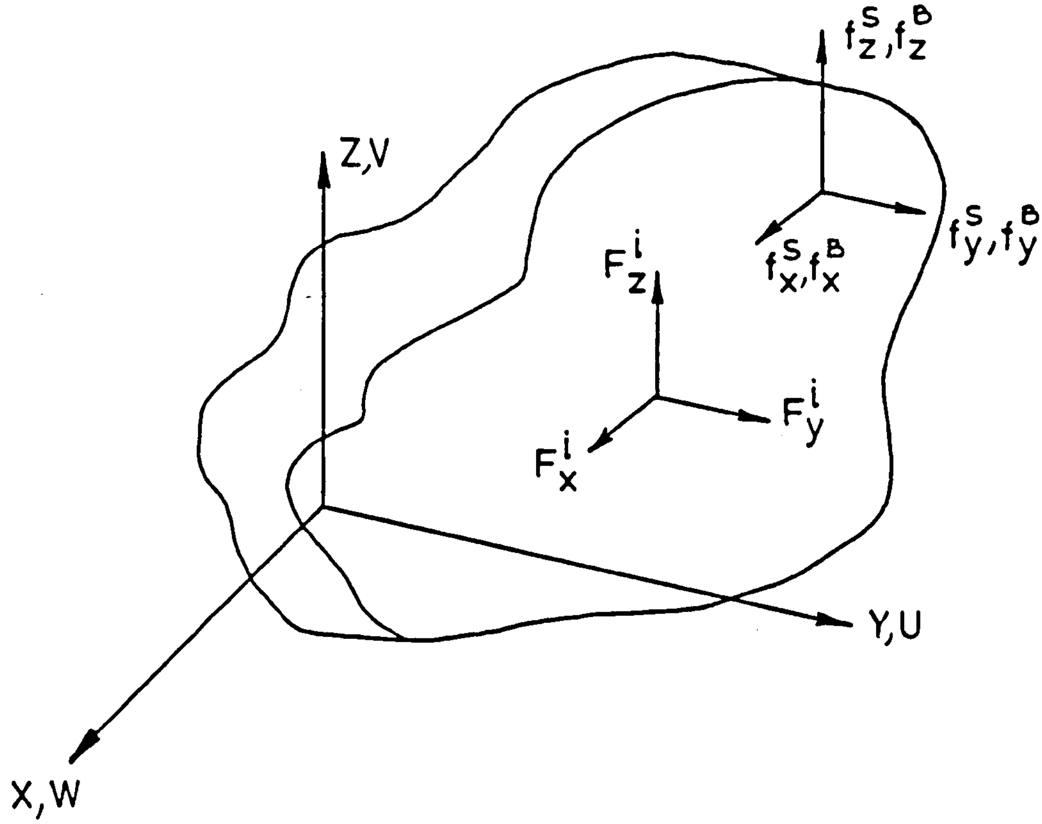
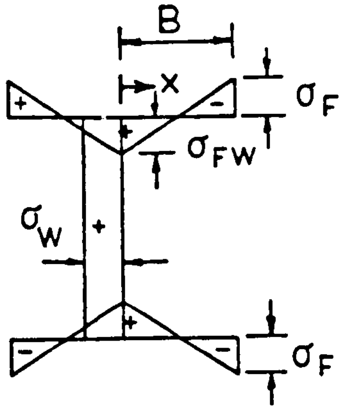


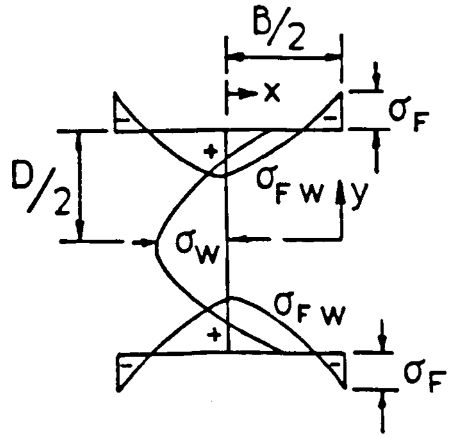
FIG.3.5 GENERAL THREE DIMENSIONAL BODY



$$\sigma = \frac{\sigma_F - \sigma_{FW}}{B} X + \sigma_{FW} \quad \text{(For flanges)}$$

$$\sigma = \sigma_W \quad \text{(For web)}$$

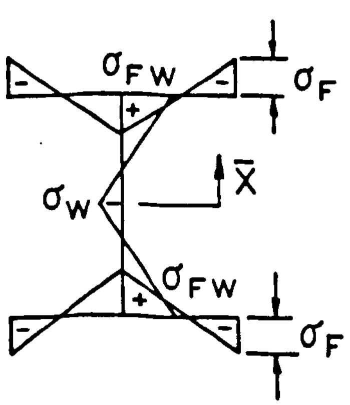
(a) Lehigh distribution



$$\sigma = \frac{(\sigma_{FW} - \sigma_W)}{(D/2)^2} y^2 + \sigma_W \quad \text{(Web)}$$

$$\sigma = \frac{\sigma_F - \sigma_{FW}}{(B/2)^2} x^2 + \sigma_F \quad \text{(Flange)}$$

(b) Parabolic distribution



$$\sigma = \frac{\sigma_F - \sigma_{FW}}{B} X + \sigma_{FW} \quad \text{(For flange)}$$

$$\sigma = \frac{\sigma_{FW} - \sigma_W}{D/2} \bar{X} + \sigma_{FW} \quad \text{(Web)}$$

(c) Triangular distribution

FIG. 3.6 TYPICAL PATTERNS OF RESIDUAL STRESS ADJUSTED TO ENSURE FULL 3-D EQUILIBRIUM

Chapter 4

Derivation of Stiffness Matrices

4.1 Introduction

The most important feature of the matrix stiffness analysis of structures is the formulation of the stiffness matrix for a discrete element. This is required in order that the real structure may be represented by a system with a finite number of degrees of freedom upon which the actual analysis can be performed.

These stiffness matrices are usually formulated by assuming that the displacements within an element vary in a suitable fashion; normally polynomials, whose coefficients are equal to the number of element nodal point displacements, are selected. It is then necessary to introduce these interpolation functions to the equilibrium equations in order to evaluate the element stiffness matrices.

Many previous investigators have derived stiffness matrices for a beam-column element in three dimensions in the form of a 12x12 stiffness matrix for a member having 6 degrees of freedom at each node. Tecan and Mahapatra (1969), produced a tangent stiffness matrix for a space frame member using a Taylor expansion (series). Przemieniecki (1968) gave two methods to obtain the linear tangential and geometrical stiffnesses. The first was based on the displacement method, while the second employed the force method. Based on considerations of nonlinear displacements and the second variation of the strains, Roberts and Azizian (1983a) developed both the geometrical and the tangential stiffnesses.

A 14x14 element stiffness matrix for a beam-column has been developed by Tebedge and Tall (1973), Barsoum and Gallagher (1970), Rajasekaran and Murry (1973), Yang and Mcguire (1986), and Kitiporncahi and Chan (1987) by using the strain displacement relationships. The element nodal displacements were taken to be adequately represented by a linear polynomial for the axial displacements, by a cubic polynomial for lateral displacements and twists, and by a quadratic polynomial for the rotations and warping.

4.2- Interpolation Function

In the finite element approach the displacements are approximated by forming shape functions. The functions chosen to represent the solution within the element are most commonly taken as polynomial series for the following reasons :

a- It is straightforward to perform differentiation or integration and to computerise the processes.

b- It is straightforward to increase the accuracy of the results by increasing the order of the function.

The approximate function chosen for the displacement model must be capable of reproducing certain features of the true displacements if convergence to the true solution is to be obtained Zienkiewicz (1977), Bathe and Wilson (1976), Richards (1977), Rao (1982). It must be continuous within the element and there must be compatibility between adjacent elements . In general the number of terms in the chosen series must be directly related to the total number of degrees of freedom for the element.

In a one dimensional situation (extensional or flexural behaviour of bars), the required functions for the longitudinal and flexural displacements may be approximated respectively as

$$\Phi_i(x) = a_0 + a_1\frac{X}{L} \qquad (4.1)$$

$$\Phi_j(x) = a_0 + a_1\frac{X}{L} + a_2\frac{X^2}{L} + a_3\frac{X^3}{L} \qquad (4.2)$$

where i=1,2 and j=1,2,3,and 4

Applying the boundary conditions enables the arbitrary constants a_0 , a_1 , a_2 ,and a_3 to be obtained leading to the following equations, which are illustrated by Richards (1977), Zienkiewciez (1977):

$$\Phi_1(X)= 1 - 3\frac{X^2}{L^2} + 2\frac{X^3}{L^3} \qquad (4.3a)$$

$$\Phi_2(X) = X - 2\frac{X^2}{L} + \frac{X^3}{L^2} \qquad (4.3b)$$

$$\Phi_3(X) = -2\frac{X^3}{L^3} + 3\frac{X^2}{L^2} \qquad (4.3c)$$

$$\Phi_4(X) = \frac{X^3}{L^2} - \frac{X^2}{L} \qquad (4.3d)$$

Axial displacement can be represented in matrix form as

$$[W] = [(1-\frac{X}{L}) \quad \frac{X}{L}]\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \qquad (4.4)$$

Equation 4.3 represents the flexural displacements (about the X, Y, and Z axes), it can be re-arranged in matrix form as:

$$[V],\ [U],\ \text{and}\ [\Phi] = [\Phi_1(X) \quad \Phi_2(X) \quad \Phi_3(X) \quad \Phi_4(X)]\begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} \qquad (4.5)$$

4.3- Stiffness Matrices

The element stiffness matrices play an important part in the analysis of beam-columns having an arbitrary open cross-section. These stiffnesses can be explained as; tangential stiffness matrix $[K_E^L]$ is the combination of axial stiffness K_{WW}, flexural stiffnesses K_{UU} and K_{VV} in the U and V directions respectively, torsional stiffness $K_{\phi\phi}$, stiffness caused by coupling between W and V K_{WV}, and similarly for K_{WV}, $K_{W\phi}$, K_{UV}, $K_{U\phi}$, and $K_{V\phi}$. The element geometrical stiffness matrix $[K_G^L]$, also represents the combination of the effects of the element forces such as : axial forces F_x, shear forces Q_y and Q_z, bending moments M_y and M_z and torsion M_x, etc..

The nonlinear element stiffness and geometrical matrices $[K_E^{NL}]$ and $[K_G^{NL}]$ incorporate the higher terms and the coupling between the displacements. Full details are covered later in this section.

The employment of these matrices in assembling the element stiffnesses, imposing boundary conditions, solving the equilibrium equations for nodal displacements, and evaluating the element stresses is explained in full in chapter 5. This section concentrates on the development of the stiffness matrices (geometrical and tangential) of a beam-column in space. These stiffnesses are developed by either the principle of virtual work or by using total potential energy.

The following notation will used in determining the total linear , and nonlinear stiffnesses:

$$[n^i_{N,I}] = [\; n^i_{1,I} \quad n^i_{2,I} \quad n^i_{3,I} \quad n^i_{4,I} \;] \tag{4.6}$$

i = the derivative degree with respect to X

N =1 to 4

I = kind of function (0 linear and 1 cubic)

Example

$$[K_{VU\phi}^{ijk}] = \int_0^L n_{N,I}^{i} \cdot n_{N,J}^{j} \cdot n_{N,K}^{k} dx \qquad (4.7)$$

k , j , i = the derivative degree with respect to X

K , J , and I =1 cubic function

0 linear function

N = 1,2,3,4

4.3.1- Principle of Virtual Work

The stiffness matrices explained herein are obtained by substituting equation 4.5 into equation 3.56 and then carrying out the integration by assuming that parameters such as EI_y, EI_z, GK, M_y, M ,..., etc. are calculated at each node. The averages of the values evaluated at the nodes of the element are used in the stiffnesses. Some rearrangement needs to be introduced into equation 3.56 in order to obtain both linear and nonlinear geometrical and tangential stiffness matrices which are classified as:

1- Linear tangential stiffness matrix $[K_E^l]$

2- Linear geometrical stiffness matrix $[K_G^l]$

3- Nonlinear tangential stiffness matrix $[K_E^{Nl}]$

4- Nonlinear geometrical stiffness matrix $[K_G^{NL}]$

4.3.1.1 Linear Tangential Stiffness Matrix

The Linear tangential stiffness matrix $[K_E{}^L]$ incorporates terms such as GK, EI_Y, EI_z, warping EI which represent torsional, flexural and warping rigidities for the member. It is defined by equation 4.8. This equation is expressed in Table 4.1.

$$[K_E^L] = \begin{bmatrix} a & b & c & d \\ & e & f & g \\ & & h & j \\ & & & i \end{bmatrix} \tag{4.8}$$

where

$a=K_{WW}^{11}$, $b=\bar{Y}K_{UW}^{12}$, $e=K_{UU}^{22}$, $c=\bar{Z}K_{UW}^{12}$

$f=K_{UV}^{22}$, $d=S_{\omega}K_{\Phi W}^{12}$, $g=K_{\Phi U}^{22}$

$j=K_{\Phi V}^{22}$, $h=K_{VV}^{22}$, $i=K_{\Phi\Phi}^{22}$

4.3.1.2 Linear Geometrical Stiffness Matrix

The Linear geometrical matrix [$K_G{}^L$]) includes parameters for the shear forces (Q_y, Q_z), bending moments (M_y, M_z), and torsional moment (M_x), etc.. The full definition of the terms of equation 4.9 is presented in Table 4.2.

$$[K_G^L] = \begin{bmatrix} A & B & \\ C & D & E \\ & F & G \\ & & H \end{bmatrix} \tag{4.9}$$

where

$A= Q_y K_{UW}^{11}$, $B=Q_z K_{VW}^{11}$, $C=F_x K_{UU}^{11}$

$D=M_x(K_{VW}^{11} - K_{VU}^{12})$, $F=F_x K_{VV}^{11}$, $H=M_\rho K_{V\Phi}^{10}$

$E= (Z_s F_x + M_y)K_{U\Phi}^{11} + Q_z K_{U\Phi}^{10}$

$\upsilon = (Y_s F_x + M_z) K^{11}_{V\phi} + Q_y K^{10}_{V\phi}$

4.3.1.3 Nonlinear Tangential Stiffness Matrix

The nonlinear tangential stiffness matrix $[K_E^{NL}]$ is given in equation 4.10. It incorporates the influence of the higher order terms and their coupling. Details of the terms are presented in Table 4.3.

$$[K_E^{NL}] = \begin{bmatrix} \overline{A} & \overline{B} & \overline{C} \\ \overline{D} & \overline{E} & \overline{F} \\ & \overline{G} & \overline{H} \\ & & \overline{I} \end{bmatrix} \qquad (4.10)$$

where

$\overline{A} = EAK^{111}_{wvu}$, $B_1 = EAK^{111}_{wuv}$

$\overline{C} = EA\{-\frac{\overline{Y}}{2} K^{120}_{wv\phi} + \overline{Z} K^{120}_{wv\phi} - K^{111}_{wu\phi}$

$\overline{D} = \frac{EA\overline{Y}}{2} K^{112}_{uuu} - EI_{yz} K^{202}_{u\phi u} + EI_{yz} K^{202}_{u\phi u}$

$\overline{E} = \frac{EA\overline{Y}}{2} K^{112}_{vvu} - \frac{EA\overline{Z}}{2} K^{112}_{uuv} + EI_{yz} K^{202}_{u\phi v}$

$\overline{F} = EA\overline{Y}[Z_s K^{111}_{\phi vu} + Y_s K^{112}_{\phi\phi u}] + \frac{1}{2} EA(Y_s^2 + Z_s^2) K^{112}_{\phi\phi u} +$

$ES_\omega K^{112}_{uu\phi} - EI_{yz}[K^{211}_{uv\phi} + K^{112}_{\phi\phi u} + K^{220}_{uu\phi}] +$

$EI_z[K^{211}_{uv\phi} - K^{220}_{uv\phi} - Y_s K^{112}_{\phi\phi u} + \quad EI_y[K^{121}_{uv\phi} + Z_s K^{112}_{\phi\phi u} - K^{220}_{uv\phi}]$

$\overline{H} = EA\overline{Z}[Z_s K^{111}_{\phi uv} + Y_s K^{112}_{\phi uv}] + \frac{1}{2} EA(Y_s^2 + Z_s^2) K^{112}_{\phi\phi v}$

$$+ \; ES_{\omega}K^{112}_{vv\phi} - EI_{yz}[K^{211}_{vv\phi} + Z_sK^{112}_{\phi\phi v} + K^{220}_{uv\phi}]$$

$$+ \; EI_z[K^{211}_{vu\phi} - K^{220}_{vu\phi} - Y_sK^{112}_{\phi\phi v}] + EI_y[K^{211}_{vu\phi} + Z_sK^{112}_{\phi\phi v} - K^{220}_{vu\phi}]$$

$$\bar{C} = EA\frac{\bar{I}}{2}K^{112}_{vvv} - EI_yK^{211}_{v\phi v} + EI_{yz}K^{202}_{v\phi v}$$

$$\bar{I} = ES_{\omega}[Y_sK^{112}_{\phi v\phi} - Z_sK^{112}_{\phi u\phi} + \frac{1}{2}(Y_s^2 + Z_s^2)K^{202}_{\phi\phi\phi}]$$

$$+ \; EI_{y\omega}[K^{220}_{\phi v\phi} + K^{211}_{\phi v\phi} + Y_sK^{112}_{\phi\phi\phi}]$$

$$+ \; EI_{\omega z}[K^{220}_{\phi u\phi} + K^{211}_{\phi u\phi} + Z^{112}_{\phi\phi\phi}]$$

4.3.1.4 Nonlinear Geometrical Stiffness Matrix

The nonlinear geometrical stiffness matrix $[K_G^{Nl}]$ includes the full higher order terms which are described in equation 4.11. The expressions for the terms of equation 4.11 are fully explained in Table 4.4.

$$[K_G^{NL}] = \begin{bmatrix} A_1 & B_1 & C_1 \\ D_1 & E_1 & F_1 \\ & G_1 & H_1 \\ & & I_1 \end{bmatrix} \qquad (4.11)$$

where

$$A_1 = -Q_zK^{101}_{w\phi v} \; , \; B_1 = -Q_yK^{101}_{w\phi v} \; , \; C_1 = -Q_yK^{110}_{wv\phi} - Q_zK^{110}_{wu\phi}$$

$$D_1 = -Q_yZ_s[K^{102}_{u\phi u} + K^{211}_{u\phi u} + K^{101}_{u\phi u}]$$

$$-Q_z Y_s\{K^{111}_{u\phi u} + K^{211}_{u\phi u}\} + M_x\{K^{201}_{u\phi u} + K^{120}_{u\phi u} - k^{111}_{u\phi u}\}$$

$$C_1 = -Q_z Y_s[K^{102}_{v\phi v} + K^{211}_{v\phi v} + K^{101}_{v\phi v}]$$

$$-Q_z Y_s\{K^{111}_{v\phi v} + K^{211}_{v\phi v}\} + M_x\{K^{201}_{v\phi v} + K^{120}_{v\phi v} - k^{111}_{v\phi v}\}$$

$$E = -Q_y Z_s K^{211}_{u\phi v} - Q_z Y_s K^{211}_{u\phi v}$$

$$F = [PY_s - M_z]K^{101}_{\phi\phi u} + Q_z Y_s[K^{120}_{uu\phi} + K^{111}_{uv\phi}]$$

$$-Q_y Z_s\{K^{121}_{uv\phi} + K^{121}_{uu\phi} + K^{211}_{uv\phi}] + M_x\{K^{210}_{uu\phi} - K^{120}_{uu\phi} + K^{111}_{uu\phi}\}$$

$$H = [PZ_s - M_y]K^{101}_{\phi\phi v} + Q_z Y_s[K^{120}_{vv\phi} + K^{121}_{vu\phi}]$$

$$-Q_y Z_s\{K^{121}_{vu\phi} + K^{121}_{vu\phi} - K^{120}_{uv\phi}\} + M_x(K^{210}_{vv\phi} - K^{120}_{vv\phi} + K^{111}_{vv\phi})$$

The subassemblage of the element stiffnesses stated in equations 4.8 - 4.11 is the total stiffness matrix which is employed in the analysis of beam-columns under different load patterns together with various boundary conditions in the elastic and inelastic ranges as presented in equation 4.12.

$$[K_t] = [K^L_E] + [K^L_G] + [K^{NL}_E] + [K^{NL}_G] \qquad (4.12)$$

Further details of the assembly of the full geometrical linear and nonlinear stiffness matrices and the element tangent stiffnesses are explained in chapter 5 and the validity of their implementation in the analysis is demonstrated in chapters 6 and 7 respectively.

4.3.2 Total Potential Energy

Recall equation 3.62 from chapter 3 which represents the equilibrium equation of a beam-column of thin-walled open cross-section in three dimensions. Substituting equations 4.5 and 4.7 into 3.62, and carrying out the integrations the equilibrium condition can be obtained as a stationary point of the total potential energy as:

$$\delta\Pi = \frac{\partial\Pi}{\partial W}dW + \frac{\partial\Pi}{\partial U}dU + \frac{\partial\Pi}{\partial U}dV + \frac{\partial\Pi}{\partial\Phi}d\Phi \tag{4.13}$$

or

$$\delta\Pi = \frac{\partial\Pi}{\partial r} \tag{4.14}$$

where

$$r = [W_1 \quad U_1 \quad \theta_{u1} \quad V_1 \quad \theta_{v1} \quad \Phi_1 \quad X_1 \quad W_2 \quad U_2 \quad \theta_{u2} \quad V_2 \quad \theta_{v2} \quad \Phi_2 \quad X_2]^T$$

After some rearrangement the general form of the equation can be written as:

$$\delta\Pi = \int_0^1 \Big[[EA[\langle W\rangle[K^{11}_{ww}] - Y(\langle V\rangle[K^{201}_{v\phi w}] + \langle U\rangle[K^{21}_{uw}]+\langle V\rangle[K^{111}_{v\phi w}])$$

$$+ \frac{1}{2}(\langle V\rangle([K^{111}_{vvw}] + \langle U\rangle[K^{111}_{uuw}] + \frac{\rho_o^2}{EA}\langle\Phi\rangle[K^{111}_{\phi\phi w}])$$

$$- Z(\langle V\rangle[K^{21}_{vw}] - \langle U\rangle[K^{201}_{u\phi w}]-\langle U\rangle[K^{111}_{u\phi w}]) - \frac{ES_\omega}{EA}\langle\Phi\rangle[K^{21}_{\phi w}]$$

$$- Y_s\langle\Phi\rangle[K^{111}_{\phi uw}] + Z_s\langle\Phi\rangle[K^{111}_{\phi vw}] - \langle V\rangle\{[K^{201}_{v\phi w}]-[K^{21}_{vw}]-[K^{111}_{v\phi w}]\}$$

$$- \langle U\rangle\{[K^{201}_{u\phi w}]-[K^{21}_{uw}]+[K^{111}_{u\phi w}]\} - M_\omega\langle\Phi\rangle[K^{21}_{\phi w}$$

$$- M_y(\langle V\rangle[K^{21}_{vw}] - \langle U\rangle[K^{201}_{U\phi w}] - \langle U\rangle[K^{111}_{u\phi w}])$$

$$- M_z(\langle U\rangle[K^{21}_{uw}] + \langle V\rangle[K^{201}_{v\phi w}] + \langle V\rangle[K^{111}_{v\phi w}])$$

$$+ \frac{Q_y}{2}(\langle V\rangle[K^{11}_{vw}] - \langle\Phi\rangle[K^{021}_{\phi uw}] - \frac{Q_z}{2}(\langle U\rangle[K^{11}_{uw}] + \langle\Phi\rangle[K^{021}_{\phi vw}] + \overline{P}]\{W\}$$

$$+ EA\langle W\rangle\{[K^{111}_{wvv}]-\overline{Y}([K^{102}_{w\phi v}] + [K^{111}_{w\phi v}])$$

$$- \overline{Z}[K^{12}_{wv}] + Z_s[K^{111}_{w\phi v}] - [K^{102}_{w\phi v}] - Y_s[K^{111}_{w\phi v}] + [K^{12}_{wv}]\}$$

$$+ EA\overline{Y}\{Y_s\langle U\rangle([K^{211}_{u\phi v} + [K^{211}_{uvv}])\}$$

$$+ EA\{\overline{Z}Y_s\langle V\rangle([K^{112}_{v\phi v} + [K^{211}_{v\phi v}]) - Z_s\langle U\rangle[K^{112}_{u\phi v}]$$

$$+ \frac{1}{2}(\langle U\rangle[K^{112}_{uuv}] + \langle V\rangle[K^{112}_{vvv}]$$

$$+ \langle V\rangle[K^{211}_{vvv}]\} + ES_\omega\{\langle\Phi\rangle[K^{211}_{\phi\phi v}] + \langle\Phi\rangle[K^{211}_{\phi vv}]\}$$

$$+ EI_y\{\langle V\rangle[K^{22}_{vv}] -\langle U\rangle[K^{202}_{u\phi v}] - \langle U\rangle[K^{112}_{u\phi v}]$$

$$- \langle V\rangle[K^{112}_{v\phi v}] - \langle V\rangle[K^{211}_{v\phi v}]\}$$

$$+ EI_z\{\langle U\rangle[K^{202}_{u\phi v}] + \langle U\rangle[K^{211}_{u\phi v}] + \overline{\beta}_z\langle\Phi\rangle[K^{112}_{\phi\phi v}]\}$$

$$+ EI_{yz}\{\langle U\rangle[K^{22}_{uv}] + \langle V\rangle[K^{202}_{v\phi v}] - \langle U\rangle[K^{112}_{u\phi v}]$$

$$+ \langle V\rangle[K^{112}_{v\phi v}] + \langle V\rangle[K^{202}_{u\phi v}] + \langle V\rangle[K^{211}_{v\phi v}] + \langle U\rangle[K^{211}_{u\phi v}]\}$$

$$+ EI_{\omega y}\langle\Phi\rangle\{[K^{202}_{\phi\phi v}] - [K^{211}_{\phi\phi v}] + [K^{121}_{\phi\phi v}]\}$$

$$+ EI_{\omega z}\langle\Phi\rangle\{[K^{22}_{\phi v}] + [K^{121}_{\phi\phi v}]\}$$

$$+ P\langle V\rangle([K^{11}_{vv}] - Y_s\langle\Phi\rangle[K^{11}_{\phi v}] + Z_s\langle\Phi\rangle[K^{101}_{\phi\phi v}])$$

$$- \frac{1}{2}Q_y\{\langle W\rangle[K^{11}_{wv}] + 2(\langle\Phi\rangle[K^{001}_{\phi\phi v}] +\langle\Phi\rangle[K^{01}_{\phi v}])\} + \frac{1}{2}Q_z\langle W\rangle[K^{101}_{w\phi v}]$$

$$- M_y\{\langle W\rangle[K^{12}_{wv}] + \langle\Phi\rangle[K^{011}_{\phi\phi v}]\}$$

$$+ M_z\{\langle\Phi\rangle[K^{11}_{\phi v}] - \langle W\rangle[K^{102}_{w\phi v}] - \langle W\rangle[K^{111}_{w\phi v}]\}$$

$$+ \frac{1}{2}M_x\{\langle U\rangle[K^{21}_{uv}] - \langle U\rangle[K^{12}_{uv}] + 2\langle\Phi\rangle[K^{111}_{\phi vv}]$$

$$+ 2(\langle V\rangle[K^{102}_{v\phi v}] + \langle V\rangle[K^{201}_{v\phi v}])\}]\{V\}$$

$$+ EA\langle W\rangle\{[K^{111}_{wuu}] - \overline{Z}([K^{102}_{w\phi u}] + [K^{111}_{w\phi u}])$$

$$- \overline{Y}[K^{12}_{wu}] - Y_s[K^{111}_{w\phi u}] - [K^{102}_{w\phi u}] - Z_s[K^{111}_{w\phi u}] + [K^{12}_{wu}]\}$$

$$+ EA\overline{Z}\{-Z_s\langle V\rangle([K^{211}_{v\phi u} + [K^{211}_{vuu}])\}$$

$$- EA\overline{Y}\{Z_s\langle U\rangle([K^{112}_{u\phi u} + [K^{211}_{u\phi u}]) - Y_s\langle V\rangle[K^{112}_{v\phi u}]$$

$$+ \frac{1}{2}(\langle U\rangle[K^{112}_{uuu}] + \langle V\rangle[K^{112}_{vvu}]\ (Y_s^2+Z_s^2)\langle\Phi\rangle[K^{112}_{\phi\phi u}]))\}$$

$$+ \langle U\rangle[K^{211}_{uuu}]\} + ES_\omega\{-Z_s\langle\Phi\rangle[K^{211}_{\phi\phi u}] + \langle\Phi\rangle[K^{211}_{\phi uu}]\}$$

$$+ EI_z\{\langle U\rangle[K^{22}_{uu}] - \langle V\rangle[K^{202}_{v\phi u}] + \langle V\rangle[K^{112}_{v\phi u}]$$

$$+ \langle U\rangle[K^{112}_{u\phi u}] + \langle U\rangle[K^{211}_{u\phi u}]\}$$

$$- EI_y\{\langle V\rangle[K^{202}_{v\phi u}] + \langle V\rangle[K^{211}_{v\phi u}] - \overline{\beta}_y\langle\Phi\rangle[K^{112}_{\phi\phi v}]\}$$

$$+ EI_{yz}\{\langle V\rangle[K^{22}_{vu}] + \langle V\rangle[K^{202}_{v\phi u}] - \langle U\rangle[K^{112}_{u\phi u}]$$

$$- \langle U\rangle[K^{112}_{u\phi u}] - \langle V\rangle[K^{202}_{v\phi u}] - \langle V\rangle[K^{211}_{v\phi u}] + \langle U\rangle[K^{211}_{u\phi u}]\}$$

$$+ EI_{\omega z}\langle\Phi\rangle\{[K^{202}_{\phi\phi u}] + [K^{211}_{\phi\phi u}] - [K^{121}_{\phi\phi u}]\}$$

$$+ EI_{\omega y}\langle\Phi\rangle\{[K^{22}_{\phi u}] - [K^{121}_{\phi\phi u}\}$$

$$+ P\langle V\rangle\{[K^{11}_{uu}] + Z_s\langle\Phi\rangle[K^{11}_{\phi u}] + Y_s\langle\Phi\rangle[K^{101}_{\phi\phi u}])$$

$$+ \frac{1}{2}Q_z\{\langle W\rangle[K^{11}_{wu}] - 2(\langle\Phi\rangle[K^{001}_{\phi\phi u}] - \langle\Phi\rangle[K^{01}_{\phi u}])\} - \frac{1}{2}Q_y\langle W\rangle[K^{101}_{w\phi u}]$$

$$- M_z\{\langle W\rangle[K^{12}_{wu}] + \langle\phi\rangle[K^{011}_{\phi\phi u}]\}$$

$$+ M_y\{\langle\Phi\rangle[K^{11}_{\phi u}] - \langle W\rangle[K^{102}_{w\phi u} + \langle W\rangle[K^{111}_{w\phi u}]\}$$

$$+ \frac{1}{2}M_x\{\langle V\rangle[K^{21}_{vu}] - \langle V\rangle[K^{12}_{vu}] + 2\langle\Phi\rangle[K^{111}_{\phi uu}]$$

$$- 2(\langle U\rangle[K^{102}_{u\phi u}] + \langle V\rangle[K^{201}_{u\phi u}])\}]\{U\}$$

$$+ EA\langle W\rangle\{-\overline{Y}([K^{120}_{wv\phi}]-[K^{111}_{wv\phi}]) + \overline{Z}([K^{120}_{wu\phi}]-[K^{111}_{wu\phi}])$$

$$- Y_s[K^{111}_{wu\phi}] + Z_s[K^{111}_{wv\phi}] - [K^{111}_{wu\phi}] - [K^{111}_{wv\phi}]$$

$$-[K^{120}_{wu\phi}] + K^{120}_{wv\phi}] + \frac{ES_\omega}{EA}[K^{12}_{w\phi}]\}$$

$$+ ES_\omega\{\langle W\rangle(Z_s[K^{111}_{wu\phi}] - Y_s[K^{111}_{wv\phi}])\}$$

$$+ EA\overline{Y}\{Y_s\langle U\rangle[K^{211}_{uv\phi}-Z_s\langle U\rangle[K^{211}_{uu\phi}]+(Y_s^2+Z_s^s)\langle U\rangle[K^{211}_{u\phi\phi}]\}$$

$$+ EA\overline{Z}\{Y_s\langle U\rangle[K^{211}_{vv\phi}-Z_s\langle V\rangle[K^{211}_{vu\phi}]+(Y_s^2+Z_s^s)\langle V\rangle[K^{211}_{v\phi\phi}]\}$$

$$+ ES_\omega\{Y_s(\langle V\rangle[K^{112}_{v\phi\phi}] + \langle\Phi\rangle[K^{211}_{\phi v\phi}]) - Z_s(\langle U\rangle[K^{112}_{u\phi\phi}] + \langle\Phi\rangle[K^{211}_{\phi u\phi}])$$

$$+ \frac{1}{2}(Y_s^2+Z_s^2)\langle\Phi\rangle[^{112}_{\phi\phi\phi}] + (Y_s^2+Z_s^2)\langle\Phi\rangle[^{211}_{\phi\phi\phi}]\}$$

$$+ EI_\omega\langle\Phi\rangle\{[K^{22}_{\phi\phi}] + \overline{\beta}_\omega\{[K^{112}_{\phi\phi\phi}]$$

$$+ 2[K^{211}_{\phi\phi\phi}]\} + GK\langle\Phi\rangle[^{11}_{\phi\phi}]\}$$

$$+ EI_z\{\langle V\rangle([K^{220}_{vu\phi}] + [K^{211}_{vv\phi}] + [K^{211}_{vu\phi}]) + 2\overline{\beta}_y\langle U\rangle[K^{211}_{u\phi\phi}]\}$$

$$+ EI_y\{\langle U\rangle([K^{220}_{uv\phi}] + [K^{211}_{uu\phi}] + [K^{211}_{uv\phi}]) + 2\overline{\beta}_y\langle V\rangle[K^{211}_{v\phi\phi}]\}$$

$$+ EI_{yz}\{\langle U\rangle([K^{220}_{uu\phi}] + [K^{211}_{uv\phi}]) + \langle V\rangle([K^{220}_{vv\phi}] + [K^{211}_{vu\phi}]$$

$$+ EI_{\omega y}\{\langle U\rangle([K^{22}_{u\phi}] - [K^{121}_{u\phi\phi}])$$

$$+ \langle V\rangle([K^{202}_{v\phi\phi}] - [K^{112}_{v\phi\phi}] + [K^{121}_{v\phi\phi}]) + \frac{1}{2}\langle V\rangle[K^{112}_{v\phi\phi}])$$

$$+ \langle\Phi\rangle(-Y_s[K^{112}_{\phi\phi\phi}] + [K^{220}_{\phi v\phi}] - 2Y_s[K^{211}_{\phi\phi\phi}] + [K^{112}_{\phi v\phi}])\}$$

$$+ EI_{\omega z}\{\langle V\rangle([K^{22}_{v\phi}] + [K^{121}_{v\phi\phi}])$$

$$- \langle U\rangle([K^{202}_{u\phi\phi}] + [K^{112}_{u\phi\phi}] - [K^{121}_{u\phi\phi}]) + \frac{1}{2}[K^{112}_{u\phi\phi}])$$

$$- \langle\Phi\rangle(Z_s[K^{112}_{\phi\phi\phi}] - [K^{220}_{\phi u\phi}] + 2Z_s[K^{211}_{\phi\phi\phi}] + [K^{112}_{\phi u\phi}])\}$$

$$+ M_\rho\langle\Phi\rangle[K^{11}_{\phi\phi}] + P\{Z_s\langle U\rangle[K^{11}_{u\phi}] - Y_s\langle V\rangle[K^{11}_{v\phi}]$$

$$+ Z_s\langle V\rangle[K^{110}_{v\phi\phi}] + Y_s\langle U\rangle[K^{110}_{u\phi\phi}] + Z_s\langle V\rangle[K^{101}_{v\phi\phi}] + Y_s\langle U\rangle[K^{101}_{u\phi\phi}]$$

$$+ \frac{1}{2}Q_y\{(-2\langle V\rangle[K^{10}_{v\phi}] - \langle W\rangle[K^{110}_{wu\phi}] - 2\langle V\rangle([K^{100}_{v\phi\phi}]$$

$$+ Z_s\langle\Phi\rangle(-\frac{1}{2}[K^{001}_{\phi\phi\phi}] + [K^{100}_{\phi\phi\phi}] + 2[K^{010}_{\phi\phi\phi}]$$

$$- Y_s\langle\Phi\rangle([K^{01}_{\phi\phi}] + [K^{10}_{\phi\phi}]))\}$$

$$- \frac{1}{2}Q_z\{(-2\langle U\rangle[K^{10}_{u\phi}] + \langle W\rangle[K^{110}_{wv\phi}] + 2\langle U\rangle([K^{100}_{u\phi\phi}]$$

$$+ Y_s\langle\Phi\rangle(\frac{1}{2}[K^{001}_{\phi\phi\phi}] + [K^{100}_{\phi\phi\phi}] - 2[K^{010}_{\phi\phi\phi}]$$

$$- Z_s\langle\Phi\rangle([K^{01}_{\phi\phi}] + [K^{10}_{\phi\phi}]))\}$$

$$+ M_y\{\langle U\rangle[K^{11}_{u\phi} - \langle V\rangle[K^{110}_{v\phi\phi}]$$

$$- \langle V\rangle[K^{101}_{v\phi\phi}] + \langle W\rangle[K^{120}_{wu\phi}] + \langle W\rangle[K^{111}_{wu\phi}]\}$$

$$- M_z\{\langle V\rangle[K^{11}_{v\phi} - \langle U\rangle[K^{110}_{u\phi\phi}]$$

$$- \langle U\rangle[K^{101}_{u\phi\phi}] + \langle W\rangle[K^{120}_{wv\phi}] + \langle W\rangle[K^{111}_{wv\phi}]\}$$

$$- M_\omega\langle W\rangle[K^{12}_{w\phi}] + \frac{M_x}{2}\{\langle U\rangle[K^{111}_{uu\phi}] - \langle V\rangle[K^{111}_{vv\phi}]$$

$$+ \langle\Phi\rangle[K^{001}_{\phi\phi\phi}] + 2\langle\Phi\rangle[K^{100}_{\phi\phi\phi}] - 2\langle U\rangle[K^{120}_{uu\phi}] + 2\langle V\rangle[K^{120}_{vv\phi}]]]\{\delta\Phi\}$$

$$- M_y\Phi_o[\{V''\} - \langle U\rangle[K^{11}_{u\phi}]\{\Phi\} - \langle\Phi\rangle[K^{11}_{\phi u}]\{U\}]$$

$$+ M_z\Phi_o[\{U''\} + \langle V\rangle[K^{11}_{v\phi}]\{\Phi\} + \langle\Phi\rangle[K^{11}_{\phi v}]\{V\}]$$

$$+ \frac{1}{2}P\Phi_o[Z_s(\langle V\rangle[K^{11}_{v\phi}]\{\Phi\} + \langle\Phi\rangle[K^{11}_{\phi v}]\{V\})]$$

$$+ \frac{1}{2}P\Phi_o[Y_s(\langle U\rangle[K^{11}_{u\phi}]\{\Phi\} + \langle\Phi\rangle[K^{11}_{\phi u}]\{U\})]$$

$$- Q_y\Phi_o[\{V'\} + Y_s\{\Phi'\} + 2Z_s\{\Phi\}$$

$$+ Q_z\Phi_o[\{U'\} + Y_s\{\Phi'\} + 2Y_s\{\Phi\}\Big]dx$$

$$+ \int_0^1 \Big[[D_1\langle U\rangle[K^{22}_{uu}] + D_5\langle V\rangle[K^{22}_{vu}] + D_6\langle\Phi\rangle[K^{22}_{\phi u}]]\{\delta U\}$$

$$+ [D_2\langle V\rangle[K^{22}_{vu}] + D_4\langle U\rangle[K^{22}_{uv}] + D_7\langle\Phi\rangle[K^{22}_{\phi v}]]\{\delta V\}$$

$$+ [D_3\langle V\rangle[K^{22}_{\phi\phi}] + D_6\langle U\rangle[K^{22}_{u\phi}] + D_7\langle\Phi\rangle[K^{22}_{v\phi}]]\{\delta\Phi\}\Big]\,\mathbf{dx} \qquad (4.16)$$

where D_1, D_2,...., D_7 were defined in Appendix C

Equation 4.16 can be written in a simple form as:

$$\delta\Pi = \langle r\rangle[K_{T2}]\{r\}] + \{\overline{P}\} + \{r_0\overline{P}\} \qquad (4.17)$$

where

$$\{r_o\} = [W_{01}\ U_{01}\ \theta_{01}\ V_{01}\ \Phi_{01}\ X_1\ W_{02}\ U_{02}\ \theta_{02}\ V_{02}\ \Phi_{02}\ X_{02}]^T \quad (4.18)$$

are the initial deflections.

$$\{\overline{P}\} = [F_{x1}\ Q_{z1}\ M_{y1}\ Q_{y1}\ M_{x1}\ M_{w1}\ F_{x2}\ Q_{z2}\ M_{y2}\ Q_{y2}\ M_{x2}\ M_{w2}]^T \quad (4.19)$$

$$[K_{T2}] = [K_E^L + K_G^L + K_E^{NL} + K_G^{NL} + K_o] \qquad (4.20)$$

Carry out the integration of equation 4.16 by assuming the terms EI_y, EI_z, $\overline{Y}$, M_y, M_ω, etc. to vary linearly in the form

$$EI_y = EI_{y1} + (EI_{y2} - EI_{y1})\frac{X}{l} \qquad (4.20a)$$

or

$$M_y = M_{y1} + (M_{y2} - M_{y1})\frac{X}{l} \qquad (4.20b)$$

where 1 and 2 are nodal member numbers of the element

The terms of equation 4.20 can be described as:

1- Linear tangential stiffness matrix $[K_E^l]$

2- Linear geometrical stiffness matrix $[K_G^l]$

3- Nonlinear tangential stiffness matrix $[K_E^{Nl}]$

4- Nonlinear geometrical stiffness matrix $[K_G^{NL}]$

5- Initial stiffness matrix $[K_o]$

4.3.2.1 Linear Tangential Stiffness Matrix

The linear tangential stiffness matrix $[K_E^L]$ contains terms such as EI_y, ES_ω, $EI_{\omega y}$, etc. These expressions are fully presented in Table 4.5.

4.3.2.2 Linear Geometrical Stiffness Matrix

Table 4.6 gives the terms of the linear geometrical stiffness matrix $[K_G^L]$. The Wagner effect together with internal forces such as M_y, Q_z, etc are included.

4.3.2.3 **Nonlinear Tangential Stiffness Matrix**

The higher order terms such as $EI_{y',x}V_{,xx}$, $EI_{\omega y}\Phi_{,x}V_{,xx}$. are accounted for,and are given Table 4.7. These expressions are detailed in Appendices G and J.

4.3.2.4 **Nonlinear Geometrical Stiffness Matrix**

The nonlinear geometrical stiffness matrix $[K_G^{NL}]$ for a beam-element in three dimensions is presented in Table 4.8. The terms of that Table are given in full detail in Appendices H and J.

4.3.2.5 **Initial Stiffness Matrix**

The inclusion of the initial deflections in the stiffness matrix is explained in Table 4.9.

4.3.2.6 **Assembly**

The assembly of the submatrices presented in Tables 4.5 to 4.9 corresponding to the linear geometrical and tangential stiffness matrix and the nonlinear terms together with the initial stiffness matrix is demonstrated by several illustrative examples in Chapters 6 & 7 respectively.

4.4- **Transformation Matrix**

The transformation matrix has been explained in many references such as Weaver (1967), Poresi and Lynn (1977), and Chen and Atsuta (1977). The transformation presented herein is based on using vector notation. The element displacements with respect to the local coordinates can be related to those in the global coordinate system by the transformation [T].

$$\{r_E\}_L = [T]\{r_E\}_G \qquad (4.21)$$

where [T] is the element transformation matrix whic can be obtained in the following using vector notation.

Let us consider a beam element which is defined by joint numbers II and JJ, a third point KK is used to locate the principal axes as shown in Fig 4.1. The transformation from local to global coordinates is most conveniently developed using vector notation.

The unit vectors in the $\overline{s1}$ and $\overline{g1}$ directions are given by

$$\overline{s1} = S1X\ \bar{i} + S1Y\ \bar{j} + S1Z\ \bar{k} \qquad (4.22)$$

$$\overline{g1} = GX\ \bar{i} + GY\ \bar{j} + GZ\ \bar{k} \qquad (4.23)$$

where $\bar{i}$, $\bar{j}$, and $\bar{k}$ are unit vectors in the $\bar{X}$, $\bar{Y}$, and $\bar{Z}$ directions respectively, the direction cosines are

$$S1X = \frac{X(JJ)-X(II)}{l}, \quad S1Y = \frac{Y(JJ)-Y(II)}{l}, \quad S1Z = \frac{Z(JJ)-Z(II)}{l}$$

$$GX = \frac{X(KK)-X(II)}{l}, \quad GY = \frac{Y(KK)-Y(II)}{l}, \quad GZ = \frac{Z(KK)-Z(II)}{l}$$

where

$$l = \sqrt{[X(JJ)-X(II)]^2 + [Y(JJ)-Y(II)]^2 + [Z(JJ)-Z(II)]^2}$$

$$G = \sqrt{[X(KK)-X(II)]^2 + [Y(KK)-Y(II)]^2 + [Z(KK)-Z(II)]^2}$$

The unit vectors in the $\overline{s3}$ and $\overline{s2}$ directions are given respectively by

$$\overline{s3} = \frac{\overline{s1} \times \overline{g}}{|\overline{s3}|} = S3X\ \bar{i} + S3Y\ \bar{j} + S3Z\ \bar{k} \tag{4.24}$$

$$\overline{s2} = \overline{s3} \times \overline{s1} = S2X\ \bar{i} + S2Y\ \bar{j} + S2Z\ \bar{k} \tag{4.25}$$

Equation 4.22, 3.24, and 4.25 can be written in matrix form as:

$$\begin{bmatrix} \overline{s1} \\ \overline{s2} \\ \overline{s3} \end{bmatrix} = [S_A] \begin{bmatrix} \bar{i} \\ \bar{j} \\ \bar{k} \end{bmatrix} \tag{4.26}$$

where

$$[S_A]_{3x3} = \begin{bmatrix} S1X & S1Y & S1Z \\ S2X & S2Y & S2Z \\ S3X & S3Y & S3Z \end{bmatrix} \tag{4.27}$$

The transformation matrix of a beam-column having 7 degrees of free dom at each node can be written as:

$$[S] = \begin{bmatrix} [S_A] & & \\ & [S_A] & \\ & & 1 \end{bmatrix} \tag{4.28}$$

The assembly of the total transformation matrix in 3-dimensions for a beam element is performed in equation 4.29, the diagonal terms corresponding to the warping are set to unity. So the transformation stiffness to a beam element can be written as:

$$[T] = \begin{bmatrix} [S] & \\ & [S] \end{bmatrix} \tag{4.29}$$

4.5 Strain Displacement Matrices

In this section a full explanation of the expressions for the strain displacement matrices are presented. The linear and higher order terms are included. The validity and the accuracy of the inclusion of both linear and nonlinear strains are demonstrated for several problems in the examples given in chapters 6 and 7.

4.5.1 Linear Strain Matrix

The axial strain due to displacement in the X-direction is $W_{,x}$. $\overline{\Phi}_z$, $\overline{\Phi}_y$, and $\overline{\Phi}_\omega$ are the curvatures produced from the displacements U, V, and Φ, which are known as the strains due to curvature. They produce additional axial strains in the beam whose values vary linearly with the depth of the beam for the biaxial and warping curvature. The total linear axial strain Σ^l_{xx} due to the deformations is given by

$$\Sigma^l_{xx} = W_{,x} - \overline{\Phi}_y - \overline{\Phi}_z + \overline{\Phi}_\omega \tag{4.30}$$

where

$$\overline{\Phi}_y = U_{,xx}Z \tag{4.31a}$$

$$\overline{\Phi}_z = U_{,xx}Y \tag{4.31b}$$

$$\overline{\Phi}_\omega = \Phi_{,xx}\overline{\omega}_n \tag{4.31c}$$

in which Y and Z are the coordinates of the observed point on the cross section (position where the strain needs to be evaluated). $\overline{\omega}_n$ is the normalized function explained in the previous chapter. The curvatures $V_{,xx}$, $U_{,xx}$, and $\Phi_{,xx}$ are the second derivatives of the in-plane and out-of-plane displacements, and rotations with respect to X respectively.

Equation 4.30 can be written in anther form as:

$$\Sigma^{1}_{xx} = \Sigma_x - \Phi_y Z - \Phi_z Y - \quad \Phi_{\omega}\bar{\omega}_n \tag{4.32}$$

where

$$\Phi_y = U_{,xx} \tag{4.33a}$$

$$\Phi_z = V_{,xx} \tag{4.33b}$$

$$\Phi_{\omega} = \Phi_{,xx} \tag{4.33d}$$

Substituting equation 4.4 into equation 4.33a and applying the two boundary conditions which are; X=0 and X=1 (at the two nodes) yields

$$\begin{bmatrix} \Sigma^1_x \\ \Sigma^2_x \end{bmatrix} = \begin{bmatrix} \frac{1}{1} & -\frac{1}{1} \\ -\frac{1}{1} & \frac{1}{1} \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \tag{4.34}$$

$$[\Sigma_x] = \begin{bmatrix} \Sigma^1_x \\ \Sigma^2_x \end{bmatrix} \tag{4.35}$$

where Σ^1_x, Σ^2_x are the axial strains at node 1 and 2, 1 is the segment beam length, W_1 and W_2 are longitudinal displacements at node 1 and 2.

The value of the Curvature Φ_y can be obtained by substituting equation 4.3 into equation 4.33a and applying the same boundary conditions described and stated above to obtain

$$\begin{bmatrix} \Phi_{y1} \\ \Phi_{y2} \end{bmatrix} = \begin{bmatrix} -\frac{6}{1^2} & -\frac{4}{1} & \frac{6}{1^2} & -\frac{2}{1} \\ \frac{6}{1^2} & \frac{2}{1} & -\frac{6}{1^2} & \frac{4}{1} \end{bmatrix} \begin{bmatrix} U_1 \\ \theta_{u1} \\ U_2 \\ \theta_{u2} \end{bmatrix} \tag{4.36}$$

where

$$[\Phi_y] = \begin{bmatrix} \Phi_{y1} \\ \Phi_{y2} \end{bmatrix} \tag{4.37}$$

where Φ_{y1} and Φ_{y2} are the curvatures at nodes 1 and 2 respectively. Similar procedures may be applied for Φ_z and Φ_ω, which can be written as:

$$[\Phi_z] = \begin{bmatrix} \Phi_{z1} \\ \Phi_{z2} \end{bmatrix} \tag{4.38}$$

$$[\Phi_\omega] = \begin{bmatrix} \Phi_{\omega 1} \\ \Phi_{\omega 2} \end{bmatrix} \tag{4.39}$$

Substituting for equations 4.35, 4.37, 4.38, and 4.39 enables the complete result to be written in matrix from as:

$$\begin{bmatrix} \Sigma_x \\ \Phi_y \\ \Phi_z \\ \Phi_\omega \end{bmatrix} = [\Sigma_{x1} \quad \Sigma_{x2} \quad \Phi_{y1} \quad \Phi_{y2} \quad \Phi_{z1} \quad \Phi_{z2} \quad \Phi\omega 1 \quad \Phi_{\omega 2}]^T \tag{4.40}$$

Equation 4.40 after some arrangements can be expressed as

$$\begin{bmatrix} \Sigma_{x1} \\ \Phi_{y1} \\ \Phi_{z1} \\ \Phi_{\omega 1} \\ \Sigma_{x2} \\ \Phi_{y2} \\ \Phi_{z2} \\ \Phi_{\omega 2} \end{bmatrix} = [\bar{X}]\{r\} \tag{4.41}$$

where

$\bar{X}$ is the element linear strain matrix, which is given in Table 4.10.

4.5.2 **Nonlinear Strain Matrix** Nonlinear strains occur due to coupling between the displacements (U, V, W, and ϕ). Those strains are, axial nonlinear strains ($\bar{\Sigma}$) and the nonlinear strain due to the curvatures (ϕ_y and ϕ_z) about the Y and Z axes. The total nonlinear strain can be expressed as:

$$\Sigma_x^n = \bar{\Sigma} + \phi_y^n Y + \phi_z^n Z \qquad (4.42)$$

where

$$\bar{\Sigma} = \frac{1}{2}[V_{,x}^2 + U_{,x}^2 + \rho\phi_{,x}^2] + Z_s\phi_{,x}[U_{,x} + V_{,x}\phi] - Y_s\phi_{,x}[V_{,x} - U_{,x}\phi] \qquad (4.43)$$

$$\phi_y^n = U_{,xx}\phi + \frac{V_{,xx}\phi^2}{2} \qquad (4.44)$$

$$\phi_z^n = -V_{,xx}\phi + \frac{U_{,xx}\phi^2}{2} \qquad (3.45)$$

Equations 4.43 to 4.45 can be written in matrix form as:

$$\begin{bmatrix} \bar{\Sigma} \\ \phi_y^n \\ \phi_z^n \\ \phi_\omega^n \end{bmatrix} = [\bar{\Sigma}_1 \quad \bar{\Sigma}_2 \phi_{y1}^n \quad \phi_{y2}^n \quad \phi_{z1}^n \quad \phi_{z2}^n \quad \phi_{\omega 1}^n \quad \phi_{\omega 2}^n]^T \qquad (4.46)$$

Substituting equation 4.3 into equation 4.46 by employing some mathematical calculations yields

$$\begin{bmatrix} \overline{\Sigma}_1 \\ \Phi^n_{y1} \\ \Phi^n_{z1} \\ \Phi^n_{\omega 1} \\ \overline{\Sigma}_2 \\ \Phi^n_{y2} \\ \Phi^n_{z1} \\ \Phi^n_{\omega 1} \end{bmatrix} = [X^n]\{r\} \qquad (4.47)$$

Where $\overline{\Sigma}_1$ and $\overline{\Sigma}_2$ are nonlinear axial curvatures at node 1 and 2. Additional axial strains are induced by the curvatures at the two nodes, which are Φ^n_{y1}, Φ^n_{y2}, Φ^n_{z1},etc. The terms of the matrix X^n of the nonlinear axial strains are presented in Table 4.11.

The total strain matrix $[\overline{B}]$ for a beam-column in three dimensions is the summation of the linear and nonlinear strain matrices, which have been given in equations 4.41 and 4.47 respectively. $B^=$ is the total axial strain. Both $\overline{B}$ and $B^=$ are illustrated as:

$$[B^=] = [\overline{B}]\{r\} \qquad (4.48)$$

where

$$\{B^{=}\} = \begin{bmatrix} \Sigma_{x1} + \overline{\Sigma}_1 \\ \Phi_{y1} + \Phi^n_{y1} \\ \Phi_{z1} + \Phi^n_{z1} \\ \Phi_{\omega 1} + \Phi^n_{\omega 1} \\ \Sigma_{x2} + \overline{\Sigma}_2 \\ \Phi_{y2} + \Phi^n_{y2} \\ \Phi_{z2} + \Phi^n_{z2} \\ \Phi_{\omega 2} + \Phi^n_{\omega 2} \end{bmatrix} \qquad (4.49)$$

and

$$[\overline{B}] = [\overline{X} = X^n]_{4\times 14} \qquad (4.50)$$

4.6 Conclusions

The element stiffness matrices of a beam-column of thin-walled open cross-section in three dimensions were derived. The interpolation functions employed were linear polynomials for the axial displacements, cubic polynomials for lateral displacements and twists, and quadratic polynomials for the rotations and warping. Linear and nonlinear geometrical and tangential stiffnesses have been obtained by two methods which are the principle of virtual work and total potential energy. In the latter method the initial deformation stiffness matrix was included. The terms of both linear and nonlinear stiffness matrices are calculated based on the average of the values of the two nodes i.e. $(M_{y1} + M_{y2})$ when using virtual work, while for the potential energy the evaluation was based on a linear function $(a + a_1\frac{X}{L})$ between the two nodes. The transformation matrix for the element displacements with respect to the local coordinate and the global coordinate systems was developed together with linear and nonlinear strain displacement matrices. The important concepts of these stiffnesses are:

1- The sections have arbitrary shapes such [, I, T, L, etc.

2- The effect of non-coincident shear centre and centroid are accounted for i.e. Wagner effect is incorporated.

3- Different load patterns are included.

4- The higher order terms are involved.

5- Strains vary linearly along the cross-section.

6- The section and sectorial properties and forces were evaluated for an arbitrary axis system.

Those matrices can deal with the analysis of a member in space under eccentric load, uniaxial and biaxial bending, and bending and torsion or any combination of these loads. These matrices can be arranged to perform several analyses such as Linear, partial Nonlinear, and Full nonlinear analysis. In "Linear" the basic stiffness matrices (tangent and geometrical which includes flexural rigidities, torsional moments and/ or shear forces) and second order strains are used. In "Partial non-linear" the the linear stiffnesses matrices and higher order terms of the strain-displacement relations are included. In "Full non-linear" incorporation of both the contributions of the higher order terms to the stiffness matrices and strain-displacement relations is included. Further details of these analyses can be found in the papers by El-Khenfas and Nethercot (1987b,c) and in Chapters 6 and 8.

Table 4.1 Linear Tangential Stiffness Matrix

O													
	A												
$-O\bar{Y}$	B	C											
	a_1	b_1	$\bar{A}$										
$-O\bar{Z}$	b_1	c_1	$\bar{B}$	$\bar{C}$									
	$\bar{a}$	$\bar{b}$	a	b	E								
$-X$	$\bar{b}$	$\bar{c}$	b	c	F	G							
$-O$		$O\bar{Y}$		$O\bar{Z}$		X	$-O$						
	$-A$	$-B$	$-a_1$	$-b_1$	$\bar{a}$	$-\bar{b}$		A					
$-O\bar{Y}$	B	D	$+b_1$	$c1/2$	$\bar{b}$	$\bar{c}1/2$	$-O\bar{Y}$	$-B$	$-2C$				
	$-a_1$	$-b_1$	$-\bar{A}$	$-\bar{B}$	$-a$	$-b$		a	b_1	$\bar{A}$			
$O\bar{Z}$	$-b_1$	$c1/2$	$\bar{B}$	$\bar{D}$	b	$c/2$	$-O\bar{Z}$	$-b_1$	c_1	$-\bar{B}$	$-2\bar{C}$		
	$-\bar{a}$	$-\bar{b}$	$-a$	$-b$	$-E$	$-F$		$\bar{a}$	$-\bar{b}$	a	$-b$	E	
X	$\bar{b}$	$\bar{c}/2$	b	$c/2$	F	T	$-x$	$-\bar{b}$	$\bar{c}$	b	c	$-F$	M

$X = \frac{ES_w}{\ell}$ $\quad B = \frac{6EI_z}{\ell^2}$ $\quad a_1 = \frac{12EI_{yz}}{\ell^3}$ $\quad E = \frac{12EI_w}{\ell} + \frac{6}{5\ell}GK$ $\quad a = \frac{12EI_{wz}}{\ell^3}$

$O = \frac{EA}{\ell}$ $\quad C = \frac{4EI_z}{\ell}$ $\quad b_1 = \frac{6EI_{yz}}{\ell}$ $\quad F = \frac{6EI_w}{\ell^2} + \frac{1}{10}GK$ $\quad b = \frac{6EI_{wz}}{\ell^2}$

$A = \frac{12EI_z}{\ell^3}$ $\quad D = \frac{2EI_z}{\ell}$ $\quad c_1 = \frac{4EI_{yz}}{\ell}$ $\quad T = \frac{2EI_w}{\ell} - \frac{\ell}{30}GK$ $\quad c = \frac{4EI_{wz}}{\ell^2}$

For $\bar{A}$, $\bar{B}$ etc. Change Y to Z and vise versa in the expressions A, B etc.

