Partial cluster-tilted algebras via twin cotorsion pairs, quasi-abelian categories and Auslander-Reiten theory

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Abstract

In this thesis we study *partial cluster-tilted algebras*. These algebras are opposite endomorphism rings of rigid objects in cluster categories, and they are a generalisation of cluster-tilted algebras. The key motivation for the work we present here is to understand the representation theory of a partial cluster-tilted algebra. In our study of how the Auslander-Reiten theory of a partial cluster-tilted algebra is induced by the Auslander-Reiten theory of the corresponding cluster category, we use twin cotorsion pairs on triangulated categories to extract quasi-abelian categories from cluster categories, and develop Auslander-Reiten theory in quasi-abelian and Krull-Schmidt categories.

We prove that, under a mild assumption, the heart $\overline{\mathcal{H}}$ of a twin cotorsion pair $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ on a triangulated category $\mathcal{C}$ is a quasi-abelian category. If $\mathcal{C}$ is also Krull-Schmidt and $\mathcal{T}=\mathcal{U}$, we show that the heart of the cotorsion pair $(\mathcal{S},\mathcal{T})$ is equivalent to the Gabriel-Zisman localisation of $\overline{\mathcal{H}}$ at the class of its regular morphisms. In particular, suppose $\mathcal{C}$ is a cluster category with a rigid object R and let $[\mathcal{X}_R]$ denote the ideal of morphisms factoring through $\mathcal{X}_R=\operatorname{Ker}(\operatorname{Hom}_{\mathcal{C}}(R,-))$. Then applications of our results show that $\mathcal{C}/[\mathcal{X}_R]$ is a quasi-abelian category. We also obtain a new proof of an equivalence between the localisation of this category at its class of regular morphisms and a certain subfactor category of $\mathcal{C}$.

We generalise some of the theory developed for abelian categories in papers of Auslander and Reiten to semi-abelian and quasi-abelian categories. In addition, we generalise some Auslander-Reiten theory results of S. Liu for Hom-finite,

Krull-Schmidt categories by removing the Hom-finite and indecomposability restrictions. As a main result, we give equivalent characterisations of Auslander-Reiten sequences in a skeletally small, quasi-abelian, Krull-Schmidt category.

Lastly, we construct partial cluster-tilted algebras of arbitrarily large finite global dimension coming from cluster categories associated to Dynkin-type $\mathbb{A}$ quivers. In particular, this shows that there is an infinite family of partial cluster-tilted algebras that are not cluster-tilted. Then we consider how the Auslander-Reiten theory of the algebra $(\mathrm{End}_{\mathcal{C}} R)^{\mathrm{op}}$, where R is a basic rigid object of a Hom-finite, Krull-Schmidt, triangulated k-category $\mathcal{C}$ with Serre duality, is induced by the Auslander-Reiten theory of $\mathcal{C}$ via the functor $\mathrm{Hom}_{\mathcal{C}}(R,-)$. Let $\mathcal{C}(R)$ denote the subcategory of $\mathcal{C}$ consisting of objects X for which there is a triangle $R_0 \to R_1 \to X \to \Sigma R_0$ with $R_i \in \operatorname{\mathsf{add}} R$. We show that if $f \colon X \to Y$ is an irreducible morphism in $\mathcal{C}$ with $X \in \mathcal{C}(R)$, then $\mathrm{Hom}_{\mathcal{C}}(R,f)$ is irreducible if Y also lies in $\mathcal{C}(R)$, or $\mathrm{Hom}_{\mathcal{C}}(R,f)$ is split otherwise. If X does not lie in $\mathcal{C}(R)$, we provide partial results dependent on properties of the morphism $f + [\mathcal{X}_R](X,Y)$ in the quotient $\mathcal{C}/[\mathcal{X}_R]$.

Contents

Chapter I

Introduction

Representation theory is the study of *algebras* (i.e. rings having a module structure over a central ground ring) via the modules they admit. In particular, if S is a k-algebra and the ground ring k is a field, then the S-modules in which we are interested are also k-vector spaces and S acts on these vector spaces by k-linear transformations. Therefore, trying to understand the more general algebra S in this way allows us to use techniques from linear algebra, which is a relatively well understood area.

On the other hand, the category of modules over a given ring is an example of an *abelian category* (see Definition II.1.36) and so one might also use category theory to help understand this category. For instance, if S is an artin algebra, then it was shown in [AR75] that the category $S-\mathsf{mod}$ of finitely generated left S-modules has *almost split sequences* or *Auslander-Reiten sequences* (see Definition II.4.11). To study the *Auslander-Reiten theory* of an additive category $\mathcal{A}$ is to study these sequences and morphisms such as almost split or irreducible morphisms (see Definitions II.3.11 and II.3.3). *Irreducible* morphisms are non-split morphisms admitting no non-trivial factorisation, and knowing these allows one to build the *Auslander-Reiten quiver* of $\mathcal{A}$ (see Definition II.3.14), which is a directed graph with the indecomposable objects (up to isomorphism) of $\mathcal{A}$ as vertices and irreducible morphisms (up to a scalar) as arrows. In sufficiently nice cases, the Auslander-Reiten quiver of $\mathcal{A}$ completely determines $\mathcal{A}$.

Let k be a field. A finite-dimensional k-algebra is called *hereditary* if it has *global dimension* (see Theorem II.4.3) at most 1. Hereditary algebras are of great significance because any basic finite-dimensional algebra is isomorphic to a quotient of a hereditary algebra (see [ASS06, Thm. II.3.7]). Therefore, one can use the representation theory of an appropriate hereditary k-algebra in order to study the module category of a given k-algebra A. Another approach to understanding the representation theory of A is to use *tilting theory*. Broadly, this involves comparing the representation theory of an algebra to the representation theory of the (opposite) endomorphism ring of a tilting module.

Let us make this more precise. Let A be a finite-dimensional k-algebra and denote by $A-\mathsf{mod}$ the category of finitely generated (equivalently, finite-dimensional) left A-modules. A *torsion theory* in $A-\mathsf{mod}$ is a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $A-\mathsf{mod}$, such that $\operatorname{Hom}_{A-\mathsf{mod}}(\mathcal{T}, \mathcal{F}) = 0$, $\operatorname{Hom}_{A-\mathsf{mod}}(X, \mathcal{F}) = 0$ implies $X \in \mathcal{T}$, and $\operatorname{Hom}_{A-\mathsf{mod}}(\mathcal{T}, Y) = 0$ implies $Y \in \mathcal{F}$ (see [ASS06, Def. VI.1.1]). If $(\mathcal{T}, \mathcal{F})$ is a torsion theory in $A-\mathsf{mod}$, then for any object $X \in A-\mathsf{mod}$ there exists an exact sequence $0 \to T_X \to X \to F_X \to 0$ in $A-\mathsf{mod}$ with $T_X \in \mathcal{T}$ and $F_X \in \mathcal{F}$. Thus, it can be seen that torsion theories of $A-\mathsf{mod}$ can be used to better understand $A-\mathsf{mod}$ itself. A tilting module M over A induces torsion theories relating the category $A-\mathsf{mod}$ to the module category of $\Gamma := (\operatorname{End}_{A-\mathsf{mod}} M)^{\mathrm{op}}$. Recall that a basic module $M \in A-\mathsf{mod}$ is called a *tilting* module if M has projective dimension $\operatorname{p.dim}_A M$ at most 1, $\operatorname{Ext}^1_A(M, M) = 0$ and there exists an exact sequence $0 \to A \to M_1 \to M_2 \to 0$, where M_1, M_2 lie in $\mathsf{add}\, M$ (see Definition V.2.7). There is a torsion theory $(\operatorname{Fac} M, \mathcal{F})$ in $A-\mathsf{mod}$, where an object in $\operatorname{Fac} M$ is a quotient of a finite direct sum of copies of M, and a torsion theory $(\mathcal{X}, \mathcal{Y})$ in $\Gamma-\mathsf{mod}$ such that there are equivalences $\operatorname{Hom}_{A-\mathsf{mod}}(M, -)\colon \operatorname{Fac} M \xrightarrow{\simeq} \mathcal{Y}$ and $\operatorname{Ext}^1_A(M, -)\colon \mathcal{F} \xrightarrow{\simeq} \mathcal{X}$ (see [BB80]). See, for example, [Rei07] for more details.

If $A = H$ is a hereditary algebra, then $\Gamma = (\mathrm{End}_{H\,\text{-}\,\mathsf{mod}}\, M)^{\mathrm{op}}$ is called a *tilted algebra* (see Definition V.2.8). Furthermore, the equivalences above induce a triangle equivalence

$$\mathrm{RHom}_{H\,\text{-}\,\mathsf{mod}}(M,-)\colon \mathsf{D}^b(H\,\text{-}\,\mathsf{mod}) \xrightarrow[\triangle]{\simeq} \mathsf{D}^b(\Gamma\,\text{-}\,\mathsf{mod})$$

between bounded derived categories; see [Hap88, §III.2]. Thus, we see that tilted algebras are closely related to hereditary algebras, so one can use the better understood representation theory of hereditary algebras in studying tilted algebras.

A connection between tilting theory and cluster algebras was found in [MRZ03]. *Cluster algebras* are defined by an iterative process and are commutative subrings of rational function fields over $\mathbb{Q}$. They were introduced in [FZ02] to aid the search for an explicit description of the dual canonical basis associated to a quantised enveloping algebra of a symmetrisable Kac-Moody Lie algebra. However, only in a few cases is a complete description known; see, for example, [CM00], [HYY03], [Lus90], [Lus93], [Mar98], [Xi99a], [Xi99b]. They have since seen links to various fields, such as Poisson geometry, integrable systems and even particle physics; see e.g. [BZ05], [FG09b], [FG09a], [FG09c], [FZ03], [GLS13], [Kim12], [Kim17], [Gli11], [GM19]. See, for example, [Mar13], [Rei10] and the references therein for more comprehensive surveys.

The motivation for the work in this thesis comes from the strong representation-theoretic link to cluster algebras; see, for example, [CC06], [CCS06], [BMRRT], [BMR08], [Pal08], [CK08], [BMRT07]. In order to better understand the *decorated quiver representations* defined in [MRZ03], cluster categories were introduced in [BMRRT] for finite-dimensional hereditary algebras, and have been shown to give a natural model for the combinatorics of the corresponding cluster algebra. (Independently, for Dynkin-type $\mathbb{A}$ quivers, an equivalent category defined via geometric means was given in [CCS06].) Indeed, for a finite, acyclic quiver Q there is a bijection between the set of cluster variables in the cluster algebra associated to

Q and the set of isoclasses of indecomposables without self-extensions in the cluster category associated to Q (see [BMRRT], [CK06], [BMRT07]). A generalised cluster category for algebras of global dimension at most 2 is given in [Ami09]. See, for example, the surveys [BM06], [Kel10] for more details.

For a finite-dimensional hereditary k-algebra H, the *cluster category* $\mathcal{C}_H$ of H (as defined in [BMRRT]) is the *orbit category* $\mathsf{D}^b(H)/(\tau^{-1} \circ [1])$, where $\mathsf{D}^b(H) := \mathsf{D}^b(H-\mathsf{mod})$ is the bounded derived category of finitely generated left H-modules, τ^{-1} is a quasi-inverse to the Auslander-Reiten translate τ and $[1]$ is the standard suspension (or shift) functor on $\mathsf{D}^b(H)$ (see Definitions II.8.1 and II.8.4). Then, the k-category $\mathcal{C}_H$ is Hom-finite, Krull-Schmidt, triangulated, has Serre duality and has Auslander-Reiten triangles (see [BMRRT], [Kel05], or Theorem II.8.5 and Proposition II.8.8).

Let T be an object of $\mathcal{C}_H$. Then T is said to be: *rigid* if $\mathrm{Ext}^1_{\mathcal{C}_H}(T,T) = 0$ (see [BMRRT, §3]); *maximal rigid* if T is rigid and has a maximal number of non-isomorphic indecomposable direct summands with respect to this property; and *cluster-tilting* if $\mathsf{add}\, T$ is functorially finite and ${}^{\perp_1}(\mathsf{add}\, T) = \mathsf{add}\, T = (\mathsf{add}\, T)^{\perp_1}$ (see Definitions III.2.1 and III.5.1, and [ZZ11, Def. 2.1]). Moreover, in a cluster category $\mathcal{C}_H$, an object T is maximal rigid if and only if it is cluster-tilting (see [BIRS09, Thm. II.1.8]). If T is cluster-tilting, then the ring $\Lambda_T := (\mathrm{End}_{\mathcal{C}_H} T)^{\mathrm{op}}$ is called a *cluster-tilted algebra* (see [BMR07]) and in cluster theory is the analogue of a tilted algebra. We remark here that for arbitrary triangulated categories maximal rigid objects are not necessarily cluster-tilting. Indeed, each cluster-tilting object is maximal rigid, but the converse is not true in general; see, for example, [BIKR08] or [BMV10].

One phenomenon that arises in classical tilting theory is that the opposite endomorphism ring of a partial tilting module is again a tilted algebra (see [Hap88, Cor. III.6.5]). Recall that a module $M' \in H-\mathsf{mod}$ is called a *partial tilting* module if ($\mathrm{p.dim}_H M' \leqslant 1$ and) $\mathrm{Ext}^1_H(M', M') = 0$ (see [Hap88, §III.6]). This motivates the following definition.

Definition. [Definition V.1.1] If T' is a rigid object of a cluster category $\mathcal{C}_H$, then we call $\Lambda_{T'} := (\operatorname{End}_{\mathcal{C}_H} T')^{\mathrm{op}}$ a *partial cluster-tilted algebra.*

This terminology is further justified by the fact that any rigid object can be extended to a cluster-tilting object in a cluster category; see [BMRRT, Prop. 3.2].

It was observed in [BMR08] that, in contrast to the classical setting, the class of partial cluster-tilted algebras does not coincide with the class of cluster-tilted algebras (see Remark V.2.6). We reinforce this further by providing a family of partial cluster-tilted algebras with arbitrarily large finite global dimension that are not cluster-tilted in Chapter V. Recall that a cluster-tilted algebra must have global dimension $0, 1$ or ∞ (see [KR07]).

Theorem. [Theorem V.2.4] *For each* $n \in \mathbb{Z}_{\geqslant 0}$*, there exists a partial cluster-tilted algebra* Λ *with* $\operatorname{gl.dim} \Lambda = n$*.*

This demonstrates that not all of the results from classical tilting theory will translate over to the cluster-tilting setting. Moreover, unlike the cluster-tilting case, extracting the Auslander-Reiten quiver of a partial cluster-tilted algebra directly from the Auslander-Reiten quiver of the corresponding cluster category is much less clear.

Let us recall how this works for the cluster-tilting case. Suppose T is a cluster-tilting (equivalently, maximal rigid) object in a cluster category $\mathcal{C}_H$. For an additive category $\mathcal{A}$ with a subcategory $\mathcal{B}$ that is closed under finite direct sums, we will denote by $[\mathcal{B}]$ the *ideal* of $\mathcal{A}$ (see Definition II.1.29) consisting of morphisms that factor through objects of $\mathcal{B}$, and by $\mathcal{A}/[\mathcal{B}]$ the corresponding additive quotient category. It was shown in [BMR07] that the functor

$$\operatorname{Hom}_{\mathcal{C}_H/[\mathcal{X}_T]}(T, -) \colon \mathcal{C}_H/[\mathcal{X}_T] \xrightarrow{\simeq} \Lambda_T - \mathsf{mod}$$

is an equivalence, where $\mathcal{X}_T := \mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}_H}(T, -))$ is the full subcategory of $\mathcal{C}_H$ consisting of objects that are annihilated by the functor $\mathrm{Hom}_{\mathcal{C}_H}(T, -)$. Hence, the Auslander-Reiten quiver of Λ_T can be obtained directly from the Auslander-Reiten quiver of $\mathcal{C}_H$, which is well-understood, by simply deleting the vertices $[X]$ where $X \in \mathcal{X}_T$ and any arrows incident to such vertices; see [BMR07] and [KZ08], and also [Liu10].

If T' is rigid but not necessarily cluster-tilting, then $\mathcal{C}_H/[\mathcal{X}_{T'}]$ may not be equivalent to $\Lambda_{T'}-\mathsf{mod}$. A desire to understand whether the Auslander-Reiten quiver of the cluster category determines the Auslander-Reiten quiver of the partial cluster-tilted algebra has motivated the results in this thesis.

Assume still that k is a field, and suppose that $\mathcal{C}$ is a skeletally small, Hom-finite, Krull-Schmidt, triangulated k-category (with suspension functor Σ) that has Serre duality. Fix a *rigid* object $R \in \mathcal{C}$ (i.e. $\mathrm{Hom}_{\mathcal{C}}(R, \Sigma R) = 0$), and set $\Lambda_R := (\mathrm{End}_{\mathcal{C}} R)^{\mathrm{op}}$ and $\mathcal{X}_R = \mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(R, -))$. In §V.3 we determine what happens to certain arrows in the Auslander-Reiten quiver of $\mathcal{C}$ under the functor $\mathrm{Hom}_{\mathcal{C}}(R, -)\colon \mathcal{C} \to \Lambda_R-\mathsf{mod}$. In our direct approach, the subcategory $\mathcal{C}(R)$ of $\mathcal{C}$ consisting of objects Z admitting a triangle $R_0 \to R_1 \to Z \to \Sigma R_0$ in $\mathcal{C}$ with $R_i \in \mathsf{add}\, R$ has been useful. For example, if the domain of an irreducible morphism $f\colon X \to Y$ lies in $\mathcal{C}(R)$ we know precisely what happens.

Theorem. [Propositions V.3.22 and V.3.23] *Let X, Y be objects of $\mathcal{C}$ with $\mathsf{add}\, X \cap \mathcal{X}_R = 0 = \mathsf{add}\, Y \cap \mathcal{X}_R$ and $X \in \mathcal{C}(R)$. Suppose $f\colon X \to Y$ is an irreducible morphism in $\mathcal{C}$.*

(i) *If $Y \in \mathcal{C}(R)$, then $\mathrm{Hom}_{\mathcal{C}}(R, f)$ is irreducible in $\Lambda_R-\mathsf{mod}$.*

(ii) *If $Y \notin \mathcal{C}(R)$, then $\mathrm{Hom}_{\mathcal{C}}(R, f)$ is a section (so not irreducible) in $\Lambda_R-\mathsf{mod}$.*

On the other hand, if X is not in $\mathcal{C}(R)$, then it has been more difficult to understand how f behaves under $\mathrm{Hom}_{\mathcal{C}}(R, -)$. In this case, we have been inspired by work of

Buan and Marsh. It was shown in [BM12] (see also Proposition V.3.7) that there is, up to natural isomorphism, a commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(R,-)"] \arrow[d, "Q"'] & \Lambda_R - \mathsf{mod} \\ \mathcal{C}/[\mathcal{X}_R] \arrow[ur, "\mathrm{Hom}_{\mathcal{C}/[\mathcal{X}_R]}(R,-)"] \arrow[r, "L_{\mathcal{R}}"] & (\mathcal{C}/[\mathcal{X}_R])_{\mathcal{R}} \arrow[u, "\simeq"', "\mathrm{Hom}_{(\mathcal{C}/[\mathcal{X}_R])_{\mathcal{R}}}(R,-)"] \end{tikzcd} \tag{I.0.1}$$

where Q is the canonical quotient functor, and $L_{\mathcal{R}} \colon \mathcal{C}/[\mathcal{X}_R] \to (\mathcal{C}/[\mathcal{X}_R])_{\mathcal{R}}$ is the localisation functor associated to the Gabriel-Zisman localisation (see §II.5) of $\mathcal{C}/[\mathcal{X}_R]$ at the class $\mathcal{R}$ of its *regular* (i.e. simultaneously monic and epic) morphisms. See also [BM13] and [Bel13].

Therefore, in determining the irreducibility of $\mathrm{Hom}_{\mathcal{C}}(R, f)$, we can pass to the quotient $\mathcal{C}/[\mathcal{X}_R]$ first and use properties of $Q(f)$ to tell us more. An example of the results of this kind we obtain is the following.

Proposition. [Proposition V.3.30] *Let $X \notin \mathcal{C}(R)$, $Y \in \mathcal{C}(R)$ with $\mathsf{add}\, X \cap \mathcal{X}_R = 0 = \mathsf{add}\, Y \cap \mathcal{X}_R$. Let $f \colon X \to Y$ be an irreducible morphism in $\mathcal{C}$. If $Q(f)$ is right almost split and monic, then $\mathrm{Hom}_{\mathcal{C}}(R, f)$ is irreducible.*

That conditions on the image of f in $\mathcal{C}/[\mathcal{X}_R]$ can be imposed to determine if $\mathrm{Hom}_{\mathcal{C}}(R, f)$ is irreducible suggests that a better comprehension of the Auslander-Reiten theory of $\mathcal{C}/[\mathcal{X}_R]$ will be useful. As an application of one of the main results from Chapter III, we show that $\mathcal{C}/[\mathcal{X}_R]$ arises as a quasi-abelian heart of a twin cotorsion pair (in the sense of [Nak13]) on $\mathcal{C}$, and hence $\mathcal{C}/[\mathcal{X}_R]$ has sufficient structure in which to study Auslander-Reiten theory (see Chapter IV).

We now give some more details on how we show $\mathcal{C}/[\mathcal{X}_R]$ is a quasi-abelian category in Chapter III. A *cotorsion pair* on $\mathcal{C}$ is a pair $(\mathcal{U}, \mathcal{V})$ of full additive subcategories of $\mathcal{C}$ that are closed under isomorphisms and direct summands, satisfying $\mathrm{Ext}^1_{\mathcal{C}}(\mathcal{U}, \mathcal{V}) = 0$ and $\mathcal{C} = \mathcal{U} * \Sigma\mathcal{V}$ (see Definitions III.2.2 and III.2.3). An example of a cotorsion pair is $(\Sigma\mathcal{C}^{\leqslant 0}, \Sigma^{-1}\mathcal{C}^{\geqslant 0})$ whenever $(\mathcal{C}^{\leqslant 0}, \mathcal{C}^{\geqslant 0})$ is a *t-structure* (in the sense of [BBD82]) on $\mathcal{C}$ (see [Nak11, Exam. 2.5]). The heart of a t-structure is an abelian category and, analogously, Nakaoka extracted an abelian category from a triangulated category with a cotorsion pair, which is also known as the *heart* (see Definition III.2.21).

A *twin cotorsion pair* on $\mathcal{C}$ consists of two cotorsion pairs $(\mathcal{S}, \mathcal{T})$ and $(\mathcal{U}, \mathcal{V})$ on $\mathcal{C}$ satisfying $\mathcal{S} \subseteq \mathcal{U}$ (see Definition III.2.8). As for cotorsion pairs, Nakaoka defined the heart of a twin cotorsion pair as follows. First, we need the subcategories:

$$\mathcal{W} \coloneqq \mathcal{T} \cap \mathcal{U}, \quad \mathcal{C}^- \coloneqq \Sigma^{-1}\mathcal{S} * \mathcal{W}, \quad \mathcal{C}^+ \coloneqq \mathcal{W} * \Sigma\mathcal{V}$$

of $\mathcal{C}$. Then the *heart* of the twin cotorsion pair $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ on $\mathcal{C}$ is $\overline{\mathcal{H}} \coloneqq (\mathcal{C}^- \cap \mathcal{C}^+)/[\mathcal{W}]$ (see Definition III.2.18). In general, $\overline{\mathcal{H}}$ is not abelian but semi-abelian (see [Nak13, Thm. 5.4]).

A *semi-abelian* category is a *preabelian* category (i.e. every morphism has a kernel and a cokernel) in which the canonical morphism $\widetilde{f}\colon \operatorname{Coim} f \to \operatorname{Im} f$ is regular for every morphism f (see Definitions II.9.1 and II.9.12). A semi-abelian category is called *integral* if monomorphisms and epimorphisms are stable under both pullback and pushout (see Definition II.9.19). Nakaoka showed that the heart of a twin cotorsion pair is always semi-abelian (see [Nak13, Thm. 5.4]) and, under certain assumptions, is integral (see [Nak13, Thm. 6.3]). With $\mathcal{C}$ and R as above, the pair $\mathcal{P} = ((\mathsf{add}\,\Sigma R, \mathcal{X}_R), (\mathcal{X}_R, {\mathcal{X}_R}^{\perp_1}))$, where ${\mathcal{X}_R}^{\perp_1} = \operatorname{Ker}(\operatorname{Ext}^1_{\mathcal{C}}(\mathcal{X}_R, -))$, is a twin cotorsion pair on $\mathcal{C}$ with $\overline{\mathcal{H}} = \mathcal{C}/[\mathcal{X}_R]$ (see Lemma III.5.6). In particular, $\mathcal{P}$ satisfies the hypotheses of [Nak13, Thm. 6.3], so $\mathcal{C}/[\mathcal{X}_R]$ is integral and [BM12, Cor. 3.10] is recovered (see [Nak13, Exam. 6.10 (2)]).

In §III.3, we prove that the heart of a twin cotorsion pair $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$, satisfying a mild assumption, is quasi-abelian. A *quasi-abelian* category is a semi-abelian category in which kernels and cokernels are stable under both pullback and pushout (see Definition II.9.18). Classical examples of such a category include: any abelian category; the category of filtered modules over a filtered ring; the category of topological abelian groups; the category of Λ-lattices for an order Λ over a noetherian integral domain; and the torsion class and the torsion-free class of any torsion theory in an abelian category (see Example II.9.26); see [Rum01, §2] for more details. We show the following.

Theorem. [Theorem III.3.5] *Suppose* $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ *is a twin cotorsion pair on* $\mathcal{C}$. *If* $\mathcal{C}^{-}\subseteq\mathcal{C}^{+}$ *or* $\mathcal{C}^{+}\subseteq\mathcal{C}^{-}$, *then* $\overline{\mathcal{H}}$ *is quasi-abelian.*

The condition of this Theorem is satisfied if $\mathcal{U}\subseteq\mathcal{T}$ or $\mathcal{T}\subseteq\mathcal{U}$ (see Corollary III.3.6), and hence the heart $\mathcal{C}/[\mathcal{X}_R]$ of $\mathcal{P}=((\operatorname{\mathsf{add}}\Sigma R,\mathcal{X}_R),(\mathcal{X}_R,{\mathcal{X}_R}^{\perp_1}))$ is quasi-abelian (see Theorem III.5.7). As seen above in (I.0.1), one can localise $\mathcal{C}/[\mathcal{X}_R]$ to get to (a category equivalent to) the module category $\Lambda_R-\mathsf{mod}$, so knowing that $\mathcal{C}/[\mathcal{X}_R]$ is quasi-abelian is potentially of great value since quasi-abelian categories have enough structure in which to study Auslander-Reiten theory.

In Chapter IV we focus on Auslander-Reiten theory in categories more general than abelian ones and in Krull-Schmidt categories. For example, we show that any irreducible morphism in a semi-abelian category must be a proper monomorphism or a proper epimorphism or both (see Proposition IV.2.2). (Note that in an abelian category, this latter phenomenon cannot happen.) But in general it is not known how much Auslander-Reiten theory can be developed in semi-abelian, or even integral categories, because a lot of the arguments needed involve the pullback or pushout of kernels and cokernels. We show in §IV.2 that many of the general results proved in [AR77a] and [AR77b] by Auslander and Reiten for abelian categories still hold in quasi-abelian categories. Thus, these generalisations can be used to more fully understand $\mathcal{C}/[\mathcal{X}_R]$ and may have implications in comprehending the representation theory of partial cluster-tilted algebras.

Furthermore, since $\mathcal{C}$, and hence $\mathcal{C}/[\mathcal{X}_R]$, is a Hom-finite, Krull-Schmidt category, one can utilise techniques from a different perspective. Liu initiated the study of Auslander-Reiten theory in Krull-Schmidt categories that are Hom-finite over commutative artinian rings in [Liu10]. In particular, Liu introduced the notion of an *admissible* ideal (see Definition IV.3.10), an example of which is the ideal $[\mathcal{X}_R]$ (see Example IV.3.11). It was shown in [Liu10, §1] that if $\mathcal{A}$ is a Hom-finite, Krull-Schmidt category and $\mathcal{I}$ is an admissible ideal of $\mathcal{A}$, then, under suitable assumptions, irreducible morphisms (between indecomposables) and minimal left/right almost split morphisms behave well under the quotient functor $\mathcal{A} \to \mathcal{A}/\mathcal{I}$. We extend these results of Liu by establishing a body of Auslander-Reiten theory results for Krull-Schmidt categories that are not necessarily Hom-finite (see §IV.3). We also prove new observations in this setting, inspired by work of Auslander and Reiten. Moreover, in case $\mathcal{A}$ is quasi-abelian and Krull-Schmidt, we can also draw upon our work on Auslander-Reiten theory for quasi-abelian categories and are able to provide a characterisation result for *Auslander-Reiten sequences* (in the sense of Definition IV.3.6) in $\mathcal{A}$.

Theorem. [Theorem IV.3.19] *Let $\mathcal{A}$ be a skeletally small, quasi-abelian, Krull-Schmidt category. Let $\xi\colon X \xrightarrow{f} Y \xrightarrow{g} Z$ be a short exact sequence in $\mathcal{A}$. Then statements* (i)–(vi) *are equivalent.*

(i) *ξ is an Auslander-Reiten sequence.*

(ii) *$\mathrm{End}_{\mathcal{A}} X$ is local and g is right almost split.*

(iii) *$\mathrm{End}_{\mathcal{A}} Z$ is local and f is left almost split.*

(iv) *f is minimal left almost split.*

(v) *g is minimal right almost split.*

(vi) *f* and *g are irreducible.*

In particular, this characterisation applies to the category $\mathcal{C}/[\mathcal{X}_R]$.

Outline

In Chapter II we provide much of the background material needed to understand later parts of this thesis. At the beginning of Chapter II we state some assumptions we make throughout the whole thesis. In the rest of the chapter, we: set up our category-theoretic language and notation; recall notions from the theory of quivers; recall the Auslander-Reiten theory of a finite-dimensional algebra, and in particular recall the definition of the Auslander-Reiten translate; recall how one localises a category; provide the basics of triangulated categories that we use throughout; define the derived category, showing how its Auslander-Reiten theory is induced from the corresponding abelian category in case of a finite-dimensional hereditary algebra; and recall the definition of the cluster category and how its Auslander-Reiten theory is induced from the derived category. Lastly in Chapter II, we recall types of additive categories that have less structure than abelian categories in §II.9, and how one may define a first extension group for such categories in §II.10.

In Chapter III, our main result provides a criterion for the heart $\overline{\mathcal{H}}$ of a twin cotorsion pair $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ on a triangulated category $\mathcal{C}$ to be a quasi-abelian category. If $\mathcal{C}$ is also Krull-Schmidt and $\mathcal{T}=\mathcal{U}$, we relate $\overline{\mathcal{H}}$ to the heart of the cotorsion pair $(\mathcal{S},\mathcal{T})$ by localising $\overline{\mathcal{H}}$ at the class of its regular morphisms. Lastly in this chapter, we apply our results to a cluster category $\mathcal{C}$ with a rigid object R.

In Chapter IV, we focus on Auslander-Reiten theory for quasi-abelian and (not necessarily Hom-finite) Krull-Schmidt categories. For a skeletally small, quasi-abelian, Krull-Schmidt category, we provide equivalent criteria for a short exact sequence to be an Auslander-Reiten sequence.

In Chapter V, we first give a recipe to produce a partial cluster-tilted algebra of global dimension n for each $n \in \mathbb{N}$. We then consider what happens to an irreducible morphism $f\colon X \to Y$ in a Hom-finite, Krull-Schmidt, triangulated k-category $\mathcal{C}$ with Serre duality under the functor $\operatorname{Hom}_{\mathcal{C}}(R,-)$ for a rigid object

R in $\mathcal{C}$. If $X \in \mathcal{C}(R)$, then the irreducibility of $\mathrm{Hom}_{\mathcal{C}}(R, f)$ is determined by the membership of Y in $\mathcal{C}(R)$. If $X \notin \mathcal{C}(R)$, then we know the irreducibility of $\mathrm{Hom}_{\mathcal{C}}(R, f)$ in some situations.

The results in Chapters III, IV and V are new, except where otherwise indicated. The results of Chapter III have been published in [Sha19], and the results of Chapter IV have been published in [Sha20].

Chapter II

Preliminaries

We use this first chapter to recall the background material we will need in the rest of this thesis. It is also used to set up notation and define terminology that we will then use in Chapters III–V. The general assumptions and conventions we make are as follows.

(i) The set of natural numbers for us is $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.

(ii) We assume the Axiom of Choice.

(iii) Rings are assumed to be associative and with multiplicative identity (not necessarily non-zero).

(iv) Any module considered is unital, and typically we work with finitely generated left modules.

(v) For simplicity, we assume that any field is algebraically closed.

II.1 Category theory

We need the language of category theory throughout this thesis, so in this section we recall the fundamental notions. Our choice of notation and conventions usually follow that of [Alu09], which gives a great introduction to some of the theory presented. We also use [ML98], [Bor94a], [ASS06] and [Kra15].

II.1.1 The fundamentals

In this section, we recall what we mean by a category and some properties categories may have. We also recall the definition of a functor and some properties they may have.

Definition II.1.1. [Alu09, Def. I.3.1] A *category* $\mathcal{A}$ consists of: a class obj$(\mathcal{A})$, elements of which are called *objects*; for any $X, Y \in \mathrm{obj}(\mathcal{A})$ a class $\mathrm{Hom}_{\mathcal{A}}(X, Y)$, elements of which are called *morphisms*; and for any $X, Y, Z \in \mathrm{obj}(\mathcal{A})$ a binary operation $\circ := \circ_{(X,Y),(Y,Z)} \colon \mathrm{Hom}_{\mathcal{A}}(X, Y) \times \mathrm{Hom}_{\mathcal{A}}(Y, Z) \to \mathrm{Hom}_{\mathcal{A}}(X, Z)$ called *composition* mapping a pair (f, g) to $g \circ f$, such that:

(i) (associativity) for all $W, X, Y, Z \in \mathrm{obj}(\mathcal{A})$ and for all $f \in \mathrm{Hom}_{\mathcal{A}}(W, X)$, $g \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$, $h \in \mathrm{Hom}_{\mathcal{A}}(Y, Z)$, we have $h \circ (g \circ f) = (h \circ g) \circ f$; and

(ii) (identity) for each $X \in \mathrm{obj}(\mathcal{A})$ there exists a distinguished morphism $1_X \in \mathrm{Hom}_{\mathcal{A}}(X, X)$ such that for all $f \in \mathrm{Hom}_{\mathcal{A}}(W, X)$, $g \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$ we have $1_X \circ f = f$ and $g \circ 1_X = g$.

For the remainder of this section, $\mathcal{A}$ will denote an arbitrary category. The distinguished morphism 1_X for an object X of $\mathcal{A}$ is called the *identity* morphism. To simplify notation, we will often denote $g \circ f$ by gf, usually use $g \colon X \to Y$ and $g \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$ interchangeably, and just write $X \in \mathcal{A}$ to really mean X is an object in the category $\mathcal{A}$. Furthermore, we denote by $\mathrm{End}_{\mathcal{A}} X$ the collection $\mathrm{Hom}_{\mathcal{A}}(X, X)$ of morphisms $X \to X$.

Knowing when objects in a given category are essentially the same is often helpful, which gives us the following definition.

Definition II.1.2. [Alu09, Def. I.4.1] A morphism $f\colon X \to Y$ in $\mathcal{A}$ is an *isomorphism* if there exists $g\colon Y \to X$ in $\mathcal{A}$ such that $gf = 1_X$ and $fg = 1_Y$. That is, $f\colon X \to Y$ is an isomorphism if it has a two-sided inverse, and in this case we say that X and Y are *isomorphic* and denote this by $X \cong Y$.

We call the subclass of $\mathrm{obj}(\mathcal{A})$ consisting of all objects isomorphic to an object $X \in \mathcal{A}$ the *isoclass* of X. And we denote by $\mathrm{Aut}_{\mathcal{A}} X$ the class of all *automorphisms* of X, i.e. the subclass of $\mathrm{End}_{\mathcal{A}} X$ consisting of all isomorphisms.

From a category, other categories can immediately be considered. The opposite category gives us the language to formulate definitions and results in a concise way (as we will see later), hence we introduce it here.

Definition II.1.3. [Alu09, Exer. I.3.1] The *opposite category* $\mathcal{A}^{\mathrm{op}}$ of $\mathcal{A}$ is the category with the same objects as $\mathcal{A}$ and, for all $X, Y \in \mathcal{A}^{\mathrm{op}}$, we set

$$\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}(X, Y) := \{f^{\mathrm{op}} \mid f \in \mathrm{Hom}_{\mathcal{A}}(Y, X)\}.$$

The composition $\circ^{\mathrm{op}}$ in $\mathcal{A}^{\mathrm{op}}$ of morphisms $f^{\mathrm{op}}\colon X \to Y$ and $g^{\mathrm{op}}\colon Y \to Z$ is given by $g^{\mathrm{op}} \circ^{\mathrm{op}} f^{\mathrm{op}} := (f \circ g)^{\mathrm{op}}\colon X \to Z$.

We may also start to consider categories contained in $\mathcal{A}$.

Definition II.1.4. [Alu09, Exer. I.3.8] A *subcategory* $\mathcal{B}$ of a category $\mathcal{A}$ consists of:

(i) a subcollection $\mathrm{obj}(\mathcal{B})$ of $\mathrm{obj}(\mathcal{A})$; and

(ii) for all $X, Y \in \mathrm{obj}(\mathcal{B})$, a subcollection $\mathrm{Hom}_{\mathcal{B}}(X, Y)$ of $\mathrm{Hom}_{\mathcal{A}}(X, Y)$, such that $1_X \in \mathrm{Hom}_{\mathcal{B}}(X, X)$ for all X, and for any pair of composable morphisms $f \in \mathrm{Hom}_{\mathcal{B}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{B}}(Y, Z)$ we have $gf \in \mathrm{Hom}_{\mathcal{B}}(X, Z)$.

Subcategories come equipped with canonical inclusion functors. Let us recall first the definition of a covariant and a contravariant functor.

Definition II.1.5. [Alu09, Def. VIII.1.1] Let $\mathcal{A}, \mathcal{B}$ be categories. A *covariant functor* $\mathscr{F} \colon \mathcal{A} \to \mathcal{B}$ is an assignment of an object $\mathscr{F}(X) \in \operatorname{obj}(\mathcal{B})$ for each $X \in \operatorname{obj}(\mathcal{A})$, and for each pair of objects $X, Y \in \operatorname{obj}(\mathcal{A})$ an assignment of a morphism $\mathscr{F}_{X,Y}(f) \in \operatorname{Hom}_{\mathcal{B}}(\mathscr{F}(X), \mathscr{F}(Y))$ for each $f \in \operatorname{Hom}_{\mathcal{A}}(X, Y)$, such that $\mathscr{F}_{X,X}(1_X) = 1_{\mathscr{F}(X)}$ for any $X \in \operatorname{obj}(\mathcal{A})$ and that for any morphisms $f \in \operatorname{Hom}_{\mathcal{A}}(X, Y), g \in \operatorname{Hom}_{\mathcal{A}}(Y, Z)$ we have $\mathscr{F}_{X,Z}(gf) = \mathscr{F}_{Y,Z}(g)\mathscr{F}_{X,Y}(f)$.

Notation. When $\mathscr{F}$ is a functor, we will often write just $\mathscr{F}(f)$ instead of $\mathscr{F}_{X,Y}(f)$ in order to avoid multiple subscripts.

Definition II.1.6. [Alu09, Def. VIII.1.1] By a *contravariant functor* $\mathscr{F} \colon \mathcal{A} \to \mathcal{B}$ we mean a covariant functor $\mathscr{F} \colon \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$.

Now let us recall some properties that a functor may have, followed by some important examples.

Definition II.1.7. [Alu09, Def. VIII.1.6, Def. VIII.1.7] Let $\mathscr{F} \colon \mathcal{A} \to \mathcal{B}$ be a covariant functor. We say $\mathscr{F}$ is:

(i) *full* if the assignment $\operatorname{Hom}_{\mathcal{A}}(X, Y) \to \operatorname{Hom}_{\mathcal{B}}(\mathscr{F}(X), \mathscr{F}(Y))$ is surjective for all objects $X, Y \in \operatorname{obj}(\mathcal{A})$;

(ii) *faithful* if the assignment $\operatorname{Hom}_{\mathcal{A}}(X, Y) \to \operatorname{Hom}_{\mathcal{B}}(\mathscr{F}(X), \mathscr{F}(Y))$ is injective for all objects $X, Y \in \operatorname{obj}(\mathcal{A})$;

(iii) *fully faithful* if $\mathscr{F}$ is full and faithful; and

(iv) *essentially surjective*, or *dense*, if for each $Y \in \operatorname{obj}(\mathcal{B})$ there exists some $X \in \operatorname{obj}(\mathcal{A})$ such that $\mathscr{F}(X) \cong Y$ in $\mathcal{B}$.

Example II.1.8. [Alu09, Exer. I.3.8] Let $\mathcal{B}$ be a subcategory of $\mathcal{A}$. Consider the *canonical inclusion* functor $\mathscr{I} \colon \mathcal{B} \to \mathcal{A}$ given by $\mathscr{I}(X) = X$ for all $X \in \operatorname{obj}(\mathcal{B})$ and $\mathscr{I}(f) = f$ for any morphism f in $\mathcal{B}$. Note that $\mathscr{I}$ is always faithful. We call

$\mathcal{B}$ a *full* subcategory precisely when $\mathscr{I}$ is full, i.e. when we have $\mathrm{Hom}_{\mathcal{B}}(X, Y) = \mathrm{Hom}_{\mathcal{A}}(X, Y)$ for all $X, Y \in \mathcal{B}$.

Definition II.1.9. [ML98, p. 131], [Bor94a, Def. 1.2.3] We call $\mathcal{A}$ *locally small* if for all $X, Y \in \mathcal{A}$ we have that $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is a set. Furthermore, if $\mathcal{A}$ is locally small and $\mathrm{obj}(\mathcal{A})$ is also a set, then $\mathcal{A}$ is said to be *small*.

Example II.1.10. [Alu09, pp. 486–487] Fix an object X of $\mathcal{A}$. Assume for simplicity that $\mathcal{A}$ is locally small, and let Set denote the category of sets. For each $A \in \mathrm{obj}(\mathcal{A})$ consider the *set* $\mathrm{Hom}_{\mathcal{A}}(X, A)$. This assignment of objects extends to a covariant functor as follows. Let $f\colon A \to B$ be a morphism in $\mathcal{A}$. Then $\mathrm{Hom}_{\mathcal{A}}(X, f)\colon \mathrm{Hom}_{\mathcal{A}}(X, A) \to \mathrm{Hom}_{\mathcal{A}}(X, B)$ is given by $\mathrm{Hom}_{\mathcal{A}}(X, f)(g) := f \circ g$ for each $g \in \mathrm{Hom}_{\mathcal{A}}(X, A)$. Checking that identity morphisms are preserved and that this assignment respects composition is routine. Therefore, for each $X \in \mathcal{A}$ we have a covariant functor $\mathrm{Hom}_{\mathcal{A}}(X, -)$ from $\mathcal{A}$ to Set.

Dually, for each $X \in \mathcal{A}$, there is a contravariant functor $\mathrm{Hom}_{\mathcal{A}}(-, X)\colon \mathcal{A} \to \mathsf{Set}$ that sends an object $A \in \mathcal{A}$ to $\mathrm{Hom}_{\mathcal{A}}(A, X)$, and sends a morphism $f\colon A \to B$ to the function $\mathrm{Hom}_{\mathcal{A}}(f, X)\colon \mathrm{Hom}_{\mathcal{A}}(B, X) \to \mathrm{Hom}_{\mathcal{A}}(A, X)$ given by $\mathrm{Hom}_{\mathcal{A}}(f, X)(g) := g \circ f$ for each $g \in \mathrm{Hom}_{\mathcal{A}}(B, X)$.

Example II.1.11. [Bor94a, p. 6] For any category $\mathcal{A}$ we will denote the *identity functor* on $\mathcal{A}$ by $\mathbb{1}_{\mathcal{A}}$. This functor acts, as its name suggests, as the identity on objects and morphisms. That is, $\mathbb{1}_{\mathcal{A}}(X) = X$ for each $X \in \mathcal{A}$ and $\mathbb{1}_{\mathcal{A}}(f) = f$ for each morphism f in $\mathcal{A}$. It is easy to see that this is a covariant functor $\mathcal{A} \to \mathcal{A}$.

Lastly in this section we recall how functors may be composed and compared.

Definition II.1.12. [ML98, p. 14] Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be categories and suppose we have covariant functors $\mathscr{F}\colon \mathcal{A} \to \mathcal{B}$ and $\mathscr{G}\colon \mathcal{B} \to \mathcal{C}$. The *composite* functor $\mathscr{G} \circ \mathscr{F}\colon \mathcal{A} \to \mathcal{C}$ is the covariant functor that has $(\mathscr{G} \circ \mathscr{F})(A) = \mathscr{G}(\mathscr{F}(A))$ for each object A of $\mathcal{A}$, and $(\mathscr{G} \circ \mathscr{F})(f) = \mathscr{G}(\mathscr{F}(f))$ for each morphism f in $\mathcal{A}$.

It turns out that considering categories $\mathcal{A}, \mathcal{B}$ to be the same only if they are *isomorphic*, i.e. there are covariant functors $\mathscr{F} \colon \mathcal{A} \to \mathcal{B}$ and $\mathscr{G} \colon \mathcal{B} \to \mathcal{A}$ such that $\mathscr{G} \circ \mathscr{F} = \mathbb{1}_{\mathcal{A}}$ and $\mathscr{F} \circ \mathscr{G} = \mathbb{1}_{\mathcal{B}}$, is too strong a requirement. To define what we mean by an equivalence of categories, we need the following definition.

Definition II.1.13. [Alu09, Def. VIII.1.15] Let $\mathcal{A}, \mathcal{B}$ be categories and suppose we have covariant functors $\mathscr{F}, \mathscr{G} \colon \mathcal{A} \to \mathcal{B}$. A *natural transformation* $\nu \colon \mathscr{F} \rightsquigarrow \mathscr{G}$ is a collection of morphisms $\nu = \{\nu_X \colon \mathscr{F}(X) \to \mathscr{G}(X)\}_{X \in \mathcal{A}}$ in $\mathcal{B}$, such that for all $X, Y \in \mathcal{A}$ and every morphism $f \colon X \to Y$ in $\mathcal{A}$ the diagram

$$\begin{tikzcd} \mathscr{F}(X) \arrow[r, "\mathscr{F}(f)"] \arrow[d, "\nu_X"'] \arrow[dr, phantom, "\circ"] & \mathscr{F}(Y) \arrow[d, "\nu_Y"] \\ \mathscr{G}(X) \arrow[r, "\mathscr{G}(f)"] & \mathscr{G}(Y) \end{tikzcd}$$

commutes in $\mathcal{B}$. If $\nu_X \colon \mathscr{F}(X) \to \mathscr{G}(X)$ is an isomorphism in $\mathcal{B}$ for each $X \in \mathcal{A}$, then ν is called a *natural isomorphism*.

Definition II.1.14. [Alu09, Def. VIII. 1.7] Let $\mathcal{A}, \mathcal{B}$ be categories and suppose $\mathscr{F} \colon \mathcal{A} \to \mathcal{B}$ is a covariant functor. Then $\mathscr{F}$ is said to be an *equivalence of categories* if there exists a covariant functor $\mathscr{G} \colon \mathcal{B} \to \mathcal{A}$ and natural isomorphisms $\nu \colon \mathscr{G} \circ \mathscr{F} \rightsquigarrow \mathbb{1}_{\mathcal{A}}$ and $\mu \colon \mathscr{F} \circ \mathscr{G} \rightsquigarrow \mathbb{1}_{\mathcal{B}}$. In this case, we say that $\mathcal{A}$ and $\mathcal{B}$ are *equivalent* and denote this by $\mathcal{A} \simeq \mathcal{B}$.

The following result gives a useful criterion for a functor to be an equivalence.

Theorem II.1.15. [ML98, Thm. IV.4.1] *Let $\mathcal{A}, \mathcal{B}$ be categories and $\mathscr{F} \colon \mathcal{A} \to \mathcal{B}$ a covariant functor. The functor $\mathscr{F}$ is an equivalence of categories if and only if it is fully faithful and essentially surjective.*

II.1.2 Additive categories

For much of this thesis we deal with triangulated categories, which are additive categories with some more structure. In this section, we recall what is meant by an additive category, a k-category and an ideal. A basic example of an additive category is the category Ab of all abelian groups. We will see that this motivates many definitions in this section.

Throughout, let $\mathcal{A}$ be a category.

Definition II.1.16. [Alu09, Exer. IX.1.3] We call $\mathcal{A}$ *preadditive* if $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ has the structure of an abelian group for all X,Y in $\mathcal{A}$, such that the composition function $\mathrm{Hom}_{\mathcal{A}}(X,Y)\times \mathrm{Hom}_{\mathcal{A}}(Y,Z)\to \mathrm{Hom}_{\mathcal{A}}(X,Z)$ is $\mathbb{Z}$-bilinear for all objects X,Y,Z in $\mathcal{A}$.

Remark II.1.17. We observe here that whenever we impose that $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ is an abelian group for each X,Y in $\mathcal{A}$, we are implicitly saying that $\mathcal{A}$ is locally small (see Definition II.1.9). Indeed, (for us) a group must be a set.

An example of a preadditive category is Ab. In this category, a/the trivial group $\{e\}$ has a distinguished property. Namely, for each abelian group G there is a unique group homomorphism $\{e\}\to G$ and a unique group homomorphism $G\to\{e\}$. These properties have names in the literature.

Definition II.1.18. [Alu09, Def. I.5.1, p. 561] Let X be an object in $\mathcal{A}$. Then X is called *initial* if $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ is a singleton for all $Y\in\mathcal{A}$. That is, for each $Y\in\mathcal{A}$ there is one and only one morphism $X\to Y$ in $\mathcal{A}$.

Dually, X is said to be *final* if X is initial in the opposite category $\mathcal{A}^{\mathrm{op}}$. If X is both initial and final, then we call X a *zero object*.

Proposition II.1.19. [Alu09, Prop. I.5.4] *Zero objects, where they exist, are unique up to unique isomorphism.*

Thus, we usually speak of *the* zero object when such an object exists in a category. Zero objects are also known as *null* objects; see, for example [ML98].

We may form the product of abelian groups, which satisfies the universal property in the next definition.

Definition II.1.20. [Alu09, §I.5.4] For $X, Y \in \mathcal{A}$, a *product* of X and Y is an object $X \Pi Y$ in $\mathcal{A}$ endowed with morphisms $\pi_X \colon X \Pi Y \to X$ and $\pi_Y \colon X \Pi Y \to Y$ satisfying the following universal property: given $Z \in \mathcal{A}$ and morphisms $f_X \colon Z \to X$, $f_Y \colon Z \to Y$, there exists a *unique* morphism $\sigma \colon Z \to X \Pi Y$ such that the diagram

$$\begin{tikzcd}
 & & X \\
Z \arrow[urr, bend left, "f_X"] \arrow[r, dotted, "\exists!\sigma"] \arrow[drr, bend right, "f_Y"'] & X \Pi Y \arrow[ur, "\pi_X"'] \arrow[dr, "\pi_Y"] \arrow[u, phantom, "\circ"] \arrow[d, phantom, "\circ"] & \\
 & & Y
\end{tikzcd}$$

commutes in $\mathcal{A}$.

Definition II.1.21. [Alu09, §I.5.5] For $X, Y \in \mathcal{A}$, a *coproduct* of X and Y is an object $X \amalg Y$ in $\mathcal{A}$ endowed with morphisms $i_X \colon X \to X \amalg Y$ and $i_Y \colon Y \to X \amalg Y$, such that $X \amalg Y$ with ${i_X}^{\mathrm{op}}$ and ${i_Y}^{\mathrm{op}}$ is a product in $\mathcal{A}^{\mathrm{op}}$.

As with all objects defined by universal properties of this kind, products and coproducts (where they exist) are unique up to unique isomorphism. We are now in a position to define an additive category.

Definition II.1.22. [Alu09, Def. IX.1.1] We call $\mathcal{A}$ an *additive* category if $\mathcal{A}$:

(i) is preadditive;

(ii) has a zero object; and

(iii) has finite products and finite coproducts.

Remark II.1.23. Let $\mathcal{A}$ be an additive category. For objects X, Y of $\mathcal{A}$, we have that $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is an abelian group and we will denote the group operation by $+$, and the identity element by 0 if no confusion may arise or by $0_{X,Y}$ if it adds clarity.

We denote the zero object in $\mathcal{A}$ by 0. Then for $X \in \mathcal{A}$, the set $\mathrm{Hom}_{\mathcal{A}}(X, 0)$ is a singleton, and hence the trivial group with element 0. Similarly, $\mathrm{Hom}_{\mathcal{A}}(0, X) =$

$\{0\}$. Furthermore, it is easy to show that X is isomorphic to the zero object if and only if $1_X = 0 \in \operatorname{End}_{\mathcal{A}} X$ (see [ML98, §VIII.2, Prop. 1]).

In an additive category, we have the following phenomenon.

Proposition II.1.24. [ML98, Exer. VIII.2.1] *Let X, Y be objects in an additive category $\mathcal{A}$. Then the product $X \amalg Y$ and coproduct $X \amalg Y$ are isomorphic.*

This tells us that *finite* products and coproducts coincide (really, are isomorphic) in an additive category. Therefore, we denote this object by $X \oplus Y$ and call it the *direct sum*. This is also known as the *biproduct* in the literature; see, for example, [Büh10].

We will often deal with additive categories that have additional structure on their Hom-sets. The following notion captures this.

Definition II.1.25. [ASS06, Def. A.1.4] Let S be a commutative ring. An additive category $\mathcal{A}$ is an *S-category* if $\operatorname{Hom}_{\mathcal{A}}(X, Y)$ is an S-module for all objects X, Y in $\mathcal{A}$, and composition of morphisms is S-bilinear.

Example II.1.26. Let S be a commutative ring. The category $S-\mathsf{Mod}$ of all S-modules is an S-category. See [Alu09, §III] for more details.

Notice then that a category which is additive in the sense of Definition II.1.22 is just a $\mathbb{Z}$-category in the sense of Definition II.1.25.

Definition II.1.27. [Gre85, §0] A covariant functor $\mathscr{F} \colon \mathcal{A} \to \mathcal{B}$ between two S-categories is called *S-additive* if $\mathscr{F}_{X,Y} \colon \operatorname{Hom}_{\mathcal{A}}(X, Y) \to \operatorname{Hom}_{\mathcal{B}}(\mathscr{F}(X), \mathscr{F}(Y))$ is an S-module homomorphism for all $X, Y \in \mathcal{A}$.

An S-category is also known as an *S-linear* category and an S-additive functor as an *S-linear* functor in the literature; see, for example, [Kel08]. As for categories, a $\mathbb{Z}$-additive functor is just what is normally known as an *additive* functor.

The following property of an S-additive functor will be used without reference in the remainder of this thesis. It follows from [HS97, Prop. II.9.5].

Proposition II.1.28. *An S-additive functor between S-categories preserves the zero object and finite direct sums.*

Quotient categories will play an important role later and for this we first need the notion of an ideal of a category.

Definition II.1.29. [ASS06, Def. A.3.1] Suppose that $\mathcal{A}$ is an S-category for some commutative ring S. A *(two-sided) ideal* $\mathcal{I}$ of $\mathcal{A}$ is sub-bifunctor

$$\mathcal{I}(-,-) \subseteq \operatorname{Hom}_{\mathcal{A}}(-,-)\colon \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to S-\mathsf{Mod}$$

of the bifunctor $\operatorname{Hom}_{\mathcal{A}}(-,-)$, which assigns to each pair (X,Y) of objects of $\mathcal{A}$ an S-submodule $\mathcal{I}(X,Y)$ of $\operatorname{Hom}_{\mathcal{A}}(X,Y)$, such that:

(i) for all $f \in \mathcal{I}(X,Y)$ and for all $g \in \operatorname{Hom}_{\mathcal{A}}(Y,Z)$, we have $gf \in \mathcal{I}(X,Z)$; and

(ii) for all $f \in \mathcal{I}(X,Y)$ and for all $g \in \operatorname{Hom}_{\mathcal{A}}(W,X)$, we have $fg \in \mathcal{I}(W,Y)$.

Definition II.1.30. [ASS06, §A.3] If $\mathcal{A}$ is an S-category and $\mathcal{I}$ an ideal of $\mathcal{A}$, then we define the *quotient category* $\mathcal{A}/\mathcal{I}$ to be the category with objects $\operatorname{obj}(\mathcal{A}/\mathcal{I}) \coloneqq \operatorname{obj}(\mathcal{A})$ and morphism sets

$$\operatorname{Hom}_{\mathcal{A}/\mathcal{I}}(X,Y) \coloneqq \operatorname{Hom}_{\mathcal{A}}(X,Y)/\mathcal{I}(X,Y)$$

for each $X, Y \in \operatorname{obj}(\mathcal{A}/\mathcal{I})$.

It can be easily shown that if $\mathcal{A}$ is an S-category and $\mathcal{I}$ an ideal of $\mathcal{A}$, then the quotient $\mathcal{A}/\mathcal{I}$ is also an S-category. The quotient category $\mathcal{A}/\mathcal{I}$ comes equipped with a canonical quotient functor $Q_{\mathcal{I}}\colon \mathcal{A} \to \mathcal{A}/\mathcal{I}$, which is S-additive, is the identity on objects and takes a morphism $f \in \mathrm{Hom}_{\mathcal{A}}(X,Y)$ to its coset $f+\mathcal{I}(X,Y) \in \mathrm{Hom}_{\mathcal{A}/\mathcal{I}}(X,Y)$. Furthermore, $Q_{\mathcal{I}}$ satisfies the following universal property: for any S-additive functor $\mathscr{F}\colon \mathcal{A} \to \mathcal{B}$ such that $\mathscr{F}(i) = 0$ for all $X,Y \in \mathrm{obj}(\mathcal{A})$ and for all $i \in \mathcal{I}(X,Y)$, there exists a unique S-additive functor $\mathscr{G}\colon \mathcal{A}/\mathcal{I} \to \mathcal{B}$ with $\mathscr{F} = \mathscr{G} \circ Q_{\mathcal{I}}$.

Example II.1.31. Let $\mathcal{A}$ be an S-category. Suppose $\mathcal{B}$ is a subcategory of $\mathcal{A}$ that is closed under finite direct sums. For objects X,Y of $\mathcal{A}$, let $[\mathcal{B}](X,Y)$ denote the subset of $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ consisting of morphisms that factor through an object of $\mathcal{B}$. That is, $f \in [\mathcal{B}](X,Y)$ if and only if there exists $B \in \mathcal{B}$ and morphisms $g\colon X \to B$ and $h\colon B \to Y$ with $f = hg$. Then $[\mathcal{B}]$ is an ideal of $\mathcal{A}$. See, for example, [Pre09, p. 401].

Furthermore, if $\mathcal{B}$ is a full additive subcategory of $\mathcal{A}$ that is closed under isomorphisms and direct summands, then $[\mathcal{B}]$ coincides with the ideal generated by the collection of all identity morphisms 1_B such that $B \in \mathcal{B}$.

II.1.3 Abelian categories

As we saw above, Ab is an example of an additive category, but it is also the prototypical example of an abelian category. Furthermore, module categories are also examples of abelian categories. We recall some basics here.

Definition II.1.32. [Alu09, Def. IX.1.2] Let $\mathcal{A}$ be an additive category and let $f\colon X \to Y$ be a morphism in $\mathcal{A}$. A *kernel* of f is a morphism $i\colon K \to X$ in $\mathcal{A}$ such that $fi = 0$, satisfying the following universal property: for any $g\colon W \to X$ with $fg = 0$, there exists a unique morphism $\widehat{g}\colon W \to K$ such that $g = i\widehat{g}$.

Dually, a *cokernel* of f is a morphism $q\colon Y \to C$ such that $q^{\mathrm{op}}\colon C \to Y$ is a kernel of f^{op} in $\mathcal{A}^{\mathrm{op}}$.

If a kernel of f exists, then we will sometimes say f *admits* or *has* a kernel. Similarly, if a cokernel of f exists, then f is said to *admit* or *have* a cokernel.

If $\mathcal{A}$ is a category in which every morphism has a kernel in $\mathcal{A}$, then we will say $\mathcal{A}$ *has kernels*. Similarly, if every morphism in $\mathcal{A}$ has a cokernel in $\mathcal{A}$, then we say $\mathcal{A}$ *has cokernels*.

Remark II.1.33. Again, a kernel, if it exists, is unique up to a unique isomorphism, and so we will typically talk of *the* kernel. Similarly for cokernels.

Notation. If $f\colon X \to Y$ admits a kernel, then we will denote such a morphism by $\ker f\colon \operatorname{Ker} f \to X$. Similarly, the cokernel of f, if it exists, will be denoted by $\operatorname{coker} f\colon Y \to \operatorname{Coker} f$. Kernels and cokernels have key properties which can deduced from the uniqueness condition in their definition.

Definition II.1.34. [Alu09, Def. I.4.7, Def. I.4.8] Let $f\colon X \to Y$ be a morphism in $\mathcal{A}$. Then f is said to be *monic* or a *monomorphism* if, for any $g, h \in \operatorname{Hom}_{\mathcal{A}}(W, X)$, we have $f \circ g = f \circ h$ implies $g = h$.

Dually, we call f *epic* or an *epimorphism* if f^{op} is a monomorphism in $\mathcal{A}^{\mathrm{op}}$.

It is then easy to show the following.

Proposition II.1.35. [ML98, p. 191] *If $\mathcal{A}$ is an additive category, then any kernel $i\colon K \to X$ is a monomorphism and any cokernel $q\colon Y \to C$ is an epimorphism.*

Notation. We will often use the arrow $\hookrightarrow$ to indicate that a morphism is a monomorphism, and the arrow $\twoheadrightarrow$ to indicate an epimorphism.

Note that in general it is not true that every monomorphism is a kernel or that every epimorphism if a cokernel. It is true in the category Ab, however, which motivates the main definition of this section.

Definition II.1.36. [Alu09, Def. IX.1.6] A category $\mathcal{A}$ is said to be *abelian* if:

(i) $\mathcal{A}$ is additive;

(ii) $\mathcal{A}$ has kernels and cokernels; and

(iii) in $\mathcal{A}$ every monomorphism is the kernel of some morphism, and every epimorphism is the cokernel of some morphism.

Example II.1.37. Let S be a (not necessarily commutative) ring. Then the category $S-\mathsf{Mod}$ of all left S-modules is abelian. Moreover, if S is left noetherian, then the full subcategory $S-\mathsf{mod}$ of $S-\mathsf{Mod}$ consisting of all finitely generated left S-modules is also an abelian category. See [Alu09, §IX.1] for more details.

We note also that if S is a finite-dimensional k-algebra for a field k, then a left S-module is finitely generated (as an S-module) if and only if it is finite-dimensional over k. Thus, $S-\mathsf{mod}$ is, equivalently, the category of all finite-dimensional left S-modules.

There is a canonical *exact* structure (see [Büh10]) that one can put on an abelian category. We will only need the definition of a short exact sequence in an abelian category. For this we need the following.

Definition II.1.38. [Pop73, p. 23] Given a morphism $f\colon X \to Y$ in an additive category, the *coimage* $\operatorname{coim} f\colon X \to \operatorname{Coim} f$, if it exists, is the cokernel $\operatorname{coker}(\ker f)$ of the kernel of f. Dually, the *image* $\operatorname{im} f\colon \operatorname{Im} f \to Y$ is the kernel $\ker(\operatorname{coker} f)$ of the cokernel of f.

Definition II.1.39. [Alu09, §III.7.1] Let $\mathcal{A}$ be an abelian category. A sequence

$$\cdots \longrightarrow X^{i-1} \xrightarrow{f^{i-1}} X^{i} \xrightarrow{f^{i}} X^{i+1} \longrightarrow \cdots$$

of composable morphisms in $\mathcal{A}$ is said to be *exact at* X^i if $\operatorname{im} f^{i-1} = \ker f^i$. The sequence is *exact* if it is exact at X^i for all i.

An exact sequence of the form $0 \to X \to Y \to Z \to 0$ is known as a *short exact sequence*.

Definition II.1.40. [Alu09, Exam. VIII.1.18] Let $\mathscr{F}\colon \mathcal{A} \to \mathcal{B}$ be a covariant additive functor between two abelian categories. We call $\mathscr{F}$ *left exact* if for each exact sequence $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$, we have that $0 \longrightarrow \mathscr{F}(X) \xrightarrow{\mathscr{F}(f)} \mathscr{F}(Y) \xrightarrow{\mathscr{F}(g)} \mathscr{F}(Z)$ is exact in $\mathcal{B}$.

Similarly, $\mathscr{F}$ is said to be *right exact* if for each exact sequence $X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ in $\mathcal{A}$, we have that $\mathscr{F}(X) \xrightarrow{\mathscr{F}(f)} \mathscr{F}(Y) \xrightarrow{\mathscr{F}(g)} \mathscr{F}(Z) \longrightarrow 0$ is exact in $\mathcal{B}$.

Lastly, $\mathscr{F}$ is called *exact* if it is both left exact and right exact.

Example II.1.41. Suppose $\mathcal{A}$ is an abelian category and fix an object $X \in \mathcal{A}$. The functors $\operatorname{Hom}_{\mathcal{A}}(X,-)\colon \mathcal{A} \to \mathsf{Ab}$ and $\operatorname{Hom}_{\mathcal{A}}(-,X)\colon \mathcal{A}^{\mathrm{op}} \to \mathsf{Ab}$ are both left exact.

II.1.4 Krull-Schmidt categories

In Chapter IV we will develop some theory for Krull-Schmidt categories (see Definition II.1.45). In such a category, every object is isomorphic to a finite direct sum of objects with local endomorphism rings.

Definition II.1.42. [AF92, p. 144] A ring S is said to be *local* if the sum of any two non-units is again a non-unit.

There are some useful criteria for a ring to be local. For this we need to recall the following definition.

Definition II.1.43. [Jac45, Def. 2, Cor. 2 to Thm. 18] Let S be a ring. The *Jacobson radical* $J(S)$ of S is defined to be the (two-sided) ideal that is the intersection of all the maximal right ideals of S.

From [AF92, Prop. 15.15], we obtain the following characterisation.

Proposition II.1.44. *Let S be a ring. Then the following are equivalent.*

(i) *S is a local ring.*

(ii) *The Jacobson radical $J(S)$ is the unique maximal ideal of S.*

(iii) *For any element $x \in S$, either x is a unit or $1 - x$ is a unit in S.*

(iv) *The set J' of all elements in S that do not have a left inverse is closed under addition in S.*

In this case, $J(S) = J'$.

Let us now give the main definition of this section.

Definition II.1.45. [Kra15, p. 544] An additive category $\mathcal{A}$ is called *Krull-Schmidt* if for each object X of $\mathcal{A}$ there exists a finite direct sum decomposition $X = X_1 \oplus \cdots \oplus X_n$, where $\operatorname{End}_{\mathcal{A}}(X_i)$ is a local ring for all $1 \leqslant i \leqslant n$.

Remark II.1.46. As noted in [Kra15], in a Krull-Schmidt category, the uniqueness (up to permutation and isomorphism) of a decomposition $X = X_1 \oplus \cdots \oplus X_n$ of an object X, where $\operatorname{End}_{\mathcal{A}}(X_i)$ is a local ring for all $1 \leqslant i \leqslant n$, follows from the existence and uniqueness of projective covers over semi-perfect rings. See [Kra15, Thm. 4.2].

Definition II.1.47. [Kra15, p. 537] An object X in an additive category $\mathcal{A}$ is called *indecomposable* if $X \neq 0$, and $X \cong X_1 \oplus X_2$ implies $X_1 = 0$ or $X_2 = 0$.

It is well-known that an object in an additive category with local endomorphism ring is indecomposable; see, for example, [Har74, §1.4]. We end this section by noting that the converse holds in a Krull-Schmidt category, and this readily follows from the definition.

Proposition II.1.48. *In a Krull-Schmidt category $\mathcal{A}$, an object $X \in \mathcal{A}$ is indecomposable if and only if $\operatorname{End}_{\mathcal{A}} X$ is local.*

II.2 Quivers, path algebras and translations

Any finite-dimensional algebra is the quotient of a path algebra by some ideal, so the theory of quivers has become a core part of the study of finite-dimensional algebras. In this section, we recall the definitions of a quiver, a representation of a quiver, a path algebra and a translation quiver. For more detailed introductions to these concepts, we direct the reader to §1, §2 and §3.4 of [Sch14], and §II.1, §II.2, §III.1 and §IV.4 of [ASS06]. See also [Ben98], [Rie80] and [Hap88].

For this section (including its subsections), let k be a field.

II.2.1 Quivers

A quiver is nothing other than a directed graph, but its use in representation theory is of great value. Quivers allow us to more easily imagine how modules of the corresponding path algebra look. Indeed, the more detailed examples we give in this thesis will be given through the language of quivers and paths algebras. We also need this theory in order to define the Auslander-Reiten quiver of a category, which gives a pictorial description of the category.

Definition II.2.1. [Sch14, Def. 1.1] A *quiver* $Q = (Q_0, Q_1, s, t)$ consists of a set Q_0 of *vertices*, a set Q_1 of *arrows*, a *source* mapping $s\colon Q_1 \to Q_0$ that maps an arrow to its start vertex, and a *target* mapping $t\colon Q_1 \to Q_0$ that maps an arrow to its end vertex.

Suppose for the remainder of this section that $Q = (Q_0, Q_1, s, t)$ is a quiver. An arrow α in Q will sometimes be denoted by $s(\alpha) \xrightarrow{\alpha} t(\alpha)$.

