Equivariant Gorenstein Duality

Dimitar Kodjabachev

A thesis submitted for the degree of
Doctor of Philosophy

School of Mathematics and Statistics
University of Sheffield

September 21, 2018

To my father

This thesis concerns the study of two flavours of duality that appear in stable homotopy theory and their equivariant reformulations. Concretely, we look at the Gorenstein duality framework introduced by Dwyer, Greenlees and Iyengar in [17] and the more classical notion of Anderson duality introduced by Anderson in [1]. We study examples of ring spectra that exhibit these duality phenomena, both non-equivariantly and equivariantly, coming from the ring spectra of topological modular forms. Along the way, connecting the work of [29], [52] and [35], we make contact with Serre duality phenomena that arise in derived algebraic geometry and record an unexpected interlace of Anderson, Gorenstein and Serre duality.

Acknowledgements

I would like to thank my supervisor, Professor John Greenlees, for accepting me as a student, holding my hand and patiently guiding me during my attempts to overcome my ignorance and see beyond it. It was a privilege to work under his supervision.

It would be an understatement to say that I have learned everything I know about topological modular forms from Lennart Meier. He has answered even the questions I did not know I should ask. Without his work on Anderson duality this thesis would have been much thinner.

A big thank you goes towards the crowd in the J14A/B tandem of offices in the School of Mathematics and Statistics at the University of Sheffield. The cyclic permutation of mathematics, football and pub crawls during my four years there kept me sane.

I would like to thank Luca Pol and Jordan Williamson for reading bits and pieces of this document and for helping me understand how little I know by constantly asking questions.

To Vladimir Mitankin I thank for carefully and pedantically reading a draft of this document and for being a true friend who always has my back.

Last but not least I would like to thank my family. Without their constant support none of this would have been possible.

Contents

Introduction

Equivariant Duality and Topological Modular Forms

The Gorenstein duality framework, introduced by Dwyer, Greenlees and Iyengar in [17], can be seen as a vast generalization of the notion of a Gorenstein ring from classical commutative algebra to the algebra of ring spectra. One of the main contributions of this framework is that it provides a unifying language which allows one to view a number of dualities in homotopy theory such as Poincaré duality for manifolds, Gorenstein duality for commutative rings, Benson-Carlson duality for cohomology rings of finite groups and Poincaré duality for groups as instances of a single phenomenon. Gorenstein duality was explored in a non-equivariant context in [17] and [29] and equivariantly in the case of $G = C_2$ in [27]. In this thesis we set up an equivariant reformulation of the framework which allows one to study Gorenstein duality for commutative ring G-spectra where G is an arbitrary finite group.

We also recall the more familiar notion of Anderson duality. Introduced in [1] by Anderson in his study of universal coefficient theorems for K-theory, this duality was explored non-equivariantly in [63] and [29] and equivariantly in [59], [32] and [35]. In [29], a peculiar relationship between Gorenstein duality for a connective ring spectrum and Anderson self-duality for its non-connective counterpart is recorded – Gorenstein duality implies automatically Anderson self-duality and the converse holds under some connectivity conditions.

A rich source of ring G-spectra which exhibit equivariant Anderson self-duality and Gorenstein duality are the ring spectra of topological modular forms with level n structure. These are commutative ring spectra that arise as the global sections of a certain sheaf $\mathcal{O}^{\mathsf{top}}$ on the moduli stacks of elliptic curves with level n structure $\mathcal{M}_1(n)$, or their Deligne-Mumford compactifications $\overline{\mathcal{M}}_1(n)$, and they come in three variants distinguished by the peculiar capitalization of their names.

$$\begin{aligned}
\mathrm{TMF}_1(n) &\overset{\mathsf{def}}{=} \mathcal{O}^{\mathsf{top}}(\mathcal{M}_1(n)),\\
\mathrm{Tmf}_1(n) &\overset{\mathsf{def}}{=} \mathcal{O}^{\mathsf{top}}(\overline{\mathcal{M}}_1(n)),\\
\mathrm{tmf}_1(n) &\overset{\mathsf{def}}{=} \mathrm{Tmf}_1(n)\langle 0\rangle.
\end{aligned}$$

In [52] Meier studies the non-equivariant Anderson self-duality picture for the non-connective ring spectra $\mathrm{Tmf}_1(n)$. He shows that $\mathrm{Tmf}_1(n)$ is Anderson self-dual if and only if the moduli stack $\overline{\mathcal{M}}_1(n)$ has Serre duality with a dualizing sheaf of a very specific form and establishes a link between duality phenomena in stable homotopy theory and derived algebraic geometry.

We look at the complementing story of Gorenstein duality for the connective ring spectra $\mathrm{tmf}_1(n)$. These can be constructed as genuine ring $(\mathbb{Z}/n)^\times$-spectra and are thus natural candidates for equivariant Gorenstein duality, so we ask the following question.

Question Does $\mathrm{tmf}_1(n)$ exhibit *equivariant* Gorenstein duality?

As a first approximation we look at the non-equivariant picture where we have a complete answer given by the following theorem.

Theorem A. *The ring spectrum* $\mathrm{tmf}_1(n)$ *has non-equivariant Gorenstein duality if and only if* $n \in \{1, \ldots, 8, 11, 14, 15\}$ *or potentially* $n = 23$ *with non-equivariant Gorenstein duality shifts* a *as follows.*

n	1	2	3	4	5	6	7	8	11	14	15	23
a	-22	-14	-10	-8	-6	-6	-4	-4	-2	-2	-2	0

Equivariantly much less is known and computations are considerably more difficult. Equivariant Gorenstein duality implies non-equivariant Gorenstein duality and therefore the list of ring spectra that *could* exhibit the duality equivariantly is the one from Theorem A. Levels 1 and 2 are not interesting equivariantly as the group acting is the trivial group. Topological modular forms with level 3 structure were studied equivariantly by Hill and Meier in [35] and by Greenlees and Meier in [27]. This leaves levels 4 and 6 as the natural candidates to look at next and in this thesis we obtain the following results.

Theorem B. *The ring* C_2*-spectrum* $\mathrm{tmf}_1(4)$ *is Gorenstein of shift* $\alpha = -5 - 3\sigma$*.*

Conjecture C. $\mathrm{tmf}_1(4)$ *has equivariant Gorenstein duality with shift* $\alpha = -7 - \sigma$*.*

Theorem D. *The ring* C_2*-spectrum* $\mathrm{tmf}_1(6)$ *is Gorenstein of shift* $\alpha = -4 - 2\sigma$*.*

Conjecture E. $\mathrm{tmf}_1(6)$ *has equivariant Gorenstein duality with shift* $\alpha = -6$*.*

Overview

We now give a bird's eye view of the structure of the thesis.

Chapter 1 collects the necessary background material on equivariant stable homotopy theory. We set up a black box version of the category of ring G-spectra using the language of ∞-categories and describe a number of computational techniques available for recovering the $RO(G)$-graded homotopy type of a G-spectrum such as the homotopy fixed point spectral sequence, the Tate square and a strategy specific to the case of $G = C_4$ called the iterated Tate argument.

Chapter 2 contains a blitz introduction to the language of stacks along with a description of the moduli stacks of elliptic curves with level structure and the moduli stack of formal groups which make an appearance later in the document.

Chapter 3 contains the equivariant reformulations of the Gorenstein duality framework and makes up the heart of this thesis.