Table 4.2 Linear Geometrical Stiffness Matrix

	a												
QY/ℓ	b	c											
		$-T$	a										
$-QZ/\ell$	$-T$		b	c									
	A	C	$-\bar{A}$	$-\bar{C}$	a_1								
	B	D	$-\bar{B}$	$\bar{D}$	b_1	c_1							
		$-QY/\ell$		QZ/ℓ									
	$-a$	$-b$		$-T$	$-A$	$-B$		a					
$-QY/\ell$	b	d	$-T$	M	B	F	QY/ℓ	$-b$	c				
			$-a$	$-b$	$\bar{A}$	$\bar{B}$			$-T$	a			
QZ/ℓ	$-$	$-M$	b	d	$-\bar{B}$	$-\bar{F}$	QZ/ℓ	T		$-b$	c		
	$-A$	$-E$	$\bar{A}$	$\bar{E}$	$-a_1$	$-b_1$						a_1	$-b_1$
	E	F	$-\bar{E}$	$-\bar{F}$	b_1	d_1						$-b_1$	$-c_1$

$a = \frac{6}{5\ell} P$ $\quad a_1 = \frac{6}{5\ell} M_p$ $\quad T = \frac{M_x}{\ell}$ $\quad C = \frac{1}{10} M_y - \frac{1}{10} PZ_s - \frac{1}{10} Q_y$

$b = \frac{1}{10} P$ $\quad b_1 = \frac{1}{10} M_p$ $\quad M = \frac{M_x}{2}$ $\quad D = \frac{1}{10} M_y - \frac{1}{10} PZ_s + \frac{1}{10} \ell Q_y$

$c = \frac{2}{15} \ell P$ $\quad c_1 = \frac{2\ell M}{15} p$ $\quad A = -\frac{6}{5\ell} M_y - \frac{6}{5\ell} PZ_s - \frac{1Q}{2}_y$ $\quad E = -\frac{1}{10} M_y + \frac{1}{10} PZ_s + \frac{1}{10} \ell Q_y$

$d = -\frac{\ell}{30} P$ $\quad d_1 = -\frac{\ell}{30} M_p$ $\quad B = -\frac{2}{15} \ell M_y + \frac{2}{15} \ell Z_s$ $\quad F = -\frac{1}{30} M_y - \frac{\ell}{30} \ell Z_s - \frac{\ell^2}{60} Q_y$

For $\bar{A}$, $\bar{B}$ etc. Change Y to Z and vice versa in the expressions for A, B etc

Table 4.3 Non-Linear Stiffness Matrix

For definitions of terms see Appendix E

U												
V												
$\bar{U}$	a_1	c_1										
$\bar{V}$	b_1	d_1										
X_1	$\bar{a}$	$\bar{c}$	a	c								
Z_1	$\bar{b}$	$\bar{d}$	b	d								
	$-U$	$-V$	$-\bar{U}$	$-\bar{V}$	$-X_1$	$-Z_1$						
$\bar{W}$			N	O	J	K	$-\bar{W}$					
$\bar{Y}$			Q	R	L	M	$-\bar{Y}$					
$\bar{W}$	e_1	g_1			$\bar{J}$	$\bar{K}$	$-W$	i_1	k_1			
$\bar{Y}$	f_1	h_1			$\bar{I}$	$\bar{M}$	$-Y$	i_1	m_1			
X_2	$\bar{e}$	$\bar{g}$	e	g			$-X_2$	$\bar{i}$	$\bar{k}$	i	k	
Z_2	$\bar{f}$	$\bar{h}$	f	h			$-X_2$	$\bar{I}$	$\bar{m}$	l	m	

$a_1 = EI_z(RO11 + SO11) - EI_y(SO11 + RO11)$

$b_1 = EI_z(RO12 + SO12) - EI_y(SO12 + RO21)$

$c_1 = EI_z(RO21 + SO12) - EI_y(SO12 + RO12)$

$d_1 = EI_z(RO22 + SO22) - EI_y(SO22 + RO22)$

$N = EI_z(RO31 + SO13) - EI_y(SO13 + RO13)$

$O = EI_z(RO32 + SO23) - EI_y(SO23 + RO23)$

$Q = EI_z(RO41 + SO14) - EI_y(SO14 + RO14)$

$R = EI_z(RO42 + SO24) - EI_y(SO24 + RO24)$

$e_1 = EI_z(RO13 + SO13) - EI_y(SO13 + RO31)$

$f_1 = EI_z(RO14 + SO14) - EI_y(SO14 + RO41)$

$g_1 = EI_z(RO23 + SO23) - EI_y(SO23 + RO32)$

$h_1 = EI_z(RO24 + SO24) - EI_y(SO24 + RO42)$

$i_1 = EI_z(RO33 + SO33) - EI_y(SO33 + RO33)$

$j_1 = EI_z(RO34 + SO34) - EI_y(SO34 + RO43)$

$k_1 = EI_z(RO43 + SO34) - EI_y(SO34 + RO34)$

$m_1 = EI_z(RO44 + SO44) - EI_y(SO44 + RO44)$

$\bar{a} = EI_y(ZV11 + YV11) + Y_s EI_z RO11$

$\bar{b} = EI_y(ZV12 + YV12) + Y_s EI_z RO12$

$\bar{c} = EI_y(ZV21 + YV21) + Y_s EI_z RO21$

$\bar{d} = EI_y(ZV22 + YV22) + Y_s EI_z RO22$

$\bar{e} = EI_y(ZV13 + YV13) + Y_s EI_z RO13$

$\bar{f} = EI_y(ZV14 + YV14) + Y_s EI_z RO14$

$\bar{g} = EI_y(ZV23 + YV23) + Y_s EI_z RO23$

$\bar{h} = EI_y(ZV24 + YV24) + Y_s EI_z RO24$

$\bar{i} = EI_y(ZV33 + YV33) + Y_s EI_z RO33$

$\bar{j} = EI_y(ZV34 + YV34) + Y_s EI_z RO34$

$\bar{k} = EI_y(ZV43 + YV43) + Y_s EI_z RO43$

$\bar{m} = EI_y(ZV44 + YV44) + Y_s EI_z RO44$

$\bar{J} = -EI_y(YV31 + ZV13) + z_s EI_z RO13$

$\bar{K} = -EI_y(YV41 + ZV14) + z_s EI_z RO14$

$\bar{M} = -EI_y(YV42 + ZV24) + z_s EI_z RO24$

For a, b, c,....etc. change z, YO11, YV11, Y_s RO11 etc. to Z, UO11, YU11, Z_s RO11 etc.

$U = -\frac{EA}{2l}AU1 \qquad V = -\frac{EA}{2l}AU2$

$W = -\frac{EA}{2l}AU3 \qquad Y = -\frac{EA}{2l}AU4$

$\bar{U} = -\frac{EA}{2l}AV1 \qquad \bar{V} = -\frac{EA}{2l}AV2$

$\bar{W} = -\frac{EA}{2l}AV3 \qquad \bar{Y} = -EA/2lAV4$

$X_1 = -\frac{EA}{2l}(Y_s^2 + Z_s^2)AO1 - \frac{1}{2l}(EI_y + EI_z)AO1$

$+ Y_s\frac{EA}{l}AV1 - Z_s\frac{EA}{l}AU1$

$X_2 = -\frac{EA}{2l}(Y_s^2 + Z_s^2)AO3 - \frac{1}{2l}(EI_y + EI_z)AO3$

$+ Y_s\frac{EA}{l}AV3 - Z_s\frac{EA}{l}AU3$

$Z_1 = -\frac{EA}{2l}(Y_s^2 + Z_s^2)AO2 - \frac{1}{2l}(EI_y + EI_z)AO2$

$+ Ynybn\frac{EA}{l}AV2 - Z_s\frac{EA}{l}AU2$

$Z_2 = -\frac{EA}{2l}(Y_s^2 + Z_s^2)AO4 - \frac{1}{2l}(EI_y + EI_z)AO4$

$+ Y_s\frac{EA}{l}AV4 - Z_s\frac{EA}{l}AU4$

Table 4.4 Non-Linear Geometrical Stiffness Matrix

U	A											
V	B	AA										
$\bar{U}$			− A									
$\bar{V}$			− B	− AA								
	a	c	$\bar{a}$	$\bar{c}$								
	b	d	$\bar{b}$	$\bar{d}$								
	− U	− V	− $\bar{U}$	− $\bar{V}$								
W	C	E			m	o	− W	G				
Y	D	F			n	p	− Y	H	I			
$\bar{W}$			− C	− E	m	O	− W			− G		
$\bar{Y}$			− D	− F	n	p	− Y			H	− I	
	e	g	$\bar{e}$	$\bar{g}$				i	k	$\bar{i}$	$\bar{k}$	
	f	h	$\bar{f}$	$\bar{h}$				j	l	$\bar{j}$	$\bar{l}$	

$U = \frac{Q_z}{1} WO1$ $\qquad V = \frac{Q_z}{1} WO2$

$W = \frac{Q_z}{1} WO3$ $\qquad Y = \frac{Q_z}{1} WO4$

For $\bar{U}$, $\bar{V}$, $\bar{W}$ and $\bar{Y}$ replace z by y

in the expression for U, V, W, and Y.

$A = \frac{M_x}{2}\ XO11$ $\quad B = \frac{M_x}{2}\ XO12$ $\quad C = \frac{M_x}{2}\ XO13$ $\quad AA = \frac{M_x}{2}\ XO22$

$D = \frac{M_x}{2}\ XO14$ $\quad E = \frac{M_x}{2}\ XO23$ $\quad F = \frac{M_x}{2}\ XO24$

$G = \frac{M_x}{2}\ XO33$ $\quad H = \frac{M_x}{2}\ XO34$ $\quad I = \frac{M_x}{2}\ XO44$

$a = -M_y[CX11 + BO11] + \frac{M_z}{2}XU11 + \frac{1\,Q_z}{2\,1}(v_2-v_1)$

$b = -M_y[CX21 + BO12] + \frac{M_z}{2}XU12 + .1Q_z(v_2-v_1)$

$c = -M_y[CX12 + BO12] + \frac{M_z}{2}XU12 - .1Q_z(v_2-v_1)$

$d = -M_y[CX22 + BO22] + \frac{M_z}{2}XU22$

$e = -M_y[CX31 + BO13] + \frac{M_z}{2}XU13 + \frac{1}{2}Q_z/1(v_2-v_1)$

$f = -M_y[CX41 + BO14] + \frac{M_z}{2}XU14 - .1Q_z(v_2-v_1)$

$g = -M_y[CX32 + BO23] + \frac{M_z}{2}XU23 + .1Q_z(v_2-v_1)$

$h = -M_y[CX42 + BO24] + \frac{M_z}{2}XU24 - \frac{1}{60.}Q_z(v_2-v_1)$

$i = -M_y[CX33 + BO33] + \frac{M_z}{2}XU33 - \frac{1\,Q_z}{2\,1}(v_2-v_1)$

$j = -M_y[CX43 + BO34] + \frac{M_z}{2}XU34 - .1Q_z(v_2-v_1)$

$k = -M_y[CX34 + BO34] + \frac{M_z}{2}XU34 - .1Q_z(v_2-v_1)$

$l = -M_y[CX44 + BO44] + \frac{M_z}{2}XU44$

$m = -M_y[CX13 + BO13] + \frac{M_z}{2}XU13 - \frac{1\,Q_z}{2\,1}(v_2-v_1)$

$n = -M_y[CX14 + BO14] + \frac{M_z}{2}XU14 - .1Q_z(v_2-v_1)$

$o = -M_y[CX23 + BO23] + \frac{M_z}{2}XU23 - .1Q_z(v_2-v_1)$

$p = -M_y[CX24 + BO12] + \frac{M_z}{2}XU12 + .1Q_z(v_2-v_1)$

For $\bar{a}$, $\bar{b}$ etc change y to z and vice versa in expression for a, b, etc.

For definitions of terms see Appendix F

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Table 4.5 Linear Tangential Stiffness Matrix

a_1													
B_1	A_z												
C_1	B_z	C_z											
$\bar{b}_1$	A_{yz}	B_{yz}	A_y										
$\bar{C}_1$	B_{yz}	C_{yz}	B_y	C_y									
$+e_1$	A_{wy}	B_{wy}	A_{wz}	B_{wz}	m								
f_1	B_{wy}	C_{wy}	B_{wz}	C_{wz}	n	O							
$-a_1$	$-b_1$	$-C_1$	$-\bar{b}_1$	$-c_1$	$-e_1$	f_1	a_1						
$-b_1$	$-A_z$	$-B_z$	$-A_{yz}$	$-B_{yz}$	$-A_{wy}$	$-B_{wy}$	b_1	A_z					
$-d_1$	F_z	D_z	F_{yz}	D_{yz}	$+F_{wy}$	$-D_{wy}$	$+d_1$	$-F_z$	E_z				
$-\bar{b}_1$	$-A_{yz}$	$-B_{yz}$	$-A_y$	$-B_y$	$-A_z$	$-B_{wz}$	$\bar{b}_1$	A_{yz}	$-F_{yz}$	A_y			
$-\bar{d}_1$	F_{yz}	D_{yz}	F_y	D_y	f_{wz}	$-D_{wz}$	$\bar{d}_1$	$-F_{zy}$	E_{yz}	$-F_y$	E_y		
$-e_1$	$-A_{wy}$	$-B_{wy}$	$-A_{wz}$	$-B_{wz}$	$-m$	$-n$	e_1	A_{wy}	$-F_{wy}$	A_{wz}	$-F_{wz}$	m	
g_1	F_{wy}	D_{wy}	F_{wz}	D_{wz}	Q	R	$-g_1$	$-F_{wy}$	E_{wy}	$-F_{wz}$	E_{wz}	$-Q$	N

$\bar{Y} = \bar{Y}_2 - \bar{Y}_1 \qquad A = EA_2 - EA_1 \qquad Y_s = Y_{s2} - Y_{s1}$

$A_z = EI_{z2} - EI_{z1} \quad A_\omega\ EI_{\omega 2} - EI_{\omega 1} \quad A_{GK} = GK_2 - GK_1$

$ES_\omega = E_{\omega 2} - ES_{\omega 1}$

$a_1 = \frac{EA_1}{l} + \frac{A}{2l} \qquad b_1 = \frac{A}{L^2}(\bar{Y} - Y_s)$

$c_1 = -\frac{EA_1}{l}(Y_{s1} - \bar{Y}_1) \qquad d_1 + = \frac{EA_1}{l}(Y_{s1} - \bar{Y}_1) - \frac{A}{l}(Y_s - \bar{Y})$

$e_1 = -\frac{ES_\omega}{L^2} \qquad f_1 = \frac{ES_\omega}{l} \qquad g_1 = \frac{ES_{\omega 1}}{l} - \frac{ES_\omega}{l}$

For the terms $\bar{a}_1$, $\bar{b}_1$, $\bar{c}_1$, and $\bar{d}_1$ change Y to Z or vice versa.

$A_z = 12\frac{EI_z}{L^3} + 6\frac{A_z}{l^3} \qquad B_z = 6\frac{EI_z}{L^2} + 2\frac{A_z}{l^2}$

$C_z = 4\frac{EI_z}{L} + \frac{A_z}{l} \qquad D_z = 2\frac{EI_z}{L} + \frac{A_z}{l}$

$E_z = 4\frac{EI_z}{L} + 3\frac{A_z}{l} \qquad F_z = 6\frac{EI_z}{L^2} + 4\frac{A_z}{l^2}$

Change the subscript of the following terms; $\bar{A}_z$, $\bar{B}_z$, $\bar{C}_z$, $\bar{D}_z$, E_z, F_z and EI_{z1}, A_z to y, yz, yw, and wz respectively in the form of $\bar{A}_y$, ...and EI_{y1}, and A_y

$m = 12\frac{EI_{\omega 1}}{L^3} + 6\frac{A_\omega}{l^3} + \frac{6}{51}GK_1 + .6\frac{GK}{l} \qquad n = 6\frac{EI_{\omega 1}}{L^2} + 2\frac{A_\omega}{l^2} + \frac{1}{10}GK_1 - \frac{1}{10}A_{GK}$

$o = 4\frac{EI_{\omega 1}}{l} + A_\omega + \frac{2}{15}lGK_1 + \frac{1}{30}A_{GK} \qquad Q = 6\frac{EI_{\omega 1}}{L^2} + 4\frac{A_\omega}{L^2} + \frac{1}{10}GK_1$

$R = 2\frac{EI_{\omega 1}}{l} + \frac{A_\omega}{l} - \frac{1}{30}GK_1 - \frac{1}{60}A_{GK} \qquad N = 4\frac{EI_{\omega 1}}{l} + 3\frac{A_\omega}{l} - \frac{4l}{30}GK_1 + \frac{1}{10}A_{GK}$

Table 4.6 **Linear Geometrical Stiffness Matrix**

D	a												
E	b	c											
F		A	a										
G	-A		b	c									
H	a_1	b_1	$-\bar{a}_1$	$-\bar{b}_1$	g								
I	c_1	d_1	$-\bar{c}_1$	$-\bar{d}_1$	h	i							
	-D	-E	-F	-G	-H	-I							
-D	-a	-b		A	t_1	l_1	D	a					
J	d	e	-B	-C	w_1	x_1	-J	-d	f				
-F		-A	-a	-b	$-\bar{t}_1$	$-\bar{l}_1$	F		B	a			
M	B	C	d	e	$-\bar{w}_1$	$-\bar{x}_1$	-M	-B		-d	f		
-H	e_1	j_1	$-\bar{e}_1$	$-\bar{j}_1$	-g	-h	H	r_1	m_1	$-\bar{r}_1$	$-\bar{m}_1$	g	
N	f_1	h_1	$-\bar{f}_1$	$-\bar{h}_1$	m	r	O	n_1	a_1	$-\bar{n}_1$	$-\bar{o}_1$	-m	O

$a = \frac{6}{5l}F_1 + \frac{.6}{l}\Delta F$ $\quad b = \frac{1}{10}F_1 + \frac{1}{10}\Delta F$

$c = \frac{2}{15}lF_1 + 1/30\,l\Delta F$ $\quad d = \frac{1}{10}F_1$

$e = -\frac{1}{30}lF_1 - \frac{1}{60}l\Delta F$ $\quad f = \frac{4}{30}lF_1 + \frac{1}{60}l\Delta F$

$g = \frac{6}{5l}M_{p1} + \frac{.6}{l}\Delta M_p$ $\quad h = \frac{1}{10}M_{p1} + \frac{1}{10}\Delta M_p$

$i = \frac{2}{15}lM_{p1} + \frac{1}{30}l\Delta M_p$ $\quad o = \frac{2}{15}lM_{p1} + \frac{1}{10}l\Delta M_p$

$m = \frac{1}{10}M$ $\quad n = -\frac{1}{10}M_{p1} - \frac{1}{10}\Delta M_p$ $\quad r = -\frac{1}{30}lM_{p1} - \frac{1}{60}l\Delta M_p$

$A = \frac{M_{z1}}{l} - \frac{1\Delta M_z}{4l}$ $\quad B = \frac{M_{z1}}{l} - \frac{3\Delta M_z}{4l}$

$\Delta F = F^2 - F^1$ $\quad \Delta M_z = M_{z2} - M_{z1}$ $\quad \Delta M_y = M_{y2} - M_{y1}$

$\Delta Q_y = Q_{y2} - Q_{y1}$ $\quad \Delta Q_z = Q_{z2} - Q_{z1}$ $\quad \Delta Y_s = Y_{s2} - Y_{s1}$

$\Delta M_z = M_{z2} - M_{z1}$ $\quad M_\omega = M_{\omega 2} - M_{\omega 1}$ $\quad \Delta M_p = M_{p2} - M_{p1}$

$\Delta Z_s = Z_{s2} + Z_{s1}$ $\quad M_p = \int_A \sigma\sigma^2 dA$

$I = \frac{M_{\omega 1}}{l}$ $\quad J = \frac{\Delta Q_z}{24} - \frac{M_{z1}}{l} + \frac{\Delta M_z}{l}$ $\quad C = \frac{1}{2}M_{z1} - \frac{31}{210}\Delta M_z$ $\quad D = \frac{Q_{z1}}{21} + \frac{\Delta Q_z}{41} + \frac{\Delta M_z}{l^2}$

$M = \frac{\Delta Q_y}{24} - \frac{M_{y1}}{l} + \frac{\Delta M_y}{l}$ $\quad N = \frac{M_{\omega 1}}{l} + \frac{\Delta M_\omega}{l}$ $\quad E = \frac{\Delta Q_z}{24} + \frac{M_{y1}}{l}$ $\quad H = \frac{\Delta M_\omega}{l^2}$

$a_1 = \frac{6}{5l}M_{y1} + \frac{.6}{l}\Delta M_y - .5lQ_{z1} - \frac{13}{70}l\Delta Q_{z1} + \frac{6}{5l}Z_{s1}F_1 + \frac{.6}{l}\Delta F\Delta Z_s$

$b_1 = \frac{1}{10}M_{y1} + \frac{\Delta M_y}{10} - \frac{1}{10}lQ_{z1} + \frac{3}{70}lQ_z + \frac{1}{10}Z_sF_1 + \frac{1}{10}\Delta F\Delta Z_s$

$c_1 = \frac{1}{10}M_{y1} + \frac{\Delta M_y}{10} - \frac{1}{10}lQ_{z1} + \frac{103}{210}lQ_z + \frac{1}{10}Z_sF_1 + \frac{1}{10}\Delta F\Delta Z_s$

$d_1 = \frac{2}{15}lM_{y1} + \frac{1}{30}l\Delta M_y + .5l^2Q_{z1} + \frac{17}{105}l^2\Delta Q_z + \frac{2}{15}lZ_{s1}F_1 + \frac{1}{30}l\Delta F\Delta Z_s$

$e_1 = -\frac{6}{5l}M_{y1} - \frac{.6}{l}\Delta M_y + \frac{1}{10}lQ_{z1} + \frac{1}{10}\Delta Q_{z1} - \frac{6}{5l}Z_{s1}F_1 - \frac{.6}{l}\Delta F\Delta Z_s$

$f_1 = \frac{1}{10}M_{y1} - \frac{1}{70}lQ_{z1} - \frac{1}{420}l\Delta Q_z + \frac{1}{10}F_1Z_{s1}$

$h_1 = -\frac{1}{10}M_{y1} - \frac{\Delta M_y}{10} - \frac{1}{10}lQ_{z1} + \frac{3}{70}lQ_z - \frac{1}{10}Z_sF_1 - \frac{1}{10}\Delta F\Delta Z_s$

$r_1 = \frac{6}{5l}M_{y1} + \frac{.6}{l}\Delta M_y - .5lQ_{z1} - \frac{26}{70}l\Delta Q_{z1} + \frac{6}{5l}Z_{s1}F_1 + \frac{.6}{l}\Delta F\Delta Z_s$

$n_1 = -\frac{1}{10}M_{y1} + \frac{1}{10}lQ_{z1} + \frac{23}{210}l\Delta Q_z - \frac{1}{10}F_1Z_{s1}$

$m_1 = -\frac{1}{10}M_{y1} + \frac{1}{10}lQ_{z1} + \frac{2}{35}l\Delta Q_z - \frac{1}{10}F_1Z_{s1}$

$o_1 = \frac{2}{15}lM_{y1} - \frac{l^2}{210}\Delta Q_z + \frac{2}{15}lF_1Z_{s1}$

$t_1 = -\frac{6}{5l}M_{y1} - \frac{.6}{l}\Delta M_y - .5lQ_{z1} + \frac{1}{2}l\Delta Q_z - \frac{6}{5l}Z_{s1}F_1 + \frac{.6}{l}\Delta F\Delta Z_s$

$w_1 = -\frac{1}{10}M_{y1} + \frac{1}{10}lQ_{z1} + \frac{2}{15}l\Delta Q_z - \frac{1}{10}Z_sF_1$

$x_1 = -\frac{l}{30}M_{y1} - \frac{1}{60}l^2Q_{z1} + \frac{1}{84}l^2\Delta Q_z - \frac{1}{30}F_1Z_{s1}$

For $\bar{a}_1, \bar{b}_1, \ldots, \bar{x}_1$ instead of $a_1, b_1, \ldots, x_1$ replace Y by Z and vice versa.

Table 4.7 Nonlinear Tangential Stiffness Matrix

Details to these terms are given in appendices G, I and J respectively

a	a_1												
b	a_2	a_3											
c	d_1	d_3	b_1										
d	d_2	d_4	b_2	b_3									
e	e_1	e_2	f_1	f_2	c_1								
f	e_3	e_4	f_3	f_4	c_2	c_3							
	$-a$	$-b$	$-c$	$-d$	$-e$	$-f$							
J	a_4	a_6	d_5	d_7	e_5	e_6	$-J$	a_8					
g	a_5	a_7	d_6	d_8	e_7	e_8	$-g$	a_9	a_{10}				
h	d_5	d_7	b_4	b_6	f_5	f_6	$-h$	d_9	d_{11}	b_8			
1°	d_6	d_8	b_5	b_7	f_7	f_8	$-1°$	d_{10}	d_{12}	b_9	b_{10}		
0	e_5	e_6	f_5	f_6	c_4	c_6	-0	e_0	e_{11}	f_9	f_{11}	c_8	
N	e_7	e_8	f_7	f_8	c_5	c_7	$-N$	e_{10}	e_{12}	f_{10}	f_{12}	c_9	c_{10}

Table 4.8 Nonlinear Geometrical Matrices

E_1	A_1												
E_2	A_2	A_3											
E_3			$-A_1$										
E_4			$-A_2$	$-A_3$									
E_5	B_1	B_3	C_1	C_3	D_1								
E_6	B_2	B_4	C_2	C_4	D_2	D_3							
	$-E_1$	$-E_2$	$-E_3$	$-E_4$	$-E_5$	$-E_6$							
E_7	A_4	A_6			B_{13}	B_{14}	$-E_7$	A_8					
E_8	A_5	A_7			B_{15}	B_{16}	$-E_8$	A_9	A_{10}				
E_9			$-A_4$	$-A_6$	C_{13}	C_{14}	$-E_9$			$-A_8$			
E_{10}			$-A_5$	$-A_7$	C_{15}	C_{16}	$-E_{10}$			$-A_9$	$-A_{10}$		
E_{11}	B_5	B_7	C_5	C_7	D_4	D_5	$-E_{11}$	B_9	B_{11}	C_9	C_{11}	D_8	
E_{12}	B_6	B_8	C_6	C_8	D_6	D_7	$-E_{12}$	B_{10}	B_{12}	C_{10}	C_{12}	D_9	D_{10}

Full definition of terms see appendices G, I and J respectively

Table 4.9 Initial Stiffness Matrix

	A_1	A_2	$\bar{A}_1$	$\bar{A}_2$									
	A_3	A_4	$\bar{A}_3$	$\bar{A}_4$									
					A_9	A_{11}							
					A_{10}	A_{12}							
					$\bar{A}_9$	$\bar{A}_{11}$							
					$\bar{A}_{10}$	$\bar{A}_{12}$							
	A_5	A_6	$-\bar{A}$	$-\bar{A}_2$				A_7	A_8	$\bar{A}_{13}$	$\bar{A}_{14}$		
	A_7	A_8	$\bar{A}_5$	$\bar{A}6$				A_9	A_{10}	$\bar{A}_{15}$	$\bar{A}_{16}$		

$A_1 = [\frac{6}{51}(M_{y1} + P_1Z_s) + \frac{.6}{1}(\Delta M_y + \Delta P\Delta Z_s)]\Phi_{o1}$

$A_2 = [\frac{1}{10}(M_{y1} + P_1Z_s) + \frac{1}{10}(\Delta M_y + \Delta P\Delta Z_s)]\chi_{o1}$

$A_3 = [\frac{1}{10}(M_{y1} + P_1Z_s) + \frac{1}{10}(\Delta M_y + \Delta P\Delta Z_s)]\Phi_{o1}$

$A_4 = [\frac{21}{15}(M_{y1} + P_1Z_s) + \frac{1}{30}(\Delta M_y + \Delta P\Delta Z_s)]\chi_{o1}$

$A_5 = [\frac{1}{10}(M_{y1} + P_1Z_s0]\Phi_{o1}$

$A_6 = [-\frac{1}{30}(M_{y1} + P_1Z_s) - \frac{1}{30}(\Delta M_y + \Delta P\Delta Z_s)]\Phi_{o1}$

$A_7 = [\frac{6}{51}(M_{y1} + P_1Z_s) + \frac{.6}{1}(\Delta M_y + \Delta P\Delta Z_s)]\Phi_{o2}$

$A_8 = [\frac{1}{10}(M_{y1} + P_1Z_s0]\chi_{o2}$

$A_9 = -[\frac{1}{10}(M_{y1} + P_1Z_s0]\Phi_{o2}$

$A_{10} = [\frac{21}{15}(M_{y1} + P_1Z_s) + \frac{1}{30}(\Delta M_y + \Delta P\Delta Z_s)]\chi_{o1}$

$A_{11} = [\frac{1}{10}(M_{y1} + P_1Z_s) - \frac{1}{10}(\Delta M_y + \Delta P\Delta Z_s)]\Phi_{o1}$

$A_{12} = [-\frac{1}{30}(M_{y1} + P_1Z_s) - \frac{1}{60}(\Delta M_y + \Delta P\Delta Z_s)]\Phi_{o1}$

For $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_3$, etc. instead of A_1, A_2, vice versa, etc. replace y by z and

Table 4.10 Linear Strain Matrix

$-\frac{1}{\ell}$							$\frac{1}{\ell}$						
	$-\frac{6}{\ell^2}$	$-\frac{4}{\ell}$						$\frac{6}{\ell^2}$	$-\frac{2}{\ell}$				
			$-\frac{6}{\ell^2}$	$-\frac{4}{\ell}$						$\frac{6}{\ell^2}$	$-\frac{2}{\ell}$		
					$-\frac{6}{\ell^2}$	$-\frac{4}{\ell}$						$\frac{6}{\ell^2}$	$-\frac{2}{\ell}$
$-\frac{1}{\ell}$							$\frac{1}{\ell}$						
	$\frac{6}{\ell^2}$	$\frac{4}{\ell}$						$-\frac{6}{\ell^2}$	$-\frac{4}{\ell}$				
			$\frac{6}{\ell^2}$	$\frac{4}{\ell}$						$-\frac{6}{\ell^2}$	$\frac{4}{\ell}$		
					$\frac{6}{\ell^2}$	$\frac{4}{\ell}$						$-\frac{6}{\ell^2}$	$\frac{4}{\ell}$

Linear strain can be written as

$$\varepsilon^{\ell} = \varepsilon^{\ell}_{xxo} - Y\,\Phi_z - Z\,\Phi_z - \bar{w}\,\Phi_w$$

where

$\varepsilon^{\ell}_{xxo} = W_{,x}$, $\Phi_y = V_{,xx}$

$\Phi_z = U_{,xx}$, and $\Phi_{\bar{w}} = \Phi_{,xx}$

Table 4.11 Non-Linear Strain Matrix

		$\theta_{u1}/2$		$\theta_{v1}/2$									
			a_{vo}		$-\bar{A}/2$								
	a_{uo}				$-A/2$								
									$\theta_{u2}/2$		$\theta_{v2}/2$		
					$-\bar{C}/2$					a_{v1}		$-\bar{B}/2$	
					$-C/2$			$-a_{u1}$				$-B/2$	

$$A = -\frac{6}{l^2}\Phi_1 V_1 - \frac{4}{l}\Phi_1\theta_{v1} + \frac{6}{l^2}\Phi_1 V_2 - \frac{2}{l}\Phi_1\theta_{v2}$$

$$B = \frac{6}{l^2}\Phi_2 V_1 + \frac{2}{l}\Phi_2\theta_{v1} - \frac{6}{l^2}\Phi_2 V_2 + \frac{4}{l}\Phi_2\theta_{v2}$$

$$\bar{A} = \frac{6}{l^2}\Phi_1 V_1 - \frac{4}{l}\Phi_1\theta_{v1} + \frac{6}{l^2}\Phi_1 V_2 - \frac{2}{l}\Phi_1\theta_{v2}$$

$$C = \frac{4}{l^2}\Phi_1 V_1 + \frac{48}{l^2}\Phi_1 V_2$$

$$\bar{B} = \frac{6}{l^2}\Phi_2 U_1 + \frac{2}{l}\Phi_2\theta_{u1} - \frac{6}{l^2}\Phi_2 U_2 + \frac{4}{l}\Phi_2\theta_{u2}$$

$$\bar{C} = -\frac{4}{l^2}\Phi_1 U_1 + \frac{48}{l^2}\Phi_1 U_2$$

$$a_{vo} = -\frac{6}{l^2}V_1 - \frac{4}{l}\theta_{v1} + \frac{6}{l^2}V_2 - \frac{2}{l}\theta_{v2}$$

$$a_{v1} = \frac{6}{l^2}V_1 + \frac{2}{l}\theta_{v1} - \frac{6}{l^2}V_2 + \frac{4}{l}\theta_{v2}$$

$$a_{uo} = -\frac{6}{l^2}U_1 - \frac{4}{l}\theta_{u1} + \frac{6}{l^2}U_2 - \frac{2}{l}\theta_{u2}$$

$$a_{u1} = \frac{6}{l^2}U_1 + \frac{2}{l}\theta_{u1} - \frac{6}{l^2}U_2 + \frac{4}{l}\theta_{u2}$$

Nonlinear strain

$$\varepsilon^{NL} = \frac{1}{2}\left(U^2_{,x} + V^2_{,x}\right) - Y_s\Phi_{,x}V_{,x} + Z_s\Phi_{,x}U_{,x} + V_{,xx}\Phi + U_{,xx}\frac{\Phi^2}{2} - U_{,xx}\Phi - V_{,xx}\frac{\Phi^2}{2}$$

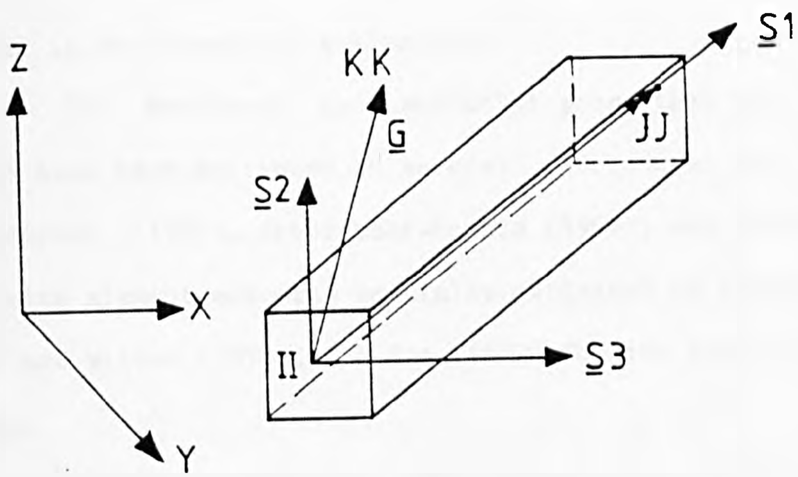


FIG. 4.1 TRANSFORMATION OF BEAM-ELEMENT

Chapter 5

Analytical Procedure and Computer Program Structure

5.1 Introduction

Using finite element concepts a comprehensive analytical procedure and computer implementation of the general formulation for a beam-column of thin-walled open cross-section in space is presented. The program TDA (Three Dimensional Analysis) can be used in both the elastic and inelastic resulting analytical processes are systematically arranged in for numerical evaluation.

The sectional and sectorial properties for I-section beam-columns have been mentioned in several references such as Chan and Kitipornchai (1987), Zhibrroski-Koscia (1967), and Zienkiewicz (1977). The finite element concepts are fully explained by Zienkiewicz (1977), Bathe and Wilson (1976), and Rao (1982) for the analysis of structural members.

A full coverage of the analytical procedure and general descriptions for the program TDCP is presented herein. Details of the results obtained by applying TDCP to selected problems covering different aspects of elastic and inelastic behaviour are contained in chapters 6, 7, and 8 and in references (El-Khenfas 1987c,b)

5.2 Finite Element Method

The first step in the finite element analysis is equivalent to replacing the domain having an infinite number of degrees of freedom by a system having a finite number of degrees of freedom.

5.2.1 Number of Elements (NEL)

If the beam has no change in geometry, material properties and external conditions (like applied loads), then it can be divided into equal lengths as shown in Fig. 5.1a. If discontinuities are present Figs (5.1b and 5.1c), nodes have to be introduced at these discontinuities. The number of elements to be chosen for idealization is related to the accuracy desired with an increase in the number of elements generally providing more accurate results, but at the cost of greater computation.

5.2.2 Number of Segments over the Cross-Section (NELS)

In order to evaluate the cross-sectional properties appearing in the various element stiffness matrices (El-Khenfas and Nethercot 1987a), it is convenient to divide each plate element of the cross-section into a number of small segments. The size of these segments must be chosen with care for both the elastic and inelastic analysis. The number required for reasonable accuracy depends on the pattern of applied loads (which affects the type of response) and the residual stress patterns (which affects the spread of yield). Any strain distribution over the cross-section leading to a mix of elastic and yielded regions may be conveniently allowed for by considering the member's cross-section to be divided as shown in Fig. 5.2a. The flanges and the web are divided into NY segments along each plate length and NZ through its thickness. $\Delta A_{i,j}$ is the segment area, where i=1,NY and j=1,NZ and its centroidal coordinate are $Y_{i,j}$ and $Z_{i,j}$ as given in Fig. 5.2b.

For elastic analysis the number of segments over the cross-section can simply reflect the natural division i.e. web, half flange etc. For the elastic-plastic analysis, however, the sub-division must be selected with care, with the number of segments depending on the loads and initial geometrical imperfections, e.g. if the load applied is uniaxial and residual stresses are absent then the flange thickness may be divided into a small number of segments (NZ equal to 10 and NY taken between 20 and 30). Other load patterns which cause the spread of yield to be unsymmetrical as illustrated in Fig. 5.2b, require a much finer subdivision e.g. NY and NZ taken as 80 and 10.

5.2.3 **Section Properties**

The accurate determination of section properties is one of the most important aspects of the whole analytical process. It depends principally on locating the instantaneous positions of the centroid and the shear centre for the partially yielded cross-section.

1- Centroid

Fig 5.3 shows the cross-section of a thin walled steel beam. Every element is divided into a number of plate segments. The overall cross-section centroid coordinates Y_c and Z_c with respect to reference axes Y and Z can be evaluated as:

$$Y_c = \frac{\sum_{n=1}^{NELS} \sum_{i,j=1}^{i=NY,J=NZ} Y_{i,j} \Delta A_{i,j}}{\sum_{n=1}^{NELS} \sum_{i,j=1}^{i=NY,J=NZ} \Delta A_{i,j}}$$

$$Z_c = \frac{\sum_{n=1}^{NELS} \sum_{i,j=1}^{i=NY,J=NZ} Z_{i,j} \Delta A_{i,j}}{\sum_{n=1}^{NELS} \sum_{i,j=1}^{i=NY,J=NZ} \Delta A_{i,j}}$$

The moments of inertia about the Y and Z axes and the product moment of inertia about these axes can be expressed as

$$I_Y = \sum_{n=1}^{NELS} \sum_{i,j=1}^{i=NY,j=NZ} Z^2_{i,j} \Delta A_{i,j}$$

$$I_Z = \sum_{n=1}^{NELS} \sum_{i,j=1}^{i=NY,j=NZ} Y^2_{i,j} \Delta A_{i,j}$$

$$I_{YZ} = \sum_{n=1}^{NELS} \sum_{i,j=1}^{i=NY,j=NZ} Z_{i,j} Y_{i,j} \Delta A_{i,j}$$

The above properties with respect to the y and z axes can be rewritten as

$$I_y = I_Y + \sum_{n=1}^{NELS} \sum_{i,j=1}^{i=NY,j=NZ} (Z_{i,j} - Z_c)^2 \Delta A_{i,j}$$

$$I_z = I_Z + \sum_{n=1}^{NELS} \sum_{i,j=1}^{i=NY,j=NZ} (Y_{i,j} - Y_c)^2 \Delta A_{i,j}$$

$$I_{yz} = I_{YZ} + \sum_{n=1}^{NELS} \sum_{i,j=1}^{i=NY,j=NZ} (Z_{i,j} - Z_c)(Y_{i,j} - Y_c) \Delta A_{i,j}$$

$$TAN\beta = \frac{I_{yz}}{I_z - I_y}$$

where β is the inclination of the principal axes (η and ζ) to the y and z axes and is positive if anticlockwise from the y-axis). The principal moments of inertia about η and ζ can be written as:

$$I_\zeta = I_y COS^2\beta + I_z SIN^2\beta - I_{yz} SIN 2\beta$$

$$I_\eta = I_z COS^2\beta + I_y SIN^2\beta - I_{yz} SIN 2\beta$$

ii- Shear Centre

Sectorial coordinates (ω) of points s_1 and s_2 can be written as $\omega_{Ds_1s_2}$, where D denotes the ' pole ', s_1 and s_2 are points of intersection of the radius, ds_{12} is the distance between the two points, and $h_{s_1s_2}$ is the distance of pole D from a line passing through points s_1 and s_2. To determine the sectorial coordinates of a beam having an arbitrary thin-walled open cross-section as illustrated in Fig. 5.4, by definition $\omega_{Ds_1s_2}$ is equal to the algebraic sum of twice $\delta\omega Ds_1s_2$ The increment of this value is equal to the product of

$$\delta\omega_{Ds_1s_2} = \delta ds_{12} x h_{s_1s_2}$$

if $\delta\omega_{Ds_1s_2}$ is equal to $\delta\omega$, and $\delta ds_{12}$is equal to ds. Substituting for these in the above equation and integrating from 0 to s_n

$$\omega = \int_{o}^{s_n} h_{s_1s_2} ds$$

where s_n is the length of the path between s_1 and s_2 for any segment in the cross-section

The sectorial properties with respect to an arbitrary pole D are (S_ω) sectorial moment of a section from pole D, product warping moment of inertia about Y and Z axes ($I_{\omega y}$) and ($I_{\omega z}$), and warping moment of inertia (I_ω). They can be obtained from

$$S_\omega = \sum_{n=1}^{NELS} \sum_{i,j=1}^{i=NY,j=NZ)} \omega_{i,j} \Delta A_{i,j}$$

$$I_{\omega y} = \sum_{n=1}^{NELS} \sum_{i,j=1}^{i=NY,j=NZ)} Z_{i,j} \omega_{i,j} \Delta A_{i,j}$$

$$I_{\omega z} = \sum_{n=1}^{NELS} \sum_{i,j=1}^{i=NY,j=NZ)} Y_{i,j} \omega_{i,j} \Delta A_{i,j}$$

$$I_{\omega z} = \sum_{n=1}^{NELS} \sum_{i,j=1}^{i=NY,j=NZ)} \omega^2_{i,j} \Delta A_{i,j}$$

Those properties can used to determine the location of shear centre. Further details are given in several references such Chen and Atsuta (1977) and Zhibrorski-Koscia (1967)

$$Y_s = \frac{I_{zz} I_{y\omega} - I_{yz} I_{\omega z}}{I_{yy} I_{zz} - I_{yz}^2}$$

$$Z_s = \frac{I_{yy} I_{z\omega} - I_{yz} I_{\omega y}}{I_{yy} I_{zz} - I_{yz}^2}$$

Once the shear centre coordinates (Y_s and Z_s) have been determined, then it may be used as a ' pole' to recalculate the previous sectorial properties which can be written in the following form:

$$\omega_{i,j} = \sum_{i,j=1}^{i=NZ,j=NY} \rho_{i,j} B_{i,j}$$

where $\rho_{i,j}$ is the distance between the centroid of the segment and the shear centre and $B_{i,j}$ is the segment width.

$$(\omega_o)_{i,j} = \sum_{i,j=1}^{i=NZ,j=NY} \rho_{oi,j} B_{i,j}$$

$$(\omega_n)_{i,j} = \frac{1}{A}\sum_{i,j=1}^{i=NZ,j=NY} [(\omega)_{i,j} t_{i,j} B_{i,j}]$$

$$I_{\omega y} = \sum_{n=1}^{n=NELS} \sum_{i,j=1}^{j=NY,i=NZ} \omega_{i,j} Y_{i,j} t_{i,j} B_{i,j}$$

$$I_{\omega z} = \sum_{n=1}^{n=NELS} \sum_{i,j=1}^{j=NY,iNZ} \omega_{i,j} Z_{i,j} t_{i,j} B_{i,j}$$

$$I_{\omega} = \sum_{n=1}^{n=NELS} \sum_{i,j=1}^{j=NY,iNZ} \omega_{i,j}^2 t_{i,j} B_{i,j}$$

In which A is the cross-section area and $t_{i,j}$ is the segment thickness. $(\omega_n)_{i,j}$ is the segment normalized function, $I_\omega y$,$I_\omega z$ are the product warping moment of inertia with respect to y and z axes. The warping moment of inertia is denoted by I_ω. whilst $(\rho_o)_{i,j}$ is the distance between the centroid of the segment and sectional centroid.

5.2.4 Internal Forces

At any stage in the analysis it is assumed that the full cross-sectional strain distributions will be known (from the deformation). Increments of the internal forces corresponding to changes of the strain distribution due to a change in the applied load parameter may then be determined from the stresses calculated in each plate segment at its centroid. The stress resultants may be evaluated numerically as:

$$P = \sum_{n=1}^{NELS} \sum_{i,j=1}^{i=NY,j=NZ} \sigma_{i,j} \Delta A_{i,j}$$

$$M_y = \sum_{n=1}^{NELS} \sum_{i,j=1}^{i=NY,j=NZ} \sigma_{i,j} Z_{i,j} \Delta A_{i,j}$$

$$M_z = \sum_{n=1}^{NELS} \sum_{i,j=1}^{i=NY,j=NZ} \sigma_{i,j} Y_{i,j} \Delta A_{i,j}$$

$$M_\omega = \sum_{n=1}^{NELS} \sum_{i,j=1}^{i=NY,j=NZ} \sigma_{i,j} \omega_{i,j} \Delta A_{i,j}$$

$$M_\rho = \sum_{n=1}^{NELS} \sum_{i,j=1}^{i=NY,j=NZ} \sigma_{i,j} \rho_{i,j}^2 \Delta A_{i,j}$$

where $\Delta A_{i,j} = B_{i,j} t_{i,j}$

5.3 Assembly of the Stiffness Matrices

The element stiffness matrices of equation 4.17 were evaluated with respect to nodal displacements in the local coordinates. For an illustration equation 4.14 can be written as:

$$\{r\}_L^T[K_{T2}^E]\{r\}_L + \{r\}_L\{P^t\} = 0 \tag{5.1}$$

where

$$\{\bar{P}_E\} = \{\bar{P} + r_o\bar{P}\}$$

The transformation of the element stiffness matrices from local principal generalized coordinates to the global coordinates can take place as

$$\{r_E\}_L^T[K_T^E]\{r_E\}_L + \{r_E\}_L\{P^E\} = 0 \tag{5.2}$$

where

$$\{r_E\}_L = [T]\{r_L\}_G \tag{5.3}$$

where the transformation matrix [T] is given and explained in chapter 4 equation 4.29. $\{r\}_L$ and $\{r\}_G$ are the element displacements with respect to local and global respectively. P_E is the element applied loads and $[K_T^E]$ is the assembly to the element stiffness matrices. Equation 5.1 can be written in the local system as

$$\delta\Pi = \{r\}^T[K_T^E]\{r\}_L + \{\bar{P}_E\}^T\{r_E\}L \tag{5.4}$$

Substituting equation 5.3 into equation 5.4 yields

$$\Xi(x) = [T]^T[K_T^E][T]\{r\}_G + \{\bar{P}_E\}_G \tag{5.5}$$

or

$$[K_T^E]_G]\{r\}_G + \{\bar{P}_E\} = 0 \tag{5.6}$$

For equilibrium equation 5.6 equals zero as

$$\Xi(x) = \frac{\delta \Pi}{\delta \{r\}_G} = 0 \tag{5.7}$$

If the beam is divided into a number of elements then the assembly of these members can be presented in the following form

$$[K_T]\{R\} + \{P\} = 0 \tag{5.8}$$

where $\{R\}$, $\{P\}$, and $[K_T]$ are the assembly to the elements displacements, applied loads, and to the stiffness matrices (tangential, geometrical linear and nonlinear and initial displacement matrix respectively). Their assembly arrangement are given in equations 5.9 to 5.11.

$$[K_T] = \begin{bmatrix} [K_T^E] & [K_T^E] & \ddots & & \\ & & & \cdot & \\ & & & & [K_T^E] \end{bmatrix} \tag{5.9}$$

$$\{R\} = \begin{bmatrix} r_E \\ \vdots \\ r_E \end{bmatrix} \tag{5.10}$$

$$\{P\} = \begin{bmatrix} \bar{P}_E \\ \vdots \\ \bar{P}_E \end{bmatrix} \qquad 5.11)$$

equation 5.8 represents the finite element form of the total equilibrium equation.

5.4 Method of Solution

The complexity of the analysis is related to a number of factors. It depends upon the nature of the loading, the material properties, and the kinds of assumptions made in deriving the stiffness matrices (Chapter 4). A solution can be obtained by using the Newton-Raphson technique, which has been employed by many investigators. Fig. 5.5 illustrates this method.

The discretised system of algebraic equation takes the form

$$\Xi(R) = [K_T]\{R\} + \{P\} = 0 \tag{5.12}$$

If $R = R_n$ is a solution to the above equation, it can be written, by using Taylor's expression as

$$\Xi(R_{n+1}) = \Xi(R_n) + \left(\frac{d\Xi}{dR}\right)_n \Delta R_n = 0 \tag{5.13}$$

with

$$R_{n+1} = R_n + \Delta R^n \tag{5.14}$$

From equation No.5.12 we obtain

$$\frac{d\Xi}{dR} = \frac{dP}{dR} = K_T$$

The steps of the Newton-Raphson procedure employed are:

1- Calculate the out of balance forces

$$\{\Delta P\}_{n+1} = \{P\} - [K_T]_n \{R\}_n$$

2- Compute the increments in displacements to equilibrate the unbalanced forces.

$$\{\Delta R\}_n = [K_T]^{-1}\{\Delta P\}_n$$

3- Update the displacements

$$\{R\}_n = \{R\}_{n-1} + \{R\}_n$$

4- Repeat 1 to 3 until the imbalance of forces is sufficienttl small.

5- Apply another load increment and repeat steps 1 to 4.

5.5 Convergence Criteria

The numerical solution scheme outlined above is based on a step by step approach which involves the combination of an incremental and an iterative process until the desired convergence limit is achieved. Possible convergence criteria can be classified under:

1- Force criteria

2- Displacement criteria

3- Stress criteria

Bergan and Clough (1972) suggested three alternative norms for measuring the tolerance γ in order to achieve convergence which were

1- " Modified Absolute Norm :;

$$|\Sigma| = \left[\left[\frac{1}{N} \sum \frac{\Delta R_i}{R_{i,ref}} \right] \right]$$

2- " Modified Euclidean Norm ";

$$|\Sigma| = \left[\frac{1}{N} \sum \frac{\Delta R_i}{R_{i,ref}} \right]^{\frac{1}{2}}$$

3- " Maximum Norm "

$$|\Sigma| = \max_i \frac{\Delta R_i}{R_{i,ref}}$$

They employed the following criterion to define those norms as

$$|\Sigma| < \gamma$$

where γ ranges between 10^{-2} to 10^{-6} depending on the desired accuracy, N is the total number of unknown components, and ΔR_i is the change in displacement component i. Every component is scaled by a reference displacement quantity $R_{i,ref}$.

Crisfield (1981) modified Riks's approach to be suitable for the finite element method and to produce a fast incremental solution. The procedure was applied in conjunction with the modified Newton-Raphson technique. He reported that his approach not only allowed limit points to be passed but also improved the convergence characteristics. He provided illustrative examples which covered large deflection analysis of shallow elastic shells and the collapse analysis of a stiffened steel diaphragm from a box-girder. His criterion may be stated as:

$$\frac{|\bar{g}|}{\max.(\lambda|P|, |\bar{r}|)} < 10^{-4}$$

$$|\bar{g}| = \sqrt{g^T K_D^{-1} g}$$

Where $|\bar{g}|$ is proposed by Peano and Riccioni (1978). $\bar{g}$ is the out-of-balance force, P the total applied force, $\bar{r}$ the reaction vector and K_D is a diagonal matrix containing the leading diagonal elements of the tangent stiffness matrix.

To ensure convergence to the correct result certain simple requirements have to be satisfied, the most obvious of which is clearly that the displacement function should be able to represent the displacement distribution as closely as possible. In general the larger and more complex the numerical problem the more difficult it will be to achieve satisfactory convergence.

In the present analysis two methods were applied together to meet the necessary convergence limit; these can be explained as:

i- Out of balance forces

ii- Iterative process

5.5.1 Out of Balance Forces

If the sum of the total forces at each node is not in equilibrium i.e. the summation of the internal forces does not balance the external forces, the difference between them is termed the out of balance force. These residual forces should be re-applied until both balance each other. To explain this phenomenon, the illustrative example of Fig. 5.6 shows the load position, beam segment length, and support conditions. The out-of-balance forces are

$$\Delta Q_{yi} = [\frac{M_i^L - M_i^R}{L_i} + \frac{M_{i+1}^L - M_{i+1}^R}{L_{i+1}}] - Q$$

$$\Delta M_{\omega i} = (M_{\omega i}^L - M_{\omega i}^R - M_\omega$$

where M_{yi}^L / M_{yi}^R and $M_{\omega i}^R / M_{\omega i}^R$ are the bending moments about the Y-axis and bimoment at node i from left and right. Δ Q sub i and Δ $M_{\omega i}$ are out-of-balance forces for the shear and bimoment respectively, L_i is the element segment length, and the subscript i varies from 1 to NJ (NJ= number of joints). If the residual forces $\{\Delta P_i\}$ obtained from the equilibrium of node i do not have small values. $\{\Delta P_i\}$ should be re-applied as an increment of force and this process repeated until a balance is achieved. A full description is given in Fig. 5.6.

where

$$\{\Delta P_i\} = \lfloor \Delta P_i \quad \Delta M_{yi} \quad \Delta Q_{zi} \quad \Delta M_{zi} \quad \Delta Q_i \quad \Delta M_{ti} \quad \Delta M_\omega i \rfloor$$

where $\{\Delta P_i\}$ is an array to the out-of-balance forces at node i.

5.5.2 Iterative Process

In this method both the unbalanced or residual forces within the structure nodes and the increment of displacements are implemented at the same time. The out of balance force and the variation of the deformations are described as:

1- Out of balance force

$$\Sigma = \left[\frac{[\Delta P_I]^T [K_D][\Delta P_I]}{[P_T]^T [K_D][P_T]} \right]^{1/2} < \gamma$$

ii- Increment of Displacements

$$\Sigma = \left[\frac{[\Delta R_I]^T [K_D][\Delta R_I]}{[R_T]^T [K_D][R_T]} \right]^{1/2} < \gamma$$

where ΔR_I are the incremental displacements at the iteration, R_t are the updated total displacements, ΔP_I is the out-of-balance force and P_T are the total applied forces. K_D is the diagonal stiffness of the whole stiffness matrix. Based on the accuracy desired the tolerance γ is chosen in this analysis to be equal to 0.001, for both case i and case ii.

5.6 Computer Program Description

A very general finite element program (TDCP) has been written in the Fortran 77 Language. It contains many subroutines, the main functions of which are to follow the loss of stiffness due to spread of yield and hence to trace the three dimensional load-deflection response up to collapse of a beam-column having almost any open cross-section composed of a series of flat plates. The Newton-Raphson technique has been used for both the the elastic and the inelastic ranges. The power and versatility of the program has been demonstrated by its application to a range of selected problems; these are presented in chapters 6, 7, and 8 respectively.

5.6.1 Program Structure

The full TDCP details together with its listing are given in ref. (El-Khenfas 1987a). This program uses many subroutines to provide a solution for the elastic and inelastic response of beam-columns. The more important routines and their functions are briefly detailed below.

MAIN	Reads the input data, controls the calling sequences.
OUT1,2,3	Presents the output of loading, displacements, stresses, internal forces, etc.
STIFF1,2	Forms the linear and nonlinear tangential stiffness matrices.
GEOMET1,2	Forms the linear and nonlinear geometrical stiffness matrices.
TRANS	Transforms the elements stiffness and loads,and displacements from local to global. (or to the references axis).
ASSEM	Controls the assembly of the total stiffness matrices or the applied loads.
ITLOAD	Controls the load or displacement increments.
STRAN1,2	Evaluates linear and nonlinear curvatures
BOUND	Modifies the total stiffness matrix of the structure according to the boundary conditions.
PROPER	Calls subroutines to calculate the section and sectorial properties, trace spread of yield, and internal forces.

ELASPL	Perform the elastic-plastic analysis using Newton-Raphson technique.
RESD	Calls several routines to generate different residual stress distributions.
INDEF	Forms the initial crookedness by sinusoidal or polynomial functions.
GAUSS	Used to solve the equilibrium equation by Gauss elimination.
START	Restart facility from a previous equilibrium solution.
STORE	Store the whole of the information obtained from the previous iteration.

A basic flow chart is given in Fig. 5.7.

5.7 Computational Steps

The computational processes of TDCP can be explained in the following steps

STEP 0 Initialization

In this step initialisations has to be made to the whole arrays

STEP 1 Input Data

a) Beam Details

NJ	number of nodes
NEL	number of elements
NDF	number of degree of freedom

SLEND	slenderness ratio $(\frac{L}{r_y})$

b) (Cross-Section Details

NES	number of elements in the cross-section
NJS	number of joints in the section
T(M)	plate thickness
NOSI(M)	node I number
NOSJ(M)	node J number
SEY1(M)	coordinates of node 1 with respect to Y-axis
SEY2(M)	coordinates of node 2 with respect to Y-axis
SEZ1(M)	coordinates of node 1 with respect to Z-axis
SEZ2(M)	coordinates of node 2 with respect to Z-axis
M	is the number of elements in the section

c) Applied Loads

ECY, ECZ, ECR	load eccentricities
APL(7xN-6)	Axial load (F_{x1})
APL(7xN-5)	Shear force Q_{z1}
APL(7xN-4)	Bending moment M_{y1}
APL(7xN-3)	Shear force Q_{y1}
APL(7xN-2)	Bending moment M_{z1}
APL(7xN-1)	Torsion M_{x1}
APL(7xN)	Bimoment $M_{\omega 1}$
APL(7xN+1)	Axial load (F_{x2})
APL(7xN+2)	Shear force Q_{z2}
APL(7xN+3)	Bending moment M_{y1}
APL(7xN+4)	Shear force Q_{y1}

APL(7xN+5)	Bending moment M_{z1}
APL(7xN+6)	Torsion M_{x1}
APL(7xN+7)	Bimoment $M_{\omega 1}$
N	Number of elements (1,2,...,NEL)
INCL	Type of load increment
1	Axial load
2	Bending moment about Y-axis
3	Bending moment about Z-axis
12	Axial + Bending about Y-axis
13	Axial + Bending about z-axis
123	Biaxial bending

Different combination can be performed

d) Initial Deflection (R_o)

INDEF	0 No initial deflection
	1 Initial deflection present
U_o, V_o, Φ_o	Out-of-plane and in-plane deflections and twisting respectively (they are given in TDCP by ACY, ACZ, ACR with respect to L)
INDEFT	0 Sine wave
INDEFT	1 polynomial (you have to insert to the program the wanted function)

e) Type of Analysis

IANAL	0 Elastic
IANAL	1 Inelastic

Analysis options

INC	1 Linear

2 Non-linear

3 Full non-linear

f) Stresses

IRES	0 No residual stress
	1 Residual stress present
KIND	1 Leigh distribution
	2 Parabolic distribution
	3 Triangular Distribution
	4 ~ 6 Other types
σ_f	Flange tip residual stress
σ_{fw}	Web-flange junction residual stress
σ_w	Web residual stress
NMAT	Number of material type
ESHD	Strain hardening (values)
E	Young's modulus
G	Shear Modulus

g) Boundary Conditions (B.C.)

IRR	Number of restrained nodes (B.C)
KKM	0 Degree of freedom restrained
	1 Degree of freedom unrestrained
JJM	Number of the degree of freedom restrained

STEP 2 Load Increment

PAX = AP1 + AFX1

Where PAX is the total load, AP1 is the load increment (usually given as ratio e.g. with respect the squash load or Euler load in the case of a compressive column load), and AFX1 is equal to the value of the internal load of the previous step.

PAMY,Z = AP1Y,Z + PAMY3

Where PAMY or Z is the total bending moment, AP1Y and AP1Z are either the increment of bending moment about the Y or Z-axis,and PAMY3 is the internal moment of the previous step.

STEP 3 Displacements

Displacements have to be updated after each load increment as

$R = R + \Delta R$

STEP 4 Strains

Strains have to be calculated at each load increment ,and then used to calculated the internal forces of to check spread of yield.

STEP 5 Section and Sectorial Properties

Section and sectorial properties have to be calculated each time of calculating curvatures (Φ_y, Φ_z, Φ_ω, Σ_o).