Definition II.2.2. [Sch14, Def. 2.1] For $n \in \mathbb{N}$, a *path of length* n is a sequence $p = \alpha_n \cdots \alpha_2\alpha_1$ of n arrows in Q_1, such that $t(\alpha_i) = s(\alpha_{i+1})$ for each $i = 1, \dots, n-1$. (Note that we read paths from right to left.) We will also abuse notation by defining the source of a path $p = \alpha_n \cdots \alpha_1$ to be $s(p) \coloneqq s(\alpha_1)$ and the target to be $t(p) \coloneqq t(\alpha_n)$. A path of length 0 is just a constant path at a given vertex.

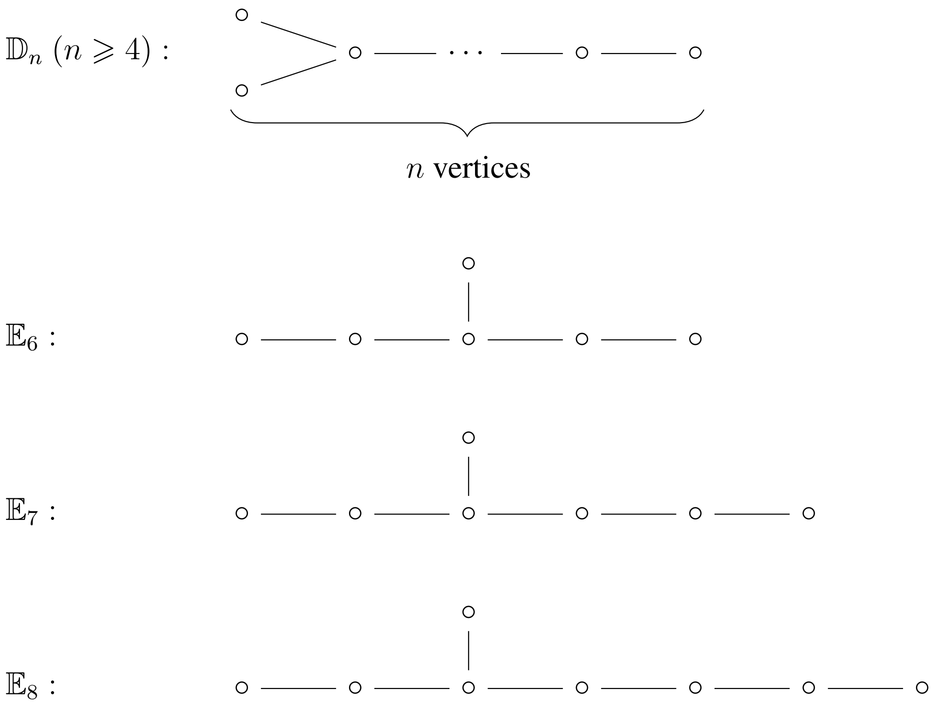
We will often consider quivers satisfying nice properties. We recall them here.

Definition II.2.3. [Sch14, Exam. 2.2], [ASS06, p. 43] An *(oriented) n-cycle* in Q is a path $\alpha_n \cdots \alpha_1$ of length $n \geqslant 1$ with $t(\alpha_n) = s(\alpha_1)$. A 1-cycle is simply called a *loop*. We call Q *acyclic* if it contains no n-cycles for all $n \geqslant 1$.

Definition II.2.4. [Sch14, p. 4] We call Q *finite* if both Q_0 and Q_1 are finite sets.

Definition II.2.5. [ASS06, p. 42] We call Q *connected* if its underlying undirected graph $\overline{Q}$ is connected, i.e. there is an unoriented path between any two vertices in $\overline{Q}$.

Example II.2.6. [ASS06, p. 252] Below is a collection of (undirected) graphs known as the *Dynkin* graphs. Note that any orientation of these graphs gives a finite, connected and acyclic quiver. That is, any *Dynkin-type* quiver, i.e. a quiver that has underlying graph one of Dynkin graphs, is finite, connected and acyclic.



We recall what is meant by a morphism of quivers as we will need this notion later in order to say how one Auslander-Reiten quiver is induced from another.

Definition II.2.7. [Ben98, Def. 4.15.1] Let $Q = (Q_0, Q_1, s, t)$ and $Q' = (Q'_0, Q'_1, s', t')$ be quivers. A *morphism of quivers* $f = (f_0, f_1)\colon Q \to Q'$ *from* Q *to* Q' is a pair of assignments $f_0\colon Q_0 \to Q'_0$ and $f_1\colon Q_1 \to Q'_1$, such that for any arrow $\alpha \in Q_1$ we have $s'(f_1(\alpha)) = f_0(s(\alpha))$ and $t'(f_1(\alpha)) = f_0(t(\alpha))$.

We call $f = (f_0, f_1)\colon Q \to Q'$ an *isomorphism of quivers* if there exists a morphism of quivers $g = (g_0, g_1)\colon Q' \to Q$ such that $f \circ g$ is the identity morphism of quivers on Q' and $g \circ f$ is the identity on Q.

Definition II.2.8. [Ben98, Def. 4.15.1] Suppose $Q = (Q_0, Q_1, s, t)$ is a quiver. For a vertex $v \in Q_0$, we define two subsets of Q_0:

$$v^+ := \left\{ w \in Q_0 \;\middle|\; \text{there exists an arrow } v \xrightarrow{\alpha} w \right\}$$

and

$$v^- := \left\{ u \in Q_0 \;\middle|\; \text{there exists an arrow } u \xrightarrow{\alpha} v \right\}.$$

That is, v^+ is the collection of all immediate successors of v and v^- is the collection of all immediate predecessors of v.

If v^+ and v^- are both finite for all $v \in Q_0$, then Q is called *locally finite*.

Now suppose $Q = (Q_0, Q_1, s, t)$ is a locally finite quiver and $\varphi\colon Q \to Q$ is an automorphism of quivers of Q. We describe how φ induces an equivalence relation on Q_0 and a quotient quiver Q/φ of Q. We were unable to find a reference containing this well-known construction, so we include the details here.

We will say that vertices $u, v \in Q_0$ are *equivalent*, denoted $u \sim_\varphi v$, if $v = \varphi^n(u)$ for some $n \in \mathbb{Z}$. This is an equivalence relation on Q_0 as φ is an automorphism. The quiver Q/φ will have as its vertex set the collection $(Q/\varphi)_0$ of $\sim_\varphi$-equivalence classes $[u]_{\sim_\varphi}$.

Fix a vertex $X \in (Q/\varphi)_0$ and suppose $v \in Q_0$ is a representative of the equivalence class X. As Q is locally finite, we have that v^+ is finite, so

$$v^+ = \{w_1^v, \dots, w_{m_v}^v\}$$

for some $m_v \in \mathbb{N}$. For $w_i^v \in v^+$, suppose there are $n_i^v \in \mathbb{N}$ arrows $v \to w_i^v$ labelled by $\alpha_{i,1}^v, \dots, \alpha_{i,n_i^v}^v$. Let Y be a vertex in $(Q/\varphi)_0$. The number $n_{X,Y}$ of arrows $X \to Y$ is given by

$$n_{X,Y} = \sum_{w_i^v \in Y} n_i^v,$$

and they are canonically labelled by elements in the set

$$\bigcup_{w_i^v \in Y} \{\alpha_{i,1}^v, \ldots, \alpha_{i,n_i^v}^v\}.$$

One can check that this is independent of the choice of representative v of the equivalence class X since φ is an automorphism. Therefore, for each $\sim_\varphi$-equivalence class $X \in (Q/\varphi)_0$ we fix a representative $v_X \in X$.

The *quotient quiver Q/φ of Q by φ* is then the graph with vertex set $(Q/\varphi)_0$, and arrows $X \to Y$ as determined and labelled above using the set $\{v_X \mid X \in (Q/\varphi)_0\}$ of fixed representatives.

II.2.2 Path algebras

In this section we recall how one can get to a k-algebra from a quiver.

Definition II.2.9. [ASS06, Def. II.1.2] Let Q be a quiver. The *path algebra kQ* of Q is the k-algebra whose underlying k-vector space has basis all the possible paths of length $n \geqslant 0$ in Q, with the following multiplication on basis elements extended k-linearly. Given paths $p = \alpha_n \cdots \alpha_1$ and $q = \beta_m \cdots \beta_1$ in Q, we set

$$q \cdot p = \begin{cases} \beta_m \cdots \beta_1 \alpha_n \cdots \alpha_1 & \text{if } t(\alpha_n) = s(\beta_1) \\ 0 & \text{otherwise.} \end{cases}$$

Now we will see why we isolated certain properties of quivers in §II.2.1.

Proposition II.2.10. [ASS06, Lem. II.1.4] *Let Q be a quiver. Then*

(i) *kQ is an associative algebra;*

(ii) *kQ has a multiplicative identity if and only if Q_0 is finite; and*

(iii) *kQ is finite-dimensional if and only if Q is finite and acyclic.*

Certain quotients of path algebras are important in the theory (see [ASS06, Thm. II.3.7]) and to define these we need some more definitions first.

Definition II.2.11. [ASS06, Def. II.1.9] Let Q be a finite quiver. We denote by kQ_+ the two-sided ideal of kQ, which is called the *arrow ideal*, generated by all paths of length at least 1 in Q.

Definition II.2.12. [ASS06, Def. II.2.1] Let Q be a finite quiver with corresponding arrow ideal $kQ_+ \trianglelefteq kQ$. A two-sided ideal $I \trianglelefteq kQ$ is called *admissible* if there exists an integer $n \geqslant 2$ such that $(kQ_+)^n \subseteq I \subseteq (kQ_+)^2$.

If I is admissible in kQ, then we call the pair (Q, I) a *bound quiver* and the quotient k-algebra kQ/I is known as a *bound quiver algebra*.

If Q is a finite quiver and I an admissible ideal of kQ, then [ASS06, Cor. II.2.9] tells us that I is generated by a finite set of elements of a particular form—these are known as relations.

Definition II.2.13. [ASS06, Def. II.2.3] Let Q be a quiver. A *relation in* Q is a k-linear combination $\rho = \sum_{i=1}^{r} \lambda_i p_i \in kQ$ of paths p_i in Q, such that each p_i has length at least 2, and $p_1, \ldots, p_r$ all have the common start vertex $s(p_1) = \cdots = s(p_r)$ and the common end vertex $t(p_1) = \cdots = t(p_r)$.

II.2.3 Quiver representations

Let Q denote a finite quiver in this section. In §II.2.2 we saw how one defines an algebra kQ from Q. It is then natural to study the category of (finite-dimensional) left kQ-modules. In this section, we will see how the category of (finite-dimensional) quiver representations of Q can be utilised in this study.

Definition II.2.14. [Sch14, Def. 1.2] A *(quiver) representation* $M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ of Q is a collection $\{M_i\}_{i \in Q_0}$ of k-vector spaces, together with a collection $\{\varphi_\alpha \colon M_{s(\alpha)} \to M_{t(\alpha)}\}_{\alpha \in Q_1}$ of linear maps.

A quiver representation $M = (M_i, \varphi_\alpha)_{i\in Q_0, \alpha\in Q_1}$ is said to be *finite-dimensional* if M_i is a finite-dimensional k-vector space for each $i \in Q_0$.

Notation. We will often denote a quiver representation $M = (M_i, \varphi_\alpha)_{i\in Q_0,\alpha\in Q_1}$ of Q by (M_i, φ_α), or even just M, and assume it is understood that the index i, respectively, α, runs over the vertex set Q_0, respectively, the arrow set Q_1.

Definition II.2.15. [Sch14, Def. 1.3] Suppose $M = (M_i, \varphi_\alpha)$ and $N = (N_i, \psi_\alpha)$ are representations of Q. A *morphism* $(f_i)_{i\in Q_0}$ *of quiver representations* is a collection $\{f_i : M_i \to N_i\}_{i\in Q_0}$ of k-linear maps, such that the "arrow-square"

$$\begin{tikzcd} M_{s(\alpha)} \arrow[r, "\varphi_\alpha"] \arrow[d, "f_{s(\alpha)}"'] \arrow[dr, phantom, "\circ"] & M_{t(\alpha)} \arrow[d, "f_{t(\alpha)}"] \\ N_{s(\alpha)} \arrow[r, "\psi_\alpha"'] & N_{t(\alpha)} \end{tikzcd}$$

commutes for each arrow $\alpha \in Q_1$.

An *isomorphism* of quiver representations is a morphism $(f_i)_{i\in Q_0}$ in which each f_i is an isomorphism of k-vector spaces.

For a quiver representation $M = (M_i, \varphi_\alpha)$ of Q, it is clear that we have an identity morphism $(1_{M_i})_{i\in Q_0}$, which is the collection $\{1_{M_i} : M_i \to M_i\}_{i\in Q_0}$ of identity linear maps. Composition of morphisms of quiver representations is also clear.

We denote by $\operatorname{Rep}_k Q$ the category that has $\operatorname{obj}(\operatorname{Rep}_k Q)$ equal to the class of all quiver representations of Q, and for any two quiver representations M, N of Q the morphism set $\operatorname{Hom}_{\operatorname{Rep}_k Q}(M, N)$ is the collection of all morphisms of quiver representations $M \to N$. The full subcategory of $\operatorname{Rep}_k Q$ consisting of all finite-dimensional representations of Q is denoted by $\operatorname{rep}_k Q$.

One may also define representations for quivers with relations and we recall the definition here.

Definition II.2.16. [ASS06, Def. III.1.4] Let Q be a finite quiver and suppose $p = \alpha_n \cdots \alpha_1$ is a non-constant path in Q. Let $M = (M_i, \varphi_\alpha)$ be a representation of Q. The *evaluation of* M *on* p is the composition $\varphi_p := \varphi_{\alpha_n} \circ \cdots \varphi_{\alpha_1}$ of k-linear maps.

Definition II.2.17. [ASS06, §III.1] Let Q be a finite quiver and let $M = (M_i, \varphi_\alpha)$ be a quiver representation of Q. Suppose I is an admissible ideal of kQ. We say M is *bound by* I if, for each relation $\rho = \sum_{i=1}^{r} \lambda_i p_i$ in I, we have that M evaluated on ρ is the zero map, i.e. $\varphi_\rho = \sum_{i=1}^{r} \lambda_i \varphi_{p_i} = 0$.

Suppose Q is a finite quiver and I an admissible ideal of kQ. In the terminology above, we denote by $\mathrm{Rep}_k(Q, I)$ the full subcategory of $\mathrm{Rep}_k Q$ consisting of all quiver representations of Q that are bound by I. Similarly, $\mathrm{rep}_k(Q, I)$ denotes the full subcategory of $\mathrm{Rep}_k(Q, I)$ consisting of all finite-dimensional quiver representations of Q that are bound by I. See [ASS06, §III.1] for more details.

The following makes precise the relationship between the category $kQ/I - \mathsf{Mod}$ (respectively, $kQ/I - \mathsf{mod}$) and the category $\mathrm{Rep}_k(Q, I)$ (respectively, $\mathrm{rep}_k(Q, I)$).

Theorem II.2.18. [ASS06, Thm. III.1.6] *Let k be a field. Let Q be a finite, connected quiver and let I be an admissible ideal of kQ. There is a k-additive equivalence $kQ/I - \mathsf{Mod} \simeq \mathrm{Rep}_k(Q, I)$ of categories, which restricts to a k-additive equivalence $kQ/I - \mathsf{mod} \simeq \mathrm{rep}_k(Q, I)$.*

With $I = 0$ in the above, we get the following immediate corollary.

Corollary II.2.19. [ASS06, Cor. III.1.7] *Let k be a field and let Q be a finite, connected, acyclic quiver. There is a k-additive equivalence $kQ - \mathsf{Mod} \simeq \mathrm{Rep}_k Q$ of categories, which restricts to a k-additive equivalence $kQ - \mathsf{mod} \simeq \mathrm{rep}_k Q$.*

Therefore, it follows from the above result that the categories $\mathrm{Rep}_k(Q, I)$ and $\mathrm{rep}_k(Q, I)$ are abelian k-categories, since both $kQ/I - \mathsf{Mod}$ and $kQ/I - \mathsf{mod}$ are abelian k-categories themselves as kQ/I is a noetherian ring (see also [ASS06, Lem. III.1.3]). In fact, $\operatorname{rep}_k Q$ has more structure. The next result follows from Theorem II.4.4, which is a more general result; see also [Sch14, Thm. 1.2].

Theorem II.2.20. *For a finite acyclic quiver Q the category $\operatorname{rep}_k Q$ is a Krull-Schmidt category.*

Hence, using Corollary II.2.19, we see that the category $kQ-\mathsf{mod}$ is also Krull-Schmidt for a finite, connected, acyclic quiver Q.

Many examples in this thesis come from path algebras, and we choose to describe their modules by the corresponding quiver representations. Given a finite, acyclic quiver Q, there are three particular quiver representations one has at each vertex $i \in Q_0$, which we recall now.

Definition II.2.21. [Sch14, Def. 2.2] Let Q be a finite, acyclic quiver. Fix a vertex i of Q.

(i) The *simple representation* $S_i = ((S_i)_j, \varphi_\alpha)_{j\in Q_0, \alpha\in Q_1}$ *at* i is the quiver representation of Q with $(S_i)_j = k$ if $i = j$ and $(S_i)_j = 0$ if $i \neq j$, and with $\varphi_\alpha = 0$ for each arrow $\alpha \in Q_1$.

(ii) For each $j \in Q_0$, define $(P_i)_j$ to be the k-vector space with basis all the possible paths from i to j. For an arrow α in Q and a basis element $\beta_n \cdots \beta_1$ of $(P_i)_{s(\alpha)}$ (where $\beta_l \in Q_1$ for all $1 \leqslant l \leqslant n$), i.e. a path from i to $s(\alpha)$, we set $\psi_\alpha(\beta_n \cdots \beta_1) := \alpha\beta_n \cdots \beta_1 \in (P_i)_{t(\alpha)}$. Extending k-linearly gives us a well-defined linear map $\psi_\alpha \colon (P_i)_{s(\alpha)} \to (P_i)_{t(\alpha)}$. The representation $P_i = ((P_i)_j, \psi_\alpha)$ is called the *projective representation at* i.

(iii) For each $j \in Q_0$, define $(I_i)_j$ to be the k-vector space with basis all the possible paths from j to i. For an arrow α in Q and a basis element $\gamma_m \cdots \gamma_1$ of $(I_i)_{s(\alpha)}$ (where $\gamma_l \in Q_1$ for all $1 \leqslant l \leqslant m$), i.e. a path from $s(\alpha)$ to i, we set

$$\rho_\alpha(\gamma_m \cdots \gamma_1) := \begin{cases} \gamma_m \cdots \gamma_2 & \text{if } \gamma_1 = \alpha \\ 0 & \text{else.} \end{cases}$$

Extending k-linearly gives us a well-defined linear map $\rho_\alpha \colon (I_i)_{s(\alpha)} \to (I_i)_{t(\alpha)}$. The representation $I_i = ((I_i)_j, \rho_\alpha)$ is called the *injective representation at* i.

We will use details from the following example without reference in the rest of the thesis.

Example II.2.22. Let k be a field. Consider the Dynkin-type $\mathbb{A}_3$ quiver $Q\colon\ 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$. By Corollary II.2.19, we know the module category $kQ-\mathsf{mod}$ of the path algebra kQ is equivalent to the category $\operatorname{rep}_k Q$ of quiver representations of Q. Up to isomorphism, there are six indecomposable left kQ-modules, which we describe below using their quiver representation counterparts. See [Sch14, p. 58, Exam. 2.8].

$$\begin{aligned}
P_1 = I_3 :&\quad k \xrightarrow{1} k \xrightarrow{1} k \\
P_2 :&\quad 0 \xrightarrow{0} k \xrightarrow{1} k \\
P_3 = S_3 :&\quad 0 \xrightarrow{0} 0 \xrightarrow{0} k \\
I_1 = S_1 :&\quad k \xrightarrow{0} 0 \xrightarrow{0} 0 \\
I_2 :&\quad k \xrightarrow{1} k \xrightarrow{0} 0 \\
S_2 :&\quad 0 \xrightarrow{0} k \xrightarrow{0} 0
\end{aligned}$$

The projective kQ-modules in the list above are P_1, P_2, P_3, the injectives are I_1, I_2, I_3 and the simples are S_1, S_2, S_3.

There is shorthand notation for describing the quiver representations above that is common in literature. Consider the representation $P_2 :\quad 0 \xrightarrow{0} k \xrightarrow{1} k$. We will denote this representation by $\substack{2\\3}$. This notation means that the representation has one copy of k at vertices 2 and 3, and no copies of k at vertex 1. Furthermore, the stacking of the numbers indicates to us the linear maps between the vector spaces at each vertex. As 2 is stacked on top of 3 in the notation $\substack{2\\3}$, this means there is a non-zero linear map (the identity in this case) from the vector space at vertex 2 to the vector space at vertex 3. As another example, the representation $0 \xrightarrow{0} k \xrightarrow{0} k$ is denoted by $2 \oplus 3$, because a copy of k appears at vertices 2 and 3, but there is a zero map between those copies of k. That is, the representation $0 \xrightarrow{0} k \xrightarrow{0} k$ is decomposable and it is the direct sum $S_2 \oplus S_3$.

One can also ask which morphisms exist between the six indecomposables listed above. For example, there is a monomorphism $a\colon 3 \to \begin{smallmatrix}2\\3\end{smallmatrix}$ and an epimorphism $b\colon \begin{smallmatrix}2\\3\end{smallmatrix} \to 2$:

$$\begin{tikzcd}
3 \arrow[d, "a"'] & & 0 \arrow[r, "0"] \arrow[d, "0"] \arrow[dr, phantom, "\alpha\text{-square}"] & 0 \arrow[r, "0"] \arrow[d, "0"] \arrow[dr, phantom, "\beta\text{-square}"] & k \arrow[d, "1"] \\
\begin{smallmatrix}2\\3\end{smallmatrix} \arrow[d, "b"'] & & 0 \arrow[r, "0"] \arrow[d, "0"] \arrow[dr, phantom, "\alpha\text{-square}"] & k \arrow[r, "1"] \arrow[d, "1"] \arrow[dr, phantom, "\beta\text{-square}"] & k \arrow[d, "0"] \\
2 & & 0 \arrow[r, "0"] & k \arrow[r, "0"] & 0
\end{tikzcd}$$

That is, $a = (0, 0, 1)$ and $b = (0, 1, 0)$. It is easy to see that each α-square and each β-square commutes as each one either starts at 0 or ends at 0.

In Example II.2.22 above, we see that the projective representation and the simple representation coincide at certain vertices, and similarly for the injective representation and simple representation.

Remark II.2.23. [Sch14, Rem. 2.2] Let Q be a quiver finite, acyclic quiver.

(i) A vertex i is called a *source* if for every arrow α in Q we have $t(\alpha) \neq i$. Then we have i is a source if and only if $I_i = S_i$.

(ii) A vertex i is a called a *sink* if for every arrow α in Q we have $s(\alpha) \neq i$. Then we have i is a sink if and only if $P_i = S_i$.

II.2.4 Translation quivers

For this section, we use [ASS06] and [Hap88] as our main references. See also [Rie80] from where these concepts originate.

As we will see later, the Auslander-Reiten quiver of the category of finitely generated left modules over a finite-dimensional algebra is a translation quiver. In addition, the Auslander-Reiten quiver of the bounded derived category of $kQ-\mathsf{mod}$, where Q is Dynkin-type, is a stable translation quiver. In this section, we recall the definitions of a translation quiver, a stable translation quiver, and the mesh category associated to a translation quiver.

Definition II.2.24. [ASS06, p. 131] Suppose Γ is a locally finite quiver with no loops. Suppose $\tau\colon D_0 \to \tau(D_0)$ is a bijection, where $D_0, \tau(D_0)$ are both subsets of Γ_0. The pair (Γ, τ) is called a *translation quiver* if, for every vertex $w \in D_0$ and each $v \in w^-$, the number of arrows of the form $v \to w$ is equal to the number of arrows of the form $\tau w \to v$.

In this case, τ is called the *translation*. Furthermore, a vertex v in $\Gamma_0 \setminus D_0$ is called *projective*, and a vertex w in $\Gamma_0 \setminus \tau(D_0)$ is called *injective*.

In the Auslander-Reiten quiver of a path algebra of a quiver of Dynkin-type, we will see that the projective vertices correspond to the indecomposable projective modules (or indecomposable projective representations as defined in Definition II.2.21), and that the injective vertices correspond to the indecomposable injective modules. See Example II.4.25.

Definition II.2.25. [Hap88, p. 41] Let (Γ, τ) be a translation quiver. Denote by D_1 the subset of Γ_1 consisting of all arrows α with $t(\alpha) \in D_0$ a non-projective vertex. A *polarisation* of Γ is an injection $\sigma\colon D_1 \to \Gamma_1$ such that for all $\alpha \in D_1$ we have

$s(\sigma(\alpha)) = \tau(t(\alpha))$ and $t(\sigma(\alpha)) = s(\alpha)$, i.e.

$$\begin{tikzcd} & s(\alpha) \arrow[dr, "\alpha"] & \\ \tau(t(\alpha)) \arrow[ur, "\sigma(\alpha)"] & & t(\alpha) \end{tikzcd}$$

Remark II.2.26. Note that an arbitrary translation quiver (Γ, τ) may have many polarisations. However, if Γ has no multiple arrows, then there is a unique polarisation on Γ. See [Hap88, p. 41].

Definition II.2.27. [Hap88, p. 41] Suppose (Γ, τ) is a translation quiver. If Γ has no multiple arrows and the domain D_0 of τ is equal to Γ_0, then (Γ, τ) is called *stable*.

As mentioned above, we will see that the Auslander-Reiten quiver of the bounded derived category of $kQ - \mathsf{mod}$ where Q is Dynkin-type is an example of a stable translation quiver; see Example II.7.39. In order to arrive at this we need the notion of a path category and the corresponding mesh category. We conclude this section by recalling these definitions here.

Definition II.2.28. [ASS06, p. 131] Suppose (Γ, τ) is a translation quiver and let $w \in \Gamma_0$ be a non-projective vertex. The full subquiver of Γ consisting of $w, \tau w$ and the vertices in $w^- = (\tau w)^+$ is called the *mesh starting at τw and ending at w*.

Definition II.2.29. [Hap88, p. 54] Let (Γ, τ) be a translation quiver. The *path category* $k(\Gamma, \tau)$ of (Γ, τ) is defined to have $\mathrm{obj}(k(\Gamma, \tau)) = \Gamma_0$, and for each pair v, w of vertices in Γ_0 we define $\mathrm{Hom}_{k(\Gamma,\tau)}(v, w)$ to be the k-vector space with basis all possible paths from v to w. Composition in this category is induced from concatenation of paths in Γ.

Definition II.2.30. [Hap88, pp. 54–55] Let $k(\Gamma, \tau)$ be the path category of a translation quiver (Γ, τ) where Γ has no multiple arrows. Let σ be the unique polarisation of Γ (see Remark II.2.26). Let $w \in \Gamma_0$ be a non-projective vertex.

The *mesh relation at w* is

$$\rho_w := \sum_{\alpha_i : v_i \to w} \sigma(\alpha_i) \circ \alpha_i = \sum_{\alpha_i : v_i \to w} \alpha_i \sigma(\alpha_i),$$

where $\alpha_i \sigma(\alpha_i)$ denotes the path $\sigma(\alpha_i)$ then α_i. The *mesh ideal* is the ideal $\mathcal{I}_{\text{mesh}}$ of $k(\Gamma, \tau)$ generated by the mesh relations ρ_w at w as w varies over all non-projective vertices of Γ.

Definition II.2.31. [Hap88, p. 55] Let $k(\Gamma, \tau)$ be the path category of a translation quiver (Γ, τ) where Γ has no multiple arrows. The *mesh category* of (Γ, τ) is the quotient category $\overline{k(\Gamma, \tau)} := k(\Gamma, \tau)/\mathcal{I}_{\text{mesh}}$.

The mesh category associated to a translation quiver (Γ, τ) (where Γ is without multiple arrows) is denoted by $k(\Gamma)$ in [Hap88]. However, we denote the path algebra of Γ by $k\Gamma$, so we prefer to make a larger distinction in our notation.

II.3 Auslander-Reiten theory in additive categories

Auslander-Reiten theory is the study of irreducible morphisms, almost split morphisms and Auslander-Reiten sequences. Some of these concepts were introduced for very general categories in [AR75] and [AR77a], but others were given only for categories with more structure, e.g. abelian categories.

In this section, we briefly recall what irreducible, almost split and minimal morphisms are so that we can keep our treatment quite general here. We will specialise to the abelian, triangulated and Krull-Schmidt settings later. Our eventual goal is to show, for a finite-dimensional hereditary algebra H, how the Auslander-Reiten theory of the module category $H - \mathsf{mod}$ induces the Auslander-Reiten theory of the bounded derived category $\mathsf{D}^b(H - \mathsf{mod})$, and how this in turn induces the Auslander-Reiten theory of the cluster category $\mathcal{C}_H$.

II.3.1 The basics

For this section, suppose $\mathcal{A}$ is an arbitrary category. We will impose conditions on $\mathcal{A}$ as they are needed.

Definition II.3.1. [ML98, p. 19] Let $f\colon X \to Y$ be a morphism in $\mathcal{A}$. We say that f is a *section*, or a *split monomorphism*, if there exists $g\colon Y \to X$ such that $gf = 1_X$.

We call f a *retraction*, or a *split epimorphism*, if there exists $g\colon Y \to X$ such that $fg = 1_Y$.

Thus, a section is a morphism with a left inverse, and a retraction is a morphism with a right inverse. Immediately from this observation, we have the following.

Lemma II.3.2. *Let $f\colon X \to Y$ be a morphism in an arbitrary category. Then f is an isomorphism if and only if f is both a section and a retraction.*

We study morphisms of the following kind in later chapters and they are a central point of focus in Chapter V.

Definition II.3.3. [AR77a, §2] A morphism $f\colon X \to Y$ of $\mathcal{A}$ is *irreducible* if the following conditions are satisfied:

(i) f is not a section;

(ii) f is not a retraction; and

(iii) if $f = hg$, for some $g\colon X \to Z$ and $h\colon Z \to Y$, then either h is a retraction or g is a section.

The following lemma is easily verifiable.

Lemma II.3.4. *Let $f\colon X \to Y$, $g\colon Y \to Z$ be morphisms in $\mathcal{A}$.*

(i) *If f is an isomorphism and g is irreducible, then gf is irreducible.*

(ii) *If f is irreducible and g is an isomorphism, then gf is irreducible.*

Proposition II.3.5. *Suppose $\mathcal{A}$ is an additive category and let X, Y be arbitrary objects in $\mathcal{A}$. If $f\colon X \to Y$ is an irreducible morphism, then $f \neq 0$.*

Proof. Suppose $f\colon X \to Y$ is irreducible but that $f = 0$. We can then write $f = 0_{0,Y} \circ 0_{X,0}$, where $0_{X,0}\colon X \to 0$ and $0_{0,Y}\colon 0 \to Y$ are the zero morphisms. If $0_{X,0}$ were a section, then there would exist $r\colon 0 \to X$ (necessarily the zero morphism also), such that $1_X = r \circ 0_{X,0} = 0$ whence $X = 0$. However, this would mean that f is section, which is not true as f is irreducible. Therefore, $0_{X,0}$ is not a section. Similarly, $0_{0,Y}$ cannot be a retraction. But then $f = 0_{0,Y} \circ 0_{X,0}$ contradicts (iii) of Definition II.3.3. Hence, $f \neq 0$. ■

The following corollary of Proposition II.3.5 is known, e.g. see [AR77a, p. 458]. We include a short proof for completeness.

Corollary II.3.6. *Suppose $\mathcal{A}$ is an additive category and let X, Y be arbitrary objects in $\mathcal{A}$. If $f\colon X \to Y$ is an irreducible morphism, then $X, Y \neq 0$.*

Proof. If X or Y were zero, then f would be the zero morphism, contradicting Proposition II.3.5. ■

Let S be a commutative ring.

Definition II.3.7. [Kel64] Let $\mathcal{A}$ be an S-category. The *radical* $\operatorname{rad}_{\mathcal{A}}(-,-)$ of $\mathcal{A}$ is the ideal of $\mathcal{A}$ given by

$$\operatorname{rad}_{\mathcal{A}}(X,Y) := \{\, f \in \operatorname{Hom}_{\mathcal{A}}(X,Y) \mid 1_X - gf \text{ is invertible for all } g\colon Y \to X \,\}$$

for any two objects $X, Y \in \mathcal{A}$.

By a *radical* morphism $f\colon X \to Y$, we mean an element of $\operatorname{rad}_{\mathcal{A}}(X,Y)$. For $n \in \mathbb{Z}_{>0}$, $\operatorname{rad}^n_{\mathcal{A}}(X,Y)$ denotes the S-submodule of $\operatorname{Hom}_{\mathcal{A}}(X,Y)$ generated by morphisms that are a composition of n radical morphisms. We also have that $\operatorname{rad}_{\mathcal{A}}(X,Y)$ is equal to

$$\{ f \in \operatorname{Hom}_{\mathcal{A}}(X,Y) \mid 1_Y - fg \text{ is invertible for all } g\colon Y \to X \}$$

for any two objects $X,Y \in \mathcal{A}$ by [Kra15, Cor. 2.10]. Furthermore, $\operatorname{rad}_{\mathcal{A}}(X,X) \subseteq \operatorname{End}_{\mathcal{A}} X$ coincides with the Jacobson radical $J(\operatorname{End}_{\mathcal{A}} X)$ of the ring $\operatorname{End}_{\mathcal{A}} X$. See [Kra15, §2] for more details.

Lemma II.3.8. [Kel64, Lem. 1] *Let $\mathcal{A}$ be an S-category. Let $X_1,\dots,X_n,Y_1,\dots,Y_m$ be objects of $\mathcal{A}$. A morphism*

$$f = \begin{pmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{m1} & \cdots & f_{mn} \end{pmatrix} \colon \bigoplus_{i=1}^{n} X_i \longrightarrow \bigoplus_{j=1}^{m} Y_j$$

in $\mathcal{A}$ is in $\operatorname{rad}_{\mathcal{A}}(\bigoplus_{i=1}^n X_i, \bigoplus_{j=1}^m Y_j)$, if and only if $f_{ji}\colon X_i \to Y_j$ lies in $\operatorname{rad}_{\mathcal{A}}(X_i,Y_j)$ for each $i = 1,\dots,n$ and $j = 1,\dots,m$.

The following result characterises irreducible morphisms between indecomposables. Although this is stated in [Bau82] as Proposition 2.4, it is not proved. However, this is entirely similar to the module case presented as, for example, [ASS06, Lem. IV.1.6].

Proposition II.3.9. [Bau82, Prop. 2.4] *If X,Y are indecomposable objects in a Krull-Schmidt S-category $\mathcal{A}$, then a morphism $f\colon X \to Y$ is irreducible if and only if $f \in \operatorname{rad}_{\mathcal{A}}(X,Y) \setminus \operatorname{rad}^2_{\mathcal{A}}(X,Y)$.*

The next two types of morphism are key in the study of a category's Auslander-Reiten theory.

Definition II.3.10. [AR77a, §2] Let $f\colon X \to Y$ be a morphism in $\mathcal{A}$. We say that f is *right minimal* (respectively, *left minimal*) if, for any endomorphism $g\colon X \to$

X (respectively, $g\colon Y \to Y$), $fg = f$ (respectively, $gf = f$) implies g is an automorphism.

Note that any monomorphism is right minimal and that any epimorphism is left minimal.

Definition II.3.11. [AR77a, §2] Let $f\colon X \to Y$ be a morphism in $\mathcal{A}$. We call f *right almost split* if

(i) f is not a retraction; and

(ii) for any non-retraction $u\colon U \to Y$ there exists $\widehat{u}\colon U \to X$ such that $f\widehat{u} = u$.

And f is said to be *left almost split* if

(i) f is not a section; and

(ii) for any non-section $v\colon X \to V$ there exists $\widehat{v}\colon Y \to V$ such that $\widehat{v}f = v$.

If f is both right (respectively, left) minimal and right (respectively, left) almost split, then f is called *minimal right* (respectively, *minimal left*) *almost split*.

It is worth commenting here that any two minimal right almost split morphisms $f\colon X \to Y, f'\colon X' \to Y$ with common codomain are isomorphic in the following sense: there exists an isomorphism $g\colon X \to X'$ with $f = f'g$. Dually for minimal left almost split morphisms. See [AR77a, Prop. 2.2].

The following lemma is also useful.

Lemma II.3.12. [AR77a, Lem. 2.3] *Suppose $\mathcal{A}$ is an additive category.*

1. *If $f\colon X \to Y$ is left almost split in $\mathcal{A}$, then $\operatorname{End}_{\mathcal{A}} X$ is local.*
2. *If $f\colon X \to Y$ is right almost split in $\mathcal{A}$, then $\operatorname{End}_{\mathcal{A}} Y$ is local.*

Remark II.3.13. It is easy to show that morphisms such as irreducible, almost split and minimal morphisms are preserved under equivalence of categories.

II.3.2 The Auslander-Reiten quiver of a Krull-Schmidt k-category

In order to understand a category, we need to understand two things: the objects and the morphisms. In a Krull-Schmidt category $\mathcal{A}$, an object X admits a finite direct sum decomposition $X = X_1 \oplus \cdots \oplus X_n$, where $\mathrm{End}_{\mathcal{A}}(X_i)$ is a local ring (equivalently, X_i is indecomposable) for all $1 \leqslant i \leqslant n$, so the morphisms between arbitrary objects are built up from the morphisms between indecomposables. Hence, it is enough to determine the indecomposable objects of $\mathcal{A}$ and the morphisms between them. As we will see, the associated Auslander-Reiten quiver (see Definition II.3.14) keeps track of this information.

Let k be a field and suppose $\mathcal{A}$ is a Krull-Schmidt k-category. Define a quiver $\Gamma_{\mathrm{AR}}(\mathcal{A})$ associated to $\mathcal{A}$ in the following way. The vertices of $\Gamma_{\mathrm{AR}}(\mathcal{A})$ are the isoclasses $[X]$ of indecomposable objects in $\mathcal{A}$. For two vertices $[X]$, $[Y]$, the arrows $[X] \to [Y]$ are in bijection with and labelled by the elements of a basis of the k-vector space

$$\mathrm{Irr}_{\mathcal{A}}(X,Y) \coloneqq \mathrm{rad}_{\mathcal{A}}(X,Y)/\,\mathrm{rad}^2_{\mathcal{A}}(X,Y).$$

Definition II.3.14. The quiver $\Gamma_{\mathrm{AR}}(\mathcal{A})$ is called the *Auslander-Reiten quiver* of $\mathcal{A}$.

Remark II.3.15. In the above we did not assume that $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ is finite-dimensional over k for $X, Y \in \mathcal{A}$, so it may be the case that the Auslander-Reiten quiver $\Gamma_{\mathrm{AR}}(\mathcal{A})$ is not finite or even locally finite.

Remark II.3.16. We also remark here that Liu has defined an Auslander-Reiten quiver for a Krull-Schmidt category that is Hom-finite over a commutative artinian ring; see [Liu10, §2].

Notation. In practice we will label vertices of Auslander-Reiten quivers by a chosen representative of an equivalence class $[X]$, rather than by the equivalence class $[X]$ itself.

II.4 Module categories

The cluster category as introduced in [BMRRT], to which we apply many of our results, is defined only for a finite-dimensional hereditary algebra. (A generalised cluster category was introduced in [Ami09] for finite-dimensional algebras of global dimension at most 2 and quivers with potential.)

In this section, we first recall the global dimension of a ring so that we may recall the definition of a finite-dimensional hereditary algebra. Our aim is to recall the Auslander-Reiten theory of such an algebra and describe its Auslander-Reiten quiver, which we do in §II.4.3. For this we will need to know what the Auslander-Reiten translation for the category of finitely generated left modules of a finite-dimensional algebra is, so we recall this in §II.4.2.

II.4.1 Dimensions of rings

The left global dimension of a ring S keeps track of how long projective resolutions of left S-modules may be, and the right global dimension is defined similarly. If S is noetherian then these dimensions agree and this common number is called the global dimension. We give more details below, using [Aus55] as our main reference.

Definition II.4.1. [ML98, p. 118] In a category $\mathcal{A}$, an object $P \in \mathcal{A}$ is said to be *projective* if for every epimorphism $f\colon X \twoheadrightarrow Y$ and every morphism $g\colon P \to Y$, there exists $\widehat{g}\colon P \to X$ such that $g = f\widehat{g}$.

An object $I \in \mathcal{A}$ is called *injective* if for every monomorphism $f\colon X \hookrightarrow Y$ and every morphism $g\colon X \to I$, there exists $\widehat{g}\colon Y \to I$ such that $g = \widehat{g}f$.

Note that in an abelian category $\mathcal{A}$, an object P is projective if and only if $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is an exact functor (see Definition II.1.40). Dually, $I \in \mathcal{A}$ is injective if and only if $\mathrm{Hom}_{\mathcal{A}}(-, I)$ is exact. See [Alu09, §VIII.6.1].

Definition II.4.2. [Aus55, p. 67] Let S be a ring and let M be a left S-module. The *projective dimension* $\mathrm{p.dim}_S\, M$ of M is the least $n \in \mathbb{N}$ for which there exists a projective resolution (by left projective S-modules) of length n of M, and ∞ if no such n exists. The *left global dimension* $\mathrm{l.gl.dim}\, S$ of S is the supremum of projective dimensions $\mathrm{p.dim}_S\, M$ over all left S-modules M, and ∞ if $\mathrm{p.dim}_S\, M = \infty$ for some M. We similarly define projective dimension for right S-modules and the *right global dimension* of S, which is denoted $\mathrm{r.gl.dim}\, S$.

Theorem II.4.3. [Aus55, Cor. 5] *If S is a noetherian ring, then*

$$\mathrm{l.gl.dim}\, S = \mathrm{r.gl.dim}\, S.$$

In this case, we define these equal values to be the global dimension $\mathrm{gl.dim}\, S$ *of S.*

II.4.2 The Auslander-Reiten translation for a finite-dimensional algebra

In this section we recall how the Auslander-Reiten translation is defined for a finite-dimensional algebra. We use [ASS06, §IV] as our main reference, but see also [AR75].

Let k be an algebraically closed field, and suppose A is a finite-dimensional k-algebra. Notice that these conditions on A imply that A is both left noetherian and right noetherian, hence noetherian. (Similarly, A is also an artinian ring.) Moreover, the category $A-\mathsf{mod}$ of finitely generated left A-modules is an abelian category. Recall that $A-\mathsf{mod}$ is also the category of all finite-dimensional left A-modules (see Example II.1.37). Furthermore, we observe here that $A-\mathsf{mod}$ (and indeed $A-\mathsf{Mod}$) is a k-category. In fact, $A-\mathsf{mod}$ is a Krull-Schmidt k-category, which follows from the next result.

Theorem II.4.4. [ASS06, Thm. I.4.10] *Let A be a finite-dimensional k-algebra and X a finitely generated left A-module. Then $X \cong X_1 \oplus \cdots \oplus X_n$, where*

$\operatorname{End}_{A-\mathsf{mod}}(X_i)$ *is local for each* $i = 1, \ldots, n$. *Moreover, this decomposition is unique up to isomorphism of the summands and a permutation of the summands.*

Let $(-)^t$ denote the contravariant left exact functor $\operatorname{Hom}_{A-\mathsf{mod}}(-, A)$ on $A-\mathsf{mod}$. Given a finite-dimensional left A-module X, we can take a minimal projective presentation $P^{-1} \xrightarrow{f} P^0 \to X \to 0$ of X (which is unique up to isomorphism) and apply $(-)^t$ to get an exact sequence $0 \to X^t \to (P^0)^t \xrightarrow{f^t} (P^{-1})^t$. Then we can take the cokernel of f^t and get an exact sequence $0 \to X^t \to (P^0)^t \xrightarrow{f^t} (P^{-1})^t \to \operatorname{Tr}(X) \to 0$, where $\operatorname{Tr}(X) \coloneqq \operatorname{Coker}(f^t)$, of right A-modules or left A^{op}-modules.

Definition II.4.5. [ASS06, p. 107], [AR75, §2] We call the module $\operatorname{Tr}(X)$, obtained in the way just described, the *transpose*, or *transposition*, *of* X.

Remark II.4.6. It is important to note here that $\operatorname{Tr}(-)$ does not necessarily define a functor $A-\mathsf{mod} \to A^{\mathrm{op}}-\mathsf{mod}$. We must instead pass to appropriate quotient categories to get a functor, and in doing so we get that Tr is actually a duality between the projectively stable module categories $A-\underline{\mathsf{mod}}$ and $A^{\mathrm{op}}-\underline{\mathsf{mod}}$. See [ASS06, §IV.2] for more details.

Definition II.4.7. [ASS06, Def. III.2.8] We let $D(-) \coloneqq \operatorname{Hom}_{k-\mathsf{mod}}(-, k)$ denote the standard duality on $A-\mathsf{mod}$, which is an exact contravariant functor, and define the *Nakayama functor* to be

$$\nu(-) \coloneqq D(-^t) = D(\operatorname{Hom}_{A-\mathsf{mod}}(-, A)).$$

Consider the exact sequence $0 \to X^t \to (P^0)^t \xrightarrow{f^t} (P^{-1})^t \to \operatorname{Tr}(X) \to 0$ (as acquired above) for an object $X \in A-\mathsf{mod}$. Applying the exact functor $D(-)$ to this sequence, we obtain the exact sequence

$$0 \longrightarrow D(\operatorname{Tr} X) \longrightarrow D((P^{-1})^t) \xrightarrow{D(f^t)} D((P^0)^t) \longrightarrow D(X^t) \longrightarrow 0,$$

which is

$$0 \longrightarrow \tau X \longrightarrow \nu P^{-1} \xrightarrow{\nu f} \nu P^0 \longrightarrow \nu X \longrightarrow 0$$

in terms of the Nakayama functor ν and where $\tau X := D(\operatorname{Tr} X)$.

Definition II.4.8. [ASS06, Def. IV.2.3], [AR75, §3] The *Auslander-Reiten translation* (on the module category) is defined to be $\tau := D(\operatorname{Tr}(-))$. The *inverse Auslander-Reiten translation* (on the module category) is defined to be $\tau^{-1} := \operatorname{Tr}(D(-))$.

The object τX is then called the *Auslander-Reiten translate of* X.

Proposition II.4.9. [ASS06, Prop. IV.2.10] *Let X, Y be indecomposable objects in* $A-\mathsf{mod}$.

(i) $\tau X = 0 \iff X$ *is projective.*

(ii) *If X is not projective, then τX is indecomposable non-injective and* $\tau^{-1}\tau X \cong X$.

(iii) *If X, Y are non-projective, then* $X \cong Y \iff \tau X \cong \tau Y$.

(iv) $\tau^{-1} X = 0 \iff X$ *is injective.*

(v) *If X is not injective, then $\tau^{-1}X$ is indecomposable non-projective and* $\tau\tau^{-1}X \cong X$.

(vi) *If X, Y are non-injective, then* $X \cong Y \iff \tau^{-1}X \cong \tau^{-1}Y$.

Therefore, we see that τ yields a bijection between isoclasses of indecomposable non-projective left A-modules and isoclasses of indecomposable non-injective left A-modules. Furthermore, it can be shown that the Nakayama functor takes care of the missing indecomposables. For a ring S, we denote by $S-\mathsf{Proj}$ the full subcategory of $S-\mathsf{Mod}$ whose objects are projective left S-modules, and by $S-\mathsf{proj}$ the full subcategory of $S-\mathsf{mod}$ consisting of all finitely generated projective left S-modules. Similarly, $\mathsf{Inj}\, S$ and $S-\mathsf{inj}$ denote the full subcategories of $S-\mathsf{Mod}$ and $S-\mathsf{mod}$, respectively, consisting of all injective objects.

Proposition II.4.10. [ASS06, Prop. III.2.10] *The Nakayama functor* $\nu = D(-^t)$ *induces an equivalence of categories* $A-\mathsf{proj} \xrightarrow{\simeq} A-\mathsf{inj}$ *with quasi-inverse* $\nu^{-1} = \mathrm{Hom}_{A-\mathsf{mod}}(DA_A, -)$.

Almost split sequences were introduced by Auslander and Reiten in [AR75] for module categories. Since then this notion has been transported to a variety of other contexts. For example, these sequences appear in [Rin84] for Krull-Schmidt categories with short exact sequences, in [GR97] for exact categories and in [KR05] for Hom-finite abelian categories. We adopt the same definition for an arbitrary abelian category.

Definition II.4.11. Suppose $\mathcal{A}$ is an abelian category. A short exact sequence $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ is called an *Auslander-Reiten sequence*, or an *almost split sequence*, if f is minimal left almost split and g is minimal right almost split.

The following result establishes the existence of an Auslander-Reiten sequence starting at any indecomposable non-injective and one ending at any indecomposable non-projective.

Theorem II.4.12. [ASS06, Thm. IV.3.1]

(i) *Let* $Z \in A-\mathsf{mod}$ *be an indecomposable non-projective module. Then there exists an Auslander-Reiten sequence* $0 \to \tau Z \to Y \to Z \to 0$ *in* $A-\mathsf{mod}$.

(ii) *Let* $X \in A-\mathsf{mod}$ *be an indecomposable non-injective module. Then there exists an Auslander-Reiten sequence* $0 \to X \to Y' \to \tau^{-1}X \to 0$ *in* $A-\mathsf{mod}$.

The following gives a characterisation of Auslander-Reiten sequences in the module category of a finite-dimensional algebra. We show this result is valid in more general settings later (see Theorem IV.3.19).

Theorem II.4.13. [ASS06, Thm. IV.1.13] *Let* $\xi\colon\ 0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ *be a short exact sequence in* $A-\mathsf{mod}$*. Then the following are equivalent.*

(i) ξ *is an Auslander-Reiten sequence.*

(ii) X *is indecomposable and* g *is right almost split.*

(iii) Z *is indecomposable and* f *is left almost split.*

(iv) f *is minimal left almost split.*

(v) g *is minimal right almost split.*

(vi) X *and* Z *are indecomposable, and* f *and* g *are irreducible.*

We conclude this section by stating a result that, along with its dual, demonstrates how irreducible morphisms and Auslander-Reiten sequences are related; see [ASS06] for the full statement. Recall that $\operatorname{Irr}_{\mathcal{A}}(X,Y)$ is the k-vector space $\operatorname{rad}_{\mathcal{A}}(X,Y)/\operatorname{rad}^2_{\mathcal{A}}(X,Y)$ for indecomposable objects X, Y in a Krull-Schmidt k-category $\mathcal{A}$ (see §II.3.2).

Theorem II.4.14. [ASS06, Cor. IV.4.4] *Let* $\xi\colon\ 0 \longrightarrow X \xrightarrow{f} \bigoplus_{i=1}^{r} Y_i^{m_i} \longrightarrow Z \longrightarrow 0$ *be a short exact sequence in* $A-\mathsf{mod}$*, where* X, Z *and* Y_i $(\forall 1 \leqslant i \leqslant r)$ *are indecomposable, and* $Y_i \ncong Y_j$ *for all* $i \neq j$*. Write* $f = (\, f_1 \ \cdots \ f_r \,)^{\mathrm{T}}$ *where* $f_i = (\, f_{i1} \ \cdots \ f_{im_i} \,)^{\mathrm{T}}\colon X \to Y_i^{m_i}$*. Then the following are equivalent.*

(i) ξ *is an Auslander-Reiten sequence.*

(ii) *For each* $1 \leqslant i \leqslant r$, *the morphism* f_i *lies in* $\operatorname{rad}_{A-\mathsf{mod}}(X, Y_i^{m_i})$, *the set* $\{f_{i1} + \operatorname{rad}^2_{A-\mathsf{mod}}(X, Y_i), \ldots, f_{im_i} + \operatorname{rad}^2_{A-\mathsf{mod}}(X, Y_i)\}$ *forms a basis of* $\operatorname{Irr}_{A-\mathsf{mod}}(X, Y_i)$, *and if* Y' *is an indecomposable such that* $\operatorname{Irr}_{A-\mathsf{mod}}(X, Y') \neq 0$, *then we must have* $Y' \cong Y_i$ *for some* $i \in \{1, \ldots, r\}$.

Therefore, for an indecomposable non-injective left A-module X, there is an Auslander-Reiten sequence starting at X by Theorem II.4.12, and, moreover, this sequence keeps track of *all* the irreducible morphisms from X to any other indecomposable left A-module. And dually for an indecomposable non-projective.

II.4.3 Auslander-Reiten theory of a finite-dimensional hereditary algebra

We are now in a position to specialise to the hereditary case. We remark here again that any finite-dimensional algebra can be studied via an appropriate finite-dimensional hereditary algebra (see [ASS06, Thm. II.3.7]).

Definition II.4.15. [ARS95, p. 17] A ring H is called *left hereditary* if all left ideals of H are projective left H-modules.

Let $k = \overline{k}$ be an algebraically closed field. It can be shown that a finite-dimensional k-algebra is left hereditary if and only if its global dimension is at most 1 (see [ASS06, Thm. VII.1.4]).

Definition II.4.16. [ASS06, p. 246] A finite-dimensional k-algebra H is called *hereditary* if $\operatorname{gl.dim} H \leqslant 1$.

For the remainder of this section, let H denote a finite-dimensional hereditary k-algebra. The aim of this section is to recall the description of the Auslander-Reiten quiver of $H-\mathsf{mod}$. In order to reduce to simpler cases, we need a few more definitions.

Let S be a ring. Recall that an element e of S is called an *idempotent* if $e^2 = e$ (see [ASS06, §I.4]). Two idempotents $e, f \in S$ are called *orthogonal* if $ef = 0 = fe$. An idempotent $e \in S$ is called *primitive* if $e = f + g$ implies $f = 0$ or $g = 0$ for all pairs of orthogonal idempotents $f, g \in S$.

A set $E = \{e_i\}_{i=1}^n$ of idempotents of S is called *complete* if $e_1 + \cdots + e_n = 1$ is the multiplicative identity in S. A set E of idempotents of a ring S is called a *complete set of primitive orthogonal idempotents* if each element of E is primitive, elements of E are pairwise orthogonal, and E is complete. See [ASS06, p. 18] for more details.

Definition II.4.17. [ASS06, Def. I.6.1] Let A be a k-algebra with a complete set $\{e_1, \ldots, e_n\}$ of primitive orthogonal idempotents. We call A *basic* if $Ae_i \ncong Ae_j$ for all $i \neq j$.

Theorem II.4.18. *Let A be a finite-dimensional k-algebra. Then there is a basic k-algebra A^b such that there is a k-additive equivalence $A - \mathsf{mod} \simeq A^b - \mathsf{mod}$ of modules categories.*

Proof. See [ASS06, Def. I.6.3] for how to define A^b, and [ASS06, Cor. I.6.10] for the equivalence. ■

Therefore, in studying the category $H - \mathsf{mod}$, we may assume H is basic. Furthermore, if $H = H_1 \times H_2$ is a direct product of two other k-algebras, then it is enough to study H_1 and H_2 individually in order to understand the representation theory of H. This leads us to the following definition.

Definition II.4.19. [ASS06, p. 18] An algebra A is called *connected* if it is not the direct product of two non-zero algebras.

Hence, we may make one more reduction and, in summary, suppose that H is a finite-dimensional, basic, connected, hereditary k-algebra. The following result then says that H arises (up to isomorphism) as a path algebra.

Theorem II.4.20. [ASS06, Thm. VII.1.7] *If H is a finite-dimensional, basic, connected, hereditary k-algebra, then there is a finite, connected, acyclic quiver Q_H such that H is isomorphic to the path algebra kQ_H as k-algebras.*

If A is a finite-dimensional k-algebra, then we have seen that $A-\mathsf{mod}$ is a Krull-Schmidt k-category (see Theorem II.4.4), so we may define its Auslander-Reiten quiver (see §II.3.2). See also [ASS06, §IV.4].

Definition II.4.21. Let A be a basic, connected, finite-dimensional k-algebra. The *Auslander-Reiten quiver* of A is the quiver $\Gamma_{\mathrm{AR}}(A-\mathsf{mod})$ as defined in Definition II.3.14.

Proposition II.4.22. [ASS06, pp. 129–131] *Let A be a basic, connected, finite-dimensional k-algebra. Let $\Gamma_{\mathrm{AR}}(A-\mathsf{mod})$ be its Auslander-Reiten quiver.*

(i) *The quiver $\Gamma_{\mathrm{AR}}(A-\mathsf{mod})$ has no loops and is locally finite.*

(ii) *Let $\Gamma_{\mathrm{AR}}(A-\mathsf{proj})_0$ (respectively, $\Gamma_{\mathrm{AR}}(A-\mathsf{inj})_0$) be the subset of $\Gamma_{\mathrm{AR}}(A-\mathsf{mod})_0$ consisting of vertices, each of which corresponds to a projective (respectively, an injective) module. Then the Auslander-Reiten translation τ (as defined in* Definition II.4.8*) on $A-\mathsf{mod}$ induces a bijection*

$$\tau\colon \Gamma_{\mathrm{AR}}(A-\mathsf{mod})_0 \setminus \Gamma_{\mathrm{AR}}(A-\mathsf{proj})_0 \longrightarrow \Gamma_{\mathrm{AR}}(A-\mathsf{mod})_0 \setminus \Gamma_{\mathrm{AR}}(A-\mathsf{inj})_0.$$

(iii) *The pair $(\Gamma_{\mathrm{AR}}(A-\mathsf{mod}), \tau)$ is a translation quiver, where the translation is given by $\tau[X] = [\tau X]$ for all vertices $[X]$ such that X is not a projective A-module.*

(iv) *If M is an indecomposable non-projective A-module, then by Theorem II.4.12 there is an almost split sequence $0 \to \tau M \to \bigoplus_{i=1}^{r} E_i^{m_i} \to M \to 0$, where the E_i are pairwise non-isomorphic. Then in $\Gamma_{\mathrm{AR}}(A-\mathsf{mod})$ there is* *a corresponding mesh*

$$
\begin{tikzcd}[column sep=large]
 & {[E_1]} \arrow[dr, shift left=2, "g_{11}"] \arrow[dr, shift right=2, "g_{1m_1}"'] & \\
{[\tau M]} \arrow[ur, shift left=2, "f_{11}"] \arrow[ur, shift right=2, "f_{1m_1}"'] \arrow[dr, shift left=2, "f_{r1}"] \arrow[dr, shift right=2, "f_{rm_r}"'] \arrow[rr, dashed] & \vdots & {[M]} \\
 & {[E_r]} \arrow[ur, shift left=2, "g_{r1}"] \arrow[ur, shift right=2, "g_{rm_r}"'] &
\end{tikzcd}
$$

where, for each $1 \leqslant i \leqslant r$, the set $\{f_{i1}, \ldots, f_{im_i}\}$ is a basis of the space $\operatorname{Irr}_{A-\mathsf{mod}}(\tau M, E_i)$ and the set $\{g_{i1}, \ldots, g_{im_i}\}$ is a basis of the space $\operatorname{Irr}_{A-\mathsf{mod}}(E_i, M)$.

When $H-\mathsf{mod}$ has only finitely many isoclasses of indecomposable objects we can say more.

Definition II.4.23. [ASS06, Def. I.4.11] Let A be a finite-dimensional k-algebra. Then A is said to be of *finite representation type*, or simply *representation-finite*, if the number of isoclasses of indecomposable left A-modules is finite. Otherwise, A is said to be of *infinite representation type*, or simply *representation-infinite*.

Theorem II.4.24 (Gabriel's Theorem)**.** [Gab72], [ASS06, Thm. VII.5.10] *Let Q be a finite, connected, acyclic quiver. Then kQ is representation-finite if and only if the underlying graph of Q is a Dynkin graph.*

Thus, if Q is a Dynkin-type quiver, then the Auslander-Reiten quiver $\Gamma := \Gamma_{\mathrm{AR}}(kQ-\mathsf{mod})$ completely describes the category $kQ-\mathsf{mod}$. That is, the mesh category $\overline{k(\Gamma, \tau)}$ of (Γ, τ) (see Definition II.2.31), where τ is the Auslander-Reiten translation for kQ, is equivalent to any full subcategory of $kQ-\mathsf{mod}$ whose object set contains precisely one representative of each isoclass. See [SS07, §X] for more details.

Example II.4.25. Let k be a field. Consider again the quiver $Q\colon\ 1 \longrightarrow 2 \longrightarrow 3$. The Auslander-Reiten quiver $\Gamma := \Gamma_{\mathrm{AR}}(kQ-\mathsf{mod})$ of kQ is then:

$$\begin{tikzcd}
 & & P_1 = \begin{smallmatrix}1\\2\\3\end{smallmatrix} = I_3 \arrow[dr, "d"] & & \\
 & P_2 = \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[ur, "c"] \arrow[dr, "b"] \arrow[rr, dashed, no head] & & I_2 = \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[dr, "f"] & \\
P_3 = 3 \arrow[ur, "a"] \arrow[rr, dashed, no head] & & S_2 = 2 \arrow[ur, "e"] \arrow[rr, dashed, no head] & & I_1 = 1
\end{tikzcd}$$

where the morphisms of quiver representations above are: $a = (0,0,1)$, $b = (0,1,0)$, $c = (0,1,1)$, $d = (1,1,0)$, $e = (0,1,0)$ and $f = (1,0,0)$. The dashed lines indicate the mesh relations (see Definition II.2.30): $ba = 0$, $dc + eb = 0$ and $fe = 0$. Thus, there are 3 almost split sequences:

$$0 \longrightarrow 3 \xrightarrow{\ a\ } \begin{smallmatrix}2\\3\end{smallmatrix} \xrightarrow{\ b\ } 2 \longrightarrow 0,$$

$$0 \longrightarrow \begin{smallmatrix}2\\3\end{smallmatrix} \xrightarrow{\binom{c}{b}} \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus 2 \xrightarrow{(d\ e)} \begin{smallmatrix}1\\2\end{smallmatrix} \longrightarrow 0,$$

$$0 \longrightarrow 2 \xrightarrow{\ e\ } \begin{smallmatrix}1\\2\end{smallmatrix} \xrightarrow{\ f\ } 1 \longrightarrow 0.$$

In particular, we can read off the Auslander-Reiten translate for each non-projective: $\tau(2) = 3$, $\tau(\begin{smallmatrix}1\\2\end{smallmatrix}) = \begin{smallmatrix}2\\3\end{smallmatrix}$ and $\tau(1) = 2$.

Moreover, as Q is a Dynkin-type quiver, we have that any object in $kQ-\mathsf{mod}$ is isomorphic to a finite direct sum of the indecomposables appearing in the Auslander-Reiten quiver Γ, and if X, Y are in $kQ-\mathsf{mod}$ then any morphism $X \to Y$ is (up to isomorphism) a sum of compositions of the irreducible morphisms appearing in Γ.

II.5 Localisation of categories

The process of localising a category allows the formal inversion of a collection of morphisms, producing a new category but with the same objects as the original. We need localisation to define the cluster category associated to a hereditary algebra H, which is an orbit category of the derived category of $H-\mathsf{mod}$. The bounded derived category itself is a certain localisation of the homotopy category of H, so this is the first instance where we will need the theory of localisation of categories.

In addition, in Chapters III and V, we need localisation to go from an integral category to an abelian one. In these situations, localisation allows us to invert morphisms that "should be" isomorphisms.

We follow the development in [GZ67, §I] to recall the category of fractions and a calculus of fractions. See also [Kra10].

II.5.1 Gabriel-Zisman localisation: the category of fractions

Let $\mathcal{A}$ be a category and suppose $\mathcal{M}$ is a class of morphisms in $\mathcal{A}$. Informally, the *Gabriel-Zisman localisation* $\mathcal{A}_{\mathcal{M}}$ of $\mathcal{A}$ at $\mathcal{M}$ is the category $\mathcal{A}$ with a formal inverse s^{-1} for each morphism $s \in \mathcal{M}$ thrown in. The category $\mathcal{A}_{\mathcal{M}}$ was originally called *the category of fractions of $\mathcal{A}$ for $\mathcal{M}$* in [GZ67] and denoted by $\mathcal{A}[\mathcal{M}^{-1}]$. However, in order to distinguish between the various localisations of triangulated categories, we follow the terminology used in, for example, [Sim06], [BM13] and [BM12].

Definition II.5.1. [GZ67, §I.1] Suppose $\mathscr{F}\colon \mathcal{A} \to \mathcal{B}$ is functor and s is a morphism in $\mathcal{A}$. We say $\mathscr{F}$ *makes s invertible* if $\mathscr{F}(s)$ is invertible in $\mathcal{B}$. In this case, we will also sometimes say that $\mathscr{F}$ *inverts s*.

Definition II.5.2. [GZ67, §I.1] A *category of fractions of $\mathcal{A}$ for $\mathcal{M}$*, if it exists, is a category $\mathcal{A}_{\mathcal{M}} = \mathcal{A}[\mathcal{M}^{-1}]$ together with a functor $L_{\mathcal{M}}\colon \mathcal{A} \to \mathcal{A}_{\mathcal{M}}$ such that:

(i) $L_{\mathcal{M}}$ inverts all morphisms in $\mathcal{M}$; and

(ii) for any functor $\mathscr{F}\colon \mathcal{A} \to \mathcal{B}$ that makes all morphisms in $\mathcal{M}$ invertible, there exists a unique functor $\widehat{\mathscr{F}}\colon \mathcal{A}_{\mathcal{M}} \to \mathcal{B}$ such that $\mathscr{F} = \widehat{\mathscr{F}} \circ L_{\mathcal{M}}$.

In this setting, we also call the category $\mathcal{A}_{\mathcal{M}}$ a *Gabriel-Zisman localisation of $\mathcal{A}$ at* (or *with respect to*) $\mathcal{M}$, and we call $L_{\mathcal{M}}$ a *localisation functor*. Note also that if $\mathcal{A}_{\mathcal{M}}$ exists, then it is unique up to unique isomorphism.

We now recall how one may construct $\mathcal{A}_{\mathcal{M}}$. The objects of $\mathcal{A}_{\mathcal{M}}$ are the same as those of $\mathcal{A}$, but the morphisms are more complicated to describe. We follow [Kra10, §2.2]; see also [GZ67, §I.1] and [PP79, §1.13].

First, set $\operatorname{obj}(\mathcal{A}_{\mathcal{M}}) = \operatorname{obj}(\mathcal{A})$. For the morphisms, first consider

$$\mathcal{M}^{-1}(Y,X) := \left\{ s^{-1}\colon X \to Y \;\middle|\; s\colon Y \to X \text{ and } s \in \mathcal{M} \right\}$$

for each pair of objects X, Y in $\operatorname{obj}(\mathcal{A})$. Define a quiver $Q'_{\mathcal{A}}$ with vertices $\operatorname{obj}(\mathcal{A})$, and for vertices X, Y the arrows $X \to Y$ are in bijection and labelled by elements of the coproduct $\operatorname{Hom}_{\mathcal{A}}(X,Y) \amalg \mathcal{M}^{-1}(Y,X)$. By abuse of notation, the same symbol for an element of $\operatorname{Hom}_{\mathcal{A}}(X,Y)$ or $\mathcal{M}^{-1}(Y,X)$ will be used to denote the corresponding element in $\operatorname{Hom}_{\mathcal{A}}(X,Y) \amalg \mathcal{M}^{-1}(Y,X)$. Let $Q_{\mathcal{A}}$ be the quiver just like $Q'_{\mathcal{A}}$ but with the loops corresponding to identity morphisms and arrows corresponding to composite morphisms removed (see [ML98, p. 48]).

Let $\mathcal{P} = \mathcal{P}(Q_{\mathcal{A}})$ be the *path category* of $Q_{\mathcal{A}}$, i.e. the objects of $\mathcal{P}$ are the vertices of $Q_{\mathcal{A}}$ (denoted by the same symbols), and the morphisms between two vertices are all the possible paths in $Q_{\mathcal{A}}$ between them such that composition of morphisms is induced by path concatenation. Note that for $X \in \mathcal{P}$ the identity morphism 1_X on X in the category $\mathcal{P}$ is the constant path at X. We will denote composition in $\mathcal{P}$ by $*$.

There is a canonical functor $\mathscr{D}\colon \mathcal{A} \to \mathcal{P}$ which is the identity on objects, and sends a non-identity morphism $f\colon X \to Y$ of $\mathcal{A}$ to the path $\mathscr{D}(f) = f$ of length 1 in $\mathrm{Hom}_{\mathcal{P}}(X,Y)$, or if $f = 1_X$ then $\mathscr{D}(f)$ is the constant path at X in $\mathcal{P}$. Let $\mathcal{E}$ be the smallest (composition respecting) equivalence relation on $\mathcal{P}$ generated by the relations $s^{-1} * \mathscr{D}(s) = 1_Y$ in $\mathcal{E}(Y,Y)$ and $\mathscr{D}(s) * s^{-1} = 1_X$ in $\mathcal{E}(X,X)$ for every $s\colon Y \to X$ in $\mathcal{M}$. Then we define $\mathrm{Hom}_{\mathcal{A}_{\mathcal{M}}}(X,Y) := \mathrm{Hom}_{\mathcal{P}}(X,Y)/\mathcal{E}(X,Y)$. This defines the category $\mathcal{A}_{\mathcal{M}}$ in which composition of morphisms is induced by the composition $*$ in $\mathcal{P}$.

Lastly, we recall how the localisation functor $L_{\mathcal{M}}\colon \mathcal{A} \to \mathcal{A}_{\mathcal{M}}$ is defined. For each object $X \in \mathcal{A}$, we set $L_{\mathcal{M}}(X) = X$. For any objects $X, Y \in \mathrm{obj}(\mathcal{A})$, the mapping $(L_{\mathcal{M}})_{X,Y}\colon \mathrm{Hom}_{\mathcal{A}}(X,Y) \to \mathrm{Hom}_{\mathcal{A}_{\mathcal{M}}}(X,Y)$ is defined to be the composition

$$\mathrm{Hom}_{\mathcal{A}}(X,Y) \xhookrightarrow{\mathscr{D}_{X,Y}} \mathrm{Hom}_{\mathcal{P}}(X,Y) \xrightarrow{/\mathcal{E}(X,Y)} \mathrm{Hom}_{\mathcal{A}_{\mathcal{M}}}(X,Y).$$

With this construction above, there are set-theoretic issues that may arise. The existence of $\mathcal{A}_{\mathcal{M}}$ can be ensured in some cases. Recall that a category $\mathcal{A}$ is called *small* if $\mathrm{obj}(\mathcal{A})$ is a set and $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ is a set for all objects $X, Y \in \mathrm{obj}(\mathcal{A})$; see Definition II.1.9.