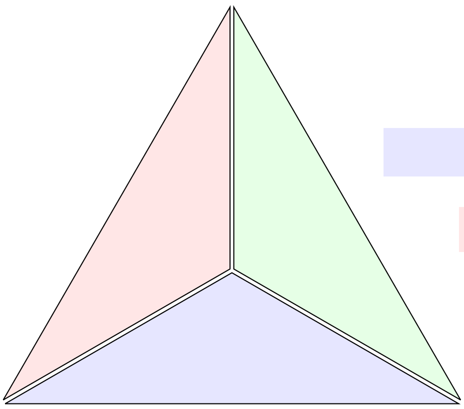
Chapter 4 recalls the classical definition of modular forms in number theory and describes two methods that can be used to identify the $(\mathbb{Z}/n)^\times$-action on the graded ring of modular forms with respect to the congruence subgroup $\Gamma_1(n)$, the Tate normal form and the Eisenstein methods.

Chapter 5 introduces the main objects of study, the ring spectra of topological modular forms with level structure. We sketch the definitions and collect the relevant properties with emphasis on the equivariant structure of the objects.

Chapter 6 contains the calculations of two toys examples which potentially exhibit Gorenstein duality in an equivariant sense, namely the ring spectra of topological modular forms with level 4 and level 6 structure leading to Theorem B and D and Conjectures C and E above.

Chapter 7 concludes with an outline of the future directions. We record a couple of questions which were partially explored in the thesis and report on their current progress.

Finally, in Appendix A, we provide a summary of the duality picture for the ring spectra of topological modular forms.



$$\mathrm{Cell}_{\mathbf{k}}(\mathbf{r}) \simeq \Sigma^a k^{\mathbf{r}}$$

$$k^{\mathbf{R}} \simeq \Sigma^b \mathbf{R}$$

$$H^0(\mathcal{X}, \mathcal{F}) \cong H^1(\mathcal{X}, \mathcal{F}^\vee \otimes w^{\otimes -m})^\vee$$

Chapter 1

Background

In this chapter we collect the necessary background material on equivariant stable homotopy theory. Along the way we introduce a number of conventions and fix the notation that will be in force throughout the rest of the document.

We begin with an overview of the theory of arithmetic fracture squares in the context of a presentable, stable, symmetric monoidal ∞-category in Section 1.2. In Section 1.3 we introduce the category of G-spectra as the symmetric monoidal ∞-category associated to the symmetric monoidal model category of orthogonal G-spectra. Mackey functors and the various fixed points functors of G-spectra are recalled in Section 1.4 and Section 1.5, respectively followed by a definition of strongly even G-spectra and a discussion on real Real orientations in Section 1.6. In the last three sections of this chapter we describe several computational techniques that are available in the category of G-spectra. We look at the homotopy fixed points spectral sequence in Section 1.7 and the Tate square in Section 1.8. Finally, in Section 1.9 we introduce the iterated Tate argument, a strategy for recovering the $RO(C_4)$-graded homotopy type of a connective ring C_4-spectrum where C_4 is the cyclic group of order 4.

1.1. Conventions and notation

Throughout the document G will *always* denote a finite group. The theory of G-spectra can be developed for an arbitrary compact Lie group and in the setting of G-spaces things work equally well for any topological group, but we will not need this level of generality. In all of the examples we encounter G will be finite.

We write Top_G for the category with objects all compactly generated weak Hausdorff topological spaces equipped with a G-action and with morphisms all G-equivariant maps. The category of *pointed* compactly generated weak Hausdorff topological spaces equipped with a G-action (fixing the basepoint) and basepoint preserving G-equivariant maps will be denoted by $\mathrm{Top}_{G,*}$. If we want to convert an unpointed G-space X into a pointed one we take the disjoint union of X and a G-fixed basepoint and denote the result by X_+.

We will use the theory of ∞-categories as developed in [44] and the theory of symmetric monoidal ∞-categories and rings and modules in them developed in [43] as a black box and simply collect the necessary results referring to [44] and [43] for the full details. Some of the key points for choosing the framework of symmetric monoidal ∞-categories are the following. Suppose $\mathcal{C}$ is a symmetric monoidal ∞-category.

- The homotopy category of $\mathcal{C}$ is an ordinary symmetric monoidal category, see [43, Remark 2.1.2.20] and [30, Remark 4.28].
- One can make sense of strictly commutative ring objects in $\mathcal{C}$ and their associated theory, see [43, Definition 2.1.3.1 and Chapter 3].
- Given a strictly commutative ring object R in $\mathcal{C}$ one can construct the category of modules over R which is again a symmetric monoidal category, see [43, Section 3.3 and Section 3.4].

1.2. Arithmetic fracture squares

We start by recalling the bare bones of the theory of arithmetic fracture squares following [48, Part 1]. We will apply these constructions to the category of G-spectra introduced in Section 1.3.9 below and to the category of modules over a ring G-spectrum in Section 3.2 of Chapter 3. We work under the following hypotheses.

Hypotheses 1.2.1. Let $(\mathcal{C}, \otimes, 1)$ be a presentable, symmetric monoidal, stable ∞-category such that the monoidal product commutes with arbitrary colimits in each variable. Let A be an associative algebra object in $\mathcal{C}$ and suppose that

1. the unit 1 is a compact object,
2. $\mathcal{C}$ is generated as a localizing subcategory by a set of dualizable objects,
3. A is dualizable.

We identify three full subcategories of $\mathcal{C}$, the subcategories of A-null, A-complete and A-cellular objects, and a certain cartesian square which allows one to assemble any object in $\mathcal{C}$ from its A-complete and A-null parts.

Definition 1.2.2. An object X of $\mathcal{C}$ is called

(i) A-*null* if $\mathrm{Hom}_{\mathcal{C}}(A, X) \simeq *$,

(ii) A-*complete* if for every A-null Y we have $\mathrm{Hom}_{\mathcal{C}}(Y, X) \simeq *$.

We write $\mathcal{C}^{A\text{-null}}$ and $\mathcal{C}^{A\text{-comp}}$ for the full subcategories of $\mathcal{C}$ spanned by the A-null and A-complete objects, respectively.

Definition 1.2.3. A morphism $X \to Y$ in $\mathcal{C}$ is called an A-*equivalence* if the induced map $\mathrm{Hom}_{\mathcal{C}}(A, X) \to \mathrm{Hom}_{\mathcal{C}}(A, Y)$ is an equivalence.

Definition 1.2.4. An object X of $\mathcal{C}$ is called *A-cellular* if for every A-null Y we have $\mathrm{Hom}_{\mathcal{C}}(X,Y) \simeq *$. Equivalently, X is A-cellular, if for every A-equivalence $Y \to Z$ the induced map $\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}}(X,Z)$ is an equivalence. We write $\mathcal{C}^{A\text{-cell}}$ for the full subcategory of $\mathcal{C}$ spanned by the A-cellular objects.

Our interest in these subcategories of $\mathcal{C}$ is motivated by the following results.

Lemma 1.2.5. *There exist functors denoted* Null_A, $(-)^\wedge_A$ *and* Cell_A, *called* A-nullification, A-completion *and* A-cellularization, *fitting in a diagram of adjunctions*

$$\begin{array}{ccccc} & & \mathcal{C}^{A\text{-comp}} & & \\ & & (-)^\wedge_A \uparrow\downarrow & & \\ \mathcal{C}^{A\text{-cell}} & \underset{\longrightarrow}{\overset{\mathrm{Cell}_A}{\longleftarrow}} & \mathcal{C} & \underset{\longleftarrow}{\overset{\mathrm{Null}_A}{\longrightarrow}} & \mathcal{C}^{A\text{-null}}, \end{array}$$

where the unlabelled functors are the relevant inclusions and the labelled functors are their left or right adjoints following the convention that the left adjoint is written on the left or on the top of the right adjoint.