STEP 6 Store

The complete results obtained have to be saved by storing them

STEP 7 Start

In this step the program can be restarted from where it stopped

Step 8 Control Cards

IYYY	0 Start th program from the beginning
	1 Restart from where it stopped
MAXIT	Number of cycles of loading required
ISTEP=ISTEP+1	Number of steps required
ISTEP=MAXIT	When the number of steps (ISTEP) is equal to number of cycles of loading required, the program store the complete data and stop executions.

For other cycles of loads repeat step 2 up to step 8 again and again until failure (collapse occured).

5.8 Conclusions

A very general TDCP (Three Dimensional Computer Program) based on finite element concepts has been developed. It was written in the Fortran 77 Language. The main function of its subroutines is to follow the loss of stiffness due to spread of yield and hence to trace the full load -deflection response of beam-column of an arbitrary open thin-walled cross-section subjected to different loading and boundary conditions up to collapse. This program can deal with three types of analysis which are Linear, Partial non-linear, or Full non-linear. These methods are fully explained in Chapter 4.

The section and sectorial properties were evaluated at each load increment together with the internal forces based of the strains which in turn depend upon the type of analysis required. The convergence to the correct result employed two methods at the same time. The first one used the Out-of-balance forces which represent the differences between the applied loads and the internal loads, while the second is based on the increments of the displacements with respect to the total displacements.

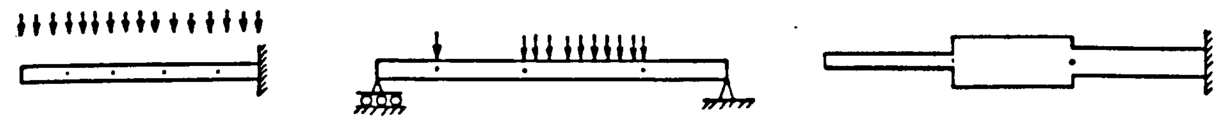
The main points drawn from this study may be summarised as:

1- A TDCP has been developed for the ultimate strength analysis of members to incorporating several types of open thin-walled sections (], T, L, I,etc.).

2- The linear and non-linear strains are used to generate the stress-results and to trace the spread of yield.

3- The section and sectorial properties can be evaluated in the elastic and inelastic ranges with respect to instantaneous shear centre and centroid.

4- The convergence to a correct result is based on the incremental displacements and the out-of-balance forces.

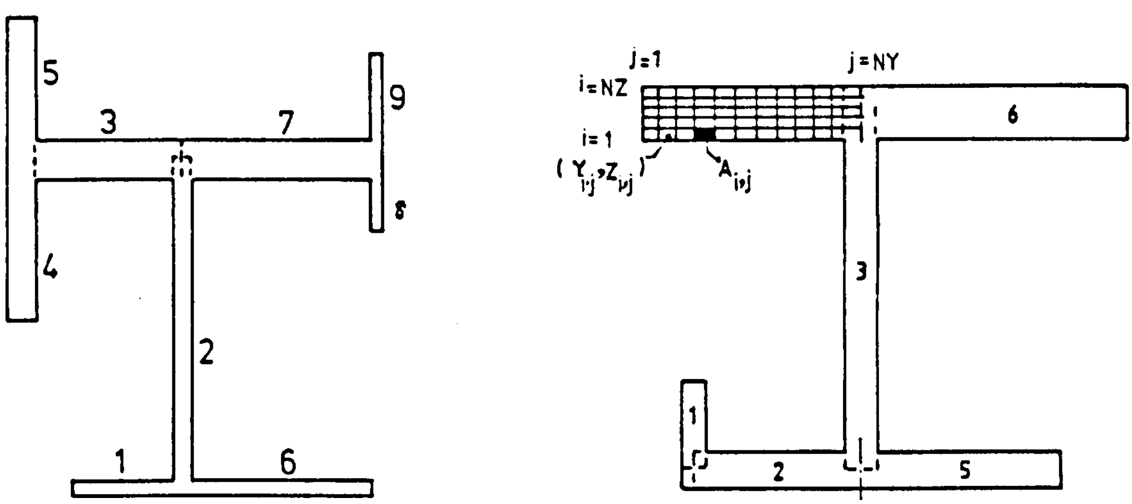


a. Beam divided to equal elements

b. Nodes to be introduced at changing loads

c. Node to be introduced at changing thickness

FIG.5.1 LOCATION OF NODES AT DISCONTINUITIES



a. Number of elements in the cross-section

b Number of plates in the cross-sectional element

FIG.5.2 NUMBER OF ELEMENTS AND SEGMENTS IN THE CROSS-SECTION

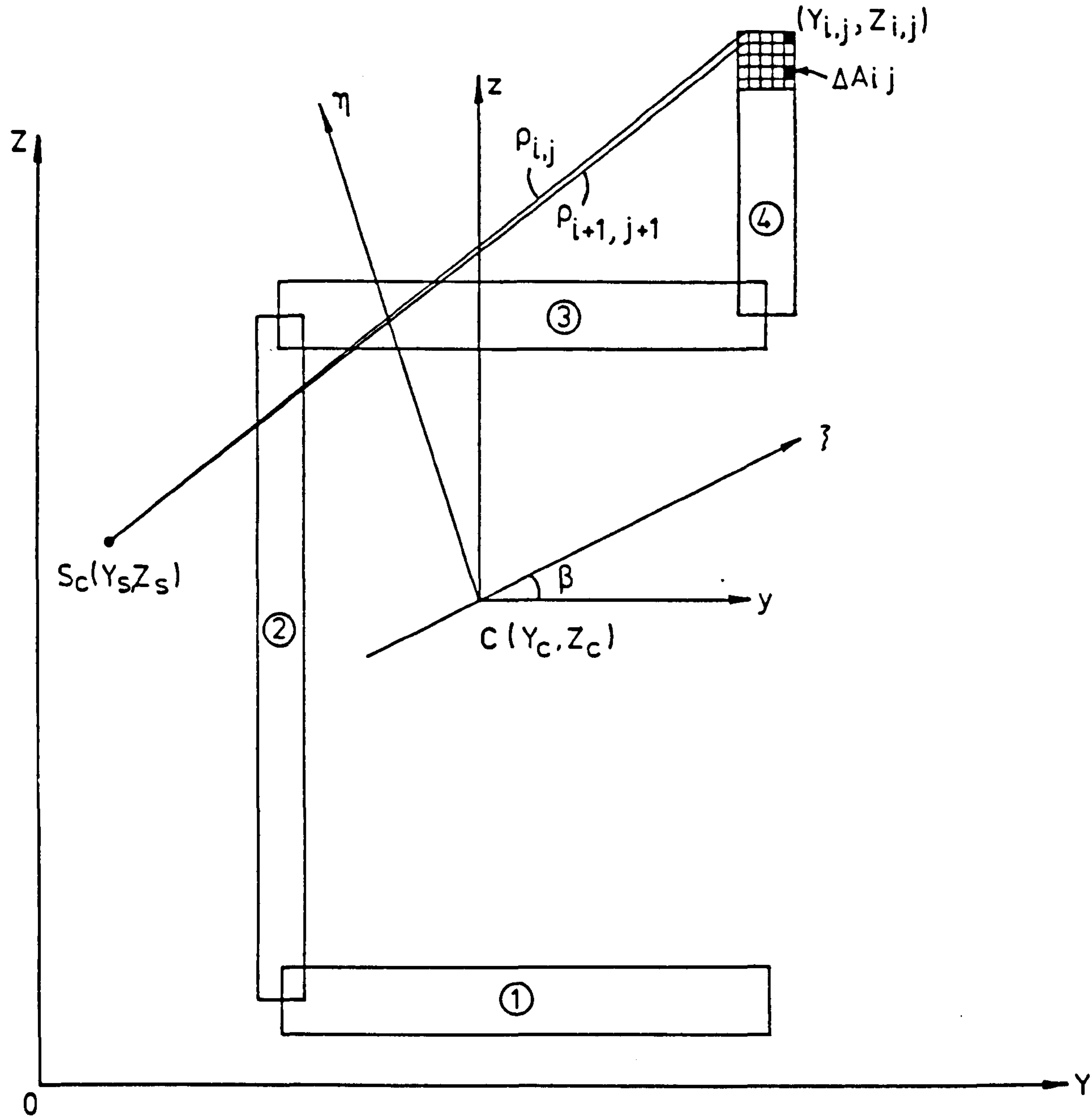


FIG.5.3 CROSS-SECTION ILLUSTRATING THE PROCESS OF EVALUATING SECTION PROPERTIES

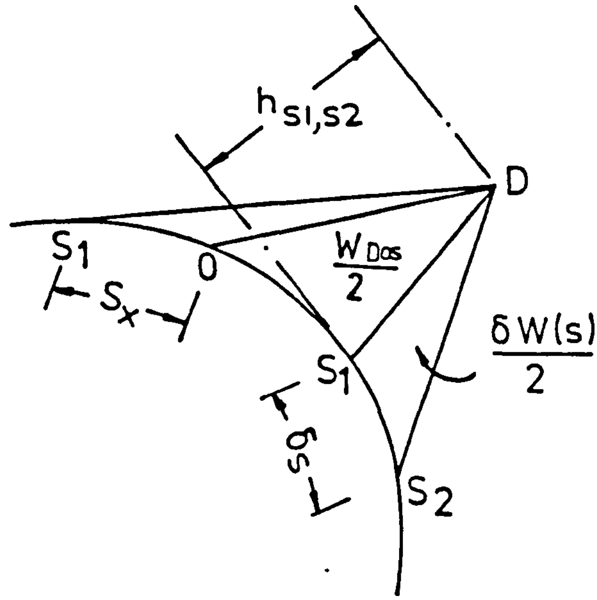
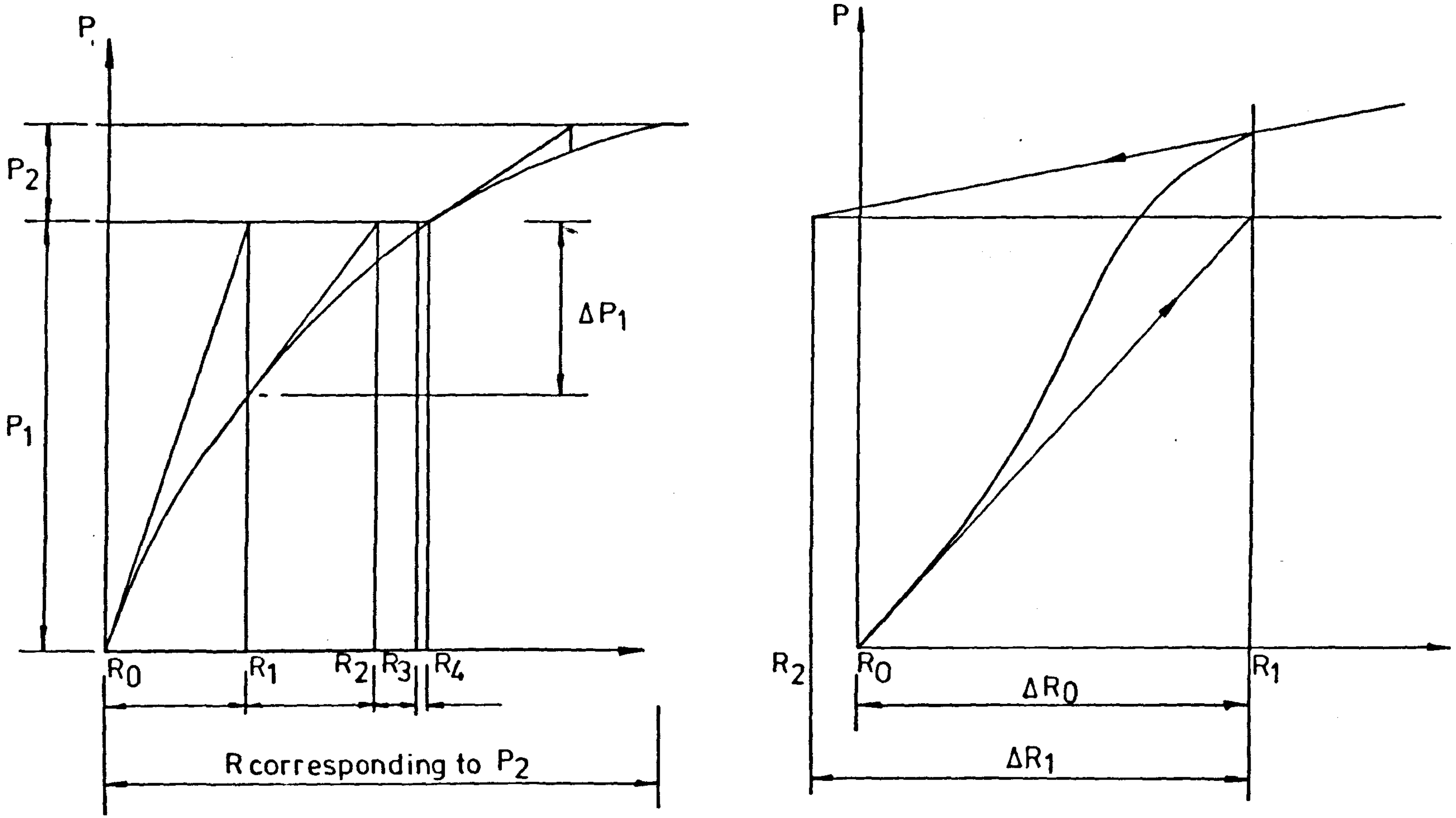
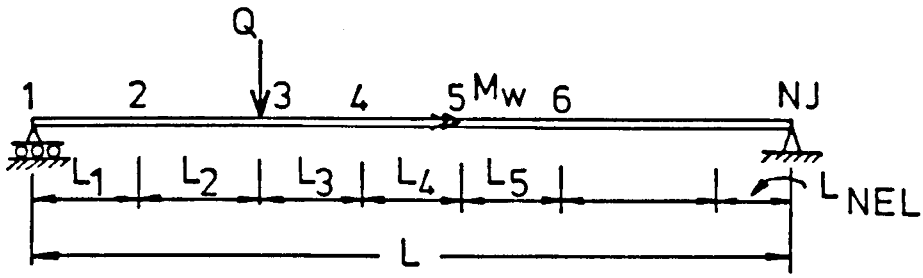


FIG. 5.4 SECTORIAL COORDINATES OF THIN WALLED SECTION

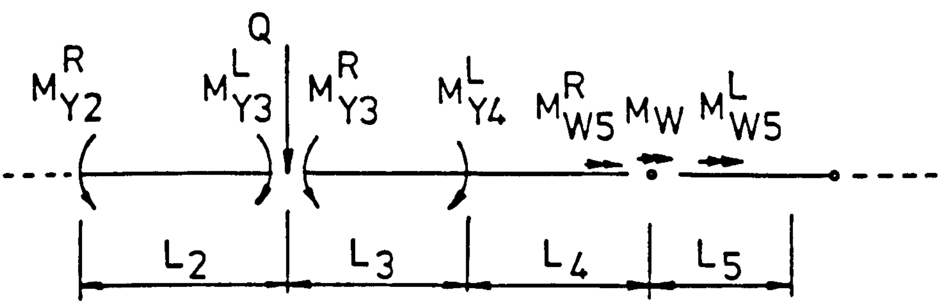


(a) Possible convergence (b) Possible divergence(ref.23)

FIG.5.5 NEWTON RAPHSON METHOD

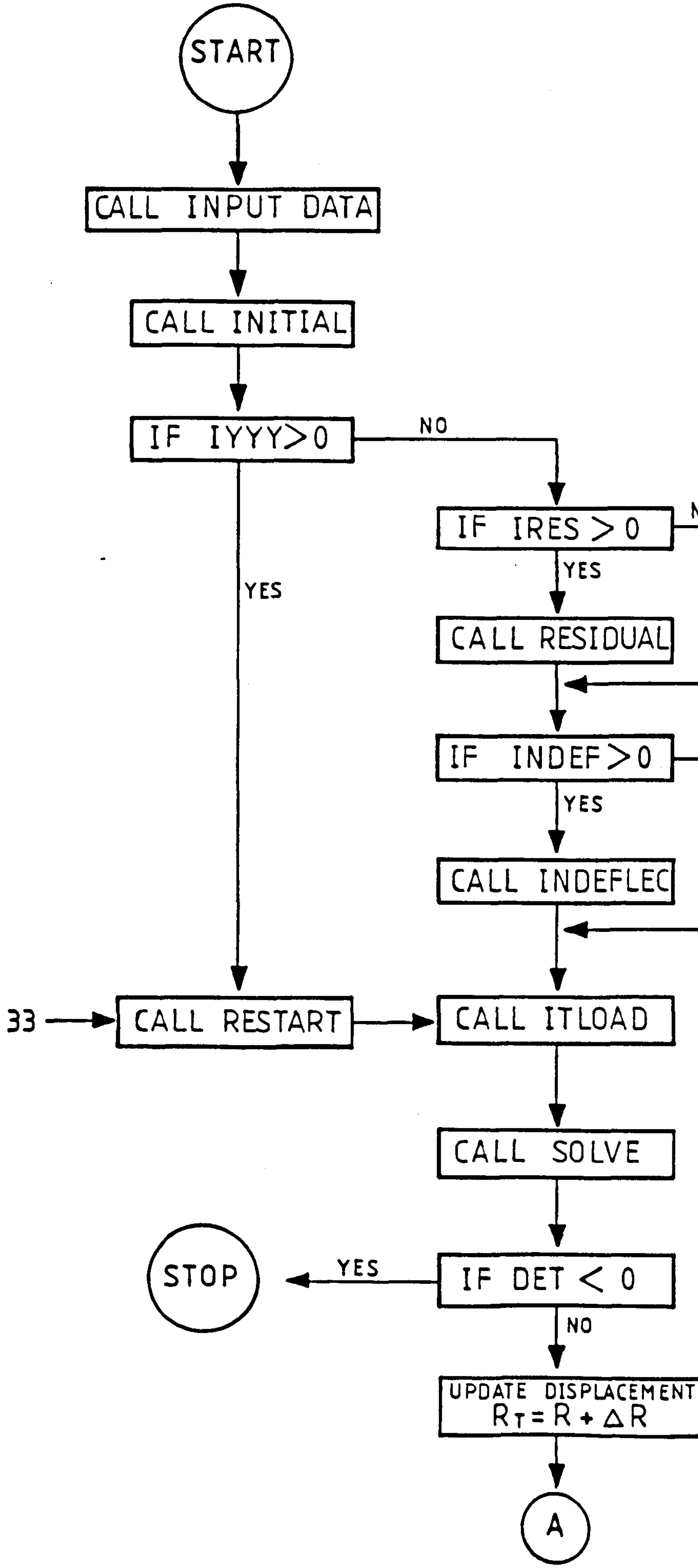


a. Illustration of the number of elements (NEL), number of joints (NJ) and applied loads



b. Distribution of internal and external forces

FIG.5.6 EXAMPLE SHOWING NUMBER OF ELEMENTS AND INTERNAL AND EXTERNAL FORCES


START
CALL INPUT DATA
CALL INITIAL
IF IYYY>0
NO
YES
IF IRES > 0
NO
YES
CALL RESIDUAL
IF INDEF>0
NO
YES
CALL INDEFLEC
33
CALL RESTART
CALL ITLOAD
CALL SOLVE
STOP
YES
IF DET < 0
NO
UPDATE DISPLACEMENT
$R_T = R + \Delta R$
A

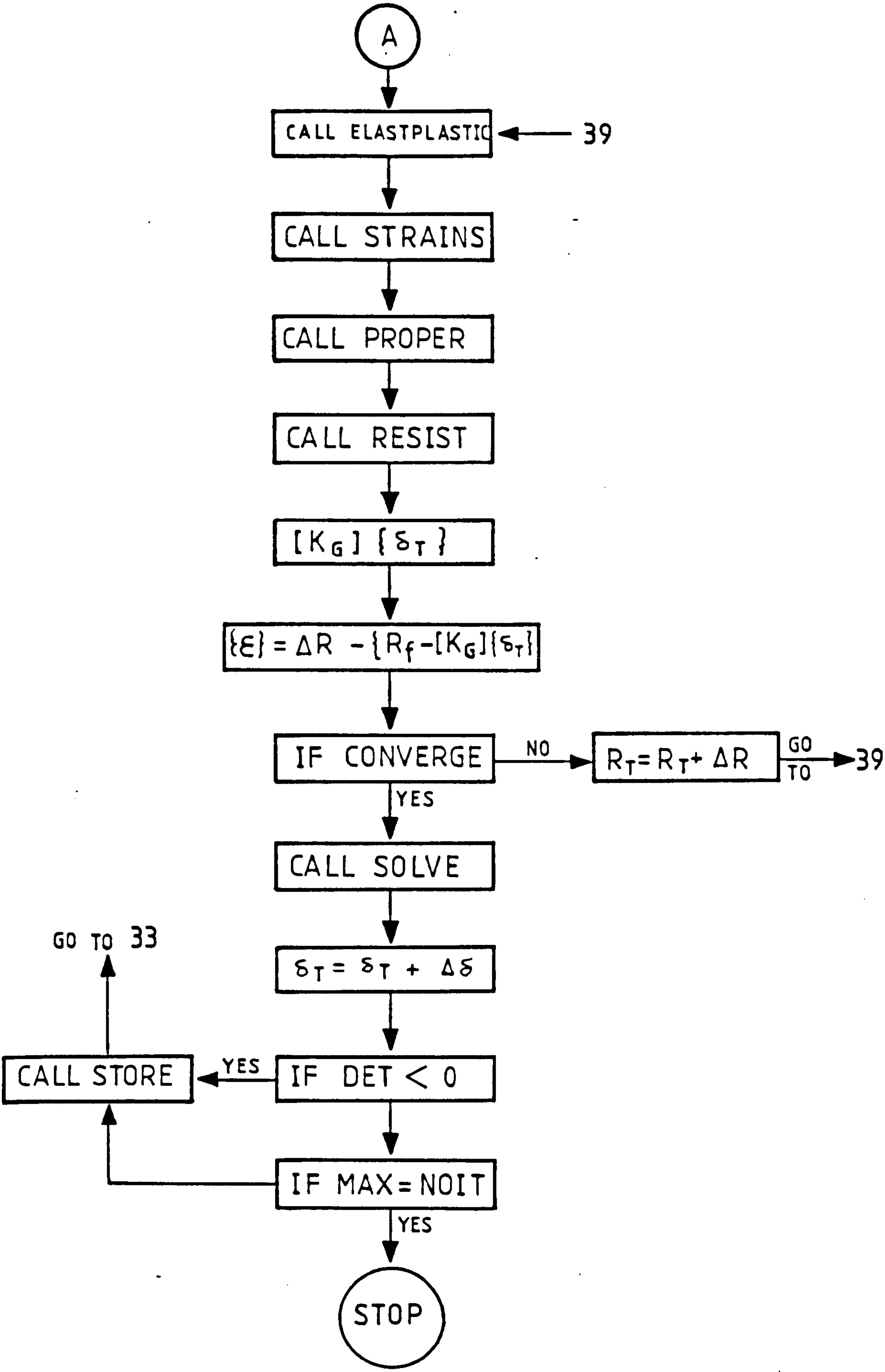


FIG. 5·7 FLOW CHART TO TDCP (ELKENFAS 1987a)

Chapter 6

Elastic Analysis of Beam-Column in Three-Dimensional

6.1- Introduction

Beam-columns are defined as structural members which are subjected to combined axial compression and bending. The bending may result from eccentric loading, transverse loads, applied moments, or combinations of loads.

The classical linear buckling theories for elastic beams and columns were investigated by Bleich(1933) who considered I-beams with unequal flanges subjected to axial load and equal and opposite end moments, then he applied his theory to the problem of beams of I-section with the tension flange restrained against lateral displacements. Johnston (1941), Massonnet (1947), Horne (1954) and Salvadori (1955,1956) studied doubly symmmetric sections under combined axial thrusts and equal end moments.

Differential equations governing the flexural torsional buckling of thin-walled member subjected to eccentric thrusts have been used by Goodier (1942), Timoshenko and Gere (1961) and Vlasov (1961). Their results were confirmed by tests done by Hill and Clark (1951a,b). Goodier (1942,1956) examined the stability of a general open section under axial load and end moments; Anderson and Trahair (1972) concentrated on lateral torsional buckling of monosymmetric sections.

Flexural-torsional behaviour has been studied by, Vlasov (1961), and Trahair (1966,1969). They investigated the effect of elastic end restraints, while the effect of continuous diaphram restrained has been considered by Hancock and Trahair (1978,1979). A BSCA publication (1968,1970) investigated members of different cross sections such as (Tee, Channel, Z, L, etc. sections) under bending and torsion. Moore (1986) used the finite difference method to solve his developed differential equation in order to study nonuniform torsional buckling of Z-sections under lateral loads.

The behaviour of beam-columns depends principally on: their slenderness, the shape and dimensions of the cross-section, the exact form of the applied loading, and the conditions of support provided. These will control not only the load carrying capacity but also the form of structural response e.g. in-plane flexure, biaxial bending etc. (Chen and Atsuta(1977)). For slender members elastic analysis will provide a close approximation to their true behaviour; more stocky members, however, failing by inelastic action, will need to be analysed in a manner which accounts for the gradual development of plasticity. Even for such cases the basis for an adequate analytical approach must be developed from an elastic treatment of the problem. Such a formulation has been presented in chapter 3. In this chapter the application of this approach to a series of elastic problems, which cover a range of structural phenomena, is reported.

6.2 Analysis

The analysis used herein employs the set of assumptions normally associated with analytical studies of the behaviour of thin-walled beam-columns in the elastic range. These are:

1 - The member may have any arbitrary open cross-section with each plate element having a thickness significantly less than its width.

2- The member's cross-section is constant along the span.

3- Both shear deformation and distortion of the cross-section (apart from warping) are ignored.

4- Different patterns of initial deflections (U_o, V_0, and Φ_o), may be present.

Full details of the problem formulation, leading to a series of stiffness matrices for use in a nonlinear finite element approach, are available in chapter 3. A feature of this formulation is the inclusion of several levels of so-called higher order terms in the geometric strain-displacement relationships resulting in a hierarchy of stiffness matrices. This permits different types of combined bending and torsion problem to be treated with different levels of sophistication. A computer program based on the material of chapter 3 has been prepared (El-Khenfas 1987a), using a Newton Raphson solution scheme.

The numerical solution scheme is based on a step by step approach which involves the combination of incremental and iterative processes until the desired convergence is achieved; details were given in chapter 5. Some judgement is necessary when deciding on:

1- Number of elements along member length

If the beam has no change in the geometry, material properties and external load and restraint conditions, then it can be divided into equal lengths. If discontinuities are present, nodes have to be introduced at these points.

ii- Number of segments in the cross-section.

The cross-section is divided into a number of plate elements, each element containing several small segments. These segments are then used to evaluate the cross-sectional and sectorial properties such as A, I_y, I_Z, I_ω, K, etc.. Both the number of plate elements and the number of segments used will affect the accuracy with an increase generally providing more accurate results, but at the cost of greater computation.

6.3 Numerical Results

Several cases have been chosen to cover the range of problem types outlined in the Introduction; these are listed in Table 6.1

6.3.1 Linear Bending and Torsion

The first class of problem considered is the linear combination of bending and torsion of the type covered in design guides by Terrington (1968,1970). Because direct comparison is made against the results of Terrington, which are themselves based on the solution of the governing differential equations, the example is quoted in the original imperial system of units.

A simply supported beam of 30 ft. span subjected to lateral loads Q_z=1 ton and Q_y =0.1xQ_z applied at midspan on the top of a compound section, consisting of a 18x7$\frac{1}{2}$x 150lb RSJ with a 10x3x19 lb channel welded to the top flange has been selected. Table 6.2 compares longitudinal stresses due to vertical and horizontal bending with those given by Terrington. In determining these values the contributions to the member stiffness matrices arising from the nonlinear strains included in the formulation (chapter 3) have been suppressed. It is clear from the results of Table 6.2 that the analysis is capable of dealing adequately with this type of behaviour, the stress values obtained being almost indistinguishable from those of Terrington 1970.

6.3.2 Flexural and Flexural-Torsional Buckling

The analysis may also be applied to bifurcation type problems, involving the determination of elastic critical loads. In this case a slightly different solution technique, in which the lowest eigenvalue of the stiffness matrix including destabilising effects is determined, must be employed.

The example selected for study is a simply supported beam of monosymmetric I-section previously analysed by Anderson and Trahair (1972). This type of problem is inherently more complex than the more usual doubly symmetrical section due to the need to include the Wagner effect in the torsional aspects of the behaviour, a feature that is accentuated if loading is applied remote from the shear centre axis.

Anderson and Trahair(1972) checked their theory against experimental evidence, and they confirmed that the Wagner effect results in either an increase or decrease in the effective torsional stiffness for monosymmetric open cross sections. Because the smaller flange is further from the shear centre, the stresses in this flange have a greater lever arm and predominate in the Wagner effect. Thus when the smaller flange is in tension the Wagner effect provides an increase in torsional stiffness of the beam. The reverse is true when the smaller flange is in compression.

Two examples illustrating the accuracy of the author's approach have been selected for comparison. For both central load (cases A). and distributed load (Cases B). the transverse loads were applied at distances of up to $7.6(\frac{D}{2})$ above or below the shear centre axis.

Tables 6.3 and 6.4 compare the results obtained by calculation for both doubly symmetrical and monosymmetric I-sections with those of Anderson and Trahair. Generally good agreement has been achieved with the largest differences being associated with loads applied below the level of the bottom flange. These are thought to be a result of the approximate form for displacements used by Anderson and Trahair; the present study makes no prior assumptions about the buckled shape.

A general equilibrium equation has been developed by Roberts and Azizian (1983a) governing the geometrically nonlinear behaviour of thin-walled open cross-sections by assuming small displacements and linear elastic material behaviour. The solution to this equation, when applied to beam-columns under equal end moments and axial load in the presence of in-plane initial deflection, has been checked against existing test data.

A simply supported I-section column subjected to a compressive load, possessing an initial deflection in the form of a half sine wave of maximum amplitude U_o at mid-height has been chosen for comparison. Boundary conditions for in-plane and out-of-plane bending and twisting are fully restrained at both ends, while warping is permitted at both ends. (It should be pointed out, however, that since a pure in-plane response was expected the out-of-plane and torsional conditions will have no direct effect on the result.). Numerical results have been obtained, which allow for the effect of in-plane deflection on flexural buckling. These require suppression of some aspects of the full analysis corresponding to the equilibrium equation stated in chapter 3 in order to match Azizian's assumptions. Fig. 6.1 illustrates the load-deflection behaviour obtained from both analyses in which the difference between the failure loads given by the two analyses is approximately 1 per cent. A further comparison is presented in Fig. 6.2, which deals with the lateral- torsional response of an initially curved and twisted beam. In this case a full three-dimensional response is obtained as the in-plane moments act through the initial out-of-plane deformations. Failure loads for the three levels of imperfection considered differ from those of Roberts and Azizian by a maximum of 4 per cent, the percentage differences being 4, 2.5, and 0.5 for the three cases respectively.

Results for the elastic critical moment of a simply supported T-section under equal end moments applied about the strong axis have been compared against those reported by Roberts and Azizian (1983b). For the case when the flange is in tension the differences between the two results are not more than 3%, while when the flange is in compression large differences between the two solutions were obtained. This case was therefore checked against the separate solution given by Kitipornchai and Chan (1987) which agreed with the authors to within 1 percent. They previously identified an error in certain aspects of the work of Robert and Azizian.

6.3.3 Biaxial bending

Tables 6.5, 6.6, 6.7, and 6.8 compare results for the biaxial bending of doubly symmetrical I-sections with the 'exact' theoretical solution of Culver (1966a,b). In both cases the member is assumed free from residual stresses. For the results of Tables 6.5 and 6.6 no initial lack of straightness has been assumed; in the case of Tables 6.7 and 6.8 initial bow (both planes) and initial twist have been included as indicated in the first entry of each table. The comparison between the results obtained by the author and those of Culver show excellent agreement, the percentage differences being not more than 2% for lateral and transverse displacements and a maximum 6.8% for the twists. The larger differences in rotations occured because of the inclusion of torsional moment and shear forces in the linear geometrical stiffness matrix as explained in chapter 4. All of the results of Tables 6.5 - 6.8 were obtained after applying some limitations (neglect of higher order terms, and using the second order strain in determining the internal forces), to the full formulation of chapter 3 in order to provide an analysis that was similar in concept to that employed by Culver.

6.3.4 Nonlinear Bending and Torsion

Sections for which the principal axes are oblique e.g. a Z-section, will, even if loaded vertically, deflect both vertically and horizontally. Because of deflection in the horizontal direction, the applied load will also move with the beam so that it no longer acts in the same plane as the vertical reactions at both ends. As a result the section will also twist. The additional stresses caused by this twisting will reduce the load-carrying capacity of the member.

The problem of nonlinear bending and torsion has been extensively studied by Moore (1986), who concentrated on the effect of movement of the point of application of the load for Z-sections. He extended the original theory of Vlasov (1961) and Timoshenko (1965) by solving the augmented differential equations using a finite difference method. The validity and accuracy of his theory were assessed by comparing its results with those obtained from experiments.

The example selected for study comprises a beam subjected to a pair of lateral point loads applied to the top flange at its 1/3 and 2/3 points, either passing through the shear centre or with eccentricities e_y=-25 mm, or e_y=+25 mm. Fig. 6.3 and Table 6.9 provide full details of the load-deflection (in-plane and out-of-plane) relations. The vertical displacement was evaluated at point 1 and point 2 and the vertical displacements were calculated at point 2. These values were obtained for a simply supported Z-section beam under two point loads placed at the 1/3 and 2/3 positions on the top flange passing through the shear centre. The values of the lateral displacements at the top and bottom flanges which are indicated by points 1 and 2 are closer to the experimental value than those of Moore, whilst for the in-plane behaviour deflections are closer to those conducted by Moore with little difference from the experimental data. Figure 6.4 represents the load-rotation curves of the results of Moore's tests, Vlasov ,and Moore's theory and those obtained by calculations which are based on the approach presented in chapter 3.

The author's values are generally rather closer to the experimental data reported by Moore than are the predictions of Moore's own theory.

6.3.5 Biaxial Bending and Torsion

It is possible to implement the theoretical formulation of chapter 3 at a variety of levels by suppressing various of the higher order contributions to the stiffness expressions. In particular, three classes of approach may be employed:

i- Linear - using only the basic stiffness matrix (tangent and geometrical which includes torsional moments and/ or shear forces) and second order strains.

ii- Nonlinear - using only the same stiffness matrix as i but including the higher order strain terms such as $\frac{1}{2}(V^2_{,x}+U^2_{,x}+\rho^2_o\Phi^2_{,x})$, $V_{,xx}\Phi + U_{,xx}\frac{\Phi^2}{2}$, and $U_{,xx}\Phi + V_{,xx}\frac{\Phi^2}{2}$ to $V_{,xx}$ to the axial strain, bending strain about the Y and Z axis. Further details are given in chapters 4 and 5.

iii- Full nonlinear - using all higher order contributions to the stiffness matrices and strain-displacement relations as described in chapter 4.

Some quantitative indication of the effect of adopting these different degrees of sophistication may be obtained by considering specific examples. A simply supported beam-column of W12x14 section, free to warp at both ends with an initial lack of straightness $U_o = \frac{L}{1000}$ at the mid-span subjected to an axial load applied with eccentricities at both ends $\frac{e_{y1}}{B} = \frac{e_{y2}}{B} = 0.025$ and $\frac{e_{z1}}{D} = \frac{e_{z2}}{D} = 0.25$ has been considered. Several analyses have been performed in which this loading is assumed to act in conjunction with a constant applied mid-span torsional moment M_t. The load path assumed is full torsional loading (up to a selected value) followed by end loading to failure. Results have been obtained for a series of values of M_t using each of the 3 solution schemes outlined above.

For the case of $\frac{M_t}{M_y} = 0.023$; the difference between the 3 approaches is clearly illustrated by the set of load deflection curves given in Fig. 6.5 ' Linear' analysis overestimates the maximum load by 24.2 per cent, whilst partial nonlinear analysis overestimates the failure load by 11.4 per cent as compared with the full nonlinear analysis. In the early stages of loading the differences between the three analyses are negligable, because the degree of coupling between the displacements is smaller. In other word the contributions of the nonlinear stiffness matrices which depend on the products of derivatives of displacements and rotations become insignificant. As the loads increase followed by change in displacements, then the differences between the analyses are noticeable and become more and more significant. Finally the findings for the full range of M_t/M_y values considered in the form of an interaction diagram of combinations of P and M_t at failure are described in detail in chapter 8.

6.4 Conclusions

An elastic analysis for thin-walled open cross-sections of general shape subjected to any combination of compression, biaxial bending and torsion has been presented. The approach utilises a nonlinear finite element formulation, which incorporates any form of initial deformations. A selection of illustrative studies, designed to demonstrate something of the range of problem types that may be treated has been reported. In each case excellent agreement with previously obtained theoretical results and / or experimental data has been obtained. Because of its general nature, the method can deal with

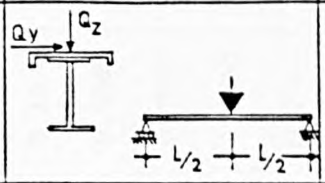
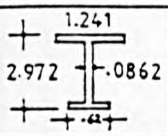
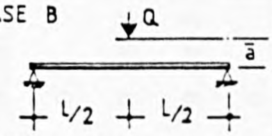
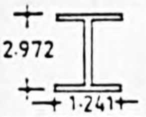
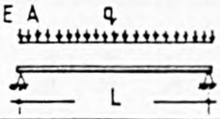
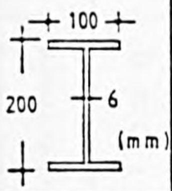
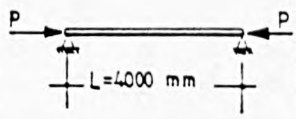
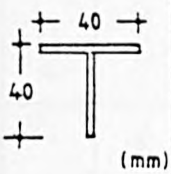
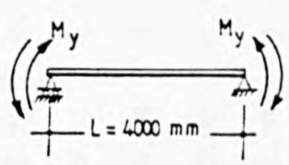
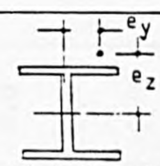
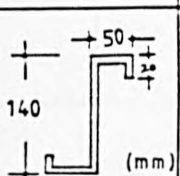
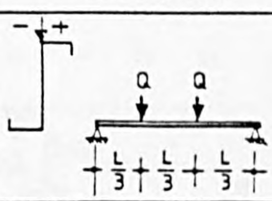
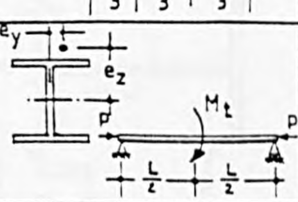
1- Arbitrary cross-sections, including those with no axes of symmetry.

2- Any arbitrary set of applied loads.

3- Any arrangement of flexural and torsional support conditions.

It therefore represents a powerful tool for studying the response of beam-columns in a very general manner.

Table 6.1 Examples Selected for Study

Reference	Section Type	Load Arrangement	λ	Warping at the ends	Initial Deflection U/B	Initial Deflection V/D	Initial Deflection ϕ rad
BSCA(1970)			88	R	0	0	0
Anderson & Trahair (1972)		CASE B 	368	R	0	0	0
		CASE A 	241				
Roberts & Azizian (1983)			490	R	0	0.01	0.01
					0	0.02	0.02
					0	0.04	0.04
Roberts & Azizian (1983) Kitiporchai & Chan (1987)			78	R	0.	0.	0.
Culver (1966 a,b)	14WF43	 at both ends	140&60	U,R	0.	0.	0.
			60 & 140	U,R	0.015	0.089	.0173
					0.035	0.02	0.04
Moore(1986)			134	R	0.	0.	0.
	W 12x.14		157	R	$\frac{L}{1000}$	0.	0.

R= Warping restrained U= Warping unrestrained
B,D = Width of flange and Depth of the beam λ = Slenderness ratio

Table 6.2 Comparison between Authors results and those by BSCA (1970)

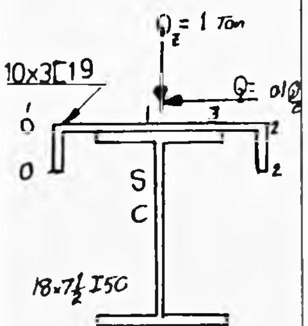
Load and section description	Points	Co-ordinates (in.)		σ_y (T/in²)		σ_z (T/in²)		$\sigma_T=\sigma_y+\sigma_z$ (T/in²)	
		Y	Z	BSCA	Authors	BCSA	Authors	BCSA	Authors
	0'	-5.0	3.96	0.3230	0.3231	-0.3810	-0.3812	-0.058	-0.0581
	0	-5.0	6.96	0.5680	0.5679	-0.3810	-0.3812	0.1870	0.1867
	1	0.0	6.96	0.5680	0.5679	0.0	0.0	0.5680	0.5679
	2	5.0	3.96	0.3230	0.3231	0.3810	0.3812	0.7040	0.7042
	2'	5.0	6.96	0.5680	0.5679	0.3810	0.3812	0.9490	0.9491
	3	0.0	3.38	0.2760	0.2758	0.0	0.0	0.2760	0.2758
	4	3.75	-11.36	-0.9270	-0.9269	0.2860	0.2859	-0.6410	-0.6411
	5	0.0	-11.36	-0.9270	-0.0269	0.0	0.0	-0.927	-0.9269
	6	-3.75	-11.36	-0.9270	-0.9269	-0.2860	-0.2859	-1.2130	-1.218

Table 6.3 Comparison of calculated results with Anderson and Trahair (1971) for stability of double symmetric beams

Type of loading	δ	ε	a_z	γ	Failure load: Anderson and Trahair (1971) $\times 10^{-3}$	Failure load: Calculated $\times 10^{-3}$	% error
	K=0.3						
CASE B	0	0.6	8.923	13.97	0.7853	0.758[1]	1.0
CASE B	0	0.3	4.49125	19.84	1.12	1.10[1]	0.1
CASE B	0	0.0	0	29.63	1.67	1.690[1]	0.1
CASE B	0	-0.3	-4.491	44.09	2.48	2.40[1]	1.6
CASE B	0	-0.6	-8.983	61.09	3.49	3.61[1]	5.4
CASE A	0	0.6	8.983	7.0	39.12	37.0[2]	2.5
CASE A	0	0.3	4.491	10.67	59.63	58.0[2]	2.3
CASE A	0	0.0	0	17.78	99.36	99.0[2]	0.3
CASE A	0	-0.3	-4.491	28.62	159.94	154.0[2]	3.1
CASE A	0	-0.6	-8.983	39.67	221.69	220.0[2]	4.5

$\varepsilon = 0.068\ a_z$

$\gamma_c = 184.16\ Q_z$

$\gamma_o = 18305.5\ q_z$

Table 6.4 Comparison of Authors results with those of Anderson and Trahair (1971) for stability of monosymmetric beams

Type of loading	K=0.3						
	δ	ε	a_z	γ	Failure load: Anderson and Trahair (1971) $\times 10^{-3}$	Failure load: Calculated $\times 10^{-3}$	% error
Case B	0.204	0.6	8.9825	15.855	3.61	3.57[1]	1.8
		0.3	4.4913	23.06	5.25	5.10[1]	1.5
		0.0	0.0	34.763	7.92	7.80[1]	1.2
		-0.3	-4.4913	51.19	11.67	11.88[1]	2.1
		-0.6	-8.9825	70.84	16.45	16.50[1]	3.1
Case A		0.6	8.9825	7.78	93.63	100.0[2]	2.8
		0.3	4.4913	12.16	146.44	143.5[2]	2.1
		0.0	0.0	20.24	243.72	243.0[2]	1.3
		-0.3	-4.4913	31.53	379.796	363.0[2]	6.0
		-0.6	-8.9825	43.12	519.28	475.0[2]	7.2

$\varepsilon = 0.07\ a_z$

$\gamma_A = 76.46\ Q$

$\gamma_B = 4015.4\ q_z$

[1] concentrated load Q kips

[2] distributed load q kip in

Table 6.5 Comparison of calculated results with exact solution of Culver (1966a), for biaxial loading, no initial deflection

$\frac{L}{r_y} = 60$ $\frac{e_y}{B} = 0.625$, $\frac{e_z}{D} = 0.366$ 14WF43 Warping unrestrained

$\frac{P}{P_y}$	$U/B \times 10^{-2}$		$V/D \times 10^{-2}$		ϕ - Rotation (rad) $\times 10^{-3}$	
	Culver	Calculated	Culver	Calculated	Culver	Calculated
0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.048	1.515	1.590	0.092	0.092	0.05	0.058
0.096	3.093	3.095	0.184	0.184	0.20	0.22
0.144	4.738	4.993	0.276	0.276	0.40	0.48
0.194	6.454	6.466	0.368	0.368	0.80	0.87
0.246	8.273	8.248	0.461	0.460	1.30	1.39

Table 6.6 Comparison of calculated results with exact solution of Culver (1966a) for beam-column under biaxial bending, no initial deflection

$\frac{L}{r_y} = 140$ $\frac{e_y}{B} = 0.0625$, $\frac{e_z}{D} = 0.366$ 14WF43 Warping unrestrained

$\frac{P}{P_y}$	$U/B \times 10^{-2}$		$V/D \times 10^{-3}$		ϕ -Rotation (rad) $\times 10^{-2}$	
	Culver	Calculated	Culver	Calculated	Culver	Calculated
0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.048	9.06	9.06	5.03	5.03	0.12	0.12
0.096	20.68	20.66	10.15	10.15	0.57	0.57
0.144	36.11	36.04	15.27	15.30	1.50	1.55
0.194	57.80	57.33	20.33	20.40	3.43	3.25
0.246	89.73	90.63	25.04	25.25	6.96	6.48

Table 6.7 Comparison of calculated results with exact solution of Culver (1966b) for beam-column subjected to biaxial bending, initial deflection included

$\frac{L}{r_z} = 60$	$\frac{e_y}{B} = 0.625$,	$\frac{e_z}{D} = 0.366$	14WF43	Warping unrestrained		
$\frac{P}{P_y}$	$U/B \times 10^{-2}$		$V/D \times 10^{-2}$		ϕ - Rotation (rad) $\times 10^{-3}$	
	Culver	Calculated	Culver	Calculated	Culver	Calculated
0.0	1.475	1.475	0.864	0.864	0.173	0.173
0.048	1.566	1.565	0.922	0.921	0.43	0.33
0.096	3.199	3.196	1.845	1.844	1.31	1.10
0.144	4.903	4.899	2.767	2.769	2.67	2.70
0.192	6.688	6.680	3.689	3.697	4.59	3.88

Table 6.8 Comparison of calculated results with exact solution of Culver (1966b) for beam-column subjected to biaxial bending, initial deflection included

$\frac{L}{r_z} = 140$	$\frac{e_y}{B} = 0.625$,	$\frac{e_z}{D} = 0.366$	14WF43	Warping Unrestrained		
$\frac{P}{P_y}$	$U/B \times 10^{-2}$		$V/D \times 10^{-2}$		ϕ - Rotation (rad) $\times 10^{-3}$	
	Culver	Calculated	Culver	Calculated	Culver	Calculated
0.0	3.45	3.45	2.02	2.02	0.403	0.403
0.048	9.80	9.84	5.07	5.08	0.51	0.40
0.096	22.51	22.40	10.14	10.22	2.05	1.54
0.144	39.99	39.52	14.97	15.15	5.36	4.05
0.192	66.15	64.45	19.07	19.65	12.01	13.80

Table 6.9 Comparison of Author's results with Moore (1986) analysis and tests for beams of Z-section subjected to bearing and torsion

$\frac{P}{P_{ey}}$	U_2 - Displacement (mm)			U - Displacement (mm)			V - Displacement (mm)		
	Moore	Expt.	Authors	Moore	Expt.	Authors	Moore	Expt.	Authors
0	0	0	0	0	0	0	0	0	0
0.045	1.056	1.060	1.136	1.074	1.06	1.127	0.704	0.693	0.700
0.089	1.972	2.071	2.205	2.183	1.914	2.309	1.309	1.32	1.403
0.134	2.958	2.817	3.261	3.380	2.535	3.523	1.97	2.754	2.108
0.178	3.803	3.83	4.293	4.578	4.085	4.782	2.747	3.804	1.817
0.223	4.648	4.859	5.30	5.915	5.634	6.091	3.521	4.859	3.535
0.267	5.211	5.915	6.288	7.324	6.901	7.454	4.155	5.915	4.261
0.357	5.958	6.972	7.258	8.873	8.521	8.876	4.93	7.042	4.999
0.356	6.761	8.169	8.214	10.592	10.282	10.363	5.775	8.169	5.75
0.401	7.324	9.437	9.157	12.535	12.092	11.921	6.549	9.437	6.518
0.446	7.845	10.985	10.092	14.788	13.944	13.558	7.324	10.989	7.307

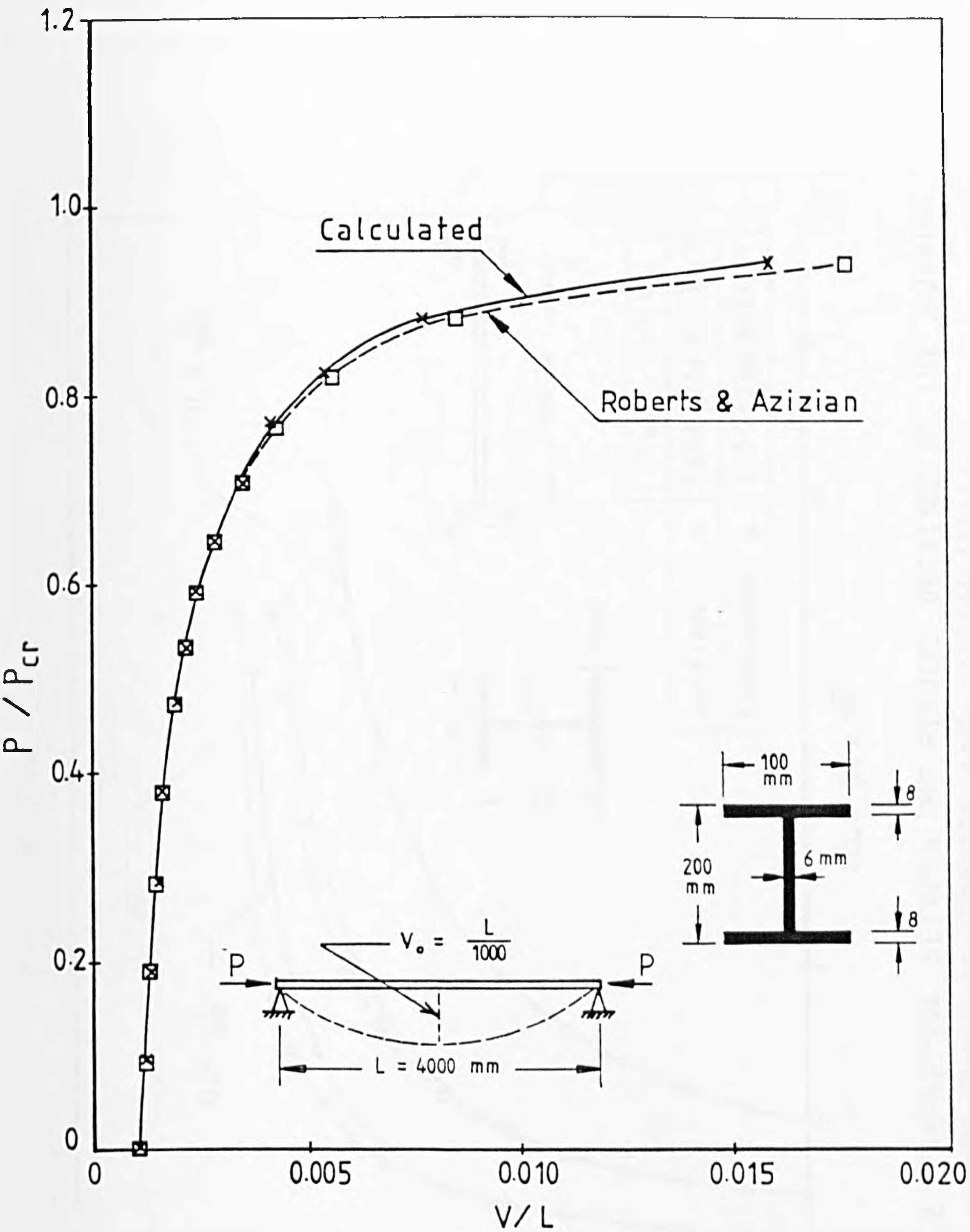


FIG. 6.1 COMPARISON BETWEEN AUTHOR'S RESULTS AND ROBERTS & AZIZIAN (1983) FOR I SECTIONS SUBJECTED TO AXIAL LOAD AND VERTICAL DEFLECTION

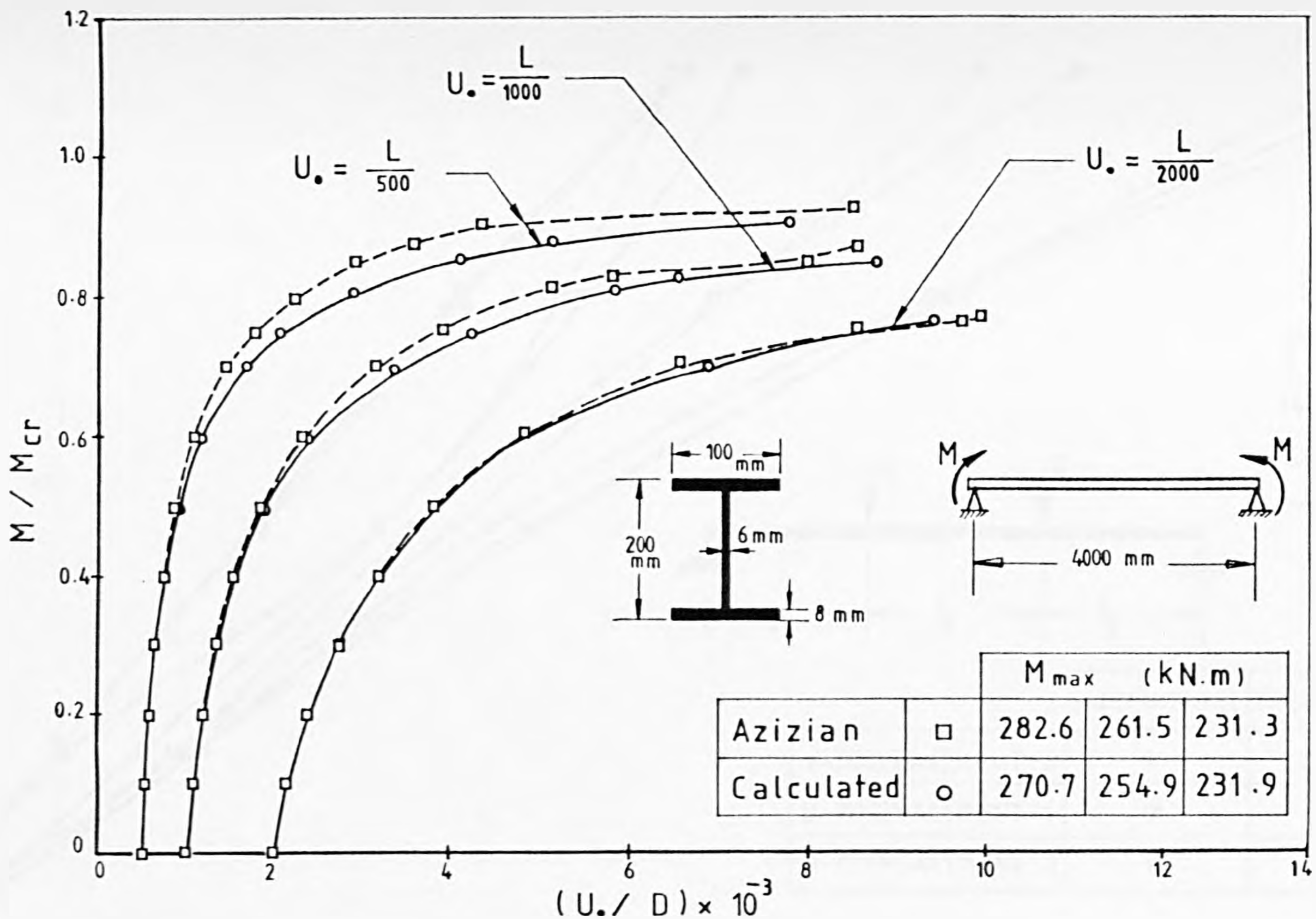


FIG. 6.2 COMPARISON BETWEEN THE RESULTS OBTAINED BY THE AUTHOR AND THOSE OF ROBERTS AND AZIZIAN (1983).