Theorem II.5.3. [PP79, Thm. 1.13.3] *If $\mathcal{A}$ is a small category and $\mathcal{M}$ is a collection of morphisms in $\mathcal{A}$, then the Gabriel-Zisman localisation $\mathcal{A}_{\mathcal{M}}$ exists.*

II.5.2 Calculus of fractions

Let $\mathcal{A}$ be a category. Although the Gabriel-Zisman localisation $\mathcal{A}_{\mathcal{M}}$ of $\mathcal{A}$ at a class $\mathcal{M}$ of morphisms may exist, the morphisms may not have a particularly nice or explicit description. We will often be in a situation in which we can give a relatively simple description of the morphisms in the localisation under consideration because the morphisms in $\mathcal{M}$ satisfy some conditions.

The following notion was introduced in [GZ67, §I.2], but we follow the labelling as in [Kra10, §3.1].

Definition II.5.4. [Kra10, §3.1] A class $\mathcal{M}$ of morphisms in $\mathcal{A}$ is said to *admit a calculus of left fractions* if the conditions (LF1)–(LF3) below are satisfied.

(LF1) The identity morphism 1_X is in $\mathcal{M}$ for all $X \in \mathcal{A}$, and $\mathcal{M}$ is closed under composition of morphisms.

(LF2) Any diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"'] & Y \\ X' & \end{tikzcd}$$

in $\mathcal{A}$ in which $s \in \mathcal{M}$ may be completed to a commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"'] \arrow[dr, phantom, "\circ"] & Y \arrow[d, dotted, "t"] \\ X' \arrow[r, dotted, "g"'] & Y' \end{tikzcd}$$

where $t \in \mathcal{M}$.

(LF3) For any pair of morphisms $f, g \colon X \to Y$ in $\mathcal{A}$, if there is $s \colon W \to X$ in $\mathcal{M}$ such that $fs = gs$, then there exists $t \colon Y \to Z$ in $\mathcal{M}$ such that $tf = tg$.

There is a dual notion.

Definition II.5.5. [BM12, §4] A class of morphisms $\mathcal{M}$ in $\mathcal{A}$ is said to *admit a calculus of right fractions* if the conditions (RF1)–(RF3) below are satisfied.

(RF1) The identity morphism 1_X is in $\mathcal{M}$ for all $X \in \mathcal{A}$, and $\mathcal{M}$ is closed under composition of morphisms.

(RF2) Any diagram

$$\begin{tikzcd} & Y \arrow[d, "t"] \\ X' \arrow[r, "g"'] & Y' \end{tikzcd}$$

in $\mathcal{A}$ in which $t \in \mathcal{M}$ may be completed to a commutative square

$$\begin{tikzcd} X \arrow[r, dotted, "f"] \arrow[d, dotted, "s"'] \arrow[dr, phantom, "\circ"] & Y \arrow[d, "t"] \\ X' \arrow[r, "g"'] & Y' \end{tikzcd}$$

where $s \in \mathcal{M}$.

(RF3) For any pair of morphisms $f, g\colon X \to Y$ in $\mathcal{A}$, if there is $t\colon Y \to Z$ in $\mathcal{M}$ such that $tf = tg$, then there exists $s\colon W \to X$ in $\mathcal{M}$ such that $fs = gs$.

Note that a class $\mathcal{M}$ of morphisms admitting a calculus of left fractions (respectively, right fractions) was called a *left calculable system* (respectively, *right calculable system*) in [PP79]. If $\mathcal{M}$ admits a calculus of left fractions and a calculus of right fractions, then $\mathcal{M}$ is said to be a *multiplicative system*; see [Kra10, §3.1] and [Ver96, §II.2].

Suppose $\mathcal{M}$ is a class of morphisms in $\mathcal{A}$ such that $\mathcal{M}$ admits a calculus of left fractions in $\mathcal{A}$. We will now recall the construction of the *category $\mathcal{M}^{-1}\mathcal{A}$ of left fractions of $\mathcal{A}$ with respect to $\mathcal{M}$*; see [GZ67, §I.2], [Kra10, §3.1] and [PP79, §1.14]. We set $\operatorname{obj}(\mathcal{M}^{-1}\mathcal{A}) = \operatorname{obj}(\mathcal{A})$. The collection $\operatorname{Hom}_{\mathcal{M}^{-1}\mathcal{A}}(X, Y)$ of morphisms $X \to Y$, for $X, Y \in \operatorname{obj}(\mathcal{M}^{-1}\mathcal{A})$, will be given by equivalence classes of a certain relation on the collection of all diagrams in $\mathcal{A}$ of the form

$$\begin{tikzcd} & A & \\ X \arrow[ur, "f"] & & Y \arrow[ul, "s"'] \end{tikzcd}$$

where $s \in \mathcal{M}$. We call such a diagram (f, A, s) a *(left) triple from X to Y*; see [Zim14, §3.5.3]. We say that two triples (f, A, s) and (g, B, t) from X to Y are *equivalent*, denoted $(f, A, s) \sim_{\mathrm{LF}} (g, B, t)$, if there is a commutative diagram

$$\begin{tikzcd} & A \arrow[d] & \\ X \arrow[ur, "f"] \arrow[r, "h"] \arrow[dr, "g"'] & C & Y \arrow[ul, "s"'] \arrow[l, "u"'] \arrow[dl, "t"] \\ & B \arrow[u] & \end{tikzcd}$$

where $u \in \mathcal{M}$.

Proposition II.5.6. [GZ67, p. 13] *For $X, Y \in \operatorname{obj}(\mathcal{M}^{-1}\mathcal{A})$, the relation $\sim_{\mathrm{LF}}$ is an equivalence relation on the triples from X to Y.*

For a triple (f, A, s) from X to Y, we denote its $\sim_{\mathrm{LF}}$-equivalence class by $[f, s]_{\mathrm{LF}}$ which we call a *left fraction (from X to Y)*.

For $X, Y \in \operatorname{obj}(\mathcal{M}^{-1}\mathcal{A})$, we define $\operatorname{Hom}_{\mathcal{M}^{-1}\mathcal{A}}(X, Y)$ to be the collection of all left fractions from X to Y. Given $[f, s]_{\mathrm{LF}} \in \operatorname{Hom}_{\mathcal{M}^{-1}\mathcal{A}}(X, Y)$ and $[g, t]_{\mathrm{LF}} \in \operatorname{Hom}_{\mathcal{M}^{-1}\mathcal{A}}(Y, Z)$, we define the composition $[g, t]_{\mathrm{LF}} \circ [f, s]_{\mathrm{LF}}$ as follows. Let $X \xrightarrow{f} A \xleftarrow{s} Y$ and $Y \xrightarrow{g} B \xleftarrow{t} Z$ be representatives of $[f, s]_{\mathrm{LF}}$ and $[g, t]_{\mathrm{LF}}$, respectively. Then using (LF2) we obtain a commutative diagram

$$\begin{tikzcd} & & C & & \\ & A \arrow[ur, dotted, "h"] & & B \arrow[ul, dotted, "u"'] & \\ X \arrow[ur, "f"] & & Y \arrow[ul, "s"'] \arrow[ur, "g"] & & Z \arrow[ul, "t"'] \end{tikzcd}$$

where $u \in \mathcal{M}$. Note that ut lies in $\mathcal{M}$ by (LF1). Thus, we set $[g, t]_{\mathrm{LF}} \circ [f, s]_{\mathrm{LF}} := [hf, ut]_{\mathrm{LF}}$. Checking that this is well-defined and associative is routine. The identity morphism for an object $X \in \operatorname{obj}(\mathcal{M}^{-1}\mathcal{A})$ is the left fraction $[1_X, 1_X]_{\mathrm{LF}}$, and therefore we have constructed the category $\mathcal{M}^{-1}\mathcal{A}$ of left fractions.

Finally, we recall that there is a canonical functor $F_{\mathcal{M}} \colon \mathcal{A} \to \mathcal{M}^{-1}\mathcal{A}$ given by

$F_{\mathcal{M}}(X) = X$ for $X \in \mathcal{A}$, and $F_{\mathcal{M}}(f) = [f, 1_Y]_{\mathrm{LF}}$ for a morphism $f: X \to Y$ in $\mathcal{A}$. Moreover, for any morphism $s: Y \to A$ in $\mathcal{M}$, the left fraction $F_{\mathcal{M}}(s) = [s, 1_A]_{\mathrm{LF}}$ is invertible with inverse $[1_A, s]_{\mathrm{LF}}$ in $\mathcal{M}^{-1}\mathcal{A}$.

Hence, if a class $\mathcal{M}$ of morphisms admits a calculus of left fractions, we have seen that we are able to construct two other categories $\mathcal{A}_{\mathcal{M}}$ and $\mathcal{M}^{-1}\mathcal{A}$, by formally inverting the morphisms in $\mathcal{M}$. The next result tells us these constructions agree.

Proposition II.5.7. [GZ67, I.2.4] *Suppose $\mathcal{M}$ admits a calculus of left fractions in a category $\mathcal{A}$. Consider the functor $\pi: \mathcal{M}^{-1}\mathcal{A} \to \mathcal{A}_{\mathcal{M}}$ defined by $\pi(X) = X$ for $X \in \mathcal{M}^{-1}\mathcal{A}$, and $\pi([f, s]_{\mathrm{LF}}) = (L_{\mathcal{M}}(s))^{-1} \circ L_{\mathcal{M}}(f)$. Then π is an isomorphism of categories.*

Furthermore, if $\mathcal{A}$ was additive to begin with then the localisations inherit this structure.

Proposition II.5.8. [GZ67, I.3.3], [PP79, Thm. 4.7.5] *Suppose $\mathcal{M}$ admits a calculus of left fractions in an additive category $\mathcal{A}$. Then the Gabriel-Zisman localisation $\mathcal{A}_{\mathcal{M}} \cong \mathcal{M}^{-1}\mathcal{A}$ is also additive. Furthermore, the localisation functors $L_{\mathcal{M}}: \mathcal{A} \to \mathcal{A}_{\mathcal{M}}$ and $F_{\mathcal{M}}: \mathcal{A} \to \mathcal{M}^{-1}\mathcal{A}$ are both additive.*

In order to give another situation in which the existence of the localisation is guaranteed, we need the following notion.

Definition II.5.9. [ML98] A *skeleton* of $\mathcal{A}$ is a full subcategory $\mathcal{B}$ such that the (fully faithful) canonical inclusion $\mathscr{F}: \mathcal{B} \hookrightarrow \mathcal{A}$ is essentially surjective and, for all $X, Y \in \mathrm{obj}(\mathcal{B})$, if $X \neq Y$ then X is not isomorphic to Y in $\mathcal{B}$.

Note that $\mathcal{A}$ will in general have many skeletons, but they are all isomorphic (see [ML98, Exer. IV.4.1]). Therefore, if we need to make use of a skeleton of $\mathcal{A}$, then we will choose one $\mathcal{A}^{\mathrm{skel}}$ with a fixed collection $\{\varphi_X: X \to \widetilde{X}\}_{X \in \mathcal{A}}$ of isomorphisms, where $\widetilde{X} \in \mathcal{A}^{\mathrm{skel}}$ and $\varphi_{\widetilde{X}} = 1_{\widetilde{X}}$ for each $\widetilde{X} \in \mathcal{A}^{\mathrm{skel}}$. Following

[Pre09], we say that a category $\mathcal{A}$ is *skeletally small* if $\mathcal{A}^{\text{skel}}$ is small, i.e. $\operatorname{obj}(\mathcal{A}^{\text{skel}})$ and $\operatorname{Hom}_{\mathcal{A}^{\text{skel}}}(\widetilde{X}, \widetilde{Y})$ are sets for all $\widetilde{X}, \widetilde{Y} \in \mathcal{A}^{\text{skel}}$.

Suppose $\mathcal{A}^{\text{skel}}$ (with $\{\varphi_X \colon X \to \widetilde{X}\}_{X \in \mathcal{A}}$ as the fixed collection of isomorphisms) is our chosen skeleton for $\mathcal{A}$. Then the inclusion $\mathscr{F} \colon \mathcal{A}^{\text{skel}} \to \mathcal{A}$ is an equivalence of categories by Theorem II.1.15. A quasi-inverse $\mathscr{G} \colon \mathcal{A} \to \mathcal{A}^{\text{skel}}$ for $\mathscr{F}$ is given as follows. For each $X \in \mathcal{A}$ set $\mathscr{G}(X) := \widetilde{X}$, and for any $f \in \operatorname{Hom}_{\mathcal{A}}(X, Y)$ put $\mathscr{G}(f) := \varphi_Y f \varphi_X^{-1}$. See [ML98, p. 93].

Furthermore, if $\mathcal{A}$ is additive then it is straightforward to show that the skeleton $\mathcal{A}^{\text{skel}}$ is also an additive category and that $\mathscr{F}, \mathscr{G}$ are additive functors.

The following observation is well-known (see, for example, [BM12, §4]) and follows immediately from [Kra10, Lem. 3.3.1].

Corollary II.5.10. *Suppose $\mathcal{M}$ admits a calculus of left fractions in a skeletally small category $\mathcal{A}$. Then the Gabriel-Zisman localisation $\mathcal{A}_{\mathcal{M}} \cong \mathcal{M}^{-1}\mathcal{A}$ exists.*

Dually, if $\mathcal{M}$ admits a calculus of right fractions in $\mathcal{A}$, we can construct the *category of right fractions of $\mathcal{A}$ with respect to $\mathcal{M}$*. Furthermore, there are duals of Propositions II.5.6, II.5.7 and II.5.8, and Corollary II.5.10. Hence, if $\mathcal{M}$ is a multiplicative system, i.e. admits a calculus of left fractions and a calculus of right fractions, then the Gabriel-Zisman localisation $\mathcal{A}_{\mathcal{M}}$ of $\mathcal{A}$ at $\mathcal{M}$, the category of left fractions of $\mathcal{A}$ with respect to $\mathcal{M}$ and the category of right fractions of $\mathcal{A}$ with respect to $\mathcal{M}$ are all isomorphic. In particular, in this case a morphism in $\mathcal{A}_{\mathcal{M}}$ can be viewed in two ways: as a left fraction $[f, s]_{\text{LF}}$ or as a right fraction $[t, g]_{\text{RF}}$, and we have $[f, s]_{\text{LF}} = [t, g]_{\text{RF}}$ if and only if $ft = sg$.

II.6 Triangulated categories

Many of the categories we will come across in this thesis will have more structure than an additive category but will not be abelian. The categories we often consider are triangulated categories, and examples include the derived category and the cluster category.

In this section we recall the definition and some basic properties of a triangulated category. We use [Hap88] and [HJ10] as the main references for this section.

Throughout this section we assume that $\mathcal{C}$ is an additive category endowed with an additive *automorphism* Σ.

Definition II.6.1. [HJ10, p. 11] A *triangle* in $\mathcal{C}$ is a sequence of composable morphisms in $\mathcal{C}$ of the form

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X.$$

Definition II.6.2. [HJ10, p. 11] A *morphism of triangles* from a triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ to a triangle $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X'$ in $\mathcal{C}$ is a triple (f, g, h) of morphisms $f\colon X \to X'$, $g\colon Y \to Y'$ and $h\colon Z \to Z'$, such that the following diagram commutes in $\mathcal{C}$:

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & \Sigma X'
\end{tikzcd}$$

Further, if f, g, h are all isomorphisms then we say (f, g, h) is an *isomorphism* of triangles.

For the formal definition of a triangulated category, we follow the labelling of axioms as in [HJ10, §3], except that our axiom (TR3) is slightly weaker than the one given in [HJ10]. This is because the converse of the axiom may be deduced from axioms (TR0)–(TR3) as labelled below; see Lemma II.6.7.

Definition II.6.3. [HJ10, Def. 3.1], [Hap88, pp. 2–3] A collection $\mathcal{T}$ of triangles in $\mathcal{C}$ is called a *triangulation* (for $\mathcal{C}$), and the elements of $\mathcal{T}$ are then called *distinguished triangles*, if the following conditions are satisfied.

(TR0) Any triangle that is isomorphic to a distinguished triangle is also a distinguished triangle.

(TR1) For each $X \in \mathcal{C}$, the triangle $X \xrightarrow{1_X} X \xrightarrow{0} 0 \xrightarrow{0} \Sigma X$ is distinguished.

(TR2) For any morphism $f\colon X \to Y$ in $\mathcal{C}$, there exists a distinguished triangle $X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X$ for some $Z \in \mathcal{C}$.

(TR3) If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is a distinguished triangle, then the triangle $Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$ is also distinguished.

(TR4) Given distinguished triangles $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ and $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X'$, and morphisms $f\colon X \to X'$ and $g\colon Y \to Y'$ such that $gu = u'f$, there exists morphism $h\colon Z \to Z'$ such that

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, dotted, "\exists h"] & \Sigma X \arrow[d, "\Sigma f"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & \Sigma X'
\end{tikzcd}$$

commutes.

(TR5) (Octahedral Axiom) Given distinguished triangles $X \xrightarrow{u} Y \longrightarrow Z' \longrightarrow \Sigma X$, $Y \xrightarrow{v} Z \longrightarrow X' \longrightarrow \Sigma Y$ and $X \xrightarrow{vu} Z \longrightarrow Y' \longrightarrow \Sigma X$, there exists a distinguished triangle $Z' \longrightarrow Y' \longrightarrow X' \longrightarrow \Sigma Z'$ such that the diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal, "1_X"'] & Y \arrow[r] \arrow[d, "v"] & Z' \arrow[r] \arrow[d] & \Sigma X \arrow[d, equal, "1_{\Sigma X}"] \\
X \arrow[r, "vu"] \arrow[d, "u"'] & Z \arrow[r] \arrow[d, equal, "1_Z"] & Y' \arrow[r] \arrow[d] & \Sigma X \arrow[d, "\Sigma u"] \\
Y \arrow[r, "v"] \arrow[d] & Z \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d, equal, "1_{X'}"] & \Sigma Y \arrow[d] \\
Z' \arrow[r] & Y' \arrow[r] & X' \arrow[r] & \Sigma Z'
\end{tikzcd}$$

commutes.

Definition II.6.4. [Hap88, p. 3] A *triangulated category* is a triple $(\mathcal{C}, \Sigma, \mathcal{T})$ where $\mathcal{C}$ is an additive category, Σ is an automorphism of $\mathcal{C}$ and $\mathcal{T}$ is a triangulation for $\mathcal{C}$.

If $(\mathcal{C}, \Sigma, \mathcal{T})$ is a triangulated category, then we typically refer to $\mathcal{C}$ as the triangulated category and assume the existence of Σ and $\mathcal{T}$ is understood. Furthermore, in this setting the functor Σ is called the *suspension* (or *shift*) functor, and we refer to elements of $\mathcal{T}$ as just *triangles* rather than distinguished triangles each time.

For the rest of this section, let $(\mathcal{C}, \Sigma, \mathcal{T})$ be a triangulated category.

Definition II.6.5. [Hap88, p. 4] Suppose $\mathcal{A}$ is an abelian category. A covariant additive functor $\mathscr{F}\colon \mathcal{C} \to \mathcal{A}$ is called *cohomological* if for each distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ in $\mathcal{C}$, there is a long exact sequence

$$\cdots \xrightarrow{\mathscr{F}(\Sigma^{i-1}w)} \mathscr{F}(\Sigma^i X) \xrightarrow{\mathscr{F}(\Sigma^i u)} \mathscr{F}(\Sigma^i Y) \xrightarrow{\mathscr{F}(\Sigma^i v)} \mathscr{F}(\Sigma^i Z) \xrightarrow{\mathscr{F}(\Sigma^i w)} \mathscr{F}(\Sigma^{i+1} X) \xrightarrow{\mathscr{F}(\Sigma^{i+1}u)} \cdots$$

in $\mathcal{A}$.

A contravariant additive functor $\mathscr{F}\colon \mathcal{C} \to \mathcal{A}$ is called *cohomological* if for each distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ in $\mathcal{C}$, there is a long exact sequence

$$\cdots \xrightarrow{\mathscr{F}(\Sigma^{i+1}u)} \mathscr{F}(\Sigma^{i+1} X) \xrightarrow{\mathscr{F}(\Sigma^i w)} \mathscr{F}(\Sigma^i Z) \xrightarrow{\mathscr{F}(\Sigma^i v)} \mathscr{F}(\Sigma^i Y) \xrightarrow{\mathscr{F}(\Sigma^i u)} \mathscr{F}(\Sigma^i X) \xrightarrow{\mathscr{F}(\Sigma^{i-1}w)} \cdots$$

in $\mathcal{A}$.

We now present some easy results that will be used often when dealing with triangulated categories. For a more extensive list of properties of a triangulated category, see [Hap88, §I]. Recall that Ab denotes the category of all abelian groups.

Lemma II.6.6. [Hap88, Prop. I.1.2 (b)] *Let $X \in \mathcal{C}$ be an arbitrary object. Then the covariant functor* $\mathrm{Hom}_{\mathcal{C}}(X, -)\colon \mathcal{C} \to \mathsf{Ab}$ *is cohomological and the contravariant functor* $\mathrm{Hom}_{\mathcal{C}}(-, X)\colon \mathcal{C} \to \mathsf{Ab}$ *is cohomological.*

Lemma II.6.7. [Hap88, Lem. I.1.3] *Suppose* $Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$ *is a triangle. Then* $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ *is also a triangle.*

Proposition II.6.8. [HJ10, Prop. 4.3] *Suppose we have a morphism*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & \Sigma X' \end{tikzcd}$$

of triangles in $\mathcal{C}$. *If* f *and* g *are isomorphisms, then* h *is an isomorphism.*

Lemma II.6.9. [BM12, Lem. 3.2] *Suppose*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & \Sigma X' \end{tikzcd}$$

is a morphism of triangles in $\mathcal{C}$.

(i) *If* $gu = 0$*, then there exist morphisms* $\varphi_1 \colon Z \to Y'$ *and* $\varphi_2 \colon \Sigma X \to Z'$ *such that* $h = v'\varphi_1 + \varphi_2 w$.

(ii) *If* $w'h = 0$*, then there exist morphisms* $\psi_1 \colon Y \to X'$ *and* $\psi_2 \colon Z \to Y'$ *such that* $g = u'\psi_1 + \psi_2 v$.

Later we will be considering the action of a triangle autoequivalence on a triangulated category. A triangle functor is an additive functor between triangulated categories that is compatible with the triangulated structures, and a triangle equivalence is an equivalence of categories that is also a triangle functor. We give precise definitions below.

Definition II.6.10. [Kel07, §5.4] Let $\mathcal{C} = (\mathcal{C}, \Sigma, \mathcal{T})$ and $\mathcal{C}' = (\mathcal{C}', \Sigma', \mathcal{T}')$ be triangulated categories. An additive covariant functor $\mathscr{F} \colon \mathcal{C} \to \mathcal{C}'$ is called a *triangle* functor, or a *triangulated* functor, if the following conditions are satisfied.

(i) There exists a natural isomorphism $\alpha = \{\alpha_X\}_{X \in \mathcal{C}} \colon \mathscr{F} \circ \Sigma \rightsquigarrow \Sigma' \circ \mathscr{F}$.

(ii) If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is a distinguished triangle in $\mathcal{C}$, then

$$\mathscr{F}(X) \xrightarrow{\mathscr{F}(u)} \mathscr{F}(Y) \xrightarrow{\mathscr{F}(v)} \mathscr{F}(Z) \xrightarrow{\alpha_X \circ \mathscr{F}(w)} \Sigma'(\mathscr{F}X)$$

is distinguished in $\mathcal{C}'$.

Definition II.6.11. [Kel07, §5.4] A triangle functor $\mathscr{F} : \mathcal{C} \to \mathcal{C}'$ is called a *triangle* equivalence if $\mathscr{F}$ is an equivalence of categories (in the sense of Definition II.1.14). In this case $\mathcal{C}$ and $\mathcal{C}'$ are said to be *triangle* equivalent.

Notation. We will sometimes denote a triangle functor (respectively, triangle equivalence) $\mathscr{F} : \mathcal{C} \to \mathcal{C}'$ by $\mathscr{F} : \mathcal{C} \underset{\triangle}{\longrightarrow} \mathcal{C}'$ (respectively, $\mathscr{F} : \mathcal{C} \underset{\triangle}{\xrightarrow{\simeq}} \mathcal{C}'$) in order to emphasise the kind of functor it is.

The following is standard notation. See, for example, [BMRRT, p. 576].

Definition II.6.12. For $X, Y \in \mathcal{C}$ and $i \in \mathbb{Z}$, we define

$$\mathrm{Ext}^i_{\mathcal{C}}(X, Y) := \mathrm{Hom}_{\mathcal{C}}(X, \Sigma^i Y).$$

We remark here that $\mathrm{Ext}^i_{\mathcal{C}}(X, Y) \cong \mathrm{Hom}_{\mathcal{C}}(\Sigma^{-i} X, Y)$ as Σ is an automorphism of $\mathcal{C}$.

We will see later that the cluster category has some Ext-symmetry. To formalise this we will need the definitions of a Serre functor and a Calabi-Yau category. See [Kel08, §2.6] for more details.

Definition II.6.13. [Kra15, p. 547] An additive category $\mathcal{A}$ is called Hom-*finite* if $\mathcal{A}$ is an S-category, for some commutative ring S, and $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is a finite length S-module for every $X, Y \in \mathcal{A}$. In this case, we will also say that $\mathcal{A}$ is a Hom-*finite S-category* in order to emphasise that the Hom-sets are finite length over S.

Definition II.6.14. [Kel08, §2.6] Let k be a field and let $\mathcal{C}$ be a triangulated, Hom-finite k-category with suspension functor Σ. A *right Serre functor* for $\mathcal{C}$ is a triangle functor $\mathbb{S}\colon \mathcal{C} \xrightarrow[\triangle]{} \mathcal{C}$ such that for any $X, Y \in \mathcal{C}$ we have

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \cong D\,\mathrm{Hom}_{\mathcal{C}}(Y,\mathbb{S}X),$$

which is functorial in both arguments and where $D(-) := \mathrm{Hom}_{k-\mathsf{mod}}(-,k)$. If $\mathbb{S}$ is also a triangle autoequivalence, then $\mathbb{S}\colon \mathcal{C} \xrightarrow[\triangle]{\simeq} \mathcal{C}$ is called a *Serre functor* and we say $\mathcal{C}$ *has Serre duality*.

Definition II.6.15. [Kel08, §2.6] Let k be a field and let $\mathcal{C}$ be a triangulated, Hom-finite k-category with suspension functor Σ. For $n \in \mathbb{N}$, we say $\mathcal{C}$ is *weakly n-Calabi-Yau* if $\mathcal{C}$ admits a Serre functor $\mathbb{S}$ such that there is a natural isomorphism $\mathbb{S} \cong \Sigma^n$ (as k-additive functors). Furthermore, if this isomorphism is as k-additive triangle functors, then we say $\mathcal{C}$ is *n-Calabi-Yau*.

II.7 Derived categories

One of the most well-known examples of a triangulated category is the derived category of an abelian category. The derived category is a localisation of the homotopy category at the class of quasi-isomorphisms, so we will see the work of our previous sections coming together here.

In §II.7.1 we swiftly get to the definition of the derived category $\mathsf{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$ and recall how one may give $\mathsf{D}(\mathcal{A})$ a triangulated structure. In §II.7.2, we specialise to the case where $\mathcal{A} = H-\mathsf{mod}$ for a finite-dimensional hereditary algebra H and recall how the Auslander-Reiten theory of the bounded derived category $\mathsf{D}^b(\mathcal{A})$ is induced by the Auslander-Reiten theory of $\mathcal{A}$.

II.7.1 The definition

First, we recall how the homotopy category associated to an additive category is defined. After this, we will be able to consider a special type of morphism—namely, a quasi-isomorphism—in the category of complexes of an abelian category. We will need an abelian category as we will make use of the existence of kernels and cokernels, and we will use the sufficiently nice structure an abelian category has in order to do some homological algebra. We use [Alu09, §§IX.3–IX.5], [Yek15, §4.2] and [Zim14, §3.5] as our main references for this section.

Let $\mathcal{A}$ be an additive category.

Definition II.7.1. [Alu09, §IX.3.1] A *(cochain) complex* $(X^\bullet, d_X^\bullet)$ in $\mathcal{A}$ is a sequence

$$\cdots \xrightarrow{d_X^{i-2}} X^{i-1} \xrightarrow{d_X^{i-1}} X^i \xrightarrow{d_X^i} X^{i+1} \xrightarrow{d_X^{i+1}} \cdots$$

of composable morphisms in $\mathcal{A}$ such that $d_X^i \circ d_X^{i-1} = 0$ for all $i \in \mathbb{Z}$. The morphisms d_X^i are known as the *differentials* of $(X^\bullet, d_X^\bullet)$. Typically we will denote $(X^\bullet, d_X^\bullet)$ just by $X^\bullet$ if no confusion may arise.

Given complexes $(X^\bullet, d_X^\bullet)$ and $(Y^\bullet, d_Y^\bullet)$, a *morphism of complexes* $\alpha^\bullet\colon (X^\bullet, d_X^\bullet) \to (Y^\bullet, d_Y^\bullet)$ is a collection $\{\alpha^i\colon X^i \to Y^i\}_{i\in\mathbb{Z}}$ of morphisms in $\mathcal{A}$, such that the square

$$\begin{tikzcd} X^i \arrow[r, "d_X^i"] \arrow[d, "\alpha^i"'] \arrow[dr, phantom, "\circ"] & X^{i+1} \arrow[d, "\alpha^{i+1}"] \\ Y^i \arrow[r, "d_Y^i"'] & Y^{i+1} \end{tikzcd}$$

commutes in $\mathcal{A}$ for each $i \in \mathbb{Z}$.

Definition II.7.2. [Alu09, §IX.3.2] The *category of complexes in $\mathcal{A}$* is the category $\mathsf{C}(\mathcal{A})$ whose objects are the complexes in $\mathcal{A}$, and for complexes $X^\bullet, Y^\bullet$ we set $\mathrm{Hom}_{\mathsf{C}(\mathcal{A})}(X^\bullet, Y^\bullet)$ to be the collection of all morphisms of complexes $X^\bullet \to Y^\bullet$.

It is easy to show that the category $\mathsf{C}(\mathcal{A})$ is additive. We will need the following subcategories of $\mathsf{C}(\mathcal{A})$ in order to obtain an autoequivalence on the bounded derived category later.

Definition II.7.3. [Kra07, §1.7] We make the following definitions.

(i) We put $\mathsf{C}^{\emptyset}(\mathcal{A}) := \mathsf{C}(\mathcal{A})$. (This notation will allow us to simplify some statements later.)

(ii) $\mathsf{C}^{+}(\mathcal{A})$ is the full subcategory of $\mathsf{C}(\mathcal{A})$ whose objects are *bounded below* complexes, i.e. complexes $X^{\bullet}$ for which there exists $N \in \mathbb{Z}$ such that for all $i \in \mathbb{Z}$ with $i < N$ we have $X^i = 0$.

(iii) $\mathsf{C}^{-}(\mathcal{A})$ is the full subcategory of $\mathsf{C}(\mathcal{A})$ whose objects are *bounded above* complexes, i.e. complexes $X^{\bullet}$ for which there exists $N \in \mathbb{Z}$ such that for all $i \in \mathbb{Z}$ with $i > N$ we have $X^i = 0$.

(iv) $\mathsf{C}^{b}(\mathcal{A}) = \mathsf{C}^{-}(\mathcal{A}) \cap \mathsf{C}^{+}(\mathcal{A})$ is the full subcategory of $\mathsf{C}(\mathcal{A})$ whose objects are *bounded* complexes, i.e. complexes $X^{\bullet}$ for which there exists $N \in \mathbb{N}$ such that for all $i \in \mathbb{Z}$ with $|i| > N$ we have $X^i = 0$.

The homotopy category $\mathsf{K}(\mathcal{A})$ is a quotient of the category $\mathsf{C}(\mathcal{A})$ of complexes by a certain ideal.

Definition II.7.4. [Alu09, Def. IX.4.8] Let $\alpha^{\bullet}, \beta^{\bullet} \colon X^{\bullet} \to Y^{\bullet}$ be morphisms of complexes in $\mathsf{C}(\mathcal{A})$. A *homotopy between $\alpha^{\bullet}$ and $\beta^{\bullet}$* is a collection $h = \{h^i : X^i \to Y^{i-1}\}_{i \in \mathbb{Z}}$ of morphisms in $\mathcal{A}$ such that $\beta^i - \alpha^i = h^{i+1} d_X^i + d_Y^{i-1} h^i$ for all $i \in \mathbb{Z}$. If a homotopy between $\alpha^{\bullet}$ and $\beta^{\bullet}$ exists, we say that $\alpha^{\bullet}$ is *homotopic* to $\beta^{\bullet}$ and denote this by $\alpha^{\bullet} \sim \beta^{\bullet}$.

Definition II.7.5. [Yek15, Def. 4.2.1] If a morphism of complexes $\alpha^{\bullet}$ in $\mathsf{C}(\mathcal{A})$ is homotopic to the zero morphism 0, then we say $\alpha^{\bullet}$ is *null-homotopic*.

For objects $X^\bullet, Y^\bullet$ in $\mathsf{C}(\mathcal{A})$, let $\mathcal{N}(X^\bullet, Y^\bullet)$ be the subcollection of $\mathrm{Hom}_{\mathsf{C}(\mathcal{A})}(X^\bullet, Y^\bullet)$ consisting of all null-homotopic morphisms. Then $\mathcal{N}$ forms an ideal of $\mathsf{C}(\mathcal{A})$ (see [Yek15, Prop. 4.2.3]).

Definition II.7.6. [Yek15, Def. 4.2.4] The *homotopy category* $\mathsf{K}(\mathcal{A})$ of complexes in $\mathcal{A}$ is the quotient category $\mathsf{C}(\mathcal{A})/\mathcal{N}$. That is, $\mathrm{obj}(\mathsf{K}(\mathcal{A})) = \mathrm{obj}(\mathsf{C}(\mathcal{A}))$, and for $X^\bullet, Y^\bullet \in \mathrm{obj}(\mathsf{K}(\mathcal{A}))$ we have

$$\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X^\bullet, Y^\bullet) = \mathrm{Hom}_{\mathsf{C}(\mathcal{A})}(X^\bullet, Y^\bullet)/\mathcal{N}(X^\bullet, Y^\bullet).$$

For $x \in \{\emptyset, +, -, b\}$, we define $\mathsf{K}^x(\mathcal{A})$ to be the full subcategory of $\mathsf{K}(\mathcal{A})$ whose objects are the same as those of $\mathsf{C}^x(\mathcal{A})$.

In order to show that the homotopy category is triangulated, we need an automorphism of $\mathsf{K}(\mathcal{A})$.

Definition II.7.7. [Alu09, p. 596] For $n \in \mathbb{Z}$, the *shift by n functor* $(-)[n]\colon \mathsf{C}(\mathcal{A}) \to \mathsf{C}(\mathcal{A})$ is defined by $X^\bullet \mapsto X[n]^\bullet$, where $X[n]^i := X^{i+n}$ and $d^i_{X[n]} := (-1)^n d^{i+n}_X$, and $\alpha[n]^i := \alpha^{i+n}$ for a morphism $\alpha^\bullet$ of complexes.

Example II.7.8. Let $X^\bullet$ be the complex:

$$\cdots \longrightarrow X^{-2} \xrightarrow{d_X^{-2}} X^{-1} \xrightarrow{d_X^{-1}} X^0 \xrightarrow{d_X^0} X^1 \longrightarrow \cdots$$

Then $X[1]^\bullet$ is the complex:

$$\cdots \longrightarrow X^{-1} \xrightarrow{-d_X^{-1}} X^0 \xrightarrow{-d_X^0} X^1 \xrightarrow{-d_X^1} X^2 \longrightarrow \cdots$$

degree -2 degree -1 degree 0 degree 1

That is, if $n \geqslant 0$ then $X[n]^\bullet$ is the complex $X^\bullet$ shifted left by n and with the same differentials up to a sign change of $(-1)^n$. And if $n < 0$ then we shift right by n instead, but still have a sign change of $(-1)^n = (-1)^{-n}$ on the differential maps.

It is clear that $(-)[1]\colon \mathsf{C}(\mathcal{A}) \to \mathsf{C}(\mathcal{A})$ is an automorphism of $\mathsf{C}(\mathcal{A})$, with inverse $(-)[-1]$. Moreover, for $x \in \{\emptyset, +, -, b\}$, these mutually inverse automorphisms induce automorphisms on $\mathsf{K}^x(\mathcal{A})$, which we also denote by $(-)[1]$ and $(-)[-1]$, respectively.

We need the following definition in order to describe the triangulations we will put on the homotopy category and on the derived category.

Definition II.7.9. [Alu09, p. 606] Let $\alpha^\bullet\colon X^\bullet \to Y^\bullet$ be a morphism in $\mathsf{C}(\mathcal{A})$. The *mapping cone* $\mathrm{MC}(\alpha)^\bullet$ *of* $\alpha^\bullet$ is the complex with $\mathrm{MC}(\alpha)^i := X[1]^i \oplus Y^i$ and with differentials

$$d^i_{\mathrm{MC}(\alpha)} := \begin{pmatrix} -d_X^{i+1} & 0 \\ \alpha^{i+1} & d_Y^i \end{pmatrix}.$$

Proposition II.7.10. [Zim14, p. 313] *Suppose $\alpha^\bullet\colon X^\bullet \to Y^\bullet$ is a morphism in $\mathsf{C}(\mathcal{A})$. Then there is a morphism of complexes $\iota_\alpha^\bullet\colon Y^\bullet \to \mathrm{MC}(\alpha)^\bullet$, where*

$$\iota_\alpha^i\colon Y^i \to X[1]^i \oplus Y^i = \mathrm{MC}(\alpha)^i$$

is the canonical inclusion; and there is also a morphism of complexes $\pi_\alpha^\bullet\colon \mathrm{MC}(\alpha)^\bullet \to X[1]^\bullet$, where

$$\pi_\alpha^i\colon \mathrm{MC}(\alpha)^i = X[1]^i \oplus Y^i \to X[1]^i$$

is the canonical projection.

Theorem II.7.11. [Zim14, Prop. 3.5.25] *Let $\mathcal{A}$ be an additive category. For $x \in \{\emptyset, +, -, b\}$, the category $\mathsf{K}^x(\mathcal{A})$ is a triangulated category with suspension functor $(-)[1]\colon \mathsf{K}^x(\mathcal{A}) \to \mathsf{K}^x(\mathcal{A})$ and triangulation consisting of every triangle isomorphic to one of the form*

$$\Delta_\alpha\colon\ X^\bullet \xrightarrow{\alpha^\bullet} Y^\bullet \xrightarrow{\iota_\alpha^\bullet} \mathrm{MC}(\alpha)^\bullet \xrightarrow{\pi_\alpha^\bullet} X[1]^\bullet,$$

in $\mathsf{K}^x(\mathcal{A})$, for some $\alpha^\bullet \in \mathrm{Hom}_{\mathsf{K}^x(\mathcal{A})}(X^\bullet, Y^\bullet)$.

For the remainder of this section, suppose further that $\mathcal{A}$ is an abelian category.

Suppose $(X^\bullet, d^\bullet)$ is a complex in $\mathsf{C}(\mathcal{A})$, and fix $i \in \mathbb{Z}$. Then the condition $d^i \circ d^{i-1} = 0$ implies that the domain $\operatorname{Im} d^{i-1}$ of the image morphism of d^{i-1} can be considered as a subobject of the domain $\operatorname{Ker} d^i$ of the kernel of d^i, via a morphism $s^i \colon \operatorname{Im} d^{i-1} \hookrightarrow \operatorname{Ker} d^i$, say. We define the i^{th} *cohomology* of $X^\bullet$ to be the object

$$H^i(X^\bullet) \coloneqq \frac{\operatorname{Ker} d^i}{\operatorname{Im} d^{i-1}} \coloneqq \operatorname{Coker}(s^i)$$

of $\mathcal{A}$. It can also be shown that, for any morphism $\alpha^\bullet \colon X^\bullet \to Y^\bullet$ of complexes, there is an induced morphism $H^i(\alpha) \colon H^i(X^\bullet) \to H^i(Y^\bullet)$ in $\mathcal{A}$ on cohomology. See [Alu09, §IX.3] for more details.

Definition II.7.12. [Alu09, Def. IX.4.3] A morphism of complexes $\alpha^\bullet \colon X^\bullet \to Y^\bullet$ is called a *quasi-isomorphism* if the induced morphism $H^i(\alpha) \colon H^i(X^\bullet) \to H^i(Y^\bullet)$ on cohomology is an isomorphism for all $i \in \mathbb{Z}$.

Since homotopic morphisms of complexes induce the same morphism in cohomology (see [Alu09, Prop. IX.4.10]), it makes sense to speak of quasi-isomorphisms in the homotopy category $\mathsf{K}(\mathcal{A})$ as well. Let Qis denote the class of all quasi-isomorphisms in $\mathsf{K}(\mathcal{A})$. The next three results show that Qis satisfies the conditions (LF1) = (RF1), (RF2), and (RF3) and (LF3) from §II.5.2. See [Ver96, §III.1], [Zim14], or [Kra07, §3.1].

Proposition II.7.13. *The identity morphisms in $\mathsf{K}(\mathcal{A})$ are quasi-isomorphisms, and the composition of any two quasi-isomorphisms is again a quasi-isomorphism.*

Proposition II.7.14. [Zim14, Lem. 3.5.33], [Kra07, §3.1] *Suppose $\alpha^\bullet \colon X^\bullet \to Z^\bullet$ is a morphism of complexes and $\nu^\bullet \colon Y^\bullet \to Z^\bullet$ is a quasi-isomorphism both in $\mathsf{K}(\mathcal{A})$. Then there exist an object $Z'^\bullet \in \mathsf{K}(\mathcal{A})$, a morphism of complexes $\alpha'^\bullet \colon Z'^\bullet \to Y^\bullet$*

and a quasi-isomorphism $\nu'^{\bullet}\colon Z'^{\bullet} \to X^{\bullet}$ such that

$$\begin{tikzcd} Z'^{\bullet} \arrow[r, dotted, "\alpha'^{\bullet}"] \arrow[d, dotted, "\nu'^{\bullet}"'] \arrow[dr, phantom, "\circ"] & Y^{\bullet} \arrow[d, "\nu^{\bullet}"] \\ X^{\bullet} \arrow[r, "\alpha^{\bullet}"'] & Z^{\bullet} \end{tikzcd}$$

commutes in $\mathsf{K}(\mathcal{A})$.

Proposition II.7.15. [Kra07, §3.1] *Let $f^{\bullet}\colon X^{\bullet} \to Y^{\bullet}$ be a morphism in* $\mathsf{K}(\mathcal{A})$. *Then there exists a quasi-isomorphism $\nu^{\bullet}\colon Y^{\bullet} \to Z^{\bullet}$ such that $\nu^{\bullet} f^{\bullet} = 0$ if and only if there exists a quasi-isomorphism $\mu^{\bullet}\colon W^{\bullet} \to X^{\bullet}$ such that $f^{\bullet}\mu^{\bullet} = 0$.*

One can also show that the dual of Proposition II.7.14 holds, and hence Qis admits both a calculus of left fractions and a calculus of right fractions in $\mathsf{K}(\mathcal{A})$.

Definition II.7.16. [Ver96, Def. III.1.2.2] Let $\mathcal{A}$ be an abelian category. Let Qis denote the collection of quasi-isomorphisms in the homotopy category $\mathsf{K}(\mathcal{A})$ of $\mathcal{A}$. The *derived category* $\mathsf{D}(\mathcal{A})$ of $\mathcal{A}$ is the Gabriel-Zisman localisation $\mathsf{K}(\mathcal{A})[\mathrm{Qis}^{-1}]$ of $\mathsf{K}(\mathcal{A})$ at Qis.

We will denote the localisation functor $\mathsf{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ by L_{Qis}.

Definition II.7.17. [Kra07, §1.7] For $x \in \{\emptyset, +, -, b\}$, we define $\mathsf{D}^x(\mathcal{A})$ to be the full subcategory of $\mathsf{D}(\mathcal{A})$ with $\mathrm{obj}(\mathsf{D}^x(\mathcal{A})) = \mathrm{obj}(\mathsf{K}^x(\mathcal{A})) = \mathrm{obj}(\mathsf{C}^x(\mathcal{A}))$.

The category $\mathsf{D}^b(\mathcal{A})$ is known as the *bounded derived category* of $\mathcal{A}$, and this will have a large role to play in our studies.

Example II.7.18. For a ring S, its associated derived category

$$\mathsf{D}(S\text{–}\mathsf{Mod}) = \mathsf{K}(S\text{–}\mathsf{Mod})[\mathrm{Qis}^{-1}]$$

is locally small. Furthermore, if S is noetherian, then

$$\mathsf{D}(S\text{–}\mathsf{mod}) = \mathsf{K}(S\text{–}\mathsf{mod})[\mathrm{Qis}^{-1}]$$

is also defined and is again locally small. See [Kra10, §4.14] or [Wei94, §§10.3–10.4] for details.

Note that, for $x \in \{\emptyset, +, -, b\}$, the automorphism $(-)[1]\colon \mathsf{K}^x(\mathcal{A}) \to \mathsf{K}^x(\mathcal{A})$ induces an automorphism of $\mathsf{D}^x(\mathcal{A})$, which is also denoted by $(-)[1]$.

Theorem II.7.19. [Zim14, Prop. 3.5.40] *Let $\mathcal{A}$ be an abelian category. For $x \in \{\emptyset, +, -, b\}$, the category $\mathsf{D}^x(\mathcal{A})$ is a triangulated category with suspension functor $(-)[1]\colon \mathsf{D}^x(\mathcal{A}) \to \mathsf{D}^x(\mathcal{A})$ and triangulation consisting of every triangle isomorphic to one of the form*

$$\Delta_\alpha\colon\ X^\bullet \xrightarrow{L_{\mathrm{Qis}}(\alpha^\bullet)} Y^\bullet \xrightarrow{L_{\mathrm{Qis}}(\iota_\alpha^\bullet)} \mathrm{MC}(\alpha)^\bullet \xrightarrow{L_{\mathrm{Qis}}(\pi_\alpha^\bullet)} X[1]^\bullet,$$

in $\mathsf{D}^x(\mathcal{A})$, for some $\alpha^\bullet \in \mathrm{Hom}_{\mathsf{K}^x(\mathcal{A})}(X^\bullet, Y^\bullet)$.

A complex $X^\bullet$ is said to have *bounded cohomology* if there exists $N \in \mathbb{N}$ such that $H^i(X^\bullet) = 0$ for all $i \in \mathbb{Z}$ with $|i| > N$. For an additive subcategory $\mathcal{B}$ of $\mathcal{A}$ and $x \in \{+, -\}$, we let $\mathsf{K}^{x,\mathrm{coh}b}(\mathcal{B})$ denote the full subcategory of $\mathsf{K}^x(\mathcal{B})$ whose objects are the complexes that have bounded cohomology.

Theorem II.7.20. [Zim14, Prop. 3.5.43] *If S is a noetherian ring, then we have triangle equivalences $\mathsf{D}^-(S-\mathsf{mod}) \underset{\triangle}{\simeq} \mathsf{K}^-(S-\mathsf{proj})$ and $\mathsf{D}^b(S-\mathsf{mod}) \underset{\triangle}{\simeq} \mathsf{K}^{-,cohb}(S-\mathsf{proj})$.*

From this, we obtain the following well-known result.

Corollary II.7.21. *If S is a noetherian ring with finite global dimension, then we have a triangle equivalence $\mathsf{D}^b(S-\mathsf{mod}) \underset{\triangle}{\simeq} \mathsf{K}^b(S-\mathsf{proj})$.*

One also has duals to Theorem II.7.20 and Corollary II.7.21 (see [Zim14, Rem. 3.5.45]):

Corollary II.7.22. *Let S be a noetherian ring. Then there are triangle equivalences* $\mathsf{D}^+(S-\mathsf{mod}) \underset{\triangle}{\simeq} \mathsf{K}^+(S-\mathsf{inj})$ *and* $\mathsf{D}^b(S-\mathsf{mod}) \underset{\triangle}{\simeq} \mathsf{K}^{+,cohb}(S-\mathsf{inj})$. *Furthermore, if S has finite global dimension, then we have a triangle equivalence* $\mathsf{D}^b(S-\mathsf{mod}) \underset{\triangle}{\simeq} \mathsf{K}^b(S-\mathsf{inj})$.

The next lemma is often useful and follows from [Zim14, Lem. 3.5.49]. Recall that a complex $X^\bullet$ is said to be a *stalk complex concentrated in degree* n if $X^i = 0$ for all $i \neq n$ and the *stalk* X^n is non-zero.

Lemma II.7.23. *If S is a noetherian ring, then $S-\mathsf{mod}$ is a full subcategory of* $\mathsf{D}^b(S-\mathsf{mod})$. *More precisely, each $X \in S-\mathsf{mod}$ is identified with the stalk complex concentrated in degree 0 with stalk X.*

By abuse of notation, in the situation of Lemma II.7.23, for $X \in S-\mathsf{mod}$ we denote by X the stalk complex $\cdots \to 0 \to X \to 0 \to \cdots$ concentrated in degree 0 as well.

II.7.2 Auslander-Reiten theory of the derived category of a finite-dimensional hereditary algebra

For this section, let $k = \overline{k}$ be an algebraically closed field and suppose H is a finite-dimensional, basic, connected, hereditary k-algebra. Our aim is to describe the Auslander-Reiten quiver of the bounded derived category $\mathsf{D}^b(H) := \mathsf{D}^b(H-\mathsf{mod})$ of the abelian category $H-\mathsf{mod}$ of finitely generated left H-modules. Recall that $\mathsf{D}^b(H)$ is a triangulated category with suspension functor $(-)[1]$, which shifts a complex by 1 degree to the left; see Theorem II.7.19.

The analogue for an Auslander-Reiten sequence in a Krull-Schmidt, triangulated category is the following.

Definition II.7.24. [Hap88, I.4.1] A triangle $X \overset{u}{\to} Y \overset{v}{\to} Z \overset{w}{\to} \Sigma X$ in a Krull-Schmidt, triangulated category $\mathcal{C}$ (with suspension functor Σ) is called an *Auslander-Reiten triangle*, or an *almost split triangle*, if

(i) X, Z are indecomposable;

(ii) $w \neq 0$; and

(iii) If $f: W \to Z$ is a non-retraction, then there exists $\widehat{f}: W \to Y$ such that $v\widehat{f} = f$.

Definition II.7.25. [RVdB02, §I.2] Let $\mathcal{C}$ be a Krull-Schmidt, triangulated category with suspension functor Σ. If, for every indecomposable $Z \in \mathcal{C}$, there exists an Auslander-Reiten triangle $X \to Y \to Z \to \Sigma X$, then we say that $\mathcal{C}$ *has right Auslander-Reiten triangles*. Similarly, $\mathcal{C}$ *has left Auslander-Reiten triangles* if, for every indecomposable $X \in \mathcal{C}$, there exists an Auslander-Reiten triangle $X \to Y \to Z \to \Sigma X$. And, we say $\mathcal{C}$ *has Auslander-Reiten triangles* if $\mathcal{C}$ has left and right Auslander-Reiten triangles.

The following lemma connects condition (ii) in Definition II.7.24 to split morphisms.

Lemma II.7.26. [Hap88, Lem. I.1.4] *Let $\mathcal{C}$ be a triangulated category with suspension functor Σ. Suppose $X \overset{u}{\to} Y \overset{v}{\to} Z \overset{w}{\to} \Sigma X$ is a triangle in $\mathcal{C}$. Then the following are equivalent.*

(i) *The morphism w is zero.*

(ii) *The morphism u is a section.*

(iii) *The morphism v is a retraction.*

There is also a corresponding relationship between irreducible morphisms and Auslander-Reiten triangles in a triangulated category.

Theorem II.7.27. [Hap88, Lem. I.4.8] *Suppose $\mathcal{C}$ is a Krull-Schmidt, triangulated, Hom-finite k-category that has Auslander-Reiten triangles. Let Σ denote the suspension functor of $\mathcal{C}$. Suppose $X \longrightarrow \bigoplus_{i=1}^{r} Y_i^{m_i} \longrightarrow Z \longrightarrow \Sigma X$ is an Auslander-Reiten triangle in $\mathcal{C}$, where X, Z and Y_i ($\forall 1 \leqslant i \leqslant r$) are indecomposable, and $Y_i \ncong Y_j$ for all $i \neq j$. Then for any indecomposable object Y' of $\mathcal{C}$, we have that $\mathrm{Irr}_{\mathcal{C}}(X, Y') \neq 0$ if and only if $Y' \cong Y_i$ for some $i \in \{1, \ldots, r\}$.*

Thus, for a Krull-Schmidt, triangulated, Hom-finite k-category with Auslander-Reiten triangles, the Auslander-Reiten quiver (in the sense of Definition II.3.14) will again record information about all the indecomposable objects and all the irreducible morphisms between them. We now recall the results needed to see that $\mathsf{D}^b(H)$ is a category of this kind. Firstly, since $\mathsf{D}^b(H)$ is (triangle) equivalent to $\mathsf{K}^b(H-\mathsf{proj})$ (see Corollary II.7.21), it follows that $\mathsf{D}^b(H)$ is a k-category.

The following is well-known.

Theorem II.7.28. *The category $\mathsf{D}^b(H)$ is Krull-Schmidt.*

Proof. This is an application of [LC07, Cor. B]. ∎

Hence, the *Auslander-Reiten quiver* $\Gamma_{\mathrm{AR}}(\mathsf{D}^b(H))$ of $\mathsf{D}^b(H)$ can be defined as in Definition II.3.14. Furthermore, indecomposable objects in $\mathsf{D}^b(H)$ have a simple description.

Lemma II.7.29. [Hap88, Lem. I.5.2] *An indecomposable object in $\mathsf{D}^b(H)$ is isomorphic to a stalk complex with indecomposable stalk.*

Since H is a finite-dimensional, basic, connected, hereditary k-algebra, we may assume that $H = kQ$ for some finite, acyclic, connected quiver Q by Theorem II.4.20.

Theorem II.7.30. [Hap88, Thm. I.4.6, §I.5.4] *The category $\mathsf{D}^b(H) = \mathsf{D}^b(H-\mathsf{mod})$ has Auslander-Reiten triangles. Moreover, any Auslander-Reiten triangle in $\mathsf{D}^b(H)$ is isomorphic to a triangle of the form:*

(i) $X \to Y \to Z \to X[1]$, *where* $0 \to X \to Y \to Z \to 0$ *is an Auslander-Reiten sequence in* $H-\mathsf{mod}$*; or*

(ii) $I_i[-1] \to (I_i/S_i)[-1] \oplus \operatorname{rad} P_i \to P_i \to I_i$ *for some* $i \in Q_0$*, where* I_i *is the injective,* P_i *is the projective and* S_i *is the simple at vertex* i *(see* Definition II.2.21*).*

Let us now see how we can extract an endofunctor of $\mathsf{D}^b(H)$, in such a way that it acts as an analogue of the Auslander-Reiten translation τ on $H-\mathsf{mod}$, by considering the Auslander-Reiten triangles of the two forms in Theorem II.7.30.

First, let us note that the Nakayama functor $\nu\colon H-\mathsf{proj} \xrightarrow{\simeq} H-\mathsf{inj}$ (see Definition II.4.7) induces an equivalence between certain subcategories of the bounded homotopy category.

Proposition II.7.31. [Hap88, p. 37] *The Nakayama functor* ν *on* $H-\mathsf{mod}$ *induces a triangle equivalence* $\mathsf{K}^b(H-\mathsf{proj}) \xrightarrow[\triangle]{\simeq} \mathsf{K}^b(H-\mathsf{inj})$*, which we also denote by* ν*.*

Since $\mathsf{D}^b(H)$ is triangle equivalent to both $\mathsf{K}^b(H-\mathsf{proj})$ and $\mathsf{K}^b(H-\mathsf{inj})$ by Corollaries II.7.21 and II.7.22, respectively, we have that ν induces a triangle autoequivalence on $\mathsf{D}^b(H)$ from Proposition II.7.31. By abuse of notation, we denote this induced functor by ν as well. That is, we have a triangle autoequivalence $\nu\colon \mathsf{D}^b(H) \xrightarrow[\triangle]{\simeq} \mathsf{D}^b(H)$.

Consider an Auslander-Reiten triangle of the form $X \to Y \to Z \to X[1]$ in $\mathsf{D}^b(H)$, where $0 \to X \to Y \to Z \to 0$ is an Auslander-Reiten sequence in $H-\mathsf{mod}$. Notice that $\tau Z \cong X$ where τ is the Auslander-Reiten translation on $H-\mathsf{mod}$ (as in Definition II.4.8). The module Z is an indecomposable non-projective left H-module and can be identified in $\mathsf{D}^b(H)$ with the stalk complex

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow Z \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

concentrated in degree 0. Recall from §II.4.2 that there is an exact sequence

$$0 \longrightarrow \tau Z \longrightarrow \nu P^{-1} \xrightarrow{\nu f} \nu P^0 \longrightarrow \nu Z \longrightarrow 0,$$

where $P^{-1} \xrightarrow{f} P^0 \longrightarrow Z \longrightarrow 0$ is a minimal projective presentation of Z. In fact, since H is hereditary, the complex $0 \longrightarrow P^{-1} \xrightarrow{f} P^0 \longrightarrow Z \longrightarrow 0$ will be a minimal projective *resolution* of Z. Thus, in $\mathsf{D}^b(H)$ we can replace Z with the truncated minimal projective resolution

$$P^\bullet\colon \quad \cdots \longrightarrow 0 \longrightarrow P^{-1} \xrightarrow{f} P^0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

with P^0 in degree 0. This is just the equivalence $\mathsf{D}^b(H) \simeq \mathsf{K}^b(H-\mathsf{proj})$.

Now we can apply the Nakayama functor $\nu\colon \mathsf{K}^b(H-\mathsf{proj}) \longrightarrow \mathsf{K}^b(H-\mathsf{inj})$ and shift by one to the right, i.e. apply the composite functor $[-1]\circ\nu$, yielding the complex

$$(\nu P^\bullet)[-1]\colon \ \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \nu P^{-1} \xrightarrow{\nu f} \nu P^0 \longrightarrow 0 \longrightarrow \cdots$$

with νP^{-1} in degree 0. This is now a minimal injective resolution of the cohomology in degree 0 (see [Rin84, p. 73]), which is the object we denoted by τZ, the Auslander-Reiten translate of Z, in §II.4.2. Thus, in $\mathsf{D}^b(H)$ the complex $(\nu P^\bullet)[-1]$ is isomorphic to the stalk complex $\cdots \to 0 \to \tau Z \to 0 \to \cdots$ concentrated in degree 0 using Corollary II.7.22.

Now suppose we have an Auslander-Reiten triangle $I_i[-1] \to (I_i/S_i)[-1] \oplus \operatorname{rad} P_i \to P_i \to I_i$ of the second kind, where i is some vertex of Q. Then $\nu P_i = I_i$ by [Sch14, Prop. 2.29], and so $I_i[-1] = (\nu P_i)[-1]$.

These considerations lead to the following.

Definition II.7.32. [Hap88, pp. 37, 42] The *Auslander-Reiten translation* on $\mathsf{D}^b(H)$ is defined to be the endofunctor $[-1]\circ\nu$, denoted also by τ.

The Auslander-Reiten translation $\tau \colon \mathsf{D}^b(H) \to \mathsf{D}^b(H)$ is a triangle autoequivalence as $[-1]$ is an automorphism of $\mathsf{D}^b(H)$ and ν is a triangle autoequivalence of $\mathsf{D}^b(H)$. Let ν^{-1} denote a quasi-inverse to the autoequivalence $\nu \colon \mathsf{D}^b(H) \to \mathsf{D}^b(H)$. A quasi-inverse for τ is then $\tau^{-1} := \nu^{-1} \circ [1]$, which we call the *inverse Auslander-Reiten translation* on $\mathsf{D}^b(H)$.

Proposition II.7.33. [Hap88, Cor. I.4.9] *The Auslander-Reiten quiver* $\Gamma_{\mathrm{AR}}(\mathsf{D}^b(H))$ *of* $\mathsf{D}^b(H)$ *equipped with the Auslander-Reiten translation* τ *is a stable translation quiver.*

We now recall how one may "stitch" together copies of the Auslander-Reiten quiver of $H-\mathsf{mod}$ to get an explicit description of the Auslander-Reiten quiver of $\mathsf{D}^b(H)$.

Let $\Gamma_0 := \Gamma_{\mathrm{AR}}(H-\mathsf{mod})$ be the Auslander-Reiten quiver of H. For each $i \in \mathbb{Z}$, we set Γ_i to be the quiver $\Gamma_{\mathrm{AR}}(H-\mathsf{mod})$ but where a vertex has label $X[i]$ where it was X before. We define a quiver Γ to be the disjoint union $\amalg_{i\in\mathbb{Z}}\Gamma_i$, such that for each $i \in \mathbb{Z}$ there is an additional 'connecting' arrow

$$I_x[i] \longrightarrow P_y[i+1]$$

connecting Γ_i to Γ_{i+1} whenever there is an arrow $x \longrightarrow y$ in Q.

Proposition II.7.34. [Hap88, Prop. I.5.5] *The Auslander-Reiten quiver* $\Gamma_{\mathrm{AR}}(\mathsf{D}^b(H))$ *of* $\mathsf{D}^b(H)$ *is the quiver* Γ *constructed above.*

Definition II.7.35. Suppose $\mathcal{A}$ is an additive category with skeleton $\mathcal{A}^{\mathrm{skel}}$. We denote by $\mathsf{ind}\,\mathcal{A}$ the full subcategory of $\mathcal{A}^{\mathrm{skel}}$ with $\mathrm{obj}(\mathsf{ind}\,\mathcal{A})$ consisting of only indecomposable objects.

Corollary II.7.36. [Hap88, pp. 54–55] *Let kQ be the path algebra of a Dynkin-type quiver Q.*

(i) *The Auslander-Reiten quiver of* $\mathsf{D}^b(kQ)$ *is isomorphic to* $\mathbb{Z}Q$.

(ii) *The mesh category* $\overline{k(\mathbb{Z}Q,\tau)}$ *is equivalent to* $\operatorname{ind} \mathsf{D}^b(kQ)$.

Recall that Theorem II.4.13 gave us a characterisation of almost split sequences in $A-\mathsf{mod}$, where A is a finite-dimensional algebra. In particular, it says that in any Auslander-Reiten sequence $0 \longrightarrow X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \longrightarrow 0$, the morphisms f', g' must be irreducible. An analogous statement is made in [Hap88] for an Auslander-Reiten triangle in a triangulated category, but incorrectly. More precisely, part (ii) of [Hap88, Prop. I.4.3] states that if $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is an Auslander-Reiten triangle, then u, v are irreducible. However, this is incorrect as stated because the middle term Y may be 0, but irreducible morphisms must be non-zero (see Proposition II.3.5). Auslander-Reiten triangles of the form $X \to 0 \to Z \to \Sigma X$ appear, for example, in the bounded derived category of the category of finitely generated modules over the path algebra of a quiver with one vertex and no arrows (see §IV.3 also). We provide a criterion below to detect when the morphisms u, v are irreducible in an Auslander-Reiten triangles.

Lemma II.7.37. *Let*

$$X \xrightarrow{\quad u \quad} Y \xrightarrow{\quad v \quad} Z \xrightarrow{\quad w \quad} \Sigma X \tag{II.7.1}$$

be an Auslander-Reiten triangle in a triangulated category $\mathcal{C}$ *with suspension functor* Σ. *The morphisms* u, v *are irreducible if and only if* Y *is non-zero.*

Proof. ($\Rightarrow$) Suppose u, v are irreducible. Then Y is non-zero by Corollary II.3.6.

($\Leftarrow$) Suppose Y is non-zero. We only show that u is irreducible, as the proof that v is irreducible is similar. Since (II.7.1) is an Auslander-Reiten triangle, $w \neq 0$ by definition (see Definition II.7.24). Thus, u is not a section by Lemma II.7.26. Assume, for contradiction, that u is a retraction. By Lemma II.6.7, $\Sigma^{-1}Z \xrightarrow{-\Sigma^{-1}w} X \xrightarrow{\quad u \quad} Y \xrightarrow{\quad v \quad} Z$ is a triangle, which is split since u is a retraction, and so $v = 0$ (using Lemma II.7.26). But v is minimal right almost split (see [Hap88, §I.4.5]), and in particular right minimal, so $v = 0 = v0$ implies that $0 \in \operatorname{End}_{\mathcal{C}} Y$ is an automorphism. This, in turn, yields $Y = 0$ but this is a contradiction with our assumption on Y. So u is not a retraction.

We refer the reader to [Hap88, p. 34] in order to show that $u = h_2h_1$ implies h_1 is a section or h_2 is a retraction. ■

The following proposition is useful when calculating triangles in $\mathsf{D}^b(H)$, and we will use it without reference in examples.

Proposition II.7.38. [RS07, §5] *Suppose $X, Y \in H-\mathsf{mod}$ are indecomposable and that $f\colon X \to Y$ is morphism in $H-\mathsf{mod}$. Considering f as a morphism in $\mathsf{D}^b(H)$, we have that*

$$\mathrm{MC}(f) \cong (\operatorname{Ker} f)[1] \oplus \operatorname{Coker} f.$$

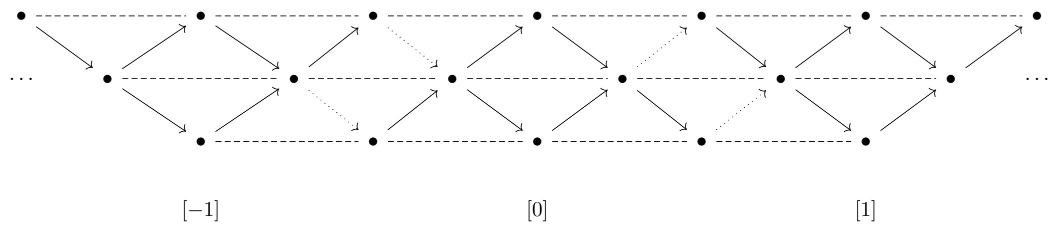
We conclude this section with an example of how one "stitches" together copies of the Auslander-Reiten quiver of a specific representation-finite path algebra to get the Auslander-Reiten quiver of the corresponding bounded derived category.

Example II.7.39. In Example II.4.25 we saw that the Auslander-Reiten quiver of $kQ-\mathsf{mod}$, where $Q\colon\ 1 \longrightarrow 2 \longrightarrow 3$, is:

$$\begin{tikzcd}
 & & P_1 = I_3 \arrow[dr] & & \\
 & P_2 \arrow[ur] \arrow[dr] \arrow[rr, dashed, no head] & & I_2 \arrow[dr] & \\
P_3 \arrow[ur] \arrow[rr, dashed, no head] & & S_2 \arrow[ur] \arrow[rr, dashed, no head] & & I_1
\end{tikzcd}$$

Since Q is a Dynkin-type quiver, Proposition II.7.34 tells us that the Auslander-

Reiten quiver of $\mathsf{D}^b(kQ)$ has the form:



where the dotted arrows are the 'connecting' arrows described just above the statement of Proposition II.7.34. See also Corollary II.7.36.

II.8 Cluster categories

The cluster category was introduced in [BMRRT] and it was shown that it models the combinatorics of the corresponding cluster algebra in certain cases. In Chapter V we will be studying the Auslander-Reiten theory of opposite endomorphism rings of certain objects in the cluster category. Furthermore, our work in Chapters III and IV has applications to the cluster category and a quotient of the cluster category, respectively.

The cluster category is a specific orbit category. In §II.8.1 we recall the definitions and some basic properties of these categories. In §II.8.2 we show how the Auslander-Reiten theory of the bounded derived category induces the Auslander-Reiten theory of the cluster category.

II.8.1 The definition

Given a category $\mathcal{A}$ with an automorphism $G\colon \mathcal{A} \to \mathcal{A}$, one may want to consider the "quotient" of $\mathcal{A}$ by the action of G—that is, a category with G-orbits as objects. The formal construction is as follows.

Definition II.8.1. [Kel05, §1], [CM06, Def. 2.3] Let $\mathcal{A}$ be an additive category and suppose G is an automorphism of $\mathcal{A}$. The *orbit category* $\mathcal{A}/G$ is the category which has the same objects as $\mathcal{A}$, and for $X, Y \in \mathrm{obj}(\mathcal{A}/G) = \mathrm{obj}(\mathcal{A})$ we set

$$\mathrm{Hom}_{\mathcal{A}/G}(X,Y) := \coprod_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(G^i X, Y).$$

The composition of morphisms in $\mathcal{A}/G$ is as follows. Given homogeneous morphisms $f_i \in \mathrm{Hom}_{\mathcal{A}}(G^i X, Y)$ and $g_j \in \mathrm{Hom}_{\mathcal{A}}(G^j Y, Z)$, the composition $g_j \circ f_i \in \mathrm{Hom}_{\mathcal{A}}(G^{i+j} X, Z)$ is defined to be $g_j G^j(f_i)\colon G^{i+j}X \to Z$. Thus, given arbitrary morphisms $f = (f_i) \in \mathrm{Hom}_{\mathcal{A}/G}(X,Y)$ and $g = (g_j) \in \mathrm{Hom}_{\mathcal{A}/G}(Y,Z)$, we define $g \circ f$ to be the morphism $h = (h_m)$, where

$$h_m = \sum_{m=i+j} g_j \circ f_i = \sum_{m=i+j} g_j G^{i+j}(f_i).$$

Note that this sum makes sense since f_i, g_j are non-zero for only finitely many i, j.

