Multiple vector bundles and linear generalised complex structures

Malte Alexander Heuer

The University of Sheffield
School of Mathematics and Statistics

A thesis submitted for the degree of
Doctor of Philosophy
in Mathematics

June 2019

Supervisors: Professor Madeleine Jotz Lean
Professor Sarah Whitehouse

Advisor: Dr Evgeny Shinder

Abstract

In this thesis we first study multiple vector bundles, which we define as certain functors from an indexing cube category to the category of smooth manifolds. We describe in detail the cores of n-fold vector bundles and we define an n-pullback of an n-fold vector bundle, as well as n-fold analogues of the core sequences for double vector bundles. We prove the existence of splittings and decompositions of multiple vector bundles, thus showing an equivalent definition in terms of n-fold vector bundle atlases. Furthermore, we define multiply linear sections of an n-fold vector bundle and the category of symmetric n-fold vector bundles as n-fold vector bundles equipped with a certain signed action of the symmetric group S_n.

Secondly, we study linear generalised complex structures on vector bundles. We show the existence of adapted Dorfman connections, which then give adapted linear splittings. This allows to lift the side morphism on $TM \oplus E^*$ to the generalised complex structure in $TE \oplus T^*E$. We describe under which conditions on the side morphism and the Dorfman connection they induce a linear generalised complex structure, furthermore we show the equivalent description in terms of complex VB-Dirac structures in $T_{\mathbb{C}}E \oplus T^*_{\mathbb{C}}E$. Then we study the compatibility of a linear generalised complex structure with an additional Lie algebroid structure and we recover the conditions for morphisms of 2-term representations up to homotopy. We prove that the side and core of the aforementioned complex VB-Dirac structures form complex Lie bialgebroids and we study the induced Drinfeld doubles. In the special case of a complex structure we show that these can be recovered from matched pairs of Courant algebroids. Finally, we translate our results to the abstract setting of VB-Courant algebroids, describing in a splitting the compatibility with the corresponding split Lie 2-algebroid.

Acknowledgements

First I would like to thank my supervisor Madeleine Jotz Lean for the inspiration, guidance and support I have received over the course of the last four years. I am very grateful for all her suggestions and help whenever I was stuck on a problem, and also for her feedback whenever I did write up some of my work. Thank you that you were always enthusiastic about and interested in my project and progress. I am convinced that you will have many other successful students in the years to come.

I also thank Simon Willerton for coordinating my viva, in particular since it took place in Göttingen. And I especially thank the examiners Ping Xu and Chenchang Zhu for reading the thesis carefully and for giving helpful advice, suggestions and comments about past and future research.

Ich danke meinen Eltern von ganzem Herzen, dass ihr mich immer unterstützt habt und mir zwar Rat gegeben, es mir aber vor allem ermöglicht habt, frei und unabhängig meinen Weg zu gehen und meine Entscheidungen zu treffen. Auch meinen Schwestern Milena und Leonie bin ich sehr dankbar für ihre moralische Unterstützung.

Szeretnék köszönetet mondani, Andi, hogy motiválsz engem, és várom a jövőt. Szeretlek!

I want to say thanks to all my friends both from Sheffield and from Germany, for making the time during my PhD studies very enjoyable. Thank you for all the pub evenings, discussions, trips, football matches, card games etc. Luca, apart from that you were also a great housemate and I am glad that you decided to stay in Sheffield. I am also very grateful for all of you who have come to visit me in Sheffield, making all of these visits memorable (some even more so). Especially I would like to thank Robin and Verena for accommodating me so frequently in Göttingen, making it a lot easier for me to meet my supervisor regularly in person.

Finally, I would like to thank the School of Mathematics and Statistics in Sheffield and the German Foreign Exchange Service (DAAD) for the financial funding I have received during my time in Sheffield.

Contents

Chapter 1

Introduction

The fundamental notion connecting both parts of this thesis is the concept of a *double vector bundle*. Double vector bundles were first introduced by Jean Pradines as a structural tool in the study of nonholonomic jets in [64, 62, 63, 65]. Pradines' original definition was given in terms of *double vector bundle charts and atlases*. Later, in [48], Kirill Mackenzie gave a more modern definition of double vector bundles as a square of compatible vector bundle structures. The equivalence of these two definitions relies on the existence of *linear splittings* and *decompositions* for Mackenzie's double vector bundles. The existence of linear splittings was first stated as part of the definition but was later shown to be redundant, a proof can be found for example in Fernando del Carpio-Marek's thesis [17].

The most immediate examples for double vector bundles are the tangent bundle TE and the cotangent bundle T^*E of a vector bundle E. General double vector bundles were studied for example for their application to Poisson geometry and the integration of Lie bialgebroids by Kirill Mackenzie and Ping Xu in [48] and [55]. More recently, double vector bundles were also studied for their connection with representation theory of Lie algebroids by Alfonso Gracia-Saz and Rajan Mehta in [28], by Thiago Drummond, Madeleine Jotz Lean and Cristian Ortiz in [19] and by Gracia-Saz, Jotz Lean, Mackenzie and Mehta in [25]. Henrique Bursztyn, Alejandro Cabrera and Mathias del Hoyo made use of double vector bundles in the study of vector bundles over Lie groupoids and Lie algebroids in [7]. Another interesting application of double vector bundles is Jotz Lean's geometrisation of graded manifolds of degree 2 in [37], also extended to Lie 2-algebroids in [39]. Double vector bundles also play a role in the definition of infinitesimal ideal systems, a notion of ideal in Lie algebroids introduced by Jotz Lean and Ortiz in [40].

The first part of this thesis is concerned with the higher order generalisation of double vector bundles to *multiple vector bundles*. First triple vector bundles and then n-fold vector bundles were introduced by Kirill Mackenzie in [50] and studied jointly with Alfonso Gracia-Saz in [26, 27]. In these papers n-fold vector bundles are defined as n-dimensional cube diagrams of compatible vector bundle structures, together with a splitting condition. Higher order vector bundle were also studied by Janusz Grabowski and Mikołaj Rotkievicz in [24], who gave a definition of an n-fold vector bundle as a manifold equipped with n commuting Euler vector fields, generating the scalar multiplications of the different vector bundle structures. Grabowski and Rotkiewicz also stated the existence of local charts for their n-fold vector bundles, and thus the equivalence of the two definitions, and they gave a proof in the case of $n = 2$.

The main ingredient for the second part of this thesis is the concept of *generalised complex geometry*, which is a unification of complex and symplectic geometry introduced by Nigel Hitchin in [34] and further studied by his student Marco Gualtieri in [30, 31, 32]. Both complex geometry and symplectic geometry have been highly important aspects of differential geometry over the course of the last century. The study of complex manifolds naturally arises from algebraic geometry, whereas symplectic geometry arose from the Hamiltonian formulation of analytical mechanics. An almost complex structure equips all tangent spaces with the structure of complex vector spaces, and gives rise to holomorphic charts if and only if the Nijenhuis tensor vanishes, as proved by August Newlander and Louis Nirenberg in [57]. A 2-form ω on a manifold M defines a presymplectic structure if it is closed, that is if $\mathbf{d}\omega = 0$, whereas a bi-vector π on M defines a Poisson structure if the corresponding Poisson bracket on $C^\infty(M)$ satisfies the Jacobi identity. The idea of generalised geometry is to work with the generalised tangent bundle $TM \oplus T^*M$ instead of considering only the tangent bundle or the cotangent bundle independently. Irene Dorfman in [18] and independently Ted Courant in [13] introduced the Courant-Dorfman bracket on $TM \oplus T^*M$, which is also independently defined and studied in [9] by Antonella Cabras and Vinogradov. Later this bracket gave rise to the definition of a Courant algebroid by Zhang-Ju Liu, Alan Weinstein and Ping Xu in [45]. Ted Courant showed how the integrability conditions for Poisson and presymplectic structures can be unified in terms of the Courant-Dorfman bracket, and defined Dirac structures on M as a common generalisation. Later, Nigel Hitchin introduced generalised complex structures in [34] as a unification of symplectic and complex geometry. Generalised complex structures on a manifold M can be equivalently defined via complex Dirac structures in the complexified generalised tangent bundle $T_{\mathbb{C}}M \oplus T_{\mathbb{C}}^*M$ or via an automorphism $\mathcal{J}$ of $TM \oplus T^*M$. Marco Gualtieri proved a generalised Darboux theorem for generalised complex structures in [30, 31] and also defined generalised

Kähler structures in [30, 32].

Since then, generalised complex geometry has been a highly active field of research. They have been studied for instance for their relation to T-duality – a concept arising in string theory – by Gil Cavalcanti and Marco Gualtieri in [11]. Generalised Kähler structures in have been studied for example by Hitchin in [35] and by Henrique Bursztyn, Cavalcanti and Gualtieri in [8].

The relation between generalised complex geometry and Lie algebroids and Lie groupoids was first studied by Marius Crainic in [14]. In [41], Madeleine Jotz Lean, Mathieu Stiénon and Ping Xu defined the compatibility of generalised complex structures on Lie groupoids with the groupoid structure, and of generalised complex structures on Lie algebroids with the algebroid structure. Lie groupoids and Lie algebroids with such generalised complex structures were there called "Glanon groupoids" and "Glanon algebroids", respectively. The authors furthermore proved a correspondence between compatible generalised complex structures on Lie groupoids and compatible generalised complex structures on Lie algebroids.

Results and structure of this thesis

After recalling definitions and properties of double vector bundles, Lie algebroids, Courant algebroids, Dirac structures, generalised complex structures, VB-Courant algebroids and graded manifolds in **Chapter 2**, this thesis is divided in two relatively independent parts.

Multiple vector bundles

First, in **Chapter 3** we study multiple vector bundles. Most of this chapter can be found on the arXiv as joint work with my supervisor Madeleine Jotz Lean in [33]. We give a definition of n-fold vector bundles in the modern language of category theory which is equivalent to the definition of Kirill Mackenzie and Alfonso Gracia-Saz in [27] to. We hence define an n-fold vector bundle as a special contravariant functor from an indexing category $\square^n$ to the category of smooth manifolds (**Definition 3.1.1**). The indexing category is here given by the poset of the subsets of $\underline{n} := \{1, \ldots, n\}$. Unlike the definition in [27] and the definition of Janusz Grabowski and Mikołaj Rotkievicz in [24], our definition can easily be extended to a definition of ∞-fold vector bundles without a total space. These ∞-fold vector bundles might in the future prove to be a useful object leading to a better understanding of Lie ∞-algebroids. Here we give a detailed description of the different cores in a multiple vector bundles, which generalises the notion of the core of a double vector bundle. We also describe several prototypes of multiple vector bundles, notably the trivial or decomposed ones which are given as fibred products of ordinary vector bundles. Then we define linear splittings and decompositions of multiple vector bundles. A decomposition is an isomorphism between a given multiple vector bundle and an associated decomposed one. Such a decomposition therefore allows to define local charts for an n-fold vector bundle.

We define the n-pullback of an n-fold vector bundle and show that it defines an n-fold vector bundle itself (**Definition 3.1.12** and **Theorem 3.1.13**). This n-pullback plays an important role in the definition of the ultracore sequences, the higher order generalisation of Mackenzie's core sequences for a double vector bundle. We prove that the ultracore sequences are indeed short exact sequences of vector bundles. Making use of these ultracore sequences we then prove inductively the existence of local linear splittings using a method similar to the proof of Fernando del Carpio-Marek for double vector bundles in [17]. Given local linear splittings, they can be extended to global linear splittings with a partition of unity (**Theorem 3.2.3**). We furthermore show how in the n-fold case the existence of splittings is equivalent to the existence of decompositions. More precisely, a decomposition of an n-fold vector bundle $\mathbb{E}$ can be constructed from a linear splitting of $\mathbb{E}$ and additional linear splittings of all the underlying faces and cores. Thus we obtain our main result of that chapter that every n-fold vector bundle admits a decomposition (**Corollary 3.2.4**). We then define n-fold vector bundle atlases and demonstrate how it follows from the existence of decompositions that both definitions of n-fold vector bundles are equivalent.

Furthermore, we show how to obtain decompositions of ∞-fold vector bundles as a colimit of decompositions of n-fold vector bundles. This is not possible with the definition of higher order vector bundles given by Grabowski and Rotkiewicz in [24].

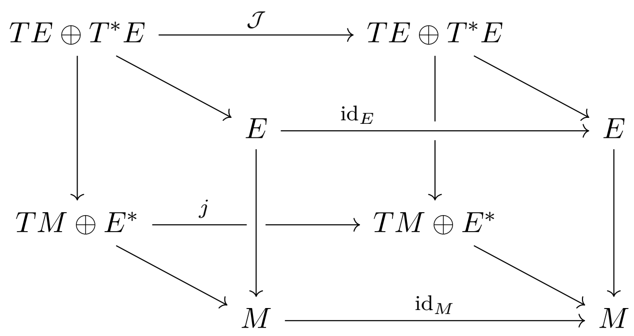
We also define multiply linear sections of an n-fold vector bundle, as a straightforward generalisation of linear sections for $n = 2$. Similarly to the case of $n = 2$ we obtain a short exact sequence of $C^\infty(M)$-modules, where M is the absolute base of the n-fold vector bundle.

Moreover, we define symmetric n-fold vector bundles as n-fold vector bundles equipped with a symmetric structure, which is given by a certain action of the symmetric group. This is a generalisation of the concept of involutive double vector bundles to higher orders and will be important for the geometrisation of graded manifolds of degree n analogously to the degree 2 case, which was carried out by Madeleine Jotz Lean in [37]. Extending this geometrisation to the case of a general degree n is an ongoing joint project with Madeleine Jotz Lean. We will give an explicit equivalence of categories between the category of symmetric n-fold vector bundles and the category of $[n]$-manifolds.

For convenience of the reader we finally demonstrate in detail our results in the special case of $n = 3$, that is for triple vector bundles, since the general notation is fairly technical. In this case we will also demonstrate how a decomposition of a triple vector bundle is equivalent to a horizontal lift, that is a splitting of the short exact sequence of $C^\infty(M)$-modules given by doubly linear sections, together with splittings of the side and core double vector bundle.

Linear generalised complex structures

Second, in **Chapter 4** we study linear generalised complex structures on vector bundles and in particular on Lie algebroids. A linear generalised structure on a vector bundle $E \to M$ is a generalised complex structure $\mathcal{J} \colon TE \oplus T^*E \to TE \oplus T^*E$, which is furthermore a double vector bundle morphism



.

We use linear splittings of the generalised tangent bundle $TE \oplus T^*E \to E$ of a vector bundle $E \to M$ in order to describe the generalised complex structures in terms of the side morphism $j \colon TM \oplus E^* \to TM \oplus E^*$. In order to do so, we use the correspondence of linear splittings of $TE \oplus T^*E$ with $(TM \oplus E^*)$-Dorfman connections on $E \oplus T^*M$ described by Jotz Lean in [36]. Dorfman connections can be thought of as the Courant algebroid analogue of linear connections. We show that for any linear generalised complex structure $\mathcal{J}$ on E with side morphism j there is an adapted Dorfman connection such that the corresponding horizontal lift $\sigma \colon \Gamma(TM \oplus E^*) \to \Gamma_E^\ell(TE \oplus T^*E)$ is compatible with $\mathcal{J}$, in the sense that it satisfies $\sigma(j\nu) = \mathcal{J}\sigma(\nu)$ for any $\nu \in \Gamma(TM \oplus E^*)$ (**Proposition 4.1.15**).

We give equivalent conditions on the side morphism j and the Dorfman connection Δ under which these define a linear generalised complex structure on E (**Theorem 4.1.19**). To describe the integrability condition we define a bracket $\mathbb{A}$ on $TM \oplus E^*$ which does not admit an anchor and thus does not define a Lie algebroid or dull algebroid structure. This bracket $\mathbb{A}$ is defined in terms of j and

Δ but we can show that it is in fact independent of the choice of the adapted Dorfman connection Δ.

We also show that in the case of a linear generalised Kähler structure on E, which involves two commuting linear generalised complex structures on E, we can find a Dorfman connection which is simultaneously adapted to both linear generalised complex structures.

Then we show that the linearity of the generalised complex structure is equivalent to the linearity of the corresponding Dirac structures. Hence a linear generalised complex structure is equivalent to a pair of complex conjugated, transversal complex VB-Dirac structures in $T_{\mathbb{C}}E \oplus T^*_{\mathbb{C}}E$.

After having described the linear structure, we consider a Lie algebroid $A \to M$ and describe the conditions on j and Δ, under which a linear generalised complex structure $\mathcal{J}\colon TA \oplus T^*A \to TA \oplus T^*A$ is additionally a Lie algebroid morphism over j. The latter is the definition of Glanon algebroids given in [41]. We show that this is equivalent to the property of the corresponding Dirac structures to be additionally Lie subalgebroids of $T_{\mathbb{C}}A \oplus T^*_{\mathbb{C}}A \to T_{\mathbb{C}}M \oplus A^*_{\mathbb{C}}$ and therefore such a generalised complex structure is equivalent to a pair of complex conjugated, transversal complex LA-Dirac structures in $T_{\mathbb{C}}A \oplus T^*_{\mathbb{C}}A$. This correspondence has already been stated in [41]. The proof of this equivalence makes use of the description of LA-Dirac structures in terms of a linear splitting, which is given by Jotz Lean in [38]. We also recover the conditions for a morphism of 2-term representations up to homotopy which represent the VB-algebroid structure of $TA \oplus T^*A$ in the linear splitting (**Theorem 4.3.11**).

Afterwards we show that the core morphism of a generalised complex structure on a Lie algebroid defines itself a generalised complex structure in the degenerate Courant algebroid $A \oplus T^*M$, the definition of which was given by Madeleine Jotz Lean in [38].

Subsequently, we show that the $\pm i$-eigenbundles $U_\pm$ of the complexified side morphism $j_{\mathbb{C}}$ of a linear generalised complex structure give rise to a complex A-Manin pair, the complexified version of the A-Manin pair defined by Jotz Lean in [38]. This defines in particular Courant algebroids $C_\pm$ such that $U_\pm$ is a Dirac structure in $C_\pm$. We show that these Courant algebroids are isomorphic to the Drinfeld doubles of the Lie bialgebroids $(U_\pm, K_\mp)$, where $K_\pm$ is the $\pm i$-eigenbundle of the core morphism of the generalised complex structure. Drinfeld doubles of Lie bialgebroids were defined by Zhang-Ju Liu, Alan Weinstein and Ping Xu in [45] and also studied by Kirill Mackenzie in [49]. It was shown in [45] that a Lie bialgebroid is equivalent to two transversal Dirac structures in a Courant algebroid, namely the Drinfeld double.

We also investigate in detail the two extremal cases of generalised complex structures on a Lie algebroid, that is the case of a generalised complex structure induced by either a complex structure or a symplectic structure. In the case of a holomorphic Lie algebroid we find that the two Courant algebroids C_+ and C_- can be decomposed in a direct sum of Courant algebroids $C_T^{1,0} \oplus C_A^{0,1}$ and $C_T^{0,1} \oplus C_A^{1,0}$, respectively. We thus show that these Courant algebroids form matched pairs and $C_\pm$ is given by the matched sum (**Theorem 4.8.9**), which are notions defined for Courant algebroids by Mathieu Stiénon and Melchior Grützmann in [29]. For that we first show that in the complex case our adapted Dorfman connection can be taken to be a standard Dorfman connection, which is induced by an ordinary TM-connection ∇ on E. This connection ∇ is then adapted to the complex structure in the sense that the corresponding horizontal lift $\sigma\colon \Gamma(TM) \to \Gamma_E^\ell(TE)$ satisfies $\sigma(J_M X) = J_E \sigma(X)$. To our knowledge, this is a new insight. In the case of a symplectic Lie algebroid we find that the complex Lie algebroids $U_\pm$ are isomorphic to the complexified tangent bundle $T_\mathbb{C} M$ and thus also to $A_\mathbb{C}^*$. The dual $K_\mp$ is therefore isomorphic to $T_\mathbb{C}^* M$ and to $A_\mathbb{C}$.

Finally, we extend our results to the more abstract setting of a VB-Courant algebroid. We therefore replace the generalised tangent bundle $TE \oplus T^*E$ of a vector bundle E with a general VB-Courant algebroid. Here we obtain adapted Lagrangian splittings. Making use of the correspondence between decomposed VB-Courant algebroids and split Lie 2-algebroids described in [39], we extend the conditions for generalised complex structures to the setting of split Lie 2-algebroids.

Appendix

In the appendix we assume that we want to work with a fixed Dorfman connection instead of choosing an adapted one. This can be useful in the case where we have previously fixed a linear splitting, for example adapted to a different geometric structure on $TE \oplus T^*E$. In general it might not be possible in the presence of independent geometric structures to find one splitting which is adapted to both simultaneously. With a general linear splitting the computations are much more complicated. We describe how a linear generalised complex structure on E is then equivalent to the side morphism j and a 2-form $\Psi \in \Omega(TM \oplus E^*, E^*)$, which depends on the linear splitting, both satisfying certain properties (**Theorem A.1.6**).

Ongoing work and future projects

In the future, the two main parts of this thesis can hopefully be connected via graded manifolds. The geometrisation of graded manifolds of degree n is an ongoing joint project with my supervisor Madeleine Jotz Lean. We will prove an equivalence of categories with such $[n]$-manifolds and the category of symmetric n-fold vector bundles which is described here. This equivalence will be constructed analogously to the degree 2 case in [37] and will rely on the existence of decompositions proved in this thesis. The second goal is to describe generalised complex structures in Lie 2-algebroids without the choice of a splitting. Generalised complex structures naturally live in Courant algebroids which were shown to be equivalent to symplectic Lie 2-algebroids by Dimitry Roytenberg in [66]. Building up on this equivalence David Li-Bland gave an equivalence between VB-Courant algebroids and Lie 2-algebroids in [44]. Later, Madeleine Jotz Lean described how this correspondence can be obtained from her geometrisation of [2]-manifolds. She gave an equivalence between [2]-manifolds and metric double vector bundles and showed how in this picture Lie 2-algebroids correspond to VB-Courant algebroids, Poisson Lie 2-algebroids correspond to LA-Courant algebroids and symplectic Lie 2-algebroids correspond to tangent prolongations of Courant algebroids. Translating our results for generalised complex structures in split Lie 2-algebroids to Lie 2-algebroids as graded manifolds of degree 2 equipped with a cohomological vector field might also lead to a definition of generalised complex structures in Lie n-algebroids.

Chapter 2

Background

In this chapter we will collect definitions and recall properties of the basic objects that we will work with later. We will go through the definitions of double vector bundles, Lie algebroids, Courant algebroids, Dirac structures, generalised complex structures, VB-Courant algebroids and graded manifolds. The expert reader may skip the sections about structures they are already familiar with and focus on the other ones.

2.1 Double vector bundles

In this section we will recapitulate the definition and properties of a double vector bundle. Double vector bundles were first introduced by Jean Pradines in [65] in his study of nonholonomic jets. They were further studied for their application to Poisson geometry and in particular the integration of Lie bialgebroids for example by Kirill Mackenzie [48], also together with Ping Xu in [55]. For a comprehensive overview about the general theory of double vector bundles we can recommend Mackenzie's book [51].

2.1.1 Definitions and examples

Definition 2.1.1. *A **double vector bundle** $(D; A, B; M)$ consists of vector bundle structures $D \to A$, $D \to B$, $A \to M$, $B \to M$:*

$$\begin{array}{ccc} D & \xrightarrow{q_B^D} & B \\ {\scriptstyle q_A^D}\downarrow & & \downarrow{\scriptstyle q_B} \\ A & \xrightarrow{q_A} & M \end{array}, \tag{2.1}$$

such that the structure maps (bundle projection, addition, scalar multiplication and zero section) of D over A are vector bundle morphisms over the corresponding structure maps of $B \to M$ and the other way around.

The compatibility of the vector bundle additions is equivalent to the **interchange law**: for any tuples $d_1, d_2, d_3, d_4 \in D$ such that $q_A^D(d_1) = q_A^D(d_2)$, $q_A^D(d_3) = q_A^D(d_4)$, $q_B^D(d_1) = q_B^D(d_3)$ and $q_B^D(d_2) = q_B^D(d_4)$ we have the equality

$$(d_1 +_A d_2) +_B (d_3 +_A d_4) = (d_1 +_B d_3) +_A (d_2 +_B d_4)\,. \tag{2.2}$$

To obtain a category of double vector bundles, we have to define morphisms between them. This can also be found in [51].

Definition 2.1.2. *A* ***morphism of double vector bundles*** *from $(D; A, B; M)$ to $(D'; A', B'; M')$ consists of maps $\Psi\colon D \to D'$, $\psi_A\colon A \to A'$, $\psi_B\colon B \to B'$ and $\psi_0\colon M \to M'$, such that (Ψ, ψ_A), (Ψ, ψ_A), (ψ_A, ψ_0), (ψ_B, ψ_0) are vector bundle morphisms and such that the following cube commutes*

$$\begin{array}{ccccccc} D & & & \xrightarrow{\Psi} & D' & & \\ & \searrow & & & \downarrow & \searrow & \\ \downarrow & & B & \xrightarrow{\psi_B} & & & B' \\ & & \downarrow & & \downarrow & & \downarrow \\ A & & & \xrightarrow{\psi_A} & A' & & \\ & \searrow & & & & \searrow & \\ & & M & \xrightarrow{\psi_0} & & & M' \end{array} \quad . \tag{2.3}$$

The restriction of such a morphism to core elements induces a morphism of vector bundles $\psi_C\colon C \to C'$, called the core morphism of Ψ.

We will need the following properties of vector bundles, first to prove the surjectivity of the double projection in Lemma 2.1.6 following an argument of David Li-Bland and Pavol Ševera in [43], and later in the generalisation to multiple vector bundles in Theorem 3.1.13. These results are straightforward and well-known, but we want to be able to reference them in the proof of Theorem 3.1.13 in order to keep that proof as concise as possible.

Lemma 2.1.3. *Let $f\colon M \to N$ be a smooth map, and let $q_E\colon E \to N$ be a smooth vector bundle. Then the inclusion $f^!E \hookrightarrow E \times M$ is a smooth embedding.*

Proof. This follows directly from the definition of the pullback vector bundle. As a topological space $f^!E$ is given as the pullback $E \times_{(q_E, N, f)} M$, which is an embedded submanifold of $E \times M$ since q_E is a surjective submersion and therefore f and q_E are immediately transversal. □

Lemma 2.1.4. *Let $A \to M$ and $B \to N$ be two smooth vector bundles, and let $\phi\colon A \to B$ be a homomorphism of vector bundles over a surjective submersion $f\colon M \to N$. Assume that ϕ is surjective in each fibre. Then the pullback homomorphism $f^!\phi\colon A \to f^!B$, $a_m \mapsto (\phi(a_m), m)$ over the identity on M is surjective in each fibre.*

Proof. This statement is immediate, as the fibre of $f^!B$ over m is given exactly by the fibre of B over $f(m)$ and the restriction of $f^!\phi$ to the fibre over m is the restriction of ϕ to the fibre and therefore surjective. □

The following Lemma uses the technique of Li-Bland and Ševera in [43].

Lemma 2.1.5. *Let $A \to M$ and $B \to N$ be two smooth vector bundles, and let $\phi\colon A \to B$ be a homomorphism of vector bundles over a smooth map $f\colon M \to N$. Then ϕ is a surjective submersion if and only if ϕ is surjective in each fiber and f is a surjective submersion.*

Proof. Choose $a_m \in A$. Then it is easy to see in local coordinates that the tangent space $T_{a_m}A$ splits as $T_{a_m}A \simeq T_mM \oplus A_m$, and the tangent space $T_{\phi(a_m)}B$ splits as $T_{f(m)}N \oplus B_{f(m)}$. In those splittings, the map $T_{a_m}\phi\colon T_{a_m}A \to T_{\phi(a_m)}B$ reads

$$T_{a_m}\phi = T_mf \oplus \phi|_{A_m}\colon T_mM \oplus A_m \to T_{f(m)}N \oplus B_{f(m)}.$$

Therefore, $T_{a_m}\phi$ is surjective if and only if $T_mf\colon T_mM \to T_{f(m)}N$ is surjective and $\phi|_{A(m)}\colon A_m \to B_{f(m)}$ is surjective.

Thus if ϕ is a surjective submersion, clearly f is surjective and by the argument above f is a submersion and ϕ is surjective in every fibre. Conversely, if f is a surjective submersion and ϕ is surjective in every fibre, by the previous argument, $T_{a_m}\phi$ is surjective for any $a_m \in A$ and thus ϕ is a surjective submersion. □

The following property is in the literature sometimes stated as an additional condition in the definition of a double vector bundle, but David Li-Bland and Pavol Ševera showed that it is redundant in [43]. This is the case of $n = 2$ of Theorem 3.1.13.

Lemma 2.1.6. *Let $(D; A, B; M)$ be a double vector bundle. Then the double projection $(q_A^D, q_B^D)\colon D \to A \times_M B$ is a surjective submersion and a double vector bundle morphism.*

Proof. The vector bundle projections $q_B\colon B \to M$ and $q_A^D\colon D \to A$ are by definition surjective submersions. Hence Lemma 2.1.5 shows that q_A^D is surjective in each fiber. Now we can identify $A \times_M B$ with the pullback vector bundle $q_B^! A$, and then $(q_A^D, q_B^D)\colon D \to A \times_M B$ is identified with the pullback $q_B^! q_A^D\colon D \to q_B^! A$ as morphism of vector bundles over B. By Lemma 2.1.4 it is hence surjective in each fibre, and so $(q_A^D, q_B^D)\colon D \to A \times_M B$ is a surjective submersion again by Lemma 2.1.5.

That it is a morphism of double vector bundles follows directly from the definition of a double vector bundle. □

Lemma 2.1.7. *Let $q_A\colon A \to M$ be a smooth vector bundle, and let $B \subseteq A$ and $N \subseteq M$ be embedded submanifolds with $q_A(B) = N$ and such that for each $n \in N$, $B(n) \subseteq A(n)$ is a vector subspace. Then $B \to N$ has a unique smooth vector bundle structure, such that the smooth embeddings build a vector bundle homomorphism into $A \to M$.*

Proof. From the definition of a vector subbundle it is immediate that B can be equipped with the structure of a vector subbundle of $A|_N$. With this structure the embedding is clearly a vector bundle homomorphism. Moreover, the embeddings determine the vector bundle structure on $q_A|_B\colon B \to N$ uniquely. □

An important role in the theory of double vector bundles plays the core of a double vector bundle, which is defined in [51] as follows.

Definition 2.1.8. *The common kernel of both projections of a double vector bundle is called the **core** of the double vector bundle:*

$$C = \{c \in D \mid \exists m \in M\colon q_B^D(c) = 0_m^B,\ q_A^D(c) = 0_m^A\}\,. \tag{2.4}$$

Note that $m \in M$ is just the projection $q_A \circ q_A^D(c) = q_B \circ q_B^D(c)$ of c to M.

Lemma 2.1.9. *The core C is itself a vector bundle over M with the vector bundle structure inherited from either the structure of $D \to A$ or of $D \to B$, which coincide on the subset $C \subseteq D$.*

Proof. That C is a smooth submanifold of D follows from the fact that it is the preimage of $0_M^A \times_M 0_M^B$ under the surjective submersion $(q_A^D, q_B^D)\colon D \to A \times_M B$, and that $0_M^A \times_M 0_M^B$ is the image of the zero section in the vector bundle $A \oplus B$ and is thus a smooth submanifold of $A \times_M B$.

To show that the additions over A and B coincide on C we first note that for any $m \in M$ we have $0^D_{0_m^A} = 0^D_{0_m^B} =: 0_m^D$, as the zero section of $D \to A$ is a vector bundle morphism over the zero section of $B \to M$. Take now $c_1, c_2 \in C$ projecting to $m \in M$, then we use the interchange law (2.2) to show

$$\begin{aligned} c_1 +_A c_2 &= (c_1 +_B 0^D_m) +_A (0^D_m +_B c_2) \\ &= (c_1 +_A 0^D_m) +_B (0^D_m +_A c_2) \\ &= c_1 +_B c_2 \,. \end{aligned}$$

That C is closed under this addition is immediate as both projections q^D_B and q^D_A are vector bundle morphisms. That C is closed under both scalar multiplications $\cdot_A$ and $\cdot_B$ follows from the same observation. That both scalar multiplications coincide follows for integers and then rational numbers from the fact that the additions coincide and then by continuity of the multiplications for all real numbers. Therefore we have a well-defined vector bundle structure in $C \to M$ with the structure given by either the restriction of the structure of $D \to A$ or of the structure of $D \to B$. □

Definition 2.1.10. *Let $(D; A, B; M)$ be a double vector bundle. We will view the core C simultaneously as a vector bundle $C \to M$ and as a subset of D. A given section $c \in \Gamma(C)$ induces a **core section** $c^\dagger$ of $D \to A$, defined by*

$$c^\dagger(a_m) = 0^D_{a_m} +_B c(m) \,. \tag{2.5}$$

***Linear sections** of D over A are sections $\xi \in \Gamma_A(D)$ that are vector bundle morphisms over some section $b \in \Gamma(B)$. The set of core sections is denoted by $\Gamma^c_A(D)$ and the set of linear sections by $\Gamma^\ell_A(D)$. For an element $\varphi \in \Gamma(\mathrm{Hom}(A, C))$ we obtain the **core-linear section** $\widetilde{\varphi}$ of D over A, which is a linear section over the zero section of B. It is defined by*

$$\widetilde{\varphi}(a_m) = 0^D_{a_m} +_B \varphi(a_m) \,. \tag{2.6}$$

for any $a_m \in A$ over $m \in M$. In the same manner we define linear sections, core sections and core-linear sections of D over B.

Given a double vector bundle $(D; A, B; M)$ with core C, there are two short exact sequences of vector bundles, called the core sequences as in the following proposition (see also [51]).

Proposition 2.1.11. *The following are short exact sequences of vector bundles over A and B, respectively.*

$$\begin{aligned} 0 \to q^!_A C \xrightarrow{\iota_A} D \xrightarrow{(q^D_A, q^D_B)} q^!_A B \to 0 \\ 0 \to q^!_B C \xrightarrow{\iota_B} D \xrightarrow{(q^D_A, q^D_B)} q^!_B A \to 0 \,. \end{aligned} \tag{2.7}$$

The maps ι_A and ι_B are the core inclusions, defined by $\iota_A((a,c)) := 0^D_a +_B c$ and $\iota_B((b,c)) := 0^D_b +_A c$ for $a \in A$, $b \in B$, $c \in C$ over the same $m \in M$.

Proof. The injectivity of the maps ι_A and ι_B is immediate. Since the addition in C is defined as restriction of $+_A$ and $+_B$, both maps define vector bundle morphisms. As a manifold $q^!_B A = A \times_M B = q^!_A B$ and the map (q^D_A, q^D_B) is surjective according to Lemma 2.1.6. That it defines a vector bundle morphism over A and over B follows directly from the fact that q^D_A and q^A_B are vector bundle morphisms over q_B and q_A, respectively.

Given any $(a,c) \in q^!_A C$ we have that $(q^D_A, q^D_B)(\iota_A(a,c)) = (q^D_A, q^D_B)(0^D_a +_B c) = (a, 0^B_m)$ and thus $\iota_A(a,c)$ is in the kernel of (q^D_A, q^D_B) as a vector bundle morphism over A. Conversely, given $d \in D$ in the kernel of (q^D_A, q^D_B) as vector bundle morphism over A, we have that $d -_B 0^D_{q^D_A(d)}$ projects to 0^A_m and 0^B_m and is therefore by definition an element of the core,$c \in C$. Thus $d = 0^D_{q^D_A(d)} +_B c = \iota_A q^D_A(d)$ and the first sequence is exact. Completely analogously it follows that the second sequence is exact. □

Definition 2.1.12. *Given a double vector bundle $(D; A, B; M)$, the **flip** of D is a double vector bundle $(D^{flip}; B, A; M)$ with the same vector bundle structures as D but exchanged role of A and B. This is a different double vector bundle since the position of A and B is important in the definition of morphisms of double vector bundles, given in Definition 2.1.2.*

We will now give some basic examples of double vector bundles.

Example 2.1.13. *For three vector bundles A, B, C over M there is the **trivial double vector bundle** with sides A and B and with core C, given by*

$$\begin{array}{ccc} A \times_M B \times_M C & \xrightarrow{\mathrm{pr}_B} & B \\ {\scriptstyle \mathrm{pr}_A}\downarrow & & \downarrow \\ A & \longrightarrow & M \end{array} , \tag{2.8}$$

where the additions are defined as $(a_m, b_m, c_m) +_A (a_m, b'_m, c'_m) = (a_m, b_m + b'_m, c_m + c'_m)$ and analogously for the addition over B. Here the core section $c^\dagger \in \Gamma_A(D)$ is thus given by $c^\dagger(a_m) = (a_m, 0^B_m, c(m))$, whilst core-linear sections are given by $\widetilde{\varphi}(a_m) = (a_m, 0^B_m, \varphi(a_m))$ for $\varphi \in \Gamma(\mathrm{Hom}(A, C))$.

For any manifold M we will denote by $p_M : TM \to M$ the projection of the tangent bundle to the manifold and by $c_M : T^*M \to M$ the cotangent projection.

Example 2.1.14. *For any vector bundle $q_E\colon E \to M$ the* ***tangent double***

$$\begin{array}{ccc} TE & \xrightarrow{p_E} & E \\ {\scriptstyle Tq_E}\big\downarrow & & \big\downarrow{\scriptstyle q_E} \\ TM & \xrightarrow{p_M} & M \end{array}\ , \tag{2.9}$$

is a double vector bundle with core isomorphic to E. Core sections of TE over E are given by vertical vector fields

$$e^{\uparrow}(e'_m) = \frac{d}{dt}\Big|_{t=0} e'_m + te(m)\,. \tag{2.10}$$

Vertical vector fields act on linear functions ℓ_ε for $\varepsilon \in \Gamma(E^*)$ by $e^{\uparrow}(\ell_\varepsilon) = q_E^*\langle\varepsilon, e\rangle$ since

$$e^{\uparrow}(\ell_\varepsilon)(e'_m) = \frac{d}{dt}\Big|_{t=0} \ell_\varepsilon(e'_m + e(m)) = \frac{d}{dt}\Big|_{t=0} \langle\varepsilon, e'_m + te(m)\rangle = \langle\varepsilon, e\rangle(m)$$

and on pullbacks of functions $f \in C^\infty(M)$ by $e^{\uparrow}(q_E^* f) = 0$ since

$$e^{\uparrow}(q_E^* f)(e'_m) = \frac{d}{dt}\Big|_{t=0} q_E^* f(e'_m + e(m)) = \frac{d}{dt}\Big|_{t=0} f(m) = 0\,.$$

Linear vector fields $\xi \in \Gamma_E^\ell(TE)$ over $X \in \Gamma(TM)$ send linear functions ℓ_ε to linear functions and pullbacks of functions on M to pullbacks and therefore correspond to derivations $D \in \mathrm{Der}(\Gamma(E))$ over X with the correspondence given by

$$\xi(\ell_\epsilon) = \ell_{D^*(\epsilon)} \quad \text{and} \quad \xi(q_E^*(f)) = q_E^*(X(f))\,. \tag{2.11}$$

Given a derivation D as above, we write $\widehat{D}$ for the corresponding linear vector field. See for example [51] and also [25] or [36]. We will also write $\mathfrak{X}^\ell(E)$ for the space of linear vector fields and $\mathfrak{X}^c(E)$ for the space of vertical vector fields.

A special case of this is the double tangent bundle of a manifold.

Example 2.1.15 (Double tangent bundle)**.** *For any smooth manifold M the double tangent bundle*

$$\begin{array}{ccc} TTM & \xrightarrow{p_{TM}} & TM \\ {\scriptstyle Tp_M}\big\downarrow & & \big\downarrow{\scriptstyle p_M} \\ TM & \xrightarrow{p_M} & M \end{array}\ ,$$

is a double vector bundle with core TM. This double vector bundle comes with a canonical flip, an isomorphism of double vector bundles $J\colon TTM \to TTM^{\mathit{flip}}$, exchanging the two vector bundle structures Tp_M and p_{TM}.

2.1.2 Dualising double vector bundles

For every double vector bundle (D, A, B, M) with core C there are two ways of dualising a double vector bundle, we can dualise D either as a vector bundle over A or as vector bundle over B. The dual bundle of D over A will be denoted by D^*_A and is a double vector bundle with sides A and C^* and with core B^*.

$$\begin{array}{ccc} D^*_A & \xrightarrow{r^{D^*_A}_{C^*}} & C^* \\ {\scriptstyle q^{D^*}_A}\big\downarrow & & \big\downarrow{\scriptstyle q_{C^*}} \\ A & \xrightarrow{q_A} & M \end{array} .$$

The vertical projections are the ordinary projections of the dual vector bundles, whereas the projection of D^*_A to C^* is given by

$$r^{D^*_A}_{C^*}(\delta)(c_m) := \langle \delta, 0^D_{a_m} +_B c_m \rangle_A \,,$$

for $\delta \in (D^*_A)_{a_m}$. The identification of the core of D^*_A with B^* is given by

$$\langle \beta, b_m \rangle := \langle \beta, 0^D_{b_m} \rangle \,,$$

for any element $\beta \in D^*_A$ over 0^A_m projecting to $0^{C^*}_m$ under $r^{C^*}_{D^*}$.

In Mackenzie's book [51] there is a detailed description of the following properties of the duals of double vector bundles, which we just recall for the sake of completeness, as it later proves to be one way of defining the Lie algebroid structure on the generalised tangent bundle of a Lie algebroid.

Proposition 2.1.16. *There is a non-degenerate pairing between the vector bundles $D^*_A)^*_{C^*} \to C^*$ and $D^*_B \to C^*$ given for $\Phi \in D^*_A$ and $\Psi \in D^*_B$ over the same $\gamma \in \Gamma(C^*)$ by*

$$\langle \Phi, \Psi \rangle := \langle \Phi, d \rangle_A - \langle \Psi, d \rangle_B$$

for any $d \in D$ such that $q^D_A(d) = q^{D^*}_A(\Phi)$ and $q^D_B(d) = q^{D^*}_B(\Psi)$. It is also shown that this is a pairing of double vector bundles and induces a double vector bundle isomorphism between $(D^*_A)^*_{C^*}$ and the flip of D^*_B.

Example 2.1.17. *Dualising the tangent double of a vector bundle (TE, E, TM, M) over E gives the **cotangent double**, a double vector bundle*

$$\begin{array}{ccc} T^*E & \xrightarrow{c_E} & E \\ {\scriptstyle r_E}\big\downarrow & & \big\downarrow{\scriptstyle q_E} \\ E^* & \xrightarrow{q_{E^*}} & M \end{array} ,$$

with core T^*M. Sections of $T^*E \to E$ are 1-forms on E. The space of linear sections is generated by exact 1-forms of the form $\mathbf{d}\ell_\varepsilon$, linear over a section

$\varepsilon \in \Gamma(E^*)$. Core sections are given by pullbacks $q_E^*\theta$ of 1-forms $\theta \in \Omega^1(M)$. This example is covered in detail in Mackenzie's book [51].

Kirill Mackenzie and Ping Xu proved in [53] that the double vector bundles T^*E and T^*E^* are isomorphic. We now give a brief recollection of this isomorphism.

Proposition 2.1.18. *Given a vector bundle $E \to M$ and its dual vector bundle E^*, there is an isomorphism $R\colon T^*E^* \to T^*E$ over the sides id_E and id_{E^*} with core morphism $-\mathrm{id}_{T^*M}$. This map is called the* ***reversal isomorhism*** *and is given on core sections and prototypical linear sections of $T^*E^* \to E^*$ as follows:*

$$\begin{aligned} R(q_{E^*}^*\theta)(\varepsilon_m) &= \mathbf{d}_{0_m^E}\ell_\varepsilon - q_E^*\theta(0_m^E) \\ R(\mathbf{d}\ell_e)(\varepsilon_m) &= \mathbf{d}_{e(m)}\big(\ell_\varepsilon - q_E^*\langle\varepsilon, e\rangle\big)\,. \end{aligned} \tag{2.12}$$

Thus core sections of $T^*E \to E^*$ are given by $R(-q_{E^*}^*\theta)$ and typical linear sections are given by $a^R := R(\mathbf{d}\ell_e)$.

Alternatively, the inverse isomorphism $R^{-1}\colon T^*E \to T^*E^*$ can be defined as composition of the following isomorphisms. First, applying the tangent functor T to the pairing of E with E^* induces a map $TE \times_{TM} TE^* \to T\mathbb{R}$, the projection to the fibre in $T\mathbb{R} \to \mathbb{R}$ gives a non-degenerate pairing between TE and TE^* as vector bundles over TM and thus an isomorphism $I\colon TE^* \to (TE)^*_{TM}$. This is in fact an isomorphism of double vector bundles, and the dual of I as a morphism over E^* gives an isomorphism $I^t_{E^*}\colon ((TE)^*_{TM})^*_{E^*} \to T^*E^*$. Precomposing this isomorphism with the isomorphism between the iterated dual to the dual of TE over E described in Proposition 2.1.16 gives the desired isomorphism of double vector bundles $TE^* \to T^*E^*$.

Example 2.1.19. *Another important example of double vector bundles, in particular in the study of linear generalised complex structures, is the generalised tangent bundle of a vector bundle E. It is given as the direct sum of vector bundles over E of the tangent double and the cotangent double:*

$$\begin{array}{ccc} TE \oplus T^*E & \longrightarrow & E \\ \downarrow & E \oplus T^*M & \downarrow \\ TM \oplus E^* & \longrightarrow & M \end{array} \quad (E \oplus T^*M \searrow M).$$

We will describe this double vector bundle in more detail in Section 2.3.3 about VB-Courant algebroids.

2.1.3 Decompositions and splittings

This section is concerned with decompositions and linear splittings of double vector bundles, which shows that Definition 2.1.1 of double vector bundles is equivalent to the original definition in terms of double vector bundle charts given by Pradines in [65].

Definition 2.1.20. *A **linear splitting** of a double vector bundle $(D; A, B; M)$ is an injective morphism of double vector bundles*

$$\Sigma\colon A \times_M B \to D$$

which is the identity on the sides A and B.

Definition 2.1.21. *A **decomposition** of a double vector bundle $(D; A, B; M)$ is an isomorphism of double vector bundles*

$$\mathcal{S}\colon A \times_M B \times_M C \to D$$

from the trivial double vector bundle $A \times_M B \times_M C$ to D which is the identity on the sides and on the core.

The following property is straight forward and mentioned in several places, for example in [25].

Lemma 2.1.22. *Linear splittings and decompositions of a double vector bundle are equivalent to each other.*

Proof. Let $(D; A, B; M)$ be a double vector bundle with core C. Given a linear splitting Σ, we define a decomposition by $\mathcal{S}(a_m, b_m, c_m) := \Sigma(a_m, b_m) +_B (0^D_{b_m} + c_m) = \Sigma(a_m, b_m) +_A (0^D_{a_m} + c_m)$. Conversely, given a decomposition $\mathcal{S}$ of D, we define the splitting by $\Sigma(a_m, b_m) := \mathcal{S}(a_m, b_m, 0^C_m)$. These constructions are inverse to each other. □

Definition 2.1.23. *Let $(D; A, B; M)$ be a double vector bundle with core C. Given two different decompositions $\mathcal{S}_1, \mathcal{S}_2\colon A \times_M B \times_M C \to D$, the composition $\mathcal{S}_1^{-1} \circ \mathcal{S}_2$ is an automorphism of $A \times_M B \times_M C$ over the identity on A, B and C. Such a morphism is necessarily of the form $(a_m, b_m, c_m) \mapsto (a_m, b_m, c_m + \phi(a_m, b_m))$ for some $\phi \in \Gamma(A^* \otimes B^* \otimes C)$. We call ϕ the **change of splittings**.*

Conversely, given one decomposition $\mathcal{S}$ and such an automorphism of $A \times_M B \times_M C$, the composition defines a second decomposition of D.

We will now give a proof of the well-known fact that every double vector bundle admits a decomposition. This has been stated in several places, however rarely with a full proof. Fernando del Carpio-Marek gave a proof in his thesis [17]. We first need the following short exact sequences of $C^\infty(M)$-modules, which can be found for example in [25].

Lemma 2.1.24. *The following is a short exact sequence of $C^\infty(M)$-modules:*

$$0 \to \Gamma(\mathrm{Hom}(A,C)) \xrightarrow{\widetilde{(\cdot)}} \Gamma_A^\ell(D) \xrightarrow{\pi} \Gamma(B) \to 0\,, \tag{2.13}$$

where π sends a linear section to its base section and $\widetilde{(\cdot)}$ denotes the map that sends a section of $\mathrm{Hom}(A,C)$ to the corresponding core linear section of $D \to A$.

There is an analogous sequence for $\Gamma_B^\ell(D)$.

Proof. Injectivity of $\widetilde{(\cdot)}$ and exactness at $\Gamma_A^\ell(D)$ follow directly from the exactness of the core sequence of vector bundles over A from Proposition 2.1.11. Given $\xi \in \Gamma_A^\ell(D)$ projecting to $0\colon M \to B$ we define a section $\phi \in \Gamma(\mathrm{Hom}(A,C)$ by setting for any $a_m \in A$ over $m \in M$ $\phi_m(a_m) := \xi(a_m) -_B 0^D_{a_m}$. Now every short exact sequence of vector bundles is non-canonically split which can be seen by choosing a Riemannian metric on the vector bundle in the middle of the sequence. Choose thus a splitting $s\colon q_A^! B \to D$. Then we define for any section $b \in \Gamma(B)$ a section $\hat{b}$ of $D \to A$ by $\hat{b}(a_m) = s(a_m, b(m))$. Now $\hat{b}$ is a vector bundle morphism over b and therefore an element of $\Gamma_A^\ell(D)$ projecting to b. This shows that the map π is surjective and the sequence exact. □

The following equivalence is also described in [25].

Lemma 2.1.25. *Splittings of double vector bundles are equivalent to horizontal lifts σ_B, which are splittings of the exact sequence of $C^\infty(M)$-modules* (2.13).

Proof. Given a linear splitting Σ, we define the lift for $b \in \Gamma(B)$ and $a_m \in A$ over $m \in M$ by $\sigma_B(b)(a_m) := \Sigma(a_m, b(m))$. Since Σ is a vector bundle morphism over A, this defines a linear section in $\Gamma_A^\ell(D)$. Since Σ is additionally a vector bundle morphism over B we have also $\sigma_B(b+b') = \sigma_B(b) + \sigma_B(b')$ and $\sigma_B(fb) = (q_A)^* f \sigma_B(b)$ and σ_B is indeed a morphism of $C^\infty(M)$-modules.

Conversely, given a horizontal lift σ_B splitting the sequence (2.13), we define for any $(a_m, b_m) \in A \times_M B$ the splitting by $\Sigma(a_m, b_m) := \sigma_B(b)(a_m)$, where $b \in \Gamma(B)$ is any section such that $b(m) = b_m$. To show that this does not depend on the choice of the section b we choose a neighbourhood U of $m \in M$ such that $B|_U$ is a trivial vector bundle. Choosing local basis sections $\beta_1, \ldots, \beta^k$ for $B|_U$ we can write any two sections b_1, b_2 of B with $b_1(m) = b_2(m) = b_m$ as a sum $b_i = \sum_{j=1}^{k} f_j^i \beta_j$ for suitable functions $f_j^i \in C^\infty(U)$. Using that σ_B is a morphism of $C^\infty(M)$-modules it then follows that $\sigma_B(b_1)(a_m) = \sigma_B(b_2)(a_m)$ and Σ is well-defined. That Σ is a vector bundle morphism over B follows directly from the fact that σ_B is a morphism of $C^\infty(M)$-modules. Additionally, Σ is a vector bundle morphism over A since $\sigma_B(b)$ is a linear section of $D \to A$ for any $b \in \Gamma(B)$. Thus Σ is a double vector bundle morphism and a linear splitting. □

Because of the symmetry of Σ thus a horizontal lift σ_B is therefore also equivalent to a lift $\sigma_A \colon \Gamma(A) \to \Gamma_B^\ell(D)$, splitting the sequence

$$0 \to \Gamma(\mathrm{Hom}(B, C)) \xrightarrow{\widetilde{(\cdot)}} \Gamma_B^\ell(D) \xrightarrow{\pi} \Gamma(A) \to 0\,.$$

Given two linear splittings Σ_1, Σ_2 with change of splitting given by $\phi \in \Gamma(A^* \otimes B^* \otimes C) \cong \Gamma(\mathrm{Hom}(B, A^* \otimes C)) \cong \Gamma(\mathrm{Hom}(A, B^* \otimes C))$ the corresponding lifts are related by $\sigma_B^1(b) - \sigma_B^1(b) = \widetilde{\phi(b)}$ and $\sigma_A^1(a) - \sigma_A^1(a) = \widetilde{\phi(a)}$.

The following is an important example of a linear splitting.

Example 2.1.26. *Consider the tangent double of a vector bundle $(TE; TM, E; M)$. As seen in Example 2.1.14 a linear section of $TE \to E$ over $X \in \Gamma(TM)$ corresponds to a derivation on $\Gamma(E)$ with symbol X. A horizontal lift $\sigma_{TM} \colon \Gamma(TM) \to \Gamma_E^\ell(TE)$ is thus equivalent to a choice of a derivation ∇_X of $\Gamma(E)$ with symbol X for every $X \in \Gamma(TM)$, depending linearly on X. Thus linear splittings of TE correspond to TM-connections ∇ on E. The corresponding lift $\sigma^\nabla := \sigma_{TM}$ is given for $X \in \Gamma(TM)$ and $e_m \in E_m$ by the formula*

$$\sigma^\nabla(X)(e_m) = T_m e X(m) -_E \left.\frac{d}{dt}\right|_{t=0} (e_m + t\nabla_X e)\,, \tag{2.14}$$

where e is any section of E such that $e(m) = e_m$.

This allows us to express properties of TE, such as the Lie bracket in terms of the Lie bracket on TM and the connection. Let us fix a connection ∇ as above and denote the lift by σ^∇. We write $R_\nabla \in \Gamma(\mathrm{Hom}(TM \otimes TM, E))$ for the curvature of the connection. Then for any $X, Y \in \Gamma(TM)$ and $e, e_1, e_2 \in \Gamma(E)$ we have

$$[\sigma^\nabla(X), \sigma^\nabla(Y)] = \sigma^\nabla([X, Y]) - R_\nabla(X, Y)^\uparrow \tag{2.15}$$
$$[\sigma^\nabla(X), e^\uparrow] = (\nabla_X e)^\uparrow \tag{2.16}$$
$$[e_1^\uparrow, e_2^\uparrow] = 0\,, \tag{2.17}$$

which follows directly from the description of linear and core sections in Example 2.1.14. See also [25].

Example 2.1.27. *Given a decomposition $\mathcal{S}\colon A \times_M B \times_M C \to D$ of a double vector bundle $(D; A, B; M)$, the dual of $\mathcal{S}$ as a vector bundle morphism over A is again an isomorphism of double vector bundles $\mathcal{S}_A^t\colon D_A^* \to A \times_M C^* \times_M B^*$. The inverse of this is a decomposition of D_A^*. For details on duals of double vector bundle morphisms we refer to [51].*

This observation together with Example 2.1.26 shows in particular that also splittings of $(T^*E; E, E^*; M)$ are in one-to-one correspondence with linear TM-connections on E.

With the previous results, it is simple to prove the existence of decompositions as follows.

Proposition 2.1.28. *Every double vector bundle is non-canonically isomorphic to a trivial double vector bundle $A \times_M B \times_M C$ with sides A and B and core C.*

Proof. Let $(D; A, B; M)$ be a double vector bundle with core C. as shown by Lemma 2.1.22 and Lemma 2.1.25 a decomposition of D is equivalent to a splitting of the short exact sequence of sheaves of $C^\infty(M)$-modules

$$0 \to \Gamma(\operatorname{Hom}(A, C)) \xrightarrow{\tilde{(\cdot)}} \Gamma_A^\ell(D) \xrightarrow{\pi} \Gamma(B) \to 0\,.$$

Since both B and $\operatorname{Hom}(A, C)$ are vector bundles over M, the sheaves $\Gamma(B)$ and $\Gamma(\operatorname{Hom}(A, C))$ are locally free and finitely generated. For every point $p \in M$ there is an open neighbourhood U such that $\Gamma_U(B)$ and $\Gamma_U(\operatorname{Hom}(A, C))$ are free $C^\infty(U)$-modules. Free modules are especially projective and an R-module P over a commutative ring R is projective if and only if for every surjective R-module homomorphism $\pi\colon M \to P$ there is a right inverse $\sigma\colon P \to M$, that is $\pi \circ \sigma = \mathrm{id}_P$. Thus there is a splitting $\sigma_{B|_U}$ of the sequence above restricted to U. As $C^\infty(U)$ is a commutative ring, the category of $C^\infty(U)$-modules is abelian and the splitting of the short exact sequence is equivalent to $\Gamma^\ell_{A|_U}(D)$ being isomorphic to a direct sum of the free $C^\infty(U)$-modules $\Gamma_U(\operatorname{Hom}(A, C))$ and $\Gamma_U(B)$ by the splitting lemma. Hence $\Gamma^\ell_{A|_U}(D)$ is itself a free $C^\infty(U)$-module and the sheaf $\Gamma^\ell_A(D)$ is therefore locally free and finitely generated and thus isomorphic to the sheaf sections of a vector bundle $\hat{B}$. Now a splitting σ_B of the sequence is equivalent to a splitting of the following short exact sequence of vector bundles:

$$0 \to \operatorname{Hom}(A, C)) \longrightarrow \hat{B} \longrightarrow B \to 0\,.$$

Such a splitting can always be found, for example by choosing a Riemannian metric on $\hat{B}$ and identifying B with the orthogonal complement of the image of $\operatorname{Hom}(A, C)$.

Thus for every double vector bundle $(D; A, B; M)$ there exists a horizontal lift $\sigma_A\colon \Gamma(B) \to \Gamma^{\ell}_A(D)$ and therefore a decomposition $\mathcal{S}\colon A \times_M B \times_M C \to D$. □

2.2 Lie algebroids

Lie groupoids and Lie algebroids are the many-point versions of Lie groups and Lie algebras. Groupoids were first introduced to differential geometry by Charles Ehresmann ([20, 21, 22] and then subsequently studied by Jean Pradines [58, 59, 61, 60], Lie groupoids have later been studied due to their close connection with Poisson geometry for example by Alain Coste, Pierre Dazord and Alan Weinstein in [12] and by Weinstein in [70]. A comprehensive overview of the theory of Lie groupoids and algebroids can be found in Mackenzie's books, [47, 51]. More recently, the problem of integrability of Lie algebroids, that is the existence of a corresponding Lie groupoid or Lie's third theorem for algebroids, has been solved by Marius Crainic and Rui Loja Fernandes in [15, 16]. In this thesis we will mainly work with Lie algebroids. However, we include the definition of a Lie groupoid here since one main motivation behind the study of Lie algebroids is that they arise as infinitesimal version of Lie groupoids. Furthermore, the results about generalised complex Lie algebroids in Section 4.3 can be applied to analogous results about the integrating Lie groupoids making use of results of Madeleine Jotz Lean, Mathieu Stiénon and Ping Xu in [41]. This is future work.

2.2.1 Definitions and examples

Definition 2.2.1. *A **Lie groupoid** over a smooth manifold M is a smooth manifold G equipped with two surjective submersions $\mathsf{s}\colon G \to M$ and $\mathsf{t}\colon G \to M$, called source and target projection, with a smooth inclusion $\mathbf{1}\colon M \to G$ and with a smooth partial multiplication $\mu\colon G \times_M G \to G$, where $G \times_M G := \{(g,h) \in G \times G \mid \mathsf{s}(g) = \mathsf{t}(h)\} \subseteq G \times G$, such that the following axioms are satisfied:*

1. $\mathsf{s}(\mu(g,h)) = \mathsf{s}(h)$ *and* $\mathsf{t}(\mu(g,h)) = \mathsf{t}(g)$ *for all* $(g,h) \in G \times_M G$,

2. μ *is associative,*

3. $\mathsf{s}(\mathbf{1}(x)) = \mathsf{t}(\mathbf{1}(x)) = x$ *for all* $x \in M$,

4. $\mu(g, \mathbf{1}(\mathsf{s}(g))) = g = \mu(\mathbf{1}(\mathsf{t}(g)), g)$ *for all* $g \in G$,

5. *for every* $g \in G$ *there is* $g^{-1} \in G$, *s.t.* $\mathsf{s}(g^{-1}) = \mathsf{t}(g)$, $\mathsf{t}(g^{-1})$, $\mu(g, g^{-1}) = \mathbf{1}(\mathsf{t}(g))$ *and* $\mu(g^{-1}, g) = \mathbf{1}(\mathsf{s}(g))$.

We think of the elements of M as the objects and of elements of G as arrows between those objects, which we can compose if the source and target coincide.

Definition 2.2.2. *A* ***Lie algebroid*** *over a smooth manifold M is a vector bundle $A \to M$, together with a vector bundle morphism*

$$\rho\colon A \to TM\,, \tag{2.18}$$

called the anchor, a $\mathbb{R}$-bilinear, skew-symmetric bracket on sections of A

$$[\cdot,\cdot]\colon \Gamma(A)\times\Gamma(A) \to \Gamma(A)\,, \tag{2.19}$$

that satisfies the Jacobi identity and the Leibniz identity

$$[a, fb] = f[a,b] + \rho(a)(f)b\,, \tag{2.20}$$

for all $a, b \in \Gamma(A)$ and $f \in C^\infty(M)$.

The following property is sometimes stated as part of the definition of a Lie algebroid, however it already follows from the other axioms.

Lemma 2.2.3. *Let A be a Lie algebroid over M. For all $a, b \in \Gamma(A)$ we then have*

$$\rho([a,b]) = [\rho(a),\rho(b)]\,.$$

Proof. Fix $a, b \in \Gamma(A)$. Since the bracket satisfies the Jacobi identity, we have for any $c \in \Gamma(A)$ and any $f \in C^\infty(M)$

$$\begin{aligned} 0 &= [[a,b], fc] + [[fc,a],b] + [[b,fc],a] \\ &= f[[a,b],c] + \rho([a,b])(f)c \\ &\quad + [f[c,a] - \rho(a)(f)c, b] + [f[b,c] + \rho(b)(f)c, a] \\ &= f\big([[a,b],c] + [[c,a],b] + [[b,c],a]\big) \\ &\quad + \rho([a,b])(f)c - \rho(b)(f)[c,a] - \rho(a)(f)[c,b] + \rho(b)\big(\rho(a)(f)\big)c \\ &\quad - \rho(a)(f)[b,c] + \rho(b)(f)[c,a] - \rho(a)\big(\rho(b)(f)\big)c \\ &= \big(\rho([a,b]) - [\rho(a),\rho(b)]\big)(f)c\,, \end{aligned}$$

where we made use of the Leibniz identity, skew-symmetry and again the Jacobi identity. Since c and f were arbitrary, it follows that $\rho([a,b]) = [\rho(a),\rho(b)]$. □

The following two examples are in some sense the extremal cases of Lie algebroids, with the anchor being the identity or vanishing.

Example 2.2.4. *The tangent bundle TM of a manifold M is a Lie algebroid, where the anchor is the identity and the Lie bracket is the Lie bracket of vector fields.*

Example 2.2.5. *A Lie algebra bundle $M \times \mathfrak{g}$ is a Lie algebroid, with trivial anchor and fibrewise bracket.*

A Lie algebra is the same as a Lie algebroid over a point, i.e. where $M = \{*\}$.

In [51] and also in [16] morphisms of Lie algebroids are defined in the general case of Lie algebroids over different base manifolds. This gives rise to the category of Lie algebroids. The problem with defining such a morphism is that in general we can not push forward any section. However, in this thesis we will only deal with the special case of morphisms where the underlying vector bundle morphism is already an isomorphism over a diffeomorphism on the base. This case is naturally considerably easier since we can push forward any section. We give here only the definition in this special case and refer to [51] or [16] for the general case.

Definition 2.2.6. *Given two Lie algebroids $A \to M$ and $B \to N$ with anchor maps ρ_A and ρ_B and brackets $[\cdot,\cdot]_A$ and $[\cdot,\cdot]_B$ and a vector bundle morphism $(\Phi, \varphi)\colon A \to B$. Assume that any section a of A can be pushed forward to a section $\Phi_! a$ of B, such that $\Phi \circ a = (\Phi_! a) \circ \varphi$. Then (Φ, φ) is a **morphism of Lie algebroids** from A to B if and only if for any $a_1, a_2 \in \Gamma(A)$ we have*

1. $\rho_B \circ \Phi = T\varphi \circ \rho_A\,,$
2. $\Phi \circ ([a_1, a_2]_A) = [a_1', a_2'] \circ \varphi\,.$

Example 2.2.7. *Given a Lie algebroid A, the **tangent prolongation** Lie algebroid of A is the following Lie algebroid structure on the vector bundle $TA \to TM$. Since TA is a double vector bundle as in Example 2.1.14, the space of sections $\Gamma_{TM}(TA)$ is generated by core sections and linear sections. Linear sections are given by $Ta\colon TM \to TA$ for sections $a \in \Gamma(A)$, whereas core sections are given by $a^\dagger\colon TM \to TA$ for $a \in \Gamma(A)$ defined by*

$$a^\dagger(v_m) = 0^{TA}_{v_m} +_A \left.\frac{d}{dt}\right|_{t=0} ta(m)\,. \tag{2.21}$$

The anchor ρ_{TA} of the Lie algebroid $TA \to TM$ is a vector bundle morphism from $TA \to TM$ to $T(TM) \to TM$, where on $T(TM)$ the vector bundle structure is the canonical projection p_{TM} of the tangent bundle over the smooth manifold TM. It is given on linear sections by $\rho_{TA}(Ta) = \widehat{[\rho(a), \cdot]}$ and on core sections by $\rho_{TA}(a^\dagger) = (\rho(a))^\uparrow$. Note that $[\rho(a), \cdot]$ is a derivation of $\Gamma(TM)$ and $\rho_{TA}(Ta)$ is the corresponding linear vector field (see also Example 2.1.14). The Lie bracket is defined by

$$\begin{aligned} [Ta_1, Ta_2] &= T[a_1, a_2]\,, \\ [Ta_1, a_2^\dagger] &= [a_1, a_2]^\dagger\,, \\ [a_1^\dagger, a_2^\dagger] &= 0\,. \end{aligned} \tag{2.22}$$

More details and proofs on this construction can be found again in Mackenzie's book [51].

Equivalently, the tangent prolongation Lie algebroid is obtained by applying the tangent functor to all the structure maps in the Lie algebroid A and for the anchor composing it with the canonical flip $J\colon TTM \to TTM$ interchanging the two vector bundle structures on $TTM \to TM$.

Lie algebroids are closely related to Poisson geometry. For instance we have the following example of a Lie algebroid.

Example 2.2.8. *Let P be a Poisson manifold with Poisson bracket $\{\cdot,\cdot\}$ on $C^\infty(P)$. The induced morphism $\pi^\sharp\colon T^*P \to TP$ is the anchor for a Lie algebroid structure on the cotangent bundle T^*P defined by additive and Leibniz extension of the bracket*

$$[df, dg] = d\{f, g\}\,,$$

where $f, g \in C^\infty(P)$. Equivalently, with the bi-vector field π the bracket can be described as follows for $\omega_1, \omega_2 \in \Gamma(T^*P)$

$$[\omega_1, \omega_2] = \mathcal{L}_{\pi^\sharp\omega_1}\omega_2 - \mathcal{L}_{\pi^\sharp\omega_2}\omega_1 - \mathbf{d}(\pi(\omega_1, \omega_2)\,.$$

This classical example can be found in [12] and is also described in [53] and [51].

Furthermore, it is shown in [53] that Lie algebroid structures on $A \to M$ are equivalent to linear Poisson structures on $A^* \to M$ as follows.

Example 2.2.9. *A Poisson structure on a vector bundle E is called **linear** if $\pi^\sharp\colon T^*E \to TE$ is a vector bundle morphism over some $\rho\colon E^* \to TM$. Then ρ is the anchor for a Lie algebroid structure on E^* with bracket given by*

$$\langle[\varepsilon_1, \varepsilon_2], e\rangle := \Big(\pi^\sharp(\mathbf{d}\ell_{\varepsilon_1})\ell_{\varepsilon_2}\Big)(e)\,.$$

Conversely, a Lie algebroid structure $A \to M$ with anchor ρ induces a linear Poisson structure $\pi^\sharp\colon T^*A^* \to TA^*$ on A^* by setting for $a \in \Gamma(A)$ and $\theta \in \Gamma(T^*M)$

$$\pi^\sharp(\mathbf{d}\ell_a) := \widehat{[a,\cdot]}\,, \qquad \pi^\sharp(q_{A^*}^*\theta) := (\rho^t(\theta))^\uparrow\,.$$

As before $\widehat{[a,\cdot]}$ denotes the linear vector field on A^* corresponding to the derivation $[a,\cdot]$ of $\Gamma(A)$ over $\rho(a)$.

The previous examples allow to define a Lie algebroid structure on the vector bundle $T^*A \to A^*$ for a Lie algebroid A. This has been done by Mackenzie and Xu in [54] and plays an important role in the study of Lie bialgebroids and Poisson groupoids.

Example 2.2.10. *Let $A \to M$ be a Lie algebroid. Consider A^* as a Poisson manifold with the linear Poisson structure described in Example 2.2.9. Then $T^*A^* \to A^*$ is a Lie algebroid with the cotangent Lie algebroid structure described in Example 2.2.8. The reversal isomorphism of double vector bundles $R\colon T^*A^* \to T^*A$ from Proposition 2.1.18 thus defines a Lie algebroid structure on $T^*A \to A^*$. On sections of the types mentioned in Proposition 2.1.18 the anchor and bracket is then given by extension of the previous examples as follows.*

$$\begin{aligned} &\rho_{T^*A}(a^R) = \widehat{\mathcal{L}_a}\,, \qquad \rho_{T^*A}(\theta^\dagger) = (\rho^t\theta)^\uparrow\,, \\ &[a_1^R, a_2^R] = [a_1, a_2]^R\,, \quad [a^R, \theta^\dagger] = (\mathcal{L}_{\rho(a)}\theta)^\dagger\,, \quad [\theta_1^\dagger, \theta_2^\dagger] = 0\,. \end{aligned} \tag{2.23}$$

This description can also be found in the appendix of [36].

Now we can combine the structure of TA and T^*A in order to define a Lie algebroid structure on the generalised tangent bundle over a Lie algebroid. This is also described for example in [36].

Example 2.2.11. *Given a Lie algebroid $A \to M$, then the generalised tangent bundle $TA \oplus T^*A$ has a Lie algebroid structure over $TM \oplus A^*$. This is obtained from the Lie algebroid $TA \to TM$ (Example 2.2.7) and the Lie algebroid $T^*A \to A^*$ (Example 2.2.10) by taking the direct product $TA \times T^*A \to TM \times A^*$ and then the pullback to the diagonal $\Delta_A \to \Delta_M$ to obtain a Lie algebroid structure on the fibred product $TA \times_A T^*A \to TM \times_M A^*$.*

Analogously to the Lie derivative, interior and exterior derivative in the case of the tangent bundle and differential forms we can define a Lie derivative, interior derivative and Lie algebroid differential for general Lie algebroids. The Lie algebroid differential then gives rise to a cochain complex defining Lie algebroid cohomology. Details about these constructions and properties of these maps can again be found in [51]. We recall the definitions.

Definition 2.2.12. *Given a Lie algebroid $A \to M$ with anchor ρ and bracket $[\cdot,\cdot]$, we write $\Omega(A) := \Gamma(\wedge A^*)$. We define the **Lie algebroid differential** $\mathbf{d}_A\colon \Omega^k(A) \to \Omega^{k+1}(A)$ by setting for $\omega \in \Omega^k(A)$*

$$\begin{aligned} (\mathbf{d}_A\omega)(a_0, \ldots, a_k) := &\sum_i (-1)^i \rho(a_i)\omega(a_0, \ldots, \widehat{a_i}, \ldots a_k) \\ &+ \sum_{i<j} (-1)^{i+j}\omega([a_i, a_j], a_0, \ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, a_k)\,. \end{aligned} \tag{2.24}$$

*For a section $a \in \Gamma(A)$ we furthermore define the **Lie derivative** $\mathcal{L}_a^A\colon \Omega^k(A) \to \Omega^k(A)$ for $\omega \in \Omega^k(A)$ by*

$$(\mathcal{L}_a\omega)(a_1, \ldots, a_k) := \rho(a)\omega(a_1, \ldots, a_k) - \sum_i \omega(a_1, \ldots, [a, a_i], \ldots, a_k)\,. \tag{2.25}$$

*The **interior derivative** $\iota_a\colon \Omega^{k+1}(A) \to \Omega^k(A)$ is given for $\omega \in \Omega^{k+1}(A)$ by*

$$(\iota_a \omega)(a_1, \dots, a_k) := \omega(a, a_1, \dots, a_k)\,. \tag{2.26}$$

Another concept arising from Poisson geometry is Lie bialgebroids. They were introduced by Kirill Mackenzie and Ping Xu in [53] as the infinitesimal counterpart to Poisson groupoids.

Definition 2.2.13. *Given Lie algebroids A and A^* which are dual as vector bundles. Then (A, A^*) is a **Lie bialgebroid** if for all $X, Y \in \Gamma(A)$*

$$\mathbf{d}_*[X,Y] = \mathcal{L}_X \mathbf{d}_* Y - \mathcal{L}_Y \mathbf{d}_* X\,, \tag{2.27}$$

where d_* is the Lie algebroid differential of A^*.

Later it was shown in [45] by Liu, Weinstein and Xu that Lie bialgebroids are equivalent to two transversal Dirac structures in a Courant algebroid, the Drinfeld double of the Lie bialgebroid. We recall this structure in Example 2.3.4.

2.2.2 Linear connections

In the studies of Lie algebroids linear connections play an important role as flat connections define representations of Lie algebroids. In this section we recall some standard definitions about linear connections. Since we will later also need connections of only anchored vector bundles we define connections in greater generality. All the definitions in this sections can be found in several places, for example in [1].

Definition 2.2.14. *Given a vector bundle $Q \to M$ with an anchor map $\rho\colon Q \to TM$ and a vector bundle $B \to M$, then a **linear Q-connection on** B is an $\mathbb{R}$-bilinear map*

$$\begin{aligned} \nabla\colon \Gamma(Q) \times \Gamma(B) &\to \Gamma(Q) \\ (q, b) &\mapsto \nabla_q b\,, \end{aligned} \tag{2.28}$$

such that for any $q \in \Gamma(Q)$, $b \in \Gamma(B)$ and $f \in C^\infty(M)$

1. $\nabla_{fq} b = f \nabla_q b\,,$
2. $\nabla_q f b = \rho(q)(f) b + f \nabla_q b\,.$

*If Q is additionally a Lie algebroid (or later dull algebroid as in Definition 2.3.14), the **curvature of** ∇ is $R_\nabla \in \Gamma(Q^* \otimes Q^* \otimes B^*)$ defined by*

$$R_\nabla(q_1, q_2)(b) = \nabla_{q_1} \nabla_{q_2} b - \nabla_{q_2} \nabla_{q_1} b - \nabla_{[q_1, q_2]} b\,. \tag{2.29}$$

Since the bracket of a Lie algebroid is skew-symmetric we have in that case $R_\nabla \in \Omega^2(Q, B^*)$. *In the case of a dull algebroid as in Definition 2.3.14, this is not guaranteed. A connection is called* ***flat*** *if* $R_\nabla = 0$.

Definition 2.2.15. *A* ***representation of a Lie algebroid*** *A on a vector bundle E is a flat A-connection on E.*

The following notion was defined by Tahar Mokri in [56] and also studied by Jiang-Hia Lu in [46] and Kirill Mackenzie in [52].

Definition 2.2.16. *A* ***matched pair of Lie algebroids*** *consists of two Lie algebroids* $(A\,\rho_A, [\cdot,\cdot]_A)$ *and* $(B, \rho_B, [\cdot,\cdot]_B)$ *over* M, *with representations* $\overrightarrow{\nabla}$ *of* A *on* B *and* $\overleftarrow{\nabla}$ *of* B *on* A, *satisfying for all* $a, a_1, a_2 \in \Gamma(A)$ *and* $b, b_1, b_2 \in \Gamma(B)$ *the following relations:*

1. $[\rho_A(a), \rho_B(b)]_{TM} = -\rho_A(\overleftarrow{\nabla}_b a) + \rho_B(\overrightarrow{\nabla}_a b)\,,$
2. $\overrightarrow{\nabla}_a[b_1, b_2]_B = [\overrightarrow{\nabla}_a b_1, b_2]_B + [b_1, \overrightarrow{\nabla}_a b_2]_B + \overrightarrow{\nabla}_{\overleftarrow{\nabla}_{b_2} a} b_1 - \overrightarrow{\nabla}_{\overleftarrow{\nabla}_{b_1} a} b_2\,,$
3. $\overleftarrow{\nabla}_b[a_1, a_2]_A = [\overleftarrow{\nabla}_b a_1, a_2]_A + [a_1, \overleftarrow{\nabla}_b a_2]_A + \overleftarrow{\nabla}_{\overrightarrow{\nabla}_{a_2} b} a_1 - \overleftarrow{\nabla}_{\overrightarrow{\nabla}_{a_1} b} a_2\,.$

The importance of the notion of matched pairs of Lie algebroids stems from the following theorem, proved in [56].