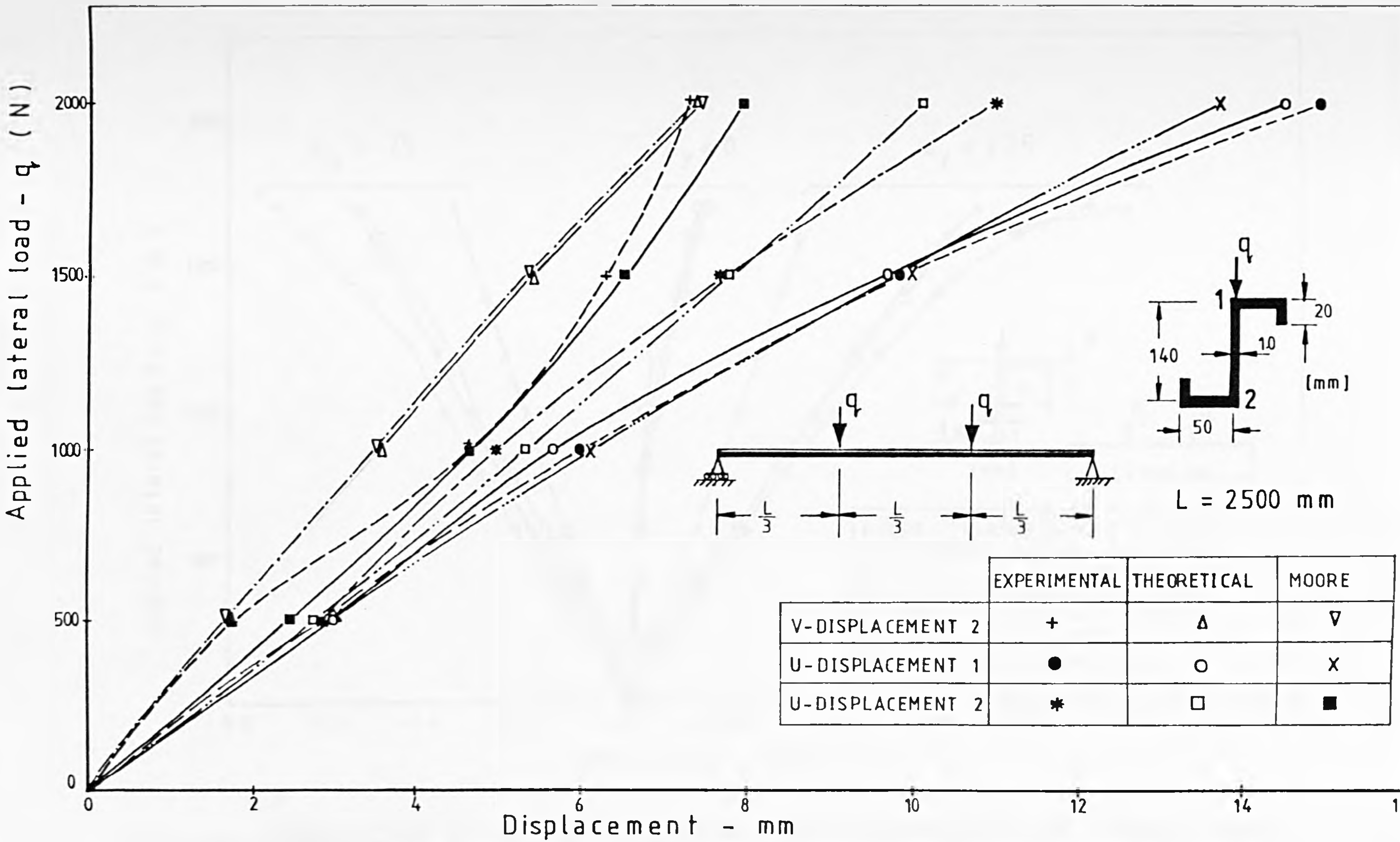


FIG. 6.3 COMPARISON OF AUTHOR RESULTS WITH ANALYSIS OF MOORE (1986) AND TEST DATA LATERAL LOAD APPLIED THROUGH SHEAR CENTRE

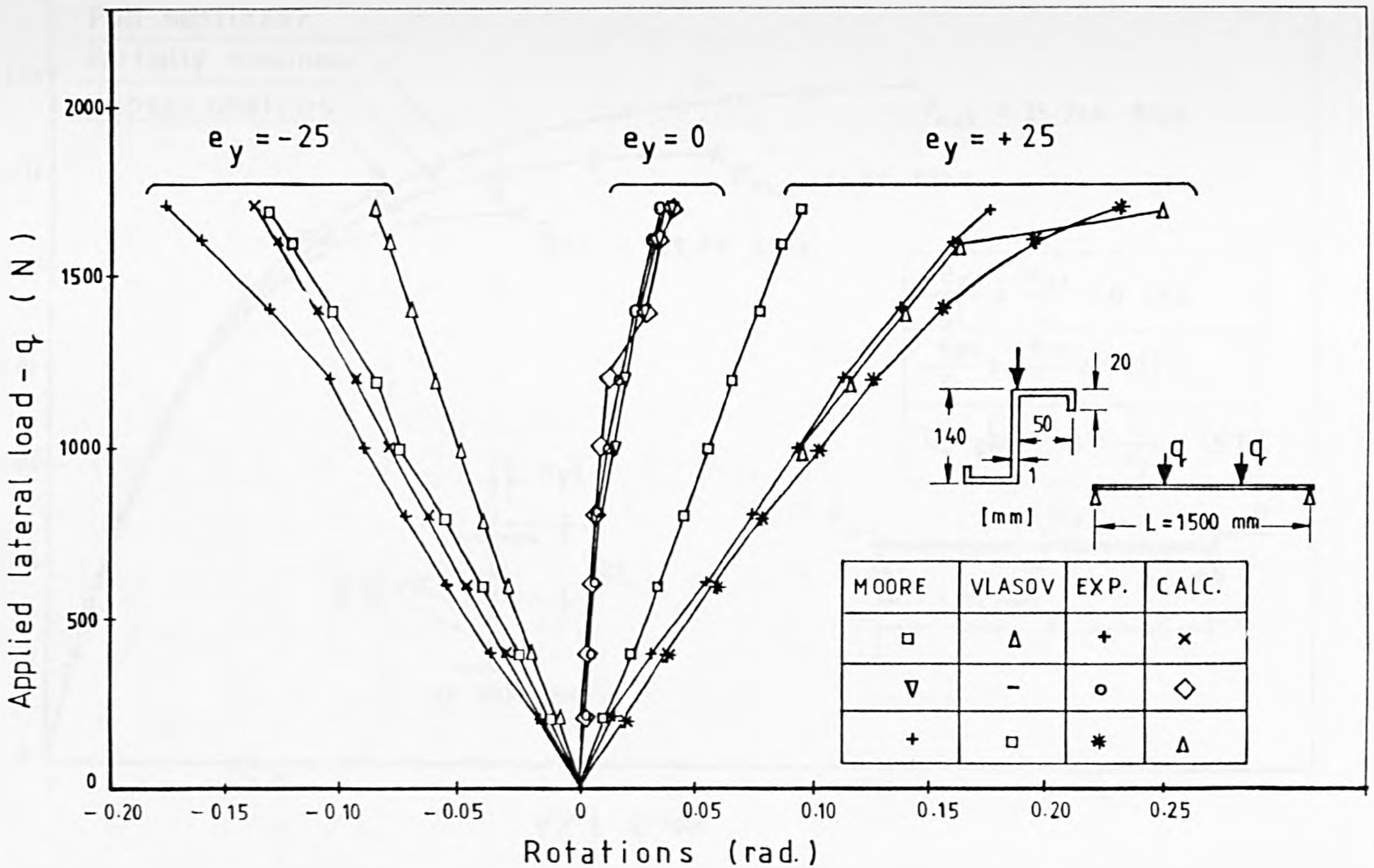


FIG. 6.4 COMPARISON OF AUTHOR'S RESULTS WITH MOORE(1986) AND VLASOV(1961)

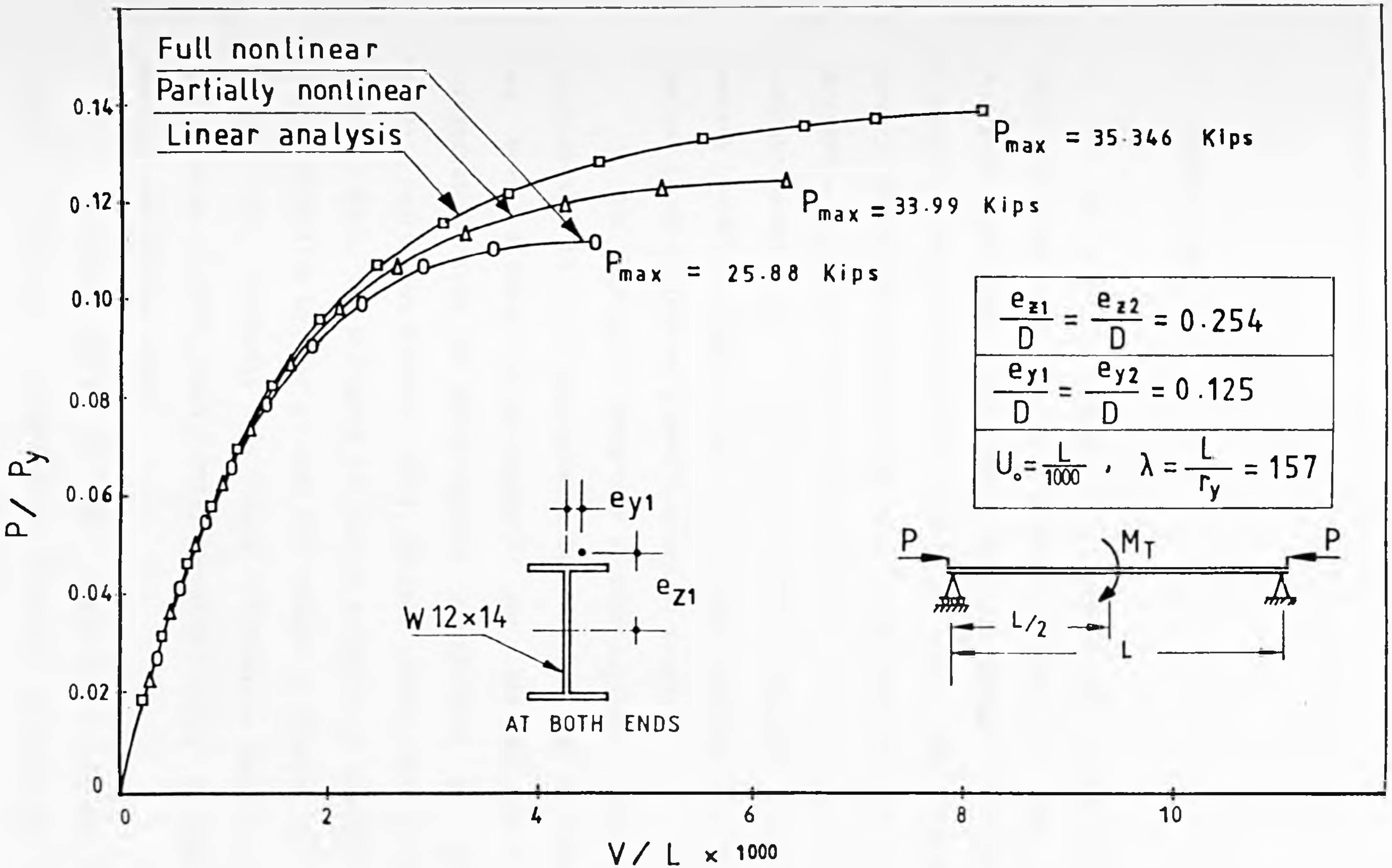


FIG. 6.5 LOAD - DEFLECTION CURVE FOR THREE TYPES OF ANALYSIS OF A SIMPLY SUPPORTED I - BEAM SUBJECTED TO BIAXIAL BENDING AND CONSTANT TORSION AT MID - SPAN

Chapter 7

Inelastic Analysis of Beam-Column in Space

7.1 Introduction

In order to study the response of a beam-column in the inelastic range it is necessary to allow correctly for the losses in stiffness that occur due to spread of yield through the cross-section, with due account being taken of variations along the length of the member. The presence of initial geometrical imperfections and residual stresses will affect both the initiation and propagation of yielding. Accurate monitoring of the full three-dimensional response of such members therefore requires that very careful consideration be given to the way in which gradual plastification is treated.

The inelastic behaviour of beam-columns of thin-walled open cross-section in three dimensions under various types of applied loads has been studied experimentally and analytically by several investigators such as Nethercot (1973a,b),Trahair and Kitipornchai (1972), Fujita and Yoshida (1972), Matthey (1984), Anslijn (1983), and Lindner (1981). The influence of initial geometrical imperfections has been incorporated by some of them and ignored by others. Reviews of the subject are available in many references such as Chen and Santathadaporn (1968), Chen (1977), Johnston (1976), Massonet (1976), and Chen and Atsuta (1977)

The application of the general finite element formulation presented in chapter 3 to assess the inelastic behaviour of thin-walled steel members deforming in space is described herein. This approach is applicable to various types of open cross-section, including those with no degree of symmetry.

In this chapter the accuracy and versatility of the theoretical approach which has been presented in chapter 3 is demonstrated by application to a wide range of selected problems covering flexural, flexural-torsional buckling, and bending and torsion.

7.2 Assumptions

In addition to the basic assumptions of Vlasov(1961) and those presented in chapter 3, the following are taken into consideration:

I- The distribution of elastic and yielded regions within the cross section is determined from the strain distribution and the material stress-strain curve.

II- The moduli of the material are E and G in the elastic regions while in the yielded regions E_t and G_t are used, with E_t and G_t being dependant on the type of stress-strain curve employed. Any monotonically increasing curve may be assumed.

III- Any reasonable pattern of residual stresses and virtually any distribution of initial crookedness may be included.

IV- Control of the nonlinear numerical solution is as described in section 4.6 of chapter 4 with accuracy being based on a combined displacement change/ out of balance load check.

V- Yield is controlled by normal stresses only (although the inclusion of more sophisticated yield criteria is possible. with little extra work).

VI- The sectional and sectorial properties were evaluated after each load increment in the inelastic range.

Various possible load types are given in Table 7.1., which lists the example problems considered herein. In the case of transverse loads, these may be either distributed or concentrated and may be applied at any eccentricity to the shear centre axis. Allowance for the effect of spread of yield in any region on cross-sectional properties makes no assumptions concerning the location of the shear centre, existence of symmetry etc.. Equilibrium of internal and external forces is maintained at all stages of the solution, see section 5.5 of chapter 5.

The sectorial properties such as warping moment of inertia (I_{ω}), warping second moments of inertia ($I_{y\omega}$ and ($I_{z\omega}$), and normalized function ($\bar{\omega}_n$) are not taken as the elastic values in the inelastic range as was the case in most previous work (Rajasekaran and Murray (1973), Yang and McGuire (1986), and Kitipornchai and Trahair (1975)) but are evaluated accounting for the loss of stiffness due to spread of yield in a similar fashion to the other sectional properties such as I_y, $\bar{Y}$, I_{yz}, etc.

7.3 **Numerical Results**

Several illustrative examples have been selected for study. Different aspects of behaviour such as flexural buckling, flexural-torsional buckling, and biaxial bending are covered by these cases. The influence of initial deflection and residual stresses are incorporated. The particular features of each type of problem are listed in Table 7.1.

In each case the number of elements and the number of plate segments within the cross-section were chosen so as to achieve accurate solutions without increasing the computational effect unnecessarily as described in Chapter 5. The convergence criterion of section 5.5 was employed.

7.3.1 Column with Initial Deflections

A simply supported I-section under axial load assuming various values of initial deformations, and including the effect of material strain-hardening (E_{sh} = E/10) has been considered. The initial crookedness is represented by a sinusoidal function with its maximum value at midspan. Residual stress was not included so as to permit comparison with the previous solution of Fujita and Yoshida (1972).

Some limitations are imposed on the suggested formulation of chapter 3 in order to achieve comparable results; these are the influence of the higher order terms of the strain-displacement relationship and the nonlinear tangential and geometrical stiffness matrices which incorporate the derivative of the displacements such as $PU_{,x}\Phi$, $M_y V_{,xx}\Phi$, $EI_y U_{,xx}\frac{\Phi^2}{2}$, $Y_s EI_z U_{,xx}\Phi_{,x}$ etc.. Four examples are chosen from those studied both theoretically and experimentally by Fujita and Yoshida (1972) and the results obtained are given in Table 7.2. The percentage differences between the present solution and the earlier results is in no case greater than 4 per cent.

7.3.2 Flexural & Flexural Torsional Buckling

Fig. 7.1 shows a monosymmetric I- section together with the residual stresses produced by welding; these have adjusted so as to meet the equilibrium condition of equations 3.43, 3.44, and 3.45. This pattern has been implemented by Fukuomto et al. (1971) in the inelastic analysis of a simply supported beam under equal end moments applied about the strong axis containing initial out-of-plane displacements (U_o). Warping was assumed to be unrestrained at both ends, while the in-plane (V_o) and out-of-plane deflection together with the twist (Φ) were fixed at the two ends. The out-of-plane displacement is represented by a sine wave with its maximum amplitude at mid-span is equal to 0.5 inch. The residual stress at the flange tips (σ_f) was equal to the yield stress (σ_y). From assumption V of section 7.2 this means that the spread of yield begins as soon as any load is applied. The correlation between the two results is good, the differences between them being of the order of 7 per cent, despite the different type of solution procedure employed.

Lindner (1981) presented a numerical simulation of some Berlin tests on beam-columns. Table 7.3 contains 5 of the test specimens selected on the basis of an attempt to cover a reasonably wide range of different variables. Different values of the initial deflections as represented by a sine wave are listed in Table 7.3. The values of end eccentricity used to generate moments in accordance with Lindner's investigation are also listed. In each case calculated solutions were obtained by applying the axial load with small increments until collapse occured. Comparison of the ultimate loads of the Berlin tests with those obtained by the Author are listed in Table 7.4. Generally very good agreement has been achieved, in which the percentage errors range between -4 to 6 % and -3 to 8% with respect to test data and analytical solution respectively.

7.3.3 Biaxial Bending

Several comparisons have been made with the tests on I-sections subjected to biaxially eccentric compression conducted by Anslijn (1983). Table 7.5 presents the input data for four tests chosen for the purposes of comparison, whilst Table 7.6 compares the calculated ultimate loads with the experimental values. Differences of less than 5% were obtained in each case. The test displacements were measured at the tips of the two flanges for vertical deflections and the top and the bottom of the web for horizontal deflections, the average being taken for both the in-plane V and out-of-plane U displacements. A typical comparison between these average values and those calculated by the program is given in Figs 7.2 and 7.3 for test No. 75 and No. 48. This shows excellent agreement for both in-plane and out-of-plane deflections. The differences found in the latter stages is thought to be due to cross-sectional distortion, leading to changes in flange tip movement that occur near the collapse.

Many cases from the comprehensive parametric study of biaxially loaded I-section beam-columns presented by Matthey (1984) have been selected for examination. For each analysis axial load was applied up to a predetermined limit followed by end moment loading to failure. Thus the actual discrepancies between the two solution given in Table 7.7 are of course, smaller than the numerical values would suggest since they relate to only a part of the applied loading. Nonetheless agreement is considered excellent with the 3 examples selected representing very different sets of parameters.

7.4 Conclusions

The inelastic three-dimensional behaviour of beam-columns under different loading conditions has been studied. The presence of initial imperfections such as residual stresses and initial deflections together with the effects of strain hardening were incorporated in the analysis. Results from several previous theoretical and experimental studies were used to check the Author's formulation and computer program which is capable of dealing with such problems. The validity and the accuracy of both the approach and the computer program shows an excellent agreement with those problems previously investigated, thereby confirming that the analysis represents a powerful tool for the study of this class of problem.

Table 7.1 Problems Chosen for Study.

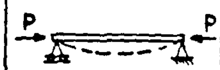
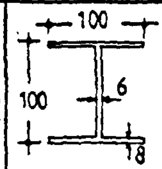
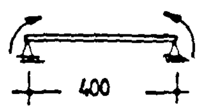
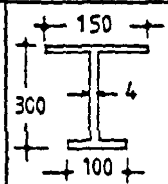
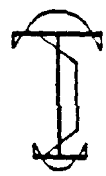
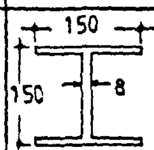
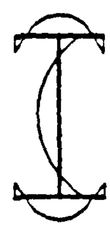
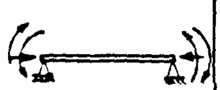
Reference	λ	loading Arrangement	Cross-Section	σ_y	σ_r	Initial Deflection		
						V/L	v /L	o
Fujita & Yashida (1972)	60			28.6 Kg/mm²	—	—	0.0067	—
						—	0.0112	—
						—	0.0152	—
						—	0.0215	—
Fukumoto & et al (1971)	103			3666 kN/mm²		0.00125	—	—
Lindner (1981)	19,28 42,52	The eccentric load applied with various eccentricit-ies ,Table 3 and Table 5		—	—	Several values of initial deflections are employed see Table 7.5 and Table 7.7		
Anslijn (1983)	60,90		HEA200					
Mathey (1984)	141, 47, 94		HEA200	235.0 N/mm²		L/1000	L/100	.001

Table 7.2 Comparison between Author's result and those obtained by Fujita and Yashida (1972) for compressive column with initial deflection

$\delta = \frac{V_o}{L}$	P/P_y			% error between	
	Fujita and Yoshida		Author	1 and 3	2 and 3
	Experimental 1	Theoretical 2	3		
0.0067	0.603	0.607	0.616	1.6	2.1
0.0112	0.507	0.525	0.517	1.9	1.6
0.0157	0.442	0.432	0.432	2.3	3.7
0.0215	0.367	0.362	0.366	0	1.0

Table 7.3 Data for problems chosen for purpose of comparison with Lindner (1981)

Test No.	Yield stress (kN/cm^2)	Initial crockedness			Eccentricities (cm)	
		U_o (cm)	V_o (cm)	ϕ_o "rad"	Top	Bottom
					e_{z1}	e_{z2}
1-3-13	65.0	0.030	0.041	0	-14.0	-9.5
2-1-11	65.0	-0.004	0.0035	0.003	2.5	2.5
3-1-33	69.0	-0.046	0.003	-0.006	-2.0	+ 2.0
3-2-21	68.0	0.042	0.010	-0.003	+6.5	0.0
5-3-22	68.0	-0.089	0.049	0.003	1.0	15.0

Table 7.4 Comparison of selected tests and analysis conducted by Lindner (1981) against those obtained by Author

Test No	$\frac{L}{\Upsilon_y}$	Collapse load "kN"			% error cal.	
		Author	Test	Lindner	Test	Lindner
1-3-13	19	1124.0	1209.0	1178.0	7	4.5
2-1-11	28	2524.0	2492.0	2461.0	1	2.5
3-1-33	42	2322.0	2469.0	2254.0	6	-3
3-2-21	42	1715.0	1644.0	1860.0	-4	8
5-3-22	52	1120.8	1128.0	1170.0	1	4

Table 7.5 Data for Problems Chosen for Comparison, Anslijn(1981)

Test No. (8)	Yield Stress [N/mm^2]	Initial deflections (m)			Residual Stresses [N/mm^2]			Eccentricities [mm]				Moment at end (Applied)	
		U_o 'cm'	V_o 'cm'	$\phi_o \times 10^{-3}$ 'rad'	Tip Flange	Flange web	Web	Top		Bottom		M_y	M_z
								e_y	e_z	e_y	e_z		
22	278.0	0.7	0.5	-1.11	100.0	-55.5	62.1	21.63	-1.24	21.63	1.24		
24	249.0	2.55	1.5	.50	101.0	-55.5	62.1	3.61	7.44	3.605	-7.44		
48	249.0	0.8	0.9	1.11	100.0	-55.5	62.1	7.21	0	0	-2.48		
75	223.0	1.5	1.0	1.11	100.0	-55.5	62.1	21.63	7.44	-21.4	7.44		

Table 7.6 Comparison of Author's Result with Experimental Data of Anslijn (1981)

Test No.	$\frac{L}{r_y}$	Collapse Load		% difference
		PT/Ps	Pc/Ps	
22	60	0.229	0.243	5.48
24	60	0.409	0.428	4.44
118	60	0.516	0.526	1.84
75	96	0.220	0.224	1.82

Table 7.7 Comparison of Author's Result with Solution of Matthey (1984)

σ_y = 235.0 N/mm^2 $U_o = V_o = \frac{\ell}{1000}$ & ϕ_o = .001 red, Residual stress Parabolic distribution								
Case No.	$\frac{\ell}{r_y}$	Bending moments		$m_y = \frac{M_y}{M_{py}}$		$m_z = \frac{M_z}{M_{pz}}$		% age differences
		M_y	M_z	Matthey	Authors	Matthey	Calculated	
11-324	141	X=0  X=L		0.343	0.351	0.479	0.490	2.2
14-324	47	 .5M_y		0.441	0.466	0.618	0.653	5.4
36-242	94			0.103	0.110	0.481	0.514	6.4

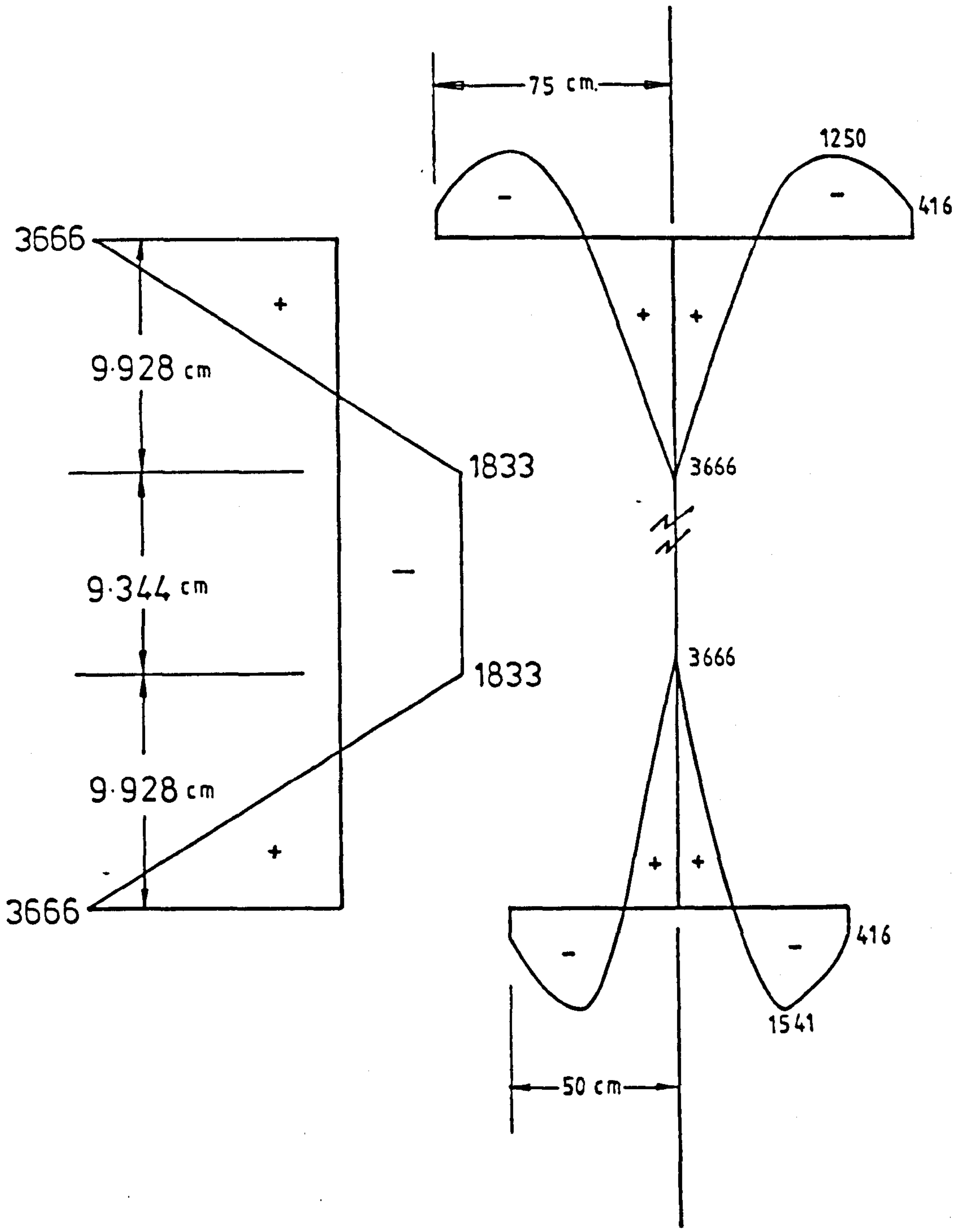


FIG. 71 ADJUSTED RESIDUAL STRESS DISTRIBUTION PRESENTED BY FUKUMOTO ET AL (1972)

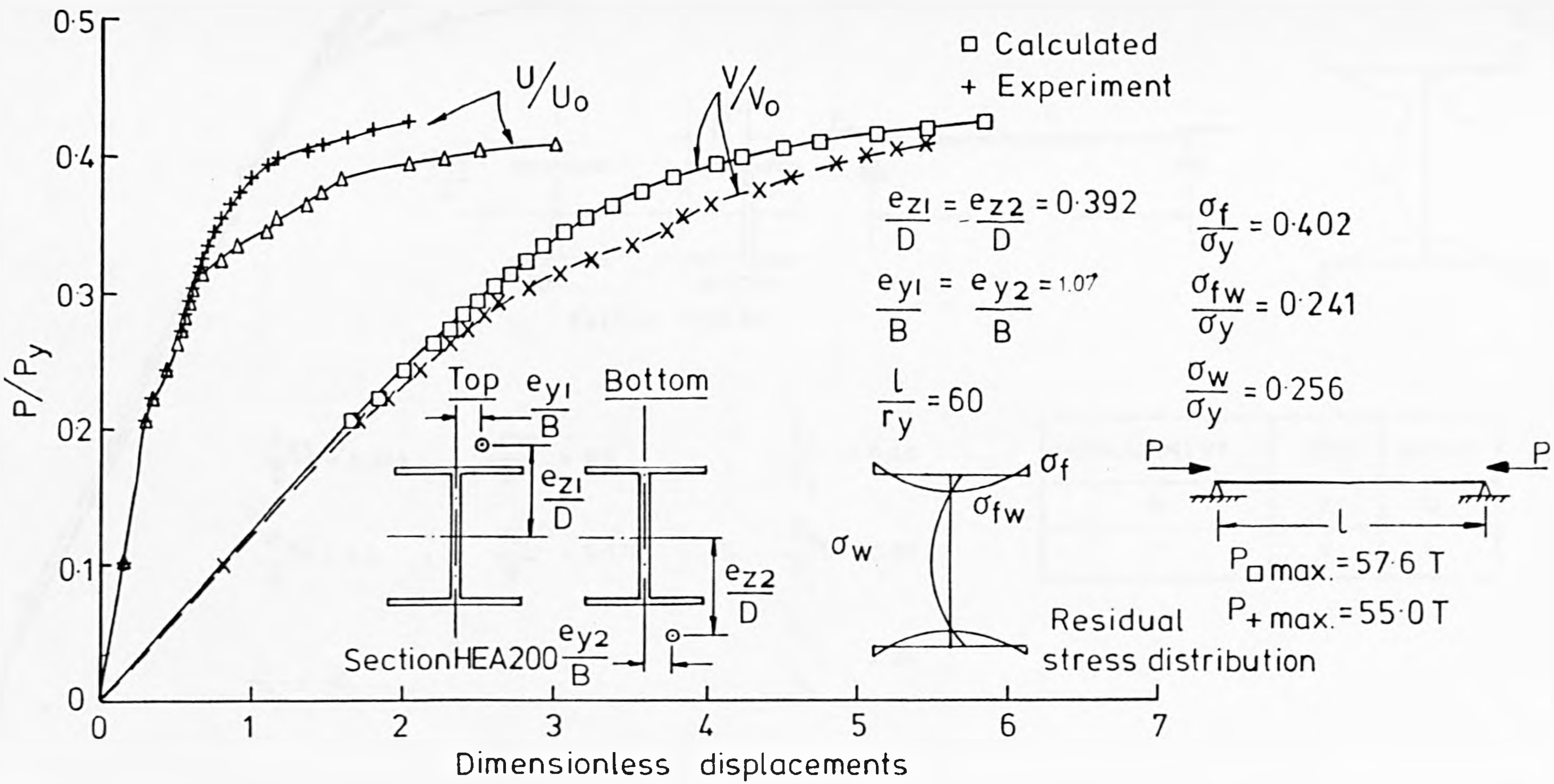


FIG.7.2 COMPARISON BETWEEN AUTHOR'S RESULTS AND EXPERIMENTS (ANSLIJN, 1981) ON BEAM-COLUMN UNDER BIAXIAL BENDING

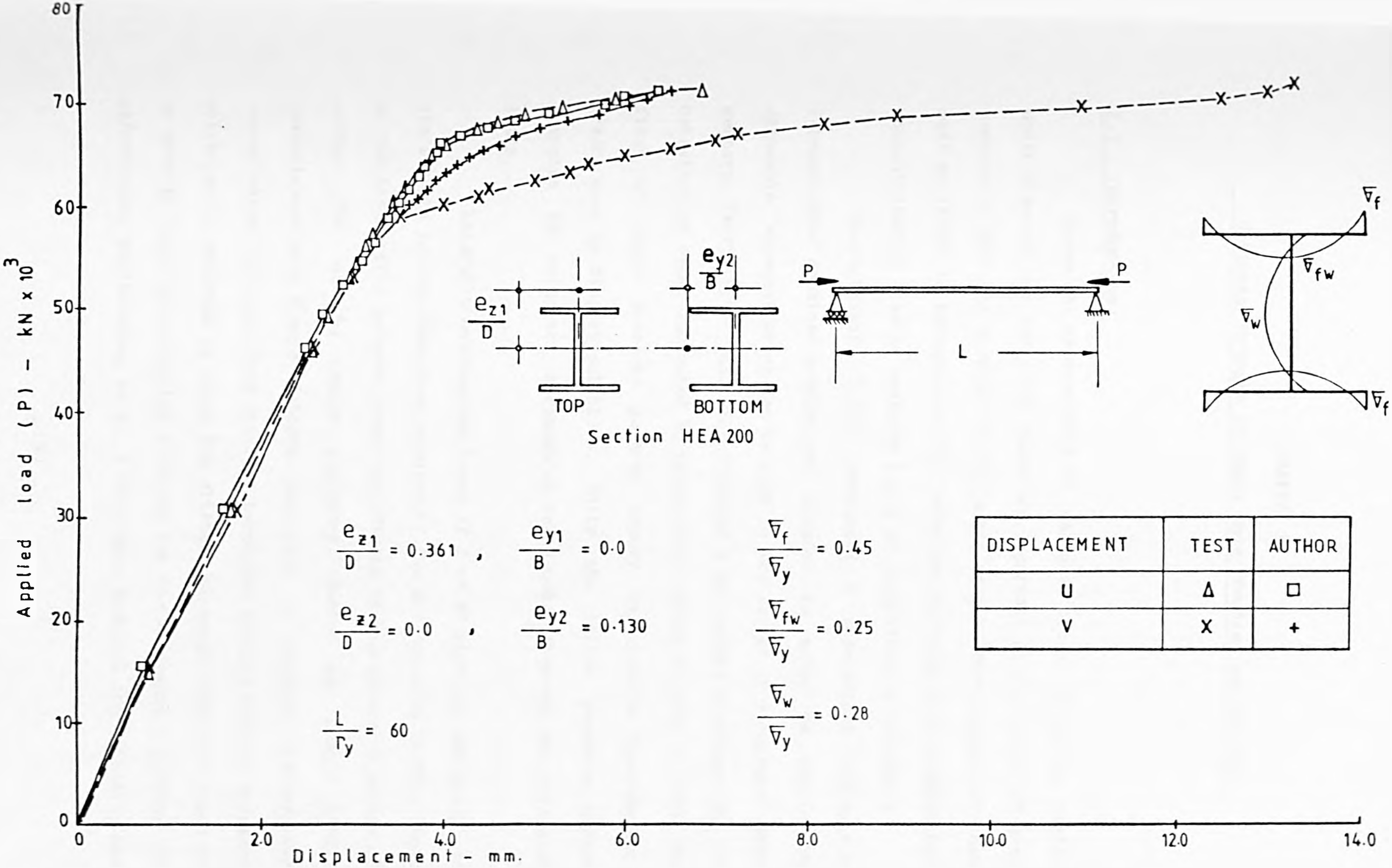


FIG. 7.3 COMPARSION OF AUTHOR'S RESULTS WITH ANSLIJN (1981) TESTS, IN-PLANE & OUT-OF-PLANE DISPLACEMENT, TEST 48

CHAPTER 8

Ultimate Strength of Beams under Bending and Torsion

8.1 Introduction

Beams and beam-columns of various types of cross section under flexural and torsional loads are important in the design of steel members in both the elastic and inelastic ranges. Many researchers have made an effort to investigate this behaviour although most studies have concentrated on elastic analysis based on simplified calculations.

Sourochnikoff (1951) presented a theoretical analysis of I-beams under combined bending and torsion in order to obtain the allowable stresses which can be used in the design of structural steel members. Terrington (1968, 1970) provided a two-volume treatise on the behaviour of beams subjected to transverse loading causing torsion. The first of these presents general theory and formulae together with graphs used to display solutions, while the latter presents worked examples to calculate stresses in beams under flexural and torsional loads.

Laterally unsupported beams of I or WF sections subjected to bending and torsion have been studied by Chu and Johnston (1974). Based on the interaction between these two effects they presented a series of curves for use by design engineers. Pastor and DeWolf (1979) investigated wide flange I-beams subjected to uniform flexure and concentrated torsion. They provided tabulated results for the stresses which can be employed to check the ultimate strength behaviour based on an elastic limit condition but allowing for the interactive effects of deformation. Kollbrunner, et al. (1979) have examined theoretically and experimentally the elastic-plastic behaviour of thin-walled fixed ended I-beams under bending and torsion.

The design of beams having W-shapes under combined bending and torsion has been investgated by Johnston (1982). He presented charts to permit rapid design checks for a variety of load types and end conditions. These graphs were based on solutions tabulated by Heins and Seaburg (1963). Moore (1986) studied the stability of Z-sections subjected to bending and torsion. All the above studies were in the elastic range. Not much work has been done towards bending and torsion in the inelastic range.

The aim of this study is to provide an analysis in the elastic and inelastic ranges for beams of thin walled open cross section subjected to biaxial bending together with concentrated torsion applied at mid span, and also to use the results in a preliminary study of possible simplified interactive approaches to design.

8.2 Analysis

The formulation presented in chapter 3 and used as the basis for the inelastic ultimate strength analysis procedure described in chapters 4 and 5 has been applied to a series of problems involving various combinations of compression, bending about the principal axes and direct torsional loading. Problems of this type, involving the interaction of flexural-torsional buckling effects and applied torsion in the inelastic range, have not, to the author's's knowledge, previously been studied. In making these studies some attention has been given to the different possibilities - in terms of degree of nonlinearity included - presented by the formulation of chapter 3.

1- Linear analysis

In this analysis the usual linear stiffness matrices (tangential $[K_t^L]$ and geometrical $[K_G^L]$) and the second order strain-displacement relations ($\Sigma = W_{,x} - U_{,xx}Z - V_{,xx}Y + \Phi_{,xx}\bar{\omega}_n$) are used.

2- Nonlinear Analysis

The higher order terms of axial strain ($\frac{1}{2}(U^2_{,x}+V^2_{,x}+\rho_o^2\Phi^2_{,x})$, and the bending strains about the Y and Z axes ($U_{,xx}\Phi - V_{,xx}\frac{\Phi^2}{2}$ and $V_{,xx}\Phi + U_{,xx}\frac{\Phi^2}{2}$) are added to the second order strain as:

$$\Sigma = W_{,x}+\frac{1}{2}(U^2_{,x}+V^2_{,x}+\rho\Phi^2_{,x})$$

$$-(U_{,xx}+V_{,xx}\Phi-U_{,xx}\frac{\Phi}{2})Z - (V_{,xx}+U_{,xx}\Phi+V_{,xx}\frac{\Phi}{2})Y - \Phi_{,xx}\bar{\omega}_n$$

and the same stiffness matrices stated in 1 are employed.

3- Full nonlinear analysis

Both the nonlinear strains described in 2 and the nonlinear geometrical $[K_G^{NL}]$ and nonlinear tangential $[K_t^{NL}]$ stiffness matrices are employed. Further details are given in Chapter 4.

The differences between these three types of analyses are demonstrated by several examples for both the elastic and the inelastic ranges.

8.3 Numerical Results

Numerical results for a simply supported beam of W12x14 section subjected to biaxial bending and torsion in the form of equal or unequal end moments about the strong axis together with a concentrated torque at midspan are presented. The boundary conditions are $U = V = \Phi'' = 0$ at both ends with warping permitted. The effects of residual stress and initial deflections are not considered in the elastic analysis, although these imperfections are considered in the inelastic cases.

Several examples are presented in this study, including some comparisons with results obtained previously by other workers.

8.3.1 Comparison with Previous Analysis

Pastor and DeWolf (1979) presented an elastic analysis for I-beams under equal end moments and constant torsion. Based on a small deflection theory, so that the terms other than those of first order were neglected, they employed the following differential equations for a beam under bending and torsion:

$$EI_y V'' = M_y - M_x U'$$

$$EI_z U'' = M_y \Phi - M_x V'$$

$$GK\Phi' - EI_\omega \Phi'' = M_x - M_y U'$$

in which E is the modulus of elasticity, I_y and I_z are the moments of inertia about the major and minor axes. G is the elastic shear modulus, K is the torsion constant, (GK is the torsional rigidity) and $I_\omega = I_z \frac{H^2}{4}$.

They provided general solutions (complementary and particular) to the above equations in terms of Φ. Their study was directed at evaluating the bending (σ_y and σ_z) and warping (σ_ω) stresses and the torsional rotations at mid-span. They considered several slenderness ratios with the torsional moment M_t equal to $\frac{M_{cr}}{100}$ fixed whilst the end moments were varied in steps of $M_p/17$. Tabulated results for stresses and rotations were provided. By imposing some limitations on the present approach such as neglecting the higher order terms in the strain-displacement relations ($U_{,xx}\Phi$, $V_{,xx}\Phi$, etc.) and the coupling between the displacements which are presented in the nonlinear tangential and geometrical stiffness matrices (chapter 4) it is possible to match the basis of this approximate solution. An example is chosen for the purpose of comparison so as to check the validity of the present formulation. In this comparison the load increment was taken similar to those used by Pastor and DeWolf. At each load step the convergence to the correct result is based on the out-of-balance force and displacement increments respectively. Further details of this method were given in Chapter 5.

Table 8.1 summarizes the comparison between the Author's results and those of Pastor and DeWolf for the bending stresses σ_y and σ_z about the Y and Z axes respectively, and the warping stresses σ_ω at midspan. It is clear that in the early stages of loading the differences between the stresses are small, being between 2 and 6 per cent. As soon as the applied load approaches the failure load, however, differences between the two sets of results increase, reaching 20% in some instances. Table 8.1 presents a comparison between the total stresses and the rotations in which the percentage differences are not more than 13.7 and 10 per cent for total stress (σ_t) and rotations at mid-span respectively. Fig. 8.1 presents the comparison of the three stress components in graphical form, whilst Fig. 8.2 shows the variation of the total peak stress and twist.

It is believed that the differences between the two sets of results at high load levels are attributable to the choice of unsuitable approximations for the degree of twisting by Pastor and

DeWolf. The author's solution makes no approximations and places no constraints on the deflected form of the member.

8.3.2 Elastic Analysis

This section presents some quantitative indication of the variations obtained by adopting the three types of analysis stated in section 8.2. Several examples have been considered in which the biaxial flexural loading is assumed to act in conjunction with a constant applied mid-span torsional moment (M_t). The load path assumed is full torsional load (up to the selected value) followed by flexural loading to failure. Results have been obtained for a simply supported beam-column of W12*14 section, free to warp at both ends with or without an initial lack of straightness $U_o=\frac{L}{1000}$ at mid-span.

8.3.2.1 Biaxial Bending and Torsion

A simply supported I-beam subjected to an axial load applied with eccentricities at both ends $\frac{e_{y1}}{B} = \frac{e_{y2}}{B} = 0.025$ and $\frac{e_{z1}}{D} = \frac{e_{z2}}{D} = 0.25$ has been considered. Fig. 8.3 and Table 8.2 summarise the findings for the full range of $\frac{M_t}{M_y}$ values in the form of interactive combinations of P and M_t at failure. As the problem approaches the case of pure torsional loading (P—> 0) so the differences between the results obtained from the three approaches become less. In this region the degree of coupling between bending and torsional action is smaller with the result that those contributions to the stiffness matrices that depend on the product of derivatives of displacements and rotations become less significant. Conversely geometrical nonlinearity is of more significance when there is greater scope for the various interactive effects produced by the compressive load acting through the various deformations.

Figs. 8.4a and 8.4b present a load-rotation and load-deflection response for the case in which M_t is zero. The maximum load obtained from the linear analysis overestimates that given by the full nonlinear analysis by 25 per cent, whilst partial nonlinear analysis overestimates the failure load by 15 per cent as compared with the full nonlinear analysis. Load-rotation responses for $\frac{M_t}{M_p}$=0.035 and 0.069 are shown in Figs. 8.5a and 8.5b. In the first case the failure load given by the linear analysis exceeds those of the partial and full nonlinear analyses by 10% and 17% respectively.

Load rotation curves and load deflection responses for the case of $\frac{M_t}{M_p}$ kept constant and equal to 0.138, while the axial load is applied with equal eccentricities ($\frac{e_{y1}}{B}=\frac{e_{y2}}{B}=0.125$ and $\frac{e_{z1}}{D}=\frac{e_{z2}}{D}=0.257$) at both ends for the three types of analyses (linear, nonlinear analysis, and full nonlinear) are presented in Fig. 8.6. In each type of analysis the eccentric load is increased gradually by a percentage of the squash load ($\bar{P}_y$). Its increment is made smaller as soon as the deflections rapidly increased, sometimes reaching $\frac{\bar{P}_y}{1000}$ or less near failure.

Fig. 8.6a represents load twisting curves in non-dimensional form as $P/\bar{P}_y$ vs Φ/Φ_{max}, in which $\bar{P}_y$ and Φ_{max} are 203 KIP and 38.8° respectively. The load-deflection response is displayed in Fig. 8.6b. which describes the interaction of $P/\bar{P}_y$ and V/V_{max} where V_{max} is the maximum in-plane displacement at failure (V_{max}=22.6 mm). In both Figs the torsional moment is kept unchanged ($\frac{M_t}{M_p}$=0.138) followed by increasing the axial load until collapse. It is clear from Figs 8.6a and 8.6b at the early stages of loading, that the differences between the linear, nonlinear, and full nonlinear analysis are negligible. As the eccentric load is increased the variations of the three analyses can be easily recognised. These discrepancies in the results occured due to taking into account the nonlinear strain-displacement relations with the linear stiffnesses in the nonlinear analysis and also due to the coupling between the displacements and rotations in the nonlinear tangential and geometrical stiffness matrices together with the nonlinear strains in the full nonlinear analysis. The ratios between the maximum load (P_{max}) to the yield load ($\overline{P}_y = A \times \overline{\sigma}_y$) for the linear, nonlinear, and full nonlinear analysis at failure are 0.0395, 0.0347, and 0.033 respectively.

In each load increment the equilibrium condition of the convergence criteria must be satisfied, in other words the square root of the change of the displacement (ΔR) and out-of-balance force (ΔP) should be less than the tolerance error which was chosen equal to 0.001; this was explained in detail in Chapter 5.

8.3.3 Inelastic Analysis

In the inelastic range, the critical loads decrease from the elastic values because the stiffnesses of the member are reduced by spread of yield. Several illustrative examples with different slendernesses covering bending and torsion and compression have been considered.

8.3.3.1 Torsional moment (M_t) applied at mid-span

A simply supported beam having a length given by the slenderness ratio $\overline{\lambda}_1 = \sqrt{M_p / M_{cr}} = .4$ and initial imperfections under torsional moment applied at mid-span is investigated. The imperfections are out-of-plane displacement (U_o) and residual stress. The lack of straightness is represented by a sine wave and its maximum amplitude is equal to L/1000 at mid-span. The residual stress (σ_r) distribution is given in Fig. 8.7, in which its values at the flange tips (σ_f), flange-web junctionsns (σ_{fw}), and along the web (σ_w) take the value $0.3\sigma_y$, where σ_y is the yield stress.

This example contains two cases; in the first the effect of the residual stress is incorporated, in the second it is ignored. The two cases are solved by linear analysis, in which the influence of both the higher order terms of the strain-displacement relations and the nonlinear stiffness matrices (tangential and geometrical) are not accounted. In both cases the torsional moment (M_t) is increased monotonically until failure. The results, in the form of a moment-twist relation, are given in Fig. 8.7. It shows in the early stages of loading (elastic range) that both approaches give very similar results. But as soon as yield occurs the differences become noticeable as seen in the figure. Fig. 8.8 presents spread of yield at the initiation, part way to failure and at the final stages. Use of the more rigorous approach causes a reduction in the load at which yield starts of 25 per cent and a decrease in the collapse load of 9 per cent.

8.3.3.2 **Bending and Torsion**

Three values of non-dimensional slenderness $\sqrt{\frac{M_p}{M_{cr}}}$ given by 0.4, 0.7, and 1.2 are considered herein. In each case the applied loads are equal end moments followed by torsional moment (M_t) at mid-span vary from zero up to failure.

The applied end moments are equal ($M_{y1}=M_{y2}$) and are set; at the same time for every value the torsional moment M_t is varied from zero to the collapse load. Results obtained by linear analysis are provided in Fig. 8.9, which summarizes the interaction between the ultimate torsional and bending moments in the form of $\frac{M_t}{M_p}$ vs $\frac{M_y}{M_y}$. These curves illustrate the real response of the structural member subjected to this type of loading. Fig 8.9 shows that in each example the values of the ultimate torsional moment depends on the values of the applied end moments. Its value decreases when the end moments are increasing because of early yield in the cross-section

8.3.3.3 **Nonuniform Bending and Torsion**

A beam-column of I-section (W12x14) subjected to unequal end moments and torsion (M_t) at mid-span has been investigated. The beam length is give by slenderness ratio ($\bar{\lambda}_3$) equal to 1.2. The value of M_{y2} which represents the moment at end 2 is kept equal to zero, while M_{y1} adopts a series of values. For every value of M_{y1} the torsional moment is increased from zero to maximum (failure) load. A full set of results is presented in Fig. 8.10.

The results of these analyses are compared with the equivalent set from the previous study using equal end moments in the form of interaction curves of M_y and M_t in Fig. 8.10. It shows that as bending increases the differences between the ultimate torsional moments is decreased. Traces of the spread of yield through the cross section for both cases are presented in Figs 8.11 for unequal end moments and 8.12 for equal end moments.

Figs 8.11a, 8.11b, and 8.11c show the yield spread for 3 different values of end moments at 3 different levels of applied torque representing first yield, an intermediate stage and failure. The ultimate torsional moments are greater than those at first yield by 7, 9, and 9.2 per cent.

Figs. 8.12a, 8.12b, and 8.12c present the yield spread for the case of equal end moments and torsion applied at mid-span. The values of the torsional moments at first yield are less than those at collapse by 8, 8, and 6.4 per cent. The differences between the torsional moments at first yield for the case of applied equal and unequal end moments are 63, 49, and 21 per cent, and at the ultimate load are 63, 49, and 22 per cent.

8.3.3.4 **Biaxial Bending and Torsion**

Several examples of the response of a simply supported I-Beam (W12x14) subjected to biaxial bending and torsion are presented herein. The boundary conditions employed are; end displacements (in-plane and out-of-plane) restrained, together with twist prevented at the two ends, while warping is permitted. The influence of residual stresses, assuming a Lehigh distribution, is considered in the case of torsional moment only.

The linear and nonlinear analysis of a beam-column subjected to eccentric compressive loads together with a torsional moment at midspan are investigated for two illustrative examples. In both the torsional moment is kept constant, while the eccentric axial load is varied from zero up to collapse. The compressive load is applied with eccentricities $\frac{e_{y1}}{B}=0.125$ / $\frac{e_{z1}}{D}=0.686$ at node A and $\frac{e_{y2}}{B}=0.0$ / $\frac{e_{z2}}{d}=0.686$ at node B with the torsional moment being held at 3.0 Kip.in for the first example, while for the second case the axial load has eccentricities $\frac{e_{y1}}{B}= 0.063$ / $\frac{e_{z1}}{D}=0.172$ and $\frac{e_{y2}}{B}$ 0.063 / $\frac{e_{z2}}{D}=0.127$ at nodes A and B respectively and $M_t = 9.0$ Kip.in. The slenderness ratio ($\frac{L}{r_y}$) for both cases is equal to 25.

Figures 8.13a and 8.13b give load-rotation curves and load-deflection (in-plane and out-of-plane) curves for the results obtained from linear and nonlinear analysis of the first example. The loss of stiffnesses due to yield spread is demonstrated in Fig. 8.14 The differences in the first yield loads for the two analyses is 6 per cent. The value of $\frac{P}{P_y}$ at failure is 0.098 for linear analysis and 0.090 for the second solution; thus the simpler linear analysis overestimates the failure load by some 9 per cent.

The load-deflection responses for the second example are given in Fig. 8.15. Loss of stiffness due to spread of yield is presented in Fig. 8.16. First yield occured at loads corresponding to $\frac{P}{P_y}$=0.078 and $\frac{P}{P_y}$=0.08 respectively, while the difference at failure is 6 per cent.

8.4 Examination of the Failure State of Members under Bending and Torsion

For a simply supported I-beam subjected to a twisting moment (M_t) at mid-span the total twisting moment is the sum of the St. Venant torsion (T_{sv}) and warping torsion (T_w); the torque diagram is given in Fig. 8.17 and the overall differential equation describing the twisting is:

$$M_t = T_{sv} + T_w \tag{8.1}$$

where

$$T_w = GK\Phi' \quad , \quad T_{sv} = -EI_\omega \Phi''' \tag{8.2}$$

so equation 8.1 can be written as

$$M_t = GK\Phi' - EI_\omega \Phi''' \tag{8.3}$$

equation 8.3 can be written in another form as

$$A = \lambda^2\Phi' - \Phi''' \tag{8.4}$$

where

$$A=\frac{M_t}{EI_\omega} \quad \text{and} \quad \lambda^2=\frac{GK}{EI_\omega}$$

Assume the general solution of equation 8.4 is

$$\Phi = A_1 + A_2 \text{SINH}\lambda X + A_3 \text{COSH}\lambda X + \frac{A}{\lambda^2}X \tag{8.5}$$

where A_1, A_2, and A_3 are constants of integration; these constants are evaluated to comply with the torsional boundary conditions. X is the distance along the longitudinal axis. The boundary conditions for a simply supported I-beam used herein are stated below:

$$\Phi\Big|_{X=0,1} = 0 \tag{8.6a}$$

$$\Phi'\Big|_{X=\frac{1}{2}} = 0 \tag{8.6b}$$

Substituting equations 8.6a and 8.6b in equation 8.5 and making some manipulation yields

$$\Phi = \frac{M_t}{EI_\omega\lambda^3}\left[-\frac{1}{\text{COSH}\frac{\lambda}{21}}\text{SINH}\lambda X + \lambda X\right] \tag{8.7}$$

$$\Phi'' = \frac{M_t}{EI_\omega\lambda}\left[-\frac{1}{\text{COSH}\frac{\lambda}{21}}\text{SINH}\lambda X\right] \tag{8.8}$$

The equations presented by Matthey (1984) and Chen and Atsuta (1977) for calculating the first yield moment and plastic moment about the Y and Z axes are employed as:

a) At first yield

$$M_{yy} = \frac{I_y}{\frac{D}{2}}\sigma_y = \sigma_y \cdot Z_z$$

$$M_{yz} = \frac{I_z}{\frac{B}{2}}\sigma_y = \sigma_y \cdot Z_y$$

where Z_y and Z_z are the section modulii about the Y and Z axes

b) At full plasticity

$$M_{py} = S_y \cdot \sigma_y$$

$$M_{pz} = S_z \cdot \sigma_y$$

where

$$S_y = [BT(h-t) + T_w(\frac{h}{2}-T)^2]$$

$$S_z = [B\frac{T_2}{2} + 1/4T_w^2(h-2T)]$$

For Section WF12x14

$S_y = 17.2285in^3$ $I_y = 87.177in^4$ $K = 0.0629in^4$

$S_z = 1.95415in^3$ $I_z = 2.4531n^4$ $r_y = 4.5728in$

$\sigma_y = 50KSI$ $A = 4.169in^2$ $r_z = 0.7910in$

M_{py} = 861.425kip.in $\qquad M_{yy}$ = 734.122kip.in

M_{pz} = 97.7075kip.in $\qquad M_{yz}$ = 62.325kip.in

8.4.1 Stability of I-Section under Flexural and Torsional Loading

based on a simple design check; Pastor and DeWolf (1979) suggested that the summation of the stresses caused by flexural and torsional loads on an I-section (W12x14) be limited to the yield stress (σ_y). This can be illustrated as:

$$\sigma_{by} + \sigma_{bz} + \sigma_\omega \ngtr \sigma_y \tag{8.9}$$

in which σ_{by} and σ_{bz} are the bending stresses about the Y and Z axes, and σ_ω is the warping stress. If we substitute for bending stress by

$$\sigma_{by} = \frac{M_y}{I_y}Z = \frac{M_y}{Z_y} \tag{8.10a}$$

$$\sigma_{bz} = \frac{M_y}{I_z}Y = \frac{M_z}{Z_z} \tag{8.10b}$$

Substituting equations 8.10a and 8.10b into equation 8.9 yields

$$\frac{M_y}{Z_y\sigma_y} + \frac{M_z}{Z_z\sigma_y} + \frac{\sigma_\omega}{\sigma_y} \ngtr 1 \tag{8.11}$$

or

$$\frac{M_y}{M_{yy}} + \frac{M_z}{M_{yz}} + \frac{\sigma_\omega}{\sigma_y} \ngtr 1 \tag{8.12}$$

where

$$M_{yy} = Z_z.\sigma_y \quad \text{and} \quad M_{yz} = Z_z.\sigma_y$$

in which M_{yy} and M_{yz} are the first yield moments about the Y and Z axes.

The value of M_z is taken equal to the twist times the applied end moments in the form of $M_z=\Phi M_y$, where Φ is the twist calculated by equation 8.7. The warping stress (σ_ω) in terms of the normalized function ($\bar{\omega}_n$), and the second derivative of Φ with respect to the X-axis (Φ'') is given by equation 8.8 and it can be written as

$$\sigma_\omega = E\bar{\omega}_n\Phi'' \tag{8.13}$$

Substitute for σ_ω and M_z in equation 8.12, which yields

$$\frac{M_y}{M_{yy}} + \frac{M_z}{M_{yz}} + \frac{E\bar{\omega}_n\Phi''}{\sigma_y} > 1 \tag{8.14}$$

An investigation was conducted of three cases which represent simply supported beams. Their end displacements (U and V) together with the twist (Φ) are restrained, whilst the warping is permitted at the two ends. The beam lengths are given by the slendernesses $\bar{\lambda}_1$, $\bar{\lambda}_2$, and $\bar{\lambda}_3$ equal to 0.4, 0.7, and 1.2 respectively.

Substituting for warping flexural rigidity (EI_ω), λ which is equal to square root of the torsional rigidity divided by warping flexural rigidity ($\lambda=\sqrt{GK/EI_\omega}$), and for the beam length (L) in equations 8.7 and 8.8, this yields:

Case A $\quad \bar{\lambda}_1=0.4$

$$\Phi = 0.0001799\ M_t \tag{8.15a}$$

$$\Phi'' = 0.275344 \times 10^{-5} M_t \tag{8.15b}$$

Case B $\quad \bar{\lambda}_1 = 0.7$

$$\Phi = 0.0010132\ M_t \tag{8.16a}$$

$$\Phi'' = 0.480 \times 10^{-6} M_t \tag{8.16b}$$

Case C $\quad \bar{\lambda}_1 = 1.2$

$$\Phi = 0.0046412\ M_t \tag{8.17a}$$

$$\Phi'' = 0.744934 \times 10^{-5} M_t \tag{8.17b}$$

8.4.1.1 Stability of Beam-Column of I-Section under Bending and Torsion Based on Pastor and DeWolf (1979) Suggestions

Stresses caused by flexural and torsional loads based on the Author's full finite element (f.e.) analysis and the use of the equations 8.15 to 8.17 to calculate the interaction stresses are presented in Tables 8.3, 8.4, and 8.5 for each load combination in each case. The end moments are applied first incrementally to a set limit followed by incrementing the torsional moment (M_t) to failure. For zero torsional moment the end moments are simply increased to collapse and a similar procedure is used when the uniform moments are zero. The interaction between torsion and bending at the ultimate condition has been discussed in full in section 8.3.3.2. Equation 8.14 is used to calculate the stresses at first yield. The ultimate values of the bending moments about the strong axis and the corresponding torsional moment at mid-span, the maximum twist at mid-span obtained from the analysis and from equation 8.7 and 8.8 are presented in Tables 8.3, 8.4, and 8.5 for case A, case B, and case C respectively.

Table 8.3 contains the results for a stocky beam-column of slenderness $\bar{\lambda}_1$. The summation of stresses calculated by equation 8.14 are less than 1 for $\frac{M_y}{M_p}>0.35$ and the corresponding $\frac{M_t}{M_{yw}}>0.043$. When $\frac{M_y}{M_p}<0.35$ the summmation of $\frac{M_y}{M_{yy}}$, $\frac{M_z}{M_{yz}}$, and $\frac{\sigma_\omega}{\sigma_y}$ are greater than 1. So according to the assumptions of Pastor and DeWolf (1979) the results are not safe. Table 8.4 presents the interaction of case B in which the summation of the stresses are less than 1, except when the $\frac{M_t}{M_{yw}}$ and $\frac{M_y}{M_p}$ are equal to 0.070/ .070 and 0.115/ 0.172 respectively. So the last two examples are not safe according to Pastor and DeWolf. For the slender member the summation of the interaction of stresses is fully presented in Table 8.5, in which A_{fy} is less than 1 so the examples investgated in this case are considered to be safe.

Comparison of the results provided in Tables 8.3 - 8.5 shows that the assumption made by Pastor and DeWolf are applicable for some problems and not good for others. For a slender member the application of their assumptions is in good agreement, but for both the intermediate and more stocky members, it seems the results of some examples are not acceptable. In the interaction equation employed by Pastor and DeWolf, the torsional moment is not incorporated and instead they used warping stress. The inclusion of σ_ω in their analysis it seems to be not applicable in the inelastic range because the warping stress increases rapidly in the yielded regions, so the interaction equation which used the ratio of σ_ω / σ_y must be less than one is not valid. In the next section an attempt to incorporate this effect in another more realistic form is made.

8.4.1.2 Stability of Beam-Column of Thin-Walled I-Section Subjected to Combined Bending and Torsion Based on Author's Suggestions

In this section equation 8.12 has been used with some modification for two separate methods as:

i- Based on Initial Yielding

ii- Based on Full Plasticity

i- Based on Initial Yielding

In this case the interaction equation which represent the summation of stresses calculated after first yield is given by

$$A_{fy} = \frac{M_y}{M_{yy}} + \frac{M_z}{M_{yz}} + \frac{M_t}{M_{y\omega}} \tag{8.18}$$

in which A_{fy} is the summation of the ratio of moments at initial yield. $M_{y\omega}$ is the first yield warping moment (Chen and Atsuta 1977) and is given by

$$M_{y\omega} = \frac{I_\omega}{\frac{BD}{4}} . \sigma_y \tag{8.19}$$

where I_ω is the warping moment of inertia, B and D are cross-section width and depth respectively.

ii- Based of Full Plasticity

The contribution to fully plastic interaction is given in the form:

$$A_{fp} = \frac{M_y}{M_{py}} + \frac{M_z}{M_{pz}} + \frac{M_t}{M_{p\omega}} \qquad (8.20)$$

A_{fp} is equal to the summations of M_y/ M_{py}, M_z/ M_{pz}, and $M_t/ M_{p\omega}$ at full plasticity. M_{py}, M_{pz}, and $M_{p\omega}$ are the plastic moments about Y and Z axes are calculated using Matthey (1984) equations and plastic warping torque. The last term ($M_{p\omega}$) is that torque required to cause full plastification of flanges due to tensile or compressive stresses. These stresses are illustrated in Fig. 8.17 previously presented by (Salmon and Johnson 1980), in which the warping moment can be written as:

$$M_{p\omega} = \frac{B^2}{L} T_f (D-T_f) \sigma_y = \frac{2185}{L}(W12x14) \qquad (8.21)$$

where T_f is the flange thickness. The warping moments calculated by the above formula for the three cases described in section 8.4.1 ($\overline{\lambda}_1$=0.4, $\overline{\lambda}_2$=0.7, and $\overline{\lambda}_2$=1.2) are 43.2, 24.8, and 19.2 kip.in respectively.