Suppose $G\colon \mathcal{A} \to \mathcal{A}$ is an automorphism of an additive category $\mathcal{A}$. It is easy to check that $\mathcal{A}/G$ is an additive category. There is a canonical additive functor $\pi\colon \mathcal{A} \to \mathcal{A}/G$ that is the identity on objects, and for any X, Y in $\mathcal{A}$ the mapping

$$\pi_{X,Y}\colon\ \mathrm{Hom}_{\mathcal{A}}(X,Y) \to \mathrm{Hom}_{\mathcal{A}/G}(X,Y)$$

is the canonical embedding

$$\mathrm{Hom}_{\mathcal{A}}(G^0 X, Y) \to \coprod_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(G^i X, Y).$$

Theorem II.8.2. [Kel05, Thm. 1] *Let k be a field. Suppose $\mathcal{D} = \mathsf{D}^b(H-\mathsf{mod})$ is the bounded derived category of a finite-dimensional hereditary k-algebra H. Let $(-)[1]$ denote the suspension functor of $\mathcal{D}$. Suppose G is a standard automorphism of $\mathcal{D}$ satisfying:*

(i) *for any object* $X \in \mathsf{ind}(H-\mathsf{mod})$*, the set* $\{G^iX \in \mathrm{obj}(\mathcal{D}) \mid G^iX \in H-\mathsf{mod}\}$ *is finite; and*

(ii) *there exists* $N \in \mathbb{N}$ *such that the set*

$$\{\, X[n] \in \mathrm{obj}(\mathcal{D}) \mid X \in \mathsf{ind}(H-\mathsf{mod}), 0 \leqslant n \leqslant N \,\}$$

meets the G*-orbit* $\{G^iY \mid i \in \mathbb{Z}\}$ *of any object* $Y \in \mathsf{ind}\,\mathcal{D}$ *non-trivially.*

Then the orbit category $\mathcal{D}/G$ *is triangulated with suspension functor induced by the suspension functor* $(-)[1]$ *of* $\mathcal{D}$*, and the canonical functor* $\pi\colon \mathcal{D} \xrightarrow[\triangle]{} \mathcal{D}/G$ *is a triangle functor.*

We observe here that, in the situation of Theorem II.8.2, one may obtain an orbit category $\mathcal{D}/G$ that is triangulated with only an autoequivalence G of $\mathcal{D}$ (but still satisfying the conditions in the Theorem). Thus, let $\mathcal{D}$ be as in Theorem II.8.2, and suppose $G\colon \mathcal{D} \to \mathcal{D}$ is an autoequivalence satisfying conditions (i) and (ii) above. By considering the skeleton $\mathcal{D}^{\mathrm{skel}}$ of $\mathcal{D}$, one obtains a triangulated category $\mathcal{D}' := \mathcal{D}^{\mathrm{skel}}$ (via "transport of structure"), a triangle equivalence $\mathscr{G}\colon \mathcal{D} \xrightarrow[\triangle]{\simeq} \mathcal{D}'$, and an automorphism $G'\colon \mathcal{D}' \to \mathcal{D}'$ induced by G (see [Kel05, §1]). In this situation we define the category $\mathcal{D}/G$ to be the orbit category $\mathcal{D}'/G'$. Note that there is still a canonical triangle functor $\pi\colon \mathcal{D} \to \mathcal{D}/G$ in this case. Indeed, if $\pi'\colon \mathcal{D}' \to \mathcal{D}'/G'$ is the canonical triangle functor of Theorem II.8.2, then

$$\pi := \pi' \circ \mathscr{G}\colon \mathcal{D} \xrightarrow[\triangle]{} \mathcal{D}'/G' = \mathcal{D}/G$$

is a triangle functor.

Remark II.8.3. Suppose $\mathcal{A}$ is an additive category and G is an automorphism of $\mathcal{A}$. We collect a few basic observations about $\mathcal{A}/G$ here.

(i) Let X be an object of $\mathcal{A}/G$. The identity morphism in $\mathrm{Hom}_{\mathcal{A}/G}(X, X)$ is the morphism (f_i) with $f_0 = 1_X \in \mathrm{Hom}_{\mathcal{A}}(X, X)$ and $f_i = 0$ for all $i \neq 0$. We will denote this identity morphism also by 1_X.

(ii) If X is an object in an orbit category $\mathcal{A}/G$, then $X \cong GX$. Indeed, consider the morphism $G(1_X) = 1_{GX} \in \mathrm{Hom}_{\mathcal{A}}(G^1X, GX)$ as a morphism in $\coprod_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(G^iX, GX) = \mathrm{Hom}_{\mathcal{A}/G}(X, GX)$, and the morphism $1_X \in \mathrm{Hom}_{\mathcal{A}}(X, X) = \mathrm{Hom}_{\mathcal{A}}(G^{-1}(GX), X)$ as a morphism in $\coprod_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(G^i(GX), X) = \mathrm{Hom}_{\mathcal{A}/G}(GX, X)$. Then in $\mathcal{A}/G$ we have

$$1_X \circ 1_{GX} = 1_X \circ G^{-1}(1_{GX}) = 1_X \circ G^{-1}(G(1_X)) = 1_X \circ 1_X = 1_X,$$

and similarly we have

$$1_{GX} \circ 1_X = 1_{GX} \circ G^1(1_X) = 1_{GX} \circ 1_{GX} = 1_{GX}.$$

This shows that X and GX are isomorphic in $\mathcal{A}/G$. This, of course, implies $X \cong G^iX$ for any $i \in \mathbb{Z}$.

(iii) The automorphism G induces an endofunctor $\widetilde{G}$ of $\mathcal{A}/G$ as follows. For an object $X \in \mathcal{A}/G$ we put $\widetilde{G}(X) = GX$, and $\widetilde{G}(f) = (Gf_i)$ for a morphism $f = (f_i)$. This functor $\widetilde{G}\colon \mathcal{A}/G \to \mathcal{A}/G$ is naturally isomorphic to the identity functor $\mathbb{1}_{\mathcal{A}/G}$.

With all this terminology, we are now able to define the categories that play a major role in the motivation of this thesis.

Definition II.8.4. [BMRRT, p. 576] Let k be a field. Suppose H is a finite-dimensional hereditary k-algebra. The *cluster category* of H is the orbit category

$$\mathcal{C}_H \coloneqq \mathsf{D}^b(H\,\text{–}\,\mathsf{mod})/F,$$

where $F \coloneqq \tau^{-1} \circ [1] \cong \nu^{-1} \circ [2]$.

We now recall some key properties of the cluster category; see [BMRRT] for more details.

Theorem II.8.5. *Let $\mathcal{C}_H$ be the cluster category of a finite-dimensional hereditary k-algebra H. Then $\mathcal{C}_H$ is a* Hom*-finite, Krull-Schmidt, triangulated k-category that is also 2-Calabi-Yau.*

Proof. Checking that $\mathcal{C}_H$ is a k-category is an easy exercise. The Hom-finiteness of $\mathcal{C}_H$ is noted in [BMRRT, p. 576] and $\mathcal{C}_H$ is Krull-Schmidt by [BMRRT, Prop. 1.2]. The autoequivalence F (really the induced automorphism of a skeleton of $\mathsf{D}^b(H-\mathsf{mod})$) satisfies the conditions of Theorem II.8.2 (see [Kel05, §7.2]), so $\mathcal{C}_H$ is a triangulated category with suspension induced from $\mathsf{D}^b(H-\mathsf{mod})$ and also denoted by $(-)[1]$.

It follows from [BMRRT, Prop. 1.4] that the functor τ^2 is a right Serre functor for $\mathcal{C}_H$. As $\tau^{-1}[1] = F \cong \mathbb{1}_{\mathcal{C}_H}$ on $\mathcal{C}_H$, we have $\tau \cong [1]$ so τ is a triangle autoequivalence of $\mathcal{C}_H$. Thus, τ^2 is also a triangle autoequivalence on $\mathcal{C}_H$ and hence τ^2 is a Serre functor (see Definition II.6.14). Lastly, note that $\tau^2 \cong [1]^2$ on $\mathcal{C}_H$, and thus $\mathcal{C}_H$ is 2-Calabi-Yau (see Definition II.6.15). See also [Kel05, §7.2]. ∎

II.8.2 Auslander-Reiten theory of a cluster category

Let $\mathcal{C}_H = \mathsf{D}^b(H)/F$ be the cluster category of a finite-dimensional hereditary k-algebra H, where $\mathsf{D}^b(H)$ denotes the bounded derived category $\mathsf{D}^b(H-\mathsf{mod})$ and $F = \tau^{-1}[1]$. In this section we will first recall the description of a fundamental domain for the action of F on $\mathsf{ind}\,\mathsf{D}^b(H)$. For this, we need the following standard notation.

Definition II.8.6. Let $\mathcal{C}$ be a triangulated category with suspension functor Σ. Suppose $\mathcal{B}$ is a full subcategory of $\mathcal{C}$ and $n \in \mathbb{Z}$. We denote by $\Sigma^n\mathcal{B}$ the full subcategory of $\mathcal{C}$ with $\mathrm{obj}(\Sigma^n\mathcal{B}) := \{\Sigma^n B \mid B \in \mathcal{B}\}$.

Consider the subcategory $(H-\mathsf{proj})[1]$ of $\mathcal{C}_H$. Each object in $(H-\mathsf{proj})[1]$ is of the form $P[1]$ for a projective left H-module $P \in \mathrm{obj}(H-\mathsf{mod})$, where we are viewing $H-\mathsf{mod}$ as a full subcategory of $\mathsf{D}^b(H)$. Set

$$\mathcal{E} := \mathrm{obj}(\mathsf{ind}(H-\mathsf{mod})) \cup \mathrm{obj}(\mathsf{ind}((H-\mathsf{proj})[1])) \subseteq \mathrm{obj}(\mathcal{C}_H).$$

Proposition II.8.7. [BMRRT, Prop. 1.6] *Consider the set* $\mathrm{obj}(\mathsf{ind}\,\mathcal{C}_H)$ *consisting of isoclass representatives, one for each isoclass of indecomposable objects in* $\mathcal{C}_H$*. Then* $\mathrm{obj}(\mathsf{ind}\,\mathcal{C}_H)$ *is in bijection with* $\mathcal{E}$*.*

Consider the Auslander-Reiten quiver $\Gamma_{\mathrm{AR}}(\mathsf{D}^b(H))$ of $\mathsf{D}^b(H)$. The autoequivalence $F\colon \mathsf{D}^b(H) \to \mathsf{D}^b(H)$ induces a quiver automorphism φ_F of $\Gamma_{\mathrm{AR}}(\mathsf{D}^b(H))$, which takes a vertex $[X]$ to the vertex $[FX]$. Recall that from a quiver with a quiver automorphism we can define a quotient quiver (see §II.2.1). See also [BMRRT, p. 577].

Proposition II.8.8. [BMRRT, Prop. 1.3] *The cluster category* $\mathcal{C}_H$ *has Auslander-Reiten triangles and these are induced by those in* $\mathsf{D}^b(H)$*. Furthermore, the Auslander-Reiten quiver* $\Gamma_{\mathrm{AR}}(\mathcal{C}_H)$ *of* $\mathcal{C}_H$ *is the quotient quiver* $\Gamma_{\mathrm{AR}}(\mathsf{D}^b(H))/\varphi_F$*, where* φ_F *is the quiver automorphism of* $\Gamma_{\mathrm{AR}}(\mathsf{D}^b(H))$ *induced by* F*.*

Example II.8.9. Let Q be the quiver $1 \longrightarrow 2 \longrightarrow 3$. In Example II.7.39, we saw that the Auslander-Reiten quiver of $\mathsf{D}^b(kQ) = \mathsf{D}^b(kQ-\mathsf{mod})$ is a disjoint union (indexed by $\mathbb{Z}$) of copies of the Auslander-Reiten quiver $\Gamma_{\mathrm{AR}}(kQ-\mathsf{mod})$ of kQ, with some additional arrows connecting one copy of $\Gamma_{\mathrm{AR}}(kQ-\mathsf{mod})$ to the next. Using Proposition II.8.8, the Auslander-Reiten quiver $\Gamma_{\mathrm{AR}}(\mathcal{C}_{kQ})$ of the cluster category $\mathcal{C}_{kQ}$ is:

$$\begin{tikzcd}[column sep=small]
 & & P_1[1] \arrow[dr] \arrow[rr, dash, dashed] & & P_1 = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[dr] \arrow[rr, dash, dashed] & & P_3[1] \arrow[dr] & & \\
 & P_2[1] \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & P_2 = \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & I_2 = \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & P_2[1] \arrow[dr] & \\
P_3[1] \arrow[ur] \arrow[rr, dash, dashed] & & P_3 = 3 \arrow[ur] \arrow[rr, dash, dashed] & & S_2 = 2 \arrow[ur] \arrow[rr, dash, dashed] & & S_1 = 1 \arrow[ur] \arrow[rr, dash, dashed] & & P_1[1]
\end{tikzcd}$$

where the lefthand copy of $P_i[1]$ is identified with the corresponding righthand copy for $i = 1, 2, 3$.

II.9 Preabelian categories

Later in this thesis, we bring together aspects of Auslander-Reiten theory, which we have been recalling in the previous sections, and categories that are in general not abelian. The main result of Chapter III shows how one can obtain a quasi-abelian category (see Definition II.9.18) from a triangulated category. In addition, we also study semi-abelian categories (see Definition II.9.12) in §IV.2. A category of either kind is a preabelian category (see Definition II.9.1) with some additional structure.

In this section, we recall the notion of a preabelian category, and provide some basic results that will be helpful later. For more details, we direct the reader to [Rum01], [Bor94a] and [Pop73].

Definition II.9.1. [BD68, §5.4] A *preabelian* category is an additive category in which every morphism has a kernel and a cokernel.

Example II.9.2. Any abelian category (see Definition II.1.36) is preabelian.

Example II.9.3. [GM03, §II.5.17] The category of filtered abelian groups is preabelian and *not* abelian.

Recall that in a Krull-Schmidt category an object is indecomposable if and only if it has a local endomorphism ring. In a Hom-finite category, the same is true if and only if all idempotents split. Let us recall what this means now.

Definition II.9.4. [Aus74, p. 188], [Büh10, Def. 6.1] Let $\mathcal{A}$ be a category. We will say e *is an idempotent in* $\mathcal{A}$ if there exists an object X of $\mathcal{A}$ for which e is an idempotent of the ring $\mathrm{End}_{\mathcal{A}} X$. An idempotent e in $\mathcal{A}$ with $e \in \mathrm{End}_{\mathcal{A}} X$ *splits* if there exist morphisms $\pi\colon X \to Y$ and $\iota\colon Y \to X$ in $\mathcal{A}$ with $e = \iota\pi$ and $\pi\iota = 1_Y$. The category $\mathcal{A}$ *has split idempotents* (or is *idempotent complete*) if all idempotents in $\mathcal{A}$ split.

With the notation as in Definition II.9.4, we see that if $e = \iota\pi$ is a split idempotent then π is a retraction and ι is a section.

Proposition II.9.5. [Büh10, Rmk. 6.2] *Let $\mathcal{A}$ be an additive category. Then the following are equivalent.*

(i) *$\mathcal{A}$ has split idempotents.*

(ii) *Each idempotent in $\mathcal{A}$ admits a kernel in $\mathcal{A}$.*

(iii) *Each idempotent in $\mathcal{A}$ admits a cokernel in $\mathcal{A}$.*

Remark II.9.6. By Proposition II.9.5, any preabelian category $\mathcal{A}$ has split idempotents because every morphism in $\mathcal{A}$ admits a kernel, so in particular every idempotent does. See also [Bor94a, Prop. 6.5.4] and [Aus74, p. 188].

The following lemma is a standard result and often useful; see, for example, [ASS06, pp. 106–107]

Lemma II.9.7. *Suppose $\mathcal{A}$ is an additive category with split idempotents. Let $f\colon X \to Y$ be a morphism in $\mathcal{A}$.*

(i) *Suppose X is indecomposable and $Y \neq 0$. If f is a retraction, then f is an isomorphism.*

(ii) *Suppose Y is indecomposable and $X \neq 0$. If f is a section, then f is an isomorphism.*

The next lemma may be found as an exercise in [Osb00]. For a proof, see the proof of [Alu09, Lem. IX.1.8], which is the corresponding result in an abelian category; the proof in [Alu09] is sufficient since only the existence of (co)kernels is needed.

Lemma II.9.8. [Osb00, Exer. 7.13] *In a preabelian category:*

(i) *every kernel is the kernel of its cokernel; and*

(ii) *every cokernel is the cokernel of its kernel.*

Recall that for a morphism $f\colon X \to Y$ in an additive category, we denote the kernel (respectively, cokernel) of f, if it exists, by $\ker f\colon \operatorname{Ker} f \to X$ (respectively, $\operatorname{coker} f\colon Y \to \operatorname{Coker} f$).

The following result provides helpful criteria for detecting isomorphisms in a category.

Proposition II.9.9. *Let $\mathcal{A}$ be a category and suppose $f\colon X \to Y$ is a morphism in $\mathcal{A}$. Consider the following statements.*

(i) *f is an isomorphism.*

(ii) *f is a section and a retraction.*

(iii) *f is an epimorphism and a section.*

(iv) *f is a monomorphism and a retraction.*

(v) *f is an epimorphism and a kernel.*

(vi) *f is a monomorphism and a cokernel.*

Then statements (i)–(iv) *are equivalent. If $\mathcal{A}$ is a preabelian category, then all statements* (i)–(vi) *are equivalent.*

Proof. The equivalence of (i) and (ii) follows from the definitions; see Lemma II.3.2. The equivalence of (i) and (iii) follows from [Bor94a, Prop. 1.9.3], and the equivalence of (i) and (iv) follows from [Bor94a, Exer. 1.11.9]. Hence, (i)–(iv) are equivalent in any category.

Now suppose $\mathcal{A}$ is a preabelian category. If f is an isomorphism, then it is easy to see that f is epic and thus its cokernel is $Y \longrightarrow 0$ by [Alu09, Lem. IX.1.5]. Then it quickly follows that $f = \ker(\operatorname{coker} f)$. Thus, (i) implies (v). The converse follows from [Bor94a, Prop. 2.4.5].

Similarly, (i) and (vi) are equivalent, which establishes the equivalence of (i)–(vi) in case $\mathcal{A}$ is preabelian. ■

In a preabelian category one can factorise a morphism f in a canonical way. Recall that the coimage of f is the cokernel of the kernel of f, and the image of f is the kernel of the cokernel of f; see Definition II.1.38.

Proposition II.9.10. [Pop73, p. 24] *Let $\mathcal{A}$ be a preabelian category and $f\colon X \to Y$ a morphism in $\mathcal{A}$. Then f decomposes as*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, two heads, "\operatorname{coim} f"'] \arrow[dr, phantom, "\circ"] & Y \\ \operatorname{Coim} f \arrow[r, "\widetilde{f}"'] & \operatorname{Im} f \arrow[u, hook, "\operatorname{im} f"'] \end{tikzcd}$$

Definition II.9.11. [Pop73, p. 24] The morphism $\widetilde{f}$ in Proposition II.9.10 above is called the *parallel of f*. Furthermore, if $\widetilde{f}$ is an isomorphism then f is said to be *strict*.

A preabelian category is abelian if and only if every morphism is strict; see [Rum01, p. 167]. However, in an arbitrary preabelian category, there is no reason the parallel

$\widetilde{f}$ of a morphism f would be a monomorphism or an epimorphism. This yields the following definitions.

Definition II.9.12. [Rum01, p. 167] Let $\mathcal{A}$ be a preabelian category. We call $\mathcal{A}$ *left semi-abelian* if each morphism $f\colon X \to Y$ factorises as $f = ip$ for some monomorphism i and cokernel p. We call $\mathcal{A}$ *right semi-abelian* if instead each morphism f decomposes as $f = ip$ with i a kernel and p some epimorphism. If $\mathcal{A}$ is both left and right semi-abelian, then it is simply called *semi-abelian*.

Definition II.9.13. A morphism in a category is called *regular* if it is both a monomorphism and an epimorphism.

Remark II.9.14. Note that a preabelian category is semi-abelian if and only if, for every morphism f, the parallel $\widetilde{f}$ of f is regular (see [Rum01, pp. 167–168]).

One can show that pullbacks and pushouts exist in a preabelian category from the existence of kernels and cokernels, respectively; see [Pop73, Prop. 2.2]. We recall how these are defined now as they are used throughout this thesis.

Definition II.9.15. [Alu09, p. 567], [Rum01, p. 172] Let $f\colon X \to Z$ and $g\colon Y \to Z$ be two morphisms in a category $\mathcal{A}$ with a common codomain. The *pullback* (or *fibered product*) *of* f *and* g is an object $X \Pi_Z Y$ equipped with two morphisms $g'\colon X \Pi_Z Y \to X$ and $f'\colon X \Pi_Z Y \to Y$ in $\mathcal{A}$ such that $gf' = fg'$, and satisfying the following universal property: given any $\varphi_X\colon A \to X$, $\varphi_Y\colon A \to Y$ with $g\varphi_Y = f\varphi_X$, there exists a unique morphism $\sigma\colon A \to X \Pi_Z Y$ such that

$$\begin{tikzcd}
A \arrow[dr, dotted, "\exists!\sigma"] \arrow[drr, bend left, "\varphi_Y"] \arrow[ddr, bend right, "\varphi_X"'] & & \\
& X \Pi_Z Y \arrow[r, "f'"] \arrow[d, "g'"] \arrow[dr, phantom, "\square", near start] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Z
\end{tikzcd}$$

commutes. We use the symbol '□' in the corner of the pullback object to denote that the square above commutes and that it is a pullback square.

Let $f\colon Z \to X$ and $g\colon Z \to Y$ be two morphisms in $\mathcal{A}$ with a common domain. The *pushout* (or *cofibered product*) of f and g is an object $X \amalg_Z Y$ equipped with two morphisms $g'\colon X \to X \amalg_Z Y$ and $f'\colon Y \to X \amalg_Z Y$ in $\mathcal{A}$, such that $(X \amalg_Z Y, g'^{\mathrm{op}}, f'^{\mathrm{op}})$ is a pullback of f^{op} and g^{op} in $\mathcal{A}^{\mathrm{op}}$. The corresponding commutative diagram is

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, "f"] & Y \arrow[d, "f'"] \arrow[ddr, bend left, "\psi_Y"] & \\
X \arrow[r, "g'"] \arrow[drr, bend right, "\psi_X"'] & X \amalg_Z Y \arrow[ul, phantom, "\square" very near start] \arrow[dr, dotted, "\exists! \sigma"] & \\
& & A
\end{tikzcd}$$

and similarly we use a ‘□’ in the corner of the pushout object.

In addition, a square

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "b"'] \arrow[dr, phantom, "\square"] & B \arrow[d, "c"] \\
C \arrow[r, "d"'] & D
\end{tikzcd}$$

is called *exact* if it simultaneously a pullback square and a pushout square, in which case the ‘□’ is placed in the middle of the square.

Definition II.9.16. Suppose

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "b"'] & B \arrow[d, "c"] \\
C \arrow[r, "d"'] & D
\end{tikzcd}$$

is a commutative diagram in a category $\mathcal{A}$. Let $\mathcal{P}$ be a class of morphisms in $\mathcal{A}$ (e.g. the class of all kernels in $\mathcal{A}$). We say that $\mathcal{P}$ is *stable under pullback* (respectively, *stable under pushout*) if, in any diagram above that is a pullback (respectively, pushout) square, d is in $\mathcal{P}$ implies a is in $\mathcal{P}$ (respectively, a is in $\mathcal{P}$ implies d is in $\mathcal{P}$).

Remark II.9.17. Note that the class of monomorphisms is stable under pullback in any category, and the class of kernels is stable under pullback in any additive category. Dually, the class of epimorphisms is always stable under pushout, and the class of cokernels is stable under pushout in an additive setting. See [Kel69, Prop. 5.2].

This motivates the next two definitions.

Definition II.9.18. [Rum01, p. 168] Let $\mathcal{A}$ be a preabelian category. We call $\mathcal{A}$ *left quasi-abelian* if cokernels are stable under pullback in $\mathcal{A}$. If kernels are stable under pushout in $\mathcal{A}$, then we call $\mathcal{A}$ *right quasi-abelian*. If $\mathcal{A}$ is left and right quasi-abelian, then $\mathcal{A}$ is called *quasi-abelian*.

Recall that the class of monomorphisms does not necessarily coincide with the class of kernels in general. Similarly for epimorphisms and cokernels.

Definition II.9.19. [Rum01, p. 168] Let $\mathcal{A}$ be a preabelian category. We call $\mathcal{A}$ *left integral* if epimorphisms are stable under pullback in $\mathcal{A}$. If monomorphisms are stable under pushout in $\mathcal{A}$, then we call $\mathcal{A}$ *right integral*. If $\mathcal{A}$ is both left and right integral, then $\mathcal{A}$ is called *integral*.

Remark II.9.20. The history of the term 'quasi-abelian' category is not straightforward. We use the terminology as in [Rum08], but note that such categories were called 'almost abelian' in [Rum01]. We refer the reader to the 'Historical remark' in [Rum08] for more details.

We recall an observation from [Rum01].

Proposition II.9.21. [Rum01, p. 169, Cor. 1]

(i) *Every left (respectively, right) quasi-abelian category is left (respectively, right) semi-abelian.*

(ii) *Every left (respectively, right) integral category is left (respectively, right) semi-abelian.*

We finish this section with some examples. The first example (due to Rump) is of a semi-abelian category that is not quasi-abelian. The second example is of a quasi-abelian and integral category with a quasi-abelian (proper) subcategory. The third and fourth examples are of quasi-abelian categories that are not abelian. The last example explores how one shows the existence of kernels and cokernels in a torsion class, which is a quasi-abelian category, in an abelian category.

Example II.9.22. [Rum08, Exam. 1] Let k be a field. Let Q be the quiver

$$\begin{tikzcd} 1 \arrow[r, "\alpha"] \arrow[d, "\gamma"] & 2 \arrow[d, "\delta"] & 3 \arrow[l, "\beta"'] \arrow[d, "\varepsilon"] \\ 4 \arrow[r, "\zeta"'] & 5 & 6 \arrow[l, "\eta"] \end{tikzcd}$$

and consider the bound quiver algebra $A := kQ/(\delta\alpha - \zeta\gamma, \delta\beta - \eta\varepsilon)$. Then the full subcategory $A-\mathsf{proj}$ of $A-\mathsf{mod}$ is semi-abelian, but neither left nor right quasi-abelian.

Example II.9.23. [Rum01, §2.2] Let TAb denote the category of topological abelian groups which has continuous group homomorphisms as the morphisms. Then TAb is both quasi-abelian and integral. The full subcategory HAb consisting of Hausdorff topological abelian groups is quasi-abelian.

Example II.9.24. [Rum01, §2.3] Let $k \in \{\mathbb{R}, \mathbb{C}\}$. The category of all *Banach spaces over* k, i.e. complete normed k-vector spaces, is quasi-abelian but not abelian.

Example II.9.25. [Pro00] The category of locally convex topological $\mathbb{C}$-vector spaces with continuous linear maps as morphisms is quasi-abelian but not abelian.

Example II.9.26. [Rum01, §4] Let $\mathcal{A}$ be an abelian category. Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory in $\mathcal{A}$. Recall that a *torsion theory* in $\mathcal{A}$ is a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\mathcal{A}$, such that $\operatorname{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{F}) = 0$, $\operatorname{Hom}_{\mathcal{A}}(X, \mathcal{F}) = 0$ implies $X \in \mathcal{T}$, and $\operatorname{Hom}_{\mathcal{A}}(\mathcal{T}, Y) = 0$ implies $Y \in \mathcal{F}$. Furthermore, for any object $X \in \mathcal{A}$ there exists an exact sequence $0 \to T_X \to X \to F_X \to 0$ in $\mathcal{A}$ with $T_X \in \mathcal{T}$ and $F_X \in \mathcal{F}$.

It is straightforward to check that $\mathcal{T}$ is additive. Let us show how kernels and cokernels are obtained in $\mathcal{T}$. (The existence of kernels and cokernels in $\mathcal{F}$ is dual.) To this end, suppose $t\colon T \to U$ is an arbitrary morphism in $\mathcal{T}$. Let $k\colon K \to T$ (respectively, $c\colon U \to C$) be the kernel (respectively, cokernel) of t in $\mathcal{A}$. As $(\mathcal{T}, \mathcal{F})$ is a torsion theory, there exists a short exact sequence $0 \to T_K \xrightarrow{a} K \xrightarrow{b} F_K \to 0$ in $\mathcal{A}$ with $T_K \in \mathcal{T}$ and $F_K \in \mathcal{F}$. We claim that the monomorphism $ka\colon T_K \to T$ is a kernel for t in $\mathcal{T}$. First, $t(ka) = 0$ as $tk = 0$. Now suppose that we have a morphism $v\colon V \to T$, with $V \in \mathcal{T}$, such that $tv = 0$. Then there exists $w\colon V \to K$ such that $v = kw$ as $k\colon K \to T$ is a kernel of t. The morphism $bw\colon V \to F_K$ vanishes as $\mathrm{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{F}) = 0$, and hence there is a morphism $x\colon V \to T_K$ such that $w = ax$. We see that $(ka)x = kw = v$, and so ka is a kernel of t in $\mathcal{T}$ as the uniqueness of x follows from ka being monic.

Lastly, we claim that the cokernel morphism $c\colon U \to C$ is already a morphism in the torsion class $\mathcal{T}$; that is, we claim $C \in \mathcal{T}$. Note that is is enough to show $\mathrm{Hom}_{\mathcal{A}}(C, \mathcal{F}) = 0$ as $(\mathcal{T}, \mathcal{F})$ is a torsion theory. Thus, let $f\colon C \to F$ be a morphism in $\mathcal{A}$ with $F \in \mathcal{F}$. Then the morphism $fc \in \mathrm{Hom}_{\mathcal{A}}(U, F)$ is the zero map as $U \in \mathcal{T}$ and $F \in \mathcal{F}$. But c is an epimorphism as it is a cokernel, so $fc = 0$ yields $f = 0$. Therefore, $C \in \mathcal{T}$ and $c\colon U \to C$ is a cokernel of t in $\mathcal{T}$.

II.10 Ext in a preabelian category

In order to avoid some Hom-finiteness restrictions in later arguments, we recall in this section how a first extension group (see Definition II.10.11) may be defined in a preabelian category in such a way that it is a bimodule (see Theorem II.10.16). Although we follow the development in [RW77], there is an error in their Theorem 4 ([RW77, p. 523]) that is corrected in [Coo80]. However, we also believe there should be more (set-theoretic) assumptions in place to ensure that the first extension group is indeed a group (see Remark II.10.12).

Let $\mathcal{A}$ denote an additive category. Recall that a sequence $\xi\colon\ X \xrightarrow{f} Y \xrightarrow{g} Z$ of morphisms in $\mathcal{A}$ is called a *complex* if $gf = 0$. We note that what we call a complex is called a 'sequence' in [RW77].

Definition II.10.1. [RW77, p. 523] A complex $\xi\colon X \overset{f}{\to} Y \overset{g}{\to} Z$ in $\mathcal{A}$ is called:

(i) *left exact* if $f = \ker g$;

(ii) *right exact* if $g = \operatorname{coker} f$; and

(iii) *short exact* if it is both left exact and right exact.

Definition II.10.2. Suppose $\nu\colon A \overset{a}{\to} B \overset{b}{\to} C$ and $\xi\colon X \overset{f}{\to} Y \overset{g}{\to} Z$ are complexes in $\mathcal{A}$. A *morphism* $(u, v, w)\colon \nu \to \xi$ *of complexes* is a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{a} & B & \xrightarrow{b} & C \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

in $\mathcal{A}$. An *isomorphism of complexes* is a morphism (u, v, w) of complexes for which the morphisms u, v, w are all isomorphisms themselves in $\mathcal{A}$.

We state a version of the well-known Splitting Lemma for an additive category, which is normally stated for an abelian category (see, for example, [ML95, Prop. I.4.3] or [Bor94b, Prop. 1.8.7]). In addition, we do not assume initially that the sequence of morphisms is short exact, since this is a consequence of the equivalent conditions. We omit the proof because the one in [Bor94b] works essentially unchanged.

Proposition II.10.3 (Splitting Lemma)**.** *Let $\mathcal{A}$ be an additive category with a sequence $\xi\colon X \overset{f}{\to} Y \overset{g}{\to} Z$ of morphisms. Then the following are equivalent.*

(i) *There is an isomorphism* $Y \cong X \oplus Z$*, where* f *corresponds to the canonical inclusion* $X \hookrightarrow X \oplus Z$ *and* g *to the canonical projection* $X \oplus Z \twoheadrightarrow Z$*.*

(ii) *The morphism* f *is a section and* ξ *is right exact.*

(iii) *The morphism* g *is a retraction and* ξ *is left exact.*

In this case, $X \xrightarrow{f} Y \xrightarrow{g} Z$ *is short exact.*

Definition II.10.4. A short exact sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$ is called *split* if it satisfies any of the equivalent conditions of Proposition II.10.3. Otherwise, the sequence is said to be *non-split*.

In a non-split short exact sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$, we have that f is not a section and g is not a retraction by Proposition II.10.3. However, more can be said as we see now.

Lemma II.10.5. *Let* $\mathcal{A}$ *be an additive category, and suppose* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *is a non-split short exact sequence in* $\mathcal{A}$*. Then both* f *and* g *are neither sections nor retractions.*

Proof. As noted above, we need only show that f is not a retraction and that g is not a section. Assume, for contradiction, that f is a retraction. Then f is an epimorphism and so $Z \cong \operatorname{Coker} f \cong 0$ by [Alu09, Lem. IX.1.5]. However, this implies that g is a retraction which is a contradiction. Therefore, f cannot be a retraction. Showing that g is not a section is dual. ■

Throughout the remainder of this section, $\mathcal{A}$ is assumed to be a preabelian category and we suppose X, Z are objects of $\mathcal{A}$.

Definition II.10.6. [RW77, p. 523] Let $\xi \colon X \xrightarrow{f} Y \xrightarrow{g} Z$ be a complex in $\mathcal{A}$. Let $a \colon X \to X'$ be any morphism in $\mathcal{A}$. We define a new complex $a\xi$ as follows. First, we form the pushout $X' \amalg_X Y$ of a along f with morphisms $f' \colon X' \to X' \amalg_X Y$

and $a'\colon Y \to X' \amalg_X Y$. Then we obtain a unique morphism $g'\colon X' \amalg_X Y \to Z$ using the universal property of the pushout with the morphisms $0\colon X' \to Z$ and $g\colon Y \to Z$ as in the following commutative diagram.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "a"] & Y \arrow[r, "g"] \arrow[d, "a'"] & Z \arrow[dd, equal] \\
X' \arrow[r, "f'"] \arrow[drr, bend right, "0"'] \arrow[ur, phantom, "\square"] & X' \amalg_X Y \arrow[dr, dotted, "\exists! g'"] & \\
 & & Z
\end{tikzcd}$$

The complex $a\xi$ is then defined to be $X' \xrightarrow{f'} X' \amalg_X Y \xrightarrow{g'} Z$.

Dually, we define ξb for a morphism $b\colon Z' \to Z$. The commutative diagram

$$\begin{tikzcd}
X \arrow[dd, equal] \arrow[dr, dotted, "\exists! f'"] \arrow[drr, bend left, "0"] & & \\
 & Y \Pi_Z Z' \arrow[r, "g'"] \arrow[d, "b'"] \arrow[dr, phantom, "\square"] & Z' \arrow[d, "b"] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

summarises the process and ξb is the complex $X \xrightarrow{f'} Y \Pi_Z Z' \xrightarrow{g'} Z'$.

Proposition II.10.7. [RW77, Thm. 4] *Suppose $\xi\colon X \xrightarrow{f} Y \xrightarrow{g} Z$ is a complex in $\mathcal{A}$. Let $a_1\colon X \to X_1, a_2\colon X_1 \to X_2, b_1\colon Z_1 \to Z, b_2\colon Z_2 \to Z_1$ be arbitrary morphisms in $\mathcal{A}$. Then $(a_2a_1)\xi \cong a_2(a_1\xi)$ and $(\xi b_1)b_2 \cong \xi(b_1b_2)$.*

Thus, for a complex ξ we may write $a_2a_1\xi$ and ξb_1b_2 without ambiguity.

Suppose $\nu\colon A \xrightarrow{a} B \xrightarrow{b} C$ and $\xi\colon X \xrightarrow{f} Y \xrightarrow{g} Z$ are short exact sequences in $\mathcal{A}$. A *morphism $\nu \to \xi$ of short exact sequences* is a just a morphism of complexes $(u, v, w)\colon \nu \to \xi$ as defined in Definition II.10.2.

If $A = X$ and $C = Z$, then a morphism of short exact sequences of the form $(1_X, v, 1_Z)$ in which $v\colon B \to Y$ is an isomorphism is called an *isomorphism of* *short exact sequences with the same end-terms*, and we denote this by $\nu \;{}_X{\cong}_Z\; \xi$. This is clearly an equivalence relation on the class of short exact sequences of the form $X \to - \to Z$.

Recall that in a preabelian category, kernels are stable under pullback but may not be stable under pushout, and dually for cokernels (see Remark II.9.17). Thus, Richman and Walker make the following definitions.

Definition II.10.8. [RW77, p. 524] Let $\xi\colon X \xrightarrow{f} Y \xrightarrow{g} Z$ be a short exact sequence in $\mathcal{A}$. We say that ξ is *stable* if $a\xi$ and ξb are short exact for all $a\colon X \to X'$, $b\colon Z' \to Z$. In this case, we call $f = \ker g$ a *stable kernel* and $g = \operatorname{coker} f$ a *stable cokernel*. We will sometimes also call ξ *stable exact* in this case to emphasise the exactness of ξ.

Theorem II.10.9. [Coo80, Thm. 2] *Suppose $\xi\colon X \xrightarrow{f} Y \xrightarrow{g} Z$ is a stable exact sequence in $\mathcal{A}$. Then there is an isomorphism $a(\xi b) \;{}_{X'}{\cong}_{Z'}\; (a\xi)b$ for all $a\colon X \to X'$, $b\colon Z' \to Z$ in $\mathcal{A}$.*

Remark II.10.10. Recall that we are trying to define a first extension group that is also a bimodule. It can readily be seen that a statement like Theorem II.10.9 is needed to give a definition of an extension group $\operatorname{Ext}^1_{\mathcal{A}}(Z, X)$ that is also an $(\operatorname{End}_{\mathcal{A}} X, \operatorname{End}_{\mathcal{A}} Z)$-bimodule. Theorem II.10.9 was claimed to hold for all sequences in [RW77] (see [RW77, Thm. 4]), but a counterexample was given in [Coo80]. Cooper presents the corrected statement as Theorem II.10.9, presenting one half of the argument and suggesting a diagram to use for the dual argument. However, the suggested dual diagram is not the right one to consider.

First, let us recall the set-up in [Coo80, Thm. 2]. Let $E\colon A \xrightarrow{f} B \xrightarrow{g} C$ be a stable exact sequence, and suppose we have morphisms $\alpha\colon A \to A'$ and $\beta\colon C' \to C$.

Then, as obtained in [Coo80, p. 266], there is a commutative diagram

$$\begin{tikzcd}
\alpha(E\beta)\colon \arrow[d, "{(1_{A'},\varphi_2,1_{C'})}"'] & A' \arrow[r, "f_2"] \arrow[d, equal] & B_2 \arrow[r, "g_2"] \arrow[d, "\varphi_2"] & C' \arrow[d, equal] \\
(\alpha E)\beta\colon & A' \arrow[r, "f_3"] & B_3 \arrow[r, "g_3"] & C'
\end{tikzcd}$$

It is shown in detail that $g_3 = \operatorname{coker} f_3$ and φ_2 is an epimorphism. It is then suggested that the diagram

$$\alpha(E\beta) \to (\alpha E)\beta \to \alpha E \to f_3\alpha E$$

with a dual proof strategy will yield $f_2 = \ker g_2$ and φ_2 is a monomorphism. However, this diagram should be replaced by

$$\alpha(E\beta) \to (\alpha E)\beta \to \alpha E \to f_2\alpha E.$$

Furthermore, we note that it is straightforward to find a morphism $\alpha(E\beta) \to (\alpha E)\beta$ of the form $(1_{A'}, \varphi_2, 1_{C'})$, but to then show that φ_2 is an isomorphism requires the stability of E (see [RW77, Cor. 7]). In an abelian category, the fact that φ_2 is an isomorphism would follow quickly from, for example, the Five Lemma.

Therefore, if ξ is stable exact, then we may write $a\xi b$ without ambiguity.

Following [ML95], we introduce some notation to help the reading of the sequel. Let A, B, C, D be objects in $\mathcal{A}$. We denote by ∇_A the *codiagonal* morphism $(\,1_A \;\; 1_A\,)\colon A \oplus A \to A$, and denote by Δ_A the *diagonal* morphism $\left(\begin{smallmatrix} 1_A \\ 1_A \end{smallmatrix}\right)\colon A \to A \oplus A$. For two morphisms $a\colon A \to C$ and $b\colon B \to D$, we let $a \oplus b$ denote the morphism $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)\colon A \oplus B \to C \oplus D$. We are now in a position to define $\operatorname{Ext}^1_{\mathcal{A}}(Z, X)$.

Definition II.10.11. [RW77, §4] Let $\mathcal{A}$ be a preabelian category. Define $\operatorname{Ext}^1_{\mathcal{A}}(Z, X)$ to be the class of equivalence classes under ${}_X{\cong}_Z$ of stable exact sequences of the form $X \to - \to Z$ in $\mathcal{A}$.

By abuse of terminology/notation, by an element ξ of $\operatorname{Ext}^1_{\mathcal{A}}(Z,X)$ we will really mean the equivalence class $[\xi]_{{}_X\cong_Z}$ of ξ in $\operatorname{Ext}^1_{\mathcal{A}}(Z,X)$. If $\xi\colon X \xrightarrow{f} Y \xrightarrow{g} Z$, $\xi'\colon X \xrightarrow{f'} Y' \xrightarrow{g'} Z$ are elements of $\operatorname{Ext}^1_{\mathcal{A}}(Z,X)$, then we define the *Baer sum* of ξ and ξ' to be (the equivalence class of)

$$\xi + \xi' \coloneqq \nabla_X(\xi \oplus \xi')\Delta_Z.$$

Note that by [RW77, Thm. 8] and Theorem II.10.9, $\xi + \xi'$ is stable exact and the Baer sum $+$ is a closed binary operation on $\operatorname{Ext}^1_{\mathcal{A}}(Z,X)$.

Remark II.10.12. It is observed in [RW77] that one may then follow [ML95, pp. 70–71] in order to show that $\operatorname{Ext}^1_{\mathcal{A}}(Z,X)$ is an abelian group. However, $\operatorname{Ext}^1_{\mathcal{A}}(Z,X)$ may not form a set and hence may not be a group. A similar issue arises in [Coo80].

Note, however, that if $\mathcal{A}$ is skeletally small, then $\operatorname{Ext}^1_{\mathcal{A}}(Z,X)$ will be a set. Indeed, for objects Y, Y' in $\mathcal{A}$, the sets $\{\xi\colon X \to Y \to Z \mid \xi \text{ is short exact}\}$ and $\{\xi'\colon X \to Y' \to Z \mid \xi' \text{ is short exact}\}$ are in bijection whenever Y is isomorphic to Y'. So, up to equivalence with respect to ${}_X\cong_Z$, the collection of all short exact sequences of the form $X \xrightarrow{f} Y \xrightarrow{g} Z$ is determined only by the isomorphism class of Y and the morphisms f, g since the end-terms X and Z are fixed. Therefore, the collection of all ${}_X\cong_Z$-equivalence classes will form a set, and hence restricting our attention to the classes of stable exact sequences will also yield a set.

These set-theoretic considerations lead us to the next theorem.

Theorem II.10.13. [RW77, §4] *Suppose $\mathcal{A}$ is a preabelian category with objects X, Z, and suppose $\operatorname{Ext}^1_{\mathcal{A}}(Z,X)$ is a set. Then $\operatorname{Ext}^1_{\mathcal{A}}(Z,X)$ is an abelian group with the group operation given by the Baer sum defined in* Definition II.10.11. *The class of the split extension $\xi_0\colon X \to X \oplus Z \to Z$ is the identity element, and the inverse of $\xi \in \operatorname{Ext}^1_{\mathcal{A}}(Z,X)$ is $(-1_X)\xi$.*

Therefore, if $\mathcal{A}$ is preabelian, $\operatorname{Ext}^1_{\mathcal{A}}(Z,X)$ is known as a *first extension group*.

Example II.10.14. For a commutative ring S and $M, N \in S-\mathsf{Mod}$, one normally defines extension groups $\operatorname{Ext}^i_S(M,N)$ homologically for $i \geqslant 0$ (see, for example, [Alu09, Thm. IX.8.14]). For $i=1$, $\operatorname{Ext}^1_S(M,N)$ is isomorphic to $\operatorname{Ext}^1_{S-\mathsf{Mod}}(M,N)$ (in the sense of Definition II.10.11 above) as abelian groups by, for example, [Wei94, Cor. 3.4.5] (see also [Yon54]). Therefore, we will choose to denote this abelian group by $\operatorname{Ext}^1_{S-\mathsf{Mod}}(M,N)$ throughout the remainder of this thesis.

We state without proof one corresponding result from [ML95] that is needed for the next theorem of this section.

Lemma II.10.15. [ML95, p. 71] *Let ξ_0 denote the split short exact sequence $X \to X \oplus Z \to Z$ and let $\xi \colon X \xrightarrow{f} Y \xrightarrow{g} Z$ be an arbitrary stable exact sequence. Let 0_X (respectively, 0_Z) denote the zero morphism in $\operatorname{End}_{\mathcal{A}} X$ (respectively, $\operatorname{End}_{\mathcal{A}} Z$). Then $0_X\xi \; {}_X{\cong}_Z \; \xi_0$ and $1_X\xi = \xi$, and $\xi 0_Z \; {}_X{\cong}_Z \; \xi_0$ and $\xi 1_Z = \xi$.*

Theorem II.10.16. *Let $\mathcal{A}$ be a preabelian category with objects X, Z, and suppose $\operatorname{Ext}^1_{\mathcal{A}}(Z,X)$ is a set. Then $\operatorname{Ext}^1_{\mathcal{A}}(Z,X)$ is an $(\operatorname{End}_{\mathcal{A}} X, \operatorname{End}_{\mathcal{A}} Z)$-bimodule.*

Proof. This follows from Theorem II.10.13, Proposition II.10.7, Lemma II.10.15 and Theorem II.10.9. ∎

In [AR77a] in the context of abelian categories, Auslander and Reiten use that $\operatorname{Ext}^1_{\mathcal{A}}(Z,X)$ is an $(\operatorname{End}_{\mathcal{A}} X, \operatorname{End}_{\mathcal{A}} Z)$-bimodule in order to avoid some Hom-finite (see Definition II.6.13) restrictions. We do the same in Chapter IV in a more general context.

The last result of this section tells us that $\operatorname{Ext}^1_{\mathcal{A}}(-,-)$ is also an additive bifunctor (see [RW77, §4]).

Theorem II.10.17. [RW77, §4], [Coo80, p. 267] *Let $\mathcal{A}$ be a preabelian category with objects X, Z. Assume that $\operatorname{Ext}^1_{\mathcal{A}}(Z,Y)$ and $\operatorname{Ext}^1_{\mathcal{A}}(Y,X)$ are sets for all objects Y of $\mathcal{A}$. Then $\operatorname{Ext}^1_{\mathcal{A}}(Z,-) \colon \mathcal{A} \to \mathsf{Ab}$ is a covariant additive functor and*

$\operatorname{Ext}^1_{\mathcal{A}}(-,X)\colon \mathcal{A} \to \mathsf{Ab}$ *is a contravariant additive functor, where* Ab *denotes the category of all abelian groups.*

Chapter III

Quasi-abelian hearts of twin cotorsion pairs on triangulated categories

III.1 Introduction

Cotorsion pairs were first defined specifically for the category of abelian groups in [Sal79] as an analogue of the torsion theories introduced in [Dic66], which were themselves used to generalise the notion of torsion in abelian groups. Torsion theories for triangulated categories were introduced in [IY08] and used in the study of rigid Cohen-Macaulay modules over specific Veronese subrings. Analogously, Nakaoka [Nak11] defined cotorsion pairs for triangulated categories as follows. Let $\mathcal{C}$ be a triangulated category with suspension functor Σ. A *cotorsion pair* on $\mathcal{C}$ is a pair $(\mathcal{U}, \mathcal{V})$ of full additive subcategories of $\mathcal{C}$ that are closed under isomorphisms and direct summands, satisfying $\operatorname{Ext}^1_{\mathcal{C}}(\mathcal{U}, \mathcal{V}) = 0$ and $\mathcal{C} = \mathcal{U} * \Sigma\mathcal{V}$ (see Definitions III.2.2 and III.2.3). This allowed Nakaoka to extract an abelian category, known as the *heart* of the cotorsion pair [Nak11, Def. 3.7], from the triangulated category. The key motivating examples for Nakaoka were the following.

(i) A t-structure $(\mathcal{C}^{\leqslant 0}, \mathcal{C}^{\geqslant 0})$ on a triangulated category $\mathcal{C}$, in the sense of [BBD82], can be interpreted as a cotorsion pair $(\Sigma\mathcal{C}^{\leqslant 0}, \Sigma^{-1}\mathcal{C}^{\geqslant 0})$. In this case the heart $\mathcal{C}^{\leqslant 0} \cap \mathcal{C}^{\geqslant 0}$ of the t-structure coincides with the heart of the cotorsion pair.

(ii) Suppose $\mathcal{C}$ is a triangulated category, with a tilting subcategory $\mathcal{T}$ (see [Iya07, Def. 2.2]). It was shown in [KZ08] (see also [BMR07] and [KR07]) that $\mathcal{C}/[\mathcal{T}]$ is an abelian category. The corresponding cotorsion pair in this setting is $(\mathcal{T}, \mathcal{T})$ and has $\mathcal{C}/[\mathcal{T}]$ as its heart.

In [BM12], Buan and Marsh generalised the results of [KZ08] and [BMR07] in the following way. Assume k is a field, and suppose $\mathcal{C}$ is a skeletally small, Hom-finite, Krull-Schmidt, triangulated k-category that has Serre duality (see Definition II.6.14). Fix an object R of $\mathcal{C}$ that is rigid (see Definition III.5.1). Let $\mathcal{X}_R$ denote the full additive subcategory of $\mathcal{C}$ that consists of objects X such that $\mathrm{Hom}_{\mathcal{C}}(R, X) = 0$, and consider the (Gabriel-Zisman) localisation $(\mathcal{C}/[\mathcal{X}_R])_{\mathcal{R}}$ (see §II.5) of $\mathcal{C}/[\mathcal{X}_R]$ at the class $\mathcal{R}$ of regular morphisms (see Definition II.9.13). Then the following was shown in [BM12].

Theorem III.1.1. [BM12, Thm. 5.7] *There is an equivalence*

$$(\mathcal{C}/[\mathcal{X}_R])_{\mathcal{R}} \simeq (\mathrm{End}_{\mathcal{C}}\, R)^{\mathrm{op}} - \mathsf{mod}\,.$$

Beligiannis further developed these ideas in [Bel13].

Nakaoka was then able to put this into a more general context by introducing the following concept in [Nak13]. A *twin cotorsion pair on* $\mathcal{C}$ consists of two cotorsion pairs $(\mathcal{S}, \mathcal{T})$ and $(\mathcal{U}, \mathcal{V})$ on $\mathcal{C}$ which satisfy $\mathcal{S} \subseteq \mathcal{U}$. As for cotorsion pairs, Nakaoka defined the *heart* of a twin cotorsion pair as a certain subfactor category of $\mathcal{C}$ (see Definition III.2.18). By setting $\mathcal{S} = \mathcal{U}$ and $\mathcal{T} = \mathcal{V}$, one recovers the original cotorsion pair theory: the heart of the twin cotorsion pair $((\mathcal{U}, \mathcal{V}), (\mathcal{U}, \mathcal{V}))$ coincides with the heart of the cotorsion pair $(\mathcal{U}, \mathcal{V})$ (see [Nak13, Exam. 2.10]).

For a twin cotorsion pair, the associated heart $\overline{\mathcal{H}}$ is shown [Nak13, Thm. 5.4] to be semi-abelian (see Definition II.9.12). Furthermore, Nakaoka showed [Nak13, Thm. 6.3] that if $\mathcal{U} \subseteq \mathcal{S} * \mathcal{T}$ or $\mathcal{T} \subseteq \mathcal{U} * \mathcal{V}$, then $\overline{\mathcal{H}}$ is integral (see Definition II.9.19), so that localising at the class of regular morphisms produces an abelian category (see [Rum01]). With $\mathcal{C}$ and R as above, and setting $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})) = ((\mathsf{add}\,\Sigma R, \mathcal{X}_R),(\mathcal{X}_R, {\mathcal{X}_R}^{\perp_1}))$, where ${\mathcal{X}_R}^{\perp_1} = \operatorname{Ker}(\operatorname{Ext}^1_{\mathcal{C}}(\mathcal{X}_R, -))$, one recovers Theorem III.1.1 (see Lemma III.5.6).

The main result of this chapter concerns quasi-abelian categories. Recall that a *quasi-abelian* category is an additive category that has kernels and cokernels, and in which kernels are stable under pushout and cokernels are stable under pullback (see Definition II.9.16). Important examples of such categories include: any abelian category; the category of topological abelian groups; and the torsion class and torsion-free class in any torsion theory of an abelian category (see Example II.9.26); see [Rum01, §2] for more details. In this chapter, we prove that the heart of a twin cotorsion pair, satisfying a different mild assumption, is quasi-abelian (see Theorem III.3.5). This assumption is satisfied if $\mathcal{U} \subseteq \mathcal{T}$ or $\mathcal{T} \subseteq \mathcal{U}$, and hence is met in the setting of [BM12] discussed above (see Corollary III.3.6) where $\mathcal{T} = \mathcal{U}$.

Let $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ be a twin cotorsion pair with heart $\overline{\mathcal{H}}$ on a Krull-Schmidt, triangulated category. We show in §III.4 that if $\mathcal{T}$ coincides with $\mathcal{U}$, then the heart $\overline{\mathscr{H}}_{(\mathcal{S},\mathcal{T})}$ of $(\mathcal{S},\mathcal{T})$ (see Definition III.2.21 and also [Nak11, Def. 3.7]) is equivalent to the localisation $\overline{\mathcal{H}}_{\mathcal{R}}$ of $\overline{\mathcal{H}}$ at the class $\mathcal{R}$ of its regular morphisms (see Theorem III.4.8). Since $\mathcal{T} = \mathcal{U}$ when $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})) = ((\mathsf{add}\,\Sigma R, \mathcal{X}_R),(\mathcal{X}_R, {\mathcal{X}_R}^{\perp_1}))$, the results of §III.4 also apply in the setting of [BM12] as we explain in §III.5. Our methods are also related to work of Marsh and Palu: in [MP17], equivalences are found from subfactor categories of a Krull-Schmidt, Hom-finite, triangulated category to localisations of module (and hence abelian) categories, whereas we localise not necessarily abelian categories. We also note that Theorem III.4.8 may be obtained from results of [Bel13] in a different way (see Remark III.4.10).

The motivation for our results comes from cluster theory (see Example III.5.11). The cluster category $\mathcal{C} = \mathcal{C}_H$ (see §II.8) of a finite-dimensional hereditary k-algebra H is an example of a Hom-finite, Krull-Schmidt, triangulated k-category that has Serre duality. It is especially interesting that $\mathcal{C}/[\mathcal{X}_R]$ is quasi-abelian in this case, as many aspects of Auslander-Reiten theory for abelian categories (developed in [AR77a], [AR77b]) remain valid in the case of quasi-abelian categories (see Chapter IV).

This chapter is organised in the following way. We first recall the definition and some properties of twin cotorsion pairs, and prove some new observations in §III.2. In §III.3 we prove our main result: the case when the heart of a twin cotorsion pair becomes quasi-abelian. In §III.4 we relate the heart of a twin cotorsion pair $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ to the heart of the cotorsion pair $(\mathcal{S},\mathcal{T})$ whenever $\mathcal{T} = \mathcal{U}$. Lastly, we explore our main motivating example in §III.5, namely the setting of [BM12].

III.2 Twin cotorsion pairs on triangulated categories

Throughout this section, let $\mathcal{C}$ denote a fixed triangulated category with suspension functor Σ. We follow [Nak11] and [Nak13] in order to recall some of the definitions and theory concerning twin cotorsion pairs on triangulated categories, but first we need some notation.

Definition III.2.1. Let $\mathcal{U} \subseteq \mathcal{C}$ be a full additive subcategory of $\mathcal{C}$ that is closed under isomorphisms and direct summands. By $\mathrm{Ext}^i_{\mathcal{C}}(\mathcal{U}, X) = 0$ (respectively, $\mathrm{Ext}^i_{\mathcal{C}}(X,\mathcal{U}) = 0$) we mean $\mathrm{Ext}^i_{\mathcal{C}}(U, X) = 0$ (respectively, $\mathrm{Ext}^i_{\mathcal{C}}(X, U) = 0$) for all $U \in \mathcal{U}$. We define the following full additive subcategories of $\mathcal{C}$ where $i \in \mathbb{N}$:

$$\mathcal{U}^{\perp_i} := \left\{ X \in \mathcal{C} \mid \mathrm{Ext}^i_{\mathcal{C}}(\mathcal{U}, X) = 0 \right\},$$

$$^{\perp_i}\mathcal{U} := \left\{ X \in \mathcal{C} \mid \mathrm{Ext}^i_{\mathcal{C}}(X,\mathcal{U}) = 0 \right\}.$$

Definition III.2.2. [IY08, p. 122] Let $\mathcal{U}, \mathcal{V} \subseteq \mathcal{C}$ be full additive subcategories of $\mathcal{C}$ that are closed under isomorphisms and direct summands. By $\mathcal{U} * \mathcal{V}$ we denote the full subcategory of $\mathcal{C}$ consisting of objects $X \in \mathcal{C}$ for which there exists a triangle $U \to X \to V \to \Sigma U$ in $\mathcal{C}$ with $U \in \mathcal{U}$, $V \in \mathcal{V}$.

Definition III.2.3. [Nak11, Def. 2.1] Let $\mathcal{U}, \mathcal{V} \subseteq \mathcal{C}$ be full additive subcategories of $\mathcal{C}$ that are closed under isomorphisms and direct summands. We call (the ordered pair) $(\mathcal{U}, \mathcal{V})$ a *cotorsion pair (on $\mathcal{C}$)* if $\mathrm{Ext}^1_{\mathcal{C}}(\mathcal{U}, \mathcal{V}) = 0$ and $\mathcal{C} = \mathcal{U} * \Sigma\mathcal{V}$.

As pointed out in [Nak11, Rem. 2.2], a pair $(\mathcal{U}, \mathcal{V})$ is a cotorsion pair on a Krull-Schmidt, Hom-finite, triangulated k-category $\mathcal{C}'$ (with suspension Σ') if and only if $(\Sigma'^{-1}\mathcal{U}, \mathcal{V})$ is a torsion theory in $\mathcal{C}'$ as defined in [IY08]. Recall that a *torsion theory in $\mathcal{C}'$* (in the sense of [IY08, Def. 2.2]) is a pair $(\mathcal{X}, \mathcal{Y})$ of full additive subcategories $\mathcal{X}, \mathcal{Y}$ of $\mathcal{C}'$ that are closed under isomorphisms and direct summands, such that $\mathrm{Hom}_{\mathcal{C}'}(\mathcal{X}, \mathcal{Y}) = 0$ and $\mathcal{C}' = \mathcal{X} * \mathcal{Y}$. We note that in [IY08] all categories are assumed to be Krull-Schmidt and all triangulated categories are also assumed to be Hom-finite k-categories (see [IY08, pp. 121–122]). Therefore, some of the results from [IY08] may not translate directly over to the more general setting considered in [Nak13].

Definition III.2.4. [AR91, pp. 113–114] Let $\mathcal{X} \subseteq \mathcal{C}$ be a full subcategory, closed under isomorphisms and direct summands.

(i) A *right $\mathcal{X}$-approximation* of A in $\mathcal{C}$ is a morphism $X \to A$ in $\mathcal{C}$ with $X \in \mathcal{X}$, such that for any object $X' \in \mathcal{X}$ we have an exact sequence

$$\mathrm{Hom}_{\mathcal{A}}(X', X) \to \mathrm{Hom}_{\mathcal{A}}(X', A) \to 0.$$

A right $\mathcal{X}$-approximation is called a *minimal* right $\mathcal{X}$-approximation if it is also right minimal.

(ii) A *left $\mathcal{X}$-approximation* of A in $\mathcal{C}$ is a morphism $A \to X$ in $\mathcal{C}$ with $X \in \mathcal{X}$, such that for any object $X' \in \mathcal{X}$ we have an exact sequence

$$\mathrm{Hom}_{\mathcal{A}}(X, X') \to \mathrm{Hom}_{\mathcal{A}}(A, X') \to 0.$$

A left $\mathcal{X}$-approximation is called a *minimal* left $\mathcal{X}$-approximation if it is also left minimal.

The terminology of approximations was introduced in [AR91], but the same notions were established independently by Enochs [Eno81] specifically for the subcategories of injective objects and projective objects in a module category. The term 'preenvelope' (respectively, 'precover') in [Eno81] corresponds to the notion of left (respectively, right) approximation.

We say that an object $A \in \mathcal{C}$ *has a right* (respectively, *left*) *$\mathcal{X}$-approximation* if there exists a right (respectively, left) $\mathcal{X}$-approximation $X \to A$ (respectively, $A \to X$) of A in $\mathcal{C}$ for some $X \in \mathcal{X}$.

Lemma III.2.5 (Triangulated Wakamatsu's Lemma)**.** [Jør09, Lem. 2.1] *Let $\mathcal{X}$ be an extension-closed full subcategory of $\mathcal{C}$ that is closed under isomorphisms and direct summands.*

(i) *Suppose $X \xrightarrow{x} A$ is a minimal right $\mathcal{X}$-approximation of A in $\mathcal{C}$, which completes to a triangle $\Sigma^{-1}A \xrightarrow{w} Y \longrightarrow X \xrightarrow{x} A$. Then $w \colon \Sigma^{-1}A \to Y$ is a left $\mathcal{X}^{\perp_1}$-approximation of $\Sigma^{-1}A$.*

(ii) *Suppose $A \xrightarrow{x'} X'$ is a minimal left $\mathcal{X}$-approximation of A in $\mathcal{C}$, which completes to a triangle $A \xrightarrow{x'} X' \longrightarrow Z \xrightarrow{z} \Sigma A$. Then $z \colon Z \to \Sigma A$ is a right ${}^{\perp_1}\mathcal{X}$-approximation of ΣA.*

Although the notion of a contravariantly (respectively, covariantly) finite subcategory (see below) is related to the idea of right (respectively, left) approximations, it dates back to [AS80, p. 81] in which these concepts were defined in the context of module categories.

Definition III.2.6. [AR91, pp. 114, 142] Let $\mathcal{X} \subseteq \mathcal{C}$ be a full subcategory, closed under isomorphisms and direct summands. We say $\mathcal{X}$ is *contravariantly* (respectively, *covariantly*) *finite* if A has a right (respectively, left) $\mathcal{X}$-approximation for each $A \in \mathcal{C}$. If $\mathcal{X}$ is both contravariantly finite and covariantly finite, then $\mathcal{X}$ is called *functorially finite*.

The next proposition collects some elementary properties about cotorsion pairs that will be very useful in the sequel; see for example [IY08] or [Nak11].

Proposition III.2.7. *Let* $(\mathcal{U}, \mathcal{V})$ *be a cotorsion pair* on $\mathcal{C}$.

(i) [IY08, p. 123], [Nak11, Rem. 2.3] *We have* $\mathcal{U} = {}^{\perp_1}\mathcal{V}$ *and* $\mathcal{V} = \mathcal{U}^{\perp_1}$.

(ii) [IY08, p. 123], [Nak13, Lem. 2.14] *Let* X *be an object in* $\mathcal{C}$. *Since* $(\mathcal{U}, \mathcal{V})$ *is a cotorsion pair, there is a triangle* $U \xrightarrow{u} X \xrightarrow{v} \Sigma V \xrightarrow{w} \Sigma U$, *where* $U \in \mathcal{U}$ *and* $V \in \mathcal{V}$. *Then the morphism* $u \colon U \to X$ *is a right* $\mathcal{U}$*-approximation of* X *and the morphism* $v \colon X \to \Sigma V$ *is a left* $\Sigma\mathcal{V}$*-approximation of* X.

(iii) *The subcategory* $\mathcal{U}$ *is contravariantly finite and the subcategory* $\mathcal{V}$ *is covariantly finite.*

(iv) [Nak11, Rem. 2.4] *The subcategories* $\mathcal{U}$ *and* $\mathcal{V}$ *are extension-closed.*

Definition III.2.8. [Nak13, Def. 2.7] Let $(\mathcal{S}, \mathcal{T})$ and $(\mathcal{U}, \mathcal{V})$ be two cotorsion pairs on $\mathcal{C}$. The ordered pair $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ is called a *twin cotorsion pair (on* $\mathcal{C}$*)* if $\operatorname{Ext}^1_{\mathcal{C}}(\mathcal{S}, \mathcal{V}) = 0$.

The following easily verifiable result is often useful.

Proposition III.2.9. [Nak13, p. 198] *Let* $(\mathcal{S}, \mathcal{T})$ *and* $(\mathcal{U}, \mathcal{V})$ *be cotorsion pairs on* $\mathcal{C}$. *Then the following are equivalent.*

(i) $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ *is a twin cotorsion pair.*

(ii) $\mathcal{S} \subseteq \mathcal{U}$.

(iii) $\mathcal{V} \subseteq \mathcal{T}$.

Throughout the remainder of this section, let $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ be a twin cotorsion pair on $\mathcal{C}$.

Definition III.2.10. [Nak13, Def. 2.8] We define full subcategories of $\mathcal{C}$ as follows:

$$\mathcal{W} := \mathcal{T} \cap \mathcal{U}, \quad \mathcal{C}^- := \Sigma^{-1}\mathcal{S} * \mathcal{W}, \quad \mathcal{C}^+ := \mathcal{W} * \Sigma\mathcal{V}, \quad \mathcal{H} := \mathcal{C}^- \cap \mathcal{C}^+.$$

From this definition, we immediately see that $\mathcal{W}$ is contained in the subcategories $\mathcal{C}^-$, $\mathcal{C}^+$ and $\mathcal{H}$; and that $\mathcal{W}$ is extension-closed as $\mathcal{T}$ and $\mathcal{U}$ are extension-closed. It is also clear that all four subcategories $\mathcal{W}$, $\mathcal{C}^-$, $\mathcal{C}^+$ and $\mathcal{H}$ are additive and closed under isomorphisms.

Proposition III.2.11. *The subcategories $\mathcal{W}$, $\mathcal{C}^-$, $\mathcal{C}^+$ and $\mathcal{H}$ are closed under direct summands.*

Proof. Since $\mathcal{T}$ and $\mathcal{U}$ are assumed to be closed under direct summands (see Definition III.2.3), we immediately see that $\mathcal{W}$ is also closed under direct summands. That $\mathcal{H}$ is closed under direct summands will follow from $\mathcal{C}^-$ and $\mathcal{C}^+$ having this property. We will give the proof just for $\mathcal{C}^-$ as the proof for $\mathcal{C}^+$ is similar.

Suppose $X = X_1 \oplus X_2 \in \mathcal{C}^-$, then there is a distinguished triangle $\Sigma^{-1}S \xrightarrow{s} X \xrightarrow{x} W \xrightarrow{t} S$ with $S \in \mathcal{S}$ and $W \in \mathcal{W}$. Since $\mathcal{C} = \mathcal{S} * \Sigma\mathcal{T}$, there exists a triangle $\Sigma^{-1}S_1 \xrightarrow{a} X_1 \xrightarrow{b} T_1 \xrightarrow{c} S_1$ where $S_1 \in \mathcal{S}$ and $T_1 \in \mathcal{T}$. Thus, it suffices to show that $T_1 \in \mathcal{U}$ as then we will have $T_1 \in \mathcal{T} \cap \mathcal{U} = \mathcal{W}$, and hence $X_1 \in \Sigma^{-1}\mathcal{S} * \mathcal{W} = \mathcal{C}^-$.

First, we claim that $x\colon X \to W$ is a left $\mathcal{T}$-approximation of X. Indeed, if $T \in \mathcal{T}$ then we get an exact sequence $\operatorname{Hom}_{\mathcal{C}}(W,T) \to \operatorname{Hom}_{\mathcal{C}}(X,T) \to \operatorname{Hom}_{\mathcal{C}}(\Sigma^{-1}S,T)$ since $\operatorname{Hom}_{\mathcal{C}}(-,T)$ is a cohomological functor (see Lemma II.6.6), where $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}S, T) \cong \mathrm{Hom}_{\mathcal{C}}(S, \Sigma T) = \mathrm{Ext}^1_{\mathcal{C}}(S, T) = 0$ since $(\mathcal{S}, \mathcal{T})$ is a cotorsion pair. Thus, $\mathrm{Hom}_{\mathcal{C}}(W, T) \to \mathrm{Hom}_{\mathcal{C}}(X, T)$ is surjective for $T \in \mathcal{T}$ and $x\colon X \to W$ is a left $\mathcal{T}$-approximation of X.