Proof. See the discussion before [48, Definition 2.19] as well as [48, Definition 3.10] and [48, Construction 3.2]. □

Corollary 1.2.6. *For every object X in $\mathcal{C}$ we have a cofibre sequence*

$$\mathrm{Cell}_A(X) \to X \to \mathrm{Null}_A(X). \tag{1.1}$$

□

Lemma 1.2.7. *For every object X in $\mathcal{C}$ there exits a cartesian square*

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{Null}_A(X) \\ \downarrow & \lrcorner & \downarrow \\ (X)^\wedge_A & \longrightarrow & \mathrm{Null}_A((X)^\wedge_A), \end{array} \tag{1.2}$$

called the A-arithmetic fracture square *of X.*

Proof. See [48, Construction 3.16, Proposition 3.18]. □

1.3. The category of G-spectra

We continue with a minimalistic, ∞-categorical introduction to the category of G-spectra following [48, Section 5]. In our exposition we intentionally try to be as model agnostic as possible and concentrate on the characteristic features of the objects and constructions.

When we have to resort to a specific model of G-spectra we will use the category of orthogonal G-spectra of Mandell and May [46] which we recall below.

We start with a brief discussion on how one can associate a (symmetric monoidal) ∞-category to any (symmetric monoidal) model category. Let $\mathcal{M}$ be a model category and let $\mathcal{M}^c$ denote the full subcategory of $\mathcal{M}$ spanned by the cofibrant objects. Write $\mathrm{N}\colon \mathrm{Cat} \to \mathrm{sSet}$ for the ordinary nerve functor.

Definition 1.3.1 ([43, Definition 1.3.4.15])**.** Define the *underlying ∞-category* of the model category $\mathcal{M}$ to be the ∞-category $\mathrm{N}(\mathcal{M}^c)[\mathcal{W}^{-1}]$, where on the left we mean localization in the ∞-categorical sense.

Example 1.3.2. *The category Top_G can be equipped with a model structure where a map $X \to Y$ is a weak equivalence (respectively a fibration) if and only if $X^H \to Y^H$ is a weak equivalence (respectively a Serre fibration) of spaces for each subgroup $H \leq G$. Define the* ∞-category of G-spaces S_G *as the underlying ∞-category of Top_G.*

Suppose now that $(\mathcal{M}, \otimes, \mathbf{1})$ is a symmetric monoidal model category, i.e. a closed symmetric monoidal category equipped with a model structure such that the pushout-product axiom is satisfied. The subcategory $\mathcal{M}^c$ inherits a symmetric monoidal structure and moreover, since the class of weak equivalences $\mathcal{W}$ in $\mathcal{M}^c$ is stable under tensoring with objects of $\mathcal{M}^c$, the underlying ∞-category $\mathrm{N}(\mathcal{M}^c)[\mathcal{W}^{-1}]$ of $\mathcal{M}$ inherits a symmetric monoidal structure.

Definition 1.3.3 ([43, Example 4.1.7.6])**.** Define the *underlying symmetric monoidal ∞-category* of the symmetric monoidal model category $\mathcal{M}$ to be the symmetric monoidal ∞-category $\mathrm{N}(\mathcal{M}^c)[\mathcal{W}^{-1}]$.

Note that this construction is functorial. If $(\mathcal{M}, \otimes, \mathbf{1}_{\mathcal{M}})$ and $(\mathcal{N}, \otimes, \mathbf{1}_{\mathcal{N}})$ are symmetric monoidal model categories and $F\colon \mathcal{M} \to \mathcal{N}$ is a symmetric monoidal left Quillen functor, then by the universal properties of localization F induces a symmetric monoidal functor between the underlying symmetric monoidal ∞-categories of $\mathcal{M}$ and $\mathcal{N}$.

Example 1.3.4. *The category $\mathrm{Top}_{G,*}$ is a symmetric monoidal model category with a model structure defined by detecting weak equivalences and fibrations on fixed points for subgroups $H \leq G$ and monoidal product given by the smash product of pointed G-spaces. Define the* symmetric monoidal ∞-category of pointed G-spaces $\mathrm{S}_{G,*}$ *as the underlying symmetric monoidal ∞-category of $\mathrm{Top}_{G,*}$.*

To define the category of G-spectra as an ∞-category we look at one particular classical model of G-spectra, namely the category of orthogonal G-spectra of Mandell and May [46]. This is a closed symmetric monoidal category that can be equipped with a model structure, called the *stable* model structure, making it into a closed symmetric monoidal model category. We give a short overview of the model following [33, Appendix A.2].

Write $\underline{\mathrm{Top}}_{G,*}$ for the category with objects G-spaces and morphisms all continuous, but not necessarily equivariant, maps. This is a closed symmetric monoidal category enriched in $\mathrm{Top}_{G,*}$. For G-spaces X and Y the space of morphisms from X to Y in $\underline{\mathrm{Top}}_{G,*}$ is a G-space with G acting by conjugation and we have the relation

$$\mathrm{Hom}_{\underline{\mathrm{Top}}_{G,*}}(X,Y)^G \simeq \mathrm{Hom}_{\mathrm{Top}_{G,*}}(X,Y).$$

Let V and W be real orthogonal representations of G. We write $O(V)$ for the orthogonal group of non-equivariant linear isometries of V and $O(V,W)$ for the Stiefel manifold of linear isometric embeddings of V into W.

Definition 1.3.5 ([33, Definition A.10])**.** Let I_G be the category where

- objects are all finite dimensional real orthogonal representations of G, and
- morphisms are given by the Thom space of the complementary bundle $W - V$ over $O(V,W)$

$$I_G(V,W) \overset{\text{def}}{=} \mathrm{Thom}(O(V,W); W - V).$$

Definition 1.3.6 ([33, Definition A.13])**.** An *orthogonal G-spectrum* X is a functor

$$X : I_G \longrightarrow \underline{\mathrm{Top}}_{G,*}.$$

We write $\mathrm{Sp}_G^{\text{orth}}$ for the category of orthogonal G-spectra. If we unpack the definitions we see that an orthogonal G-spectrum X consists of a collection of G-spaces $\{X_V\}$ for each finite dimensional real orthogonal representation V of G and for each $V \to W$ an equivariant map $S^{W-V} \wedge X_V \to X_W$ compatible with the composition in I_G and varying continuously in $V \to W$.

Definition 1.3.7 ([33, Definition 2.14])**.** Let X be an orthogonal G-spectrum, k be an integer and H be a subgroup of G. The *H-equivariant k-th homotopy group* of X is the group

$$\pi_k^H(X) \overset{\text{def}}{=} \mathrm{colim}_{V>-k}\, \pi_{V+k}^H(X_V),$$

where the colimit is taken over the partially ordered set of real orthogonal representations V of G such that $V > -k$. The partially ordered set of representations of G is a class and one must check that this colimit exists. For details see the discussion after [33, Definition 2.16].

Definition 1.3.8 ([33, Definition 2.17])**.** A map of orthogonal G-spectra is called a *stable weak equivalence* if it induces an isomorphism of homotopy groups π_k^H for all $k \in \mathbb{Z}$ and all $H \leq G$.

The stable weak equivalences participate as the weak equivalence in a model structure on $\mathrm{Sp}_G^{\text{orth}}$ called the stable model structure defined in [46, Chapter III]. As mentioned earlier, the category of orthogonal G-spectra equipped with this model is a closed symmetric monoidal model category.