Tables 8.6, 8.7, and 8.8 contains the results of the whole series based on the use of equations 8.18 and 8.20 for the first yield and full plasticity respectively. The interaction of stresses A_{fy} and A_{py} are fully listed. It shows that the summation of warping and bending stress according to the first yield criterion or the plastic criterion are less than 1. The form of interaction gives a safe prediction of failure. Tables 8.6, 8.7, and 8.8 show that A_{fy} is increased by increasing the uniform end moments.

The square root of the sum of the terms in the interaction formulae of equations 8.18 and 8.20 squared can be written as:

$$C_1 = \sqrt{(\frac{M_y}{M_{yy}})^2 + (\frac{M_z}{M_{yz}})^2 + (\frac{M_t}{M_{y\omega}})^2} \tag{8.22a}$$

$$C_2 = \sqrt{(\frac{M_y}{M_{py}})^2 + (\frac{M_z}{M_{pz}})^2 + (\frac{M_t}{M_{p\omega}})^2} \tag{8.22b}$$

The above two equations are based on initial and full plastic moments and are presented in a parabolic form. The reasons for incorporating them in this fashion, are first to investigate their interaction with the ratio of torsional moment with respect to either initial warpin moment or to full plastic warping moment ($\frac{M_t}{M_{yw}}$ and $\frac{M_t}{M_{pw}}$), and second to compare the results obtained against those of equations 8.18 and 8.20. Tabulated numerical calculations using equations 8.18, 8.20, 8.22a, and 8.22b involving series of slendernesses are fully presented in Table 8.9. It shows that for $\frac{L}{r_y}$=35 the values of C_1 and A_{fy} are increasing when the ultimate torsional moment decreases, and similarly for A_{fp}. The term C_2 is gradually decreasing when $\frac{M_t}{M_{pw}}$ is beteen 0.15 and 0.085, and then it starts increasing. The value of the two terms A_{fp} and C_2 are equal when $\frac{M_y}{M_{pw}}$=0/ $\frac{M_t}{M_{pw}}$=0.154 and $\frac{M_y}{M_{py}}$=0.836/ $\frac{M_y}{M_{py}}$=0, because twisting is a function of M_t and $M_z = \Phi M_y$. The value of Φ according to equation 8.15 is equal to zero when the applied torque is zero and M_z is equal to zero when the uniform end moments is zero. Similar behaviour occurd for the intermediate and the more slender member. The ratio of $\frac{C_2}{A_{fp}}$ ranges from 0.75 to 1.0, 0.7 to 1.0, and 64 to 1 for the three slendernesses ($\bar{\lambda}_1$, $\bar{\lambda}_2$, and $\bar{\lambda}_3$) respectively. Fig. 8.18 illustrates the interaction curves for three slendernesses $\bar{\lambda}_1$, $\bar{\lambda}_2$, and $\bar{\lambda}_3$ of beam-column subjected to flexural and torsional loads. The uniform end moments are applied about the strong axis and at the same time as the torsional moment at the mid-span. Details of load incrementation have been given in section 8.4.1.1. These curves indicate that it is possible to provide a practical prediction formula which can then be used to determine the ultimate loads. Further explanations will be provided in section 8.5.

8.4.1.3 Stability of Beam-Column under Biaxial Bending and Torsion

In this case selected values of the mid-span torsional moment (M_t) are applied followed by an axial load with eccentricities $\frac{e_{y1}}{D}$, $\frac{e_{y2}}{D}$=0.167 and $\frac{e_{z1}}{B}$, $\frac{e_{z2}}{B}$=0.257 about both ends. At every value of M_t the end load is increased up to failure. The interaction of the compressive force P and torsional moment M_t is given in Table 8.8 where it is compared with the results generated from the following equation.

$$AA = \frac{P}{P_{pl}} + \frac{M_t}{M_{\omega p}} + \frac{M_y}{M_{pyl}} + \frac{M_z}{M_{pzl}}$$

where $P_{pl} = \sigma_y.A$, and M_t, M_y, M_z, $M_{\omega p}$, M_{pyl}, and M_{pzl} are defined in section 8.4.2. The results of the above equation shows that the term AA is less than 1.

8.5 Prediction of Stability of Beam-Column under Flexural Bending and Torsion Using a Regression Analysis

The ultimate bending and torsion values presented in Table 8.9 and illustrated in the interaction curves of Fig. 8.19 have been used herein to develop a practical formula based on the results previously discussed concerning beams subjected to combined bending and torsion. Three equations are developed based on:

1- Interaction of M_t/M_{pw} and A_{fy}

2- Interaction of M_t/M_{pw} and C_1

3- Combination of 1 and 2

where A_{fy} and C_1 have been defined in equations 8.18 and 8.22a respectively.

The development of these equations are based on the application of regression techniques (Draper and Smith, 1961, Neville and Kennedy, 1964) A statistical package SPSS-X21, available at USCC (University of Sheffield Computer Centre), is employed to calculate different parameters such as, mean value, maximum/ minimum values, standard deviation, correlation, mean square, residual standard errors, etc. and then these variables are used in deriving regression equations. The equations conducted for the three slendernesses $\bar{\lambda}_1=0.4$, $\bar{\lambda}_2=0.7$, and $\bar{\lambda}_3=1.2$ are summarized in the following three cases:

Case i M_t/M_{pw} is function of A_{fy} and L

Case ii M_t/M_{pw} is function of C_1 and L

Case iii M_t/M_{pw} is function of A_{fy}, C_1, and L

in which $\frac{M_t}{M_{pw}}$, A_{fy}, C_1, and L can represent any values of moments and slendernesses. In other words the equations can be applied to any case previously explained.

The prediction equation of case i in which $\frac{M_t}{M_{pw}} = F(A_{fy}, L)$ can be written as:

$$\frac{M_t}{M_{pw}} = 0.167937 - 0.1217A_{fy} - 0.001037L \tag{8.23}$$

the other equation which based on case ii is

$$\frac{M_t}{M_{pw}} = 0.164857 - 0.119876C_1 - 0.001118L \qquad (8.24)$$

The last equation which employs $\frac{M_t}{M_{pw}}$ as a function of A_{fy}, C_1, L can be written as:

$$\frac{M_t}{M_{pw}} = 0.164835 - 0.083002C_1 - 0.03828A_{fy} - 0.001094L \qquad (8.25)$$

Equations 8.23 - 8.25 can be used to determine the ultimate moment for any set limit of torsion or vice versa. They can also be used for slendernesses between $0.4 \leqslant \bar{\lambda} \leqslant 1.2$. These equations can play an important part in the design of members having thin-walled open cross-section in the design of steel structures and leads the way for further work in this area.

8.6 General Features of the Analysis

Different cases of loading and slenderness have been addressed to provide some idea of the interaction of these variable on the failure condition. Before a fully validate ultimate strength approach to the design of slender members under any form of combined torsional/ flexural/ compressive loading can be produced a considerable amount of further work is required. This should cover:

1- Different slenderness ratio.

2- Different loading conditions.

3- Different boundary conditions.

4- Variable compressive load eccentricities.

5- Including initial imperfections (such as residual stress and initial deflections).

6 Interaction equation on the form

$$\frac{P}{P_{pl}} + \frac{M_t}{M_{\omega p}} + \frac{M_y}{M_{pyl}} + \frac{M_z}{M_{pzl}} \leqslant 1$$

NOTE An extension to some of this work is now in progress and the results are encouraging.

8.7 Conclusions

The elastic and inelastic 3-dimensional behaviour of beam-columns under bending and torsional loading has been studied. The presence of initial imperfections such as residual stresses and initial deflections together with monitoring of the yield spread through the cross-section have been considered.Several example problems, which have not been previously solved, have been investigated.

The solutions for these examples were obtained by three types of analysis; 'Linear', 'Nonlinear', and Full Nonlinear. The results show the importance of including of the so call 'higher order terms' of the strain displacement relations and the nonlinear geometrical and tangential stiffness matrices in the analysis. From these problems, several important observations are possible:

1- The ability of the formulation presented in chapter 3 to cover the full range of analysis of beam-columns in 3-dimensions.

2- The need to include higher order terms in the strain-displacement relations and stiffness matrices in order to avoid overestimates of load carrying capacity.

Table 8.1 Comparison of Author's result against those obtained by Pastor and de Wolf (1979), for a beam subjected to constant torsional moment and equal end moments, ($\bar{\sigma}_y$ = 50 ksi, w12x43)

$\frac{M}{M_{cr}}$	$\sigma_y/\bar{\sigma}_y$		% age difference	$\sigma_z/\bar{\sigma}_y$		% age error	$\sigma_w/\bar{\sigma}_y$		% age error	$\sigma_t/\bar{\sigma}_y$		% age error	Rotations "rad" 10^{-2}		% age error
	Author	Pastor		Author	Pastor		Author	Pastor		Author	Pastor		Author	Pastor	
0.0	0.0	0.0	–	0.0	0.0	–	0.284	0.282	0.7	0.284	0.282	0.7	2.95	2.96	0.3
0.1	0.025	0.026	3.8	0.067	0.068	1.5	0.292	0.284	2.7	0.386	0.378	2.1	3.04	3.00	1.3
0.2	0.050	0.052	3.8	0.134	0.136	1.5	0.300	0.290	3.3	0.484	0.478	1.2	3.12	3.09	1.0
0.3	0.078	0.082	4.8	0.201	0.204	1.5	0.313	0.304	3.0	0.592	0.588	0.7	3.29	3.25	1.2
0.4	0.118	0.120	1.7	0.271	0.270	0.4	0.335	0.322	4.0	0.718	0.712	0.8	3.57	3.51	1.7
0.5	0.156	0.166	6.0	0.345	0.338	2.1	0.369	0.354	4.2	0.870	0.858	1.4	4.01	3.93	2.0
0.6	0.218	0.232	0.6	0.430	0.406	6.0	0.426	0.402	6.0	1.076	1.040	3.3	4.70	4.57	2.8
0.7	0.346	0.338	2.4	0.552	0.474	14.1	0.547	0.486	12.5	1.444	1.298	10.1	5.91	5.71	3.4
0.8	0.580	0.542	7.0	0.707	0.554	21.6	0.749	0.656	14.2	2.016	1.74	13.7	8.98	8.00	10.1

Table 8.2 Results obtained for three analyses of beam-column under biaxial bending and torsional moment at mid-span

$\frac{M}{M_{cr}}$ 10^{-2}	$P/\bar{P}_y \times 10^{-2}$ Type of analysis 10^{-2}			Differences between $\times 10^{-2}$			% age error between		
	Linear 1	Non-linear 2	Full non-linear 3	1 & 2 A	1 & 3 B	2 and 3 C	1&2	1&3	2&3
0.0	19.30	16.40	14.50	2.90	4.80	1.90	15.0	24.9	13.1
2.30	13.98	12.55	11.26	1.43	2.72	1.29	10.2	19.5	10.3
3.45	11.88	10.66	9.90	1.22	1.98	0.76	10.3	16.7	7.1
4.60	10.23	9.26	8.34	0.97	1.89	0.92	9.5	18.5	9.9
6.90	7.58	6.82	6.06	0.76	1.52	0.76	10.0	20.1	11.1
9.20	6.06	5.12	4.55	0.94	1.51	0.57	15.5	29.5	11.1
11.50	4.90	4.11	3.47	0.79	1.43	0.64	16.1	29.2	15.6
13.79	3.95	3.47	3.32	0.48	0.63	0.15	12.15	15.9	4.3
16.09	2.33	2.21	2.05	0.12	0.25	0.16	5.2	10.7	7.2
18.22	0.0	0.0	0.0	0	0	0	0	0	0

Table 8.3 Presentation of Calculation of Stresses Caused by Bending and Torsion of Case A

$\phi \times 10^{-3}$ f.e. 'rad'	M_y kip.in	M_t kip.in	$\phi_c \times 10^{-3}$ eqn. 8.9a 'rad'	$\phi'' \times 10^{-4}$ eqn. 8.9b	$M_z = \phi_c M_y$ kip.in	$\sigma_w = E \bar{w}_n \phi''$	First Yield			A_{fy} = A+B+C
							A	B	C	
9.017	0.0	53.785	9.676	1.481	0.0	52.759	0.0	0.0	1.055	1.055
8.264	100.0	45.117	8.117	1.243	0.812	44.256	0.136	0.013	0.885	1.034
7.461	150.0	40.889	7.356	1.126	1.103	40.109	0.204	0.018	0.802	1.024
6.908	200.0	37.332	6.715	1.028	1.343	36.620	0.272	0.022	0.732	1.026
5.564	300.0	29.697	5.342	0.818	1.603	29.131	0.409	0.026	0.583	1.018
4.162	400.0	21.841	3.929	0.601	1.572	21.424	0.545	0.025	0.429	0.999
3.121	500.0	15.14	2.724	0.417	1.362	14.851	0.681	0.022	0.297	1.00
0.409	720.8	0.0	0.0	0.0	0.0	0.0	0.981	0.0	0.0	0.981

$A = M_y/M_{yy}$ $B = M_z/M_{yz}$ $C = \sigma_w/\sigma_y$

Table 8.4 Presentation of Calculation of Stresses, Caused by Bending and Torsion of Case B

$\phi \times 10^{-3}$ f.e. 'rad'	M_y kip.in	M_t kip.in	$\phi_c \times 10^{-3}$ eqn. 8.9a 'rad'	$\phi_c'' \times 10^{-6}$ eqn. 8.9b	$M_z = \phi_c M_z$ kip.in	$\sigma_w = E \bar{w}_n \phi''$	First Yield			A_{fy} = A+B+C
							A	B	C	
30.166	0.0	29.521	29.91	141.701	0.0	49.50	0.0	0.0	0.990	0.990
27.615	50.0	26.789	27.139	128.568	1.357	44.915	0.068	0.022	0.898	0.989
25.743	100.0	24.635	24.960	118.248	2.496	41.309	0.136	0.041	0.826	1.003
23.625	150.0	22.110	22.401	106.128	3.360	37.076	0.204	0.055	0.742	1.002
21.294	200.0	19.742	20.003	94.762	4.001	33.105	0.273	0.065	0.662	0.999
13.240	400.0	10.754	10.896	51.619	4.358	18.033	0.545	0.072	0.361	0.977
10.19	500.0	7.330	7.427	35.184	3.713	12.292	0.681	0.061	0.256	0.988
3.147	630.0	0.0	0.0	0.0	0.0	0.778	0.731	0.0	0.0	0.731

$A = M_y/M_{yy}$ $B = M_z/M_{yz}$ $C = \sigma_w/\sigma_y$

Table 8.5 Presentation of Calculation of Stresses Caused by Bending and Torsion of Case C

$\phi \times 10^{-2}$ f.e. rad	M_y kip.in	M_t kip.in	$\phi_c \times 10^{-2}$ eqn. 8.9a	$\phi_c'' \times 10^{-5}$ eqn. 8.9b	$M_z = \phi_c M_y$ kip·in	$\sigma_w = E \bar{w}_n \phi''$	First Yield			A_{fy} = A+B+C
							A	B	C	
9.381	0.0	19.30	8.962	14.377	0.0	37.415	0.0	0.0	0.748	0.748
7.018	100.0	13.96	6.479	10.399	6.479	36.329	0.136	0.106	0.727	0.969
6.199	150.0	11.64	5.400	8.696	8.100	30.380	0.204	0.132	0.608	0.944
5.450	200.0	9.55	4.43	7.114	8.860	24.853	0.273	0.145	0.497	0.914
4.866	250.0	7.77	3.606	5.788	9.015	20.220	0.341	0.147	0.404	0.892
4.318	300.0	6.19	2.873	4.615	8.619	16.109	0.409	0.141	0.322	0.871
3.835	350.0	4.70	2.181	3.501	7.635	12.231	0.477	0.125	0.245	0.846
1.779	562.0	0.0	0.0	0.0	0.0	0.0	0.766	0.0	0.0	0.766

$A = M_y/M_{yy}$ $B = M_z/M_{yz}$ $C = \sigma_w/\sigma_y$

Table 8.6 Presentation of Calculation of Stress Caused by Torsion of Case A

$A = M_y/M_{yy}$ $B = M_z/M_{yz}$ $C = M_t/M_{pw}$ $\bar{A} = M_y/M_{py}$ $\bar{B} = M_z/M_{pz}$ $\bar{C} = M_t/M_{pw}$

$\phi \times 10^{-3}$ f.e.	M_y kip.in	M_t kip.in	$\phi_c \times 10^{-3}$ eqn. 8.9a 'rad'	$\phi_c'' \times 10^{-4}$ eqn. 8.9b	$M_z = \phi_c M_y$ kip.in	First Yield			$A_{fy} =$ (A+B+C)	Full Plasticity			$A_{fp} =$ $(\bar{A}+\bar{B}+\bar{C})$
						A	B	C		$\bar{A}$	$\bar{B}$	$\bar{C}$	
9.017	0.0	53.785	9.676	1.481	0.0	0.0	0.0	0.154	0.154	0.0	0.0	0.689	0.689
8.264	100.0	45.117	8.117	1.243	0.812	0.136	0.013	0.129	0.278	0.117	0.008	0.578	0.703
7.461	150.0	40.889	7.356	1.126	1.103	0.204	0.018	0.117	0.339	0.174	0.011	0.524	0.709
6.908	200.0	37.332	6.715	1.028	1.343	0.272	0.022	0.107	0.401	0.232	0.014	0.478	0.724
5.564	300.0	29.697	5.342	0.818	1.603	0.409	0.026	0.085	0.52	0.348	0.016	0.381	0.745
4.162	400.0	21.841	3.929	0.601	1.572	0.545	0.025	0.062	0.632	0.464	0.016	0.280	0.760
3.121	500.0	15.14	2.724	0.417	1.362	0.681	0.022	0.043	0.746	0.580	0.014	0.194	0.783
0.409	720.8	0.0	0.0	0.0	0.0	0.981	0.0	0.0	0.981	0.882	0.0	0.0	0.882

$$A_{fy} = \frac{M_y}{M_{yy}} + \frac{M_z}{M_{yz}} + \frac{M_t}{M_{yw}} \qquad A_{fp} = \frac{M_y}{M_{py}} + \frac{M_z}{M_{pz}} + \frac{M_t}{M_{pw}}$$

Table 8.7 Presentation of Calculation of Stress Caused by Bending and Torsion of Case B

$A = M_y/M_{yy}$ $B = M_z/M_{yz}$ $C = M_t/M_{yw}$ $\bar{A} = M_y/M_{py}$ $\bar{B} = M_z/M_{pz}$ $\bar{C} = M_t/M_{pw}$

$\phi\ x10^{-3}$ f.e. 'rad'	M_y kip.in	M_t kip.in	$\phi_c\ x10^{-3}$ eqn. 8.9a 'rad'	$\phi_c''x10^{-4}$ eqn. 8.9b	$M_z = \phi_c . M_y$ kip.in	First Yield			A_{fy} = (A+B+C)	Full Plasticity			A_{fp} = $(\bar{A}+\bar{B}+\bar{C})$
						A	B	C		$\bar{A}$	$\bar{B}$	$\bar{C}$	
30.166	0.0	29.521	29.91	141.701	0.0	0.0	0.0	0.084	0.084	0.0	0.0	0.681	0.681
27.615	50.0	26.789	27.139	128.568	1,357	0.068	0.022	0.077	0.167	0.058	0.014	0.619	0.691
25.743	100.0	24.635	24.960	118.245	2.496	0.136	0.041	0.070	0.247	0.116	0.026	0.569	0.711
23.625	150.0	22.110	22.401	106.128	3.360	0.204	0.055	0.064	0.323	0.174	0.035	0.511	0.720
21.294	200.0	19.742	20.003	94.762	4.001	0.273	0.065	0.056	0.394	0.232	0.041	0.456	0.729
13.24	400.0	10.754	10.896	51.619	4.359	0.595	0.071	0.031	0.647	0.464	0.045	0.248	0.757
10.19	500.0	7.330	7.427	35.184	3.713	0.681	0.061	0.021	0.763	0.580	0.038	0.169	0.787
3.147	630.0	0.0	0.0	0.0	0.0	0.858	0.0	0.0	0.858	0.731	0.0	0.0	0.731

$$A_{fy} = \frac{M_y}{M_{yy}} + \frac{M_z}{M_{yz}} + \frac{M_t}{M_{yw}} \qquad A_{fp} = \frac{M_y}{M_{py}} + \frac{M_z}{M_{pz}} + \frac{M_t}{M_{pw}}$$

Table 8.8 Presentation of Calculation of Stress Caused by Bending and Torsion of Case C

$A = M_y/M_{yy}$ $B = M_z/M_{yz}$ $C = M_t/M_{yw}$ $\bar{A} = M_y/M_{py}$ $\bar{B} = M_z/M_{pz}$ $\bar{C} = M_z/M_{wz}$

$\phi x10^{-2}$ f.e. 'rad'	M_y kip.in	M_t kip.in	$\phi_c x10^{-2}$ eqn.8.9a 'rad'	$\phi''_c x10^{-5}$ eqn. 8.9b	$M_z = \phi_c M_y$ kip.in	First Yield			(A+B+C)	Full Plasticity			$(\bar{A}+\bar{B}+\bar{C})$
						A	B	C		$\bar{A}$	$\bar{B}$	$\bar{C}$	
9.381	0.0	19.30	8.962	14.377	0.0	0.0	0.0	0.055	0.055	0.0	0.0	0.773	0.773
7.018	100.0	13.96	6.479	10.394	6.479	0.136	0.106	0.040	0.282	0.116	0.066	0.562	0.744
6.200	150.0	11.64	5.400	8.696	8.100	0.204	0.132	0.033	0.369	0.174	0.083	0.469	0.726
5.450	200.0	9.55	4.430	7.114	8.860	0.273	0.145	0.027	0.445	0.232	0.091	0.385	0.708
4.866	250.0	7.77	3.606	5.788	9.015	0.341	0.147	0.022	0.510	0.290	0.092	0.313	0.695
4.318	300.0	6.19	2.873	4.615	8.619	0.409	0.141	0.018	0.568	0.348	0.088	0.249	0.685
3.835	350.0	4.70	2.181	3.501	7.635	0.477	0.125	0.013	0.615	0.468	0.078	0.189	0.735
1.779	562.0	0.0	0.0	0.0	0.0	0.766	0.0	0.0	0.766	0.652	0.0	0.0	0.652

$$A_{fy} = \frac{M_y}{M_{yy}} + \frac{M_z}{M_{yz}} + \frac{M_t}{M_{yw}} \qquad A_{fp} = \frac{M_y}{M_{py}} + \frac{M_z}{M_{pz}} + \frac{M_t}{M_{pw}}$$

Table 8.9 Interaction of Stresses of the Three Cases (A, B and C)

L/r_z	M_t/M_{pw}	M_y/M_{py}	$\times 10^{-2}$ f.e.	A_{fy}	A_{fp}	C_1	C_2
	0.154	0.0	0.902	0.154	0.689	0.154	0.689
	0.129	0.116	0.826	0.278	0.703	0.187	0.590
	0.117	0.174	0.746	0.339	0.709	0.235	0.552
35.0	0.107	0.232	0.691	0.401	0.724	0.292	0.531
	0.085	0.348	0.556	0.42	0.745	0.418	0.516
	0.062	0.464	0.416	0.632	0.760	0.549	0.542
	0.043	0.580	0.312	0.746	0.783	0.683	0.612
	0.0	0.836	0.041	0.981	0.881	0.981	0.882
	0.084	0.0	3.017	0.084	0.681	0.084	0.681
	0.076	0.058	2.762	0.167	0.691	0.105	0.622
	0.070	0.116	2.574	0.247	0.711	0.158	0.581
64.0	0.063	0.174	2.363	0.323	0.720	0.221	0.794
	0.056	0.232	2.129	0.394	0.729	0.286	0.513
	0.031	0.464	1.324	0.647	0.757	0.551	0.528
	0.021	0.580	1.019	0.763	0.787	0.684	0.605
	0.0	0.731	0.315	0.858	0.731	0.858	0.731
	0.055	0.0	9.381	0.055	0.773	0.055	0.773
	0.040	0.116	7.018	0.282	0.744	0.177	0.578
	0.033	0.174	6.200	0.369	0.726	0.245	0.507
	0.027	0.232	5.45	0.445	0.708	0.310	0.459
111.0	0.022	0.290	4.866	0.510	0.695	0.372	0.437
	0.018	0.348	4.318	0.568	0.685	0.433	0.437
	0.013	0.406	3.835	0.615	0.735	0.493	0.511
	0.0	0.652	1.779	0.766	0.652	0.766	0.652

Table 8.10 Ultimate Strength of Beam-Column under Biaxial Bending and Torsion

$\frac{e_{y1}}{B} = \frac{e_{y2}}{B} = 0.167$ $\frac{e_{z1}}{D} = \frac{e_{z2}}{D} = 0.257$ $\frac{L}{r_y} = 64$

M_t kip.in	P kip	M_y kip.in	M_z kip.in	First Yield A	First Yield B	First Yield C	First Yield D	A+B+C+D	A^2	B^2	C^2	D^2	$\sqrt{A^2+B^2+C^2+D^2}$
0.0	36.05	72.09	9.01	0.173	0.098	0.145	0.0	0.416	0.020	0.010	0.021	0.0	0.247
5.0	29.12	58.24	7.28	0.140	0.079	0.117	0.264	0.60	0.020	0.006	0.014	0.070	0.332
9.0	22.64	45.28	5.66	0.109	0.062	0.091	0.476	0.738	0.012	0.004	0.008	0.227	0.501
14.0	8.32	16.64	2.08	0.040	0.023	0.033	0.74	0.836	0.002	0.001	0.001	0.548	0.743
15.925	0.0	0.0	0.0	0.0	0.0	0.0	0.842	0.842	0.0	0.0	0.0	0.709	0.842

$A = P/P_L$

$B = M_y/M_{yy}$

$C = M_z/M_{yz}$

$D = M_t/M_{yw}$

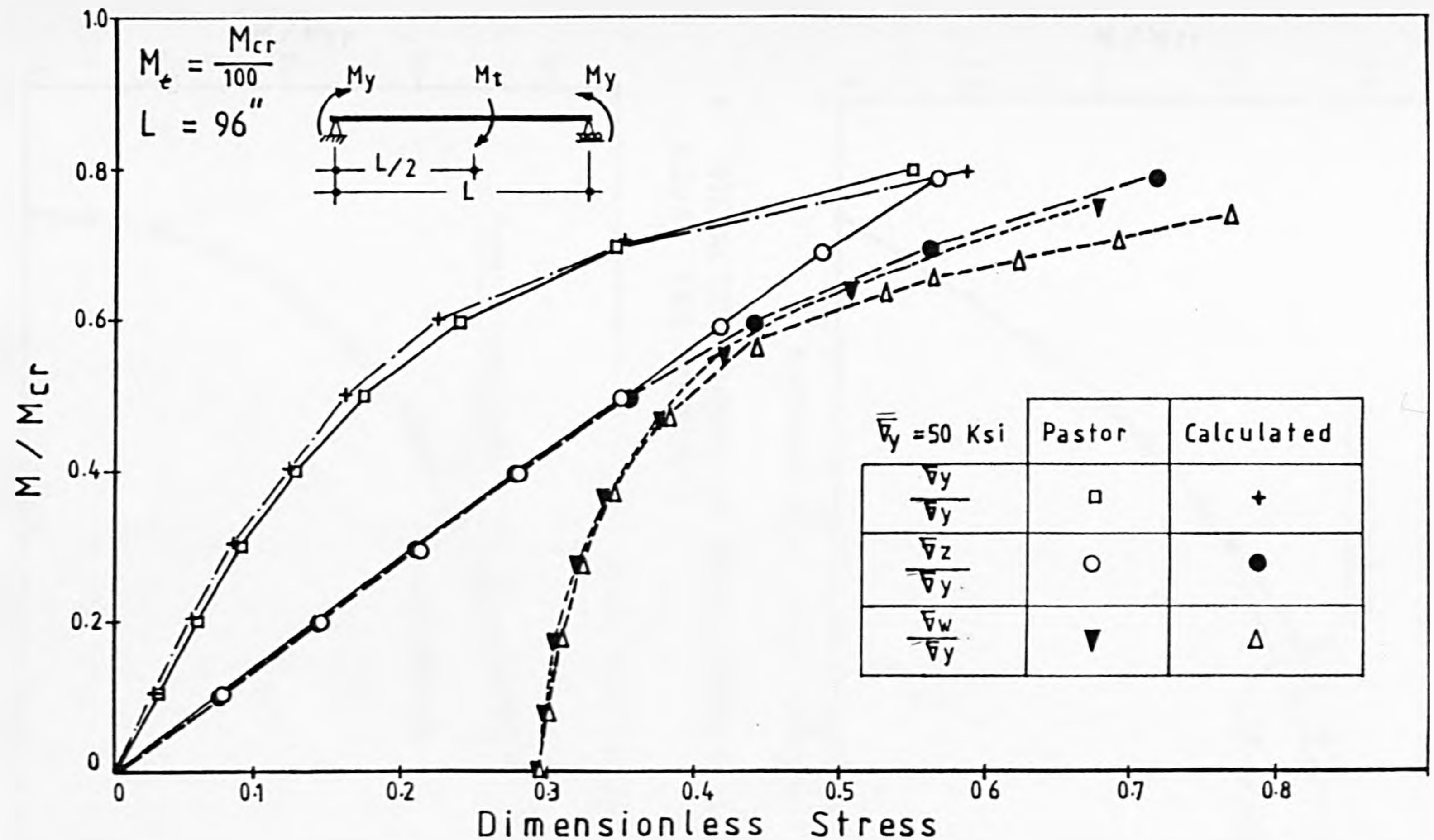
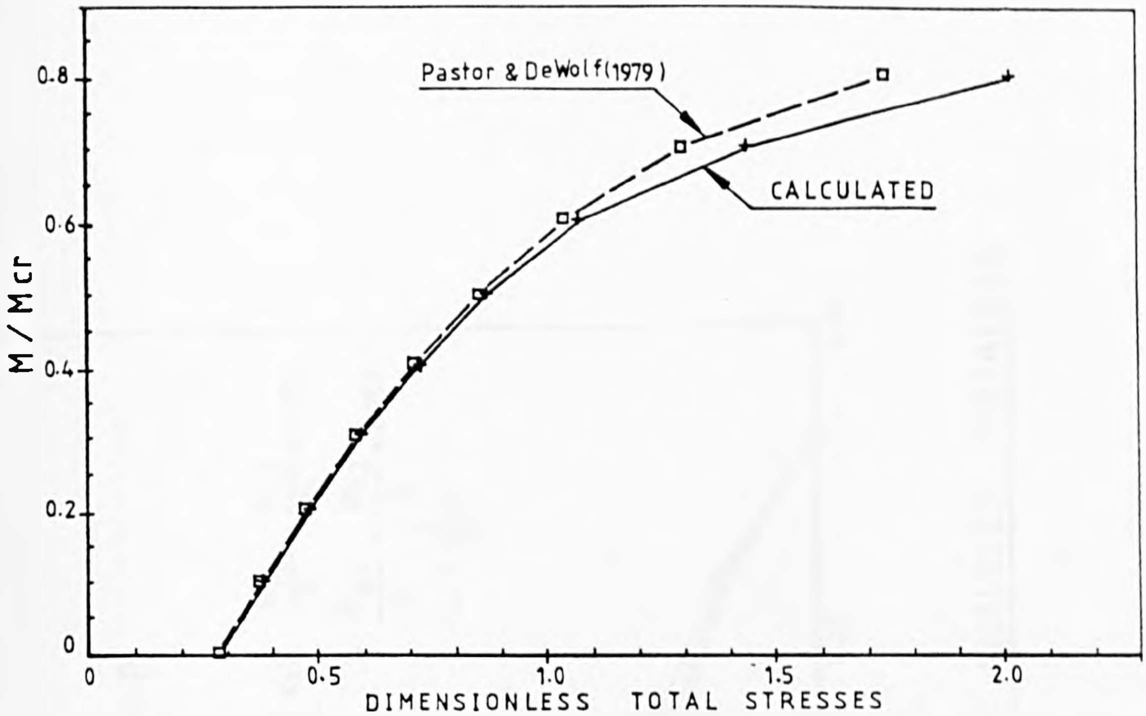
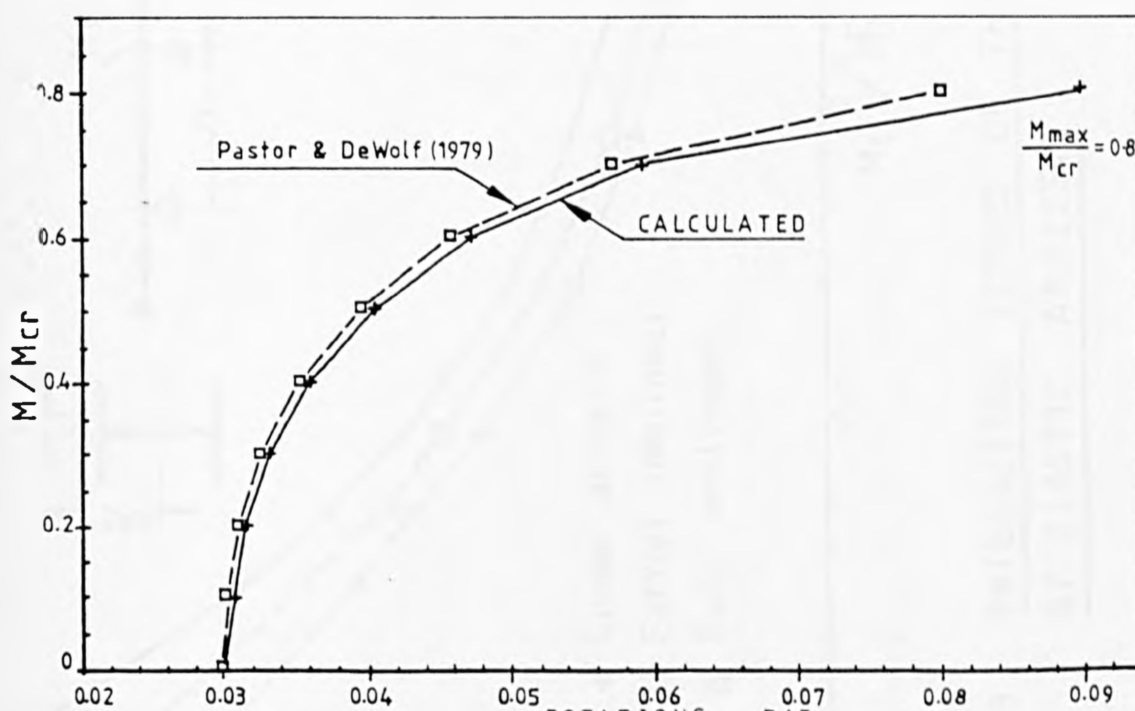


FIG. 8.1 COMPARISON BETWEEN THE BENDING AND WARPING STRESSES OBTAINED BY PASTOR AND DeWOLF (1979) AGAINST THOSE CONDUCTED BY AUTHOR'S APPROACH



a - INTERACTION CURVES OF TOTAL STRESSES AND APPLIED EQUAL END MOMENTS



b - LOAD ROTATION CURVE

FIG. 8.2 COMPARISON BETWEEN TOTAL STRESS AND ROTATION OF I-BEA UNDER CONSTANT TORSIONAL MOMENT AT MID-SPAN AND EQUAL END MOMENTS TO THOSE OBTAINED BY AUTHER AND PASTOR (1979).

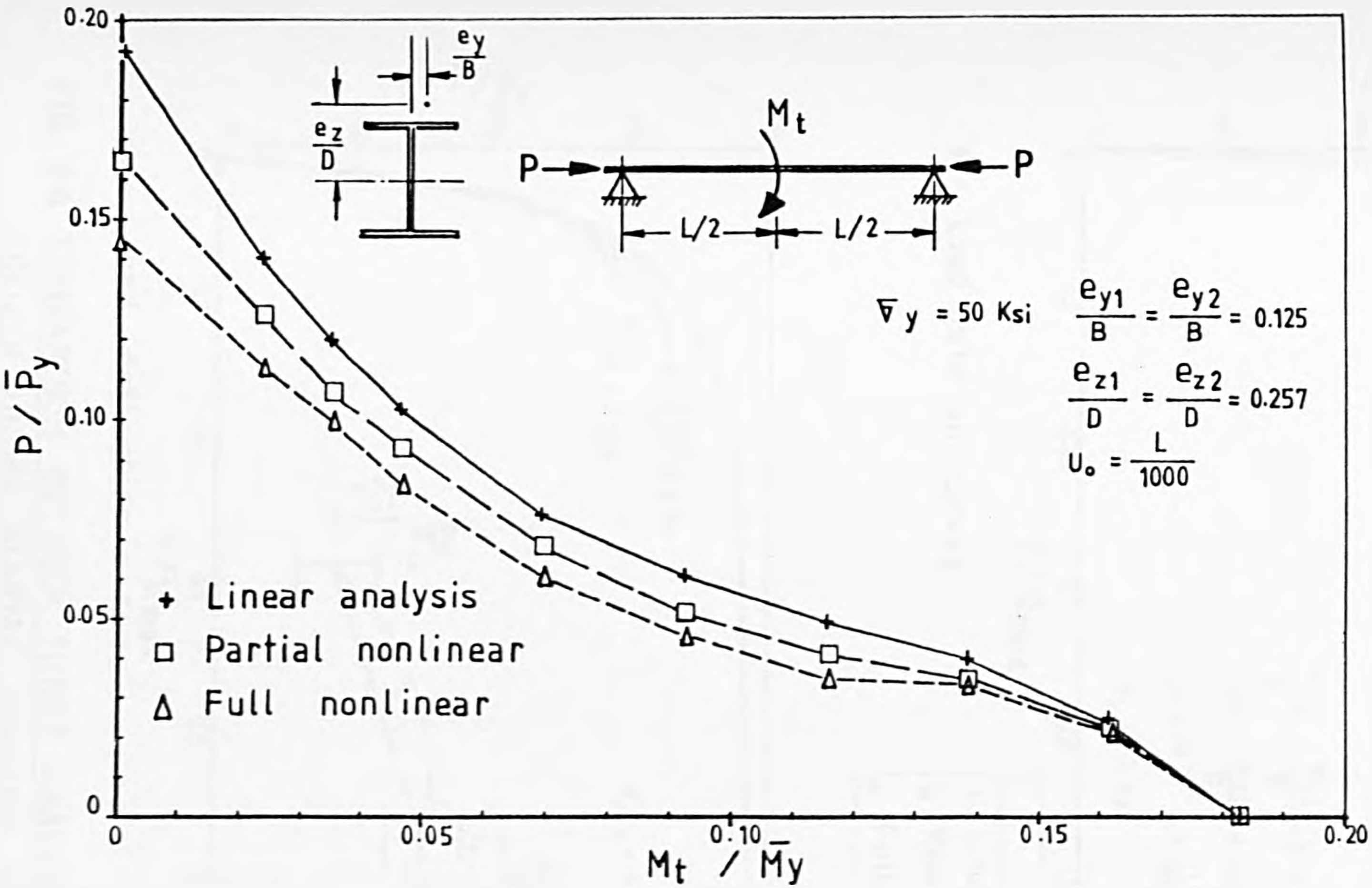
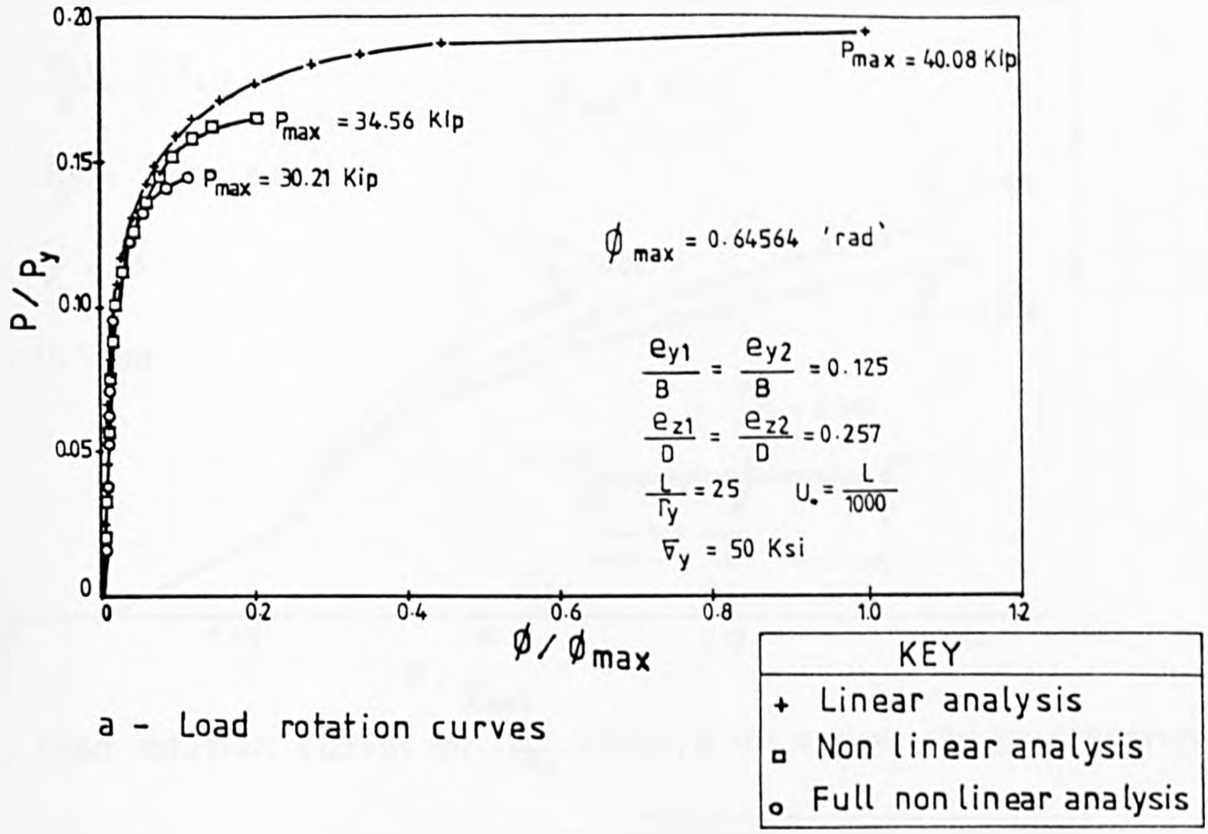
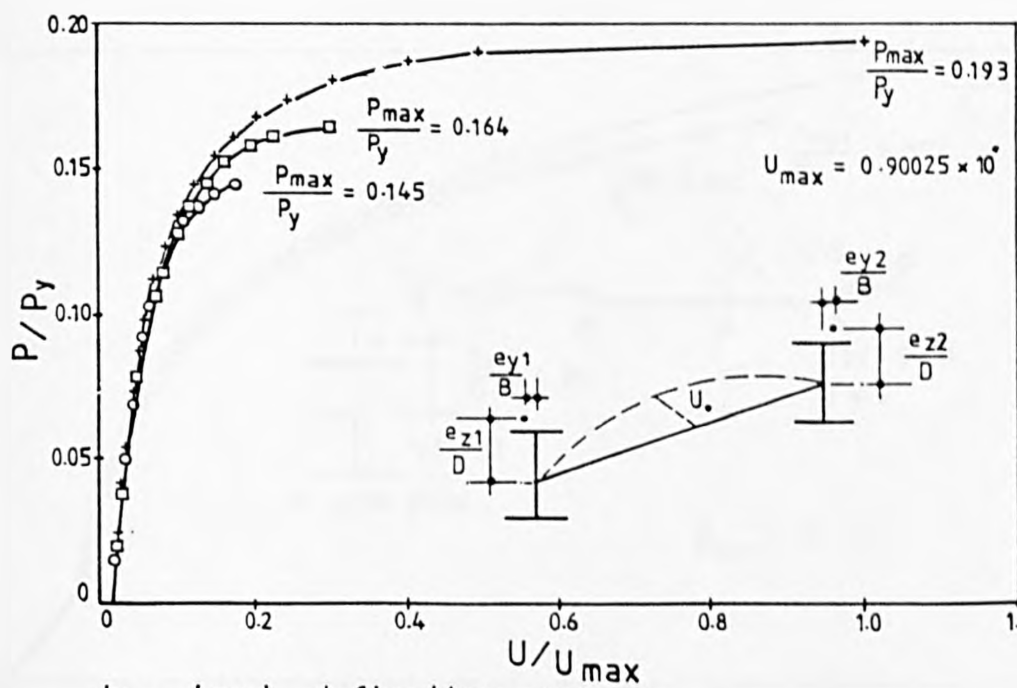


FIG. 8.3 INTERACTION CURVES OF THE THREE ANALYSES OBTAINED BY ELASTIC ANALYSIS

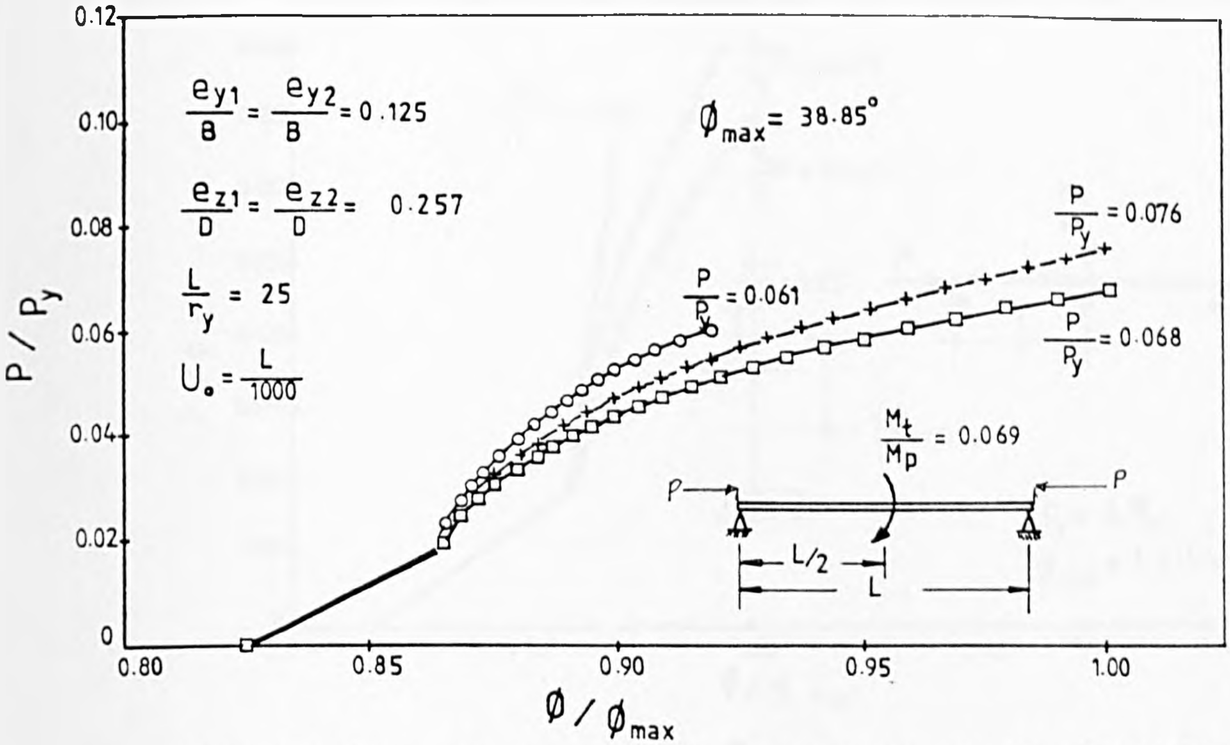


a - Load rotation curves



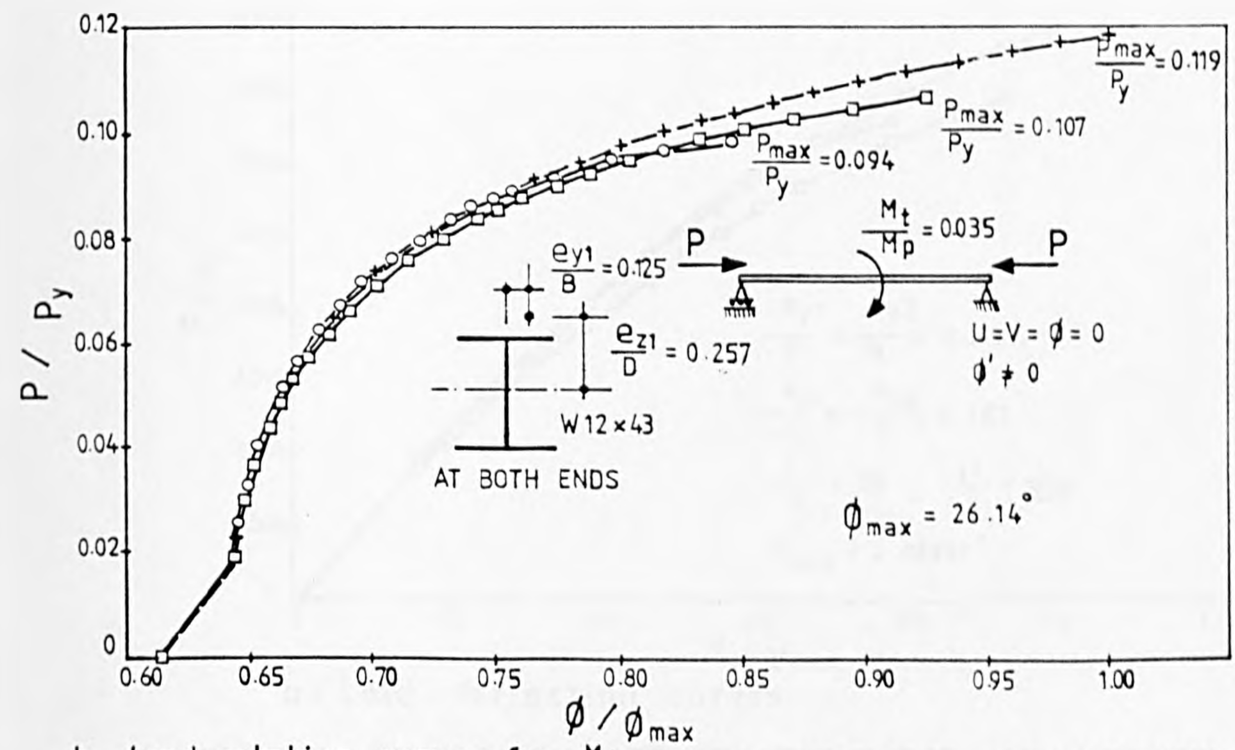
b - Load deflection curves

FIG. 8.4 COMPARISON BETWEEN THREE ANALYSES OF A BEAM-COLUMN UNDER BIAXIAL BENDING AND INITIAL DEFLECTION $U_o = \frac{L}{1000}$ (ELASTIC ANALYSIS)



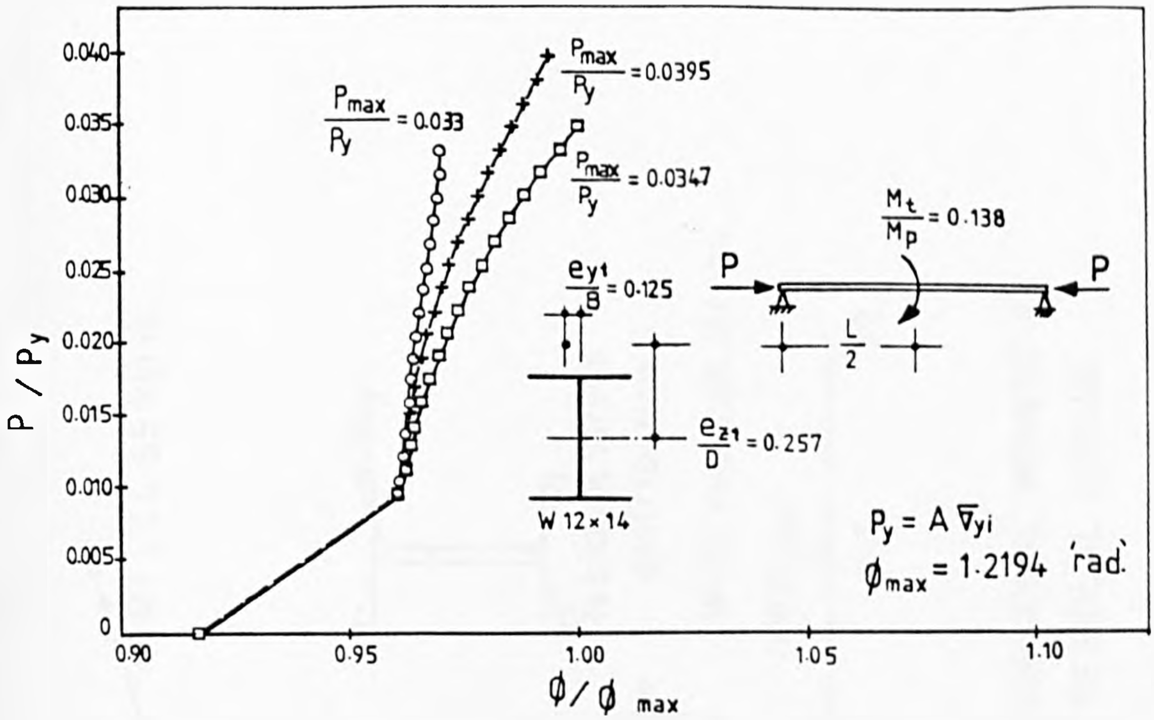
a - Load rotation curves for $\frac{M_t}{M_p} = 0.069$, P APPLIED LOAD WITH ECCENTRICITIES

FOR KEY SEE FIG. 8.4



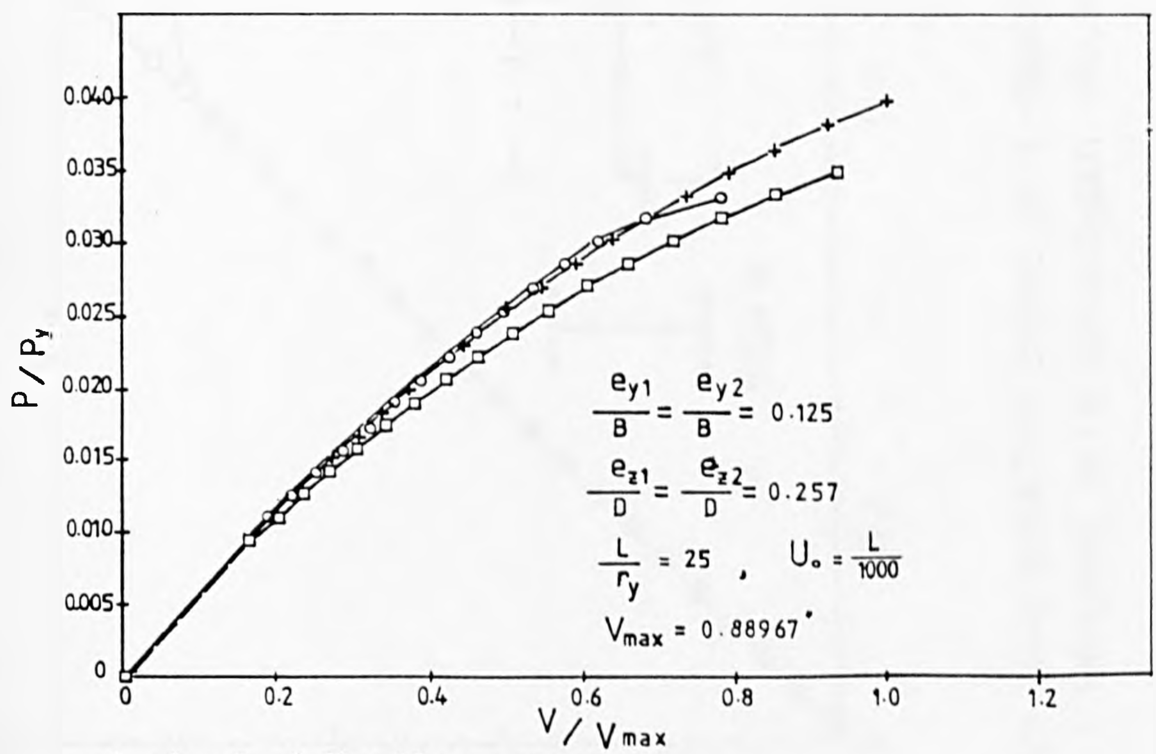
b - Load rotation curves for $\frac{M_t}{M_p} = 0.035$ AND COMPRESSIVE LOAD WITH ECCENTRICITIES

FIG. 8.5 INTERACTION CURVES OF ROTATIONS AND LOADS OF BEAM-COLUMN SUBJECTED TO BIAXIAL BENDING AND CONSTANT TORSION AT MID-SPAN



a - Load-rotation curves ($\frac{M_t}{M_p}$ = 0.138)

FOR KEY SEE FIG. 8.4



b - Load - deflection curves

FIG. 8.6 COMPARISON BETWEEN THE THREE ANALYSES OF A BEAM-COLUMN UNDER BIAXIAL BENDING AND CONSTANT TORSIONAL MOMENT AT MID-SPAN