In order to show $T_1 \in \mathcal{U}$, it is enough to show that any $v\colon T_1 \to \Sigma V$ is in fact the zero map as $\mathcal{U} = {}^{\perp_1}\mathcal{V}$ (see Proposition III.2.7). Let $v\colon T_1 \to \Sigma V$ be arbitrary. Since $b\pi_1\colon X = X_1 \oplus X_2 \to T_1$ is a morphism with codomain in $\mathcal{T}$, where $\pi_1\colon X \to X_1$ is the canonical projection, it must factor through the left $\mathcal{T}$-approximation $x\colon X \to W$. That is, there exists $d\colon W \to T_1$ such that $dx = b\pi_1$. We then have $(vb)\pi_1 = vdx = 0$, because $vd\colon W \to \Sigma V$ vanishes as $W \in \mathcal{W} \subseteq \mathcal{U} = {}^{\perp_1}\mathcal{V}$. This in turn implies $vb = 0$ as π_1 is an epimorphism. Since $\Sigma^{-1}S_1 \xrightarrow{a} X_1 \xrightarrow{b} T_1 \xrightarrow{c} S_1$ is a triangle, we see that $v\colon T_1 \to \Sigma V$ must factor through $c\colon T_1 \to S_1$. Thus, $v = fc$ for some $f \in \mathrm{Hom}_{\mathcal{C}}(S_1, \Sigma V) = \mathrm{Ext}^1_{\mathcal{C}}(S_1, V)$. But $\mathrm{Ext}^1_{\mathcal{C}}(S_1, V) = 0$ as $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ is a twin cotorsion pair, so $v = fc = 0$ and we are done. ■

We now recall some notions from [Nak13] needed for the remainder of this section.

Definition III.2.12. [Nak13, Def. 3.1] For $X \in \mathcal{C}$, we define $K_X \in \mathcal{C}$ and a morphism $k_X\colon K_X \to X$ as follows. Since $\mathcal{S} * \Sigma\mathcal{T} = \mathcal{C} = \mathcal{U} * \Sigma\mathcal{V}$, we have two triangles $\Sigma^{-1}S \longrightarrow X \xrightarrow{a} T \longrightarrow S$ $(S \in \mathcal{S},\ T \in \mathcal{T})$ and $U \longrightarrow T \xrightarrow{b} \Sigma V \longrightarrow \Sigma U$ $(U \in \mathcal{U},\ V \in \mathcal{V})$. Then we may complete the composition $ba\colon X \to \Sigma V$ to a triangle

$$V \longrightarrow K_X \xrightarrow{k_X} X \xrightarrow{ba} \Sigma V.$$

Definition III.2.13. [Nak13, Def. 3.4] For $X \in \mathcal{C}$, we define $Z_X \in \mathcal{C}$ and $z_X\colon X \to Z_X$ as follows. Since $\mathcal{S} * \Sigma\mathcal{T} = \mathcal{C} = \mathcal{U} * \Sigma\mathcal{V}$, we have two triangles $V \longrightarrow U \xrightarrow{c} X \longrightarrow \Sigma V$ $(U \in \mathcal{U},\ V \in \mathcal{V})$ and $\Sigma^{-1}T \longrightarrow \Sigma^{-1}S \xrightarrow{d} U \longrightarrow T$ $(S \in \mathcal{S},\ T \in \mathcal{T})$. Then we may complete the composition $cd\colon \Sigma^{-1}S \to X$ to a triangle

$$\Sigma^{-1}S \xrightarrow{cd} X \xrightarrow{z_X} Z_X \longrightarrow S.$$

Definition III.2.14. [Nak13, Def. 4.1] Let $f\colon X \to Y$ be a morphism in $\mathcal{C}$ with $X \in \mathcal{C}^-$. We define $M_f \in \mathcal{C}$ and $m_f\colon Y \to M_f$ as follows. Since $X \in \mathcal{C}^-$, there is a triangle $\Sigma^{-1}S \xrightarrow{s} X \longrightarrow W \longrightarrow S$ $(S \in \mathcal{S}, W \in \mathcal{W})$. Then we may complete $fs\colon \Sigma^{-1}S \to Y$ to a triangle

$$\Sigma^{-1}S \xrightarrow{fs} Y \xrightarrow{m_f} M_f \longrightarrow S.$$

Definition III.2.15. [Nak13, Rem. 4.3] Let $f\colon X \to Y$ be a morphism in $\mathcal{C}$ with $Y \in \mathcal{C}^+$. We define $L_f \in \mathcal{C}$ and $l_f\colon L_f \to X$ as follows. Since $Y \in \mathcal{C}^+$, there is a triangle $W \longrightarrow Y \xrightarrow{v} \Sigma V \longrightarrow \Sigma W$ $(W \in \mathcal{W}, V \in \mathcal{V})$. Then we may complete $vf\colon X \to \Sigma V$ to a triangle

$$V \longrightarrow L_f \xrightarrow{l_f} X \xrightarrow{vf} \Sigma V.$$

We now present strengthened versions of [Nak13, Claim 3.2] and [Nak13, Claim 3.5].

Proposition III.2.16. *Suppose C is an arbitrary object of $\mathcal{C}$. Then*

(i) $K_C \in \mathcal{C}^-$;

(ii) $C \in \mathcal{C}^+ \iff K_C \in \mathcal{C}^+ \iff K_C \in \mathcal{H}$;

(iii) $Z_C \in \mathcal{C}^+$; *and*

(iv) $C \in \mathcal{C}^- \iff Z_C \in \mathcal{C}^- \iff Z_C \in \mathcal{H}$.

Proof. The proofs for (i) and (iii) are [Nak13, Claim 3.2 (1)] and [Nak13, Claim 3.5 (1)], respectively. Since $K_C \in \mathcal{C}^-$, we immediately see that $K_C \in \mathcal{C}^+$ if and only if $K_C \in \mathcal{H}$. For (ii), the proof that $C \in \mathcal{C}^+$ implies $K_C \in \mathcal{H}$ is [Nak13, Claim 3.2 (2)]. Thus, we show the converse. There is a triangle $V \to K_C \to C \to \Sigma V$ where $V \in \mathcal{V}$, so if $K_C \in \mathcal{C}^+$ then $C \in \mathcal{C}^+$ using [Nak13, Lem. 2.13 (2)]. The proof of statement (iv) is similar. ■

The next proposition follows from [Nak13, Prop. 3.6] and [Nak13, Prop. 3.7], but we state it in the language of approximations.

Proposition III.2.17. *Suppose C is an arbitrary object of $\mathcal{C}$.*

(i) *The morphism $k_C \colon K_C \to C$ is a right $\mathcal{C}^-$-approximation of C.*

(ii) *The morphism $z_C \colon C \to Z_C$ is a left $\mathcal{C}^+$-approximation of C.*

Recall that for a subcategory $\mathcal{A} \subseteq \mathcal{C}$ that is closed under finite direct sums, we denote by $[\mathcal{A}]$ the two-sided ideal of $\mathcal{C}$ such that $[\mathcal{A}](X,Y)$ consists of all morphisms $X \to Y$ that factor through an object in $\mathcal{A}$ (see Example II.1.31). Recall also that if $\mathcal{A}$ is a full additive subcategory that is closed under isomorphisms and direct summands, then $[\mathcal{A}]$ coincides with the ideal generated by all identity morphisms 1_A such that $A \in \mathcal{A}$. With this notation we are in position to recall the definition of the heart associated to a twin cotorsion pair.

Definition III.2.18. [Nak13] Recall that $\mathcal{W}$ is a subcategory of $\mathcal{C}^+$, $\mathcal{C}^-$ and $\mathcal{H}$. We define the following additive quotients $\overline{\mathcal{C}^+} := \mathcal{C}^+/[\mathcal{W}]$, $\overline{\mathcal{C}^-} := \mathcal{C}^-/[\mathcal{W}]$ and $\overline{\mathcal{H}} := \mathcal{H}/[\mathcal{W}]$. We call the category $\overline{\mathcal{H}}$ the *heart* of $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ by analogy with [Nak11].

Suppose $\mathcal{I}$ an ideal of $\mathcal{C}$. We will denote by $\overline{f}$ the coset $f + \mathcal{I}(X,Y)$ in $\operatorname{Hom}_{\mathcal{C}/\mathcal{I}}(X,Y)$ of $f \in \operatorname{Hom}_{\mathcal{C}}(X,Y)$. The next result is a combination of [Nak13, Cor. 4.5] and [Nak13, Cor. 4.6], and most of the proof can be found there. We provide the missing link.

Proposition III.2.19. *Let $f \in \mathrm{Hom}_{\mathcal{H}}(A, B)$ be a morphism in the subcategory $\mathcal{H}$, which completes to a triangle $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \Sigma A$ in $\mathcal{C}$. Then the following are equivalent:*

(i) $\overline{f} \in \mathrm{Hom}_{\overline{\mathcal{H}}}(A, B)$ *is an epimorphism;*

(ii) $Z_{M_f} \in \mathcal{W}$*, i.e.* $Z_{M_f} \cong 0$ *in* $\overline{\mathcal{H}}$*;*

(iii) $M_f \in \mathcal{U}$*; and*

(iv) $g \colon B \to C$ *factors through* $\mathcal{U}$*.*

Proof. The equivalence of (i)–(iii) is [Nak13, Cor. 4.5], and (iv) implies (i) is [Nak13, Cor. 4.6]. We prove (iii) implies (iv). To this end, suppose $M_f \in \mathcal{U}$. From Definition III.2.14, there are triangles $\Sigma^{-1}S_A \xrightarrow{s_A} A \xrightarrow{w_A} W_A \longrightarrow S_A$ and $\Sigma^{-1}S_A \xrightarrow{fs_A} B \xrightarrow{m_f} M_f \longrightarrow S_A$, where $S_A \in \mathcal{S}$, $W_A \in \mathcal{W}$. Then $g(fs_A) = 0$ as $gf = 0$, so g factors through $m_f \colon B \to M_f$ where $M_f \in \mathcal{U}$ by assumption. Hence, g admits a factorisation through $\mathcal{U}$ as desired. ∎

To be explicit, we state the dual in full.

Proposition III.2.20. *Let $f \in \mathrm{Hom}_{\mathcal{H}}(A, B)$ be a morphism in the subcategory $\mathcal{H}$, which completes to a triangle $\Sigma^{-1}C \xrightarrow{h} A \xrightarrow{f} B \to C$ in $\mathcal{C}$. Then the following are equivalent:*

(i) $\overline{f} \in \mathrm{Hom}_{\overline{\mathcal{H}}}(A, B)$ *is a monomorphism;*

(ii) $K_{L_f} \in \mathcal{W}$*, i.e.* $K_{L_f} \cong 0$ *in* $\overline{\mathcal{H}}$*;*

(iii) $L_f \in \mathcal{T}$*; and*

(iv) $h \colon \Sigma^{-1}C \to A$ *factors through* $\mathcal{T}$*.*

The last result of this section is an application of these previous two propositions to the case of a *degenerate* twin cotorsion pair; that is, a twin cotorsion pair $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ where $(\mathcal{S},\mathcal{T}) = (\mathcal{U},\mathcal{V})$. One may recover results from [Nak11] about cotorsion pairs on triangulated categories through the theory of twin cotorsion pairs developed in [Nak13] using such degenerate twin cotorsion pairs (see [Nak13, Exam. 2.10 (1)]). We recall some definitions, which we will also need later, from [Nak11] for convenience.

Definition III.2.21. [Nak11] Given a cotorsion pair $(\mathcal{U},\mathcal{V})$ on a triangulated category $\mathcal{C}$ (with suspension functor Σ), define $\mathscr{W} := \mathcal{U}\cap\mathcal{V}$, $\mathscr{C}^- := \Sigma^{-1}\mathcal{U} * \mathscr{W}$ and $\mathscr{C}^+ := \mathscr{W} * \Sigma\mathcal{V}$. The *heart* of the (individual) cotorsion pair $(\mathcal{U},\mathcal{V})$ is defined to be $\overline{\mathscr{H}}_{(\mathcal{U},\mathcal{V})} := (\mathscr{C}^- \cap \mathscr{C}^+)/[\mathscr{W}]$.

It is easy to see that $\mathscr{W} = \mathcal{W}$, $\mathscr{C}^- = \mathcal{C}^-$ and $\mathscr{C}^+ = \mathcal{C}^+$ for a degenerate twin cotorsion pair $((\mathcal{U},\mathcal{V}),(\mathcal{U},\mathcal{V}))$, and that $\overline{\mathscr{H}}_{(\mathcal{U},\mathcal{V})}$ coincides with the heart $\overline{\mathcal{H}}$ of the twin cotorsion pair $((\mathcal{U},\mathcal{V}),(\mathcal{U},\mathcal{V}))$.

Corollary III.2.22. *Suppose we have a degenerate twin cotorsion pair $((\mathcal{U},\mathcal{V}),(\mathcal{U},\mathcal{V}))$ on $\mathcal{C}$ and objects $X,Y \in \mathcal{H}$. Assume $Z \longrightarrow X \xrightarrow{f} Y \longrightarrow \Sigma Z$ is a triangle in $\mathcal{C}$.*

(i) *If $\overline{f}$ is epic in $\overline{\mathcal{H}}$, then $Z \in \mathcal{C}^-$.*

(ii) *If $\overline{f}$ is monic in $\overline{\mathcal{H}}$, then $\Sigma Z \in \mathcal{C}^+$.*

Proof. We only prove (i) as (ii) is similar. Since $\overline{f}$ is epic in $\overline{\mathcal{H}}$, by Proposition III.2.19 we have that $M_f \in \mathcal{U} = \mathcal{S}$. Recall from Definition III.2.14 that M_f is obtained by taking a triangle $\Sigma^{-1}S \xrightarrow{s} X \to W \to S$ ($S \in \mathcal{S}$, $W \in \mathcal{W}$), which exists as $X \in \mathcal{H} \subseteq \mathcal{C}^-$, and then completing the composition fs to a triangle $\Sigma^{-1}S \xrightarrow{fs} Y \xrightarrow{m_f} M_f \longrightarrow S$. Then applying the octahedral axiom (TR5), we get a commutative diagram

$$\begin{tikzcd}
\Sigma^{-1}S \arrow[r, "s"] \arrow[d, equal] & X \arrow[r] \arrow[d, "f"] & W \arrow[r] \arrow[d] & S \arrow[d, equal] \\
\Sigma^{-1}S \arrow[r, "fs"] \arrow[d] & Y \arrow[r, "m_f"] \arrow[d, equal] & M_f \arrow[r] \arrow[d] & S \arrow[d] \\
X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d] & \Sigma Z \arrow[r] \arrow[d, equal] & \Sigma X \arrow[d] \\
W \arrow[r] & M_f \arrow[r] & \Sigma Z \arrow[r] & \Sigma W
\end{tikzcd}$$

where the rows are triangles. Therefore, there is a triangle $\Sigma^{-1}M_f \to Z \to W \to M_f$ by Lemma II.6.7, where $\Sigma^{-1}M_f \in \Sigma^{-1}\mathcal{U}$ and $W \in \mathcal{W}$, so $Z \in \Sigma^{-1}\mathcal{U} * \mathcal{W} = \Sigma^{-1}\mathcal{S} * \mathcal{W} = \mathcal{C}^-$ as $\mathcal{U} = \mathcal{S}$ for a degenerate twin cotorsion pair. ■

III.3 Main result: the case when $\overline{\mathcal{H}}$ is quasi-abelian

Let $\mathcal{C}$ be a fixed triangulated category with suspension functor Σ, and suppose $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ is a twin cotorsion pair on $\mathcal{C}$. No other assumptions are made on $\mathcal{C}$ in this section. We recall two key results from [Nak13] concerning the factor category $\overline{\mathcal{H}} = \mathcal{H}/[\mathcal{W}]$. First, it is shown that $\overline{\mathcal{H}}$ is semi-abelian [Nak13, Thm. 5.4], and that, if $\mathcal{U} \subseteq \mathcal{S} * \mathcal{T}$ or $\mathcal{T} \subseteq \mathcal{U} * \mathcal{V}$, then $\overline{\mathcal{H}}$ is also integral [Nak13, Thm. 6.3].

In this section we prove our main result: if $\mathcal{H}$ is equal to $\mathcal{C}^-$ or $\mathcal{C}^+$, then $\overline{\mathcal{H}} = \mathcal{H}/[\mathcal{W}]$ is quasi-abelian. In order to prove this, we need the following lemma.

Lemma III.3.1. *Let $\mathcal{A}$ be a left semi-abelian category. Suppose*

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "b"'] \arrow[dr, phantom, "\square"] & B \arrow[d, "c"] \\
C \arrow[r, "d"'] & D
\end{tikzcd}$$

is a pullback diagram in $\mathcal{A}$. Suppose we also have morphisms $x_B \colon X \to B$ and

$x_C\colon X \to C$ such that x_B is a cokernel and

$$\begin{tikzcd}
X \arrow[r, "x_B"] \arrow[d, "x_C"'] \arrow[dr, phantom, "\circ"] & B \arrow[d, "c"] \\
C \arrow[r, "d"'] & D
\end{tikzcd}$$

commutes. Then $a\colon A \to B$ is also a cokernel in $\mathcal{A}$.

Proof. From the assumptions, we obtain the following commutative diagram

$$\begin{tikzcd}
X \arrow[drr, bend left, "x_B"] \arrow[dr, dotted, "\exists! e"] \arrow[ddr, bend right, "x_C"'] & & \\
& A \arrow[r, "a"] \arrow[d, "b"'] \arrow[dr, phantom, "\square"] & B \arrow[d, "c"] \\
& C \arrow[r, "d"'] & D
\end{tikzcd}$$

using the universal property for the pullback because $cx_B = dx_C$. Thus, $ae = x_B$ is a cokernel, and hence a is cokernel in the left semi-abelian category $\mathcal{A}$ by [Rum01, Prop. 2]. ■

Dually, we also have the following.

Lemma III.3.2. *Let $\mathcal{A}$ be a right semi-abelian category. Suppose*

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "b"'] \arrow[dr, phantom, "\square"] & B \arrow[d, "c"] \\
C \arrow[r, "d"'] & D
\end{tikzcd}$$

is a pushout diagram in $\mathcal{A}$. Suppose we also have morphisms $x_B\colon B \to X$ and $x_C\colon C \to X$ such that x_C is a kernel and

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "b"'] \arrow[dr, phantom, "\circ"] & B \arrow[d, "x_B"] \\
C \arrow[r, "x_C"'] & X
\end{tikzcd}$$

commutes. Then $d\colon C \to D$ *is also a kernel in* $\mathcal{A}$*.*

We will also need the following notion and easy lemma in the proof of our main theorem.

Definition III.3.3. [Fre66, p. 99] Suppose $f\colon X \to Y$ is a morphism in an additive category $\mathcal{A}$. A *weak kernel* of f is a morphism $w\colon W \to X$ such that $fw = 0$, with the following property: for every morphism $g\colon V \to X$ with $fg = 0$, there exists $\widehat{g}\colon V \to W$ such that $w\widehat{g} = g$. A *weak cokernel* is defined dually.

Lemma III.3.4. [BM12, Lem. 2.5] *In any additive category, we have the following.*

(i) *A monomorphism that is a weak kernel is a kernel.*

(ii) *An epimorphism that is a weak cokernel is a cokernel.*

We now show that, under the right conditions, $\overline{\mathcal{H}}$ is quasi-abelian. We note that $\mathcal{H} = \mathcal{C}^- \iff \mathcal{C}^- \subseteq \mathcal{C}^+ \iff \Sigma^{-1}\mathcal{S} * \mathcal{W} \subseteq \mathcal{W} * \Sigma\mathcal{V}$, and dually that $\mathcal{H} = \mathcal{C}^+ \iff \mathcal{C}^+ \subseteq \mathcal{C}^- \iff \mathcal{W} * \Sigma\mathcal{V} \subseteq \Sigma^{-1}\mathcal{S} * \mathcal{W}$.

Theorem III.3.5. *Let* $\mathcal{C}$ *be a triangulated category with a twin cotorsion pair* $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$*. If* $\mathcal{H} = \mathcal{C}^-$ *or* $\mathcal{H} = \mathcal{C}^+$*, then* $\overline{\mathcal{H}} = \mathcal{H}/[\mathcal{W}]$ *is quasi-abelian.*

Proof. Since $\overline{\mathcal{H}}$ is semi-abelian [Nak13, Thm. 5.4], we have that $\overline{\mathcal{H}}$ is left quasi-abelian if and only if $\overline{\mathcal{H}}$ is right quasi-abelian [Rum01, Prop. 3]. Therefore, we will show that if $\mathcal{H} = \mathcal{C}^-$ then $\overline{\mathcal{H}}$ is left quasi-abelian. Showing $\overline{\mathcal{H}}$ is right quasi-abelian whenever $\mathcal{H} = \mathcal{C}^+$ is similar.

Suppose $\mathcal{H} = \mathcal{C}^-$ and that we have a pullback diagram

$$\begin{tikzcd} A \arrow[r, "\overline{a}"] \arrow[d, "\overline{b}"'] \arrow[dr, phantom, "\square"] & B \arrow[d, "\overline{c}"] \\ C \arrow[r, "\overline{d}"'] & D \end{tikzcd}$$

in $\overline{\mathcal{H}}$, where $\overline{d}$ is a cokernel. We need to show that $\overline{a}$ is also a cokernel.

By [Nak13, Lem. 5.1], we may assume that $d \in \operatorname{Hom}_{\mathcal{H}}(C, D)$ is a morphism for which there is a distinguished triangle $\Sigma^{-1}S \longrightarrow C \xrightarrow{d} D \xrightarrow{e} S$ in $\mathcal{C}$ with $S \in \mathcal{S}$. An application of (TR3) yields a triangle $C \xrightarrow{d} D \xrightarrow{e} S \longrightarrow \Sigma C$.

We can complete c to a triangle $\Sigma^{-1}E \longrightarrow B \xrightarrow{c} D \xrightarrow{f} E$ and complete the composition $fd\colon C \to E$ to another triangle $C \xrightarrow{fd} E \longrightarrow \Sigma X \xrightarrow{\Sigma x_C} \Sigma C.$ We obtain a triangle $D \xrightarrow{f} E \longrightarrow \Sigma B \xrightarrow{-\Sigma c} \Sigma D$ using (TR3). Then by the octahedral axiom (TR5), there is a commutative diagram

$$\begin{tikzcd}
C \arrow[r, "d"] \arrow[d, equal] & D \arrow[r, "e"] \arrow[d, "f"] & S \arrow[r] \arrow[d, "-\Sigma g"'] & \Sigma C \arrow[d, equal] \\
C \arrow[r, "fd"] \arrow[d, "d"'] & E \arrow[r] \arrow[d, equal] & \Sigma X \arrow[r, "\Sigma x_C"] \arrow[d, "-\Sigma x_B"'] & \Sigma C \arrow[d, "\Sigma d"] \\
D \arrow[r, "f"] \arrow[d, "e"'] & E \arrow[r] \arrow[d] & \Sigma B \arrow[r, "-\Sigma c"] \arrow[d, equal] & \Sigma D \arrow[d, "\Sigma e"] \\
S \arrow[r, "-\Sigma g"'] & \Sigma X \arrow[r, "-\Sigma x_B"'] & \Sigma B \arrow[r, "-\Sigma(ec)"'] & \Sigma S
\end{tikzcd} \tag{III.3.1}$$

in $\mathcal{C}$ where the rows are triangles. There is a triangle $\Sigma^{-1}S \xrightarrow{g} X \xrightarrow{x_B} B \xrightarrow{ec} S$ using Lemma II.6.7. Since $S \in \mathcal{S}$ and $B \in \mathcal{H} = \mathcal{C}^-$, we have $X \in \mathcal{C}^- = \mathcal{H}$ by [Nak13, Lem. 2.12 (2)].

Since $B \in \mathcal{H} \subseteq \mathcal{C}^+ = \mathcal{W} * \Sigma\mathcal{V}$, there is a triangle $W \xrightarrow{h} B \xrightarrow{i} \Sigma V \longrightarrow \Sigma W$ with $W \in \mathcal{W}$ and $V \in \mathcal{V}$. Consider the composition $(ec) \circ h\colon W \to S$ and apply the octahedral axiom (TR5) to get the following commutative diagram

$$\begin{tikzcd}
W \arrow[r, "h"] \arrow[d, equal] & B \arrow[r, "i"] \arrow[d, "ec"] & \Sigma V \arrow[r] \arrow[d, "-\Sigma l"] & \Sigma W \arrow[d, equal] \\
W \arrow[r, "ech"] \arrow[d, "h"'] & S \arrow[r, "-\Sigma j"] \arrow[d, equal] & \Sigma Y \arrow[r, "-\Sigma k"] \arrow[d, "\Sigma y"] & \Sigma W \arrow[d, "\Sigma h"] \\
B \arrow[r, "ec"] \arrow[d, "i"'] & S \arrow[r, "-\Sigma g"] \arrow[d, "-\Sigma j"] & \Sigma X \arrow[r, "-\Sigma x_B"] \arrow[d, equal] & \Sigma B \arrow[d, "\Sigma i"] \\
\Sigma V \arrow[r, "-\Sigma l"'] & \Sigma Y \arrow[r, "\Sigma y"'] & \Sigma X \arrow[r, "-\Sigma(ix_B)"'] & \Sigma^2 V
\end{tikzcd} \tag{III.3.2}$$

in $\mathcal{C}$ with rows as triangles. Then we see that $Y \xrightarrow{-y} X \xrightarrow{ix_B} \Sigma V \xrightarrow{-\Sigma l} \Sigma Y$ is also a triangle using Lemma II.6.7 on the bottom triangle of (III.3.2). Moreover, this implies that $Y \xrightarrow{y} X \xrightarrow{ix_B} \Sigma V \xrightarrow{\Sigma l} \Sigma Y$ is a triangle in $\mathcal{C}$, using the triangle isomorphism $(-1_Y, 1_X, 1_{\Sigma V})$.

We claim that $\overline{x_B}\colon X \to B$ is a cokernel for $\overline{y}\colon Y \to X$ in $\overline{\mathcal{H}}$. The morphism $x_B \in \operatorname{Hom}_{\mathcal{H}}(X, B)$ embeds in the triangle $X \xrightarrow{x_B} B \xrightarrow{ec} S \xrightarrow{-\Sigma g} \Sigma X$ where $S \in \mathcal{S} \subseteq \mathcal{U}$, and hence $\overline{x_B}$ is epic in $\overline{\mathcal{H}}$ by Proposition III.2.19 as ec factors through $\mathcal{U}$. Thus, by Lemma III.3.4 it suffices to show that $\overline{x_B}$ is a weak cokernel for $\overline{y}$. First, from (III.3.2) we see that $x_B y = hk$ factors through $\mathcal{W}$ as $W \in \mathcal{W}$. Thus, in the factor category $\overline{\mathcal{H}}$ we have $\overline{x_B y} = 0$. Now suppose that there is some $m\colon X \to M$ such that $\overline{my} = 0$ in $\overline{\mathcal{H}}$. Then my factors through $\mathcal{W}$, so there is a commutative diagram

$$\begin{tikzcd}
Y \arrow[r, "y"] \arrow[d, "n"'] \arrow[dr, phantom, "\circ"] & X \arrow[d, "m"] \\
W_0 \arrow[r, "q"'] & M
\end{tikzcd}$$

in $\mathcal{C}$, where $W_0 \in \mathcal{W}$. Thus, we have a morphism of triangles

$$\begin{tikzcd}
Y \arrow[r, "y"] \arrow[d, "n"] & X \arrow[r, "ix_B"] \arrow[d, "m"] & \Sigma V \arrow[r, "\Sigma l"] \arrow[d, dotted, "\exists p"] & \Sigma Y \arrow[d, "\Sigma n"] \\
W_0 \arrow[r, "q"] & M \arrow[r] & N \arrow[r, "r"] & \Sigma W_0
\end{tikzcd}$$

where p exists using (TR4). From the commutativity of (III.3.2), we have another morphism

$$\begin{tikzcd}
\Sigma^{-1}S \arrow[r, "g"] \arrow[d, "j"] & X \arrow[r, "x_B"] \arrow[d, equal] & B \arrow[r, "ec"] \arrow[d, "i"] & S \arrow[d, "\Sigma j"] \\
Y \arrow[r, "y"] & X \arrow[r, "ix_B"] & \Sigma V \arrow[r, "\Sigma l"] & \Sigma Y
\end{tikzcd}$$

of triangles, where $(-\Sigma l) \circ i = (-\Sigma j) \circ ec$ and $(\Sigma y) \circ (-\Sigma j) = -\Sigma g$ yield $(\Sigma l) \circ i = (\Sigma j) \circ ec$ and $yj = g$, respectively. Therefore, composing these two morphisms of triangles we get a commutative diagram

$$\begin{tikzcd}
\Sigma^{-1}S \arrow[r, "g"] \arrow[d, "nj"] & X \arrow[r, "x_B"] \arrow[d, "m"] & B \arrow[r, "ec"] \arrow[d, "pi"] & S \arrow[d, "\Sigma(nj)"] \\
W_0 \arrow[r, "q"] & M \arrow[r] & N \arrow[r, "r"] & \Sigma W_0
\end{tikzcd}$$

where the two rows are triangles. Notice that $\Sigma(nj) \in \operatorname{Hom}_{\mathcal{C}}(S, \Sigma W_0) = \operatorname{Ext}^1_{\mathcal{C}}(S, W_0) = 0$ as $W_0 \in \mathcal{W} \subseteq \mathcal{T} = \mathcal{S}^{\perp_1}$. This implies $r \circ pi$ vanishes, so there exists $\varphi_1 \colon X \to W_0$ and $\varphi_2 \colon B \to M$ such that $m = q\varphi_1 + \varphi_2 x_B$ by Lemma II.6.9. Finally, in $\overline{\mathcal{H}}$ we have $\overline{m} = \overline{q\varphi_1} + \overline{\varphi_2 x_B} = \overline{\varphi_2 x_B}$, as $W_0 \in \mathcal{W}$ and so $\overline{q\varphi_1} = 0$. This says that $\overline{x_B}$ is indeed a weak cokernel, and hence a cokernel, of $\overline{y}$.

In (III.3.1) we see $(-\Sigma c)(-\Sigma x_B) = (\Sigma d)(\Sigma x_C)$, so $cx_B = dx_C$ in $\mathcal{H}$ and $\overline{cx_B} = \overline{dx_C}$ in the (left) semi-abelian category $\overline{\mathcal{H}}$. Therefore, since $\overline{x_B}$ is a cokernel, it follows from Lemma III.3.1 that $\overline{a} \colon A \to B$ must also be a cokernel.

Hence, $\overline{\mathcal{H}}$ is left quasi-abelian, and thus quasi-abelian by [Rum01, Prop. 3]. ■

Corollary III.3.6. *Let $\mathcal{C}$ be a triangulated category with a twin cotorsion pair $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$. If $\mathcal{U} \subseteq \mathcal{T}$ or $\mathcal{T} \subseteq \mathcal{U}$, then $\overline{\mathcal{H}} = \mathcal{H}/[\mathcal{W}]$ is quasi-abelian.*

Proof. If $\mathcal{U} \subseteq \mathcal{T}$ then $\mathcal{H} = \mathcal{C}^-$, or if $\mathcal{T} \subseteq \mathcal{U}$ then $\mathcal{H} = \mathcal{C}^+$. Therefore, in either case we may apply Theorem III.3.5 to get that $\overline{\mathcal{H}}$ is quasi-abelian. ■

Note that if $\mathcal{U} \subseteq \mathcal{T}$ or $\mathcal{T} \subseteq \mathcal{U}$, then $\overline{\mathcal{H}}$ is also integral: this follows from [Nak13, Thm. 6.3] since, for example, $\mathcal{U} \subseteq \mathcal{T}$ implies $\mathcal{U} \subseteq \mathcal{S} * \mathcal{T}$. We also remark that in

[Liu13] there is the corresponding result for exact categories.

III.4 Localisation of an integral heart of a twin cotorsion pair

For this section, we fix a skeletally small, Krull-Schmidt, triangulated category $\mathcal{C}$ and a twin cotorsion pair $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ on $\mathcal{C}$ with $\mathcal{T}=\mathcal{U}$. In this setting, we have that the heart of $(\mathcal{S},\mathcal{T})$ is $\overline{\mathscr{H}} := \overline{\mathscr{H}}_{(\mathcal{S},\mathcal{T})} = (\Sigma^{-1}\mathcal{S} * \mathcal{S})/[\mathcal{S}]$ and the heart of $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ is $\overline{\mathcal{H}} = \mathcal{C}/[\mathcal{W}]$, where $\mathcal{W}=\mathcal{T}=\mathcal{U}$ (see Definitions III.2.21 and III.2.18, respectively). We will show that there is an equivalence $\overline{\mathscr{H}} \simeq \overline{\mathcal{H}}_{\mathcal{R}}$ (see Theorem III.4.8), where $\overline{\mathcal{H}}_{\mathcal{R}}$ is the (Gabriel-Zisman) localisation of $\overline{\mathcal{H}}$ at the class $\mathcal{R}$ of regular morphisms in $\overline{\mathcal{H}}$.

Our line of proof will be as follows: first, we obtain a canonical functor $F\colon \overline{\mathscr{H}} \to \overline{\mathcal{H}}$; then, composing with the localisation functor $L_{\mathcal{R}}\colon \overline{\mathcal{H}} \to \overline{\mathcal{H}}_{\mathcal{R}}$, we get a functor $\overline{\mathscr{H}} \to \overline{\mathcal{H}}_{\mathcal{R}}$ that we show is fully faithful and dense. The proofs in this section are inspired by methods used in [BM13], [BM12] and [MP17].

For the convenience of the reader, we recall some details of the description of the localisation of an integral category at its regular morphisms (see §II.5 for more details). To this end, suppose $\mathcal{A}$ is an integral category and let $\mathcal{R}_{\mathcal{A}}$ be the class of regular morphisms in $\mathcal{A}$. In this case, $\mathcal{R}_{\mathcal{A}}$ admits a *calculus of left fractions* (see §II.5.2 or [GZ67, §I.2]) by [Rum01, Prop. 6]. The objects of the localisation $\mathcal{A}_{\mathcal{R}_{\mathcal{A}}}$ are the objects of $\mathcal{A}$. A morphism in $\mathcal{A}_{\mathcal{R}_{\mathcal{A}}}$ from X to Y is a *left fraction* of the form

$$\begin{tikzcd} & A & \\ X \arrow[ur, "f"] & & Y \arrow[ul, "r"'] \end{tikzcd}$$

denoted $[f,r]_{\mathrm{LF}}$, up to a certain equivalence (see §II.5.2 for more details), where f is any morphism in $\mathcal{A}$ and r is in $\mathcal{R}_{\mathcal{A}}$. The localisation functor $L_{\mathcal{R}_{\mathcal{A}}}\colon \mathcal{A} \to \mathcal{A}_{\mathcal{R}_{\mathcal{A}}}$

is the identity on objects and takes a morphism $f\colon X \to A$ to the left fraction $[f] \coloneqq L_{\mathcal{R}_{\mathcal{A}}}(f) = [f, 1_A]_{\mathrm{LF}}$. If $r\colon Y \to A$ is in $\mathcal{R}_{\mathcal{A}}$, then the morphism $[r]$ in $\mathcal{A}_{\mathcal{R}_{\mathcal{A}}}$ is invertible with inverse $[r]^{-1}$ equal to the left fraction $[1_A, r]_{\mathrm{LF}}$. An exposition of the morphisms as right fractions may be found in [BM12, §4].

Proposition III.4.1. *There is an additive functor $F\colon \overline{\mathscr{H}} \to \overline{\mathcal{H}}$ that is the identity on objects and, for a morphism $f\colon X \to Y$ in $\Sigma^{-1}\mathcal{S} * \mathcal{S}$, maps the coset $f + [\mathcal{S}](X,Y)$ to the coset $f + [\mathcal{W}](X,Y)$.*

Proof. The full subcategory $\mathscr{H} = \Sigma^{-1}\mathcal{S} * \mathcal{S} \subseteq \mathcal{C} = \mathcal{H}$ comes equipped with an inclusion functor $\iota\colon \mathscr{H} \to \mathcal{H}$, which may be composed with the additive quotient functor $Q_{[\mathcal{W}]}\colon \mathcal{H} \to \mathcal{H}/[\mathcal{W}]$ to get a functor $Q_{[\mathcal{W}]} \circ \iota\colon \mathscr{H} \to \overline{\mathcal{H}}$. Note that this functor maps any morphism in the ideal $[\mathcal{S}]$ to 0 as $\mathcal{S} \subseteq \mathcal{U} = \mathcal{W}$, and therefore we get the following commutative diagram

$$\begin{tikzcd}
\mathscr{H} = \Sigma^{-1}\mathcal{S} * \mathcal{S} \arrow[rr, "\iota"] \arrow[d, "Q_{[\mathcal{S}]}"'] \arrow[drr, phantom, "\circ"] & & \mathcal{C} = \mathcal{H} \arrow[d, "Q_{[\mathcal{W}]}"] \\
\overline{\mathscr{H}} = (\Sigma^{-1}\mathcal{S} * \mathcal{S})/[\mathcal{S}] \arrow[rr, dotted, "\exists! F"'] & & \mathcal{C}/[\mathcal{W}] = \overline{\mathcal{H}}
\end{tikzcd}$$

of additive categories using the universal property of the additive quotient $\overline{\mathscr{H}}$. Furthermore, we see that $F(X) = F(Q_{[\mathcal{S}]}(X)) = Q_{[\mathcal{W}]}(\iota(X)) = X$ and

$$F(f + [\mathcal{S}](X,Y)) = F(Q_{[\mathcal{S}]}(f)) = Q_{[\mathcal{W}]}(\iota(f)) = Q_{[\mathcal{W}]}(f) = f + [\mathcal{W}](X,Y).$$

■

The next result below is a characterisation of the regular morphisms in $\overline{\mathcal{H}} = \mathcal{C}/[\mathcal{W}]$, and is a special case of [Bel13, Lem. 4.1]. Note that $\Sigma^{-1}\mathcal{S}$ is a contravariantly finite and *rigid* (i.e. $\mathrm{Ext}^1_{\mathcal{C}}(\Sigma^{-1}\mathcal{S}, \Sigma^{-1}\mathcal{S}) = 0$) subcategory of $\mathcal{C}$, because $\mathcal{S} \subseteq \mathcal{U} = \mathcal{T} = \mathcal{S}^{\perp_1}$, and that $(\Sigma^{-1}\mathcal{S})^{\perp_0} = \mathcal{S}^{\perp_1} = \mathcal{T} = \mathcal{U} = \mathcal{W}$.

Proposition III.4.2. *Suppose* $\Sigma^{-1}Z \xrightarrow{h} X \xrightarrow{f} Y \xrightarrow{g} Z$ *is a triangle in* $\mathcal{C}$. *Denote by* $\overline{f}$ *the morphism* $f + [\mathcal{W}](X,Y) \in \mathrm{Hom}_{\overline{\mathcal{H}}}(X,Y)$ *in* $\overline{\mathcal{H}}$.

(i) *The morphism* $\overline{f}$ *is monic if and only if* h *factors through* $\mathcal{W}$.

(ii) *The morphism* $\overline{f}$ *is epic if and only if* g *factors through* $\mathcal{W}$.

(iii) *The morphism* $\overline{f}$ *is regular if and only if* h *and* g *factor through* $\mathcal{W}$.

The following lemma is a generalisation of [BM13, Lem. 3.3]. The proof of Buan and Marsh easily generalises, so we omit the proof of our statement. One is able to recover the result of Buan and Marsh by putting the appropriate restrictions on $\mathcal{C}$ and by setting $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})) = ((\mathsf{add}\,\Sigma T, \mathcal{X}_T), (\mathcal{X}_T, {\mathcal{X}_T}^{\perp_1}))$, where T is a rigid object in $\mathcal{C}$.

Lemma III.4.3. *Let* Y *be an arbitrary object of* $\mathcal{C}$. *Then there exists* $X \in \Sigma^{-1}\mathcal{S} * \mathcal{S} = \mathscr{H}$ *and a morphism* $\overline{r}\colon X \to Y$ *in the class* $\mathcal{R}$ *of regular morphisms in* $\overline{\mathcal{H}}$.

By Proposition III.4.1, we have an additive functor $F\colon \overline{\mathscr{H}} \to \overline{\mathcal{H}}$. Define $G := L_{\mathcal{R}} \circ F$, where $L_{\mathcal{R}}\colon \overline{\mathcal{H}} \to \overline{\mathcal{H}}_{\mathcal{R}}$ is the additive localisation functor (see Proposition II.5.8), to obtain the following commutative diagram of additive functors

$$\begin{tikzcd} \overline{\mathscr{H}} \arrow[r, "F"] \arrow[dr, "G"'] & \overline{\mathcal{H}} \arrow[d, "L_{\mathcal{R}}"] \\ & \overline{\mathcal{H}}_{\mathcal{R}} \end{tikzcd}$$

Note that $G(X) = X$ and $G(f + [\mathcal{S}](X,Y)) = [\overline{f}] = [f + [\mathcal{W}](X,Y), 1_Y]_{\mathrm{LF}}$. The remainder of this section is dedicated to showing that G is an equivalence of categories.

Proposition III.4.4. *The functor* $G\colon \overline{\mathscr{H}} \to \overline{\mathcal{H}}_{\mathcal{R}}$ *is dense.*

Proof. Recall that the objects of $\overline{\mathcal{H}}_{\mathcal{R}}$ are the objects of $\mathcal{H} = \mathcal{C}$. Let $Y \in \overline{\mathcal{H}}_{\mathcal{R}}$ be arbitrary. Then by Lemma III.4.3, there exists a morphism $r\colon X \to Y$ in $\mathcal{C}$ with

$X \in \Sigma^{-1}\mathcal{S} * \mathcal{S} = \mathscr{H}$, such that $\overline{r}$ is regular in $\overline{\mathcal{H}}$. Hence, in $\overline{\mathcal{H}}_{\mathcal{R}}$ we have that $L_{\mathcal{R}}(\overline{r})\colon X \to Y$ is an isomorphism so that $Y \cong X = L_{\mathcal{R}}F(X) = G(X)$, and G is a dense functor. ■

To show G is faithful we need the following observation due to Beligiannis.

Lemma III.4.5. [Bel13, Rem. 4.3 (iii)] *Suppose $X \in \Sigma^{-1}\mathcal{S} * \mathcal{S}$ and $f\colon X \to Y$ is a morphism in $\mathcal{C}$. If f factors through $\mathcal{W}$, then f factors through $\mathcal{S}$.*

Proposition III.4.6. *The functor $G\colon \overline{\mathscr{H}} \to \overline{\mathcal{H}}_{\mathcal{R}}$ is faithful.*

Proof. Suppose $\overline{f} = f + [\mathcal{S}](X,Y)\colon X \to Y$ is a morphism in $\overline{\mathscr{H}} = (\Sigma^{-1}\mathcal{S} * \mathcal{S})/[\mathcal{S}]$ such that $G(\overline{f}) = L_{\mathcal{R}}(F(\overline{f})) = 0$ in $\overline{\mathcal{H}}_{\mathcal{R}} = (\mathcal{C}/[\mathcal{W}])_{\mathcal{R}}$. Then $f + [\mathcal{W}](X,Y) = F(\overline{f}) = 0$ in $\overline{\mathcal{H}} = \mathcal{C}/[\mathcal{W}]$ because $L_{\mathcal{R}}\colon \overline{\mathcal{H}} \to \overline{\mathcal{H}}_{\mathcal{R}}$ is faithful by [BM12, Lem. 4.4]. Hence, f factors through $\mathcal{W}$ in $\mathcal{C}$. Note that $X \in \Sigma^{-1}\mathcal{S} * \mathcal{S}$, so f factors through $\mathcal{S}$ by Lemma III.4.5 and $\overline{f}$ is the zero morphism in $\overline{\mathscr{H}}$. Therefore, the functor G is faithful. ■

Proposition III.4.7. *The functor $G\colon \overline{\mathscr{H}} \to \overline{\mathcal{H}}_{\mathcal{R}}$ is full.*

Proof. Let X, Y be objects in $\overline{\mathscr{H}}$ and consider the mapping

$$\mathrm{Hom}_{\overline{\mathscr{H}}}(X,Y) \longrightarrow \mathrm{Hom}_{\overline{\mathcal{H}}_{\mathcal{R}}}(G(X), G(Y)) = \mathrm{Hom}_{\overline{\mathcal{H}}_{\mathcal{R}}}(X,Y).$$

Let $X \xrightarrow{\overline{f}} A \xleftarrow{\overline{r}} Y$ be an arbitrary morphism in $\mathrm{Hom}_{\overline{\mathcal{H}}_{\mathcal{R}}}(G(X), G(Y))$. Since $r\colon Y \to A$ is a morphism in $\mathcal{C}$ such that $\overline{r}$ is regular in $\overline{\mathcal{H}}$, there is a triangle $\Sigma^{-1}Z \xrightarrow{s} Y \xrightarrow{r} A \xrightarrow{t} Z$ such that s, t factor through $\mathcal{W}$ by Proposition III.4.2. As $X \in \operatorname{obj}\overline{\mathscr{H}} = \operatorname{obj}\mathscr{H} = \operatorname{obj}(\Sigma^{-1}\mathcal{S} * \mathcal{S})$, there exists a triangle $\Sigma^{-1}S_1 \xrightarrow{a} \Sigma^{-1}S_0 \xrightarrow{b} X \xrightarrow{c} S_1$ in $\mathcal{C}$ with $S_0, S_1 \in \mathcal{S}$. Suppose $t\colon A \to Z$ factors as ed for some $d\colon A \to T$, $e\colon T \to Z$ with $T \in \mathcal{W} = \mathcal{T}$. Then the morphism $dfb \in \mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}S_0, T) \cong \mathrm{Ext}^1_{\mathcal{C}}(S_0, T) = 0$ vanishes, and hence $tfb = edfb$ is the zero map too. Thus, there exists $g\colon \Sigma^{-1}S_0 \to Y$ such that $rg = fb$. Applying (TR4), we obtain a morphism

$$\begin{tikzcd}
\Sigma^{-1}S_1 \arrow[r, "a"] \arrow[d, dotted, "h"] & \Sigma^{-1}S_0 \arrow[r, "b"] \arrow[d, "g"] & X \arrow[r, "c"] \arrow[d, "f"] & S_1 \arrow[d, dotted, "\Sigma h"] \\
\Sigma^{-1}Z \arrow[r, "s"] & Y \arrow[r, "r"] & A \arrow[r, "t"] & Z
\end{tikzcd}$$

of triangles in $\mathcal{C}$, in which $ga = sh$ vanishes as $S_1 \in \mathcal{S}$ and s factors through $\mathcal{W} = \mathcal{T}$. Hence, by Lemma II.6.9, there are morphisms $u \in \operatorname{Hom}_{\mathcal{C}}(X,Y) = \operatorname{Hom}_{\mathscr{H}}(X,Y)$ and $v \in \operatorname{Hom}_{\mathcal{C}}(S_1,A)$ such that $f = ru + vc$ in $\mathcal{C}$. Therefore, in $\overline{\mathcal{H}} = \mathcal{C}/[\mathcal{W}]$ we have $\overline{f} = \overline{ru} + \overline{vc} = \overline{ru}$, as $\overline{v} = 0$ because $S_1 \in \mathcal{S} \subseteq \mathcal{U} = \mathcal{W}$. This implies that $[\overline{f}] = [\overline{ru}] = [\overline{r}][\overline{u}]$, and hence $[u + [\mathcal{W}](X,Y)] = [\overline{u}] = [\overline{r}]^{-1}[\overline{f}] = [f,r]_{\mathrm{LF}}$ in $\overline{\mathcal{H}}_{\mathcal{R}}$. Finally, we see that

$$\begin{aligned}
[f,r]_{\mathrm{LF}} &= [u + [\mathcal{W}](X,Y)] \\
&= [F(u + [\mathcal{S}](X,Y))] \\
&= (L_{\mathcal{R}} \circ F)(u + [\mathcal{S}](X,Y)) \\
&= G(u + [\mathcal{S}](X,Y)),
\end{aligned}$$

and the map $\operatorname{Hom}_{\overline{\mathscr{H}}}(X,Y) \to \operatorname{Hom}_{\overline{\mathcal{H}}_{\mathcal{R}}}(G(X),G(Y))$ is surjective, i.e. G is a full functor. ■

Theorem III.4.8. *Suppose $\mathcal{C}$ is a skeletally small, Krull-Schmidt, triangulated category, and assume $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ is a twin cotorsion pair on $\mathcal{C}$ that satisfies $\mathcal{T} = \mathcal{U}$. Let $\mathcal{R}$ denote the class of regular morphisms in the heart $\overline{\mathcal{H}}$ of $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$. Then the Gabriel-Zisman localisation $\overline{\mathcal{H}}_{\mathcal{R}}$ is equivalent to the heart $\overline{\mathscr{H}}_{(\mathcal{S},\mathcal{T})}$ of the cotorsion pair $(\mathcal{S},\mathcal{T})$.*

Proof. By Theorem II.1.15, a fully faithful, dense functor is an equivalence. Hence, the functor $G = L_{\mathcal{R}} \circ F$ gives an equivalence $\overline{\mathscr{H}}_{(\mathcal{S},\mathcal{T})} \xrightarrow{\simeq} \overline{\mathcal{H}}_{\mathcal{R}}$ using Propositions III.4.4, III.4.6 and III.4.7 above. ■

Remark III.4.9. Notice that although Theorem III.4.8 looks somewhat independent of the cotorsion pair $(\mathcal{U}, \mathcal{V})$, we have that the pair $(\mathcal{S}, \mathcal{T})$ determines $(\mathcal{U}, \mathcal{V})$, and vice versa, using Proposition III.2.7 and that $\mathcal{T} = \mathcal{U}$.

Remark III.4.10. We show now that the conclusion of Theorem III.4.8 also follows from results of Beligiannis. Let $\mathcal{C}$ be a category as in the statement of Theorem III.4.8 above. Let $\mathcal{X}$ be a contravariantly finite and rigid subcategory of $\mathcal{C}$. Suppose further that $\mathcal{X}^{\perp_0}$ is contravariantly finite. Beligiannis shows (see Remark 4.3, Lemma 4.4 and Theorem 4.6 in [Bel13]) that there are equivalences

$$(\mathcal{X} * \Sigma\mathcal{X})/[\Sigma\mathcal{X}] = (\mathcal{X} * \Sigma\mathcal{X})/[\mathcal{X}^{\perp_0}] \xrightarrow{\simeq} \operatorname{mod}\mathcal{X} \xleftarrow{\simeq} (\mathcal{C}/[\mathcal{X}^{\perp_0}])_{\mathcal{R}},$$

where $\operatorname{mod}\mathcal{X}$ is the category of coherent functors over $\mathcal{X}$ (see [Aus66]) and $\mathcal{R}$ is the class of regular morphisms in the category $\mathcal{C}/[\mathcal{X}^{\perp_0}]$. In the situation of Theorem III.4.8, we have that $\Sigma^{-1}\mathcal{S}$ is a contravariantly finite, rigid subcategory, and that $(\Sigma^{-1}\mathcal{S})^{\perp_0} = \mathcal{W} = \mathcal{U}$ is also contravariantly finite; see the discussion above Proposition III.4.2 for more details. Therefore, with $\mathcal{X} = \Sigma^{-1}\mathcal{S}$ one obtains

$$(\mathcal{X} * \Sigma\mathcal{X})/[\Sigma\mathcal{X}] = (\Sigma^{-1}\mathcal{S} * \mathcal{S})/[\mathcal{S}] = \overline{\mathscr{H}}_{(\mathcal{S},\mathcal{T})}$$

and

$$(\mathcal{C}/[\mathcal{X}^{\perp_0}])_{\mathcal{R}} = (\mathcal{C}/[\mathcal{W}])_{\mathcal{R}} = \overline{\mathcal{H}}_{\mathcal{R}}.$$

Hence, one may deduce that $\overline{\mathscr{H}}_{(\mathcal{S},\mathcal{T})}$ and $\overline{\mathcal{H}}_{\mathcal{R}}$ are equivalent from the results in [Bel13]. However, the proof method is different: Beligiannis makes use of adjoint functors and obtains a functor $(\mathcal{C}/[\mathcal{X}^{\perp_0}])_{\mathcal{R}} \to (\mathcal{X} * \Sigma\mathcal{X})/[\mathcal{X}^{\perp_0}]$, which is stated to be an equivalence, using the universal property of the localisation $(\mathcal{C}/[\mathcal{X}^{\perp_0}])_{\mathcal{R}}$; on the other hand, we construct an explicit equivalence in the other direction.

III.5 An application to the cluster category

In this section, we assume k is a field and that, unless otherwise stated, $\mathcal{C}$ is a Hom-finite, Krull-Schmidt, triangulated k-category with a Serre functor ν. As usual, we will denote the suspension functor of $\mathcal{C}$ by Σ.

For an object $X \in \mathcal{C}$, we denote by $\mathsf{add}\, X$ the full additive subcategory of $\mathcal{C}$ consisting of objects that are isomorphic to direct summands of finite direct sums of copies of X. We observe that the subcategories $\Sigma(\mathsf{add}\, X)$ (in the sense of Definition II.8.6) and $\mathsf{add}\, \Sigma X$ coincide.

We recall some terminology. See, for example, [BMRRT, p. 583], [XO15, Def. 2.2], [ZZ11, Def. 2.1], [KR07, §2], [Iya07, Def. 2.2].

Definition III.5.1. An object R of $\mathcal{C}$ is called *rigid* if $\mathrm{Ext}^1_{\mathcal{C}}(R, R) = 0$. An object $T \in \mathcal{C}$ is *maximal rigid* if T is rigid and has a maximal number of non-isomorphic indecomposable direct summands or, equivalently, if T is rigid and $\mathrm{Ext}^1_{\mathcal{C}}(T \oplus X, T \oplus X) = 0$ if and only if $X \in \mathsf{add}\, T$ for $X \in \mathcal{C}$.

For the remainder of this section, we assume that R is a fixed rigid object of $\mathcal{C}$.

Definition III.5.2. We denote by $\mathcal{X}_R \coloneqq (\mathsf{add}\, R)^{\perp_0}$ the full additive subcategory of $\mathcal{C}$ that consists of objects X such that $\mathrm{Hom}_{\mathcal{C}}(R, X) = 0$.

Lemma III.5.3. [BM13, §2] *For any object $R \in \mathcal{C}$, $\mathcal{X}_R$ is equal to $(\mathsf{add}\, \Sigma R)^{\perp_1}$.*

Proof. Note that $\mathrm{Hom}_{\mathcal{C}}(R, X) \cong \mathrm{Hom}_{\mathcal{C}}(\Sigma R, \Sigma X) = \mathrm{Ext}^1_{\mathcal{C}}(\Sigma R, X)$, so $\mathrm{Hom}_{\mathcal{C}}(R, X) = 0$ if and only if $\mathrm{Ext}^1_{\mathcal{C}}(\Sigma R, X) = 0$. That is, $X \in \mathcal{X}_R$ if and only if $X \in (\mathsf{add}\, \Sigma R)^{\perp_1}$. ∎

The next proposition collects some easily verifiable observations, some of which may be found in [BM13].

Proposition III.5.4. *For any rigid object $R' \in \mathcal{C}$, the subcategories $\operatorname{add} R'$ and $\mathcal{X}_{R'}$ are closed under isomorphisms and direct summands. Moreover, these subcategories are also extension-closed.*

Proof. Suppose R' is rigid. That $\operatorname{add} R'$ is closed under isomorphisms and direct summands follows immediately from its definition. Thus, assume $R'_1 \longrightarrow X \longrightarrow R'_2 \xrightarrow{h} \Sigma R'_1$ is a triangle in $\mathcal{C}$ with $R'_1, R'_2 \in \operatorname{add} R'$. Then $\operatorname{Hom}_{\mathcal{C}}(R'_2, \Sigma R'_1) = \operatorname{Ext}^1_{\mathcal{C}}(R'_2, R'_1) = 0$ as R' is rigid, so h is the zero morphism. Hence, by Lemma II.7.26, this triangle splits and $X \cong R'_1 \oplus R'_2 \in \operatorname{add} R'$.

If $X, Y \in \mathcal{C}$ are isomorphic then $\operatorname{Hom}_{\mathcal{C}}(R', X) = 0 \iff \operatorname{Hom}_{\mathcal{C}}(R', Y) = 0$, since $\operatorname{Hom}_{\mathcal{C}}(R', X)$ and $\operatorname{Hom}_{\mathcal{C}}(R', Y)$ are in bijection.

Let $X \in \mathcal{X}_{R'}$ be arbitrary and X_1 some direct summand of X. Since $\operatorname{Hom}_{\mathcal{C}}(R', -)$ is an additive functor, we have that $\operatorname{Hom}_{\mathcal{C}}(R', X_1)$ is a direct summand of $\operatorname{Hom}_{\mathcal{C}}(R', X)$. Moreover, $\operatorname{Hom}_{\mathcal{C}}(R', X) = 0$ as $X \in \mathcal{X}_{R'}$, so $\operatorname{Hom}_{\mathcal{C}}(R', X_1) = 0$ and hence $X_1 \in \mathcal{X}_{R'}$.

Lastly, suppose we have a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ with $X, Z \in \mathcal{X}_{R'}$. We wish to show that this implies $Y \in \mathcal{X}_{R'}$. Applying $\operatorname{Hom}_{\mathcal{C}}(R', -)$ we get the exact sequence

$$0 = \operatorname{Hom}_{\mathcal{C}}(R', X) \xrightarrow{f_*} \operatorname{Hom}_{\mathcal{C}}(R', Y) \xrightarrow{g_*} \operatorname{Hom}_{\mathcal{C}}(R', Z) = 0,$$

where $f_* = \operatorname{Hom}_{\mathcal{C}}(R', f)$ and $g_* = \operatorname{Hom}_{\mathcal{C}}(R', g)$. Hence, $\operatorname{Hom}_{\mathcal{C}}(R', Y) = \ker g_* = \operatorname{im} f_* = 0$ and $Y \in \mathcal{X}_{R'}$. ■

Remark III.5.5. Since $\mathcal{C}$ is a Krull-Schmidt category, if an object X of $\mathcal{C}$ has a right $\mathcal{X}$-approximation, for some subcategory $\mathcal{X} \subseteq \mathcal{C}$, then X has a minimal right $\mathcal{X}$-approximation (see Definition II.3.10) by [KS98, Cor. 1.4]. Dually, the existence of a left $\mathcal{X}$-approximation implies the existence of a minimal such one under our assumptions.

The next result is stated in [Nak13], but we include the details to illustrate where the various assumptions on $\mathcal{C}$ are needed. See also [BM12].

Lemma III.5.6. [Nak13, Exam. 2.10 (2)] *The pair* $((\mathsf{add}\,\Sigma R, \mathcal{X}_R), (\mathcal{X}_R, {\mathcal{X}_R}^{\perp_1}))$ *is a twin cotorsion pair with heart* $\overline{\mathcal{H}} = \mathcal{C}/[\mathcal{X}_R]$.

Proof. First, we show that $(\mathsf{add}\,\Sigma R, \mathcal{X}_R)$ is a cotorsion pair on $\mathcal{C}$. Since $\mathcal{C}$ is assumed to be Hom-finite, we have that $\mathsf{add}\,\Sigma R$ is contravariantly finite, so for any $X \in \mathcal{C}$ there exists a triangle $\Sigma R_0 \xrightarrow{f} X \longrightarrow Y \longrightarrow \Sigma^2 R_0$, where $f\colon \Sigma R_0 \to X$ is a minimal right $\mathsf{add}\,\Sigma R$-approximation of X because $\mathcal{C}$ is also Krull-Schmidt. Since $\mathsf{add}\,\Sigma R$ is extension-closed (Proposition III.5.4), by Lemma III.2.5 we have that $Y \in (\mathsf{add}\,\Sigma R)^{\perp_0} = \Sigma\mathcal{X}_R$. Therefore, $\mathcal{C} = \mathsf{add}\,\Sigma R * \Sigma\mathcal{X}_R$. We also have $\mathrm{Ext}^1_{\mathcal{C}}(\Sigma R, \mathcal{X}_R) = \mathrm{Hom}_{\mathcal{C}}(\Sigma R, \Sigma\mathcal{X}_R) \cong \mathrm{Hom}_{\mathcal{C}}(R, \mathcal{X}_R) = 0$. Comparing with Definition III.2.3, we see that $(\mathcal{S}, \mathcal{T}) := (\mathsf{add}\,\Sigma R, \mathcal{X}_R)$ is indeed a cotorsion pair.

To see that $(\mathcal{U}, \mathcal{V}) := (\mathcal{X}_R, {\mathcal{X}_R}^{\perp_1})$ is a cotorsion pair, take a minimal left $\mathsf{add}\,\nu R$-approximation $r\colon X \to \nu R_1$ of X and complete it to a triangle $Z \xrightarrow{s} X \xrightarrow{r} \nu R_1 \longrightarrow \Sigma Z$. Then by Lemma III.2.5 again, we have $Z \in \mathcal{X}_R$ and so $\mathcal{C} = \mathcal{X}_R * \Sigma(\Sigma^{-1}\,\mathsf{add}\,\nu R)$. In addition,

$$\mathrm{Ext}^1_{\mathcal{C}}(\mathcal{X}_R, \Sigma^{-1}\,\mathsf{add}\,\nu R) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{X}_R, \mathsf{add}\,\nu R) \cong \mathrm{Hom}_{\mathcal{C}}(\mathsf{add}\,R, \mathcal{X}_R) = 0,$$

so $(\mathcal{X}_R, \mathsf{add}\,\Sigma^{-1}\nu R)$ is a cotorsion pair. Therefore, by Proposition III.2.7, we see that $(\mathcal{U}, \mathcal{V}) = (\mathcal{X}_R, {\mathcal{X}_R}^{\perp_1}) = (\mathcal{X}_R, \mathsf{add}\,\Sigma^{-1}\nu R)$ is a cotorsion pair.

Furthermore, we have $\mathrm{Hom}_{\mathcal{C}}(R, \Sigma R) = \mathrm{Ext}^1_{\mathcal{C}}(R, R) = 0$ as R is rigid, so $\mathcal{S} = \mathsf{add}\,\Sigma R \subseteq \mathcal{X}_R = \mathcal{U}$. Hence, $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})) = ((\mathsf{add}\,\Sigma R, \mathcal{X}_R), (\mathcal{X}_R, {\mathcal{X}_R}^{\perp_1}))$ is a twin cotorsion pair on $\mathcal{C}$ (see Definition III.2.8). In particular, $\mathcal{T} = \mathcal{X}_R = \mathcal{U}$, so $\mathcal{W} = \mathcal{T} = \mathcal{U} = \mathcal{X}_R$, and

$$\mathcal{C}^- = \Sigma^{-1}\mathcal{S} * \mathcal{W} = \Sigma^{-1}\mathcal{S} * \mathcal{T} = \mathcal{C} = \mathcal{U} * \Sigma\mathcal{V} = \mathcal{W} * \Sigma\mathcal{V} = \mathcal{C}^+.$$

Therefore, $\mathcal{H} = \mathcal{C}^- \cap \mathcal{C}^+ = \mathcal{C}$ and the heart associated to $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ is $\overline{\mathcal{H}} = \mathcal{H}/[\mathcal{W}] = \mathcal{C}/[\mathcal{X}_R]$. ■

Theorem III.5.7. *Suppose $\mathcal{C}$ is a* Hom*-finite, Krull-Schmidt, triangulated k-category that has Serre duality, and assume R is a rigid object of $\mathcal{C}$. Then $\mathcal{C}/[\mathcal{X}_R]$ is quasi-abelian.*

Proof. Consider the twin cotorsion pair $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})) = ((\mathsf{add}\,\Sigma R, \mathcal{X}_R),(\mathcal{X}_R, {\mathcal{X}_R}^{\perp_1}))$. As $\mathcal{T} = \mathcal{U}$ in this case, by Corollary III.3.6, $\overline{\mathcal{H}} = \mathcal{C}/[\mathcal{X}_R]$ is quasi-abelian. ■

Assume also that $\mathcal{C}$ is skeletally small. Let $\mathcal{R}$ be the class of regular morphisms in $\mathcal{C}/[\mathcal{X}_R]$, and denote by $\mathcal{C}(R)$ the subcategory $\mathsf{add}\,R * \mathsf{add}\,\Sigma R$ considered in [KR07, §5.1]; see also [IY08, Prop. 6.2], [BM13] and [BM12]. An equivalence between $\mathcal{C}(R)/[\mathsf{add}\,\Sigma R]$ and $(\mathcal{C}/[\mathcal{X}_R])_{\mathcal{R}}$ exists by combining [IY08, Prop. 6.2] with Theorem III.1.1 (or results of [Bel13] as discussed in Remark III.4.10) as follows

$$\mathcal{C}(R)/[\mathsf{add}\,\Sigma R] \xrightarrow{\simeq} \Lambda_R\text{-}\mathsf{mod} \xleftarrow{\simeq} (\mathcal{C}/[\mathcal{X}_R])_{\mathcal{R}},$$

where $\Lambda_R := (\operatorname{End}_{\mathcal{C}} R)^{\mathrm{op}}$. We now give a new proof that $\mathcal{C}(R)/[\mathsf{add}\,\Sigma R]$ and $(\mathcal{C}/[\mathcal{X}_R])_{\mathcal{R}}$ are equivalent, which avoids going via the module category $\Lambda_R\text{-}\mathsf{mod}$ altogether.

Theorem III.5.8. *Let $\mathcal{C}$ be a skeletally small,* Hom*-finite, Krull-Schmidt, triangulated k-category, and assume R is a rigid object of $\mathcal{C}$. Let $\mathcal{R}$ be the class of regular morphisms in $\mathcal{C}/[\mathcal{X}_R]$ and let $L_{\mathcal{R}}\colon \mathcal{C}/[\mathcal{X}_R] \to (\mathcal{C}/[\mathcal{X}_R])_{\mathcal{R}}$ be the localisation functor. Then there is an additive functor $F\colon \mathcal{C}(R)/[\mathsf{add}\,\Sigma R] \to \mathcal{C}/[\mathcal{X}_R]$ such that the composition*

$$L_{\mathcal{R}} \circ F\colon \mathcal{C}(R)/[\mathsf{add}\,\Sigma R] \xrightarrow{\simeq} (\mathcal{C}/[\mathcal{X}_R])_{\mathcal{R}}$$

is an equivalence.

Proof. Let $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})) = ((\mathsf{add}\,\Sigma R,\mathcal{X}_R),(\mathcal{X}_R,{\mathcal{X}_R}^{\perp_1}))$. The heart (see Definition III.2.21) of the cotorsion pair $(\mathcal{S},\mathcal{T}) = (\mathsf{add}\,\Sigma R,\mathcal{X}_R)$ is $\overline{\mathscr{H}}_{(\mathcal{S},\mathcal{T})} = \mathcal{C}(R)/[\mathsf{add}\,\Sigma R]$, and the heart of the twin cotorsion pair $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ is $\overline{\mathcal{H}} = \mathcal{C}/[\mathcal{X}_R]$. By Proposition III.4.1 there is an additive functor $F\colon \mathcal{C}(R)/[\mathsf{add}\,\Sigma R] \to \mathcal{C}/[\mathcal{X}_R]$ that is the identity on objects and maps a morphism $f+[\mathsf{add}\,\Sigma R](X,Y)$ to $f+[\mathcal{X}_R](X,Y)$, which is well-defined as $\mathsf{add}\,\Sigma R \subseteq \mathcal{X}_R$ since R is a rigid. Then an application of Theorem III.4.8 yields an equivalence

$$\mathcal{C}(R)/[\mathsf{add}\,\Sigma R] = \overline{\mathscr{H}}_{(\mathcal{S},\mathcal{T})} \xrightarrow{\simeq} \overline{\mathcal{H}}_{\mathcal{R}} = (\mathcal{C}/[\mathcal{X}_R])_{\mathcal{R}}.$$

■

We make two last observations before giving an example to demonstrate this theory.

Proposition III.5.9. *Let $\mathcal{C}$ be a* Hom*-finite, triangulated k-category which is 2-Calabi-Yau. Suppose $(\mathcal{S},\mathcal{T})$ and $(\mathcal{U},\mathcal{V})$ are cotorsion pairs on $\mathcal{C}$. Then $\mathcal{T}=\mathcal{U}$ if and only if $\mathcal{S}=\mathcal{V}$.*

Proof. Assume $\mathcal{T}=\mathcal{U}$. Then we have the following chain of equalities

$$\mathcal{S} = {}^{\perp_1}\mathcal{T} = {}^{\perp_1}\mathcal{U} = \mathcal{U}^{\perp_1} = \mathcal{V},$$

using Proposition III.2.7 and that $\mathcal{C}$ is 2-Calabi-Yau. The converse is proved similarly. ■

Corollary III.5.10. *Let $\mathcal{C}$ be a* Hom*-finite, Krull-Schmidt, 2-Calabi-Yau, triangulated k-category, and assume R is a rigid object of $\mathcal{C}$. Then the subcategory* $\mathsf{add}\,\Sigma R$ *coincides with ${\mathcal{X}_R}^{\perp_1}$.*

Proof. Since $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})) = ((\mathsf{add}\,\Sigma R,\mathcal{X}_R),(\mathcal{X}_R,{\mathcal{X}_R}^{\perp_1}))$ is a pair of cotorsion pairs on $\mathcal{C}$ with $\mathcal{T}=\mathcal{U}$, we must have $\mathsf{add}\,\Sigma R = \mathcal{S} = \mathcal{V} = {\mathcal{X}_R}^{\perp_1}$ by Proposition III.5.9. ■

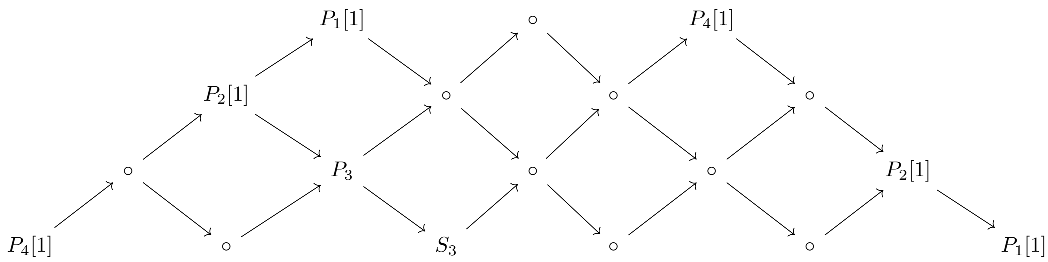
It can also be shown that Corollary III.5.10 follows from [BM13, Lem. 2.2] using $T = \Sigma R$ and the 2-Calabi-Yau property.