We can now define the ∞-category of G-spectra.

Definition 1.3.9 ([48, Definition 5.10])**.** The *symmetric monoidal ∞-category of G-spectra* is the underlying symmetric monoidal ∞-category of $\mathrm{Sp}_G^{\mathrm{orth}}$ which we denote by Sp_G. We write $\wedge$ for the monoidal product and S^0 for the unit (the sphere spectrum). Given G-spectra X and Y we write $F(X, Y)$ for the internal Hom object or function spectrum in Sp_G.

Proposition 1.3.10 ([48, Proposition 5.14])**.** *Let $H \leq G$ be a subgroup. There is a symmetric monoidal colimit preserving functor called* restriction

$$\mathrm{res}_H^G \colon \mathrm{Sp}_G \longrightarrow \mathrm{Sp}_H$$

which admits a left adjoint ind_H^G, called induction*, and a right adjoint coind_H^G called* coinduction*. Moreover, there is a natural equivalence*

$$\mathrm{ind}_H^G \simeq \mathrm{coind}_H^G .$$

1.4. Mackey functors

The homotopy groups of a G-spectrum exhibit a structure much richer than that of an abelian group, namely they form a Mackey functor. We recall this notion below by looking and comparing two definitions of Mackey functors.

Definition 1.4.1. The Burnside category of G is defined as the full subcategory $\widetilde{\mathcal{B}}_G$ of Sp_G spanned by the objects $\Sigma_+^\infty X$ for X a finite G-set.

An equivalent and more concrete description of the Burnside category can be given in terms of the category of spans of the category $\mathrm{Set}_G^{\mathrm{fin}}$ of finite G-sets and G-equivariant maps.

Definition 1.4.2. Let $\mathcal{C}$ be a category with all finite limits and colimits. Define $\mathrm{Span}(\mathcal{C})$, the *category of spans* of $\mathcal{C}$, to be the category with objects the objects of $\mathcal{C}$ and morphisms equivalence classes of diagrams of the form $X \leftarrow A \rightarrow Y$ called spans where two spans $X \leftarrow A \rightarrow Y$ and $X \leftarrow B \rightarrow Y$ are equivalent if there is an isomorphism $A \xrightarrow{\cong} B$ such that the diagram

$$\begin{array}{ccc} & A & \\ \swarrow & \downarrow \cong & \searrow \\ X & & Y \\ \nwarrow & & \nearrow \\ & B & \end{array}$$

commutes. Composition is defined by taking pullback, given spans $X \leftarrow A \rightarrow Y$ and

$Y \leftarrow B \rightarrow Z$ their composition is

$$\begin{array}{ccccc} & & A \times_Y B & & \\ & \swarrow & & \searrow & \\ & A & & B & \\ \swarrow & & \searrow \quad \swarrow & & \searrow \\ X & & Y & & Z. \end{array}$$

Observe that $\mathrm{Hom}_{\mathrm{Span}(\mathcal{C})}(X, Y)$ is an abelian monoid under the operation which takes spans $X \leftarrow A \rightarrow Y$ and $X \leftarrow B \rightarrow Y$ to the span $X \leftarrow A \coprod B \rightarrow Y$. To turn $\mathrm{Span}(\mathcal{C})$ into an additive category we let $\mathrm{Span}^+(\mathcal{C})$ denote the preadditive completion of $\mathrm{Span}(\mathcal{C})$, i.e.

$$\mathrm{Hom}_{\mathrm{Span}^+(\mathcal{C})}(X, Y) \stackrel{\mathsf{def}}{=} K(\mathrm{Hom}_{\mathrm{Span}(\mathcal{C})}(X, Y))$$

where $K(-)$ on the right denotes the Grothendieck group completion.

Definition 1.4.3. Define the *Burnside category* of G as

$$\mathcal{B}_G \stackrel{\mathsf{def}}{=} \mathrm{Span}^+(\mathrm{Set}_G^{\mathsf{fin}}).$$

It is a classical statement that Definition 1.4.1 and Definition 1.4.3 are equivalent.

Proposition 1.4.4. *Let $\widetilde{\mathcal{B}}_G$ and $\mathcal{B}_G$ be as above. Then the categories $\pi_0(\widetilde{\mathcal{B}}_G)$ and $\mathcal{B}_G$ are equivalent.*

Proof. See [50, Chapter XIX, Section 3]. □

We are now ready to give the first definition of a Mackey functor.

Definition 1.4.5. A *Mackey functor* is an additive contravariant functor $\mathcal{B}_G^{\mathsf{op}} \rightarrow \mathrm{Ab}$ from the Burnside category of G to the category of abelian groups.

We write Mack_G for the category of Mackey functors and natural transformations between them. Individual Mackey functors will be denoted by underlined letters like $\underline{M}$. From the definition above it is clear that the category Mack_G is an abelian category.

Example 1.4.6. *As a first simple but important example of a Mackey functor we look at the* Burnside Mackey functor $\mathcal{A}$ *defined as the functor represented by the G-orbit G/G, i.e.*

$$\mathcal{A}(-) = \mathrm{Hom}_{\mathcal{B}_G}(-, G/G).$$

Its value on an orbit G/H can be identified with the classical Burnside ring $A(H)$ of H.

We can equip the category Mack_G with a closed symmetric monoidal structure as follows.

Definition 1.4.7. Let $\underline{M}$ and $\underline{N}$ be Mackey functors and define their *box product*, denoted $\underline{M}\Box\underline{N}$, by the left Kan extension

$$\begin{array}{ccccc} \mathcal{B}_G \times \mathcal{B}_G & \xrightarrow{\underline{M}\times\underline{N}} & \mathrm{Ab}\times\mathrm{Ab} & \xrightarrow{-\otimes -} & \mathrm{Ab}\,. \\ \big\downarrow{\scriptstyle -\times -} & & & \nearrow_{\underline{M}\Box\underline{N}} & \\ \mathcal{B}_G & & & & \end{array}$$

Given a Mackey functor $\underline{M}$ the functor $-\Box\underline{M}$ has a right adjoint $\mathrm{Hom}_{\mathrm{Mack}_G}(-,\underline{M})$ which is the internal Hom object in Mack_G.

Theorem 1.4.8. *The category* Mack_G *is a closed symmetric monoidal category with product given by* $\Box$*, unit the Burnside Mackey functor and internal Hom object as above.*

A more algebraic description of Mackey functors is given by Dress in [15]. Recall that the *orbit category* of G is the full subcategory $\mathcal{O}_G$ of Top_G on the objects G/H.

Definition 1.4.9. A *Mackey functor* is a pair of additive functors $(\underline{M}_*, \underline{M}^*)$ from the orbit category of G to the category of abelian groups. The functor $\underline{M}_*$ is covariant, $\underline{M}^*$ is contravariant and they have the same object function which we denote by $\underline{M}$. The two functors take pullback diagrams in $\mathcal{O}_G$ of the form

$$\begin{array}{ccc} P & \xrightarrow{\delta} & X \\ {\scriptstyle\gamma}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle\alpha} \\ Y & \xrightarrow{\beta} & Z \end{array}$$

to commutative squares

$$\begin{array}{ccc} \underline{M}(P) & \xrightarrow{\underline{M}_*(\delta)} & \underline{M}(X) \\ {\scriptstyle \underline{M}^*(\gamma)}\big\uparrow & & \big\uparrow{\scriptstyle \underline{M}^*(\alpha)} \\ \underline{M}(Y) & \xrightarrow{\underline{M}_*(\beta)} & \underline{M}(Z). \end{array}$$

Remark. Because any finite G-set can be written as a disjoint union of orbits of the form G/H and Mackey functors are additive, a Mackey functor $\underline{M}$ is completely determined by its values $\underline{M}(G/H)$ on orbits.