Theorem 2.2.17. *Given a matched pair of Lie algebroids A and B as above, then there is a Lie algebroid structure* $A \bowtie B$ *on* $A \oplus B$, *called the* ***matched sum Lie algebroid****, which is defined as follows. The anchor is given for* $a \in \Gamma(A)$ *and* $b \in \Gamma(B)$ *by*

$$\rho((a, b)) := \rho_A(a) + \rho_B(b) \tag{2.30}$$

and the bracket is given for $a_1, a_2 \in \Gamma(A)$ *and* $b_1, b_2 \in \Gamma(B)$ *by*

$$[(a_1, b_1), (a_2, b_2)] := \Big([a_1, a_2]_A + \overleftarrow{\nabla}_{b_1} a_2 - \overleftarrow{\nabla}_{b_2} a_1, [b_1, b_2]_B + \overrightarrow{\nabla}_{a_1} b_2 - \overrightarrow{\nabla}_{a_2} b_1\Big)\,. \tag{2.31}$$

Conversely, given a Lie algebroid structure on $A \oplus B$ *such that both* $A \cong A \oplus 0$ *and* $B \cong 0 \oplus B$ *are Lie subalgebroids, then the equation*

$$[(a, 0), (0, b)] =: (\overleftarrow{\nabla}_b a, \overrightarrow{\nabla}_a b)\,, \tag{2.32}$$

for $a \in \Gamma(A)$ *and* $b \in \Gamma(A)$ *defines representations* $\overrightarrow{\nabla}$ *of A on B and* $\overleftarrow{\nabla}$ *of B on A, making A and B into a matched pair of Lie algebroids.*

Definition 2.2.18. *Let Q be an anchored vector bundle over M and ∇ a linear Q-connection ∇ on a vector bundle $B \to M$, the* ***dual connection*** ∇^* *is a Q-connection on B^*, defined by the equation*

$$\langle \nabla^*_q \beta, b\rangle = \rho(q)\langle \beta, b\rangle - \langle \beta, \nabla_q b\rangle\,, \tag{2.33}$$

for any $q \in \Gamma(Q)$, $b \in \Gamma(B)$, $\beta \in \Gamma(B^*)$.

Definition 2.2.19. *Let A be a Lie algebroid over M and E be a vector bundle over M, the* ***space of E-valued forms on A*** *is given by $\Omega^\bullet(A; E) := \Gamma(\wedge^\bullet A^* \otimes E)$, and an A-connection on E induces a differential $\mathbf{d}_\nabla \colon \Omega^k(A; E) \to \Omega^{k+1}(A; E)$ by setting for $\omega \in \Omega^k(A; E)$*

$$\begin{aligned}(\mathbf{d}_\nabla \omega)(a_0, \dots, a_k) := &\sum_i (-1)^i \nabla_{a_i}\Big(\omega(a_0, \dots, \widehat{a_i}, \dots a_k)\Big)\\ &+ \sum_{i<j} (-1)^{i+j} \omega([a_i, a_j], a_0, \dots, \widehat{a_i}, \dots, \widehat{a_j}, \dots, a_k)\,,\end{aligned} \tag{2.34}$$

for any $a_0, \dots, a_k \in \Gamma(A)$.

Definition 2.2.20. *Let Q be an anchored vector bundle and ∇^B and ∇^C linear Q-connections on B and C, respectively. Then this induces a linear Q-connection ∇^{Hom} on $\mathrm{Hom}(B, C)$ by*

$$\nabla^{\mathrm{Hom}}_q(\phi) = \phi \circ \nabla^B_q - \nabla^C_q \circ \phi\,,$$

for any $q \in \Gamma(Q)$ and $\phi \in \Gamma(\mathrm{Hom}(B, C))$.

2.2.3 Representations up to homotopy

Representations up to homotopy of Lie algebroids were introduced by Camilo Arias Abad and Marius Crainic in [1] as a generalisation of Lie algebra representations to general Lie algebroids. The idea of defining representations up to homotopy goes back already to Sam Evans, Jiang-Hua Lu and Alan Weinstein, who defined similar representations in [23]. It was shown by Alfonso Gracia-Saz and Rajan Mehta in [28] that after the choice of a decomposition VB-algebroid structures on a double vector bundle correspond to 2-term representations up to homotopy of the side algebroid. Later Thiago Drummond, Madeleine Jotz Lean and Cristian Ortiz proved in [19] that this correspondence also holds on the level of morphisms. We recall in this section the definition of representations up to homotopy and the aforementioned correspondence. Let in this section A always be a Lie algebroid over M.

The following definitions are given in [1] and recalled in [19]. Given a $\mathbb{Z}$-graded vector bundle $E = \bigoplus_{j\in\mathbb{Z}} E_j$, the space $\Omega(A;E) := \Gamma(\wedge A^* \otimes E)$ has a total grading defined by

$$\Omega(A;E)_k = \bigoplus_{i+j=k} \Gamma(\wedge^i A^* \otimes E_j)\,. \tag{2.35}$$

Definition 2.2.21. *Let A be a Lie algebroid over M. A **representation up to homotopy** of A is a graded vector bundle E and an operator*

$$D\colon \Omega(A;E)_\bullet \to \Omega(A;E)_{\bullet+1}\,, \tag{2.36}$$

satisfying $D^2 = 0$ and the graded derivation rule

$$D(\omega\wedge\eta) = \mathbf{d}_A(\omega)\wedge\eta + (-1)^k\omega\wedge D(\eta)\,, \tag{2.37}$$

for any $\omega \in \Omega^k(A)$ and $\eta \in \Omega(A;E)$.

Definition 2.2.22. *Given two such representations up to homotopy (E, D_E) and (F, D_F) up to homotopy of A, a morphism of representations up to homotopy is a degree zero, $\Omega(A)$-linear map*

$$\Phi\colon \Omega(A;E) \to \Omega(A;F)\,,$$

such that $\Phi\circ D_E = D_F\circ\Phi$.

Because of the results of [28] and [19], 2-term representations up to homotopy are of particular relevance to us, that is when the graded vector bundles are concentrated in degrees 0 and 1. It is also shown in [1] that such a 2-term representation up to homotopy of A on $E_0 \oplus E_1$ is equivalent to the following data. We use here the sign convention of [19].

1. a vector bundle morphism $\partial\colon E_0 \to E_1$,
2. linear A-connections ∇^0 and ∇^1 on E_0 and E_1 such that $\partial\circ\nabla^0 = \nabla^1\circ\partial$,
3. an element $K \in \Omega^2(A; \mathrm{Hom}(E_1, E_0))$, such that
   - $\mathbf{d}_{\nabla^{\mathrm{Hom}}} K = 0$,
   - $R_{\nabla^0} = -K\circ\partial$ and
   - $R_{\nabla^1} = -\partial\circ K$.

Such a 2-term representation up to homotopy will thus also be denoted by the tuple $(\partial, \nabla^0, \nabla^1, K)$ as above, called the **structure operators** of the 2-term representation up to homotopy. According to [19] and [1] a morphism between two such 2-term representations up to homotopy $(\partial_E, \nabla^0_E, \nabla^1_E, K_E)$ on $E = E_0 \oplus E_1$ and $(\partial_F, \nabla^0_F, \nabla^1_F, K_F)$ on $F = F_0 \oplus F_1$ of the same Lie algebroid $A \to M$ consists of the following data: A triple $(\varphi_0, \varphi_1, \Phi)$, where $\varphi_k \colon E_k \to F_k$ for $k = 0, 1$ are bundle maps and $\Phi \in \Omega^1(A; \mathrm{Hom}(E_1, F_0))$, satisfying for all $a \in \Gamma(A)$

1. $\varphi_1 \circ \partial_E = \partial_F \circ \varphi_0\,,$
2. $\nabla^{\mathrm{Hom}}_a(\varphi_0) = \Phi_a \circ \partial_E\,,$
3. $\nabla^{\mathrm{Hom}}_a(\varphi_1) = \partial_F \circ \Phi_a\,,$
4. $\mathbf{d}_{\nabla^{\mathrm{Hom}}}\Phi = \varphi_0 \circ K_E - K_F \circ \varphi_1\,.$

Now we will recall the correspondence between VB-algebroid structures on a decomposed double vector bundle and 2-term representations up to homotopy shown in [28, 19]. The following definition of VB-algebroids can be found in [28] and the definition of morphisms can be found in [19].

Definition 2.2.23. *A* ***VB-algebroid*** *is a double vector bundle $(D; A, B; M)$, where additionally $D \to B$ is a Lie algebroid, such that the anchor $\rho_D \colon D \to TB$ is a vector bundle homomorphism over some $\rho_A \colon A \to TM$ and the Lie bracket of D is linear, that is*

1. $[\Gamma^\ell_B(D), \Gamma^\ell_B(D)] \subseteq \Gamma^\ell_B(D)\,,$
2. $[\Gamma^\ell_B(D), \Gamma^c_B(D)] \subseteq \Gamma^c_B(D)\,,$
3. $[\Gamma^c_B(D), \Gamma^c_B(D)] = 0\,.$

A ***morphism of VB-algebroids*** *between two VB-algebroids $(D; A, B; M)$ and $(D'; A', B'; M)$ over the same base M is a morphism $(\Psi; \psi_A, \psi_B; \mathrm{id}_M)$ of double vector bundles, such that $(\Psi; \psi_B)$ is additionally a Lie algebroid morphism.*

The Lie algebroid structure on D then also induces a Lie algebroid structure on $A \to M$ with anchor ρ_A and bracket defined as follows: For linear sections $\xi, \zeta \in \Gamma^\ell_B(D)$ over $a, b \in \Gamma(A)$ then $[\xi, \zeta]$ is linear over $[a, b]_A$. With this structure A is called the **side algebroid** of D.

In [28] it is shown that there is a correspondence between VB-algebroid structures on a decomposed double vector bundle $A \times_M B \times_M C$ with side algebroid A and 2-term representations up to homotopy of A onto $C_{[0]} \oplus B_{[1]}$, the graded vector bundle with C in degree 0 and B in degree 1. In [19], it is shown that this also holds for morphisms and thus gives an equivalence of categories. More precisely, the following statements are proved.

Theorem 2.2.24. *Let $A \to M$ be a Lie algebroid, $B \to M$ and $C \to M$ be vector bundles. There is a one-to-one correspondence between VB-algebroid structures on $A \times_M B \times_M C$ with core C and side algebroid A and 2-term representations up to homotopy of A on $C_{[0]} \oplus B_{[1]}$.*

Let $(D = A \times_M B \times_M C; A, B; M)$ and $(D' = A' \times_M B' \times_M C'; A', B'; M)$ be two VB-algebroids. A double vector bundle morphism $(\Psi; \psi_A, \psi_B; \mathrm{id}_M)$ from D to D' with core morphism ψ_C is a VB-algebroid morphism if and only if ψ_A is a Lie algebroid morphism and (ψ_C, ψ_B, Φ) are the components of a morphism of the corresponding 2-term representations up to homotopy. Here $\Phi \in \Omega^1(A, \mathrm{Hom}(B, C'))$ is defined by the observation that any double vector bundle morphism Ψ as above is given for $a \in A$, $b \in B$ and $c \in C$ over $m \in M$ by

$$(a, b, c) \mapsto (\psi_A(a), \psi_B(b), \psi_C(c) + \Phi(a, b)) ,$$

for some $\Phi \in \Gamma(A^* \otimes B^* \otimes C')$.

This correspondence is described in [19] as follows. Given a 2-term representation up to homotopy $(\partial, \nabla^C, \nabla^B, K)$ of A on $C_{[0]} \oplus B_{[1]}$, let $D = A \times_M B \times_M C$ and let $\sigma\colon \Gamma(A) \to \Gamma^\ell_B(D)$ be the canonical lift $\sigma = \sigma_A$ corresponding to the decomposition id_D according to Lemma 2.1.25. Then the Lie algebroid structure of $D \to B$ is given by the anchor

$$\begin{aligned} \rho_D(\sigma(a)) &= \widehat{\nabla^B_a} \in \mathfrak{X}^\ell(B) , \\ \rho_D(c^\dagger) &= \partial(c)^\uparrow \in \mathfrak{X}^c(B) , \end{aligned} \tag{2.38}$$

and the bracket

$$\begin{aligned} [\sigma(a_1), \sigma(a_2)] &= \sigma([a_1, a_2]) + \widetilde{K(a_1, a_2)} , \\ [\sigma(a), c^\dagger] &= (\nabla^C_a c)^\dagger , \\ [c_1^\dagger, c_2^\dagger] &= 0 , \end{aligned} \tag{2.39}$$

where $a, a_1, a_2 \in \Gamma(A)$ and $c, c_1, c_2 \in \Gamma(C)$.

Conversely, given a VB-algebroid structure on $D = A \times_M B \times_M C$, (2.38) defines a linear A-connection ∇^1 on B and a vector bundle morphism $\partial\colon C \to B$. Furthermore (2.39) defines a linear A-connection ∇^0 on C and a form $K \in \Omega^2(A, \mathrm{Hom}(B, C))$ and together, these form the structure operators of a 2-term representation up to homotopy of A on $C_{[0]} \oplus B_{[1]}$.

2.3 Courant algebroids and Dirac structures

Irene Dorfman ([18]) and Theodor Courant ([13]) introduced independently a bracket on the generalised tangent bundle $TM \oplus T^*M$. Courant showed how symplectic and Poisson structures can be interpreted as special cases of Dirac structures, which depend on this Courant-Dorfman bracket. In this section we will recapitulate the basic definitions and examples of Courant algebroids and Dirac structures.

2.3.1 Courant algebroids

Definition 2.3.1. *A **Courant algebroid** over a smooth manifold B is a vector bundle $\mathbb{E} \to B$ endowed with an anchor $\rho\colon \mathbb{E} \to TB$, an $\mathbb{R}$-bilinear bracket*

$$[\![\cdot,\cdot]\!]\colon \Gamma(\mathbb{E}) \times \Gamma(\mathbb{E}) \to \Gamma(\mathbb{E}) \tag{2.40}$$

and an $\mathbb{R}$-bilinear symmetric non-degenerate fibrewise pairing

$$\langle\cdot,\cdot\rangle\colon \mathbb{E} \times_B \mathbb{E} \to \mathbb{R}\,, \tag{2.41}$$

such that for all $e_1, e_2, e_3 \in \Gamma(\mathbb{E})$

1. $[\![e_1,[\![e_2,e_3]\!]]\!] = [\![[\![e_1,e_2]\!],e_3]\!] + [\![e_2,[\![e_1,e_3]\!]]\!]\,,$
2. $\rho(e_1)\langle e_2,e_3\rangle = \langle[\![e_1,e_2]\!],e_3\rangle + \langle e_2,[\![e_1,e_3]\!]\rangle\,,$
3. $[\![e_1,e_2]\!] + [\![e_2,e_1]\!] = \mathcal{D}\langle e_1,e_2\rangle\,,$ *where* $\mathcal{D} = \rho^t \circ \mathbf{d}\colon C^\infty(B) \to \Gamma(\mathbb{E})$.

The two properties in the following lemma are often stated as part of the definition of a Courant algebroid, but they already follow from the previous axioms and are therefore redundant.

Lemma 2.3.2. *Given a Courant algebroid $\mathbb{E} \to B$ with bracket $[\![\cdot,\cdot]\!]$ and anchor ρ we have for all $e_1, e_2 \in \Gamma(\mathbb{E})$ and $f \in C^\infty(B)$*

4. $\rho([\![e_1,e_2]\!]) = [\rho(e_1),\rho(e_2)]\,,$
5. $[\![e_1,fe_2]\!] = f[\![e_1,e_2]\!] + \rho(e_1)(f)e_2\,.$

Proof. To show the Leibniz identity 5 we use property 2 for sections e_1, fe_2 and e_3:

$$\begin{aligned}
0 &= -\rho(e_1)\langle fe_2,e_3\rangle + \langle[\![e_1,fe_2]\!],e_3\rangle + \langle fe_2,[\![e_1,e_3]\!]\rangle\\
&= f\big(-\rho(e_1)\langle e_2,e_3\rangle + \langle[\![e_1,e_2]\!],e_3\rangle + \langle e_2,[\![e_1,e_3]\!]\rangle\big)\\
&\quad -\rho(e_1)(f)\langle e_2,e_3\rangle + \langle[\![e_1,fe_2]\!],e_3\rangle - f\langle[\![e_1,e_2]\!],e_3\rangle\\
&= \langle[\![e_1,fe_2]\!] - f[\![e_1,e_2]\!] - \rho(e_1)(f)e_2, e_3\rangle\,.
\end{aligned}$$

Since this holds for all sections e_3, we conclude 5. To show 4 we use 1 with sections e_1, e_2 and fe_3 and make use of the Leibniz identity:

$$\begin{aligned}
0 &= [\![e_1, [\![e_2, fe_3]\!]]\!] - [\![[\![e_1, e_2]\!], fe_3]\!] - [\![e_2, [\![e_1, fe_3]\!]]\!] \\
&= f([\![e_1, [\![e_2, e_3]\!]]\!] - [\![[\![e_1, e_2]\!], e_3]\!] - [\![e_2, [\![e_1, e_3]\!]]\!]) \\
&\quad + \rho(e_1)(\rho(e_2)(f))e_3 + \rho(e_1(f))[\![e_2, e_3]\!] + \rho(e_2)(f)[\![e_1, e_3]\!] \\
&\quad - \rho(e_2)(\rho(e_1)(f))e_3 - \rho(e_2(f))[\![e_1, e_3]\!] - \rho(e_1)(f)[\![e_2, e_3]\!] \\
&\quad - \rho([\![e_1, e_2]\!])(f)e_3 \\
&= ([\rho(e_1), \rho(e_2)] - \rho([\![e_1, e_2]\!]))(f)e_3 \,.
\end{aligned}$$

Since this holds true again for all $f \in C^\infty(B)$ and all e_3, we conclude that 4 holds. □

The standard example is the bundle $\mathbb{T}M = TM \oplus T^*M$ over M, where the Courant algebroid structure is defined as follows:

Example 2.3.3. *The* ***standard Courant algebroid*** *over a smooth manifold M is the generalised tangent bundle $\mathbb{T}M = TM \oplus T^*M$ with the anchor $\rho = \mathrm{pr}_{TM}$, the Courant-Dorfman bracket, defined by*

$$[\![(X, \theta), (Y, \eta)]\!] := ([X, Y], \mathcal{L}_X\eta - \iota_Y d\theta)\,, \tag{2.42}$$

and the canonical symmetric pairing, defined by

$$\langle (X, \theta), (Y, \eta)\rangle := \theta(Y) + \eta(X)\,, \tag{2.43}$$

where $X, Y \in \Gamma(TM)$ are vector fields on M and $\theta, \eta \in \Omega^1(M)$ 1-forms on M.

The following was introduced in [45].

Example 2.3.4. *Let (A, A^*) be a Lie bialgebroid. The* ***Drinfeld double*** *of (A, A^*) is the vector bundle $C := A \oplus A^*$ equipped with a Courant algebroid structure as follows. The anchor for $a \in \Gamma(A)$ and $\alpha \in \Gamma(A^*)$ is given by $\rho(a, \alpha) := \rho_A(a) + \rho_{A^*}(\alpha)$. The pairing is defined to be $\langle (a, \alpha), (b, \beta)\rangle = \langle a, \beta\rangle + \langle b, \alpha\rangle$ and the bracket is given by*

$$[\![(a, \alpha), (b, \beta)]\!] = \Big([a, b]_A + \mathcal{L}^{A^*}_\alpha b - \iota_\beta \mathbf{d}_{A^*} a, [\alpha, \beta]_{A^*} + \mathcal{L}^A_a \beta - \iota_b \mathbf{d}_A \alpha\Big) \tag{2.44}$$

Note that Example 2.3.3 is a special case of this, where T^*M is equipped with the trivial Lie algebroid structure and TM with the canonical one.

The standard Courant algebroid over M is a special case of the previous example, where the Lie algebroid is $A = TM$ with the standard Lie algebroid structure and the Lie algebroid structure on $A^* = T^*M$ is the trivial one.

2.3.2 Dirac structures

Now we will give the definition of Dirac structures in Courant algebroids, for a more detailed review of Dirac geometry we refer to Henrique Bursztyn's brief introduction in [6].

Definition 2.3.5. *A* ***Dirac structure*** *in a Courant algebroid* $\mathbb{E}$ *is a subbundle* $D \subseteq \mathbb{E}$ *such that*

1. *D is maximally isotropic with respect to the pairing, i.e.* $D = D^{\perp}$.
2. *the space of sections is closed under the Courant bracket, i.e.*

$$[\![\Gamma(D), \Gamma(D)]\!] \subseteq \Gamma(D)\,.$$

In the case of the standard Courant algebroid $\mathbb{T}M$ over M we speak of a Dirac structure on M. Special cases of those are presymplectic structures, Poisson structures and regular foliations. This is another example of how different integrability conditions can be unified using generalised geometry.

Example 2.3.6. *Consider a presymplectic structure, that is a closed 2-form* $\omega \in \Omega^2(M)$. *This is equivalent to a vector bundle morphism* $\omega^\flat \colon TM \to T^*M$ *which is skew-symmetric in the sense that* $(\omega^\flat)^t = -\omega^\flat$. *The correspondence is given by* $\omega^\flat(X) := \iota_X\omega$. *The graph of* $\omega^\flat$ *is a Dirac structure* D_ω *in* $TM \oplus T^*M$.

Example 2.3.7. *A Poisson structure is a skew-symmetric vector bundle morphism* $\pi^\sharp \colon T^*M \to TM$, *such that the corresponding Poisson bracket satisfies the Jacobi identity. The graph of* $\pi^\sharp$ *is then again a Dirac structure* D_π *in* $TM \oplus T^*M$.

Example 2.3.8. *Given a regular distribution on* M, *that is a subbundle* $F \subseteq TM$, *denote the annihilator of* F *in* T^*M *by* F°. *Then* $D_F := F \oplus F^\circ$ *defines a Dirac structure if and only if* F *is involutive, that is the space of sections* $\Gamma(F)$ *is closed under the Lie bracket on* TM. *According to Frobenius' theorem this is equivalent to the existence of a foliation integrating* F.

2.3.3 VB-Courant algebroids

Definition 2.3.9. *A* ***metric double vector bundle*** *is a double vector bundle* $(D; A, B; M)$ *equipped with a symmetric, non-degenerate fibrewise pairing* $D \times_B D \to \mathbb{R}$, *such that the induced map* $D \to D^*_B$ *is an isomorphism of double vector bundles. In particular the core is isomorphic to* A^*.

Definition 2.3.10. *A **VB-Courant algebroid** $(\mathbb{E}; Q, B; M)$ is a metric double vector bundle*

$$\begin{array}{ccc} \mathbb{E} & \longrightarrow & B \\ \downarrow & Q^* \searrow & \downarrow \\ Q & \longrightarrow & M \end{array} \,, \qquad (2.45)$$

such that $\mathbb{E} \to B$ is a Courant algebroid, the anchor $\rho_{\mathbb{E}} \colon \mathbb{E} \to TB$ is linear, i.e. a morphism of double vector bundles over some morphism $\rho_Q \colon Q \to TM$ and the Courant bracket is linear, that is

1. $[\![\Gamma^\ell_B(\mathbb{E}), \Gamma^\ell_B(\mathbb{E})]\!] \subseteq \Gamma^\ell_B(\mathbb{E})\,,$
2. $[\![\Gamma^\ell_B(\mathbb{E}), \Gamma^c_B(\mathbb{E})]\!] \subseteq \Gamma^c_B(\mathbb{E})\,,$
3. $[\![\Gamma^c_B(\mathbb{E}), \Gamma^c_B(\mathbb{E})]\!] = 0\,.$

Example 2.3.11. *The standard Courant algebroid over a vector bundle E is a VB-Courant algebroid where $\mathbb{E} = TE \oplus T^*E$, $Q = TM \oplus E^*$ and $B = E$.*

Example 2.3.12. *The tangent double of a Courant algebroid $E \to M$ is a VB-Courant algebroid, where $\mathbb{E} = TE$, $Q = E$ and $B = TM$. The anchor of TE is given by $J \circ T\rho_E \colon TE \to T(TM)$, where J is the canonical flip, exchanging the two vector bundle structures Tp_M and p_{TM} of $TTM \to TM$. Compare also the tangent prolongation Lie algebroid in Example 2.2.7.*

Definition 2.3.13. *Given a VB-Courant algebroid $(\mathbb{E}; Q, B; M)$, a **VB-Dirac structure** in $\mathbb{E}$ is a sub-double vector bundle $(D; U, B; M)$ with $U \subseteq Q$ such that $D \to B$ is a Dirac structure in $\mathbb{E} \to B$.*

In the study of VB-Courant algebroids also the side bundle Q inherits a bracket on its space of sections depending on a choice of linear splitting. However, this bracket fails to satisfy the axioms of a Lie algebroid, in particular the bracket is not necessarily skew-symmetric and does not have to satisfy the Jacobi identity. For this reason, Madeleine Jotz Lean defined a weaker form of algebroid, called a dull algebroid in [36].

Definition 2.3.14. *A **dull algebroid** over M is a vector bundle $Q \to M$ endowed with an anchor $\rho \colon Q \to TM$ and an $\mathbb{R}$-bilinear bracket*

$$[\cdot,\cdot] \colon \Gamma(Q) \times \Gamma(Q) \to \Gamma(Q)\,, \qquad (2.46)$$

that is compatible with the anchor, that is $\rho([q_1, q_2]) = [\rho(q_1), \rho(q_2)]$ *for all* $q_1, q_2 \in \Gamma(Q)$ *and satisfies the Leibniz identity in both terms, that is for all* $q_1, q_2 \in \Gamma(Q)$ *and* $f, g \in C^\infty(M)$ *we have*

$$[fq_1, gq_2] = fg[q_1, q_2] + f\rho(q_1)(g)q_2 - g\rho(q_2)(f)q_1 \,. \tag{2.47}$$

Note that a dull algebroid is a Lie algebroid if and only if the bracket is additionally skew-symmetric and satisfies the Jacobi identity. In the case of a Lie algebroid the compatibility with the anchor follows from the Leibniz and the Jacobi identity as shown in Lemma 2.2.3. Since the dull bracket is not required to satisfy the Jacobi identity, we have to impose the compatibility of the anchor with the bracket here as an extra condition.

Dual to dull brackets are Dorfman connections, which play an important role in the study of Courant algebroids. In some sense they are the Courant algebroid equivalent of linear connections for Lie algebroid. For instance linear splittings of the standard Courant algebroid over a vector bundle E are in correspondence with $(TM \oplus E^*)$-Dorfman connections on $E \oplus T^*M$. This is the content of Theorem 2.3.18, which Madeleine Jotz Lean proved in [36]. We first recall the definition of a Dorfman connection of an anchored vector bundle on its dual, which is a slightly less general case than the definition of [36] where also preduals are considered. However, for our purposes the definition with the dual bundle is sufficient.

Definition 2.3.15. *Let* $Q \to M$ *be an anchored vector bundle with anchor* ρ *and* Q^* *its dual vector bundle. A* ***Q-Dorfman connection on Q^**** *is an* $\mathbb{R}$*-bilinear map*

$$\Delta \colon \Gamma(Q) \times \Gamma(Q^*) \to \Gamma(Q^*)\,, \tag{2.48}$$

such that for all $q \in \Gamma(Q)$*,* $\tau \in \Gamma(Q^*)$ *and* $f \in C^\infty(M)$ *we have*

1. $\Delta_{fq}\tau = f\Delta_q\tau + \langle q, \tau\rangle \rho^t \mathbf{d}f$,
2. $\Delta_q(f\tau) = f\Delta_q\tau + \rho(q)(f)\tau$,
3. $\Delta_q(\rho^t\mathbf{d}f) = \rho^t\mathbf{d}(\mathcal{L}_{\rho(q)}f)$.

The following lemma shows the duality between Q-Dorfman connections on Q^* and dull algebroid structures on Q and can be found in [36].

Lemma 2.3.16. *Given a vector bundle* Q *with an anchor* $\rho\colon Q \to TM$*, then a* Q*-Dorfman connection on* Q^* *is equivalent to a dull algebroid structure on* Q*, with the correspondence given by*

$$\langle \Delta_{q_1}\tau, q_2\rangle = \rho(q_1)\langle \tau, q_2\rangle - \langle \tau, [q_1, q_2]_\Delta\rangle\,, \tag{2.49}$$

for $q_1, q_2 \in \Gamma(Q)$ *and* $\tau \in \Gamma(Q^*)$*.*

Proof. Given such a Dorfman connection Δ, (2.49) defines a bracket $[\cdot,\cdot]_\Delta$ on $\Gamma(Q^*)$ since the pairing is non-degenerate. We can easily compute that for any $q_1, q_2 \in \Gamma(Q)$ and $f, g \in C^\infty(M)$ and $\tau \in \Gamma(Q^*)$ we have:

$$\begin{aligned}\langle [fq_1, gq_2]_\Delta, \tau\rangle &= \rho(fq_1)\langle gq_2, \tau\rangle - \langle gq_2, \Delta_{fq_1}\tau\rangle \\ &= fg\rho(q_1)\langle q_2, \tau\rangle + f\langle q_2, \tau\rangle\rho(q_1)(g) \\ &\quad - fg\langle q_2, \Delta_{q_1}\tau\rangle - g\langle q_1, \tau\rangle\langle q_2, \rho^t\mathbf{d}f\rangle \\ &= \langle fg[q_1, q_2]_\Delta, \tau\rangle + \langle f\rho(q_1)(g)q_2, \tau\rangle - \langle g\rho(q_2)(f)q_1, \tau\rangle\,.\end{aligned}$$

Since the pairing is non-degenerate this shows that the bracket satisfies the Leibniz identity in both terms. The third axiom of a Dorfman connection is dual to the compatibility of the anchor with the bracket, that is

$$\rho(q_1)\circ\rho(q_2) - \rho(q_2)\circ\rho(q_1) = \rho([q_1, q_2])\,.$$

Thus (2.49) defines a dull algebroid on Q.

Conversely, given a dull bracket $[\cdot,\cdot]_Q$ on $\Gamma(Q)$, (2.49) defines again by non-degeneracy of the pairing a map $\Delta\colon \Gamma(Q)\times\Gamma(Q^*)\to\Gamma(Q^*)$. The first two axioms for a Dorfman connection follow immediately from the Leibniz identity for the bracket by the computation above and the third axiom follows directly from the compatibility of the anchor with the dull bracket. □

The curvature of a Dorfman connection is defined in [36] analogously to the curvature of a linear connection.

Definition 2.3.17. *Given a Q-Dorfman connection Δ on Q^* as in Definition 2.3.15, the **curvature of** Δ is*

$$R_\Delta\colon \Gamma(Q)\times\Gamma(Q)\to\Gamma(\mathrm{Hom}(Q^*, Q^*))\,, \tag{2.50}$$

defined by $R_\Delta(q_1, q_2)(\tau) := \Delta_{q_1}\Delta_{q_2}\tau - \Delta_{q_2}\Delta_{q_1}\tau - \Delta_{[q_1,q_2]_\Delta}\tau$.

Dorfman connections are of interest to us since $(TM\oplus E^*)$-Dorfman connections on $E\oplus T^*M$ are in one-to-one correspondence with linear splittings of the generalised tangent bundle $TE\oplus T^*E$ of a vector bundle E. Jotz Lean proved the following theorem in [36].

Theorem 2.3.18. *Let $E\to M$ be a vector bundle over a smooth manifold M. A linear splitting Σ of $TE\oplus T^*E$ defines a Dorfman connection $\Delta\colon \Gamma(TM\oplus E^*)\times\Gamma(E\oplus T^*M)\to\Gamma(E\oplus T^*M)$ by*

$$\Sigma((X,\varepsilon)(m), e(m)) = (T_m eX(m), \mathbf{d}_{e_m}\ell_\varepsilon) - (\Delta_{(X,\varepsilon)}(e,0))^\uparrow(e(m)) \tag{2.51}$$

and $\Delta_{(X,\varepsilon)}(0,\theta) = (0, \mathcal{L}_X\theta)$ *for all sections* $X \in \Gamma(TM)$, $\varepsilon \in \Gamma(E^*)$, $\theta \in \Gamma(T^*M)$ *and* $e \in \Gamma(E)$. *Conversely, given such a Dorfman connection,* (2.51) *defines a linear splitting and the constructions are inverse to each other.*

After the choice of a suitable linear splitting of the underlying double vector bundle of a VB-Courant algebroid, the additional structure can equivalently be defined in terms of split Lie 2-algebroids. This has been done by Madeleine Jotz Lean in [39]. We give here a recollection of this correspondence. First we define the suitable splittings.

Definition 2.3.19. *A linear splitting* $\Sigma\colon A \times_M B \to D$ *of a metric double vector bundle* $(D; A, B; M)$ *is called* ***Lagrangian*** *if the image of* Σ *is isotropic in* D*.*

Jotz Lean furthermore proved that a change of Lagrangian splittings corresponds to a skew-symmetric section $\Phi \in \Omega^2(Q, B^*) \subseteq \Gamma(Q^* \otimes B^* \otimes Q^*)$.

Split Lie n-algebroids have been defined and studied by Yunhe Sheng and Chenchang Zhu in [68]. However, we give here the definition of split Lie 2-algebroids of [39]. The equivalence between their definitions for $n = 2$ is easy to see. The definition involves linear Q-connections and their curvature, which are both defined in Definition 2.2.14.

Definition 2.3.20. *Let* $Q \to M$ *and* $B \to M$ *be vector bundles. A split Lie 2-algebroid structure on* $Q \oplus B^*$ *is a tuple* $(\rho_Q, l, [\![\cdot,\cdot]\!], \nabla, \omega)$*, where* $\rho_Q\colon Q \to M$ *is an anchor map,* $[\![\cdot,\cdot]\!]$ *is a skew-symmetric dull bracket on* $\Gamma(Q)$ *anchored by* ρ_Q*,* $l\colon B^* \to Q$ *is a vector bundle morphism,* ∇ *is a linear* Q*-connection on* B *and* $\omega \in \Omega^3(Q, B^*)$*, such that for any* $\beta, \beta_1, \beta_2 \in \Gamma(B^*)$ *and* $q, q_1, q_2 \in \Gamma(Q)$ *we have*

1. $\nabla^*_{l(\beta_1)}\beta_2 + \nabla^*_{l(\beta_2)}\beta_1 = 0\,,$
2. $[\![q, l(\beta)]\!] = l(\nabla^*_q\beta)\,,$
3. $\mathbf{Jac}_{[\![\cdot,\cdot]\!]} = l \circ \omega \in \Omega^3(Q, Q)\,,$
4. $R_\nabla(q_1, q_2)b = l^t\langle \iota_{q_2}\iota_{q_1}\omega, b\rangle\,,$
5. $\mathbf{d}_{\nabla^*}\omega = 0\,.$

In [39] Jotz Lean proved the following correspondence between VB-Courant algebroid structures and split Lie 2-algebroids when given a Lagrangian splitting of the underlying metric double vector bundle.

Theorem 2.3.21. *Let* $(\mathbb{E}; Q, B; M)$ *be a VB-Courant algebroid and* Σ *a Lagrangian splitting with corresponding horizontal lift* $\sigma_Q\colon \Gamma(Q) \to \Gamma^\ell_B(\mathbb{E})$*. Then there is a* *split Lie 2-algebroid structure on* $Q \oplus B^*$ *such that for any* $q, q_1, q_2 \in \Gamma(Q)$ *and* $\tau \in \Gamma(Q^*)$ *we have*

$$\begin{aligned} \rho_{\mathbb{E}}(\sigma_Q(q)) &= \widehat{\nabla_q} \in \Gamma(TB)\,, \\ [\![\sigma_Q(q), \tau^\dagger]\!] &= (\Delta_q \tau)^\dagger\,, \\ [\![\sigma_Q(q_1), \sigma_Q(q_1)]\!] &= \sigma_Q([\![q_1, q_2]\!]) - \widetilde{R_\omega(q_1, q_2)}\,, \end{aligned} \tag{2.52}$$

where Δ *is the Dorfman connection which is dual to the dull bracket on* Q*, and* $R_\omega \in \Omega^2(Q, \mathrm{Hom}(B, Q^*))$ *is defined by* $R_\omega(q_1, q_2)(b) = \langle \iota_{q_2}\iota_{q_1}\omega, b\rangle$*.*

Conversely, given a Lagrangian splitting Σ *of a metric double vector bundle* $(\mathbb{E}; Q, B; M)$ *and a split Lie 2-algebroid structure on* $Q \oplus B^*$*, then* (2.52) *defines a VB-Courant algebroid structure on* $\mathbb{E}$*.*

2.4 Generalised complex structures

Generalised complex geometry was introduced by Nigel Hitchin in [34] as a unification of symplectic and complex geometry. It was further developed by his student Marco Gualtieri in his thesis [30, 31]. Since they simultaneously unify symplectic and complex structures, generalised complex structures have been studied for their relation to T-duality – a concept arising in string theory – by Gil Cavalcanti and Gualtieri in [11]. The relation between generalised complex geometry and Lie algebroids and Lie groupoids was first studied by Marius Crainic in [14]. Gualtieri also defined generalised Kähler structures in [30, 32]. These have been studied for example by Hitchin in [35] and by Henrique Bursztyn, Cavalcanti and Gualtieri in [8].

In this section we will give a brief overview of the basic definitions in generalised complex geometry. Here $\mathbb{E} \to M$ always denotes a Courant algebroid as defined in Definition 2.3.1.

Definition 2.4.1. *A **generalised almost complex structure** in* $\mathbb{E}$ *is a vector bundle morphism* $J\colon \mathbb{E} \to \mathbb{E}$ *over* id_M *such that* $J^2 = -1$ *and* J *is orthogonal with respect to the pairing, i.e.*

$$\langle J(e_1), J(e_2)\rangle = \langle e_1, e_2\rangle\,, \tag{2.53}$$

for all sections $e_1, e_2 \in \mathbb{E}$*.*

Definition 2.4.2. *A generalised almost complex structure* $J\colon \mathbb{E} \to \mathbb{E}$ *is called **generalised complex structure**, if and only if the Nijenhuis tensor of* J *vanishes,* *that is*

$$0 = N_J(e_1, e_2) := [\![e_1, e_2]\!] - [\![J(e_1), J(e_2)]\!] \\ + J([\![J(e_1), e_2]\!] + [\![e_1, J(e_2)]\!]) \,, \tag{2.54}$$

for all sections $e_1, e_2 \in \Gamma(\mathbb{E})$.

In the case of a generalised complex structure in the standard Courant algebroid $TM \oplus T^*M$ we speak of a generalised complex structure on the manifold M.

Example 2.4.3. *Given an almost complex structure* $J\colon TM \to TM$ *the map* $\mathcal{J}\colon \mathbb{T}M \to \mathbb{T}M$

$$\mathcal{J}_J = \begin{pmatrix} J & 0 \\ 0 & -J^t \end{pmatrix} \tag{2.55}$$

is a generalised almost complex structure. For the integrability condition a simple computation shows that

$$N_{\mathcal{J}_J}\big((X,\theta),(Y,\eta)\big) = \big(N_J(X,Y), -\langle N_J(X,\cdot),\eta\rangle + \langle N_J(Y,\cdot),\theta\rangle\big) \,,$$

for any $X, Y \in \Gamma(TM)$ *and* $\theta, \eta \in \Gamma(T^*M)$*. Thus* $\mathcal{J}_J$ *is a generalised complex structure if and only if* J *is a complex structure on* M*.*

Example 2.4.4. *Let* ω *be a non-degenerate 2-form on* M*. This induces a skew-symmetric isomorphism* $\omega^\flat\colon TM \to T^*M$*. Then the map* $\mathcal{J}_\omega\colon \mathbb{T}M \to \mathbb{T}M$*, defined by*

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -(\omega^\flat)^{-1} \\ \omega^\flat & 0 \end{pmatrix} \tag{2.56}$$

is a generalised almost complex structure. Using classic properties of the Lie derivative, interior and exterior derivative, in particular $\mathcal{L}_X \circ \iota_Y = \iota_Y \mathcal{L}_X + \iota_{[X,Y]}$ *and* $\mathcal{L}_X = \iota_X \circ \mathbf{d} + \mathbf{d} \circ \iota_X$ *we can compute the Nijenhuis tensor for* $\mathcal{J}_\omega$ *and obtain*

$$N_{\mathcal{J}_\omega}\big((X,\omega^\flat V),(Y,\omega^\flat W)\big) = \Big((\omega^\flat)^{-1}\big(\iota_X\iota_Y\mathbf{d}\omega - \iota_V\iota_W\mathbf{d}\omega\big), \iota_W\iota_X\mathbf{d}\omega + \iota_Y\iota_V\mathbf{d}\omega\Big) \,,$$

for any $X, Y, V, W \in \Gamma(TM)$*. Thus* $\mathcal{J}_\omega$ *is a generalised complex structure if and only if the 2-form* ω *is closed, or in other words defines a symplectic structure.*

The two previous examples show how the different integrability conditions for complex and symplectic structures are simultaneously encoded in terms of the Courant-Dorfman bracket in generalised geometry.

In his thesis [30], later published in [31], Marco Gualtieri described an equivalence between generalised complex structures and pairs of transversal, complex conjugated Dirac structures in $T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$. In fact, Nigel Hitchin originally defined generalised complex structures in [34] as such a pair of complex Dirac structures in $T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$.

The correspondence is the following. For a generalised complex structure $\mathcal{J}$ in a Courant algebroid $\mathbb{E}$, the $\pm i$-eigenbundles of the complexification $\mathcal{J}_{\mathbb{C}}$ are complex conjugated Dirac structures in the complexified Courant algebroid $\mathbb{E}_{\mathbb{C}}$. Similarly every pair of complex conjugated, transversal Dirac structures $D_{\pm}$ in $\mathbb{E}_{\mathbb{C}}$ gives rise to a generalised complex structure $\mathcal{J}$ on $\mathbb{E}$ such that the $\pm i$-eigenbundles of $\mathcal{J}_{\mathbb{C}}$ are given by $D_{\pm}$.

2.5 Complexifications

All the definitions in the previous sections can be extended complex linearly. We then still consider (real) smooth manifolds and thus consider smooth functions and bundles, not holomorphic ones. For the sake of completeness we write down here the straight forward definitions of complex Lie algebroids, Courant algebroids and complex Dirac structures as we will later need them when considering generalised complex structures on vector bundles and Lie algebroids. We write $\mathbf{d}_{\mathbb{C}}$ for the complex linear extension of $\mathbf{d}\colon C^\infty(M) \to \Gamma(T^*M)$.

2.5.1 Algebroids and connections

Definition 2.5.1. *A* ***complex Lie algebroid*** *over a smooth manifold M is a complex vector bundle $A \to M$ endowed with a complex linear anchor map*

$$\rho\colon A \to T_{\mathbb{C}}M\,, \tag{2.57}$$

a $\mathbb{C}$-bilinear, skew-symmetric bracket on the space of sections of A,

$$[\cdot,\cdot]\colon \Gamma(A) \times \Gamma(A) \to \Gamma(A)\,, \tag{2.58}$$

that satisfies the Jacobi identity and the Leibniz identity

$$[a, fb] = f[a,b] + \rho(a)(f)b\,, \tag{2.59}$$

for all $a, b \in \Gamma(A)$ and $f \in C^\infty(M,\mathbb{C})$.

Definition 2.5.2. *A* ***complex Courant algebroid*** *over a smooth manifold M is a complex vector bundle $C \to M$ endowed with a complex linear anchor map*

$$\rho_C\colon C \to T_{\mathbb{C}}M\,, \tag{2.60}$$

a $\mathbb{C}$-bilinear bracket

$$[\![\cdot,\cdot]\!]\colon \Gamma(C)\times\Gamma(C)\to\Gamma(C)\,, \tag{2.61}$$

and a $\mathbb{C}$-bilinear symmetric non-degenerate fibrewise pairing

$$\langle\cdot,\cdot\rangle\colon C\times_M C\to\mathbb{C}\,, \tag{2.62}$$

such that for all $c_1, c_2, c_3 \in \Gamma(C)$

1. $[\![c_1,[\![c_2,c_3]\!]]\!] = [\![[\![c_1,c_2]\!],c_3]\!] + [\![c_2,[\![c_1,c_3]\!]]\!]$
2. $\rho(c_1)\langle c_2,c_3\rangle = \langle [\![c_1,c_2]\!],c_3\rangle + \langle c_1,[\![c_2,c_3]\!]\rangle$
3. $[\![c_1,c_2]\!] + [\![c_2,c_1]\!] = \mathcal{D}\langle c_1,c_2\rangle$ *, where* $\mathcal{D} = \rho_C^t\circ \mathbf{d}_{\mathbb{C}}\colon C^\infty(M,\mathbb{C})\to\Gamma(C)$.

Example 2.5.3. *Given a (real) Lie algebroid $A\to M$, the complexification $A_{\mathbb{C}}$ is a complex Lie algebroid with the anchor and bracket extended $\mathbb{C}$-linearly. Similarly, for a (real) Courant algebroid $E\to M$, the complexification $E_{\mathbb{C}}$ is a complex Courant algebroid with the anchor, the bracket and the pairing extended $\mathbb{C}$-linearly.*

Definition 2.5.4. *Let C be a complex Courant algebroid over M. A* ***complex Dirac structure*** *in C is a complex subbundle $D\subseteq C$, such that*

1. *D is maximally isotropic with respect to the pairing, i.e. $D = D^\perp$.*
2. *The space of sections is closed under the Courant bracket, i.e.*

$$[\![\Gamma(D),\Gamma(D)]\!]\subseteq\Gamma(D)\,.$$

Analogously to Definition 2.3.14 we define a complex dull algebroid.

Definition 2.5.5. *A* ***complex dull algebroid*** *Q over M is a complex vector bundle $Q\to M$ endowed with a complex linear anchor*

$$\rho\colon Q\to T_{\mathbb{C}}M\,, \tag{2.63}$$

and a $\mathbb{C}$-bilinear bracket

$$[\cdot,\cdot]\colon\Gamma(Q)\times\Gamma(Q)\to\Gamma(Q)\,, \tag{2.64}$$

that satisfies the Leibniz identity in both terms, that is

$$[fq_1,gq_2] = fg[q_1,q_2] + f\rho(q_1)(g)q_2 - g\rho(q_2)(f)q_1\,, \tag{2.65}$$

for all $q_1,q_2\in\Gamma(Q)$ and $f,g\in C^\infty(M)$.

Analogously to Definition 2.3.15 we define a complex Q-Dorfman connection on Q^* for a complex dull algebroid Q as follows.

Definition 2.5.6. *Let Q be a complex dull algebroid over M and Q^* its dual. A* ***complex Q-Dorfman connection on*** Q^* *is a $\mathbb{C}$-bilinear map*

$$\Delta\colon \Gamma(Q)\times\Gamma(Q^*)\to\Gamma(Q^*)\,, \tag{2.66}$$

such that for all $q\in\Gamma(Q)$, $\tau\in\Gamma(Q^*)$ and $f\in C^\infty(M,\mathbb{C})$

1. $\Delta_{fq}\tau = f\Delta_q\tau + \langle q,\tau\rangle\rho^t\mathbf{d}_{\mathbb{C}}f$,
2. $\Delta_q(f\tau) = f\Delta_q\tau + \rho(q)(f)\tau$,
3. $\Delta_q(\rho^t\mathbf{d}_{\mathbb{C}}f) = \rho^t\mathbf{d}_{\mathbb{C}}(\mathcal{L}^{\mathbb{C}}_{\rho(q)}f)$.

The ***complex curvature*** *of such a complex Dorfman connection Δ is defined by the same formula as for a real Dorfman connection, that is $R_\Delta(q_1,q_2)(\tau) := \Delta_{q_1}\Delta_{q_2}\tau - \Delta_{q_2}\Delta_{q_1}\tau - \Delta_{[q_1,q_2]}\tau$.*

Example 2.5.7. *If Q is a (real) dull algebroid over M and Δ is a Q-Dorfman connection on Q^*, the complexification $Q_{\mathbb{C}}$ is a complex dull algebroid with the structure extended by $\mathbb{C}$-linearity and the complexification $\Delta^{\mathbb{C}}$ defined by extending Δ in both arguments $\mathbb{C}$-linearly is a complex Dorfman connection.*

2.5.2 Double vector bundles

Now we will consider double vector bundles and their complexifications as vector bundles over one side. Let from now on D be a double vector bundle as follows

$$\begin{array}{ccc} D & \xrightarrow{q^D_B} & B \\ {\scriptstyle q^D_A}\downarrow & & \downarrow{\scriptstyle q_B} \\ A & \xrightarrow{q_A} & M \end{array}\,.$$

Proposition 2.5.8. *The complexification of D as a vector bundle over A is again a double vector bundle with sides A, $B_{\mathbb{C}}$ and core $C_{\mathbb{C}}$. We will denote this double vector bundle by $D^A_{\mathbb{C}}$.*

Proof. The projection $q^{D^A_{\mathbb{C}}}_{B_{\mathbb{C}}}\colon D^A_{\mathbb{C}}\to B_{\mathbb{C}}$ is given by the complexification of the vector bundle morphism q^D_B over q_A. The addition and scalar multiplication are then also given by complexification. It is easy to see that the core can be identified with $C_{\mathbb{C}}$. □

Lemma 2.5.9. *For a vector bundle $E \to M$ there is a canonical isomorphism of double vector bundles between $TE_{\mathbb{C}}^{TM}$ and $T(E_{\mathbb{C}})$.*

Proof. The bundle $E_{\mathbb{C}}$ can be written as $E \oplus iE$ and thus the tangent bundle of $E_{\mathbb{C}}$ can be identified with $TE \oplus T(iE)$ where we consider the direct sum as the sum of vector bundles over TM. This is identified with the complexification $TE \oplus iTE$ of TE as bundle over TM. □

Proposition 2.5.10. *Let now additionally $D \to B$ and $A \to M$ be Lie algebroids, such that D is a VB-algebroid, then the complexification $D_{\mathbb{C}}^{A}$ is also a VB-algebroid, with the structure of the Lie algebroid $D_{\mathbb{C}}^{A} \to B_{\mathbb{C}}$ defined componentwise in real and imaginary part of $D_{\mathbb{C}}^{A}$.*

Proof. Every section of $D_{\mathbb{C}}^{A} \to B_{\mathbb{C}}$ can be written uniquely as $d_1 +_A id_2$ for sections $d_1, d_2 \in \Gamma_B(D)$. After the identification of $T(B_{\mathbb{C}})$ with $(TB)_{\mathbb{C}}^{TM}$ according to Lemma 2.5.9, we can define the anchor and bracket componentwise. It is easy to check that this defines a Lie algebroid again. □

Definition 2.5.11. *A **complex VB-Courant algebroid** is a (real) metric double vector bundle $(\mathbb{E}; Q, B; M)$, such that $\mathbb{E} \to Q$ is additionally a complex Courant algebroid with linear anchor and bracket. That is $\rho_{\mathbb{E}} \colon \mathbb{E} \to T_{\mathbb{C}}B$ is a morphism of complex vector bundles and the bracket is linear in the same sense as for a real VB-Courant algebroid.*

Definition 2.5.12. *Given a complex VB-Courant algebroid $(\mathbb{E}; Q, B; M)$, a **complex VB-Dirac structure** in $\mathbb{E}$ is a sub-double vector bundle $(D; U, B; M)$, such that $D \to M$ is a complex Dirac structure in $\mathbb{E} \to B$ and $U \to M$ is a complex subbundle of $Q \to M$.*

2.6 Graded manifolds

In this section we briefly recall the definition of $\mathbb{N}$-graded manifolds and Poisson structures, symplectic structures and Lie algebroid structures on these graded manifolds. Lie n-algebroids have been studied for example by Giuseppe Bonavolontà and Norbert Poncin in [5]. In the case of $n = 2$ Dimitry Roytenberg established a one-to-one correspondence between symplectic Lie 2-algebroids and Courant algebroids in his thesis [66] and later in [67]. David Li-Bland further studied this correspondence in [44] and showed furthermore an equivalence between Lie 2-algebroids and VB-Courant algebroids. Later, Madeleine Jotz Lean gave a different description of this correspondence, first proving an equivalence of categories between graded manifolds of degree 2 and metric double vector bundles in [37] and then showing how the additional Lie structure translates to the structure of a VB-Courant algebroid in [39]. For an introduction to graded manifolds see also [10].

Definition 2.6.1. *An* $\mathbb{N}$***-graded manifold*** $\mathcal{M}$ ***of degree*** n *(short* $[n]$*-manifold) is a manifold* M *equipped with a sheaf* $C^\infty(\mathcal{M})$ *of* $\mathbb{N}$*-graded, graded-commutative, associative, unital* $C^\infty(M)$*-algebras, which is locally freely generated by finitely many elements of degrees* $1, \ldots, n$*. Here locally freely generated means that for any* $m \in M$ *there is an open neighbourhood* $U \subseteq M$*, such that* $C^\infty(\mathcal{M})(U)$ *is a freely and finitely generated* $C^\infty(U)$*-module.*

The following example is a trivial way of defining an n-manifold from a graded vector bundle. It can be found in [5] and is also explained in [37].

Example 2.6.2. *Given a graded vector bundle* $E_{-1} \oplus E_{-2} \oplus \ldots \oplus E_{-n}$*, this induces a* ***split*** $[n]$***-manifold****, where the local generators in degree* k *of the sheaf* $C^\infty(\mathcal{M})$ *are given by local basis sections of* E^*_{-k}*. This graded manifold is then written as* $E_{-1}[-1] \oplus \ldots \oplus E_{-n}[-n]$*.*

An important theorem in super-geometry is Batchelor's theorem, stating that every supermanifold is non-canonically isomorphic to a special type of supermanifold and was proved by Marjorie Batchelor in [4, 2, 3]. We give here the corresponding theorem for $\mathbb{N}$-graded manifolds, which can be found for example in [5].

Theorem 2.6.3. *Every* $[n]$*-manifold is non-canonically isomorphic to a split* $[n]$*-manifold.*

We will also need vector fields, Poisson structures and symplectic structures on graded manifolds, which are defined in [37] as follows.

Definition 2.6.4. *A* ***vector field*** Φ ***of degree*** d *on an* $[n]$*-manifold is a derivation of* $C^\infty(\mathcal{M})$ *of degree* d*, that is*

$$|\Phi(\xi)| = d + |\xi| \tag{2.67}$$

for a homogeneous element $\xi \in C^\infty(\mathcal{M})$ *of degree* $|\xi|$*.*

Definition 2.6.5. *A* ***Poisson structure*** *on an* $\mathbb{N}$*-graded manifold* $\mathcal{M}$ *is a bracket of degree* -2 *on the sheaf of functions* $C^\infty(\mathcal{M})$*:*

$$\{\cdot,\cdot\}\colon C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})\,, \tag{2.68}$$

that is graded skew-symmetric, that is

$$\{\xi_1, \xi_2\} = -(-1)^{|\xi_1||\xi_2|}\{\xi_2, \xi_1\}\,, \tag{2.69}$$

satisfies the graded Leibniz identity, that is

$$\{\xi_1, \xi_2 \cdot \xi_3\} = \{\xi_1, \xi_2\} \cdot \xi_3 + (-1)^{|\xi_1||\xi_2|}\xi_2 \cdot \{\xi_1, \xi_3\} \tag{2.70}$$

and the graded Jacobi identity, that is

$$\{\xi_1, \{\xi_2, \xi_3\}\} = \{\{\xi_1, \xi_2\}, \xi_3\} + (-1)^{|\xi_1||\xi_2|}\{\xi_2, \{\xi_1, \xi_3\}\}\,, \tag{2.71}$$

where $\xi_1, \xi_2, \xi_3 \in C^\infty_{\mathcal{M}}(U)$ are always homogenous elements of degree $|\xi_1|$, $|\xi_2|$ and $|\xi_3|$, respectively.

Definition 2.6.6. *A Poisson structure on $\mathcal{M}$ is called* ***symplectic*** *if the image of*

$$\begin{aligned} \sharp\colon C^\infty(\mathcal{M}) &\to \mathrm{Der}(C^\infty(\mathcal{M})) \\ f &\mapsto \{f, \cdot\} \end{aligned} \tag{2.72}$$

generates $\mathrm{Der}(C^\infty(\mathcal{M}))$ as $C^\infty(\mathcal{M})$-module.

Definition 2.6.7. *Given a graded manifold $\mathcal{M}$, a vector field $\mathcal{Q}$ on $\mathcal{M}$ is called* ***cohomological*** *if $\mathcal{Q}^2 = 0$. A* ***Lie algebroid structure*** *on $\mathcal{M}$ is a cohomological vector field $\mathcal{Q}$ of degree 1. An $\mathbb{N}$-manifold of degree n with a Lie algebroid structure is called a* ***Lie-n-algebroid****.*

Clearly, [1]-manifolds are equivalent to ordinary vector bundles, where the sheaf of functions corresponds to $\Gamma(\bigwedge E^*)$. Lie algebroids then correspond to Lie 1-algebroids, with the cohomological vector field given by the Lie algebroid differential described in Definition 2.2.12.

As mentioned above, Roytenberg gave a correspondence between symplectic Lie 2-algebroids and Courant algebroids in [67]. The equivalence of categories between metric double vector bundles and [2]-manifolds is described in [37]. Here we will briefly sketch the correspondence of split Lie 2-algebroids in the sense of Definition 2.3.20 and the split version of the definition above, that is as a split graded manifold equipped with a cohomological vector field. This can be found in [39].

Proposition 2.6.8. *Given a split [2]-manifold $Q[-1] \oplus B^*[-2]$, then the structure of a split Lie 2-algebroid over $Q \oplus B^*$ in the sense of Definition 2.3.20 is equivalent to a cohomological vector field $\mathcal{Q}$ on the split [2]-manifold $Q[-1] \oplus B^*[-2]$.*

Proof. Given the split Lie 2-algebroid structure $(\rho_Q, l, [\![\cdot,\cdot]\!], \nabla, \omega)$, the cohomological vector field is given on functions $f \in C^\infty(M)$ by $\mathcal{Q}(f) = \rho_Q^* \mathbf{d} f \in \Gamma(Q^*)$, on generating sections of degree 1, that is $\tau \in \Gamma(Q^*)$ by $\mathcal{Q}(\tau) = l^t\tau + \mathbf{d}_Q\tau \in \Gamma(B) \oplus \Omega^2(Q)$ and on the generating sections of degree 2, that is $b \in \Gamma(B)$ by $\mathcal{Q}(b) = \mathbf{d}_\nabla b - \langle \omega, b\rangle \in \Omega^1(Q, B) \oplus \Omega^3(Q)$. For the proof and computations that the condition that $\mathcal{Q}^2 = 0$ is equivalent to the properties of a split Lie 2-algebroid we refer to [39]. □

Chapter 3

Multiple vector bundles

Double vector bundles have been intensively studied in differential geometry in the last 50 years. For a brief review on the history, definition and properties of double vector bundles see Section 2.1. The generalisation to triple vector bundles, and later n-fold vector bundles, was introduced by Kirill Mackenzie in [50] and studied jointly with Alfonso Gracia-Saz in [26] and [27]. There, n-fold vector bundles are defined as diagrams of vector bundle structures with an extra splitting condition, which we will prove here to be redundant. Furthermore, in [27] the authors study the duality of higher order vector bundles. In [24] Janusz Grabowski and Mikołaj Rotkievicz define n-fold vector bundles as a manifold equipped with n commuting Euler vector fields which generate the scalar multiplications of the different vector bundle structures. Grabowski and Rotkiewicz also state the existence of local charts for their n-fold vector bundles and thus the equivalence of the two definitions and give a proof in the case of $n = 2$. Relying on this existence of local charts is also Elizaveta Vishnyakova's recent work on n-fold vector bundles and graded manifolds with associated weight systems in [69].

In this chapter, most of which can be found on the arXiv as joint work with my supervisor Madeleine Jotz Lean in [33], we study multiple vector bundles. In Section 3.1 we define multiple vector bundles as special functors from an indexing category to the category of smooth manifolds. This allows not only to define n-fold vector bundles but also ∞-fold vector bundles which do not have a total space. We give several prototypes of multiple vector bundles, notably the trivial or decomposed ones. Then we define the n-pullback corresponding to an n-fold vector bundle in Definition 3.1.12 and show that it admits an n-fold vector bundle structure itself in Theorem 3.1.13. It plays the role of $A \times_M B$ for a double vector bundle $(D; A, B; M)$. Making use of this n-pullback, we prove the higher order generalisation of Mackenzie's core sequences in Proposition 3.1.21. Finally, we study in detail the cores of a multiple vector bundle.

In Section 3.2, we define linear splittings and decompositions of multiple vector bundles and show in Theorem 3.2.2 how we can construct a decomposition of an n-fold vector bundle from a given linear splitting and linear splittings of its highest order cores. In Theorem 3.2.3 we prove inductively the existence of linear splittings for any n-fold vector bundle. Together with 3.2.2 this proves also the existence of decompositions. In Definition 3.2.9 we then define n-fold vector bundle atlases and show how it follows directly from the existence of decompositions that both definitions are equivalent. We then show how the existence of decompositions of ∞-fold vector bundles can be deduced from the existence of decompositions of n-fold vector bundles via a colimit argument.

Then, in Section 3.3, we define multiply linear sections of an n-fold vector bundle, as a generalisation of linear sections for $n = 2$ and doubly linear sections for $n = 3$. We construct again a short exact sequence of $C^\infty(M)$-modules, analogously to the case of $n = 2$.

In Section 3.4 we then define the category of symmetric n-fold vector bundles as n-fold vector bundles equipped with a certain action of the symmetric group S_n. This is a generalisation of the concept of involutive double vector bundles to higher orders and will play an important role in the geometrisation of graded manifolds of degree n analogously to the degree 2 case which was carried out by Madeleine Jotz Lean in [37]. This geometrisation of graded manifolds is work in progress, joint with Madeleine Jotz Lean.

Finally, in Section 3.5, we demonstrate our results of the previous sections in the special case of $n = 3$ in detail for convenience of the reader. Since the general notation is fairly technical, we hope that this special treatment of triple vector bundles will make it easier to understand the results and proofs in the general case. In particular we here present splittings, decompositions and doubly linear sections of triple vector bundles. We also repeat the short exact sequence of $C^\infty(M)$-modules obtained in Section 3.3 here in more accessible notation. In this case of $n = 3$ we will also demonstrate how a decomposition of a triple vector bundle is equivalent to a splitting of this sequence, called a horizontal lift, and splittings of the side and core double vector bundles.

3.1 Definition and properties

In this section we introduce multiple vector bundles and discuss some of their properties. The novelty of our definition is that instead of considering an n-fold vector bundle as a smooth manifold with n-commuting vector bundle structures, we see a multiple vector bundle as a special functor from a cube category to smooth manifolds. In particular, the "total space" of an n-fold vector bundle does not play that central a role anymore, and we can even define ∞-fold vector bundles, with no total space at all.

In the following, we write $\mathbb{N}$ for the set of positive integers: $\mathbb{N} = \{1, 2, \ldots\}$. For $n \in \mathbb{N}$, we write $\underline{n}$ for the set $\{1, \ldots, n\}$.

3.1.1 Multiple vector bundles

We define the **∞-cube category** $\square^{\mathbb{N}}$ to be the opposite poset category of the finite subsets of $\mathbb{N}$. In other words, the objects are *finite* subsets $I \subseteq \mathbb{N}$ and there is exactly one arrow from I to J if $J \subseteq I$. Every morphism can of course be written as a composition of generating arrows of the form

$$I \to I \setminus \{i\} \quad \text{for} \quad I \subseteq \mathbb{N} \text{ finite and } i \in I\,.$$

Every subset $I \subseteq \mathbb{N}$ of cardinality k is the source of k generating arrows.

In a similar manner, we define the **n-cube category** $\square^n$ with subsets I of $\underline{n}$ as objects and with arrows $I \to J \Leftrightarrow J \subseteq I$. This category can be thought of as an indexing of an n-dimensional cube, where every vertex corresponds to a finite subset of $\underline{n}$ and the edges correspond to generating arrows as defined above.

Definition 3.1.1. *A* ***∞-fold vector bundle****, and respectively an* ***n-fold vector bundle****, is a covariant functor $\mathbb{E}\colon \square^{\mathbb{N}} \to \mathbf{Man}^\infty$ – respectively a covariant functor $\mathbb{E}\colon \square^n \to \mathbf{Man}^\infty$ – to the category of smooth manifolds, such that, writing E_I for $\mathbb{E}(I)$ and $p^I_J := \mathbb{E}(I \to J)$,*

(a) for all $I \subseteq \mathbb{N}$ (respectively $I \subseteq \underline{n}$) and all $i \in I$, $p^I_{I\setminus\{i\}}\colon E_I \to E_{I\setminus\{i\}}$ has a smooth vector bundle structure, and

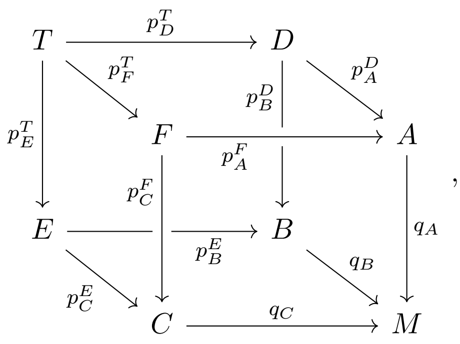
(b) for all $I \subseteq \mathbb{N}$ (respectively $I \subseteq \underline{n}$) and $i \neq j \in I$,

$$\begin{array}{ccc} E_I & \xrightarrow{p^I_{I\setminus\{i\}}} & E_{I\setminus\{i\}} \\ \downarrow{\scriptstyle p^I_{I\setminus\{j\}}} & & \downarrow{\scriptstyle p^{I\setminus\{i\}}_{I\setminus\{i,j\}}} \\ E_{I\setminus\{j\}} & \xrightarrow{p^{I\setminus\{j\}}_{I\setminus\{i,j\}}} & E_{I\setminus\{i,j\}} \end{array}$$

is a double vector bundle.

Note that item (a) is only necessary in order to imply that this definition for $n = 1$ just gives an ordinary vector bundle. For $n \geq 2$ item (b) already implies the former and in the case of $n = 2$ it just gives a double vector bundle as defined in Definition 2.1.1.

Example 3.1.2. *A triple vector bundle, or 3-fold vector bundle is a functor* $\mathbb{E}\colon \square^3 \to \mathbf{Man}^\infty$ *and can be viewed as a diagram*



, (3.1)

where we wrote $T := \mathbb{E}(\{1,2,3\})$*,* $D := \mathbb{E}(\{1,2\})$*,* $E := \mathbb{E}(\{2,3\})$*,* $F := \mathbb{E}(\{1,3\})$*,* $A := E_{\{1\}}$*,* $B := E_{\{2\}}$ *and* $C := E_{\{3\}}$*.*

We will generally say *multiple vector bundle* for an n-fold or ∞-fold vector bundle, when the dimension of the underlying cube diagram does not need to be specified. Our definition of n-fold vector bundles is different but equivalent notation to the definition in [27].

For better readability we will often write for the vector bundle projections $p^I_i := p^I_{I\setminus\{i\}}$ and in the case of an n-fold vector bundle also $p_i := p^n_{\underline{n}\setminus\{i\}}$. The smooth manifold $E_\emptyset =: M$ will be called the *absolute base* of $\mathbb{E}$. If $\mathbb{E}$ is an n-fold vector bundle, the smooth manifold $\mathbb{E}(\underline{n}) =: E$ is called its *total space.* In the case of an ∞-fold vector bundle there is no total space. Given a finite subset $I \subseteq \mathbb{N}$ and $i \in I$, we write $+_{I\setminus\{i\}}$ for the addition and $\cdot_{I\setminus\{i\}}$ for the scalar multiplication of the vector bundle $E_I \to E_{I\setminus\{i\}}$. This notation is omissive since it only specifies the base space of the vector bundle in the fibers of which the addition or scalar multiplication is taken. However, it is always clear from the summands or factors which fibre space is considered.

Remark 3.1.3. *There is a canonical functor* $\pi^n_k\colon \square^n \to \square^k$ *for* $k \leq n$ *defined by* $\pi^n_k(I) = I \cap \underline{k}$ *and* $\pi^n_k(I \to J) = (I \cap \underline{k}) \to (J \cap \underline{k})$*. The canonical functor* $\pi^{\mathbb{N}}_n\colon \square^{\mathbb{N}} \to \square^n$ *is defined in the same manner by* $\pi^{\mathbb{N}}_n(I) = I \cap \underline{n}$*. Furthermore there are inclusion functors of full subcategories* $\iota^n_k\colon \square^k \to \square^n$ *and* $\iota^{\mathbb{N}}_n\colon \square^n \to \square^{\mathbb{N}}$ *sending a subset* $I \subseteq \underline{k}$ *to the same set viewed as a subset of* $\underline{n}$ *or* $\mathbb{N}$*, respectively.*

The functor ι^n_k *is right-inverse (and therefore right-adjoint) to the functor* π^n_k*, that is* $\pi^n_k \circ \iota^n_k = \mathrm{id}_{\square^k}$*. Similarly, the functor* $\iota^{\mathbb{N}}_k$ *is right-inverse and right-adjoint to* $\pi^{\mathbb{N}}_k$*.*

Given now a k-fold vector bundle $\mathbb{E} : \square^k \to \mathbf{Man}^\infty$*, the composition* $\mathbb{E} \circ \pi_k^n$ *is an n-fold vector bundle whereas the composition* $\mathbb{E} \circ \pi_k^{\mathbb{N}}$ *is an* ∞*-fold vector bundle.*

In this light, an n-fold vector bundle $\mathbb{E} \colon \square^n \to \mathbf{Man}^\infty$ *can be viewed as a special case of an* ∞*-fold vector bundle* $\mathbb{E} \colon \square^{\mathbb{N}} \to \mathbf{Man}^\infty$ *where additionally* $\mathbb{E} = \mathbb{E} \circ \iota_n^{\mathbb{N}} \circ \pi_n^{\mathbb{N}}$*:*

$$\begin{array}{ccc} \square^{\mathbb{N}} & \xrightarrow{\mathbb{E}} & \mathbf{Man}^\infty \\ {\scriptstyle \pi_n^{\mathbb{N}}}\downarrow & & \uparrow{\scriptstyle \mathbb{E}} \\ \square^n & \xrightarrow[\iota_n^{\mathbb{N}}]{} & \square^{\mathbb{N}} \end{array} \quad . \tag{3.2}$$

In other words $\mathbb{E}(I) = \mathbb{E}(I \cap \underline{n})$ *for all* $I \subseteq \mathbb{N}$ *and* $\mathbb{E}$ *is completely determined by its values on all the subsets of* $\underline{n}$ *already.*

Remark 3.1.4. *Sometimes we will want to use a different indexing category* $\Diamond^n$ *isomorphic to the n-cube category* $\square^n$*, especially when defining the faces and cores of a multiple vector bundle. However, in order to define the morphisms between two n-fold vector bundles we will need to fix the indexing category to be* $\square^n$*. With the greater generality we would not obtain a category of n-fold vector bundles.*

But we can use an indexing category $\Diamond^n$ *as above together with an explicit choice of isomorphism* $\mathbf{i} \colon \square^n \to \Diamond^n$ *and with slight abuse of notation call a functor* $\mathbb{E} \colon \Diamond^n \to \mathbf{Man}^\infty$ *an n-fold vector bundle instead of using more precisely* $\mathbb{E} \circ \mathbf{i}^{-1}$*.*

With this remark in mind the following lemma will allow us to use a different indexing category for the faces of a multiple vector bundle.

Lemma 3.1.5. *For each pair of subsets* $J \subseteq I \subseteq \mathbb{N}$ *with* J *finite, the finite sets* $K \subset \mathbb{N}$ *such that* $J \subseteq K \subseteq I$ *form a full subcategory* $\Diamond^{I,J}$ *of* $\square^{\mathbb{N}}$*, which is itself isomorphic to the* $(\#I - \#J)$*-cube category* $\square^{\#I-\#J}$*. We will from now on fix the isomorphism* $\mathbf{i}^{I,J} \colon \square^{\#I-\#J} \to \Diamond^{I,J}$ *which is induced by the canonical ordering on* $\mathbb{N}$*.*

Proof. The objects in $\Diamond^{I,J}$ are all subsets of I that contain J. This is clearly a full subcategory of $\square^{\mathbb{N}}$. The ordering on $\mathbb{N}$ induces an ordering of the $(\#I - \#J)$ elements of $I \setminus J$. This then defines an isomorphism of categories $\mathbf{i}^{I,J} \colon \square^{\#I-\#J} \to \Diamond^{I,J}$. □

We will now prove the following straightforward proposition which establishes the faces of a multiple vector bundle.

Proposition 3.1.6. *Let* $\mathbb{E} \colon \square^{\mathbb{N}} \to \mathbf{Man}^\infty$ *be a multiple vector bundle.*

(a) *The restriction of* $\mathbb{E}$ *to* $\Diamond^{I,J}$ *to the full subcategory* $\Diamond^{I,J}$ *of* $\square^n$ *is a* $(\#I - \#J)$*-fold vector bundle with total space* E_I *(if* I *is finite) and absolute base* E_J*, denoted by* $\mathbb{E}^{I,J}$*. We call this the* (I,J)*-face of* $\mathbb{E}$*.*

(b) *In particular, if* $I = \emptyset$ *we obtain a* $(\#I)$*-fold vector bundle* $\mathbb{E}^{I,\emptyset}$ *with total space* E_I *and absolute base* M*. We call* $\mathbb{E}^{I,\emptyset}$ *the* I*-face of* $\mathbb{E}$*.*

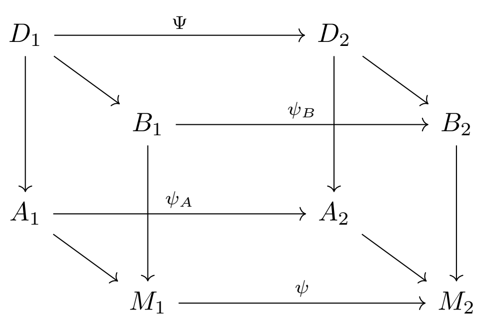
Proof. Since $\mathbb{E}^{I,J}$ is defined via the restriction of $\mathbb{E}$ to $\Diamond^{I,J}$, all arrows $\mathbb{E}^{I,J}(K \to K \setminus \{k\}) = p^K_{K\setminus\{k\}}$ for $J \subseteq K \subseteq I$ and $k \in K \setminus J$ are smooth vector bundles and all possible squares

$$\begin{array}{ccc} \mathbb{E}^{I,J}(K) & \xrightarrow{p^K_{K\setminus\{k_1\}}} & \mathbb{E}^{I,J}(K \setminus \{k_1\}) \\ \downarrow{\scriptstyle p^K_{K\setminus\{k_2\}}} & & \downarrow{\scriptstyle p^{K\setminus\{k_1\}}_{K\setminus\{k_1,k_2\}}} \\ \mathbb{E}^{I,J}(K \setminus \{k_2\}) & \xrightarrow{p^{K\setminus\{k_2\}}_{K\setminus\{k_1,k_2\}}} & \mathbb{E}^{I,J}(K \setminus \{k_1,k_2\}) \end{array}$$

are double vector bundles. Therefore $\mathbb{E}^{I,J}$ is an $(\#I - \#J)$-fold vector bundle. In the case of finite I the object I is an initial object in $\Diamond^{I,J}$ and E_I is the total space of $\mathbb{E}^{I,J}$. The object J is terminal in $\Diamond^{I,J}$ and therefore E_J is the absolute base of $\mathbb{E}^{I,J}$. □

Given an ∞-fold vector bundle $\mathbb{E}\colon \square^{\mathbb{N}} \to \mathbf{Man}^\infty$ and an open subset $U \subseteq M$, we define the *restriction of* $\mathbb{E}$ *to* U to be the ∞-fold vector bundle $\mathbb{E}|_U\colon \square^{\mathbb{N}} \to \mathbf{Man}^\infty$, $\mathbb{E}|_U(I) = \left(p^I_\emptyset\right)^{-1}(U)$ and $\mathbb{E}|_U(I \to J) = \mathbb{E}(I \to J)|_{(p^I_\emptyset)^{-1}(U)}\colon \left(p^I_\emptyset\right)^{-1}(U) \to \left(p^J_\emptyset\right)^{-1}(U)$. The absolute base of $\mathbb{E}|_U$ is U. In the same manner, if $\mathbb{E}\colon \square^n \to \mathbf{Man}^\infty$ is an n-fold vector bundle, and U an open subset of M, then its restriction $\mathbb{E}|_U$ to U is an n-fold vector bundle with total space $(p^{\underline{n}}_\emptyset)^{-1}(U)$ and with absolute base U.

Recall that a morphism of double vector bundles from $(D_1; A_1, B_1; M_1)$ to $(D_2; A_2, B_2; M_2)$ is a commutative cube



, (3.3)

all faces of which are vector bundle morphisms. This morphism will be written as tuple $(\Psi; \psi_A, \psi_B; \psi)$. Similarly we define morphisms of multiple vector bundles.

Definition 3.1.7. *Let $\mathbb{E}\colon \square^{\mathbb{N}} \to \mathbf{Man}^\infty$ and $\mathbb{F}\colon \square^{\mathbb{N}} \to \mathbf{Man}^\infty$ be two multiple vector bundles. A* ***morphism of multiple vector bundles*** *from $\mathbb{E}$ to $\mathbb{F}$ is a natural transformation $\tau\colon \mathbb{E} \to \mathbb{F}$ such that for all objects I of $\square^{\mathbb{N}}$ and for all $i \in I$, the commutative diagram*

$$\begin{array}{ccc} E_I & \xrightarrow{\tau(I)} & F_I \\ \downarrow{\scriptstyle p^I_{I\setminus\{i\}}} & & \downarrow{\scriptstyle p^I_{I\setminus\{i\}}} \\ E_{I\setminus\{i\}} & \xrightarrow{\tau(I\setminus\{i\})} & F_{I\setminus\{i\}} \end{array} \tag{3.4}$$

is a homomorphism of vector bundles.