Example III.5.11. Let k be a field. Consider the cluster category $\mathcal{C} := \mathcal{C}_{kQ}$ associated to the linearly oriented Dynkin-type quiver $Q: \quad 1 \to 2 \to 3 \to 4$. Its Auslander-Reiten quiver, with the mesh relations omitted, is

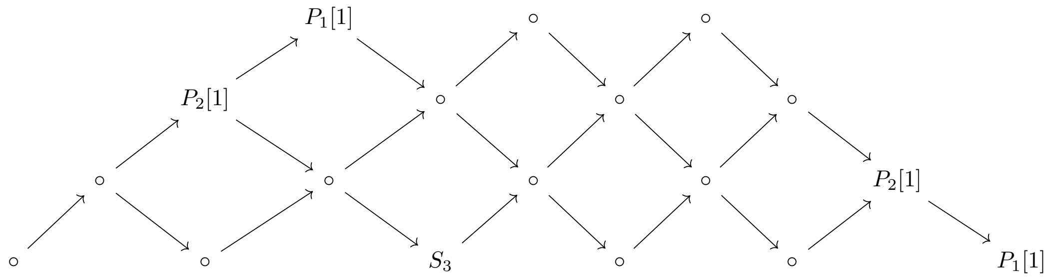
$$\begin{tikzcd}[column sep=tiny, row sep=small]
 & & & P_1[1] \arrow[dr] & & P_1 = \begin{smallmatrix}1\\2\\3\\4\end{smallmatrix} \arrow[dr] & & P_4[1] \arrow[dr] & & & \\
 & & P_2[1] \arrow[ur] \arrow[dr] & & P_2 = \begin{smallmatrix}2\\3\\4\end{smallmatrix} \arrow[ur] \arrow[dr] & & I_3 = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[ur] \arrow[dr] & & P_3[1] \arrow[dr] & & \\
 & P_3[1] \arrow[ur] \arrow[dr] & & P_3 = \begin{smallmatrix}3\\4\end{smallmatrix} \arrow[ur] \arrow[dr] & & M = \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[ur] \arrow[dr] & & I_2 = \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[ur] \arrow[dr] & & P_2[1] \arrow[dr] & \\
P_4[1] \arrow[ur] & & P_4 = 4 \arrow[ur] & & S_3 = 3 \arrow[ur] & & S_2 = 2 \arrow[ur] & & S_1 = 1 \arrow[ur] & & P_1[1]
\end{tikzcd}$$

where the lefthand copy of $P_i[1]$ is identified with the corresponding righthand copy (for $i = 1, 2, 3, 4$); see, for example, [Sch14, §3.1]. Recall that an object X of a Krull-Schmidt category is called *basic* if, when $X = X_1 \oplus \cdots \oplus X_r$ is a direct sum decomposition of X into indecomposables, we have $X_i \ncong X_j$ for all $i \neq j$. We set $R := P_1 \oplus P_2 \oplus S_2$, which is a basic, rigid object of $\mathcal{C}$. Note that since R has just 3 non-isomorphic indecomposable direct summands, it is not maximal rigid (see [BMR07, Cor. 2.3]) and hence not cluster-tilting. Denote by Λ_R the ring $(\operatorname{End}_{\mathcal{C}} R)^{\mathrm{op}}$. We describe the twin cotorsion pair $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})) = ((\operatorname{\mathsf{add}} R[1], \mathcal{X}_R), (\mathcal{X}_R, {\mathcal{X}_R}^{\perp_1}))$ pictorially below, where "$\circ$" denotes that the corresponding object does not belong to the subcategory. Since the cluster category is 2-Calabi-Yau (see Theorem II.8.5), that $\mathcal{S}$ coincides with $\mathcal{V}$ below is not unexpected (see Corollary III.5.10).

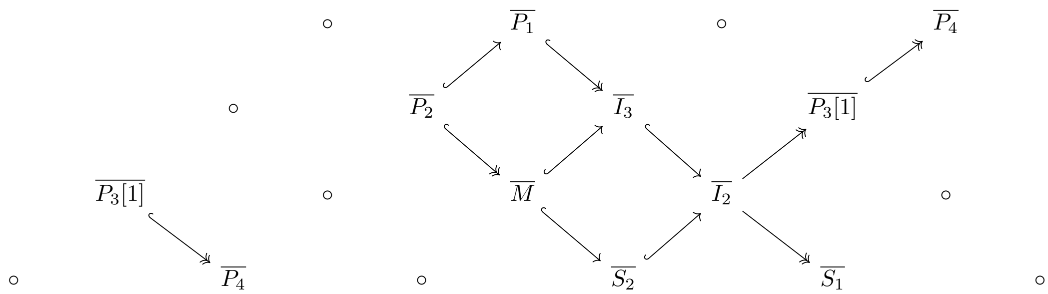
$$\mathcal{T} = \mathcal{U} = \mathcal{W} = \mathcal{X}_R$$



$$\mathsf{add}\, R[1] = \mathcal{S} = \mathcal{V} = \mathcal{X}_R{}^{\perp_1}$$



By [Liu10, Prop. 2.9], the quasi-abelian heart $\overline{\mathcal{H}} = \mathcal{H}/[\mathcal{W}] = \mathcal{C}/[\mathcal{X}_R]$ for this twin cotorsion pair then has the following Auslander-Reiten quiver (ignoring the objects denoted by a "$\circ$" that lie in $\mathcal{X}_R$, and again with the mesh relations omitted).



We have denoted by $\overline{X}$ the image of the object X of $\mathcal{C}$ in $\mathcal{C}/[\mathcal{X}_R]$, monomorphisms by "$\hookrightarrow$" and epimorphisms by "$\twoheadrightarrow$". The extra righthand copy of $\overline{P_4}$ is included to illustrate that this quiver really is connected and similar in shape to the Auslander-Reiten quiver of $\Lambda_R - \mathsf{mod}$ (see below). In this example there are precisely three irreducible morphisms between indecomposables that are also regular morphisms—namely the morphisms $\overline{P_1} \to \overline{I_3}$, $\overline{P_2} \to \overline{M}$ and $\overline{P_3[1]} \to \overline{P_4}$. As noted in §III.1, one may show that various aspects of Auslander-Reiten theory are still applicable in quasi-abelian categories; see Chapter IV for more details. However, one noticeable difference is that in a quasi-abelian category there exist irreducible morphisms that are regular. On the other hand, in an abelian category an irreducible morphism cannot be regular, since a morphism is regular if and only if it is an isomorphism in such a category, and irreducible morphisms cannot be isomorphisms by definition.

In addition, one may obtain the Auslander-Reiten quiver of $\Lambda_R - \mathsf{mod}$ by localising $\overline{\mathcal{H}}$ at the regular morphisms as shown in [BM12]. In this case, one obtains the Auslander-Reiten quiver

$$\begin{tikzcd}
 & \begin{smallmatrix}1'\\2'\end{smallmatrix} \arrow[dr] & & 3' \\
2' \arrow[ur] \arrow[dr] \arrow[rr, dashed, no head] & & \begin{smallmatrix}1' & & 3'\\ & 2' & \end{smallmatrix} \arrow[ur] \arrow[dr] & \\
 & \begin{smallmatrix}3'\\2'\end{smallmatrix} \arrow[ur] & & 1'
\end{tikzcd}$$

where Λ_R is isomorphic to the path algebra of the quiver $1' \to 2' \leftarrow 3'$.

Chapter IV

Auslander-Reiten theory in quasi-abelian and Krull-Schmidt categories

IV.1 Introduction

As is well-known, the work of Auslander and Reiten on *almost split sequences* (which later also became known as *Auslander-Reiten sequences*), introduced in [AR75], has played a large role in comprehending the representation theory of artin algebras. In trying to understand these sequences, it became apparent that two types of morphisms would also play a fundamental role (see [AR77a]). Irreducible morphisms and minimal left/right almost split morphisms (see Definitions II.3.3 and II.3.11) were defined in [AR77a], and the relationship between these morphisms and Auslander-Reiten sequences was studied. In fact, many of the abstract results of Auslander and Reiten were proven for an arbitrary abelian category, not just a module category, and in this chapter we show that much of this Auslander-Reiten theory also holds in a more general context—namely in that of a quasi-abelian category (see §IV.2).

Auslander-Reiten theory has also been studied in other contexts, including in triangulated, exact and extriangulated categories (see, for example, [RVdB02], [Hap88], [GR97], [INP18]). In another context, Liu developed Auslander-Reiten theory in Krull-Schmidt categories that are Hom-finite over a commutative artinian ring in [Liu10]. However, there are interesting examples of categories that are Krull-Schmidt yet not necessarily Hom-finite; see Example IV.3.20. We extend the theory developed by Liu by removing the Hom-finite assumption for all the results in [Liu10, §1]; see §IV.3. Our main result in this context provides a list of equivalent criteria for a short exact sequence to be an *Auslander-Reiten sequence* (see Definition IV.3.6) in a skeletally small, Krull-Schmidt, quasi-abelian category (see Theorem IV.3.19).

Although not every quasi-abelian category is Krull-Schmidt (e.g. the category of Banach spaces over $\mathbb{R}$), the quotient category $\mathcal{C}/[\mathcal{X}_R]$ is an example of a category that is both quasi-abelian and Krull-Schmidt, where $\mathcal{C}$ is a cluster category, $R \in \mathcal{C}$ is a rigid object and $\mathcal{X}_R = \mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(R, -))$ (see Theorem III.5.7, Example III.5.11 and Remark IV.3.5). The nature of the results in §IV.2 and §IV.3 are in some ways quite different. For quasi-abelian categories we study more the properties that irreducible morphisms possess, whereas for Krull-Schmidt categories we say more about how these morphisms behave under categorical quotients. Therefore, in a Krull-Schmidt quasi-abelian category we can apply both types of results and the strongest conclusions can be drawn (see Theorem IV.3.19 for example). In particular, these conclusions apply to $\mathcal{C}/[\mathcal{X}_R]$.

This chapter is structured as follows. In §IV.2 we develop Auslander-Reiten theory for semi-abelian and quasi-abelian categories. At the beginning of §IV.3, we switch focus to Auslander-Reiten theory in Krull-Schmidt categories, and then present a characterisation theorem for Auslander-Reiten sequences in a Krull-Schmidt quasi-abelian category. Lastly, in §IV.4 we explore an example coming from the cluster category, which demonstrates the theory we develop in the earlier sections.

IV.2 Auslander-Reiten theory in quasi-abelian categories

In this section, we explore some Auslander-Reiten theory in connection with semi-abelian and quasi-abelian categories. We remark here that quasi-abelian categories carry a canonical exact structure: a quasi-abelian category endowed with the class of all its short exact sequences forms an exact category in the sense of Quillen [Qui73] (see [Sch99, Rem. 1.1.11]). Some Auslander-Reiten theory for exact categories was developed in [GR97], but our results are different in nature: we explore properties of the morphisms involved in Auslander-Reiten sequences, whereas [GR97] focuses more on the existence and construction of such sequences. See [INP18] also.

Remark IV.2.1. Recall that every left (respectively, right) quasi-abelian category is left (respectively, right) semi-abelian (see Proposition II.9.21). Furthermore, in a left (respectively, right) semi-abelian category, if a morphism f factorises as $f = ip$ with i monic and p a cokernel (respectively, i a kernel and p epic), then $p = \operatorname{coim} f$ (respectively, $i = \operatorname{im} f$) up to unique isomorphism.

For the results presented here that are analogues of those in known work, we omit the proofs that carry over or that are easy generalisations. Instead, we focus on those arguments that need significant modification or that have been omitted in previous work. Furthermore, many of the results in the remainder of the chapter have duals, which we state but do not prove.

The next proposition is a version of [AR77a, Prop. 2.6 (a)] for the semi-abelian setting. Recall that a monomorphism (respectively, epimorphism) that is not an isomorphism is called a *proper monomorphism* (respectively, *proper epimorphism*).

Proposition IV.2.2. *Suppose a category $\mathcal{A}$ is left or right semi-abelian. If $f\colon X \to Y$ is irreducible in $\mathcal{A}$, then it is a proper monomorphism or a proper epimorphism.*

Proof. Suppose $f\colon X \to Y$ is an irreducible morphism with coimage $\operatorname{coim} f\colon X \twoheadrightarrow \operatorname{Coim} f$. Note that f cannot be an isomorphism since it is not, for example, a section.

Suppose now that $\mathcal{A}$ is left semi-abelian. Then we have a factorisation $f = i \circ \operatorname{coim} f$ where i is monic (see Remark IV.2.1). If f is a proper monomorphism then we are done, so suppose not. Then $f = i \circ \operatorname{coim} f$ is irreducible implies i is a retraction or $\operatorname{coim} f$ is a section. The latter implies $\operatorname{coim} f$ is monic and this in turn yields that f is monic, which is contrary to our assumption that f is not a proper monomorphism. Thus, i must be a retraction and hence an epimorphism. Then f is the composition of two epimorphisms and is thus epic itself, i.e. f is a proper epimorphism.

The case when $\mathcal{A}$ is right semi-abelian is proved similarly. ∎

In an abelian category, we have that a morphism f is an isomorphism if and only if it is regular. Therefore, in an abelian setting an irreducible morphism is either a proper monomorphism or a proper epimorphism, but never both simultaneously. However, this need not hold in an arbitrary category. In particular, we will see in Example IV.4.1 irreducible morphisms (so non-isomorphisms) that are regular.

The following two results give a version of [AR77a, Prop. 2.7] in a more general setting. For Proposition IV.2.3, we assume the category $\mathcal{A}$ is quasi-abelian, but we only really need that $\mathcal{A}$ is right semi-abelian and left quasi-abelian. However, these are equivalent. Recall that by Proposition II.9.21, a left (respectively, right) quasi-abelian category is left (respectively, right) semi-abelian. Thus, a quasi-abelian category is immediately seen to be right semi-abelian and left quasi-abelian. Conversely, if $\mathcal{A}$ is right semi-abelian and left quasi-abelian, then $\mathcal{A}$ is left semi-abelian and hence semi-abelian. Then, by [Rum01, Prop. 3], if $\mathcal{A}$ is semi-abelian, then it is left quasi-abelian if and only if it is right quasi-abelian if and only if it is quasi-abelian. Dually, for the second result we only require that $\mathcal{A}$ is left semi-abelian and right quasi-abelian. The proof we give is inspired by that of Auslander and Reiten; however, since regular morphisms may not be isomorphisms or, for example, monomorphisms may not be kernels in the categories we are dealing with, we must consider some different short exact sequences in the proof.

Proposition IV.2.3. *Suppose $\mathcal{A}$ is a quasi-abelian category and that $f\colon X \to Y$ is a morphism in $\mathcal{A}$ with cokernel $c\colon Y \to C$. If f is irreducible, then for all $v\colon V \to C$ either there exists $v_1\colon V \to Y$ such that $cv_1 = v$ or there exists $v_2\colon Y \to V$ such that $c = vv_2$. Furthermore, if $X \xrightarrow{f} Y \xrightarrow{c} C$ is a non-split short exact sequence, then the converse also holds.*

Proof. First, suppose that $f\colon X \to Y$ is irreducible and that $v\colon V \to C$ is arbitrary. As $\mathcal{A}$ is quasi-abelian and thus semi-abelian, we may decompose $(\operatorname{im} f) \circ p$, where $\operatorname{im} f\colon L \to Y$ is the image of f and $p\colon X \to L$ is an epimorphism. Since $\mathcal{A}$ is quasi-abelian we may consider the following commutative diagram

$$\begin{tikzcd}
K \arrow[d, "w", "\cong"'] \arrow[dr, "\ker a"] & & \\
L \arrow[r, "(\ker a)w^{-1}"'] \arrow[d, equal] & E \arrow[r, "a"] \arrow[d, "u"] \arrow[dr, phantom, "\square"] & V \arrow[d, "v"] \\
L \arrow[r, "\operatorname{im} f"] & Y \arrow[r, "c"] & C \\
X \arrow[u, two heads, "p"] \arrow[ur, "f"'] & &
\end{tikzcd}$$

where E is the pullback of c along v, a is a cokernel since cokernels are stable under pullback in (left) quasi-abelian categories and the induced morphism $w\colon K \to L$ is an isomorphism by [Rum01, Lem. 1]. Then $f = (\operatorname{im} f) \circ p = u \circ ((\ker a)w^{-1}p)$, so either $(\ker a)w^{-1}p$ is a section or u is a retraction as f is irreducible.

If $(\ker a)w^{-1}p$ is a section then there exists $v_1'\colon E \to X$ such that $v_1'(\ker a)w^{-1}p = 1_X$. Then $pv_1'(\ker a)w^{-1}p = p = 1_L p$ implies $pv_1'(\ker a)w^{-1} = 1_L$ as p is epic. Thus $(\ker a)w^{-1}$ is a section, and hence $\ker a$ is a section as w^{-1} is an isomorphism. Since a is a cokernel, we have $a = \operatorname{coker}(\ker a)$ by Lemma II.9.8. Therefore, $\ker a$

is a section implies a is a retraction by the Splitting Lemma (Proposition II.10.3). That is, there exists $v_1''\colon V \to E$ such that $av_1'' = 1_V$. Define $v_1 := uv_1''$ and note that $cv_1 = cuv_1'' = vav_1'' = v$. Otherwise, u is a retraction and so there exists $v_2'\colon Y \to E$ with $uv_2' = 1_Y$. Setting $v_2 := av_2'$ we see that $vv_2 = vav_2' = cuv_2' = c$. This concludes the proof of the first statement.

For the converse, we assume that $X \xrightarrow{f} Y \xrightarrow{c} C$ is a non-split short exact sequence and, further, that for all $v\colon V \to C$ either there exists $v_1\colon V \to Y$ such that $cv_1 = v$ or there exists $v_2\colon Y \to V$ such that $c = vv_2$. Then f is not a section or a retraction, by Lemma II.10.5, as $X \xrightarrow{f} Y \xrightarrow{c} C$ is non-split. It remains to show part (iii) of Definition II.3.3. To this end, suppose $f = hg$ for some $g\colon X \to U$ and $h\colon U \to Y$. Since $hg = f = \ker c$ is a kernel, g is also a kernel by [Rum01, Prop. 2] as $\mathcal{A}$ is quasi-abelian and so, in particular, right semi-abelian. Thus, $g = \ker(\operatorname{coker} g)$ (by Lemma II.9.8) and $X \xrightarrow{g} U \xrightarrow{\operatorname{coker} g} V$ is short exact. Consider the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, equal] & U \arrow[r, "\operatorname{coker} g"] \arrow[d, "h"] & V \arrow[d, "v"] \\
X \arrow[r, "f"] & Y \arrow[r, "c"] & C
\end{tikzcd}$$

where v exists since $(ch)g = cf = 0$. Since 1_X is an isomorphism, $\operatorname{coker} g$ and c are cokernels, and $\mathcal{A}$ is quasi-abelian, by [Rum01, Prop. 5] we have that the right square is exact (see Definition II.9.15). By assumption, either there exists $v_1\colon V \to Y$ such that $cv_1 = v$ or there exists $v_2\colon Y \to V$ such that $c = vv_2$. In the first case, we have the following situation

$$\begin{tikzcd}
V \arrow[dr, dotted, "\exists! a"] \arrow[drr, bend left, "1_V"] \arrow[ddr, bend right, "v_1"'] & & \\
& U \arrow[r, "\operatorname{coker} g"] \arrow[d, "h"'] \arrow[dr, phantom, "\square"] & V \arrow[d, "v"] \\
& Y \arrow[r, "c"'] & C
\end{tikzcd}$$

since $cv_1 = v = v1_V$, and hence there exists a (unique) morphism $a\colon V \to U$ such that $(\operatorname{coker} g) \circ a = 1_V$ (and $v_1 = ha$). Thus, $\operatorname{coker} g$ is a retraction and by the Splitting Lemma (Proposition II.10.3) we have that g is a section. Otherwise, in the case where v_2 exists, we have that there is a (unique) morphism $b\colon Y \to U$ such that $hb = 1_Y$ (and $(\operatorname{coker} g) \circ b = v_2$), in which case h is seen to be a retraction. The following diagram summarises this case.

$$
\begin{tikzcd}
Y \arrow[drr, bend left, "v_2"] \arrow[dr, dotted, "\exists! b"] \arrow[ddr, bend right, "1_Y"'] & & \\
& U \arrow[r, "\operatorname{coker} g"] \arrow[d, "h"'] \arrow[dr, phantom, "\square"] & V \arrow[d, "v"] \\
& Y \arrow[r, "c"'] & C
\end{tikzcd}
$$

Therefore, f is irreducible and the proof is complete. ■

The dual statement is as follows.

Proposition IV.2.4. *Suppose $\mathcal{A}$ is a quasi-abelian category. Suppose $f\colon X \to Y$ is a morphism in $\mathcal{A}$ with kernel* $\ker f\colon \operatorname{Ker} f \to X$. *If f is irreducible, then for all $u\colon \operatorname{Ker} f \to U$ either there exists $u_1\colon X \to U$ such that $u_1 \ker f = u$ or there exists $u_2\colon U \to X$ such that $\ker f = u_2 u$. Furthermore, if $\operatorname{Ker} f \xrightarrow{\ker f} X \xrightarrow{f} Y$ is a non-split short exact sequence, then the converse also holds.*

The next two propositions together give a combined version of [AS80, Lem. 3.8] and [ASS06, Lem. IV.1.9] valid for any S-category. We provide a full proof as the proof of the corresponding result for module categories is omitted in [AS80]. We also note that the equivalence of (i) and (iii) in each statement below has already appeared in the proof of [AR77a, Thm. 2.4] in the setting of an additive category with split idempotents.

Proposition IV.2.5. *Let $f\colon X \to Y$ be a morphism in an S-category $\mathcal{A}$, where $\operatorname{End}_{\mathcal{A}} X$ is a local ring. Then the following are equivalent:*

(i) *f is not a section;*

(ii) *$f \in \operatorname{rad}_{\mathcal{A}}(X, Y)$;*

(iii) *$\operatorname{Im}(\operatorname{Hom}_{\mathcal{A}}(f, X)) \subseteq J(\operatorname{End}_{\mathcal{A}} X) = \operatorname{rad}_{\mathcal{A}}(X, X)$; and*

(iv) *for all $Z \in \mathcal{A}$, the image $\operatorname{Im}(\operatorname{Hom}_{\mathcal{A}}(f, Z))$ of the map $\operatorname{Hom}_{\mathcal{A}}(f, Z)\colon \operatorname{Hom}_{\mathcal{A}}(Y, Z) \to \operatorname{Hom}_{\mathcal{A}}(X, Z)$ is contained in $\operatorname{rad}_{\mathcal{A}}(X, Z)$.*

Proof. First, assume $f = 0$. If f were a section then we would have $1_X = gf = 0$ for some $g\colon Y \to X$, but this is impossible as $\operatorname{End}_{\mathcal{A}} X$ is local so $X \neq 0$. Thus, f is not a section and (i) holds true. Furthermore, for any $Z \in \mathcal{A}$, we have $\operatorname{Im}(\operatorname{Hom}_{\mathcal{A}}(f, Z)) = 0$ if $f = 0$, and so is contained in $\operatorname{rad}_{\mathcal{A}}(X, Z)$. That is, (iii) and (iv) are satisfied. Since $\operatorname{rad}_{\mathcal{A}}$ is an ideal of $\mathcal{A}$, $f = 0 \in \operatorname{rad}_{\mathcal{A}}(X, Y)$ and (ii) also holds in this case.

Therefore, we may now assume $f \neq 0$. It is clear that (iv) implies (iii).

(iii) $\Rightarrow$ (i). If f is a section, then there exists $g\colon Y \to X$ with $\operatorname{Hom}_{\mathcal{A}}(f, X)(g) = gf = 1_X \in \operatorname{Im}(\operatorname{Hom}_{\mathcal{A}}(f, X)) \backslash J(\operatorname{End}_{\mathcal{A}} X)$, so $\operatorname{Im}(\operatorname{Hom}_{\mathcal{A}}(f, X)) \nsubseteq J(\operatorname{End}_{\mathcal{A}} X)$.

(i) $\Rightarrow$ (ii). Suppose f is not a section. Since $\operatorname{End}_{\mathcal{A}} X$ is local, $\operatorname{rad}_{\mathcal{A}}(X, X) = J(\operatorname{End}_{\mathcal{A}} X)$ is the set of all non-left invertible elements of $\operatorname{End}_{\mathcal{A}} X$ by Proposition II.1.44. Let $g\colon Y \to X$ be arbitrary and consider $gf\colon X \to X$. Notice that gf cannot have a left inverse because we are assuming f is not a section. Therefore, $gf \in \operatorname{rad}_{\mathcal{A}}(X, X)$ and $1_X - gf$ is invertible. This is precisely the requirement for f to be radical.

(ii) $\Rightarrow$ (iv). Suppose $f \in \operatorname{rad}_{\mathcal{A}}(X, Y)$, and let $Z \in \mathcal{A}$ and $g\colon Y \to Z$ be arbitrary. Since $\operatorname{rad}_{\mathcal{A}}$ is an ideal of $\mathcal{A}$, we immediately see that $\operatorname{Hom}_{\mathcal{A}}(f, Z)(g) = gf \in \operatorname{rad}_{\mathcal{A}}(X, Z)$ and so $\operatorname{Im}(\operatorname{Hom}_{\mathcal{A}}(f, Z)) \subseteq \operatorname{rad}_{\mathcal{A}}(X, Z)$. ■

Proposition IV.2.6. *Let $f\colon X \to Y$ be a morphism in an S-category $\mathcal{A}$, where $\operatorname{End}_{\mathcal{A}} Y$ is a local ring. Then the following are equivalent:*

(i) *f is not a retraction;*

(ii) $f \in \operatorname{rad}_{\mathcal{A}}(X, Y)$*;*

(iii) $\operatorname{Im}(\operatorname{Hom}_{\mathcal{A}}(Y, f)) \subseteq J(\operatorname{End}_{\mathcal{A}} Y) = \operatorname{rad}_{\mathcal{A}}(Y, Y)$*; and*

(iv) *for all $Z \in \mathcal{A}$, the image $\operatorname{Im}(\operatorname{Hom}_{\mathcal{A}}(Z, f))$ of the map $\operatorname{Hom}_{\mathcal{A}}(Z, f)\colon \operatorname{Hom}_{\mathcal{A}}(Z, X) \to \operatorname{Hom}_{\mathcal{A}}(Z, Y)$ is contained in $\operatorname{rad}_{\mathcal{A}}(Z, Y)$.*

Immediately from the above two results, we have the following.

Corollary IV.2.7. *Let $\mathcal{A}$ be an S-category and $f\colon X \to Y$ a morphism in $\mathcal{A}$, and suppose $\operatorname{End}_{\mathcal{A}} X$ is local or $\operatorname{End}_{\mathcal{A}} Y$ is local. If f is neither a section nor a retraction, then $f \in \operatorname{rad}_{\mathcal{A}}(X, Y)$.*

So far we have only studied irreducible morphisms and, as mentioned earlier, we will also be concerned with (minimal) left/right almost split morphisms.

Proposition IV.2.8. *Let $\mathcal{A}$ be an additive category with split idempotents.*

(i) *If $f\colon X \to Y$ is minimal left almost split and $Y \neq 0$, then f is irreducible.*

(ii) *If $f\colon X \to Y$ is minimal right almost split and $X \neq 0$, then f is irreducible.*

Proof. For (i), notice that f satisfies the criterion in [AR77a, Thm. 2.4 (b)]. Statement (ii) is dual. ■

The next proposition is an observation that we may generalise [BSS11, Prop. 2.18] to a category with split idempotents that is not necessarily Krull-Schmidt, e.g. the category of all left R-modules for a ring R, or the category of all Banach spaces (over $\mathbb{R}$, for example). This result generalises [AR77a, Cor. 2.5] since an irreducible morphism with a domain or codomain that has local endomorphism ring is radical by Corollary IV.2.7. We omit the proof as the one given in [BSS11] holds in our generality using [Büh10, Rem. 7.4]. See [Bau82, Prop. 3.2] also.

Proposition IV.2.9. *Let $\mathcal{A}$ be an additive category with split idempotents. Suppose $f\colon X \to Y$ is a radical irreducible morphism in $\mathcal{A}$.*

(i) *If $0 \neq g\colon W \to X$ is a section, then $fg\colon W \to Y$ is irreducible.*

(ii) *If $0 \neq h\colon Y \to Z$ is a retraction, then $hf\colon X \to Z$ is irreducible.*

The following is a version of [AR77a, Prop. 2.10] for the quasi-abelian setting. We will see that the idea behind the proof is the same, but we have to negotiate around the fact that the class of kernels (respectively, cokernels) does not necessarily coincide with the class of monomorphisms (respectively, epimorphisms) in the category.

Proposition IV.2.10. *Suppose $\mathcal{A}$ is a quasi-abelian category.*

(i) *If $f\colon X \to Y$ is an irreducible monomorphism, Y is indecomposable and $v\colon V \to \operatorname{Coker} f$ is any irreducible morphism, then v is epic.*

(ii) *If $f\colon X \to Y$ is an irreducible epimorphism, X is indecomposable and $u\colon \operatorname{Ker} f \to X$ is any irreducible morphism, then u is monic.*

Proof. We only prove (i) as (ii) is dual. Suppose $f\colon X \to Y$ is an irreducible monomorphism, and that Y is an indecomposable object. First, if $C \coloneqq \operatorname{Coker} f = 0$ then any morphism $v\colon V \to C$ is trivially epic as $\mathcal{A}$ is additive, so we may suppose $C \neq 0$.

Consider the (not necessarily short exact) sequence $X \xrightarrow{f} Y \overset{c:=\operatorname{coker} f}{\twoheadrightarrow} C$. Let $v\colon V \to C$ be an irreducible morphism in $\mathcal{A}$. By Proposition IV.2.3 either there exists $v_1\colon V \to Y$ such that $cv_1 = v$ or there exists $v_2\colon Y \to V$ such that $c = vv_2$. In the latter case, as c is epic, v would also be epic and we would be done. Thus, suppose no such v_2 exists. Then there exists $v_1\colon V \to Y$ with $cv_1 = v$. But v is irreducible, and so either c is a retraction or v_1 is a section. If $c\colon Y \to C$ is a retraction, then by Lemma II.9.7 we have that c is an isomorphism since Y is indecomposable and $C \neq 0$. However, this implies $f = c^{-1}cf = 0$ and in turn yields $1_X = 0$, since $f \circ 1_X = f = 0$ and f is monic. Thus, we would have $X = 0$ and f is in fact a section, which contradicts that f is irreducible. Hence, c cannot be a retraction and so $v_1 \colon V \to Y$ must be a section.

If $V = 0$ then $v \colon 0 = V \to C$ is a section, which is impossible as v is assumed to be irreducible. Therefore, $V \neq 0$ and hence, by Lemma II.9.7 again, $v_1 \colon V \to Y$ must be an isomorphism and, in particular, an epimorphism. Finally, we observe that $v = cv_1$ is the composition of two epimorphisms and hence an epimorphism itself. ■

Definition IV.2.11. [RW77, p. 522] Let $\mathcal{A}$ be an additive category. A kernel (respectively, cokernel) is called *semi-stable* if every pushout (respectively, pullback) of it is again a kernel (respectively, a cokernel).

Example IV.2.12. Consider a stable exact sequence $\xi \colon X \xrightarrow{f} Y \xrightarrow{g} Z$ (see Definition II.10.8). Let $a \colon X \to X'$ be a morphism, and form the sequence $a\xi$ as in Definition II.10.6. Since ξ is stable, the sequence $a\xi$ is short exact and hence the pushout of f along a is again a kernel. Thus, f is a semi-stable kernel. Dually, g is a semi-stable cokernel.

Remark IV.2.13. All kernels are semi-stable in a right quasi-abelian category and all cokernels are semi-stable in a left quasi-abelian category. In particular, all short exact sequences are stable in a quasi-abelian category.

Theorem IV.2.14. *Suppose $\xi \colon X \xrightarrow{f} Y \xrightarrow{g} Z$ and $\xi' \colon X \xrightarrow{f'} Y' \xrightarrow{g'} Z$ are two complexes in a preabelian category $\mathcal{A}$.*

(i) *If ξ is short exact with g semi-stable, ξ' is left exact and $(1_X, v, 1_Z) \colon \xi \to \xi'$ is a morphism of complexes, then v is an isomorphism.*

(ii) *If ξ' is short exact with f' semi-stable, ξ is right exact and $(1_X, v, 1_Z) \colon \xi \to \xi'$ is a morphism of complexes, then v is an isomorphism.*

Proof. Statement (i) is just [RW77, Thm. 6], and (ii) is a dual version. ■

Lemma IV.2.15. *Let $\mathcal{A}$ be a preabelian category. Suppose $\xi\colon X \xrightarrow{f} Y \xrightarrow{g} Z$ is a short exact sequence and that f or g is semi-stable. Suppose there is a morphism $(u, v, w)\colon \xi \to \xi$ of short exact sequences for which $u \in \operatorname{Aut}_{\mathcal{A}} X$ and $w \in \operatorname{Aut}_{\mathcal{A}} Z$. Then $v \in \operatorname{Aut}_{\mathcal{A}} Y$.*

Proof. Since u and w are automorphisms, and $(u, v, w)\colon \xi \to \xi$ is a morphism of short exact sequences, we obtain a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "fu^{-1}"] \arrow[d, equal] & Y \arrow[r, "wg"] \arrow[d, "v"] & Z \arrow[d, equal] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

with exact rows. If f is semi-stable, then v is an automorphism using Theorem IV.2.14 (ii). If g is semi-stable, then it follows that wg is also semi-stable as w is an isomorphism. Thus, by Theorem IV.2.14 (i), we have that v is an automorphism of Y. ■

Our main theorem in §IV.3 generalises Theorem II.4.13, and part of the proof uses tools to detect when an endomorphism (u, v, w) of a short exact sequence is in fact an *isomorphism*, i.e. u, v, w are all isomorphisms. We present generalisations of these tools now, and we will see the work of §II.10 used below. We will assume for simplicity that a preabelian category $\mathcal{A}$ is skeletally small whenever our proofs require the use of an extension group. However, we only really need that the first extension group is a set in the relevant arguments (see Remark II.10.12).

Recall that an additive category $\mathcal{A}$ is called Hom-*finite* if $\mathcal{A}$ is an S-category, for some commutative ring S, and $\operatorname{Hom}_{\mathcal{A}}(X, Y)$ is a finite length S-module for any $X, Y \in \mathcal{A}$; see Definition II.6.13.

Proposition IV.2.16. *Let $\mathcal{A}$ be a* Hom*-finite category. Suppose we have a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[r, "g"] \arrow[d, "v"] & Z \arrow[d, "w"] \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

in $\mathcal{A}$ with non-split short exact rows. If $\operatorname{End}_{\mathcal{A}} X$ (respectively, $\operatorname{End}_{\mathcal{A}} Z$) is local and w (respectively, u) is an automorphism, then u (respectively, w) is an automorphism.

Further, if $\mathcal{A}$ is also preabelian and if f or g is semi-stable, then v is also an automorphism in this case.

Proof. Suppose that $\operatorname{End}_{\mathcal{A}} X$ is local and w is an automorphism of Z. Showing that u is an automorphism in this case is the same as in [ASS06, Lem. IV.1.12].

Now assume $\mathcal{A}$ is also preabelian, and that f is a semi-stable kernel or g is a semi-stable cokernel. Consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "fu^{-1}"] \arrow[d, equal] & Y \arrow[r, "wg"] \arrow[d, "v"] & Z \arrow[d, equal] \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

that has short exact rows. Then v is an automorphism by Lemma IV.2.15. ■

The following corollary is a generalisation of [ASS06, Lem. IV.1.12] to a quasi-abelian Hom-finite setting, and it follows quickly from Proposition IV.2.16 in light of Remark IV.2.13.

Corollary IV.2.17. *Let $\mathcal{A}$ be a* Hom*-finite category, which is left or right quasi-*

abelian. Suppose we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[r, "g"] \arrow[d, "v"] & Z \arrow[d, "w"] \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

in $\mathcal{A}$ with non-split short exact rows. If $\operatorname{End}_{\mathcal{A}} X$ (respectively, $\operatorname{End}_{\mathcal{A}} Z$) is local and w (respectively, u) is an automorphism, then u (respectively, w) is an automorphism. Furthermore, v is also an automorphism.

The next proposition is a generalisation of [AR77a, Lem. 2.13] for preabelian categories. However, note that we need to assume the short exact sequence in question is stable.

Proposition IV.2.18. *Let $\mathcal{A}$ be a skeletally small, preabelian category. Suppose we have a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[r, "g"] \arrow[d, "v"] & Z \arrow[d, "w"] \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

in $\mathcal{A}$, where $\xi\colon X \xrightarrow{f} Y \xrightarrow{g} Z$ is a non-split stable exact sequence. If $\operatorname{End}_{\mathcal{A}} X$ (respectively, $\operatorname{End}_{\mathcal{A}} Z$) is local and w (respectively, u) is an automorphism, then u (respectively, w) and hence v are automorphisms.

Proof. Suppose $\operatorname{End}_{\mathcal{A}} X$ is local and w lies in $\operatorname{Aut}_{\mathcal{A}} Z$. The proof is that of [AR77a] with the following adjustments. In order to show u is an automorphism, one needs that $u\xi \cong \xi 1_Z = \xi$ and that $\operatorname{Ext}^1_{\mathcal{A}}(Z, X)$ is a left $\operatorname{End}_{\mathcal{A}} X$-module, which follow from [RW77, Cor. 7] and Theorem II.10.16, respectively. Lastly, an application of Lemma IV.2.15 yields that v is also an automorphism. ■

Since all short exact sequences are stable in a quasi-abelian category (see Remark

IV.2.13), we obtain a direct generalisation of [AR77a, Lem. 2.13] as follows.

Corollary IV.2.19. *Let $\mathcal{A}$ be a skeletally small, quasi-abelian category. Suppose we have a commutative diagram*

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} \\ X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z \end{array}$$

in $\mathcal{A}$, where $X \overset{f}{\to} Y \overset{g}{\to} Z$ is a non-split short exact sequence. If $\operatorname{End}_{\mathcal{A}} X$ (respectively, $\operatorname{End}_{\mathcal{A}} Z$) is local and w (respectively, u) is an automorphism, then u (respectively, w) and hence v are automorphisms.

The following definition has been given by Auslander for abelian categories (see [Aus74, p. 292]), but we may adopt the same definition for additive categories and we are able to derive some of the same consequences.

Definition IV.2.20. [Aus74, p. 292] Let $\mathcal{A}$ be an additive category with an object X. Suppose $\mathscr{F}\colon \mathcal{A} \to \mathsf{Ab}$ is a covariant additive functor to the category Ab of all abelian groups. An element $x \in \mathscr{F}(X)$ is said to be *minimal* if $x \neq 0$ (where 0 is the identity element of the abelian group $\mathscr{F}(X)$), and if for all proper epimorphisms $f\colon X \to Y$ in $\mathcal{A}$, we have that $\mathscr{F}(f)\colon \mathscr{F}(X) \to \mathscr{F}(Y)$ satisfies $\mathscr{F}(f)(x) = 0$.

A definition of minimal can be made for a contravariant functor $\mathscr{G}\colon \mathcal{A} \to \mathsf{Ab}$ by considering $\mathscr{G}$ as a covariant functor $\mathcal{A}^{\mathrm{op}} \to \mathsf{Ab}$.

An immediate result is a version of [Aus74, p. 292, Lem. 3.2 (a)] for additive categories.

Proposition IV.2.21. *Let $\mathcal{A}$ be an additive category with an object X. Suppose $\mathscr{F}\colon \mathcal{A} \to \mathsf{Ab}$ is a covariant additive functor. If $\mathscr{F}(X)$ has a minimal element, then X is indecomposable in $\mathcal{A}$.*

Proof. Assume $x \in \mathscr{F}(X)$ is minimal, and that $X = X_1 \oplus X_2$ with X_1, X_2 both non-zero. Let $\iota_i \colon X_i \hookrightarrow X$ and $\pi_i \colon X \twoheadrightarrow X_i$ be the canonical inclusion and projection morphisms, respectively, for $i = 1, 2$. Note that π_i is a proper epimorphism for $i = 1, 2$ since X_1 and X_2 are non-zero. Therefore, $\mathscr{F}(\pi_i)(x) = 0$ for $i = 1, 2$ as x is minimal. However, this implies

$$\begin{aligned} x &= 1_{\mathscr{F}(X)}(x) = \mathscr{F}(1_X)(x) && \text{as } \mathscr{F} \text{ is a functor} \\ &= \mathscr{F}(\iota_1\pi_1 + \iota_2\pi_2)(x) && \text{as } X = X_1 \oplus X_2 \\ &= \mathscr{F}(\iota_1\pi_1)(x) + \mathscr{F}(\iota_2\pi_2)(x) && \text{as } \mathscr{F} \text{ is additive} \\ &= \mathscr{F}(\iota_1)(\mathscr{F}(\pi_1)(x)) + \mathscr{F}(\iota_2)(\mathscr{F}(\pi_2)(x)) && \text{as } \mathscr{F} \text{ is covariant} \\ &= 0 && \text{since } \mathscr{F}(\pi_i)(x) = 0. \end{aligned}$$

This is a contradiction because $x \neq 0$ since it is minimal. Hence, X must be indecomposable. ■

The next proposition generalises [AR77a, Prop. 2.6 (b)] to a semi-abelian setting. The strategy in the proof is the same, but we need a technical result from [Rum01] in order to work in a category with less structure. Recall that if $\mathcal{A}$ is skeletally small, then $\operatorname{Ext}^1_{\mathcal{A}}(-,-)$ is an additive bifunctor (see Theorem II.10.17).

Proposition IV.2.22. *Let $\mathcal{A}$ be a skeletally small, semi-abelian category with objects X, Z. Consider the covariant additive functor $\operatorname{Ext}^1_{\mathcal{A}}(Z,-)\colon \mathcal{A} \to \mathsf{Ab}$ and the contravariant additive functor $\operatorname{Ext}^1_{\mathcal{A}}(-,X)\colon \mathcal{A} \to \mathsf{Ab}$. Suppose $\xi \colon X \xrightarrow{f} Y \xrightarrow{g} Z$ is an element of $\operatorname{Ext}^1_{\mathcal{A}}(Z,X)$.*

(i) *If f is irreducible, then $\xi \in \operatorname{Ext}^1_{\mathcal{A}}(-,X)(Z)$ is minimal, and hence Z is indecomposable.*

(ii) *If g is irreducible, then $\xi \in \operatorname{Ext}^1_{\mathcal{A}}(Z,-)(X)$ is minimal, and hence X is indecomposable.*

Proof. We prove (ii); the proof for (i) is similar. Suppose g is irreducible in the stable exact sequence $\xi\colon X \xrightarrow{f} Y \xrightarrow{g} Z$. Since g is not a retraction, by the Splitting Lemma (Proposition II.10.3) we know ξ is not split and hence $\xi \neq 0$ in $\operatorname{Ext}^1_{\mathcal{A}}(Z, X)$. Suppose $a\colon X \to X_1$ is a proper epimorphism. We will show that $\operatorname{Ext}^1_{\mathcal{A}}(Z, a)(\xi) = a\xi = 0$, i.e. the short exact sequence $a\xi$ is split. By definition, $a\xi$ comes with some commutative diagram

$$\begin{tikzcd}
\xi\colon & X \arrow[r, "f"] \arrow[d, "a"] & Y \arrow[r, "g"] \arrow[d, "b"] & Z \arrow[d, equal] \\
a\xi\colon & X_1 \arrow[r, "f_1"] \arrow[ur, phantom, "\square"] & Y_1 \arrow[r, "g_1"] & Z
\end{tikzcd}$$

where the left square is a pushout square. Thus, $g = g_1 b$ and so g_1 is a retraction or b is a section as g is assumed to be irreducible.

Assume, for contradiction, that b is a section. Then there exists $r\colon Y_1 \to Y$ such that $rb = 1_Y$. This yields $(rf_1)a = rbf = f = \ker g$. As $\mathcal{A}$ is (right) semi-abelian, we have that a is also a kernel by [Rum01, Prop. 2]. Therefore, a is an epic kernel and hence an isomorphism by Proposition II.9.9, which contradicts that a is a proper epimorphism. Hence, b cannot be a section.

Thus, g_1 must be a retraction, whence $a\xi\colon X_1 \xrightarrow{f_1} Y_1 \xrightarrow{g_1} Z$ is split (Proposition II.10.3 again) and $a\xi = 0$ in $\operatorname{Ext}^1_{\mathcal{A}}(Z, -)(X_1)$. Since $a\colon X \to X_1$ was an arbitrary proper epimorphism, we see that ξ is a minimal element of $\operatorname{Ext}^1_{\mathcal{A}}(Z, -)(X)$ and that X is indecomposable by Proposition IV.2.21. ■

We also observe that [AR77a, Prop. 2.11] remains valid in a more general situation.

Proposition IV.2.23. *Let $\mathcal{A}$ be a skeletally small, preabelian category. Suppose $f\colon X \to Y$ is an irreducible morphism in $\mathcal{A}$ and let Z be an object of $\mathcal{A}$.*

(i) *If $\operatorname{Hom}_{\mathcal{A}}(Y, Z) = 0$, then* $0 \longrightarrow \operatorname{Ext}^1_{\mathcal{A}}(Z, X) \xrightarrow{\operatorname{Ext}^1_{\mathcal{A}}(Z, f)} \operatorname{Ext}^1_{\mathcal{A}}(Z, Y)$ *is exact.*

(ii) *If* $\operatorname{Hom}_{\mathcal{A}}(Z,X)=0$, *then* $0 \longrightarrow \operatorname{Ext}^1_{\mathcal{A}}(Y,Z) \xrightarrow{\operatorname{Ext}^1_{\mathcal{A}}(f,Z)} \operatorname{Ext}^1_{\mathcal{A}}(X,Z)$ *is exact.*

Proof. This is an arrow-theoretic translation of the proof from [AR77a]. ■

The next result is an analogue of [AR77a, Prop. 2.8] for which we need the theory of subobjects in an abelian category. Recall that two monomorphisms $i_1\colon X_1 \to X$ and $i_2\colon X_2 \to X$ in an abelian category are said to be *equivalent* if there is an isomorphism $f\colon X_1 \xrightarrow{\cong} X_2$ such that $i_1 = i_2 \circ f$. Then a *subobject* of an object X in an abelian category is an equivalence class of monomorphisms into X. Furthermore, if $i\colon V \to X$ and $j\colon W \to X$ are representatives of subobjects of X, then we write $V \subseteq W$ if there exists a morphism $g\colon V \to W$ such that $i = j \circ g$. See [Kra15] or [Fre03] for more details.

Proposition IV.2.24. *Let* $\mathcal{A}$ *be a skeletally small, quasi-abelian category and suppose* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *is a non-split short exact sequence in* $\mathcal{A}$.

(i) *The morphism* f *is irreducible if and only if, for any subobject* $\mathscr{F}$ *of* $\operatorname{Hom}_{\mathcal{A}}(-,Z)$, *we have either* $\mathscr{F}$ *contains or is contained in* $\operatorname{Im}(\operatorname{Hom}_{\mathcal{A}}(-,g))$, *the image of the natural transformation* $\operatorname{Hom}_{\mathcal{A}}(-,g)\colon \operatorname{Hom}_{\mathcal{A}}(-,Y) \to \operatorname{Hom}_{\mathcal{A}}(-,Z)$.

(ii) *The morphism* g *is irreducible if and only if, for any subobject* $\mathscr{F}$ *of* $\operatorname{Hom}_{\mathcal{A}}(X,-)$, *we have either* $\mathscr{F}$ *contains or is contained in* $\operatorname{Im}(\operatorname{Hom}_{\mathcal{A}}(f,-))$, *the image of the natural transformation* $\operatorname{Hom}_{\mathcal{A}}(f,-)\colon \operatorname{Hom}_{\mathcal{A}}(Y,-) \to \operatorname{Hom}_{\mathcal{A}}(X,-)$.

Proof. Note that the category of functors $\mathcal{A} \to \mathsf{Ab}$ is abelian as $\mathcal{A}$ is skeletally small (see [Pre09, Thm. 10.1.3]). The proof is identical to that for [AR77a, Prop. 2.8], noting that we may use [Pre09, Prop. 10.1.13], and Propositions IV.2.3 and IV.2.4. ■

Proposition IV.2.25 and its dual, Proposition IV.2.26, below give analogues of one direction of parts (a) and (b) of [AR77a, Cor. 2.9] for quasi-abelian categories. There is an obvious method to prove (ii) in light of Proposition IV.2.3, but the details are omitted in [AR77a] so we include them here for completeness.

Proposition IV.2.25. *Let $\mathcal{A}$ be a quasi-abelian category and suppose $\xi\colon X \overset{f}{\to} Y \overset{g}{\to} Z$ is a non-split short exact sequence in $\mathcal{A}$. Suppose f is irreducible. Then the following statements hold.*

(i) *For any proper subobject $\iota\colon Y' \hookrightarrow Y$ such that $\operatorname{Im} f \subseteq Y'$ given by a monomorphism $s\colon \operatorname{Im} f \hookrightarrow Y'$, we have that s is a section.*

(ii) *For any short exact sequence $\xi'\colon A \overset{a}{\to} B \overset{b}{\to} Z$, either there exists $j\colon A \to X$ such that $j\xi' = \xi$ or there exists $i\colon X \to A$ such that $\xi i = \xi'$.*

Proof. Suppose f is irreducible. To show (i) holds, we assume $\iota\colon Y' \to Y$ is a proper monomorphism and $s\colon \operatorname{Im} f \to Y'$ is a monomorphism such that $\operatorname{im} f = \iota s$. Since $\mathcal{A}$ is preabelian and ξ is short exact, we have that $f = \ker g = \ker(\operatorname{coker} f) = \operatorname{im} f$ by Lemma II.9.8. Therefore, $f = \iota s$ and hence either ι is a retraction or s is a section. If ι is a retraction then it would be a monic retraction, and hence an isomorphism by Proposition II.9.9. But this contradicts our assumption on ι, so we must have that s is a section.

For (ii), if $A \overset{a}{\to} B \overset{b}{\to} Z$ is a short exact sequence, then by Proposition IV.2.3 either there exists $v_1\colon B \to Y$ with $gv_1 = b$ or there exists $v_2\colon Y \to B$ with $g = bv_2$. This will yield one of the two morphisms of short exact sequences indicated in the following diagram.

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, dotted, bend right, "\exists! i_1"'] \arrow[dr, phantom, "\mathrm{I}"] & B \arrow[r, "b"] \arrow[d, bend right, "v_1"'] & Z \arrow[d, equal, "1_Z"] \\
X \arrow[r, "f"] \arrow[u, dotted, bend right, "\exists! i_2"'] & Y \arrow[r, "g"] \arrow[u, bend right, "v_2"'] & Z
\end{tikzcd}$$

Therefore, we need only show that the left square I is a pushout in either case. However, this follows immediately from the dual of [Rum01, Prop. 5] since a, f are kernels and 1_Z is an isomorphism. ■

Proposition IV.2.26. *Let $\mathcal{A}$ be a quasi-abelian category and suppose $\xi\colon X \xrightarrow{f} Y \xrightarrow{g} Z$ is a non-split short exact sequence in $\mathcal{A}$. Suppose g is irreducible. Then the following statements hold.*

(i) *For any non-zero subobject $X' \overset{\iota}{\hookrightarrow} X$, the induced morphism $r\colon Y/X' = \operatorname{Coker}(f\iota) \to Z$ is a retraction.*

(ii) *For any short exact sequence $\xi'\colon X \xrightarrow{b} B \xrightarrow{c} C$, either there exists $j\colon Z \to C$ such that $\xi' j = \xi$ or there exists $i\colon C \to Z$ such that $i\xi = \xi'$.*

IV.3 Auslander-Reiten theory in Krull-Schmidt categories

Let $\mathcal{A}$ denote an S-category, for some commutative ring S, and let $\mathcal{I}$ denote an ideal of $\mathcal{A}$. For a morphism $f\colon X \to Y$ in $\mathcal{A}$, we will denote by $\overline{f}$ the morphism $f + \mathcal{I}(X, Y)$ in the additive quotient S-category $\mathcal{A}/\mathcal{I}$. For most of this section we will study Krull-Schmidt categories that are not necessarily Hom-finite. There are very interesting examples of Hom-infinite generalised cluster categories (see [Ami09], [Pla11a], [KY11]) coming from quivers with potential. In these examples, a certain Krull-Schmidt category has been used in [Pla11b] to show the existence of cluster characters for Hom-infinite cluster categories. We provide an example of this kind at the end of this section (see Example IV.3.20).

Before we begin our study of Auslander-Reiten theory in Krull-Schmidt categories, we present a series of lemmas inspired by [AR77b, Lem. 1.1]. The proofs are omitted since they are easy generalisations of those in [AR77b].

Lemma IV.3.1. *Suppose $X, Y \in \mathcal{A}$ and $f \in \mathcal{I}(X,Y)$. If $1_X \notin \mathcal{I}(X,X)$ or $1_Y \notin \mathcal{I}(Y,Y)$, then $f\colon X \to Y$ is not an isomorphism.*

Lemma IV.3.2. *Suppose $X = \bigoplus_{i=1}^n X_i$ in $\mathcal{A}$, with $\operatorname{End}_{\mathcal{A}} X_i$ local and $1_{X_i} \notin \mathcal{I}(X_i, X_i)$ for each $i = 1, \dots, n$. For an endomorphism $f\colon X \to X$, if $f \in \mathcal{I}(X,X)$ then $f \in \operatorname{rad}_{\mathcal{A}}(X,X)$.*

Lemma IV.3.3. *Suppose $X = \bigoplus_{i=1}^n X_i$ and $Y = \bigoplus_{j=1}^m Y_j$ in $\mathcal{A}$, with $\operatorname{End}_{\mathcal{A}} X_i$ and $\operatorname{End}_{\mathcal{A}} Y_j$ local for all i, j. Let $f\colon X \to Y$ be a morphism in $\mathcal{A}$.*

(i) *If $1_{X_i} \notin \mathcal{I}(X_i, X_i)$ $\forall 1 \leqslant i \leqslant n$, then f is a section in $\mathcal{A}$ $\iff$ $\overline{f}$ is a section in $\mathcal{A}/\mathcal{I}$.*

(ii) *If $1_{Y_j} \notin \mathcal{I}(Y_j, Y_j)$ $\forall 1 \leqslant j \leqslant m$, then f is a retraction in $\mathcal{A}$ $\iff$ $\overline{f}$ is a retraction in $\mathcal{A}/\mathcal{I}$.*

(iii) *If $1_{X_i} \notin \mathcal{I}(X_i, X_i)$ and $1_{Y_j} \notin \mathcal{I}(Y_j, Y_j)$ for all i, j, then f is an isomorphism in $\mathcal{A}$ $\iff$ $\overline{f}$ is an isomorphism in $\mathcal{A}/\mathcal{I}$.*

The forward direction of the next lemma can be found in [Liu10] just above [Liu10, Def. 1.6], but it is a short argument so we include it here for completeness.

Lemma IV.3.4. *Suppose $X = \bigoplus_{i=1}^n X_i$ in $\mathcal{A}$, with $\operatorname{End}_{\mathcal{A}} X_i$ local and $1_{X_i} \notin \mathcal{I}(X_i, X_i)$ for each $i = 1, \dots, n$. Then $\operatorname{End}_{\mathcal{A}} X$ is local if and only if $\operatorname{End}_{\mathcal{A}/\mathcal{I}} X$ is local.*

Proof. ($\Rightarrow$) If $\operatorname{End}_{\mathcal{A}} X$ is local, then $\mathcal{I}(X,X)$ is contained in the Jacobson radical $J(\operatorname{End}_{\mathcal{A}} X)$, which is the unique maximal ideal of $\operatorname{End}_{\mathcal{A}} X$, since $1_X \notin \mathcal{I}(X,X)$. Then $\operatorname{End}_{\mathcal{A}/\mathcal{I}} X$ has unique maximal ideal $J(\operatorname{End}_{\mathcal{A}} X)/\mathcal{I}(X,X)$.

($\Leftarrow$) Conversely, assume $\operatorname{End}_{\mathcal{A}/\mathcal{I}} X$ is local and let $u\colon X \to X$ be a non-unit in $\operatorname{End}_{\mathcal{A}} X$. We will show that $1_X - u$ is a unit in $\operatorname{End}_{\mathcal{A}} X$ (and thus conclude that $\operatorname{End}_{\mathcal{A}} X$ is local by Proposition II.1.44). If $\overline{u}$ is a unit in $\operatorname{End}_{\mathcal{A}/\mathcal{I}} X$ then $\overline{1_X} = \overline{uv}$

for some $v\colon X \to X$. Then $1_X - uv \in \mathcal{I}(X,X)$, so $1_X - uv$ is radical by Lemma IV.3.2. Then $uv = 1_X - (1_X - uv)$ is a unit, so that u has a right inverse. A similar argument also shows u has a left inverse and so u is a unit, contrary to our assumption on u. Thus, $\overline{u}$ is not invertible and, as $\operatorname{End}_{\mathcal{A}/\mathcal{I}} X$ is local, $\overline{u}$ must be radical. Therefore, $\overline{1_X} - \overline{u}$ is a unit in $\operatorname{End}_{\mathcal{A}/\mathcal{I}} X$. So for some $w\colon X \to X$ we have $\overline{w}(\overline{1_X} - \overline{u}) = \overline{1_X}$. This shows that $1_X - w(1_X - u) \in \mathcal{I}(X,X)$ must be radical using Lemma IV.3.2 as before, so $w(1_X - u) = 1_X - (1_X - w(1_X - u))$ is invertible and $1_X - u$ has a left inverse. Again, a similar argument shows $1_X - u$ has a right inverse, and hence a two-sided inverse. ■

Remark IV.3.5. As noted in [Liu10, p. 431], this implies that if $\mathcal{A}$ is a Krull-Schmidt S-category then $\mathcal{A}/\mathcal{I}$ is also a Krull-Schmidt S-category.

Recall that a weak (co)kernel is just like a (co)kernel but without the uniqueness condition in the universal property (see Definition III.3.3). It is easy to show that a morphism is a pseudo-(co)kernel, in the sense of [Liu10], if and only if it is a weak (co)kernel. We call a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ *short weak exact* if f is a weak kernel of g and g is a weak cokernel of f.

Definition IV.3.6. We call a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$ an *Auslander-Reiten sequence (in an additive category)* if the following conditions are satisfied.

(i) The sequence is short weak exact.

(ii) The morphism f is minimal left almost split.

(iii) The morphism g is minimal right almost split.

In an almost identical way, Liu defines an Auslander-Reiten sequence for a Hom-finite, Krull-Schmidt category in [Liu10]. However, we do not impose the condition that the middle term be non-zero, because Auslander-Reiten sequences of the form $X \to 0 \to Z$ do appear, for example, in the bounded derived category

$\mathsf{D}^b(k\mathbb{A}_1-\mathsf{mod})$ of the path algebra $k\mathbb{A}_1$, where k is an algebraically closed field, and $\mathbb{A}_1$ is the quiver with one vertex and no arrows. As we will see now, the results of [Liu10, §1] can be generalised to the not necessarily Hom-finite setting. First, we note that [Liu10, Lem. 1.1] is still valid for an arbitrary additive category.

Next, we state a more general version of [Liu10, Prop. 1.5]; the proof from [Liu10] carries over to our setting.

Proposition IV.3.7. *Suppose $\mathcal{A}$ is a preabelian category, and let $\xi\colon X \xrightarrow{f} Y \xrightarrow{g} Z$ be an Auslander-Reiten sequence in $\mathcal{A}$ with $Y \neq 0$. Then ξ is short exact.*

Recall that an additive category $\mathcal{A}$ is *Krull-Schmidt* if, for any object X of $\mathcal{A}$, there exists a finite direct sum decomposition $X = X_1 \oplus \cdots \oplus X_n$ where $\mathrm{End}_{\mathcal{A}}(X_i)$ is a local ring for $i = 1, \ldots, n$ (see Definition II.1.45).

Throughout the remainder of this section, we further assume that $\mathcal{A}$ is a Krull-Schmidt category unless otherwise stated. In particular, $\mathcal{A}$ has split idempotents (see Definition II.9.4), and an object $X \in \mathcal{A}$ is indecomposable if and only if $\mathrm{End}_{\mathcal{A}} X$ is local. We still assume $\mathcal{I}$ is an ideal of $\mathcal{A}$.

The following lemma is a generalisation of [Liu10, Lem. 1.2] that will be needed to prove a uniqueness result about Auslander-Reiten sequences in a Krull-Schmidt category (see Theorem IV.3.9). Part of the proof in [Liu10] uses heavily that the category is Hom-finite, so the corresponding part of the proof below is quite different in nature.

Lemma IV.3.8. *Suppose*

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ Y & \xrightarrow{g} & Z \end{array}$$

is a commutative diagram in $\mathcal{A}$, with f, g both non-zero.

(i) *If f, g are minimal right almost split, then $u \in \mathrm{Aut}_{\mathcal{A}} Y \iff v \in \mathrm{Aut}_{\mathcal{A}} Z$.*

(ii) *If f, g are minimal left almost split, then* $u \in \operatorname{Aut}_{\mathcal{A}} Y \iff v \in \operatorname{Aut}_{\mathcal{A}} Z$.

Proof. We prove only (i) as the proof for (ii) is dual. Assume f, g are non-zero, minimal right almost split morphisms with $vf = gu$. Note that the argument in [Liu10] that u is an automorphism of Y whenever v is an automorphism of Z works here as well, so we only show the converse. We observe for later use that Y, Z are both non-zero since there exists a non-zero morphism between them.

Therefore, suppose $u \in \operatorname{Aut}_{\mathcal{A}} Y$ with inverse u^{-1}. Since f is right almost split, we have that $\operatorname{End}_{\mathcal{A}} Z$ is local, so Z is indecomposable. Assume, for contradiction, that v is not a retraction. Then v factors through the right almost split morphism g as, say, $v = ga$ for some $a \colon Z \to Y$. In particular, we see that $g = guu^{-1} = vfu^{-1} = gafu^{-1}$ and hence afu^{-1} is an automorphism of Y as g is right minimal. This means that a is a retraction. Then, by Lemma II.9.7, we have that a is an isomorphism because Z is indecomposable and $Y \neq 0$. However, this yields that $f = a^{-1}(afu^{-1})u$ is an isomorphism, and hence a retraction, which contradicts that f is right almost split. Hence, v must be a retraction, and thus also an isomorphism by Lemma II.9.7. ∎

Now we generalise [Liu10, Thm. 1.4] to a not necessarily Hom-finite (but still Krull-Schmidt) setting.

Theorem IV.3.9. *Let $\mathcal{A}$ be a Krull-Schmidt category, and suppose $X \xrightarrow{f} Y \xrightarrow{g} Z$ is an Auslander-Reiten sequence in $\mathcal{A}$ with $Y \neq 0$.*

(i) *Up to isomorphism, $X \xrightarrow{f} Y \xrightarrow{g} Z$ is the unique Auslander-Reiten sequence starting at X and the unique one ending at Z.*

(ii) *Any irreducible morphism $f_1 \colon X \to Y_1$ or $g_1 \colon Y_1 \to Z$ fits into an Auslander-Reiten sequence* $X \xrightarrow{\left(\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right)} Y_1 \oplus Y_2 \xrightarrow{(g_1\ g_2)} Z$.

Proof. Follow the proof in [Liu10], replacing the use of [Liu10, Lem. 1.2] with Lemma IV.3.8. ∎

For a Hom-finite, Krull-Schmidt category, Liu identifies a nice class of ideals—admissible ideals. It is observed in [Liu10] that, for such an ideal $\mathcal{I}$ of a Hom-finite, Krull-Schmidt category $\mathcal{A}$, irreducible morphisms (between indecomposables) and minimal left/right almost split morphisms remain, respectively, so under the quotient functor $\mathcal{A} \to \mathcal{A}/\mathcal{I}$. We adopt the same definition but without the Hom-finite restriction.

Definition IV.3.10. [Liu10, Def. 1.6] Suppose $\mathcal{A}$ is a Krull-Schmidt S-category. An ideal $\mathcal{I}$ of $\mathcal{A}$ is called *admissible* if it satisfies the following.

(i) Whenever $X, Y \in \mathcal{A}$ are indecomposable such that $1_X \notin \mathcal{I}(X,X)$ and $1_Y \notin \mathcal{I}(Y,Y)$, then $\mathcal{I}(X,Y) \subseteq \mathrm{rad}^2_{\mathcal{A}}(X,Y)$.

(ii) If $f\colon X \to Y$ is minimal left almost split, where $1_X \notin \mathcal{I}(X,X)$, and $g \in \mathcal{I}(X,M)$, then we can express $g = hf$ for some $h \in \mathcal{I}(Y,M)$.

(iii) If $f\colon X \to Y$ is minimal right almost split, where $1_Y \notin \mathcal{I}(Y,Y)$, and $g \in \mathcal{I}(M,Y)$, then we can express $g = fh$ for some $h \in \mathcal{I}(M,X)$.

Example IV.3.11. Suppose $\mathcal{B} \subseteq \mathcal{A}$ is a full subcategory closed under direct sums and direct summands. Then the ideal $[\mathcal{B}]$ of morphisms factoring through objects of $\mathcal{B}$ is admissible. See [Liu10, Prop. 1.9].

The next result follows quickly from the definition of an admissible ideal.

Lemma IV.3.12. *Suppose $\mathcal{I}$ is an admissible ideal of $\mathcal{A}$. Suppose $X = \bigoplus_{i=1}^n X_i$ and $Y = \bigoplus_{j=1}^m Y_j$ are decompositions into indecomposables in $\mathcal{A}$ with $1_{X_i} \notin \mathcal{I}(X_i,X_i)$, $1_{Y_j} \notin \mathcal{I}(Y_j,Y_j)$ for all i,j. Then $\mathcal{I}(X,Y) \subseteq \mathrm{rad}^2_{\mathcal{A}}(X,Y)$.*

Proof. Let $f \in \mathcal{I}(X,Y)$ be arbitrary and write

$$f = \begin{pmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{m1} & \cdots & f_{mn} \end{pmatrix}$$

where $f_{ji}\colon X_i \to Y_j$. Then for each i,j we have $f_{ji} = \pi_j f \iota_i \in \mathcal{I}(X_i, Y_j)$, where $\pi_j\colon Y \to Y_j$ is the natural projection and $\iota_i\colon X_i \to X$ is the natural inclusion. Since $\mathcal{I}$ is admissible and $1_{X_i} \notin \mathcal{I}(X_i, X_i), 1_{Y_j} \notin \mathcal{I}(Y_j, Y_j)$, we have that $f_{ji} \in \mathcal{I}(X_i, Y_j) \subseteq \operatorname{rad}^2_{\mathcal{A}}(X_i, Y_j)$ for each i,j. Therefore, f is a sum of morphisms in $\operatorname{rad}^2_{\mathcal{A}}(X,Y)$ and hence $f \in \operatorname{rad}^2_{\mathcal{A}}(X,Y)$ as desired. ■

The next lemma generalises [Liu10, Lem. 1.7 (1)]. The proof in [Liu10] makes use of a specific characterisation of irreducible morphisms between indecomposables (see [Bau82, Prop. 2.4]), which we cannot use since we make no indecomposability assumptions on the domain and codomain of the morphism. See also [AR77b, Prop. 1.2].

Proposition IV.3.13. *Suppose $\mathcal{I}$ is an admissible ideal of $\mathcal{A}$. Suppose $X = \bigoplus_{i=1}^n X_i$ and $Y = \bigoplus_{j=1}^m Y_j$ are decompositions into indecomposables in $\mathcal{A}$ with $1_{X_i} \notin \mathcal{I}(X_i, X_i), 1_{Y_j} \notin \mathcal{I}(Y_j, Y_j)$ for all i,j. Then $f\colon X \to Y$ is irreducible in $\mathcal{A}$ if and only if $\overline{f} = f + \mathcal{I}(X,Y)$ is irreducible in $\mathcal{A}/\mathcal{I}$.*

Proof. ($\Rightarrow$) Assume $f\colon X \to Y$ is an irreducible morphism in $\mathcal{A}$. By Lemma IV.3.3, $\overline{f}$ is neither a section nor a retraction in $\mathcal{A}/\mathcal{I}$. Now suppose $\overline{f} = \overline{hg}$ in $\mathcal{A}/\mathcal{I}$ for some morphisms $g\colon X \to Z$, $h\colon Z \to Y$ of $\mathcal{A}$. Then $f - hg \in \mathcal{I}(X,Y) \subseteq \operatorname{rad}^2_{\mathcal{A}}(X,Y)$ by Lemma IV.3.12. Therefore, there is an object $W \in \mathcal{A}$ and morphisms $a \in \operatorname{rad}_{\mathcal{A}}(X,W), b \in \operatorname{rad}_{\mathcal{A}}(W,Y)$ such that $f - hg = ba$. This yields $f = hg + ba = (\,h\;\; b\,)\left(\begin{smallmatrix} g \\ a \end{smallmatrix}\right)$, so that either $(\,h\;\; b\,)$ is a retraction or $\left(\begin{smallmatrix} g \\ a \end{smallmatrix}\right)$ is a section because f is irreducible. First, assume $(\,h\;\; b\,)$ is a retraction. Then there is a morphism $\left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right)\colon Y \to Z \oplus W$ such that $1_Y = (\,h\;\; b\,)\left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right) = hs + bt$. Now

$b \in \mathrm{rad}_{\mathcal{A}}(W, Y)$, so $bt \in \mathrm{rad}_{\mathcal{A}}(Y, Y)$ as $\mathrm{rad}_{\mathcal{A}}$ is an ideal of $\mathcal{A}$. Then $hs = 1_Y - bt$ is invertible, so h is a retraction and hence $\overline{h}$ is also a retraction. In the other case, we find that $\overline{g}$ is a section in a similar fashion. Thus, $\overline{f}$ is an irreducible morphism.