To translate between the two definitions, suppose we are given a Mackey functor $\underline{M}$ in terms of Definition 1.4.5. We can define a pair of functors, one covariant and one contravariant, from the orbit category to the Burnside category by restricting to spans of the form $X \xleftarrow{=} X \to Y$ and $X \leftarrow Y \xrightarrow{=} Y$. Composing these functors with $\underline{M}$ we obtain functors $\underline{M}^*\colon \mathcal{O}_G^{\mathsf{op}} \to \mathrm{Ab}$ and $\underline{M}_*\colon \mathcal{O}_G \to \mathrm{Ab}$.

Definition 1.4.10. Let $\underline{M}$ be a Mackey functor and $K \leq H$ are subgroups of G. The maps

$$\mathrm{res}_K^H \stackrel{\mathsf{def}}{=} \underline{M}^*(G/H \to G/K),$$
$$\mathrm{tr}_K^H \stackrel{\mathsf{def}}{=} \underline{M}_*(G/K \to G/H)$$

are called *restriction* and *transfer* maps, respectively.

Example 1.4.11. *Consider $\mathbb{Z}$ as a $\mathbb{Z}[G]$-module with a trivial G-action. Define the* constant Mackey functor *at $\mathbb{Z}$*

$$\begin{aligned} \underline{\mathbb{Z}}\colon \mathcal{B}_G^{op} &\longrightarrow \mathrm{Ab} \\ G/H &\longmapsto \mathbb{Z}. \end{aligned}$$

All restriction maps are necessarily the identity and for $K \leq H$ the transfer map $\mathrm{tr}_K^H\colon \mathbb{Z} \to \mathbb{Z}$ is multiplication by $|H/K|$.

Example 1.4.12. *Let M be a $\mathbb{Z}[G]$-module. Define the* fixed point Mackey functor *of M by*

$$\begin{aligned} \underline{M}\colon \mathcal{B}_G^{op} &\longrightarrow \mathrm{Ab} \\ G/H &\longmapsto M^H. \end{aligned}$$

The restriction maps are given by inclusion of fixed points and for $K \leq H$ the transfer map $\mathrm{tr}_K^H\colon M^K \to M^H$ is given by the formula

$$\mathrm{tr}_K^H(x) = \sum_{\gamma \in W_H(K)} \gamma \circ x,$$

where $W_H(K)$ denotes the Weyl group $N_H(K)/K$.

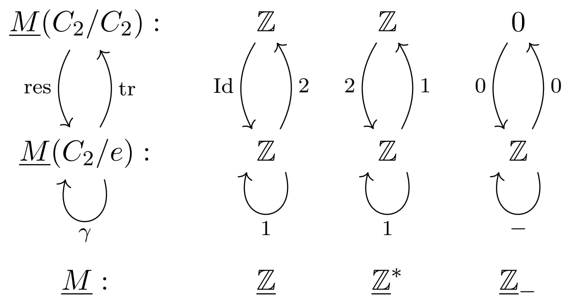
Example 1.4.13. *Let X be a G-spectrum and n be an integer. Consider the functor*

$$\begin{aligned} \underline{\pi}_n^{(-)}(X)\colon \mathcal{B}_G^{op} &\longrightarrow \mathrm{Ab} \\ G/H &\longmapsto \pi_n^H(X). \end{aligned}$$

This is a Mackey functor called the homotopy Mackey functor *of X.*

A common way of spelling out explicitly the data of a Mackey functor $\underline{M}$ is a *Lewis diagram*, introduced in [42]. We place the value of $\underline{M}$ at G/G on the top and the value of $\underline{M}$ on G/e on the bottom of the diagram. Thus the restriction maps are going downwards and the transfers maps are going upwards. We illustrate this with a few examples.

Example 1.4.14. *Let $G = C_2$. Consider the Mackey functors $\underline{\mathbb{Z}}$, $\underline{\mathbb{Z}}^*$ and $\underline{\mathbb{Z}}_-$ defined as* *follows.*



Construction 1.4.15. We end this section with two constructions.

(1) Let $\underline{M}$ be a Mackey functor. One can define an *Eilenberg-MacLane* G-spectrum $H\underline{M}$ associated to $\underline{M}$ which, by analogue with its non-equivariant counterpart, is characterized by

$$\underline{\pi}_i(H\underline{M}) = \begin{cases} \underline{M} & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

(2) Let I be an injective abelian group and $\underline{M}$ be a Mackey functor. Consider the functor $D_I(\underline{M})\colon \mathcal{B}_G^{\mathrm{op}} \to \mathrm{Ab}$ defined by

$$D_I(\underline{M})(G/H) = \mathrm{Hom}_{\mathbb{Z}}(\underline{M}(G/H), I)).$$

Using Dress's definition one can check that this is again a Mackey functor. We write D_I for the *duality functor* $D_I\colon \mathrm{Mack}_G \to \mathrm{Mack}_G$ sending a Mackey functor $\underline{M}$ to $D_I(\underline{M})$.

1.5. Fixed points functors

There are three types of fixed points functors on G-spectra, the categorical, homotopy and geometric fixed points, that will all be relevant for us later. We review the corresponding definitions and properties still following [48].

The category of G-spectra is a presentable, symmetric monoidal, stable ∞-category and as such receives a canonical symmetric monoidal, colimit-preserving functor from the category Sp, see [43, Corollary 1.4.2.23], which we denote by $\iota_*\colon \mathrm{Sp} \to \mathrm{Sp}_G$.

Definition 1.5.1 ([48, Definition 6.2])**.** The functor ι_* has a lax symmetric monoidal right adjoint

$$(-)^G\colon \mathrm{Sp}_G \longrightarrow \mathrm{Sp}$$

called *categorical fixed points*. More generally, for H a subgroup of G, we can define the *categorical H-fixed points* as the composition

$$\mathrm{Sp}_G \xrightarrow{\mathrm{res}^G_H} \mathrm{Sp}_H \xrightarrow{(-)^H} \mathrm{Sp},$$

which we again denote by $(-)^H$.

Proposition 1.5.2 ([48, Section 6.2])**.** *The categorical fixed points functor has the following properties.*

(1) $(-)^G$ *is lax symmetric monoidal.*

(2) $(-)^G$ *commutes with colimits.*

The categorical fixed points functor does not have the properties that one might expect or want by analogy with the fixed points of G-spaces. For example it does not in general commute with smash products or suspensions. A better behaved fixed points functor and in particular one that is symmetric monoidal and commutes with suspensions and the formation of Thom spectra is the geometric fixed points functor. In order to define it we need the notion of a *family of subgroups of* G.

Definition 1.5.3. A *family of subgroups of* G is a collection $\mathcal{F}$ of subgroups closed under passage to conjugates and subgroups.

Examples 1.5.4. *Some examples of families of subgroups of* G *are:*

- *The trivial family* $\{e\}$.
- *The family* $\mathcal{P}$ *of all proper subgroups of* G.
- *The family* $\mathcal{F}\langle p\rangle$ *of* p*-subgroups of* G.
- *The family* $\mathcal{F}(V)$ *of subgroups of* G *such that* $V^H \neq 0$ *for a* G*-representation* V.
- *The family* $\mathcal{F}[N]$ *of subgroups of* G *which do not contain a given normal subgroup* N *of* G.