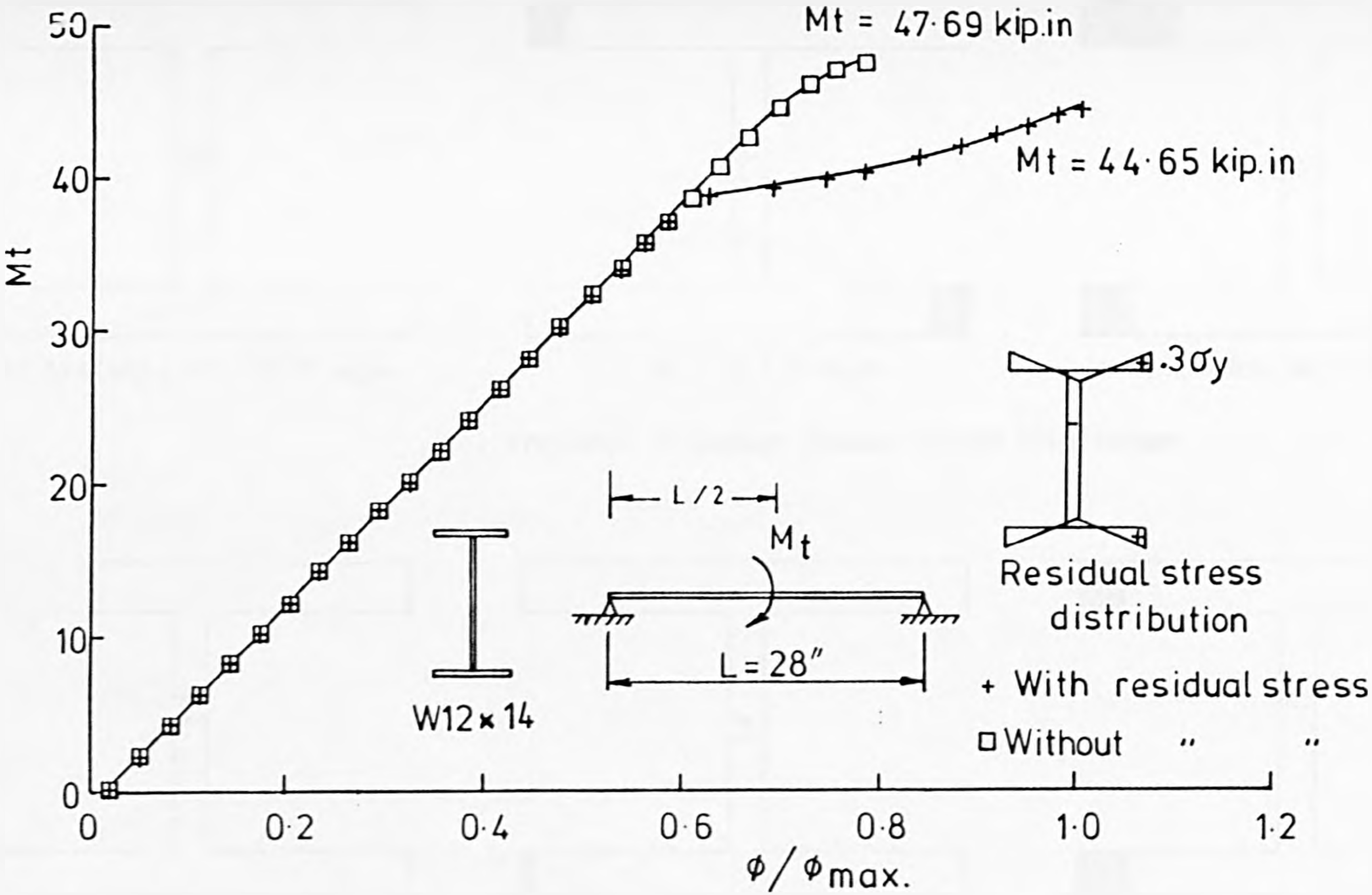


FIG.8.7 LOAD ROTATION CURVE OF I-BEAM UNDER TORSIONAL MOMENT AT MIDSPAN, WITH OR WITHOUT INCLUSION OF RESIDUAL STRESS

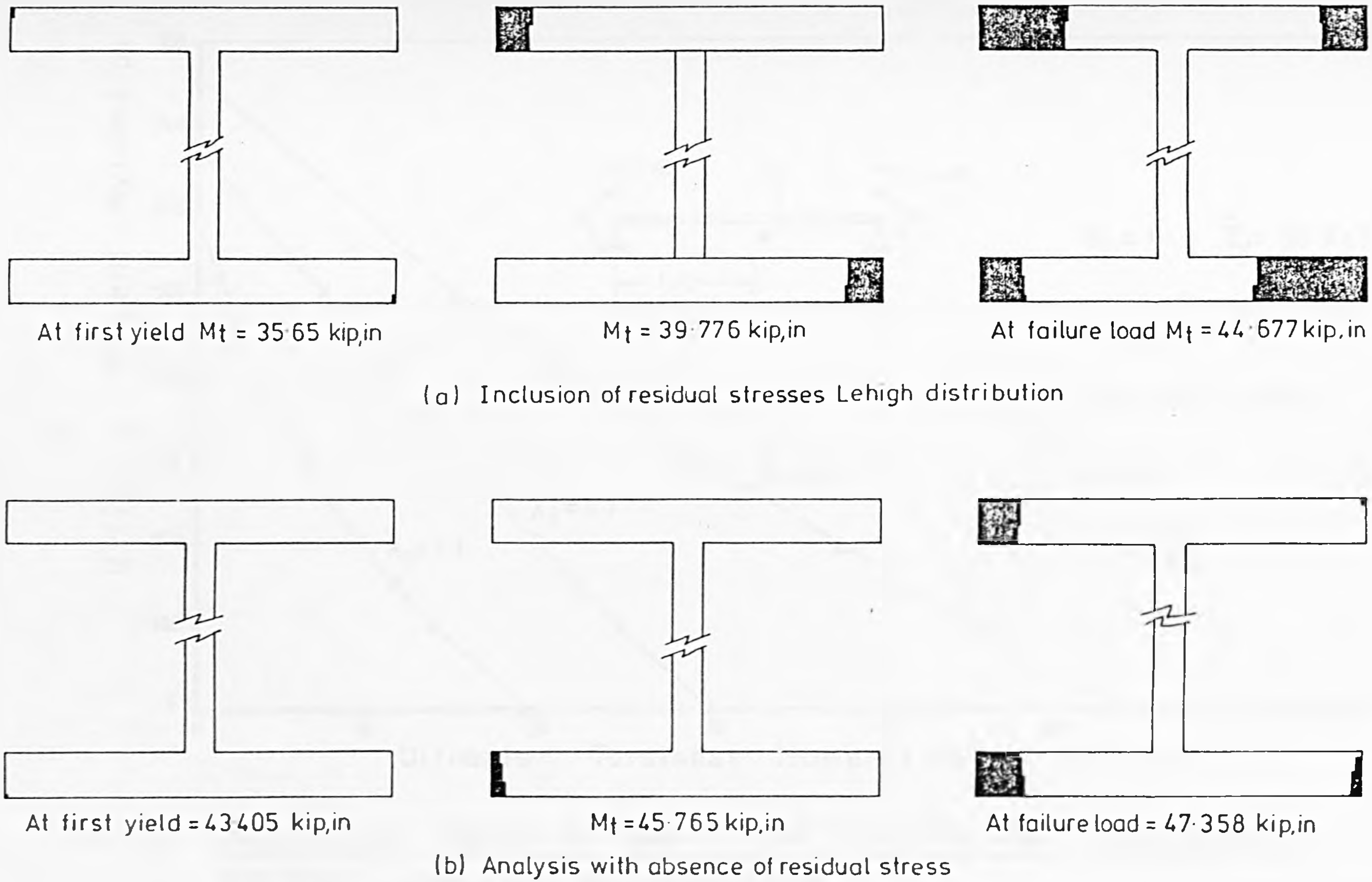


FIG. 8.8 TYPICAL SPREAD OF YIELD OF BEAM - COLUMN UNDER CONSTANT TORSION AT MID-SPAN

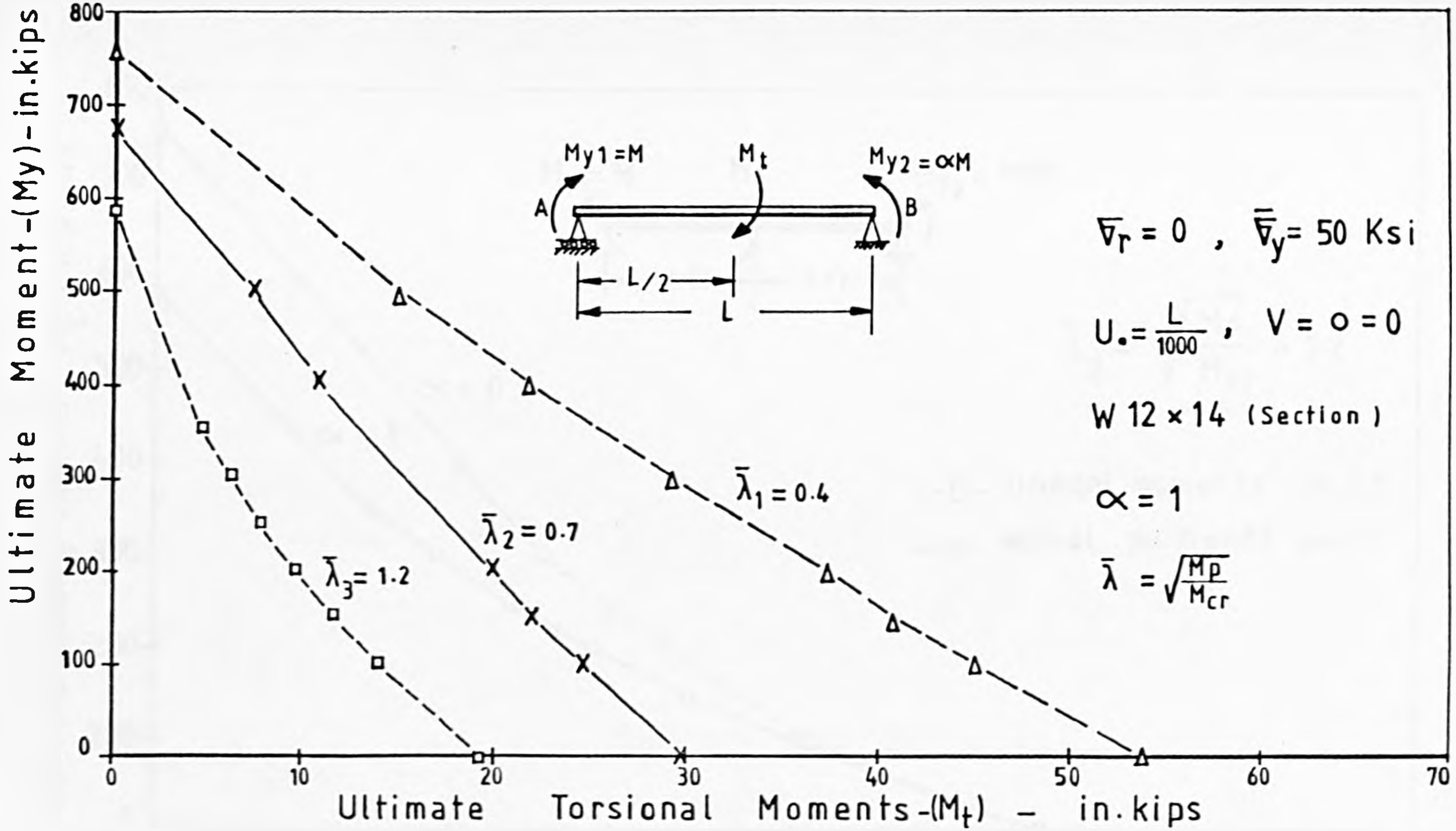


FIG. 8.9 INTERACTION CURVES OF EQUAL END MOMENTS AND CONCENTRATED TORSION APPLIED AT MID-SPAN

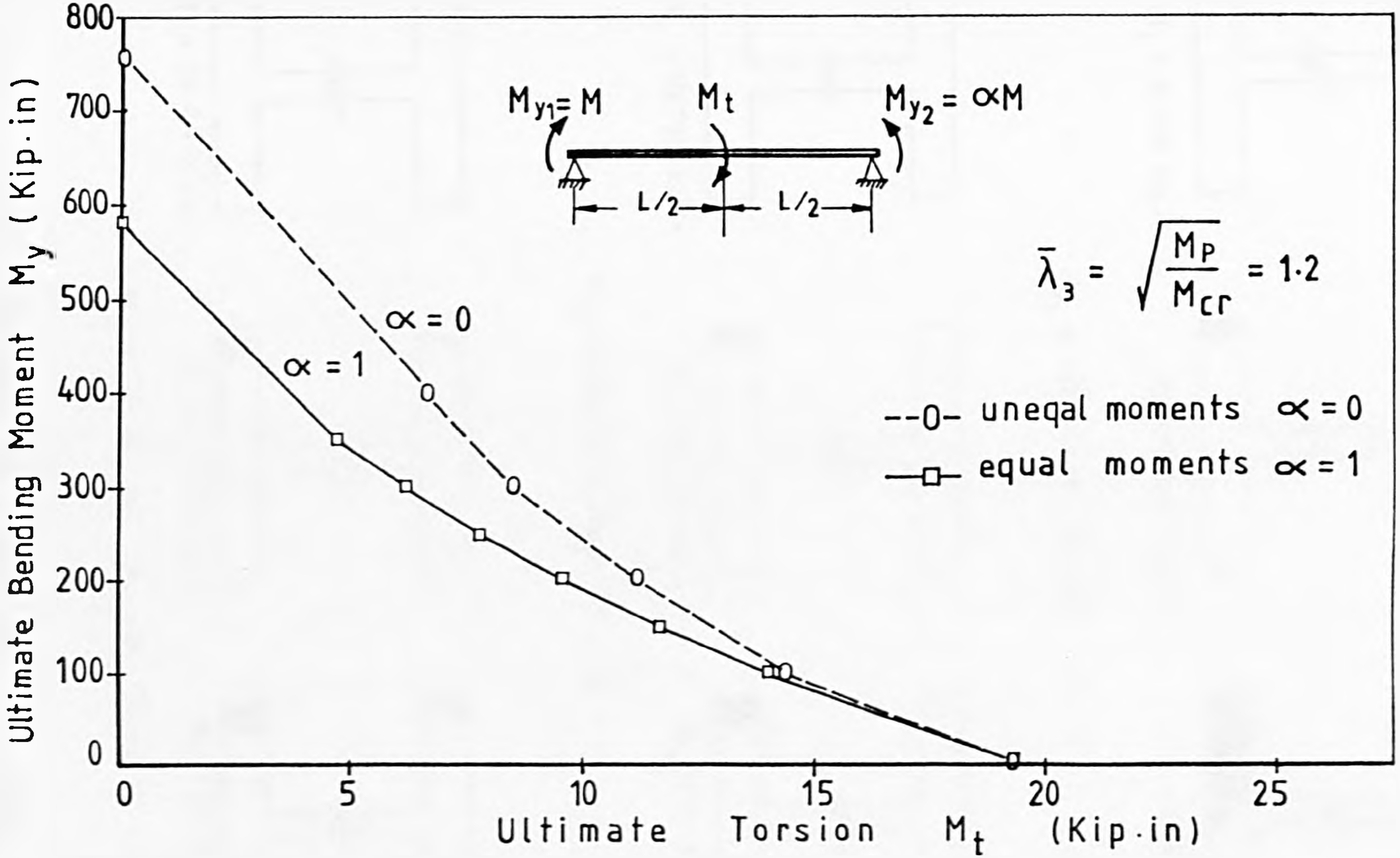


FIG. 8·10 <u>INTERACTION BETWEEN BENDING AND TORSION FOR A BEAM-COLUMN SUBJECTED TO UNIFORM OR NONUNIFORM ENDS MOMENT AND TORSION AT MID-SPAN.</u>

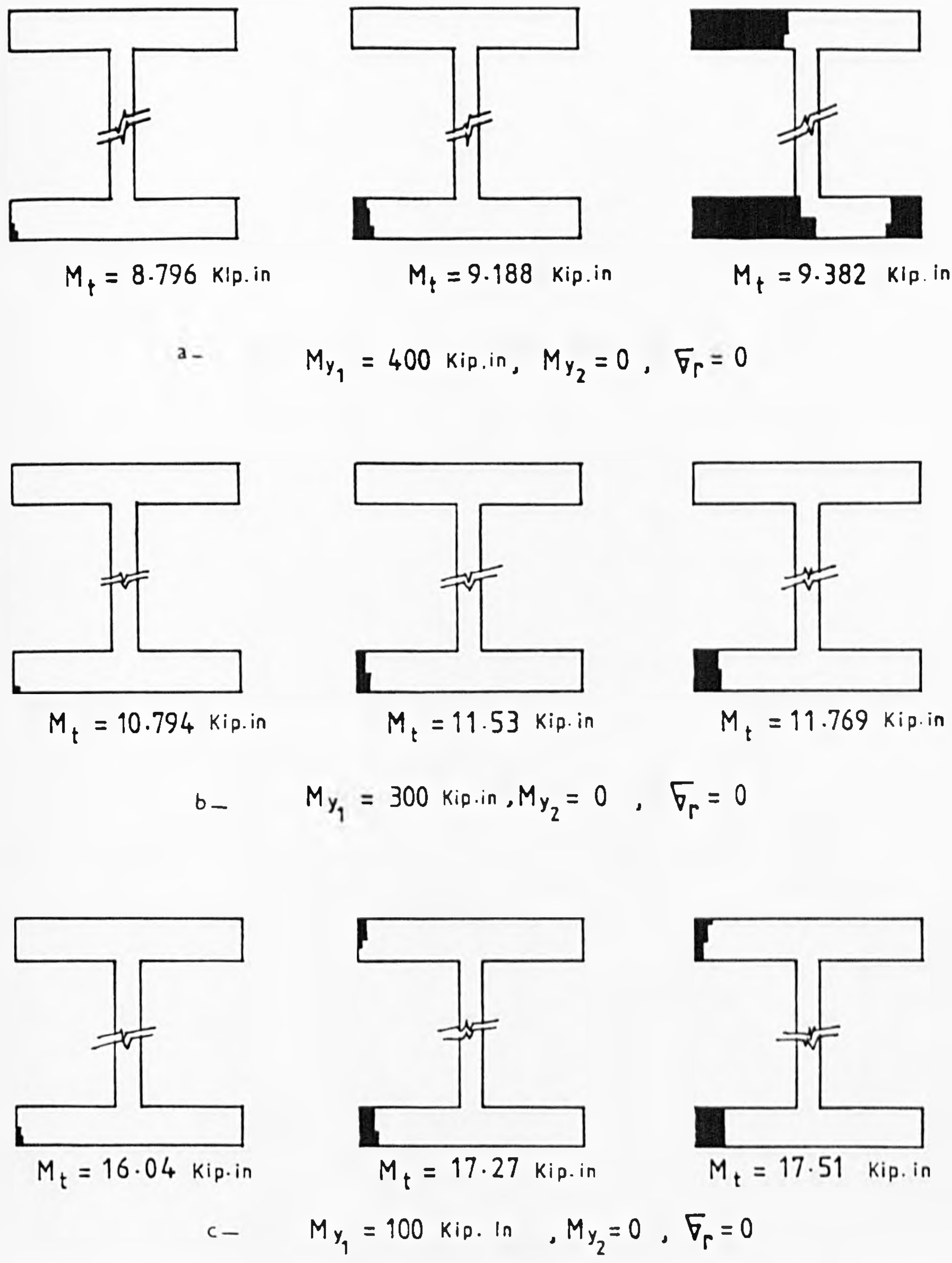


FIG. 8.11 <u>TYPICAL SPREAD OF YIELD AT MID-SPAN OF BEAM-COLUMN UNDER UNEQUAL END MOMENT AND TORSION, NO RESIDUAL STRESS INCLUDED</u>

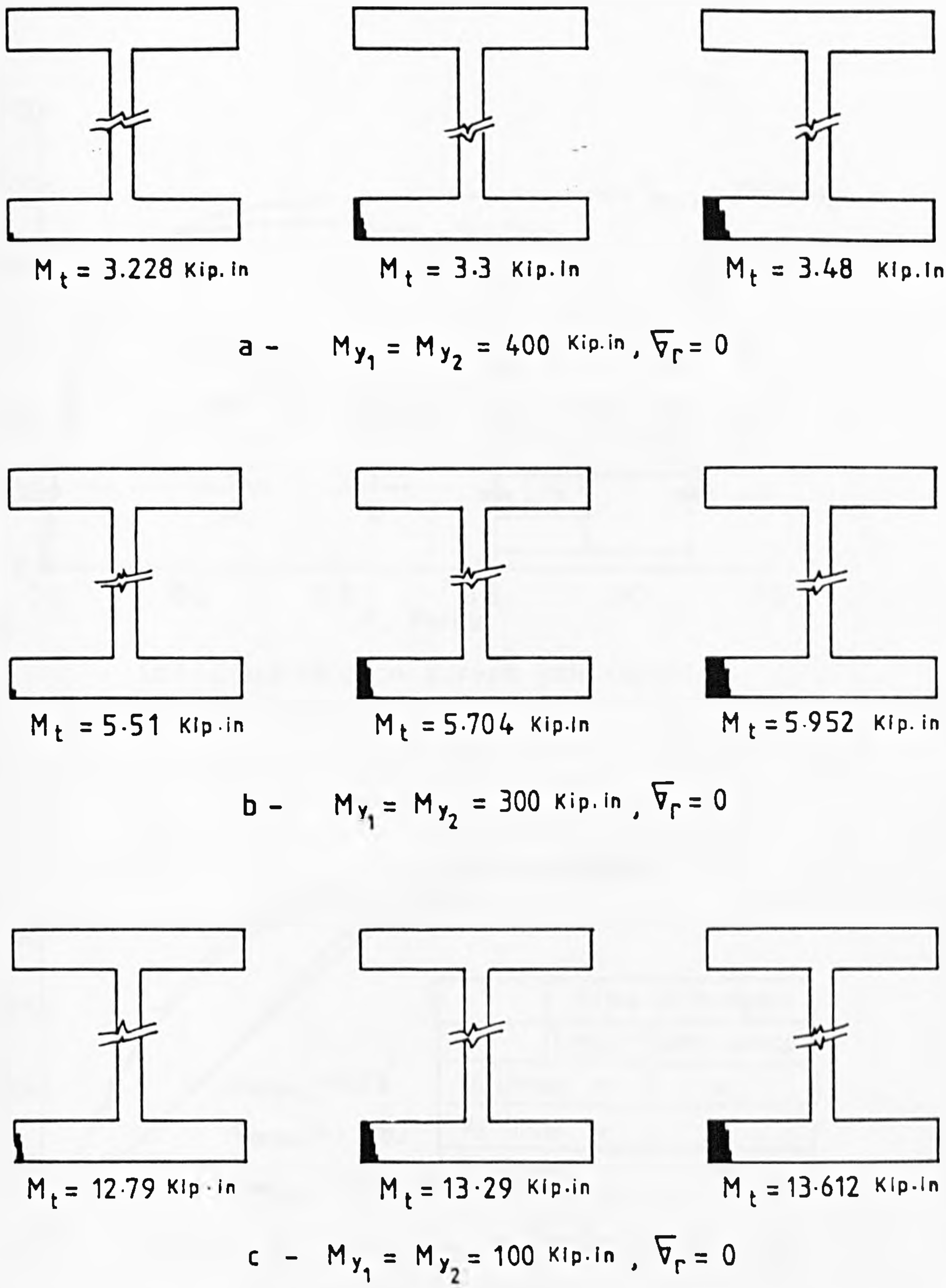
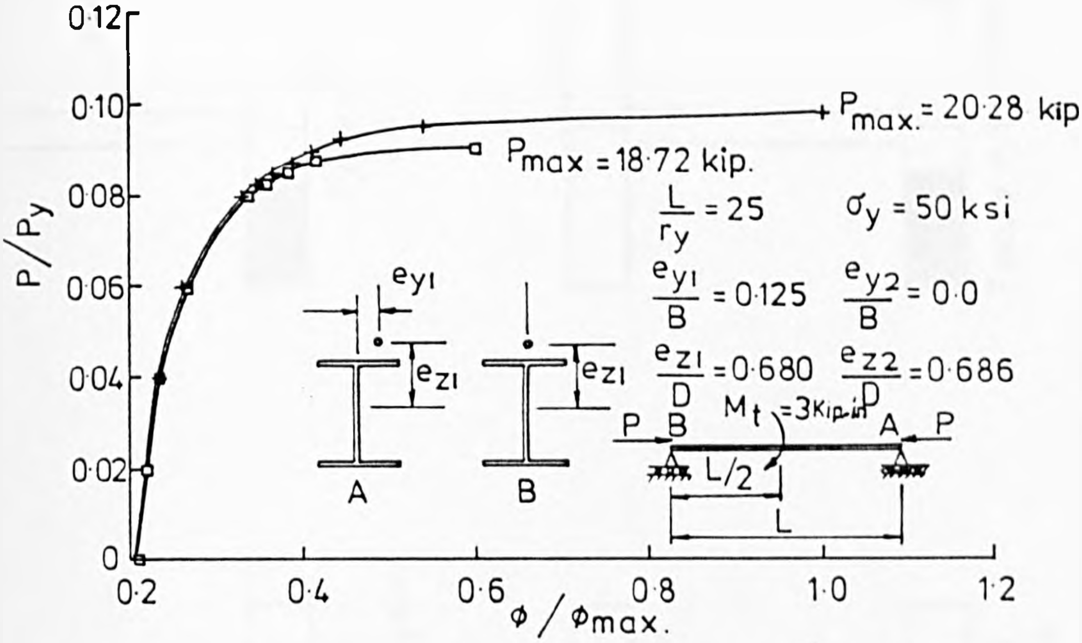
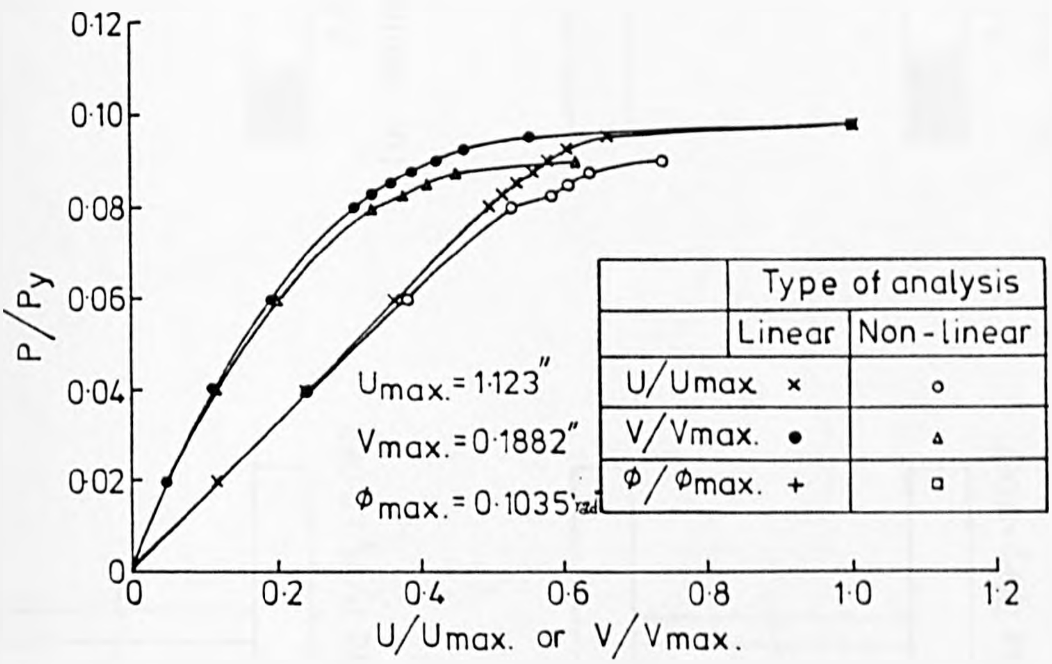


FIG. 8·12 TYPICAL SPREAD OF YIELD (MID-SPAN) OF BEAM-COLUMN SUBJECTED TO EQUAL CONSTANT END MOMENTS AND TORSIONAL MOMENT AT MID-SPAN



(a) Load rotation curves (mid-span)



(b) Load deflections (in-plane and out-of-plane) curves (mid-span)

FIG. 8.13 COMPARISON OF AUTHOR'S RESULTS FOR LINEAR AND NONLINEAR ANALYSIS OF BEAM-COLUMN SUBJECTED TO COMPRESSIVE LOAD WITH ECCENTRICITY AND CONSTANT TORSION APPLIED AT MIDSPAN

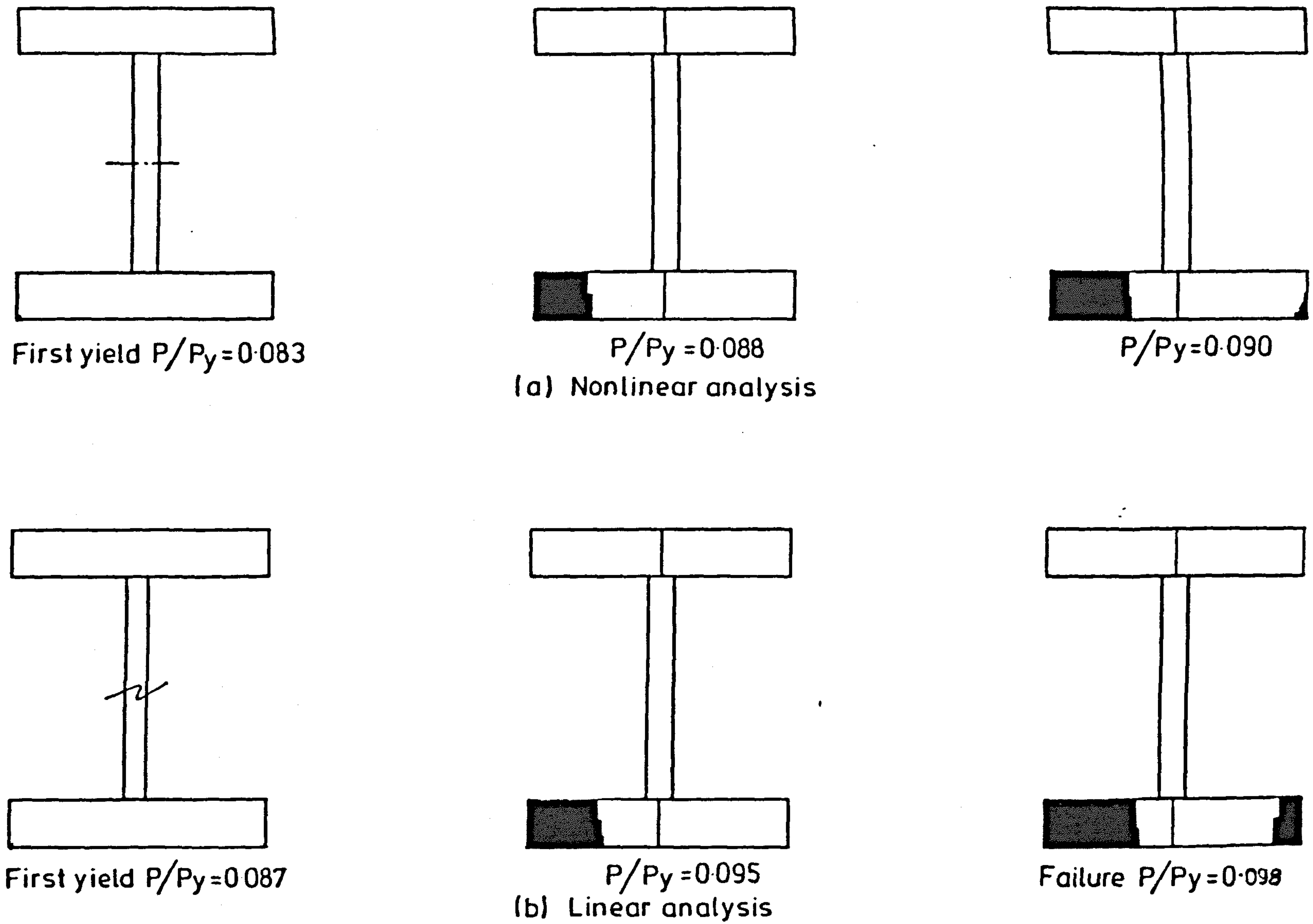
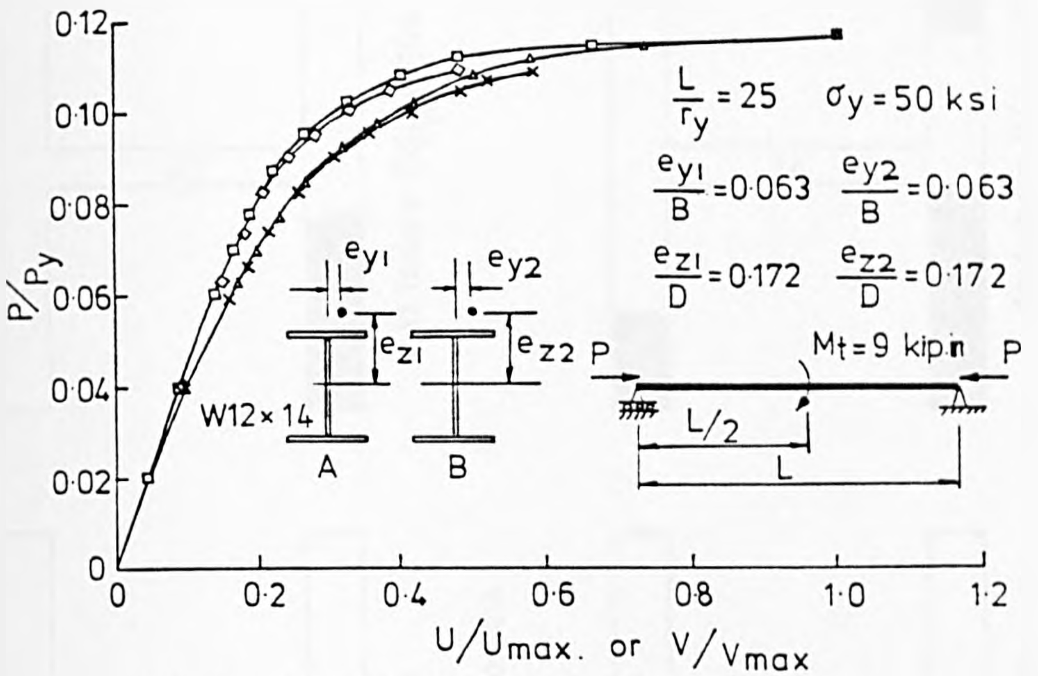


FIG.8.14 SPREAD OF YIELD FOR EXAMPLE OF FIG. 8.13



(a) Load– deflection curves (midspan)

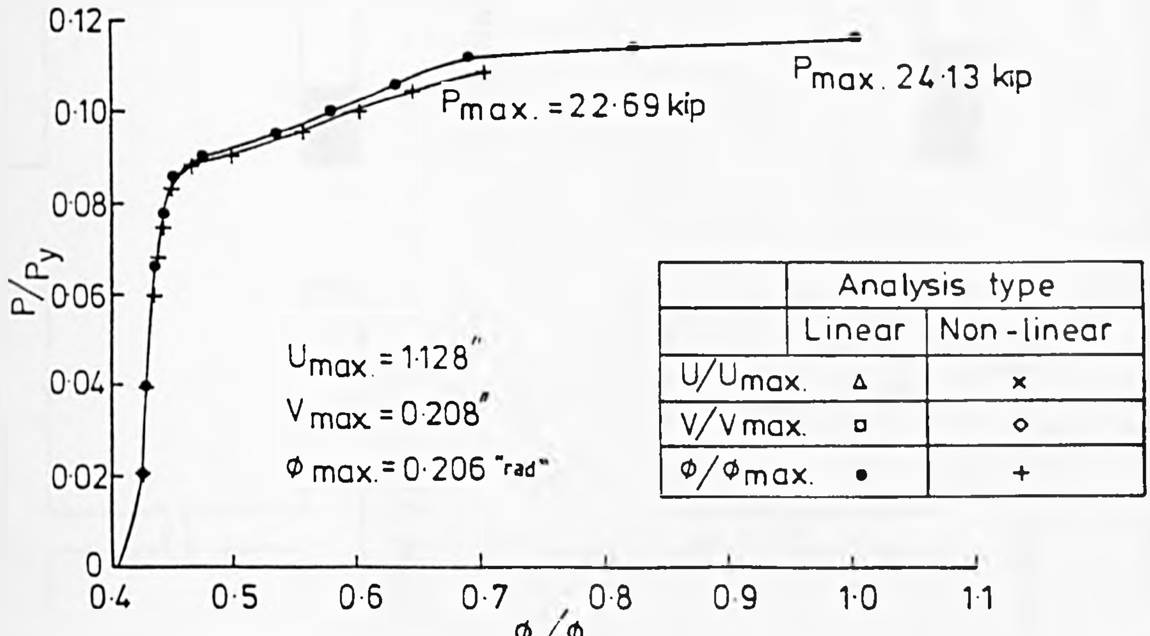
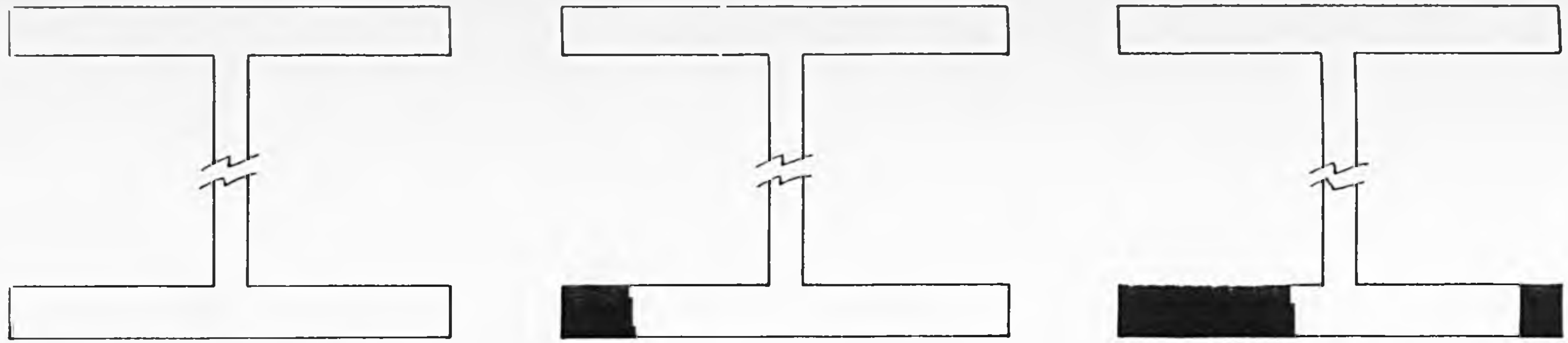
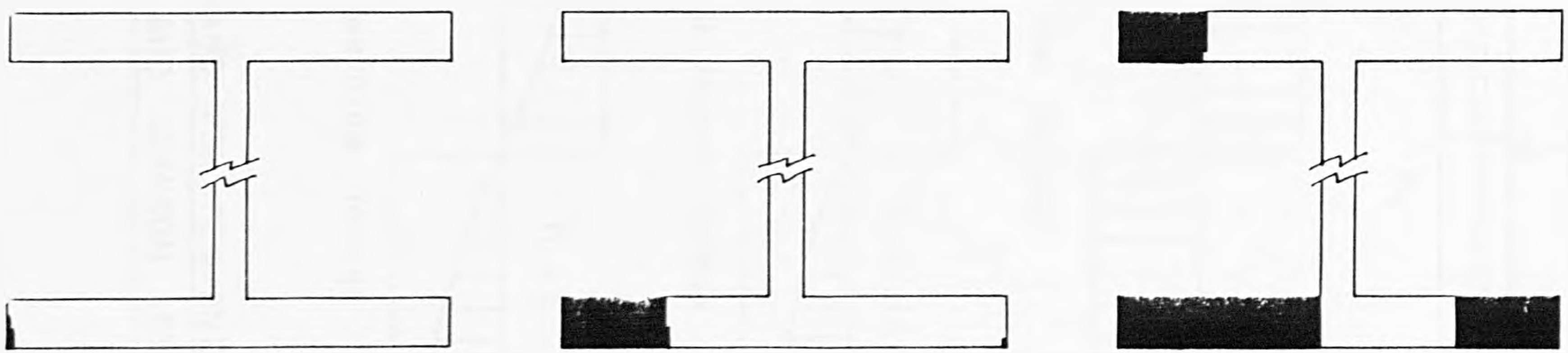


FIG 8.J5 COMPARISON BETWEEN LINEAR AND NON-LINEAR ANALYSIS OF BEAM-COLUMN UNDER BIAXIAL BENDING AND CONSTANT TORSION AT MIDSPAN



First yield at $P/P_y = 0.078$ | $P/P_y = 0.093$ | At failure $P/P_y = 0.109$

(a) Non-linear analysis



First yield $P/P_y = 0.08$ | $P/P_y = 0.104$ | At failure $P/P_y = 0.116$

(b) Linear analysis

FIG. 8.16 TYPICAL SPREAD OF YIELD OF BEAM-COLUMN UNDER BIAXIAL BENDING AND TORSION

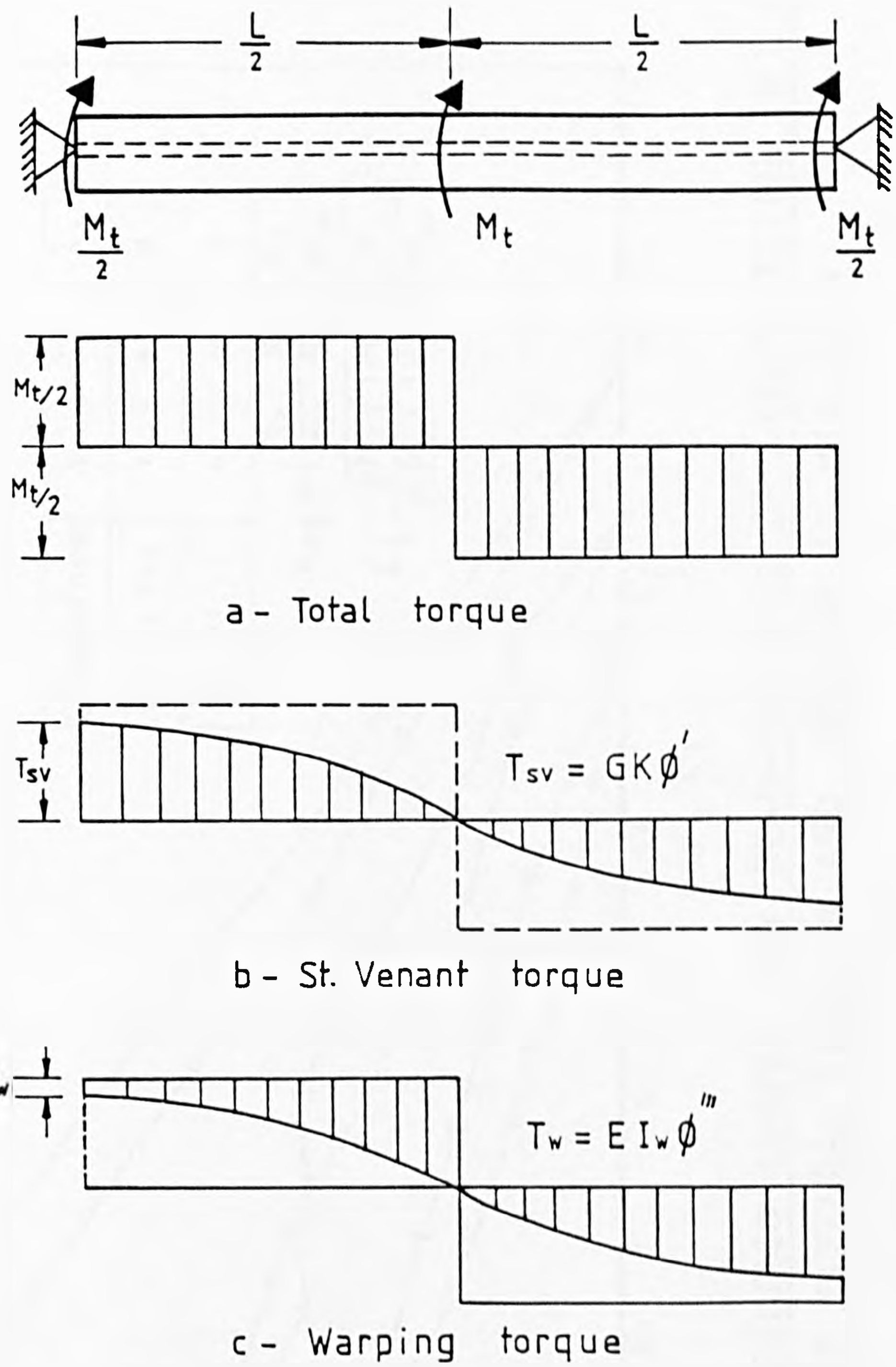


FIG. 8.17 TORQUE DIAGRAMS FOR PIN-ENDED BEAM (SALMON AND JOHNSON, 1980)

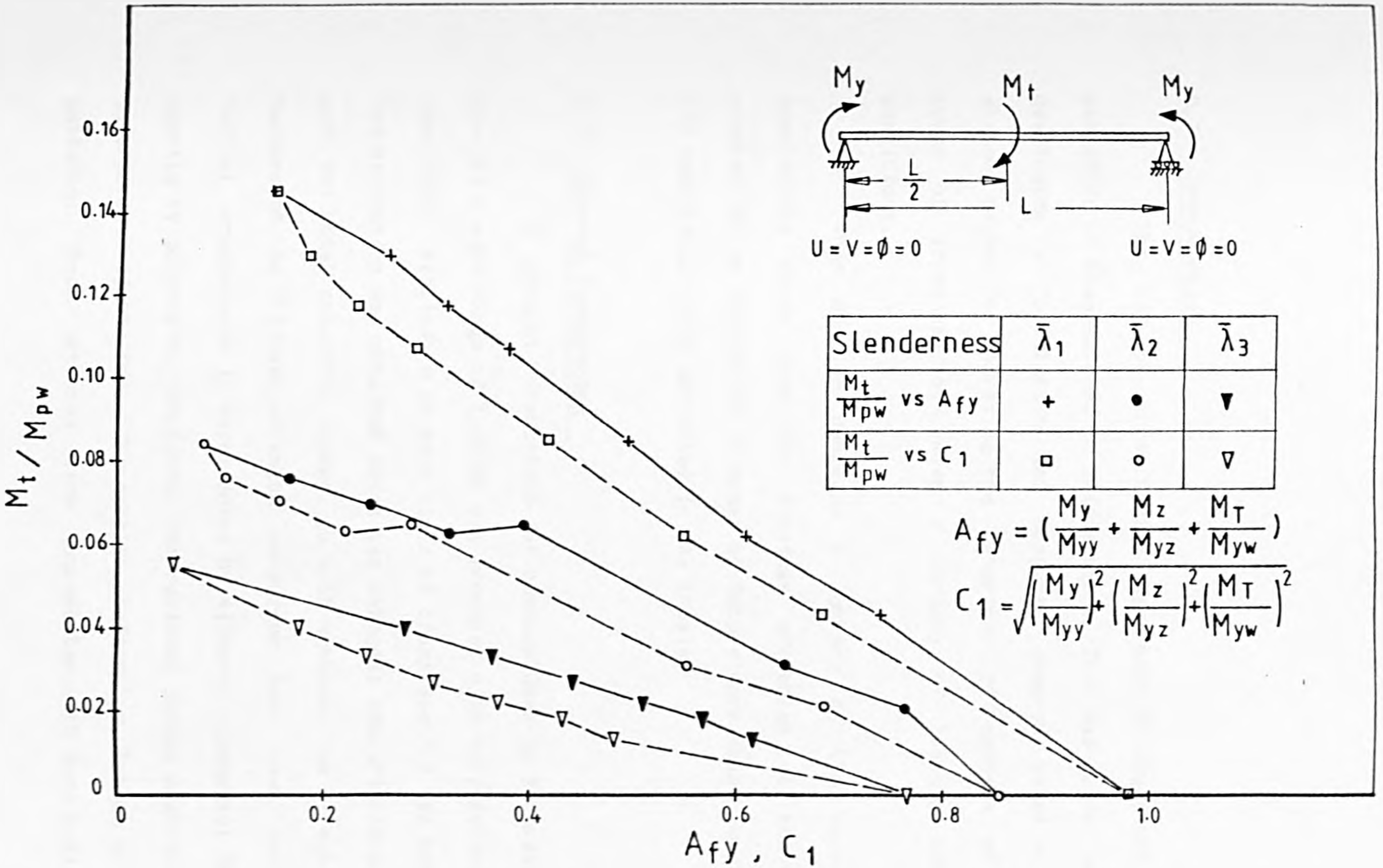


FIG. 8 . 18 INTERACTION CURVES FOR BEAM - COLUMNS OF I-BEAM SECTION (W 12 × 14) UNDER FLEXURAL BENDING AND TORSION AT MID - SPAN

Chapter 9

Conclusions and Further Work

9.1 Introduction

The aim of this thesis has been to study the full range behaviour of beam-columns in 3-dimensions. This has been achieved by developing a formulation and computer program based on the finite element method for analysing the behaviour of members of different types of cross-section under a variety of loading and boundary conditions.

This chapter contains a summary of the more important conclusions drawn from the findings presented at the end of each chapter and an indication of areas of future work which would augment the analytical study presented in the thesis.

9.2 General Formulation

A general formulation for beam-columns in 3-dimensions acted upon by a wide range of loading and provided with very general support conditions applicable to many kinds of cross-section has been derived. The derivation was obtained using two methods: the principle of virtual work and total potential energy. In both methods, the effect of several factors on the ultimate strength behaviour have been included. The initial crookedness is represented by either a sinusoidal form or more exactly by polynomial functions. The residual stress distributions were incorporated in the form of parabolic, triangular, Lehigh or any other patterns. These patterns were implemented with some modification in order to satisfy the equilibrium conditions for three-dimensional response.

The main reason for formulating the governing equations by two separate methods was to provide a check on the large number of complicated terms which resulted due to coupling between the displacements and rotations.

9.3 Derivation of Stiffness Matrices

The element stiffness matrices and the strain-displacement matrices for a beam-column of thin-walled open cross-section in 3-dimensions were derived. The linear and nonlinear tangential and geometrical stiffness matrices correspond to an element with 14 D.O.F.(seven degrees of freedom at each node: axial, in-plane, and out-of-plane displacements, three rotations about X, Y, and Z axes and warping). The interpolation functions employed were linear polynomials for the axial displacements, cubic polynomial for lateral displacements and twists, and quadratic polynomial for rotations and warping.

The terms in both the linear and nonlinear stiffness matrices were calculated by two different methods. The first is based on the average of the values of the sectorial/ sectional properties and internal forces for the two nodes, while the second uses a linear function to evaluate these values between the two nodes. The transformation matrix for the element displacements with respect to local coordinate and the global coordinate systems was developed in a sophisticated yet straightforward fashion.

9.4 Analysis Type Options

The inclusion of the higher terms in the formulations have been arranged to form at least three types of analyses; 'Linear' uses the linear tangential and geometrical stiffnesses together with linear strains, 'Nonlinear' employs linear tangential and geometrical stiffness matrices and the higher order strain-displacement relationships, and ' Full Nonlinear" which incorporates the contributions of both higher order strain terms and nonlinear stiffness matrices.

9.5 Development of Computer Program (TDCP)

A three dimensional computer program (TDCP) which implements the formulation has been developed. TDCP is based on finite element techniques and is written in the Fortran 77 Language. The program contains many subroutines, the functions of which can be summarized as:

1- Calculating the sectional and sectorial properties in the elastic and inelastic ranges for complex cross-sections.

2- Calculating the internal strains and stress resultants.

3- Controlling convergence to the correct result based on the incremental displacements and out-of-balance forces.

4- Tracing the full load-deflection response and loss of stiffness due to yield spread.

9.6 Comparison with Previous Work

The validity and versatility of both the rigorous formulation and TDCP have been tested against previously obtained experimental and theoretical results. Several examples covering flexure, flexural-torsional action, biaxial bending, and bending and torsion have been checked in the elastic and inelastic ranges. They contain a wide range of parameters e.g. different types of cross-section, loading, boundary conditions and initial geometrical imperfections. In all cases excellent agreement has been achieved when some limitations were imposed on the present formulation in order to match the more restricted approaches used elsewhere.

9.7 Ultimate Strength Behaviour of Members under Bending and Torsion

Preliminary results from a new investigation of the ultimate strength behaviour of beam-columns subjected to biaxial bending and torsion have been reported, problems in which direct torsional loading is present have received little attention previously.

Several problems of bending and torsion have been solved. The solutions were obtained by three types of analyses; 'Linear', 'Nonlinear', and Full Nonlinear. The results show the significance of the inclusion of the so called 'higher order terms' in the strain-displacement relations and the nonlinear geometrical and tangential stiffness matrices in the analysis.

Several important observations are possible:

1- The ability of the formulation presented to cover the full range of analysis of beam-columns in 3-dimensions.

2- The need to include higher order terms in the strain-displacement relations and stiffness matrices in order to avoid overestimates of load carrying capacity.

3- This analysis represents a powerful tool for the further study of problems in each of the classes previously identified.

9.8 Future Work

The work undertaken in this thesis has established a general formulation and a program TDCP which can be used for determining the ultimate strength of beam-columns. This work can be easily extended to incorporate many aspects such as:

1- Semi-Rigid Connections

This work is now under investigation and the preliminary results are encouraging (Wang et al. 1987).

2- Other types of material such as Aluminium can be included. In such case the characteristic of the material is nonlinear; this extension is also in progress.

3- Analysis of Frames in Three-dimensions; with rigid and semi-rigid connections

4- Considerations of requirements for members deforming in space to permit more rational strength/ stiffness requirements to be determined.

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Appendix A

The equilibrium equation for a member having an arbitrary cross-section can be obtained by substituting equations 3.1, 3.2, 3.24, and 3.25 into equation 3.38. It can be converted to a finite element procedure by integrating over the area of the section and making use of the definition of the stress resultants .

Recall again equation 3.38

$$\int_V [E\epsilon_{xx}\delta W_{,x}+\frac{1}{2}\sigma_{xx}\delta\{W^2_{,x}+V^2_{,x}+U^2_{,x}\}]dV$$

$$\int_V [G(W_{,y}+U_{,x})\delta(W_y+U_{,x}) + (W_{,z}+V_{,x})\delta(W_{,z}+V_{,x})]dV$$

$$+\frac{1}{2}\int_V \sigma_{xy}\delta[W_{,x}W_{,y}+V_{,x}V_{,y} + U_{,x}U_y]dV$$

$$+\frac{1}{2}\int_V \sigma_{xz}\delta[W_{,x}W_{,z}+V_{,x}V_{,z}+U_xU_z]dV = \int_s F_1\delta U_1 ds \qquad (3.38)$$

which can also be written as

$$I_1 + I_2 + I_3 + I_4 + I_5 + - I_6 = 0. \qquad (A\text{-}1)$$

where

$$I_1 = \int_V [E\varepsilon_{xx}\delta W_{,x}dV \qquad (A\text{-}2a)$$

$$I_2 = \int_V [G\{W_{,y}+U_{,x}\}\delta(W_y+U_{,x}) + \{W_{,z}+V_{,x}\}\delta(W_{,z}+V_{,x})]dV \qquad (A\_2b)$$

$$I_3 = +\frac{1}{2}\int_V \sigma_{xx}\delta\{W^2_{,x}+V^2_{,x}+U^2_{,x}\}]dV \qquad (A\_2c)$$

$$I_4 = \frac{1}{2}\int_V \sigma_{xy}\delta[W_{,x}W_{,y}+V_{,x}V_{,y}+U_{,x}U_{,y}]dV \qquad (A\_2d)$$

$$I_5 = \frac{1}{2}\int_V \sigma_{xz}\delta[W_{,x}W_{,z}+V_{,x}V_{,z}+U_{,x}U_{,z}]dV \qquad (A\_2e)$$

$$I_6 = \frac{1}{2}\int_v F_i\delta U_i ds \qquad (A\_2f)$$

Substitute equations 3.1, 3.2 and 3.24 into equation A_2a to yield

$$\int_o^L \Bigg[[EA\{W_{,x} + \frac{1}{2}(U^2_{,x} + V^2_{,x} + \rho^2\phi^2_{,x})$$

$$- \overline{Y}(V_{,xx}\Phi + U_{,xx}) - \overline{Z}(V_{,xx}-U_{,xx}\Phi)$$

$$- Y_s\Phi_{,x} + Z_sV_{,x}\Phi_{,x} - \frac{ES_\omega}{EA}\}\delta W_{,x}$$

$$- EA\overline{Y}[W_{,x}\delta(V_{,xx}\Phi+U_{,xx}) +\{ \frac{1}{2}(U^2_{,x} + V^2_{,x})$$

$$+ Y_s\Phi_{,x}V_{,x} - Z_s\Phi_{,x}U_{,x})]\delta U_{,xx}$$

$$+ W_{,x}\delta(V_{,x}\Phi_{,x}) + \frac{1}{2}(Y_s^2 + Z_s^2)\phi^2_{,x}\delta U_{,xx}]$$

$$- EA\overline{Z}[W_{,x}\delta(V_{,xx}-U_{,xx}\Phi) +\{ \frac{1}{2}(U^2_{,x} + V^2_{,x})$$

$$+ Y_s\Phi_{,x}V_{,x} - Z_s\Phi_{,x}U_{,x})\}\delta V_{,xx}$$

$$- W_{,x}\delta(U_{,x}\Phi_{,x}) + \frac{1}{2}(Y_s^2 + Z_s^2)\phi^2_{,x}\delta V_{,xx}]$$

$$+ EI_\omega\Phi_{,xx}\delta\Phi_{,xx} + EI_y[V_{,xx}\delta(V_{,xx}-U_{,x}\Phi_{,x})$$

$$- \delta(U_{,xx}V_{,xx}\Phi) - Z_s\Phi^2_{,x}\delta V_{,xx}]$$

$$+ EI_z[U_{,xx}\delta(U_{,xx}+V_{,x}\Phi_{,x}) - \delta(U_{,xx}V_{,xx}\Phi) - Y_s\Phi^2_{,x}\delta U_{,xx}]$$

$$+EI_{yz}[\delta(U_{,xx}V_{,xx} - U_{,xx}\Phi U_{,xx} + V_{,xx}\Phi V_{,xx})$$

$$+ V_{,xx}\delta(\Phi_{,x}V_{,x})$$

$$- U_{,xx}\delta(\Phi_{,x}U_{,x}) - \Phi^2_{,x}(Z_s\delta U_{,xx} + Y_s\delta V_{,xx})]$$

$$+ EI_{\omega y}[\delta\{\Phi_{,xx}(V_{,xx}\Phi + U_{,xx})\}$$

$$+ \Phi_{,xx}\delta(\Phi_{,x}V_{,x}) - Y_s\Phi^2_{,x}\delta\Phi,xx]$$

$$+ EI_{\omega z}[\delta\{\Phi_{,xx}(V_{,xx} - U_{,xx}\Phi)\}$$

$$- \Phi_{,xx}\delta(\Phi_{,x}U_{,x}) - Z_s\Phi^2_{,x}\delta\Phi,xx]$$

$$-ES_\omega[W_{,x} + \frac{1}{2}\{U^2_{,x} + V^2_{,x}\} + Y_s\Phi_{,x}V_{,x} - Z_s\Phi,xU_{,x}$$

$$+ \frac{1}{2}(Y^2_s + Z^2_s)]\delta\Phi,xx$$

$$+H_\omega\Phi^2_{,x}\delta\Phi_{,xx} + H_y\delta(\Phi^2_{,x}U_{,xx}) + H_z\delta(\Phi^2_{,x}V_{,xx}) + \Phi_{,x}\delta\Phi_{,x}\Big]dx \qquad (A\_3)$$

where

$$\rho^2 = (Y-Y_s)^2 + (Z-Z_s)^2 \qquad (A\_4a)$$

$$H_y = \frac{1}{2}\int_A Y(Y^2+Z^2)dA \qquad (A\_4b)$$

$$H_z = \frac{1}{2}\int_A Z(Y^2+Z^2)dA \qquad (A\_4c)$$

$$H_y = \frac{1}{2}\int_A Y\omega(Y^2+Z^2)dA \qquad \text{(A\_4d)}$$

which may be written as:

$$I_2 = \int_0^1 Gk\Phi_{,x}\delta\Phi_{,x}dx \qquad \text{(A\_5)}$$

$$K = \int_A [(Z-Z_s)^2+(Y-Y_s)^2-2(Y-Y_s)\frac{\partial\bar{\omega}}{\partial Z}$$

$$-2(Z-Z_s)]\frac{\partial\bar{\omega}}{\partial y}+(\frac{\partial\bar{\omega}}{\partial y})^2+(\frac{\partial\omega}{\partial z})^2] \qquad \text{(A\_6)}$$

in which K has been defined (Love 1944 and Nishida & Fukumoto 1985) as the torsional rigidity

$$I_3 = \int_0^1 F_x[U_{,x}\delta U_{,x}+V_{,x}\delta V_{,x}+\rho^2\Phi_{,x}\delta\Phi_{,x}$$

$$+\int_0^1 [Z_s\delta\{\Phi_{,x}(\Phi V_{,x}+U_{,x})\} - Y_s\delta\{\Phi_{,x}(V_{,x}-U_{,x}\Phi)\}]dx$$

$$-\int_0^1 [M_y\delta\{\Phi_{,x}(V_{,x}\Phi+U_{,x})\} +M_z\delta\{\Phi_{,x}(V_{,x}-\Phi U_{,x})\}]dx \qquad \text{(A\_7)}$$

In the same manner $I_4 + I_5$ can be written as

$$I_4+I_5=\int_0^1 Q_y[\delta\{V_{,x}\Phi-w_{,x}(V_{,x}\Phi+U_{,x})]dx$$

$$\int_0^1 Q_z[\delta\{U_{,x}\Phi+W_{,x}(V_{,x}+\Phi U_{,x})\}dx]dx$$

$$+\int_0^1 [M_x \delta\{V_{,xx}(V_{,x}\Phi+U_{,x})+(U_{,x}\Phi_{,x}-U_{,x})U_{,x}\}]dx$$

$$-\int_0^1 [M_x \delta\{V_{,x}(V_{,xx}\Phi+V_{,x}\Phi_{,x})+(U_{,x}\Phi+V_{,x})U_{,xx}\}]dx + A \qquad (A\_8)$$

where

$$A = \int_0^1 [\sigma_{xz}\frac{d\overline{\omega}_n}{dy} + \sigma_{xy}\frac{d\overline{\omega}_n}{dy}]\Phi_{,x}\overline{X}dx \qquad (A\_9)$$

The above equation may be shown, after some manipulation to be equal to zero.