($\Leftarrow$) Conversely, suppose $\overline{f}\colon X \to Y$ is irreducible in $\mathcal{A}/\mathcal{I}$. By Lemma IV.3.3, f cannot be a section or a retraction. Assume $f = hg$ for some $g\colon X \to Z$, $h\colon Z \to Y$ of $\mathcal{A}$. Then in $\mathcal{A}/\mathcal{I}$ we have $\overline{f} = \overline{hg}$ and so either $\overline{h}$ is a retraction or $\overline{g}$ is a section, since $\overline{f}$ is irreducible. Therefore, h is a retraction or g is a section, respectively, by Lemma IV.3.3 again. Hence, f is irreducible. ■

In a not necessarily Hom-finite, Krull-Schmidt category, the results [Liu10, Lem. 1.7 (2), (3)], [Liu10, Prop. 1.8] and [Liu10, Lem. 1.9 (2)] all hold using the same proofs that Liu provides. This concludes our work on generalisations of results from [Liu10, §1]. We now recall some last definitions from [Liu10] and prove some new results.

Definition IV.3.14. [Liu10, Def. 2.2] An object $X \in \mathcal{A}$ is called *pseudo-projective* (respectively, *pseudo-injective*) if there exists a minimal right almost split monomorphism $W \to X$ (respectively, minimal left almost split epimorphism $X \to Y$).

Definition IV.3.15. [Liu10, Def. 2.6] Suppose $\mathcal{A}$ is a Krull-Schmidt S-category. We call $\mathcal{A}$ a *left Auslander-Reiten* category if, for every indecomposable $Z \in \mathcal{A}$, either Z is pseudo-projective or it is the last term of an Auslander-Reiten sequence in $\mathcal{A}$. Dually, $\mathcal{A}$ is a *right Auslander-Reiten* category if, for every indecomposable $X \in \mathcal{A}$, either X is pseudo-injective or it is the first term of an Auslander-Reiten sequence. If $\mathcal{A}$ is both a left and right Auslander-Reiten category, then we simply call $\mathcal{A}$ an *Auslander-Reiten* category.

Remark IV.3.16. Let $\mathcal{C}$ be a Krull-Schmidt triangulated category with suspension functor Σ. Then $\mathcal{C}$ is said to *have right Auslander-Reiten triangles* if for every indecomposable Z there is an Auslander-Reiten triangle ending at Z (see Definition

II.7.25). That is, for each indecomposable Z there is a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ with f minimal left almost split and g minimal right almost split. Therefore, a Krull-Schmidt, Hom-finite, triangulated S-category that has right Auslander-Reiten triangles is immediately seen to be a left Auslander-Reiten category in light of a result of Liu: [Liu10, Lem. 6.1] shows that $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is an Auslander-Reiten triangle if and only if $X \xrightarrow{f} Y \xrightarrow{g} Z$ is an Auslander-Reiten sequence as in Definition IV.3.6. The Hom-finite assumption may be removed by noting that one can use Theorem IV.3.9 in the proof of [Liu10, Lem. 6.1]. (There is, unfortunately, a clash in the usage of 'left' and 'right' between [Liu10] and [RVdB02].)

The following two propositions generalise [AR77b, Prop. 1.2], and the last theorem of this section is an analogue of Theorem II.4.13 (see also [AR77a, Thm. 2.14]). For the most part, the proofs are straightforward generalisations of those for the abelian case, using the more general results from this chapter and [Liu10] as appropriate. Thus, we only outline the proofs indicating the required generalised results where it is clear what needs to be done, and provide more details otherwise.

Proposition IV.3.17. *Suppose $\mathcal{A}$ is a left Auslander-Reiten category. Let $f\colon X \to Y$ be a morphism in $\mathcal{A}$ and let $\mathcal{I}$ be an admissible ideal of $\mathcal{A}$. Suppose $X = \bigoplus_{i=1}^{n} X_i$ and $Y = \bigoplus_{j=1}^{m} Y_j$ are decompositions into indecomposables in $\mathcal{A}$ with $1_{X_i} \notin \mathcal{I}(X_i, X_i), 1_{Y_j} \notin \mathcal{I}(Y_j, Y_j)$ for all i, j. Then $\overline{f} = f + \mathcal{I}(X, Y)\colon X \to Y$ is irreducible and right almost split in $\mathcal{A}/\mathcal{I}$, if and only if there exists $g\colon X' \to Y$ in $\mathcal{A}$ with $1_{X'} \in \mathcal{I}(X', X')$ such that $(\, f \;\; g\,)\colon X \oplus X' \to Y$ is minimal right almost split in $\mathcal{A}$.*

Proof. ($\Rightarrow$) Use: Proposition IV.3.13 instead of [AR77b, Prop. 1.2 (a)] to show f is irreducible; Lemma II.3.12 and Lemma IV.3.4 to show $\mathrm{End}_{\mathcal{A}} Y$ is local; and [AR77a, Thm. 2.4] and that $\mathcal{A}$ is a left Auslander-Reiten category to obtain a minimal right almost split morphism $(\, f \;\; g\,)\colon X \oplus X' \to Y$.

By [Liu10, Lem. 1.7], the morphism $(\,\overline{f} \;\; \overline{g}\,)\colon X \oplus X' \to Y$ is minimal right almost split, and hence a non-retraction, in $\mathcal{A}/\mathcal{I}$. Since $\overline{f}\colon X \to Y$ is right almost split, there exists $(\,\overline{a}\ \ \overline{b}\,)\colon X \oplus X' \to X$ such that $(\,\overline{fa}\ \ \overline{fb}\,) = \overline{f} \circ (\,\overline{a}\ \ \overline{b}\,) = (\,\overline{f}\ \ \overline{g}\,)$. We now deviate from the proof given in [AR77b]. This implies

$$(\,\overline{f}\ \ \overline{g}\,)\begin{pmatrix}\overline{a} & \overline{b}\\ 0 & 0\end{pmatrix} = (\,\overline{fa}\ \ \overline{fb}\,) = (\,\overline{f}\ \ \overline{g}\,),$$

so $\left(\begin{smallmatrix}\overline{a} & \overline{b}\\ 0 & 0\end{smallmatrix}\right)$ is an automorphism of $X \oplus X'$ in $\mathcal{A}/\mathcal{I}$ as $(\,\overline{f}\ \ \overline{g}\,)$ is right minimal. Hence, there is $\left(\begin{smallmatrix}\overline{r} & \overline{s}\\ \overline{t} & \overline{u}\end{smallmatrix}\right) \in \operatorname{End}_{\mathcal{A}/\mathcal{I}}(X \oplus X')$ such that

$$\begin{pmatrix}\overline{r} & \overline{s}\\ \overline{t} & \overline{u}\end{pmatrix}\begin{pmatrix}\overline{a} & \overline{b}\\ 0 & 0\end{pmatrix} = \begin{pmatrix}\overline{1_X} & 0\\ 0 & \overline{1_{X'}}\end{pmatrix} = \begin{pmatrix}\overline{a} & \overline{b}\\ 0 & 0\end{pmatrix}\begin{pmatrix}\overline{r} & \overline{s}\\ \overline{t} & \overline{u}\end{pmatrix} = \begin{pmatrix}\overline{ar}+\overline{bt} & \overline{as}+\overline{bu}\\ 0 & 0\end{pmatrix}.$$

Therefore, $\overline{1_{X'}} = 0$ and hence $1_{X'} \in \mathcal{I}(X', X')$.

($\Leftarrow$) Use [Liu10, Lem. 1.7] to get that $(\,\overline{f}\ \ \overline{g}\,)$ is minimal right almost split in $\mathcal{A}/\mathcal{I}$, and that $\overline{\iota_X}\colon X \hookrightarrow X \oplus X'$ is an isomorphism in the factor category as $1_{X'} \in \mathcal{I}(X', X')$. ■

Dually, the following is also true.

Proposition IV.3.18. *Suppose $\mathcal{A}$ is a right Auslander-Reiten category. Let $f\colon X \to Y$ be a morphism in $\mathcal{A}$ and let $\mathcal{I}$ be an admissible ideal of $\mathcal{A}$. Suppose $X = \bigoplus_{i=1}^{n} X_i$ and $Y = \bigoplus_{j=1}^{m} Y_j$ are decompositions into indecomposables in $\mathcal{A}$ with $1_{X_i} \notin \mathcal{I}(X_i, X_i), 1_{Y_j} \notin \mathcal{I}(Y_j, Y_j)$ for all i, j. Then $f + \mathcal{I}(X, Y)\colon X \to Y$ is irreducible and left almost split in $\mathcal{A}/\mathcal{I}$, if and only if there exists $g\colon X \to Y'$ in $\mathcal{A}$ with $1_{Y'} \in \mathcal{I}(Y', Y')$ such that $\left(\begin{smallmatrix}f\\ g\end{smallmatrix}\right)\colon X \to Y \oplus Y'$ is minimal left almost split in $\mathcal{A}$.*

Our main result of this section is the following characterisation of Auslander-Reiten sequences, which is a more general version of [ASS06, Thm. IV.1.13]. Furthermore, statement (f) in [ASS06] has stronger assumptions than the corresponding statement (vi) below: more precisely, in (vi) we do not assume any indecomposability assumptions on the first and last term of the short exact sequence.

Theorem IV.3.19. *Let $\mathcal{A}$ be a skeletally small, preabelian category. Let $\xi\colon X \xrightarrow{f} Y \xrightarrow{g} Z$ be a stable exact sequence in $\mathcal{A}$, i.e. $\xi \in \operatorname{Ext}^1_{\mathcal{A}}(Z, X)$. Then statements* (i)–(iii) *are equivalent.*

(i) *ξ is an Auslander-Reiten sequence.*

(ii) *$\operatorname{End}_{\mathcal{A}} X$ is local and g is right almost split.*

(iii) *$\operatorname{End}_{\mathcal{A}} Z$ is local and f is left almost split.*

Suppose further that $\mathcal{A}$ is quasi-abelian and Krull-Schmidt. Then (i)–(vi) *are equivalent.*

(iv) *f is minimal left almost split.*

(v) *g is minimal right almost split.*

(vi) *f* and *g are irreducible.*

Proof. From Definition IV.3.6 and Lemma II.3.12, (ii) and (iii) follow from (i). To show (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i), use Proposition IV.2.18 instead of [ASS06, Lem. IV.1.12]. And (iii) $\Rightarrow$ (ii) is dual to (ii) $\Rightarrow$ (iii), so this establishes the equivalence of (i)–(iii).

Now suppose further that $\mathcal{A}$ is quasi-abelian and Krull-Schmidt. Statements (iv) and (v) follow from (i) by definition, and (iv) $\Rightarrow$ (iii) is dual to (v) $\Rightarrow$ (ii).

First, we claim that if g is right almost split then Y is non-zero. Indeed, if $Y = 0$ then $1_Z \circ g = g = 0$, which implies $1_Z = 0$ as $g = \operatorname{coker} f$ is an epimorphism (since ξ is short exact). However, if g is right almost split, then $\operatorname{End}_{\mathcal{A}} Z$ is local by Lemma II.3.12 and hence 1_Z cannot be the zero morphism.

(v) $\Rightarrow$ (ii). Since g is right almost split, we may use our claim above to conclude that g is irreducible by Proposition IV.2.8 (ii) (as g is also right minimal). Then X is indecomposable by Proposition IV.2.22, which is equivalent to $\operatorname{End}_{\mathcal{A}} X$ being local as $\mathcal{A}$ is Krull-Schmidt.

For (i) implies (vi), use Proposition IV.2.8 (noting again that Y is non-zero if g is right almost split).

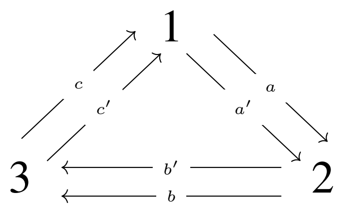
(vi) $\Rightarrow$ (ii). Suppose that f, g are irreducible. First we show that g is right almost split. Note that g is not a retraction as it is irreducible by assumption. Thus, let $h\colon M \to Z$ be a non-retraction. Since $\mathcal{A}$ is Krull-Schmidt we may write $M = \bigoplus_{i=1}^{n} M_i$, for some indecomposable objects M_i, and $h = \begin{pmatrix} h_1 & \cdots & h_n \end{pmatrix}$ where $h_i\colon M_i \to Z$. Since h is not a retraction, it follows that no h_i may be a retraction either. Fix $i \in \{1, \ldots, n\}$. As f is irreducible, the criterion from Proposition IV.2.3 tells us that either there exists $v_{i,1}\colon M_i \to Y$ such that $gv_{i,1} = h_i$ or there exists $v_{i,2}\colon Y \to M_i$ such that $g = h_i v_{i,2}$. Suppose we are in the latter case and that $g = h_i v_{i,2}$ for some $v_{i,2}\colon Y \to M_i$. Then, as g is irreducible and h_i is not a retraction, we have that $v_{i,2}$ is section. But M_i is indecomposable and $Y \neq 0$ (as, for example, g is irreducible), so $v_{i,2}$ is in fact an isomorphism by Lemma II.9.7. In this case, we then get $h_i = g \circ {v_{i,2}}^{-1}$. Therefore, for all $1 \leqslant i \leqslant n$ we have that $h_i = g \circ w_i$ for some $w_i\colon M_i \to Y$. Hence, $h = \begin{pmatrix} h_1 & \cdots & h_n \end{pmatrix} = g \circ \begin{pmatrix} w_1 & \cdots & w_n \end{pmatrix}$ and g is seen to be right almost split. Dually, we have that f is left almost split and hence $\operatorname{End}_{\mathcal{A}} X$ is local by Lemma II.3.12.

This shows (i)–(vi) are equivalent and finishes the proof. ■

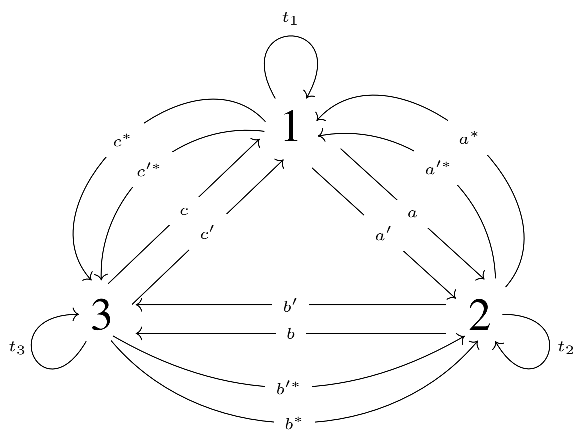
We conclude this section with an example of a Hom-infinite, Krull-Schmidt category. The author is grateful to P.-G. Plamondon for communicating the following example and answering several questions.

Example IV.3.20. Let k be a field. Consider the quiver with potential (Q, W) where

Q is the quiver



and $W = cba + c'b'a'$ is the potential. Following [KY11, §2.6], we recall the construction of the *complete Ginzburg dg algebra* $G := \widehat{\Gamma}(Q, W)$ associated to Q. From Q, consider the quiver $\widetilde{Q}$:



The quiver $\widetilde{Q}$ is given the following grading: arrows x, x' have degree 0 and arrows x^*, x'^* have degree -1 for $x \in \{a, b, c\}$, and the loop t_i has degree -2 for $1 \leqslant i \leqslant 3$. Then G has underlying graded algebra given by the completion of the graded path algebra $k\widetilde{Q}$ with respect to the ideal generated by the arrows of $\widetilde{Q}$ in the category of graded k-vector spaces. Furthermore, G is a dg algebra, equipped with a differential of degree $+1$.

Let $\mathsf{mod}-G$, $\mathsf{K}(G)$ and $\mathsf{D}(G)$ denote the category of right dg G-modules, the homotopy category of right dg G-modules and the corresponding derived category, respectively. The *perfect derived category* $\mathsf{per}\, G$ is the smallest full subcategory of $\mathsf{D}(G)$ that contains G, and is closed under shifts, extensions and direct summands. Let $J(Q, W)$ denote the *Jacobian algebra* associated to (Q, W). Then $J(Q, W)$ is the complete path algebra $\widehat{kQ}$ modulo the closure of the ideal generated by $\partial_x(W)$ and $\partial_{x'}(W)$ for $x \in \{a, b, c\}$, where

$$\partial_x(W) \coloneqq \sum_{W=yxz} zy,$$

where the sum is over all decompositions of W with y, z (possibly constant) paths. The sum $\partial_{x'}(W)$ is defined similarly. It is easy to check that $J(Q, W)$ is infinite-dimensional over k.

The category $\mathsf{per}\, G$ is Krull-Schmidt by [KY11, Lem. 2.17]. Furthermore, we have

$$\begin{aligned}
\operatorname{End}_{\mathsf{per}\, G} G &= \operatorname{End}_{\mathsf{D}(G)} G && \text{since } \mathsf{per}\, G \text{ is a full subcategory} \\
&\cong \operatorname{Hom}_{\mathsf{K}(G)}(G, G) && \text{since } G \text{ is cofibrant} \\
& && \text{(see [KY11, pp. 2126–2127])} \\
&= H^0(\mathcal{H}om_{\mathsf{mod}-G}(G, G)) \\
&\cong H^0(G) \\
&= J(Q, W) && \text{by [KY11, Lem. 2.8].}
\end{aligned}$$

It follows that $\mathsf{per}\, G$ is a Hom-infinite k-category and is also Krull-Schmidt. We remark that, by [Pla11b, Lem. 2.9], the corresponding cluster category is also a Hom-infinite k-category in this case.

IV.4 An example from cluster theory

Let k be a field. In this section, we will present an example coming from cluster theory that encapsulates some of the theory we have explored.

Recall that the Auslander-Reiten quiver $\Gamma_{\mathrm{AR}}(\mathcal{A})$ of a Krull-Schmidt k-category $\mathcal{A}$ has isoclasses of indecomposable objects as its vertices and irreducible morphisms (up to a scalar) as the arrows. See also Remark II.3.16. This gives a complete pictorial description of $\mathcal{A}$ in sufficiently nice cases, e.g. $\mathcal{A} = A\text{-}\mathsf{mod}$

for a representation-finite, finite-dimensional algebra A, and a nearly complete description in some other cases, e.g. $\mathcal{A} = A-\mathsf{mod}$ for a tame finite-dimensional algebra A.

Example IV.4.1. Let k be a field. Consider the cluster category $\mathcal{C} \coloneqq \mathcal{C}_{kQ}$ associated to the linearly oriented Dynkin-type quiver

$$Q: \quad 1 \to 2 \to 3.$$

Recall that $\mathcal{C}$ is a Krull-Schmidt, triangulated k-category (see Theorem II.8.5). Let $(-)[1]$ denote the suspension functor of $\mathcal{C}$. Its Auslander-Reiten quiver, with the mesh relations omitted, is

$$\begin{tikzcd}[column sep=small]
 & & P_1[1] \arrow[dr] & & P_1 = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[dr, "b"] & & P_3[1] \arrow[dr] & & \\
 & P_2[1] \arrow[ur] \arrow[dr] & & P_2 = \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[ur, "a"] \arrow[dr, "c"'] & & I_2 = \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[ur] \arrow[dr] & & P_2[1] \arrow[dr] & \\
P_3[1] \arrow[ur] & & P_3 = 3 \arrow[ur] & & S_2 = 2 \arrow[ur, "d"'] & & S_1 = 1 \arrow[ur] & & P_1[1]
\end{tikzcd}$$

where the lefthand copy of $P_i[1]$ is identified with the corresponding righthand copy (for $i = 1, 2, 3$); see Example II.8.9. We set $R \coloneqq P_1 \oplus P_2$, which is a basic, rigid object of $\mathcal{C}$. By $\mathsf{add}\, R[1]$ we denote the full subcategory of $\mathcal{C}$ consisting of objects that are isomorphic to direct summands of finite direct sums of copies of $R[1]$. The full subcategory $\mathcal{X}_R$ consists of objects X for which $\mathrm{Hom}_{\mathcal{C}}(R, X) = 0$. Then the pair $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})) = ((\mathsf{add}\, R[1], \mathcal{X}_R), (\mathcal{X}_R, \mathsf{add}\, R[1]))$ is a twin cotorsion pair on $\mathcal{C}$ with heart $\overline{\mathcal{H}} = \mathcal{C}/[\mathcal{X}_R]$ (see Lemma III.5.6 and Corollary III.5.10, or [Nak13, Exam. 2.10], for more details), where $[\mathcal{X}_R]$ is the ideal of morphisms factoring through objects of $\mathcal{X}_R$. Note that $[\mathcal{X}_R]$ is an admissible ideal by Example IV.3.11 as $\mathcal{X}_R$ is closed under direct summands (see Proposition III.5.4).

The subcategory $\mathcal{X}_R$ is described pictorially below, where "$\circ$" denotes that the corresponding object does not belong to the subcategory.

$$\mathcal{T} = \mathcal{X}_R = \mathcal{U}$$

$$\begin{tikzcd}[column sep=small]
 & & P_1[1] \arrow[dr] & & \circ \arrow[dr] & & P_3[1] \arrow[dr] & & \\
 & P_2[1] \arrow[ur] \arrow[dr] & & \circ \arrow[ur] \arrow[dr] & & \circ \arrow[ur] \arrow[dr] & & P_2[1] \arrow[dr] & \\
P_3[1] \arrow[ur] & & P_3 \arrow[ur] & & \circ \arrow[ur] & & \circ \arrow[ur] & & P_1[1]
\end{tikzcd}$$

The heart $\overline{\mathcal{H}} = \mathcal{C}/[\mathcal{X}_R]$ for this twin cotorsion pair is quasi-abelian by Theorem III.5.7, and is Krull-Schmidt (see Remark IV.3.5). By [Liu10, Prop. 2.9], $\mathcal{C}/[\mathcal{X}_R]$ has the following Auslander-Reiten quiver (ignoring the objects denoted by a "$\circ$" that lie in $\mathcal{X}_R$).

$$\overline{\mathcal{H}} = \mathcal{C}/[\mathcal{X}_R]$$

$$\begin{tikzcd}[column sep=small]
 & & \circ & & \overline{P_1} \arrow[dr, hook, two heads, "\overline{b}"] & & \circ & & \\
 & \circ & & \overline{P_2} \arrow[ur, hook, "\overline{a}"] \arrow[dr, hook, two heads, "\overline{c}"'] & & \overline{I_2} \arrow[dr, two heads, "\overline{e}"] & & \circ & \\
\circ & & \circ & & \overline{S_2} \arrow[ur, hook, "\overline{d}"'] & & \overline{S_1} & & \circ
\end{tikzcd}$$

Again we have omitted the mesh relations. Furthermore, we have denoted by $\overline{X}$ the image in $\mathcal{C}/[\mathcal{X}_R]$ of the object X of $\mathcal{C}$, monomorphisms by "$\hookrightarrow$" and epimorphisms by "$\twoheadrightarrow$". In this example, we notice that there are precisely two irreducible morphisms (up to a scalar) that are regular (monic and epic simultaneously)—namely, $\overline{b}$ and $\overline{c}$.

Consider the Auslander-Reiten triangle $P_2 \xrightarrow{\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right)} P_1 \oplus S_2 \xrightarrow{(b\ d)} I_2 \longrightarrow P_2[1]$ in $\mathcal{C}$, and note that the minimal left almost split morphism $\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right)$ is irreducible by Proposition IV.2.8. Therefore, by Proposition IV.3.13, $\left(\begin{smallmatrix} \overline{a} \\ \overline{c} \end{smallmatrix}\right) \colon \overline{P_2} \to \overline{P_1} \oplus \overline{S_2}$ is also irreducible. Similarly, $(\,\overline{b}\;\;\overline{d}\,)\colon \overline{P_1}\oplus\overline{S_2}\to\overline{I_2}$ is irreducible in $\mathcal{C}/[\mathcal{X}_R]$. We remark that one cannot use [Liu10, Lem. 1.7 (1)] since the morphisms are not between indecomposable objects.

One can check that $\left(\begin{smallmatrix}\overline{a}\\ \overline{c}\end{smallmatrix}\right)=\ker(\,\overline{b}\;\;\overline{d}\,)$ and $(\,\overline{b}\;\;\overline{d}\,)=\operatorname{coker}\left(\begin{smallmatrix}\overline{a}\\ \overline{c}\end{smallmatrix}\right)$ by, for example, using the construction of (co)kernels as in [BM12, Lem. 3.4]. So, we have that

$$\overline{P_2}\xrightarrow{\left(\begin{smallmatrix}\overline{a}\\ \overline{c}\end{smallmatrix}\right)}\overline{P_1}\oplus\overline{S_2}\xrightarrow{(\,\overline{b}\;\overline{d}\,)}\overline{I_2}$$

is a short exact sequence in the quasi-abelian, Krull-Schmidt category $\mathcal{C}/[\mathcal{X}_R]$. Hence, by Theorem IV.3.19, the sequence is an Auslander-Reiten sequence because it satisfies statement (vi) in the Theorem. Note that we could also have established this fact using [Liu10, Prop. 1.8] and Proposition IV.3.7.

Furthermore, this example also shows that the indecomposability conditions in Proposition IV.2.10 cannot be removed. The morphism $\left(\begin{smallmatrix}\overline{a}\\ \overline{c}\end{smallmatrix}\right)$ is an irreducible monomorphism, but has decomposable codomain, and the morphism $\overline{d}$ is an irreducible morphism with codomain the cokernel of $\left(\begin{smallmatrix}\overline{a}\\ \overline{c}\end{smallmatrix}\right)$ that is not epic. Indeed, $\overline{e}\overline{d}=0$ but $\overline{e}\neq 0$ so $\overline{d}$ cannot be an epimorphism.

Chapter V

Partial cluster-tilted algebras

V.1 Introduction

Let k be a field. The *cluster category* of a finite-dimensional hereditary k-algebra (see Definition II.8.4) was defined in [BMRRT], and shown to model the combinatorics of the corresponding cluster algebra in case the algebra is the path algebra of a Dynkin-type quiver. Inspired by the close connection between the representation theory of a hereditary k-algebra H and the representation theory of its *tilted algebras*, which are algebras of the form $(\operatorname{End}_{H-\mathsf{mod}} M)^{\mathrm{op}}$ for a tilting module M (see Definitions V.2.7 and V.2.8), Buan, Marsh and Reiten introduced the class of cluster-tilted algebras in [BMR07]. By definition, a *cluster-tilted algebra* is an algebra of the form $(\operatorname{End}_{\mathcal{C}_H} T)^{\mathrm{op}}$, where T is a cluster-tilting (or, equivalently, maximal rigid) object in some cluster category $\mathcal{C}_H$. Recall that a basic object T' in a triangulated category $\mathcal{C}$ is *maximal rigid* if T' is *rigid* (i.e. $\operatorname{Ext}^1_{\mathcal{C}}(T', T') = 0$) and has a maximal number of non-isomorphic indecomposable direct summands with respect to this property (see Definition III.5.1).

In classical tilting theory, every algebra of the form $(\operatorname{End}_{H-\mathsf{mod}} M')^{\mathrm{op}}$ for some *partial tilting* module M' (i.e. $\operatorname{Ext}^1_{H-\mathsf{mod}}(M', M') = 0$) arises as a tilted algebra (see [Hap88, Cor. III.6.5]). The focus of this chapter is on the analogue of the opposite endomorphism ring of a partial tilting module in cluster-tilting theory.

Definition V.1.1. A *partial cluster-tilted algebra* is an algebra of the form $(\mathrm{End}_{\mathcal{C}_H} T')^{\mathrm{op}}$, for some rigid object T' in a cluster category $\mathcal{C}_H$ of a finite-dimensional hereditary k-algebra H.

Unlike the classical setting, there exist partial cluster-tilted algebras that are not cluster-tilted (see [BMR08, p. 158] and Remark V.2.6). Therefore, it is of interest to understand this more general class of algebras. As shown in [KR07], each cluster-tilted algebra must have global dimension 0, 1 or ∞. In §V.2, we construct a partial cluster-tilted algebra of global dimension n for each non-negative integer n (see Theorem V.2.4).

Suppose T is a cluster-tilting object in a cluster category $\mathcal{C}_H$, and $\Lambda_T := (\mathrm{End}_{\mathcal{C}_H} T)^{\mathrm{op}}$ is the corresponding cluster-tilted algebra. It was shown in [BMR07] that the functor $\mathrm{Hom}_{\mathcal{C}_H}(T,-)\colon \mathcal{C}_H \to \Lambda_T\text{-}\mathsf{mod}$ induces an equivalence $\mathcal{C}_H/[\mathsf{add}\,\tau T] \xrightarrow{\simeq} \Lambda_T\text{-}\mathsf{mod}$ (see [BMR07, Thm. 2.2]). (Note that for T cluster-tilting, the subcategory $\mathsf{add}\,\tau T$ coincides with $\mathcal{X}_T = (\mathsf{add}\,T)^{\perp_0} = \mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}_H}(T,-))$; see Definition III.5.2.) Consequently, the Auslander-Reiten quiver of Λ_T can be obtained directly from the Auslander-Reiten quiver of $\mathcal{C}_H$, by deleting the vertices $[X]$ where $X \in \mathsf{add}\,\tau T$ and any arrows incident to such vertices; see [BMR07] and [KZ08], and also [Liu10].

One can then ask if the Auslander-Reiten quiver of a partial cluster-tilted algebra can also be easily obtained from the Auslander-Reiten quiver of the corresponding cluster category. However, if R is rigid and not cluster-tilting in a cluster category $\mathcal{C} := \mathcal{C}_H$, then $\mathcal{C}/[\mathcal{X}_R]$ is not necessarily equivalent to $\Lambda_R\text{-}\mathsf{mod}$, where $\Lambda_R := (\mathrm{End}_{\mathcal{C}} R)^{\mathrm{op}}$. In §V.3 we consider how irreducible morphisms in $\mathcal{C}$ behave under the functor $\mathrm{Hom}_{\mathcal{C}}(R,-)\colon \mathcal{C} \to \Lambda_R\text{-}\mathsf{mod}$. That is, given an irreducible morphism $f\colon X \to Y$ in $\mathcal{C}$, we investigate the irreducibility of $\mathrm{Hom}_{\mathcal{C}}(R,f)$ in $\Lambda_R\text{-}\mathsf{mod}$. In doing so, we make use of the subcategory $\mathcal{C}(R)$ consisting of objects Z which admit a triangle $R_0 \to R_1 \to Z \to R_0[1]$ with $R_i \in \mathsf{add}\,R$ (see Definition V.3.5). If X

lies in $\mathcal{C}(R)$, the irreducibility of $\mathrm{Hom}_{\mathcal{C}}(R, f)$ is determined by whether Y also lies in $\mathcal{C}(R)$. On the other hand, if X is not in $\mathcal{C}(R)$, then we know the irreducibility of $\mathrm{Hom}_{\mathcal{C}}(R, f)$ in some cases. See §V.3.3 for more details.

This chapter is organised as follows. In §V.2, for any $n \in \mathbb{N}$, we construct a partial cluster-tilted algebra of global dimension n. In §V.3 we give results that can be used to obtain information on the Auslander-Reiten quiver of a partial cluster-tilted algebra using the Auslander-Reiten quiver of the corresponding cluster category. We conclude this chapter with some examples, illustrating the theory we develop in earlier sections.

V.2 Partial cluster-tilted algebras of arbitrary global dimension

It was proven in [KR07, §2.1] that any cluster-tilted algebra Λ_T is *Gorenstein* of dimension at most 1, that is every finitely presented projective has injective dimension at most 1 and every finitely presented injective has projective dimension at most 1. A corollary of this is that Λ_T is then either a hereditary algebra or has infinite global dimension. In contrast, our main result in this section is that there are partial cluster-tilted algebras of arbitrarily large finite global dimension (see Theorem V.2.4). For this we need some preliminary results, which we give now.

Throughout this section, let $k = \overline{k}$ be an algebraically closed field. The following proposition was observed in [BMRRT] as part of the proof for their Proposition 4.2.

Proposition V.2.1. [BMRRT, p. 591] *Let H be a finite-dimensional hereditary k-algebra, and let $\mathcal{C}_H$ be the cluster category of H. For all objects $X \cong P, Y \cong P'$ in $\mathcal{C}$, where $P, P' \in H-\mathsf{proj}$, we have $\mathrm{Ext}^1_{\mathcal{C}_H}(X, Y) = 0$.*

Proof. From [BMRRT, Prop. 1.7], we have

$$\mathrm{Ext}^1_{\mathcal{C}_H}(X,Y) \cong \mathrm{Ext}^1_{\mathcal{C}_H}(P,P') \cong \mathrm{Ext}^1_{H-\mathsf{mod}}(P,P') \oplus \mathrm{Ext}^1_{H-\mathsf{mod}}(P',P) = 0$$

as P, P' are projective. ■

The following is well-known, but we provide a simple proof for convenience. Recall that for a quiver Q we denote by S_i (respectively, P_i) the simple representation (respectively, projective representation) at vertex i (see Definition II.2.21).

Proposition V.2.2. *Let A be the bound quiver algebra associated to the following quiver with the indicated relations:*

$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n \longrightarrow n+1.$$

That is, $A = k\overrightarrow{\mathbb{A}}_{n+1}/J(k\overrightarrow{\mathbb{A}}_{n+1})^2$, where $\overrightarrow{\mathbb{A}}_{n+1}$ is the Dynkin-type $\mathbb{A}$ quiver with $n+1$ vertices and is linearly oriented. Then A has global dimension precisely n.

Proof. Note that the global dimension of a finite-dimensional k-algebra is the supremum of the projective dimensions of all simple modules (see [Rin84, p. 67]). Let S_i denote the simple representation at vertex i. Therefore, we have

$$\begin{aligned}
\mathrm{gl.dim}\, A &= \sup_{M \in A-\mathsf{mod}} \{\mathrm{p.dim}\, M\} \\
&= \sup_{\substack{S \in A-\mathsf{mod} \\ S \text{ simple}}} \{\mathrm{p.dim}\, S\} && \text{using the observation above} \\
&= \sup\{\mathrm{p.dim}\, S_1, \mathrm{p.dim}\, S_2, \ldots, \mathrm{p.dim}\, S_{n+1}\} \\
&= \sup\{n, n-1, \ldots, 0\} \\
&= n.
\end{aligned}$$

■

One last ingredient that we need for our main result is the following.

Lemma V.2.3. [ASS06, §VII, Cor. 5.14] *Let M be an indecomposable module over a finite-dimensional, representation-finite, hereditary k-algebra H. Then* $\operatorname{End}_{H\text{-}\mathsf{mod}} M \cong k$ *and* $\operatorname{Ext}^1_{H\text{-}\mathsf{mod}}(M, M) = 0$.

Theorem V.2.4. *There exist partial cluster-tilted algebras of arbitrarily large finite global dimension. Moreover, for all $n \in \mathbb{N}$ with $n \geqslant 1$ there is a partial cluster-tilted algebra Λ_R coming from a cluster category of type $\mathbb{A}_{2n}$ such that* $\operatorname{gl.dim} \Lambda_R = n$.

Proof. We will treat the cases $n = 0, 1$ separately first. If $n = 0$ then take $Q = \mathbb{A}_1$ and let $R = P_1 \in \mathcal{C}_{kQ}$. Note that R is rigid by Proposition V.2.1. As R is an object induced from a projective kQ-module, we have that $\Lambda_R = (\operatorname{End}_{\mathcal{C}_{kQ}} R)^{\mathrm{op}} = (\operatorname{End}_{\mathcal{C}_{kQ}} P_1)^{\mathrm{op}} \cong (\operatorname{End}_{kQ\text{-}\mathsf{mod}} P_1)^{\mathrm{op}}$ by [BMRRT, Prop. 1.7 (d)]. Furthermore, since R is indecomposable and kQ is a representation-finite hereditary algebra (Theorems II.4.20 and II.4.24) we have that $\Lambda_R \cong k^{\mathrm{op}} = k$ (by Lemma V.2.3), which has global dimension $n = 0$.

If $n = 1$, we set $Q = \overrightarrow{\mathbb{A}}_2 \colon 1 \to 2$ and $R = R_1 \oplus R_2 \in \mathcal{C}_{kQ}$ with $R_1 = P_2$ and $R_2 = P_1$. Again, R is rigid by Proposition V.2.1, but this time we have Λ_R is isomorphic to the path algebra of

$$1' \longleftarrow 2'.$$

The isomorphism is given by mapping the idempotent $\widetilde{1_{R_i}}^{\mathrm{op}} := (\iota_{R_i} \circ \pi_{R_i})^{\mathrm{op}}$ to constant path at vertex i', where $\iota_{R_i} \colon R_i \hookrightarrow R$, respectively, $\pi_{R_i} \colon R \twoheadrightarrow R_i$, is the natural inclusion, respectively, projection; and mapping the opposite endomorphism $\widetilde{\varphi}^{\mathrm{op}}$ to the unique path of length 1, where $\widetilde{\varphi} := \iota_{P_1} \varphi \pi_{P_2}$ and $\varphi \colon P_2 \hookrightarrow P_1$ is the inclusion. Since Λ_R is the path algebra of a finite, acyclic quiver without relations, it is hereditary by Theorem II.4.20, so $\operatorname{gl.dim} \Lambda_R \leqslant 1$. Note that the simple Λ_R-

module $S_{2'}$, which has quiver representation

$$0 \longleftarrow k,$$

has projective dimension 1 and so in fact $\mathrm{gl.dim}\,\Lambda_R = 1$.

Finally, we show how to produce a partial cluster-tilted algebra of global dimension n for each $n \geqslant 2$. For this, we let $Q = \overrightarrow{\mathbb{A}}_{2n}$ be the quiver

$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow 2n-1 \longrightarrow 2n,$$

and consider its path algebra $kQ \coloneqq k\overrightarrow{\mathbb{A}}_{2n}$ and its bounded derived category $\mathsf{D} \coloneqq \mathsf{D}^b(kQ-\mathsf{mod})$. Set R to be the basic object

$$P_{2n} \oplus P_1 \oplus S_1 \oplus S_3 \oplus \cdots \oplus S_{2n-5} \oplus S_{2n-3}$$

of the cluster category $\mathcal{C} \coloneqq \mathcal{C}_{kQ}$ of kQ. Notice that R has $n+1$ non-isomorphic indecomposable direct summands, consisting of $n-1$ non-projective simples S_{2r-1} for $r = 1, \ldots, n-1$, one non-simple projective P_1 and one simple projective $S_{2n} = P_{2n}$. In order to show that R is rigid, we must show that $\mathrm{Ext}^1_{\mathcal{C}}(R, R) = 0$, but as $\mathrm{Ext}^1_{\mathcal{C}}(X, -)$ is an additive functor and it suffices to show $\mathrm{Ext}^1_{\mathcal{C}}(R', R'') = 0$ for all indecomposable direct summands R', R'' of R. By [BMRRT, Prop. 1.7], for indecomposable kQ-modules X, Y, we have

$$\mathrm{Ext}^1_{\mathcal{C}}(X, Y) \cong \mathrm{Ext}^1_{kQ-\mathsf{mod}}(X, Y) \oplus \mathrm{Ext}^1_{kQ-\mathsf{mod}}(Y, X),$$

so it is enough to show that $\mathrm{Ext}^1_{kQ-\mathsf{mod}}(R', R'') = 0$. Since P_{2n} and P_1 are projective kQ-modules, we have that $\mathrm{Ext}^1_{kQ-\mathsf{mod}}(P_l, R'') = 0$ for $l = 1, 2n$. Thus, it remains to show $\mathrm{Ext}^1_{kQ-\mathsf{mod}}(S_i, S_j) = 0$ and $\mathrm{Ext}^1_{kQ-\mathsf{mod}}(S_i, P_l) = 0$ for all $i, j \in \{1, 3, \ldots, 2n-3\}$ and $l \in \{1, 2n\}$.

For $i, j \in \{1, 3, \ldots, 2n-3\}$ and $l \in \{1, 2n\}$, using Corollary II.7.36 and the

Auslander-Reiten quiver of $kQ = k\overrightarrow{\mathbb{A}}_{2n}$ (see for example [Sch14, §3.1]), we see that

$$\operatorname{Hom}_{\mathsf{D}}(S_i, P_l[1]) = \begin{cases} k & \text{if } l = i+1 \\ 0 & \text{else} \end{cases}$$

and

$$\operatorname{Hom}_{\mathsf{D}}(S_i, S_j[1]) = \begin{cases} k & \text{if } j = i+1 \\ 0 & \text{else.} \end{cases}$$

From this and [Hap88, p. 30], we deduce that

$$\operatorname{Ext}^1_{kQ-\mathsf{mod}}(S_i, P_l) \cong \operatorname{Ext}^1_{\mathsf{D}}(S_i, P_l) = \operatorname{Hom}_{\mathsf{D}}(S_i, P_l[1]) = \begin{cases} k & \text{if } l = i+1 \\ 0 & \text{else} \end{cases}$$

and, similarly,

$$\operatorname{Ext}^1_{kQ-\mathsf{mod}}(S_i, S_j) \cong \operatorname{Ext}^1_{\mathsf{D}}(S_i, S_j) = \operatorname{Hom}_{\mathsf{D}}(S_i, S_j[1]) = \begin{cases} k & \text{if } j = i+1 \\ 0 & \text{else.} \end{cases}$$

Since $R = P_{2n} \oplus P_1 \oplus S_1 \oplus S_3 \oplus \cdots \oplus S_{2n-5} \oplus S_{2n-3}$, we can see that we have no pairs (i, l) (respectively, (i, j)) of subscripts with $l = i + 1$ (respectively, $j = i + 1$), and hence $\operatorname{Ext}^1_{kQ-\mathsf{mod}}(S_i, S_j) = 0 = \operatorname{Ext}^1_{kQ-\mathsf{mod}}(S_i, P_l)$ for all $i, j \in \{1, 3, \ldots, 2n-3\}$ and $l \in \{1, 2n\}$. Therefore, R is a rigid object of $\mathcal{C}$ as claimed.

We also claim that $\operatorname{End}_{\mathcal{C}} R$ is isomorphic to the bound quiver algebra $A = k\overrightarrow{\mathbb{A}}_{n+1}/J(k\overrightarrow{\mathbb{A}}_{n+1})^2$ given by the following quiver

$$1 \xrightarrow{a_1} 2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} n \xrightarrow{a_n} n+1, \qquad \text{(V.2.1)}$$

modulo the indicated relations. By [BMRRT, Prop. 1.5], for indecomposable kQ-

modules X, Y, we have

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) = \mathrm{Hom}_{\mathsf{D}}(X,Y) \oplus \mathrm{Hom}_{\mathsf{D}}(X, \tau^{-1}Y[1]),$$

and [BMRRT, Prop. 1.7(d)] states that if X is a projective kQ-module then

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \cong \mathrm{Hom}_{kQ-\mathsf{mod}}(X,Y).$$

Using results from [Bon84] to compute dimensions of Hom-spaces (see also [Sch14, §3.1]), we then have

$$\begin{array}{lllll}
\mathrm{Hom}_{\mathcal{C}}(P_{2n}, P_1) & = & \mathrm{Hom}_{kQ-\mathsf{mod}}(P_{2n}, P_1) & \cong & k, \\
\mathrm{Hom}_{\mathcal{C}}(P_{2n}, S_i) & = & \mathrm{Hom}_{kQ-\mathsf{mod}}(P_{2n}, S_i) & = & \begin{cases} 0 & \text{if } i \neq 2n \\ k & \text{if } i = 2n. \end{cases} \\
\mathrm{Hom}_{\mathcal{C}}(P_1, P_{2n}) & = & \mathrm{Hom}_{kQ-\mathsf{mod}}(P_1, P_{2n}) & = & 0, \\
\mathrm{Hom}_{\mathcal{C}}(P_1, S_i) & = & \mathrm{Hom}_{kQ-\mathsf{mod}}(P_1, S_i) & = & \begin{cases} 0 & \text{if } i \neq 1 \\ k & \text{if } i = 1. \end{cases}
\end{array}$$

Suppose $i, j \in \{1, 3, \ldots, 2n-3\}$. Next, we determine $\mathrm{Hom}_{\mathcal{C}}(S_i, S_j)$ for each $i \neq j$. Observe that

$$\begin{array}{rcll}
\mathrm{Hom}_{\mathcal{C}}(S_i, S_j) & = & \mathrm{Hom}_{\mathsf{D}}(S_i, S_j) \oplus \mathrm{Hom}_{\mathsf{D}}(S_i, \tau^{-1}S_j[1]) & \\
& = & 0 \oplus \mathrm{Hom}_{\mathsf{D}}(S_i, \tau^{-1}S_j[1]) & \text{as } i \neq j \\
& \cong & \mathrm{Hom}_{\mathsf{D}}(S_i, \tau^{-1}S_j[1]), &
\end{array}$$

and that there are two cases: (a) $j < i$, or (b) $i < j$.

(a) If $j < i$, then we claim that $3 \leqslant 2n - i + j - 1 \leqslant 2n - 2$. As $i \leqslant 2n-3$, we have $3-2n \leqslant -i$ and so $3+j \leqslant 2n-i+j$. Thus, $3 \leqslant 2+j \leqslant 2n-i+j-1$.

Also, the assumption $j < i$ implies $-i + j < 0$, so $2n - i + j - 1 < 2n - 1$. Hence, $3 \leqslant 2n - i + j - 1 \leqslant 2n - 2$.

Therefore, $\tau^{2n-i-1}(S_j) = S_{2n-i+j-1}$ is a simple module in $kQ-\mathsf{mod}$. Note then that

$$\begin{aligned} \mathrm{Hom}_{\mathsf{D}}(S_i, \tau^{-1}S_j[1]) &\cong \mathrm{Hom}_{\mathsf{D}}(\tau^{2n-i}S_i, \tau^{2n-i-1}S_j[1]) \\ &= \mathrm{Hom}_{\mathsf{D}}(P_{2n}, S_{2n-i+j-1}[1]) \\ &= 0, \end{aligned}$$

as the only indecomposable objects of D to which P_{2n} has a non-zero morphism are of the form P_m for some $m \in \{1, \ldots, 2n\}$.

(b) On the other hand, if $i < j$ then we have $1 \leqslant i < j \leqslant 2n - 3$, and note $1 \leqslant j - 1$ as $1 \leqslant i < j$ and $j \in \{1, 3, \ldots, 2n - 3\}$. Then

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(S_i, S_j) &\cong \mathrm{Hom}_{\mathsf{D}}(S_i, \tau^{-1}S_j[1]) \\ &= \mathrm{Hom}_{\mathsf{D}}(S_i, S_{j-1}[1]) \\ &= \begin{cases} k & \text{if } j - 1 = i + 1 \\ 0 & \text{else} \end{cases} \\ &= \begin{cases} k & \text{if } j = i + 2 \\ 0 & \text{else.} \end{cases} \end{aligned}$$

Finally we wish to show that $\mathrm{Hom}_{\mathcal{C}}(S_{2r-1}, P_l) = 0$ for $1 \leqslant r \leqslant n-1$ and $l = 1, 2n$.

Using a similar method to the above, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(S_{2r-1}, P_l) &= \mathrm{Hom}_{\mathsf{D}}(S_{2r-1}, P_l) \oplus \mathrm{Hom}_{\mathsf{D}}(S_{2r-1}, \tau^{-1}P_l[1]) \\
&= 0 \oplus \mathrm{Hom}_{\mathsf{D}}(S_{2r-1}, \tau^{-1}P_l[1]) && \text{as } 2r-1 \neq 2n \\
&\cong \mathrm{Hom}_{\mathsf{D}}(S_{2r-1}, \tau^{-1}P_l[1]) \\
&\cong \mathrm{Hom}_{\mathsf{D}}(\tau S_{2r-1}, P_l[1]) \\
&= \mathrm{Hom}_{\mathsf{D}}(S_{2r}, P_l[1]) && \text{as } 1 \leqslant 2r-1 \leqslant 2n-3 \\
&= \begin{cases} k & \text{if } l = 2r+1 \\ 0 & \text{else.} \end{cases}
\end{aligned}$$

Therefore, for $1 \leqslant r \leqslant n-1$ and $l \in \{1, 2n\}$ we have $\mathrm{Hom}_{\mathcal{C}}(S_{2r-1}, P_l) = 0$.

Hence, we may choose non-zero morphisms (i.e. basis elements of the corresponding Hom-space) $\varphi_{2n}\colon P_{2n} \to P_1, \varphi_1\colon P_1 \to S_1$ and $\psi_r\colon S_{2r-1} \to S_{2r+1}$ for each $r = 1, \dots, n-2$. Given these we can consider the following non-zero morphisms of $\mathrm{End}_{\mathcal{C}} R$:

(i) $\widetilde{1_{R'}} \coloneqq \iota_{R'} \circ \pi_{R'}$, for each indecomposable direct summand R' of R, where $\iota_{R'}\colon R' \to R$, respectively, $\pi_{R'}\colon R \to R'$, is the canonical inclusion, respectively, projection; and

(ii) $\widetilde{\varphi_{2n}} \coloneqq \iota_{P_1}\varphi_{2n}\pi_{P_{2n}}, \widetilde{\varphi_1} \coloneqq \iota_{S_1}\varphi_1\pi_{P_1}$, and $\widetilde{\psi_r} \coloneqq \iota_{S_{2r+1}}\psi_r\pi_{S_{2r-1}}$ for each $r = 1, \dots, n-2$.

It is easy to see these generate $\mathrm{End}_{\mathcal{C}} R$ and an easy calculation shows they are k-linearly independent. Using the computations above we see that $\beta\alpha = 0$ for composable $\alpha, \beta \in \mathrm{End}_{\mathcal{C}} R$ amongst the endomorphisms of R detailed in (ii). Noting that a basis for $A = k\overrightarrow{\mathbb{A}}_{n+1}/J(k\overrightarrow{\mathbb{A}}_{n+1})^2$ is given by the set $\{e_1, \dots, e_{n+1}, a_1, \dots, a_n\}$, where e_i is the constant path at vertex i and a_i is the

i^{th} arrow in (V.2.1), we obtain a well-defined, surjective k-algebra homomorphism $\Phi\colon \operatorname{End}_{\mathcal{C}} R \to A$ defined by $\Phi(\widetilde{1_{P_{2n}}}) = e_1, \Phi(\widetilde{1_{P_1}}) = e_2, \Phi(\widetilde{\varphi_{2n}}) = a_1, \Phi(\widetilde{\varphi_1}) = a_2$ and $\Phi(\widetilde{1_{S_{2r-1}}}) = e_{r+2}, \Phi(\widetilde{\psi_r}) = a_{r+2}$ for each $r = 1, \dots, n-2$, and extended k-linearly. For injectivity, we note that if $0 \neq \alpha = \sum \lambda_i \widetilde{1_{R_i}} + \sum \mu_i \widetilde{\varphi_i} + \sum \nu_i \widetilde{\psi_i}$, then $(\lambda_1, \dots, \lambda_n, \mu_{2n}, \mu_1, \nu_1, \dots, \nu_{n-2}) \neq (0, \dots, 0)$, and so we have $\Phi(\alpha) = \sum \lambda_i e_i + \mu_{2n} a_1 + \mu_1 a_2 + \sum \nu_i a_{i+2} \neq 0$. Therefore, Φ yields a k-algebra isomorphism $\operatorname{End}_{\mathcal{C}} R$ to A. Moreover, $\operatorname{gl.dim} \Lambda_R = \operatorname{gl.dim} \operatorname{End}_{\mathcal{C}} R = \operatorname{gl.dim} A = n$ by Proposition V.2.2, finishing the proof. ■

Corollary V.2.5. *There exist partial cluster-tilted algebras that are not cluster-tilted.*

Proof. In Theorem V.2.4 we obtain partial cluster-tilted algebras of finite global dimension $n \geqslant 2 > 1$, and so these algebras are not Gorenstein of dimension 1. Therefore, they cannot arise as cluster-tilted algebras, since by [KR07, §2.1] we know that cluster-tilted algebras must be hereditary if they have finite global dimension. ■

Remark V.2.6. The opposite endomorphism ring of a partial tilting module is always a tilted algebra (see [Hap88, Cor. III.6.5]). In [BMR08] it was observed that the analogous result in cluster-tilting theory does not hold.

We explain our interpretation of their observation. It is noted that the path algebra $\Lambda_T = (\operatorname{End}_{\mathcal{C}} T)^{\text{op}}$ of an oriented 4-cycle modulo its radical cubed arises as a cluster-tilted algebra of Dynkin-type $\mathbb{D}_4$ (see [BMR08, p. 158]). Recall that if Λ is a cluster-tilted algebra coming from a cluster category $\mathcal{C}_{kQ}$, where Q is a Dynkin-type quiver, then we say Λ is *of Dynkin-type* Q. We may assume T is basic. By [BMR07, Cor. 2.3], T must have 4 non-isomorphic indecomposable direct summands. Then one can obtain a partial cluster-tilted algebra $\Lambda_R = (\operatorname{End}_{\mathcal{C}} R)^{\text{op}}$ by taking R to be the direct sum of any 3 distinct direct summands of T. Thus, by the symmetry of the quiver of Λ_T, the algebra Λ_R is isomorphic to the bound quiver algebra of the following quiver with the indicated relations

$$\begin{tikzcd}
& 2 \arrow[dr] & \\
1 \arrow[ur] & & 3 \arrow[ll]
\end{tikzcd}$$

By [BIRS11, Thm. 2.3] (see also [BMR06, Cor. 4.3]), a cluster-tilted algebra is determined by its *ordinary quiver* (in the sense of [ASS06, Def. II.3.1]). So if Λ_R were cluster-tilted, then it would be of Dynkin-type $\mathbb{A}_3$. However, this cannot be the case by [BV08, Prop. 3.1], an application of which implies that a cluster-tilted algebra of Dynkin-type $\mathbb{A}_3$ is isomorphic to the path algebra of an oriented 3-cycle modulo the square of its radical or to the path algebra of a quiver of Dynkin-type $\mathbb{A}_3$.

Note that Λ_R above has global dimension 3, so Corollary V.2.5 also follows from this observation in [BMR08]. Our Theorem V.2.4, on the other hand, provides a systematic way to construct partial cluster-tilted algebras of any finite global dimension.

We finish this section by showing that Theorem V.2.4 implies that the class of partial cluster-tilted algebras does not coincide with the class of tilted algebras.

Definition V.2.7. [Hap88, §III.4] Let A be a finite-dimensional k-algebra and T a finitely generated left A-module. We call T *tilting* if:

(i) the projective dimension of T is at most 1;

(ii) $\operatorname{Ext}^1_{A-\mathsf{mod}}(T,T) = 0$, i.e. T is rigid; and

(iii) there is an exact sequence $0 \to A \to T_1 \to T_2 \to 0$ with $T_i \in \mathsf{add}\, T$.

Definition V.2.8. [Hap88, §III.5] If an algebra B is of the form $B = \operatorname{End}_{H-\mathsf{mod}} T$ for some finite-dimensional hereditary k-algebra H, where T is a tilting left H-module, then we call B a *tilted algebra*.

Lemma V.2.9. [ASS06, Lem. VIII.3.2 (e)] *Let H be a finite-dimensional hereditary k-algebra ($k = \overline{k}$), T a tilting module over H and $B = \mathrm{End}_{H-\mathsf{mod}} T$. Then* $\mathrm{gl.dim}\, B \leqslant 2$.

Corollary V.2.10. *There exist partial cluster-tilted algebras that are not tilted algebras.*

Proof. In Theorem V.2.4, partial cluster-tilted algebras of finite global dimension $n \geqslant 3 > 2$ were constructed, and so these cannot be tilted algebras in view of Lemma V.2.9. ■

Remark V.2.11. We remark here that Corollary V.2.10 also follows from the observation in [BMR08, p. 158] (see Remark V.2.6).

V.3 The Auslander-Reiten quiver of $\Lambda_R - \mathsf{mod}$

Throughout this section (and its subsections), we let k be a field and $\mathcal{C}$ a skeletally small, Hom-finite, Krull-Schmidt, triangulated k-category (with suspension functor Σ) that has Serre duality. Suppose also that R is a basic rigid object of $\mathcal{C}$, and set $\Lambda_R := (\mathrm{End}_{\mathcal{C}}\, R)^{\mathrm{op}}$. In this section, we investigate how the Auslander-Reiten quiver of $\Lambda_R - \mathsf{mod}$ can be extracted from the Auslander-Reiten quiver of $\mathcal{C}$ via the functor $\mathrm{Hom}_{\mathcal{C}}(R, -)\colon \mathcal{C} \to \Lambda_R - \mathsf{mod}$. We begin by recalling known results in §V.3.1 and, in particular, recalling some categories to which $\Lambda_R - \mathsf{mod}$ is equivalent. In §V.3.2 we provide some results that will be useful to us in §V.3.3, where we then look at how irreducible morphisms behave under $\mathrm{Hom}_{\mathcal{C}}(R, -)$.

V.3.1 $\Lambda_R - \mathsf{mod}$ as localisations

In [BM13] Buan and Marsh showed that one can localise $\mathcal{C}$ at a certain class of morphisms to obtain a category equivalent to $\Lambda_R - \mathsf{mod}$, and in [BM12] the same authors showed that a localisation of $\mathcal{C}/[\mathcal{X}_R]$ is also equivalent to $\Lambda_R-\mathsf{mod}$, where $\mathcal{X}_R = (\mathsf{add}\, R)^{\perp_0}$ is the full subcategory of $\mathcal{C}$ consisting of objects X such that $\mathrm{Hom}_{\mathcal{C}}(R, X) = 0$ (see Definition III.5.2). Recall that for a full additive subcategory $\mathcal{U} \subseteq \mathcal{C}$ that is closed under isomorphisms and direct summands, we set

$$\mathcal{U}^{\perp_i} := \left\{X \in \mathcal{C} \mid \mathrm{Hom}_{\mathcal{C}}(\mathcal{U}, \Sigma^i X) = 0\right\},$$

where $i \in \mathbb{N}$ (see Definition III.2.1).

Let us recall some of the definitions and results from [BM13] and [BM12]. The following is often helpful.

Lemma V.3.1. [BM13, Lem. 2.3] *Let $f\colon X \to Y$ be an arbitrary morphism in $\mathcal{C}$. Then $\mathrm{Hom}_{\mathcal{C}}(R, f) = 0$ if and only if f factors through $\mathcal{X}_R$.*

We now recall two important classes of morphisms considered by Buan and Marsh.

Definition V.3.2. [BM13] We denote by $\widetilde{\mathcal{S}}$ the class of morphisms $f\colon X \to Y$ in $\mathcal{C}$ such that when $\Sigma^{-1}Z \xrightarrow{h} X \xrightarrow{f} Y \xrightarrow{g} Z$ is the completion of f to a triangle, both h and g factor through $\mathcal{X}_R$.

Let $\mathcal{S}$ denote the subclass of $\widetilde{\mathcal{S}}$ consisting of those f in $\mathcal{C}$ for which, when completed to a triangle as in the notation above, g factors through $\mathcal{X}_R$ and $\Sigma^{-1}Z \in \mathcal{X}_R$.

Remark V.3.3. We remark here that the class $\widetilde{\mathcal{S}}$ (as defined in Definition V.3.2 and in [BM13]) was denoted by $\mathcal{S}$ in [BM12] (see [BM12, §7]).

We recall two characterisations of morphisms in $\widetilde{\mathcal{S}}$ observed by Buan and Marsh.

Lemma V.3.4. *Let $f\colon X \to Y$ be a morphism in $\mathcal{C}$.*

(i) [BM13, Lem. 2.5] *The morphism f lies in $\widetilde{\mathcal{S}}$ if and only if $\mathrm{Hom}_{\mathcal{C}}(R, f)$ is an isomorphism in $\Lambda_R-\mathsf{mod}$.*

(ii) [BM12, p. 167] *The morphism f lies in $\widetilde{\mathcal{S}}$ if and only if Q(f) is regular in $\mathcal{C}/[\mathcal{X}_R]$, where $Q\colon \mathcal{C} \to \mathcal{C}/[\mathcal{X}_R]$ is the quotient functor.*

From this, it is then natural to compare the (Gabriel-Zisman) localisation $\mathcal{C}_{\widetilde{\mathcal{S}}}$, respectively, $\mathcal{C}_{\mathcal{S}}$, of $\mathcal{C}$ at the class $\widetilde{\mathcal{S}}$, respectively, $\mathcal{S}$, with the category $\Lambda_R-\mathsf{mod}$. Let $L_{\mathcal{S}}\colon \mathcal{C} \to \mathcal{C}_{\mathcal{S}}$ and $L_{\widetilde{\mathcal{S}}}\colon \mathcal{C} \to \mathcal{C}_{\widetilde{\mathcal{S}}}$ be the corresponding localisation functors (see Definition II.5.2). It was shown in [BM13] that the induced functor $G\colon \mathcal{C}_{\widetilde{\mathcal{S}}} \xrightarrow{\simeq} \Lambda_R-\mathsf{mod}$ with $\mathrm{Hom}_{\mathcal{C}}(R,-) = GL_{\widetilde{\mathcal{S}}}$ (see Lemma V.3.4 (i)) is an equivalence. Furthermore, using that $\mathcal{S} \subseteq \widetilde{\mathcal{S}}$, there is a canonical functor $J\colon \mathcal{C}_{\mathcal{S}} \xrightarrow{\cong} \mathcal{C}_{\widetilde{\mathcal{S}}}$ with $L_{\widetilde{\mathcal{S}}} = JL_{\mathcal{S}}$, which was proven to be an isomorphism of categories in [BM13]. See [BM13, p. 2855] for more details.

Definition V.3.5. By $\mathcal{C}(R)$ we denote the full subcategory of $\mathcal{C}$ whose objects are those $X \in \mathcal{C}$ for which there exists a triangle $R_0 \to R_1 \to X \to \Sigma R_0$ in $\mathcal{C}$ with $R_0, R_1 \in \mathsf{add}\, R$.

As mentioned in §V.1, the subcategory $\mathcal{C}(R)$ will play a large role in our considerations later, but it has already appeared in the literature; see, for example, [KR07], [IY08] and [BM13]. Note that $\mathcal{C}(R)$ is equal to the extension subcategory $\mathsf{add}\, R * \mathsf{add}\, \Sigma R$ (see Definition III.2.2 and also §III.5). We recall here a summary result from [BM13].

Theorem V.3.6. [BM13, Thm. 5.4] *There are equivalences of categories*

$$\mathcal{C}(R)/[\mathsf{add}\, \Sigma R] \xrightarrow{\simeq} \mathcal{C}(R)/[(\mathsf{add}\, \Sigma R)^{\perp_1}] \xrightarrow{\simeq} \Lambda_R-\mathsf{mod} \xrightarrow{\simeq} \mathcal{C}_{\widetilde{\mathcal{S}}},$$

where the equivalence $\mathcal{C}(R)/[(\mathsf{add}\, \Sigma R)^{\perp_1}] \xrightarrow{\simeq} \Lambda_R-\mathsf{mod}$ is induced by the restriction of the functor $\mathrm{Hom}_{\mathcal{C}}(R,-)$ to $\mathcal{C}(R)$.

Note that $(\mathsf{add}\, \Sigma R)^{\perp_1}$ is equal to $(\mathsf{add}\, R)^{\perp_0} = \mathcal{X}_R$.

Another perspective that will be useful for us is the one taken in [BM12]. The quotient category $\mathcal{C}/[\mathcal{X}_R]$ is an integral category (see [BM12, Cor. 3.10] and

[Nak13, Thm. 6.3]), so by [Rum01, Prop. 6] the class $\mathcal{R}$ of regular morphisms in $\mathcal{C}/[\mathcal{X}_R]$ admits a calculus of left fractions and a calculus of right fractions (see §II.5.2). In particular, the localisation $(\mathcal{C}/[\mathcal{X}_R])_{\mathcal{R}}$ of $\mathcal{C}/[\mathcal{X}_R]$ at the class $\mathcal{R}$ is an abelian category (see [Rum01, p. 173]). We summarise the functors and equivalences we need in the following statement.

Proposition V.3.7. *There is commutative diagram of functors*

$$\begin{tikzcd}
 & \mathcal{C}/[\mathcal{X}_R] \arrow[r, "L_{\mathcal{R}}"] & (\mathcal{C}/[\mathcal{X}_R])_{\mathcal{R}} \arrow[dr, "H'", "\simeq"'] & \\
\mathcal{C} \arrow[ur, "Q"] \arrow[rrr, "\mathrm{Hom}_{\mathcal{C}}(R,-)"] \arrow[dr, "L_{\mathcal{S}}"'] & & & \Lambda_R-\mathsf{mod} \\
 & \mathcal{C}_{\mathcal{S}} \arrow[r, "\cong", "J"'] & \mathcal{C}_{\widetilde{\mathcal{S}}} \arrow[ur, "\simeq", "G"'] &
\end{tikzcd}$$

where H' is naturally isomorphic to $\mathrm{Hom}_{(\mathcal{C}/[\mathcal{X}_R])_{\mathcal{R}}}(R,-)$ and is an equivalence, J is an isomorphism of categories and G is an equivalence.

Proof. This follows from [BM13, p. 2855], [BM13, Thm. 4.4], [BM12, p. 167] and [BM12, Lem. 7.3]. ■

We conclude this section with one last observation that will be needed later and follows from Proposition V.3.7 and [BM13, Lem. 3.5 (b)].

Lemma V.3.8. *Let X, Y be objects in $\mathcal{C}$. Suppose $Y \in \mathcal{X}_R$ and consider the direct sum $X \oplus Y$. Let $\iota_X : X \hookrightarrow X \oplus Y$ be the canonical inclusion and let $\pi_X : X \oplus Y \twoheadrightarrow X$ be the canonical projection. Then $\mathrm{Hom}_{\Lambda_R-\mathsf{mod}}(R, \iota_X)$ and $\mathrm{Hom}_{\Lambda_R-\mathsf{mod}}(R, \pi_X)$ are mutually inverse in $\Lambda_R-\mathsf{mod}$.*

V.3.2 The subcategory $\mathcal{C}(R)$

Let k, $\mathcal{C}$ and R be as before. In this section, we give some preliminary results all involving the subcategory $\mathcal{C}(R)$. Recall that for a functor $\mathscr{F} : \mathcal{A} \to \mathcal{B}$ and objects

$X, Y \in \mathcal{A}$, we denote by $\mathscr{F}_{X,Y}$ the map $\operatorname{Hom}_{\mathcal{A}}(X,Y) \to \operatorname{Hom}_{\mathcal{B}}(\mathscr{F}(X), \mathscr{F}(Y))$ (see Definition II.1.5). Although a localisation functor is not full in general, Buan and Marsh establish the following.

Proposition V.3.9. [BM13, Prop. 3.7] *Let X, Y be objects in $\mathcal{C}$ with $X \in \mathcal{C}(R)$. Then $(L_{\mathcal{S}})_{X,Y}\colon \operatorname{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \operatorname{Hom}_{\mathcal{C}_{\mathcal{S}}}(X,Y)$ is surjective.*

The following corollary is a special case of [HJ15, Lem. 1.11], but we include a short proof using the theory developed in [BM13] and [BM12].

Corollary V.3.10. *Let X, Y be objects in $\mathcal{C}$. If $X \in \mathcal{C}(R)$, then*

$$\operatorname{Hom}_{\mathcal{C}}(R,-)_{X,Y}\colon \operatorname{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \operatorname{Hom}_{\Lambda_R-\mathsf{mod}}(\operatorname{Hom}_{\mathcal{C}}(R,X), \operatorname{Hom}_{\mathcal{C}}(R,Y))$$

is surjective.

Proof. By Proposition V.3.7, there is an equivalence $G\colon \mathcal{C}_{\widetilde{\mathcal{S}}} \to \Lambda_R-\mathsf{mod}$ and an isomorphism $J\colon \mathcal{C}_{\mathcal{S}} \to \mathcal{C}_{\widetilde{\mathcal{S}}}$ such that $\operatorname{Hom}_{\mathcal{C}}(R,-) = GL_{\widetilde{\mathcal{S}}} = GJL_{\mathcal{S}}$. If $X \in \mathcal{C}(R)$, then the mapping $\operatorname{Hom}_{\mathcal{C}}(R,-)_{X,Y} = G_{X,Y} \circ J_{X,Y} \circ (L_{\mathcal{S}})_{X,Y}$ is surjective because $(L_{\mathcal{S}})_{X,Y}$ is surjective by Proposition V.3.9, and $G_{X,Y}$ and $J_{X,Y}$ are bijective. ∎

Let $Q\colon \mathcal{C} \to \mathcal{C}/[\mathcal{X}_R]$ be the quotient functor as in Proposition V.3.7. For $f\colon X \to Y$ in $\mathcal{C}$, we will denote by $\overline{f}$ the image $Q(f) = f + [\mathcal{X}_R](X,Y)$ in the quotient $\operatorname{Hom}_{\mathcal{C}/[\mathcal{X}_R]}(X,Y) = \operatorname{Hom}_{\mathcal{C}}(X,Y)/[\mathcal{X}_R](X,Y)$. In addition, for many of the results in the sequel, we consider objects that do not have indecomposable direct summands lying in $\mathcal{X}_R$. Given an object $X = \bigoplus_{i=1}^{n} X_i$ in the Krull-Schmidt category $\mathcal{C}$ with X_i indecomposable, we have $X_i \notin \mathcal{X}_R$ for all $1 \leqslant i \leqslant n$, if and only if $1_{X_i} \notin [\mathcal{X}_R]$ for all $1 \leqslant i \leqslant n$, if and only if $\mathsf{add}\, X \cap \mathcal{X}_R = 0$.

The next three results give a way to detect if a morphism with codomain in $\mathcal{C}(R)$ splits under $\operatorname{Hom}_{\mathcal{C}}(R,-)$.