To any family $\mathcal{F}$ of subgroups of G we can associate a universal G-space $E\mathcal{F}$ and a pointed G-space $\tilde{E}\mathcal{F}$ fitting into a cofibre sequence

$$E\mathcal{F}_+ \to S^0 \to \tilde{E}\mathcal{F}$$

and characterized up to a weak equivalence by

$$E\mathcal{F}^H = \begin{cases} * & \text{if } H \in \mathcal{F} \\ \emptyset & \text{otherwise} \end{cases}, \quad \tilde{E}\mathcal{F}^H = \begin{cases} S^0 & \text{if } H \in \mathcal{F} \\ * & \text{otherwise.} \end{cases}$$

Given a family $\mathcal{F}$ we write $A_{\mathcal{F}}$ for the commutative ring G-spectrum

$$A_{\mathcal{F}} \overset{\text{def}}{=} \prod_{H\in\mathcal{F}} F(G/H_+, S^0).$$

The category of G-spectra together with the ring G-spectrum $A_{\mathcal{F}}$ satisfy Hypotheses 1.2.1 and we can therefore apply the theory of arithmetic fracture squares to form the subcategories of $A_{\mathcal{F}}$-null, $A_{\mathcal{F}}$-complete and $A_{\mathcal{F}}$-cellular objects. Furthermore, we can describe the localization, nullification and cellularization functors explicitly.

Proposition 1.5.5. *Let X be a G-spectrum.*

(1) The $A_{\mathcal{F}}$-nullification of X is given by

$$\mathrm{Null}_{A_{\mathcal{F}}}(X) = \tilde{E}\mathcal{F} \wedge X.$$

(2) The $A_{\mathcal{F}}$-completion of X is given by

$$(X)^{\wedge}_{A_{\mathcal{F}}} = F(E\mathcal{F}_+, X).$$

(3) The $A_{\mathcal{F}}$-cellularization of X is given by

$$\mathrm{Cell}_{A_{\mathcal{F}}}(X) = E\mathcal{F}_+ \wedge X.$$

Proof. Parts (1) and (2) are [48, Proposition 6.5]. Part (3) is [48, Proposition 6.6]. □

We can now define the geometric fixed points functor. Let $\mathcal{P}$ be the family of all proper subgroups of G.

Definition 1.5.6 ([48, Definition 6.12])**.** Define the *geometric fixed points functor* Φ^G as the composition

$$\mathrm{Sp}_G \xrightarrow{\mathrm{Null}_{A_{\mathcal{P}}}} \mathrm{Sp}_G^{A_{\mathcal{P}}\text{-null}} \subseteq \mathrm{Sp}_G \xrightarrow{(-)^G} \mathrm{Sp}\,.$$

More generally, for H a subgroup of G, we define the *geometric H-fixed points functor* as the composition

$$\mathrm{Sp}_G \xrightarrow{\mathrm{res}^G_H} \mathrm{Sp}_H \xrightarrow{\Phi^H} \mathrm{Sp},$$

which we again denote by Φ^H.

Proposition 1.5.7 ([48, Section 6.2])**.** *The geometric fixed points functor has the following properties.*

(1) $\Phi^G(\Sigma^\infty A) \simeq \Sigma^\infty A^G$ for a G-space A.

(2) Φ^G is symmetric monoidal.

(3) Φ^G commutes with filtered colimits.

Remark. Both the geometric and the categorical fixed points functors detect weak equivalences in the category of G-spectra in the sense that a map $f\colon X \to Y$ of G-spectra is a weak equivalence if and only if f^G, respectively $\Phi^G(f)$ is a weak equivalence in Sp.

We end the section with a short digression on connective covers. Recall that there is an adjunction between the ∞-categories of spectra Sp and connective spectra $\mathrm{Sp}_{\geq 0}$

$$\mathrm{Sp}_{\geq 0} \underset{\tau_{\geq 0}}{\overset{\iota}{\rightleftarrows}} \mathrm{Sp},$$

with left adjoint given by the inclusion and right adjoint given by the connective cover. The usual connective cover of a spectrum Y is given by the composition $Y\langle 0\rangle = \iota\tau_{\geq 0}Y$. The G-spectra we study later in this document arise as the output of a derived algebraic geometry machine and come as naive G-spectra. We promote them to genuine G-spectra by taking the associated cofree construction, i.e. for a naive G-spectrum X we look at $X^h = F(EG_+, X)$. Given such a cofree object there are two equally natural ways to construct its equivariant connective cover. We can either first promote X and then take its equivariant connective cover or we can take its non-equivariant connective cover, promote and then take equivariant connective cover again. The following argument shows that the two constructions are equivalent.

Lemma 1.5.8. *Let X be a naive G-spectrum. Then the spectra $X^h\langle 0\rangle$ and $(X\langle 0\rangle)^h\langle 0\rangle$ are equivalent.*

Proof. It suffices to check the claim on fixed points. Right adjoints commute with homotopy limits and so for all subgroups H of G we have an equivalence

$$\tau_{\geq 0}(X^{hH}) \to (\tau_{\geq 0}X)^{hH}.$$

Taking homotopy fixed points is a homotopy limit and homotopy limits in connective spectra are computed by first including into spectra, computing the homotopy limit there and then taking connective cover. The target of the map above is therefore $\tau_{\geq 0}(\iota\tau_{\geq 0}X)^{hH}$. Applying ι we get the desired equivalence

$$X^{hH}\langle 0\rangle \to X\langle 0\rangle^{hH}\langle 0\rangle.$$

□

1.6. Strongly even G-spectra and Real orientations

Let ρ_G and $\bar{\rho}_G$ denote the regular and reduced regular representation of G, respectively.

Definition 1.6.1. A G-spectrum X is called *even* if

(i) $\pi^e_{2k+1}(X) = 0$ and

(ii) $\pi^H_{k\rho_H - 1}(X) = 0$ for all $e \neq H \leq G$ and $k \in \mathbb{Z}$.

X is called *strongly even* if, in addition, all of the restriction maps

$$\pi^H_{k\rho_H}(X) \to (\pi^e_{k|H|}(X))^H$$

are isomorphisms.

Specializing this definition to the case of $G = C_2$ we recover the notion of even and strongly even C_2-spectra introduced by Hill and Meier in [35] and explored in [27] and [52]. In this case we can also characterize strongly even C_2-spectra in the following equivalent ways.

Proposition 1.6.2 ([52, Proposition 6.8])**.** *For a C_2-spectrum X the following conditions are equivalent.*

(1) X is strongly even,

(2) $\pi^{C_2}_{*\rho_{C_2}-i}(X) = 0$ for $0 < i < 3$,

(3) The odd slices $P^{2i+1}_{2i+1}(X)$ in the slice spectral sequence of X are zero and the even slices $P^{2i}_{2i}(X)$ are equivalent to $\Sigma^{i\rho_{C_2}} H\underline{\pi}_{2i}(X)$.

The following propositions are due to Meier.