Given two n-fold vector bundles $\mathbb{E}\colon \square^n \to \mathbf{Man}^\infty$ and $\mathbb{F}\colon \square^n \to \mathbf{Man}^\infty$, a ***morphism of n-fold vector bundles*** *from $\mathbb{E}$ to $\mathbb{F}$ is a natural transformation $\tau\colon \mathbb{E} \to \mathbb{F}$ such that the diagram above is a vector bundle homomorphism for all $I \subseteq \underline{n}$ and $i \in I$. The morphism τ is surjective (resp. injective) if each of its components $\tau(I)$, $I \subseteq \underline{n}$ is surjective (resp. injective).*

Together with the definition of multiple vector bundles in Definition 3.1.1 this defines for $n \in \mathbb{N}$ the category **nVB** of n-fold vector bundles and also the category ∞**VB** of ∞-fold vector bundles.

Remark 3.1.8. *Note that if we would not demand a fixed isomorphism between the indexing category and the corresponding cube category, we could not define morphisms. It is important in which position the respective faces are. For example morphisms out of a double vector bundle $(D; A, B; M)$ are different from morphisms out of the corresponding flip $(D^{\mathit{flip}}; B, A; M)$.*

However, if we have given two n-fold vector bundles $\mathbb{E}$ and $\mathbb{F}$ defined with the same indexing category $\lozenge^n$ together with a chosen isomorphism $\mathbf{i}\colon \square^n \to \lozenge^n$ we will also consider a natural transformation $\tau\mathbb{E} \to \mathbb{F}$ to be a morphism of n-fold vector bundles if $\tau \circ \mathrm{id}_{\mathbf{i}}$ is a morphism of n-fold vector bundles from $\mathbb{E} \circ \mathbf{i}$ to $\mathbb{F} \circ \mathbf{i}$. With the same indexing category this makes the notation a lot less cumbersome than precomposing everything with $\mathbf{i}$.

3.1.2 Prototypes

In this section, we describe a few standard examples of multiple vector bundles, that will be relevant in the formulation of our main theorem.

Decomposed multiple and n-fold vector bundles

Consider a smooth manifold M and a collection of vector bundles $\mathcal{A} = (q_J\colon A_J \to M)_{J\subseteq\mathbb{N},\, \#J<\infty}$, with $A_\emptyset = M$. We define a functor $\mathbb{E}^{\mathcal{A}}\colon \square^{\mathbb{N}} \to \mathbf{Man}^\infty$ as follows.

Each finite subset $I \subseteq \mathbb{N}$ is sent to $E_I := \prod_{J\subseteq I}^M A_J$, the fibered product of vector bundles over M.

For $I \subseteq \mathbb{N}$ with $1 \leq \#I < \infty$ and for $k \in I$, the arrow $I \to I \setminus \{k\}$ is sent to the canonical vector bundle projection

$$p_k^I \colon \prod_{J\subseteq I}^{M} A_J \to \prod_{J\subseteq I\setminus\{k\}}^{M} A_J.$$

In particular, the arrow $\{i\} \to \emptyset$ for $i \in \mathbb{N}$ is sent to the vector bundle projection $p_\emptyset^{\{i\}} = q_{\{i\}} \colon E_{\{i\}} = A_{\{i\}} \to E_\emptyset = M$. A multiple vector bundle $\mathbb{E}^{\mathcal{A}} \colon \square^{\mathbb{N}} \to \mathbf{Man}^\infty$ constructed in this manner is called a **decomposed multiple vector bundle**. A decomposed n-fold vector bundle $\mathbb{E}^{\mathcal{A}} \colon \square^n \to \mathbf{Man}^\infty$ is defined accordingly. In that case we will write $E^{\mathcal{A}} := \mathbb{E}^{\mathcal{A}}(\underline{n})$ for the total space. Decomposed n-fold vector bundles are also defined in [27].

Example 3.1.9. *A* trivial or decomposed triple vector bundle *is given by*

$$E_{\{1,2,3\}} = A_{\{1\}} \times_M A_{\{2\}} \times_M A_{\{3\}} \times_M A_{\{1,2\}} \times_M A_{\{1,3\}} \times_M A_{\{2,3\}} \times_M A_{\{1,2,3\}},$$

with decomposed sides

$$E_{\{1,2\}} = A_{\{1\}} \times_M A_{\{2\}} \times_M A_{\{1,2\}}\,, \quad E_{\{1,3\}} = A_{\{1\}} \times_M A_{\{3\}} \times_M A_{\{1,3\}}\,,$$
$$E_{\{2,3\}} = A_{\{2\}} \times_M A_{\{3\}} \times_M A_{\{2,3\}}\,,$$

where A_I, $I \subseteq \underline{n}$ are all vector bundles over M, the projections are the appropriate projections to the factors and the additions are defined in an obvious manner in the fibers.

Vacant multiple and n-fold vector bundles

As a special case of this, if $\overline{\mathcal{A}} = (q_i \colon A_i \to M)_{i\in\mathbb{N}}$ is a collection of vector bundles over M, we construct the multiple vector bundle $\mathbb{E}^{\overline{\mathcal{A}}} \colon \square^{\mathbb{N}} \to \mathbf{Man}^\infty$ as follows:

$$I \mapsto \prod_{i\in I}^{M} A_i, \qquad (I \to I \setminus \{k\}) \mapsto \left(p_k^I \colon \prod_{i\in I}^{M} A_i \to \prod_{i\in I\setminus\{k\}}^{M} A_i \right).$$

Such a multiple vector bundle is called a **vacant decomposed** multiple vector bundle. We will see later that all *cores* of these multiple vector bundles are trivial.

Given a collection of vector bundles $\mathcal{A} = (q_J \colon A_J \to M)_{J\subseteq\mathbb{N},\,\#J<\infty}$, with $A_\emptyset = M$, we can define $\overline{\mathcal{A}} = (q_i \colon A_i \to M)_{i\in\mathbb{N}}$ by $A_i = A_{\{i\}}$. We get then a monomorphism of multiple vector bundles

$$\iota \colon \mathbb{E}^{\overline{\mathcal{A}}} \to \mathbb{E}^{\mathcal{A}} \tag{3.5}$$

defined by $\iota(I)\colon \prod_{i\in I}^M A_{\{i\}} \to \prod_{J\subseteq I}^M A_J$, $\iota(I)((v_i)_{i\in I}) = (w_J)_{J\subseteq I}$, $w_{\{i\}} = v_i$ for $i \in I$, $w_\emptyset = v_\emptyset := m \in M$ and $w_J = 0_m^{A_J}$ for $\#J \geq 2$. In particular, $\iota(\{i\}) = \mathrm{id}_{A_{\{i\}}}$ for all $i \in \mathbb{N}$.

In the case of an n-fold vector bundle we write $\overline{E} := \overline{\mathbb{E}}(\underline{n})$ for the total space.

"Diagonal" decomposed and vacant k-fold vector bundles

More generally, consider a collection $\mathcal{A} = (q_I\colon A_I \to M)_{I\subseteq \underline{n}}$ of vector bundles, with $A_\emptyset = M$, and a partition $\rho = \{I_1, \ldots, I_k\}$ of $\underline{n}$ with $I_j \neq \emptyset$, for $j = 1, \ldots, k$. Then we can define a category $\lozenge^\rho$ with objects the subsets $\nu \subseteq \rho$ and with morphisms $\nu_1 \to \nu_2 \Leftrightarrow \nu_2 \subseteq \nu_1$. This category is isomorphic to the k-cube category $\square^k$ by the isomorphism $\mathbf{i}^\rho\colon \square^k \to \lozenge^\rho$ defined by sending the object $\{j\}$ of $\square^k$ to the object $\{I_k\}$ of $\lozenge^\rho$. Fixing this isomorphism, we can use the indexing category $\lozenge^\rho$ to define a vacant k-fold vector bundle $\overline{\mathbb{E}_\rho^{\mathcal{A}}}\colon \lozenge^\rho \to \mathbf{Man}^\infty$ as follows

$$\nu \mapsto \prod_{K\in\nu}^M A_K, \qquad (\nu \to \nu \setminus \{I\}) \mapsto \left(p^\nu_{\nu\setminus\{I\}}\colon \prod_{K\in\nu}^M A_K \to \prod_{K\in\nu\setminus\{I\}}^M A_K \right).$$

We will from now on write $[\nu] := \cup_{K\in\nu} K$ for $\nu \subseteq \rho$. In a similar manner, we define a decomposed k-fold vector bundle $\mathbb{E}_\rho^{\mathcal{A}}\colon \lozenge^\rho \to \mathbf{Man}^\infty$ as follows

$$\nu \mapsto \prod_{\nu'\subseteq\nu}^M A_{[\nu']}, \qquad (\nu \to \nu \setminus \{I\}) \mapsto \left(\prod_{\nu'\subseteq\nu}^M A_{[\nu']} \to \prod_{\nu'\subseteq\nu\setminus\{I\}}^M A_{[\nu']} \right),$$

where the map on the right-hand side is the canonical projection. We get as before an obvious monomorphism of k-fold vector bundles $\iota^\rho\colon \overline{\mathbb{E}_\rho^{\mathcal{A}}} \to \mathbb{E}_\rho^{\mathcal{A}}$. For each $\nu \subseteq \rho$ we have furthermore the obvious canonical injections

$$\eta^\rho(\nu)\colon \mathbb{E}_\rho^{\mathcal{A}}(\nu) = \prod_{\nu'\subseteq\nu}^M A_{[\nu']} \hookrightarrow \mathbb{E}^{\mathcal{A}}([\nu]) = \prod_{J\subseteq[\nu]}^M A_J\,.$$

The tangent prolongation of a multiple vector bundle

Given an n-fold vector bundle $\mathbb{E}\colon \square^n \to \mathbf{Man}^\infty$ we define an $(n+1)$-fold vector bundle $T\mathbb{E}\colon \square^{n+1} \to \mathbf{Man}^\infty$, the **tangent prolongation of** $\mathbb{E}$, as follows. Given $I \subseteq \underline{n}$, we set $T\mathbb{E}(I) := E_I$ and $T\mathbb{E}(I\cup\{n+1\}) := TE_I$. Furthermore, for $i \in I \subseteq \underline{n}$ we set

$$T\mathbb{E}(I \to I \setminus \{i\}) := p_i^I\colon E_I \to E_{I\setminus\{i\}}\,,$$
$$T\mathbb{E}(I \cup \{n+1\} \to (I \cup \{n+1\}) \setminus \{i\}) := T(p_i^I)\colon TE_I \to TE_{I\setminus\{i\}}\,,$$
$$T\mathbb{E}(I \cup \{n+1\} \to I) := p_{E_I}\colon TE_I \to E_I\,,$$

where the last map is the canonical projection.

Similarly, given an ∞-fold vector bundle $\mathbb{E}\colon \square^n \to \mathbf{Man}^\infty$ we define an ∞-fold vector bundle $T\mathbb{E}\colon \square^n \to \mathbf{Man}^\infty$, the **tangent prolongation of** $\mathbb{E}$, as follows. We define the category $\lozenge^T$ to be the poset of finite subsets of $\mathbb{N}\cup\{0\}$, together with the isomorphism $\mathbf{i}^T : \square^{\mathbb{N}} \to \lozenge^T$ given by shifting by 1, that is $\mathbf{i}^T(I) := \{i-1 \mid i \in I\}$. Now we define the functor $T\mathbb{E} \circ (\mathbf{i}^T)^{-1} \colon \lozenge^T \to \mathbf{Man}^\infty$ as follows. Given any $I \subseteq \mathbb{N}$, we set $T\mathbb{E} \circ (\mathbf{i}^T)-1(I) := E_I$ and $T\mathbb{E} \circ (\mathbf{i}^T)^{-1}(I \cup \{0\}) := TE_I$. Furthermore, for $i \in I \subseteq \underline{n}$ we set

$$\begin{aligned}
&T\mathbb{E} \circ (\mathbf{i}^T)^{-1}(I \to I \setminus \{i\}) := p_i^I \colon E_I \to E_{I\setminus\{i\}}\,, \\
&T\mathbb{E} \circ (\mathbf{i}^T)^{-1}(I \cup \{0\} \to (I \cup \{0\}) \setminus \{i\}) := T(p_i^I)\colon TE_I \to TE_{I\setminus\{i\}}\,, \\
&T\mathbb{E} \circ (\mathbf{i}^T)^{-1}(I \cup \{0\} \to I) := p_{E_I} \colon TE_I \to E_I\,,
\end{aligned}$$

where the last map is the canonical projection.

Of course one can define the tangent prolongation of an n-fold vector bundle with a similar shift in order to be consistent with the infinite case. However, the given definition without such a shift seems more natural in the finite case, but does unfortunately not extend to the infinite case.

Multiple vector spaces

A multiple vector bundle where the absolute base $\mathbb{E}(\emptyset) = \{*\}$ is a point is called a multiple vector space. Let $\mathbb{E}$ be an n-fold vector space in this sense. Corollary 3.2.4 which we prove later then states in this case that $\mathbb{E}$ is isomorphic to a direct product of vector spaces in the following way. $\mathbb{E}(J) = \prod_{I\subseteq J} V_I$ for any $J \subseteq \underline{n}$, where V_I is a finite dimensional real vector space.

Lemma 3.1.10. *Let us consider two decomposed multiple vector spaces $\mathbb{V}(J) = \prod_{I\subseteq J} V_I$ and $\mathbb{W}(J) = \prod_{I\subseteq J} W_I$, with finite dimensional real vector spaces V_I, W_I for $I \subseteq \underline{n}$ and a morphism between them, $\tau\colon \mathbb{V} \to \mathbb{W}$. Then*

$$\tau(J)\big((v_I)_{I\subseteq J}\big) = \Big(\sum_{\rho=(I_1,\ldots,I_k)\in\mathcal{P}(K)} \omega_\rho(v_{I_1},\ldots,v_{I_k})\Big)_{K\subseteq J}\,, \tag{3.6}$$

with $\omega_\rho \in \mathrm{Hom}(V_{I_1} \otimes \ldots \otimes V_{I_k}, W_K)$.

Proof. Given an element $(v_I)_{I\subseteq J} \in \mathbb{V}(J)$ with $v_{I_1} \neq 0$ and $v_{I_2} \neq 0$ for $i \in I_1 \cap I_2$ we can write $(v_I)_{I\subseteq\underline{n}} = (u_I)_{I\subseteq\underline{n}} +_{\underline{n}\setminus i} (w_I)_{I\subseteq\underline{n}}$ where $u_I = v_I$ for $I \neq I_2$, $w_{I_2} = 0$ and $u_I = v_I$ for $I \subseteq \underline{n} \setminus \{i\}$, $u_{I_2} = v_{I_2}$ and $u_I = 0$ otherwise. Doing this repeatedly, we can write every element $(v_I)_{I\subseteq J}$ as a consecutive sum of elements of the form where for all non-disjoint I_1 and I_2 either $v_{I_1} = 0$ or $v_{I_2} = 0$. Thus the map $\tau(J)$

is determined by its action on elements of this form. Define for $\rho = (I_1, \ldots, I_k) \in \mathcal{P}(K)$ then $\omega_\rho(v_{I_1}, \ldots, v_{I_k}) := \mathrm{pr}_K\, \tau(K)((w_I)_{I \subseteq K})$ where $w_{I_j} = v_{I_j}$ for $j = 1, \ldots, k$ and $w_I = 0$ otherwise. Now $\tau(K)$ is a vector bundle morphism over all $\tau(K \setminus \{k\})$ for $k \in K$ and thus ω_ρ has to be linear in every argument and therefore an element of $\mathrm{Hom}(V_{I_1} \otimes \ldots \otimes V_{I_k}, W_K)$.

□

Multiple homomorphism vector bundles

Given two n-fold vector bundles $\mathbb{E}$ and $\mathbb{F}$ with the same absolute base $\mathbb{E}(\emptyset) = \mathbb{F}(\emptyset) = M$ we construct an n-fold vector bundle $\mathrm{Hom}_n(\mathbb{E}, \mathbb{F})$, which is the n-fold analogon of the bundle $\mathrm{Hom}(E, F)$ for ordinary vector bundles E and F over M.

For $m \in M$ the restrictions $\mathbb{E}|_m$ and $\mathbb{F}|_m$ define n-fold vector bundles over a single point as absolute base. With this we can define a space $\mathrm{Hom}_n(\mathbb{E}, \mathbb{F})$ as

$$\mathrm{Hom}_n(\mathbb{E}, \mathbb{F}) := \Big\{ \Phi_m \colon \mathbb{E}|_m \to \mathbb{F}|_m \;\Big|\; m \in M, \Phi_m \text{ is a morphism of } n\text{-fold vector bundles} \Big\}. \tag{3.7}$$

This space is equipped with the obvious projection to M mapping an element Φ_m to m. Since n-fold vector bundle morphisms have by definition underlying $(n-1)$-fold vector bundle morphisms between the faces there are additionally projections $\mathrm{Hom}_n(\mathbb{E}, \mathbb{F}) \to \mathrm{Hom}_{n-1}(\mathbb{E}^{\underline{n} \setminus \{k\}, \emptyset}, \mathbb{F}^{\underline{n} \setminus \{k\}, \emptyset})$ for all $k \in \underline{n}$. Each of these projections carries a vector bundle structure, with the addition of two elements Φ_m and Ψ_m which project to the same base $\phi\colon \mathbb{E}^{\underline{n} \setminus \{k\}}|_m \to \mathbb{F}^{\underline{n} \setminus \{k\}}|_m$ in $\mathrm{Hom}_{n-1}(\mathbb{E}^{\underline{n} \setminus \{k\}, \emptyset}, \mathbb{F}^{\underline{n} \setminus \{k\}, \emptyset})$ defined as $(\Phi_m +_{\underline{n} \setminus \{k\}} \Psi_m)(e) := \Phi_m(e) +_{\underline{n} \setminus \{k\}} \Psi_m(e)$. That this is indeed a vector bundle follows from the fact that Now we define a functor $\mathrm{Hom}(\mathbb{E}, \mathbb{F}) \colon \square^n \to \mathbf{Man}^\infty$ by setting $\mathrm{Hom}(\mathbb{E}, \mathbb{F})(I) := \mathrm{Hom}_{\#I}(\mathbb{E}^{I,\emptyset}, \mathbb{F}^{I,\emptyset})$. That this is indeed an n-fold vector bundle $\mathrm{Hom}(\mathbb{E}, \mathbb{F})$ with total space $\mathrm{Hom}_n(\mathbb{E}, \mathbb{F})$ and absolute base M is an immediate application of the existence of n-fold vector bundle charts which we prove later in Corollary 3.2.10 since that allows to reduce the problem to the fibres. The desired interchange laws in $\mathrm{Hom}(\mathbb{E}, \mathbb{F})$ follow from the interchange laws in $\mathbb{E}$ and $\mathbb{F}$ as the addition is defined fibrewise with the addition in $\mathbb{F}$.

Every morphism of n-fold vector bundles $\mathbb{E} \to \mathbb{F}$ over the identity on M corresponds then to a smooth map $M \to \mathrm{Hom}_n(\mathbb{E}, \mathbb{F})$ which is a section of the projection to M.

Let us consider the case of $n = 2$ as an example of this construction. For that purpose consider two double vector bundles $(D_1; A_1, B_1; M)$ and $(D_2; A_2, B_2; M)$

over the same base manifold M. Then this gives rise to a homomorphism double vector bundle

$$\begin{array}{ccc} \mathrm{Hom}_{DVB}(D_1, D_2) & \longrightarrow & \mathrm{Hom}_{VB}(B_1, B_2) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{VB}(A_1, A_2) & \longrightarrow & M \end{array},$$

and a double vector bundle homomorphism from D_1 to D_2 fixing the base manifold M corresponds to a map $M \to \mathrm{Hom}_{DVB}(D_1, D_2)$ which is right-inverse to the projection to M.

In particular, let $F \to M$ be an ordinary vector bundle and consider the n-fold vector bundle $\mathbb{F}$ defined by $\mathbb{F}(\underline{n}) = F$ and $\mathbb{F}(I) = M$ for all $I \subsetneq \underline{n}$. Then we write $\mathrm{Mor}_n(\mathbb{E}, F)$ for the space of n-fold vector bundle morphisms from $\mathbb{E}$ to $\mathbb{F}$ over id_M.

Lemma 3.1.11. *Let $\mathbb{E}$ be an n-fold vector bundle over M and F be a vector bundle over M. Then the space $\mathrm{Mor}_n(\mathbb{E}, F)$ is a $C^\infty(M)$-module.*

Proof. An element τ of $\mathrm{Mor}_n(\mathbb{E}, F)$ necessarily satisfies $\tau(I)\colon E(I) \to M$, $\tau(I)(e) = p^I_\emptyset(e)$ for all $e \in \mathbb{E}(I)$, $I \subsetneq \underline{n}$. Take $f_1, f_2 \in C^\infty(M)$ and $\tau_1, \tau_2 \in \mathrm{Mor}_n(\mathbb{E}, F)$. Then $(f_1 \cdot \tau_1 + f_2 \cdot \tau_2)\colon \mathbb{E} \to F$ is defined by $(f_1 \cdot \tau_1 + f_2 \cdot \tau_2)(I)(e) = p^I_\emptyset(e)$ for all $e \in \mathbb{E}(I)$, $I \subsetneq \underline{n}$ and $(f_1 \cdot \tau_1 + f_2 \cdot \tau_2)(\underline{n})(e) = f_1(p^I_\emptyset(e)) \cdot \tau_1(e) + f_2(p^I_\emptyset(e)) \cdot \tau_2(e)$ for $e \in \mathbb{E}(\underline{n})$.

By construction, $(f_1 \cdot \tau_1 + f_2 \cdot \tau_2)(\underline{n})$ is smooth and

$$\begin{array}{ccc} \mathbb{E}(\underline{n}) & \xrightarrow{(f_1\tau_1 + f_2\tau_2)(\underline{n})} & F \\ \Big\downarrow p^{\underline{n}}_{\underline{n}\setminus\{i\}} & & \Big\downarrow q_F \\ \mathbb{E}(\underline{n} \setminus \{i\}) & \xrightarrow{\tau(\underline{n}\setminus\{i\})} & M \end{array}$$

is a morphism of vector bundles for all $i \in \underline{n}$. For $I \subsetneq \underline{n}$ and $i \in I$, the map $(f_1 \cdot \tau_1 + f_2 \cdot \tau_2)(I)\colon \mathbb{E}(I) \to M$ is obviously a vector bundle morphism over $\tau(I \setminus \{i\})\colon \mathbb{E}(I \setminus \{i\}) \to M$. □

3.1.3 The n-pullback of an n-fold vector bundle

Definition 3.1.12. *Let $\mathbb{E}$ be an n-fold vector bundle. We define the **n-pullback of** $\mathbb{E}$ to be the set*

$$P = \left\{ (e_1, \ldots, e_n) \,\middle|\, e_i \in E_{\underline{n}\setminus\{i\}} \text{ and } p^{\underline{n}\setminus\{i\}}_j(e_i) = p^{\underline{n}\setminus\{j\}}_i(e_j) \text{ for } i, j \in \underline{n} \right\}. \quad (3.8)$$

In the case of $n = 2$, the 2-pullback of the double vector bundle $(D; A, B; M)$ is the space $A \times_M B$, which is a smooth manifold and a double vector bundle itself. Furthermore, as we have seen in Lemma 2.1.6 the double projection $D \to A \times_M B$

is a surjective submersion. We will prove in the following theorem that both of these statements are true in greater generality for any n-fold vector bundle. This is a crucial part of the proof of the existence of linear decompositions of multiple vector bundles.

Theorem 3.1.13. *Let $\mathbb{E}\colon \square^n \to \mathbf{Man}^\infty$ be an n-fold vector bundle. Then*

(a) The n-pullback of $\mathbb{E}$, denoted by P, is a smooth embedded submanifold of the product $E_{\underline{n}\setminus\{1\}} \times \ldots \times E_{\underline{n}\setminus\{n\}}$.

(b) The functor $\mathbb{P}$ defined by $\mathbb{P}(\underline{n}) = P$, $\mathbb{P}(S) = E_S$ for all $S \subsetneq \underline{n}$ and the vector bundle projections $p_i^S\colon E_S \to E_{S\setminus\{i\}}$ for all $S \subsetneq \underline{n}$ and $i \in S$ and $p_i'\colon P \to E_{\underline{n}\setminus\{i\}}$, $(e_1, \ldots, e_n) \mapsto e_i$ is an n-fold vector bundle.

(c) The map $\pi(\underline{n})\colon E \to P$ given by $\pi(\underline{n})\colon e \mapsto (p_1(e), \ldots, p_n(e))$, defines together with $\pi(J) = \mathrm{id}_{E_J}$ for $J \subsetneq \underline{n}$, a surjective n-fold vector bundle morphism $\pi\colon \mathbb{E} \to \mathbb{P}$.

Note that for each $i \in \underline{n}$, the top map $\pi(\underline{n})\colon E \to P$ of π is then necessarily a vector bundle morphism over the identity on $E_{\underline{n}\setminus\{i\}}$.

Proof. For the proof of this theorem, we will make use of several easy lemmas which are proved in the background section. These are Lemma 2.1.7, Lemma 2.1.5, Lemma 2.1.4 and Lemma 2.1.3.

We prove this by induction over n. The case of $n = 1$ is trivially satisfied since in that case $\mathbb{E}$ is an ordinary vector bundle $E = E_{\{1\}} \to E_\emptyset = M$ and so $P = M$. Let us now take $n \in \mathbb{N}$ with $n \geq 2$ and assume that all three claims are true for any $(n-1)$-fold vector bundle $\mathbb{E}$.

Recall from Proposition 3.1.6 that $\mathbb{E}^{\underline{n},\{k\}}$ is an $(n-1)$-fold vector bundle. The corresponding $(n-1)$-pullback is

$$P_k^{\mathbf{up}} := \Big\{(e_1, \ldots, \widehat{k}, \ldots, e_n) \;\Big|\; e_i \in E_{\underline{n}\setminus\{i\}} \colon p_j^{\underline{n}\setminus\{i\}}(e_i) = p_i^{\underline{n}\setminus\{j\}}(e_j) \text{ for } i, j \in \underline{n} \setminus \{k\}\Big\}. \tag{3.9}$$

By the induction hypothesis (b), this is the total space of an $(n-1)$-fold vector bundle $\mathbb{P}_k^{\mathbf{up}}$ with underlying nodes E_J for $k \in J \subsetneq \underline{n}$. The absolute base of this $(n-1)$-fold vector bundle is $E_{\{k\}}$, and by (c) we have a smooth morphism $\pi_k^{\mathbf{up}}\colon \mathbb{E}^{\underline{n},\{k\}} \to \mathbb{P}_k^{\mathbf{up}}$ of $(n-1)$-fold vector bundles that is surjective. In a similar manner, $\mathbb{E}^{\underline{n}\setminus\{k\},\emptyset}$ is an $(n-1)$-fold vector bundle. The corresponding $(n-1)$-pullback is

$$P_k^{\mathbf{low}} := \Big\{(b_1,\dots,\widehat{k},\dots,b_n) \;\Big|\; b_i \in E_{\underline{n}\setminus\{k,i\}} \colon p_j^{\underline{n}\setminus\{k,i\}}(b_i) = p_i^{\underline{n}\setminus\{k,j\}}(b_j) \text{ for } i,j \in \underline{n}\setminus\{k\}\Big\}. \tag{3.10}$$

Again by the induction hypothesis (b) this is the total space of an $(n-1)$-fold vector bundle $\mathbb{P}_k^{\mathbf{low}}$ with underlying nodes E_J for $J \subsetneq \underline{n}\setminus\{k\}$. By (c) we have a smooth surjective morphism $\pi_k^{\mathbf{low}} \colon \mathbb{E}^{\underline{n}\setminus\{k\},\emptyset} \to \mathbb{P}_k^{\mathbf{low}}$ of $(n-1)$-fold vector bundles.

By the induction hypothesis (a), $P_k^{\mathbf{up}}$ and $P_k^{\mathbf{low}}$ are embedded submanifolds of $\prod_{\substack{i=1\\i\neq k}}^n E_{\underline{n}\setminus\{i\}}$ and $\prod_{\substack{i=1\\i\neq k}}^n E_{\underline{n}\setminus\{i,k\}}$, respectively. Since for each $i \neq k$ in $\underline{n}$, we have the smooth vector bundle $p_k^{\underline{n}\setminus\{i\}} \colon E_{\underline{n}\setminus\{i\}} \to E_{\underline{n}\setminus\{i,k\}}$, the product $\prod_{\substack{i=1\\i\neq k}}^n E_{\underline{n}\setminus\{i\}}$ has a smooth vector bundle structure over $\prod_{\substack{i=1\\i\neq k}}^n E_{\underline{n}\setminus\{i,k\}}$, the projection of which we denote by q_k. Using the surjectivity of $\pi_k^{\mathbf{low}}(\underline{n}\setminus\{k\}) \colon E_{\underline{n}\setminus\{k\}} \to P_k^{\mathbf{low}}$, the surjectivity of $p_k \colon E \to E_{\underline{n}\setminus\{k\}}$, as well as the identities $p_i^{\underline{n}\setminus\{k\}} \circ p_k = p_k^{\underline{n}\setminus\{i\}} \circ p_i$ for $i \neq k$, we find easily that $q_k(P_k^{\mathbf{up}}) = P_k^{\mathbf{low}}$. Further, $P_k^{\mathbf{up}}$ is clearly closed under the addition of $\prod_{\substack{i=1\\i\neq k}}^n E_{\underline{n}\setminus\{i\}} \to \prod_{\substack{i=1\\i\neq k}}^n E_{\underline{n}\setminus\{i,k\}}$. Lemma 2.1.7 yields then that $q_k \colon P_k^{\mathbf{up}} \to P_k^{\mathbf{low}}$ is a smooth vector bundle.

Next let us set for simplicity $\delta_k := \pi_k^{\mathbf{low}}(\underline{n}\setminus\{k\}) \colon E_{\underline{n}\setminus\{k\}} \to P_k^{\mathbf{low}}$. Recall that it is defined by

$$\delta_k \colon e_k \mapsto \left(p_1^{\underline{n}\setminus\{k\}}(e_k),\dots,\hat{k},\dots,p_n^{\underline{n}\setminus\{k\}}(e_k)\right).$$

Since $n \geq 2$ we can choose $i \in \underline{n}\setminus\{k\}$. Then $\delta_k \colon E_{\underline{n}\setminus\{k\}} \to P_k^{\mathbf{low}}$ is a surjective smooth vector bundle homomorphism over the identity on $E_{\underline{n}\setminus\{i,k\}}$. By Lemma 2.1.5, it is a surjective submersion. We consider the pullback vector bundles $(\delta_k)^! P_k^{\mathbf{up}}$ over $E_{\underline{n}\setminus\{k\}}$, for each $k \in \underline{n}$. As a set, each $(\delta_k)^! P_k^{\mathbf{up}}$can easily be identified with P.

Denote by φ_k the inclusion of $P_k^{\mathbf{up}}$ in $E_{\underline{n}\setminus\{1\}} \times \dots \hat{k} \dots \times E_{\underline{n}\setminus\{n\}}$. Then P is embedded into $E_{\underline{n}\setminus\{1\}} \times \dots \times E_{\underline{n}\setminus\{n\}}$ via the composition

$$P \hookrightarrow P_k^{\mathbf{up}} \times E_{\underline{n}\setminus\{k\}} \xhookrightarrow{\varphi_k \times \mathrm{id}_{E_{\underline{n}\setminus\{k\}}}} (E_{\underline{n}\setminus\{1\}} \times \dots \hat{k} \dots \times E_{\underline{n}\setminus\{n\}}) \times E_{\underline{n}\setminus\{k\}} \quad ,$$

where the map on the left is the embedding as in Lemma 2.1.3. It is easy to see that up to the obvious reordering of the factors on the right, the embeddings obtained for $k = 1,\dots,n$ are the same map. Therefore, all the obtained smooth structures on P are compatible and so P is a smooth manifold and all its projections are smooth. In particular, we have proved (a).

The compatibility of the vector bundle structures of P over $E_{\underline{n}\setminus\{i\}}$ and $E_{\underline{n}\setminus\{j\}}$ for $i \neq j \in \underline{n}$ follows from the compatibility of the structures in $\mathbb{E}^{\underline{n}\setminus\{k\},\emptyset}$. More precisely, the interchange law in the double vector bundle $(P; E_{\underline{n}\setminus\{i\}}, E_{\underline{n}\setminus\{j\}}; E_{\underline{n}\setminus\{i,j\}})$

follows immediately from the interchange laws in all the double vector bundles $(E_{\underline{n}\setminus\{k\}}; E_{\underline{n}\setminus\{k,i\}}, E_{\underline{n}\setminus\{k,j\}}; E_{\underline{n}\setminus\{k,i,j\}})$ for $k \in \underline{n} \setminus \{i,j\}$, since the different additions in P are all defined component-wise. Hence we can define $\mathbb{P}\colon \square^n \to \mathbf{Man}^\infty$ and we obtain an n-fold vector bundle.

For each $k = 1,\dots,n$, $\pi_k^{\mathbf{up}}(\underline{n})\colon E \to P_k^{\mathbf{up}}$ is a vector bundle morphism over $\delta_k\colon E_{\underline{n}\setminus\{k\}} \to P_k^{\mathbf{low}}$. The pullback of $\pi_k^{\mathbf{up}}(\underline{n})$ via the map δ_k is hence a vector bundle morphism $E \to (\delta_k)^! P_k^{\mathbf{up}}$ over the identity on $E_{\underline{n}\setminus\{k\}}$, and it is easy to see that it coincides – via the identification of P with $(\delta_k)^! P_k^{\mathbf{up}}$ – with the n-fold projection $\pi(\underline{n})$ from E to P. Hence $\pi\colon \mathbb{E} \to \mathbb{P}$ is an n-fold vector bundle morphism.

As before choose $i \in \underline{n} \setminus \{k\}$. Since $\pi_k^{\mathbf{up}}(\underline{n})\colon E \to P_k^{\mathbf{up}}$ is a surjective vector bundle morphism over the identity on $E_{\underline{n}\setminus\{i\}}$, it is a surjective submersion by Lemma 2.1.5. But since $\delta_k\colon E_{\underline{n}\setminus\{k\}} \to P_k^{\mathbf{low}}$ is a surjective submersion and $\pi_k^{\mathbf{up}}(\underline{n})$ is a vector bundle morphism over δ_k, by Lemma 2.1.5 it must be surjective in each fiber of $p_k\colon E \to E_{\underline{n}\setminus\{k\}}$. By Lemma 2.1.4, the pullback $\pi(\underline{n}) = \delta_k^! \pi_k^{\mathbf{up}}(\underline{n})\colon E \to P$ is then surjective in each fiber of $p_k\colon E \to E_{\underline{n}\setminus\{k\}}$. Since the base map is the identity on $E_{\underline{n}\setminus\{k\}}$, $\pi(\underline{n})$ is surjective. $\square$

Note that we have proved as well the following result.

Corollary 3.1.14. *In the situation of Theorem 3.1.13, the projection $\pi(\underline{n})\colon E \to P$ is a surjective submersion.*

3.1.4 Cores of a multiple vector bundle

Given a double vector bundle $(D; A, B; M)$, recall from Definition 2.1.8 and Lemma 2.1.9 that the intersection $C = (p_B^D)^{-1}(0_M^B) \cap (p_A^D)^{-1}(0_M^A)$ is called the core of the double vector bundle $(D; A, B; M)$ and has a natural vector bundle structure over M, which we will denote by $q_C\colon C \to M$. In this section, we explain the cores of multiple vector bundles. These cores have also been defined using a different notation by Alfonso Gracia-Saz and Kirill Mackenzie in [27].

Let $\mathbb{E}$ be a multiple vector bundle with absolute base $M := E_\emptyset$. For each $S \subseteq \mathbb{N}$ and each $k \in S$, we have the zero section $0_{S\setminus\{k\}}^{\mathbb{E},S}\colon E_{S\setminus\{k\}} \to E_S$, $e \mapsto 0_e^{E_S}$. For each $R \subseteq S \subseteq \mathbb{N}$, all compositions of $\#S - \#R$ composable zero sections, starting with some $0_R^{R\cup\{i\}}\colon E_R \to E_{R\cup\{i\}}$, for some $i \in S \setminus R$, and ending into E_S, are equal and the obtained map is written $0_R^{\mathbb{E},S}\colon E_R \to E_S$. In particular, we set $0_S^{\mathbb{E},S} = \mathrm{id}_{E_S}$. If it is clear from the context, which multiple vector bundle we are considering, we write $0_R^S := 0_R^{\mathbb{E},S}$. The image of $e \in E_R$ under 0_R^S is denoted by $\mathbf{0}_e^S$, and the image of E_R under 0_R^S is written $\mathbf{0}_R^S$. For better readability we sometimes write $\mathbf{0}_M^S := \mathbf{0}_\emptyset^S$ and $\mathbf{0}_R^E := \mathbf{0}_R^{\underline{n}}$.

Choose a subset $S \subseteq \mathbb{N}$ and $j, k \in S$ with $j \neq k$. Then

$$\begin{array}{ccc} E_S & \xrightarrow{p_k^S} & E_{S\setminus\{k\}} \\ \big\downarrow p_j^S & & \big\downarrow p_j^{S\setminus\{k\}} \\ E_{S\setminus\{j\}} & \xrightarrow{p_k^{S\setminus\{j\}}} & E_{S\setminus\{j,k\}} \end{array}$$

is a double vector bundle, which has therefore a core

$$E^S_{\{j,k\}} := (p^S_{S\setminus\{j\}})^{-1}\left(\mathbf{0}^{S\setminus\{j\}}_{S\setminus\{j,k\}}\right) \cap (p^S_{S\setminus\{k\}})^{-1}\left(\mathbf{0}^{S\setminus\{k\}}_{S\setminus\{j,k\}}\right). \tag{3.11}$$

This core has then an induced vector bundle structure over $E_{S\setminus\{j,k\}}$ with projection $(p^{S\setminus\{j\}}_{S\setminus\{k\}} \circ p^S_{S\setminus\{j\}})|_{E^S_{\{j,k\}}}$, which we denote by $c^S_{\{j,k\}} \colon E^S_{\{j,k\}} \to E_{S\setminus\{j,k\}}$. This is a special case of the side cores, as the following proposition shows.

Proposition 3.1.15. *Let $\mathbb{E}$ be a multiple vector bundle, $S \subseteq \mathbb{N}$ a finite subset and $J \subseteq S$ non-empty. The (S, J)-core*

$$E^S_J := \bigcap_{j\in J} (p^S_j)^{-1}\left(\mathbf{0}^{S\setminus\{j\}}_{S\setminus J}\right), \tag{3.12}$$

is a smooth embedded submanifold of E_S and inherits a vector bundle structure over $E_{S\setminus J}$ with projection $c^S_J := (\mathbb{E}(S \to S \setminus J))|_{E^S_J} \colon E^S_J \to E_{S\setminus J}$. In particular, for $J = \{s\}$ of cardinality 1, we get $E^S_J = E_S$ and $c^S_J = p^S_s$.

Proof. That E^S_J is a submanifold of E_S follows from Theorem 3.1.13: Consider the $(S, S\setminus J)$-face of $\mathbb{E}$, the $\#J$-fold vector bundle $\mathbb{E}^{S,S\setminus J}$. We denote the corresponding $\#J$-pullback by P^S_J. This is the total space of an $\#J$-fold vector bundle $\mathbb{P}^S_J$ with absolute base $E_{S\setminus J}$. The image of $E_{S\setminus J}$ under any $\#J$ composable zero sections of P^S_J, $Z := \mathbf{0}^{P^S_J}_{E_{S\setminus J}}$ is an embedded submanifold of P^S_J. By Corollary 3.1.14 the $\#J$-fold projection $\pi^S_J \colon E_S \to P^S_J$ is a surjective submersion. E^S_J is the preimage of Z under π^S_J and is thus a smooth embedded submanifold of E_S.

The vector bundle structure is similar to the case $n = 2$. Any two elements $e, e' \in E^S_J$ with $c^S_J(e) = c^S_J(e') =: b$ can be added over any p^S_j, for $j \in J$, since $p^S_j(e) = 0^{S\setminus\{j\}}_b = p^S_j(e')$. All the additions clearly preserve E^S_J. For any $j \in J$, $\mathbf{0}^{S\setminus\{j\}}_{S\setminus J}$ is an embedded submanifold of $E_{S\setminus\{j\}}$ and we get a unique vector bundle structure $E^S_J \to \mathbf{0}^{S\setminus\{j\}}_{S\setminus J}$ according to Lemma 2.1.7. The interchange laws in all the double vector bundles $(E_S; E_{S\setminus\{j_1\}}, E_{S\setminus\{j_2\}}; E_{S\setminus\{j_1,j_2\}})$ imply that after identification of $\mathbf{0}^{S\setminus\{j\}}_{S\setminus J}$ with $E_{S\setminus J}$ all the additions coincide: Since we have $\mathbf{0}^S_{\mathbf{0}^{S\setminus\{j_1\}}_b} = \mathbf{0}^S_b = \mathbf{0}^S_{\mathbf{0}^{S\setminus\{j_2\}}_b}$, we find easily

$$\begin{aligned} e \underset{S\setminus\{j_1\}}{+} e' &= \left(e \underset{S\setminus\{j_2\}}{+} \mathbf{0}^S_{\mathbf{0}^{S\setminus\{j_2\}}_b}\right) \underset{S\setminus\{j_1\}}{+} \left(\mathbf{0}^S_{\mathbf{0}^{S\setminus\{j_2\}}_b} \underset{S\setminus\{j_2\}}{+} e'\right) \\ &= \left(e \underset{S\setminus\{j_1\}}{+} \mathbf{0}^S_{\mathbf{0}^{S\setminus\{j_1\}}_b}\right) \underset{S\setminus\{j_2\}}{+} \left(\mathbf{0}^S_{\mathbf{0}^{S\setminus\{j_1\}}_b} \underset{S\setminus\{j_1\}}{+} e'\right) = e \underset{S\setminus\{j_2\}}{+} e'. \end{aligned} \tag{3.13}$$

Therefore, E_J^S has a well-defined vector bundle structure over $E_{S\setminus J}$. $\square$

We begin by proving that a side core can be constructed 'by stages'.

Lemma 3.1.16. *Let $\mathbb{E}$ be a multiple vector bundle and $S \subseteq \mathbb{N}$. Choose $K \subseteq J \subseteq S$. Then*

$$E_J^S = \left\{ e \in E_K^S \mid p_j^S(e) \in \mathbf{0}_{S\setminus J}^{S\setminus\{j\}}, j \in J \setminus K, \text{ and } c_K^S(e) \in \mathbf{0}_{S\setminus J}^{S\setminus K} \right\}. \tag{3.14}$$

Proof. For simplicity, we denote here by X the set on the right-hand side of the equation. First, take $e \in E_J^S$. Then since $p_j^S(e) \in \mathbf{0}_{S\setminus J}^{S\setminus\{j\}}$ for all $j \in J$, and since $K \subseteq J$, we have for $k \in K$: $p_k^S(e) = \mathbf{0}_{e_k}^{S\setminus\{k\}}$ for some $e_k \in E_{S\setminus J}$. Since $\mathbf{0}_{e_k}^{S\setminus\{k\}} = \mathbf{0}_{\mathbf{0}_{e_k}^{S\setminus K}}^{S\setminus\{k\}}$, we find $p_k^S(e) \in \mathbf{0}_{S\setminus K}^{S\setminus\{k\}}$ for all $k \in K$. Therefore $e \in E_K^S$ with $p_j^S(e) \in \mathbf{0}_{S\setminus J}^{S\setminus\{j\}}$ for $j \in J \setminus K$ and we only need to check that $c_K^S(e) \in \mathbf{0}_{S\setminus J}^{S\setminus K}$ in order to find that $e \in X$. But for any choice of $k \in K$, we find $c_K^S(e) = p_{S\setminus K}^S(e) = p_{S\setminus K}^{S\setminus\{k\}}(p_k^S(e)) = p_{S\setminus K}^{S\setminus\{k\}}(\mathbf{0}_{e_k}^{S\setminus\{k\}}) = 0_{e_k}^{S\setminus K}$ with $e_k \in E_{S\setminus J}$.

Conversely, take $e \in X$. Then since $e \in E_K^S$ we find for each $k \in K$ an element $e_k \in E_{S\setminus K}$ such that $p_k^S(e) = 0_{e_k}^{S\setminus\{k\}}$. But then $e_k = p_{S\setminus\{K\}}^{S\setminus\{k\}}(0_{e_k}^{S\setminus\{k\}}) = p_{S\setminus\{K\}}^{S\setminus\{k\}}(p_k^S(e)) = p_{S\setminus\{K\}}^S(e) = c_K^S(e) \in \mathbf{0}_{S\setminus J}^{S\setminus K}$ shows that $e \in (p_k^S)^{-1}\left(\mathbf{0}_{S\setminus J}^{S\setminus\{k\}}\right)$. Since $k \in K$ was arbitrary and also $e \in (p_j^S)^{-1}\left(\mathbf{0}_{S\setminus J}^{S\setminus\{k\}}\right)$ for all $j \in J \setminus K$, we find that $e \in E_J^S$. $\square$

Using this, we prove the following theorem.

Theorem 3.1.17. *Let $\mathbb{E}$ be a multiple vector bundle. For each $S \subset \mathbb{N}$ and $J \subseteq S$ non-empty, the space E_J^S is the total space of an $(\#S - \#J + 1)$-fold vector bundle in the following way.*

The partition $\rho_J^S = \{J, \{s_1\}, \ldots, \{s_{(\#S-\#J+1)}\}\}$ of S into the set J and sets with one element which we order in the natural ordering induced by the ordering of $\mathbb{N}$ gives rise to a category $\lozenge_J^S := \lozenge^{\rho_J^S}$ isomorphic to the $(\#S - \#J + 1)$-cube category as discussed in Section 3.1.2. Let us now fix the isomorphism of indexing categories described there by $\mathbf{i}_J^S := \mathbf{i}^{\rho_J^S} \colon \square^{\#S-\#J+1} \to \lozenge^{\rho_J^S}$. We will again write $[\nu] := \cup_{K\in\nu} K$ for any subset $\nu \subseteq \rho_J^S$. Now we define $\mathbb{E}_J^S \colon \lozenge_J^S \to \mathbf{Man}^\infty$ by setting $\mathbb{E}_J^S(\nu) = E_J^{[\nu]}$ if $J \in \nu$ and $\mathbb{E}_J^S(\nu) = E_{[\nu]}$ if $J \notin \nu$ and define the morphisms by

$$\mathbb{E}_J^S(\nu_1 \to \nu_2) = \mathbb{E}([\nu_1] \to [\nu_2])|_{E_J^{[\nu_1]}} \colon E_J^{[\nu_1]} \to E_J^{[\nu_2]}, \qquad \text{if } J \in \nu_2 \subseteq \nu_1,$$

$$\mathbb{E}_J^S(\nu_1 \to \nu_2) = \mathbb{E}([\nu_1] \to [\nu_2]) \colon E_{[\nu_1]} \to E_{[\nu_2]}, \qquad \text{if } \nu_2 \subseteq \nu_1 \not\ni J$$

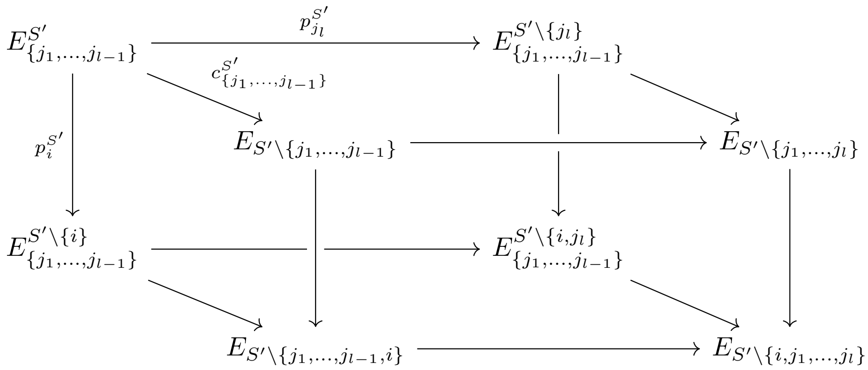
$$\mathbb{E}_J^S(\nu_1 \to \nu_2) = \mathbb{E}([\nu_1] \setminus J \to [\nu_2]) \circ c_J^{[\nu_1]} \colon E_J^{[\nu_1]} \to E_{[\nu_2]}, \quad \text{if } \nu_2 \subseteq \nu_1,\ J \in \nu_1 \setminus \nu_2.$$

Then $\mathbb{E}_J^S$ is a $(\#S - \#J + 1)$-fold vector bundle.

Proof. The nodes of $\mathbb{E}^S_J$ are given by $E^{S'}_J$ for $J \subseteq S' \subseteq S$ and E_I for $I \subseteq S \setminus J$. The generating arrows are given by $p^I_i \colon E_I \to E_{I\setminus\{i\}}$ for $i \in I \subseteq S \setminus J$ and $c^{S'}_J : E^{S'}_J \to E_{S'\setminus J}$ and $p^{S'}_i|_{E^{S'}_J} : E^{S'}_J \to E^{S'\setminus\{i\}}_J$ for $i \in S' \setminus J$. In the following we just write $p^{S'}_i$ for the restriction $p^{S'}_i|_{E^{S'}_J}$.

For $\#J < \#S$ we prove by induction over $\#J =: l$ that this defines a multiple vector bundle. For $J = \{s\}$ of cardinality 1 it is easy to see that $\mathbb{E}^S_J = \mathbb{E}^{S,\emptyset}$, which is an $\#S$-fold vector bundle by Proposition 3.1.6.

Now assume that $E^S_{\{j_1,\dots,j_{l-1}\}}$ is the total space of a $(\#S - l + 2)$-fold vector bundle. Choose $j_l \in S \setminus \{j_1,\dots,j_{l-1}\}$, $S' \subseteq S$ with $\{j_1,\dots,j_l\} =: J \subseteq S'$, and choose $i \in S' \setminus J$. Then by the induction hypothesis and Proposition 3.1.6,



,

is a triple vector bundle, and by (3.14), its upper side core is

$$\begin{array}{ccc} E^{S'}_J & \xrightarrow{c^{S'}_J} & E_{S'\setminus J} \\ \big\downarrow p^{S'}_i & & \big\downarrow p^{S'\setminus J}_i \\ E^{S'\setminus\{i\}}_J & \xrightarrow{c^{S'\setminus\{i\}}_J} & E_{S'\setminus(J\cup\{i\})}. \end{array}$$

Hence this diagram is a double vector bundle (see for example [51]) and, as before, all commutative squares in our $(\#S - l + 1)$-cube diagram are double vector bundles. □

If $l = \#S$, then $J = S$ and E^S_S has a vector bundle structure over M with projection $c^S_S = \mathbb{E}(S \to \emptyset)|_{E^S_S}$. The nodes at the source of only one arrow of $\mathbb{E}^S_J$ are the nodes $E_{\{i\}}$ of $\mathbb{E}$ for $i \in S \setminus J$, and the (J,J)-core $c^J_J \colon E^J_J \to M$ of $\mathbb{E}$.

We have then for each $\nu \subseteq \rho^S_J$ an inclusion $\eta^J(\nu) \colon \mathbb{E}^S_J(\nu) \hookrightarrow E_{[\nu]}$, since $\mathbb{E}^S_J(\nu)$ is an embedded submanifold of $E_{[\nu]}$ for all $\nu \subseteq \rho^S_J$.

Example 3.1.18. *Given the n-fold vector bundle $\mathbb{E}^{\mathcal{A}}$ defined in Section 3.1.2, its (S,J)-core $(\mathbb{E}^{\mathcal{A}})^S_J$ has nodes $(\mathbb{E}^{\mathcal{A}})^S_J \circ (\mathbf{i}^S_J)^{-1}(\nu) = \prod^M_{\nu' \subseteq \nu} A_{[\nu']}$ for $\nu \subseteq \rho^S_J$ and can thus be identified with $\mathbb{E}^{\mathcal{A}}_{\rho^S_J}$ defined as in Section 3.1.2. In particular, $(E^{\mathcal{A}})^S_S = A_S$.*

For instance, for $n = 3$ (see Example 3.1.9) we have decomposed cores

$$E_{\{1,2\}}^{\{1,2,3\}} = A_{\{3\}} \times_M A_{\{1,2\}} \times_M A_{\{1,2,3\}}\,, \quad E_{\{2,3\}}^{\{1,2,3\}} = A_{\{1\}} \times_M A_{\{2,3\}} \times_M A_{\{1,2,3\}}\,,$$
$$E_{\{1,3\}}^{\{1,2,3\}} = A_{\{2\}} \times_M A_{\{1,3\}} \times_M A_{\{1,2,3\}}\,.$$

Remark 3.1.19. *Let $\mathbb{E}$ be an n-fold vector bundle.*

(a) It follows directly from the definitions that the cores of the faces of $\mathbb{E}$ are given by the faces of the cores of $\mathbb{E}$. That is, $(\mathbb{E}^{S,\emptyset})_J^S = (\mathbb{E}_J^S)^{\rho_J^S,\emptyset}$ for $J \subseteq S \subseteq \underline{n}$. Note that both these multiple vector bundles canonically use the indexing category $\lozenge_J^S$.

(b) Note also that (3.14) can now be written $E_J^S = (\mathbb{E}_K^S)_{\rho_K^J}^{\rho_K^S}$.

(c) For $I, J \subseteq S$ with $I \cap J = \emptyset$ the intersection of the cores $E_I^S \cup E_J^S$ is the iterated core, the total space of $(\mathbb{E}_J^S)_{\{\{i\}_{i\in I}\}}^{\rho_J^S}$, or equivalently the total space of $(\mathbb{E}_I^S)_{\{\{j\}_{j\in J}\}}^{\rho_I^S}$. This is since the intersection of the two cores consists of all elements of E_I^S which project additionally to the right zeros in all J-directions. Since the $(\#S - \#I + 1)$-fold vector bundle structure on $\mathbb{E}_I^S$ was given by restricting the structure of $\mathbb{E}$, this intersection is then precisely the corresponding core in $\mathbb{E}_I^S$.

(d) In the case of $I \cap J \neq \emptyset$ the intersection of the cores $E_I^S \cap E_J^S$ is given by $E_{I\cup J}^S$ instead since it consists precisely of all elements of E_S which project to the correct zeros in all $I \cup J$-directions.

Proposition 3.1.20. *Given a morphism $\tau\colon \mathbb{E} \to \mathbb{F}$ of multiple vector bundles, we have for any $J \subseteq S \subseteq \mathbb{N}$ an induced core morphism of the $(\#S - \#J + 1)$-fold vector bundles $\tau_J^S\colon \mathbb{E}_J^S \to \mathbb{F}_J^S$ defined by*

$$\tau_J^S(\nu) = \tau([\nu])|_{E_J^{[\nu]}}\colon E_J^{[\nu]} \to F_J^{[\nu]} \qquad \text{for } \nu \subseteq \rho_J^S \text{ with } J \in \nu \tag{3.15}$$
$$\tau_J^S(\nu) = \tau([\nu])\colon E_{[\nu]} \to F_{[\nu]} \qquad \text{for } \nu \subseteq \rho_J^S \text{ with } J \notin \nu\,, \tag{3.16}$$

where we consider $E_J^{[\nu]}$ and $F_J^{[\nu]}$ as subsets of $E_{[\nu]}$ and $F_{[\nu]}$, respectively. Furthermore, $(\cdot)_J^S$ is a covariant functor from multiple vector bundles to multiple vector bundles.

Proof. For $J \notin \nu$ there is nothing to show as $\mathbb{E}_J^S(\nu) = \mathbb{E}([\nu])$ and $\mathbb{F}_J^S(\nu) = \mathbb{F}([\nu])$ and thus all the maps are well defined vector bundle morphisms.

For $J \in \nu$ it remains to be shown that τ_J^S is well defined, that is $\tau([\nu])(E_J^{[\nu]}) \subseteq F_J^{[\nu]}$. Linearity follows then directly from linearity of τ. The manifold $E_J^{[\nu]}$ is defined as the set of all elements of $E_{[\nu]}$ that project to $\mathbf{0}^{\mathbb{E},[\nu]\setminus\{j\}}_{[\nu]\setminus J}$ for all $j \in J$. Since for all $I \subseteq \underline{n}$, $\tau(I)\colon E_I \to F_I$ is a vector bundle homomorphism over $\tau(I \setminus \{i\})$ for all $i \in I$, the image of $e \in E^{[\nu]}_J$ under $\tau([\nu])$ thus projects to $\mathbf{0}^{\mathbb{F},[\nu]\setminus\{j\}}_{[\nu]\setminus J}$ in $F_{[\nu]\setminus\{j\}}$ and is an element of $F^{[\nu]}_J$.

Functoriality follows directly from the definition: in the case of $J \notin \nu$ we have

$$(\sigma \circ \tau)^S_J(\nu) = (\sigma \circ \tau)([\nu]) = \sigma([\nu]) \circ \tau([\nu]) = \sigma^S_J(\nu) \circ \tau^S_J(\nu)\,,$$

whereas for $J \in \nu$ we have

$$(\sigma \circ \tau)^S_J(\nu) = (\sigma \circ \tau)([\nu])|_{E^{[\nu]}_J} = \sigma([\nu])|_{F^{[\nu]}_J} \circ \tau([\nu])|_{E^{[\nu]}_J} = \sigma^S_J(\nu) \circ \tau^S_J(\nu)\,. \qquad \square$$

From Theorem 3.1.13 we obtain easily the following proposition; the n-fold analogon of the core sequences for double vector bundles, which were defined by Kirill Mackenzie in [51]. They are important in the proof of the existence of decompositions of n-fold vector bundles. We call them the ultracore sequences of $\mathbb{E}$.

Proposition 3.1.21. *Let $\mathbb{E}$ be an n-fold vector bundle. For each $k \in \underline{n}$, we have a short exact sequence*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (p^{\underline{n}\setminus\{k\}}_{\emptyset})^! E^{\underline{n}}_{\underline{n}} & \xrightarrow{\ \iota\ } & E & \xrightarrow{\ \pi(\underline{n})\ } & P & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & E_{\underline{n}\setminus\{k\}} & & E_{\underline{n}\setminus\{k\}} & & E_{\underline{n}\setminus\{k\}} & & \end{array} \tag{3.17}$$

of vector bundles over $E_{\underline{n}\setminus\{k\}}$, where P is the n-pullback defined in Theorem 3.1.13.

Proof. By Theorem 3.1.13, the map $\pi(\underline{n})\colon E \to P$ is a surjective vector bundle morphism over $\mathrm{id}_{E_{\underline{n}\setminus\{k\}}}$.

Take any e in the kernel of $\pi(\underline{n})$ considered as vector bundle morphism over $E_{\underline{n}\setminus\{k\}}$. Denote its projection in E_J for any $J \subseteq \underline{n} \setminus \{k\}$ by e_J, with $m := e_\emptyset \in M$. Write $\underline{n} \setminus \{k\} = \{j_1, \ldots, j_{n-1}\}$. Define now recursively

$$f^0 := e\,, \qquad f^l := f^{l-1} \underset{\underline{n}\setminus\{j_l\}}{-} \mathbf{0}^E_{e_{\underline{n}\setminus\{k,j_1,\ldots,j_{l-1}\}}}\,.$$

Then it is easy to show by induction that $p^{\underline{n}}_I(f^l) = \mathbf{0}^I_{e_{I\cap(\underline{n}\setminus\{k,j_1,\ldots,j_l\})}}$. The above implies that f^{n-1} projects to $\mathbf{0}^I_m$ for all $I \subseteq \underline{n}$. It is thus an element of the ultracore $E^{\underline{n}}_{\underline{n}}$, and we denote it by $z := f^{n-1}$.

Now

$$e = \left(\left(\left(z \underset{\underline{n}\setminus\{j_{n-1}\}}{+} \mathbf{0}^E_{e_{\{j_{n-1}\}}}\right) \underset{\underline{n}\setminus\{j_{n-2}\}}{+} \mathbf{0}^E_{e_{\{j_{n-1},j_{n-2}\}}}\right) \underset{\underline{n}\setminus\{j_3\}}{+} \cdots\right) \underset{\underline{n}\setminus\{j_1\}}{+} \mathbf{0}^E_{e_{\underline{n}\setminus\{k\}}} \tag{3.18}$$
$$=: \iota(z, e_{\underline{n}\setminus\{k\}})\,.$$

and the defined map $\iota\colon E^{\underline{n}}_{\underline{n}} \times_M E_{\underline{n}\setminus\{k\}} \to E$ is clearly an injective morphism of vector bundles over $E_{\underline{n}\setminus\{k\}}$, by construction making the sequence exact. That the map ι does not depend on the chosen order of the set $\underline{n} \setminus \{k\}$ follows from the interchange laws in all the top double vector bundles since all we do is adding zero elements in different directions. This together with (3.18) also shows that the recursive definition of the ultracore-element z is independent of the chosen ordering. □

3.2 Splittings of multiple vector bundles

In this section we achieve our main goal in this paper: we prove that any n-fold vector bundle admits a (non-canonical) linear splitting. We begin by discussing the notions of linear splitting versus linear decomposition. Then we prove inductively our main theorem, and finally we explain how n-fold vector bundles can now be defined using *n-fold vector bundle atlases*.

3.2.1 Splittings and decompositions of n-fold vector bundles

Let $\mathbb{E}$ be an n-fold vector bundle. This gives rise to a family $\mathcal{A}$ of smooth vector bundles $\mathcal{A} = (q_J\colon A_J \to M)_{J\subseteq \underline{n},\, \#J<\infty}$ over $M = \mathbb{E}(\emptyset)$ defined by $A_{\{i\}} = E_{\{i\}}$ for $i = 1,\dots,n$ and $A_J = E^J_J$ for $\#J \geq 2$. By Example 3.1.18, if $\mathbb{E}$ is already a decomposed n-fold vector bundle, then each element of the family of vector bundles defining it appears as one of the cores of $\mathbb{E}$. This is why we call the vector bundles $A_J = E^J_J$ the **building bundles of** $\mathbb{E}$.

We can then consider the decomposed n-fold vector bundles $\mathbb{E}^{\mathcal{A}}$ and $\overline{\mathbb{E}} := \mathbb{E}^{\overline{\mathcal{A}}}$ defined in Section 3.1.2. We call $\mathbb{E}^{\mathcal{A}}$ the decomposed n-fold vector bundle associated to $\mathbb{E}$ and $\overline{\mathbb{E}}$ the vacant, decomposed n-fold vector bundle associated to $\mathbb{E}$.

Definition 3.2.1. *A **linear splitting** of the n-fold vector bundle $\mathbb{E}$ is a monomorphism $\Sigma\colon \overline{\mathbb{E}} \to \mathbb{E}$ of n-fold vector bundles, such that for $i = 1,\dots,n$, $\Sigma(\{i\})\colon E_{\{i\}} \to E_{\{i\}}$ is the identity.*

*A **decomposition** of the n-fold vector bundle $\mathbb{E}$ is a natural isomorphism $\mathcal{S}\colon \mathbb{E}^{\mathcal{A}} \to \mathbb{E}$ of n-fold vector bundles over the identity maps $\mathcal{S}(\{i\}) = \mathrm{id}_{E_{\{i\}}}\colon A_{\{i\}} \to E_{\{i\}}$ such that additionally the induced core morphisms $\mathcal{S}^I_I(\{I\})$ are the identities $\mathrm{id}_{E^I_I}$ for all $I \subseteq \underline{n}$.*

Linear splittings and decompositions of double vector bundles are equivalent to each other. Given a splitting Σ, define the decomposition by $\mathcal{S}(a_m, b_m, c_m) := \Sigma(a_m, b_m) +_B (0^D_{b_m} +_A c_m) = \Sigma(a_m, b_m) +_A (0^D_{a_m} +_B c_m)$. Conversely, given a decomposition $\mathcal{S}$ define the splitting by $\Sigma(a_m, b_m) := \mathcal{S}(a_m, b_m, 0^C_m)$. These two constructions are obviously inverse to each other. We prove here that a similar equivalence holds true in the general case of n-fold vector bundles.

A linear splitting Σ of an n-fold vector bundle $\mathbb{E}$ and decompositions $\mathcal{S}^I$ of the highest order cores – the $(n-1)$-fold vector bundles $\mathbb{E}^{\underline{n}}_I$ for all $I \subseteq \underline{n}$ with $\#I = 2$ – are called **compatible** if they coincide on all possible intersections. That is, $\mathcal{S}^I(\{\{k\}_{k\in\underline{n}\setminus I}\})|_{\overline{\mathbb{E}}(\underline{n}\setminus I)} = \Sigma(\underline{n}\setminus I)$ and $\mathcal{S}^I(\rho^{\underline{n}}_I)|_{(E^{\mathcal{A}})^{\underline{n}}_I \cap (E^{\mathcal{A}})^{\underline{n}}_J} = \mathcal{S}^J(\rho^{\underline{n}}_J)|_{(E^{\mathcal{A}})^{\underline{n}}_I \cap (E^{\mathcal{A}})^{\underline{n}}_J}$ for all $I, J \subseteq \underline{n}$ of cardinality 2. Note that we view here the total spaces of $(\mathbb{E}^{\mathcal{A}})^{\underline{n}}_I$, $(\mathbb{E}^{\mathcal{A}})^{\underline{n}}_J$ and of $\mathbb{E}^{\underline{n}}_I$ and $\mathbb{E}^{\underline{n}}_J$ as embedded in $E^{\mathcal{A}} = \mathbb{E}^{\mathcal{A}}(\underline{n})$ and $E = \mathbb{E}(\underline{n})$, respectively. Also recall that $\mathbb{E}^{\underline{n}}_I(\{k\}_{k\in\underline{n}\setminus I}) = \mathbb{E}(\underline{n}\setminus I)$ by definition.

Theorem 3.2.2. *Let $\mathbb{E}$ be an n-fold vector bundle.*

(a) Let $\mathcal{S}$ be a decomposition of $\mathbb{E}$. Then the composition $\Sigma = \mathcal{S}\circ\iota\colon \overline{\mathbb{E}} \to \mathbb{E}$, with ι defined as in (3.5), is a splitting of $\mathbb{E}$. Furthermore, the core morphisms $\mathcal{S}^{\underline{n}}_J\colon \mathbb{E}^{\rho_J} \to \mathbb{E}^{\underline{n}}_J$ are decompositions of $\mathbb{E}^{\underline{n}}_J$ for all $J \subseteq \underline{n}$ and these decompositions and the linear splitting are compatible.

(b) Conversely, given a linear splitting Σ of $\mathbb{E}$ and compatible decompositions of the highest order cores $\mathbb{E}^{\underline{n}}_J$ with top maps $\mathcal{S}^J\colon (E^{\mathcal{A}})^{\underline{n}}_J \to E^{\underline{n}}_J$, for $J \subseteq \underline{n}$ with $\#J = 2$, there exists a unique decomposition $\mathcal{S}$ of $\mathbb{E}$ such that $\Sigma = \mathcal{S}\circ\iota$ and such that the core morphisms of $\mathcal{S}$ are given by $\mathcal{S}^{\underline{n}}_J(\rho^{\underline{n}}_J) = \mathcal{S}^J$ for all J.

Proof. Let us consider a decomposition $\mathcal{S}\colon \mathbb{E}^{\mathcal{A}} \to \mathbb{E}$. Then the composition $\Sigma = \mathcal{S}\circ\iota$ is clearly a monomorphism of n-fold vector bundles, with $\Sigma(\{i\}) = \mathcal{S}(\{i\})\circ\iota(\{i\}) = \mathrm{id}_{E_{\{i\}}}\circ\mathrm{id}_{E_{\{i\}}} = \mathrm{id}_{E_{\{i\}}}$. Furthermore, Proposition 3.1.20 implies that the restrictions $\mathcal{S}^{\underline{n}}_J$ are isomorphisms of multiple vector bundles. Since for any $\nu \subseteq \rho^{\underline{n}}_J$ the (ν,ν)-core of $\mathbb{E}^{\underline{n}}_J$ equals $E^{[\nu]}_{[\nu]}$ which follows from Remark 3.1.19 for $J \in \nu$ and directly from the definition for $J \notin \nu$, these are all the building bundles of $\mathbb{E}^{\underline{n}}_J$. Now $\mathcal{S}^{[\nu]}_{[\nu]} = \mathrm{id}_{E^{[\nu]}_{[\nu]}}$ and thus $\mathcal{S}^{\underline{n}}_J$ induces the identity on all building bundles of $\mathbb{E}^{\underline{n}}_J$ and is therefore a decomposition. Since all $\mathcal{S}^{\underline{n}}_J$ and Σ are defined as restrictions of the same map $\mathcal{S}$ they are clearly compatible.

Conversely, assume that we have a splitting Σ of $\mathbb{E}$ and compatible decompositions $\mathcal{S}^J$ of the cores $\mathbb{E}^{\underline{n}}_J$ with $J \subseteq \underline{n}$, $\#J = 2$ as in (b). We prove that there is a unique decomposition $\mathcal{S}$ of $\mathbb{E}$ that restricts in the sense of (b) to Σ and the $\mathcal{S}^J$.

Let now $J_1, \ldots, J_{\binom{n}{2}}$ denote the subsets of $\underline{n}$ with $\#J_k = 2$. We define now an increasing chain of $\binom{n}{2}$ decomposed n-fold vector bundles as follows. For $k = 0, \ldots, \binom{n}{2}$ define a family of vector bundles over M, $\mathcal{A}^k = (B_I)_{I\subseteq\underline{n}}$ with $B_I = A_I$ for all I with either $\#I = 1$ or if there is $i \le k$ such that $J_i \subseteq I$; and

$B_I = M$ otherwise. Now let $\mathbb{E}^k := \mathbb{E}^{\mathcal{A}^k}$ with total space $E^k := \mathbb{E}^{\mathcal{A}^k}(\underline{n})$. There are obvious inclusions $\overline{\mathbb{E}}(\underline{n}) = E^0 \hookrightarrow E^1 \hookrightarrow \ldots \hookrightarrow E^{\binom{n}{2}} = \mathbb{E}^{\mathcal{A}}(\underline{n})$. We thus view the E^k as submanifolds of $\mathbb{E}^{\mathcal{A}}(\underline{n})$. Note that additionally $(E^{\mathcal{A}})^{\underline{n}}_{J_i} \subseteq E^k$ for all $i \leq k$. Now we show that we can define a decomposition $\mathcal{S}$ of $\mathbb{E}$ inductively on the E^k for $k = 0, \ldots, \binom{n}{2}$ and that it is unique with respect to the given linear splittings.

Since $\mathbb{E}^0 = \overline{\mathbb{E}}$ we set $\mathcal{S}^0 := \Sigma$ and this is clearly unique in the sense of (b). By the compatibility condition it also restricts to $\mathcal{S}^{J_i}$ on $E^0 \cap (E^{\mathcal{A}})^{\underline{n}}_{J_i}$ for $i = 1, \ldots, \binom{n}{2}$. Take now $k \geq 0$ and assume that we have a uniquely defined injective morphism of n-fold vector bundles $\mathcal{S}^k : \mathbb{E}^k \to \mathbb{E}$ that restricts to Σ on E^0 and to $\mathcal{S}^{J_i}$ on $E^k \cap (E^{\mathcal{A}})^{\underline{n}}_{J_i}$ for $i = 1, \ldots, \binom{n}{2}$. Take $\mathbf{x} = (a_I)_{I \subseteq \underline{n}} \in E^{k+1}$. Then in particular $a_I = 0^{A_I}_m$ if $\#I \geq 2$ and there is no $i \leq k+1$ with $J_i \subseteq I$. Set $\mathbf{y} := (b_I)_{I \subseteq \underline{n}}$ with $b_I = a_I$ if either $\#I = 1$ or there is $i \leq k$ such that $J_i \subseteq I$ and $b_I = 0^{A_I}_m$ otherwise. Set furthermore $\mathbf{z} := (c_I)_{I \subseteq \underline{n}}$ where $c_I = b_I$ whenever $I \subseteq \underline{n} \setminus J_{k+1}$, $c_I = a_I$ whenever $J_{k+1} \subseteq I$ and there is no $i \leq k$ with $J_i \subseteq I$, and $c_I = 0^{A_I}_m$ otherwise. Then $\mathbf{y} \in E^k$ and $\mathbf{z} \in (\mathbb{E}^{\mathcal{A}})^{\underline{n}}_{J_{k+1}}$. Furthermore, writing $J_{k+1} = \{s, t\}$, it is easy to check that

$$\mathbf{x} = \mathbf{y} \underset{\underline{n}\setminus\{s\}}{+} \left(\mathbf{0}^{\underline{n}}_{p_s(\mathbf{y})} \underset{\underline{n}\setminus\{t\}}{+} \mathbf{z} \right) = \mathbf{y} \underset{\underline{n}\setminus\{t\}}{+} \left(\mathbf{0}^{\underline{n}}_{p_t(\mathbf{y})} \underset{\underline{n}\setminus\{s\}}{+} \mathbf{z} \right). \tag{3.19}$$

The last equality follows directly from the interchange law in the double vector bundle $(E; E_{\underline{n}\setminus\{s\}}, E_{\underline{n}\setminus\{t\}}; E_{\underline{n}\setminus\{s,t\}})$ since $\mathcal{S}^{J_{k+1}}(\mathbf{z})$ is in the core of this double vector bundle. Thus we can define

$$\begin{aligned} \mathcal{S}^{k+1}(\mathbf{x}) :=& \mathcal{S}^k(\mathbf{y}) \underset{\underline{n}\setminus\{s\}}{+} \left(\mathbf{0}^{\underline{n}}_{p_s(\mathcal{S}^k(\mathbf{y}))} \underset{\underline{n}\setminus\{t\}}{+} \mathcal{S}^{J_{k+1}}(\mathbf{z}) \right) \\ =& \mathcal{S}^k(\mathbf{y}) \underset{\underline{n}\setminus\{t\}}{+} \left(\mathbf{0}^{\underline{n}}_{p_t(\mathcal{S}^k(\mathbf{y}))} \underset{\underline{n}\setminus\{s\}}{+} \mathcal{S}^{J_{k+1}}(\mathbf{z}) \right). \end{aligned} \tag{3.20}$$

It is easy to check that this defines an injective morphism of n-fold vector bundles $\mathcal{S}^{k+1} : \mathbb{E}^{k+1} \to \mathbb{E}$. Linearity over $E_{\underline{n}\setminus\{j\}}$ follows directly from linearity of $\mathcal{S}^k$ and $\mathcal{S}^{J_{k+1}}$ and the interchange laws in the double vector bundles $(E; E_{\underline{n}\setminus\{j\}}, E_{\underline{n}\setminus\{s\}}; E_{\underline{n}\setminus\{j,s\}})$ and $(E; E_{\underline{n}\setminus\{j\}}, E_{\underline{n}\setminus\{t\}}; E_{\underline{n}\setminus\{j,t\}})$ since the construction of $\mathbf{y}$ and $\mathbf{z}$ from $\mathbf{x}$ is linear. If now $\mathbf{x}$ was already in E^k, then $\mathbf{y} = \mathbf{x}$ and thus $\mathcal{S}^{k+1}$ restricts to $\mathcal{S}^k$ on E^k and therefore also to Σ. If $\mathbf{x}$ was in $(E^{\mathcal{A}})^{\underline{n}}_J$ for any $J \subseteq \underline{n}$ with $\#J = 2$, then $\mathbf{y} \in E^k \cap (E^{\mathcal{A}})^{\underline{n}}_J$ and by induction hypothesis $\mathcal{S}^k(\mathbf{y}) = \mathcal{S}^J(\mathbf{y})$. Furthermore, $\mathbf{z} \in (E^{\mathcal{A}})^{\underline{n}}_{J_{k+1}} \cap (E^{\mathcal{A}})^{\underline{n}}_J$ and by the compatibility of $\mathcal{S}^{J_{k+1}}$ with $\mathcal{S}^J$ we get that $\mathcal{S}^{J_{k+1}}(\mathbf{z}) = \mathcal{S}^J(\mathbf{z})$. Thus clearly $\mathcal{S}^{k+1}$ restricts to all $\mathcal{S}^J$ on the intersection $E^{k+1} \cap (E^{\mathcal{A}})^{\underline{n}}_J$. Also it clearly is the only morphism from $\mathbb{E}^{k+1}$ to $\mathbb{E}$ restricting to $\mathcal{S}^k$ on E^k and to all $\mathcal{S}^J$ and thus by the induction hypothesis the only morphism restricting to Σ and all $\mathcal{S}^J$. Thus we find eventually a unique injective morphism $\mathcal{S} := \mathcal{S}^{\binom{n}{2}} : \mathbb{E}^{\mathcal{A}} \to \mathbb{E}$ that restricts to Σ and all $\mathcal{S}^J$ for $\#J = 2$. That $\mathcal{S}$ is surjective now follows from linearity and a dimension count. □

3.2.2 Existence of splittings

In this section, we finally state and prove our main theorem. We prove by induction that every n-fold vector bundle is non-canonically isomorphic to a decomposed one.