Lemma V.3.11. *Let X, Y be objects in $\mathcal{C}$ with $Y \in \mathcal{C}(R)$ and* $\mathsf{add}\, X \cap \mathcal{X}_R = 0$*, and let $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ be arbitrary. Then the following are equivalent.*

(i) *The morphism $f\colon X \to Y$ is a section in $\mathcal{C}$.*

(ii) *The morphism $\overline{f}\colon X \to Y$ is a section in $\mathcal{C}/[\mathcal{X}_R]$.*

(iii) *The morphism $\mathrm{Hom}_{\mathcal{C}}(R, f)\colon \mathrm{Hom}_{\mathcal{C}}(R, X) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(R, Y)$ is a section in* $\Lambda_R - \mathsf{mod}$.

Proof. The equivalence of (i) and (ii) follows from Lemma IV.3.3 and does not require $Y \in \mathcal{C}(R)$.

(i) $\Rightarrow$ (iii). This is immediate as $\mathrm{Hom}_{\mathcal{C}}(R, -)$ is a covariant functor.

(iii) $\Rightarrow$ (i). Suppose now that $\mathrm{Hom}_{\mathcal{C}}(R, f)$ is a section, so that there is some $\beta\colon \mathrm{Hom}_{\mathcal{C}}(R, Y) \to \mathrm{Hom}_{\mathcal{C}}(R, X)$ with $1_{\mathrm{Hom}_{\mathcal{C}}(R,X)} = \beta\, \mathrm{Hom}_{\mathcal{C}}(R, f)$. Since $Y \in \mathcal{C}(R)$, we know that

$$\mathrm{Hom}_{\mathcal{C}}(R, -)_{Y,X}\colon \mathrm{Hom}_{\mathcal{C}}(Y, X) \to \mathrm{Hom}_{\Lambda_R - \mathsf{mod}}(\mathrm{Hom}_{\mathcal{C}}(R, Y), \mathrm{Hom}_{\mathcal{C}}(R, X))$$

is surjective by Corollary V.3.10. Therefore, there exists $g\colon Y \to X$ in $\mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(R, g) = \beta$. Hence, $\mathrm{Hom}_{\mathcal{C}}(R, 1_X) = 1_{\mathrm{Hom}_{\mathcal{C}}(R,X)} = \beta\, \mathrm{Hom}_{\mathcal{C}}(R, f) = \mathrm{Hom}_{\mathcal{C}}(R, g)\, \mathrm{Hom}_{\mathcal{C}}(R, f) = \mathrm{Hom}_{\mathcal{C}}(R, gf)$. So $\mathrm{Hom}_{\mathcal{C}}(R, 1_X - gf) = 0$ and so $1_X - gf \in \mathrm{End}_{\mathcal{C}}\, X$ factors through $\mathcal{X}_R$ by Lemma V.3.1. As $\mathsf{add}\, X \cap \mathcal{X}_R = 0$, by Lemma IV.3.2, we have that $1_X - gf$ is radical. Consequently, $gf = 1_X - (1_X - gf)$ is an isomorphism and so f is a section as claimed. ■

Similarly, we have the following.

Lemma V.3.12. *Let X, Y be objects in $\mathcal{C}$ with $Y \in \mathcal{C}(R)$ and* $\mathsf{add}\, Y \cap \mathcal{X}_R = 0$*, and let $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ be arbitrary. Then the following are equivalent.*

(i) *The morphism $f\colon X \to Y$ is a retraction in $\mathcal{C}$.*

(ii) *The morphism* $\overline{f}\colon X \to Y$ *is a retraction in* $\mathcal{C}/[\mathcal{X}_R]$.

(iii) *The morphism* $\mathrm{Hom}_{\mathcal{C}}(R, f)\colon\ \mathrm{Hom}_{\mathcal{C}}(R, X) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(R, Y)$ *is a retraction in* $\Lambda_R-\mathsf{mod}$.

Proof. The equivalence of (i) and (ii) follows from Lemma IV.3.3 and does not require $Y \in \mathcal{C}(R)$.

(i) $\Rightarrow$ (iii). This is immediate as $\mathrm{Hom}_{\mathcal{C}}(R, -)$ is a covariant functor.

(iii) $\Rightarrow$ (i). Now assume $\mathrm{Hom}_{\mathcal{C}}(R, f)\colon \mathrm{Hom}_{\mathcal{C}}(R, X) \to \mathrm{Hom}_{\mathcal{C}}(R, Y)$ is a retraction, so that there is some section $\alpha\colon \mathrm{Hom}_{\mathcal{C}}(R, Y) \to \mathrm{Hom}_{\mathcal{C}}(R, X)$ with $\mathrm{Hom}_{\mathcal{C}}(R, f)\alpha = 1_{\mathrm{Hom}_{\mathcal{C}}(R,Y)} = \mathrm{Hom}_{\mathcal{C}}(R, 1_Y)$. Since $Y \in \mathcal{C}(R)$, by Corollary V.3.10 there exists $g\colon Y \to X$ such that $\mathrm{Hom}_{\mathcal{C}}(R, g) = \alpha$. Thus, $\mathrm{Hom}_{\mathcal{C}}(R, 1_Y) = \mathrm{Hom}_{\mathcal{C}}(R, f)\,\mathrm{Hom}_{\mathcal{C}}(R, g) = \mathrm{Hom}_{\mathcal{C}}(R, fg)$ and $1_Y - fg \in \mathrm{End}_{\mathcal{C}}\, Y$ factors through $\mathcal{X}_R$. As $\mathsf{add}\, Y \cap \mathcal{X}_R = 0$, by Lemma IV.3.2, we know $1_Y - fg$ is radical and so fg is an isomorphism, which shows f is a retraction as desired. ■

This yields the following immediate corollary.

Corollary V.3.13. *Let* X, Y *be objects in* $\mathcal{C}$ *with* $Y \in \mathcal{C}(R)$ *and* $\mathsf{add}\, X \cap \mathcal{X}_R = 0 = \mathsf{add}\, Y \cap \mathcal{X}_R$*, and let* $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ *be arbitrary. Then the following are equivalent.*

(i) *The morphism* $f\colon X \to Y$ *is an isomorphism in* $\mathcal{C}$.

(ii) *The morphism* $\overline{f}\colon X \to Y$ *is an isomorphism in* $\mathcal{C}/[\mathcal{X}_R]$.

(iii) *The morphism* $\mathrm{Hom}_{\mathcal{C}}(R, f)\colon\ \mathrm{Hom}_{\mathcal{C}}(R, X) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(R, Y)$ *is an isomorphism in* $\Lambda_R-\mathsf{mod}$.

Proof. This follows quickly from Proposition II.9.9, and Lemmas V.3.11 and V.3.12. ■

Remark V.3.14. Lemma V.3.11 and Lemma V.3.12 appear to be dual results in some sense, but we give a warning: in both results we need the codomain Y of the morphism f to be in the $\mathcal{C}(R)$. As seen in the proofs, this is because we are required to lift a morphism that has domain $\operatorname{Hom}_{\mathcal{C}}(R, Y)$ and for that we need $Y \in \mathcal{C}(R)$ in order to use Corollary V.3.10.

We recall a useful characterisation of objects in $\mathcal{C}(R)$.

Lemma V.3.15. [BM13, Lem. 3.2] *For an object X in $\mathcal{C}$, the following are equivalent.*

(i) *X is in $\mathcal{C}(R)$.*

(ii) *If f is a right $\mathsf{add}\, R$-approximation in the triangle $U \to R_0 \xrightarrow{f} X \to \Sigma U$, then U is also in $\mathsf{add}\, R$.*

(iii) *If f is a minimal right $\mathsf{add}\, R$-approximation in the triangle $U \to R_0 \xrightarrow{f} X \to \Sigma U$, then U is also in $\mathsf{add}\, R$.*

It is worth noting here that given an object X in $\mathcal{C}(R)$ with corresponding triangle $R_0 \longrightarrow R_1 \xrightarrow{r} X \longrightarrow \Sigma R_0$ (as in Definition V.3.5), the morphism $r\colon R_1 \to X$ is a right $\mathsf{add}\, R$-approximation of X.

The following lemma was proved in [MP17] (see also [IY08, Prop. 2.1]), but we give an alternative proof. Note that $\mathsf{add}\, R$ is functorially finite (see Definition III.2.6) as $\mathcal{C}$ is both Hom-finite and Krull-Schmidt.

Lemma V.3.16. [MP17, Lem. 3.17] *The subcategory $\mathcal{C}(R)$ is closed under direct summands and isomorphisms.*

Proof. Suppose $A \oplus B \in \mathcal{C}(R)$. We will prove that $A \in \mathcal{C}(R)$. Let $U \to R_A \xrightarrow{f} A \to \Sigma U$, respectively, $V \to R_B \xrightarrow{g} B \to \Sigma V$, be the completion to a triangle of an arbitrary right $\mathsf{add}\, R$-approximation of A, respectively, B, and

$$W \longrightarrow R_A \oplus R_B \xrightarrow{f \oplus g} A \oplus B \longrightarrow \Sigma W$$

the completion of the morphism $f \oplus g$ to a triangle. Since $\mathsf{add}\, R$-approximations are additive, we have that $f \oplus g$ is a right $\mathsf{add}\, R$-approximation of $A \oplus B$. Thus, by Lemma V.3.15 we know $W \in \mathsf{add}\, R$. Consider the following commutative diagram, in which g, h exist by use of the (TR4) axiom for triangulated categories and ι_X, respectively, π_X, is the canonical inclusion, respectively, projection for $X \in \{R_A, A\}$.

$$\begin{tikzcd}
U \arrow[r] \arrow[d, dotted, "g"'] & R_A \arrow[r, "f"] \arrow[d, hook, "\iota_{R_A}"'] \arrow[dr, phantom, "\circ"] & A \arrow[r] \arrow[d, hook, "\iota_A"] & \Sigma U \arrow[d, dotted, "\Sigma g"] \\
W \arrow[r] \arrow[d, dotted, "h"'] & R_A \oplus R_B \arrow[r, "f\oplus g"] \arrow[d, two heads, "\pi_{R_A}"'] \arrow[dr, phantom, "\circ"] & A \oplus B \arrow[r] \arrow[d, two heads, "\pi_A"] & \Sigma W \arrow[d, dotted, "\Sigma h"] \\
U \arrow[r] & R_A \arrow[r, "f"] & A \arrow[r] & \Sigma U
\end{tikzcd}$$

Notice, however, that $\pi_{R_A} \circ \iota_{R_A} = 1_{R_A}$ and $\pi_A \circ \iota_A = 1_A$ are isomorphisms so hg is also an isomorphism by Proposition II.6.8, and hence U is a direct summand of W. Therefore, $U \in \mathsf{add}\, R$ and $A \in \mathcal{C}(R)$ by Lemma V.3.15 again.

Suppose now that we have an isomorphism $x\colon X \xrightarrow{\cong} Y$, and that $X \in \mathcal{C}(R)$ with triangle $R_0 \to R_1 \xrightarrow{r} X \to \Sigma R_0$ ($R_i \in \mathsf{add}\, R$) witnessing this. Then we have a morphism of triangles

$$\begin{tikzcd}
R_0 \arrow[r] \arrow[d, dotted, "y"'] & R_1 \arrow[r, "r"] \arrow[d, equal, "1_{R_1}"'] \arrow[dr, phantom, "\circ"] & X \arrow[r] \arrow[d, "\cong"', "x"] & \Sigma R_0 \arrow[d, dotted, "\Sigma y"] \\
T \arrow[r] & R_1 \arrow[r, "xr"] & Y \arrow[r] & \Sigma T,
\end{tikzcd}$$

where the bottom row is the triangle completion of xr, and y exists by an application of axiom (TR4). Since 1_{R_1} and x are isomorphisms, we have that y is also an isomorphism by Proposition II.6.8. Thus, $T \in \mathsf{add}\, R$ and $Y \in \mathcal{C}(R)$. ■

Corollary V.3.17. *The subcategory $\mathcal{C}(R)$ is Krull-Schmidt.*

Proof. Note that as $\mathcal{C}(R)$ is a full subcategory of $\mathcal{C}$, it is preadditive. Clearly, the zero object of $\mathcal{C}$ lies in $\mathcal{C}(R) = \mathsf{add}\, R * \mathsf{add}\, \Sigma R$ as $0 \in \mathsf{add}\, R$. Given objects $X, Y \in \mathcal{C}(R)$, taking the direct sum of the triangles witnessing that X and Y belong to $\mathcal{C}(R)$ gives a triangle showing $X \oplus Y$ also belongs to $\mathcal{C}(R)$. Thus, $\mathcal{C}(R)$ is an additive category.

Let X be an object of $\mathcal{C}(R)$. By Lemma V.3.16, a Krull-Schmidt decomposition of X in $\mathcal{C}$ gives a Krull-Schmidt decomposition of X in $\mathcal{C}(R)$. ■

We now recall two useful results from [BM13].

Lemma V.3.18. [BM13, Lem. 3.3] *Let X be an object in $\mathcal{C}$. Then there exists a morphism $s\colon W \to X$ in $\mathcal{S}$ with $W \in \mathcal{C}(R)$.*

Lemma V.3.19. [BM13, Lem. 3.6] *Let $s\colon W \to X$ be a morphism in $\mathcal{S}$. Then, for any morphism $f\colon Y \to X$ with $Y \in \mathcal{C}(R)$ there exists $\widehat{f}\colon Y \to W$ such that $f = s\widehat{f}$.*

Notice that these two lemmas above say that any object X in $\mathcal{C}$ admits a right $\mathcal{C}(R)$-approximation $s\colon W \to X$ with the additional property that $s \in \mathcal{S} \subseteq \widetilde{\mathcal{S}}$. In particular, $\mathrm{Hom}_{\mathcal{C}}(R, s)$ is an isomorphism (see Lemma V.3.4). We present a modified version of Lemma V.3.18.

Lemma V.3.20. *If $X \in \mathcal{C}$, then there exists a morphism $\widetilde{s}\colon \widetilde{W} \to X$ in $\widetilde{\mathcal{S}}$ with $\widetilde{W} \in \mathcal{C}(R)$ and $\mathsf{add}\,\widetilde{W} \cap \mathcal{X}_R = 0$.*

Proof. Let $U \to W \xrightarrow{s} X \to \Sigma U$ be the completion of a morphism $s\colon W \to X$, where $W \in \mathcal{C}(R)$ and $s \in \mathcal{S}$ as obtained using Lemma V.3.18. Then we can express $W = \widetilde{W} \oplus W_0$, where $\mathsf{add}\,\widetilde{W} \cap \mathcal{X}_R = 0$ and $W_0 \in \mathcal{X}_R$, and $s = (\,\widetilde{s}\;\; s_0\,)$ where $\widetilde{s}\colon \widetilde{W} \to X$ and $s_0\colon W_0 \to X$. The morphism $\mathrm{Hom}_{\mathcal{C}}(R, s)$ is an isomorphism since s is a morphism in $\mathcal{S} \subseteq \widetilde{\mathcal{S}}$ (see Lemma V.3.4) and $\mathrm{Hom}_{\mathcal{C}}(R, \iota_{\widetilde{W}})$ is an isomorphism by Lemma V.3.8, where $\iota_{\widetilde{W}}\colon \widetilde{W} \to \widetilde{W} \oplus W_0 = W$ is the canonical inclusion morphism. Thus, $\mathrm{Hom}_{\mathcal{C}}(R, \widetilde{s}) = \mathrm{Hom}_{\mathcal{C}}(R, s\iota_{\widetilde{W}}) = \mathrm{Hom}_{\mathcal{C}}(R, s) \circ \mathrm{Hom}_{\mathcal{C}}(R, \iota_{\widetilde{W}})$ is also an isomorphism (as the composition of isomorphisms). Then, by Lemma V.3.4, we have that $\widetilde{s}\colon \widetilde{W} \to X$ is a morphism in $\widetilde{S}$ where $\widetilde{W} \in \mathcal{C}(R)$ by Lemma V.3.16. ■

Remark V.3.21. Let X be an object of $\mathcal{C}$. Let $\widetilde{s}\colon \widetilde{W} \to X$ be a morphism in $\widetilde{\mathcal{S}}$, where $\widetilde{W} \in \mathcal{C}(R)$ and $\mathsf{add}\,\widetilde{W} \cap \mathcal{X}_R = 0$. Then $\mathrm{End}_{\mathcal{C}/[\mathcal{X}_R]}\widetilde{W}$ is local if and only if $\mathrm{Hom}_{\mathcal{C}}(R, X)$ is indecomposable. Indeed, $\mathrm{End}_{\mathcal{C}/[\mathcal{X}_R]}\widetilde{W}$ is local, if and only if $\widetilde{W}$ is indecomposable in $\mathcal{C}/[\mathcal{X}_R]$ (see e.g. Proposition II.1.48), if and only if $\mathrm{Hom}_{\mathcal{C}}(R, \widetilde{W})$ is indecomposable (using Theorem V.3.6). Since $\widetilde{s}$ is a morphism in the class $\widetilde{\mathcal{S}}$, the morphism $\mathrm{Hom}_{\mathcal{C}}(R, \widetilde{s})$ is an isomorphism and hence $\mathrm{Hom}_{\mathcal{C}}(R, \widetilde{W})$ is indecomposable if and only if $\mathrm{Hom}_{\mathcal{C}}(R, X)$ is indecomposable.

This gives us a way to use the Auslander-Reiten quiver of $\mathcal{C}$ to detect when an object $X \in \mathcal{C}$ will become decomposable under the functor $\mathrm{Hom}_{\mathcal{C}}(R, -)$.

V.3.3 Irreducible morphisms and the functor $\mathrm{Hom}_{\mathcal{C}}(R, -)$

Let k, $\mathcal{C}$ and R be as before. Although the results in §V.3.1 of Buan and Marsh yield the category $\Lambda_R-\mathsf{mod}$ as different localisations, they do not tell us explicitly how the irreducible morphisms and indecomposable objects of $\mathcal{C}$ behave under the localisation functors. In [BMR07] it was shown how to obtain the Auslander-Reiten quiver of $\Lambda_R-\mathsf{mod}$ from the Auslander-Reiten quiver of $\mathcal{C}$ when $R = T$ is maximal rigid. In this section, we investigate how much can be said about the irreducibility of a morphism $\mathrm{Hom}_{\mathcal{C}}(R, f)$ when $f\colon X \to Y$ is irreducible in $\mathcal{C}$ in the case where R is rigid and not necessarily maximal rigid. We approach this via two cases:

(a) when $X \in \mathcal{C}(R)$; and

(b) when $X \notin \mathcal{C}(R)$.

The case when $X \in \mathcal{C}(R)$

For an irreducible morphism $f\colon X \to Y$ in $\mathcal{C}$ with $X \in \mathcal{C}(R)$, we show below that $\mathrm{Hom}_{\mathcal{C}}(R, f)$ is irreducible if $Y \in \mathcal{C}(R)$, and that $\mathrm{Hom}_{\mathcal{C}}(R, f)$ is a section if $Y \notin \mathcal{C}(R)$.

Proposition V.3.22. *Let X, Y be objects in $\mathcal{C}$ with $X, Y \in \mathcal{C}(R)$ and* $\mathsf{add}\, X \cap \mathcal{X}_R = 0 = \mathsf{add}\, Y \cap \mathcal{X}_R$. *If $f\colon X \to Y$ is irreducible, then $\mathrm{Hom}_{\mathcal{C}}(R, f)\colon \mathrm{Hom}_{\mathcal{C}}(R, X) \to \mathrm{Hom}_{\mathcal{C}}(R, Y)$ is irreducible. Furthermore, if f factors only through $\mathcal{C}(R)$, then f is irreducible whenever $\mathrm{Hom}_{\mathcal{C}}(R, f)$ is irreducible.*

Proof. Suppose X, Y satisfy the conditions in the statement and that $f\colon X \to Y$ is an irreducible morphism in $\mathcal{C}$. Note that this implies f is an irreducible morphism of $\mathcal{C}(R)$ as $X, Y \in \mathcal{C}(R)$. Recall that the ideal $[\mathcal{X}_R]$ is an admissible ideal (see Example IV.3.11). Therefore, $\overline{f} \in \mathrm{Hom}_{\mathcal{C}(R)/[\mathcal{X}_R]}(X, Y)$ is irreducible by Proposition IV.3.13, since $\mathcal{C}(R)$ is Krull-Schmidt (see Corollary V.3.17), $\mathsf{add}\, X \cap \mathcal{X}_R = 0 = \mathsf{add}\, Y \cap \mathcal{X}_R$ and f is irreducible. By Theorem V.3.6 we then see that $\mathrm{Hom}_{\mathcal{C}}(R, f)$ is irreducible in $\Lambda_R - \mathsf{mod}$ (see also Remark II.3.13).

Now assume that f only factors through $\mathcal{C}(R)$ and that $\mathrm{Hom}_{\mathcal{C}}(R, f)$ is irreducible. We wish to show that $f\colon X \to Y$ is irreducible. Using Lemmas V.3.11 and V.3.12, we have that f is neither a section nor a retraction. Thus, suppose we have a factorisation $f = hg$ for some $g\colon X \to Z$ and $h\colon Z \to Y$. Since f only factors through $\mathcal{C}(R)$, we know that $Z \in \mathcal{C}(R)$. Applying $\mathrm{Hom}_{\mathcal{C}}(R, -)$ we see that $\mathrm{Hom}_{\mathcal{C}}(R, f) = \mathrm{Hom}_{\mathcal{C}}(R, h) \circ \mathrm{Hom}_{\mathcal{C}}(R, g)$, and so $\mathrm{Hom}_{\mathcal{C}}(R, h)$ is a retraction or $\mathrm{Hom}_{\mathcal{C}}(R, g)$ is a section as $\mathrm{Hom}_{\mathcal{C}}(R, f)$ is irreducible. In the first case, h is a retraction by Lemma V.3.12 as $Y \in \mathcal{C}(R)$ and $\mathsf{add}\, Y \cap \mathcal{X}_R = 0$, and in the second case we have that g is a section by Lemma V.3.11 since $Z \in \mathcal{C}(R)$ and $\mathsf{add}\, X \cap \mathcal{X}_R = 0$. Hence, f is irreducible and the proof is complete. ■

Note that for the next result we cannot appeal to the equivalence of $\mathcal{C}(R)/[\mathcal{X}_R] \simeq \Lambda_R - \mathsf{mod}$ as we did in Proposition V.3.22 above.

Proposition V.3.23. *If $X \in \mathcal{C}(R)$ and $Y \notin \mathcal{C}(R)$, then for any irreducible morphism $f\colon X \to Y$ we have that $\mathrm{Hom}_{\mathcal{C}}(R, f)$ is a section.*

Proof. By Lemma V.3.18 there exists $W \in \mathcal{C}(R)$ and $s\colon W \to Y$ in $\mathcal{S}$. Since $X \in \mathcal{C}(R)$, by Lemma V.3.19 we have that f factors through s so that there exists

$g\colon X \to W$ with $f = sg$. As f is irreducible, we must have that s is a retraction or g is a section. If s were a retraction then $Y \in \mathsf{add}\, W \subseteq \mathcal{C}(R)$ and so $Y \in \mathcal{C}(R)$, which is contrary to our assumptions. Thus, it must be the case that g is a section, and consequently $\mathrm{Hom}_{\mathcal{C}}(R, g)$ is also a section since $\mathrm{Hom}_{\mathcal{C}}(R, -)$ is a covariant functor. Since s is a morphism in $\mathcal{S}$, we know $\mathrm{Hom}_{\mathcal{C}}(R, s)$ is an isomorphism (by Lemma V.3.4), and hence $\mathrm{Hom}_{\mathcal{C}}(R, f) = \mathrm{Hom}_{\mathcal{C}}(R, sg) = \mathrm{Hom}_{\mathcal{C}}(R, s)\,\mathrm{Hom}_{\mathcal{C}}(R, g)$ is a section, as claimed. ∎

Note that it is not clear in general whether $\mathrm{Hom}_{\mathcal{C}}(R, f)$ must be either irreducible or split for an arrow $f\colon X \to Y$ in the Auslander-Reiten quiver of $\mathcal{C}$. The previous two results tell us this is the case if X belongs to $\mathcal{C}(R)$.

Corollary V.3.24. *If $f\colon X \to Y$ is an arrow in the Auslander-Reiten quiver of $\mathcal{C}$ with $X \in \mathcal{C}(R)$, then $\mathrm{Hom}_{\mathcal{C}}(R, f)$ is either irreducible or is split.*

Proof. Suppose $f\colon X \to Y$ is an arrow in the Auslander-Reiten quiver of $\mathcal{C}$. Then X and Y are indecomposable and f is an irreducible morphism. If X lies in $\mathcal{X}_R$, then $\mathrm{Hom}_{\mathcal{C}}(R, f)\colon 0 \to \mathrm{Hom}_{\mathcal{C}}(R, Y)$ is a section. Dually, if Y belongs to $\mathcal{X}_R$ then $\mathrm{Hom}_{\mathcal{C}}(R, f)$ is a retraction.

Thus, we may suppose $X, Y \notin \mathcal{X}_R$. If $Y \in \mathcal{C}(R)$, then $\mathrm{Hom}_{\mathcal{C}}(R, f)$ is irreducible by Proposition V.3.22; and if $Y \notin \mathcal{C}(R)$, then $\mathrm{Hom}_{\mathcal{C}}(R, f)$ splits by Proposition V.3.23. ∎

The case when $X \notin \mathcal{C}(R)$

For an irreducible morphism $f\colon X \to Y$ where $X \notin \mathcal{C}(R)$, we are able to determine the irreducibility of $\mathrm{Hom}_{\mathcal{C}}(R, f)$ in some special cases when $Y \in \mathcal{C}(R)$ (see Propositions V.3.29, V.3.30 and V.3.31). Our strategy is to use Lemma V.3.20 to produce a morphism $\widetilde{s}$, such that $f\widetilde{s}$ is irreducible in the subcategory $\mathcal{C}(R)$, and then use the equivalence $\mathcal{C}(R)/[\mathcal{X}_R] \simeq \Lambda_R-\mathsf{mod}$. For this we need the following lemma.

Lemma V.3.25. *Suppose $X \in \mathcal{C}$ and $Y \in \mathcal{C}(R)$ with $\operatorname{add} X \cap \mathcal{X}_R = 0 = \operatorname{add} Y \cap \mathcal{X}_R$. Let $\widetilde{s}\colon \widetilde{W} \to X$ be a morphism in the class $\widetilde{\mathcal{S}}$, where $\widetilde{W} \in \mathcal{C}(R)$ and $\operatorname{add} \widetilde{W} \cap \mathcal{X}_R = 0$. If $f\colon X \to Y$ is irreducible in $\mathcal{C}$ and $f\widetilde{s}\colon \widetilde{W} \to Y$ is irreducible in the subcategory $\mathcal{C}(R)$, then $\operatorname{Hom}_{\mathcal{C}}(R, f)\colon \operatorname{Hom}_{\mathcal{C}}(R, X) \to \operatorname{Hom}_{\mathcal{C}}(R, Y)$ is irreducible in $\Lambda_R-\mathsf{mod}$.*

Proof. Since $f\widetilde{s}\colon \widetilde{W} \to Y$ is an irreducible morphism in $\mathcal{C}(R)$ with $\operatorname{add} \widetilde{W} \cap \mathcal{X}_R = 0$ and $\operatorname{add} Y \cap \mathcal{X}_R = 0$, we may apply Proposition IV.3.13 to conclude that $\overline{f\widetilde{s}} \in \operatorname{Hom}_{\mathcal{C}(R)/[\mathcal{X}_R]}(\widetilde{W}, Y)$ is irreducible. By Theorem V.3.6 we see that $\operatorname{Hom}_{\mathcal{C}}(R, f\widetilde{s}) = \operatorname{Hom}_{\mathcal{C}}(R, f) \operatorname{Hom}_{\mathcal{C}}(R, \widetilde{s})$ is irreducible in $\Lambda_R-\mathsf{mod}$. As $\operatorname{Hom}_{\mathcal{C}}(R, \widetilde{s})$ is an isomorphism (by Lemma V.3.4), we have that $\operatorname{Hom}_{\mathcal{C}}(R, f)$ is irreducible by Lemma II.3.4. ■

The following example shows it is, unfortunately, not possible to prove in general that fs is irreducible in $\mathcal{C}(R)$ if $X \notin \mathcal{C}(R), Y \in \mathcal{C}(R)$, $f\colon X \to Y$ is irreducible in $\mathcal{C}$, $s\colon W \to X$ is a morphism in $\mathcal{S}$ and $W \in \mathcal{C}(R)$ (even with X, Y indecomposable).

Example V.3.26. Let Q be the quiver $1 \to 2 \to 3 \to 4$ and set $R := P_1 \oplus I_2 \oplus S_2$. Consider the following portion of the Auslander-Reiten quiver of $\mathcal{C} := \mathcal{C}_{kQ}$

$$\begin{tikzcd}
P_1 \arrow[dr, "g"] & & \\
& I_3 \arrow[dr, "f"] & \\
{\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}} \arrow[ur, "j"] \arrow[dr, "m"'] & & I_2 \\
& S_2 \arrow[ur, "n"'] &
\end{tikzcd}$$

in which $fj + nm = 0$. Setting $X := I_3$ and $Y := I_2$, we see that we have an irreducible morphism $f\colon X \to Y$. Furthermore, $Y \in \operatorname{add} R \subseteq \mathcal{C}(R)$, but

$X \notin \mathcal{C}(R)$ since there is a triangle

$$P_4 \longrightarrow P_1 \xrightarrow{r} X \longrightarrow P_4[1],$$

where r is a right $\mathsf{add}\, R$-approximation but $P_4 \notin \mathsf{add}\, R$ (see Lemma V.3.15).

Now consider the morphism $s\colon W := P_1 \oplus {\scriptstyle \frac{2}{3}} \to X$ given by $s := (\, g \;\; j\,)$. The completion of s to a triangle is then

$$P_2 \longrightarrow W \xrightarrow{s} X \xrightarrow{h} P_2[1].$$

We have that $W \in \mathcal{C}(R)$ as $P_1 \in \mathsf{add}\, R$ and ${\scriptstyle \frac{2}{3}} \in \mathsf{add}\, R[1]$. Furthermore, $P_2 \in \mathcal{X}_R$ whilst the morphism $h\colon X \to P_2[1]$ factors through $P_4[1] \in \mathcal{X}_R$, and so s is a morphism in the class $\mathcal{S} \subseteq \widetilde{\mathcal{S}}$.

Finally, we claim that the composition $fs\colon W \to Y$ is not an irreducible morphism in the full subcategory $\mathcal{C}(R)$. We have

$$\begin{aligned} fs &= f(\, g \;\; j\,) \\ &= (\, fg \;\; fj\,) \\ &= (\, fg \;\; -nm\,) \\ &= (\, fg \;\; -n\,)\begin{pmatrix} 1_{P_1} & 0 \\ 0 & m \end{pmatrix}, \end{aligned}$$

and this is a genuine factorisation in $\mathcal{C}(R)$ as $fg\colon P_1 \to Y, -n\colon S_2 \to Y, m\colon {\scriptstyle \frac{2}{3}} \to S_2$ and $P_1, Y, S_2, {\scriptstyle \frac{2}{3}} \in \mathcal{C}(R)$. Since $\mathcal{C}(R)$ is Krull-Schmidt, we see that $(\, fg \;\; -n\,)$ cannot be a retraction as $Y = I_2$ is not a direct summand of $P_1 \oplus S_2$, and $\begin{pmatrix} 1_{P_1} & 0 \\ 0 & m \end{pmatrix}$ cannot be a section as $P_1 \oplus {\scriptstyle \frac{2}{3}}$ is not a summand of $P_1 \oplus S_2$. Therefore, although fs itself is neither a section nor a retraction, fs fails (iii) of Definition II.3.3 and so is not irreducible in $\mathcal{C}(R)$.

Notice, however, that $f\widetilde{s}$ is irreducible in $\mathcal{C}(R)$, where $\widetilde{s} = g\colon \widetilde{W} = P_1 \to X$ is a morphism in $\widetilde{\mathcal{S}}$ and $\widetilde{W} \in \mathcal{C}(R) \setminus \mathcal{X}_R$; see Remark V.3.32.

In the series of technical propositions below that describe the irreducibility of $\operatorname{Hom}_{\mathcal{C}}(R, f)$ in certain cases, we make use of the following lemma.

Lemma V.3.27. *Let $f\colon X \to Y$ be an irreducible morphism in $\mathcal{C}$ with* $\mathsf{add}\, X \cap \mathcal{X}_R = 0$ *and $Y \in \mathcal{C}(R)$. For any morphism $\widetilde{s}\colon \widetilde{W} \to X$ is a morphism in the class $\widetilde{\mathcal{S}}$, the composition $f\widetilde{s}$ is neither a section nor a retraction.*

Proof. If $f\widetilde{s}$ were a retraction then this would imply f is a retraction, but this is not the case as f is irreducible. If $f\widetilde{s}$ were a section, then $\operatorname{Hom}_{\mathcal{C}}(R, f\widetilde{s}) = \operatorname{Hom}_{\mathcal{C}}(R, f) \circ \operatorname{Hom}_{\mathcal{C}}(R, \widetilde{s})$ would be a section. But this would imply $\operatorname{Hom}_{\mathcal{C}}(R, f)$ is a section as $\operatorname{Hom}_{\mathcal{C}}(R, \widetilde{s})$ is an isomorphism (see Lemma V.3.4). However, by Lemma V.3.11, we would then have that f is a section, which again cannot happen. ∎

For our first case, we need the following.

Lemma V.3.28. [Kra00, Lem. 2.4] *Let $f\colon X \to Y$ be a morphism in a preadditive category $\mathcal{C}$ such that $f \neq 0$. If $\operatorname{End}_{\mathcal{C}} X$ (respectively, $\operatorname{End}_{\mathcal{C}} Y$) is local, then f is right (respectively, left) minimal.*

Proposition V.3.29. *Let $X \notin \mathcal{C}(R)$ and $Y \in \mathcal{C}(R)$ with* $\mathsf{add}\, X \cap \mathcal{X}_R = 0 = \mathsf{add}\, Y \cap \mathcal{X}_R$*. Let $\widetilde{s}\colon \widetilde{W} \to X$ be a morphism in $\widetilde{\mathcal{S}}$ (with $\widetilde{W} \in \mathcal{C}(R)$ and* $\mathsf{add}\, \widetilde{W} \cap \mathcal{X}_R = 0$*) as obtained in* Lemma V.3.20*. Suppose $f\colon X \to Y$ is an irreducible morphism in $\mathcal{C}$. If $\overline{f}$ is right almost split in $\mathcal{C}/[\mathcal{X}_R]$ and $\operatorname{End}_{\mathcal{C}/[\mathcal{X}_R]} \widetilde{W}$ is local, then $\operatorname{Hom}_{\mathcal{C}}(R, f)$ is irreducible in Λ_R*$-\mathsf{mod}$.

Proof. We will show $f\widetilde{s}$ is irreducible in $\mathcal{C}(R)$ and apply Lemma V.3.25. First, $f\widetilde{s}$ is neither a section nor a retraction by Lemma V.3.27. Thus, it remains to show that $f\widetilde{s}$ admits no non-trivial factorisation in $\mathcal{C}(R)$.

Assume $f\widetilde{s} = ba$ for some $a\colon \widetilde{W} \to A, b\colon A \to Y$ where $A \in \mathcal{C}(R)$. Assume that b is not a retraction, then we must show a is a section. From Lemma V.3.12, we see that $\overline{b}\colon A \to Y$ is not a retraction in $\mathcal{C}/[\mathcal{X}_R]$ as $Y \in \mathcal{C}(R)$ and $\mathsf{add}\, Y \cap \mathcal{X}_R = 0$. Therefore, there exists $\overline{c}\colon A \to X$ such that $\overline{b} = \overline{fc}$ in $\mathcal{C}/[\mathcal{X}_R]$ because $\overline{f}$ is right almost split by assumption.

We recall for convenience how we obtain the morphism $\widetilde{s} \in \widetilde{\mathcal{S}}$ as in Lemma V.3.20. By Lemma V.3.18, there is a morphism $s\colon W \to X$ in $\mathcal{S}$ with $W \in \mathcal{C}(R)$, and we may write $W = \widetilde{W} \oplus W_0$ where $\mathsf{add}\,\widetilde{W} \cap \mathcal{X}_R = 0$ and $W_0 \in \mathcal{X}_R$. Thus, we may also express $s = (\,\widetilde{s}\ \ s_0\,)$ where $\widetilde{s}\colon \widetilde{W} \to X$ and $s_0\colon W_0 \to X$. In particular, $\overline{s_0} = 0$ and $\overline{s} = (\,\overline{\widetilde{s}}\ \ \overline{s_0}\,) = (\,\overline{\widetilde{s}}\ \ 0\,)$. The morphism c has domain A which lies in $\mathcal{C}(R)$, so c must factor through s by Lemma V.3.19. That is, there exists $d = \left(\begin{smallmatrix} \widetilde{d} \\ d_0 \end{smallmatrix}\right)\colon A \to W = \widetilde{W} \oplus W_0$ such that $sd = c$. Note that in $\mathcal{C}/[\mathcal{X}_R]$ we have $\overline{c} = \overline{sd} = \overline{\widetilde{s}\widetilde{d}} + \overline{s_0 d_0} = \overline{\widetilde{s}\widetilde{d}}$ as $\overline{s_0} = 0$. So we see that

$$\overline{f\widetilde{s}} = \overline{ba} = \overline{fca} = \overline{f\widetilde{s}\widetilde{d}a} = (\overline{f\widetilde{s}})(\overline{\widetilde{d}a})$$

in $\mathcal{C}/[\mathcal{X}_R]$.

We claim that $\overline{\widetilde{d}a}$ is an automorphism of $\widetilde{W}$. The morphism $\overline{s}$ is regular by Lemma V.3.4 since $s \in \widetilde{\mathcal{S}}$, so $\overline{\widetilde{s}}$ is also regular as $\overline{s} = (\,\overline{\widetilde{s}}\ \ 0\,)$. Note that X is a non-zero object of $\mathcal{C}$ as $X \notin \mathcal{C}(R)$. In addition, $\overline{f}$ is right almost split so $\operatorname{End}_{\mathcal{C}/[\mathcal{X}_R]} Y$ is local and it follows that $\operatorname{End}_{\mathcal{C}} Y$ is local by Lemma IV.3.4. Thus, Y is also non-zero object of $\mathcal{C}$.

Assume for contradiction that $\overline{f\widetilde{s}} = 0$, then we would have $\overline{f} = 0$ as $\overline{\widetilde{s}}$ is an epimorphism. This implies that f factors through $\mathcal{X}_R$ by Lemma V.3.1, say, $f = hg$ for some $g\colon X \to Z, h\colon Z \to Y$ with $Z \in \mathcal{X}_R$. But then h is retraction or g is section as f is irreducible, which would then imply that $0 \neq Y \in \mathsf{add}\, Z \subseteq \mathcal{X}_R$ or $0 \neq X \in \mathsf{add}\, Z \subseteq \mathcal{X}_R$, respectively. However, this gives us a contradiction as $\mathsf{add}\, X \cap \mathcal{X}_R = 0 = \mathsf{add}\, Y \cap \mathcal{X}_R$. Hence, $\overline{f\widetilde{s}} \neq 0$.

Thus, $\overline{f\widetilde{s}}\colon \widetilde{W} \to Y$ is a non-zero morphism where $\operatorname{End}_{\mathcal{C}/[\mathcal{X}_R]} \widetilde{W}$ is local. So, by Lemma V.3.28, we have that $\overline{f\widetilde{s}}$ is right minimal and $\overline{f\widetilde{s}\widetilde{d}a} = \overline{f\widetilde{s}}$ implies $\overline{\widetilde{d}a}$ is an automorphism of $\widetilde{W}$. Therefore, $\overline{a}$ is a section, and hence a is a section by Lemma V.3.11 as $A \in \mathcal{C}(R)$ and $\mathsf{add}\,\widetilde{W} \cap \mathcal{X}_R = 0$. This shows that $f\widetilde{s}$ is irreducible in $\mathcal{C}(R)$.

Finally, $\mathrm{Hom}_{\mathcal{C}}(R, f)\colon\ \mathrm{Hom}_{\mathcal{C}}(R, X) \to \mathrm{Hom}_{\mathcal{C}}(R, Y)$ is irreducible in $\Lambda_R -\mathsf{mod}$ by Lemma V.3.25 and this concludes the proof. ∎

Proposition V.3.30. *Let $X \notin \mathcal{C}(R)$ and $Y \in \mathcal{C}(R)$ with* $\mathsf{add}\,X \cap \mathcal{X}_R = 0 = \mathsf{add}\,Y \cap \mathcal{X}_R$. *Let $\widetilde{s}\colon \widetilde{W} \to X$ be a morphism in $\widetilde{\mathcal{S}}$ (with $\widetilde{W} \in \mathcal{C}(R)$ and* $\mathsf{add}\,\widetilde{W} \cap \mathcal{X}_R = 0$*) as obtained in* Lemma V.3.20. *Suppose $f\colon X \to Y$ is an irreducible morphism in $\mathcal{C}$. If $\overline{f}$ is right almost split and monic in $\mathcal{C}/[\mathcal{X}_R]$, then* $\mathrm{Hom}_{\mathcal{C}}(R, f)$ *is irreducible in* $\Lambda_R -\mathsf{mod}$.

Proof. As before, $f\widetilde{s}$ is neither a section nor a retraction by Lemma V.3.27. So, assume $f\widetilde{s} = ba$ for some $a\colon \widetilde{W} \to A, b\colon A \to Y$ where $A \in \mathcal{C}(R)$. Suppose b is not a retraction. Then, arguing the same way as in Proposition V.3.29, we obtain morphisms $\overline{c}\colon A \to X$, $d = \begin{pmatrix} \widetilde{d} \\ d_0 \end{pmatrix}\colon A \to W = \widetilde{W} \oplus W_0$ such that $\overline{b} = \overline{fc}$ and $sd = c$, where $s = (\,\widetilde{s}\ \ s_0\,)\colon \widetilde{W} \oplus W_0 \to X$ is in $\mathcal{S}$ and $W_0 \in \mathcal{X}_R$. Moreover, we again have $\overline{f\widetilde{s}} = \overline{ba} = \overline{f\widetilde{s}\widetilde{d}a}$ in $\mathcal{C}/[\mathcal{X}_R]$.

Since $\overline{f}$ is monic, $\overline{f\widetilde{s}} = \overline{f}(\overline{\widetilde{s}\widetilde{d}a})$ implies $\overline{\widetilde{s}} = \overline{\widetilde{s}\widetilde{d}a}$, which in turn yields $\overline{\widetilde{d}a} = 1_{\widetilde{W}}$ as $\overline{\widetilde{s}}$ is regular (and so monic). Thus, $\overline{a}$ and hence a are again sections, and $\mathrm{Hom}_{\mathcal{C}}(R, f)$ is irreducible. ∎

Note that in the next result we replace the condition that $\overline{f}$ is right almost split with a restriction on $\widetilde{s}$.

Proposition V.3.31. *Let $X \notin \mathcal{C}(R)$ and $Y \in \mathcal{C}(R)$ with* $\mathsf{add}\,X \cap \mathcal{X}_R = 0 = \mathsf{add}\,Y \cap \mathcal{X}_R$. *Let $\widetilde{s}\colon \widetilde{W} \to X$ be a morphism in $\widetilde{\mathcal{S}}$ (with $\widetilde{W} \in \mathcal{C}(R)$ and* $\mathsf{add}\,\widetilde{W} \cap \mathcal{X}_R = 0$*) as obtained in* Lemma V.3.20. *Suppose $f\colon X \to Y$ is an irreducible morphism in $\mathcal{C}$. If $\overline{\widetilde{s}}$ is left almost split in $\mathcal{C}/[\mathcal{X}_R]$, then* $\mathrm{Hom}_{\mathcal{C}}(R, f)$ *is irreducible in* $\Lambda_R -\mathsf{mod}$.

Proof. Again we will use Lemma V.3.25. By Lemma V.3.27, the morphism $f\widetilde{s}$ is neither a section nor a retraction. It remains to show that $f\widetilde{s}$ admits no non-trivial factorisation in $\mathcal{C}(R)$. Thus, assume we have a factorisation $f\widetilde{s} = ba$ for some $a\colon \widetilde{W} \to A, b\colon A \to Y$ where $A \in \mathcal{C}(R)$. Suppose that a is not a section, so that we may show b must be a retraction.

Consider $\overline{a}\colon \widetilde{W} \to A$ in $\mathcal{C}/[\mathcal{X}_R]$, which cannot be a section by Lemma V.3.11 as $\mathsf{add}\,\widetilde{W} \cap \mathcal{X}_R = 0$ and $A \in \mathcal{C}(R)$. As $\overline{a}\colon \widetilde{W} \to A$ is a non-section and $\overline{\widetilde{s}}\colon \widetilde{W} \to X$ is left almost split, there exists $\overline{c}\colon X \to A$ such that $\overline{a} = \overline{c\widetilde{s}}$. Consequently, we have that $\overline{f\widetilde{s}} = \overline{ba} = (\overline{bc})\overline{\widetilde{s}}$. By Lemma V.3.4, $\overline{\widetilde{s}}$ is regular (so epic), and hence $\overline{f} = \overline{bc}$. Thus, $f - bc\colon X \to Y$ factors through $\mathcal{X}_R$, say $f - bc = hg$ for some $g\colon X \to Z, h\colon Z \to Y$ with $Z \in \mathcal{X}_R$. Then $f = bc + hg = \left(\begin{smallmatrix} b & h \end{smallmatrix}\right)\left(\begin{smallmatrix} c \\ g \end{smallmatrix}\right)$ is a factorisation of the irreducible morphism f in $\mathcal{C}$. Thus, we must have that either $\left(\begin{smallmatrix} c \\ g \end{smallmatrix}\right)$ is a section or $\left(\begin{smallmatrix} b & h \end{smallmatrix}\right)$ is a retraction.

If $\left(\begin{smallmatrix} c \\ g \end{smallmatrix}\right)\colon X \to A \oplus Z$ is a section, then $\left(\begin{smallmatrix} \overline{c} \\ \overline{g} \end{smallmatrix}\right)\colon X \to A \oplus Z$ is a section. But $\overline{g} = 0$, so $\left(\begin{smallmatrix} \overline{c} \\ 0 \end{smallmatrix}\right)$ is a section yields $\overline{c}$ is a section. By Lemma V.3.11, this implies $c\colon X \to A$ is a section as $A \in \mathcal{C}(R)$ and $\mathsf{add}\,X \cap \mathcal{X}_R = 0$. Thus, $X \in \mathsf{add}\,A \subseteq \mathcal{C}(R)$ contradicting our assumption on X that it does not belong to $\mathcal{C}(R)$. Therefore, $\left(\begin{smallmatrix} c \\ g \end{smallmatrix}\right)\colon X \to A \oplus Z$ cannot be a section.

Hence, $\left(\begin{smallmatrix} b & h \end{smallmatrix}\right)$ must be a retraction. Then $\left(\begin{smallmatrix} \overline{b} & \overline{h} \end{smallmatrix}\right) = \left(\begin{smallmatrix} \overline{b} & 0 \end{smallmatrix}\right)$ is a retraction implies $\overline{b}\colon A \to Y$ is a retraction. From the assumptions $Y \in \mathcal{C}(R)$ and $\mathsf{add}\,Y \cap \mathcal{X}_R = 0$, we conclude b is a retraction by Lemma V.3.12. Therefore, we have shown that $f\widetilde{s}\colon \widetilde{W} \to Y$ is an irreducible morphism in $\mathcal{C}(R)$ and so, by Lemma V.3.25, $\mathrm{Hom}_{\mathcal{C}}(R, f)\colon \mathrm{Hom}_{\mathcal{C}}(R, X) \to \mathrm{Hom}_{\mathcal{C}}(R, Y)$ is irreducible in $\Lambda_R-\mathsf{mod}$. ■

Remark V.3.32. These previous three results determine that $\mathrm{Hom}_{\mathcal{C}}(R, f)$ is irreducible in certain situations by showing that $f\widetilde{s}$ is irreducible in the subcategory $\mathcal{C}(R)$ for an appropriate $\widetilde{s}$. In Example V.3.26 we gave an example of an irreducible morphism $f\colon X \to Y$ with $X \notin \mathcal{C}(R)$ and $Y \in \mathcal{C}(R)$, and such a morphism $\widetilde{s}$ with $f\widetilde{s}$ is irreducible in $\mathcal{C}(R)$. Thus, we see this phenomenon does occur. Although we can always find a morphism $\widetilde{s} \in \widetilde{\mathcal{S}}$ with codomain X (see Lemma V.3.20), it is not clear in general if there will be a morphism $\widetilde{s}$ with the property that $f\widetilde{s}$ is irreducible in $\mathcal{C}(R)$.

Our last result of this section shows that an irreducible morphism $f \colon X \to Y$ splits under $\mathrm{Hom}_{\mathcal{C}}(R, -)$ in a particular case when neither X nor Y are in $\mathcal{C}(R)$.

Proposition V.3.33. *Let X, Y be objects in $\mathcal{C}$ with $X, Y \notin \mathcal{C}(R)$. Suppose $f \colon X \to Y$ is an irreducible morphism in $\mathcal{C}$, which fits into a triangle $Z \xrightarrow{g} X \xrightarrow{f} Y \longrightarrow \Sigma Z$, where g is also irreducible. If $Z \in \mathsf{add}\, R$, then $\mathrm{Hom}_{\mathcal{C}}(R, f)$ is a retraction.*

Proof. Note that we have an exact sequence

$$\mathrm{Hom}_{\mathcal{C}}(R, Z) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(R,g)} \mathrm{Hom}_{\mathcal{C}}(R, X) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(R,f)} \mathrm{Hom}_{\mathcal{C}}(R, Y) \longrightarrow 0$$

since $\mathrm{Hom}_{\mathcal{C}}(R, -)$ is cohomological and $\Sigma Z \in \mathsf{add}\, \Sigma R \subseteq \mathcal{X}_R$. However, $g \colon Z \to X$ is irreducible with $Z \in \mathcal{C}(R)$ and $X \notin \mathcal{C}(R)$, so $\mathrm{Hom}_{\mathcal{C}}(R, g)$ is a section by Proposition V.3.23. Therefore, the exact sequence above is actually split by the Splitting Lemma (Proposition II.10.3). In particular, $\mathrm{Hom}_{\mathcal{C}}(R, f) \colon \mathrm{Hom}_{\mathcal{C}}(R, X) \to \mathrm{Hom}_{\mathcal{C}}(R, Y)$ is a retraction. ■

V.4 Examples

In this section we work through some examples (coming from cluster categories) in detail to illustrate applications of our results. For an object X and a morphism f in a cluster category $\mathcal{C}$, we will denote the corresponding object and morphism in a quotient $\mathcal{C}/\mathcal{I}$ by $\overline{X}$ and $\overline{f}$, respectively.

Example V.4.1. Let Q be the quiver $1 \to 2 \to 3 \to 4$ and let R be the rigid object $P_1 \oplus P_2 \oplus S_2$ of the cluster category $\mathcal{C} := \mathcal{C}_{kQ}$. Recall that the Auslander-Reiten quiver for $\mathcal{C}$ (with the mesh relations omitted) is

$$
\begin{tikzcd}[column sep=small]
 & & & P_1[1] \arrow[dr] & & P_1 = \begin{smallmatrix}1\\2\\3\\4\end{smallmatrix} \arrow[dr, "g"] & & P_4[1] \arrow[dr] & & & \\
 & & P_2[1] \arrow[ur] \arrow[dr] & & P_2 = \begin{smallmatrix}2\\3\\4\end{smallmatrix} \arrow[ur, "e"] \arrow[dr, "h"] & & I_3 = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[ur] \arrow[dr, "l"] & & P_3[1] \arrow[dr] & & \\
 & P_3[1] \arrow[ur] \arrow[dr, "r"] & & P_3 = \begin{smallmatrix}3\\4\end{smallmatrix} \arrow[ur] \arrow[dr] & & M = \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[ur, "j"] \arrow[dr, "m"] & & I_2 = \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[ur, "q"] \arrow[dr, "p"] & & P_2[1] \arrow[dr] & \\
P_4[1] \arrow[ur] & & P_4 = 4 \arrow[ur] & & S_3 = 3 \arrow[ur] & & S_2 = 2 \arrow[ur, "n"] & & S_1 = 1 \arrow[ur] & & P_1[1]
\end{tikzcd}
$$

where the lefthand copy of $P_i[1]$ is identified with the corresponding righthand copy (for $i = 1, 2, 3, 4$); see Example III.5.11.

In this case, $\mathsf{ind}(\mathcal{C}(R)) = \{P_1, P_2, S_2, I_2, S_1, P_3[1], P_1[1], P_2[1], S_3\}$ and $\mathsf{ind}\,\mathcal{X}_R = \{P_1[1], P_2[1], S_3, P_3, P_4[1]\}$. Recall from Example III.5.11 that the Auslander-Reiten quiver of the quasi-abelian category $\mathcal{C}/[\mathcal{X}_R]$, which arises as the heart of the twin cotorsion pair $((\mathsf{add}\,R[1], \mathcal{X}_R), (\mathcal{X}_R, \mathsf{add}\,R[1]))$, is

$$
\begin{tikzcd}
 & \overline{P_1} \arrow[dr, hook, two heads, "\overline{g}"] & & & & \overline{P_4} \\
\overline{P_2} \arrow[ur, hook, "\overline{e}"] \arrow[dr, hook, two heads, "\overline{h}"] & & \overline{I_3} \arrow[dr, hook, "\overline{l}"] & & \overline{P_3[1]} \arrow[ur, hook, two heads, "\overline{r}"] & \\
 & \overline{M} \arrow[ur, hook, "\overline{j}"] \arrow[dr, hook, "\overline{m}"] & & \overline{I_2} \arrow[ur, two heads, "\overline{q}"] \arrow[dr, two heads, "\overline{p}"] & & \\
 & & \overline{S_2} \arrow[ur, hook, "\overline{n}"] & & \overline{S_1} &
\end{tikzcd}
$$

where we have denoted monomorphisms by "$\hookrightarrow$" and epimorphisms by "$\twoheadrightarrow$", and again we have omitted the mesh relations. We describe how our results from §V.3.3 can be used to tell what happens to an irreducible morphism $X \to Y$ in $\mathcal{C}$ with X, Y indecomposable and not in $\mathcal{X}_R$. We use the labelling of morphisms as given above.

First, a quick application of Proposition V.3.22 tells us that

$\mathrm{Hom}_{\mathcal{C}}(R,e), \mathrm{Hom}_{\mathcal{C}}(R,n), \mathrm{Hom}_{\mathcal{C}}(R,p)$ and $\mathrm{Hom}_{\mathcal{C}}(R,q)$ are all irreducible as the domain and codomain both lie in $\mathcal{C}(R)$ for e, n, p, q.

Second, we have that $\mathrm{Hom}_{\mathcal{C}}(R,g), \mathrm{Hom}_{\mathcal{C}}(R,h)$ and $\mathrm{Hom}_{\mathcal{C}}(R,r)$ are sections by Proposition V.3.23, since the domain lies in $\mathcal{C}(R)$ but the codomain does not for g, h, r. In fact, we can say more about these morphisms by looking at the morphism $\widetilde{s}_X : \widetilde{W}_X \to X$ in $\widetilde{\mathcal{S}}$ with $\widetilde{W}_X \in \mathcal{C}(R)$ and $\mathsf{add}\, \widetilde{W}_X \cap \mathcal{X}_R = 0$ (as in Lemma V.3.20) for $X \in \{I_3, M, P_4\}$. Indeed, we have $\widetilde{s}_X$ is the morphism g, h, r, respectively, for $X = I_3, M, P_4$, respectively. Thus, $g, h, r \in \widetilde{\mathcal{S}}$ and $\mathrm{Hom}_{\mathcal{C}}(R,g), \mathrm{Hom}_{\mathcal{C}}(R,h)$ and $\mathrm{Hom}_{\mathcal{C}}(R,r)$ are in fact isomorphisms by Lemma V.3.4.

Since $jh = -ge$ in $\mathcal{C}$ and $\mathrm{Hom}_{\mathcal{C}}(R,g), \mathrm{Hom}_{\mathcal{C}}(R,h)$ are isomorphisms, we see that

$$\mathrm{Hom}_{\mathcal{C}}(R,j) = -\mathrm{Hom}_{\mathcal{C}}(R,g)\,\mathrm{Hom}_{\mathcal{C}}(R,e)\,\mathrm{Hom}_{\mathcal{C}}(R,h)^{-1}$$

is irreducible by Lemma II.3.4 using that $\mathrm{Hom}_{\mathcal{C}}(R,e)$ is irreducible.

Now we will explain how $m\colon M \to S_2$ meets the hypotheses of Proposition V.3.30 but $l\colon I_3 \to I_2$ does not. Note that the morphism $\widetilde{s}_X : \widetilde{W}_X \to X$ for $X = M$, (respectively, $X = I_3$) is $h\colon P_2 \to M$ (respectively, $g\colon P_1 \to I_3$).

The short exact sequence $0 \to S_3 \to M \xrightarrow{m} S_2 \to 0$ is an Auslander-Reiten sequence in $kQ\,\text{–}\,\mathsf{mod}$ so, by Theorem II.7.30 and Proposition II.8.8, the triangle $S_3 \to M \xrightarrow{m} S_2 \to S_3[1]$ is an Auslander-Reiten triangle in $\mathcal{C}$. In particular, m is minimal right almost split in $\mathcal{C}$ and hence $\overline{m}$ is also minimal right almost split in $\mathcal{C}/[\mathcal{X}_R]$ by [Liu10, Lem. 1.7] as $[\mathcal{X}_R]$ is an admissible ideal (see Definition IV.3.10). Furthermore, $\overline{m}$ is a monomorphism by [BM12, Lem. 3.3] as $S_3 \in \mathcal{X}_R$. Hence, we may apply Proposition V.3.30 to m and conclude that $\mathrm{Hom}_{\mathcal{C}}(R,m)$ is irreducible in $\Lambda_R\,\text{–}\,\mathsf{mod}$.

We claim that $\overline{l}$ is not right almost split. Indeed, the morphism $\overline{n}\colon \overline{S_2} \to \overline{I_2}$ in $\mathcal{C}/[\mathcal{X}_R]$ cannot be a retraction, because it is not an epimorphism (the triangle to check is $P_1[1] \to S_2 \xrightarrow{n} I_2 \xrightarrow{p} S_1$, where the morphism $\overline{p}\colon \overline{I_2} \to \overline{S_1}$ is non-zero in

$\mathcal{C}/[\mathcal{X}_R]$), and it does not factor through $\overline{l}$. Thus, we cannot use Proposition V.3.30 here. However, we can use Proposition V.3.31 as the morphism $\overline{\widetilde{s}} = \overline{g}$ is left almost split in $\mathcal{C}/[\mathcal{X}_R]$. This can be seen by observing that $P_1 \xrightarrow{g} I_3 \to P_4[1] \to P_1[1]$ is an Auslander-Reiten triangle in $\mathcal{C}$, so g must be minimal left almost split and hence $\overline{g}$ is minimal left almost split by [Liu10, Lem. 1.7]. Therefore, $\mathrm{Hom}_{\mathcal{C}}(R, l)$ is also irreducible in $\Lambda_R - \mathsf{mod}$.

Recall that in the proofs of Propositions V.3.30 and V.3.31, we actually proved if $f \colon X \to Y$ is irreducible (among some other conditions), then $f\widetilde{s}$ is irreducible in $\mathcal{C}(R)$ for an appropriate morphism $\widetilde{s}$. Thus, we see that the Auslander-Reiten quiver of the subfactor category $\mathcal{C}(R)/[\mathcal{X}_R]$ is

$$\begin{tikzcd}
 & \overline{P_1} \arrow[dr, "\overline{lg}"] & & \overline{P_3[1]} \\
\overline{P_2} \arrow[ur, "\overline{e}"] \arrow[dr, "\overline{mh}"'] & & \overline{I_2} \arrow[ur, "\overline{q}"'] \arrow[dr, "\overline{p}"] & \\
 & \overline{S_2} \arrow[ur, "\overline{n}"] & & \overline{S_1}
\end{tikzcd}$$

by results of Chapter IV and [Liu10], using that $\mathcal{C}(R)$ is Krull-Schmidt (see Corollary V.3.17) and that $[\mathcal{X}_R]$ is admissible (see Example IV.3.11). Therefore, the equivalence $\mathcal{C}(R)/[\mathcal{X}_R] \simeq \Lambda_R - \mathsf{mod}$ (see Theorem V.3.6) gives us that

$$\begin{tikzcd}
 & \mathrm{Hom}_{\mathcal{C}}(R, P_1) \arrow[dr, "\mathrm{Hom}_{\mathcal{C}}(R,lg)"] & & \mathrm{Hom}_{\mathcal{C}}(R, P_3[1]) \\
\mathrm{Hom}_{\mathcal{C}}(R, P_2) \arrow[ur, "\mathrm{Hom}_{\mathcal{C}}(R,e)"] \arrow[dr, "\mathrm{Hom}_{\mathcal{C}}(R,mh)"'] & & \mathrm{Hom}_{\mathcal{C}}(R, I_2) \arrow[ur, "\mathrm{Hom}_{\mathcal{C}}(R,q)"'] \arrow[dr, "\mathrm{Hom}_{\mathcal{C}}(R,p)"] & \\
 & \mathrm{Hom}_{\mathcal{C}}(R, S_2) \arrow[ur, "\mathrm{Hom}_{\mathcal{C}}(R,n)"] & & \mathrm{Hom}_{\mathcal{C}}(R, S_1)
\end{tikzcd}$$

is the Auslander-Reiten quiver of $\Lambda_R - \mathsf{mod}$.

We can see this is the correct Auslander-Reiten quiver in a different way as follows. There are precisely three irreducible morphisms in the Auslander-Reiten quiver of $\mathcal{C}$ that become regular morphisms (and stay irreducible by Proposition IV.3.13) in $\mathcal{C}/[\mathcal{X}_R]$. In particular, when we localise $\mathcal{C}/[\mathcal{X}_R]$ at its class of regular morphisms, these morphisms will become isomorphisms and the corresponding arrows in the Auslander-Reiten quiver will "collapse". (This is how we deduced the Auslander-Reiten quiver of $\Lambda_R-\mathsf{mod}$ in Example III.5.11.)

Lastly, we show with this example that the converse to Proposition V.3.22 does not hold in general. Consider the morphisms $g\colon P_1 \to I_3$ and $l\colon I_3 \to I_2$ in $\mathcal{C}$. We have seen above that $\mathrm{Hom}_{\mathcal{C}}(R,g)$ is an isomorphism and $\mathrm{Hom}_{\mathcal{C}}(R,l)$ is irreducible. Thus, we have that $\mathrm{Hom}_{\mathcal{C}}(R,lg)\colon \mathrm{Hom}_{\mathcal{C}}(R,P_1) \to \mathrm{Hom}_{\mathcal{C}}(R,I_2)$ is irreducible in $\Lambda_R-\mathsf{mod}$ by Lemma II.3.4. However, $lg\colon P_1 \to I_2$ is certainly not irreducible in $\mathcal{C}$ by Proposition II.3.9, because it is the composition of two irreducible morphisms between indecomposables (and so belongs to $\mathrm{rad}^2_{\mathcal{C}}(P_1,I_2)$).

The next example looks at the behaviour of a morphism $f\colon X \to Y$ under $\mathrm{Hom}_{\mathcal{C}}(R,-)$ when $X,Y \notin \mathcal{C}(R)$.

Example V.4.2. Again we let $\mathcal{C} = \mathcal{C}_{kQ}$ be the cluster category of the path algebra of the quiver $Q\colon \quad 1 \to 2 \to 3 \to 4$. However, now we fix $R' := P_1 \oplus P_2 \oplus S_3$ as our rigid object. One can check that the objects S_2, I_2, M do not lie in $\mathcal{C}(R')$ by taking minimal right $\mathsf{add}\,R'$-approximations and noting that $P_3 \notin \mathsf{add}\,R'$.

It is easy to see that $\Lambda_{R'}$ is isomorphic to the path algebra of the (non-connected) quiver

$$1' \to 2' \qquad 3'$$

with $\mathrm{Hom}_{\mathcal{C}}(R',S_2) \cong S_{2'}$, $\mathrm{Hom}_{\mathcal{C}}(R',I_2) \cong {}^{1'}_{2'}$ and $\mathrm{Hom}_{\mathcal{C}}(R',M) \cong S_{2'} \oplus S_{3'}$, where $S_{i'}$ is the simple representation at vertex i'.

Consider the morphisms $n\colon S_2 \to I_2$ and $m\colon M \to S_2$ (as in the Auslander-Reiten quiver of $\mathcal{C}$ in Example V.4.1). The morphism $\mathrm{Hom}_{\mathcal{C}}(R,n)$ corresponds to the irreducible inclusion morphism $\iota_{2'}\colon S_{2'} \hookrightarrow {}^{1'}_{2'}$, whereas $\mathrm{Hom}_{\mathcal{C}}(R, m)$ corresponds to the canonical projection morphism $S_{2'} \oplus S_{3'} \twoheadrightarrow S_{2'}$ and is thus a retraction (so split). Hence, we see that for a morphism $f\colon X \to Y$ of a cluster category $\mathcal{C}$, it is entirely possible that $\mathrm{Hom}_{\mathcal{C}}(R', f)$ remains irreducible or that it splits when X and Y are both not in the subcategory $\mathcal{C}(R')$.

In this same example, we may also observe that the morphism

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\Lambda_{R'}-\mathsf{mod}}(\mathrm{Hom}_{\mathcal{C}}(R', X), \mathrm{Hom}_{\mathcal{C}}(R', Y))$$

is not necessarily surjective if X does not lie in $\mathcal{C}(R')$ (even if $Y \in \mathcal{C}(R)$). For example, since $\mathrm{Hom}_{\mathcal{C}}(R', P_2) = S_{2'} \cong \mathrm{Hom}_{\mathcal{C}}(R', S_2)$, where the isomorphism is induced by the composition $P_2 \xrightarrow{h} M \xrightarrow{m} S_2$, there is an inverse isomorphism $\alpha\colon \mathrm{Hom}_{\mathcal{C}}(R', S_2) \to \mathrm{Hom}_{\mathcal{C}}(R', P_2)$ in $\Lambda_{R'}-\mathsf{mod}$. However, there is no non-zero morphism $S_2 \to P_2$ in $\mathcal{C}$, so there is no $f \in \mathrm{Hom}_{\mathcal{C}}(S_2, P_2)$ such that $\mathrm{Hom}_{\mathcal{C}}(R', f) = \alpha \neq 0$.

Remark V.4.3. Example V.4.2 above is also interesting for another reason. Consider the induced morphism $\mathrm{Hom}_{\mathcal{C}}(R', j) = \begin{pmatrix} \iota_{2'} & 0 \\ 0 & 1_{S_{3'}} \end{pmatrix}\colon \mathrm{Hom}_{\mathcal{C}}(R', M) \cong S_{2'} \oplus S_{3'} \to {}^{1'}_{2'} \oplus S_{3'} \cong \mathrm{Hom}_{\mathcal{C}}(R', I_3)$, where $\iota_{2'}\colon S_{2'} \hookrightarrow {}^{1'}_{2'} = P_{1'}$ is the inclusion morphism and $1_{S_{3'}}\colon S_{3'} \to S_{3'}$ is the identity. This induced morphism is an example of an irreducible morphism which is not radical. Indeed, $\mathrm{Hom}_{\mathcal{C}}(R', j)$ is neither a section nor a retraction as $\Lambda_{R'}-\mathsf{mod}$ is Krull-Schmidt (see Theorem II.4.4). Now suppose that $\mathrm{Hom}_{\mathcal{C}}(R', j) = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\begin{pmatrix} g_1 & g_2 \end{pmatrix}$ for some $\begin{pmatrix} g_1 & g_2 \end{pmatrix}\colon S_{2'} \oplus S_{3'} \to Z$, $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\colon Z \to P_{1'} \oplus S_{3'}$ where $Z \in \Lambda_{R'}-\mathsf{mod}$. Then

$$\begin{pmatrix} \iota_{2'} & 0 \\ 0 & 1_{S_{3'}} \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\begin{pmatrix} g_1 & g_2 \end{pmatrix} = \begin{pmatrix} h_1 g_1 & h_1 g_2 \\ h_2 g_1 & h_2 g_2 \end{pmatrix},$$

so $h_1 g_1 = \iota_{2'}, h_2 g_2 = 1_{S_{3'}}$, and $h_1 g_2 = 0 = h_2 g_1$. Since $\iota_{2'} = h_1 g_1$ is irreducible, it must be the case that either h_1 is a retraction or g_1 is a section. If $h_1\colon Z \to P_{1'}$ is a retraction then there is a section $s\colon P_{1'} \to Z$ such that $h_1 s = 1_{P_{1'}}$. Moreover,

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \begin{pmatrix} s & g_2 \end{pmatrix} = \begin{pmatrix} h_1 s & h_1 g_2 \\ h_2 s & h_2 g_2 \end{pmatrix} = \begin{pmatrix} 1_{P_{1'}} & 0 \\ 0 & 1_{S_{3'}} \end{pmatrix} = 1_{P_{1'} \oplus S_{3'}},$$

using that $h_2 s \in \mathrm{Hom}_{\Lambda_{R'}-\mathsf{mod}}(P_{1'}, S_{3'}) = 0$. That is, $\left(\begin{smallmatrix} h_1 \\ h_2 \end{smallmatrix}\right)$ is a retraction. Similarly, if g_1 is a section then this would imply $\begin{pmatrix} g_1 & g_2 \end{pmatrix}$ is a section also. Hence, $\mathrm{Hom}_{\mathcal{C}}(R', j)$ is irreducible. However, by noting that $1_{S_{3'}}$ is not radical (as the simple $\Lambda_{R'}$-module $S_{3'}$ is non-zero) and using Lemma II.3.8, we see that $\mathrm{Hom}_{\mathcal{C}}(R', j)$ cannot be radical.

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