Proposition 1.6.3. *Let X be a G-spectrum such that $\pi^H_{*\rho_H}(X) = 0$ for all subgroups $H \leq G$. Then $X \simeq 0$.*

Proof. We will prove this by induction on the size of the group. The case of $|G| = 1$ is clear. Thus, assume that $X^H \simeq 0$ for every proper subgroup $H < G$ and that $\pi^G_{*\rho_G}(X) = 0$. The representation sphere $S^{\bar{\rho}_G}$ has a finite cellular filtration $C_\bullet$ with $C_0 = S^0$ and cofibre sequences $G/H_+ \to C_i \to C_{i+1}$, where H is a *proper* subgroup of G. As $X^H \simeq 0$ for all proper subgroups, we see that $\pi^G_{\bar{\rho}_G}(X) \cong \pi^G_0(X)$ and thus $\pi^G_i(X) \cong \pi^G_{i\rho_G}(X) = 0$ □

Proposition 1.6.4. *A map between strongly even G-spectra is an equivalence if and only if it is an equivalence of the underlying spectra.*

Proof. Let $f\colon X \to Y$ be an underlying equivalence of strongly even G-spectra. Automatically, f induces an isomorphism on all homotopy groups of the form $\pi^H_{*\rho_H}$ for H a subgroup of G. Furthermore, we have that $\pi^H_{\rho_H}(\mathrm{cof}(f)) = 0$ for all subgroups $H \leq G$ and thus $\mathrm{cof}(f) \simeq *$ by Proposition 1.6.4. □

Examples 1.6.5. *We list some examples of strongly even C_2-spectra.*

(1) $k\mathbb{R}$ or K-theory with reality, see [27, Section 11] and [21, Corollary 3.5].

(2) $M\mathbb{R}$ or real bordism, see [38, Theorem 4.11].

(3) $\mathrm{tmf}_1(3)$ or topological modular forms with level 3 structure, see [35, Corrolary 4.17] or [27, Corollary 4.6].

Next, we recall the notion of a Real orientation. Let $G = C_2$ and write ρ for the real regular representation of C_2. Consider the spaces $\mathbb{CP}^n$ and $\mathbb{CP}^\infty$ as pointed C_2-spaces with $\mathbb{CP}^0$ as a basepoint and action given by complex conjugation and fix an isomorphism $S^\rho \cong \mathbb{CP}^1$.

Definition 1.6.6. Let X be a homotopy commutative ring C_2-spectrum. A *Real orientation* for X is a class $x \in X^{\rho}_{C_2}(\mathbb{CP}^\infty) \cong [\mathbb{CP}^\infty, S^\rho \wedge X]^{C_2}$ restricting to the class in $X^{\rho}_{C_2}(\mathbb{CP}^1) \cong [\mathbb{CP}^1, S^\rho \wedge X]^{C_2}$ corresponding to the unit $1 \in [S^0, X]^{C_2} \cong [S^\rho, S^\rho \wedge X]^{C_2}$ under the chosen isomorphism $S^\rho \cong \mathbb{CP}^1$

$$\begin{aligned} X^{\rho}_{C_2}(\mathbb{CP}^\infty) &\longrightarrow X^{\rho}_{C_2}(\mathbb{CP}^1) \cong [S^\rho, S^\rho \wedge X]^{C_2} \\ x &\longmapsto e = 1. \end{aligned}$$

Theorem 1.6.7 ([38, Theorem 2.25])**.** *Real orientations of ring C_2-spectra X are in one to one correspondence with homotopy classes of ring C_2-spectra maps $M\mathbb{R} \to X$.*

Lemma 1.6.8 ([35, Lemma 3.3])**.** *Let X be an even C_2-spectrum. Then X is Real orientable.*

1.7. The homotopy fixed points spectral sequence

A fundamental computational technique in the category of G-spectra is the homotopy fixed points spectral sequence. We introduce its $RO(G)$-graded version in full generality below and then describe certain simplifications which occur in the case of $G = C_2$.

Let X be a G-spectrum. The $RO(G)$*-graded homotopy fixed points spectral sequence* has E_2-page given by

$$E_2^{*,\bigstar} = H^*(G, \pi_0^e(X \wedge S^{-\bigstar})) \Longrightarrow \pi^G_{\bigstar - *}(X^h),$$

and differentials of the form

$$d_i \colon E_i^{s,V} \to E_i^{s+i,V-1}.$$

Let now $G = C_2$ and suppose that X is even with $\pi^e_{2n}(X)$ flat over $\mathbb{Z}$ and C_2 acting on $\pi^e_{2n}(X)$ via $(-1)^n$. Let $V = (t-r) + r\sigma$ be a C_2-representation. The E_2 page of the $RO(C_2)$-graded homotopy fixed point spectral sequence then takes the form

$$\begin{aligned} E_2^{s,(t-r)+r\sigma} &= H^s(C_2, \pi_0^e(X \wedge S^{-(t-r)-r\sigma})) \\ &= H^s(C_2, \pi_t^e(X) \otimes \mathrm{sgn}^{\otimes r}) \\ &= \overline{\pi_{2*}(X)} \otimes \mathbb{Z}[u^{\pm 1}, a]/2a \\ &\Longrightarrow \pi^{C_2}_{t-s+r(\sigma-1)}(X^h), \end{aligned}$$

where $|u| = (0, 2-2\sigma)$, $|a| = (1, -\sigma)$, and $\overline{\pi_{2n}(X)}$ is isomorphic to the group $\pi^e_{2n}(X)$, but shifted in degree $(0, n\rho)$. The differentials in Adams grading are of the form

$$d_i \colon E_i^{s,(t-s-r)+r\sigma} \to E_i^{s+i,(t-s-r-1)+r\sigma}.$$

We end this section by recalling the notion of a regular homotopy fixed point spectral sequence introduced by Meier in [52]. Recall that a complex orientation on a ring spectrum R provides elements $v_i \in \pi_{2(2^n-1)}(R)$ well-defined modulo $(2, v_1, \ldots, v_n)$. Now if R is real oriented we obtain elements $\bar{v}_n$ in the E_2-term of the $RO(C_2)$-graded homotopy fixed point spectral sequence of R that are well-defined modulo $(2, \bar{v}_1, \ldots, \bar{v}_n)$.

Let X be an even real oriented commutative ring C_2-spectrum. Assume that the homotopy groups of X are 2-local, torsion-free and that C_2 acts on $\pi^e_{2n}(X)$ as $(-1)^n$.

Definition 1.7.1 ([52, Definition 6.1])**.** We say that X has a *regular homotopy fixed points spectral sequence* if

(i) $\overline{\pi_{2*}(X)}$ consists of permanent cycles,

(ii) the element u^{2^n} survives to the $E_{2^{n+2}-1}$-page and $d_{2^{n+2}-1}(u^{2^n}) = a^{2^{n+2}-1}\bar{v}_{n+1}$,

(iii) if $d_{2^{n+2}-1}(u^{2^n m}\bar{v}) = 0$ for some odd number m and some $\bar{v} \in \overline{\pi_{2*}(X)}$, then the element $u^{2^n m}\bar{v}$ is already a permanent cycle.

The following proposition is useful in checking whether a ring C_2-spectrum has a regular homotopy fixed point spectral sequence.

Proposition 1.7.2 ([52, Proposition 6.5])**.** *Let $f\colon X \to Y$ be a map of homotopy commutative ring C_2-spectra with C_2-action as above. Assume that the homotopy fixed points spectral sequence of the target Y is regular and that the map*

$$\pi^e_*(X)/(2, v_1, \ldots, v_i) \to \pi^e_*(Y)/(2, v_1, \ldots, v_i)$$

is injective for all $i \geq 0$ and that v_k is either zero or a non zero divisor in

$$\pi^e_{2*}(X)/(2, v_1, \ldots, v_{k-1})$$

for all k. Then the homotopy fixed points spectral sequence of X is regular as well.