Theorem 3.2.3. *Let $\mathbb{E}$ be an n-fold vector bundle. Then there is a linear splitting*

$$\Sigma\colon \overline{\mathbb{E}} \to \mathbb{E}\,,$$

that is a monomorphism of n-fold vector bundles from the vacant, decomposed n-fold vector bundle $\overline{\mathbb{E}}$ associated to $\mathbb{E}$, which was defined in Section 3.2.1, into $\mathbb{E}$.

Proof. We prove the following two claims by induction over n.

(a) Given an n-fold vector bundle $\mathbb{E}$, there exist n linear splittings $\Sigma_{\underline{n}\setminus\{k\}}$ of $\mathbb{E}^{\underline{n}\setminus\{k\},\emptyset}$ for $k \in \underline{n}$, such that $\Sigma_{\underline{n}\setminus\{i\}}(I) = \Sigma_{\underline{n}\setminus\{j\}}(I)$ for any $I \subseteq \underline{n} \setminus \{i,j\}$.

(b) Given a family of splittings as in (a), there exists a linear splitting of $\mathbb{E}$ with $\Sigma(I) = \Sigma_{\underline{n}\setminus\{k\}}(I)$ whenever $I \subseteq \underline{n} \setminus \{k\}$.

The case of $n = 1$ is trivial. Take now $n \geq 2$ and assume that both statements are true for l-fold vector bundles, for $l < n$. First, we prove (a). This is equivalent to having splittings Σ_I of $\mathbb{E}^{I,\emptyset}$ for all $I \subsetneq \underline{n}$ such that $\Sigma_{I_1}(J) = \Sigma_{I_2}(J)$ whenever $J \subseteq I_1 \cap I_2$. We prove that claim with an induction over $\sharp I$. For all $I \subseteq \underline{n}$ with $\sharp I = 1$ or $\sharp I = 2$, this is immediate.

Assume now that we have fixed linear splittings of $\mathbb{E}^{I,\emptyset}$ for all I with $\#I = l \leq n-2$, such that for all $J \subseteq I_1 \cap I_2$, $\Sigma_{I_1}(J) = \Sigma_J(J) = \Sigma_{I_2}(J)$. For any $I \subsetneq \underline{n}$ with $\#I = l+1$ we can then find by induction hypothesis (b) a linear splitting Σ_I of $\mathbb{E}^{I,\emptyset}$ which satisfies $\Sigma_I(J) = \Sigma_J(J)$ for all $J \subseteq I$. Now for I_1, I_2 of cardinality $l+1$ and $J \subseteq I_1 \cap I_2$, we get $\Sigma_{I_1}(J) = \Sigma_J(J) = \Sigma_{I_2}(J)$. This shows that part (a) is satisfied for every n-fold vector bundle since we eventually find linear splittings $\Sigma_{\underline{n}\setminus\{k\}}$ of all $\mathbb{E}_{\underline{n}\setminus\{k\}}$ which agree on all subsets $I \subseteq \underline{n}$ of cardinality $\#I \leq (n-2)$.

We denote in the following their top maps by

$$\Sigma_k := \Sigma_{\underline{n}\setminus\{k\}}(\underline{n} \setminus \{k\})\colon \prod_{i\in\underline{n}\setminus\{k\}}^{M} E_{\{i\}} \to E_{\underline{n}\setminus\{k\}}\,.$$

It is easy to check that given $m \in M$ and $e_i \in E_{\{i\}}$ with $p_{\emptyset}^{\{i\}}(e_i) = m$ for $i = 1, \ldots, n$, the tuple $(\Sigma_1(e_2, \ldots, e_n), \Sigma_2(e_1, e_3, \ldots, e_n), \ldots, \Sigma_n(e_1, \ldots, e_{n-1})$ is an element of P. Short exact sequences of vector bundles are always non-canonically split, so we can take a splitting θ_1 of the short exact sequence of vector bundles over $E_{\underline{n}\setminus\{1\}}$ in Proposition 3.1.21. Define $\Sigma_1^E\colon \prod_{i\in\underline{n}}^M E_{\{i\}} \to E$ by

$$\Sigma_1^E\colon (e_1,\ldots,e_n) \mapsto \theta_1\big(\Sigma_1(e_2,\ldots,e_n),\Sigma_2(e_1,e_3,\ldots,e_n),\ldots,\Sigma_n(e_1,\ldots,e_{n-1})\big)\,. \tag{3.21}$$

This is a vector bundle morphism over the linear splitting Σ_1 of $E_{\underline{n}\setminus\{1\}}$ such that

$$p_j(\Sigma_1^E(e_1,\ldots,e_n)) = \Sigma_j(e_1,\ldots,\hat{e_j},\ldots,e_n) \in E_{\underline{n}\setminus\{j\}} \tag{3.22}$$

for $j = 2,\ldots,n$. However, Σ_1^E is not necessarily linear over Σ_j as θ_1 is not a morphism of n-fold vector bundles. We will inductively construct a morphism which is linear over all sides.

First we do this locally: we choose a neighbourhood U of $m \in M$ that trivialises each of the $E_{\{i\}}$, for $i = 1,\ldots,n$. Fix smooth local frames $(b_i^1,\ldots,b_i^{l_i})$ of $E_{\{i\}}$ for $l_i = \operatorname{rk} E_{\{i\}}$. Every element of $\prod_{i\in\underline{n}}^M E_{\{i\}}$ over $m \in U$ can thus be written uniquely as

$$(e_1,\ldots,e_n) = \Big(\sum_{j=1}^{l_1} \beta_1^j b_1^j(m),\ldots,\sum_{j=1}^{l_n} \beta_n^j b_n^j(m)\Big)$$

where $\beta_i^j \in \mathbb{R}$. Assume now that we have a morphism $\Sigma_{k,U}^E\colon \overline{\mathbb{E}}|_U \to \mathbb{E}|_U$ which is linear over the splittings Σ_j for $j = 1,\ldots k$ and satisfies additionally (3.22) for all other j. We then define $\Sigma_{k+1,U}^E$ by

$$\Sigma_{k+1,U}^E(e_1,\ldots,e_n) := \sum_{j=1,\ldots,l_{k+1}}^{E\to E_{\underline{n}\setminus\{k+1\}}} \beta_{k+1}^j \cdot \Sigma_{k,U}^E(e_1,\ldots,e_k,b_{k+1}^j(m),e_{k+2},\ldots,e_n)\,,$$

where the scalar multiplication is also the one of the vector bundle $E \to E_{\underline{n}\setminus\{k+1\}}$. That this morphism is still a vector bundle morphism over Σ_j for all $j = 1,\ldots,k$ follows directly from the interchange laws in the double vector bundles $(E; E_{\underline{n}\setminus\{j\}}, E_{\underline{n}\setminus\{k+1\}}; E_{\underline{n}\setminus\{j,k+1\}})$. That it is also a vector bundle morphism over Σ_{k+1} is immediate. It furthermore still satisfies (3.22) for all other j. Starting with the restriction to U of Σ_1^E from (3.21) we get after $(n-1)$ iterations the top map of a local linear splitting Σ_U^E of $\mathbb{E}|_U$.

Now we will prove the existence of a global splitting using a partition of unity. This method was already given for double vector bundles in the original reference by Pradines [65]. Choose a locally finite cover of neighbourhoods as above, $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$, and a partition of unity $\{\varphi_\alpha\}_{\alpha\in A}$ subordinate to $\mathcal{U}$. Take then the local linear splittings $\Sigma_{U_\alpha}^E$ and define the global splitting for $(e_1,\ldots,e_n)$ over $m \in M$ by

$$\Sigma^E(e_1,\ldots,e_n) := \sum_{\{\alpha\colon m\in U_\alpha\}}^{E\to E_{\underline{n}\setminus\{1\}}} \varphi_\alpha(m) \underset{\underline{n}\setminus\{1\}}{\cdot} \Sigma_{U_\alpha}^E(e_1,\ldots,e_n)\,. \tag{3.23}$$

That this is a vector bundle morphism over all Σ_j follows again from the interchange laws in the double vector bundles $(E; E_{\underline{n}\setminus\{1\}}, E_{\underline{n}\setminus\{j\}}; E_{\underline{n}\setminus\{1,j\}})$. Injectivity follows directly from this as all Σ_k are injective. The linear splitting is then given by $\Sigma(\underline{n}) := \Sigma^E$ and $\Sigma(I) := \Sigma_{\underline{n}\setminus\{k\}}(I)$ whenever $I \subseteq \underline{n} \setminus \{k\}$. This completes the proof. □

Corollary 3.2.4. *Every n-fold vector bundle $\mathbb{E}$ is non-canonically isomorphic to the associated decomposed n-fold vector bundle defined in Section 3.1.2.*

Proof. This follows from Theorem 3.2.3 and Theorem 3.2.2. To apply Theorem 3.2.2 we have to show that we can construct compatible decompositions of all the highest order cores. This follows from a similar argument to the beginning of the proof of Theorem 3.2.3.

We have to consider all iterated highest order cores. These are firstly the $(n-1)$-fold vector bundles $\mathbb{E}^{\underline{n}}_I$ with $I \subseteq \underline{n}$ and $\#I = 2$, secondly the $(n-2)$-fold vector bundles $(\mathbb{E}^{\underline{n}}_I)^{\rho^{\underline{n}}_I}_{\nu}$ with $\nu \subseteq \rho^{\underline{n}}_I$ and $\#\nu = 2$ and so forth. Theorem 3.2.3 lets us choose linear splittings of all these multiple vector bundles. Note that the same multiple vector bundles can occur multiple times (see for example Remark 3.1.19 (c)). For these we still fix only one linear splitting. With Theorem 3.2.2 we obtain then firstly unique decompositions of all occurring double vector bundles. After fixing these, with Theorem 3.2.2 we obtain decompositions of all occurring triple vector bundles and these are all compatible by construction. Fixing these we obtain compatible decompositions of all occurring 4-fold vector bundles and so forth. Eventually after obtaining compatible decompositions of the highest order cores Theorem 3.2.2 gives us a decompositions of $\mathbb{E}$. □

Corollary 3.2.5. *For every n-fold vector bundle $\mathbb{E}$ and the associated n-pullback $\mathbb{P}$ there is an injective morphism of n-fold vector bundles $\Sigma^P \colon \mathbb{P} \to \mathbb{E}$ simultaneously splitting all the ultracore sequences from Proposition 3.1.21.*

Proof. We can choose a decomposition of $\mathbb{E}$ with top map $\mathcal{S}^E \colon \mathbb{E}^{\mathcal{A}}(\underline{n}) \to E$. This is a morphism over decompositions of the faces $E_{\underline{n}\setminus\{k\}}$ for all $k \in \underline{n}$. These decompositions induce a canonical associated decomposition of $\mathbb{P}$, the top map of which we denote by $\mathcal{S}^P \colon \prod^M_{I\subsetneq\underline{n}} E^I_I \to P$. Together with the canonical inclusion $\iota \colon \prod^M_{I\subsetneq\underline{n}} E^I_I \to \mathbb{E}^{\mathcal{A}}(\underline{n})$ we then define such a splitting with top map given by $\Sigma^P(\underline{n}) := \mathcal{S}^E \circ \iota \circ (\mathcal{S}^P)^{-1}$. □

3.2.3 n-fold vector bundle atlases

In this section we show how a change of splittings corresponds to statomorphisms of the decomposed multiple vector bundle, which were introduced in [27]. We then explain how n-fold vector bundles can alternatively be defined using smoothly compatible n-fold vector bundle charts.

For I a finite subset of $\mathbb{N}$, we denote by $\mathcal{P}(I) = \{\{I_1, \ldots, I_k\} \mid I = I_1 \sqcup \ldots \sqcup I_k\}$ the set of disjoint partitions of I. Since the elements of $\mathcal{P}(I)$ are sets, not tuples, we do not take the order into account. That is, we do not distinguish the partition $\{I_1, I_2\}$ from $\{I_2, I_1\}$.

Definition 3.2.6. *Let $\mathbb{E}$ be an n-fold vector bundle. A **statomorphism** of $\mathbb{E}$ is an isomorphism $\tau \colon \mathbb{E} \to \mathbb{E}$ that induces the identity on all building bundles E_I^I for $I \subseteq \underline{n}$. The set of statomorphisms of $\mathbb{E}$ forms a group with composition.*

Proposition 3.2.7. *Let $\mathbb{E}$ be an n-fold vector bundle and $\mathbb{E}^{\mathcal{A}}$ the corresponding decomposed n-fold vector bundle as in Definition 3.2.1. The set of global decompositions of $\mathbb{E}$ is a torsor over the group of statomorphisms of $\mathbb{E}^{\mathcal{A}}$.*

Proof. Given a decomposition $\mathcal{S} \colon \mathbb{E}^{\mathcal{A}} \to \mathbb{E}$ and a statomorphism $\tau \colon \mathbb{E}^{\mathcal{A}} \to \mathbb{E}^{\mathcal{A}}$ the composition $\mathcal{S} \circ \tau \colon \mathbb{E}^{\mathcal{A}} \to \mathbb{E}$ is again a decomposition of $\mathbb{E}$. This defines a right action of the group of statomorphisms of $\mathbb{E}^{\mathcal{A}}$ onto the set of decompositions of $\mathbb{E}$. Given two decompositions $\mathcal{S}_1, \mathcal{S}_2 \colon \mathbb{E}^{\mathcal{A}} \to \mathbb{E}$ the composition $\tau := \mathcal{S}_1^{-1} \circ \mathcal{S}_2 \colon \mathbb{E}^{\mathcal{A}} \to \mathbb{E}^{\mathcal{A}}$ defines a statomorphism of $\mathbb{E}^{\mathcal{A}}$ such that $\mathcal{S}_1 \circ \tau = \mathcal{S}_2$. This shows that the action is transitive. That it is free is immediate as $\mathcal{S} \circ \tau = \mathcal{S}$ clearly implies $\tau = \mathrm{id}$. □

The following description of statomorphisms can be found in slightly different notation in [27].

Proposition 3.2.8. *A statomorphism τ of $\mathbb{E}^{\mathcal{A}}$ is necessarily of the following form:*

$$\tau(\underline{n}) \colon (e_I)_{I \subseteq \underline{n}} \mapsto \left(\sum_{\rho = \{I_1, \ldots, I_k\} \in \mathcal{P}(I)} \varphi_\rho(e_{I_1}, \ldots, e_{I_k}) \right)_{I \subseteq \underline{n}}, \tag{3.24}$$

where $\varphi_\rho \in \Gamma(\mathrm{Hom}(E_{I_1}^{I_1} \otimes \ldots \otimes E_{I_k}^{I_k}, E_I^I))$ and for the trivial partition $\rho = \{I\}$ we additionally demand $\varphi_{\{I\}} = \mathrm{id}_{E_I^I}$.

Now we define n-fold vector bundle charts and atlases and show that our definition of n-fold vector bundles is equivalent to the definition in terms of charts.

Definition 3.2.9. *Let M be a smooth manifold and E a topological space together with a continuous map $\Pi \colon E \to M$. An **n-fold vector bundle chart** is a tuple*

$$c = (U, \Theta, (V_I)_{I \subseteq \underline{n}}), \tag{3.25}$$

where U is an open set in M, for each $I \subseteq \underline{n}$ the space V_I is a (finite dimensional) real vector space and $\Theta\colon \Pi^{-1}(U) \to U \times \prod_{I\subseteq \underline{n}} V_I$ is a homeomorphism such that $\Pi = \mathrm{pr}_1 \circ \Theta$.

*Two n-fold vector bundle charts $c = (U, \Theta, (V_I)_{I\subseteq\underline{n}})$ and $c' = (U', \Theta', (V'_I)_{I\subseteq\underline{n}})$ are **smoothly compatible** if $V_I = V'_I$ for all $I \subseteq \underline{n}$ and the "change of chart" $\Theta' \circ \Theta^{-1}$ over $U \cap U'$ has the following form:*

$$(p, (v_I)_{I\subseteq\underline{n}}) \mapsto \left(p, \left(\sum_{\rho=\{I_1,\ldots,I_k\}\in\mathcal{P}(I)} \omega_\rho(p)(v_{I_1}, \ldots, v_{I_k})\right)_{I\subseteq\underline{n}}\right) \tag{3.26}$$

with $p \in U \cap U'$, $v_I \in V_I$ and $\omega_\rho \in C^\infty(U \cap U', \mathrm{Hom}(V_{I_1} \otimes \ldots \otimes V_{I_k}, V_I))$ for $\rho = \{I_1, \ldots, I_k\} \in \mathcal{P}(I)$.

*A **smooth n-fold vector bundle atlas** $\mathfrak{A}$ on E is a set of n-fold vector bundle charts of E that are pairwise smoothly compatible and such that the set of underlying open sets in M covers M. As usual, E is then a smooth manifold and two smooth n-fold vector bundle atlases $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are **equivalent** if their union is a smooth n-fold vector bundle atlas. A smooth n**-fold vector bundle** structure on E is an equivalence class of smooth n-fold vector bundle atlases on E.*

Let $\mathbb{E}$ be an n-fold vector bundle. By Theorem 3.2.3 and Theorem 3.2.2 we have a decomposition $\mathcal{S}\colon \mathbb{E}^{\mathcal{A}} \to \mathbb{E}$ of $\mathbb{E}$, with $\mathcal{A}$ the family $(A_I)_{I\subseteq\underline{n}}$ of vector bundles over M defined by $A_{\{i\}} = \mathbb{E}(\{i\})$ for $i \in \underline{n}$ and $A_I = E_I^I$ for $I \subseteq \underline{n}$, $\#I \geq 2$. Set $\Pi = \mathbb{E}(\underline{n} \to \emptyset)\colon E \to M$. For each $I \subseteq \underline{n}$, set $V_I := \mathbb{R}^{\dim A_I}$, the vector space on which A_I is modelled. Take a covering $\{U_\alpha\}_{\alpha\in\Lambda}$ of M by open sets trivialising all the vector bundles A_I;

$$\phi_I^\alpha\colon q_I^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times V_I$$

for all $I \subseteq \underline{n}$ and all $\alpha \in \Lambda$. Then we define n-fold vector bundle charts $\Theta_\alpha\colon \Pi^{-1}(U_\alpha) \to U_\alpha \times \prod_{I\subseteq\underline{n}} V_I$ by

$$\Theta_\alpha = \left(\Pi \times (\phi_I^\alpha)_{I\subseteq\underline{n}}\right) \circ \mathcal{S}^{-1}|_{\Pi^{-1}(U_\alpha)}.$$

Given $\alpha, \beta \in \Lambda$ with $U_\alpha \cap U_\beta \neq \emptyset$, the change of chart

$$\Theta_\alpha \circ \Theta_\beta^{-1}\colon (U_\alpha \cap U_\beta) \times \prod_{I\subseteq\underline{n}} V_I \to (U_\alpha \cap U_\beta) \times \prod_{I\subseteq\underline{n}} V_I$$

is given by

$$(p, (v_I)_{I\subseteq\underline{n}}) \mapsto (p, (\rho_I^{\alpha\beta}(p)v_I)_{I\subseteq\underline{n}}), \tag{3.27}$$

with $\rho_I^{\alpha\beta} \in C^\infty(U_\alpha \cap U_\beta, \mathrm{Gl}(V_I))$ the cocycle defined by $\phi_I^\alpha \circ (\phi_I^\beta)^{-1}$. The two charts are hence smoothly compatible and we get an n-fold vector bundle atlas $\mathfrak{A} = \{(U_\alpha, \Theta_\alpha, (V_I)_{I\subseteq\underline{n}}) \mid \alpha \in \Lambda\}$ on E.

Conversely, given a space E with an n-fold vector bundle structure over a smooth manifold M as in Definition 3.2.9, we define $\mathbb{E}\colon \square^{\mathbb{N}} \to \mathbf{Man}^\infty$ as follows. Take a maximal atlas $\mathfrak{A} = \{(U_\alpha, \Theta_\alpha, (V_I)_{I\subseteq \underline{n}}) \mid \alpha \in \Lambda\}$ of E; in particular $\{U_\alpha\}_{\alpha\in\Lambda}$ is an open covering of M. For $\alpha, \beta, \gamma \in \Lambda$ we obtain from the identity $\Theta_\gamma \circ \Theta_\alpha^{-1} = \Theta_\gamma \circ \Theta_\beta^{-1} \circ \Theta_\beta \circ \Theta_\alpha^{-1}$ on $\Pi^{-1}(U_\alpha \cap U_\beta \cap U_\gamma)$ the following cocycle conditions. For $I \subseteq \underline{n}$ and $\rho = \{I_1, \ldots, I_k\} \in \mathcal{P}(I)$:

$$\begin{aligned} \omega_\rho^{\gamma\alpha}(p)(v_{I_1}, \ldots, v_{I_k}) &= \\ &\sum_{\{1,\ldots,k\}=J_1\sqcup\ldots\sqcup J_l} \omega^{\gamma\beta}_{\{I_{J_1},\ldots,I_{J_l}\}}(p)\Big(\omega^{\beta\alpha}_{\{I_j \mid j\in J_1\}}(p)\big((v_{I_j})_{j\in J_1}\big), \ldots \\ &\qquad\qquad \ldots, \omega^{\beta\alpha}_{\{I_j \mid j\in J_l\}}(p)\big((v_{I_j})_{j\in J_l}\big)\Big), \end{aligned} \tag{3.28}$$

where $I_{J_m} := \bigcup_{j\in J_m} I_j$.

We set $\mathbb{E}(\underline{n}) = E$, $\mathbb{E}(\emptyset) = M$, and more generally for $I \subseteq \underline{n}$,

$$\mathbb{E}(I) = \left(\bigsqcup_{\alpha\in\Lambda} \left(U_\alpha \times \prod_{J\subseteq I} V_J\right)\right) \Big/ \sim$$

with $\sim$ the equivalence relation defined on $\bigsqcup_{\alpha\in\Lambda}(U_\alpha \times \prod_{J\subseteq I} V_J)$ by

$$U_\alpha \times \prod_{J\subseteq I} V_J \quad \ni \quad (p, (v_J)_{J\subseteq I}) \quad \sim \quad (q, (w_J)_{J\subseteq I}) \quad \in \quad U_\beta \times \prod_{J\subseteq I} V_J$$

if and only if $p = q$ and

$$(v_J)_{J\subseteq I} = \left(\sum_{\rho=\{J_1,\ldots,J_k\}\in\mathcal{P}(J)} \omega_\rho(p)(w_{J_1}, \ldots, w_{J_k})\right)_{J\subseteq I}.$$

The relations (3.28) show the symmetry and transitivity of this relation. As in the construction of a vector bundle from vector bundle cocycles, one can show that $\mathbb{E}(I)$ has a unique smooth manifold structure such that $\Pi_I\colon \mathbb{E}(I) \to M$, $\Pi_I[p, (v_I)_{I\subseteq J}] = p$ is a surjective submersion and such that the maps

$$\Theta_\alpha^I\colon \pi_I\left(U_\alpha \times \prod_{J\subseteq I} V_J\right) \to U_\alpha \times \prod_{J\subseteq I} V_J, \qquad [p, (v_I)_{I\subseteq J}] \mapsto (p, (v_I)_{I\subseteq J})$$

are diffeomorphisms, where $\pi_I\colon \bigsqcup_{\alpha\in\Lambda}(U_\alpha \times \prod_{J\subseteq I} V_J) \to \mathbb{E}(I)$ is the projection to the equivalence classes.

We have then also $\#I$ surjective submersions

$$p^I_{I\setminus\{i\}}\colon \mathbb{E}(I) \to \mathbb{E}(I \setminus \{i\})$$

for $i \in I$, defined in charts by

$$U_\alpha \times \prod_{J \subseteq I} V_J \ni (p, (v_J)_{J \subseteq I}) \mapsto (p, (v_J)_{i \notin J \subseteq I}) \in U_\alpha \times \prod_{J \subseteq I \setminus \{i\}} V_J$$

and it is easy to see that $\mathbb{E}(I)$ is a vector bundle over $\mathbb{E}(I \setminus \{i\})$, and that for $i, j \in I$,

$$\begin{array}{ccc} \mathbb{E}(I) & \xrightarrow{p^I_{I\setminus\{i\}}} & \mathbb{E}(I \setminus \{i\}) \\ \downarrow{\scriptstyle p^I_{I\setminus\{j\}}} & & \downarrow{\scriptstyle p^{I\setminus\{i\}}_{I\setminus\{i,j\}}} \\ \mathbb{E}(I \setminus \{j\}) & \xrightarrow{p^{I\setminus\{j\}}_{I\setminus\{i,j\}}} & \mathbb{E}(I \setminus \{i,j\}) \end{array}$$

is a double vector bundle, with obvious local trivialisations given by the local charts.

The constructions above are inverse to each other and we get the following corollary of our local splitting theorem.

Corollary 3.2.10. *Definition 3.1.1 of an n-fold vector bundle as a functor from the n-cube category is equivalent to Definition 3.2.9 of an n-fold vector bundle as a space with a maximal n-fold vector bundle atlas.*

Our construction above of an n-fold vector bundle atlas on $\mathbb{E}(\underline{n})$ from an n-fold vector bundle yields an atlas with simpler changes of charts (3.27) than the most general allowed change of charts (3.26). This is due to our choice of a *global* decomposition of the n-fold vector bundle. Choosing different local or global decompositions will yield an atlas with changes of charts as in (3.26). That the equivalence class of atlases is independent of the choice of decomposition follows from Proposition 3.2.7 and (3.24). Two different decompositions will give compatible charts.

3.2.4 Decompositions of ∞-fold vector bundles

In this section we show how our proof of the existence of linear decompositions of n-fold vector bundles for all $n \in \mathbb{N}$ yields as well the existence of linear decompositions of ∞-fold vector bundles. We write here ∞**VB** for the category of ∞-fold vector bundles and ∞-fold vector bundle morphisms.

Let $\mathbb{E}$ be an ∞-fold vector bundle. Then for each $n \in \mathbb{N}$, the restriction $\mathbb{E} \circ \iota_n^{\mathbb{N}}$ defines an n-fold vector bundle, and $\mathbb{E}^n := \mathbb{E} \circ \iota_n^{\mathbb{N}} \circ \pi_n^{\mathbb{N}}$ defines again an ∞-fold vector bundle, given by $\mathbb{E}^n(I) = \mathbb{E}(I \cap \underline{n})$ for all finite $I \subseteq \mathbb{N}$. There is a sequence of monomorphisms of ∞-fold vector bundles

$$\mathbb{E}^0 \xrightarrow{\iota_0^1} \mathbb{E}^1 \xrightarrow{\iota_1^2} \mathbb{E}^2 \xrightarrow{\iota_2^3} \dots \tag{3.29}$$

defined by $\iota_k^l(I) = 0_{I\cap \underline{k}}^{I\cap \underline{l}}$ for $k \leq l \in \mathbb{N}$ and a finite subset I of $\mathbb{N}$; remember that $0_I^I = \mathrm{id}_{E_I}$. Thus we have a functor $\mathbb{E}^{\cdot}\colon \mathbb{N} \to \infty\mathbf{VB}$ sending an object $n \in \mathbb{N}$ to $\mathbb{E}^n$ and an arrow $m \leq n$ to ι_m^n. In the same manner, for each $n \in \mathbb{N}$ there is a monomorphism $\iota_n\colon \mathbb{E}^n \to \mathbb{E}$ defined by $\iota_n(I) = 0_{I\cap\underline{n}}^I\colon \mathbb{E}^n(I) \to \mathbb{E}(I)$ for all finite $I \subseteq \mathbb{N}$. It is easy to see that $\mathbb{E}$ together with the inclusions $\iota_n\colon \mathbb{E}^n \to \mathbb{E}$ defines a colimit for (3.29) in the category of ∞-fold vector bundles.

The inductive nature of the proof of Theorem 3.2.3 yields the following corollary.

Corollary 3.2.11. *Let $\mathbb{E}$ be an ∞-fold vector bundle. Define the family $\mathcal{A} = (q_I\colon A_I \to M)_{I\subseteq\mathbb{N},\#I<\infty}$ of vector bundles over M by setting $A_I = E_I^I$ for $2 \leq \#I < \infty$, $A_{\{k\}} = E_{\{k\}}$ and $A_\emptyset = \mathbb{E}(\emptyset) = M$. Then there exists a sequence of decompositions $\tilde{\mathcal{S}}^n\colon \mathbb{E}^{\mathcal{A}} \circ \iota_n^{\mathbb{N}} \to \mathbb{E} \circ \iota_n^{\mathbb{N}}$ such that the diagram of ∞-fold vector bundles*

$$\begin{array}{ccccccc} \mathbb{E}^0 & \longrightarrow & \mathbb{E}^1 & \longrightarrow & \mathbb{E}^2 & \longrightarrow & \dots \\ {\scriptstyle \mathcal{S}^0}\uparrow & & {\scriptstyle \mathcal{S}^1}\uparrow & & {\scriptstyle \mathcal{S}^2}\uparrow & & \\ (\mathbb{E}^{\mathcal{A}})^0 & \longrightarrow & (\mathbb{E}^{\mathcal{A}})^1 & \longrightarrow & (\mathbb{E}^{\mathcal{A}})^2 & \longrightarrow & \dots, \end{array} \tag{3.30}$$

commutes, where $\mathcal{S}^n(I) := \tilde{\mathcal{S}}^n(I \cap \underline{n})$ is the morphism of ∞-fold vector bundles induced by $\tilde{\mathcal{S}}^n$.

Since (3.30) commutes, and for each n, $\mathcal{S}^n$ is an isomorphism, we find that $\mathbb{E}^{\mathcal{A}}$ together with the morphisms $\tau(n) = \iota_n^{\mathcal{A}} \circ (\mathcal{S}^n)^{-1}$ for all n, is also a colimit for (3.29) in the category of ∞-fold vector bundles. Therefore there is a unique isomorphism $\mathcal{S}\colon \mathbb{E}^{\mathcal{A}} \to \mathbb{E}$ such that $\iota_n \circ \mathcal{S}^n = \mathcal{S} \circ \iota_n^{\mathcal{A}}$ for all $n \in \mathbb{N}$. We get the following theorem.

Theorem 3.2.12. *Let $\mathbb{E}$ be an ∞-fold vector bundle. Define the family $\mathcal{A} = (q_I\colon A_I \to M)_{I\subseteq\mathbb{N},\#I<\infty}$ of vector bundles over M by setting $A_I = E_I^I$ for $2 \leq \#I < \infty$, $A_{\{k\}} = E_{\{k\}}$ and $A_\emptyset = \mathbb{E}(\emptyset) = M$.*

Then $\mathbb{E}$ is non-canonically isomorphic to the associated decomposed ∞-fold vector bundle $\mathbb{E}^{\mathcal{A}}$. More precisely, given a tower of decompositions as in (3.30)*, the decomposition $\mathcal{S}\colon \mathbb{E}^{\mathcal{A}} \to \mathbb{E}$ of $\mathbb{E}$ can be uniquely chosen so that for each $n \in \mathbb{N}$, $\mathcal{S}^n\colon (\mathbb{E}^{\mathcal{A}})^n \to \mathbb{E}^n$ satisfies*

$$\mathcal{S}^n(I) = \mathcal{S}(I \cap \underline{n})\colon (\mathbb{E}^{\mathcal{A}})^n(I) = \mathbb{E}^{\mathcal{A}}(I \cap \underline{n}) \to \mathbb{E}^n(I) = \mathbb{E}(I \cap \underline{n}) \tag{3.31}$$

for all finite $I \subseteq \mathbb{N}$.

Proof. The morphism $\mathcal{S}\colon \mathbb{E}^{\mathcal{A}} \to \mathbb{E}$ is explicitly defined as follows. Choose a finite subset $I \subseteq \mathbb{N}$. Then there is $n \in \mathbb{N}$ with $I \subseteq \underline{n}$ and we can set $\mathcal{S}(I) = \mathcal{S}^n(I)$. The equalities (3.31) are now easy to check. □

3.3 Multiply linear sections

In this section we give the definition of multiply linear sections in a multiple vector bundle and show how they fit in a short exact sequence analogously to the double vector bundle case in Lemma 2.1.24. Let in this section $\mathbb{E}$ always be an n-fold vector bundle with total space $E = \mathbb{E}(\underline{n})$.

Definition 3.3.1. *An $(n-1)$**-fold linear section** from $E_{\underline{n}\setminus\{k\}}$ to E is a morphism of $(n-1)$-fold vector bundles $\tau\colon \mathbb{E}^{\underline{n}\setminus\{k\},\emptyset} \Rightarrow \mathbb{E}^{\underline{n},\{k\}}$ such that $\tau(\underline{n}\setminus\{k\})\colon E_{\underline{n}\setminus\{k\}} \to E$ is a section of the vector bundle $p_k\colon E \to E_{\underline{n}\setminus\{k\}}$. We denote the set of these $(n-1)$-fold linear sections by $\Gamma^{\ell^{n-1}}_{E_{\underline{n}\setminus\{k\}}}(E)$. This is a $C^\infty(M)$-module by $(f\cdot\tau)(I)(e_I) = f(m) \underset{I\setminus\{k\}}{\cdot} \tau(I)(e_I)$ for $f \in C^\infty(M)$, $\tau \in \Gamma^{\ell^{n-1}}_{E_{\underline{n}\setminus\{k\}}}(E)$, $I \subseteq \underline{n}\setminus\{k\}$, $e_I \in E_I$ and $m = p^I_\emptyset(e_I)$.*

Recall that we defined in the proof of Theorem 3.1.13 two $(n-1)$-fold vector bundles, $\mathbb{P}^{\mathbf{up}}_k$ and $\mathbb{P}^{\mathbf{low}}_k$ with total spaces given by (3.9) and (3.10) as follows

$$
\begin{aligned}
P^{\mathbf{up}}_k &= \Big\{(e_1,\ldots,\widehat{e_k},\ldots,e_n) \;\Big|\; e_i \in E_{\underline{n}\setminus\{i\}}\colon p^{\underline{n}\setminus\{i\}}_j(e_i) = p^{\underline{n}\setminus\{j\}}_i(e_j) \\
&\qquad\qquad \text{for } i,j \in \underline{n}\setminus\{k\}\Big\},\\
P^{\mathbf{low}}_k &= \Big\{(b_1,\ldots,\widehat{b_k},\ldots,b_n) \;\Big|\; b_i \in E_{\underline{n}\setminus\{k,i\}}\colon p^{\underline{n}\setminus\{k,i\}}_j(b_i) = p^{\underline{n}\setminus\{k,j\}}_i(b_j) \\
&\qquad\qquad \text{for } i,j \in \underline{n}\setminus\{k\}\Big\}.
\end{aligned}
$$

We showed that $P^{\mathbf{up}}_k$ is additionally a vector bundle over $P^{\mathbf{low}}_k$. It is easy to see that this in fact defines an n-fold vector bundle with total space $P^{\mathbf{up}}_k$. Thus we can consider the set of $(n-1)$-fold linear sections of $P^{\mathbf{up}}_k \to P^{\mathbf{low}}_k$, that is $\Gamma^{\ell^{n-1}}_{P^{\mathbf{low}}_k}(P^{\mathbf{up}}_k)$. These are precisely compatible tuples of $(n-2)$-fold linear sections $E_{\underline{n}\setminus\{j\}} \to E_{\underline{n}\setminus\{j,k\}}$ for $j \in \underline{n}\setminus\{k\}$.

Given now an $(n-1)$-fold linear section $\tau \in \Gamma^{\ell^{n-1}}_{E_{\underline{n}\setminus\{k\}}}(E)$, we define

$$\pi(\tau) \in \Gamma^{\ell^{n-1}}_{P^{\mathbf{low}}_k}(P^{\mathbf{up}}_k)\,, \tag{3.32}$$

by $\pi(\tau)(I)(e_I) := \tau(I)(e_I)$ for all $I \subsetneq \underline{n}\setminus\{k\}$ and

$$\pi(\tau)(\underline{n}\setminus\{k\})\big((e_{\underline{n}\setminus\{j,k\}})_{j\in\underline{n}\setminus\{k\}}\big) := \big(\tau(\underline{n}\setminus\{j,k\})(e_{\underline{n}\setminus\{j,k\}})\big)_{j\in\underline{n}\setminus\{k\}}\,. \tag{3.33}$$

This is the tuple of the underlying $(n-2)$-fold linear sections of τ of all the $E_{\underline{n}\setminus\{j\}} \to E_{\underline{n}\setminus\{j,k\}}$.

This defines a morphism of $C^\infty(M)$-modules $\pi\colon \Gamma^{\ell^{n-1}}_{E_{\underline{n}\setminus\{k\}}}(E) \to \Gamma^{\ell^{n-1}}_{P^{\mathbf{low}}_k}(P^{\mathbf{up}}_k)$.

Recall furthermore that the vector bundle $E^{\underline{n}}_{\underline{n}} \to M$ induces an $(n-1)$-fold vector bundle with indexing category $\lozenge^{\underline{n}\setminus\{k\},\emptyset}$, where the total space is given by

$E^{\underline{n}}_{\underline{n}}$ and in all other positions we have the base manifold M. We showed in Lemma 3.1.11 that the space of $(n-1)$-fold vector bundle morphisms from $\mathbb{E}^{\underline{n}\setminus\{k\},\emptyset}$ to this $(n-1)$-fold vector bundle is a $C^\infty(M)$-module, denoted by $\mathrm{Mor}_{n-1}(\mathbb{E}^{\underline{n}\setminus\{k\},\emptyset}, E^{\underline{n}}_{\underline{n}})$. We will furthermore again identify such an $(n-1)$-fold vector bundle morphism with its component on the total space, that is the smooth map $E_{\underline{n}\setminus\{k\}} \to E^{\underline{n}}_{\underline{n}}$.

With this we can define the short exact sequence of $C^\infty(M)$-modules. The case of $n = 2$ can be found in Lemma 2.1.24 and we will later demonstrate the case of $n = 3$ in Proposition 3.5.4 for the convenience of the reader in detail.

Proposition 3.3.2. *The map π defined above is surjective and fits into a short exact sequence of $C^\infty(M)$-modules*

$$0 \to \mathrm{Mor}_{n-1}(\mathbb{E}^{\underline{n}\setminus\{k\},\emptyset}, E^{\underline{n}}_{\underline{n}}) \xrightarrow{\widetilde{\cdot}} \Gamma^{\ell^{n-1}}_{E_{\underline{n}\setminus\{k\}}}(E) \xrightarrow{\pi} \Gamma^{\ell^{n-1}}_{P^{\mathbf{low}}_k}(P^{\mathbf{up}}_k) \to 0\,, \tag{3.34}$$

where the injective map is induced by the map $\iota\colon (\Pi^{\underline{n}\setminus\{k\}}_M)^! E^{\underline{n}}_{\underline{n}} \to E$ from Proposition 3.1.21, defined in (3.18)*. That is, $\widetilde{\varphi}(e_{\underline{n}\setminus\{k\}}) = \iota\big(e_{\underline{n}\setminus\{k\}}, \varphi(e_{\underline{n}\setminus\{k\}})\big)$.*

Proof. Injectivity of the map $\widetilde{\cdot}$ is clear from the injectivity of ι.

Surjectivity of π follows from the existence of linear splittings. Given a section $\xi \in \Gamma^{\ell^{n-1}}_{P^{\mathbf{low}}_k}(P^{\mathbf{up}}_k)$, we consider the pullback section $(\delta_k)^!\xi\colon E_{\underline{n}\setminus\{k\}} \to (\delta_k)^! P^{\mathbf{up}}_k = P$ which is itself a morphism of $(n-1)$-fold vector bundles. A simultaneous splitting $\Sigma^P\colon P \to E$ of all the ultracore sequences from Proposition 3.1.21 as in Corollary 3.2.5 gives then an $(n-1)$-fold linear section $\sigma_k(\xi) := \Sigma^P \circ (\delta_k)^!\xi$ of $E \to E_{\underline{n}\setminus\{k\}}$ with $\pi(\sigma_k(\xi)) = \xi$.

Given any $\varphi \in \mathrm{Mor}_{n-1}(\mathbb{E}^{\underline{n}\setminus\{k\},\emptyset}, E^{\underline{n}}_{\underline{n}})$ and $e_{\underline{n}\setminus\{k\}} \in E_{\underline{n}\setminus\{k\}}$ we find for any $j \in \underline{n} \setminus \{k\}$ that $p_j\big(\widetilde{\varphi}(e_{\underline{n}\setminus\{k\}})\big) = \mathbf{0}^{\underline{n}\setminus\{j\}}_{e_{\underline{n}\setminus\{j,k\}}}$. Thus $\widetilde{\varphi}$ is linear over the zero sections of all $E_{\underline{n}\setminus\{j\}} \to E_{\underline{n}\setminus\{j,k\}}$ and is in the kernel of π. Conversely, given an $(n-1)$-fold linear section ξ over the zero sections of all $E_{\underline{n}\setminus\{j\}} \to E_{\underline{n}\setminus\{j,k\}}$ we obtain a morphism φ by subtracting the zero elements in the reverse order. In other words,

$$\varphi(e_{\underline{n}\setminus\{k\}}) := \Big(\big(\xi(e_{\underline{n}\setminus\{k\}}) \underset{\underline{n}\setminus\{i_1\}}{-} \mathbf{0}^E_{e_{\underline{n}\setminus\{k\}}}\big) \underset{\underline{n}\setminus\{i_2\}}{-} \dots\Big) \underset{\underline{n}\setminus\{i_{n-1}\}}{-} \mathbf{0}^E_{e_{\underline{n}\setminus\{i_{n-2}\}}}$$

defines a morphism $\varphi \in \mathrm{Mor}_{n-1}(\mathbb{E}^{\underline{n}\setminus\{k\},\emptyset}, E^{\underline{n}}_{\underline{n}})$ such that $\xi = \widetilde{\varphi}$ and we obtain that the sequence is indeed exact. □

3.4 Symmetric n-fold vector bundles

In this section we define symmetric structures on n-fold vector bundles. This is the n-fold analogue of an involutive structure on double vector bundles. Symmetric

n-fold vector bundles will be important in the geometrisation of graded manifolds of degree n, in an analogue way of the geometrisation of graded manifolds of degree 2 by involutive double vector bundles in [37]. This is an ongoing joint project with Madeleine Jotz Lean. The goal is an equivalence of categories between graded manifolds of degree n and the category of symmetric n-fold vector bundles.

For the definition of these symmetric n-fold vector bundles we first need the appropriate flips of n-fold vector bundles. In the following, S_n is always the symmetric group of degree n. The n-cube category $\square^n$ is equipped with a canonical left action Φ of S_n by isomorphisms of categories in the following way. The permutation σ acts by $\Phi_\sigma\colon \square^n \to \square^n$ which maps an object $I \subseteq \underline{n}$ to the object $\sigma(I) \subseteq \underline{n}$ and morphisms in the obvious way. This action induces a right action of S_n on the category of n-fold vector bundles as follows.

Definition 3.4.1. *For $\sigma \in S_n$ we define the **σ-flip** of an n-fold vector bundle $\mathbb{E}$ to be the n-fold vector bundle $\mathbb{E}^\sigma := \mathbb{E} \circ \Phi_\sigma\colon \square^n \to \mathbf{Man}^\infty$. Since Φ is a left action of S_n on $\square^n$, we obtain that $(\mathbb{E}^\sigma)^\tau = \mathbb{E}^{\sigma\tau}$. Given a morphism of n-fold vector bundles $\psi\colon \mathbb{E} \to \mathbb{F}$, there is an obvious morphism $\psi^\sigma\colon \mathbb{E}^\sigma \to \mathbb{F}^\sigma$.*

Note that $\mathbb{E}^\sigma$ has the same underlying spaces as $\mathbb{E}$ but in different positions, which is crucial when considering morphisms of multiple vector bundles. For example the double vector bundle $(D; A, B; M)$ is different from the double vector bundle $(D; B, A; M)$.

We want the symmetric structure to carry signs on the different cores analogously to how an involutive structure on a double vector bundle induces $-\operatorname{id}$ on the core. We therefore need the following definition of the induced signs.

Definition 3.4.2. *Let $\sigma \in S_n$ and $I \subseteq \underline{n}$. Then we define the **$(\sigma, I)$-sign** by*

$$\varepsilon(\sigma, I) := (-1)^{\#\{(i,j)\in I\times I \mid \ i<j \ and \ \sigma(i)>\sigma(j)\}}\,. \tag{3.35}$$

Equivalently, it is the sign of the permutation in $S_{\#i}$ on $\{1, \ldots, \#I\}$ defined by $f^{\sigma(I)} \circ \sigma|_I \circ (f^I)^{-1}$, where $f^J\colon J \to \{1, \ldots, \#J\}$ for $J \subseteq \underline{n}$ is the unique order-preserving bijection.

Here $<$ and $>$ denote the natural order of the natural numbers. The signs above therefore measure the parity of the number of necessary transpositions to bring the set $\sigma(I)$ from the order induced by the natural order on I to the order coming from the natural order on $\sigma(I) \subseteq \underline{n}$. For example for $n \geq 4$ we have $\varepsilon\big((13), \{1, 2\}\big) = -1$ and $\varepsilon\big((13), \{1, 4\}\big) = +1$. Now we can define a symmetric structure on an n-fold vector bundle.

Definition 3.4.3. *Let $\mathbb{E}\colon \square^n \to \mathbf{Man}^\infty$ be an n-fold vector bundle such that its building bundles satisfy $E_I^I = E_J^J$ for all $I, J \subseteq \underline{n}$ with $\#I = \#J$. A* ***symmetric structure*** *on $\mathbb{E}$ is a right S_n-action $\Psi\colon \mathbb{E}(\underline{n}) \times S_n \to \mathbb{E}(\underline{n})$ on the total space $\mathbb{E}(\underline{n})$ such that*

1. *for any $\sigma \in S_n$, the action induces a morphism of n-fold vector bundles $\Psi_\sigma\colon \mathbb{E} \to \mathbb{E}^\sigma$ and*
2. $(\Psi_\sigma)_I^I = \varepsilon(\sigma, I) \cdot \mathrm{id}_{E_I^I}\colon E_I^I \to E_{\sigma(I)}^{\sigma(I)} = E_I^I$.

We call an n-fold vector bundle together with a symmetric structure a ***symmetric n-fold vector bundle****.*

Note that the definition implies $\Psi_\sigma^\tau \circ \Psi_\tau = \Psi_{\tau\sigma}\colon \mathbb{E} \to \mathbb{E}^\tau \to \mathbb{E}^{\tau\sigma}$ for all $\sigma, \tau \in S_n$. Note that (a) and (b) also already imply that all Ψ_σ are isomorphisms and thus induce isomorphisms on the building bundles. We only need to impose the condition that $E_I^I = E_J^J$ in order to be able to specify the signs of these induced morphisms on the building bundles.

It is clear that in the case of $n = 2$ this is the same as the definition of an involutive double vector bundle given in [37].

To define the category of symmetric n-fold vector bundles we also have to define morphisms between them.

Definition 3.4.4. *Let $(\mathbb{E}, \Psi^{\mathbb{E}})$ and $(\mathbb{F}, \Psi^{\mathbb{F}})$ be symmetric n-fold vector bundles. Then a* ***morphism of symmetric n-fold vector bundles*** *from $\mathbb{E}$ to $\mathbb{F}$ is a morphism of n-fold vector bundles $\Psi\colon \mathbb{E} \to \mathbb{F}$ such that additionally for any $\sigma \in S_n$ the following diagram commutes*

$$\begin{array}{ccc} \mathbb{E} & \xrightarrow{\Psi_\sigma^{\mathbb{E}}} & \mathbb{E}^\sigma \\ {\scriptstyle\Psi}\downarrow & & \downarrow{\scriptstyle\Psi^\sigma} \\ \mathbb{F} & \xrightarrow{\Psi_\sigma^{\mathbb{F}}} & \mathbb{F}^\sigma \end{array} , \tag{3.36}$$

or in other words $\Psi^\sigma \circ \Psi_\sigma^{\mathbb{E}} = \Psi \circ \Psi_\sigma^{\mathbb{F}}$.

This then defines the category **SnVB** of symmetric n-fold vector bundles. For $n = 2$ this is the category of involutive double vector bundles as for example defined in [37].

There is a forgetful functor from **SnVB** to **nVB**, given by forgetting the symmetric structure. Clearly, this functor is neither full since not every morphism of n-fold vector bundles between two symmetric n-fold vector bundles respects the symmetric structure, nor injective on objects as the same n-fold vector bundle can carry different symmetric structures. For instance, given a symmetric structure Ψ on $\mathbb{E}$ and any statomorphism $\Phi\colon \mathbb{E} \to \mathbb{E}$, then there is an induced different symmetric structure defined by

$$\Psi_\sigma^\Phi := (\Phi^\sigma)^{-1} \circ \Psi \circ \Phi\colon \mathbb{E} \to \mathbb{E}^\sigma \,. \tag{3.37}$$

Thus any n-fold vector bundle that admits one symmetric structure always admits other symmetric structures, defining different symmetric n-fold vector bundles.

The following example is the standard example for such a symmetric n-fold vector bundle.

Example 3.4.5. *Let $A_1, \dots, A_n$ be vector bundles over M. Denote by $\mathcal{A} = \{A_I\}_{I \subseteq \underline{n}}$ the family of vector bundles given by $A_I = A_{\#I}$ and let $\mathbb{E}^{\mathcal{A}}$ be the corresponding decomposed n-fold vector bundle defined in Section 3.1.2. Then there is a canonical symmetric structure on $\mathbb{E}^{\mathcal{A}}$, defined by*

$$\begin{aligned} \Psi_\sigma(\underline{n})\colon \mathbb{E}^{\mathcal{A}}(\underline{n}) &\to (\mathbb{E}^{\mathcal{A}})^\sigma(\underline{n}) \\ (a_I)_{I \subseteq \underline{n}} &\mapsto (\varepsilon(\sigma, I) a_I) \,. \end{aligned} \tag{3.38}$$

*This is called the **standard symmetric decomposed n-fold vector bundle**.*

The strategy for the geometrisation of graded manifolds of degree n is the following. First, we need to show that any symmetric n-fold vector bundle is isomorphic to an associated standard symmetric decomposed n-fold vector bundle. We need to show that we can always find symmetric decompositions and thus obtain an analogue to Corollary 3.2.4 in the symmetric case. Then we can define symmetric n-fold vector bundle atlases and obtain an equivalence between symmetric n-fold vector bundles as in Definition 3.4.3 and a definition in terms of symmetric atlases. With these atlases we can then construct a sheaf of graded commutative $C^\infty(M)$-algebras similarly to the construction of Madeleine Jotz Lean in the double vector bundle case in [37]. This sheaf defines an $\mathbb{N}$-graded manifold of degree n over M as in Definition 2.6.1. Conversely we can construct a symmetric n-fold vector bundle atlas from the sheaf of functions of such a graded manifold. We then have to show that these functors define an equivalence of categories between graded manifolds of degree n and symmetric n-fold vector bundles. Then it will also be clear why we used the signs $\varepsilon(\sigma, I)$ in Definition 3.4.3 of symmetric n-fold vector bundles.

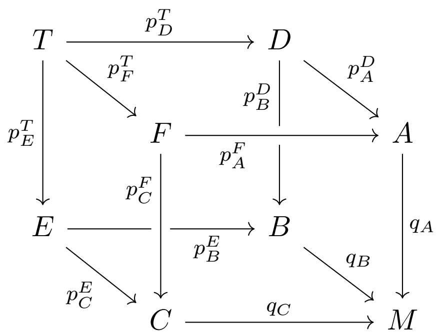
There are still some complicated technical problems and details to check in this entire construction, already in the case of $n = 3$. Solving these problems and checking all details in the general case is an ongoing joint project together with Madeleine Jotz Lean.

3.5 Example: triple vector bundles

In this section, we explain for the convenience of the reader how our results and considerations in Sections 3.1, 3.2 and 3.3 read in the case of $n = 3$. In this special case we will also show how one may obtain a decomposition from a splitting of the short exact sequence of $C^\infty(M)$-modules defined in Section 3.3, together with splittings of the side and core double vector bundles.

3.5.1 Splittings of triple vector bundles

Given a triple vector bundle $\mathbb{E}$ we will write in the following $T := \mathbb{E}(\{1,2,3\})$, $D := \mathbb{E}(\{1,2\})$, $E := \mathbb{E}(\{2,3\})$, $F := \mathbb{E}(\{1,3\})$, $A := E_{\{1\}}$, $B := E_{\{2\}}$ and $C := E_{\{3\}}$. The triple vector bundle is then a cube of vector bundle structures



, (3.39)

where all faces are double vector bundles.

We will denote the cores of the double vector bundles $(T; D, E; B)$, $(T; E, F; C)$, $(T; F, D; A)$ by L_{DE}, L_{EF} and L_{FD} and the cores of the double vector bundles $(D; A, B; M)$, $(E; B, C; M)$, $(F; C, A; M)$ by K_{AB}, K_{BC} and K_{CA}, respectively. In the general notation we would write $E^{\{1,2,3\}}_{\{2,3\}} =: L_{FD}$, $E^{\{1,2,3\}}_{\{1,3\}} =: L_{DE}$ and $E^{\{1,2,3\}}_{\{1,2\}} =: L_{EF}$ for the upper cores and $E^{\{1,2\}}_{\{1,2\}} =: K_{AB}$, $E^{\{2,3\}}_{\{2,3\}} =: K_{BC}$ and $E^{\{1,3\}}_{\{1,3\}} =: K_{CA}$ for the lower cores. The **triple core** of this triple vector bundle is $S := E^{\{1,2,3\}}_{\{1,2,3\}}$, a vector bundle over M.

The upper cores L_{DE}, L_{EF} and L_{FD} are themselves double vector bundles by Theorem 3.1.17. All three have by Lemma 3.1.16 the core S, whereas the sides of L_{DE} are given by K_{CA} and B, the sides of L_{EF} by K_{AB} and C, and the sides of L_{FD} by K_{BC} and A.

A decomposition of a triple vector bundle $(T; D, E, F; A, B, C; M)$ as above is now an isomorphism of triple vector bundles $\mathcal{S}$ from the associated decomposed triple vector bundle as in Example 3.1.9 to T over decompositions of D, E and F as double vector bundles and inducing the identity on S. In particular it is over the identities on A, B and C, and is inducing the identities on K_{AB}, K_{BC} and K_{CA}.

A linear splitting of a triple vector bundle $(T; D, E, F; A, B, C; M)$ as above is an injective morphism of triple vector bundles Σ from the vacant triple vector bundle $(A\times_M B\times_M C; A\times_M B, B\times_M C, C\times_M A; A, B, C; M)$ over linear splittings of the double vector bundles D, E and F, hence over the identities on A, B and C.

We have proved the following lemma, which is the case $n = 3$ of Theorem 3.2.2.

Lemma 3.5.1. *A decomposition of a triple vector bundle T is equivalent to a linear splitting of T and linear splittings of the three core double vector bundles L_{DE}, L_{EF} and L_{FD}.*

Note that here, starting from the splittings we get an explicit formula for the decomposition: $\mathcal{S}(a, b, c, k_{AB}, k_{BC}, k_{CA}, s)$ equals

$$\Big((\Sigma(a, b, c) +_D (0^T_{\Sigma^D(a,b)} +_F \Sigma^{L_{FD}}(a, k_{BC}))) +_F (0^T_{\Sigma^F(a,c)} +_E \Sigma^{L_{EF}}(c, k_{AB}))\Big)$$
$$+_E \Big(0^T_{\mathcal{S}^E(b,c,k_{BC})} +_D \Sigma^{L_{DE}}(b, k_{CA}) +_D (0^T_{0^D_b} +_F \overline{s})\Big)\,. \tag{3.40}$$

Now let us consider the pullback triple vector bundle associated with a triple vector bundle. Given double vector bundles $(D; A, B; M)$, $(E; B, C; M)$ and $(F; C, A; M)$, we consider the set

$$P = \{(d, e, f) \in D \times E \times F \mid p^D_A(d) = p^F_A(f),\ p^D_B(d) = p^E_B(e),\ p^E_C(e) = p^F_C(f)\}\,. \tag{3.41}$$

Then P is a triple vector bundle, with the obvious projections to D, E and F and the additions defined as follows. The space $E \times_C F$ has a vector bundle structure

$$E \times_C F \to B \times_M A, \qquad (e, f) \mapsto (p^E_B(e), p^F_A(f))\,,$$

with addition $(e_1, f_1) + (e_2, f_2) = (e_1 +_B e_2, f_1 +_A f_2)$. Since D is a double vector bundle and so non-canonically split, we have the surjective submersion $\delta^D\colon D \to B \times_M A$, given by $\delta^D(d) := (p^D_B(d), p^D_A(d))$. We define the vector bundle $P \to D$ as the pullback vector bundle structure $(\delta^D)^!(E \times_C F) \to D$. We call P the **pullback triple vector bundle defined by D, E and F** because it fills a cube in a similar manner as the pullback in category theory fills a square.

We have three short exact sequences of vector bundles over D, E and F, respectively; the one over D reads

$$0 \longrightarrow (\pi^D_M)^! S \longrightarrow T \xrightarrow{(\delta^D)^!(p^T_E, p^T_F)} P \longrightarrow 0\,, \tag{3.42}$$

where $\pi^D_M = q_A \circ p^D_A = q_B \circ p^D_B$. We are now able to state Theorem 3.2.3 in the case $n = 3$.

Theorem 3.5.2. *Every triple vector bundle is non-canonically isomorphic to a decomposed triple vector bundle.*

3.5.2 Splittings, decompositions and horizontal lifts

Let us recall here first that a decomposition of a double vector bundle is equivalent to a splitting of the short exact sequences given by its linear sections. As we have seen in Section 2.1, a splitting $\Sigma\colon A\times_M B\to D$ of D is equivalent to a homomorphism of $C^\infty(M)$-modules $\sigma_B\colon \Gamma(B)\to\Gamma^\ell_A(D)$ (a *horizontal lift*) which splits this short exact sequence. The correspondence is given by $\sigma_B(b)(a_m)=\Sigma(a_m,b(m))$ for all $b\in\Gamma(B)$ and $a_m\in A$. By symmetry of Σ a horizontal lift σ_B is therefore also equivalent to a horizontal lift $\sigma_A\colon\Gamma(A)\to\Gamma^\ell_B(D)$, splitting the sequence

$$0\to\Gamma(\operatorname{Hom}(B,C))\xrightarrow{\tilde{\cdot}}\Gamma^\ell_B(D)\xrightarrow{\pi}\Gamma(A)\to 0\,. \tag{3.43}$$

In this section, we explain how a splitting of the triple vector bundle T is equivalent to a "horizontal lift" of pairs of linear sections in $\Gamma^\ell_A(F)\times_{\Gamma(C)}\Gamma^\ell_B(E)$ to *doubly linear sections* of $T\to D$. Of course, similar results hold for doubly linear sections of $T\to E$ as lifts of elements of $\Gamma^\ell_C(F)\times_{\Gamma(A)}\Gamma^\ell_B(D)$, etc.

Definition 3.5.3. *A* ***doubly linear section*** *of T over D is a section which is a double vector bundle morphism from $(D;A,B;M)$ to $(T;F,E;C)$ over some morphisms $\xi\colon A\to F$, $\eta\colon B\to E$, $c\colon M\to C$. The morphisms ξ and η are then themselves linear sections of the double vector bundles E and F over the same section of C. We denote the set of doubly linear sections of T over D by $\Gamma^{\ell^2}_D(T)$.*

The space $\Gamma^{\ell^2}_D(T)$ is naturally a $C^\infty(M)$-module: for $f\in C^\infty(M)$ and $\xi\in\Gamma^{\ell^2}_D(T)$ doubly linear over $\xi_A\in\Gamma^\ell_A(F)$ and $\xi_B\in\Gamma^\ell_B(E)$, the section $(q_A\circ p^D_A)^*f\cdot\xi$ is doubly linear over $q_A^*f\cdot\xi_A$ and $q_B^*f\cdot\xi_B$.

Consider the double vector bundle S with sides M and core S:

$$\begin{array}{ccc} S & \xrightarrow{q_S} & M\\ {\scriptstyle q_S}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_M}\\ M & \xrightarrow{\mathrm{id}_M} & M\end{array}$$

As we have seen in Lemma 3.1.11, the space $\operatorname{Mor}_2(D,S)$ of double vector bundle morphisms $D\to S$ is a $C^\infty(M)$-module. It is easy to see that given a decomposition $A\times_M B\times_M K_{AB}\to D$, we get $\operatorname{Mor}_2(D,S)\simeq\Gamma(K^*_{AB}\otimes S)\oplus\Gamma(A^*\otimes B^*\otimes S)$. We have an obvious inclusion

$$\tilde{\cdot}\colon\ \operatorname{Mor}_2(D,S)\hookrightarrow\Gamma^{\ell^2}_D(T),$$

the images of which are exactly the doubly linear sections that project to the zero sections of $E \to A$ and $F \to B$, and so to the zero section of C.

Both $\Gamma^\ell_A(F)$ and $\Gamma^\ell_B(E)$ project onto $\Gamma(C)$, thus we can build the pullback $\Gamma^\ell_A(F) \times_{\Gamma(C)} \Gamma^\ell_B(E)$ which consists of pairs of linear sections of the respective bundles which are linear over the same section of C. Now $\Gamma^{\ell^2}_D(T)$ fits into a short exact sequence of $C^\infty(M)$-modules as in the following proposition.

Proposition 3.5.4. *Let T be a triple bundle as in* (3.39)*. We have a short exact sequence of $C^\infty(M)$-modules*

$$0 \to \mathrm{Mor}_2(D, S) \xrightarrow{\widetilde{\cdot}} \Gamma^{\ell^2}_D(T) \xrightarrow{\pi} \Gamma^\ell_A(F) \times_{\Gamma(C)} \Gamma^\ell_B(E) \to 0\,. \tag{3.44}$$

Proof. Injectivity of $\widetilde{\cdot}$ is immediate. To show surjectivity of π, choose a linear splitting $\Sigma^{E,F}$ of the double vector bundle $(T; E, F; C)$. Given $\xi = (\xi^F, \xi^E) \in \Gamma^\ell_A(F) \times_{\Gamma(C)} \Gamma^\ell_B(E)$ we can then define $\hat{\xi} \in \Gamma^{\ell^2}_D(T)$ by

$$\hat{\xi}(d) := \Sigma^{E,F}(\xi^E(p^D_B(d)), \xi^F(p^D_A(d))\,.$$

It is easy to see that this is in fact a doubly linear section. Note that the map $\hat{\cdot}$ does not define a splitting of the short exact sequence, as it is not linear over D.

Given any $\phi \in \mathrm{Mor}_2(D, S)$ and $d \in D$ over $a \in A$ and $b \in B$ it is clear that $p^T_E(\tilde{\phi}(d)) = \mathbf{0}^E_b$ and $p^T_F(\tilde{\phi}(d)) = \mathbf{0}^F_a$. Thus $\tilde{\phi}$ is linear over the zero sections of $E \to B$ and $F \to A$ and thus in the kernel of π. Conversely, given $\xi \in \Gamma^{\ell^2}_D(T)$ over the zero sections of $E \to B$ and $F \to A$, we get for any $d \in D$ over $a \in A$ and $b \in B$ that $(\xi(d) -_E \mathbf{0}^T_d) -_F \mathbf{0}^T_{\mathbf{0}^D_b}$ projects to zero in all directions and thus defines an element $\phi(d)$ of the triple core S. It is easy to check that this assignment defines a morphism $\phi \in \mathrm{Mor}_2(D, S)$. Then $\xi = \tilde{\phi}$ and the sequence is exact. □

Proposition 3.5.5. *A decomposition of a triple vector bundle T as in* (3.39) *is equivalent to linear splittings of the double vector bundles D, E, F, L_{DE} and L_{FD} and a* ***horizontal lift****, that is a splitting $\sigma\colon \Gamma^\ell_A(F) \times_{\Gamma(C)} \Gamma^\ell_B(E) \to \Gamma^{\ell^2}_D(T)$ of the short exact sequence* (3.44) *that is compatible with the splittings of the double vector bundles in the sense that for all $d \in D$ we have $\sigma(\widetilde{\phi^F}, 0^E_B)(d) = \mathbf{0}^T_d +_E \Sigma^{L_{DE}}(p^D_B(d), \phi^F(p^D_A(d)))$ for all $\phi^F \in \Gamma(\mathrm{Hom}(A, K_{CA}))$ and $\sigma(0^F_A, \widetilde{\phi^E})(d) = \mathbf{0}^T_d +_F \Sigma^{L_{FD}}(p^D_A(d), \phi^E(p^D_B(d)))$ for all $\phi^E \in \Gamma(\mathrm{Hom}(B, K_{BC}))$.*

Proof. A given decomposition $\mathcal{S}$ of T induces decompositions of all the double vector bundles by definition. These are equivalent to linear splittings and horizontal lifts $\sigma^E_C\colon \Gamma(C) \to \Gamma^\ell_B(E)$ and $\sigma^F_C\colon \Gamma(C) \to \Gamma^\ell_A(F)$. Now any two linear sections $\xi^E \in \Gamma^\ell_B(E)$ and $\xi^F \in \Gamma^\ell_A(F)$ over the same $c \in \Gamma(C)$ can be written as $\xi^E = \sigma^E_C(c) + \widetilde{\phi^E}$

and $\xi^F = \sigma_C^F(c) + \widetilde{\phi^F}$ for some $\phi^E \in \Gamma(B^* \otimes K_{BC})$ and $\phi^F \in \Gamma(A^* \otimes K_{AC})$. We define a horizontal lift by

$$\sigma(\xi^E, \xi^F)(\mathcal{S}^D(a_m, b_m, k_m)) := \mathcal{S}(a_m, b_m, c(m), k_m, \phi^F(a_m), \phi^E(b_m), 0_m^S)\,.$$

It is easy to check that this lift satisfies the additional compatibility conditions.

Conversely, given linear splittings of the double vector bundles D, E, F, L_{DE}, L_{FD} and a horizontal lift σ satisfying the extra condition, we first define a linear splitting $\Sigma^{L_{EF}}\colon C \times_M K_{AB} \to L_{EF}$ by $\Sigma^{L_{EF}}(c_m, k_{AB}) := \sigma(\sigma_C^F(c), \sigma_C^E(c))(k_{AB})$ for any section c of $C \to M$ with $c(m) = c_m$, and where we view K_{AB} as a subset of D. Then we define a linear splitting of T by

$$\Sigma(a_m, b_m, c_m) := \sigma(\sigma_C^E(c), \sigma_C^F(c))(\Sigma^D(a_m, b_m))\,,$$

where $c \in \Gamma(C)$ is any section such that $c(m) = c_m$. Together with Lemma 3.5.1 this gives a decomposition of T.

Straightforward computations show that these two constructions are indeed inverse to each other and we get the desired equivalence. □

Chapter 4

Linear generalised complex structures

In this chapter, we study linear generalised complex structures on vector bundles and in particular on Lie algebroids. Generalised complex Lie algebroids have already been studied by Madeleine Jotz Lean, Mathieu Stiénon and Ping Xu in [41], where they are called "Glanon algebroids". In [41] the authors furthermore gave a correspondence between generalised complex structures on Lie groupoids and generalised complex structures on Lie algebroids, in both cases being compatible with the respective additional structure.

In Section 4.1 we describe linear generalised complex structures on E in terms of the side morphism $j\colon TM\oplus E^* \to TM\oplus E^*$ and a linear splitting of $TE\oplus T^*E$. We prove the existence of an adapted Dorfman connection, such that the corresponding lift satisfies $\sigma(j\nu) = \mathcal{J}\sigma(\nu)$ for any $\nu \in \Gamma(TM \oplus E^*)$. To describe the integrability condition we define a non-anchored bracket $\mathbb{A}$ on $TM \oplus E^*$, which is independent of the choice of adapted Dorfman connection Δ. We also show that in the case of a linear generalised Kähler structure on E, we can find a Dorfman connection which is simultaneously adapted to both the linear generalised complex structures.

Then, in Section 4.2, we show that the linearity of the generalised complex structure is equivalent to the linearity of the corresponding Dirac structures and a linear generalised complex structure is therefore equivalent to a pair of complex conjugated, transversal complex VB-Dirac structures in $T_{\mathbb{C}}E\oplus T^*_{\mathbb{C}}E$. This correspondence has already been stated in [41] for Lie algebroids.

In Section 4.3 and Section 4.4 we consider a Lie algebroid $A \to M$ and describe the conditions on j and Δ, under which the generalised complex structure $\mathcal{J}$ is compatible with the Lie algebroid structure as defined in [41]. We show that this can be equivalently stated in terms of complex conjugated, transversal complex LA-Dirac structures in $T_{\mathbb{C}}A \oplus T^*_{\mathbb{C}}A$, using the description of LA-Dirac structures in a linear splitting given by Jotz Lean in [38]. We also recover the conditions for a morphism of 2-term representations up to homotopy.

In Section 4.5 we consider the degenerate Courant algebroid $A \oplus T^*M$, given in [38], and show that the core morphism of a generalised complex structure on a Lie algebroid defines a degenerate generalised complex structure in $A \oplus T^*M$.

Subsequently, we show in Section 4.6 that the $\pm i$-eigenbundles $U_\pm$ of the side morphism j of a linear generalised complex structure give rise to a complex A-Manin pair, and how this defines Courant algebroids $C_\pm$ with Dirac structure $U\pm$.

We show in Section 4.7 that $(U_\pm, K_\mp)$ is a Lie bialgebroid, where $K_\pm$ is the $\pm i$-eigenbundle of the core morphism of the generalised complex structure. Furthermore, we show that the Drinfeld double of this Lie bialgebroid $(U_\pm, K_\mp)$ is isomorphic to the Courant algebroids $C_\pm$ of the previous section.

Then, in Section 4.8 we look at the two extremal cases of generalised complex structures. In the case of a holomorphic Lie algebroid we find that the two Courant algebroids C_+ and C_- are given as matched sum Courant algebroids $C_T^{1,0} \oplus C_A^{0,1}$ and $C_T^{0,1} \oplus C_A^{1,0}$, respectively. In the case of a symplectic Lie algebroid we only find that the complex Lie algebroids $U_\pm$ are isomorphic to the complexified tangent bundle $T_{\mathbb{C}}M$ and thus also to $A^*_{\mathbb{C}}$. The dual $K_\mp$ is therefore isomorphic to $T^*_{\mathbb{C}}M$ and to $A_{\mathbb{C}}$.

Finally, in Section 4.9 we translate our results to general VB-Courant algebroids. We obtain adapted Lagrangian splittings and define generalised complex structures in a split Lie 2-algebroid.

In Section A of the appendix we work with a fixed Dorfman connection instead of choosing an adapted one. This might be useful when studying generalised complex vector bundles in the presence of another geometric structure on $TE \oplus T^*E$. We will describe how a linear generalised complex structure on E is then equivalent to the side morphism j and a 2-form $\Psi \in \Omega(TM \oplus E^*, E^*)$, depending on the linear splitting, both satisfying certain properties (Theorem A.1.6).

4.1 Generalised complex structures on vector bundles

In this chapter $E \to M$ is always a smooth vector bundle over a smooth manifold M. The generalised tangent bundle $\mathbb{T}E = TE \oplus T^*E$ is then a double vector bundle

$$\begin{array}{ccc} TE \oplus T^*E & \longrightarrow & E \\ \downarrow & E \oplus T^*M \searrow & \downarrow \\ TM \oplus E^* & \longrightarrow & M \end{array} \quad . \tag{4.1}$$

The vector bundle $TE \oplus T^*E \to E$ has additionally the structure of a Courant algebroid as described in Example 2.3.3, the standard Courant algebroid over the total space E.

The main definition we will work with in this chapter is the following of a linear generalised complex structure.

Definition 4.1.1. *A generalised complex structure $\mathcal{J}$ on a vector bundle $E \to M$ is called* ***linear*** *if $\mathcal{J}\colon TE \oplus T^*E \to TE \oplus T^*E$ is a morphism of double vector bundles over some side morphism $j\colon TM \oplus E^* \to TM \oplus E^*$ and core morphism $j_C\colon E \oplus T^*M \to E \oplus T^*M$.*

$$\begin{array}{ccccccc} TE \oplus T^*E & & \xrightarrow{\mathcal{J}} & & TE \oplus T^*E & & \\ \downarrow & \searrow & & & \downarrow & \searrow & \\ & & E & \xrightarrow{\mathrm{id}_E} & & & E \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ TM \oplus E^* & & \xrightarrow{j} & & TM \oplus E^* & & \downarrow \\ & \searrow & \downarrow & & & \searrow & \downarrow \\ & & M & \xrightarrow{\mathrm{id}_M} & & & M \end{array} \quad . \tag{4.2}$$

The most immediate examples of these structures are given by holomorphic vector bundles, where $\mathcal{J}$ is induced by a linear complex structure on E. In this case we have a complex structure J on the total space E and require as compatibility condition with the linear structure that J is additionally a vector bundle morphism over some complex structure j_M on M. This special case is considered in more detail in Section 4.8.1.

The other extremal case of generalised complex structures is given by symplectic structures. In this case the condition to be linear is very restrictive, any linear symplectic structure $\omega^\flat\colon TE \to T^*E$ has to be the canonical symplectic structure on $T^*M \cong E$. We recall this special case in Section 4.8.2.

Another motivating example for our studies are linear Poisson structures. Although these do not immediately give generalised complex structures, since Poisson structures can be degenerate, they showcase the idea behind our main results. Given a Poisson structure $\pi^\sharp$ on a vector bundle E, the linearity condition is that it is a double vector bundle morphism over some side morphism $\rho\colon E^* \to TM$. The Poisson structure can then be completely described in terms of E^*, it is equivalent to a Lie algebroid structure on E^*. This was shown by Kirill Mackenzie and Ping Xu in [53], and is recalled here in Example 2.2.9. In this section we show a similar result for linear generalised complex structures, where we describe the structure on $TE \oplus T^*E$ in terms of the side morphism j and a suitable linear splitting of $TE \oplus T^*E$.

Let us first consider the trivial example of a linear generalised complex structure on a vector space, that is a vector bundle over a point, $M = \{*\}$. In this case the tangent and cotangent bundle are canonically split, $TE = E \times E$ and $T^*E = E \times E^*$. The linearity condition on the generalised complex structure $\mathcal{J}$ therefore is equivalent to $\mathcal{J}$ being determined by the maps in the fibres, that is by the side morphism $j_{E^*}\colon E^* \to E^*$ and the core morphism $j_C\colon E \to E$. These have to be in negative duality to each other, that is $j = -j_C^t$, since the generalised complex structure is orthogonal with respect to the canonical pairing. Therefore a linear generalised complex structure on a vector space in this sense is equivalent to the choice of an ordinary complex structure in the vector space. This is not surprising, as the existence of a linear symplectic structure on a vector space in this sense implies that the vector space vanishes.

Now we will need some computational tools in order to study linear generalised complex structures. In particular we will use linear splittings of the generalised tangent bundle $\mathbb{T}E$. Recall from Section 2.3 that a linear splitting of the double vector bundle $TE \oplus T^*E$ is equivalent to a $(TM \oplus E^*)$-Dorfman connection on $E \oplus T^*M$ (see [36]). The lift $\sigma^\Delta\colon \Gamma(TM \oplus E^*) \to \Gamma_E(TE \oplus T^*E)$ corresponding to such a Dorfman connection Δ is defined for $X \in \Gamma(TM)$, $\varepsilon \in \Gamma(E^*)$ and $e_m \in E$ over $m \in M$ by

$$\sigma^\Delta(X,\varepsilon)(e_m) := (T_m e X(m), d_{e_m}\ell_\varepsilon) - \Delta_{(X,\varepsilon)}(e,0)^\uparrow(e_m)\,, \tag{4.3}$$

for any section $e \in \Gamma(E)$ such that $e(m) = e_m$. Here $\Delta_{(X,\varepsilon)}(e,0)^\uparrow$ denotes the core section of $TE \oplus T^*E \to E$ corresponding to $\Delta_{(X,\varepsilon)}(e,0)$. For any $\tau = (e,\theta) \in \Gamma(E \oplus T^*M)$ the corresponding core section is defined by:

$$(e,\theta)^\uparrow(e'_m) = \left(\left.\frac{d}{dt}\right|_{t=0} e'_m + te(m), (T_{e'_m} q_E)^t \theta(m) \right). \tag{4.4}$$

The lift in (4.3) defines indeed a morphism of $C^\infty(M)$-modules since we have

$$\begin{aligned}\sigma^\Delta(f(X,\varepsilon))(e_m) &= (T_m e(fX(m)), d_{e_m}\ell_{f\varepsilon}) - \Delta_{f(X,\varepsilon)}(e,0)^\uparrow(e_m)\\ &= (f(m)\cdot_E T_m e(X(m)), d_{e_m}(q_E^* f\ell_\varepsilon))\\ &\quad - (f\Delta_{(X,\varepsilon)}(e,0))^\uparrow(e_m) - (\langle e,\varepsilon\rangle(0,df))^\uparrow(e_m)\\ &= (f(m)T_m eX(m), f(m)d_{e_m}(\ell_\varepsilon) + \langle e,\varepsilon\rangle(m)d_{e_m}(q_E^* f))\\ &\quad - (f\Delta_{(X,\varepsilon)}(e,0))^\uparrow(e_m) - \langle e,\varepsilon\rangle(m)(0,(T_{e_m}q_E)^t d_{e_m}f))\\ &= f(m)\sigma^\Delta((X,\varepsilon))(e_m) + \langle e,\varepsilon\rangle(m)(0, d_{e_m}q_E^* f - (T_{e_m}q_E)^t d_{e_m}f)\\ &= f(m)\sigma^\Delta((X,\varepsilon))(e_m)\,,\end{aligned}$$

and thus $\sigma^\Delta(f\nu) = q_E^* f\sigma^\Delta(\nu)$. That the definition of σ^Δ does not depend on the choice of section $e\in\Gamma(E)$ with $e(m)=e_m$ follows then from $C^\infty(M)$-linearity completely analogously to the proof of Lemma 2.1.25.