where

$$\overline{X} = \delta[\Phi_{,xx}(-W_{,x}+(Y-Y_s)\ \{V_{,xx}\Phi+U_{,xx}+\Phi_{,x}V_{,x}\}$$

$$+(Z-Z_s)\{V_{,xx}-U_{,xx}\Phi-\Phi_{,x}U_{,x}+\overline{\omega}\Phi_{,xx}\}]$$

Appendix B

The equilibrium equation for a member having an arbitrary cross-section can be obtained by substituting equations 3.1, 3.2, 3.24, and 3.25 into equation 3.38 It can be converted to a finite element procedure by integrating over the area of the section, making use of the definition of the stress resultants. Neglecting $W_{,x}W_{,y}$, $W_{,x}W_{,z}$, and $W^2_{,x}$ from equation 3.38 yields:

$$\int_V \sigma_{xx}\delta[W_{,x} + \frac{1}{2}(U^2_{,x} + V^2_{,x})]dV$$

$$\frac{1}{2} \int_V \sigma_{xy}\delta[(W_{,y} + U_{,x}) + (W_{,z} + V_{,x})]dV$$

$$\frac{1}{2}\int_V [\sigma_{xz} \delta \{ U_{,x}U_{,y} + V_{,x}V_{,y} \} + \sigma_{xy} \delta \{U_{,x}U_{,z} + V_{,x}V_{,z}\}]dV$$

$$+\int_s F_i \delta U_i ds=0 \qquad (B\_1)$$

Equation B_1 can be written as :

$$I_1 + I_3 + I_3 + I_4 = 0 \qquad (B\_2)$$

where

$$I_1 = \int_V \sigma_{xx}\delta [U_{,x} + \frac{1}{2} (U^2_{,2} + V^2_{,x})]dV \qquad (B\_3a)$$

$$I_2 = \frac{1}{2} \int_V [\sigma_{xy}\delta \{ W_{,x} + U_{,x}\} + \{W_{,z} + V_{,x}\}]dV \qquad (B\_3b)$$

$$I_3 = \frac{1}{2}\int_V [\sigma_{xy}\delta\{U_{,x}U_{,y}\}+$$

$$+\sigma_{xy}\delta\{U_{,x}U_{,z} + V_{,x}V_{,z}\}]dV \qquad \text{(B\_3c)}$$

$$I_4 = -\int_s F_i U_i ds \qquad \text{(B\_3d)}$$

Substituting equation 3.1, 3.2 and 3.24 into equation B_3a yields:

$$I_1 = \int_V \sigma_{xx}[\delta\{W_{,x} + \frac{1}{2}(U^2_{,x} + V^2_{,x} + \rho^2\Phi^2_{,x})$$

$$-Y(V_{,xx}\Phi + U_{,xx}\frac{\Phi^2}{2}) + Z(U_{,xx}\Phi + V_{,xx}\frac{\Phi^2}{2})]$$

$$-Y\delta U_{,xx} - ZV_{,xx} - \overline{\omega}\delta\Phi_{,xx}$$

$$+\delta\{Z_s\Phi_{,x}(V_{,x}\Phi + U_{,x}) - Y_s\Phi_{,x}(V_{,x} - U_{,x}\Phi)\}]dV$$

$$= \int_V \sigma_{,xx}[\delta W_{,x} - Y\delta U_{,xx} - Z\delta V_{,xx} - \overline{\omega}\delta\Phi_{,xx}]dV$$

$$+\int_0^L F[U_{,x}\delta U_{,x} + V_{,x}\delta V_{,x} - Y_s\delta\{\Phi_{,x}(V_{,x} - U_{,x}\Phi)\}$$

$$+Z_s\delta\{\Phi_{,x}(V_{,x}\Phi + U_{,x})\} + M_\rho\Phi_{,x}\delta\Phi_{,x}$$

$$- M_z\delta\{V_{,xx}\Phi - U_{,xx}\frac{\Phi^2}{2}\} + M_y\delta\{U_{,xx} + V_{,xx}\frac{\Phi^2}{2}\}]dx$$

The above equation can be written after substituting for $\sigma_{,xx}$ as:

$$I_1 = \int_0^L \Big[EA[\{W_{,x} + \frac{1}{2}(U^2_{,x} + V^2_{,x} + \rho^2\Phi^2_{,x}\}$$

$$-EA\overline{Y}(V_{,xx}\Phi+U_{,xx})+EA\overline{Z}(U_{,xx}\Phi-V_{,xx})$$

$$-Y_sEA\Phi_{,x}V_{,x}+Z_s\Phi_{,x}U_{,x}-ES_\omega\Phi_{,x}]\delta W_{,x}$$

$$-[EA\overline{Y}\{W_{,x}+\frac{1}{2}(U^2_{,x}+V^2_{,x})-Y_s\Phi_{,x}V_{,x}+Z_z\Phi_{,x}U_{,x}\}]\delta U_{,xx}$$

$$-[EA\overline{Z}\{W_{,x}+\frac{1}{2}(U^2_{,x}+V^2_{,x})-Y_s\Phi_{,x}V_{,x}+Z_z\Phi_{,x}U_{,x}\}]\delta V_{,xx}$$

$$-[ES_\omega\{W_{,x}+\frac{1}{2}(U^2_{,x}+V^2_{,x})-Y_s\Phi_{,x}V_{,x}+Z_z\Phi_{,x}U_{,x}\}]\delta\Phi_{,xx}$$

$$+EI_z\{V_{,xx}\Phi+U_{,xx}\}\delta U_{,xx}\quad +EI_y\{U_{,xx}\Phi-V_{,xx}\}\delta V_{,xx}$$

$$+EI_\omega\Phi_{,xx}\delta\Phi_{,xx}+EI_{\omega y}\{\Phi_{,xx}\delta U_{,xx}+(V_{,xx}\Phi+U_{,xx})\delta\Phi_{,xx}\}$$

$$+EI_{\omega z}\{\Phi_{,xx}\delta V_{,xx}+(U_{,xx}\Phi+V_{,xx})\delta\Phi_{,xx}\}$$

$$-EI_{yz}[\{U_{,xx}\Phi-V_{,xx}\}\delta U_{,xx}-\{V_{,xx}\Phi+U_{,xx}\}\delta V_{,xx}]$$

$$+[EAH_y+EA\frac{\overline{Y}}{2}\{Z_s^2+Y_s^2\}-Y_sEI_z-Z_sEI_{yz}]\Phi^2_{,x}\delta U_{,xx}$$

$$+[EAH_z+EA\frac{\overline{Z}}{2}\{Z_s^2+Y_s^2\}-Y_sEI_{yz}-Z_sEI_y]\Phi^2_{,x}\delta V_{,xx}$$

$$+[EAH_\omega+\frac{ES_\omega}{2}\{Z_s^2+Y_s^2\}-Y_sEI_{y\omega}-Z_sEI_{z\omega}]\Phi^2_{,x}\delta\Phi_{,xx}$$

$$F[U_{,x}\delta U_{,x}+V_{,x}\delta V_{,x}-Y_s\delta\{\Phi_{,x}(V_{,x}-U_{,x}\Phi)\}$$

$$Z_s\delta\{\Phi_{,x}(V_{,x}\Phi+U_{,x})\}]+M_\rho\Phi_{,x}\delta\Phi_{,x}$$

$$\left.-M_z\delta\{V_{,xx}\Phi-U_{,xx}\frac{\Phi^2}{2}\}\quad +M_y\delta\{U_{,xx}\Phi+v_{,xx}\frac{\Phi^2}{2}\}\right]dx \qquad \text{(B\_5)}$$

Substituting equations 3.1 and 3.2 in equation B_3c yields the following

$$I_3 = \int_0^1 Q_y \delta\{V_{,x}\Phi - U_{,x}\frac{\Phi^2}{2}\}dx - \int_0^1 Q_z \delta\{U_{,x}\Phi + V_{,x}\frac{\Phi^2}{2}\}dx$$

$$\int_V \{\sigma_{xz}(Y-Y_s) - \sigma_{xy}(Z-Z_s)\}\delta(\Phi_{,x}\frac{\Phi^2}{2})dx$$

$$\int_V \{\sigma_{xz}(Z-Z_s) = \sigma_{xy}(Y-Y_s)\}\delta(\Phi_{,x}\Phi)dx$$

$$\int_V M_x \delta(\Phi_{,x}\frac{\Phi^2}{2})dV \qquad \text{(B\_6)}$$

Substituting equations 3.1 & 3.2 and equation 3.24 into equation. B_3b yields :

$$I_2 = -\int_0^1 [Q_y\{V_{,x}\Phi - U_{,x}\frac{\Phi^2}{2}\} + Q_z\{U_{,x}\Phi - V_{,x}\frac{\Phi^2}{2}\}]dx$$

$$- \int_V [\{\sigma_{xy}(Z-Z_s) - \sigma_{xz}(Y-Y_s)\}\Phi_{,x}$$

$$-\{\sigma_{xy}(Z-Z_s) - \sigma_{xz}(Y-Y_s)\}\Phi_{,x}\Phi dV]]dv$$

$$- \int_0^1 M_x \delta\{\frac{\Phi^2}{2}\Phi_{,x}\}dx \qquad \text{(B\_7)}$$

Adding equations B_6 and B-7 yields :

$$I_2 + I_3 = KG\Phi_{,x}\delta\Phi_{,x} \qquad \text{(B\_8)}$$

The last equation has previously been derived by several authors e.g. Vlasov (1961), Bleich (1952), Timoshenko and Gere (1961), and Rajasekaran (1971). The general equilibrium condition can be written as;

$$I_1 + I_2 + I_3 + I_4 = 0 \tag{B_9}$$

Appendix C

The term CC in equation 3.63 is explained in full detail in the following procedure viz:

$$CC = \frac{1}{2}\int_V \Big[\sigma_{xx}[Y^2\{U^2_{,xx} + 2U_{,xx}(V_{,xx}\Phi + V_{,x}\Phi_{,x})\}$$

$$+\{Z^2(V^2_{,xx} - 2V_{,xx}(U_{,xx}\Phi + U_{,x}\Phi_{,x})\} + \overline{\omega}^2_n\Phi^2_{,x}$$

$$+ YZ\{(V_{,xx}\Phi + U_{,xx} + V_{,x}\Phi_{,x})V_{,xx} - U_{,xx}(U_{,xx}\Phi + U_{,x}\Phi_{,x})\}$$

$$+ ZY\{(-U_{,xx}\Phi + V_{,xx} + U_{,x}\Phi_{,x})U_{,xx} + V_{,xx}(V_{,xx}\Phi + V_{,x}\Phi_{,x})\}$$

$$+2Y\overline{\omega}_n(V_{,xx}\Phi + U_{,xx} + V_{,x}\Phi_{,x})\Phi_{,xx}$$

$$+2Z\overline{\omega}_n(V_{,xx} - U_{,xx}\Phi - U_{,x}\Phi_{,x})\Phi_{,xx}\Big]dV \qquad \text{(cc-1)}$$

Rearranging equation cc-1 as

$$CC = \frac{1}{2}\int_V \Big[\sigma_{,xx}\{Y^2U^2_{,xx} + Z^2V^2_{,xx} + \overline{\omega}^2_n\Phi^2_{,xx}$$

$$+ YZU_{,xx}V_{,xx} + ZYV_{,xx}U_{,xx} + Y\overline{\omega}_n\Phi_{,xx}U_{,xx}$$

$$+Y\overline{\omega}_nU_{,xx}\Phi_{,xx} + Z\overline{\omega}_{\Phi_{,xx}}V_{,xx} + \overline{\omega}_nYV_{,xx}\Phi_{,xx} + H.O.T\}\Big]dv \qquad \text{(cc-2)}$$

The longitudinal stress can be represented by using interaction stress formulae plus the higher order terms (H.O.T) as:

$$\sigma_{,xx} = \frac{P}{A} + \frac{M_y}{I_y}Z + \frac{M_z}{I_z}Y + \frac{M_{\overline{\omega}}}{I_\omega}\overline{\omega}_n + H.O.T. \qquad \text{(cc-3)}$$

Substituting equation cc-3 into equation cc-2 yields

$$\frac{1}{2}\int_0^1 \Big[\frac{P}{A}[I_z U^2_{,xx} + I_y V^2_{,xx} + I_\omega \Phi^2_{,xx}$$

$$I_{yz}U_{,xx}V_{,xx} + I_{zy}V_{,xx}U_{,xx} + I_{y\omega}\Phi_{,xx}U_{,xx}$$

$$I_{y\omega}U_{,xx}\Phi_{,xx} + I_{z\omega}\Phi_{,xx}V_{,xx} + I_{z\omega}V_{,xx}\Phi_{,xx}]$$

$$+ \frac{M_y}{I_y}[a_1 U^2_{,xx} + b_1 V^2_{,xx} + a_4 \Phi^2_{,xx}$$

$$+b_2 U_{,xx}V_{,xx} + a_1 V_{,xx}U_{,xx} + 2a_3\Phi_{,xx}U_{,xx} + 2d_3 V_{,xx}\Phi_{,xx}]$$

$$+ \frac{M_z}{I_z}[a_2 U^2_{,xx} + b_2 V^2_{,xx} + c_2 \Phi^2_{,xx}$$

$$+b_2 U_{,xx}V_{,xx} + a_1 V_{,xx}U_{,xx} + 2b_2\Phi_{,xx}U_{,xx} + 2b_3 V_{,xx}\Phi_{,xx}]$$

$$+ \frac{M_\omega}{I_\omega}[a_3 U^2_{,xx} + b_3 V^2_{,xx} + c_3 \Phi^2_{,xx}$$

$$+b_2 U_{,xx}V_{,xx}+b_2 V_{,xx}U_{,xx}+2a_4\Phi_{,xx}U_{,xx} + 2b_3 V_{,xx}\Phi_{,xx}]\Big]dx \qquad \text{(cc.4)}$$

variations of equation cc.4 as

$$\delta(CC) = \int_0^1 \Big[\{(D_1 U_{,xx} + D_5 V_{,xx} + D_6\Phi_{,xx})\delta U_{,xx}$$

$$(D_2 U_{,xx} + D_4 V_{,xx} + D_7\Phi_{,xx})\delta V_{,xx}$$

$$(D_3 U_{,xx} + D_6 V_{,xx} + D_7\Phi_{,xx})\delta\Phi_{,xx}\Big]dx \qquad \text{(cc.4)}$$

where

$$D_1 = [\frac{P}{A}I_z + a_1\frac{M_y}{I_y} + a_2\frac{M_z}{I_z} + a_3\frac{M_\omega}{I_\omega}]$$

$$D_2 = [\frac{P}{A} I_y + c_4 \frac{M_y}{I_y} + b_2 \frac{M_z}{I_z} + b_3 \frac{M_\omega}{I_\omega}]$$

$$D_3 = [\frac{P}{A} I_\omega + c_1 \frac{M_y}{I_y} + c_2 \frac{M_z}{I_z} + c_3 \frac{M_\omega}{I_\omega}]$$

$$D_4 = [\frac{P}{A} I_{yz} + a_1 \frac{M_y}{I_y} + b_2 \frac{M_z}{I_z} + d_3 \frac{M_\omega}{I_\omega}]$$

$$D_5 = [\frac{P}{A} I_{zy} + b_2 \frac{M_y}{I_y} + a_1 \frac{M_z}{I_z} + d_3 \frac{M_\omega}{I_\omega}]$$

$$D_6 = [\frac{P}{A} I_{\omega y} + a_3 \frac{M_y}{I_y} + d_3 \frac{M_z}{I_z} + a_4 \frac{M_\omega}{I_\omega}]$$

$$D_7 = [\frac{P}{A} I_{\omega z} + d_3 \frac{M_y}{I_y} + b_3 \frac{M_z}{I_z} + a_4 \frac{M_\omega}{I_\omega}]$$

and

$$a_1 = \int_A Y^2 Z \, dA \quad a_2 = \int_A Y^3 \, dA \quad a_3 = \int_A \overline{\omega} Y^2 \, dA \quad a_4 = \int_A \overline{\omega}^2 Y \, dA$$

$$b_1 = \int_A Z^3 \, dA \quad b_2 = \int_A Y Z^2 \, dA \quad b_3 = \int_A \overline{\omega} Z^2 \, dA$$

$$c_2 = \int_A \overline{\omega}^2 Z \, dA \quad c_3 = \int_A \overline{\omega}^3 \, dA \quad c_3 = \int_A \overline{\omega}^3 \, dA$$

Appendix E

$$P011= \frac{.6}{1}\Phi_1 + \frac{9}{70}X_1 + \frac{.6}{1}\Phi_2 - \frac{9}{70}X_2$$

$$P012= \frac{1}{70}\Phi_1 + \frac{1}{140}X_1 + \frac{4}{35}\Phi_2 - \frac{3}{140}X_2$$

$$P014= \frac{4}{35}\Phi_1 + \frac{31}{140}X_1 - \frac{1}{70}\Phi_2 - \frac{1}{140}X_2$$

$$P022=- 43\frac{1}{420}\Phi_1 + \frac{1^2}{120}X_1 + \frac{131}{420}\Phi_2 - \frac{1^2}{168}X_2$$

$$P024=- \frac{1}{60}\Phi_1 - \frac{1^2}{840}X_1 - \frac{1}{60}\Phi_2 + \frac{1^2}{840}X_2$$

$$P044=- \frac{131}{40}\Phi_1 + \frac{1^2}{168}X_1 + 43\frac{1}{420}\Phi_2 - \frac{1^2}{120}X_2$$

P013= -P011 P023= -P012

P033= -P013 P034= -P014

$$S011 = \frac{6}{1^3}\Phi_1 + \frac{.6}{1^2}X_1 + \frac{6}{1^3}\Phi_2 - \frac{.6}{1^2}X_2$$

$$S012 = \frac{4.2}{1^2}\Phi_1 + \frac{.4}{1}X_1 + \frac{1.8}{1^2}\Phi_2 - \frac{.2}{1}X_2$$

$$S014 = \frac{1.8}{1^2}\Phi_1 + \frac{.2}{1}X_1 + \frac{4.2}{1^2}\Phi_2 - \frac{.4}{1}X_2$$

$$S024 = \frac{\Phi_1}{1} + \frac{1}{15}X_1 + \frac{\Phi_2}{1} - \frac{1}{15}X_2$$

$$S044 = \frac{.8}{1}\Phi_1 + \frac{2}{15}X_1 + \frac{3.2}{1}\Phi_2 - \frac{1}{3}X_2$$

S013= -S012 S023=-S012

S033= -S013 S034=-S014

$$YU11 = \frac{6}{l}U_1 + \frac{4.2}{l^2}\theta_{u1} - \frac{6}{l^3}U_2 + \frac{1.8}{l^2}\theta_{u2}$$

$$YU12 = -\frac{.6}{l^2}U_1 + \frac{.4}{l}\theta_{u1} + \frac{.6}{l^2}U_2 + \frac{.2}{l}\theta_{u2}$$

$$YU13 = \frac{6}{l^3}U_1 + \frac{1.8}{l^2}\theta_{u1} - \frac{6}{l^3}U_2 + \frac{4.2}{l^2}\theta_2$$

$$YU14 = -\frac{.6}{l^2}U_1 - \frac{.2}{l}\theta_{u1} + \frac{.6}{l^2}U_2 - \frac{.4}{l}\theta_{u2}$$

$$YU21 = \frac{4.2}{l^2}U_1 + \frac{3.2}{l}\theta_{u1} - \frac{4.2}{l^2}U_2 + \frac{1}{l}\theta_{u2}$$

$$YU22 = \frac{.4}{l}U_1 + \frac{1}{3}\theta_{u1} - \frac{.4}{l}U_2 + \frac{1}{15}\theta_{u2}$$

$$YU23 = \frac{1.8}{l^2}U_1 + \frac{.8}{l}\theta_{u1} - \frac{1.8}{l^2}U_2 + \frac{1}{l}\theta_{u2}$$

$$YU24 = -\frac{.2}{l}U_1 - \frac{2}{15}\theta_{u1} - \frac{.2}{l}U_2 - \frac{1}{15}\theta_{u2}$$

$$YU41 = \frac{1.8}{l^2}U_1 + \frac{1}{l}\theta_{u1} - \frac{1.8}{l^2}U_2 + \frac{.8}{l}\theta_{u2}$$

$$YU42 = \frac{.2}{l}U_1 + \frac{1}{15}\theta_{u1} - \frac{.2}{l}U_2 + \frac{2}{15}\theta_{u2}$$

$$YU43 = \frac{4.2}{l^2}U_1 + \frac{1}{l}\theta_{u1} - \frac{4.2}{l^2}U_2 + \frac{3.2}{l}\theta_{u2}$$

$$YU44 = -\frac{.4}{l}U_1 - \frac{1}{15}\theta_{u1} - \frac{.4}{l}U_2 + \frac{1}{15}\theta_{u2}$$

$YU31 = -YU11$ $\qquad$ $YU32 = -YU12$

$YU33 = -YU13$ $\qquad$ $YU34 = -YU14$

$$ZU11 = -\frac{1.2}{l^2}\theta_{u1} + \frac{1.2}{l^2}\theta_{u2}$$

$$ZU12 = \frac{.6}{l^2}U_1 + \frac{.2}{l}\theta_{u1} - \frac{.6}{l^2}U_2 + \frac{.4}{l}\theta_{u2}$$

$$ZU14 = -\frac{.6}{l^2}U_1 - \frac{.4}{l}\theta_{u1} + \frac{.6}{l^2}U_2 - \frac{.2}{l}\theta_{u2}$$

$$ZU22 = -\frac{.4}{l}U_1 + \frac{1}{3}\theta_{u1} + \frac{.4}{l}U_2 - \frac{1}{15}\theta_{u2}$$

$$ZU24 = \frac{1}{30}\, u1 - \frac{1}{30}\, u2$$

$$ZU44 = \frac{.4}{I}U1 + \frac{1}{15}\theta_{u1} - \frac{.4}{I}U2 - \frac{1}{3}\theta_{u2}$$

ZU13= -ZU11 ZU23= -ZU12

ZU33= ZU11 ZU34=-ZU14

$$AU1 = \frac{6}{5I}U1 + \frac{1}{10}\theta_{u1} - \frac{6}{5I}U2 + \frac{1}{10}\theta_{u2}$$

$$AU2 = \frac{1}{10}U1 + \frac{2I}{15}\theta_{u1} - \frac{1}{10}U2 - \frac{1}{30}\theta_{u2}$$

$$AU4 = \frac{1}{10}U1 - 1/15\theta_{u1} - \frac{1}{10}U2 - \frac{2I}{15}\theta_{u2}$$

AU13=-AU1

For YV11, YV12, etc. change U to V, θ_u to θ_v etc. in the expression for Yu11, Yu12, etc.

Appendix F

$$W01 = -.5\Phi_1 - .11\Phi_2 + .1\chi_2$$

$$W02 = .11\Phi_1 - .11\Phi_2 + \frac{1^2}{60}X2$$

$$W04 = \frac{1}{10}\Phi 1 - \frac{1^2}{60}X1 + \Phi 2$$

$$W03 = -W01$$

$$CX11= \frac{.6}{1^2}\Phi 1 - \frac{1}{70}X1 - \frac{.6}{1^2}\Phi 2 + \frac{4}{35}X2$$

$$CX12= \frac{9}{70}\Phi 1 + \frac{1}{140}X1 - \frac{9}{70}\Phi 2 + \frac{3}{140}1X2$$

$$CX13= \frac{.6}{1^2}\Phi 1 + \frac{3}{35}X1 - \frac{.6}{1^2}\Phi 2 - \frac{1}{70}X2$$

$$CX14= -\frac{9}{70}\Phi 1 - \frac{3}{140}1X1 + \frac{9}{70}\Phi 2 - \frac{1}{140}X2$$

$$CX21= -\frac{1}{70}\Phi 1 - \frac{43}{70}1X1 + \frac{1}{70}1^2\Phi 2 - \frac{1}{60}X2$$

$$CX22=\frac{1}{140}\Phi 1 + \frac{1^2}{120}X1 - \frac{1}{140}\Phi 2 - \frac{1^2}{840}X2$$

$$CX23= \frac{4}{35}\Phi 1 + \frac{13}{140}1X1 - \frac{4}{35}\Phi 2 - \frac{1}{60}X2$$

$$CX24=-\frac{3}{140}1\Phi 1 - \frac{1^2}{168}X1 - \frac{3}{140}1\Phi 2 + \frac{1^2}{840}X2$$

$$CX41= \frac{4}{35}\Phi 1 - \frac{1}{60}X1 - \frac{4}{35}\Phi 2 + \frac{13}{420}1X2$$

$$CX42=-\frac{3}{140}1\Phi 1 - \frac{1^2}{840}X1 - \frac{3}{140}1\Phi 2 + \frac{1^2}{168}X2$$

$$CX43=-\frac{1}{140}\Phi 1 + \frac{1}{60}X1 + \frac{1}{70}\Phi 2 - \frac{43}{120}1X2$$

$$CX44=-\frac{1}{140}\Phi 1 + \frac{1^2}{840}X1 + \frac{1}{140}\Phi 2 - \frac{1^2}{120}X2$$

$CX31 = -CX11 \qquad CX32 = -Cx12$

$CX33 = -CX13 \qquad CX34 = CX14$

$$B011 = \frac{.6}{l}\Phi_1 + \frac{9}{70}\chi_1 + \frac{.6}{l}\Phi_2 - \frac{9}{70}\chi_2$$

$$B012 = \frac{-1}{70}\Phi_1 + \frac{1}{140}\chi_1 + \frac{4}{35}\Phi_2 - \frac{3}{140}\chi_2$$

$$B014 = \frac{4}{35}\Phi_1 + \frac{3}{140}l\chi_1 + \frac{1}{70}\Phi_2 - \frac{1}{140}\chi_2$$

$$B022 = \frac{43}{420}l\Phi_1 + \frac{l^2}{120}l\chi_1 + \frac{13}{420}l\Phi_2 - \frac{l^2}{168}\chi_2$$

$$B024 = -\frac{1}{60}l\Phi_1 - \frac{l^2}{840}l\chi_1 - \frac{1}{60}\Phi_2 - \frac{l^2}{840}\chi_2$$

$$B044 = \frac{13}{40}l\Phi_1 + \frac{l^2}{168}l\chi_1 + \frac{43}{420}l\Phi_2 - \frac{l^2}{120}\chi_2$$

$B013 = -B011 \qquad B023 = -B012 \qquad B033 = -B013 \qquad B034 = -B014$

$$XU11 = -\frac{1.542858}{l^2}U_1 - \frac{6}{351}\theta_{u1} + \frac{1.542858}{l^2}U_2 - \frac{6}{351}\theta_{u2}$$

$$XU12 = -\frac{6}{351}U_1 - \frac{3}{35}\theta_{u1} + \frac{6}{351}u_2 + \frac{1}{70}\theta_{u2}$$

$$XU14 = -\frac{6}{351}U_1 + \frac{1}{70}\theta_{u1} + \frac{6}{351}U_2 + \frac{13}{735}\theta_{u2}$$

$$XU22 = -\frac{3}{35}u_1 + \frac{21}{35}\theta_{u1} + \frac{3}{35}u_1 + \frac{1}{105}\theta_{u2}$$

$$XU24 = \frac{1}{70}U_1 - \frac{1}{105}\theta_{u1} - \frac{1}{70}U_2 + \frac{1}{105}\theta_{u2}$$

$$XU44 = -\frac{3}{35}U_1 - \frac{1}{105}\theta_{u1} + \frac{3}{35}U_2 + \frac{21}{35}\theta_{u2}$$

$XU13 = -XU11 \qquad XU23 = -XU12 \qquad XU33 = XU11 \qquad XU34 = -XU14$

For XV11, XV12, etc. change u_1, θ_{u1} etc. to be either v_1, θ_{v1} etc. in the expression for XU11, XU12, etc. , or Φ_1, χ_1 in the expression for X011, X012, etc.

Appendix G

Coefficient of Non-linear Tangential Stiffness Matrix

The coefficients of the tangential geometrical stiffness matrix of a beam-column of thin-walled open cross-section under different loading conditions are listed in Table 4.7 and presented herein as:

$$a = a_{u1} - \bar{Z}(b_{\phi 1} - a_{\phi 1}) - Y_s a_{\phi 1} - Z_s a_{\phi 1}$$

$$b = a_{u2} - \bar{Z}(b_{\phi 2} - a_{\phi 2}) - Y_s a_{\phi 2} - Z_s a_{\phi 2}$$

$$j = a_{u3} - \bar{Z}(b_{\phi 3} - a_{\phi 3}) - Y_s a_{\phi 3} - Z_s a_{\phi 3}$$

$$g = a_{u4} - \bar{Z}(b_{\phi 4} - a_{\phi 4}) - Y_s a_{\phi 4} - Z_s a_{\phi 4}$$

$$c = a_{v1} - \bar{Y}(b_{\phi 1} - a_{\phi 1}) + Z_s a_{\phi 1} - Y_s a_{\phi 1}$$

$$d = a_{v2} - \bar{Y}(b_{\phi 2} - a_{\phi 2}) + Z_s a_{\phi 2} - Y_s a_{\phi 2}$$

$$h = a_{v3} - \bar{Y}(b_{\phi 3} - a_{\phi 3}) + Z_s a_{\phi 3} - Y_s a_{\phi 3}$$

$$i = a_{v4} - \bar{Z}(b_{\phi 4} - a_{\phi 4}) + Z_s a_{\phi 4} - Y_s a_{\phi 4}$$

$$e = -\bar{Y}(X_{v1} - a_{v1}) + \bar{Z}(X_{u1} - a_{u1}) - Y_s a_{u1} + Z_s a_{v1}$$

$$f = -\bar{Y}(X_{v2} - a_{v2}) + \bar{Z}(X_{u2} - a_{u2}) - Y_s a_{u2} + Z_s a_{v2}$$

$$o = -\bar{Y}(X_{v3} - a_{v3}) + \bar{Z}(X_{u3} - a_{u3}) - Y_s a_{u3} + Z_s a_{v3}$$

$$n = -\bar{Y}(X_{v4} - a_{v4}) + \bar{Z}(X_{u4} - a_{u4}) - Y_s a_{u4} + Z_s a_{v4}$$

$$a1 = EI_z(C_{\phi 11} + C_{\phi 11}) + EI_{zy}(f_{\phi 11} + C_{\phi 11} + C_{\phi 11})$$

$$a_2 = EI_z(C_{\phi 12} + C_{\phi 21}) + EI_{zy}(f_{\phi 12} + C_{\phi 21} + C_{\phi 12})$$

$$a_3 = EI_z(C_{\phi 22} + C_{\phi 22}) + EI_{zy}(f_{\phi 22} + C_{\phi 22} + C_{\phi 22})$$

$$a_4 = EI_z(C_{\phi 31} + C_{\phi 13}) + EI_{zy}(f_{\phi 13} + C_{\phi 31} + C_{\phi 13})$$

$$a_5 = EI_z(C_{\phi 41} + C_{\phi 14}) + EI_{zy}(f_{\phi 14} + C_{\phi 41} + C_{\phi 14})$$

$$a_6 = EI_z(C_{\phi 32} + C_{\phi 23}) + EI_{zy}(f_{\phi 23} + C_{\phi 32} + C_{\phi 23})$$

$$a_7 = EI_z(C_{\phi 42} + C_{\phi 24}) + EI_{zy}(f_{\phi 24} + C_{\phi 24} + C_{\phi 42})$$

$$a_8 = EI_z(C_{\phi 33} + C_{\phi 33}) + EI_{zy}(f_{\phi 33} + C_{\phi 33} + C_{\phi 33})$$

$$a_9 = EI_z(C_{\phi 43} + C_{\phi 34}) + EI_{zy}(f_{\phi 34} + C_{\phi 34} + C_{\phi 43})$$

$$a_{10} = -EI_y(C_{\phi 44} + C_{\phi 44}) + EI_{zy}(f_{\phi 44} + C_{\phi 44} + C_{\phi 44})$$

$$b_1 = -EI_y(C_{\phi 11} + C_{\phi 11}) + EI_{zy}(f_{\phi 11} + C_{\phi 11} + C_{\phi 11})$$

$$b_2 = -EI_y(C_{\phi 12} + C_{\phi 21}) + EI_{zy}(f_{\phi 12} + C_{\phi 21} + C_{\phi 12})$$

$$b_3 = -EI_y(C_{\phi 22} + C_{\phi 22}) + EI_{zy}(f_{\phi 22} + C_{\phi 22} + C_{\phi 22})$$

$$b_4 = -EI_y(C_{\phi 31} + C_{\phi 13}) + EI_{zy}(f_{\phi 13} + C_{\phi 31} + C_{\phi 13})$$

$$b_5 = -EI_y(C_{\phi 41} + C_{\phi 14}) + EI_{zy}(f_{\phi 14} + C_{\phi 41} + C_{\phi 14})$$

$$b_6 = -EI_y(C_{\phi 32} + C_{\phi 23}) + EI_{zy}(f_{\phi 23} + C_{\phi 32} + C_{\phi 23})$$

$$b_7 = -EI_y(C_{\phi 42} + C_{\phi 24}) + EI_{zy}(f_{\phi 24} + C_{\phi 24} + C_{\phi 42})$$

$$b_8 = -EI_y(C_{\phi 33} + C_{\phi 33}) + EI_{zy}(f_{\phi 33} + C_{\phi 33} + C_{\phi 33})$$

$$b_9 = -EI_y(C_{\phi 43} + C_{\phi 34}) + EI_{zy}(f_{\phi 34} + C_{\phi 34} + C_{\phi 43})$$

$$b_{10} = -EI_y(C_{\phi 44} + C_{\phi 44}) + EI_{zy}(f_{\phi 44} + C_{\phi 44} + C_{\phi 44})$$

$$c_1 = EI_\omega \beta_\omega (C_{\phi 11} + 2C_{\phi 11}) + EI_{\omega y}(e_{v11} + C_{v11} - Y_s C_{\phi 11} - 2Y_s C_{\phi 11})$$

$$- EI_{\omega z}(-e_{u11} + C_{u11} + Z_s C_{\phi 11} + 2Z_s C_{\phi 11})$$

$$c_2 = EI_\omega \beta_\omega (C_{\phi 21} + 2C_{\phi 12}) + EI_{\omega y}(e_{v12} + C_{v21} - Y_s C_{\phi 21} - 2Y_s C_{\phi 12})$$

$$- EI_{\omega z}(-e_{u21} + C_{u21} + Z_s C_{\phi 21} + 2Z_s C_{\phi 12})$$

$$c_3 = EI_\omega \beta_\omega (C_{\phi 22} + 2C_{\phi 22}) + EI_{\omega y}(e_{v22} + C_{v22} - Y_s C_{\phi 22} - 2Y_s C_{\phi 22})$$

$$- EI_{\omega z}(-e_{u22} + C_{u22} + Z_s C_{\phi 22} + 2Z_s C_{\phi 22})$$

$$c_4 = EI_\omega \beta_\omega (C_{\phi 31} + 2C_{\phi 13}) + EI_{\omega y}(e_{v31} + C_{v31} - Y_s C_{\phi 31} - 2Y_s C_{\phi 13})$$

$$- EI_{\omega z}(-e_{u31} + C_{u31} + Z_s C_{\phi 31} + 2Z_s C_{\phi 13})$$

$$c_5 = EI_\omega \beta_\omega (C_{\phi 41} + 2C_{\phi 14}) + EI_{\omega y}(e_{v41} + C_{v41} - Y_s C_{\phi 41} - 2Y_s C_{\phi 14})$$

$$- EI_{\omega z}(-e_{u41} + C_{u41} + Z_s C_{\phi 41} + 2Z_s C_{\phi 14})$$

$$c_6 = EI_\omega \beta_\omega (C_{\phi 32} + 2C_{\phi 23}) + EI_{\omega y}(e_{v23} + C_{v32} - Y_s C_{\phi 32} - 2Y_s C_{\phi 23})$$

$$- EI_{\omega z}(-e_{u32} + C_{u32} + Z_s C_{\phi 32} + 2Z_s C_{\phi 32})$$

$$c_7 = EI_\omega \beta_\omega (C_{\phi 42} + 2C_{\phi 24}) + EI_{\omega y}(e_{v42} + C_{v42} - Y_s C_{\phi 42} - 2Y_s C_{\phi 24})$$

$$- EI_{\omega z}(-e_{u42} + C_{u42} + Z_s C_{\phi 42} + 2Z_s C_\phi 42)$$

$$c_8 = EI_\omega \beta_\omega (C_{\phi 33} + 2C_{\phi 33}) + EI_{\omega y}(e_{v33} + C_{v33} - Y_s C_{\phi 33} - 2Y_s C_{\phi 33})$$

$$- EI_{\omega z}(-e_{u11} + C_{u33} + Z_s C_{\phi 33} + 2Z_s C_{\phi 33})$$

$$c_9 = EI_\omega \beta_\omega (C_{\phi 43} + 2C_{\phi 34}) + EI_{\omega y}(e_{v43} + C_{v43} - Y_s C_{\phi 43} - 2Y_s C_{\phi 34})$$

$$- EI_{\omega z}(-e_{u43} + C_{u43} + Z_s C_{\phi 43} + 2Z_s C_{\phi 43})$$

$$c_{10} = EI_\omega \beta_\omega (C_{\phi 44} + 2C_{\phi 44}) + EI_{\omega y}(e_{v44}+C_{v44} - Y_s C_{\phi 44} - 2Y_s C_\phi 44)$$

$$- EI_{\omega z}(-e_{u44} + C_{u44} + Z_s C_{\phi 44} + 2Z_s C_\phi 44)$$

$$d_1 = -EI_y(f_{\phi 11} + C_{\phi 11}) - EI_z(f_{\phi 11} - C_{\phi 11}) - EI_{yz}(C_{\phi 11} + f_{\phi 11})$$

$$d_2 = -EI_y(f_{\phi 12} + C_{\phi 21}) - EI_z(f_{\phi 12} - C_{\phi 21}) - EI_{yz}(C_{\phi 22} + f_{\phi 12})$$

$$d_3 = -EI_y(f_{\phi 12} + C_{\phi 21}) - EI_z(f_{\phi 12} - C_{\phi 21}) - EI_{yz}(C_{\phi 12} + f_{\phi 12})$$

$$d_4 = -EI_y(f_{\phi 22} + C_{\phi 22}) - EI_z(f_{\phi 22} - C_{\phi 22}) - EI_{yz}(C_{\phi 22} + f_{\phi 22})$$

$$d_5 = -EI_y(f_{\phi 13} + C_{\phi 31}) - EI_z(f_{\phi 13} - C_{\phi 13}) - EI_{yz}(C_{\phi 31} + f_{\phi 13})$$

$$d_6 = -EI_y(f_{\phi 14} + C_{\phi 41}) - EI_z(f_{\phi 14} - C_{\phi 14}) - EI_{yz}(C_{\phi 41} + f_{\phi 14})$$

$$d_7 = -EI_y(f_{\phi 23} + C_{\phi 32}) - EI_z(f_{\phi 23} - C_{\phi 32}) - EI_{yz}(C_{\phi 32} + f_{\phi 23})$$

$$d_8 = -EI_y(f_{\phi 24} + C_{\phi 42}) - EI_z(f_{\phi 24} - C_{\phi 24}) - EI_{yz}(C_{\phi 42} + f_{\phi 24})$$

$$d_9 = -EI_y(f_{\phi 33} + C_{\phi 33}) - EI_z(f_{\phi 33} - C_{\phi 33}) - EI_{yz}(C_{\phi 33} + f_{\phi 33})$$

$$d_{10} = -EI_y(f_{\phi 34} + C_{\phi 43}) - EI_z(f_{\phi 34} - C_{\phi 34}) - EI_{yz}(C_{\phi 43} + f_{\phi 34})$$

$$d_{11} = -EI_y(f_{\phi 34} + C_{\phi 34}) - EI_z(f_{\phi 34} - C_{\phi 43}) - EI_{yz}(C_{\phi 34} + f_{\phi 34})$$

$$d_{12} = -EI_y(f_{\phi 44} + C_{\phi 44}) - EI_z(f_{\phi 44} - C_{\phi 44}) - EI_{yz}(C_{\phi 44} + f_{\phi 44})$$

$$e_1 = -EI_{y\omega}(D_{\phi 11} + f_{\phi 11}) + EI_y(e_{v11} + C_{v11}+C_{u11} + 2\bar{B}_y C_{\phi 11})$$

$$+ EI_{yz}(e_{u11} + C_{v11}) + EI_{\omega z}(f_{\phi 11} + \frac{1}{2}C_{\phi 11} - e_{\phi 11})$$

$$e_2 = -EI_{y\omega}(D_{\phi 12} + f_{\phi 12}) + EI_y(e_{v21} + C_{v12}+C_{u21} + 2\bar{B}_y C_{\phi 12})$$

$+ EI_{yz}(e_{u12} + C_{v12}) + EI_{\omega z}(f_{\phi 12} + \frac{1}{2}C_{\phi 12} - e_{\phi 12})$

$e_3 = -EI_{y\omega}(D_{\phi 12} + f_{\phi 12}) + EI_y(e_{v21} + C_{v12}+C_{u12} + 2\bar{\beta}_y C_{\phi 12})$

$+ EI_{yz}(e_{u21} + C_{v12}) + EI_{\omega z}(f_{\phi 12} + \frac{1}{2}C_{\phi 21} - e_{\phi 21})$

$e_4 = -EI_{y\omega}(D_{\phi 22} + f_{\phi 22}) + EI_y(e_{v22} + C_{v22}+C_{u22} + 2\bar{\beta}_y C_{\phi 22})$

$+ EI_{yz}(e_{u22} + C_{v22}) + EI_{\omega z}(f_{\phi 22} + \frac{1}{2}C_{\phi 22} - e_{\phi 22})$

$e_5 = -EI_{y\omega}(D_{\phi 13} + f_{\phi 13}) + EI_y(e_{v31} + C_{v13}+C_{u13} + 2\bar{\beta}_y C_{\phi 13})$

$+ EI_{yz}(e_{u31} + C_{v13}) + EI_{\omega z}(f_{\phi 13} + \frac{1}{2}C_{\phi 31} - e_{\phi 31})$

$e_6 = -EI_{y\omega}(D_{\phi 23} + f_{\phi 23}) + EI_y(e_{v32} + C_{v23}+C_{u23} + 2\bar{\beta}_y C_{\phi 32})$

$+ EI_{yz}(e_{u32} + C_{v23}) + EI_{\omega z}(f_{\phi 23} + \frac{1}{2}C_{\phi 32} - e_{\phi 32})$

$e_7 = -EI_{y\omega}(D_{\phi 14} + f_{\phi 14}) + EI_y(e_{v41} + C_{v14}+C_{u14} + 2\bar{\beta}_y C_{\phi 14})$

$+ EI_{yz}(e_{u41} + C_{v14}) + EI_{\omega z}(f_{\phi 14} + \frac{1}{2}C_{\phi 41} - e_{\phi 41})$

$e_8 = -EI_{y\omega}(D_{\phi 24} + f_{\phi 24}) + EI_y(e_{v42} + C_{v24}+C_{u24} + 2\bar{\beta}_y C_{\phi 24})$

$+ EI_{yz}(e_{u42} + C_{v24}) + EI_{\omega z}(f_{\phi 24} + \frac{1}{2}C_{\phi 42} - e_{\phi 42})$

$e_9 = -EI_{y\omega}(D_{\phi 33} + f_{\phi 33}) + EI_y(e_{v33} + C_{v33}+C_{u33} + 2\bar{\beta}_y C_{\phi 33})$

$+ EI_{yz}(e_{u33} + C_{v33}) + EI_{\omega z}(f_{\phi 33} + \frac{1}{2}C_{\phi 33} - e_{\phi 33})$

$e_{10} = -EI_{y\omega}(D_{\phi 43} + f_{\phi 34}) + EI_y(e_{v34} + C_{v43}+C_{u43} + 2\bar{\beta}_y C_{\phi 43})$

$+ EI_{yz}(e_{u43} + C_{v34}) + EI_{\omega z}(f_{\phi 34} + \frac{1}{2}C_{\phi 43} - e_{\phi 43})$

$e_{11} = -EI_{y\omega}(D_{\phi 34} + f_{\phi 43}) + EI_y(e_{v43} + C_{v34}+C_{u34} + 2\bar{\beta}_y C_{\phi 34})$

$$+ EI_{yz}(e_{u34} + C_{v43}) + EI_{\omega z}(f_{\phi 43} + \frac{1}{2}C_{\phi 34} - e_{\phi 34})$$

$$e_{11} = -EI_{y\omega}(D_{\phi 44} + f_{\phi 44}) + EI_y(e_{v44} + C_{v44}+C_{u44} + 2\bar{\beta}_y C_{\phi 44})$$

$$+ EI_{yz}(e_{u44} + C_{v44}) + EI_{\omega z}(f_{\phi 44} + \frac{1}{2}C_{\phi 44} - e_{\phi 44})$$

$$f_1 = -EI_{y\omega}(D_{\phi 11} + f_{\phi 11}) + EI_y(e_{v11} + C_{v11}+C_{u11} + 2\bar{\beta}_y C_{\phi 11})$$

$$+ EI_{yz}(e_{u11} + C_{v11}) + EI_{\omega z}(f_{\phi 11} - \frac{1}{2}C_{\phi 11} + e_{\phi 11})$$

$$f_2 = -EI_{y\omega}(D_{\phi 12} + f_{\phi 12}) + EI_y(e_{v21} + C_{v12}+C_{u21} + 2\bar{\beta}_y C_{\phi 12})$$

$$+ EI_{yz}(e_{u12} + C_{v12}) + EI_{\omega z}(f_{\phi 12} - \frac{1}{2}C_{\phi 12} + e_{\phi 12})$$

$$f_3 = -EI_{y\omega}(D_{\phi 12} + f_{\phi 12}) + EI_y(e_{v21} + C_{v12}+C_{u12} + 2\bar{\beta}_y C_{\phi 12})$$

$$+ EI_{yz}(e_{u21} + C_{v12}) + EI_{\omega z}(f_{\phi 12} - \frac{1}{2}C_{\phi 21} + e_{\phi 21})$$

$$f_4 = -EI_{y\omega}(D_{\phi 22} + f_{\phi 22}) + EI_y(e_{v22} + C_{v22}+C_{u22} + 2\bar{\beta}_y C_{\phi 22})$$

$$+ EI_{yz}(e_{u22} + C_{v22}) + EI_{\omega z}(f_{\phi 22} - \frac{1}{2}C_{\phi 22} + e_{\phi 22})$$

$$f_5 = -EI_{y\omega}(D_{\phi 13} + f_{\phi 13}) + EI_y(e_{v31} + C_{v13}+C_{u13} + 2\bar{\beta}_y C_{\phi 13})$$

$$+ EI_{yz}(e_{u31} + C_{v13}) + EI_{\omega z}(f_{\phi 13} - \frac{1}{2}C_{\phi 31} + e_{\phi 31})$$

$$f_6 = -EI_{y\omega}(D_{\phi 23} + f_{\phi 23}) + EI_y(e_{v32} + C_{v23}+C_{u23} + 2\bar{\beta}_y C_{\phi 32})$$

$$+ EI_{yz}(e_{u32} + C_{v23}) + EI_{\omega z}(f_{\phi 23} - \frac{1}{2}C_{\phi 32} + e_{\phi 32})$$

$$f_7 = -EI_{y\omega}(D_{\phi 14} + f_{\phi 14}) + EI_y(e_{v41} + C_{v14}+C_{u14} + 2\bar{\beta}_y C_{\phi 14})$$

$$+ EI_{yz}(e_{u41} + C_{v14}) + EI_{\omega z}(f_{\phi 14} - \frac{1}{2}C_{\phi 41} + e_{\phi 41})$$

$$f_8 = -EI_{y\omega}(D_{\phi 24} + f_{\phi 24}) + EI_y(e_{v42} + C_{v24}+C_{u24} + 2\bar{\beta}_y C_{\phi 24})$$

$$+ EI_{yz}(e_{u42} + C_{v24}) + EI_{\omega z}(f_{\phi 24} - \frac{1}{2}C_{\phi 42} + e_{\phi 42})$$

$$f_9 = -EI_{y\omega}(D_{\phi 33} + f_{\phi 33}) + EI_y(e_{v33} + C_{v33} + C_{u33} + 2\overline{\beta}_y C_{\phi 33})$$

$$+ EI_{yz}(e_{u33} + C_{v33}) + EI_{\omega z}(f_{\phi 33} - \frac{1}{2}C_{\phi 33} + e_{\phi 33})$$

$$f_{10} = -EI_{y\omega}(D_{\phi 43} + f_{\phi 34}) + EI_y(e_{v34} + C_{v43} + C_{u43} + 2\overline{\beta}_y C_{\phi 43})$$

$$+ EI_{yz}(e_{u43} + C_{v34}) + EI_{\omega z}(f_{\phi 34} - \frac{1}{2}C_{\phi 43} + e_{\phi 43})$$

$$f_{11} = -EI_{y\omega}(D_{\phi 34} + f_{\phi 43}) + EI_y(e_{v43} + C_{v34} + C_{u34} + 2\overline{\beta}_y C_{\phi 34})$$

$$+ EI_{yz}(e_{u34} + C_{v43}) + EI_{\omega z}(f_{\phi 43} - \frac{1}{2}C_{\phi 34} + e_{\phi 34})$$

$$e_{12} = -EI_{y\omega}(D_{\phi 44} + f_{\phi 44}) + EI_y(e_{v44} + C_{v44} + C_{u44} + 2\overline{\beta}_y C_{\phi 44})$$

$$+ EI_{yz}(e_{u44} + C_{v44}) + EI_{\omega z}(f_{\phi 44} - \frac{1}{2}C_{\phi 44} + e_{\phi 44})$$

Appendix H

Nonlinear Geometrical Stiffness Matrix

The coefficients of the nonlinear geometrical stiffness matrix given in Table 4.8 are explained in detail as:

$A_1 = -\frac{1}{2}M_x(V_{\phi 11} + V_{\phi 11})$ $A_2 = -\frac{1}{2}M_x(V_{\phi 12} + V_{\phi 21})$

$A_3 = -\frac{1}{2}M_x(V_{\phi 22} + V_{\phi 22})$ $A_4 = -\frac{1}{2}M_x(V_{\phi 13} + V_{\phi 31})$

$A_5 = -\frac{1}{2}M_x(V_{\phi 14} + V_{\phi 41})$ $A_6 = -\frac{1}{2}M_x(V_{\phi 32} + V_{\phi 23})$

$A_7 = -\frac{1}{2}M_x(V_{\phi 24} + V_{\phi 42})$ $A_8 = -\frac{1}{2}M_x(V_{\phi 33} + V_{\phi 33})$

$A_9 = -\frac{1}{2}M_x(V_{\phi 34} + V_{\phi 43})$ $A_{10} = -\frac{1}{2}M_x(V_{\phi 44} + V_{\phi 44})$

$B_1 = FY_s(n_{\phi 11}+m_{\phi 11})-Q_zU_{\phi 11} +M_z(n_{\phi 11}+m_{\phi 11})+\frac{1}{2}M_x(W_{\phi 11}-T_{\phi 11})$

$B_2 = FY_s(n_{\phi 21}+m_{\phi 12})-Q_zU_{\phi 21} +M_z(n_{\phi 21}+m_{\phi 12})+\frac{1}{2}M_x(W_{\phi 12}-T_{\phi 21})$

$B_3 = FY_s(n_{\phi 12}+m_{\phi 21})-Q_zU_{\phi 12} +M_z(n_{\phi 21}+m_{\phi 12})+\frac{1}{2}M_x(W_{\phi 12}-T_{\phi 21})$

$B_4 = FY_s(n_{\phi 22}+m_{\phi 22})-Q_zU_{\phi 22} +M_z(n_{\phi 22}+m_{\phi 22})+\frac{1}{2}M_x(W_{\phi 22}-T_{\phi 22})$

$B_5 = FY_s(n_{\phi 31}+m_{\phi 13})-Q_zU_{\phi 31} +M_z(n_{\phi 13}+m_{\phi 31})+\frac{1}{2}M_x(W_{\phi 13}-T_{\phi 31})$

$B_7 = FY_s(n_{\phi 32}+m_{\phi 23})-Q_zU_{\phi 32} +M_z(n_{\phi 23}+m_{\phi 32})+\frac{1}{2}M_x(W_{\phi 23}-T_{\phi 32})$

$B_8 = FY_s(n_{\phi 42}+m_{\phi 24})-Q_zU_{\phi 42} +M_z(n_{\phi 24}+m_{\phi 42})+\frac{1}{2}M_x(W_{\phi 24}-T_{\phi 42})$

$B_9 = FY_s(n_{\phi 33}+m_{\phi 33})-Q_zU_{\phi 33} +M_z(n_{\phi 33}+m_{\phi 33})+\frac{1}{2}M_x(W_{\phi 33}-T_{\phi 33})$

$B_{10} = FY_s(n_{\phi 11}+m_{\phi 11})-Q_zU_{\phi 11} +M_z(n_{\phi 11}+m_{\phi 11})+\frac{1}{2}M_x(W_{\phi 11}-T_{\phi 11})$

$$B_{11} = FY_s(n_{\phi 34}+m_{\phi 43})-Q_z U_{\phi 34} +M_z(n_{\phi 43}+m_{\phi 34})+\frac{1}{2}M_x(W_{\phi 43}-T_{\phi 34})$$

$$B_{12} = FY_s(n_{\phi 44}+m_{\phi 44})-Q_z U_{\phi 44} +M_z(n_{\phi 44}+m_{\phi 44})+\frac{1}{2}M_x(W_{\phi 44}-T_{\phi 44})$$

$$B_{13} = FY_s(n_{\phi 13}+m_{\phi 31})-Q_z U_{\phi 13} +M_z(n_{\phi 31}+m_{\phi 13})+\frac{1}{2}M_x(W_{\phi 31}-T_{\phi 13})$$

$$B_{14} = FY_s(n_{\phi 23}+m_{\phi 32})-Q_z U_{\phi 23} +M_z(n_{\phi 32}+m_{\phi 23})+\frac{1}{2}M_x(W_{\phi 32}-T_{\phi 23})$$

$$B_{15} = FY_s(n_{\phi 14}+m_{\phi 41})-Q_z U_{\phi 14} +M_z(n_{\phi 41}+m_{\phi 14})+\frac{1}{2}M_x(W_{\phi 41}-T_{\phi 14})$$

$$B_{16} = FY_s(n_{\phi 24}+m_{\phi 42})-Q_z U_{\phi 24} +M_z(n_{\phi 42}+m_{\phi 24})+\frac{1}{2}M_x(W_{\phi 42}-T_{\phi 24})$$

$$C_1 = FZ_s(n_{\phi 11}+m_{\phi 11})-Q_y U_{\phi 11} -M_y(n_{\phi 11}+m_{\phi 11})-\frac{1}{2}M_x(W_{\phi 11}-T_{\phi 11})$$

$$C_2 = FZ_s(n_{\phi 21}+m_{\phi 12})-Q_y U_{\phi 21} -M_y(n_{\phi 21}+m_{\phi 12})-\frac{1}{2}M_x(W_{\phi 12}-T_{\phi 21})$$

$$C_3 = FZ_s(n_{\phi 12}+m_{\phi 21})-Q_y U_{\phi 12} -M_y(n_{\phi 21}+m_{\phi 12})-\frac{1}{2}M_x(W_{\phi 12}-T_{\phi 21})$$

$$C_4 = FZ_s(n_{\phi 22}+m_{\phi 22})-Q_y U_{\phi 22} -M_y(n_{\phi 22}+m_{\phi 22})-\frac{1}{2}M_x(W_{\phi 22}-T_{\phi 22})$$

$$C_5 = FZ_s(n_{\phi 31}+m_{\phi 13})-Q_y U_{\phi 31} -M_y(n_{\phi 13}+m_{\phi 31})-\frac{1}{2}M_x(W_{\phi 13}-T_{\phi 31})$$

$$C_7 = FZ_s(n_{\phi 32}+m_{\phi 23})-Q_y U_{\phi 32} -M_y(n_{\phi 23}+m_{\phi 32})-\frac{1}{2}M_x(W_{\phi 23}-T_{\phi 32})$$

$$C_8 = FZ_s(n_{\phi 42}+m_{\phi 24})-Q_y U_{\phi 42} -M_y(n_{\phi 24}+m_{\phi 42})-\frac{1}{2}M_x(W_{\phi 24}-T_{\phi 42})$$

$$C_9 = FZ_s(n_{\phi 33}+m_{\phi 33})-Q_y U_{\phi 33} -M_y(n_{\phi 33}+m_{\phi 33})-\frac{1}{2}M_x(W_{\phi 33}-T_{\phi 33})$$

$$C_{10} = FZ_s(n_{\phi 11}+m_{\phi 11})-Q_y U_{\phi 11} -M_y(n_{\phi 11}+m_{\phi 11})-\frac{1}{2}M_x(W_{\phi 11}-T_{\phi 11})$$

$$C_{11} = FZ_s(n_{\phi 34}+m_{\phi 43})-Q_y U_{\phi 34} -M_y(n_{\phi 43}+m_{\phi 34})-\frac{1}{2}M_x(W_{\phi 43}-T_{\phi 34})$$

$$C_{12} = FZ_s(n_{\phi 44}+m_{\phi 44})-Q_y U_{\phi 44} -M_y(n_{\phi 44}+m_{\phi 44})-\frac{1}{2}M_x(W_{\phi 44}-T_{\phi 44})$$

$$C_{13} = FZ_s(n_{\phi 13}+m_{\phi 31})-Q_y U_{\phi 13} -M_y(n_{\phi 31}+m_{\phi 13})-\frac{1}{2}M_x(W_{\phi 31}-T_{\phi 13})$$

$$C_{14} = FZ_s(n_{\phi 23}+m_{\phi 32})-Q_y U_{\phi 23} -M_y(n_{\phi 32}+m_{\phi 23})-\frac{1}{2}M_x(W_{\phi 32}-T_{\phi 23})$$

$$C_{15} = FZ_s(n_{\phi 14}+m_{\phi 41})-Q_y U_{\phi 14} -M_y(n_{\phi 41}+m_{\phi 14})-\frac{1}{2}M_x(W_{\phi 41}-T_{\phi 14})$$

$$C_{16} = FZ_s(n_{\phi 24}+m_{\phi 42})-Q_y U_{\phi 24} -M_y(n_{\phi 42}+m_{\phi 24})-\frac{1}{2}M_x(W_{\phi 42}-T_{\phi 24})$$

$$D_1 = +\frac{1}{2}Q_y Z_s[-\frac{1}{2}U_{\phi 11}+U_{\phi 11}+2R_{\phi 11}]$$

$$-\frac{1}{2}Q_z Y_s[-\frac{1}{2}U_{\phi 11}+U_{\phi 11}+2R_{\phi 11}]+M_x[U_{\phi 11}+\frac{1}{2}U_{\phi 11}]$$

$$D_2 = +\frac{1}{2}Q_y Z_s[-\frac{1}{2}U_{\phi 12}+U_{\phi 21}+2R_{\phi 21}]$$

$$-\frac{1}{2}Q_z Y_s[-\frac{1}{2}U_{\phi 12}+U_{\phi 21}+2R_{\phi 21}]+M_x[U_{\phi 21}+\frac{1}{2}U_{\phi 12}]$$

$$D_3 = +\frac{1}{2}Q_y Z_s[-\frac{1}{2}U_{\phi 22}+U_{\phi 22}+2R_{\phi 22}]$$

$$-\frac{1}{2}Q_z Y_s[-\frac{1}{2}U_{\phi 22}+U_{\phi 22}+2R_{\phi 22}]+M_x[U_{\phi 22}+\frac{1}{2}U_{\phi 11}]$$

$$D_4 = +\frac{1}{2}Q_y Z_s[-\frac{1}{2}U_{\phi 13}+U_{\phi 31}+2R_{\phi 31}]$$

$$-\frac{1}{2}Q_z Y_s[-\frac{1}{2}U_{\phi 13}+U_{\phi 31}+2R_{\phi 31}]+M_x[U_{\phi 31}+\frac{1}{2}U_{\phi 13}]$$

$$D_5 = +\frac{1}{2}Q_y Z_s[-\frac{1}{2}U_{\phi 23}+U_{\phi 32}+2R_{\phi 32}]$$

$$-\frac{1}{2}Q_z Y_s[-\frac{1}{2}U_{\phi 23}+U_{\phi 32}+2R_{\phi 32}]+M_x[U_{\phi 32}+\frac{1}{2}U_{\phi 23}]$$

$$D_6 = +\frac{1}{2}Q_y Z_s[-\frac{1}{2}U_{\phi 14}+U_{\phi 41}+2R_{\phi 41}]$$

$$-\frac{1}{2}Q_z Y_s[-\frac{1}{2}U_{\phi 14}+U_{\phi 41}+2R_{\phi 41}]+M_x[U_{\phi 41}+\frac{1}{2}U_{\phi 14}]$$

$$D_7 = +\frac{1}{2}Q_y Z_s[-\frac{1}{2}U_{\phi 24}+U_{\phi 42}+2R_{\phi 42}]$$

$$-\frac{1}{2}Q_z Y_s[-\frac{1}{2}U_{\phi 24}+U_{\phi 42}+2R_{\phi 42}]+M_x[U_{\phi 24}+\frac{1}{2}U_{\phi 42}]$$

$$D_8 = +\frac{1}{2}Q_y Z_s[-\frac{1}{2}U_{\phi 33}+U_{\phi 33}+2R_{\phi 33}]$$

$$-\frac{1}{2}Q_z Y_s[-\frac{1}{2}U_{\phi 33}+U_{\phi 33}+2R_{\phi 33}]+M_x[U_{\phi 33}+\frac{1}{2}U_{\phi 33}]$$

$$D_9 = +\frac{1}{2}Q_y Z_s[-\frac{1}{2}U_{\phi 34}+U_{\phi 43}+2R_{\phi 43}]$$

$$-\frac{1}{2}Q_z Y_s[-\frac{1}{2}U_{\phi 34}+U_{\phi 43}+2R_{\phi 43}]+M_x[U_{\phi 43}+\frac{1}{2}U_{\phi 34}]$$

$$D_{10} = +\frac{1}{2}Q_y Z_s[-\frac{1}{2}U_{\phi 44}+U_{\phi 44}+2R_{\phi 44}]$$

$$-\frac{1}{2}Q_z Y_s[-\frac{1}{2}U_{\phi 44}+U_{\phi 44}+2R_{\phi 44}]+M_x[U_{\phi 44}+\frac{1}{2}U_{\phi 44}]$$

$$E_1 = M_y(a_{\phi 1}-b_{\phi 1}) - \frac{1}{2}Q_y C_{\phi 1}$$

$$E_2 = M_y(a_{\phi 2}-b_{\phi 2}) - \frac{1}{2}Q_y C_{\phi 2}$$

$$E_3 = -M_z(a_{\phi 1}-b_{\phi 1}) + \frac{1}{2}Q_y C_{\phi 1}$$

$$E_4 = -M_z(a_{\phi 2}-b_{\phi 2}) + \frac{1}{2}Q_y C_{\phi 2}$$

$$E_5 = M_y(X_{u1}+a_{\phi 1}) - \frac{1}{2}Q_y b_{u1} - M_z(X_{v1}-a_{\phi 1}) - \frac{1}{2}Q_z B_{v1}$$

$$E_6 = M_y(X_{u2}+a_{\phi 2}) - \frac{1}{2}Q_y b_{u2} - M_z(X_{v2}-a_{\phi 2}) - \frac{1}{2}Q_z B_{v2}$$

$$E_{11} = M_y(X_{u3}+a_{\phi 3}) - \frac{1}{2}Q_y b_{u3} - M_z(X_{v3}-a_{\phi 3}) - \frac{1}{2}Q_z B_{v3}$$

$$E_{12} = M_y(X_{u4}+a_{\phi 4}) - \frac{1}{2}Q_y b_{u4} - M_z(X_{v4}-a_{\phi 4}) - \frac{1}{2}Q_z B_{v4}$$

$$E_7 = M_y(a_{\phi 3}-b_{\phi 3}) - \frac{1}{2}Q_y C_{\phi 3}$$

$$E_8 = M_y(a_{\phi 4}-b_{\phi 4}) - \frac{1}{2}Q_y C_{\phi 4}$$

$$E_9 = -M_z(a_{\phi 3}-b_{\phi 3}) + \frac{1}{2}Q_z C_{\phi 3}$$

$$E_{10} = -M_z(a_{\phi 4}-b_{\phi 4}) + \frac{1}{2}Q_z C_{\phi 4}$$

Appendix I

An illustration of the interaction between the coefficients $V_{\phi 11}$, $C_{\phi 11}$, etc., with either the section properties or the internal loads is presented. Two examples are given herein to show the procedure, which are:

example 1

$$A_l = -\frac{1}{2}M_x(V_{\phi 11}+V_{\phi 11})$$

Substituting for the term $V_{\phi 11}$ from appendix J in the above equation yields

$$A_l = -\Bigg[\left(-\frac{1}{35l^2}M_{x1}-\frac{6}{15l^2}\Delta M_x\right)\Phi_1+\left(\frac{3}{35l}\right)X_1$$

$$+\left(-\frac{27}{35l^2}M_{x1}-\frac{24}{35l^2}\Delta M_x\right)\Phi_2+\left(\frac{3}{35l}M_{x1}+\frac{3}{35l}\Delta M_x\right)X_2\Bigg]$$

example 2

$$D = EI_y[U_{\phi 21}+\frac{1}{2}U_{12}]$$

The above equation can be written, after substituting for the terms $U_{\phi 21}$ and $U_{\phi 12}$ from Appendix N as:

$$D = [(\frac{5}{42}l-\frac{5}{168}l)EI_{y1}+(\frac{1}{229}l-\frac{3}{280}l)\Delta EI_y]\Phi_1$$

$$+ \left[\left(\frac{1}{168}l^2 - \frac{1}{168}l^2\right)EI_{y1} + \left(-\frac{1}{1260}l^2 - \frac{1}{420}l^2\right)\Delta EI_y\right]\chi_1$$

$$+ \left[\left(-\frac{2}{105}l - \frac{17}{840}l\right)EI_{y1} + \left(-\frac{1}{70}l - \frac{3}{280}l\right)\Delta EI_y\right]\Phi_2$$

$$+ \left[\left(\frac{13}{840}l^2 - \frac{13}{840}l^2\right)EI_{y1} + \left(\frac{1}{90}l^2 + \frac{1}{120}l^2\right)\Delta EI_y\right]\chi_2$$

The same procedure can be applied for the rest of the terms and their interaction with the forces or the section properties.