A section $\varphi\in\Gamma(\mathrm{Hom}(E, E\oplus T^*M))$ induces a core-linear section of $TE\oplus T^*E\to E$ as defined in Definition 2.1.10. The core-linear section corresponding to φ will be denoted by $\widetilde{\varphi}$.

Furthermore, a $(TM\oplus E^*)$-Dorfman connection on $E\oplus T^*M$ with the canonical non-degenerate pairing was showed in [36] to be dual to a dull bracket $[\![\cdot,\cdot]\!]_\Delta$ on $TM\oplus E^*$ anchored by pr_{TM}. This correspondence is recapitulated in Lemma 2.3.16, and is given by the following equality for any ν_1, $\nu_2\in\Gamma(TM\oplus E^*)$ and $\tau\in\Gamma(E\oplus T^*M)$

$$\langle\Delta_{\nu_1}\tau,\nu_2\rangle = \mathrm{pr}_{TM}(\nu_1)\langle\tau,\nu_2\rangle - \langle\tau,[\![\nu_1,\nu_2]\!]_\Delta\rangle\,. \tag{4.5}$$

Let us also denote by $\mathrm{Skew}_\Delta\in\Gamma((TM\oplus E^*)^*\otimes(TM\oplus E^*)^*\otimes E^*)$ the tensor defined by

$$\mathrm{Skew}_\Delta(\nu_1,\nu_2) = \mathrm{pr}_{E^*}([\![\nu_1,\nu_2]\!]_\Delta + [\![\nu_2,\nu_1]\!]_\Delta)\,.$$

The following description of the symmetric pairing in $TE\oplus T^*E$ was proved in [36]:

Theorem 4.1.2. *Let Δ be a $TM\oplus E^*$-Dorfman connection on $E\oplus T^*M$. For sections $\nu,\nu_1,\nu_2\in\Gamma(TM\oplus E^*)$ and $\tau,\tau_1,\tau_2\in\Gamma(E\oplus T^*M)$ we have*

1. $\langle\sigma^\Delta(\nu_1),\sigma^\Delta(\nu_2)\rangle = \ell_{\mathrm{Skew}_\Delta(\nu_1,\nu_2)}\,,$
2. $\langle\sigma^\Delta(\nu),\tau^\uparrow\rangle = q_E^*\langle\nu,\tau\rangle\,,$
3. $\langle\tau_1^\uparrow,\tau_2^\uparrow\rangle = 0\,.$

Additionally, in [36] the following description of the Courant-Dorfman bracket of horizontal lifts and core sections was proved:

Theorem 4.1.3. *Let Δ be a $(TM \oplus E^*)$-Dorfman connection on $E \oplus T^*M$. For sections $\nu, \nu_1, \nu_2 \in \Gamma(TM \oplus E^*)$ and $\tau, \tau_1, \tau_2 \in \Gamma(E \oplus T^*M)$ we have*

1. $[\![\sigma^\Delta(\nu_1), \sigma^\Delta(\nu_2)]\!] = \sigma^\Delta([\![\nu_1, \nu_2]\!]_\Delta) - \widetilde{R_\Delta(\nu_1, \nu_2)(\cdot, 0)}\,,$
2. $[\![\sigma^\Delta(\nu), \tau^\uparrow]\!] = (\Delta_\nu \tau)^\uparrow\,,$
3. $[\![\tau_1^\uparrow, \tau_2^\uparrow]\!] = 0\,,$

where R_Δ denotes the curvature of Δ as in Definition 2.3.17. That is

$$R_\Delta \colon \Gamma(TM \oplus E^*) \times \Gamma(TM \oplus E^*) \to \Gamma(\mathrm{Hom}(E \oplus T^*M, E \oplus T^*M))\,, \tag{4.6}$$

defined by $R_\Delta(\nu_1, \nu_2)(\tau) := \Delta_{\nu_1}\Delta_{\nu_2}\tau - \Delta_{\nu_2}\Delta_{\nu_1}\tau - \Delta_{[\![\nu_1,\nu_2]\!]_\Delta}\tau$.

A Dorfman connection is said to be skew-symmetric, if the corresponding dull bracket is skew-symmetric. Given any dull bracket $[\![\cdot,\cdot]\!]$, we can define its skew-symmetrisation by

$$[\![\nu_1, \nu_2]\!]_{\mathrm{Skew}} = \frac{1}{2}\Big([\![\nu_1, \nu_2]\!] - [\![\nu_2, \nu_1]\!]\Big)\,. \tag{4.7}$$

The following lemma is stated in [36] and shows that we can always work with a skew-symmetric Dorfman connection. For the sake of completeness we will give here the straightforward proof.

Lemma 4.1.4. *The skew-symmetrisation of a dull algebroid is again a dull algebroid with the same anchor ρ.*

Proof. Since the Lie bracket on vector fields is skew-symmetric, we obtain

$$\begin{aligned}\rho([\![\nu_1, \nu_2]\!]_{\mathrm{Skew}}) &= \rho\Big(\frac{1}{2}\Big([\![\nu_1, \nu_2]\!] - [\![\nu_2, \nu_1]\!]\Big)\Big)\\ &= \frac{1}{2}\Big([\rho(\nu_1), \rho(\nu_2)] - [\rho(\nu_2), \rho(\nu_1)]\Big)\\ &= [\rho(\nu_1), \rho(\nu_2)]\,.\end{aligned}$$

For the Leibniz identity, we obtain for all $f_1, f_2 \in C^\infty(M)$

$$\begin{aligned}[\![f_1\nu_1, f_2\nu_2]\!]_{\mathrm{Skew}} &= \frac{1}{2}\Big([\![f_1\nu_1, f_2\nu_2]\!] - [\![f_2\nu_2, f_1\nu_1]\!]\Big)\\ &= \frac{1}{2}\Big(f_1f_2[\![\nu_1, \nu_2]\!] + f_1\rho(\nu_1)(f_2)\nu_2 - f_2\rho(\nu_2)(f_1)\nu_1\\ &\quad - (f_1f_2[\![\nu_2, \nu_1]\!] + f_2\rho(\nu_2)(f_1)\nu_1 - f_1\rho(\nu_1)(f_2)\nu_2)\Big)\\ &= f_1f_2[\![\nu_1, \nu_2]\!]_{\mathrm{Skew}} + f_1\rho(\nu_1)(f_2)\nu_2 - f_2\rho(\nu_2)(f_1)\nu_1\,.\end{aligned}$$

□

This shows that we can assume that the $(TM \oplus E^*)$-Dorfman connection on $E \oplus T^*M$ is skew-symmetric, since we can always take the skew-symmetrisation of the corresponding dull bracket and then the Dorfman connection corresponding to the skew-symmetric bracket. In that case the pairing of two horizontal lifts $\sigma^\Delta(\nu_1)$ and $\sigma^\Delta(\nu_2)$ vanishes according to Theorem 4.1.2.

In the following we fix such a skew-symmetric Dorfman connection

$$\Delta\colon \Gamma(TM \oplus E^*) \otimes \Gamma(E \oplus T^*M) \to \Gamma(E \oplus T^*M)\,, \tag{4.8}$$

and therefore a horizontal lift $\sigma^\Delta\colon \Gamma(TM \oplus E^*) \to \Gamma^\ell_E(TE \oplus T^*E)$.

Let us now consider a double vector bundle morphism $\mathcal{J}\colon TE \oplus T^*E \to TE \oplus T^*E$ as in (4.2). We want to give conditions on j and j_C in order for the morphism $\mathcal{J}$ to be a generalised complex structure.

First we describe how $\mathcal{J}$ acts on lifts $\sigma^\Delta(\nu)$ for $\nu \in \Gamma(TM \oplus E^*)$ with respect to a fixed Dorfman connection.

Lemma 4.1.5. *Given a double vector bundle morphism $\mathcal{J}$ over j as in* (4.2) *and a skew-symmetric Dorfman connection Δ as in* (4.8)*, there is a section $\Phi \in \Gamma((TM \oplus E^*)^* \otimes E^* \otimes (E \oplus T^*M))$ such that for any $\nu \in \Gamma(TM \oplus E^*)$*

$$\mathcal{J}(\sigma^\Delta(\nu)) = \sigma^\Delta(j(\nu)) + \widetilde{\Phi(\nu)}\,. \tag{4.9}$$

Here we identify $(TM \oplus E^*)^* \otimes E^* \otimes (E \oplus T^*M)$ with $\mathrm{Hom}(TM \oplus E^*, \mathrm{Hom}(E, E \oplus T^*M))$ *in order to define $\widetilde{\Phi(\nu)}$.*

Proof. Since $\mathcal{J}$ is a vector bundle morphism over j, $\mathcal{J}(\sigma^\Delta(\nu))$ is a linear section over $j(\nu)$. Thus $\mathcal{J}(\sigma^\Delta(\nu)) - \sigma^\Delta(j(\nu))$ is a core-linear section and this gives us for every ν a section $\Phi(\nu) \in \Gamma(\mathrm{Hom}(E, E \oplus T^*M))$, such that

$$\mathcal{J}(\sigma^\Delta(\nu)) = \sigma^\Delta(j(\nu)) + \widetilde{\Phi(\nu)}\,.$$

That Φ is linear and tensorial in ν, follows directly from the fact that $\mathcal{J}$ and j and vector bundle morphisms and that σ^Δ is a morphism of $C^\infty(M)$-modules. Then

$$\begin{aligned}\widetilde{\Phi(f\nu)} &= \mathcal{J}(\sigma^\Delta(f\nu)) - \sigma^\Delta(j(f\nu))\\ &= q_E^* f \mathcal{J}(\sigma^\Delta(\nu)) - q_E^* f \sigma^\Delta(j(\nu))\\ &= q_E^* f \widetilde{\Phi(\nu)}\,.\end{aligned}$$

So $\Phi(f\nu) = f\Phi(\nu)$ and $\Phi \in \Gamma((TM \oplus E^*)^* \otimes E^* \otimes (E \oplus T^*M))$. □

Now we can describe in terms of Δ, j, j_C and Φ how $\mathcal{J}$ acts on core sections, core-linear sections and linear sections.

Lemma 4.1.6. *For any sections $\tau \in \Gamma(E \oplus T^*M)$, $\nu \in \Gamma(TM \oplus E^*)$ and $\varphi \in \Gamma(\operatorname{Hom}(E, E \oplus T^*M))$ we have*

1. $\mathcal{J}(\tau^\uparrow) = j_C(\tau)^\uparrow$,
2. $\mathcal{J}(\sigma^\Delta(\nu)) = \sigma^\Delta(j(\nu)) + \widetilde{\Phi(\nu)}$,
3. $\mathcal{J}(\widetilde{\varphi}) = \widetilde{j_C \circ \varphi}$.

Proof. The first equality follows directly from the definition of the core morphism j_C whereas the second equality is directly by definition of Φ. To show the action on core-linear sections we work with local basis sections $\{\varepsilon_i\}_{i\in I}$ of E^* and $\{\tau_j\}_{j\in J}$ of $E \oplus T^*M$ for some finite index sets I and J. The section φ can then be written as $\varphi = \sum_{i,k} f_{ik}\varepsilon_i \otimes \tau_k$ for smooth functions $f_{ik} \in C^\infty(M)$ and therefore we obtain for the core-linear section $\widetilde{\varphi} = \sum_{i,k} q_E^* f_{ik}\ell_{\varepsilon_i}\tau_k^\uparrow$. Using now that $\mathcal{J}((\tau_k)^\uparrow) = (j_C\tau_k)^\uparrow$ we obtain

$$\begin{aligned} \mathcal{J}(\widetilde{\varphi}) &= \sum_{i,j} q_E^* f_{ij}\ell_{\varepsilon_i}(j_C\tau_j)^\uparrow \\ &= \widetilde{(j_C \circ \varphi)} . \end{aligned}$$ □

4.1.1 Generalised almost complex structures on vector bundles

In this section we will give a description of generalised almost complex structures without taking the integrability condition into account. Given a double vector bundle morphism $\mathcal{J}\colon TE \oplus T^*E \to TE \oplus T^*E$ as in (4.2) and a skew-symmetric $(TM \oplus E^*)$-Dorfman connection Δ on $E \oplus T^*M$, we will describe conditions on j, j_C and Φ under which $\mathcal{J}$ is a generalised almost complex structure on E.

$\mathcal{J}$ squares to $-\operatorname{id}$

The first property of a generalised almost complex structure is to square to $-\operatorname{id}$. We will describe under which conditions on j, j_C and Φ the map $\mathcal{J}\colon TE \oplus T^*E \to TE \oplus T^*E$ satisfies this property.

Lemma 4.1.7. *A double vector bundle morphism $\mathcal{J}$ as in (4.2) satisfies $\mathcal{J}^2 = -\operatorname{id}_{TE\oplus T^*E}$ if and only if for any skew-symmetric Dorfman connection Δ as in (4.8) and for all $\nu \in \Gamma(TM \oplus E^*)$ we have*

1. $j^2 = -\operatorname{id}_{TM\oplus E^*}$,
2. $j_C^2 = -\operatorname{id}_{E\oplus T^*M}$,

3. $\Phi(j(\nu)) = -j_C \circ (\Phi(\nu))$.

Proof. It is sufficient to check that $\mathcal{J}^2 = -\operatorname{id}_{TE\oplus T^*E}$ on lifts $\sigma^\Delta(\nu)$ for any $\nu \in \Gamma(TM \oplus E^*)$ and on core sections $\tau^\uparrow$ for any $\tau \in \Gamma(E \oplus T^*M)$, as those span $\Gamma_{TM\oplus E^*}(TE \oplus T^*E)$.

For core sections we obtain by definition of the core morphism j_C

$$\mathcal{J}^2(\tau^\uparrow) = \mathcal{J}((j_C(\tau))^\uparrow) = (j_C^2(\tau))^\uparrow .$$

This equals $-\tau^\uparrow$ if and only if $j_C^2 = -\operatorname{id}_{E\oplus T^*M}$.

For linear sections, we obtain by definition of Φ and Lemma 4.1.6 that

$$\begin{aligned}\mathcal{J}^2(\sigma^\Delta(\nu)) &= \mathcal{J}(\sigma^\Delta(j(\nu)) + \widetilde{\Phi(\nu)}) \\ &= \sigma^\Delta(j^2(\nu)) + \widetilde{\Phi(j(\nu))} + \widetilde{j_C \circ (\Phi(\nu))} .\end{aligned} \tag{4.10}$$

If now $\mathcal{J}^2 = -\operatorname{id}_{TE\oplus T^*E}$, then the side morphism j has to satisfy $j^2 = -\operatorname{id}_{TM\oplus E^*}$. Now (4.10) implies that

$$\Phi(j(\nu)) = -j_C \circ (\Phi(\nu)) . \tag{4.11}$$

Conversely, if $j^2 = -\operatorname{id}_{TM\oplus E^*}$ and $\Phi(j(\nu)) = -j_C \circ (\Phi(\nu))$, then by (4.10) we get immediately that $\mathcal{J}^2(\sigma^\Delta(\nu)) = -\sigma^\Delta(\nu)$ for all $\nu \in \Gamma(TM \oplus E^*)$. The lemma follows. □

$\mathcal{J}$ is orthogonal

The second property of a generalised almost complex structure is that it is orthogonal with respect to the canonical pairing. Now we describe in terms of j, j_C and Φ when a morphism $\mathcal{J}$ as in (4.2) that additionally squares to $-\operatorname{id}$ is orthogonal.

Since the image of a lift $\sigma^\Delta(\nu)$ under $\mathcal{J}$ involves a core-linear section, we need the following corollary to Theorem 4.1.2.

Corollary 4.1.8. *Given a skew-symmetric Dorfman connection as in* (4.8), $\varphi, \psi \in \Gamma(\operatorname{Hom}(E, E \oplus T^*M))$, $\nu \in \Gamma(TM \oplus E^*)$ *and* $\tau \in \Gamma(E \oplus T^*M)$ *the following equalities hold:*

1. $\langle \widetilde{\varphi}, \sigma^\Delta(\nu) \rangle = \ell_{\varphi^*(\nu)}$,
2. $\langle \widetilde{\varphi}, \tau^\uparrow \rangle = 0$,
3. $\langle \widetilde{\varphi}, \widetilde{\psi} \rangle = 0$.

Proof. Let us fix local basis sections $\{\varepsilon_i\}_{i\in I}$ of E^* and $\{\tau_k\}_{k\in K}$ of $E\oplus T^*M$ for some finite index sets I and K. The section φ can then be written as $\varphi = \sum_{i,k} f_{ik}\varepsilon_i \otimes \tau_k$ for smooth functions $f_{ik} \in C^\infty(M)$ and therefore we obtain for the core-linear section $\widetilde{\varphi} = \sum_{i,k} q_E^* f_{ik}\ell_{\varepsilon_i}\tau_k^\uparrow$. Since the pairing is bilinear and the pairing of any two core sections vanishes we get immediately that $\langle\widetilde{\varphi},\widetilde{\psi}\rangle = 0$ and $\langle\widetilde{\varphi},\tau^\uparrow\rangle = 0$. For the pairing of the core-linear section with a horizontal lift we compute with Theorem 4.1.2 for any $e_m \in E$ over $m \in M$

$$\begin{aligned}
\langle\widetilde{\varphi},\sigma^\Delta(\nu)\rangle(e_m) &= \Big\langle\sum_{i,k} q_E^* f_{ik}\ell_{\varepsilon_i}\tau_k^\uparrow, \sigma^\Delta(\nu)\Big\rangle(e_m) \\
&= \sum_{i,k} q_E^* f_{ik}\ell_{\varepsilon_i}\langle\tau_k^\uparrow,\sigma^\Delta(\nu)\rangle(e_m) \\
&= \sum_{i,k} q_E^* f_{ik}\ell_{\varepsilon_i} q_E^*\langle\tau_k,\nu\rangle(e_m) \\
&= \Big\langle\sum_{i,k} f_{ik}(m)\ell_{\varepsilon_i}(e_m)\tau_k(m), \nu(m)\Big\rangle \\
&= \langle\varphi(e_m),\nu(m)\rangle \\
&= \ell_{\varphi^*(\nu)}(e_m)\,.
\end{aligned}$$

Thus $\langle\widetilde{\varphi},\sigma^\Delta(\nu)\rangle = \ell_{\varphi^*(\nu)}(e_m)$. □

Now we can prove the following lemma.

Lemma 4.1.9. *A double vector bundle morphism $\mathcal{J}$ as in* (4.2) *such that additionally $\mathcal{J}^2 = -1$, is orthogonal if and only if for any skew-symmetric Dorfman connection Δ as in* (4.8) *and for all $\nu_1,\nu_2 \in \Gamma(TM\oplus E^*)$ we have*

1. $j = -(j_C)^t$,
2. $\Phi(\nu_2)^t(j\nu_1) = -\Phi(\nu_1)^t(j(\nu_2))$.

Proof. Again it is sufficient to check orthogonality of $\mathcal{J}$ on core sections and on horizontal lifts.

First, the equality

$$\langle\mathcal{J}(\tau_1^\uparrow),\mathcal{J}(\tau_2^\uparrow)\rangle = \langle\tau_1^\uparrow,\tau_2^\uparrow\rangle$$

for τ_1, τ_2 sections of $E\oplus T^*M$ is automatically satisfied, since $\mathcal{J}(\tau_i^\uparrow) = (j_C\tau_i)^\uparrow$ and the pairing of two core sections vanishes according to Theorem 4.1.2.

Second, for the pairing of a lift with a core section we again make use of

Theorem 4.1.2 and of Corollary 4.1.8 and obtain on one hand

$$\begin{aligned}\big\langle \mathcal{J}\sigma^\Delta(\nu), \mathcal{J}(\tau^\uparrow)\big\rangle &= \langle \sigma^\Delta(j(\nu)) + \widetilde{\Phi(\nu)}, j_C(\tau)^\uparrow\rangle \\ &= q_E^*\langle j(\nu), j_C(\tau)\rangle + \langle \widetilde{\Phi(\nu)}, j_C(\tau)^\uparrow\rangle \\ &= q_E^*\langle j(\nu), j_C(\tau)\rangle\,,\end{aligned}$$

and on the other hand

$$\langle \sigma^\Delta(\nu), \tau^\uparrow\rangle = q_E^*\langle \nu, \tau\rangle\,.$$

Hence

$$\big\langle \mathcal{J}\sigma^\Delta(\nu), \mathcal{J}(\tau^\uparrow)\big\rangle = \langle \sigma^\Delta(\nu), \tau^\uparrow\rangle$$

if and only if

$$\langle j(\nu), j_C(\tau)\rangle = \langle \nu, \tau\rangle\,.$$

According to Lemma 4.1.7 we have $j_C^2 = -1$ and therefore this is equivalent to

$$j = -(j_C)^t\,.$$

Third, for the pairing between two lifts we obtain using the skew-symmetry of the chosen Dorfman connection and Theorem 4.1.2 and Corollary 4.1.8 on one hand

$$\begin{aligned}\big\langle \mathcal{J}(\sigma^\Delta(\nu_1)), \mathcal{J}(\sigma^\Delta(\nu_2))\big\rangle &= \langle \sigma^\Delta(j(\nu_1)) + \widetilde{\Phi(\nu_1)}, \sigma^\Delta(j(\nu_2)) + \widetilde{\Phi(\nu_2)}\rangle \\ &= \langle \sigma^\Delta(j(\nu_1), \sigma^\Delta(j(\nu_2))\rangle + \langle \widetilde{\Phi(\nu_1)}, \widetilde{\Phi(\nu_2)}\rangle \\ &+ \langle \sigma^\Delta(j(\nu_2)), \widetilde{\Phi(\nu_1)}\rangle + \langle \sigma^\Delta(j(\nu_1)), \widetilde{\Phi(\nu_2)}\rangle \\ &= \ell_{\Phi(\nu_2)^t(j(\nu_1))} + \ell\Phi(\nu_1)^t(j(\nu_2))\end{aligned}$$

and on the other hand

$$\langle \sigma^\Delta(\nu_1), \sigma^\Delta(\nu_2)\rangle = 0\,.$$

Hence

$$\langle \mathcal{J}\sigma^\Delta(\nu_1), \mathcal{J}(\sigma^\Delta(\nu_2))\rangle = \langle \sigma^\Delta(\nu_1), \sigma^\Delta(\nu_2)\rangle = 0$$

if and only if

$$\Phi(\nu_2)^t(j(\nu_1)) = -\Phi(\nu_1)^t(j(\nu_2))\,. \qquad \square$$

We now introduce the following notions $\Psi, \Psi^j, (j^*\Psi) \in \Gamma\big((TM \oplus E^*)^* \otimes (TM \oplus E^*)^* \otimes E^*\big)$, defined as follows

$$\Psi(\nu_1, \nu_2) := \Phi(\nu_1)^t(\nu_2)\,, \tag{4.12}$$

$$\Psi^j(\nu_1, \nu_2) := \Phi(\nu_1)^t(j\nu_2)\,, \tag{4.13}$$

$$(j^*\Psi)(\nu_1, \nu_2) := \Psi(j\nu_1, j\nu_2)\,. \tag{4.14}$$

This lets us combine Lemma 4.1.7 and Lemma 4.1.9 in the following way:

Proposition 4.1.10. *A morphism $\mathcal{J}$ as in* (4.2) *is a generalised almost complex structure on E, if and only if for every skew-symmetric Dorfman connection Δ as in* (4.8) *we have*

1. $j^2 = -1\,,$
2. $j = -(j_C)^t\,,$
3. Ψ *is skew-symmetric, that is* $\Psi \in \Omega^2(TM \oplus E^*, E^*)\,,$
4. $\Psi(\nu_1, \nu_2) = -j^*\Psi(\nu_1, \nu_2)\,.$

Proof. The condition $\Phi(\nu_2)^t(j\nu_1) = -\Phi(\nu_1)^t(j\nu_2)$ of Lemma 4.1.9 is equivalent to Ψ^j being skew-symmetric. The conditions $j^2 = -\operatorname{id}$ and $j = -j_C^t$ imply the second condition in Lemma 4.1.7, $j_C^2 = -\operatorname{id}$. Given those properties, we can evaluate both sides of the condition $\Phi(j(\nu)) = -j_C \circ (\Phi(\nu))$ of Lemma 4.1.7 at any $e \in \Gamma(E)$ and pair with $\nu' \in \Gamma(TM \oplus E^*)$ to obtain the following equivalent condition:

$$\Psi(j\nu, \nu') = \Psi(\nu, j\nu')\,. \tag{4.15}$$

Write now $\nu = j\nu_2$. Then under the assumption of skew-symmetry of Ψ^j and of $j^2 = -1$ the equation (4.15) is equivalent to Ψ being skew-symmetric since

$$\begin{aligned} -\Psi(\nu_1, \nu_2) = \Psi(j^2\nu_1, \nu_2) &\overset{(4.15)}{=} \Psi(j\nu_1, j\nu_2) = \Psi^j(j\nu_1, \nu_2) \\ &= -\Psi^j(\nu_2, j\nu_1) = -\Psi(\nu_2, j^2\nu_1) = \Psi(\nu_2, \nu_1)\,. \end{aligned}$$

The condition that both Ψ and Ψ^j are skew-symmetric is equivalent to Ψ being skew-symmetric and satisfying $\Psi(\nu_1, j\nu_2) = \Psi(j\nu_1, \nu_2)$ or equivalently $j^*\Psi = -\Psi$. □

4.1.2 Adapted Dorfman connections

This section shows that given a linear generalised almost complex structure $\mathcal{J}$ on $E \to M$, we can always find a Dorfman connection Δ which is adapted to $\mathcal{J}$ in the sense that the corresponding lift σ^Δ satisfies $\mathcal{J}(\sigma^\Delta(\nu)) = \sigma^\Delta(j\nu)$ for any section $\nu \in \Gamma(TM \oplus E^*)$. Equivalently, the corresponding Φ defined by Lemma 4.1.5 vanishes. This will vastly simplify all the computations for the Nijenhuis tensor and later of the conditions for Lie algebroid morphisms.

First we describe the change of splittings.

Definition 4.1.11. *Given two $(TM \oplus E^*)$-Dorfman connections Δ^1 and Δ^2 on $E \oplus T^*M$ with corresponding lifts σ_1 and σ_2, the* ***change of splitting*** *from Δ^1 to Δ^2 is $\Phi_{12} \in \Gamma\big((TM \oplus E^*)^* \otimes \operatorname{Hom}(E, E \oplus T^*M)\big)$ defined by the equation*

$$\widetilde{\Phi_{12}(\nu)} := \sigma_2(\nu) - \sigma_1(\nu)\,, \tag{4.16}$$

for any $\nu \in \Gamma(TM \oplus E^*)$. The change of splitting is called ***skew-symmetric*** *if*

$$\Psi_{12}(\nu_1, \nu_2) := \Phi_{12}(\nu_1)^t(\nu_2) \tag{4.17}$$

is skew-symmetric, that is $\Psi_{12} \in \Omega^2(TM \oplus E^*, E^*)$. We will equivalently call Ψ_{12} the change of splittings.

Lemma 4.1.12. *Given two Dorfman connections Δ^1, Δ^2 as above, their corresponding dull brackets are related by*

$$[\![\nu_1, \nu_2]\!]_{\Delta^2} = [\![\nu_1, \nu_2]\!]_{\Delta^1} + (0, \Psi_{12}(\nu_1, \nu_2)) . \tag{4.18}$$

Proof. The definition of the change of splittings together with the correspondence between lifts and Dorfman connections as in (4.3) immediately gives that for any $\nu \in \Gamma(TM \oplus E^*)$ and $\tau \in \Gamma(E \oplus T^*M)$ we have

$$\Delta^2_\nu \tau = \Delta^1_\nu \tau - \Phi_{12}(\nu)(\mathrm{pr}_E\, \tau) . \tag{4.19}$$

Again dualising this equation gives the desired formula for a change of splittings for the corresponding dull brackets

$$[\![\nu_1, \nu_2]\!]_{\Delta^2} = [\![\nu_1, \nu_2]\!]_{\Delta^1} + (0, \Psi_{12}(\nu_1, \nu_2)) .$$

□

An immediate consequence is the following corollary.

Corollary 4.1.13. *If the Dorfman connection Δ_1 is skew-symmetric, then Δ_2 is skew-symmetric if and only if the change of splitting is skew-symmetric.*

We also need to describe how the 2-form Ψ_1 given by Proposition 4.1.10 behaves under the change of splitting. We prove the following.

Lemma 4.1.14. *Given a linear generalised almost complex structure $\mathcal{J}$ on E and two skew-symmetric Dorfman connections Δ_1 and Δ_2 as above with change of splitting Ψ_{12}, then we have*

$$\Psi_2(\nu_1, \nu_2) = \Psi_1(\nu_1, \nu_2) - \Psi_{12}(\nu_1, j\nu_2) - \Psi_{12}(j\nu_1, \nu_2) , \tag{4.20}$$

where Ψ_1, Ψ_2 are the 2-forms defined by Proposition 4.1.10 for Δ_1 and Δ_2, respectively.

Proof. We compute

$$\begin{aligned}
\widetilde{\Phi_2(\nu)} &= \mathcal{J}(\sigma_2(\nu)) - \sigma_2(j\nu) \\
&= \mathcal{J}(\sigma_1(\nu) + \widetilde{\Phi_{12}(\nu)}) - \sigma_1(j\nu) - \widetilde{\Phi_{12}(j\nu)} \\
&= \sigma_1(j\nu) + \widetilde{\Phi_1(\nu)} + \widetilde{j_C \circ \Phi_{12}(\nu)} - \sigma_1(j\nu) - \widetilde{\Phi_{12}(j\nu)} \\
&= \widetilde{\Phi_1(\nu)} + \widetilde{j_C \circ \Phi_{12}(\nu)} - \widetilde{\Phi_{12}(j\nu)}\,.
\end{aligned}$$

Dualising this equality leads to

$$\Psi_2(\nu_1, \nu_2) = \Psi_1(\nu_1, \nu_2) - \Psi_{12}(\nu_1, j\nu_2) - \Psi_{12}(j\nu_1, \nu_2)\,,$$

where we used that $j_C^t = -j$ according to Proposition 4.1.10. □

Using this change of splittings as in (4.20) we can now always find a skew-symmetric Dorfman connection Δ such that the corresponding 2-form Ψ vanishes, or equivalently such that $\mathcal{J}(\sigma^\Delta(\nu)) = \sigma^\Delta(j\nu)$.

Proposition 4.1.15. *For every linear generalised complex structure there is a skew-symmetric $(TM \oplus E^*)$-Dorfman connection Δ on $E^* \oplus TM$ such that $\mathcal{J}(\sigma^\Delta(\nu)) = \sigma^\Delta(j\nu)$. We call this Dorfman connection* ***adapted to*** $\mathcal{J}$*.*

Proof. Fix any skew-symmetric Dorfman connection $TM \oplus E^*$-Dorfman connection Δ_1 on $E^* \oplus TM$ and denote the corresponding lift by σ_1. Proposition 4.1.10 gives us a two-form $\Psi_1 \in \Omega^2(TM \oplus E^*, E^*)$ such that $\mathcal{J}(\sigma_1(\nu)) = \sigma_1(j\nu) + \widetilde{\Psi_1(\nu, \cdot)}$. Let now $\Psi_{12}(\nu_1, \nu_2) := -\frac{1}{2}\Psi_1(\nu_1, j\nu_2)$. By Proposition 4.1.10 this form is skew-symmetric and therefore the dull bracket defined by (4.18) is skew-symmetric again. Now according to (4.20) and using the properties from Proposition 4.1.10 the corresponding 2-form Ψ_2 vanishes. Hence we have found a Dorfman connection $\Delta := \Delta_2$ with $\Psi_2 = 0$ which by definition of Ψ_2 is equivalent to $\mathcal{J}(\sigma^\Delta(\nu)) = \sigma^\Delta(j\nu)$. □

However, the adapted Dorfman connection is not unique. We define the following notion of j-equivalence of Dorfman connections in order to take this into account.

Definition 4.1.16. *Given a vector bundle morphism $j\colon TM \oplus E^* \to TM \oplus E^*$, we call two skew-symmetric $(TM \oplus E^*)$-Dorfman connection Δ_1 and Δ_2 on $E^* \oplus TM$* ***j-equivalent****, if their change of splittings Ψ_{12} defined by* (4.18) *satisfies the following equation for all $\nu_1, \nu_2 \in \Gamma(TM \oplus E^*)$:*

$$\Psi_{12}(\nu_1, \nu_2) = \Psi_{12}(j\nu_1, j\nu_2)\,. \tag{4.21}$$

If the Dorfman connection Δ_1 is adapted to $\mathcal{J}$, then a second Dorfman connection Δ_2 is adapted to $\mathcal{J}$ if and only if they are j-equivalent. This follows from the following lemma.

Lemma 4.1.17. *Let $\mathcal{J}$ be a linear generalised almost complex structure $\mathcal{J}$ on a vector bundle $E \to M$ and Δ_1 and Δ_2 be two skew-symmetric $(TM \oplus E^*)$-Dorfman connections on $E \oplus T^*M$. Denote the two-forms given by Proposition 4.1.10 corresponding to Δ_1 and Δ_2 by Ψ_1 and Ψ_2, respectively. Then Δ_1 and Δ_2 are j-equivalent if and only if $\Psi_1 = \Psi_2$.*

Proof. Denote the change of splittings again by Ψ_{12}. Now Δ_1 is j-equivalent to Δ_2 if and only if for all $\nu_1, \nu_2 \in \Gamma(TM \oplus E^*)$ we have

$$\Psi_{12}(\nu_1, j\nu_2) = -\Psi_{12}(j(j\nu_1), j\nu_2) = -\Psi_{12}(j\nu_1, \nu_2)\,. \tag{4.22}$$

By (4.20) we have that Ψ_1 and Ψ_2 are related as follows:

$$\Psi_2(\nu_1, \nu_2) = \Psi_1(\nu_1, \nu_2) - \Psi_{12}(\nu_1, j\nu_2) - \Psi_{12}(j\nu_1, \nu_2)\,.$$

So $\Psi_1 = \Psi_2$ if and only if (4.22) holds, that is if and only if Δ_1 and Δ_2 are j-equivalent. □

4.1.3 Integrability

Now that we have characterised generalised almost complex structures for a given Dorfman connection in terms of j, j_C and Φ and have proved the existence of adapted Dorfman connections, we will consider the integrability condition of a generalised complex structure, the vanishing of the Nijenhuis tensor.

Let us now consider a linear generalised almost complex structure $\mathcal{J}$ on E as in (4.2) and fix a Dorfman connection Δ as in (4.8) which is adapted to $\mathcal{J}$. In particular j, j_C satisfy the conditions of Proposition 4.1.10 and the corresponding Ψ vanishes. That is, $\mathcal{J}(\sigma^\Delta(\nu)) = \sigma^\Delta(j\nu)$ for all $\nu \in \Gamma(TM \oplus E^*)$.

Evaluated at two core sections the Nijenhuis tensor vanishes trivially, as the Courant-Dorfman bracket of two core sections vanishes and the double vector bundle morphism $\mathcal{J}$ sends core sections to core sections.

For the Nijenhuis tensor of $\mathcal{J}$ evaluated at a horizontal lift $\sigma^\Delta(\nu)$ for $\nu \in \Gamma(TM \oplus E^*)$ and a core section $\tau^\uparrow$ for $\tau \in \Gamma(E \oplus T^*M)$ we compute using Theorem

4.1.3:

$$
\begin{aligned}
N_{\mathcal{J}}(\sigma^\Delta(\nu),\tau^\uparrow) &= [\![\sigma^\Delta(\nu),\tau^\uparrow]\!] - [\![\mathcal{J}(\sigma^\Delta(\nu)),\mathcal{J}(\tau^\uparrow)]\!] \\
&\quad + \mathcal{J}([\![\mathcal{J}(\sigma^\Delta(\nu)),\tau^\uparrow]\!] + [\![\sigma^\Delta(\nu),\mathcal{J}(\tau^\uparrow)]\!]) \\
&= (\Delta_\nu\tau)^\uparrow - [\![\sigma^\Delta(j(\nu)), j_C(\tau)^\uparrow]\!] \\
&\quad + \mathcal{J}\Big([\![\sigma^\Delta(j(\nu)),\tau^\uparrow]\!] + [\![\sigma^\Delta(\nu), j_C(\tau)^\uparrow]\!]\Big) \\
&= (\Delta_\nu\tau)^\uparrow - (\Delta_{j(\nu)} j_C(\tau))^\uparrow + \mathcal{J}\Big((\Delta_{j(\nu)}\tau)^\uparrow + (\Delta_\nu j_C(\tau))^\uparrow\Big) \\
&= (\Delta_\nu\tau)^\uparrow - (\Delta_{j(\nu)} j_C(\tau))^\uparrow + (j_C(\Delta_{j(\nu)}\tau))^\uparrow + (j_C(\Delta_\nu j_C(\tau)))^\uparrow .
\end{aligned}
$$

Thus the Nijenhuis tensor of $\mathcal{J}$ vanishes for any such pair of a lift $\sigma^\Delta(\nu)$ and a core section $\tau^\uparrow$ if and only if for all $\nu \in \Gamma(TM \oplus E^*)$ and $\tau \in \Gamma(E \oplus T^*M)$ we have

$$
\Delta_\nu\tau - \Delta_{j(\nu)} j_C(\tau) + j_C(\Delta_{j(\nu)}\tau) + j_C(\Delta_\nu j_C(\tau)) = 0 \tag{4.23}
$$

As the pairing is non-degenerate, we can dualise this condition by pairing with a second section $\nu_2 \in \Gamma(TM \oplus E^*)$. We now write $\nu_1 = (X_1, \varepsilon_1)$ instead of ν and use Proposition 4.1.10 and the duality of the Dorfman connection with the dull bracket on $TM \oplus E^*$ as described in [36] and in (4.5) to compute:

$$
\begin{aligned}
&\Big\langle \Delta_{\nu_1}\tau - \Delta_{j(\nu_1)} j_C(\tau) + j_C(\Delta_{j(\nu_1)}\tau) + j_C(\Delta_{\nu_1} j_C(\tau)), \nu_2 \Big\rangle \\
&\quad = \langle \Delta_{\nu_1}\tau, \nu_2\rangle - \langle \Delta_{j(\nu_1)} j_C(\tau), \nu_2\rangle - \langle \Delta_{j(\nu_1)}\tau, j(\nu_2)\rangle - \langle \Delta_{\nu_1} j_C(\tau), j(\nu_2)\rangle \\
&\quad = X_1\langle \tau, \nu_2\rangle - \langle \tau, [\![\nu_1,\nu_2]\!]_\Delta\rangle \\
&\qquad - \mathrm{pr}_{TM}(j(\nu_1))\langle j_C(\tau), \nu_2\rangle + \langle j_C(\tau), [\![j(\nu_1),\nu_2]\!]_\Delta\rangle \\
&\qquad - \mathrm{pr}_{TM}(j(\nu_1))\langle \tau, j(\nu_2)\rangle + \langle \tau, [\![j(\nu_1), j(\nu_2)]\!]_\Delta\rangle \\
&\qquad - X_1\langle j_C(\tau), j(\nu_2)\rangle + \langle j_C(\tau), [\![\nu_1, j(\nu_2)]\!]_\Delta\rangle \\
&\quad = \Big\langle \tau, -[\![\nu_1,\nu_2]\!]_\Delta - j([\![j(\nu_1),\nu_2]\!]_\Delta) - j([\![\nu_1, j(\nu_2)]\!]_\Delta) + [\![j(\nu_1), j(\nu_2)]\!]_\Delta \Big\rangle \\
&\quad = \Big\langle \tau, -N_{j,[\![\cdot,\cdot]\!]_\Delta}(\nu_1,\nu_2)\Big\rangle ,
\end{aligned}
$$

Thus the Nijenhuis tensor of $\mathcal{J}$ vanishes when evaluated at a pair of any lift $\sigma^\Delta(\nu)$ and any core section $\tau^\uparrow$ if and only if the Nijenhuis tensor of j with respect to the dull bracket $[\![\cdot,\cdot]\!]_\Delta$ vanishes. Here Δ is Dorfman connection adapted to $\mathcal{J}$.

We lastly need conditions for the Nijenhuis tensor of $\mathcal{J}$ evaluated at a pair of two horizontal lifts $\sigma^\Delta(\nu_1)$ and $\sigma^\Delta(\nu_2)$ vanishes. Again we make use of Lemma

4.1.6, Theorem 4.1.3 and compute for $\nu_1, \nu_2 \in \Gamma(TM \oplus E^*)$:

$$\begin{aligned}
N_{\mathcal{J}}(\sigma^\Delta(\nu_1), \sigma^\Delta(\nu_2)) &= [\![\sigma^\Delta(\nu_1), \sigma^\Delta(\nu_2)]\!] - [\![\mathcal{J}(\sigma^\Delta(\nu_1)), \mathcal{J}(\sigma^\Delta(\nu_2))]\!] \\
&\quad + \mathcal{J}\Big([\![\mathcal{J}(\sigma^\Delta(\nu_1)), \sigma^\Delta(\nu_2)]\!] + [\![\sigma^\Delta(\nu_1), \mathcal{J}(\sigma^\Delta(\nu_2))]\!]\Big) \\
&= \sigma^\Delta([\![\nu_1, \nu_2]\!]_\Delta) - \widetilde{R_\Delta(\nu_1, \nu_2)(\cdot, 0)} \\
&\quad - \sigma^\Delta([\![j(\nu_1), j(\nu_2)]\!]_\Delta) + \widetilde{R_\Delta(j(\nu_1), j(\nu_2))(\cdot, 0)} \\
&\quad + \mathcal{J}\bigg(\sigma^\Delta([\![j(\nu_1), \nu_2]\!]_\Delta) + \sigma^\Delta([\![\nu_1, j(\nu_2)]\!]_\Delta) \\
&\quad - \widetilde{R_\Delta(j(\nu_1), \nu_2)(\cdot, 0)} - \widetilde{R_\Delta(\nu_1, j(\nu_2))(\cdot, 0)}\bigg) \\
&= \sigma^\Delta([\![\nu_1, \nu_2]\!]_\Delta) - \widetilde{R_\Delta(\nu_1, \nu_2)(\cdot, 0)} \\
&\quad - \sigma^\Delta([\![j(\nu_1), j(\nu_2)]\!]_\Delta) + \widetilde{R_\Delta(j(\nu_1), j(\nu_2))(\cdot, 0)} \\
&\quad + \sigma^\Delta(j([\![j(\nu_1), \nu_2]\!]_\Delta)) + \sigma^\Delta(j([\![\nu_1, j(\nu_2)]\!]_\Delta)) \\
&\quad - \widetilde{j_C \circ R_\Delta(j(\nu_1), \nu_2)(\cdot, 0)} - \widetilde{j_C \circ R_\Delta(\nu_1, j(\nu_2))(\cdot, 0)} \\
&= \sigma^\Delta(N_{j,[\![\cdot,\cdot]\!]_\Delta}(\nu_1, \nu_2)) \\
&\quad + \widetilde{R_\Delta(j(\nu_1), j(\nu_2))(\cdot, 0)} - \widetilde{R_\Delta(\nu_1, \nu_2)(\cdot, 0)} \\
&\quad - \widetilde{j_C \circ R_\Delta(j(\nu_1), \nu_2)(\cdot, 0)} - \widetilde{j_C \circ R_\Delta(\nu_1, j(\nu_2))(\cdot, 0)}
\end{aligned}$$

Note that since the dull bracket on $TM \oplus E^*$ is anchored by pr_{TM} and $\Delta_{(X,\varepsilon)}(0, \theta) = (0, \mathcal{L}_X \theta)$ the curvature $R_\Delta(\nu_1, \nu_2)(0, \theta)$ for $\theta \in \Gamma(T^*M)$ always vanishes. Therefore the terms with R_Δ above evaluated at $(e, 0)$ vanish if and only if the corresponding terms vanish evaluated at (e, θ) for any $\theta \in \Gamma(T^*M)$. Thus the Nijenhuis tensor of $\mathcal{J}$ vanishes for all sections if and only if the Nijenhuis tensor of j with respect to $[\![\cdot,\cdot]\!]_\Delta$ vanishes and additionally the curvature of the adapted Δ satisfies

$$\begin{aligned}
0 = &R_\Delta(j(\nu_1), j(\nu_2))(\tau) - R_\Delta(\nu_1, \nu_2)(\tau) \\
&- j_C\Big(R_\Delta(j(\nu_1), \nu_2)(\tau)\Big) - j_C\Big(R_\Delta(\nu_1, j(\nu_2))(\tau)\Big),
\end{aligned} \tag{4.24}$$

for all $\nu_1, \nu_2 \in \Gamma(TM \oplus E^*)$ and $\tau \in \Gamma(E \oplus T^*M)$.

We want to dualise this equation again in order to get a condition purely depending on sections of $TM \oplus E^*$ and not additionally on a section of $E \oplus T^*M$. We will in the following write $\mathbf{Jac}_\Delta \in \Omega^3(TM \oplus E^*, TM \oplus E^*)$ for the dual of the curvature R_Δ, that is the Jacobiator of the dull bracket $[\![\cdot,\cdot]\!]_\Delta$:

$$\begin{aligned}
\mathbf{Jac}_\Delta(\nu_1, \nu_2, \nu_3) :=& R_\Delta(\nu_1, \nu_2)^t(\nu_3) \\
=& [\![[\![\nu_1, \nu_2]\!]_\Delta, \nu_3]\!]_\Delta + [\![\nu_2, [\![\nu_1, \nu_3]\!]_\Delta]\!]_\Delta - [\![\nu_1, [\![\nu_2, \nu_3]\!]_\Delta]\!]_\Delta \\
=& - [\![\nu_1, [\![\nu_2, \nu_3]\!]_\Delta]\!]_\Delta + \text{cyclic permutations},
\end{aligned} \tag{4.25}$$

where the cyclic permutations are over the set $\{1,2,3\}$. Skew-symmetry of the bracket implies that $\mathbf{Jac}_\Delta$ is indeed alternating. By pairing the right hand side of (4.24) with a third section ν_3 of $TM \oplus E$ we then get the condition

$$\begin{aligned} 0 = \mathbf{Jac}_\Delta(j\nu_1, j\nu_2, \nu_3) + \mathbf{Jac}_\Delta(j\nu_1, \nu_2, j\nu_3) + \mathbf{Jac}_\Delta(\nu_1, j\nu_2, j\nu_3) \\ - \mathbf{Jac}_\Delta(\nu_1, \nu_2, \nu_3)\,, \end{aligned} \tag{4.26}$$

Using (4.25) we can write (4.26), without writing out all cyclic permutations of ν_1, ν_2 and ν_3 explicitly, as follows

$$\begin{aligned} 0 &= [\![\nu_1, [\![\nu_2, \nu_3]\!]_\Delta]\!]_\Delta - [\![j\nu_1, [\![j\nu_2, \nu_3]\!]_\Delta]\!]_\Delta - [\![j\nu_1, [\![\nu_2, j\nu_3]\!]_\Delta]\!]_\Delta - [\![\nu_1, [\![j\nu_2, j\nu_3]\!]_\Delta]\!]_\Delta \\ &\quad + \text{cyclic permutations in 1,2,3} \\ &= [\![\nu_1, [\![\nu_2, \nu_3]\!]_\Delta - [\![j\nu_2, j\nu_3]\!]_\Delta]\!]_\Delta + [\![j\nu_1, -[\![\nu_2, j\nu_3]\!]_\Delta - [\![j\nu_2, \nu_3]\!]_\Delta]\!]_\Delta \\ &\quad + \text{cyclic permutations in 1,2,3}\,. \end{aligned} \tag{4.27}$$

Let us now define a bracket $\mathbb{A}$ on $\Gamma(TM \oplus E^*)$ by

$$\mathbb{A}(\nu_1, \nu_2) := \frac{1}{2}\Big([\![\nu_1, \nu_2]\!]_\Delta - [\![j\nu_1, j\nu_2]\!]_\Delta\Big)\,. \tag{4.28}$$

Then $N_{j,[\![\cdot,\cdot]\!]_\Delta}$ vanishes if and only if $\mathbb{A}$ satisfies

$$\mathbb{A}(\nu_1, j\nu_2) = j\mathbb{A}(\nu_1, \nu_2)\,. \tag{4.29}$$

Furthermore, (4.27) is now immediately seen to be equivalent to the Jacobi identity of this bracket $\mathbb{A}$, and the right hand side of (4.27) is the Jacobiator of $\mathbb{A}$.

Note that the bracket $\mathbb{A}$ does not admit an anchor on $TM \oplus E^*$, and is thus not a Lie algebroid bracket on $TM \oplus E^*$. However, $\mathbb{A}$ is skew-symmetric and $\mathbb{R}$-bilinear. Furthermore, it is independent of the choice of adapted Dorfman connection Δ as proved in the more general case of any Dorfman connection in the appendix (see Lemma A.1.4).

Summarising we have proved the following proposition.

Proposition 4.1.18. *Let $\mathcal{J}$ be a generalised almost complex structure on a vector bundle E and Δ an adapted Dorfman connection. Then the Nijenhuis tensor of $\mathcal{J}$ vanishes if and only if the non-anchored bracket $\mathbb{A}$ defined by* (4.28) *satisfies*

1. *the Jacobi identity,*
2. $\mathbb{A}(\nu_1, j\nu_2) = j\mathbb{A}(\nu_1, \nu_2)$ *for all* ν_1 ν_2 *in* $\Gamma(TM \oplus E^*)\,.$

The following Theorem is an immediate consequence of Proposition 4.1.10, Proposition 4.1.15 and Proposition 4.1.18.

Theorem 4.1.19. *A linear generalised complex structure on a vector bundle* $E \to M$ *is equivalent to a vector bundle morphism* $j\colon TM \oplus E^* \to TM \oplus E^*$ *and a* j*-equivalence class of skew-symmetric* $(TM \oplus E^*)$*-Dorfman connections* Δ *on* $E \oplus T^*M$ *such that* j *and the non-anchored bracket* $\mathbb{A}$ *defined by* $\mathbb{A}(\nu_1, \nu_2) := \frac{1}{2}([\![\nu_1, \nu_2]\!]_\Delta - [\![j\nu_1, j\nu_2]\!]_\Delta)$ *satisfy the following:*

1. $j^2 = -\operatorname{id}_{TM\oplus E^*}$,
2. $\mathbb{A}$ *satisfies the Jacobi identity,*
3. $\mathbb{A}(\nu_1, j\nu_2) = j\mathbb{A}(\nu_1, \nu_2)$.

Note that the bracket $\mathbb{A}$ *does not depend on the choice of representative* Δ *in the* j*-equivalence class.*

Proof. Given a linear generalised complex structure $\mathcal{J}$ on E as in (4.2) we have by Proposition 4.1.10, Proposition 4.1.15 and Proposition 4.1.18 a vector bundle morphism j and a Dorfman connection Δ, defining $\mathbb{A}$, such that j and $\mathbb{A}$ satisfy all of the properties above.

Conversely, given j and a Dorfman connection Δ as above, we define a double vector bundle morphism $\mathcal{J}\colon TE \oplus T^*E \to TE \oplus T^*E$ by setting for $\tau \in \Gamma(E \oplus T^*M)$ and $\nu \in \Gamma(TM \oplus E^*)$

$$\mathcal{J}(\tau^\uparrow) := (-j^t(\tau))^\uparrow ,$$
$$\mathcal{J}(\sigma^\Delta(\nu)) := \sigma^\Delta(j(\nu)) .$$

Again by Proposition 4.1.10 and Proposition 4.1.18 this defines a linear generalised complex structure on E. These two constructions are inverse to each other up to the choice of a different j-equivalent Dorfman connection. □

4.1.4 Generalised Kähler structures on vector bundles

Generalised Kähler structures were introduced by Marco Gualtieri first in his thesis [30] and later in [32]. In this section we will show that in the case of a generalised Kähler structure on a vector bundle E we can always find a Dorfman connection which is adapted in the sense of Proposition 4.1.15 to both the generalised complex structures simultaneously.

First let us recall the definition of a generalised Kähler structure given by Gualtieri in [30]. Recall that the generalised tangent bundle $TM \oplus T^*M$ of a manifold M is equipped with a non-degenerate pairing as described in Example 2.3.3, allowing to identify $TM \oplus T^*M$ with its dual. After this identification any automorphism G of $TM \oplus T^*M$ which is symmetric, that is $G^t = G$, and squares to id defines a symmetric metric on $TM \oplus T^*M$. Thus such an automorphism will here be called a metric. Now we can state the definition of generalised Kähler structures.

Definition 4.1.20. *A generalised Kähler structure on a manifold is a pair of commuting generalised complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ such that the symmetric non-degenerate metric $G := -\mathcal{J}_1 \circ \mathcal{J}_2$ is positive definite.*

Now let us take a vector bundle $E \to M$ equipped with such a generalised Kähler structure where both generalised complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ are linear. Denote the side morphisms on $TM \oplus E^*$ by j_1 and j_2, respectively. Take now any skew-symmetric $TM \oplus E^*$-Dorfman connection Δ on $E \oplus T^*M$. This gives rise to the corresponding 2-forms Ψ_1 and Ψ_2 as in Proposition 4.1.10. We will now prove the following compatibility between Ψ_2 and j_1.

Lemma 4.1.21. *In the setting above, we have*

$$\Psi_2(j_1\nu_1, \nu_2) + \Psi_2(\nu_1, j_1\nu_2) = \Psi_1(j_2\nu_1, \nu_2) + \Psi_1(\nu_1, j_2\nu_2)\,, \tag{4.30}$$

for all $\nu_1, \nu_2 \in TM \oplus E^*$.

Proof. Since $\mathcal{J}_1$ and $\mathcal{J}_2$ commute, so do their side morphisms j_1 and j_2. We now use the definition of Φ_1 and Φ_2 in Lemma 4.1.5 and the action of the generalised complex structure on core-linear section from Lemma 4.1.6 in order to compute for any $\nu \in \Gamma(TM \oplus E^*)$ the following:

$$\begin{aligned}
0 &= \sigma^\Delta(j_1 j_2 \nu) - \sigma^\Delta(j_2 j_1 \nu) \\
&= \mathcal{J}_1 \sigma^\Delta(j_2\nu) - \widetilde{\Phi_1(j_2\nu)} - \mathcal{J}_2\sigma^\Delta(j_1\nu) - \widetilde{\Phi_2(j_1\nu)} \\
&= \mathcal{J}_1\mathcal{J}_2\sigma^\Delta(\nu) - \mathcal{J}_1\widetilde{\Phi_2(\nu)} - \widetilde{\Phi_1(j_2\nu)} \\
&\quad - \mathcal{J}_2\mathcal{J}_1\sigma^\Delta(\nu) + \mathcal{J}_2\widetilde{\Phi_1(\nu)} + \widetilde{\Phi_2(j_1\nu)} \\
&= -\widetilde{-j_1^t \circ \Phi_2(\nu)} - \widetilde{\Phi_1(j_2\nu)} \\
&\quad + \widetilde{-j_2^t \circ \Phi_1(\nu)} + \widetilde{\Phi_2(j_1\nu)}\,,
\end{aligned}$$

since the core morphisms of $\mathcal{J}_1$ and $\mathcal{J}_2$ are $-j_1^t$ and $-j_2^t$, respectively. Thus we obtain $j_1^t \circ \Phi_2(\nu) - \Phi_1(j_2\nu) = j_2^t \circ \Phi_1(\nu) - \Phi_2(j_1\nu)$ and pairing this with a second arbitrary section $\nu_2 \in \Gamma(TM \oplus E^*)$ and dualising then gives after renaming $\nu_1 := \nu$ the desired equality

$$\Psi_2(j_1\nu_1, \nu_2) + \Psi_2(\nu_1, j_1\nu_2) = \Psi_1(j_2\nu_1, \nu_2) + \Psi_1(\nu_1, j_2\nu_2)\,. \qquad \square$$

With this lemma we can easily prove the existence of a Dorfman connection adapted to both generalised complex strctures.

Proposition 4.1.22. *Given two commuting linear generalised complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ on a vector bundle $E \to M$, there is a $TM \oplus E^*$-Dorfman connection Δ on $E \oplus T^*M$ which is adapted to both $\mathcal{J}_1$ and $\mathcal{J}_2$.*

Proof. Take a skew-symmetric Dorfman connection Δ_1, which is adapted to $\mathcal{J}_1$ as constructed in Proposition 4.1.15. Thus the 2-form Ψ_1 vanishes. The previous Lemma 4.1.21 then shows that $\Psi_2(\nu_1, \nu_2) = \Psi_2(j_1\nu_1, j\nu_2)$ for all $\nu_1, \nu_2 \in \Gamma(TM \oplus E^*)$. In order to obtain a Dorfman connection Δ_2 adapted to $\mathcal{J}_2$ we have to use as change of splitting the form $\Psi_{12} := -\frac{1}{2}\Psi_2(\cdot, j_2\cdot)$ as shown in the proof of Proposition 4.1.15. But since j_1 and j_2 commute we get immediately that also this change of splittings satisfies $\Psi_{12}(j_1\nu_1, j_1\nu_2) = \Psi_{12}(\nu_1, \nu_2)$ for all $\nu_1, \nu_2 \in TM \oplus E^*$. Thus Δ_2 is j_1-equivalent to Δ_1 as in Definition 4.1.16. Now Lemma 4.1.17 shows that Δ_2 is still adapted to $\mathcal{J}_1$ and therefore to both generalised complex structures simultaneously. □

4.2 Complex VB-Dirac structures

In this section we make use of the equivalence of a generalised complex structure and a pair of complex conjugated Dirac structures in the complexified generalised tangent bundle as described in Section 2.4. We will show that a linear generalised complex structure on a vector bundle E is equivalent to a pair of complex conjugated VB-Dirac structures.

Let us for that end consider the complexification of $\mathbb{T}E$ as a vector bundle over E. As described in Proposition 2.5.8 this is again a double vector bundle $\mathbb{T}_\mathbb{C}E \cong T_\mathbb{C}E \oplus T^*_\mathbb{C}E$ with complexified core and side bundle.

$$\begin{array}{ccccc} T_\mathbb{C}E \oplus T^*_\mathbb{C}E & \longrightarrow & & & E \\ \Big\downarrow (\pi_{TM\oplus E^*})_\mathbb{C} & & E_\mathbb{C} \oplus T^*_\mathbb{C}M & & \Big\downarrow \\ & & & \searrow & \\ T_\mathbb{C}M \oplus E^*_\mathbb{C} & \longrightarrow & & & M \end{array} \qquad (4.31)$$

We will in the following need linear splittings of this double vector bundle and in particular need the following statement which is merely a complex linear extension of Theorem 2.3.18 proved in [36].

Proposition 4.2.1. *Consider the double vector bundle $\mathbb{T}_{\mathbb{C}}E$ as in* (4.31)*. A linear splitting Σ of $\mathbb{T}_{\mathbb{C}}E$ which is additionally a morphism of complex vector bundles over the side E gives rise to a complex $(T_{\mathbb{C}}M \oplus E^*_{\mathbb{C}})$-Dorfman connection on $E_{\mathbb{C}} \oplus T^*_{\mathbb{C}}M$ by*

$$\Sigma\Big((X,\varepsilon)(m), e(m)\Big) = \Big((T_m e)_{\mathbb{C}} X(m), (\mathbf{d}_{\mathbb{C}})_{e_m}\ell_\varepsilon\Big) - \Big(\Delta_{(X,\varepsilon)}(e,0)\Big)^{\uparrow}(e(m)) \quad (4.32)$$

and $\Delta_{(X,\varepsilon)}(0,\theta) = (0, \mathcal{L}^{\mathbb{C}}_X\theta)$ for all sections $X \in \Gamma(T_{\mathbb{C}}M)$, $\varepsilon \in \Gamma(E^*_{\mathbb{C}})$, $\theta \in \Gamma(T^*_{\mathbb{C}}M)$ and $e \in \Gamma(E)$. Here we denote by $\mathbf{d}_{\mathbb{C}}$ and $\mathcal{L}^{\mathbb{C}}$ the complex linear and complex bilinear extensions of $\mathbf{d}$ and $\mathcal{L}$. Conversely, given such a complex Dorfman connection, (2.51) *defines a linear splitting which is additionally complex linear over E and these constructions are inverse to each other.*

Proof. This follows from an argument directly analogously to the proof of Theorem 4.1. in [36].

The only additional observation to be made is that defining $\Delta_\nu(e,0)$ for sections $e \in \Gamma(E)$ and $\nu \in \Gamma(T_{\mathbb{C}}M \oplus E^*_{\mathbb{C}})$ already determines Δ_ν completely on the complexification $\Gamma(E_{\mathbb{C}})$ by complex linearity. □

Lemma 4.2.2. *Let Δ be a real $(TM \oplus E^*)$-Dorfman connection on $E \oplus T^*M$ and Σ be the corresponding linear splitting. Denoting the complexification of Δ by $\Delta^{\mathbb{C}}$. Then the corresponding linear splitting according to Proposition 4.2.1 is the complexification $\Sigma^{\mathbb{C}}$ of Σ, that is the complex linear extension of Σ in $TM \oplus E^*$.*

Proof. This is immediate since both the complexified splitting and the complexified Dorfman connection are given by complex linear extensions. □

Let $\mathcal{J}\colon \mathbb{T}E \to \mathbb{T}E$ now be a linear generalised complex structure on E over $j\colon TM \oplus E^* \to TM \oplus E^*$ and with core morphism $j_C$as in Definition 4.1.1. Then we obtain get a pair of complex conjugated complex Dirac structures $D_\pm \subset \mathbb{T}_{\mathbb{C}}E$, the $\pm i$-eigenbundles of the complexification of $\mathcal{J}$, which we denote by $\mathcal{J}_{\mathbb{C}} : \mathbb{T}_{\mathbb{C}}E \to \mathbb{T}_{\mathbb{C}}E$. They are given by

$$\begin{aligned} D_+ &= \{\xi - i\mathcal{J}(\xi) \,|\, \xi \in \mathbb{T}_{\mathbb{C}}E\} \\ D_- &= \{\xi + i\mathcal{J}(\xi) \,|\, \xi \in \mathbb{T}_{\mathbb{C}}E\} \end{aligned} \quad (4.33)$$

Since $\mathcal{J}$ is a vector bundle morphism over j, elements of the form $\xi \pm i\mathcal{J}(\xi)$ project under $(\pi_{TM\oplus E^*})_{\mathbb{C}}$ to $\pi_{TM\oplus E^*}(\xi) \pm ij(\pi_{TM\oplus E^*}(\xi))$. The core of $D_\pm$ is given by all elements that project to 0 on both sides, which are all elements of the form $\tau \mp ij_C(\tau)$ for τ in the core of $\mathbb{T}E$, which is $E \oplus T^*M$.

Thus we obtain two sub-double vector bundles of $\mathbb{T}_{\mathbb{C}}E$:

$$\begin{array}{ccc} D_{\pm} & \longrightarrow & E \\ {\scriptstyle (\pi_{TM\oplus E^*})_{\mathbb{C}}|_{D_{\pm}}}\downarrow & K_{\pm} \searrow & \downarrow {\scriptstyle q_E} \\ U_{\pm} & \longrightarrow & M \end{array} ,$$

with

$$\begin{aligned} U_+ &= \{\nu - ij(\nu) |\, \nu \in T_{\mathbb{C}}M \oplus E^*_{\mathbb{C}}\}\,, \\ U_- &= \{\nu + ij(\nu) |\, \nu \in T_{\mathbb{C}}M \oplus E^*_{\mathbb{C}}\}\,, \\ K_+ &= \{\tau - ij_C(\tau) |\, \tau \in E_{\mathbb{C}} \oplus T^*_{\mathbb{C}}M\}\,, \\ K_- &= \{\tau + ij_C(\tau) |\, \tau \in E_{\mathbb{C}} \oplus T^*_{\mathbb{C}}M\}\,. \end{aligned}$$

$U_{\pm}$ is the $\pm i$-eigenbundle of the complexification of j:

$$\begin{aligned} j_{\mathbb{C}} \colon (TM \oplus E^*)_{\mathbb{C}} &\to (TM \oplus E^*)_{\mathbb{C}} \\ (\nu_1 + i\nu_2) &\mapsto j(\nu_1) + ij(\nu_2)\,, \end{aligned} \tag{4.34}$$

whereas $K_{\pm}$ is the $\pm i$-eigenbundle of the complexification of j_C:

$$\begin{aligned} j_{C,\mathbb{C}} \colon (E \oplus T^*M)_{\mathbb{C}} &\to (E \oplus T^*M)_{\mathbb{C}} \\ (\tau_1 + i\tau_2) &\mapsto j_C(\tau_1) + ij_C(\tau_2)\,. \end{aligned} \tag{4.35}$$

Lemma 4.2.3. *Consider a linear generalised almost complex structure on a vector bundle E as in* (4.2)*. Denote by $U_{\pm}$ and $K_{\pm}$ the $\pm i$-eigenbundles of $j_{\mathbb{C}}$ and $j_{C,\mathbb{C}}$, respectively. Then we have*

$$U_{\pm}^{\circ} = K_{\pm}\,, \tag{4.36}$$

where $U_{\pm}^{\circ}$ denotes the annihilator of $U_{\pm}$ in $(TM \oplus E^*)^* \cong E \oplus T^*M$.

Proof. The annihilator of $U_{\pm}$ consists of all elements $\tau_1 + i\tau_2$ of $(E \oplus T^*M)_{\mathbb{C}}$ that pair with every element of $U_{\pm}$ to zero. That is, for every $\nu \in TM \oplus E^*$

$$\begin{aligned} 0 &= \langle \tau_1 + i\tau_2, \nu \mp ij(\nu) \rangle_{\mathbb{C}} \\ &= \langle \tau_1, \nu \rangle \pm \langle \tau_2, j\nu \rangle + i\langle \tau_2, \nu \rangle \mp i\langle \tau_1, j(\nu) \rangle \end{aligned}$$

Since $j^t = -j_C$ by Proposition 4.1.10, both the equation for the real part and for the imaginary part are equivalent to the condition $\tau_1 = \pm j_C \tau_2$ or in other words that $\tau_1 + i\tau_2 = \tau_1 \mp ij_C\tau_1$ is an element of $K_{\pm}$. Thus $U_{\pm}^{\circ} = K_{\pm}$. □

Previously we have defined a bracket $\mathbb{A}$ on $TM \oplus E^*$ in (4.28) using a Dorfman connection adapted to the linear generalised complex structure. We noted that the bracket does not admit an anchor on all of $TM \oplus E^*$, since we can compute for any function $f \in C^\infty(M)$ and $\nu_1, \nu_2 \in \Gamma(TM \oplus E^*)$:

$$\begin{aligned}\mathbb{A}(\nu_1, f\nu_2) &= \frac{1}{2}\Big(-[\![j\nu_1, j(f\nu_2)]\!]_\Delta + [\![\nu_1, f\nu_2]\!]_\Delta\Big) \\ &= \frac{1}{2}\Big(-f[\![j\nu_1, j\nu_2]\!] - \mathrm{pr}_{TM}(j\nu_1)(f)j\nu_2 \\ &\quad + f[\![\nu_1, \nu_2]\!] + \mathrm{pr}_{TM}(\nu_1)(f)\nu_2\Big) \\ &= f\mathbb{A}(\nu_1, \nu_2) - \frac{1}{2}\mathrm{pr}_{TM}(j\nu_1)(f)j\nu_2 + \frac{1}{2}\mathrm{pr}_{TM}(\nu_1)(f)\nu_2\,.\end{aligned} \tag{4.37}$$

Let us now instead consider the complexification of $\mathbb{A}$, defined by extending complex linearly:

$$\mathbb{A}_\mathbb{C}(\nu_1 + i\nu_1', \nu_2 + i\nu_2') = \mathbb{A}(\nu_1, \nu_2) - \mathbb{A}(\nu_1', \nu_2') + i\big(\mathbb{A}(\nu_1, \nu_2') + \mathbb{A}(\nu_1', \nu_2)\big)\,. \tag{4.38}$$

We now show that $\mathbb{A}_\mathbb{C}$ restricted to $U_\pm$ defines a Lie algebroid structure on $U_\pm$, in particular it admits an anchor on this restriction.

Proposition 4.2.4. *Let $\mathcal{J}$ be a linear generalised complex structure on a vector bundle $E \to M$ and Δ a skew-symmetric Dorfman connection adapted to $\mathcal{J}$. Then the restriction of the complexified bracket $\mathbb{A}_\mathbb{C}$ as in* (4.38) *to the $\pm i$-eigenbundles of $j_\mathbb{C}$, denoted by $U_\pm$, coincides with the restriction of $[\![\cdot,\cdot]\!]_{\Delta^\mathbb{C}}$ and defines a complex Lie algebroid structure on $U_\pm$ with anchor $\mathrm{pr}_{T_\mathbb{C}M}|_{U_\pm}$.*

Proof. The condition $\mathbb{A}(\nu_1, j\nu_2) = j\mathbb{A}(\nu_1, \nu_2)$ of Proposition 4.1.18 ensures that we can restrict the complexified bracket $\mathbb{A}_\mathbb{C}$ to the two eigenbundles $U_\pm$ of $j_\mathbb{C}$, as for any $\nu_1, \nu_2 \in \Gamma(TM \oplus E^*)$ we have:

$$\begin{aligned}\mathbb{A}_\mathbb{C}(\nu_1 \mp ij\nu_1, \nu_2 \mp ij\nu_2) &= \mathbb{A}(\nu_1, \nu_2) - \mathbb{A}(j\nu_1, j\nu_2) \mp i\mathbb{A}(\nu_1, j\nu_2) \mp i\mathbb{A}(j\nu_1, \nu_2) \\ &= 2\mathbb{A}(\nu_1, \nu_2) \mp ij2\mathbb{A}(\nu_1, \nu_2)\,,\end{aligned}$$

which is again an element of $U_\pm$.

It follows directly from the definition of $\mathbb{A}$ in (4.28) that the restriction of $\mathbb{A}_\mathbb{C}$ to $U_\pm$ coincides with the complexification of $[\![\cdot,\cdot]\!]_\Delta$ for any adapted Dorfman connection Δ. Therefore it is anchored by the restriction of $\mathrm{pr}_{T_\mathbb{C}M}$ and satisfies the Leibniz identity. Furthermore, since Δ is skew-symmetric, so is the complexification and according to Theorem 4.1.19 $\mathbb{A}$ it satisfies the Jacobi identity on all of $\Gamma(TM \oplus E^*)$. Therefore the complexified bracket $\mathbb{A}_\mathbb{C}$ also satisfies the Jacobi identity and defines indeed a Lie algebroid structure on $U_\pm$. □

In [36], Jotz Lean described VB-Dirac structures using splittings corresponding to adapted Dorfman connections, which were defined in the same paper as follows.

Definition 4.2.5. *Given a sub-double vector bundle $D \subseteq TE \oplus T^*E$ over $U \subseteq TM \oplus E^*$ with core $K \subseteq E \oplus T^*M$, then a Dorfman connection Δ as in* (4.8) *is called **adapted to** D if D is spanned by the sections $k^\uparrow$ for $k \in \Gamma(K)$ and $\sigma^\Delta(u)$ for $u \in \Gamma(U)$.*

Analogously, for a sub-double vector bundle $D \subseteq T_\mathbb{C}E \oplus T^*_\mathbb{C}E$ which is additionally a complex subbundle over E over $U \subseteq T_\mathbb{C}M \oplus E^*_\mathbb{C}$ with core $K \subseteq E_\mathbb{C} \oplus T^*_\mathbb{C}M$, a complex $(T_\mathbb{C}M \oplus E^*_\mathbb{C})$-Dorfman connection Δ on $E_\mathbb{C} \oplus T^*_\mathbb{C}M$ is adapted to D if D is spanned by the sections $k^\uparrow$ for $k \in \Gamma(K)$ and $\sigma^\Delta(u)$ for $u \in \Gamma(U)$.

The existence of an adapted real Dorfman connection for any such sub-double vector bundle is also proved in [36]. The corresponding result for the complexified generalised tangent bundle follows from the same argument by extending everything $\mathbb{C}$-linearly.

Now we will show that the complexification of a Dorfman connection Δ which is adapted to $\mathcal{J}$ is adapted to both eigenbundles $D_\pm$ simultaneously. The brackets on $U_\pm$ given by restriction of the complex dull bracket corresponding to a complexified adapted Dorfman connection are independent of the choice of the adapted Δ, these complex dull brackets can only differ when evaluated on $\Gamma(U_+) \times \Gamma(U_-)$.

Proposition 4.2.6. *Given a linear generalised complex structure $\mathcal{J}$ on E and a Dorfman connection Δ adapted to $\mathcal{J}$, then the complexification $\Delta^\mathbb{C}$ is adapted to both D_+ and D_-.*

Proof. That the lift sends sections of $U_\pm$ to sections of $D_\pm$ follows directly from the fact that $\mathcal{J}(\sigma^\Delta(\nu)) = \sigma^\Delta(j\nu)$. As $D_\pm$ is a double vector bundle over $U_\pm$ and E with core $K_\pm$, the space of sections $\Gamma_E(D_\pm)$ is generated by core sections and the lifts of sections of $U_\pm$. Since $\sigma^{\Delta^\mathbb{C}}$ is by definition a horizontal lift for the double vector bundle $\mathbb{T}_\mathbb{C}E$ and $\sigma^{\Delta^\mathbb{C}}(\Gamma(U_\pm)) \subseteq \Gamma_E(D_\pm)$, this implies that the complexified Dorfman connection $\Delta^\mathbb{C}$ is adapted to $D_\pm$, simultaneously. □

Corollary 4.2.7. *In the situation above, $\Delta^\mathbb{C}_u$ for $u \in \Gamma(U_\pm)$ preserves $\Gamma(K_\pm)$.*

Proof. Let $k \in \Gamma(K_\pm)$ and $u, u_2 \in \Gamma(U_\pm)$. According to Lemma 4.2.3 we have $K_\pm = U_\pm^\circ$ and according to Proposition 4.2.4 the bracket $[\![\cdot,\cdot]\!]_{\Delta^\mathbb{C}}$ preserves $U_\pm$. Therefore

$$\langle \Delta^\mathbb{C}_u k, u_2 \rangle = \mathrm{pr}_{T_\mathbb{C}M}(u)(\langle k, u_2 \rangle) - \langle k, [\![u, u_2]\!]_{\Delta^\mathbb{C}} \rangle = 0\,,$$

and $\Delta^\mathbb{C}_u k$ is again a section of $K_\pm$. □

In [36] Jotz Lean proved the following description of VB-Dirac structures in the generalised tangent bundle $\mathbb{T}E$.

Theorem 4.2.8. *Let D be a sub-double vector bundle of $\mathbb{T}E$ over E and $U \subseteq TM \oplus E^*$, with core $K \subseteq E \oplus T^*M$ Let Δ be a $(TM \oplus E^*)$-Dorfman connection on $E \oplus T^*M$ which is adapted to D. Then D is a VB-Dirac structure if and only if $U = K^\circ$ and $(U, \mathrm{pr}_{TM}|_U, [\![\cdot,\cdot]\!]_\Delta|_U)$ is a Lie algebroid.*

Extending everything in this theorem complex linearly then immediately gives the following corollary about complex VB-Dirac structures.

Corollary 4.2.9. *Let D be a sub-double vector bundle of $\mathbb{T}_\mathbb{C}E$ over E and $U \subseteq T_\mathbb{C}M \oplus E^*_\mathbb{C}$, with core $K \subseteq E_\mathbb{C} \oplus T^*_\mathbb{C}M$ such that D is a complex subbundle of $\mathbb{T}_\mathbb{C}E$. Let Δ be a complex $(T_\mathbb{C}M \oplus E^*_\mathbb{C})$-Dorfman connection on $E_\mathbb{C} \oplus T^*_\mathbb{C}M$ which is adapted to D. Then D is a complex VB-Dirac structure if and only if $U = K^\circ$ and $(U, \mathrm{pr}_{T_\mathbb{C}M}|_U, [\![\cdot,\cdot]\!]_\Delta|_U)$ is a complex Lie algebroid.*

This description of complex VB-Dirac structures together with Theorem 4.1.19 then leads to the following corollary.

Corollary 4.2.10. *A linear generalised complex structure $\mathcal{J}$ on a vector bundle E is equivalent to a pair of transverse, complex conjugated complex VB-Dirac structures $D_\pm$ in $\mathbb{T}_\mathbb{C}E$.*

Proof. Given a linear generalised complex structure $\mathcal{J}$ and Δ an adapted Dorfman connection given by Proposition 4.1.15. Let $D_\pm$ be the $\pm i$-eigenbundles of $\mathcal{J}$. By Proposition 4.2.4 the restriction of $[\![\cdot,\cdot]\!]_{\Delta^\mathbb{C}}$ to $U_\pm$ defines a Lie algebroid structure. Additionally, according to Lemma 4.2.3 we have $U_\pm^\circ = K_\pm$ or equivalently $U_\pm = K_\pm^\circ$. With Theorem 4.2.8 this shows that $D_\pm$ are complex VB-Dirac structures in $\mathbb{T}_\mathbb{C}E$. That they are transverse and complex conjugated, that is $\mathbb{T}_\mathbb{C}E = D_+ \oplus D_-$ and $D_+ = \overline{D_-}$ follows directly from the definition in (4.33).

Conversely, given a pair of transverse, complex conjugated complex VB-Dirac structures $D_\pm$ in $\mathbb{T}_\mathbb{C}E$ over $U_\pm$ and with core $K_\pm$, we can write any element $\nu \in T_\mathbb{C}M \oplus E^*_\mathbb{C}$ as $\nu := u + \overline{u'}$ for $u, u' \in \Gamma(U_+)$. We then set

$$j_\mathbb{C}(\nu) := i \cdot u - i \cdot \overline{u'}\,.$$

This defines a vector bundle morphism $j_\mathbb{C}\colon T_\mathbb{C}M \oplus E^*_\mathbb{C} \to T_\mathbb{C}M \oplus E^*_\mathbb{C}$, which additionally respects the conjugation, that is $j_\mathbb{C}(\overline{\nu}) = \overline{j_\mathbb{C}\nu}$ for any $\nu \in T_\mathbb{C}M \oplus E^*_\mathbb{C}$. Thus $j_\mathbb{C}$ is the complexification of a real form $j\colon TM \oplus E^* \to TM \oplus E^*$.

Now choose any complex Dorfman connection Δ^+ adapted to D_+. Denote the corresponding lift by $\sigma^{\Delta^+}\colon \Gamma(T_\mathbb{C}M \oplus E^*_\mathbb{C}) \to \Gamma^\ell_E(T_\mathbb{C}E \oplus T^*_\mathbb{C}E)$. Then we define a new lift by setting for $u, u' \in \Gamma(U_+)$

$$\sigma(u + \overline{u'}) := \sigma^{\Delta^+}(u) + \overline{\sigma^{\Delta^+}(u')}\,.$$

Now this is still $C^\infty(M, \mathbb{C})$-linear and since the vector bundle structure of $T_{\mathbb{C}}E \oplus T^*_{\mathbb{C}}E$ over $T_{\mathbb{C}}M \oplus E^*_{\mathbb{C}}$ is given by the complexification of the corresponding real structure, it respects the complex conjugation and therefore $\sigma(u + \overline{u'})$ is indeed again a linear section over $u + \overline{u'}$. Thus σ gives rise to a second complex Dorfman connection $\Delta^{\mathbb{C}}$. This Dorfman connection is now by construction adapted to both D_+ and D_-. It is easy to see from (4.32) that this Dorfman connection additionally respects the complex conjugation in the sense that $\Delta^{\mathbb{C}}_{\overline{\nu}}\overline{\tau} = \overline{\Delta^{\mathbb{C}}_{\nu}\tau}$ and is therefore the complexification of a real Dorfman connection Δ. Now the dull bracket corresponding to $\Delta^{\mathbb{C}}$ restricts to the Lie algebroid structures on U_+ and U_- and therefore the bracket $\mathbb{A}$ defined by $\mathbb{A}(\nu_1, \nu_2) = \frac{1}{2}(\llbracket \nu_1, \nu_2 \rrbracket - \llbracket j\nu_1, j\nu_2 \rrbracket)$ satisfies all properties of Theorem 4.1.19. A different choice of adapted Dorfman connection Δ^+ will lead to a Dorfman connection $\Delta^{\mathbb{C}}$ where the dull bracket agrees with the first on $\Gamma(U_\pm) \times \Gamma(U_\pm)$ and only differs on $\Gamma(U_\pm) \times \Gamma(U_\mp)$. This is equivalent to the two Dorfman connections being j-equivalent. Thus by Theorem 4.1.19 this gives rise to a linear generalised complex structure as required.

That these two constructions are inverse to each other follows directly from the corresponding result for generalised complex structures on any manifold. □

4.3 Generalised complex structures on Lie algebroids

Now we want to consider the case where the vector bundle has the additional structure of a Lie algebroid $A \to M$. In this case the generalised tangent bundle $\mathbb{T}A$ is itself a Lie algebroid over the side $TM \oplus A^*$. This structure is described in Example 2.2.11 in the background section. We will give a description of the compatibility of a linear generalised complex structure on A with this Lie algebroid structure in terms of a linear splitting corresponding to a skew-symmetric adapted Dorfman connection

$$\Delta\colon \Gamma(TM \oplus A^*) \times \Gamma(A \oplus T^*M) \to \Gamma(A \oplus T^*M)\,. \tag{4.39}$$

To compute the compatibility with the Lie algebroid structure we will need the following definitions and results from [36] describing the Lie algebroid structure $\mathbb{T}A \to TM \oplus A^*$.

Definition 4.3.1. *Given a Lie algebroid $A \to M$ with anchor ρ and a skew-symmetric Dorfman connection Δ as in* (4.39) *we define*

$$\Omega\colon \Gamma(TM \oplus A^*) \times \Gamma(A) \to \Gamma(A \oplus T^*M) \tag{4.40}$$

and for $a \in \Gamma(A)$ *derivations* $\mathcal{L}_a$ *of* $\Gamma(TM \oplus A^*)$ *and of* $\Gamma(A \oplus T^*M)$ *over* $\rho(a)$ *by setting for* $(X, \alpha) \in \Gamma(TM \oplus A^*)$ *and* $(a', \theta) \in \Gamma(A \oplus T^*M)$

$$\Omega_{(X,\alpha)}a := \Delta_{(X,\alpha)}(a, 0) - (0, \mathbf{d}\langle \alpha, a\rangle) , \tag{4.41}$$

$$\mathcal{L}_a(X, \alpha) := ([\rho(a), X], \mathcal{L}_a\alpha) , \tag{4.42}$$

$$\mathcal{L}_a(a', \theta) := ([a, a'], \mathcal{L}_{\rho(a)}\theta) . \tag{4.43}$$

This allows to define the two basic connections associated to a Dorfman connection as given in [36].

Definition 4.3.2. *Given a Lie algebroid* $A \to M$ *with anchor* ρ *and a skew-symmetric Dorfman connection as in* (4.39) *the* ***basic connections associated to*** Δ *are maps*

$$\nabla^{\text{bas}} \colon \Gamma(A) \times \Gamma(TM \oplus A^*) \to \Gamma(TM \oplus A^*) , \tag{4.44}$$

$$\nabla^{\text{bas}} \colon \Gamma(A) \times \Gamma(A \oplus T^*M) \to \Gamma(A \oplus T^*M) . \tag{4.45}$$

They are defined for $a \in \Gamma(A)$*,* $(X, \alpha) \in \Gamma(TM \oplus A^*)$ *and* $(a', \theta) \in \Gamma(A \oplus T^*M)$ *by*

$$\nabla^{\text{bas}}_a(X, \alpha) := (\rho, \rho^t)(\Omega_{(X,\alpha)}a) + \mathcal{L}_a(X, \alpha) , \tag{4.46}$$

$$\nabla^{\text{bas}}_a(a', \theta) := \Omega_{(\rho,\rho^t)(a',\theta)}a + \mathcal{L}_a(a', \theta) . \tag{4.47}$$

The following result is also due to Madeleine Jotz Lean. Since the straight forward computational proof is left to the reader in [36], we decided to write it down here again for the convenience of the reader.

Proposition 4.3.3. *The two basic connections defined in Definition 4.3.2 are ordinary linear connections and are dual to each other, that is for* $a \in \Gamma(A)$*,* $\nu \in \Gamma(TM \oplus A^*)$ *and* $\tau \in \Gamma(A^* \oplus TM)$ *we have*

$$\langle \nabla^{\text{bas}}_a \tau, \nu\rangle = \rho(a)(\langle \tau, \nu\rangle) - \langle \kappa, \nabla^{\text{bas}}_a \nu\rangle . \tag{4.48}$$

Furthermore, they are compatible with the vector bundle morphism (ρ, ρ^t) *in the following sense:*

$$\nabla^{\text{bas}}_a(\rho, \rho^t)(a', \theta) = (\rho, \rho^t)\nabla^{\text{bas}}_a(a', \theta) . \tag{4.49}$$

Proof. Let $a \in \Gamma(A)$, $(X, \alpha) \in \Gamma(TM \oplus A^*)$, $(a', \theta) \in \Gamma(A \oplus T^*M)$ be arbitrary sections and $f \in C^\infty(M)$ be an arbitrary smooth function. Then we compute

$$
\begin{aligned}
\Omega_{f(X,\alpha)}a &= \Delta_{f(X,\alpha)}(a,0) - (0, \mathbf{d}\langle f\alpha, a\rangle) \\
&= f\Delta_{(X,\alpha)}(a,0) + \langle \alpha, a\rangle(0,\mathbf{d}f) - \langle \alpha, a\rangle(0,\mathbf{d}f) - f(0,\mathbf{d}\langle \alpha, a\rangle) \\
&= f\Omega_{(X,\alpha)}a\,. \\
\Omega_{(X,\alpha)}fa &= \Delta_{(X,\alpha)}(fa,0) - (0,\mathbf{d}\langle \alpha, fa\rangle) \\
&= f\Delta_{(X,\alpha)}(a,0) + X(f)(a,0) - \langle \alpha, a\rangle(0,\mathbf{d}f) - f(0,\mathbf{d}\langle \alpha, a\rangle) \\
&= f\Omega_{(X,\alpha)}a + \big(X(f)a, -\langle \alpha, a\rangle \mathbf{d}f\big)\,.
\end{aligned}
$$

That both $\mathcal{L}_a$ are derivations over $\rho(a)$ follows directly from the Leibniz identity for the bracket in A and TM and the fact that the standard Lie derivatives are derivations over $\rho(a)$ and X, respectively.

$$
\begin{aligned}
\mathcal{L}_a(f(X,\alpha)) &= ([\rho(a), fX], \mathcal{L}_a f\alpha) \\
&= f([\rho(a), X], \mathcal{L}_a\alpha) + \rho(a)(f)(X,\alpha) \\
&= f\mathcal{L}_a(X,\alpha) + \rho(a)(f)(X,\alpha)\,, \\
\mathcal{L}_a(f(a',\theta)) &= ([a, fa'], \mathcal{L}_{\rho(a)}f\theta) \\
&= f([a,a'], \mathcal{L}_{\rho(a)}\theta) + \rho(a)(f)(a',\theta) \\
&= f\mathcal{L}_a(a',\theta) + \rho(a)(f)(a',\theta)\,, \\
\mathcal{L}_{fa}(X,\alpha) &= ([\rho(fa), X], \mathcal{L}_{fa}\alpha) \\
&= f([\rho(a), X], \mathcal{L}_a\alpha) + \big(-X(f)\rho(a), \langle \alpha, a\rangle \rho^t \mathbf{d}f\big) \\
&= f\mathcal{L}_a(X,\alpha) + \big(-X(f)\rho(a), \langle \alpha, a\rangle \rho^t \mathbf{d}f\big)\,, \\
\mathcal{L}_{fa}(a',\theta) &= ([fa, a'], \mathcal{L}_{\rho(fa)}\theta) \\
&= f([a,a'], \mathcal{L}_{\rho(a)}\theta) + \big(-\rho(a')(f)a, \langle \theta, \rho(a)\rangle \mathbf{d}f\big) \\
&= f\mathcal{L}_a(a',\theta) + \big(-\rho(a')(f)a, \langle \theta, \rho(a)\rangle \mathbf{d}f\big)\,.
\end{aligned}
$$

Using all these properties, we can show that ∇^{bas} defines ordinary linear connections.

$$
\begin{aligned}
\nabla^{\mathrm{bas}}_{fa}(X,\alpha) &= (\rho,\rho^t)(\Omega_{(X,\alpha)}fa) + \mathcal{L}_{fa}(X,\alpha) \\
&= (\rho,\rho^t)f\Big(\Omega_{(X,\alpha)}a + \big(X(f)a, -\langle \alpha, a\rangle \mathbf{d}f\big)\Big) \\
&\quad + f\mathcal{L}_a(X,\alpha) + \Big(-X(f)\rho(a), \langle \alpha, a\rangle \rho^t \mathbf{d}f\Big) \\
&= f\nabla^{\mathrm{bas}}_a(X,\alpha)\,, \\
\nabla^{\mathrm{bas}}_a f(X,\alpha) &= (\rho,\rho^t)(\Omega_{f(X,\alpha)}a) + \mathcal{L}_a(f(X,\alpha)) \\
&= (\rho,\rho^t)(f\Omega_{(X,\alpha)}a) + f\mathcal{L}_a(X,\alpha) + \rho(a)(f)(X,\alpha) \\
&= f\nabla^{\mathrm{bas}}_a(X,\alpha) + \rho(a)(f)(X,\alpha)\,,
\end{aligned}
$$

$$\begin{aligned}
\nabla^{\text{bas}}_{fa}(a',\theta) &= \Omega_{(\rho,\rho^t)(a',\theta)}fa + \mathcal{L}_{fa}(a',\theta)\\
&= f\Omega_{(\rho,\rho^t)(a',\theta)}a + (\rho(a')(f)a, -\langle\rho^t\theta, a\rangle\mathbf{d}f)\\
&\quad + f\mathcal{L}_a(a',\theta) + \Big(-\rho(a')(f)a, \langle\theta,\rho(a)\rangle\mathbf{d}f\Big)\\
&= f\nabla^{\text{bas}}_a(a',\theta)\,,\\
\nabla^{\text{bas}}_a f(a',\theta) &= \Omega_{(\rho,\rho^t)f(a',\theta)}a + \mathcal{L}_a(f(a',\theta))\\
&= f\Omega_{(\rho,\rho^t)(a',\theta)}a + f\mathcal{L}_a(a',\theta) + \rho(a)(f)(a',\theta)\\
&= f\nabla^{\text{bas}}_a f(a',\theta) + \rho(a)(f)(a',\theta)\,.
\end{aligned}$$

To check that the two basic connections are in duality with each other we compute

$$\begin{aligned}
&\Big\langle\nabla^{\text{bas}}_a(X,\alpha),(a',\theta)\Big\rangle + \Big\langle(X,\alpha),\nabla^{\text{bas}}_a(a'\theta)\Big\rangle\\
&= \Big\langle(\rho,\rho^t)(\Omega_{(X,\alpha)}a) + \mathcal{L}_a(X,\alpha),(a',\theta)\Big\rangle + \Big\langle(X,\alpha),\Omega_{(\rho,\rho^t)(a',\theta)}a + \mathcal{L}_a(a',\theta)\Big\rangle\\
&= \Big\langle(\rho,\rho^t)(\Delta_{(X,\alpha)}(a,0) - (0,\mathbf{d}\langle\alpha,a\rangle)) + ([\rho(a),X],\mathcal{L}_a\alpha),(a',\theta)\Big\rangle\\
&\quad + \Big\langle(X,\alpha),\Delta_{(\rho,\rho^t)(a',\theta)}(a,0) - (0,\mathbf{d}\langle\rho^t\theta,a\rangle) + ([a,a'],\mathcal{L}_{\rho(a)}\theta)\Big\rangle\\
&= \Big\langle-[\![(X,\alpha),(\rho,\rho^t)(a',\theta)]\!]_\Delta,(a,0)\Big\rangle + X\langle a,\rho^t\theta\rangle\\
&\quad - \langle\rho^t\mathbf{d}\langle\alpha,a\rangle,a'\rangle + \langle[\rho(a),X],\theta\rangle + \langle\mathcal{L}_a\alpha,a'\rangle\\
&\quad + \Big\langle-[\![(\rho,\rho^t)(a',\theta),(X,\alpha)]\!]_\Delta,(a,0)\Big\rangle + \rho(a')\langle a,\alpha\rangle\\
&\quad - \langle\mathbf{d}\langle\rho^t\theta,a\rangle,X\rangle + \langle X,\mathcal{L}_{\rho(a)}\theta\rangle + \langle\alpha,[a,a']\rangle\\
&= \rho(a)\langle X,\theta\rangle + \rho(a)\langle\alpha,a'\rangle\\
&= \rho(a)\langle(X,\alpha),(a',\theta)\rangle\,.
\end{aligned}$$

For the compatibility with (ρ,ρ^t) we finally compute

$$\begin{aligned}
&\nabla^{\text{bas}}_a(\rho,\rho^t)(a',\theta) - (\rho,\rho^t)\nabla^{\text{bas}}_a(a',\theta)\\
&= (\rho,\rho^t)\Omega_{(\rho,\rho^t)(a',\theta)}a + \mathcal{L}_a(\rho,\rho^t)(a',\theta) - (\rho,\rho^t)\Omega_{(\rho,\rho^t)(a',\theta)}a - (\rho,\rho^t)\mathcal{L}_a(a',\theta)\\
&= \Big([\rho(a),\rho(a')],\mathcal{L}_a\rho^t\theta\Big) - \Big(\rho([a,a']),\rho^t\mathcal{L}_{\rho(a)}\theta\Big)\\
&= 0\,,
\end{aligned}$$

where the last equality follows from the fact that $\rho([a,a']) = [\rho(a),\rho(a')]$ and the observation that for any $a'' \in \Gamma(A)$

$$\begin{aligned}
\langle\mathcal{L}_a\rho^t\theta - \rho^t\mathcal{L}_{\rho(a)}\theta, a''\rangle &= -\langle\rho^t\theta,[a,a'']\rangle + \rho(a)\langle\rho^t\theta,a''\rangle\\
&\quad + \langle[\rho(a),\rho(a'')]),\theta\rangle - \rho(a)\langle\theta,\rho(a'')\rangle\\
&= 0\,.
\end{aligned}$$

This completes the proof. $\square$

Also in [36] Madeleine Jotz Lean defined the basic curvature associated to a Dorfman connection Δ.

Definition 4.3.4. *Given a Dorfman connection as in* (4.39) *the* ***basic curvature***

$$R_\Delta^{\mathrm{bas}}\colon \Gamma(A)\times\Gamma(A)\times\Gamma(TM\oplus A^*)\to\Gamma(A\oplus T^*M) \tag{4.50}$$

associated to Δ is defined for given $a_1,a_2\in\Gamma(A)$ and $\nu\in\Gamma(TM\oplus A^*)$ by

$$\begin{aligned} R_\Delta^{\mathrm{bas}}(a_1,a_2)\nu := &-\Omega_\nu[a_1,a_2]+\mathcal{L}_{a_1}(\Omega_\nu a_2)-\mathcal{L}_{a_2}(\Omega_\nu a_1)\\ &+\Omega_{\nabla^{\mathrm{bas}}_{a_2}\nu}a_1-\Omega_{\nabla^{\mathrm{bas}}_{a_1}\nu}a_2\,. \end{aligned} \tag{4.51}$$

The proof of the following statement is also left to the reader in [36], so we will repeat the straight forward computation here.

Proposition 4.3.5. *The basic curvature is tensorial and skew-symmetric in a_1 and a_2, in other words $R_\Delta^{\mathrm{bas}}\in\Omega^2(A,\mathrm{Hom}(TM\oplus A^*,A\oplus T^*M))$. The basic curvature encodes the curvature forms of the basic connections in the sense that $R_{\nabla^{\mathrm{bas}}}=R_\Delta^{\mathrm{bas}}\circ(\rho,\rho^t)$ and $R_{\nabla^{\mathrm{bas}}}=(\rho,\rho^t)\circ R_\Delta^{\mathrm{bas}}$.*

Proof. Skew-symmetry is immediate as the Lie bracket is skew-symmetric. To show that R_Δ^{bas} is tensorial we compute for $a_1,a_2\in\Gamma(A)$, $\nu=(X,\alpha)\in\Gamma(TM\oplus A^*)$ and $f\in C^\infty(M)$ using the properties computed in the proof of Proposition 4.3.3 the following:

$$\begin{aligned} R_\Delta^{\mathrm{bas}}(fa_1,a_2)\nu &= -\Omega_\nu[fa_1,a_2]+\mathcal{L}_{fa_1}(\Omega_\nu a_2)-\mathcal{L}_{a_2}(\Omega_\nu fa_1)\\ &\quad+\Omega_{\nabla^{\mathrm{bas}}_{a_2}\nu}fa_1-\Omega_{\nabla^{\mathrm{bas}}_{fa_1}\nu}a_2\\ &= -f\Omega_\nu[a_1,a_2]-\Big(X(f)[a_1,a_2],-\langle\alpha,[a_1,a_2]\rangle\mathbf{d}f\Big)\\ &\quad+\rho(a_2)(f)\Omega_\nu a_1+\Big(X(\rho(a_2)(f))a_1,-\langle\alpha,a_1\rangle\mathbf{d}(\rho(a_2)f)\Big)\\ &\quad+f\mathcal{L}_{a_1}\Omega_\nu a_2+\Big(-\rho(\mathrm{pr}_A\,\Omega_\nu a_2)(f)a_1,\langle\mathrm{pr}_{TM}\,\Omega_\nu a_2,\rho(a_1)\rangle\mathbf{d}f\Big)\\ &\quad-f\mathcal{L}_{a_2}\Omega_\nu a_1-\rho(a_2)(f)\Omega_\nu a_1-\mathcal{L}_{a_2}\Big(X(f)a_1,-\langle\alpha,a_1\rangle\mathbf{d}f\Big)\\ &\quad+f\Omega_{\nabla^{\mathrm{bas}}_{a_2}\nu}a_1+\Big(\mathrm{pr}_{TM}(\nabla^{\mathrm{bas}}_{a_2}\nu)(f)a_1,-\langle\mathrm{pr}_{A^*}(\nabla^{\mathrm{bas}}_{a_2}\nu),a_1\rangle\mathbf{d}f\Big)\\ &\quad-f\Omega_{\nabla^{\mathrm{bas}}_{a_1}\nu}a_2 \end{aligned}$$

$$
\begin{aligned}
&= fR_\Delta^{\mathrm{bas}}(a_1,a_2)(\nu)\\
&\quad + \Big([X,\rho(a_2)](f)a_1, \langle \alpha, [a_1,a_2]\rangle \mathbf{d}f\Big)\\
&\quad + \Big(\rho(\mathrm{pr}_A\,\Omega_\nu a_2)(f)a_1, 0\Big)\\
&\quad + \Big(0, \langle \mathrm{pr}_{T^*M}\,\Omega_\nu a_2, \rho(a_1)\rangle\Big)\\
&\quad + \Big(\mathrm{pr}_{TM}((\rho,\rho^t)\Omega_\nu a_2)(f)a_1 + \mathrm{pr}_{TM}(\mathcal{L}_{a_2}\nu)(f)a_1, 0\Big)\\
&\quad - \Big(0, \Big\langle \mathrm{pr}_{A^*}((\rho,\rho^t)\Omega_\nu a_2 + \mathcal{L}_{a_2}\nu), a_1\Big\rangle \mathbf{d}f\Big)\\
&= fR_\Delta^{\mathrm{bas}}(a_1,a_2)(\nu)\,,
\end{aligned}
$$

where we made use of the facts that $\mathrm{pr}_{TM}\circ(\rho,\rho^t) = \rho\circ\mathrm{pr}_A$ and $\mathrm{pr}_{A^*}\circ(\rho,\rho^t) = \rho\circ\mathrm{pr}_{T^*M}$, of the property of the Lie derivative $\mathcal{L}_Y\mathbf{d}f = \mathbf{d}(Y(f))$ and of skew-symmetry of the Lie bracket.

Finally we also compute

$$
\begin{aligned}
R_\Delta^{\mathrm{bas}}(a_1,a_2)f\nu &= -\Omega_{f\nu}[a_1,a_2] + \mathcal{L}_{a_1}(\Omega_{f\nu}a_2) - \mathcal{L}_{a_2}(\Omega_{f\nu}a_1)\\
&\quad + \Omega_{\nabla^{\mathrm{bas}}_{a_2}f\nu}a_1 - \Omega_{\nabla^{\mathrm{bas}}_{a_1}f\nu}a_2\\
&= -f\Omega_\nu[a_1,a_2] + f\mathcal{L}_{a_1}(\Omega_\nu a_2) + \rho(a_1(f)\Omega_\nu a_2\\
&\quad - \mathcal{L}_{a_2}(\Omega_{f\nu}a_1) - \rho(a_2)(f)\Omega_\nu a_1\\
&\quad + f\Omega_{\nabla^{\mathrm{bas}}_{a_2}\nu}a_1 + \rho(a_2)(f)\Omega_\nu a_1 - f\Omega_{\nabla^{\mathrm{bas}}_{a_1}\nu}a_2 - \rho(a_1(f)\Omega_\nu a_2\\
&= fR_\Delta^{\mathrm{bas}}(a_1,a_2)(\nu)
\end{aligned}
$$

The computations for the equalities $R_{\nabla^{\mathrm{bas}}} = R_\Delta^{\mathrm{bas}}\circ(\rho,\rho^t)$ and $R_{\nabla^{\mathrm{bas}}} = (\rho,\rho^t)\circ R_\Delta^{\mathrm{bas}}$ can be found in [36]. □

Recall from Example 2.2.11 that the generalised tangent bundle $\mathbb{T}A = TA\oplus T^*A$ over a Lie algebroid A has itself a Lie algebroid structure over $TM\oplus A^*$. In [36] Madeleine Jotz Lean proved furthermore the following description of this Lie algebroid structure in terms of core sections and linear sections.

Theorem 4.3.6. *Let $A\to M$ be a Lie algebroid $A\to M$ with anchor ρ and Δ a skew-symmetric Dorfman connection as in* (4.39)*. Write Θ for the anchor of the Lie algebroid $\mathbb{T}A$. Then we have for any $a,a_1,a_2\in\Gamma(A)$ and $\tau,\tau_1,\tau_2\in\Gamma(A\oplus T^*M)$ the following properties*

1. $[\sigma_A^\Delta(a_1),\sigma_A^\Delta(a_2)] = \sigma_A^\Delta([a_1,a_2]) - \widetilde{R_\Delta^{\mathrm{bas}}(a_1,a_2)}\,,$
2. $[\sigma_A^\Delta(a),\tau^\dagger] = (\nabla_a^{\mathrm{bas}}\tau)^\dagger\,,$
3. $[\tau_1^\dagger,\tau_2^\dagger] = 0\,,$

4. $\Theta(\sigma_A^\Delta(a)) = \widehat{\nabla_a^{\mathrm{bas}}} \in \mathfrak{X}^\ell(TM \oplus A^*)$,

5. $\Theta(\tau^\dagger) = ((\rho, \rho^t)\tau)^\uparrow \in \mathfrak{X}^c(TM \oplus A^*)$.

Equivalently, the map $(\rho, \rho^t)\colon A \oplus T^*M \to TM \oplus A^*$, the basic connections ∇^{bas} and the basic curvature R_Δ^{bas} define the 2-term representation up to homotopy corresponding to the VB-algebroid structure on $TA \oplus T^*A$ in the decomposition corresponding to Δ. For a description of this correspondence in general see [28, 19], we also recall it in Section 2.2.3.

4.3.1 Anchor preservation

Let us now consider a Lie algebroid $A \to M$ and a linear generalised complex structure $\mathcal{J}$ on A. We will describe in the following the compatibility of $\mathcal{J}$ with the Lie algebroid structure, in other words we will give conditions for A to be a generalised complex Lie algebroid as in the following definition.

Definition 4.3.7. *A **generalised complex Lie algebroid** is a Lie algebroid $A \to M$ equipped with a linear generalised complex structure $\mathcal{J}\colon \mathbb{T}A \to \mathbb{T}A$ which is additionally a Lie algebroid morphism over the side morphism $j\colon TM \oplus A^* \to TM \oplus A^*$.*

As shown before, any skew-symmetric Dorfman connection then gives rise to $\Phi \in \Gamma((TM \oplus A^*)^* \otimes A^* \otimes (A \oplus T^*M))$ by

$$\mathcal{J}(\sigma_{TM\oplus A^*}^\Delta(\nu)) - \sigma_{TM\oplus A^*}^\Delta(j\nu) = \widetilde{\Phi(\nu)} . \tag{4.52}$$

Equation (4.52) describes the compatibility of the morphism $\mathcal{J}$ with the lift $\sigma^\Delta\colon \Gamma(TM \oplus A^*) \to \Gamma_A^\ell(\mathbb{T}A)$. To give conditions on the compatibility with the Lie structure we will need the analogue for the lift of sections of A instead.

Lemma 4.3.8. *Let Φ be defined by* (4.52) *and identify $(TM\oplus A^*)\otimes A^*\otimes(A\oplus T^*M)$ with* $\mathrm{Hom}(A, \mathrm{Hom}(TM \oplus A^*, A \oplus T^*M))$. *Then we obtain for the lift $\sigma^\Delta(a) \in \Gamma_{TM\oplus A^*}^\ell(\mathbb{T}A)$ of a section $a \in \Gamma(A)$ the following equation.*

$$\mathcal{J}(\sigma^\Delta(a)) = \sigma^\Delta(a) \circ j + \widetilde{\Phi(a) \circ j^{-1}} \circ j . \tag{4.53}$$

Furthermore, on core sections in $\mathbb{T}A \to TM \oplus A^*$ we have $\mathcal{J}(\tau^\dagger) = (j_C \circ \tau)^\dagger \circ j$ for all $\tau \in \Gamma(A \oplus T^*M)$ and on core-linear sections we obtain $\mathcal{J}(\widetilde{\varphi}) = \widetilde{j_C \circ \varphi} \circ j$ for all $\varphi \in \Gamma(\mathrm{Hom}(TM \oplus A^*, A \oplus T^*M))$.

Proof. Since the linear splitting Σ corresponding to the horizontal lift is a double vector bundle homomorphism, this shows immediately that

$$\mathcal{J}(\sigma^\Delta(\nu)(a(m))) = \mathcal{J}(\Sigma(\nu(m), a(m))) = \mathcal{J}(\sigma^\Delta(a)(\nu(m)))$$

for any $a \in \Gamma(A)$ and $\nu \in \Gamma(TM \oplus A^*)$. Furthermore, by definition of core-linear sections we have $\widetilde{\Phi(\nu)}(a_m) = 0^{\mathbb{T}A}_{a_m} +_{TM\oplus A^*} \Phi(\nu(m))(a_m)$ and $\widetilde{\Phi(a)}(\nu_m) = 0^{\mathbb{T}A}_{\nu_m} +_A \Phi(\nu_m)(a(m))$ where we view the core $A \oplus T^*M$ as subset of $\mathbb{T}A$. Using the interchange law in the double vector bundle $\mathbb{T}A$ we then see that $\sigma^\Delta(j\nu)(a_m) +_A \widetilde{\Phi(\nu)}(a_m) = \sigma^\Delta(a)(j\nu_m) +_{TM\oplus A^*} \widetilde{\Phi(a) \circ j^{-1}}(j\nu_m)$.

The equality $\mathcal{J}(\tau^\dagger) = (j_C \circ \tau)^\dagger \circ j$ for all $\tau \in \Gamma(A \oplus T^*M)$ follows directly from the definition of the core morphism j_C. The equality $\mathcal{J}(\widetilde{\varphi}) = \widetilde{j_C \circ \varphi} \circ j$ for all $\varphi \in \Gamma(\mathrm{Hom}(TM \oplus A^*, A \oplus T^*M))$ follows from an argument completely analogous to the proof of Lemma 4.1.6 using local basis sections. This completes the proof. □

In particular, for a Dorfman connection Δ adapted to $\mathcal{J}$ we have $\mathcal{J}(\sigma_A^\Delta(a)) \circ j^{-1} = \sigma_A^\Delta(a)$ and $\mathcal{J}(\tau^\dagger) \circ j^{-1} = (j_C \circ \tau)^\dagger$. From now on we will work again with an adapted Dorfman connection. The corresponding results in the case of a general connection with Φ not vanishing are described in the appendix in section A.2.

Let us denote the anchor of the Lie algebroid $\mathbb{T}A \to TM \oplus A^*$ again by

$$\Theta\colon \mathbb{T}A \to T(TM \oplus A^*)\,.$$

Proposition 4.3.9. *A linear generalised complex structure $\mathcal{J}$ on a Lie algebroid $A \to M$ preserves the anchor of $\mathbb{T}A$ if and only if for all $a \in \Gamma(A)$ and for any adapted Dorfman connection Δ we have*

1. $\left(j \circ (\rho, \rho^t)\right)^t = -j \circ (\rho, \rho^t)\,,$
2. $\nabla_a^{\mathrm{bas}} \circ j = j \circ \nabla_a^{\mathrm{bas}}\,.$

Proof. The anchor preservation condition for $\mathcal{J}$ over j is the following (see [51]):

$$\Theta \circ \mathcal{J} = Tj \circ \Theta\,.$$

We compute firstly for a core section $\tau^\dagger$ using Theorem 4.3.6:

$$\begin{aligned} Tj(\Theta(\tau^\dagger)(\nu_m)) &= Tj\left(((\rho,\rho^t)\tau)^\uparrow(\nu_m)\right) \\ &= Tj\Big(\frac{d}{dt}\Big|_{t=0}(\nu_m + t(\rho,\rho^t)\tau(m))\Big) \\ &= \frac{d}{dt}\Big|_{t=0}(j\nu_m + tj((\rho,\rho^t)\tau(m))) \\ &= \left(j(\rho,\rho^t)\tau\right)^\uparrow(j\nu_m) \end{aligned}$$

On the other hand we get with Lemma 4.3.8:

$$\Theta(\mathcal{J}(\tau^\dagger(\nu_m))) = \Theta((j_C\tau)^\dagger(j\nu_m)) = ((\rho,\rho^t)j_C\tau)^\uparrow(j\nu_m)$$

Thus the anchor is preserved for every core section if and only if

$$(\rho, \rho^t) \circ j_C = j \circ (\rho, \rho^t)\,. \tag{4.54}$$

According to Proposition 4.1.10 we have $j^t = -j_C$ and thus (4.54) is equivalent to $\left(j \circ (\rho, \rho^t)\right)^t = -j \circ (\rho, \rho^t)$.

Now we consider a horizontal lift $\sigma_A^\Delta(a)$ and make use of Theorem 4.3.6 and of Lemma 4.3.8 to obtain the following:

$$\Theta \circ \mathcal{J}(\sigma_A^\Delta(a)(\nu_m)) = \Theta\Big(\sigma_A^\Delta(a)(j\nu_m)\Big) = \widehat{\nabla_a^{\mathrm{bas}}}(j\nu_m)\,, \tag{4.55}$$

and on the other hand:

$$\begin{aligned} Tj \circ \Theta(\sigma_A^\Delta(a)(\nu_m)) &= Tj(\widehat{\nabla_a^{\mathrm{bas}}}(\nu_m)) \\ &= \widehat{j \circ \nabla_a^{\mathrm{bas}} \circ j^{-1}}(j\nu_m)\,. \end{aligned}$$

The anchor preservation condition for horizontal lifts is therefore

$$\nabla_a^{\mathrm{bas}} \circ j = j \circ \nabla_a^{\mathrm{bas}}\,, \tag{4.56}$$

for all $a \in \Gamma(A)$. □

4.3.2 Bracket preservation

In this section we will give conditions, when the generalised complex structure $\mathcal{J}$ is compatible with the Lie bracket of $\mathbb{T}A \to TM \oplus A^*$. We will compute again the conditions for core sections and linear sections separately.

Proposition 4.3.10. *Let $\mathcal{J}$ be a linear generalised complex structure on a Lie algebroid A and Δ be a Dorfman connection adapted to $\mathcal{J}$. Then $\mathcal{J}$ is compatible with the Lie bracket of $\mathbb{T}A \to TM \oplus A^*$ if and only if for all sections $a, b \in \Gamma(A)$*

1. $\nabla_a^{\mathrm{bas}} \circ j = j \circ \nabla_a^{\mathrm{bas}}$,
2. $j_C \circ R_\Delta^{\mathrm{bas}}(a, b) = R_\Delta^{\mathrm{bas}}(a, b) \circ j$.

Proof. Since $\mathcal{J}$ and therefore j are invertible, every section can be pushed forward via $\mathcal{J}$. The bracket is then preserved under $\mathcal{J}$ if for all sections $\zeta, \xi \in \Gamma_{TM\oplus A^*}(\mathbb{T}A)$ we have the property (see for example [51] and [16])

$$[\mathcal{J} \circ \zeta \circ j^{-1}, \mathcal{J} \circ \xi \circ j^{-1}] = \mathcal{J} \circ [\zeta, \xi] \circ j^{-1}\,. \tag{4.57}$$

Since the bracket of two core sections $\tau_1^\dagger$ and $\tau_2^\dagger$ always vanishes and $\mathcal{J} \circ \tau^\dagger \circ j^{-1} = (j_C \circ \tau)^\dagger$ according to Lemma 4.3.8 this condition is immediately satisfied on core sections.

For the bracket of a lift with a core section we obtain on one hand again making use of Lemma 4.3.8 and Theorem 4.3.6

$$\left[\mathcal{J} \circ \sigma_A^\Delta(a) \circ j^{-1}, (j_C\tau)^\dagger\right] = \left[\sigma_A^\Delta(a), (j_C\tau)^\dagger\right] = \left(\nabla_a^{\mathrm{bas}}(j_C\tau)\right)^\dagger .$$

On the other hand we have

$$\mathcal{J} \circ [\sigma_A^\Delta(a), \tau^\dagger] \circ j^{-1} = (j_C \nabla_a^{\mathrm{bas}} \tau)^\dagger$$

and thus we obtain the condition

$$\nabla_a^{\mathrm{bas}} \circ j_C = j_C \circ \nabla_a^{\mathrm{bas}} . \tag{4.58}$$

As the Dorfman connection was chosen to be skew-symmetric, the two basic connections have to be dual to each other according to Proposition 4.3.3. Thus

$$\begin{aligned}\langle \nabla_a^{\mathrm{bas}} j\nu - j\nabla_a^{\mathrm{bas}}\nu, \tau\rangle &= -\langle j\nu, \nabla_a^{\mathrm{bas}}\tau\rangle + \rho(a)\langle j\nu, \tau\rangle \\ &\quad - \langle \nu, \nabla_a^{\mathrm{bas}} j_C\tau\rangle + \rho(a)\langle \nu, j_C\tau\rangle \\ &= -\langle \nu, \nabla_a^{\mathrm{bas}} j_C\tau - j_C\nabla_a^{\mathrm{bas}}\tau\rangle ,\end{aligned}$$

and hence equation (4.58) for a linear generalised complex structure is equivalent to (4.56), $\nabla_a^{\mathrm{bas}} \circ j = j \circ \nabla_a^{\mathrm{bas}}$ for all $a \in \Gamma(A)$.

For the bracket of two lifts we get on one hand the following:

$$\begin{aligned}\left[\mathcal{J} \circ \sigma_A^\Delta(a) \circ j^{-1}, \mathcal{J} \circ \sigma_A^\Delta(b) \circ j^{-1}\right] &= [\sigma_A^\Delta(a), \sigma_A^\Delta(b)] \\ &= \sigma_A^\Delta([a,b]) - \widetilde{R_\Delta^{\mathrm{bas}}(a,b)} .\end{aligned}$$

On the other hand we compute with Lemma 4.3.8

$$\mathcal{J} \circ [\sigma_A^\Delta(a), \sigma_A^\Delta(b)] \circ j^{-1} = \sigma_A^\Delta([a,b]) - \widetilde{j_C \circ R_\Delta^{\mathrm{bas}}(a,b) \circ j^{-1}} .$$

This gives the last condition for $\mathcal{J}$ to be a Lie algebroid morphism:

$$j_C \circ R_\Delta^{\mathrm{bas}}(a,b) = R_\Delta^{\mathrm{bas}}(a,b) \circ j . \tag{4.59}$$

□

Summarising we obtain the following Theorem:

Theorem 4.3.11. *A linear generalised complex structure $\mathcal{J}$ on A defines a generalised complex Lie algebroid if and only if for any adapted Dorfman connection Δ we have for any $a, b \in \Gamma(A)$*

1. $(\rho, \rho^t) \circ j_C = j \circ (\rho, \rho^t)$,

2. $\nabla_a^{\text{bas}} \circ j = j \circ \nabla_a^{\text{bas}}$,

3. $j_C \circ R_\Delta^{\text{bas}}(a,b) = R_\Delta^{\text{bas}}(a,b) \circ j$.

This is equivalent to the statement that $(j_C, j, 0)$ is the automorphism of 2-term representations up to homotopy of $((\rho,\rho^t), \nabla^{\text{bas}}, \nabla^{\text{bas}}, R_\Delta^{\text{bas}})$ corresponding to the VB-algebroid structure on $TA \oplus T^*A$ in the linear splitting defined by the adapted Dorfman connection Δ. For this correspondence see also [19, 36] and here recalled in Theorem 4.3.6 and Section 2.2.3.

Together with Theorem 4.1.19 this immediately gives the following characterisation of generalised complex Lie algebroids.

Corollary 4.3.12. *Let A be a Lie algebroid over M with anchor ρ. A linear generalised complex structure on A compatible with the Lie algebroid structure in the sense of Definition 4.3.7 is equivalent to a vector bundle morphism $j\colon TM \oplus A^* \to TM \oplus A^*$ and a j-equivalence class $[\Delta]$ of $TM \oplus A^*$-Dorfman connections on $A \oplus T^*M$, such that the bracket $\mathbb{A}$ defined by* (4.28) *and the corresponding basic connections and basic curvature satisfy for $\nu_1, \nu_2 \in \Gamma(TM \oplus A^*)$ and $a, b \in \Gamma(A)$*

1. $j^2 = -\operatorname{id}$,

2. $\mathbb{A}(\nu_1, j\nu_2) = j\mathbb{A}(\nu_1,\nu_2)$,

3. $(\rho,\rho^t) \circ j_C = j \circ (\rho,\rho^t)$,

4. $\nabla_a^{\text{bas}} \circ j = j \circ \nabla_a^{\text{bas}}$,

5. $j_C \circ R_\Delta^{\text{bas}}(a,b) = R_\Delta^{\text{bas}}(a,b) \circ j$.

4.4 Pair of transversal LA-Dirac structures

In Corollary 4.2.10 we showed that a linear generalised complex structure on a vector bundle is equivalent to a pair of transversal complex VB-Dirac structures $D_\pm$ in the complexified generalised tangent bundle. Given now a Lie algebroid A we obtained additional compatibility conditions of the generalised complex structure with the Lie algebroid structure described in Theorem 4.3.11. We will now show that these conditions are equivalent to $D_\pm$ defining LA-Dirac structures. For this we need a description of the Lie algebroid structure on $\mathbb{T}_\mathbb{C}A \to T_\mathbb{C}M \oplus A^*_\mathbb{C}$ in terms of a complex Dorfman connection

$$\Delta\colon \Gamma(T_\mathbb{C}M \oplus A^*_\mathbb{C}) \times \Gamma(A_\mathbb{C} \oplus T^*_\mathbb{C}M) \to \Gamma(A_\mathbb{C} \oplus T^*_\mathbb{C}M)\,. \tag{4.60}$$

This is completely analogously to the real case covered in [36] and [38]. It is noteworthy that since in the double vector bundle $\mathbb{T}_{\mathbb{C}}A$ the side A is not complexified, we also have to extend the basic connections and basic curvatures only complex linearly in the other argument. We write down the complex version of Definition 4.3.1 and of Definition 4.3.2:

Definition 4.4.1. *Given a Lie algebroid $A \to M$ with anchor ρ and a skew-symmetric complex Dorfman connection Δ as in (4.60) we set for $a \in \Gamma(A)$ and $(X, \alpha) \in \Gamma(T_{\mathbb{C}}M \oplus A^*_{\mathbb{C}})$:*

$$\Omega^{\mathbb{C}} \colon \Gamma(T_{\mathbb{C}}M \oplus A^*_{\mathbb{C}}) \times \Gamma(A) \to \Gamma(A_{\mathbb{C}} \oplus T^*_{\mathbb{C}}M)\,, \tag{4.61}$$

$$\Omega^{\mathbb{C}}_{(X,\alpha)}a := \Delta_{(X,\alpha)}(a, 0) - \big(0, \mathbf{d}_{\mathbb{C}}\langle \alpha, a\rangle_{\mathbb{C}}\big)\,. \tag{4.62}$$

*We denote by $\mathcal{L}^{\mathbb{C}}_a$ the complexifications of the $\mathcal{L}_a$ and define the **complex basic connections** by the following equations, where now $(a', \theta) \in \Gamma(A_{\mathbb{C}} \oplus T^*_{\mathbb{C}}M)$:*

$$\nabla^{\mathrm{bas}}_a(X, \alpha) := (\rho, \rho^t)_{\mathbb{C}}(\Omega^{\mathbb{C}}_{(X,\alpha)}a) + \mathcal{L}^{\mathbb{C}}_a(X, \alpha)\,, \tag{4.63}$$

$$\nabla^{\mathrm{bas}}_a(a', \theta) := \Omega^{\mathbb{C}}_{(\rho,\rho^t)_{\mathbb{C}}(a',\theta)}a + \mathcal{L}^{\mathbb{C}}_a(a', \theta)\,. \tag{4.64}$$

*The **complex basic curvature** is given by the same formula as in the real case, that is for $a, b \in \Gamma(A)$ and $\nu \in \Gamma(T_{\mathbb{C}}M \oplus A^*_{\mathbb{C}})$*

$$R^{\mathrm{bas}}_{\Delta}(a, b)\nu := -\Omega^{\mathbb{C}}_{\nu}[a, b] + \mathcal{L}^{\mathbb{C}}_a(\Omega^{\mathbb{C}}_{\nu}b) - \mathcal{L}^{\mathbb{C}}_b(\Omega^{\mathbb{C}}_{\nu}a) + \Omega^{\mathbb{C}}_{\nabla^{\mathrm{bas}}_b\nu}a - \Omega^{\mathbb{C}}_{\nabla^{\mathrm{bas}}_a\nu}b\,. \tag{4.65}$$

If Δ is the complexification of a real Dorfman connection, then the basic connections and the basic curvature defined by Δ are the complexifications of the corresponding maps defined by the real Dorfman connection. It is easy to see that the complex version of Proposition 4.3.3 also holds.

In [38] the following statement is proved.

Proposition 4.4.2. *A VB-Dirac structure $D \subseteq \mathbb{T}A$ with side $U \subseteq TM \oplus A^*$ and core $K \subseteq A \oplus T^*M$ is additionally a Lie subalgebroid of $\mathbb{T}A \to TM \oplus A^*$ if and only for an adapted Dorfman connection Δ as in (4.39) the following conditions are satisfied for all $a, b \in \Gamma(A)$, $u \in \Gamma(U)$:*

1. *$(\rho, \rho^t)(K) \subseteq U$,*
2. *$\nabla^{\mathrm{bas}}_a u \in \Gamma(U)$,*
3. *$R^{\mathrm{bas}}_{\Delta}(a, b)u \in \Gamma(K)$.*

Equivalently, the the 2-term representation up to homotopy of A on $(A \oplus T^*M)_{[0]} \oplus (TM \oplus A^*)_{[1]}$ describing the VB-algebroid structure of $\mathbb{T}A$ in the splitting corresponding to Δ restricts to a 2-term representation up to homotopy of A on $K_{[0]} \oplus U_{[1]}$.

Since according to Proposition 2.5.10 the complexified generalised tangent bundle $\mathbb{T}_{\mathbb{C}}A$ carries a Lie algebroid structure over $T_{\mathbb{C}}M \oplus A_{\mathbb{C}}^*$ which is defined componentwise, the completely analogous statement to Proposition 4.4.2 holds for complex Dorfman connections and $\mathbb{T}_{\mathbb{C}}A$:

Corollary 4.4.3. *A complex VB-Dirac structure $D \subseteq \mathbb{T}_{\mathbb{C}}A$ with side $U \subseteq T_{\mathbb{C}}M \oplus A_{\mathbb{C}}^*$ and core $K \subseteq A_{\mathbb{C}} \oplus T_{\mathbb{C}}^*M$ is additionally a Lie subalgebroid of $\mathbb{T}_{\mathbb{C}}A \to T_{\mathbb{C}}M \oplus A_{\mathbb{C}}^*$ if and only for an adapted complex $(T_{\mathbb{C}}M \oplus A_{\mathbb{C}}^*)$-Dorfman connection Δ on $A_{\mathbb{C}} \oplus T_{\mathbb{C}}^*M$ the following conditions are satisfied for all $a, b \in \Gamma(A)$, $u \in \Gamma(U)$:*

1. *$(\rho, \rho^t)_{\mathbb{C}}(K) \subseteq U$,*
2. *$\nabla_a^{\mathrm{bas}} u \in \Gamma(U)$,*
3. *$R_\Delta^{\mathrm{bas}}(a, b)u \in \Gamma(K)$.*

Note that here ∇^{bas} and R_Δ^{bas} denote the complex basic connections and complex basic curvatures as defined in Definition 4.4.1

With this description of complex LA-Dirac structures, Theorem 4.3.11 and Corollary 4.2.10 we can easily prove the following description of generalised complex Lie algebroids.

Corollary 4.4.4. *A generalised complex structure on a Lie algebroid is equivalent to a pair of transverse, complex LA-Dirac structures in $\mathbb{T}_{\mathbb{C}}A$.*

Proof. According to Corollary 4.2.10 a linear generalised complex structure $\mathcal{J}$ on A is equivalent to a pair of transversal, complex VB-Dirac structures $D_\pm$. We choose a Dorfman connection adapted to $\mathcal{J}$ as in Proposition 4.1.15. Then the complexification of Δ is adapted to D_+ and to D_- simultaneously. According to Theorem 4.3.11 $\mathcal{J}$ is a Lie algebroid morphism if and only if

1. $(\rho, \rho^t) \circ j_C = j \circ (\rho, \rho^t)\,,$
2. $\nabla_a^{\mathrm{bas}} \circ j = j \circ \nabla_a^{\mathrm{bas}}\,,$
3. $j_C \circ R_\Delta^{\mathrm{bas}}(a, b) = R_\Delta^{\mathrm{bas}}(a, b) \circ j\,.$

The first condition is equivalent to $(\rho, \rho^t)(K_\pm) \subseteq U_\pm$, the second condition is equivalent to $\nabla_a^{\mathrm{bas}} u_\pm \in \Gamma(U_\pm)$ for any $a \in \Gamma(A)$ and $u_\pm \in \Gamma(U_\pm)$ and the third condition is equivalent to $R_\Delta^{\mathrm{bas}}(a, b)(u_\pm) \in \Gamma(K_\pm)$ for all $a, b \in \Gamma(A)$ and $u_\pm \in \Gamma(U_\pm)$. According to Corollary 4.4.3 these conditions are equivalent to $D_\pm$ being complex LA-Dirac structures. □

4.5 The degenerate Courant algebroid $A \oplus T^*M$

Let in this section $A \to M$ be a generalised complex Lie algebroid with generalised complex structure $\mathcal{J}$. We recall here the structure of a degenerate Courant algebroid on $A \oplus T^*M$ described by Madeleine Jotz Lean in [38]. Then we show that the core morphism j_C of $\mathcal{J}$ is a degenerate generalised complex structure in this degenerate Courant algebroid.

The anchor of the degenerate Courant algebroid structure on $A \oplus T^*M$ is defined to be $\rho \circ \mathrm{pr}_A \colon A \oplus T^*M$. Given sections $a, b \in \Gamma(A)$ and $\theta, \eta \in \Gamma(T^*M)$, there is a bilinear pairing on $A \oplus T^*M$ defined by

$$\big\langle (a,\theta),(b,\eta)\big\rangle_d := \langle \rho(a), \eta\rangle + \langle \rho(b), \theta\rangle\,, \tag{4.66}$$

and a bracket on $\Gamma(A \oplus T^*M)$ by setting

$$\big[\!\big[(a,\theta),(b,\eta)\big]\!\big]_d = \big([a,b], \mathcal{L}_{\rho(a)}\eta - i_{\rho(b)}\mathbf{d}\theta\big)\,. \tag{4.67}$$

This anchor, bracket and pairing satisfy all properties of a Courant algebroid as in Definition 2.3.1 except for the non-degeneracy of the pairing. This will be called the degenerate Courant algebroid $A \oplus T^*M$. The bracket can be described in terms of a Dorfman connection and the corresponding basic connection as follows.

Lemma 4.5.1. *Let Δ be any skew-symmetric Dorfman connection as in* (4.39)*. Then the bracket in the degenerate Courant algebroid $A \oplus T^*M$ as defined by* (4.67) *is given for two sections $\tau_1, \tau_2 \in \Gamma(A \oplus T^*M)$ by*

$$[\![\tau_1, \tau_2]\!]_d = \Delta_{(\rho,\rho^t)\tau_1}\tau_2 - \nabla^{\mathrm{bas}}_{\mathrm{pr}_A\tau_2}\tau_1\,. \tag{4.68}$$

Proof. First we compute

$$\begin{aligned}\nabla^{\mathrm{bas}}_b(a,\theta) &= \Omega_{(\rho,\rho^t)(a,\theta)}b + \mathcal{L}_b(a,\theta)\\ &= \Delta_{(\rho,\rho^t)(a,\theta)}(b,0) - (0, \mathbf{d}\langle \rho^t\theta, b\rangle) + ([b,a], \mathcal{L}_{\rho(b)}\theta)\,,\end{aligned}$$

and

$$\begin{aligned}\langle \Delta_{(\rho a,\rho^t\theta)}(0,\eta),(X,\alpha)\rangle &= \rho(a)\langle \eta, X\rangle - \langle (0,\eta), [\![(\rho a, \rho^t\theta),(X,\alpha)]\!]_\Delta\rangle\\ &= \rho(a)\langle \eta, X\rangle - \langle \eta, [\rho a, X]\rangle\\ &= \langle \mathcal{L}_{\rho(a)}\eta, X\rangle\,.\end{aligned}$$

Thus we obtain

$$
\begin{aligned}
\Delta_{(\rho,\rho^t)(a,\theta)}(b,\eta) - \nabla_b^{\mathrm{bas}}(a,\theta) &= ([a,b], -\mathcal{L}_{\rho b}\theta) + \Delta_{(\rho a,\rho^t\theta)}(0,\eta) + (0, \mathbf{d}\langle \rho^t\theta, b\rangle) \\
&= ([a,b], \mathcal{L}_{\rho a}\eta - \mathcal{L}_{\rho b}\theta + \mathbf{d}\langle \rho^t\theta, b\rangle) \\
&= ([a,b], \mathcal{L}_{\rho a}\eta - \mathcal{L}_{\rho b}\theta + \mathbf{d}i_{\rho b}\theta) \\
&= ([a,b], \mathcal{L}_{\rho a}\eta - i_{\rho b}\mathbf{d}\theta) \\
&= [\![(a,\theta),(b,\eta)]\!]_d \,.
\end{aligned}
$$

This completes the proof. □

Now consider a generalised complex Lie algebroid structure $\mathcal{J}$ on A. This induces the core morphism j_C on $A \oplus T^*M$. It follows from the previously obtained conditions that this defines a generalised complex structure with respect to the degenerate Courant algebroid structure defined above.

Proposition 4.5.2. *The core morphism $j_C\colon A \oplus T^*M \to A \oplus T^*M$ of $\mathcal{J}$ satisfies $j_C^2 = -\operatorname{id}$, is orthogonal with respect to $\langle\cdot,\cdot\rangle_d$ and the Nijenhuis torsion of j_C with respect to $[\![\cdot,\cdot]\!]_d$ vanishes. We call this a degenerate generalised complex structure in the degenerate Courant algebroid $A \oplus T^*M$.*

Proof. Since $\mathcal{J}$ is a linear generalised complex structure we have $j_C^2 = -\operatorname{id}$ according to Proposition 4.1.10. It remains to be checked that j_C respects the pairing and that the Nijenhuis tensor of j_C vanishes. For the pairing we compute

$$
\begin{aligned}
\langle j_C\tau_1, j_C\tau_2\rangle_d &= \langle (\rho,\rho^t) j_C\tau_1, j_C\tau_2\rangle \\
&= \langle j(\rho,\rho^t)\tau_1, j_C\tau_2\rangle \\
&= \langle (\rho,\rho^t)\tau_1, \tau_2\rangle \\
&= \langle \tau_1,\tau_2\rangle_d \,,
\end{aligned}
$$

where we have used $j = -j_C^t$ according to Proposition 4.1.10 and $j \circ (\rho,\rho^t) = (\rho,\rho^t) \circ j_C$ according to Theorem 4.3.11. Hence j_C is indeed orthogonal with respect to the degenerate pairing.

Now we fix a Dorfman connection Δ that is adapted to $\mathcal{J}$. Recall that Theorem 4.3.11 shows the equality $\nabla_a^{\mathrm{bas}} \circ j_C = j_C \circ \nabla_a^{\mathrm{bas}}$ (4.58). Now we use Lemma 4.5.1

to compute the Nijenhuis torsion of j_C.

$$\begin{aligned}
N_{j_C,[\![\cdot,\cdot]\!]_d}(\tau_1,\tau_2) &= [\![\tau_1,\tau_2]\!]_d - [\![j_C\tau_1, j_C\tau_2]\!]_d + j_C\Big([\![j_C\tau_1,\tau_2]\!]_d + [\![\tau_1, j_C\tau_2]\!]_d\Big)\\
&= \Delta_{(\rho,\rho^t)\tau_1}\tau_2 - \nabla^{\text{bas}}_{\text{pr}_A\,\tau_2}\tau_1\\
&\quad - \Delta_{(\rho,\rho^t)j_C\tau_1} j_C\tau_2 + \nabla^{\text{bas}}_{\text{pr}_A\,j_C\tau_2} j_C\tau_1\\
&\quad + j_C\Delta_{(\rho,\rho^t)j_C\tau_1}\tau_2 - j_C\nabla^{\text{bas}}_{\text{pr}_A\,\tau_2} j_C\tau_1\\
&\quad + j_C\Delta_{(\rho,\rho^t)\tau_1} j_C\tau_2 - j_C\nabla^{\text{bas}}_{\text{pr}_A\,j_c\tau_2}\tau_1\\
&= \Delta_{(\rho,\rho^t)\tau_1}\tau_2 - \Delta_{(\rho,\rho^t)j_C\tau_1} j_C\tau_2\\
&\quad + j_C\Delta_{(\rho,\rho^t)j_C\tau_1}\tau_2 + j_C\Delta_{(\rho,\rho^t)\tau_1} j_C\tau_2\,,
\end{aligned}$$

which vanishes for any linear generalised complex structure according to (4.23) with $\nu = (\rho,\rho^t)\tau_1$ and $\tau = \tau_2$. Thus j_C is a degenerate generalised complex structure in the degenerate Courant algebroid $A \oplus T^*M$. □

4.6 The complex A-Manin pair

In [38] Jotz Lean defined A-Manin pairs for a given Lie algebroid A over M and constructed an equivalence between A-Manin pairs and Dirac bialgebroids over A. In this section we consider complex Courant algebroids and complex Dirac structures (see Section 2.5) and will obtain the definition of a complex A-Manin pair completely analogously to the definition over the real numbers in [38].

Let us from now on fix a generalised complex Lie algebroid A and denote the generalised complex structure by $\mathcal{J}$, the corresponding LA-Dirac structures by $D_\pm \subseteq \mathbb{T}_\mathbb{C}A$ with side bundles $U_\pm \subseteq T_\mathbb{C}M \oplus A^*_\mathbb{C}$. Let Δ be a Dorfman connection adapted to the generalised complex structure as in 4.1.15 and denote by ∇^{bas} the corresponding basic connections defined in 4.3.2. We denote by $\Delta^\mathbb{C}$ the complexification of Δ as defined before in Example 2.5.7. The results of the previous sections show that $\Delta^\mathbb{C}$ is adapted to both D_+ and D_-.

We furthermore denote here by $\nabla^{\text{bas},\mathbb{C}}$ the complexifications of ∇^{bas} with respect to both the A-argument and the $A\oplus T^*M$-argument or $TM \oplus A^*$-argument, respectively. Note that these are not the same as the basic connections corresponding to $\Delta^\mathbb{C}$ as in Definition 4.4.1 since there the A-argument is not complexified. However, in this section we will need the additional complexification in this direction.

We can now define a complex A-Manin pair as follows:

Definition 4.6.1. *A **complex A-Manin pair** consists of a complex Courant algebroid C over M, a complex Dirac structure $U \to M$, with $\iota\colon U \hookrightarrow T_\mathbb{C}M \oplus$*

$A^*_{\mathbb{C}}$ *such that* $\rho_U = \mathrm{pr}_{T_{\mathbb{C}}M} \circ \iota$ *and a morphism of (degenerate) complex Courant algebroids* $\Phi\colon A_{\mathbb{C}} \oplus T^*_{\mathbb{C}}M \to C$ *such that*

$$\Phi(A_{\mathbb{C}} \oplus T^*_{\mathbb{C}}M) + U = C$$

and $\langle u, \Phi(\tau + i\tau')\rangle_C = \langle \iota(u), \tau + i\tau'\rangle$ *for all* $(u, \tau + i\tau') \in U \times_M (A_{\mathbb{C}} \oplus T^*_{\mathbb{C}}M)$.

In [38] Jotz Lean showed that the Courant algebroid structure on C and the morphism Φ can in fact be recovered from the Lie algebroid structures on A, U and ι. All the arguments can be extended complex linearly to obtain the following straightforward consequence.

Proposition 4.6.2. *Let* $U_\pm$ *be the* $\pm i$*-eigenbundles of the side morphism* j *of a generalised complex structure. Define*

$$C_\pm := \frac{U_\pm \oplus (A_{\mathbb{C}} \oplus T^*_{\mathbb{C}}M)}{\mathrm{graph}\Big(-(\rho,\rho^t)_{\mathbb{C}}|_{K_\pm}\colon K_\pm \to U_\pm\Big)}\,, \tag{4.69}$$

and define an anchor map, a $\mathbb{C}$*-bilinear pairing and a bracket as follows. For* $u, u_1, u_2 \in \Gamma(U_\pm)$, $\tau, \tau_1, \tau_2 \in \Gamma(A_{\mathbb{C}} \oplus T^*_{\mathbb{C}}M)$ *we define the anchor by*

$$c_\pm(u \oplus \tau) := \rho_{U_\pm}(u) + (\rho_A)_{\mathbb{C}} \circ \mathrm{pr}_{A_{\mathbb{C}}} \tau\,, \tag{4.70}$$

the pairing by

$$\langle u_1 \oplus \tau_1, u_2 \oplus \tau_2\rangle_{C_\pm} := \langle u_1, \tau_2\rangle + \langle u_2, \tau_1\rangle + \langle \tau_1, (\rho,\rho^t)_{\mathbb{C}}(\tau_2)\rangle\,, \tag{4.71}$$

and the bracket by

$$\begin{aligned}\llbracket u_1 \oplus \tau_1, u_2 \oplus \tau_2\rrbracket_{C_\pm} := &\Big([u_1, u_2]_{U_\pm} + \nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_{\mathbb{C}}}\tau_1} u_2 - \nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_{\mathbb{C}}}\tau_2} u_1\Big) \\ &\oplus \Big(\llbracket \tau_1, \tau_2\rrbracket_{d,\mathbb{C}} + \Delta^{\mathbb{C}}_{u_1}\tau_2 - \Delta^{\mathbb{C}}_{u_2}\tau_1 + \big(0, \mathbf{d}_{\mathbb{C}}\langle \tau_1, u_2\rangle\big)\Big)\,.\end{aligned} \tag{4.72}$$

Then $C_\pm$ *are both complex Courant algebroids and* $(C_\pm, U_\pm)$ *together with* $\iota\colon U_\pm \hookrightarrow T_{\mathbb{C}}M \oplus A^*_{\mathbb{C}}$ *and* $\Phi\colon A_{\mathbb{C}} \oplus T^*_{\mathbb{C}}M \to C$ *the canonical inclusions are complex A-Manin pairs.*

4.6.1 The induced generalised complex structure

Now we will show that the generalised complex structure on A induces generalised complex structures $J_\pm$ in the Courant algebroids $C_\pm$ defined by Proposition 4.6.2.

Proposition 4.6.3. *Let* $u \oplus \tau \in \Gamma(C_\pm)$. *Then*

$$J_\pm(u \oplus \tau) := j_{\mathbb{C}} u \oplus j_{C,\mathbb{C}}\tau \tag{4.73}$$

is well-defined and a generalised complex structure in $C_\pm$.

Proof. Given any element $(-(\rho,\rho^t)_{\mathbb{C}}k)\oplus k$ of $\mathrm{graph}(-(\rho,\rho^t)_{\mathbb{C}}|_{K_\pm})$ we have

$$\begin{aligned} J_\pm((-(\rho,\rho^t)_{\mathbb{C}}k)\oplus k) &= (-j_{\mathbb{C}}\circ(\rho,\rho^t)_{\mathbb{C}}k)\oplus \pm ik \\ &= \pm i((-(\rho,\rho^t)_{\mathbb{C}}k)\oplus k)\,, \end{aligned}$$

which is again an element of $\mathrm{graph}(-(\rho,\rho^t)|_{K_\pm})$. Thus the map $J_\pm$ defined by (4.73) is indeed well-defined on $C_\pm$.

To show that $J_\pm$ is a generalised complex structure in $C_\pm$ we have to check that it squares to $-\,\mathrm{id}$, that it respects the pairing and that the Nijenhuis torsion with respect to the bracket on $C_\pm$ vanishes.

First, it is clear that $J_\pm^2=-1$ as J was a generalised complex structure and thus $j^2=-1$ and $j_C^2=-1$ by Proposition 4.1.10.

Second, $J_\pm$ also respects the pairing of $C_\pm$ as for $u_1\oplus\tau_1$ and $u_2\oplus\tau_2$ in $\Gamma(C_\pm)$ we compute:

$$\begin{aligned} \langle J_\pm(u_1\oplus\tau_1),J_\pm(u_2\oplus\tau_2)\rangle_{C_\pm} &= \langle j_{\mathbb{C}}u_1,j_{C,\mathbb{C}}\tau_2\rangle+\langle j_{\mathbb{C}}u_2,j_{C,\mathbb{C}}\tau_1\rangle \\ &\quad+\langle j_{C,\mathbb{C}}\tau_1,(\rho,\rho^t)_{\mathbb{C}}\circ j_{C,\mathbb{C}}(\tau_2)\rangle \\ &= \langle u_1,\tau_2\rangle+\langle u_2,\tau_1\rangle+\langle\tau_1,(\rho,\rho^t)_{\mathbb{C}}(\tau_2)\rangle \\ &= \langle u_1\oplus\tau_1,u_2\oplus\tau_2\rangle_{C_\pm}\,. \end{aligned}$$

Here we used that $j^2=-1$, that $j_{\mathbb{C}}^t=-j_{C,\mathbb{C}}$ and that $(\rho,\rho^t)_{\mathbb{C}}\circ j_{C,\mathbb{C}}=j_{\mathbb{C}}\circ(\rho,\rho^t)_{\mathbb{C}}$, which follows from Proposition 4.1.10 and Theorem 4.3.11.

The last condition remaining to be checked is that the Nijenhuis torsion of $J_\pm$ with respect to the bracket on $C_\pm$ vanishes.

$$\begin{aligned} N_{J_\pm,[\![\cdot,\cdot]\!]_{C_\pm}}(u_1\oplus\tau_1,u_2\oplus\tau_2) &= [\![u_1\oplus\tau_1,u_2\oplus\tau_2]\!]_{C_\pm} \\ &\quad-[\![J_\pm(u_1\oplus\tau_1),J_\pm(u_2\oplus\tau_2)]\!]_{C_\pm} \\ &\quad+J_\pm[\![J_\pm(u_1\oplus\tau_1),u_2\oplus\tau_2]\!]_{C_\pm} \\ &\quad+J_\pm[\![u_1\oplus\tau_1,J_\pm(u_2\oplus\tau_2)]\!]_{C_\pm}\,. \end{aligned}$$

We compute the $U_\pm$-component and the $(A_{\mathbb{C}}\oplus T^*_{\mathbb{C}}M)$-component of $N_{J_\pm,[\![\cdot,\cdot]\!]_{C_\pm}}$ independently, by writing the bracket as in (4.72). Note that there is no well-defined projection from $C_\pm$ onto these factors, but the bracket and thus the Nijenhuis torsion is well-defined, which allows us to choose the representative given by (4.72). For the $U_\pm$-component of this representative of $N_{J_\pm,[\![\cdot,\cdot]\!]_{C_\pm}}(u_1\oplus\tau_1,u_2\oplus\tau_2)$ we obtain

$$\begin{aligned}
&[u_1,u_2]_{U_\pm}+\nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_\mathbb{C}}\tau_1}u_2-\nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_\mathbb{C}}\tau_2}u_1\\
&\quad-[j_\mathbb{C}u_1,j_\mathbb{C}u_2]_{U_\pm}-\nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_\mathbb{C}}j_{C,\mathbb{C}}\tau_1}j_\mathbb{C}u_2+\nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_\mathbb{C}}j_{C,\mathbb{C}}\tau_2}j_\mathbb{C}u_1\\
&\quad+j_\mathbb{C}\Big([j_\mathbb{C}u_1,u_2]_{U_\pm}+\nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_\mathbb{C}}j_{C,\mathbb{C}}\tau_1}u_2-\nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_\mathbb{C}}\tau_2}j_\mathbb{C}u_1\Big)\\
&\quad+j_\mathbb{C}\Big([u_1,j_\mathbb{C}u_2]_{U_\pm}+\nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_\mathbb{C}}\tau_1}j_\mathbb{C}u_2-\nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_\mathbb{C}}j_{C,\mathbb{C}}\tau_2}u_1\Big)\\
&=[u_1,u_2]_{U_\pm}+\nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_\mathbb{C}}\tau_1}u_2-\nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_\mathbb{C}}\tau_2}u_1\\
&\quad-[\pm iu_1,\pm iu_2]_{U_\pm}\mp i\nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_\mathbb{C}}j_{C,\mathbb{C}}\tau_1}u_2\pm i\nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_\mathbb{C}}j_{C,\mathbb{C}}\tau_2}u_1\\
&\quad\pm i[\pm iu_1,u_2]_{U_\pm}+j_\mathbb{C}\nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_\mathbb{C}}j_{C,\mathbb{C}}\tau_1}u_2\mp ij_\mathbb{C}\nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_\mathbb{C}}\tau_2}u_1\\
&\quad\pm i[u_1,\pm iu_2]_{U_\pm}\pm ij_\mathbb{C}\nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_\mathbb{C}}\tau_1}u_2-j_\mathbb{C}\nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_\mathbb{C}}j_{C,\mathbb{C}}\tau_2}u_1\\
&=\nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_\mathbb{C}}\tau_1}u_2-\nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_\mathbb{C}}\tau_2}u_1\\
&\quad\mp i\nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_\mathbb{C}}j_{C,\mathbb{C}}\tau_1}u_2\pm i\nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_\mathbb{C}}j_{C,\mathbb{C}}\tau_2}u_1\\
&\quad\pm i\nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_\mathbb{C}}j_{C,\mathbb{C}}\tau_1}u_2-(\pm i)^2\nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_\mathbb{C}}\tau_2}u_1\\
&\quad+(\pm i)^2\nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_\mathbb{C}}\tau_1}u_2\mp i\nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_\mathbb{C}}j_{C,\mathbb{C}}\tau_2}u_1\\
&=0\,,
\end{aligned}$$

where we used that $j_\mathbb{C}u_k=\pm iu_k$ for $k=1,2$, that the bracket on $U_\pm$ is $\mathbb{C}$-bilinear and that $\nabla^{\mathrm{bas},\mathbb{C}}$ preserves $U_\pm$. The latter property follows directly from the equality $\nabla_a^{\mathrm{bas}}\circ j=j\circ\nabla^{\mathrm{bas}}$ of Corollary 4.3.12 since $U_\pm$ are the $\pm i$-eigenbundles of $j_\mathbb{C}$.

For the $A_\mathbb{C}\oplus T^*_\mathbb{C}M$-component of this representative of $N_{J_\pm,[\![\cdot,\cdot]\!]_{C_\pm}}(u_1\oplus\tau_1,u_2\oplus\tau_2)$ we compute

$$\begin{aligned}
&[\![\tau_1,\tau_2]\!]_{d,\mathbb{C}}+\Delta^\mathbb{C}_{u_1}\tau_2-\Delta^\mathbb{C}_{u_2}\tau_1+(0,\mathbf{d}_\mathbb{C}\langle\tau_1,u_2\rangle)\\
&\quad-[\![j_{C,\mathbb{C}}\tau_1,j_{C,\mathbb{C}}\tau_2]\!]_{d,\mathbb{C}}-\Delta^\mathbb{C}_{j_\mathbb{C}u_1}j_{C,\mathbb{C}}\tau_2+\Delta^\mathbb{C}_{j_\mathbb{C}u_2}j_{C,\mathbb{C}}\tau_1-(0,\mathbf{d}_\mathbb{C}\langle j_{C,\mathbb{C}}\tau_1,j_\mathbb{C}u_2\rangle)\\
&\quad+j_{C,\mathbb{C}}\Big([\![j_{C,\mathbb{C}}\tau_1,\tau_2]\!]_{d,\mathbb{C}}+\Delta^\mathbb{C}_{j_\mathbb{C}u_1}\tau_2-\Delta^\mathbb{C}_{u_2}j_{C,\mathbb{C}}\tau_1+(0,\mathbf{d}_\mathbb{C}\langle j_{C,\mathbb{C}}\tau_1,u_2\rangle)\Big)\\
&\quad+j_{C,\mathbb{C}}\Big([\![\tau_1,j_{C,\mathbb{C}}\tau_2]\!]_{d,\mathbb{C}}+\Delta^\mathbb{C}_{u_1}j_{C,\mathbb{C}}\tau_2-\Delta^\mathbb{C}_{j_\mathbb{C}u_2}\tau_1+(0,\mathbf{d}_\mathbb{C}\langle\tau_1,j_\mathbb{C}u_2\rangle)\Big)\\
&=N_{j_{C,\mathbb{C}},[\![\cdot,\cdot]\!]_{d,\mathbb{C}}}(\tau_1,\tau_2)\\
&\quad+\Delta^\mathbb{C}_{u_1}\tau_2-\Delta^\mathbb{C}_{u_2}\tau_1\mp i\Delta^\mathbb{C}_{u_1}j_{C,\mathbb{C}}\tau_2\pm i\Delta^\mathbb{C}_{u_2}j_{C,\mathbb{C}}\tau_1\\
&\quad\pm ij_{C,\mathbb{C}}\Delta^\mathbb{C}_{u_1}\tau_2-j_{C,\mathbb{C}}\Delta^\mathbb{C}_{u_2}j_{C,\mathbb{C}}\tau_1+j_{C,\mathbb{C}}\Delta^\mathbb{C}_{u_1}j_{C,\mathbb{C}}\tau_2\mp ij_{C,\mathbb{C}}\Delta^\mathbb{C}_{u_2}\tau_1\\
&=\Delta^\mathbb{C}_{u_1}(\tau_2\mp ij_{C,\mathbb{C}}\tau_2)\pm ij_{C,\mathbb{C}}\Delta^\mathbb{C}_{u_1}(\tau_2\mp ij_{C,\mathbb{C}}\tau_2)\\
&\quad-\Delta^\mathbb{C}_{u_2}(\tau_1\mp ij_{C,\mathbb{C}}\tau_1)\mp ij_{C,\mathbb{C}}\Delta^\mathbb{C}_{u_2}(\tau_1\mp ij_{C,\mathbb{C}}\tau_1)
\end{aligned}$$

$$\begin{aligned}
&= \Delta^{\mathbb{C}}_{u_1}(\tau_2 \mp i j_{C,\mathbb{C}}\tau_2) + (\pm i)^2 \Delta^{\mathbb{C}}_{u_1}(\tau_2 \mp i j_{C,\mathbb{C}}\tau_2) \\
&\quad - \Delta^{\mathbb{C}}_{u_2}(\tau_1 \mp i j_{C,\mathbb{C}}\tau_1) - (\pm i)^2 \Delta^{\mathbb{C}}_{u_2}(\tau_1 \mp i j_{C,\mathbb{C}}\tau_1) \\
&= 0\,,
\end{aligned}$$

where we used first that $j_{\mathbb{C}} u_k = \pm i u_k$ for $k = 1, 2$ and then the property that $j_{C,\mathbb{C}}$ is a degenerate generalised complex structure on the degenerate Courant algebroid $A \oplus T^*M$ according to Proposition 4.5.2 and thus the Nijenhuis torsion of the complexification vanishes. Finally we used that $\tau_l \mp i j_{C,\mathbb{C}}\tau_l$ for $l = 1, 2$ defines a section of $K_\pm$, the $\pm i$-eigenbundle of $j_{C,\mathbb{C}}$, that $\Delta^{\mathbb{C}}_{u_k}$ preserves $K_\pm$ according to Corollary 4.2.7 and hence we have $j_{C,\mathbb{C}}\Delta^{\mathbb{C}}_{u_k}\tau_l \mp i j_{C,\mathbb{C}}\tau_l = \pm i \Delta^{\mathbb{C}}_{u_k}\tau_l \mp i j_{C,\mathbb{C}}\tau_l$.

Thus we have shown that the Nijenhuis torsion of $J_\pm$ vanishes and that $J_\pm$ defines indeed a generalised complex structure in $C_\pm$. □

4.7 The Drinfeld double of the Lie bialgebroid $(U_\pm, K_\mp)$

Let again A be a generalised complex Lie algebroid over M with generalised complex structure $\mathcal{J}$ over side morphism j. Denote again by $D_\pm$, $U_\pm$ and $K_\pm$ the corresponding eigenbundles as before. In this section we show that the pair $(U_\pm, K_\mp)$ forms a Lie bialgebroid. The Drinfeld double of this Lie bialgebroid is a Courant algebroid and we will show that it is isomorphic to $C_\pm$. First we need the following identification of $K_\mp$ with the dual of $U_\pm$.