Appendix J

Details of the coefficients stated in Tables 4.7 and 4.8 are presented in this appendix. The terms can be employed in the nonlinear geometrical and tangential stiffness matrices by interchanging some parameters as described in appendix I.. Those coefficient are

$$a_{u1} = (-\frac{6}{51^2} \quad -\frac{.6}{1^2})U_1 + (-\frac{1}{10} \quad -\frac{A}{10})\theta_{u1}$$

$$+ (\frac{6}{51^2} \quad \frac{.6}{1^2}A)U_2 + (0 \quad - \frac{1}{101})\theta_{u2}$$

$$a_{u2} = (-\frac{1}{101} \quad -\frac{1}{101})U_1 + (-\frac{2}{15} \quad -\frac{1}{30})\theta_{u1}$$

$$+ (-\frac{1}{101} \quad -\frac{1}{101})U_2 + (\frac{1}{30} \quad \frac{A}{60})\theta_{u2}$$

$$a_{u3} = (\frac{6}{51^2} \quad \frac{.6}{1^2}A)U_1 + (-\frac{1}{10} \quad -\frac{1}{10})\theta_{u1}$$

$$- (\frac{6}{51^2} \quad \frac{.6}{1^2})U_2 + (\frac{1}{101} \quad 0)\theta_{u2}$$

$$a_{u4} = (-\frac{1}{101} \quad 0)U_1 + (\frac{1}{30} \quad \frac{1}{60})\theta_{u1} - (\frac{1}{101} \quad 0)U_2 + (-\frac{2}{15} \frac{1}{10})\theta_{u2}$$

$$b_{u1} = (\frac{1.2}{1^2} \quad \frac{A}{101^2})U_1 \quad (0 \quad -\frac{1}{101})\theta_{u1}$$

$$+ (-\frac{1.2}{1^2} \quad -\frac{1.1}{101^2})U_1 + (\frac{1}{101} \quad 0)\theta_{u2}$$

$$b_{u2} = (\frac{1.1}{1^2} \quad \frac{2}{101})U_1 + (\frac{2}{15} \quad \frac{1}{30})\theta_{u1}$$

$$+ (-\frac{1}{101} \quad -\frac{2}{10}1)U_2 + (0 \quad -\frac{1}{30})\theta_{u2}$$

$$b_{u3} = (-\frac{1.2}{1^2} \quad \frac{1}{101^2})U_1 \quad (o \quad -\frac{1}{101})\theta_{u1}$$

$$+ (\frac{1.2}{12} \quad \frac{1.1}{1012})U_2 - (\frac{1}{101} \quad o)\theta_{u2}$$

$$b_{u4} = (-\frac{.4}{1} \quad -\frac{2}{101})U_1 + (-\frac{1}{30} \quad -\frac{1}{30})\theta_{u1}$$

$$+ (-\frac{1.1}{1} \quad \frac{.4}{1})U_2 + (\frac{2}{15} \quad \frac{1}{10})\theta_{u2}$$

Change a_{u1}, a_{u2}, a_{u3}, a_{u4}, U_1, θ_{u1}, U_2, θ_{u2} into either a_{v1}, a_{v2}, a_{v3}, a_{v4}, V_1, θ_{v1}, V_2, θ_{v2} or $a_{\phi 1}$, $a_{\phi 2}$, $a_{\phi 3}$, $a_{\phi 4}$, Φ_1, χ_1, Φ_2, χ_2 respectively. A similar procedure can be done for b_{u1}, etc..

$$cu_{11} = (0 \quad \frac{18}{351^3})U_1 + (-\frac{1.2}{1^2} \quad -\frac{1.286}{1^2})\theta_{u1}$$

$$+ (0 \quad -\frac{18}{351^3})U_2 + (-\frac{.6}{1^2} \quad -\frac{12}{351^2})\theta_{u2}$$

$$cu_{21} = (\frac{-1.2}{1^2} \quad -\frac{1.286}{1^2})U_1 + (-\frac{4}{101} \quad \frac{1}{351})\theta_{u1}$$

$$+ (-\frac{.6}{1^2} \quad -\frac{9}{351^2})U_2 - (0 \quad \frac{1}{141})\theta_{u2}$$

$$cu_{31} = (0 \quad -\frac{18}{351^3})U_1 + (-\frac{.6}{1^2} \quad -\frac{9}{351^2})\theta_{u1}$$

$$+ (0 \quad \frac{18}{351^3})U_2 + (\frac{12}{351^2} \quad -\frac{12}{351^2})\theta_{u2}$$

$$cu_{41} = (-\frac{.6}{1^2} \quad -\frac{12}{351^2})U_1 - (0 \quad \frac{1}{141})\theta_{u1}$$

$$+ (\frac{12}{351^2} \quad \frac{12}{351^2})U_2 + (\frac{.4}{1} \quad \frac{5}{281})\theta_{u2}$$

$$cu_{12} = (\frac{1.2}{1^2} \quad -\frac{12}{351^2})U_1 + (\frac{2}{101} \quad -\frac{1}{71})\theta_{u1}$$

$$+ (\frac{1.2}{1^2} \quad \frac{12}{351^2})U_2 + (\frac{-4}{1} \quad -\frac{6}{351})\theta_{u2}$$

$$cu_{22} = (-\frac{2}{101} \quad -\frac{1}{71})U_1 + (-\frac{1}{3} \quad -\frac{2}{205}A)\theta_{u1}$$

$$+ (-\frac{2}{101} \quad -\frac{1}{351})U_2 + (\frac{1}{30} \quad -\frac{2}{205}A)\theta_{u2}$$

$$cu_{23} = (\frac{1.2}{1^2} \quad \frac{12}{351^2})U_1 - (\frac{2}{101} \quad -\frac{1}{71})\theta_{u1}$$

$$- (\frac{1.2}{1^2} \quad \frac{12}{351^2})U_2 + (\frac{4}{1} \quad + \frac{5}{351})\theta_{u2}$$

$$cu_{24} = (-\frac{.4}{1} \quad -6/351)U_1 + (\frac{1}{30} \quad -\frac{2}{205})\theta_{u1}$$

$$+ (\frac{.4}{1} \quad \frac{6}{351})U_2 + (\frac{1}{15} \quad \frac{4}{35})\theta_{u2}$$

$$cu_{13} = (0 \quad -\frac{18}{351^3})U_1 + (\frac{1.2}{1^2} \quad \frac{1.286}{1^2})\theta_{u1}$$

$$+ (0 \quad \frac{18}{351^3})U_2 + (\frac{.6}{1^2} \quad \frac{12}{351^2})\theta_{u2}$$

$$cu_{23} = (\frac{1.2}{1^2} \quad \frac{1.286}{1^2})U_1 + (\frac{4}{101} \quad -\frac{1}{351})\theta_{u1}$$

$$+ (\frac{.6}{1^2} \quad \frac{9}{351^2})U_2 + (0 \quad \frac{1}{141})\theta_2$$

$$cu_{33} = (0 \quad \frac{18}{351^3})U_1 + (\frac{.6}{1^2} + \frac{9}{351^2})\theta_{u1}$$

$$- (0 \quad \frac{18}{351^3})U_2 + (-\frac{12}{351^2} \quad -\frac{12}{351^2})\theta_{u2}$$

$$cu_{43} = (\frac{.6}{1^2} \quad \frac{12}{351^2})U_1 + (0 \quad \frac{1}{141})\theta_{u1}$$

$$- (\frac{12}{351^2} \quad + \frac{12}{351^2})U_2 - (\frac{.4}{1} \quad + \frac{5}{281})\theta_{u2}$$

$$cu_{14} = (\frac{1.2}{1^2} \quad \frac{6}{71^2})U_1 + (-\frac{1.4}{1} \quad -\frac{1.1428}{1})\theta_{u1}$$

$$- (\frac{1.2}{1^2} \quad \frac{6}{71^2})U_2 - (\frac{.2}{1} \quad -\frac{6}{351})\theta_{u2}$$

$$cu_{24} = (\frac{-1.4}{1} \quad -\frac{1.14286}{1})U_1 + (-\frac{1}{15} \quad \frac{1}{21})\theta_{u1}$$

$$+ (-\frac{.4}{101} \quad -\frac{8}{351})U_2 + (-\frac{1}{30} \quad \frac{11}{210})\theta_{u2}$$

$$cu_{34} = (-\frac{1.2}{1^2} \quad -\frac{6}{71^2})U_1 + (-\frac{4}{101} \quad -\frac{8}{351})\theta_{u1}$$

$$+ (\frac{1.2}{1^2} \quad -\frac{6}{71^2})U_2 + (\frac{.2}{1} \quad + \frac{6}{351})\theta_{u2}$$

$$cu_{44} = (-\frac{.2}{1} \quad -\frac{6}{351})U_1 + (-\frac{1}{30} \quad -\frac{11}{210})\theta_{u1}$$

$$+ (\frac{.2}{l} + \frac{6}{35l})U_2 + (\frac{l}{3} \quad \frac{11}{35})\theta_{u21}$$

Change cu_{11}, cu_{12}, ..., U_1, θ_{u1}, etc. to the form cv_{11}, cv_{12}, ..., V_1, θ_{v1}, etc. . Same procedure for the terms such as Φ_1, χ_1.

$$e_{u11} = (\frac{6}{l^3} + \frac{33}{35l^3})U_1 + (\frac{4.2}{l^2} + \frac{4}{7l^2})\theta_{u1}$$

$$- (\frac{6}{l^3} \quad \frac{33}{35l^3})U_2 + (\frac{1.8}{l^2} + \frac{13}{7l^2})\theta_{u2}$$

$$e_{u13} = (\frac{6}{l^3} + \frac{5.057}{l^3})U_1 + (\frac{1.8}{l^2} + \frac{1.42857}{l^2})\theta_{u1}$$

$$- (\frac{6}{l^3} + \frac{5.057}{l^3})U_2 + (\frac{4.2}{l^2} + \frac{3.6285}{l^2})\theta_{u1}$$

$$e_{u14} = (-\frac{.6}{l^2} - \frac{3}{7l^2})U_1 + (\frac{.2}{l} - \frac{4}{35l})\theta_{u1}$$

$$+ (\frac{.6}{l^2} + \frac{3}{7l})U_2 + (\frac{4}{10l} - \frac{11}{35l})\theta_{u2}$$

$$e_{u21} = (\frac{4.2}{l^2} + \frac{4}{l^2})U_1 + (\frac{2.2}{l} - \frac{11}{35l})\theta_{u1}$$

$$+ (-\frac{4.2}{l^2} - \frac{4}{7l^2})U_2 + (\frac{1}{l} + \frac{3}{35l})\theta_{u2}$$

$$e_{u22} + (-\frac{1}{10l} - \frac{11}{35l})U_1 + (-9.666 \quad -7.9236)\theta_{u1}$$

$$+ (\frac{1}{10l} + \frac{11}{35l})U_2 + (\frac{1}{15} - \frac{1}{105})\theta_{u2}$$

$$e_{u23} = (\frac{1.8}{l^2} + \frac{1.42857}{l^2})U_1 + (\frac{.8}{l} + \frac{18}{35l})\theta_{u1}$$

$$- (\frac{1.8}{l^2} + \frac{1.42857}{l^2})U_2 + (\frac{1}{l} + \frac{32}{35l})\theta_{u2}$$

$$e_{u24} + (-\frac{.2}{l} - \frac{4}{35l})U_1 + (-\frac{2}{15} - \frac{2}{35})\theta_{u1}$$

$$+ (\frac{.2}{l} + \frac{4}{35l})U_1 + (-\frac{1}{15} - \frac{2}{35})\theta_{u2}$$

$$e_{u31} = (-\frac{6}{l^3} - \frac{33}{35l^3})U_1 + (\frac{-4.2}{l^2} - \frac{4}{7l^2})\theta_{u1}$$

$$+ (\frac{6}{l^3} + \frac{33}{35l^3})U_2 + (-\frac{1.8}{l^2} - \frac{13}{35l^2})\theta_{u2}$$

$$e_{u32} = (-\frac{.6}{l^2} \quad -\frac{6}{35l^2})U_1 + (\frac{1}{10l} \quad +\frac{11}{35l})\theta_{u1}$$

$$+ (\frac{.6}{l^2} \quad +\frac{6}{35l^2})U_2 + (-\frac{.2}{l} \quad -\frac{3}{35l})\theta_{u2}$$

$$e_{u33} = -(\frac{6}{l^3} \quad +\frac{5.057}{l^3})U_1 - (\frac{1.8}{l^2} \quad +\frac{1.42857}{l^2})\theta_{u1}$$

$$+ (\frac{6}{l^3} \quad +\frac{5.057}{l^3})U_2 - (\frac{4.2}{l^2} \quad +\frac{3.6285}{l^2})\theta_{u1}$$

$$e_{u34} = (\frac{.6}{l^2} \quad +\frac{3}{7l^2})U_1 + (\frac{.2}{l} \quad +\frac{4}{35l})\theta_{u1}$$

$$- (\frac{.6}{l^2} \quad +\frac{3}{7l})U_2 + (\frac{4}{10l} \quad +\frac{11}{35l})\theta_{u2}$$

$$e_{u41} = (\frac{1.8}{l^2} \quad +\frac{13}{35l^2})U_1 + (\frac{1}{l} \quad +\frac{3}{35l})\theta_{u1}$$

$$- (\frac{1.8}{l^2} \quad +\frac{13}{35l^2})U_2 + (\frac{.8}{l} \quad +\frac{2}{7l})\theta_{u2}$$

$$e_{u}42 + (\frac{.2}{l} \quad +\frac{3}{35l})U_1 + (\frac{.2}{l} \quad +\frac{3}{35l})\theta_{u1}$$

$$- (\frac{.2}{l} \quad +\frac{3}{35l})U_2 + (\frac{2}{15} \quad +\frac{8}{105})\theta_{u2}$$

$$e_{u43} = (\frac{4.2}{l^2} \quad +\frac{3.6285}{l^2})U_1 + (\frac{1}{l} \quad +\frac{32}{35l})\theta_{u1}$$

$$- (\frac{4.2}{l^2} \quad +\frac{3.6285}{l^2})U_2 + (\frac{3.2}{l} \quad +\frac{2.714}{l})\theta_{u2}$$

$$e_{u}44 = (-\frac{4}{10l} \quad -\frac{11}{35l})U_1 + (-\frac{1}{15} \quad -\frac{2}{35})\theta_{u1}$$

$$+ (\frac{4}{10l} \quad +\frac{11}{35l})U_2 + (-\frac{1}{3} \quad -\frac{9}{35})\theta_{u2}$$

Change e_{u11}, e_{u12}, U_1, θ_{u1} to e_{v11}, e_{v12}, V_1, θ_{v1} to $e_{\phi 11}$, $e_{\phi 12}$, Φ_1, X_1 etc.

$$R_{\phi 11} = (-\frac{1}{3} \quad -\frac{43}{420})\Phi_1 + (\frac{5}{42}l \quad \frac{1}{229}l)X_1$$

$$+ (-\frac{1}{3} \quad -\frac{43}{420})\Phi_2 + (\frac{5}{42}l \quad \frac{1}{229}l)X_2$$

$$R_{\phi 21}=R_{\phi 12} = (-\frac{5}{84}l \quad -\frac{3}{140}l)\Phi_1 + (\frac{1}{168}l^2 \quad -\frac{1}{1260}l^2)\chi_1$$

$$+ (\frac{5}{84}l \quad +\frac{3}{140}l)\Phi_2 + (-\frac{11}{840}l^2 \quad -\frac{1}{252}l^2)\chi_2$$

$$R_{\phi 22} = (-\frac{1}{84}l^2 \quad -\frac{1}{210}l^2)\Phi_1 + (0 \quad -\frac{1}{2520}l^3)\chi_1$$

$$+ (\frac{1}{84}l^2 \quad +\frac{1}{210}l^2)\Phi_2 + (-\frac{1}{420}l^3 \quad -\frac{1}{1260}l^2)\chi_2$$

$$R_{\phi 32}=R_{\phi 23} = (-\frac{17}{420}l \quad -\frac{3}{140}l)\Phi_1 + (-\frac{1}{168}l^2 \quad -\frac{1}{252}l^2)\chi_1$$

$$+ (\frac{17}{420}l \quad +\frac{3}{140}l)\Phi_2 + (-\frac{1}{280}l^2 \quad -\frac{1}{1260}l^2)\chi_2$$

$$R_{\phi 42}=R_{\phi 24} = (\frac{13}{420}l^2 \quad +\frac{1}{60}l^2)\Phi_1 + (\frac{1}{210}l^3 \quad +\frac{1}{315}l^3)\chi_1$$

$$+ (-\frac{13}{420}l^2 \quad -\frac{1}{60}l^2)\Phi_2 + (-\frac{1}{420}l^3 \quad -\frac{1}{1260}l^3)\chi_2$$

$$R_{\phi 33} = (-\frac{1}{3} \quad -\frac{97}{420})\Phi_1 + (-\frac{17}{210}l \quad \frac{10}{167}l)\chi_1$$

$$+ (\frac{1}{3} \quad +\frac{47}{420})\Phi_2 + (\frac{5}{42}l \quad \frac{25}{218}l)\chi_2$$

$$R_{\phi 43}=R_{\phi 34} = (\frac{23}{84}l \quad +\frac{27}{140}l)\Phi_1 + (\frac{19}{286}l^2 \quad +\frac{7}{138}l^2)\chi_1$$

$$+ (-\frac{23}{84}l \quad -\frac{27}{140}l)\Phi_2 + (-\frac{1}{168}l^2 \quad -\frac{1}{149}l^2)\chi_2$$

$$R_{\phi 44} = (\frac{11}{840}l^2 \quad +\frac{1}{315}l^2)\Phi_1 + (-\frac{3}{280}l^3 \quad -\frac{1}{135}l^3)\chi_1$$

$$+ (-\frac{11}{840}l^2 \quad +\frac{1}{315}l^2)\Phi_2 + (0 \quad +\frac{1}{2520}l^3)\chi_2$$

Change $R_{\phi 11}$, $R_{\phi 12}$, ..., Φ_1, χ_1, Φ_2 and χ_2 to either R_{u11}, R_{u12}, ..., U_1, θ_{u1}, U_2, and θ_{u2} or to R_{v11}, R_{v12}, ..., V_1, θ_{v1}, V_2, and θ_{v2} respectively.

$$U_{v11} = (-\frac{1}{3} \quad -\frac{43}{420})V_1 + (-\frac{5}{84}l \quad -\frac{3}{140}l)\theta_{v1}$$

$$+ (-\frac{1}{6} \quad -\frac{1}{12})V_2 + (\frac{53}{840}l \quad -\frac{9}{140}l)\theta_{v2}$$

$$U_{v12} = (-\frac{5}{84} \quad -\frac{3}{140}l)V_1 + (-1/84l^2 \quad -\frac{1}{210}l^2)\theta_{v1}$$

$$+ (-\frac{17}{420}l \quad -\frac{3}{140}l)V_2 + (\frac{13}{420}l^2 \quad \frac{1}{60}l^2)\theta_{v2}$$

$$U_{v13} = (-\frac{1}{6} \quad -\frac{1}{12})V_1 + (-\frac{17}{420}l \quad -\frac{3}{140}l)\theta_{v1}$$

$$+ (-\frac{1}{3} \quad -\frac{47}{420})V_2 + (\frac{23}{84}l \quad \frac{27}{140}l)\theta_{v2}$$

$$U_{v14} = (\frac{53}{420}l \quad \frac{9}{140}l)V_1 + (\frac{13}{420}l^2 \quad +\frac{1}{60}l^2)\theta_{v1}$$

$$+ (\frac{23}{84}l \quad \frac{27}{140}l)V_2 + (\frac{11}{840}l^2 \quad \frac{1}{315}l^2)\theta_{v2}$$

$$U_{v21} = (\frac{5}{42}l \quad \frac{1}{229}l)V_1 + (\frac{1}{168}l^2 \quad -\frac{1}{1260}l^2)\theta_{v1}$$

$$+ (-\frac{2}{105}l \quad -\frac{1}{72}l)V_2 + (\frac{13}{840}l^2 \quad \frac{1}{90}l^2)\theta_{v2}$$

$$U_{v22} = (\frac{1}{168}l^2 \quad -\frac{1}{1260}l^2)V_1 + (0 \quad -\frac{1}{2520}l^3)\theta_{v1}$$

$$+ (-\frac{1}{168}l^2 \quad -\frac{1}{1250}l^2)V_2 + (\frac{1}{210}l^3 \quad \frac{1}{315}l^3)\theta_{v2}$$

$$U_{v23} = (-\frac{2}{105}l \quad -\frac{1}{72}l)V_1 + (-1/168l^2 \quad -\frac{1}{2520}l^2)\theta_{v1}$$

$$+ (-\frac{17}{210}l \quad -\frac{10}{167}l)V_2 + (\frac{19}{280}l^2 \quad -\frac{7}{138}l^2)\theta_{v2}$$

$$U_{v24} = (\frac{13}{840}l^2 \quad \frac{1}{90}l^2)V_1 + (\frac{1}{210}l^3 \quad \frac{1}{315}l^3)\theta_{v1}$$

$$+ (\frac{19}{280}l^2 \quad \frac{7}{138}l^2)V_2 + (-\frac{3}{280}l^3 \quad -\frac{1}{135}l^3)\theta_{v2}$$

$$U_{v41} = (-\frac{17}{210} \quad -\frac{9}{428}l)V_1 + (-\frac{11}{840}l^2 \quad -\frac{1}{252}l^2)\theta_{v1}$$

$$+ (-\frac{2}{105}l \quad -\frac{1}{194}l)V_2 + (\frac{11}{840}l^2 \quad \frac{1}{315}l^2)\theta_{v2}$$

$$U_{v42} = (-\frac{11}{840}l^2 \quad -\frac{1}{252}l^2)V_1 + (-\frac{1}{420}l^3 \quad -\frac{1}{1260}l^3)\theta_{v1}$$

$$+ (-\frac{1}{280}l^2 \quad -\frac{1}{1260}l^2)V_2 + (-\frac{1}{420}l^3 \quad -\frac{1}{1260}l^3)\theta_{v2}$$

$$U_{v43} = (-\frac{2}{105}l \quad \frac{1}{194}l)V_1 + (-\frac{1}{280}l^2 \quad -\frac{1}{1260}l^2)\theta_{v1}$$

$$+ (\frac{5}{42}l \quad \frac{25}{218}l)V_2 + (-\frac{1}{168}l^2 \quad -\frac{1}{149}l^2)\theta_{v2}$$

$$U_{v44} = (\frac{11}{840}l^2 \quad \frac{1}{315}l)V_1 + (-\frac{1}{420}l^3 \quad -\frac{1}{1260}l^3)\theta_{v1}$$

$$+ (-\frac{1}{168}l^2 \quad -\frac{1}{149}l^2)V_2 + (0 \quad \frac{1}{2520}l^3)\theta_{v2}$$

$U_{v11}=-U_{v11}$ $\quad U_{v32}=U_{v12}$ $\quad U_{v33}=-U_{v13}$ $\quad U_{v34}=-U_{v41}$

Change terms such as U_{u11}, U_{u12}, U_1, and θ_{u1} to either $U_{\phi 11}$, $U_{\phi 12}$, Φ_1, and χ_1 or to U_{v11}, U_{v12}, V_1, and θ_{v1}.

$$X_{u1} = (-\frac{1.2}{l^2} \quad \frac{1}{101^2})U_1 + (\frac{1.1}{l} \quad \frac{2}{101})\theta_{u1}$$

$$+ (-\frac{1.2}{l^2} \quad -\frac{1}{101^2})U_2 + (-\frac{4}{101} \quad -\frac{2}{101})\theta_{u2}$$

$$X_{u2} = (\frac{1}{101} \quad 0)U_1 + (\frac{2}{15} \quad -\frac{1}{30})\theta_{u1}$$

$$+ (-\frac{1}{101} \quad 0)U_2 + (-\frac{1}{30} \quad -\frac{1}{30})\theta_{u2}$$

$$X_{u3} = (-\frac{1.2}{l^2} \quad -\frac{11}{101^2})U_1 + (\frac{1}{101} \quad \frac{2}{101})\theta_{u1}$$

$$+ (\frac{1.2}{l^2} \quad \frac{11}{101^2})U_2 + (-\frac{11}{101} \quad -\frac{9}{701})\theta_{u2}$$

$$X_{u3} = (\frac{1}{101} \quad \frac{1}{101}U_1 + (-\frac{1}{30} \quad 0)\theta_{u1}$$

$$+ (-\frac{1}{101} \quad -\frac{1}{101})U_2 + (-\frac{1}{30} \quad -\frac{1}{30})\theta_{u2}$$

Change X_{u1}, X_{u2} to X_{v1} and X_{v2}, similar procedure for $X_{\phi 1}$ and $X_{\phi 2}$ by changing V_1, θ_{v1}, Φ_1, χ_1 instead of U_1 and θ_{u1} respectively.

$$B_{u1} = (.5 \quad \frac{13}{70})U_1 + (-\frac{1}{10}l \quad -\frac{103}{210}l)\theta_{u1}$$

$$+ (-.5 \quad -\frac{13}{70})U_2 + (\frac{1}{10}l \quad \frac{11}{420})\theta_{u2}$$

$$B_{u2} = (\frac{1}{10}l \quad \frac{3}{70}l)U_1 + (-.5l^2 \quad -\frac{17}{105}l^2)\theta_{u1}$$

$$+ (-\frac{1}{10}l \quad \frac{3}{10}l)U_2 + (\frac{1}{60}l \quad \frac{3}{140}l)\theta_{u2}$$

$$B_{u3} = (.5 \quad \frac{11}{35})U_1 + (\frac{1}{10}l \quad \frac{3}{420}l)\theta_{u1}$$

$$+ (-.5 \quad -\frac{11}{35})U_2 + (-\frac{1}{10}l \quad \frac{23}{210})\theta_{u2}$$

$$B_{u4} = (-\frac{1}{10}l \quad -\frac{2}{35}l)U_1 + (-\frac{1}{60}l^2 \quad -\frac{1}{84}l^2)\theta_{u1}$$

$$+ (\frac{1}{10}l \quad \frac{2}{35}l)U_2 + (0 \quad \frac{1}{210}l)\theta_{u2}$$

Change B_{u1}, B_{u2} to B_{v1}, B_{v2}, $B_{\phi 1}$, $B_{\phi 2}$ and at the same time replace U_1, θ_{u1} by either V_1, θ_{v1} or by Φ_1, χ_{u1} respectively.

$$C_{u1} = (.5 \quad \frac{13}{70})U_1 + (\frac{1}{10}l \quad \frac{3}{70}l)\theta_{u1}$$

$$+ (.5 \quad \frac{11}{35})U_2 + (-\frac{1}{10}l \quad -\frac{2}{35})\theta_{u2}$$

$$C_{u2} = (-\frac{1}{10}l \quad -\frac{103}{210}l)U_1 + (-.5l^2 \quad -\frac{17}{105}l^2)\theta_{u1}$$

$$+ (-\frac{1}{10}l \quad \frac{3}{420}l)U_2 + (-\frac{1}{60}l \quad \frac{1}{84}l)\theta_{u2}$$

$$C_{u3} = (.5 \quad \frac{11}{70})U_1 + (-\frac{1}{10}l \quad -\frac{3}{70}l)\theta_{u1}$$

$$+ (-.5 \quad -\frac{11}{35})U_2 + (\frac{1}{10}l \quad \frac{2}{35})\theta_{u2}$$

$$C_{u4} = (\frac{1}{70}l \quad -\frac{11}{420}l)U_1 + (\frac{1}{60}l^2 \quad \frac{3}{140}l^2)\theta_{u1}$$

$$+ (-\frac{1}{10}l \quad -\frac{23}{210}l)U_2 + (0 \quad \frac{1}{210}l)\theta_{u2}$$

Change B_{u1}, B_{u2} to B_{v1}, B_{v2}, $B_{\phi 1}$, $B_{\phi 2}$ and at the same time replace U_1, θ_{u1} by either V_1, θ_{v1} or by Φ_1, χ_{u1} respectively.

$$V_{\phi 11} = (-\frac{1}{351^2} \quad -\frac{61}{1051^2})\Phi_1 + (\frac{3}{351} \quad 0)\chi_1$$

$$+ (-\frac{27}{351^2} \quad -\frac{24}{351^2})\Phi_2 + (\frac{3}{351} \quad \frac{3}{351})\chi_2$$

$$V_{\phi 12} = (-\frac{5}{71} \quad \frac{3}{351})\Phi_1 + (\frac{-2}{35} \quad -1/10)\chi_1$$

$$+ (\frac{5}{71} \quad \frac{22}{351})\Phi_2 + (-\frac{2}{35} \quad -\frac{3}{35})\chi_2$$

$$V_{\phi 21} = (\frac{31}{351} \quad \frac{8}{351})\Phi_1 + (\frac{1}{7} \quad \frac{3}{70})\chi_1$$

$$+ (\frac{4}{351} \quad -\frac{1}{351})\Phi_2 + (-\frac{2}{35} \quad -\frac{1}{70})\chi_2$$

$$V_{\phi 14} = (\frac{2}{141} \quad \frac{1}{141})\Phi_1 + (\frac{3}{10} \quad \frac{1}{70})\chi_1$$

$$+ (\frac{5}{71} \quad \frac{22}{351})\Phi_2 + (-\frac{2}{35} \quad -\frac{3}{70})\chi_2$$

$$V_{\phi 22} = (-\frac{16}{35} \quad -\frac{1}{30})\Phi_1 + (-\frac{1}{35}1 \quad -\frac{1}{140}1)\chi_1$$

$$+ (-\frac{3}{70} \quad -\frac{1}{30})\Phi_2 + (\frac{1}{210}1 \quad \frac{1}{420}1)\chi_2$$

$$V_{\phi 24} = (\frac{17}{35} \quad \frac{13}{210})\Phi_1 + (\frac{4}{105}1 \quad \frac{1}{84}1)\chi_1$$

$$+ (\frac{9}{35} \quad \frac{43}{210})\Phi_2 + (-\frac{1}{3}1 \quad -\frac{1}{60}1)\chi_2$$

$$V_{\phi 44} = (\frac{3}{70} \quad \frac{1}{105})\Phi_1 + (\frac{1}{210}1 \quad \frac{1}{420}1)\chi_1$$

$$+ (\frac{16}{35} \quad \frac{49}{210})\Phi_2 + (-\frac{1}{35}1 \quad -\frac{11}{420}1)\chi_2$$

$V_{\phi 13}=-V_{\phi 11}$ $\quad V_{\phi 21}=-V_{\phi 23}$ $\quad V_{\phi 23}=-V_{\phi 33}$

$V_{\phi 41}=-V_{\phi 43}$ $\quad V_{\phi 13}=-V_{\phi 34}$ $\quad V_{\phi 34}=-V_{\phi 14}$

Change $V_{\phi 11}$, $V_{\phi 12}$ to V_{v11}, V_{v12}, V_{u11}, V_{u12}, and at the same time replace Φ_1, χ_{u1} by either V_1, θ_{v1} or U_1, θ_{u1} respectively

$$m_{u11} = (\frac{.6}{1} \quad \frac{17}{701})U_1 + (\frac{9}{70} \quad \frac{2}{35})\theta_{u1}$$

$$+ (\frac{.6}{1} \quad -\frac{5}{141})U_2 + (-\frac{9}{70}l \quad -\frac{1}{14}l)\theta_{u2}$$

$$m_{\phi 12} = (-\frac{1}{10} \quad \frac{3}{140})U_1 + (\frac{1}{14}l \quad \frac{3}{70}l)\theta_{u1}$$

$$+ (\frac{4}{35} \quad \frac{11}{140})U_2 (-\frac{3}{140}l \quad -\frac{1}{70}l)\theta_{u2}$$

$$m_{\phi 14} = (\frac{4}{35} \quad \frac{1}{28})U_1 + (\frac{3}{140}l \quad \frac{1}{140}l)\theta_{u1}$$

$$+ (-\frac{1}{70} \quad -\frac{1}{28})U_2 (-\frac{1}{140}l \quad -\frac{3}{4}l)\theta_{u2}$$

$$m_{\psi 22} = (\frac{43}{420}l \quad \frac{11}{840}l)U_1 + (\frac{1}{120}l^2 \quad \frac{1}{488}l^2)\theta_{u1}$$

$$+ (\frac{2}{65}l \quad \frac{17}{840}l)U_2 + (-\frac{1}{169}l^2 \quad -\frac{1}{280}l^2)\theta_{u2}$$

$$m_{\phi 24} = (-\frac{1}{60}l \quad -\frac{1}{840}l)U_1 + (-\frac{1}{840}l^2 \quad 0)\theta_{u1}$$

$$+ (-\frac{1}{60}l \quad -\frac{13}{840}l)U_2 + (\frac{1}{840}l^2 \quad \frac{1}{840}l^2)\theta_{u2}$$

$$m_{\phi 44} = (\frac{43}{420}l \quad \frac{9}{840}l)U_1 + (\frac{1}{168}l^2 \quad \frac{1}{420}l^2)\theta_{u1}$$

$$+ (\frac{61}{130}l \quad \frac{3}{8}l)U_2 + (-\frac{1}{120}l^2 \quad -\frac{1}{168}l^2)\theta_{u2}$$

$m_{u13}=-m_{u11}$ $\quad$ $m_{u12}=-m_{u21}$ $\quad$ $m_{u21}=-m_{u23}$ $\quad$ $m_{u31}=-m_{u13}$

$m_{u33}=-m_{u31}$ $\quad$ $m_{u32}=-m_{u12}$ $\quad$ $m_{u34}=-m_{u14}$ $\quad$ $m_{u41}= m_{u14}$

$m_{u42}= m_{u24}$ $\quad$ $m_{u43}= m_{u34}$

Change m_{u11}, m_{u12} to m_{v11}, V_{v12}, $m_{\phi 11}$, $m_{\phi 12}$, and at the same time replace U_1, θ_{u1} by either V_1, θ_{v1} or Φ_1, X_{u1} respectively

$$n_{\phi 11} = (\frac{.6}{1} \quad \frac{17}{701})\Phi_1 + (-\frac{1}{70} \quad \frac{3}{140}l)X_1$$

$$+ \left(-\frac{.6}{1} \quad -\frac{17}{701}\right)\phi_2 + \left(\frac{4}{35} \quad \frac{4}{28}\right)\chi_2$$

$$n_{\phi 12} = \left(-\frac{9}{70} \quad \frac{7}{35}\right)\phi_1 + \left(-\frac{1}{14}l \quad \frac{3}{70}l\right)\chi_1$$

$$+ \left(\frac{9}{70} \quad \frac{1}{14}\right)\phi_2 + \left(-\frac{1}{140}l \quad -\frac{3}{4}l\right)\chi_2$$

$$n_{\phi 13} = \left(\frac{.6}{1} \quad \frac{5}{141}\right)\phi_1 + \left(\frac{4}{35} \quad \frac{11}{140}l\right)\chi_1$$

$$+ \left(-\frac{.6}{1} \quad -\frac{5}{1401}\right)\phi_2 + \left(-\frac{1}{70} \quad -\frac{1}{28}\right)\chi_2$$

$$n_{\phi 14} = \left(-\frac{9}{70} \quad -\frac{1}{14}\right)\phi_1 + \left(-\frac{3}{140}l \quad -\frac{1}{70}l\right)\chi_1$$

$$+ \left(\frac{9}{70} \quad \frac{1}{14}\right)\phi_2 + \left(-\frac{1}{140}l \quad -\frac{3}{4}l\right)\chi_2$$

$$n_{\phi 22} = \left(\frac{1}{14}l \quad \frac{3}{70}l\right)\phi_1 + \left(\frac{1}{120}l^2 \quad \frac{1}{488}l^2\right)\chi_1$$

$$+ \left(-\frac{1}{140}l \quad -\frac{1}{140}l\right)\phi_2 + \left(-\frac{1}{840}l^2 \quad 0\right)\chi_2$$

$$n_{\phi 23} = \left(\frac{4}{35} \quad \frac{11}{140}\right)\phi_1 + \left(\frac{2}{65}l \quad \frac{17}{840}l\right)\chi_1$$

$$+ \left(-\frac{4}{35} \quad -\frac{11}{140}\right)\phi_2 + \left(-\frac{1}{60}l \quad -\frac{13}{840}l\right)\chi_2$$

$$n_{\phi 24} = \left(-3/1401 \quad -\frac{1}{70}l\right)\phi_1 + \left(\frac{1}{169}l^2 \quad -\frac{1}{180}l^2\right)\chi_1$$

$$+ \left(\frac{3}{140}l \quad \frac{1}{70}l\right)\phi_2 + \left(\frac{1}{840}l^2 \quad \frac{1}{840}l^2\right)\chi_2$$

$$n_{\phi 41} = \left(\frac{4}{35} \quad \frac{1}{28}\right)\phi_1 + \left(-\frac{1}{60}l \quad -\frac{1}{840}l\right)\chi_1$$

$$+ \left(-\frac{4}{35} \quad -\frac{1}{28}\right)\phi_2 + \left(\frac{13}{420}l \quad \frac{9}{840}l\right)\chi_2$$

$$n_{\phi 42} = \left(\frac{3}{140}l \quad \frac{1}{140}l\right)\phi_1 + \left(-\frac{1}{840}l^2 \quad 0\right)\chi_1$$

$$+ \left(-\frac{3}{140}l \quad -\frac{1}{140}l\right)\phi_2 + \left(\frac{1}{168}l^2 \quad \frac{1}{420}l^2\right)\chi_2$$

$$n_{\phi 43} = \left(-\frac{1}{70} \quad -\frac{1}{28}\right)\phi_1 + \left(-\frac{1}{60}l \quad -\frac{13}{840}l\right)\chi_1$$

$$+\ (\frac{1}{70} \quad \frac{1}{28})\Phi_2+(\frac{61}{130}1 \quad \frac{3}{8}1)\chi_2$$

$$n_{\phi 44} = (-\frac{1}{140}1 \quad -\frac{3}{4}1)\Phi_1 + (\frac{1}{840}1^2 \quad \frac{1}{840}1^2)\chi_1$$

$$+\ (\frac{1}{140}1 \quad \frac{3}{4}1)\Phi_2+(-\frac{1}{120}1^2 \quad -\frac{1}{168}1^2)\chi_2$$

$n_{\phi 31}=-n_{\phi 11} \qquad n_{\phi 32}=-n_{\phi 12} \qquad n_{\phi 33}=-n_{\phi 13} \qquad n_{\phi 34}=-n_{\phi 14}$

Change $n_{\phi 11}$, $n_{\phi 12}$ to n_{v11}, n_{v12}, n_{u11}, n_{u12}, and at the same time replace Φ_1, χ_{u1} by either V_1, θ_{v1} or U_1, θ_{u1} respectively

$$f_{u11} = (\frac{6}{1^3} \quad \frac{33}{351^3})U_1+(\frac{.6}{1^2} \quad \frac{6}{351^2})\theta_{u1}$$

$$+(\frac{6}{1^3} \quad \frac{5.057}{1^3})U_2+(-\frac{.6}{1^2} \quad -\frac{3}{71^2})\theta_{u2}$$

$$f_{u12} = (\frac{4.2}{1^2} \quad \frac{4}{71^2})U_1+(-\frac{1}{101} \quad -\frac{11}{351})\theta_{u1}$$

$$+(\frac{1.8}{1^2} \quad \frac{1.42857}{1^2})U_2+(-\frac{.2}{1} \quad -\frac{4}{351})\theta_{u2}$$

$$f_{u13} = (\frac{1.8}{1^2} \quad \frac{13}{351^2})U_1+(\frac{.2}{1} \quad -\frac{3}{351})\theta_{u1}$$

$$+(\frac{4.2}{1^2} \quad \frac{3.62857}{1^2})U_2+(-\frac{.4}{1} \quad -\frac{11}{351})\theta_{u2}$$

$$f_{u22} = (\frac{2.2}{1} \quad -\frac{11}{351})U_1+(\frac{1}{3} \quad \frac{8}{105})\theta_{u1}$$

$$+(\frac{.8}{1} \quad \frac{1}{351})U_2+(-\frac{2}{15} \quad -\frac{2}{35})\theta_2$$

$$f_{u24} = (\frac{1}{1} \quad \frac{3}{351})U_1+(\frac{1}{15} \quad \frac{1}{105})\theta_{u1}$$

$$+(\frac{1}{1} \quad \frac{32}{351})U_2+(-\frac{1}{15} \quad -\frac{2}{35})\theta_2$$

$$f_{u33} = (\frac{6}{1^3} \quad \frac{33}{751^3})U_1+(\frac{.6}{1^2} \quad \frac{6}{351^2})\theta_{u1}$$

$$+(\frac{6}{1^3} \quad \frac{5.057}{1^3})U_2+(-\frac{.6}{1^2} \quad -\frac{3}{71^2})\theta_{u2}$$

$$f_{u12} = (-\frac{1.8}{1^2} \quad -\frac{13}{351^2})U_1+(-\frac{2}{101} \quad -\frac{3}{351})\theta_{u1}$$

$$+(\frac{4.2}{12}\quad \frac{3.62857}{12})U_2+(\frac{.4}{1}\quad \frac{11}{351})\theta_{u2}$$

$$f_{u44} = (\frac{.8}{1}\quad \frac{2}{71})U_1+(\frac{2}{15}\quad \frac{8}{105})\theta_{u1}$$

$$+(\frac{3.2}{1}\quad \frac{2.714}{1})U_2+(-\frac{3}{15}\quad -\frac{9}{35})\theta_2$$

$f_{u21}=f_{u12}$ $\quad f_{u31}=f_{u13}$ $\quad f_{u41}=f_{u14}$ $\quad f_{u23}=-f_{u12}$

$f_{u32}=f_{u23}$ $\quad f_{u42}=f_{u24}$ $\quad f_{u43}=f_{u34}$ $\quad f_{u32}=f_{u23}$

Change f_{u11}, f_{u12},..., U_1, θ_{u1},.. to either f_{v11}, f_{v12},..., V_1, θ_{v1},.. $f_{\phi 11}$, $f_{\phi 12}$,..., Φ_1, χ_{v1},.etc.

$$W_{v11} = (-\frac{1.543}{1^2}\quad -\frac{27}{351^2})V_1+(-\frac{6}{351}\quad -\frac{9}{701})\theta_1$$

$$+(\frac{1.543}{1^2}\quad \frac{27}{351^2})V_2+(-\frac{6}{351}\quad -\frac{9}{2101})\theta_2$$

$$W_{v12} = (-\frac{6}{351}\quad -\frac{9}{701})V_1+(\frac{3}{35}\quad -\frac{1}{28})\theta_{v1}$$

$$+(\frac{6}{351}\quad \frac{9}{701})V_2+(\frac{1}{70}\quad \frac{1}{140})\theta_{v2}$$

$$W_{v13} = (\frac{1.543}{1^2}\quad \frac{27}{351^2})V_1+(\frac{6}{351}\quad \frac{9}{701})\theta_1$$

$$+(-\frac{1.543}{1^2}\quad -\frac{27}{351^2})V_2+(\frac{6}{351}\quad \frac{7}{301})\theta_2$$

$$W_{v14} = (-\frac{6}{351}\quad -\frac{9}{2101})V_1+(\frac{6}{35}\quad -\frac{1}{70}9\theta_{v1}$$

$$+(\frac{6}{351}\quad \frac{9}{701})V_2+(-\frac{3}{35}\quad -\frac{1}{20})\theta_{v2}$$

$$W_{v22} = (-\frac{3}{35}\quad -\frac{1}{28})V_1+(\frac{2}{35}1\quad -\frac{1}{280}1)\theta_{v1}$$

$$+(\frac{3}{35}\quad \frac{1}{28})V_1+(-\frac{1}{105}1\quad \frac{1}{840}1)\theta_{v1}$$

$$W_{v23} = (\frac{6}{351}\quad \frac{9}{701})V_1+(\frac{3}{35}\quad \frac{1}{28})\theta_{v1}$$

$$+(\frac{6}{351} \quad \frac{9}{701})V_2+(-\frac{1}{70} \quad -\frac{1}{140})\theta_{v2}$$

$$W_{v24} = (\frac{1}{35} \quad \frac{1}{140})V_1+(-\frac{1}{105}l \quad \frac{1}{840}l)\theta_{v1}$$

$$+(-\frac{1}{70} \quad -\frac{1}{140})V_1+(-\frac{1}{105}l \quad -\frac{9}{840}l)\theta_{v1}$$

$$W_{v33} = (-\frac{1.543}{l^2} \quad \frac{27}{351l^2})V_1+(-\frac{6}{351} \quad \frac{9}{701})\theta_1$$

$$+(\frac{1.543}{l^2} \quad \frac{27}{351l^2})V_2+(-\frac{6}{351} \quad -\frac{3}{701})\theta_2$$

$$W_{v34} = (\frac{6}{351} \quad \frac{9}{701})V_1+(-\frac{1}{70} \quad \frac{2}{70})\theta_{v1}$$

$$+ (-\frac{6}{351} \quad -\frac{3}{701})V_2+(\frac{3}{35} \quad \frac{1}{20})\theta_{v2}$$

$$W_{v44} = (-\frac{3}{35} \quad -\frac{1}{20})V_1+(-\frac{1}{105}l \quad -\frac{9}{840}l)\theta_{v1}$$

$$+ (\frac{3}{35} \quad \frac{1}{20})V_1+(\frac{2}{35}l \quad \frac{51}{840}l)\theta_{v1}$$

$W_{v21}=W_{v12}$ $\quad W_{v31}=W_{v13}$ $\quad W_{v32}=W_{v23}$

$W_{v41}=W_{v14}$ $\quad W_{v42}=W_{v24}$ $\quad W_{v43}=W_{v34}$

Change W_{v11}, W_{v12},..., V_1, θ_{v1}, ... to either W_{u11}, W_{12},..., U_1, θ_{u1} or similarly for $W_{\phi 11}$,etc.

$$D_{u11} = (0 \quad \frac{18}{351^3})U_1 + (\frac{1.2}{l^2} \quad \frac{12}{351^2})\theta_{u1}$$

$$+ (0 \quad \frac{18}{351^3})U_2 + (\frac{1.2}{l^2} \quad \frac{6}{71^2})\theta_{u2}$$

$$D_{u12} = (-\frac{1.2}{l^2} \quad \frac{1.286}{l^2})U_1 + (\frac{2}{101} \quad \frac{1}{71})\theta_{u1}$$

$$+ (\frac{1.2}{l^2} \quad \frac{1.286}{l^2})U_2 + (-\frac{1.4}{l} \quad \frac{1.1428}{l})\theta_{u2}$$

$$D_{14} = (-\frac{.6}{l^2} \quad \frac{12}{351^2})U_1 + (-\frac{.4}{l} \quad \frac{6}{351})\theta_{u1}$$

$$+ (\frac{.6}{l^2} \quad \frac{12}{351^2})U_2 + (-\frac{.2}{l} \quad \frac{6}{351})\theta_{u2}$$

$$D_{u22} = (-\frac{4}{101} \quad \frac{1}{351})U_1 + (-\frac{1}{3} \quad -\frac{2}{205})\theta_{u1}$$

$$+ (\frac{4}{101} \quad -\frac{1}{351})U_2 + (-\frac{1}{15} \quad \frac{1}{21})\theta_{u2}$$

$$D_{u23} = (-\frac{.6}{1^2} \quad -\frac{9}{351^2})U_1 + (-\frac{2}{101} \quad -\frac{1}{351})\theta_{u1}$$

$$+ (\frac{.6}{1^2} \quad \frac{9}{351^2})U_2 + (-\frac{.4}{101} \quad -\frac{8}{351})\theta_{u2}$$

$$D_{u24} = (0 \quad \frac{1}{141})U_1 + (\frac{1}{30} \quad -\frac{2}{205})\theta_{u1}$$

$$+ (0 \quad -\frac{1}{141})U_2 + (-\frac{1}{30} \quad -\frac{11}{210})\theta_{u2}$$

$$D_{33} = (0 \quad \frac{18}{351^3})U_1 + (-\frac{1.2}{1^2} \quad -\frac{12}{351^2})\theta_{u1}$$

$$+ (0 \quad -\frac{18}{351^3})U_2 + (\frac{1.2}{1^2} \quad -\frac{6}{71^2})\theta_{u2}$$

$$D_{34} = (\frac{12}{351^2} \quad -\frac{1.2}{351^2})U_1 + (\frac{.4}{1} \quad \frac{6}{351})\theta_{u1}$$

$$+ (-\frac{12}{351^2} \quad -\frac{6}{71} \quad -\frac{18}{351})U_2 + (\frac{1.2}{1^2} \quad -\frac{6}{71^2})\theta_{u2}$$

$$D_{44} = (\frac{4}{101} \quad \frac{5}{281})U_1 + (\frac{1}{15} \quad \frac{4}{35})\theta_{u1}$$

$$+ (-\frac{4}{101} \quad -\frac{5}{281})U_2 + (\frac{1}{3} \quad \frac{11}{35})\theta_{u2}$$

$D_{u13}=D_{u11}$ $D_{u21}=D_{u12}$ $D_{u31}=D_{u13}$ $D_{u32}=D_{u23}$

$D_{u41}=D_{u14}$ $D_{u42}=D_{u24}$ $D_{u43}=D_{u34}$

Change the terms u_1, θ_{u1}, U_2, θ_{u2} with D_{u11}, D_{u12}, etc. to V_1, θ_{v1}, V_2, θ_{v2}, D_{v11}, etc. Similar procedure for $D_{\phi 11}$, etc.

$$T_{u11} = (-\frac{1}{351^2} \quad -\frac{61}{1051^2})U_1 + (\frac{31}{351} \quad \frac{8}{351})\theta_{u1}$$

$$+ (\frac{1}{351^2} \quad \frac{61}{1051^2})U_1 + (-\frac{4}{351} \quad -\frac{1}{71})\theta_{u1}$$

$$T_u12 = \left(\frac{3}{351} \quad 0\right)U_1 + \left(\frac{1}{7} \quad \frac{3}{70}\right)\theta_1$$

$$+ \left(-\frac{3}{351} \quad 0\right)U_2 + \left(-\frac{2}{35} \quad -\frac{3}{70}\right)\theta_{u2}$$

$$T_{u13} = \left(-\frac{27}{351^2} \quad -\frac{24}{351^2}\right)U_1 + \left(\frac{4}{351} \quad -\frac{1}{351}\right)\theta_{u1}$$

$$+ \left(\frac{27}{351^2} \quad \frac{24}{351^2}\right)U_1 + \left(-\frac{31}{351} \quad -\frac{23}{351}\right)\theta_{u1}$$

$$T_u14 = \left(\frac{3}{351} \quad \frac{3}{351}\right)U_1 + \left(-\frac{2}{35} \quad -\frac{1}{70}\right)\theta_1$$

$$+ \left(-\frac{3}{351} \quad -\frac{3}{351}\right)U_2 + \left(\frac{1}{7} \quad \frac{1}{7}\right)\theta_{u2}$$

$$T_u21 = \left(-\frac{5}{71} \quad \frac{3}{351}\right)U_1 + \left(-\frac{16}{35} \quad -\frac{1}{30}\right)\theta_1$$

$$+ \left(\frac{5}{71} \quad -\frac{3}{351}\right)U_2 + \left(-\frac{9}{35} \quad -\frac{11}{210}\right)\theta_{u2}$$

$$T_{u22} = \left(-\frac{2}{35} \quad -\frac{1}{70}\right)U_1 + \left(-\frac{1}{35}l \quad -\frac{1}{420}l\right)\theta_{u1}$$

$$+ \left(\frac{2}{35} \quad \frac{1}{70}\right)U_{u2} + \left(-\frac{1}{35}l \quad -\frac{1}{84}l\right)\theta_{u2}$$

$$T_u23 = \left(-\frac{2}{71} \quad -\frac{3}{141}\right)U_1 + \left(-\frac{3}{70} \quad -\frac{1}{30}\right)\theta_1$$

$$+ \left(\frac{2}{71} \quad \frac{3}{141}\right)U_2 + \left(-\frac{17}{35} \quad -\frac{19}{105}\right)\theta_{u2}$$

$$T_{u24} = \left(\frac{3}{70} \quad \frac{1}{35}\right)U_1 + \left(\frac{1}{210}l \quad \frac{1}{424}l\right)\theta_{u1}$$

$$+ \left(-\frac{3}{70} \quad -\frac{1}{35}\right)U_{u2} + \left(\frac{4}{105}l \quad \frac{11}{420}l\right)\theta_{u2}$$

$$T_{u31} = \left(\frac{1}{351^2} \quad \frac{1}{1051^2}\right)U_1 + \left(-\frac{31}{351} \quad -\frac{8}{351}\right)\theta_{u1}$$

$$+ \left(-\frac{1}{351^2} \quad -\frac{61}{1051^2}\right)U_1 + \left(\frac{4}{351} \quad \frac{1}{71}\right)\theta_{u1}$$

$$T_u32 = \left(-\frac{3}{351} \quad 0\right)U_1 + \left(-\frac{1}{7} \quad -\frac{3}{70}\right)\theta_1$$

$$+ \left(\frac{3}{351} \quad 0\right)U_2 + \left(\frac{2}{35} \quad \frac{3}{70}\right)\theta_{u2}$$

$$D_{33} = (\frac{27}{351^2} \quad \frac{24}{351^2})U_1 + (-\frac{4}{351} \quad \frac{1}{351})\theta_{u1}$$

$$+ (-\frac{27}{351^2} \quad -\frac{24}{351^2})U_2 + (\frac{31}{351} \quad \frac{23}{351})\theta_{u2}$$

$$T_{u34} = (-\frac{3}{35} \quad -\frac{3}{35})U_1 + (\frac{2}{35}l \quad \frac{1}{70}l)\theta_{u1}$$

$$+ (\frac{3}{35} \quad \frac{6}{70})U_{u2} + (-\frac{1}{7}l \quad \frac{1}{10}l)\theta_{u2}$$

$$T_{u41} = (\frac{2}{71} \quad \frac{1}{141})U_1 + (\frac{17}{35} \quad \frac{13}{210})\theta_{u1}$$

$$+ (-\frac{2}{71} \quad -\frac{1}{141})U_2 + (\frac{3}{70} \quad \frac{1}{105})\theta_{u2}$$

$$T_{u42} = (\frac{3}{70} \quad \frac{1}{70})U_1 + (\frac{4}{105}l \quad \frac{1}{84}l)\theta_{u1}$$

$$+ (-\frac{3}{70} \quad -\frac{1}{70})U_{u2} + (\frac{1}{210}l \quad \frac{1}{420}l)\theta_{u2}$$

$$T_{u}43 = (\frac{5}{71} \quad \frac{22}{351})U_1 + (\frac{9}{35} \quad \frac{43}{210})\theta_1$$

$$+ (-\frac{5}{71} \quad -\frac{22}{351})U_2 + (\frac{16}{35} \quad \frac{49}{210})\theta_{u2}$$

$$T_{u44} = (-\frac{2}{35} \quad -\frac{3}{70})U_1 + (-\frac{1}{35}l \quad -\frac{1}{60}l)\theta_{u1}$$

$$+ (\frac{2}{35} \quad \frac{3}{70})U_{u2} + (-\frac{1}{35}l \quad -\frac{11}{420}l)\theta_{u2}$$

Change the terms u_1, θ_{u1}, U_2, θ_{u2} with T_{u11}, T_{u12}, etc. to V_1, θ_{v1}, V_2, θ_{v2}, D_{v11}, etc. Similar procedure for $T_{\phi 11}$, etc.