Lemma 4.7.1. *There are canonical isomorphisms $U^*_\pm \cong K_\mp$ and $K^*_\pm \cong U_\mp$.*

Proof. We have $U_\pm \subset T_{\mathbb{C}}M \oplus A^*_{\mathbb{C}}$ and furthermore a canonical isomorphism $U_+ \oplus U_- \cong T_{\mathbb{C}}M \oplus A^*_{\mathbb{C}}$ where (u_+, u_-) is sent to $u_+ + u_-$. The inverse is given by $\nu \mapsto \left(\frac{1}{2}(\nu - i j_{\mathbb{C}}\nu), \frac{1}{2}(\nu + i j_{\mathbb{C}}\nu)\right)$. Analogously we have an identification of $A_{\mathbb{C}} \oplus T^*_{\mathbb{C}}M$ with $K_+ \oplus K_-$. Now we have $K_\pm = U^\circ_\pm$, according to Lemma 4.2.3. This immediately gives the desired identification of $K_\mp$ with $U^*_\pm$ and of $U_\mp$ with $K^*_\mp$. □

Proposition 4.7.2. *The restriction of the degenerate Courant algebroid structure on $A_{\mathbb{C}} \oplus T^*_{\mathbb{C}}M$ induces a complex Lie algebroid structure on $K_\pm$.*

Proof. The Courant algebroid structure on $A_{\mathbb{C}} \oplus T^*_{\mathbb{C}}M$ is given by complexifying the structure on $A \oplus T^*M$ defined in section 4.5. According to Proposition 4.5.2 the morphism j_C is a generalised complex structure in $A \oplus T^*M$. The vanishing of the Nijenhuis tensor and $\mathbb{C}$-linearity of the complexified bracket imply that the bracket restricts to the $\pm i$-eigenbundle $K_\pm$ of $j_{C,\mathbb{C}}$ since for $k_1, k_2 \in \Gamma(K_\pm)$ we have

$$\begin{aligned}0 &= j_{C,\mathbb{C}}((N_{j_C,[\![\cdot,\cdot]\!]_d})_{\mathbb{C}}(k_1,k_2))\\ &= j_{C,\mathbb{C}}[\![k_1,k_2]\!]_{d,\mathbb{C}} - i^2 j_{C,\mathbb{C}}[\![k_1,k_2]\!]_{d,\mathbb{C}} \pm 2ij^2_{C,\mathbb{C}}[\![k_1,k_2]\!]_{d,\mathbb{C}}\\ &= 2j_{C,\mathbb{C}}[\![k_1,k_2]\!]_{d,\mathbb{C}} \mp 2i[\![k_1,k_2]\!]_{d,\mathbb{C}}\,,\end{aligned}$$

and thus $[\![k_1,k_2]\!]_{d,\mathbb{C}}$ is again a section of $K_\pm$. Since the pairing in $A \oplus T^*M$ was given by (4.66) and by Theorem 4.3.11 we have that $(\rho,\rho^t)_{\mathbb{C}}$ sends $K_\pm$ to $U_\pm = K_\pm^\circ$ the pairing restricted to $K_\pm$ vanishes and thus the restricted bracket is skew-symmetric and defines a Lie algebroid structure on $K_\pm$. □

Proposition 4.7.3. *There is an isomorphism of vector bundles*

$$\begin{aligned}F\colon U_\pm \oplus K_\mp &\to C_\pm\\ (u,k) &\mapsto u \oplus k\,.\end{aligned} \tag{4.74}$$

This equips $U_\pm \oplus K_\mp$ with the structure of a Courant algebroid. Both $U_\pm \oplus 0$ and $0 \oplus K_\mp$ are Dirac structures in this Courant algebroid and thus the pair $(U_\pm, K_\pm)$ is a complex Lie bialgebroid.

Proof. The inverse of F is given by

$$\begin{aligned}F^{-1}\colon C_\pm &\to U_\pm \oplus K_\mp\\ u\oplus\tau &\mapsto \Big(u + (\rho,\rho^t)_{\mathbb{C}}(\frac{1}{2}(\tau \mp ij_{\mathbb{C}}\tau)), \frac{1}{2}(\tau \pm ij_{\mathbb{C}}\tau)\Big)\,.\end{aligned} \tag{4.75}$$

This inverse map F^{-1} is indeed well-defined on $C_\pm$ since for any $k \in K_\pm$ we have

$$F^{-1}\Big(-(\rho,\rho^t)_{\mathbb{C}}k, k\Big) = \Big(-(\rho,\rho^t)_{\mathbb{C}}k + \frac{1}{2}(\rho,\rho^t)_{\mathbb{C}}((k \mp i(\pm ik)), \frac{1}{2}(k \pm i(\pm ik))\Big) = (0,0)\,.$$

It is easy to check that they are inverse to each other and are thus vector bundle isomorphisms.

The Courant algebroid structure of $C_\pm$ induces via this isomorphism a Courant algebroid structure on the bundle $U_\pm \oplus K_\mp$. We now show that the Lie algebroids $U_\pm$ and $K_\mp$ are Dirac structures in $C_\pm$. Liu, Weinstein and Xu showed in [45] that two transversal Dirac structures in a Courant algebroid are equivalent to a Lie bialgebroid. Thus $(U_\pm, K_\pm)$ is a Lie bialgebroid and we can define the Drinfeld double Courant algebroid on $U_\pm \oplus K_\mp$. Then we will show that the pairing and bracket of $C_\pm$ are equal to the pairing and bracket of this Drinfeld double and that they are thus isomorphic as Courant algebroids with the isomorphism given by the map F defined in (4.74).

The bracket on $C_\pm$ was defined in (4.72) as

$$\begin{aligned}[][\![u_1 \oplus \tau_1, u_2 \oplus \tau_2]\!]_{C_\pm} := &\Big([u_1, u_2]_{U_\pm} + \nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_\mathbb{C}}\tau_1} u_2 - \nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_\mathbb{C}}\tau_2} u_1\Big) \\ &\oplus \Big([\![\tau_1, \tau_2]\!]_{d,\mathbb{C}} + \Delta^\mathbb{C}_{u_1}\tau_2 - \Delta^\mathbb{C}_{u_2}\tau_1 + \big(0, \mathbf{d}_\mathbb{C}\langle \tau_1, u_2\rangle\big)\Big) .\end{aligned}$$

The bracket of two sections $(0 \oplus k_1)$, $(0 \oplus k_2)$ of $C_\pm$ where $k_1, k_2 \in \Gamma(K_\mp)$ is then given by

$$\begin{aligned}[][\![(0 \oplus k_1), (0 \oplus k_2)]\!]_{C_\pm} := &\Big([0, 0]_{U_\pm} + \nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_\mathbb{C}}k_1} 0 - \nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_\mathbb{C}}k_2} 0\Big) \\ &\oplus \Big([\![k_1, k_2]\!]_{d,\mathbb{C}} + \Delta^\mathbb{C}_0 k_2 - \Delta^\mathbb{C}_0 k_1 + \big(0, \mathbf{d}_\mathbb{C}\langle k_1, 0\rangle\big)\Big) \\ &= 0 \oplus [\![k_1, k_2]\!]_{d,\mathbb{C}} .\end{aligned}$$

Similarly, for two sections $u_1, u_2 \in \Gamma(U_\pm)$ we have $[\![(u_1 \oplus 0), (u_2 \oplus 0)]\!]_{C_\pm} = [u_1, u_2]_{U_\pm} \oplus 0$. The pairing in $C_\pm$ was defined in (4.71) as

$$\langle u_1 \oplus \tau_1, u_2 \oplus \tau_2\rangle_{C_\pm} := \langle u_1, \tau_2\rangle + \langle u_2, \tau_1\rangle + \langle \tau_1, (\rho, \rho^t)_\mathbb{C}(\tau_2)\rangle .$$

It is easy to see that both $U_\pm \oplus 0$ and $0 \oplus K_\pm$ are maximally isotropic with respect to this pairing and thus Dirac structures in $C_\pm$. Thus by the argument in [45] $(U_\pm, K_\mp)$ form complex Lie bialgebroids. □

To show that the brackets in the Drinfeld doubles are the same as the brackets in $C_\pm$ we need two technical lemmas. First we prove the following lemma.

Lemma 4.7.4. *For $u_1, u_2 \in \Gamma(U_\pm)$ and $k_1, k_2 \in \Gamma(K_\mp)$*

$$\Delta^\mathbb{C}_{u_1}k_2 - \Delta^\mathbb{C}_{u_2}k_1 + \big(0, \mathbf{d}_\mathbb{C}\langle k_1, u_2\rangle\big) - \mathcal{L}^U_{u_1}k_2 + \iota_{u_2}\mathbf{d}_U k_1 \tag{4.76}$$

is a section of $K_\pm$.

Proof. We make use of the fact that the dual bracket to the Dorfman connection $\Delta^\mathbb{C}$ restricted to $U_\pm$ is precisely the Lie bracket on $U_\pm$ and that the Lie algebroid $U_\pm$ is anchored by the restriction of $\mathrm{pr}_{T_\mathbb{C}M}$. If we pair the above section of $A_\mathbb{C} \oplus T^*_\mathbb{C}M$ with an arbitrary section u of $U_\pm$ we obtain

$$\begin{aligned}&\langle u, \Delta^\mathbb{C}_{u_1}k_2 - \Delta^\mathbb{C}_{u_2}k_1 + \big(0, \mathbf{d}_\mathbb{C}\langle k_1, u_2\rangle\big) - \mathcal{L}^U_{u_1}k_2 + \iota_{u_2}\mathbf{d}_U k_1\rangle \\ =&\, \mathrm{pr}_{T_\mathbb{C}M}(u_1)\langle k_2, u\rangle - \langle k_2, [u_1, u]\rangle - \mathrm{pr}_{T_\mathbb{C}M}(u_2)\langle k_1, u\rangle + \langle k_1, [u_2, u]\rangle \\ &+ \mathrm{pr}_{T_\mathbb{C}M}(u)\langle k_1, u_2\rangle - \mathrm{pr}_{T_\mathbb{C}M}(u_1)\langle u, k_2\rangle + \langle k_2, [u_1, u]\rangle \\ &+ \mathrm{pr}_{T_\mathbb{C}M}(u_2)\langle k_1, u\rangle - \mathrm{pr}_{T_\mathbb{C}M}(u)\langle k_1, u_2\rangle - \langle k_1, [u_2, u]\rangle \\ =&0 .\end{aligned}$$

Since $u \in \Gamma(U)$ was arbitrary and $U^\circ_\pm = K_\pm$, the term (4.76) is indeed a section of $K_\pm$. □

Now we can prove the second lemma which is necessary for the proof of the following Theorem 4.7.6.

Lemma 4.7.5. *For $u_1, u_2 \in \Gamma(U_\pm)$ and $k_1, k_2 \in \Gamma(K_\mp)$ we have*

$$\begin{aligned}\mathcal{L}^K_{k_1} u_2 - \iota_{k_2}\mathbf{d}_K u_1 = (\rho,\rho^t)_{\mathbb{C}}\Big(\Delta^{\mathbb{C}}_{u_1}k_2 - \Delta^{\mathbb{C}}_{u_2}k_1 + (0, \mathbf{d}_{\mathbb{C}}\langle k_1, u_2\rangle) - \mathcal{L}^U_{u_1}k_2 + \iota_{u_2}\mathbf{d}_U k_1\Big) \\ + \nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_{\mathbb{C}}} k_1} u_2 - \nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_{\mathbb{C}}} k_2} u_1 \,.\end{aligned} \tag{4.77}$$

Proof. The left hand side of the equation is a section of $U_\pm$ since we showed in Proposition 4.7.3 that $(U_\pm, K_\mp)$ is a Lie bialgebroid. The basic connections preserve $\Gamma(U_\pm)$ according to Corollary 4.4.3 as $U_\pm$ is the side of a complex LA-Dirac structure to which Δ is adapted. The remaining term was seen in the previous Lemma 4.7.4 to be a section of $U_\pm$. Thus also the right hand side of the equation is a section of $U_\pm$ and hence we can pair the equation with an arbitrary section k of $K_\mp = U^*_\pm$ to obtain an equivalent equation. We write now $k = (a, \theta)$, $k_i = (a_i, \theta_i)$ for $a, a_i \in \Gamma(A_{\mathbb{C}})$ and $\theta, \theta_i \in \Gamma(T^*_{\mathbb{C}}M)$ and similarly $u_i = (X_i, \alpha_i)$ for $X_i \in \Gamma(T_{\mathbb{C}}M)$ and $\alpha_i \in \Gamma(A^*_{\mathbb{C}})$. Since $(\rho,\rho^t)(K_\mp) \subseteq U_\mp = K^\circ_\mp$, the pairing of k with $(\rho,\rho^t)(-\mathcal{L}^U_{u_1}k_2 + \iota_{u_2}\mathbf{d}_U k_1)$ vanishes. With this observation, the definitions of the basic connections and of the bracket on $K_\mp$ and the property of the Dorfman connection $\Delta^{\mathbb{C}}_u(0,\theta) = (0, \mathcal{L}_{\mathrm{pr}_{T_{\mathbb{C}}M} u}\theta)$ we can compute

$$\begin{aligned}
&\Big\langle k, (\rho,\rho^t)_{\mathbb{C}}\Big(\Delta^{\mathbb{C}}_{u_1}k_2 - \Delta^{\mathbb{C}}_{u_2}k_1 + (0, \mathbf{d}_{\mathbb{C}}\langle k_1, u_2\rangle) - \mathcal{L}^U_{u_1}k_2 + \iota_{u_2}\mathbf{d}_U k_1\Big) \\
&\qquad - \mathcal{L}^K_{k_1}u_2 + \iota_{k_2}\mathbf{d}_K u_1 + \nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_{\mathbb{C}}} k_1} u_2 - \nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_{\mathbb{C}}} k_2} u_1\Big\rangle \\
&= \Big\langle k, (\rho,\rho^t)_{\mathbb{C}}\Big(\Delta^{\mathbb{C}}_{u_1}k_2 - \Delta^{\mathbb{C}}_{u_2}k_1 + (0, \mathbf{d}_{\mathbb{C}}\langle k_1, u_2\rangle)\Big) \\
&\qquad - \mathcal{L}^K_{k_1}u_2 + \iota_{k_2}\mathbf{d}_K u_1 + \mathcal{L}_{a_1}u_2 + (\rho,\rho^t)_{\mathbb{C}}\Big(\Delta^{\mathbb{C}}_{u_2}(a_1, 0) - (0, \mathbf{d}_{\mathbb{C}}\langle a_1, u_2\rangle)\Big) \\
&\qquad - \mathcal{L}_{a_2}u_1 - (\rho,\rho^t)_{\mathbb{C}}\Big(\Delta^{\mathbb{C}}_{u_1}(a_2, 0) - (0, \mathbf{d}_{\mathbb{C}}\langle a_2, u_1\rangle)\Big)\Big\rangle \\
&= \Big\langle (\rho,\rho^t)_{\mathbb{C}}k, \Delta^{\mathbb{C}}_{u_1}(0,\theta_2) - \Delta^{\mathbb{C}}_{u_2}(0,\theta_1)\Big\rangle \\
&\quad + \Big\langle (\rho,\rho^t)_{\mathbb{C}}k, (0, \mathbf{d}_{\mathbb{C}}\langle k_1, u_2\rangle) - (0, \mathbf{d}_{\mathbb{C}}\langle a_1, u_2\rangle) + (0, \mathbf{d}_{\mathbb{C}}\langle a_2, u_1\rangle)\Big\rangle \\
&\quad + \Big\langle k, -\mathcal{L}^K_{k_1}u_2 + \mathcal{L}^K_{k_2}u_1 - \mathbf{d}_K\langle k_2, u_1\rangle\Big\rangle \\
&\quad + \Big\langle k, \mathcal{L}_{a_1}u_2 - \mathcal{L}_{a_2}u_1\Big\rangle \\
&= \langle \rho(a), \mathcal{L}_{X_1}\theta_2 - \mathcal{L}_{X_2}\theta_1\rangle + \rho(a)\langle k_1, u_2\rangle \\
&\quad - \rho(a)\langle a_1, \alpha_2\rangle + \rho(a)\langle a_2, \alpha_1\rangle \\
&\quad - \rho(a_1)\langle u_2, k\rangle + \langle u_2, [k_1, k]\rangle + \rho(a_2)\langle u_1, k\rangle - \langle u_1, [k_2, k]\rangle \\
&\quad - \rho(a)\langle u_1, k_2\rangle + \langle a, \mathcal{L}_{a_1}\alpha_2 - \mathcal{L}_{a_2}\alpha_1\rangle \\
&\quad + \langle \theta, [\rho(a_1), X_2] - [\rho(a_2), X_1]\rangle
\end{aligned}$$

$$
\begin{aligned}
&= X_1\langle \rho(a), \theta_2\rangle - \langle [X_1, \rho(a)], \theta_2\rangle - X_2\langle \rho(a), \theta_1\rangle + \langle [X_2, \rho(a)], \theta_1\rangle \\
&\quad + \rho(a)(\langle a_1, \alpha_2\rangle + \langle X_2, \theta_1\rangle) - \rho(a)\langle a_1, \alpha_2\rangle + \rho(a)\langle a_2, \alpha_1\rangle \\
&\quad - \rho(a_1)(\langle X_2, \theta\rangle + \langle a, \alpha_2\rangle) + \langle \alpha_2, [a_1, a]\rangle \\
&\quad + \langle X_2, \mathcal{L}_{\rho(a_1)}\theta - \mathcal{L}_{\rho(a)}\theta_1 + \mathbf{d}\langle \rho(a), \theta_1\rangle\rangle \\
&\quad + \rho(a_2)(\langle X_1, \theta\rangle + \langle a, \alpha_1\rangle) - \langle \alpha_1, [a_2, a]\rangle \\
&\quad - \langle X_1, \mathcal{L}_{\rho(a_2)}\theta - \mathcal{L}_{\rho(a)}\theta_2 + \mathbf{d}\langle \rho(a), \theta_2\rangle\rangle \\
&\quad - \rho(a)(\langle a_2, \alpha_1\rangle + \langle X_1, \theta_2\rangle) + \langle a, \mathcal{L}_{a_1}\alpha_2 - \mathcal{L}_{a_2}\alpha_1\rangle \\
&\quad + \langle \theta, [\rho(a_1), X_2] - [\rho(a_2), X_1]\rangle \\
&= 0\,.
\end{aligned}
$$

In the last step we only used the equalities $\rho(a)\langle a', \alpha\rangle = \langle [a, a'], \alpha\rangle + \langle a', \mathcal{L}_a\alpha\rangle$ and $X\langle Y, \theta\rangle = \langle [X, Y], \theta\rangle + \langle Y, \mathcal{L}_X\theta\rangle$ for any $a, a' \in \Gamma(A)$, $\alpha \in \Gamma(A^*)$, $X, Y \in \Gamma(TM)$ and $\theta \in \Gamma(T^*M)$ in order to cancel all the remaining terms.

Since $k \in \Gamma(K_\mp)$ was arbitrary this shows the desired equation. □

Now we can finally prove that both Courant algebroid structures are isomorphic.

Theorem 4.7.6. *The Drinfeld double Courant algebroid on $U_\pm \oplus K_\mp$ is isomorphic as a complex Courant algebroid to $C_\pm$ with the isomorphism given by the isomorphism of vector bundles of Proposition 4.7.3, sending a pair (u, k) to $u \oplus k$.*

Proof. For elements of the form $u \oplus k$ the anchor in $C_\pm$ is given by $c(u \oplus k) = \rho_{U_\pm}(u) + \rho(\mathrm{pr}_{A_\mathbb{C}}\, k)$ which is the anchor of (u, k) in the Drinfeld double Courant algebroid $U_\pm \oplus K_\mp$.

The pairing in the corresponding Drinfeld doubles is given by

$$\langle (u_1, k_1), (u_2, k_2)\rangle = \langle u_1, k_2\rangle + \langle u_2, k_1\rangle\,,$$

which is easy to be seen to coincide with the pairing in $C_\pm$ since $(\rho, \rho^t)_\mathbb{C}$ sends $K_\mp$ to $U_\mp = K_\mp^\circ$ according to Theorem 4.3.11.

The bracket in the corresponding Drinfeld doubles is defined by the following equation where $u_1, u_2 \in \Gamma(U_\pm)$ and $k_1, k_2 \in \Gamma(K_\mp)$

$$
\begin{aligned}
[\![(u_1, k_1), (u_2, k_2)]\!] = \Big(&[u_1, u_2] + \mathcal{L}^K_{k_1} u_2 - \iota_{k_2}\mathbf{d}_K u_2, \\
&[k_1, k_2] + \mathcal{L}^U_{u_1} k_2 - \iota_{u_2}\mathbf{d}_U k_1 \Big)\,.
\end{aligned}
\tag{4.78}
$$

Note that in [45] the bracket in the Drinfeld double is defined slightly differently, as there the Courant bracket is skew-symmetric whereas we defined Courant algebroids with a bracket satisfying the Jacobi identity.

We make use of Lemma 4.7.5 to compute the following

$$
\begin{aligned}
[\![u_1 \oplus k_1, u_2 \oplus k_2]\!]_{C_\pm} &= \Big([u_1,u_2]_{U_\pm} + \nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_\mathbb{C}} k_1} u_2 - \nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_\mathbb{C}} k_2} u_1\Big) \\
&\quad \oplus \Big([\![k_1,k_2]\!]_{d,\mathbb{C}} + \Delta^{\mathbb{C}}_{u_1} k_2 - \Delta^{\mathbb{C}}_{u_2} k_1 + (0, \mathbf{d}_\mathbb{C}\langle k_1, u_2\rangle)\Big) \\
&= \Big([u_1,u_2]_{U_\pm} + \nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_\mathbb{C}} k_1} u_2 - \nabla^{\mathrm{bas},\mathbb{C}}_{\mathrm{pr}_{A_\mathbb{C}} k_2} u_1 \\
&\quad + (\rho,\rho^t)_\mathbb{C}\Big(\Delta^{\mathbb{C}}_{u_1} k_2 - \Delta^{\mathbb{C}}_{u_2} k_1 + (0, \mathbf{d}_\mathbb{C}\langle k_1, u_2\rangle) \\
&\quad - \mathcal{L}_{u_1} k_2 + \iota_{u_2}\mathbf{d}_U k_1\Big)\Big) \\
&\quad \oplus \Big([\![k_1,k_2]\!]_{d,\mathbb{C}} + \mathcal{L}^U_{u_1} k_2 - \iota_{u_2}\mathbf{d}_U k_1\Big) \\
&= \Big([u_1,u_2]_{U_\pm} + \mathcal{L}^K_{k_1} u_2 - \iota_{k_2}\mathbf{d}_K u_1\Big) \\
&\quad \oplus \Big([k_1,k_2]_{K_\mp} + \mathcal{L}^U_{u_1} k_2 - \iota_{u_2}\mathbf{d}_U k_1\Big) .
\end{aligned}
$$

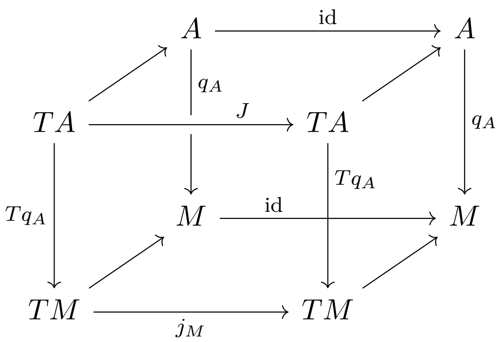
This is exactly the bracket in the Drinfeld double. Thus the isomorphism of Proposition 4.7.3 is indeed an isomorphism of Courant algebroids. □

4.8 Special cases

In this section we focus on some special cases and examples of generalised complex structures.

4.8.1 The complex case

In this section we consider the case of a holomorphic Lie algebroid. This corresponds to a linear complex structure J on A



, (4.79)

with core morphism $j_A \colon A \to A$.

The complex structure J makes the total space A into a complex manifold, whereas j_M is an almost complex structure on M. That the integrability condition of the almost complex structure on M is already implied by the integrability condition of the almost complex structure on A follows – after choice of an adapted splitting (see Corollary 4.8.6) – immediately by computations completely analogously to the computations in Section 4.1.3.

The linearity condition now guarantees that the projection $q_A\colon A \to M$ is compatible with the two complex structures, that is $Tq_A \circ J = j_M \circ Tq_A$ and thus is a holomorphic map. The core morphism j_A defines a complex structure on the fibres of A, making A into a complex vector bundle. Hence this description in terms of linear complex structures is equivalent to the usual definition of complex vector bundles.

Note that in this section we will work with a real Lie algebroid which is additionally a holomorphic vector bundle as described above. We thus have a Lie bracket which is defined on all sections and not only on the holomorphic sections as in the definition of holomorphic Lie algebroid given by Camille Laurent-Gengoux, Mathieu Stiénon and Ping Xu in [42]. Therefore we can apply the results of the previous sections about generalised complex structures on real Lie algebroids to this special case.

The corresponding generalised complex structure $\mathcal{J}$ is now given by

$$\mathcal{J} = \begin{pmatrix} J & 0 \\ 0 & -J^t \end{pmatrix}, \tag{4.80}$$

and the morphism $j\colon TM \oplus A^* \to TM \oplus A^*$ is given by

$$j = \begin{pmatrix} j_M & 0 \\ 0 & -j_A^t \end{pmatrix}. \tag{4.81}$$

The core morphism of the generalised complex structure $j_C\colon A \oplus T^*M \to A \oplus T^*M$ is given by

$$j_C = \begin{pmatrix} j_A & 0 \\ 0 & -j_M^t \end{pmatrix}. \tag{4.82}$$

Note that in this case we immediately observe that $j = -j_C^t$.

The eigenbundles of the complexified morphisms are now given by

$$\begin{aligned} U_+ &= T^{1,0}M \oplus (A^{0,1})^*\,, \\ U_- &= T^{0,1}M \oplus (A^{1,0})^*\,, \\ K_+ &= A^{1,0} \oplus (T^{0,1}M)^*\,, \\ K_- &= A^{0,1} \oplus (T^{1,0}M)^*\,, \end{aligned} \tag{4.83}$$

where we wrote $T^{1,0}M$, $T^{0,1}M$, $A^{1,0}$ and $A^{0,1}$ for the $\pm i$-eigenbundles of j_M and j_A, respectively.

These eigenbundles are Lie algebroids and thus also define Drinfeld double Courant algebroids

$$\begin{aligned} C_T^{1,0} &= T^{1,0}M \oplus (T^{1,0}M)^* , \\ C_T^{0,1} &= T^{0,1}M \oplus (T^{0,1}M)^* , \\ C_A^{1,0} &= A^{1,0} \oplus (A^{1,0})^* , \\ C_A^{0,1} &= A^{0,1} \oplus (A^{0,1})^* , \end{aligned} \tag{4.84}$$

induced by the Lie bialgebroid structure where $T^*_{\mathbb{C}}M$ and $A^*_{\mathbb{C}}$ are endowed with the trivial Lie algebroid structure with vanishing bracket and anchor. That is on $C_T^{1,0}$ and $C_T^{0,1}$ the brackets are given by

$$[\![(X,\theta),(Y,\eta)]\!] = ([X,Y], \mathcal{L}_X\eta - \iota_Y \mathbf{d}\theta) , \tag{4.85}$$

and on $C_A^{1,0}$ and $C_A^{0,1}$ by

$$[\![(a,\alpha),(b,\beta)]\!] = ([a,b], \mathcal{L}^A_a\beta - \iota_b \mathbf{d}^A\alpha) . \tag{4.86}$$

We observe that as vector bundles $C_+ = C_T^{1,0} \oplus C_A^{0,1}$ and $C_- = C_T^{0,1} \oplus C_A^{1,0}$.

In the following we will show that they in fact form matched pairs of Courant algebroids, a notion introduced by Melchior Grützmann and Mathieu Stiénon in [29]. There the authors describe all the conditions to obtain a Courant algebroid structure on the direct sum of two Courant algebroids acting on each other by connections. In the case of two Courant algebroids given as subbundles of a Courant algebroid an equivalent definition is the following.

Definition 4.8.1. *Let E be a Courant algebroid over M and E_1 and E_2 be two subbundles of E such that $E = E_1 \oplus E_2$. Assume, that $E_1 = E_2^\perp$ with respect to the pairing in E and that furthermore both E_1 and E_2 with the restriction of the anchor and the pairing and a bracket defined by the projection of the restriction of the bracket of E are Courant algebroids. Then E_1 and E_2 are a* ***matched pair of Courant algebroids****. Note that the bracket in E is not required to restrict to the brackets in E_1 or E_2.*

Naturally this definition defies the purpose of giving intrinsic conditions on E_1 and E_2 and their actions in order to retain the Courant algebroid structure in E. However, Grützmann and Stiénon proved in [29] that their conditions are equivalent to the situation described above in the case of an already given Courant algebroid structure in $E_1 \oplus E_2$ which is the case we consider here. For more details on the general case we refer to [29].

Standard Dorfman connections

In order to show that the Courant algebroid structures restrict in the way described in Definition 4.8.1 we want to give a particular description of the bracket in $C_\pm$. To do so we make use of standard Dorfman connections, defined by Jotz Lean in [36, Example 4.2] in the following way.

Definition 4.8.2. *A $(TM \oplus A^*)$-Dorfman connection Δ on $A \oplus T^*M$ is called* ***standard Dorfman connection*** *if it is induced by an ordinary connection $\nabla\colon \Gamma(TM) \times \Gamma(A) \to \Gamma(A)$ in the following way:*

$$\Delta_{X,\alpha}(a,\theta) = \Big(\nabla_X a, \mathcal{L}_X\theta + \langle \nabla^*_\bullet \alpha, a\rangle\Big)\,. \tag{4.87}$$

The dull bracket on $TM \oplus A^*$ corresponding to such a standard Dorfman connection is then given by

$$[\![(X,\alpha),(Y,\beta)]\!]_\Delta = \Big([X,Y], \nabla^*_X\beta - \nabla^*_Y\alpha\Big)\,. \tag{4.88}$$

In exactly the same way we define a complex standard Dorfman connection induced by an ordinary complex linear connection ∇.

Definition 4.8.3. *A complex $(T_{\mathbb{C}}M \oplus A^*_{\mathbb{C}})$-Dorfman connection Δ on $A_{\mathbb{C}} \oplus T^*_{\mathbb{C}}M$ is called* ***complex standard Dorfman connection*** *if it is induced by an ordinary complex connection $\nabla\colon \Gamma(T_{\mathbb{C}}M) \times \Gamma(A_{\mathbb{C}}) \to \Gamma(A_{\mathbb{C}})$ in the following way:*

$$\Delta_{X,\alpha}(a,\theta) = \Big(\nabla_X a, \mathcal{L}_X\theta + \langle \nabla^*_\bullet \alpha, a\rangle\Big)\,. \tag{4.89}$$

Now we describe standard Dorfman connections in terms of the corresponding dull bracket.

Lemma 4.8.4. *A skew-symmetric $(TM\oplus A^*)$-Dorfman connection Δ on $A\oplus T^*M$ is a standard Dorfman connection if and only if $[\![(X,0),(Y,0)]\!] = ([X,Y],0)$ for all sections $X,Y \in \Gamma(TM)$ and $[\![(0,\alpha),(0,\beta)]\!]_\Delta = 0$ for all sections $\alpha,\beta \in \Gamma(A^*)$. The analogous result holds for a complex standard Dorfman connection.*

Proof. Given a standard Dorfman connection Δ, then the corresponding bracket is given by (4.88) which clearly satisfies these properties.

Conversely, let Δ be a skew-symmetric Dorfman connection such that we have $[\![(X,0),(Y,0)]\!]_\Delta = ([X,Y],0)$ for all sections $X,Y \in \Gamma(TM)$ and $[\![(0,\alpha),(0,\beta)]\!]_\Delta = 0$ for all sections $\alpha,\beta \in \Gamma(A^*)$. Define then for any $X \in \Gamma(TM)$ and $\alpha \in \Gamma(A^*)$

$$\nabla^*_X\alpha := \mathrm{pr}_{A^*}[\![(X,0),(0,\alpha)]\!]_\Delta\,.$$

Since the bracket $[\![\cdot,\cdot]\!]_\Delta$ is anchored by pr_{TM} we immediately obtain that for any $f \in C^\infty(M)$, $X \in \Gamma(TM)$ and $\alpha \in \Gamma(A^*)$ we have $\nabla^*_{fX}\alpha = f\nabla^*_X\alpha$ and $\nabla^*_X f\alpha = f\nabla^*_X\alpha + X(f)\alpha$. Thus ∇^* defines an ordinary TM-connection on A^*. Furthermore, since the bracket is anchored by pr_{TM}, $\mathrm{pr}_{TM}[\![(X,0),(0,\alpha)]\!]_\Delta = [X,0] = 0$. Now Δ is induced by the TM-connection ∇ on A dual to ∇^*, since we have

$$\begin{aligned}[\![(X,\alpha),(Y,\beta)]\!]_\Delta &= [\![(X,0),(Y,0)]\!]_\Delta + [\![(0,\alpha),(0,\beta)]\!]_\Delta \\ &\quad + [\![(X,0),(0,\beta)]\!]_\Delta + [\![(0,\alpha),(Y,0)]\!]_\Delta \\ &= \Big([X,Y], \nabla^*_X\beta - \nabla^*_Y\alpha\Big).\end{aligned}$$

The same argument with $\mathbb{C}$-linear connections and brackets shows the analogous statement in the complex case. □

Let us from now on consider a linear generalised complex structure $\mathcal{J}$ induced by a complex structure as in (4.80).

We want to show that we can choose the adapted Dorfman connection to be a standard Dorfman connection. This allows us to write the brackets in the simpler form. We show the following statement.

Lemma 4.8.5. *Let $\mathcal{J}$ be a linear generalised complex structure on a vector bundle A, induced by a complex structure as in* (4.80)*. Let Δ be any standard Dorfman connection induced by an ordinary connection $\nabla\colon \Gamma(TM)\times\Gamma(A)\to\Gamma(A)$. Then the induced Dorfman connection Δ' given by Proposition 4.1.15 which is adapted to $\mathcal{J}$ is also a standard Dorfman connection.*

Proof. Given such a standard Dorfman connection Δ and a generalised complex structure $\mathcal{J}$ induced by a complex structure, we defined Φ by the equation

$$\mathcal{J}(\sigma^\Delta(\nu)) - \sigma^\Delta(j\nu) = \widetilde{\Phi(\nu)},$$

and set $\Psi(\nu_1,\nu_2) = \Phi(\nu_1)^t(\nu_2)$.

Because the connection Δ is induced by a connection ∇, the corresponding lift takes for $X \in \Gamma(TM)$, $\alpha \in \Gamma(A^*)$ and $a \in \Gamma(A)$ such that $a(m) = a_m$ the following form

$$\sigma^\Delta((X,\alpha))(a_m) = \Big(T_m a X(m), \mathbf{d}_{a_m}\ell_\alpha\Big) - \Big(\frac{d}{dt}\Big|_{t=0}(a_m + t\nabla_X a), (T_{a_m}q_A)^t\langle\nabla^*_\bullet\alpha, a\rangle\Big).$$

Thus

$$\begin{aligned}\widetilde{\Phi((X,\alpha))}(a_m) &= \Big(J(T_m a X(m)) - T_m a j_M X(m), \mathbf{d}_{a_m}\ell_{j_A^t\alpha} - J^t(\mathbf{d}_{a_m}\ell_\alpha)\Big) \\ &\quad + \Big(-J(\frac{d}{dt}\Big|_{t=0}(a_m + t\nabla_X a)) + \frac{d}{dt}\Big|_{t=0}(a_m + t\nabla_{j_M X} a), \\ &\qquad \langle\nabla^*_{j_M T_{a_m} q_A(\bullet)}\alpha - \nabla^*_{T_{a_m}q_A(\bullet)} j_A^t\alpha, a\rangle\Big).\end{aligned}$$

Now for any $\alpha, \beta \in \Gamma(A^*)$ and $a \in \Gamma(A)$

$$\langle \Psi((0,\alpha),(0,\beta)), a\rangle = \langle \Phi((0,\alpha))(a), (0,\beta)\rangle \,.$$

But since the TA-part of $\widetilde{\Phi((0,\alpha))}(a_m)$ vanishes, the A-part of $\Phi((0,\alpha))(a)$ has to vanish as well, since A is the core of TA. Therefore $\Psi((0,\alpha),(0,\beta)) = 0$ for all $\alpha, \beta \in \Gamma(A^*)$. Additionally we have for any $X, Y \in \Gamma(TM)$ and $a \in \Gamma(A)$

$$\langle \Psi((X,0),(Y,0)), a\rangle = \langle \Phi((X,0))(a), (Y,0)\rangle \,.$$

Since the T^*A-part of $\widetilde{\Phi((X,0))}$ vanishes, the T^*M-part of $\Phi((X,0))(a)$ has to vanish, as T^*M is the core of T^*A. Thus we also have $\Psi((X,0),(Y,0)) = 0$ for any $X, Y \in \Gamma(TM)$.

The bracket corresponding to the adapted Dorfman connection Δ' of Proposition 4.1.15 was now given for any $\nu_1, \nu_2 \in \Gamma(TM \oplus A^*)$ by

$$[\![\nu_1,\nu_2]\!]_{\Delta'} = [\![\nu_1,\nu_2]\!]_{\Delta} - \frac{1}{2}\Psi(\nu_1, j\nu_2)\,.$$

Now Δ is a standard Dorfman connection and j is given by equation (4.81). Thus also $\Psi((0,\alpha), j(0,\beta)) = 0$ and $\Psi((X,0), j(Y,0)) = 0$. With this we observe immediately that the bracket corresponding to the adapted Dorfman connection also satisfies the properties of Lemma 4.8.4 and thus the adapted Dorfman connection is also a standard Dorfman connection. □

Recall that linear TM-connections ∇ on E are in correspondence with lifts $\sigma^\nabla\colon \Gamma(TM) \to \mathfrak{X}^\ell(E)$ as we recapitulated in Example 2.1.26. The adapted standard Dorfman connection in the previous lemma is therefore induced by a linear connection that is adapted to the complex structure in the following sense.

Corollary 4.8.6. *Given a complex structure $J\colon TE \to TE$ on a vector bundle over a complex structure $j_M\colon TM \to TM$, there is always a linear TM-connection on E which is adapted to J in the sense that the lift $\sigma^\nabla\colon \Gamma(TM) \to \mathfrak{X}^\ell(E)$ induced by ∇ satisfies*

$$J(\sigma^\nabla(X)) = \sigma^\nabla(j_M X)\,, \tag{4.90}$$

for any $X \in \Gamma(TM)$.

Proof. According to Lemma 4.8.5 there is a standard Dorfman connection adapted to the induced generalised complex structure $\mathcal{J}$ as in (4.80). The TE-part of the corresponding lift is given by the lift $\sigma^\nabla\colon \Gamma(TM) \to \mathfrak{X}^\ell(E)$. Since Δ is now adapted to $\mathcal{J}$, the lift satisfies $\mathcal{J}\sigma^\Delta(X,\varepsilon) = \sigma^\Delta(j(X,\varepsilon))$. It follows immediately that the lift $\sigma^\nabla\colon \Gamma(TM) \to \mathfrak{X}^\ell(E)$ satisfies (4.90). □

Note that this result can also be obtained directly without the use of generalised complex structures and Dorfman connections. We can apply the same techniques as in the proof of Proposition 4.1.15 to splittings of the tangent bundle TE. First we choose any linear TM-connection ∇ on E, then consider the corresponding horizontal lift $\sigma^\nabla\colon \Gamma(TM) \to \mathfrak{X}^\ell(E)$. Similarly to the case of generalised complex structures, for a complex structure J on E over $j_M\colon TM \to TM$, the difference $J(\sigma^\nabla(X)) -_E \sigma^\nabla(j_M X)$ then defines a core-linear section $\widetilde{\Phi(X)}$ of $TE \to E$, where $\Phi \in \Gamma(T^*M \otimes E^* \otimes E)$. After showing the analogue to Lemma 4.1.7 in the complex case we can change the splitting with $\frac{1}{2} j_E \circ \Phi(X)$, where $j_E\colon E \to E$ is the core morphism of J and obtain an adapted lift and therefore an adapted connection as above. To our knowledge this is a new insight to almost complex structures on vector bundles.

The previous Lemma 4.8.5 allows us to choose the adapted Dorfman connection to be a standard Dorfman connection. Let us from now on fix a standard Dorfman connection Δ induced by an ordinary TM-connection ∇ on A which is adapted to $\mathcal{J}$. Furthermore, by abuse of notation we also write $\nabla\colon \Gamma(T_\mathbb{C}M) \oplus \Gamma(A_\mathbb{C}) \to \Gamma(A_\mathbb{C})$ for the complexification of ∇ in both arguments. This complexification satisfies the following.

Proposition 4.8.7. *In the setting above, we have*

$$\begin{aligned} \nabla_X a \in \Gamma(A^{1,0})\,, &\quad \textit{for all } X \in \Gamma(T^{1,0}M),\, a \in \Gamma(A^{1,0})\,, \\ \nabla_X a \in \Gamma(A^{0,1})\,, &\quad \textit{for all } X \in \Gamma(T^{0,1}M),\, a \in \Gamma(A^{0,1})\,. \end{aligned} \tag{4.91}$$

Proof. Since Δ is an adapted Dorfman connection we have by Corollary 4.2.7 that $\Delta^\mathbb{C}_u k \in \Gamma(K_\pm)$ for any $u \in \Gamma(U_\pm)$ and $k \in \Gamma(K_\pm)$. With the description of $U_\pm$ and $K_\pm$ as in equation (4.83) together with the definition of a standard Dorfman connection as in Definition 4.8.2 we immediately obtain the two properties (4.91). □

Now with the adapted standard Dorfman connection, we can describe the brackets, Lie derivatives and Lie differentials on $U_\pm$ and on $K_\pm$ as follows.

Lemma 4.8.8. *In the setting above, the Lie derivatives* $\mathcal{L}^{U_\pm}\colon \Gamma(U_\pm) \times \Gamma(K_\mp) \to \Gamma(K_\mp)$ *are given by the restriction of the complexification of* Δ*. The Lie derivatives* $\mathcal{L}^{K_\pm}\colon \Gamma(K_\pm) \times \Gamma(U_\mp) \to \Gamma(U_\mp)$ *are given by*

$$\mathcal{L}^{K_-}_{(a,\theta)}(X,\alpha) = \Big(\mathrm{pr}_{T^{1,0}M}([\rho(a), X]), \mathcal{L}_a\alpha - \mathrm{pr}_{(A^{0,1})^*}\, \rho^t i_X \mathbf{d}\theta\Big)\,, \tag{4.92}$$

$$\mathcal{L}^{K_+}_{(a,\theta)}(X,\alpha) = \Big(\mathrm{pr}_{T^{0,1}M}([\rho(a), X]), \mathcal{L}_a\alpha - \mathrm{pr}_{(A^{1,0})^*}\, \rho^t i_X \mathbf{d}\theta\Big)\,. \tag{4.93}$$

The respective Lie algebroid differential in degree 1 are given by the formulas

$$\begin{aligned}\mathbf{d}^{K_\pm}(X,\alpha)\big((a,\theta)(b,\eta)\big) &= \mathbf{d}^A\alpha(a,b) + \rho(a)\langle X,\eta\rangle - \rho(b)\langle X,\theta\rangle \\ &\quad - \langle X, \mathcal{L}_{\rho(a)}\eta - i_{\rho(b)}\mathbf{d}\theta\rangle\,,\end{aligned} \tag{4.94}$$

$$\begin{aligned}\mathbf{d}^{U_\pm}(a,\theta)\big((X,\alpha),(Y,\beta)\big) &= \mathbf{d}\theta(X,Y) + X\langle a,\beta\rangle - Y\langle a,\alpha\rangle \\ &\quad - \langle a, \nabla^*_X\beta - \nabla^*_Y\alpha\rangle\,.\end{aligned} \tag{4.95}$$

Proof. In the setting above, we have

$$[(X,\alpha),(Y,\beta)]_{U_\pm} = \Big([X,Y], \nabla^*_X\beta - \nabla^*_Y\alpha\Big)\,,$$

where X and Y are sections of $T^{1,0}M$ or $T^{0,1}M$, respectively and α and β are sections of $(A^{0,1})^*$ or $(A^{1,0})^*$, respectively.

The bracket on $K_\pm$ was defined as

$$\big[(a,\theta),(b,\eta)\big]_{K_-} = \Big([a,b], \mathcal{L}_{\rho(a)}\eta - i_{\rho(b)}\mathbf{d}\theta\Big)\,,$$

where $a,b \in \Gamma(A^{0,1})$ or $\Gamma(A^{1,0})$, respectively and $\theta,\eta \in \Gamma((T^{1,0}M)^*)$ or $\Gamma((T^{0,1}M)^*)$, respectively.

We can now compute the corresponding Lie derivatives. Since the bracket of $U_\pm$ is dual to the restriction of $\Delta^{\mathbb{C}}$, the Lie derivative $\mathcal{L}^{U_\pm} : \Gamma(U_\pm)\times\Gamma(K_\mp) \to \Gamma(K_\mp)$ is given by the adapted Dorfman connection, that is

$$\mathcal{L}^U_{(X,\alpha)}(a,\theta) = \Big(\nabla_X a, \mathcal{L}_X\theta + \langle \nabla^*_\bullet\alpha, a\rangle\Big)\,.$$

Secondly, $\mathcal{L}^K : \Gamma(K_\pm)\times\Gamma(U_\mp) \to \Gamma(U_\mp)$ can be computed as follows

$$\begin{aligned}\langle \mathcal{L}^K_{(a,\theta)}(X,\alpha),(b,\eta)\rangle &= \rho(a)\langle (X,\alpha),(b,\eta)\rangle - \langle (X,\alpha), \Big([a,b], \mathcal{L}_{\rho(a)}\eta - \iota_{\rho(b)}\mathbf{d}\theta\Big)\rangle \\ &= \rho(a)\langle\alpha,b\rangle - \langle\alpha,[a,b]\rangle + \rho(a)\langle X,\eta\rangle - \langle X, \mathcal{L}_{\rho(a)}\eta - \iota_\rho(b)\mathbf{d}\theta\rangle \\ &= \langle \mathcal{L}_a\alpha, b\rangle + \langle [\rho(a),X],\eta\rangle + \langle X, \iota_{\rho(b)}\mathbf{d}\theta\rangle\,.\end{aligned}$$

Thus the Lie derivatives are given by

$$\begin{aligned}\mathcal{L}^{K_-}_{(a,\theta)}(X,\alpha) &= \Big(\mathrm{pr}_{T^{1,0}M}([\rho(a),X]), \mathcal{L}_a\alpha - \mathrm{pr}_{(A^{0,1})^*}\,\rho^t i_X\mathbf{d}\theta\Big)\,, \\ \mathcal{L}^{K_+}_{(a,\theta)}(X,\alpha) &= \Big(\mathrm{pr}_{T^{0,1}M}([\rho(a),X]), \mathcal{L}_a\alpha - \mathrm{pr}_{(A^{1,0})^*}\,\rho^t i_X\mathbf{d}\theta\Big)\,.\end{aligned}$$

For the corresponding Lie algebroid differentials we obtain

$$\begin{aligned}\mathbf{d}^{K_\pm}(X,\alpha)\big((a,\theta),(b,\eta)\big) &= \rho(a)\langle (X,\alpha),(b,\eta)\rangle - \rho(b)\langle (X,\alpha),(a,\theta)\rangle \\ &\quad - \langle (X,\alpha), [(a,\theta),(b,\eta)]_{K_\pm}\rangle \\ &= \mathbf{d}^A\alpha(a,b) + \rho(a)\langle X,\eta\rangle - \rho(b)\langle X,\theta\rangle \\ &\quad - \langle X, \mathcal{L}_{\rho(a)}\eta - i_{\rho(b)}\mathbf{d}\theta\rangle\,.\end{aligned}$$

In the case where $X = 0$ this is the differential of A: $\mathbf{d}^K(0,\alpha) = (0, \mathbf{d}^A\alpha)$. The differential of U_+ is

$$\begin{aligned}\mathbf{d}^{U_\pm}(a,\theta)\big((X,\alpha),(Y,\beta)\big) &= X\langle(a,\theta),(Y,\beta)\rangle - Y\langle(a,\theta),(X,\alpha)\rangle \\ &\quad - \langle(a,\theta),[(X,\alpha),(Y,\beta)]_{U_\pm}\rangle \\ &= \mathbf{d}\theta(X,Y) + X\langle a,\beta\rangle - Y\langle a,\alpha\rangle - \langle a, \nabla^*_X\beta - \nabla^*_Y\alpha\rangle\,.\end{aligned}$$

In the case of $a = 0$ we get $\mathbf{d}^{U_\pm}(0,\theta) = (0,\mathbf{d}\theta)$. □

Now we can prove the following theorem.

Theorem 4.8.9. *Let A be a generalised complex Lie algebroid A with generalised complex structure induced by a complex structure as in* (4.80)*. Let $C_\pm$ be the complex Courant algebroids defined by Proposition 4.6.2 and let $C_T^{0,1}$, $C_T^{1,0}$, $C_A^{0,1}$ and $C_A^{1,0}$ be the Courant algebroids defined by* (4.84)*. Then*

1. *$C_T^{0,1}$ and $C_A^{1,0}$ form a matched pair of Courant algebroids and their matched sum Courant algebroid is C_+.*
2. *$C_T^{1,0}$ and $C_A^{0,1}$ form a matched pair of Courant algebroids and their matched sum Courant algebroid is C_-.*

Proof. By Theorem 4.7.6 the bracket on $C_\pm$ is for $u_1, u_2 \in \Gamma(U_\pm)$ and $k_1, k_2 \in \Gamma(K_\mp)$ given by

$$\begin{aligned}[\![u_1 \oplus k_1, u_2 \oplus k_2]\!]_{C_\pm} &= [u_1,u_2]_{U_\pm} + \mathcal{L}^{K_\mp}_{k_1}u_2 - \iota_{k_2}\mathbf{d}^{K_\mp}u_1 \\ &\quad \oplus [k_1,k_2]_{K_\mp} + \mathcal{L}^{U_\pm}_{u_1}k_2 - \iota_{u_2}\mathbf{d}^{U_\pm}k_1\,.\end{aligned}$$

With the formulas from Lemma 4.8.8 it is now straightforward to see that for two sections of the subbundle $C_T^{1,0} \subseteq C_+$ we have

$$\begin{aligned}[\![(X,0)\oplus(0,\theta),(Y,0)\oplus(0,\eta)]\!]_{C_+} &= \Big([X,Y], \rho^t(\mathcal{L}_X\eta - i_Y\mathbf{d}\theta)\Big) \\ &\quad \oplus \Big(0, \mathcal{L}_X\eta - i_Y\mathbf{d}\theta\Big)\,.\end{aligned}$$

Thus $C_T^{1,0}$ is not a Courant subalgebroid, as the bracket does not restrict to $C_T^{1,0}$. But the projection to $C_T^{1,0}$ of the bracket on C_+ of two such sections is precisely the bracket of these sections in $C_T^{1,0}$. Analogously, the projection to $C_T^{0,1}$ of the the bracket on C_- of two sections of $C_T^{0,1}$ equals the bracket of these sections in the Courant algebroid $C_T^{0,1}$.

Similarly for two sections of the subbundle $C_A^{0,1} \subseteq C_+$ we obtain

$$\begin{aligned}[\![(0,\alpha)\oplus(a,0),(0,\beta)\oplus(b,0)]\!]_{C_+} &= \Big(0, \mathcal{L}_a\beta - i_b\mathbf{d}^A\alpha\Big) \\ &\quad \oplus \Big([a,b], \langle\nabla^*_\bullet\alpha, b\rangle + \langle\nabla_\bullet a,\beta\rangle\Big)\,.\end{aligned}$$

Again, $C_A^{0,1}$ is not a Courant subalgebroid of C_+ since the bracket does not preserve $C_A^{0,1}$. However, the projection of this bracket to $C_A^{0,1}$ coincides again with the bracket of the Courant algebroid $C_A^{0,1}$. The analogue result holds true for $C_A^{1,0} \subset C_-$. □

Since in this case the Courant algebroid structure on the direct sum is already given, we do not need to describe the two actions of the summands on each other to obtain a matched pair. However, we still compute the two connections since by the result of [29] the Courant algebroid structure on $C_\pm$ can be recovered from the structures on the summands and these connections.

Proposition 4.8.10. *In the situation above, the action of $C_T^{1,0}$ on $C_A^{0,1}$ and the action of $C_T^{0,1}$ on $C_A^{1,0}$ are given by*

$$\overrightarrow{\nabla}_{(X,\theta)}(a,\alpha) = (\nabla_X a, \nabla_X^* \alpha)\,. \tag{4.96}$$

The action of $C_A^{0,1}$ on $C_T^{1,0}$ is given by

$$\overleftarrow{\nabla}_{(a,\alpha)}(X,\theta) = (\mathrm{pr}_{T^{1,0}M}([\rho(a),X]), \mathcal{L}_{\rho(a)}\theta)\,. \tag{4.97}$$

and the action of $C_A^{1,0}$ on $C_T^{0,1}$ by

$$\overleftarrow{\nabla}_{(a,\alpha)}(X,\theta) = (\mathrm{pr}_{T^{0,1}M}([\rho(a),X]), \mathcal{L}_{\rho(a)}\theta)\,. \tag{4.98}$$

Proof. To obtain the action of $C_T^{1,0}$ on $C_A^{0,1}$ we compute the bracket for sections of the two subbundles as follows

$$\begin{aligned}
[\![(X,0)\oplus(0,\theta),(0,\alpha)\oplus(a,0)]\!]_{C_+} &= \Big((0,\nabla_X^*\alpha) - \iota_{(a,0)}\mathbf{d}^{K_-}(X,0)\Big) \\
&\quad \oplus \Big((\nabla_X a, -\iota_{\rho(a)}\mathbf{d}\theta) - \iota_{(0,\alpha)}\mathbf{d}^{U_+}(0,\theta)\Big) \\
&= (-\,\mathrm{pr}_{T^{1,0}M}([\rho(a),X]), \nabla_X^*\alpha) \oplus (\nabla_X a, -\iota_{\rho(a)}\mathbf{d}\theta)\,,
\end{aligned}$$

where we used that $\iota_{(a,0)}\mathbf{d}^{K_\mp}(X,0) = ([\rho(a),X],0)$ and $\iota_{(0,\alpha)}\mathbf{d}^{U_\pm}(0,\theta) = 0$. The action is now given by the projection of the bracket above to $C_A^{0,1}$,

$$\overrightarrow{\nabla}_{(X,\theta)}(a,\alpha) = (\nabla_X a, \nabla_X^* \alpha)\,.$$

Since $C_T^{1,0} = (C_A^{0,1})^\circ$ the bracket of those sections is skew-symmetric and the action of $C_A^{0,1}$ on $C_T^{1,0}$ is therefore given by the negative of the projection to $C_T^{1,0}$.

$$\overleftarrow{\nabla}_{(a,\alpha)}(X,\theta) = (\mathrm{pr}_{T^{1,0}M}([\rho(a),X]), \mathcal{L}_{\rho(a)}\theta)\,.$$

Notice that $\iota_{\rho(a)}\mathbf{d}\theta = \mathcal{L}_{\rho(a)}\theta$ since $\rho(a) \in \Gamma(T^{0,1}M)$ and thus $\langle\rho(a),\theta\rangle = 0$. With these actions the Courant algebroids $C_T^{1,0}$ and $C_A^{0,1}$ form a matched pair and their matched sum is C_+.

Completely analogously we obtain for the Courant algebroid $C_- = C_T^{0,1} \oplus C_A^{1,0}$ the action of $C_T^{0,1}$ on $C_A^{1,0}$ by the same equation as (4.96) and the action of $C_A^{1,0}$ on $C_T^{0,1}$ by

$$\overleftarrow{\nabla}_{(a,\alpha)}(X,\theta) = (\mathrm{pr}_{T^{0,1}M}([\rho(a), X]), \mathcal{L}_{\rho(a)}\theta)\,, \tag{4.99}$$

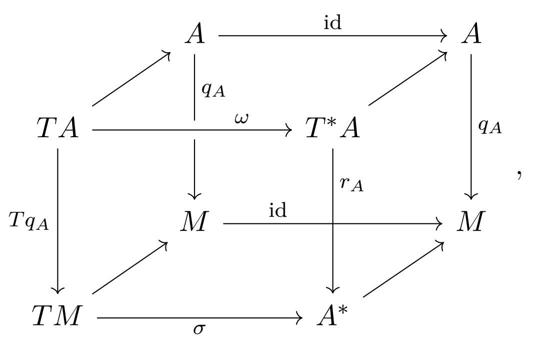
where in this case $X \in \Gamma T^{0,1}M$, $\theta \in \Gamma\big((T^{0,1}M)^*\big)$, $a \in \Gamma(A^{1,0})$ and $\alpha \in \Gamma\big((A^{1,0})^*\big)$.

By Theorem 4.8.9 these actions satisfy all properties of a matched pair of Courant algebroids as defined in [29] and the Courant algebroid structures on C_+ and C_- can be described as a matched sum. □

In [42] Camille Laurent-Gengoux, Mathieu Stiénon and Ping Xu studied holomorphic Lie algebroids in detail. They showed an equivalence between holomorphic Lie algebroid structures on $A \to M$ and linear holomorphic Poisson structures on the complex dual $\mathrm{Hom}_{\mathbb{C}}(A, \mathbb{C})$. They also showed that for a complex manifold M the Lie algebroids $T^{1,0}M$ and $T^{0,1}M$ form a matched pair of complex Lie algebroids with matched sum $T_{\mathbb{C}}M$ and more generally for a holomorphic Lie algebroid A the Lie algebroids $A^{1,0}$ and $A^{0,1}$ form a matched pair with matched sum $A_{\mathbb{C}}$. Matched pairs of Lie algebroids have been introduced by Tahar Mokri in [56] and further studied by Kirill Mackenzie in [52]. See also here Definition 2.2.16 and Theorem 2.2.17.

4.8.2 The symplectic case

In this section we consider the special case of a symplectic Lie algebroid, that is when the generalised complex structure is induced by a linear symplectic structure $\omega\colon TA \to T^*A$ on A which is a Lie algebroid morphism over some vector bundle morphism $\sigma\colon TM \to A^*$, both of which have to be invertible since the symplectic structure is an isomorphism of vector bundles over A.



(4.100)

with core morphism $\tau\colon A \to T^*M$.

The generalised complex structure $\mathcal{J}$ is then given by

$$\mathcal{J} = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}, \tag{4.101}$$

and the side morphism $j\colon TM \oplus A^* \to TM \oplus A^*$ is given by

$$j = \begin{pmatrix} 0 & -\sigma^{-1} \\ \sigma & 0 \end{pmatrix}. \tag{4.102}$$

The core morphism $j_C\colon A \oplus T^*M \to A \oplus T^*M$ is given by

$$j_C = \begin{pmatrix} 0 & -\tau^{-1} \\ \tau & 0 \end{pmatrix}. \tag{4.103}$$

The property $j = -j_C^t$ is thus equivalent to $\tau = -\sigma^t$.

The symplectic structure on A induces a Poisson structure on M, given by $\pi_M = \rho \circ (-\sigma^t)^{-1}$. This Poisson structure induces a Lie algebroid structure on T^*M and $-\sigma^t$ is an isomorphism of Lie algebroids between A and T^*M with respect to this structure ([51]). Furthermore, A^* carries a Lie algebroid structure induced by the linear Poisson structure on A or equivalently by the isomorphism $\sigma\colon TM \to A^*$. Now (A, A^*) and (T^*M, TM) are Lie bialgebroids and they are isomorphic as such. All these properties of symplectic Lie algebroids have been studied for example by Mackenzie and Xu in [53], see also [51]. These Lie bialgebroids then give rise to two isomorphic Drinfeld double Courant algebroids as defined in [45] and recapitulated in Example 2.3.4.

$$\begin{aligned} C_A &:= A \oplus A^*\,, \\ C_T &:= TM \oplus T^*M\,. \end{aligned}$$

Let us now consider the two eigenbundles of the complexification $j_{\mathbb{C}}$. They are given by

$$\begin{aligned} U_\pm &= \{(X,\alpha) \mp ij(X,\alpha) \mid (X,\alpha) \in TM \oplus A^*\} \\ &= \{(X \pm i\sigma^{-1}\alpha, \alpha \mp i\sigma X) \mid (X,\alpha) \in TM \oplus A^*\} \\ &= \{(X + iY, \pm(\sigma Y - i\sigma X) \mid (X,Y) \in TM \oplus TM\} \\ &= \{(Z \mp i(\sigma_{\mathbb{C}} Z) \mid Z \in T_{\mathbb{C}}M\}\,. \end{aligned}$$

Since the anchor of the Lie algebroid $U_\pm$ is given by the projection $\mathrm{pr}_{T_{\mathbb{C}}M}$ we immediately get from this description that $U_\pm$ is isomorphic to $T_{\mathbb{C}}M$ and thus also to $A^*_{\mathbb{C}}$ by the isomorphism $\sigma_{\mathbb{C}} \circ \mathrm{pr}_{T_{\mathbb{C}}M}\colon U_\pm \to A^*_{\mathbb{C}}$.

Similarly the two eigenbundles of $j_{C,\mathbb{C}}$ are given by

$$\begin{aligned} K_\pm &= \{(a,\theta) \mp i j_C(a,\theta) \mid (a,\theta) \in A \oplus T^*M\} \\ &= \{(a \mp i(\sigma^t)^{-1}\theta, \theta \pm i\sigma^t a) \mid (a,\theta) \in A \oplus T^*M\} \\ &= \{(a + ib, \mp(\sigma^t b - i\sigma^t a) \mid (a,b) \in A \oplus A\} \\ &= \{(c \pm i(\sigma^t_{\mathbb{C}} c) \mid c \in A_{\mathbb{C}}\}\,. \end{aligned}$$

Here the projection to $A_{\mathbb{C}}$ is a Lie algebroid morphism and by this description an isomorphism. Thus $K_\pm$ is isomorphic to $A_{\mathbb{C}}$ and therefore to $T^*_{\mathbb{C}}M$.

Summarising, we have the following isomorphisms of Lie algebroids

$$\begin{aligned} A &\cong T^*M\,, \\ TM &\cong A^*\,, \\ U_\pm &\cong T_{\mathbb{C}}M \cong A^*_{\mathbb{C}}\,, \\ K_\pm &\cong A_{\mathbb{C}} \cong T^*_{\mathbb{C}}M\,. \end{aligned}$$

Since $C_\pm$ is the Drinfeld double Courant algebroid of the Lie bialgebroid $(U_\pm, K_\mp)$ it is isomorphic to the complexification of the Courant algebroid C_A and hence also to the complexification of the Courant algebroid C_T. In this case $C_\pm$ can not be viewed as a matched sum Courant algebroid of these but as complexification instead.

4.9 Generalised complex structures in VB-Courant algebroids

Instead of considering the standard Courant algebroid over the total space of a vector bundle as in Section 4.1 we can consider a general VB-Courant algebroid as in Definition 2.3.10

$$\begin{array}{ccc} \mathbb{E} & \longrightarrow & B \\ \downarrow & Q^* & \downarrow \\ & \searrow & \\ Q & \longrightarrow & M \end{array}\,. \tag{4.104}$$

Since the anchor $\rho\colon \mathbb{E} \to TB$ is a double vector bundle morphism, it induces a morphisms of the cores $\partial_B\colon Q^* \to B$, called the **core anchor** of $\mathbb{E}$.

A linear splitting Σ of the double vector bundle $\mathbb{E}$ is called **Lagrangian** if the image of Σ is isotropic in $\mathbb{E}$. Madeleine Jotz Lean shows in [39] that a change of Lagrangian splittings corresponds to a skew-symmetric element $\Phi_{12} \in \Gamma(Q^* \otimes B^* \otimes Q^*)$.

None of the results in Section 4.1.1 does rely on the Courant-Dorfman bracket on $TE \oplus T^*E$, but only on the structure of a metric double vector bundle and the corresponding Lagrangian lifts. Thus all the results immediately translate to corresponding results for the more general case of a VB-Courant algebroid in the following way.

Let us now fix a Lagrangian splitting Σ of $\mathbb{E}$ and denote the corresponding lift by $\sigma\colon \Gamma(Q) \to \Gamma^\ell_B(\mathbb{E})$. Let us consider a double vector bundle morphism $\mathcal{J}\colon \mathbb{E} \to \mathbb{E}$ over id_B and $j\colon Q \to Q$ with core morphism $j_C\colon Q^* \to Q^*$. Then we obtain analogously to Lemma 4.1.5 the following definition of Φ depending on the choice of the splitting.

Lemma 4.9.1. *Given a double vector bundle morphism $\mathcal{J}\colon \mathbb{E} \to \mathbb{E}$ over j and id_B there is $\Phi \in \Gamma(Q^* \otimes B^* \otimes Q^*)$ defined by setting for any $q \in \Gamma(Q)$*

$$\mathcal{J}(\sigma(q)) = \sigma(jq) + \widetilde{\Phi(q)}\,. \tag{4.105}$$

Furthermore, completely analogously to the description of generalised almost complex structures on a vector bundle in Section 4.1.1 we obtain in the case of a general VB-Courant algebroid the following two lemmas.

Lemma 4.9.2. *A double vector bundle morphism $\mathcal{J}\colon \mathbb{E} \to \mathbb{E}$ satisfies $\mathcal{J}^2 = -\mathrm{id}_{TE\oplus T^*E}$ if and only if for any Lagrangian splitting and corresponding Φ we get for any $q \in \Gamma(Q)$*

1. $j^2 = -\mathrm{id}_Q\,,$
2. $j_C^2 = -\mathrm{id}_{Q^*}\,,$
3. $\Phi(j(q)) = -j_C \circ (\Phi(q))\,.$

Lemma 4.9.3. *A double vector bundle morphism $\mathcal{J}\colon \mathbb{E} \to \mathbb{E}$ such that additionally $\mathcal{J}^2 = -1$, is orthogonal if and only if for any Lagrangian splitting we have for all $b \in \Gamma(B)$ and $q_1, q_2 \in \Gamma(Q)$*

1. $j = -(j_C)^t\,,$
2. $\langle j(q_1), \Phi(q_2)(b)\rangle = -\langle j(q_2), \Phi(q_1)(b)\rangle\,.$

Now we can again define a 2-form $\Psi \in \Omega^2(Q, B^*)$ by setting

$$\Psi(q_1, q_2) := \Phi(q_1)^t(q_2)\,. \tag{4.106}$$

With this definition we obtain completely equivalent to Proposition 4.1.10 the following proposition.

Proposition 4.9.4. *A morphism $\mathcal{J}\colon \mathbb{E} \to \mathbb{E}$ is a generalised almost complex structure in $\mathbb{E}$, if and only if for any Lagrangian splitting we have*

1. $j^2 = -1$,
2. $j = -(j_C)^t$,
3. Ψ *is skew-symmetric, that is* $\Psi \in \Omega^2(Q, B^*)$,
4. $\Psi(\nu_1, \nu_2) = -j^*\Psi(\nu_1, \nu_2)$.

Also in this case we can adapt our Lagrangian splitting to the generalised almost complex structure. As mentioned before it was shown in [39] that such a change of splittings corresponds to a skew-symmetric element $\Phi_{12} \in \Gamma(Q^* \otimes B^* \otimes Q^*)$. We obtain the following

Proposition 4.9.5. *Given a generalised almost complex structure $\mathcal{J}$ in a VB-Courant algebroid $(\mathbb{E}; Q, B; M)$ with side morphism $j\colon Q \to Q$, there is a Lagrangian lift $\sigma\colon \Gamma(Q) \to \Gamma^\ell_B(\mathbb{E})$, such that for any $q \in \Gamma(Q)$ we have*

$$\mathcal{J}(\sigma(q)) = \sigma(jq) . \tag{4.107}$$

Proof. Let us fix any Lagrangian lift σ_1. This defines then by Lemma 4.9.1 a tensor $\Phi_1 \in \Gamma(Q^* \otimes B^* \otimes Q^*)$. Define another tensor $\Phi_{12} \in \Gamma(Q^* \otimes B^* \otimes Q^*)$ by setting for any $q \in \Gamma(Q)$ and $b \in \Gamma(B)$

$$\Phi_{12}(q)(b) := \frac{1}{2} j_C(\Phi_1(q)(b)) .$$

By Lemma 4.9.2 and Lemma 4.9.3 this Φ_{12} is skew-symmetric. We define a new Lagrangian lift by letting $\sigma_2(q) := \sigma_1(q) - \widetilde{\Phi_{12}(q)}$. Now we simply compute making use of Lemma 4.9.2 and Lemma 4.9.3:

$$\begin{aligned}
\mathcal{J}(\sigma_2(q)) &= \mathcal{J}\Big(\sigma_1(q) - \widetilde{\Phi_{12}(q)}\Big) \\
&= \sigma_1(jq) + \widetilde{\Phi_1(q)} - \mathcal{J}\big(\widetilde{\Phi_{12}(q)}\big) \\
&= \sigma_2(jq) + \widetilde{\Phi_{12}(jq)} + \widetilde{\Phi_1(q)} + \widetilde{j_C \circ \Phi_{12}(q)} \\
&= \sigma_2(jq) + \frac{1}{2}\widetilde{(\Phi_1(q) + j_C \circ \Phi_1(jq))} \\
&= \sigma_2(jq) .
\end{aligned}$$

Setting $\sigma := \sigma_2$ completes the proof. ∎

Using this existence of an adapted Lagrangian splitting, we can use the correspondence of VB-Courant algebroid structures to split Lie 2-algebroids proved in [39]. Let us fix such an adapted Lagrangian splitting as in Proposition 4.9.5. Then the VB-Courant algebroid structure is equivalent to a split Lie 2-algebroid structure $(\rho_Q, \partial_B^t, [\![\cdot,\cdot]\!]_\Delta, \nabla, \omega)$ on $Q \oplus B^*$, where the bracket in $\mathbb{E}$ is described by the dull bracket on Q and the dual Dorfman connection as follows.

$$\begin{aligned} [\![\sigma(q_1), \sigma(q_2)]\!] &= \sigma([\![q_1, q_2]\!]_\Delta) - \widetilde{R_\omega(q_1, q_2)} \\ [\![\sigma(q), \tau^\dagger]\!] &= (\Delta_q \tau)^\dagger \\ [\![\tau_1^\dagger, \tau_2^\dagger]\!] &= 0\,. \end{aligned}$$

Here $R_\omega(q_1, q_2) := \omega(q_1, q_2, \cdot)^t \in \Gamma(\mathrm{Hom}(B, Q^*))$

With this description of the Courant algebroid bracket we obtain similar computations and results for the Nijenhuis tensor of core sections and lifts for a linear generalised almost complex structure in the VB-Courant algebroid $\mathbb{E}$ as in Section 4.1.3 in the special case of $TE \oplus T^*E$.

First, we get analogously to the computations in Section 4.1.3 that $N_{\mathcal{J}}(\sigma(q), \tau^\dagger)$ of a generalised almost complex structure $\mathcal{J}$ in $\mathbb{E}$ vanishes for any $q \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$ if and only if $\mathbb{N}_{j,[\![\cdot,\cdot]\!]_\Delta}$ vanishes. Second, the analogous computation for the Nijenhuis tensor of two lifts gives

$$\begin{aligned} N_{\mathcal{J}}(\sigma(q_1), \sigma(q_2)) = \sigma^\Delta(N_{j,[\![\cdot,\cdot]\!]_Q}(q_1, q_2)) + \widetilde{R_\omega(j(q_1), j(q_2))} - \widetilde{R_\omega(q_1, q_2)} \\ - \widetilde{j_C \circ R_\omega(j(q_1), q_2)} - \widetilde{j_C \circ R_\omega(q_1, j(q_2))} \end{aligned} \tag{4.108}$$

Dualising the property

$$R_\omega(j(q_1), j(q_2)) - R_\omega(q_1, q_2) - j_C \circ R_\omega(j(q_1), q_2) - j_C \circ R_\omega(q_1, j(q_2)) = 0\,, \tag{4.109}$$

by evaluating at any $b \in \Gamma(B)$ and then pairing with q_3 gives as an equivalent condition on $\omega \in \Omega^3(Q, B^*)$ the following:

$$\omega(q_1, q_2, q_3) - \omega(jq_1, jq_2, q_3) - \omega(jq_1, q_2, jq_3) - \omega(q_1, jq_2, jq_3) = 0\,. \tag{4.110}$$

Thus we obtain the following proposition.

Proposition 4.9.6. *A linear generalised almost complex structure $\mathcal{J}$ in $\mathbb{E}$ over $j\colon Q \to Q$ is integrable if and only if for any adapted Lagrangian splitting we have in the corresponding split Lie 2-algebroid for any $q_1, q_2 \in \Gamma(Q)$*

1. $N_{j,[\![\cdot,\cdot]\!]_\Delta}(q_1, q_2) = 0\,,$
2. $\omega(q_1, q_2, q_3) - \omega(jq_1, jq_2, q_3) - \omega(jq_1, q_2, jq_3) - \omega(q_1, jq_2, jq_3) = 0\,.$

We can also define as before an equivalence relation on the Lagrangian splittings depending on the vector bundle morphism $j\colon Q \to Q$.

Definition 4.9.7. *Given a VB-Courant algebroid* $(\mathbb{E}; Q, B; M)$ *and a vector bundle morphism* $j\colon Q \to Q$*, two Lagrangian splittings* Σ_1 *and* Σ_2 *are* j*-**equivalent** if the corresponding change of splittings* $\Psi \in \Omega^2(Q, B^*)$ *satisfies for any* $q_1, q_2 \in \Gamma(Q)$*:*

$$\Psi(q_1, q_2) = \Psi(jq_1, jq_2)\,. \tag{4.111}$$

Analogously to Lemma 4.1.17 we obtain again that given a splitting Σ_1 which is adapted to a linear generalised almost complex structure $(\mathcal{J}, j)$, then a second splitting Σ_2 is also adapted to $(\mathcal{J}, j)$ if and only if Σ_1 and Σ_2 are j-equivalent. This then allows to formulate the analogue of Theorem 4.1.19 in the general case as follows.

Theorem 4.9.8. *A linear generalised complex structure* $\mathcal{J}$ *in a VB-Courant algebroid* $\mathbb{E}$ *is equivalent to a vector bundle morphism* $j\colon Q \to Q$ *and a* j*-equivalence class of linear splittings such that in the corresponding split Lie 2-algebroid* $(\rho_Q, \partial_B^t, [\![\cdot,\cdot]\!]_\Delta, \nabla, \omega)$ *over* $Q \oplus B^*$ *we have for any* $q_1, q_2, q_3 \in \Gamma(Q)$

1. $j^2 = -\operatorname{id}_Q$,
2. $N_{j,[\![\cdot,\cdot]\!]_\Delta} = 0$,
3. $\omega(q_1, q_2, q_3) - \omega(jq_1, jq_2, q_3) - \omega(jq_1, q_2, jq_3) - \omega(q_1, jq_2, jq_3) = 0$.

Note that we could define a bracket $\mathbb{A}$ as in the special case but we can not express the condition on the curvature terms as a Jacobi identity of this bracket. In the case of $TE \oplus T^*E$ the vanishing of the terms in R_ω on all of $\Gamma(E)$ is equivalent to the extension to R_Δ vanishing on all of $\Gamma(E \oplus T^*M)$. In the general case we do not have an analogous property.

Theorem 4.9.8 now allows us to define a generalised complex structure in a split Lie 2-algebroid as follows.

Definition 4.9.9. *A generalised complex structure in a split Lie 2-algebroid* $(\rho_Q, \partial_B^t, [\![\cdot,\cdot]\!], \nabla, \omega)$ *over* $Q \oplus B^*$ *is a vector bundle morphism* $j\colon Q \to Q$*, such that for any* $q_1, q_2, q_3 \in \Gamma(Q)$ *we have*

1. $j^2 = -\operatorname{id}_Q$,
2. $N_{j,[\![\cdot,\cdot]\!]} = 0$,
3. $\omega(q_1, q_2, q_3) - \omega(jq_1, jq_2, q_3) - \omega(jq_1, q_2, jq_3) - \omega(q_1, jq_2, jq_3) = 0$.

The translation of generalised complex structures to Lie 2-algebroids without a splitting is future work. This will make use of the one-to-one correspondence between VB-Courant algebroids and Lie 2-algebroids first described by David Li-Bland in [44], building up on Dimitry Roytenberg's correspondence between Courant algebroids and symplectic Lie 2-algebroids in [66]. Madeleine Jotz Lean described in [37] and [39] how this correspondence can be deduced from her geometrisation of graded manifolds of degree 2. Whereas such [2]-manifolds correspond to metric double vector bundles, it is shown that Lie 2-algebroids correspond to VB-Courant algebroids, Poisson Lie 2-algebroids correspond to LA-Courant algebroids and finally symplectic Lie 2-algebroid corresponding to the tangent prolongations TE of Courant algebroids E. In this light, a generalised complex structure J in a Courant algebroid gives rise to a map corresponding to TJ between two symplectic Lie 2-algebroids, which is a morphism of the underlying symplectic [2]-manifolds and additionally compatible with the cohomological vector field. Describing this compatibility in the non-split case and then extending generalised complex structures to Lie n-algebroids for arbitrary n is future work.

Appendix A

More general Dorfman connections

In this chapter of the appendix we will give computations for linear generalised complex structures $\mathcal{J}$ on a vector bundle E or a Lie algebroid A where we do not choose the Dorfman connection to be adapted to $\mathcal{J}$. This can be useful in the case where we already chose a specific Dorfman connection. For example we might want to choose such a Dorfman connection adapted to a different present geometric structure and cannot find a connection adapted to both simultaneously. Note that in the case of generalised Kähler structures we can always find a suitable connection adapted to both generalised complex structures as showed in Proposition 4.1.22.

We will also give a description of the Dorfman bracket involving core-linear sections, which might also be useful for other problems.

Since a lot of the computations in this chapter are fairly lengthy and it therefore seems a lot more reasonable to work with adapted connections, we decided to move them into the appendix.

A.1 Nijenhuis tensors

If we use any skew-symmetric Dorfman connection, which is not adapted to the generalised almost complex structure, then we will need Dorfman brackets involving core-linear sections in order to compute the Nijenhuis tensor involving lifts of sections of $TM \oplus E^*$.

In [36] the $TM \oplus E^*$-connection ∇ on E corresponding to a Dorfman connection Δ as in (4.8) was defined by $\nabla_\nu(e) := \mathrm{pr}_E \Delta_\nu(e, 0)$. The following description of the anchor in the Courant algebroid $TE \oplus T^*E \to E$, that is $\rho = \mathrm{pr}_{TE}$ is also proved in [36].

Proposition A.1.1. *Given a Dorfman connection* Δ *as in* (4.8), $\nu \in \Gamma(TM \oplus E^*)$ *and* $\tau \in \Gamma(E \oplus T^*M)$, *we have*

1. $\mathrm{pr}_{TE}(\sigma^\Delta(\nu)) = \widehat{\nabla_\nu}$,
2. $\mathrm{pr}_{TE}(\tau^\uparrow) = (\mathrm{pr}_E(\tau))^\uparrow$.

Definition A.1.2. *Given a Dorfman connection* Δ *as in* (4.8), *we define for* $\nu \in \Gamma(TM \oplus E^*)$ *a derivation* $\lozenge_\nu$ *of* $\Gamma(\mathrm{Hom}(E, E \oplus T^*M))$ *over* $\mathrm{pr}_{TM}\,\nu$ *by*

$$\lozenge_\nu \varphi := \Delta_\nu \circ \varphi - \varphi \circ \nabla_\nu \,.$$

Note that this does not define a connection on $\mathrm{Hom}(E, E \oplus T^*M)$ *as it is not* $C^\infty(M)$*-linear in* ν.

Using this definition we can now state prove the following corollary to Theorem 4.1.3 about the Dorfman brackets involving core-linear sections:

Corollary A.1.3. *For any skew-symmetric Dorfman connection* Δ *as in* (4.8), $\varphi, \psi \in \Gamma(\mathrm{Hom}(E, E \oplus T^*M))$, $\nu \in \Gamma(TM \oplus E^*)$ *and* $\tau \in \Gamma(E \oplus T^M)$ *we have*

1. $[\![\tau^\uparrow, \widetilde{\varphi}]\!] = \left(\varphi(\mathrm{pr}_E(\tau))\right)^\uparrow = -[\![\widetilde{\varphi}, \tau^\uparrow]\!]$,
2. $[\![\widetilde{\varphi}, \widetilde{\psi}]\!] = \widetilde{\psi \circ \mathrm{pr}_E \circ \varphi} - \widetilde{\varphi \circ \mathrm{pr}_E \circ \psi}$,
3. $[\![\sigma^\Delta(\nu), \widetilde{\varphi}]\!] = \widetilde{\lozenge_\nu \varphi}$,
4. $[\![\widetilde{\varphi}, \sigma^\Delta(\nu)]\!] = -\widetilde{\lozenge_\nu \varphi} + \sigma^\Delta((0, \varphi^t(\nu)) + \widetilde{\Delta_{(0,\varphi^t(\nu))}(\cdot, 0)}$.

Proof. Again we write in local coordinates $\varphi = \sum_{i,k} f_{ik}\varepsilon_i \otimes \tau_k$ and $\psi = \sum_{r,s} g_{rs}\varepsilon_r \otimes \tau_s$. This allows us to write the core-linear section $\widetilde{\varphi}$ as

$$\widetilde{\varphi} = \sum_{i,k} q_E^* f_{ik} \ell_{\varepsilon_i} \tau_k^\uparrow \,.$$

Using Theorem 4.1.3, Proposition A.1.1 and the action of vertical vector fields on pullbacks and linear functions as described in Example 2.1.14, we compute now

$$\begin{aligned} [\![\tau^\uparrow, \sum_{i,k} q_E^* f_{ik}\ell_{\varepsilon_i}\tau_k^\uparrow]\!] &= \sum_{i,k}\Big((q_E^* f_{ik}\ell_{\varepsilon_i})[\![\tau^\uparrow, \tau_k^\uparrow]\!] + \rho(\tau^\uparrow)(q_E^* f_{ik}\ell_{\varepsilon_i})\tau_k^\uparrow\Big) \\ &= \sum_{i,k} \mathrm{pr}_{TE}(\tau^\uparrow)(q_E^* f_{ik}\ell_{\varepsilon_i})\tau_k^\uparrow \\ &= \sum_{i,k} q_E^* f_{ik} q_E^* \langle \varepsilon_i, \mathrm{pr}_E(\tau)\rangle \tau_k^\uparrow \,. \end{aligned}$$

Hence

$$[\![\tau^{\uparrow}, \widetilde{\varphi}]\!] = (\varphi(\mathrm{pr}_E(\tau)))^{\uparrow},$$

and since $\langle \tau^{\uparrow}, \widetilde{\varphi}\rangle = 0$ we obtain additionally

$$[\![\widetilde{\varphi}, \tau^{\uparrow}]\!] = -[\![\tau^{\uparrow}, \widetilde{\varphi}]\!] = -(\varphi(\mathrm{pr}_E(\tau)))^{\uparrow}.$$

For the bracket of two core-linear sections $\widetilde{\varphi}$ and $\widetilde{\psi}$ as above we compute

$$\begin{aligned}
[\![\widetilde{\varphi}, \widetilde{\psi}]\!] &= \Big[\!\!\Big[\sum_{i,k} q_E^* f_{ik}\ell_{\varepsilon_i}\tau_k^{\uparrow}, \sum_{r,s} q_E^* g_{rs}\ell_{\varepsilon_r}\tau_s^{\uparrow}\Big]\!\!\Big] \\
&= \sum_{i,k,r,s}\Big(q_E^* f_{ik}\ell_{\varepsilon_i} q_E^* g_{rs}\ell_{\varepsilon_r}[\![\tau_k^{\uparrow}, \tau_s^{\uparrow}]\!] \\
&\quad + q_E^* f_{ik}\ell_{\varepsilon_i}(\mathrm{pr}_E(\tau_k))^{\uparrow}(q_E^* g_{rs}\ell_{\varepsilon_r})\tau_s^{\uparrow} \\
&\quad - q_E^* g_{rs}\ell_{\varepsilon_r}(\mathrm{pr}_E(\tau_s))^{\uparrow}(q_E^* f_{ik}\ell_{\varepsilon_i})\tau_k^{\uparrow}\Big) \\
&= \sum_{i,k,r,s}\Big(q_E^* f_{ik}\ell_{\varepsilon_i} q_E^* g_{rs} q_E^*\langle \mathrm{pr}_E\,\tau_k, \varepsilon_r\rangle\tau_s^{\uparrow} - q_E^* g_{rs}\ell_{\varepsilon_r} q_E^* f_{ik}\langle \mathrm{pr}_E(\tau_s), \varepsilon_i\rangle\tau_k^{\uparrow}\Big) \\
&= \widetilde{\psi\circ \mathrm{pr}_E \circ\varphi} - \widetilde{\varphi\circ \mathrm{pr}_E \circ\psi}.
\end{aligned}$$

Now we will compute the bracket of a lift $\sigma^{\Delta}(\nu)$ with the core-linear section $\widetilde{\varphi}$. We write $\widehat{\nabla_\nu}$ for the linear vector field on E corresponding to the derivation ∇_ν of $\Gamma(E)$ as in Example 2.1.14. We also write $X := \mathrm{pr}_{TM}\,\nu$ and use the properties from Proposition A.1.1:

$$\begin{aligned}
[\![\sigma^{\Delta}(\nu), \sum_{i,k} q_E^* f_{ik}\ell_{\varepsilon_i}\tau_k^{\uparrow}]\!] &= \sum_{i,k} \mathrm{pr}_{TE}(\sigma^{\Delta}(\nu))(q_E^* f_{ik}\ell_{\varepsilon_i})\tau_k^{\uparrow} + q_E^* f_{ik}\ell_{\varepsilon_i}[\![\sigma^{\Delta}(\nu), \tau_k^{\uparrow}]\!] \\
&= \sum_{i,k} \widehat{\nabla_\nu}(q_E^* f_{ik}\ell_{\varepsilon_i})\tau_k^{\uparrow} + q_E^* f_{ik}\ell_{\varepsilon_i}(\Delta_\nu\tau_k)^{\uparrow} \\
&= \sum_{i,k} q_E^*(X(f_{ik}))\ell_{\varepsilon_i}\tau_k^{\uparrow} + q_E^* f_{ik}\ell_{\nabla_\nu^*\varepsilon_i}\tau_k^{\uparrow} + q_E^* f_{ik}\ell_{\varepsilon_i}(\Delta_\nu\tau_k)^{\uparrow}
\end{aligned}$$

Using the definition of the dual derivation and the properties of a Dorfman connection (see Definition 2.3.15) we compute for $e \in \Gamma(E)$ on the other hand:

$$\begin{aligned}
(\Diamond_\nu\varphi)(e) &= \Delta_\nu(\varphi(e)) - \varphi(\nabla_\nu e) \\
&= \sum_{i,k} \Delta_\nu f_{ik}\langle \varepsilon_i, e\rangle\tau_k - f_{ik}\langle\varepsilon_i, \nabla_\nu e\rangle\tau_k \\
&= \sum_{i,k} f_{ik}\langle\varepsilon_i, e\rangle\Delta_\nu\tau_k + X\big(f_{ik}\langle\varepsilon_i, e\rangle\big)\tau_k - f_{ik}\Big(X(\langle\varepsilon_i, e\rangle) - \langle\nabla_\nu^*\varepsilon_i, e\rangle\Big)\tau_k \\
&= \sum_{i,k} f_{ik}\langle\varepsilon_i, e\rangle\Delta_\nu\tau_k + X(f_{ik})\langle\varepsilon_i, e\rangle\tau_k + f_{ik}\langle\nabla_\nu^*\varepsilon_i, e\rangle\tau_k.
\end{aligned}$$

Hence we have

$$\widetilde{\Diamond_\nu \varphi} = \sum_{i,k} q_E^* f_{ik} \ell_{\varepsilon_i} (\Delta_\nu \tau_k)^\uparrow + q_E^*(X(f_{ik}))\ell_{\varepsilon_i}\tau_k^\uparrow + q_E^* f_{ik}\ell_{\nabla^*_\nu \varepsilon_i}\tau_k^\uparrow$$

Comparing the terms we conclude that

$$[\![\sigma^\Delta(\nu), \widetilde{\varphi}]\!] = \widetilde{\Diamond_\nu \varphi}\,.$$

The definition of the lift as in (4.3)

$$\sigma^\Delta(X,\varepsilon)(e(m)) = (T_m e X(m), d\ell_\varepsilon(e(m))) - (\Delta_{(X,\varepsilon)}(e,0))^\uparrow$$

for $X = 0$ leads to

$$(0, d\ell_\varepsilon(e(m))) = \sigma^\Delta((0,\varepsilon)) + (\Delta_{(0,\varepsilon)}(e,0))^\uparrow\,.$$

Thus we obtain

$$\begin{aligned}
[\![\widetilde{\varphi}, \sigma^\Delta(\nu)]\!] &= -\widetilde{\Diamond_\nu \varphi} + \left(0, d\langle \widetilde{\varphi}, \sigma^\Delta(\nu)\rangle\right) \\
&= -\widetilde{\Diamond_\nu \Phi} + \left(0, d\ell_{\varphi^t \nu}\right) \\
&= -\widetilde{\Diamond_\nu \Phi} + \sigma^\Delta((0, \varphi^t \nu)) + \widetilde{\Delta_{(0,\varphi^t\nu)}(\cdot, 0)}\,.
\end{aligned}$$

□

Now we can finally compute the relevant Nijenhuis tensors. Let us now consider a linear generalised almost complex structure $\mathcal{J}$ on E as in (4.2) and fix again a Dorfman connection Δ as in (4.8). In particular j, j_C and Ψ satisfy the conditions of Proposition 4.1.10.

For two core sections the Nijenhuis tensor vanishes trivially, as the Dorfman bracket of two core sections vanishes and the double vector bundle morphism $\mathcal{J}$ sends core sections to core sections.

For the Nijenhuis tensor of $\mathcal{J}$ evaluated at a horizontal lift $\sigma^\Delta(\nu)$ for $\nu \in \Gamma(TM \oplus E^*)$ and a core section $\tau^\uparrow$ for $\tau \in \Gamma(E \oplus T^*M)$ we compute using Theorem

4.1.3 and Corollary A.1.3 the following

$$\begin{aligned}
N_{\mathcal{J}}(\sigma^{\Delta}(\nu),\tau^{\uparrow}) &= [\![\sigma^{\Delta}(\nu),\tau^{\uparrow}]\!] - [\![\mathcal{J}(\sigma^{\Delta}(\nu)),\mathcal{J}(\tau^{\uparrow})]\!] \\
&\quad + \mathcal{J}([\![\mathcal{J}(\sigma^{\Delta}(\nu)),\tau^{\uparrow}]\!] + [\![\sigma^{\Delta}(\nu),\mathcal{J}(\tau^{\uparrow})]\!]) \\
&= (\Delta_{\nu}\tau)^{\uparrow} - [\![\sigma^{\Delta}(j(\nu)) + \widetilde{\Phi(\nu)}, j_C(\tau)^{\uparrow}]\!] \\
&\quad + \mathcal{J}\Big([\![\sigma^{\Delta}(j(\nu)) + \widetilde{\Phi(\nu)},\tau^{\uparrow}]\!] + [\![\sigma^{\Delta}(\nu), j_C(\tau)^{\uparrow}]\!]\Big) \\
&= (\Delta_{\nu}\tau)^{\uparrow} - (\Delta_{j(\nu)}j_C(\tau))^{\uparrow} + \Big(\Phi(\nu)(\mathrm{pr}_E(j_C(\tau)))\Big)^{\uparrow} \\
&\quad + \mathcal{J}\Big((\Delta_{j(\nu)}\tau)^{\uparrow} - \Big(\Phi(\nu)(\mathrm{pr}_E(\tau))\Big)^{\uparrow} + (\Delta_{\nu}j_C(\tau))^{\uparrow}\Big) \\
&= (\Delta_{\nu}\tau)^{\uparrow} - (\Delta_{j(\nu)}j_C(\tau))^{\uparrow} + \Big(\Phi(\nu)(\mathrm{pr}_E(j_C(\tau)))\Big)^{\uparrow} \\
&\quad + (j_C(\Delta_{j(\nu)}\tau))^{\uparrow} - \Big(j_C(\Phi(\nu)(\mathrm{pr}_E(\tau)))\Big)^{\uparrow} + (j_C(\Delta_{\nu}j_C(\tau)))^{\uparrow}
\end{aligned}$$

Thus the Nijenhuis tensor of $\mathcal{J}$ vanishes for a lift and a core section if and only if for all $\nu \in \Gamma(TM \oplus E^*)$ and $\tau \in \Gamma(E \oplus T^*M)$ we have

$$\begin{aligned}
\Delta_{\nu}\tau - \Delta_{j(\nu)}j_C(\tau) + \Phi(\nu)\Big(\mathrm{pr}_E(j_C(\tau))\Big) + j_C(\Delta_{j(\nu)}\tau) \\
- j_C\Big(\Phi(\nu)(\mathrm{pr}_E(\tau))\Big) + j_C(\Delta_{\nu}j_C(\tau)) = 0
\end{aligned} \tag{A.1}$$

As the pairing is non-degenerate, we can dualise this condition by pairing with a second section $\nu_2 \in \Gamma(TM \oplus E^*)$. We now write $\nu_1 = (X_1, \varepsilon_1)$ instead of ν and use Proposition 4.1.10 and the duality of the Dorfman connection with the dull bracket on $TM \oplus E^*$ as described in [36] and in (4.5) to compute:

$$\begin{aligned}
&\Big\langle \Delta_{\nu_1}\tau - \Delta_{j(\nu_1)} j_C(\tau) + \Phi(\nu_1)\Big(\mathrm{pr}_E(j_C(\tau))\Big) + j_C(\Delta_{j(\nu_1)}\tau) \\
&\qquad - j_C\Big(\Phi(\nu_1)(\mathrm{pr}_E(\tau))\Big) + j_C(\Delta_{\nu_1} j_C(\tau)), \nu_2\Big\rangle \\
&\qquad = \langle \Delta_{\nu_1}\tau, \nu_2\rangle - \langle \Delta_{j(\nu_1)} j_C(\tau), \nu_2\rangle + \langle \Phi(\nu_1)\Big(\mathrm{pr}_E(j_C(\tau))\Big), \nu_2\rangle \\
&\qquad\quad - \langle \Delta_{j(\nu_1)}\tau, j(\nu_2)\rangle + \langle \Phi(\nu_1)(\mathrm{pr}_E(\tau)), j(\nu_2)\rangle - \langle \Delta_{\nu_1} j_C(\tau), j(\nu_2)\rangle \\
&\qquad = X_1\langle \tau, \nu_2\rangle - \langle \tau, [\![\nu_1, \nu_2]\!]_\Delta\rangle \\
&\qquad\quad - \mathrm{pr}_{TM}(j(\nu_1))\langle j_C(\tau), \nu_2\rangle + \langle j_C(\tau), [\![j(\nu_1), \nu_2]\!]_\Delta\rangle \\
&\qquad\quad - \mathrm{pr}_{TM}(j(\nu_1))\langle \tau, j(\nu_2)\rangle + \langle \tau, [\![j(\nu_1), j(\nu_2)]\!]_\Delta\rangle \\
&\qquad\quad - X_1\langle j_C(\tau), j(\nu_2)\rangle + \langle j_C(\tau), [\![\nu_1, j(\nu_2)]\!]_\Delta\rangle \\
&\qquad\quad + \langle \Phi(\nu_1)(\mathrm{pr}_E(j_C(\tau))), \nu_2\rangle + \langle \Phi(\nu_1)(\mathrm{pr}_E(\tau)), j(\nu_2)\rangle \\
&\qquad = \Big\langle \tau, -[\![\nu_1, \nu_2]\!]_\Delta - j([\![j(\nu_1), \nu_2]\!]_\Delta) - j([\![\nu_1, j(\nu_2)]\!]_\Delta) + [\![j(\nu_1), j(\nu_2)]\!]_\Delta \Big\rangle \\
&\qquad\quad + \Big\langle \tau, (0, \Phi(\nu_1)^t(j(\nu_2))) - j(0, \Phi(\nu_1)^t(\nu_2))) \Big\rangle \\
&\qquad = \Big\langle \tau, -N_{j,[\![\cdot,\cdot]\!]_\Delta}(\nu_1, \nu_2) + (0, \Phi(\nu_1)^t(j(\nu_2))) - j(0, \Phi(\nu_1)^t(\nu_2))) \Big\rangle \\
&\qquad = \Big\langle \tau, -N_{j,[\![\cdot,\cdot]\!]_\Delta}(\nu_1, \nu_2) + \Psi^j(\nu_1, \nu_2) - j\Psi(\nu_1, \nu_2) \Big\rangle ,
\end{aligned}$$

where we have extended the forms Ψ, Ψ^j by abuse of notation to forms $\Psi, \Psi^j \in \Omega^2(TM \oplus E^*, TM \oplus E^*)$:

$$\begin{aligned}
\Psi(\nu_1, \nu_2) &:= (0, \Psi(\nu_1, \nu_2)) = \mathrm{pr}_E^t \Psi(\nu_1, \nu_2) \\
\Psi^j(\nu_1, \nu_2) &:= (0, \Psi^j(\nu_1, \nu_2)) = \mathrm{pr}_E^t \Psi^t(\nu_1, \nu_2) .
\end{aligned}$$

Since ν_1, ν_2 and τ were arbitrary and the pairing is non-degenerate, the Nijenhuis tensor of $\mathcal{J}$ of a lift and a linear section vanishes if and only if

$$- N_{j,[\![\cdot,\cdot]\!]_\Delta} + \Psi^j - j\Psi = 0 . \tag{A.2}$$

Let us from now on assume that $\mathcal{J}$ satisfies this equation. We will now compute conditions when additionally the Nijenhuis tensor of $\mathcal{J}$ for two horizontal lifts vanishes. Again we make use of Lemma 4.1.6, Theorem 4.1.3 and of Corollary A.1.3 and compute for $\nu_1, \nu_2 \in \Gamma(TM \oplus E^*)$ the following:

$$\begin{aligned}
N_{\mathcal{J}}(\sigma^\Delta(\nu_1), \sigma^\Delta(\nu_2)) &= [\![\sigma^\Delta(\nu_1), \sigma^\Delta(\nu_2)]\!] - [\![\mathcal{J}(\sigma^\Delta(\nu_1)), \mathcal{J}(\sigma^\Delta(\nu_2))]\!] \\
&\quad + \mathcal{J}\Big([\![\mathcal{J}(\sigma^\Delta(\nu_1)), \sigma^\Delta(\nu_2)]\!] + [\![\sigma^\Delta(\nu_1), \mathcal{J}(\sigma^\Delta(\nu_2))]\!]\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \sigma^{\Delta}(\llbracket \nu_1, \nu_2 \rrbracket_{\Delta}) - \widetilde{R_{\Delta}(\nu_1, \nu_2)(\cdot, 0)} \\
&\quad - \sigma^{\Delta}(\llbracket j(\nu_1), j(\nu_2) \rrbracket_{\Delta}) + \widetilde{R_{\Delta}(j(\nu_1), j(\nu_2))(\cdot, 0)} \\
&\quad - \llbracket \sigma^{\Delta}(j(\nu_1)), \widetilde{\Phi(\nu_2)} \rrbracket - \llbracket \widetilde{\Phi(\nu_1)}, \sigma^{\Delta}(j(\nu_2)) \rrbracket - \llbracket \widetilde{\Phi(\nu_1)}, \widetilde{\Phi(\nu_2)} \rrbracket \\
&\quad + \mathcal{J}\Big(\sigma^{\Delta}(\llbracket j(\nu_1), \nu_2 \rrbracket_{\Delta}) + \sigma^{\Delta}(\llbracket \nu_1, j(\nu_2) \rrbracket_{\Delta}) \\
&\quad - \widetilde{R_{\Delta}(j(\nu_1), \nu_2)(\cdot, 0)} - \widetilde{R_{\Delta}(\nu_1, j(\nu_2))(\cdot, 0)} \\
&\quad + \llbracket \sigma^{\Delta}(\nu_1), \widetilde{\Phi(\nu_2)} \rrbracket + \llbracket \widetilde{\Phi(\nu_1)}, \sigma^{\Delta}(\nu_2) \rrbracket \Big) \\
&= \sigma^{\Delta}(\llbracket \nu_1, \nu_2 \rrbracket_{\Delta}) - \widetilde{R_{\Delta}(\nu_1, \nu_2)(\cdot, 0)} \\
&\quad - \sigma^{\Delta}(\llbracket j(\nu_1), j(\nu_2) \rrbracket_{\Delta}) + \widetilde{R_{\Delta}(j(\nu_1), j(\nu_2))(\cdot, 0)} \\
&\quad - \widetilde{\Diamond_{j(\nu_1)} \Phi(\nu_2)} + \widetilde{\Diamond_{j(\nu_2)} \Phi(\nu_1)} - \sigma^{\Delta}(0, \Phi(\nu_1)^t j(\nu_2)) \\
&\quad - \widetilde{\Delta_{(0, \Phi(\nu_1)^t j(\nu_2))}(\cdot, 0))} \\
&\quad - \widetilde{\Phi(\nu_2) \circ \mathrm{pr}_E \circ \Phi(\nu_1)} - \widetilde{\Phi(\nu_1) \circ \mathrm{pr}_E \circ \Phi(\nu_2)} \\
&\quad + \sigma^{\Delta}(j(\llbracket j(\nu_1), \nu_2 \rrbracket_{\Delta})) + \sigma^{\Delta}(j(\llbracket \nu_1, j(\nu_2) \rrbracket_{\Delta})) \\
&\quad + \widetilde{\Phi(\llbracket j(\nu_1), \nu_2 \rrbracket_{\Delta})} + \widetilde{\Phi(\llbracket \nu_1, j(\nu_2) \rrbracket_{\Delta})} \\
&\quad - \widetilde{j_C \circ R_{\Delta}(j(\nu_1), \nu_2)(\cdot, 0)} - \widetilde{j_C \circ R_{\Delta}(\nu_1, j(\nu_2))(\cdot, 0)} \\
&\quad + \widetilde{j_C \circ \Diamond_{\nu_1}(\Phi(\nu_2))} - \widetilde{j_C \circ \Diamond_{\nu_2}(\Phi(\nu_1))} \\
&\quad + \sigma^{\Delta}(j(0, \Phi(\nu_1)^t \nu_2)) + \widetilde{\Phi(0, \Phi(\nu_1)^t \nu_2)} \\
&\quad + \widetilde{j_C(\Delta_{(0, \Phi(\nu_1)^t \nu_2)}(\cdot, 0))} \\
&= \sigma^{\Delta}(N_{j, \llbracket \cdot, \cdot \rrbracket_{\Delta}}(\nu_1, \nu_2)) \\
&\quad + \widetilde{\Diamond_{j(\nu_2)} \Phi(\nu_1)} - \widetilde{\Diamond_{j(\nu_1)} \Phi(\nu_2)} \\
&\quad + \widetilde{j_C \circ \Diamond_{\nu_1}(\Phi(\nu_2)) - j_C \circ \Diamond_{\nu_2}(\Phi(\nu_1))} \\
&\quad + \widetilde{R_{\Delta}(j(\nu_1), j(\nu_2))(\cdot, 0)} - \widetilde{R_{\Delta}(\nu_1, \nu_2)(\cdot, 0)} \\
&\quad - \widetilde{j_C \circ R_{\Delta}(j(\nu_1), \nu_2)(\cdot, 0)} - \widetilde{j_C \circ R_{\Delta}(\nu_1, j(\nu_2))(\cdot, 0)} \\
&\quad - \widetilde{\Phi(\nu_2) \circ \mathrm{pr}_E \circ \Phi(\nu_1)} - \widetilde{\Phi(\nu_1) \circ \mathrm{pr}_E \circ \Phi(\nu_2)} \\
&\quad + \widetilde{\Phi(\llbracket j(\nu_1), \nu_2 \rrbracket_{\Delta})} + \widetilde{\Phi(\llbracket \nu_1, j(\nu_2) \rrbracket_{\Delta})} \\
&\quad - \sigma^{\Delta}(0, \Phi(\nu_1)^t(j(\nu_2))) + \sigma^{\Delta}(j(0, \Phi(\nu_1)^t(\nu_2))) \\
&\quad + \widetilde{\Phi((0, \Phi(\nu_1)^t(\nu_2)))} \\
&\quad + \widetilde{(j_C(\Delta_{(0, \Phi(\nu_1)^t \nu_2)}(\cdot, 0)))} - \widetilde{(\Delta_{(0, \Phi(\nu_1)^t j(\nu_2))}(\cdot, 0))}\,.
\end{aligned}
$$

The three terms involving horizontal lifts combine to the term

$$\sigma^{\Delta}\Big(N_{j, \llbracket \cdot, \cdot \rrbracket_{\Delta}}(\nu_1, \nu_2) - \Psi^j(\nu_1, \nu_2) + j\Psi(\nu_1, \nu_2)\Big)\,.$$

This is precisely the horizontal lift of the left hand side of Equation (A.2). Since we assumed that $\mathcal{J}$ satisfies (A.2) this therefore already vanishes.

Thus the Nijenhuis tensor of $\mathcal{J}$ vanishes for all sections if and only if (A.2) holds and additionally for all ν_1, $\nu_2 \in \Gamma(TM \oplus E^*)$ and $\tau \in \Gamma(E \oplus T^*M)$

$$\begin{aligned}
0 = {} & \Diamond_{j(\nu_2)}\Phi(\nu_1)(\tau) - \Diamond_{j(\nu_1)}\Phi(\nu_2)(\tau) \\
& + j_C\Big(\Diamond_{\nu_1}(\Phi(\nu_2))(\tau)\Big) - j_C\Big(\Diamond_{\nu_2}(\Phi(\nu_1))(\tau)\Big) \\
& + R_\Delta(j(\nu_1), j(\nu_2))(\tau) - R_\Delta(\nu_1,\nu_2)(\tau) \\
& - j_C\Big(R_\Delta(j(\nu_1),\nu_2)(\tau)\Big) - j_C\Big(R_\Delta(\nu_1, j(\nu_2))(\tau)\Big) \\
& - \Phi(\nu_2)(\Phi(\nu_1)(\tau)) + \Phi(\nu_1)(\Phi(\nu_2)(\tau)) + \Phi\Big((0, \Phi(\nu_1)^t(\nu_2))\Big)(\tau) \\
& + \Phi([\![j(\nu_1),\nu_2]\!]_\Delta)(\tau) + \Phi([\![\nu_1, j(\nu_2)]\!]_\Delta)(\tau) \\
& + j_C(\Delta_{(0,\Phi(\nu_1)^t\nu_2)}\tau) - \Delta_{(0,\Phi(\nu_1)^t j(\nu_2))}\tau
\end{aligned} \tag{A.3}$$

As we did for (A.1), we want to dualise this equation in order to get a condition purely depending on sections of $TM \oplus E^*$ and not also on a section of $E \oplus T^*M$. For this we will use again the non-degeneracy of the pairing and pair the left hand side of (A.3) with a third section ν_3 of $TM \oplus E$.

Firstly, we compute:

$$\begin{aligned}
\Big\langle \Diamond_{\nu_1}\Phi(\nu_2)(\tau), \nu_3\Big\rangle &= -\langle \Phi(\nu_2)(\nabla_{\nu_1}\operatorname{pr}_E(\tau)), \nu_3\rangle + \langle \Delta_{\nu_1}\Phi(\nu_2)(\tau), \nu_3\rangle \\
&= -\langle \Delta_{\nu_1}\tau, \Phi(\nu_2)^t\nu_3\rangle + \langle \Delta_{\nu_1}\Phi(\nu_2)(\tau), \nu_3\rangle \\
&= -\operatorname{pr}_{TM}(\nu_1)(\langle \tau, \Phi(\nu_2)^t\nu_3\rangle) + \langle [\![\nu_1, \Phi(\nu_2)^t\nu_3]\!]_\Delta, \tau\rangle \\
&\quad + \operatorname{pr}_{TM}(\nu_1)(\langle \Phi(\nu_2)(\tau), \nu_3\rangle) - \langle [\![\nu_1,\nu_3]\!]_\Delta, \Phi(\nu_2)(\tau)\rangle \\
&= \Big\langle [\![\nu_1, \Phi(\nu_2)^t\nu_3]\!]_\Delta - \Phi(\nu_2)^t([\![\nu_1,\nu_3]\!]_\Delta), \tau\Big\rangle
\end{aligned} \tag{A.4}$$

Using this equality we can now pair the left hand side of (A.1) with a section ν_3 of $TM \oplus E^*$ and compute the following:

$$\begin{aligned}
0 = \Big\langle & \Diamond_{j(\nu_2)}\Phi(\nu_1)(\tau) - \Diamond_{j(\nu_1)}\Phi(\nu_2)(\tau) \\
& + j_C(\Diamond_{\nu_1}(\Phi(\nu_2))(\tau)) - j_C(\Diamond_{\nu_2}(\Phi(\nu_1))(\tau)) \\
& + R_\Delta(j(\nu_1), j(\nu_2))(\tau) - R_\Delta(\nu_1,\nu_2)(\tau) \\
& - j_C(R_\Delta(j(\nu_1),\nu_2)(\tau)) - j_C(R_\Delta(\nu_1, j(\nu_2))(\tau)) \\
& - \Phi(\nu_2)(\Phi(\nu_1)(\tau)) + \Phi(\nu_1)(\Phi(\nu_2)(\tau)) + \Phi((0, \Phi(\nu_1)^t(\nu_2)))(\tau) \\
& + \Phi([\![j(\nu_1),\nu_2]\!]_\Delta)(\tau) + \Phi([\![\nu_1, j(\nu_2)]\!]_\Delta)(\tau) \\
& + j_C(\Delta_{(0,\Phi(\nu_1)^t\nu_2)}\tau) - \Delta_{(0,\Phi(\nu_1)^t j(\nu_2))}\tau, \nu_3\Big\rangle
\end{aligned}$$

$$
\begin{aligned}
= \Big\langle \tau, & \left[\!\left[j\nu_2, \Phi(\nu_1)^t \nu_3 \right]\!\right]_\Delta - \Phi(\nu_1)^t (\left[\!\left[j\nu_2, \nu_3 \right]\!\right]_\Delta) \\
& - \left[\!\left[j\nu_1, \Phi(\nu_2)^t \nu_3 \right]\!\right]_\Delta + \Phi(\nu_2)^t (\left[\!\left[j\nu_1, \nu_3 \right]\!\right]_\Delta) \\
& - \left[\!\left[\nu_1, \Phi(\nu_2)^t (j\nu_3) \right]\!\right]_\Delta + \Phi(\nu_2)^t (\left[\!\left[\nu_1, j\nu_3 \right]\!\right]_\Delta) \\
& + \left[\!\left[\nu_2, \Phi(\nu_1)^t (j\nu_3) \right]\!\right]_\Delta - \Phi(\nu_1)^t (\left[\!\left[\nu_2, j\nu_3 \right]\!\right]_\Delta) \\
& - \Phi(\nu_1)^t \Phi(\nu_2)^t (\nu_3) + \Phi(\nu_2)^t \Phi(\nu_1)^t (\nu_3) + \Phi\big(\Phi(\nu_1)^t \nu_2\big)^t (\nu_3) \\
& + \Phi(\left[\!\left[j\nu_1, \nu_2 \right]\!\right]_\Delta)^t \nu_3 + \Phi(\left[\!\left[\nu_1, j\nu_2 \right]\!\right]_\Delta)^t \nu_3 \\
& - \left[\!\left[j\nu_3, \Phi(\nu_1)^t \nu_2 \right]\!\right]_\Delta - \left[\!\left[\nu_3, \Phi(\nu_1)^t (j\nu_2) \right]\!\right]_\Delta \\
& + R_\Delta(j(\nu_1), j(\nu_2))^t (\nu_3) - R_\Delta(\nu_1, \nu_2)^t (\nu_3) \\
& + R_\Delta(j(\nu_1), \nu_2)^t (j\nu_3) + R_\Delta(\nu_1, j(\nu_2))^t (j\nu_3) \Big\rangle .
\end{aligned}
$$

Note that the properties of the dull bracket $\left[\!\left[\cdot, \cdot \right]\!\right]_\Delta$ imply that this equality is independent of $\mathrm{pr}_{T^*M}(\tau)$ and only depends on $\mathrm{pr}_E(\tau)$.

Since τ, ν_1, ν_2 and ν_3 were arbitrary sections, the vanishing of the Nijenhuis tensor is now equivalent to the term on the right hand side to vanish for all ν_1, ν_2 and ν_3. Using the notion of Ψ^j we can rewrite this condition in the following equation:

$$
\begin{aligned}
0 = & - \left[\!\left[\nu_3, \Psi^j(\nu_1, \nu_2) \right]\!\right]_\Delta - \left[\!\left[\nu_1, \Psi^j(\nu_2, \nu_3) \right]\!\right]_\Delta - \left[\!\left[\nu_2, \Psi^j(\nu_3, \nu_1) \right]\!\right]_\Delta \\
& + \left[\!\left[j\nu_3, \Psi^j(\nu_1, j\nu_2) \right]\!\right]_\Delta + \left[\!\left[j\nu_2, \Psi^j(\nu_3, j\nu_1) \right]\!\right]_\Delta + \left[\!\left[j\nu_1, \Psi^j(\nu_2, j\nu_3) \right]\!\right]_\Delta \\
& - \Psi^j(\left[\!\left[j\nu_2, \nu_3 \right]\!\right]_\Delta, j\nu_1) - \Psi^j(\left[\!\left[\nu_2, j\nu_3 \right]\!\right]_\Delta, j\nu_1) \\
& - \Psi^j(\left[\!\left[j\nu_3, \nu_1 \right]\!\right]_\Delta, j\nu_2) - \Psi^j(\left[\!\left[\nu_3, j\nu_1 \right]\!\right]_\Delta, j\nu_2) \\
& - \Psi^j(\left[\!\left[j\nu_1, \nu_2 \right]\!\right]_\Delta, j\nu_3) - \Psi^j(\left[\!\left[\nu_1, j\nu_2 \right]\!\right]_\Delta, j\nu_3) \\
& - \Psi^j(j\nu_1, \Psi^j(\nu_2, j\nu_3)) - \Psi^j(j\nu_2, \Psi^j(\nu_3, j\nu_1)) - \Psi^j(j\nu_3, \Psi^j(\nu_1, j\nu_2)) \\
& + \omega_\Delta(j\nu_1, j\nu_2, \nu_3) + \omega_\Delta(j\nu_1, \nu_2, j\nu_3) + \omega_\Delta(\nu_1, j\nu_2, j\nu_3) \\
& - \omega_\Delta(\nu_1, \nu_2, \nu_3) ,
\end{aligned}
\tag{A.5}
$$

using the notion $\omega_\Delta \in \Omega^3(TM \oplus E^*, TM \oplus E^*)$ as in (4.25).

Without writing all cyclic permutations of ν_1, ν_2 and ν_3 explicitly, (A.5) simplifies to the following:

$$
\begin{aligned}
0 = &-[\![\nu_1, \Psi^j(\nu_2,\nu_3)]\!]_\Delta + [\![j\nu_1, \Psi^j(\nu_2, j\nu_3)]\!]_\Delta \\
&- \Psi^j([\![j\nu_2,\nu_3]\!]_\Delta, j\nu_1) - \Psi^j([\![\nu_2, j\nu_3]\!]_\Delta, j\nu_1) - \Psi^j(j\nu_1, \Psi^j(\nu_2, j\nu_3)) \\
&+ [\![\nu_1, [\![\nu_2,\nu_3]\!]_\Delta]\!]_\Delta - [\![j\nu_1, [\![j\nu_2,\nu_3]\!]_\Delta]\!]_\Delta \\
&- [\![j\nu_1, [\![\nu_2, j\nu_3]\!]_\Delta]\!]_\Delta - [\![\nu_1, [\![j\nu_2, j\nu_3]\!]_\Delta]\!]_\Delta \\
&+ \text{cyclic permutations in 1,2,3} \\
= &\, [\![\nu_1, [\![\nu_2,\nu_3]\!]_\Delta - [\![j\nu_2, j\nu_3]\!]_\Delta - \Psi^j(\nu_2,\nu_3)]\!]_\Delta \\
&+ [\![j\nu_1, -[\![\nu_2, j\nu_3]\!]_\Delta - [\![j\nu_2,\nu_3]\!]_\Delta - \Psi(\nu_2,\nu_3)]\!]_\Delta \\
&- \Psi\Big(\nu_1, [\![\nu_2, j\nu_3]\!]_\Delta + [\![j\nu_2,\nu_3]\!]_\Delta + \Psi(\nu_2,\nu_3)\Big) \\
&+ \text{cyclic permutations in 1,2,3}\,.
\end{aligned}
$$

Let us now define a pseudo-bracket $\mathbb{A}$ on $\Gamma(TM \oplus E^*)$ by

$$
\begin{aligned}
\mathbb{A}(\nu_1,\nu_2) :=& \frac{1}{2}\Big([\![\nu_1,\nu_2]\!]_\Delta - [\![j\nu_1, j\nu_2]\!]_\Delta - \Psi^j(\nu_1,\nu_2)\Big) \\
=& \frac{1}{2}\Big(-j[\![j\nu_1,\nu_2]\!]_\Delta - j[\![\nu_1, j\nu_2]\!]_\Delta - j\Psi(\nu_1,\nu_2)\Big)\,.
\end{aligned}
\tag{A.6}
$$

The equality of those two expressions is immediately seen to be equivalent to the condition (A.2) obtained before. Thus this condition can now be expressed alternatively in terms of the pseudo-bracket $\mathbb{A}$ as

$$
\mathbb{A}(\nu_1, j\nu_2) = j\mathbb{A}(\nu_1,\nu_2)\,. \tag{A.7}
$$

This property is stronger than the equality $\mathbb{A}(\nu_1, j\nu_2) = \mathbb{A}(j\nu_1,\nu_2)$, which follows already from the properties of a generalised almost complex structure, since then $j^*\Psi = -\Psi$ according to Proposition 4.1.10.

Note that the pseudo-bracket $\mathbb{A}$ does not admit an anchor on $TM \oplus E^*$, and is thus not a Lie algebroid bracket on $TM \oplus E^*$. However, $\mathbb{A}$ is skew-symmetric and $\mathbb{R}$-bilinear. Furthermore, it is independent of the chosen linear splitting or equivalently the chosen Dorfman connection Δ as we prove now.

Lemma A.1.4. *Let $\mathcal{J}$ be a linear generalised almost complex structure on E. Then the pseudo-bracket $\mathbb{A}$ defined by* (A.6) *does not depend on the choice of the skew-symmetric Dorfman connection Δ.*

Proof. Let us consider two skew-symmetric Dorfman connections Δ^1 and Δ^2 with change of splittings Ψ_{12}. With (4.20) we compute for the pseudo-brackets $\mathbb{A}_1$ and

$\mathbb{A}_2$ defined by (A.6) with the respective Dorfman connections:

$$\begin{aligned}
-2\mathbb{A}_2(\nu_1,\nu_2) &= j[\![j\nu_1,\nu_2]\!]_{\Delta^2} + j[\![\nu_1,j\nu_2]\!]_{\Delta^2} + j\Psi_2(\nu_1,\nu_2)\\
&= j[\![j\nu_1,\nu_2]\!]_{\Delta^1} + j\Psi_{12}(j\nu_1,\nu_2)\\
&\quad + j[\![\nu_1,j\nu_2]\!]_{\Delta^1} + j\Psi_{12}(\nu_1,j\nu_2)\\
&\quad + j\Psi_2(\nu_1,\nu_2)\\
&= j[\![j\nu_1,\nu_2]\!]_1 + j\Psi_{12}(j\nu_1,\nu_2)\\
&\quad + j[\![\nu_1,j\nu_2]\!]_1 + j\Psi_{12}(\nu_1,j\nu_2)\\
&\quad + j\Psi_1(\nu_1,\nu_2) - j\Psi_{12}(j\nu_1,\nu_2) - j\Psi_{12}(\nu_1,j\nu_2)\\
&= -2\mathbb{A}_1(\nu_1,\nu_2)
\end{aligned}$$

Thus the pseudo-bracket $\mathbb{A}$ defined by (A.6) is indeed independent of the choice of splitting. □

With this pseudo-bracket $\mathbb{A}$ we can write the condition (A.5) now equivalently in the following way

$$\begin{aligned}
0 &= [\![\nu_1,[\![\nu_2,\nu_3]\!]_\Delta - [\![j\nu_2,j\nu_3]\!]_\Delta - \Psi^j(\nu_2,\nu_3)]\!]_\Delta\\
&\quad + [\![j\nu_1,-[\![\nu_2,j\nu_3]\!]_\Delta - [\![j\nu_2,\nu_3]\!]_\Delta - \Psi(\nu_2,\nu_3)]\!]_\Delta\\
&\quad - \Psi\Big(\nu_1,[\![\nu_2,j\nu_3]\!]_\Delta + [\![j\nu_2,\nu_3]\!]_\Delta + \Psi(\nu_2,\nu_3)\Big)\\
&\quad + \text{cyclic permutations}\\
&= [\![\nu_1,2\mathbb{A}(\nu_2,\nu_3)]\!]_\Delta\\
&\quad - [\![j\nu_1,j2\mathbb{A}(\nu_2,\nu_3)]\!]_\Delta\\
&\quad - \Psi\Big(\nu_1,j2\mathbb{A}(\nu_2,\nu_3))\Big)\\
&\quad + \text{cyclic permutations}\\
&= 2\mathbb{A}(\nu_1,2\mathbb{A}(\nu_2,\nu_3))\\
&\quad + \text{cyclic permutations}\\
&= \mathbf{Jac}_{2\mathbb{A}}(\nu_1,\nu_2,\nu_3)
\end{aligned}$$

Thus the Nijenhuis torsion for $\mathcal{J}$ vanishes, if and only if the pseudo-bracket $\mathbb{A}$ satisfies the Jacobi identity and is compatible with j in the sense of (A.7), that is

$$\mathbb{A}(\nu_1,j\nu_2) = j\mathbb{A}(\nu_1,\nu_2)\,.$$

We have proved the following proposition.

Proposition A.1.5. *Let $\mathcal{J}$ be a generalised almost complex structure on a vector bundle E. Then the Nijenhuis tensor of $\mathcal{J}$ vanishes if and only if for $\mathbb{A}$ defined by* (A.6) *we have for all ν_1, ν_2 in* $\Gamma(TM\oplus E^*)$

1. $\mathbb{A}(\nu_1, j\nu_2) = j\mathbb{A}(\nu_1, \nu_2)$,

2. $\mathbb{A}$ *satisfies the Jacobi identity.*

The following theorem is an immediate consequence of Proposition 4.1.10 and Proposition A.1.5.

Theorem A.1.6. *Let $E \to M$ be a vector bundle. Fix a skew-symmetric $TM \oplus E^*$-Dorfman connection Δ on $E \oplus T^*M$. A linear generalised complex structure on E is equivalent to the following data:*

1. *A vector bundle morphism* $j\colon TM \oplus E^* \to TM \oplus E^*$,

2. $\Psi \in \Omega^2(TM \oplus E^*, E^*)$,

such that

1. $j^2 = -\operatorname{id}_{TM\oplus E^*}$,

2. $j^*\Psi = -\Psi$,

3. $\mathbb{A}$ *defined by* (A.6) *satisfies the Jacobi identity,*

4. $\mathbb{A}(\nu_1, j\nu_2) = j\mathbb{A}(\nu_1, \nu_2)$.

Proof. Given a linear generalised complex structure $\mathcal{J}$ on E as in (4.2) we have by Lemma 4.1.5, Proposition 4.1.10 and Proposition A.1.5 a vector bundle morphism j and a form $\Psi \in \Omega^2(TM \oplus E^*, E^*)$ satisfying the properties above.

Conversely, given j and Ψ as above, we define a double vector bundle morphism $\mathcal{J}\colon TE \oplus T^*E \to TE \oplus T^*E$ by setting for $\tau \in \Gamma(E \oplus T^*M)$ and $\nu \in \Gamma(TM \oplus E^*)$

$$\mathcal{J}(\tau^\uparrow) := (-j^t(\tau))^\uparrow ,$$
$$\mathcal{J}(\sigma^\Delta(\nu)) := \sigma^\Delta(j(\nu)) + \widetilde{(\Psi(\nu, \cdot))^t} .$$

Again by Proposition 4.1.10 and Proposition A.1.5 this defines a linear generalised complex structure on E. These two constructions are inverse to each other. □

Note that whilst $\mathbb{A}$ does not depend on the choice of Δ, Ψ does depend on this choice. Given a different skew-symmetric Dorfman connection Δ_2, we get the form corresponding to the same generalised complex structure by the change of splittings as in Lemma 4.1.14.

A.2 Lie algebroid morphisms

In this section we will give the computations corresponding to the results in section 4.3 in the case where we work with a general skew-symmetric Dorfman connection instead of one which is adapted to the generalised complex structure.

Let us assume $A \to M$ is a Lie algebroid and $\mathcal{J}$ is a linear generalised complex structure. Let Δ be any skew-symmetric $TM \oplus A^*$-Dorfman connection on $A \oplus T^*M$. To compute the conditions on the Lie bracket and the anchor in terms of this splitting, we will need to describe Lie brackets of core-linear sections with other sections.

Given the basic A connections ∇^{bas} on $A \oplus T^*M$ and on $TM \oplus A^*$ described in Definition 4.3.2 and Proposition 4.3.3, we will now write ∇^{Hom} for the induced A-connection on $\Gamma(\text{Hom}(TM \oplus A^*, A \oplus T^*M))$ defined by

$$\nabla_a^{\text{Hom}} \varphi = \nabla_a^{\text{bas}} \circ \varphi - \varphi \circ \nabla_a^{\text{bas}} . \tag{A.8}$$

Now we will prove the following corollary to Theorem 4.3.6.

Corollary A.2.1. *Let A be a Lie algebroid with anchor ρ and Δ a Dorfman connection as in* (4.39). *Let φ, φ_1, $\varphi_2 \in \Gamma(\text{Hom}(TM \oplus A^*, A \oplus T^*M))$, $\tau \in \Gamma(A \oplus T^*M)$ and $a \in \Gamma(A)$. Then we obtain the following Lie brackets in $\mathbb{T}A$*

1. $\left[\tau^\dagger, \widetilde{\varphi}\right] = \varphi((\rho, \rho^t)\tau)^\dagger$,
2. $\left[\widetilde{\varphi_1}, \widetilde{\varphi_2}\right] = \widetilde{\varphi_2 \circ (\rho, \rho^t) \circ \varphi_1} - \widetilde{\varphi_1 \circ (\rho, \rho^t) \circ \varphi_2}$,
3. $\left[\sigma_A^\Delta(a), \widetilde{\varphi}\right] = \widetilde{\nabla_a^{\text{Hom}} \varphi}$,
4. $\Theta(\widetilde{\varphi}) = \widetilde{(\rho, \rho^t) \circ \varphi}$.

Proof. Using local coordinates we can write $\varphi = \sum f_{ij} \kappa_i \otimes \tau_j$ where $f_{ij} \in C^\infty(M)$, κ_i, $\tau_j \in \Gamma(A \oplus T^*M)$. Then we obtain for the corresponding core-linear section

$$\widetilde{\varphi} = \sum q^*_{TM \oplus A^*} f_{ij} \ell_{\kappa_i} \tau_j^\dagger , \tag{A.9}$$

where $\tau_j^\dagger \in \Gamma^c_{TM \oplus A^*}(\mathbb{T}A)$ is the core section corresponding to τ_j. Now we compute using Theorem 4.3.6 the following Lie bracket of a core section with a core-linear section

$$\begin{aligned}
\left[\tau^\dagger, \widetilde{\varphi}\right] &= \left[\tau^\dagger, \sum q^*_{TM\oplus A^*} f_{ij} \ell_{\kappa_i} \tau_j^\dagger\right] \\
&= \sum q^*_{TM\oplus A^*} f_{ij} \ell_{\kappa_i} [\tau^\dagger, \tau_j^\dagger] + \sum ((\rho,\rho^t)\tau)^\uparrow (q^*_{TM\oplus A^*} f_{ij} \ell_{\kappa_i}) \tau_j^\dagger \\
&= \sum ((\rho,\rho^t)\tau)^\uparrow (q^*_{TM\oplus A^*} f_{ij} \ell_{\kappa_i}) \tau_j^\dagger \\
&= \sum q^*_{TM\oplus A^*} f_{ij} q^*_{TM\oplus A^*} \langle \kappa_i, (\rho,\rho^t)\tau \rangle \tau_j^\dagger \\
&= \varphi((\rho,\rho^t)\tau)^\dagger \,.
\end{aligned}$$

For the bracket of two core-linear sections we write both $\widetilde{\varphi_1}$ and $\widetilde{\varphi_2}$ in local coordinates analogously to (A.9) and compute:

$$\begin{aligned}
\left[\widetilde{\varphi_1}, \widetilde{\varphi_2}\right] &= \left[\sum q^*_{TM\oplus A^*} f^1_{ij} \ell_{\kappa^1_i} (\tau^1_j)^\dagger, \sum q^*_{TM\oplus A^*} f^2_{kl} \ell_{\kappa^2_k} (\tau^2_l)^\dagger\right] \\
&= \sum q^*_{TM\oplus A^*} f^1_{ij} \ell_{\kappa^1_i} q^*_{TM\oplus A^*} f^2_{kl} \ell_{\kappa^2_k} [(\tau^1_j)^\dagger, (\tau^2_l)^\dagger] \\
&\quad + \sum q^*_{TM\oplus A^*} f^1_{ij} \ell_{\kappa^1_i} \varphi_2((\rho,\rho^t)\tau^1_j)^\dagger \\
&\quad - \sum q^*_{TM\oplus A^*} f^2_{kl} \ell_{\kappa^2_k} \varphi_1((\rho,\rho^t)\tau^2_l)^\dagger \\
&= \widetilde{\varphi_2 \circ (\rho,\rho^j) \circ \varphi_1} - \widetilde{\varphi_1 \circ (\rho,\rho^t) \circ \varphi_2} \,,
\end{aligned}$$

Lastly, we compute the bracket of a lift $\sigma^\Delta_A(a)$ with a core-linear section $\widetilde{\varphi}$ in local coordinates:

$$\begin{aligned}
\left[\sigma^\Delta_A(a), \widetilde{\varphi}\right] &= \left[\sigma^\Delta_A(a), \sum q^*_{TM\oplus A^*} f_{ij} \ell_{\kappa_i} \tau_j^\dagger\right] \\
&= \sum q^*_{TM\oplus A^*} f_{ij} \ell_{\kappa_i} [\sigma^\Delta_A(a), \tau_j^\dagger] \\
&\quad + \sum \widehat{\nabla^{\mathrm{bas}}_a} (q^*_{TM\oplus A^*} f_{ij} \ell_{\kappa_i}) \tau_j^\dagger \\
&= \sum q^*_{TM\oplus A^*} f_{ij} \ell_{\kappa_i} (\nabla^{\mathrm{bas}}_a \tau_j)^\dagger \\
&\quad + \sum q^*_{TM\oplus A^*} (\rho(a) f_{ij}) \ell_{\kappa_i} \tau_j^\dagger + q^*_{TM\oplus A^*} f_{ij} \ell_{\nabla^{\mathrm{bas}}_a \kappa_i} \tau_j^\dagger \\
&= \widetilde{\nabla^{\mathrm{bas}}_a \circ \varphi - \varphi \circ \nabla^{\mathrm{bas}}_a} \\
&= \widetilde{\nabla^{\mathrm{Hom}}_a \varphi} \,.
\end{aligned}$$

In the second to last step we have used that the two basic connections are dual to each other as shown in Proposition 4.3.3.

To compute the anchor of $\widetilde{\varphi}$ we just note that Θ is a vector bundle morphism from $\mathbb{T}A$ to $T(TM \oplus A^*)$ over the identity on $TM \oplus A^*$ and thus

$$\begin{aligned}
\Theta\Big(\sum q^*_{TM\oplus A^*} f_{ij} \ell_{\kappa_i} \tau_j^\dagger\Big) &= \sum q^*_{TM\oplus A^*} f_{ij} \ell_{\kappa_i} \Theta(\tau_j^\dagger) \\
&= \sum q^*_{TM\oplus A^*} f_{ij} \ell_{\kappa_i} ((\rho,\rho^t)\tau_j)^\dagger \\
&= \widetilde{(\rho,\rho^t) \circ \varphi} \,.
\end{aligned}$$

□

Now we are able to prove the following analogue of Proposition 4.3.9 for general splittings.

Proposition A.2.2. *A linear generalised complex structure $\mathcal{J}$ on a Lie algebroid $A \to M$ preserves the anchor of $\mathbb{T}A$ if and only if for all $a \in \Gamma(A)$ we have*

1. $\left(j \circ (\rho, \rho^t)\right)^t = -j \circ (\rho, \rho^t)\,,$
2. $\nabla_a^{\mathrm{bas}} \circ j - j \circ \nabla_a^{\mathrm{bas}} = (\rho, \rho^t) \circ \Phi(a)\,.$

Proof. The anchor preservation condition for $\mathcal{J}$ over j is again $\Theta \circ \mathcal{J} = Tj \circ \Theta$. The computation for core sections is the same as in the case of an adapted Dorfman connection in Proposition 4.3.9. For a horizontal lift $\sigma_A^\Delta(a)$ we now make use of Theorem 4.3.6, of Corollary A.2.1 and of Lemma 4.3.8 to obtain the following:

$$\begin{aligned}\Theta \circ \mathcal{J}(\sigma_A^\Delta(a)(\nu_m)) &= \Theta\Big(\sigma_A^\Delta(a)(j\nu_m) + \widetilde{\Phi(a) \circ j^{-1}}(j\nu_m)\Big)\\ &= \widehat{\nabla_a^{\mathrm{bas}}}(j\nu_m) + \widetilde{(\rho, \rho^t) \circ \Phi(a) \circ j^{-1}}(j\nu_m)\,,\end{aligned}$$

and on the other hand:

$$\begin{aligned}Tj \circ \Theta(\sigma_A^\Delta(a)(\nu_m)) &= Tj(\widehat{\nabla_a^{\mathrm{bas}}}(\nu_m)\\ &= \widehat{j \circ \nabla_a^{\mathrm{bas}} \circ j^{-1}}(j\nu_m)\,.\end{aligned}$$

Note that $\widetilde{(\rho, \rho^t) \circ \Phi(a) \circ j^{-1}}$ denotes here the core-linear vector field on $TM \oplus A^*$.

So the anchor preservation condition for a lift reads

$$\widehat{(\nabla_a^{\mathrm{bas}} - j \circ \nabla_a^{\mathrm{bas}} \circ j^{-1})} = \widetilde{(\rho, \rho^t) \circ \Phi(a) \circ j^{-1}}\,.$$

The linear vector field on the left is core-linear and $\left(\nabla_a^{\mathrm{bas}} - j \circ \nabla_a^{\mathrm{bas}} \circ j^{-1}\right)$ is tensorial. The anchor preservation condition for horizontal lifts is therefore equivalent to the following condition:

$$\nabla_a^{\mathrm{bas}} \circ j - j \circ \nabla_a^{\mathrm{bas}} = (\rho, \rho^t) \circ \Phi(a)\,. \tag{A.10}$$

□

Furthermore, analogous to Proposition 4.3.10 we obtain for a general linear splitting the following result.

Proposition A.2.3. *A linear generalised complex structure $\mathcal{J}$ on a Lie algebroid A which preserves the anchor of $\mathbb{T}A \to TM \oplus A^*$ is additionally compatible with the Lie bracket of $\mathbb{T}A \to TM \oplus A^*$ if and only if for all sections a, $b \in \Gamma(A)$ we have*

$$j_C \circ R^{\mathrm{bas}}_{\Delta}(a,b) - R^{\mathrm{bas}}_{\Delta}(a,b) \circ j = (\mathbf{d}_{\nabla^{\mathrm{Hom}}}\Phi)(a,b)\,.$$

Proof. Since $\mathcal{J}$ and therefore j are invertible, every section can be pushed forward via $\mathcal{J}$. The bracket is then preserved under $\mathcal{J}$ if for all sections ζ, $\xi \in \Gamma_{TM\oplus A^*}(\mathbb{T}A)$ we have the property (see for example [51] and [16])

$$[\mathcal{J}\zeta \circ j^{-1}, \mathcal{J} \circ \xi \circ j^{-1}] = \mathcal{J} \circ [\zeta, \xi] \circ j^{-1}\,. \tag{A.11}$$

For two core sections we obtain again the same as in Proposition 4.3.10. For the bracket of a lift with a core section we obtain on one hand making additionally use of Corollary A.2.1 the following

$$\begin{aligned}\left[\mathcal{J} \circ \sigma^{\Delta}_{A}(a) \circ j^{-1}, (j_C\tau)^{\dagger}\right] &= \left[\sigma^{\Delta}_{A}(a), (j_C\tau)^{\dagger}\right] + \left[\widetilde{\Phi(a) \circ j^{-1}}, (j_C\tau)^{\dagger}\right] \\ &= (\nabla^{\mathrm{bas}}_{a}(j_C\tau))^{\dagger} - \left(\Phi(a) \circ j^{-1}((\rho,\rho^t)j_C\tau)\right)^{\dagger}\end{aligned}$$

On the other hand we have again

$$\mathcal{J} \circ [\sigma^{\Delta}_{A}(a), \tau^{\dagger}] \circ j^{-1} = (j_C\nabla^{\mathrm{bas}}_{a}\tau)^{\dagger}$$

and thus we obtain the condition

$$\nabla^{\mathrm{bas}}_{a} \circ j_C - j_C \circ \nabla^{\mathrm{bas}}_{a} = \Phi(a) \circ j^{-1} \circ (\rho,\rho^t) \circ j_C = \Phi(a) \circ (\rho,\rho^t)\,, \tag{A.12}$$

where we used the anchor preservation condition (4.54) in the last step. According to Proposition 4.1.10 we know that $\Phi(a)$ has to be skew-symmetric. That is $\Phi(a)^t = -\Phi(a)$. As the Dorfman connection was chosen to be skew-symmetric, the two basic connections have to be dual to each other according to Proposition 4.3.3. Hence equation (4.58) for a linear generalised complex structure is equivalent to (A.10) which we assume to be satisfied as $\mathcal{J}$ is required to respect the anchor.

For the bracket of two lifts we get now on one hand the following:

$$\begin{aligned}[\mathcal{J} \circ \sigma^{\Delta}_{A}(a) \circ j^{-1}, \mathcal{J} \circ \sigma^{\Delta}_{A}(b) \circ j^{-1}] &= [\sigma^{\Delta}_{A}(a) + \widetilde{\Phi(a) \circ j^{-1}}, \sigma^{\Delta}_{A}(b) + \widetilde{\Phi(b) \circ j^{-1}}] \\ &= \sigma^{\Delta}_{A}([a,b]) - \widetilde{R^{\mathrm{bas}}_{\Delta}(a,b)} \\ &\quad + \widetilde{\nabla^{\mathrm{bas}}_{a} \circ \Phi(b) \circ j^{-1} - \Phi(b) \circ j^{-1} \circ \nabla^{\mathrm{bas}}_{a}} \\ &\quad - \widetilde{\nabla^{\mathrm{bas}}_{b} \circ \Phi(a) \circ j^{-1} - \Phi(a) \circ j^{-1} \circ \nabla^{\mathrm{bas}}_{b}} \\ &\quad + \widetilde{\Phi(b) \circ j^{-1} \circ (\rho,\rho^t) \circ \Phi(a) \circ j^{-1}} \\ &\quad - \widetilde{\Phi(a) \circ j^{-1} \circ (\rho,\rho^t) \circ \Phi(b) \circ j^{-1}}\,.\end{aligned}$$

On the other hand we compute with Lemma 4.3.8

$$\mathcal{J} \circ [\sigma_A^\Delta(a), \sigma_A^\Delta(b)] \circ j^{-1} = \sigma_A^\Delta([a,b]) + \widetilde{\Phi([a,b]) \circ j^{-1}} - \widetilde{j_C \circ R_\Delta^{\mathrm{bas}}(a,b) \circ j^{-1}} \,.$$

Using the property $\nabla_a^{\mathrm{bas}} \circ j = j \circ \nabla_a^{\mathrm{bas}} + (\rho, \rho^t) \circ \Phi(a)$ this gives the last condition for $\mathcal{J}$ to be a Lie algebroid morphism:

$$\begin{aligned} j_C \circ R_\Delta^{\mathrm{bas}}(a,b) - R_\Delta^{\mathrm{bas}}(a,b) \circ j &= \Phi([a,b]) + \nabla_b^{\mathrm{bas}} \circ \Phi(a) - \Phi(a) \circ \nabla_b^{\mathrm{bas}} \\ &\quad - \nabla_a^{\mathrm{bas}} \circ \Phi(b) + \Phi(b) \circ \nabla_a^{\mathrm{bas}} \\ &= (\mathbf{d}_{\nabla^{\mathrm{Hom}}} \Phi)(a,b) \,. \end{aligned} \tag{A.13}$$

□

Summarising we obtain the following Theorem:

Theorem A.2.4. *A linear generalised complex structure $\mathcal{J}$ on a Lie algebroid A is a Lie algebroid morphism if and only if for every skew-symmetric Dorfman connection Δ as in* (4.39) *and Φ defined by* (4.52) *we have for any a, $b \in \Gamma(A)$*

1. $(\rho, \rho^t) \circ j_C = j \circ (\rho, \rho^t)$ *,*
2. $\nabla_a^{\mathrm{bas}} \circ j - j \circ \nabla_a^{\mathrm{bas}} = (\rho, \rho^t) \circ \Phi(a)$ *,*
3. $$\begin{aligned} j_C \circ R_\Delta^{\mathrm{bas}}(a,b) - R_\Delta^{\mathrm{bas}}(a,b) \circ j &= \Phi([a,b]) + \nabla_b^{\mathrm{bas}} \circ \Phi(a) - \Phi(a) \circ \nabla_b^{\mathrm{bas}} \\ &\quad - \nabla_a^{\mathrm{bas}} \circ \Phi(b) + \Phi(b) \circ \nabla_a^{\mathrm{bas}} \,. \end{aligned}$$

An equivalent formulation is that the tuple (j_C, j, Φ) forms a automorphism of the 2-term representation up to homotopy $((\rho, \rho^t), \nabla^{\mathrm{bas}}, \nabla^{\mathrm{bas}}, R_\Delta^{\mathrm{bas}})$ of A onto $(A \oplus T^*M)_{[0]} \oplus (TM \oplus A^*)_{[1]}$ which defines the VB-algebroid structure of $\mathbb{T}A$ in the linear splitting corresponding to Δ. The correspondence between VB-algebroids and their morphisms after a choice of decomposition and 2-term representations up to homotopy and their morphisms was shown in [28] and in [19]. We recall this correspondence in Section 2.2.3.

Bibliography

[1] C. A. Abad and M. Crainic. Representations up to homotopy of Lie algebroids. *J. Reine Angew. Math.*, 663:91–126, 2012.

[2] M. Batchelor. The structure of supermanifolds. *Trans. Amer. Math. Soc.*, 253:329–338, 1979.

[3] M. Batchelor. Two approaches to supermanifolds. *Trans. Amer. Math. Soc.*, 258(1):257–270, 1980.

[4] M. B. Batchelor. *The structure of supermanifolds.* ProQuest LLC, Ann Arbor, MI, 1978. Thesis (Ph.D.)–Massachusetts Institute of Technology.

[5] G. Bonavolontà and N. Poncin. On the category of Lie n-algebroids. *J. Geom. Phys.*, 73:70–90, 2013.

[6] H. Bursztyn. A brief introduction to Dirac manifolds. In *Geometric and topological methods for quantum field theory*, pages 4–38. Cambridge Univ. Press, Cambridge, 2013.

[7] H. Bursztyn, A. Cabrera, and M. del Hoyo. Vector bundles over Lie groupoids and algebroids. *Adv. Math.*, 290:163–207, 2016.

[8] H. Bursztyn, G. R. Cavalcanti, and M. Gualtieri. Generalized Kähler and hyper-Kähler quotients. In *Poisson geometry in mathematics and physics*, volume 450 of *Contemp. Math.*, pages 61–77. Amer. Math. Soc., Providence, RI, 2008.

[9] A. Cabras and A. M. Vinogradov. Extensions of the Poisson bracket to differential forms and multi-vector fields. *J. Geom. Phys.*, 9(1):75–100, 1992.

[10] A. S. Cattaneo and F. Schätz. Introduction to supergeometry. *Rev. Math. Phys.*, 23(6):669–690, 2011.

[11] G. R. Cavalcanti and M. Gualtieri. Generalized complex geometry and T-duality. In *A celebration of the mathematical legacy of Raoul Bott*, volume 50

of *CRM Proc. Lecture Notes*, pages 341–365. Amer. Math. Soc., Providence, RI, 2010.

[12] A. Coste, P. Dazord, and A. Weinstein. Groupoïdes symplectiques. In *Publications du Département de Mathématiques. Nouvelle Série. A, Vol. 2*, volume 87 of *Publ. Dép. Math. Nouvelle Sér. A*, pages i–ii, 1–62. Univ. Claude-Bernard, Lyon, 1987.

[13] T. J. Courant. Dirac manifolds. *Trans. Amer. Math. Soc.*, 319(2):631–661, 1990.

[14] M. Crainic. Generalized complex structures and Lie brackets. *Bull. Braz. Math. Soc. (N.S.)*, 42(4):559–578, 2011.

[15] M. Crainic and R. L. Fernandes. Integrability of Lie brackets. *Ann. of Math. (2)*, 157(2):575–620, 2003.

[16] M. Crainic and R. L. Fernandes. Lectures on integrability of Lie brackets. In *Lectures on Poisson geometry*, volume 17 of *Geom. Topol. Monogr.*, pages 1–107. Geom. Topol. Publ., Coventry, 2011.

[17] F. del Carpio-Marek. *Geometric structures on degree 2 manifolds.* PhD thesis, IMPA, available at `www.impa.br/wp-content/uploads/2017/05/Fernando_Del_Carpio.pdf`, Rio de Janeiro, 2015.

[18] I. Y. Dorfman. Dirac structures of integrable evolution equations. *Phys. Lett. A*, 125(5):240–246, 1987.

[19] T. Drummond, M. Jotz Lean, and C. Ortiz. VB-algebroid morphisms and representations up to homotopy. *Differential Geom. Appl.*, 40:332–357, 2015.

[20] C. Ehresmann. Gattungen von lokalen Strukturen. *Jber. Deutsch. Math. Verein.*, 60(Abt. 1):49–77, 1957.

[21] C. Ehresmann. Catégories topologiques et catégories différentiables. In *Colloque Géom. Diff. Globale (Bruxelles, 1958)*, pages 137–150. Centre Belge Rech. Math., Louvain, 1959.

[22] C. Ehresmann. Catégories doubles et catégories structurées. *C. R. Acad. Sci. Paris*, 256:1198–1201, 1963.

[23] S. Evens, J.-H. Lu, and A. Weinstein. Transverse measures, the modular class and a cohomology pairing for Lie algebroids. *Quart. J. Math. Oxford Ser. (2)*, 50(200):417–436, 1999.

[24] J. Grabowski and M. Rotkiewicz. Higher vector bundles and multi-graded symplectic manifolds. *J. Geom. Phys.*, 59(9):1285–1305, 2009.

[25] A. Gracia-Saz, M. Jotz Lean, K. C. H. Mackenzie, and R. A. Mehta. Double Lie algebroids and representations up to homotopy. *J. Homotopy Relat. Struct.*, 13(2):287–319, 2018.

[26] A. Gracia-Saz and K. C. H. Mackenzie. Duality functors for triple vector bundles. *Lett. Math. Phys.*, 90(1-3):175–200, 2009.

[27] A. Gracia-Saz and K. C. H. Mackenzie. Duality functors for n-fold vector bundles. preprint available online, arXiv:1209.0027, Aug. 2012.

[28] A. Gracia-Saz and R. A. Mehta. Lie algebroid structures on double vector bundles and representation theory of Lie algebroids. *Adv. Math.*, 223(4):1236–1275, 2010.

[29] M. Grützmann and M. Stiénon. Matched pairs of Courant algebroids. *Indag. Math. (N.S.)*, 25(5):977–991, 2014.

[30] M. Gualtieri. *Generalized complex geometry.* PhD thesis, University of Oxford, Jan 2004. Available at `https://arxiv.org/abs/math/0401221v1`.

[31] M. Gualtieri. Generalized complex geometry. *Ann. of Math. (2)*, 174(1):75–123, 2011.

[32] M. Gualtieri. Generalized Kähler geometry. *Comm. Math. Phys.*, 331(1):297–331, 2014.

[33] M. Heuer and M. Jotz Lean. Multiple vector bundles: cores, splittings and decompositions. preprint available online, arXiv:1809.01484, Sept. 2018.

[34] N. Hitchin. Generalized Calabi-Yau manifolds. *Q. J. Math.*, 54(3):281–308, 2003.

[35] N. Hitchin. Instantons, Poisson structures and generalized Kähler geometry. *Comm. Math. Phys.*, 265(1):131–164, 2006.

[36] M. Jotz Lean. Dorfman connections and Courant algebroids. *J. Math. Pures Appl. (9)*, 116:1–39, 2018.

[37] M. Jotz Lean. The geometrization of $\mathbb{N}$-manifolds of degree 2. *J. Geom. Phys.*, 133:113–140, 2018.

[38] M. Jotz Lean. Dirac groupoids and Dirac bialgebroids. To appear in *Journal of Symplectic Geometry*, arXiv:1403.2934, 2019.

[39] M. Jotz Lean. Lie-2-algebroids and matched pairs of 2-representations – a geometric approach. To appear in *Pacific Journal of Mathematics*, arXiv:1712.07035, 2019.

[40] M. Jotz Lean and C. Ortiz. Foliated groupoids and infinitesimal ideal systems. *Indag. Math. (N.S.)*, 25(5):1019–1053, 2014.

[41] M. Jotz Lean, M. Stiénon, and P. Xu. Glanon groupoids. *Math. Ann.*, 364(1-2):485–518, 2016.

[42] C. Laurent-Gengoux, M. Stiénon, and P. Xu. Holomorphic Poisson manifolds and holomorphic Lie algebroids. *Int. Math. Res. Not. IMRN*, pages Art. ID rnn 088, 46, 2008.

[43] D. Li-Bland and P. Ševera. Quasi-Hamiltonian groupoids and multiplicative Manin pairs. *Int. Math. Res. Not. IMRN*, 2011(10):2295–2350, 2011.

[44] D. S. Li-Bland. *LA-Courant algebroids and their applications.* ProQuest LLC, Ann Arbor, MI, 2012. Thesis (Ph.D.)–University of Toronto (Canada).

[45] Z.-J. Liu, A. Weinstein, and P. Xu. Manin triples for Lie bialgebroids. *J. Differential Geom.*, 45(3):547–574, 1997.

[46] J.-H. Lu. Poisson homogeneous spaces and Lie algebroids associated to Poisson actions. *Duke Math. J.*, 86(2):261–304, 1997.

[47] K. Mackenzie. *Lie groupoids and Lie algebroids in differential geometry*, volume 124 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 1987.

[48] K. C. H. Mackenzie. Double Lie algebroids and second-order geometry. I. *Adv. Math.*, 94(2):180–239, 1992.

[49] K. C. H. Mackenzie. Drinfel'd doubles and Ehresmann doubles for Lie algebroids and Lie bialgebroids. *Electron. Res. Announc. Amer. Math. Soc.*, 4:74–87 (electronic), 1998.

[50] K. C. H. Mackenzie. Duality and triple structures. In *The breadth of symplectic and Poisson geometry*, volume 232 of *Progr. Math.*, pages 455–481. Birkhäuser Boston, Boston, MA, 2005.

[51] K. C. H. Mackenzie. *General theory of Lie groupoids and Lie algebroids*, volume 213 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 2005.

[52] K. C. H. Mackenzie. Ehresmann doubles and Drinfel'd doubles for Lie algebroids and Lie bialgebroids. *J. Reine Angew. Math.*, 658:193–245, 2011.

[53] K. C. H. Mackenzie and P. Xu. Lie bialgebroids and Poisson groupoids. *Duke Math. J.*, 73(2):415–452, 1994.

[54] K. C. H. Mackenzie and P. Xu. Classical lifting processes and multiplicative vector fields. *Quart. J. Math. Oxford Ser. (2)*, 49(193):59–85, 1998.

[55] K. C. H. Mackenzie and P. Xu. Integration of Lie bialgebroids. *Topology*, 39(3):445–467, 2000.

[56] T. Mokri. Matched pairs of Lie algebroids. *Glasgow Math. J.*, 39(2):167–181, 1997.

[57] A. Newlander and L. Nirenberg. Complex analytic coordinates in almost complex manifolds. *Ann. of Math. (2)*, 65:391–404, 1957.

[58] J. Pradines. Théorie de Lie pour les groupoïdes différentiables. Relations entre propriétés locales et globales. *C. R. Acad. Sci. Paris Sér. A-B*, 263:A907–A910, 1966.

[59] J. Pradines. Théorie de Lie pour les groupoïdes différentiables. Calcul différenetiel dans la catégorie des groupoïdes infinitésimaux. *C. R. Acad. Sci. Paris Sér. A-B*, 264:A245–A248, 1967.

[60] J. Pradines. Géométrie différentielle au-dessus d'un groupoïde. *C. R. Acad. Sci. Paris Sér. A-B*, 266:A1194–A1196, 1968.

[61] J. Pradines. Troisième théorème de Lie les groupoïdes différentiables. *C. R. Acad. Sci. Paris Sér. A-B*, 267:A21–A23, 1968.

[62] J. Pradines. Fibrés vectoriels doubles symétriques et jets holonomes d'ordre 2. *C. R. Acad. Sci. Paris Sér. A*, 278:1557–1560, 1974.

[63] J. Pradines. Représentation des jets non holonomes par des morphismes vectoriels doubles soudés. *C. R. Acad. Sci. Paris Sér. A*, 278:1523–1526, 1974.

[64] J. Pradines. Suites exactes vectorielles doubles et connexions. *C. R. Acad. Sci. Paris Sér. A*, 278:1587–1590, 1974.

[65] J. Pradines. *Fibres vectoriels doubles et calcul des jets non holonomes*, volume 29 of *Esquisses Mathématiques [Mathematical Sketches]*. Université d'Amiens, U.E.R. de Mathématiques, Amiens, 1977.

[66] D. Roytenberg. *Courant algebroids, derived brackets and even symplectic supermanifolds*. ProQuest LLC, Ann Arbor, MI, 1999. Thesis (Ph.D.)–University of California, Berkeley.

[67] D. Roytenberg. On the structure of graded symplectic supermanifolds and Courant algebroids. In *Quantization, Poisson brackets and beyond (Manchester, 2001)*, volume 315 of *Contemp. Math.*, pages 169–185. Amer. Math. Soc., Providence, RI, 2002.

[68] Y. Sheng and C. Zhu. Higher extensions of Lie algebroids. *Commun. Contemp. Math.*, 19(3):1650034, 41, 2017.

[69] E. Vishnyakova. Graded manifolds of type Δ and n-fold vector bundles. *Lett. Math. Phys.*, 109(2):243–293, 2019.

[70] A. Weinstein. Symplectic groupoids and Poisson manifolds. *Bull. Amer. Math. Soc. (N.S.)*, 16(1):101–104, 1987.