Example 1.7.3. *The ring C_2-spectrum of real bordism $M\mathbb{R}$ has a regular homotopy fixed point spectral sequence. See [38] or [27, Appendix].*

1.8. The Tate square

In this section we identify a construction in the category of G-spectra known as the *Tate square*, first introduced in [26], as an incarnation of the arithmetic fracture square (1.2). Let $\mathcal{F}$ be a family of subgroups of G and let $A_{\mathcal{F}}$ denote the commutative ring G-spectrum

$$A_{\mathcal{F}} = \prod_{H \in \mathcal{F}} F(G/H_+, S^0).$$

Recall the cofibre sequence (1.1) and the identification of the $A_{\mathcal{F}}$-nullification and $A_{\mathcal{F}}$-cellularization functors in Proposition 1.5.5

$$\begin{array}{ccccc} \mathrm{Cell}_{A_{\mathcal{F}}}(X) & \longrightarrow & X & \longrightarrow & \mathrm{Null}_{A_{\mathcal{F}}}(X) \\ \downarrow{\simeq} & & \| & & \downarrow{\simeq} \\ E\mathcal{F}_+ \wedge X & \longrightarrow & X & \longrightarrow & \tilde{E}\mathcal{F} \wedge X. \end{array}$$

The projection map $E\mathcal{F}_+ \to S^0$ induces a map of G-spectra

$$X = F(S^0, X) \xrightarrow{\varepsilon} F(E\mathcal{F}_+, X)$$

and smashing this map with the bottom cofibre sequence we obtain the diagram

$$\begin{array}{ccccc} X \wedge E\mathcal{F}_+ & \longrightarrow & X & \longrightarrow & X \wedge \tilde{E}\mathcal{F} \\ \downarrow{\scriptstyle \varepsilon\wedge \mathrm{Id}} & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon\wedge \mathrm{Id}} \\ F(E\mathcal{F}_+, X) \wedge E\mathcal{F}_+ & \longrightarrow & F(E\mathcal{F}_+, X) & \longrightarrow & F(E\mathcal{F}_+, X) \wedge \tilde{E}\mathcal{F}. \end{array}$$

Notation 1.8.1. We will use the following (*generalized Scandinavian*) notation for the various spectra in the diagram above.

- $\mathcal{F}(X)_h = X \wedge E\mathcal{F}_+$ and $\mathcal{F}(X)_{hG} = (\mathcal{F}(X)_h)^G$,
- $\mathcal{F}(X)^h = F(E\mathcal{F}_+, X)$ and $\mathcal{F}(X)^{hG} = (\mathcal{F}(X^h))^G$,
- $\mathcal{F}(X)^t = F(\mathcal{F}_+, X) \wedge \tilde{E}\mathcal{F}$ and $\mathcal{F}(X)^{tG} = (\mathcal{F}(X)^t)^G$,
- $\mathcal{F}(X)^\Theta = X \wedge \tilde{E}\mathcal{F}$ and $\mathcal{F}(X)^{\Theta G} = (\mathcal{F}(X)^\Theta)^G$.
- If $\mathcal{F} = \mathcal{P}$ is the family of all proper subgroups of G we will write $X^\Phi = X \wedge \tilde{E}\mathcal{P}$ and $(X)^{\Phi G} = \Phi^G(X) = (X^\Phi)^G$.

Using this notation we can rewrite the diagram in a more compact form.

$$\begin{array}{ccccc} \mathcal{F}(X)_h & \longrightarrow & X & \longrightarrow & \mathcal{F}(X)^\Theta \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}(X)_h & \longrightarrow & \mathcal{F}(X)^h & \longrightarrow & \mathcal{F}(X)^t. \end{array}$$

Specializing to the family $\mathcal{F} = \{e\}$ we find that $E\{e\} \simeq EG$ and we get the classical Tate diagram, see [26] and [21].

$$\begin{array}{ccccc} X \wedge EG_+ & \longrightarrow & X & \longrightarrow & X \wedge \tilde{E}G \\ \downarrow{\scriptstyle \varepsilon\wedge \mathrm{Id}} & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon\wedge \mathrm{Id}} \\ F(EG_+, X) \wedge EG_+ & \longrightarrow & F(EG_+, X) & \longrightarrow & F(EG_+, X) \wedge \tilde{E}G. \end{array}$$

Notation 1.8.2. There is a shorthand notation and terminology for the various spectra present in the diagram above known as the *Scandinavian notation*.

- $X_h = X \wedge EG_+$ is the *free G-spectrum associated to X*. It can be seen as a spectrum level analogue of the construction which associates to a based G-space a new based G-space whose action is free away from the basepoint.
- $X^h = F(EG_+, X)$ is the *geometric completion of X*.
- $X^t = F(EG_+, X) \wedge \tilde{E}G$ is the *Tate G-spectrum associated to X*.
- $X^\Theta = X \wedge \tilde{E}G$ is the *singular G-spectrum associated to X*. It is a spectrum level analogue of the construction which associates to a based G-space a new based G-space which has the same fixed points as the original one for non-trivial subgroups of G and is non-equivariantly contractible.

Taking categorical G-fixed points we write

- $X_{hG} = (X_h)^G$,
- $X^{hG} = (X^h)^G$,
- $X^{tG} = (X^t)^G$ and
- $X^{\Theta G} = \Theta^G X = (X^\Theta)^G$.

The Tate diagram can be again displayed in the following compact form.

$$\begin{array}{ccccc} X_h & \longrightarrow & X & \longrightarrow & X^\Theta \\ \downarrow & & \downarrow & & \downarrow \\ X_h & \longrightarrow & X^h & \longrightarrow & X^t. \end{array}$$

1.9. The iterated Tate argument

We end the chapter with a description of a procedure for recovering the full $RO(C_4)$-graded homotopy groups of a ring C_4-spectrum, where C_4 is the cyclic group of order 4. A number of simplifications which occur in this case allow an approach where a threefold application of the Tate square, consisting of two $RO(C_2)$-graded and one $RO(C_4)$-graded computations, recovers the full $RO(C_4)$-graded homotopy information. The argument was designed to attack the computation of the homotopy groups of the ring C_4-spectrum $\mathrm{tmf}_1(5)$ which we introduce in Section 5.1. We have made partial progress in the form of recovering the full $RO(C_2)$-graded homotopy groups of $\mathrm{tmf}_1(5)$. The computation is still a work in progress and is not included in this document.

Let $G = C_4$, $H = C_2$ and write $Q = G/H$ for the quotient also isomorphic to C_2. Write σ for the sign representation of C_2 or of C_4 and α for the natural one-dimensional complex representation of C_4. Let $\varepsilon, \alpha, \alpha^2, \alpha^3 = \alpha^{-1}$ denote the irreducible complex representations of C_4, all of dimension 1 and let $1, \sigma, V$ denote the irreducible real representations of C_4 of dimensions $1, 1$ and 2, respectively. The iterated Tate argument relies on a number of auxiliary facts and observations which we collect below.

Observations 1.9.1. By an observation we mean a statement which is trivial for not necessarily obvious reasons.

(1) $\tilde{E}\mathcal{P} \simeq \tilde{E}Q \simeq S^{\infty\sigma} \simeq S^{\infty(\alpha+\sigma)}$.

(2) $\tilde{E}G \simeq \tilde{E}[\supsetneq H] \simeq S^{\infty\alpha}$.

(3) $\pi_*^e(\Phi^G(X)) = \pi_*^e((X \wedge \tilde{E}\mathcal{P})^G) \cong \pi_*^G(X \wedge \tilde{E}\mathcal{P})$.

(4) The complexifications of the three real representations are $\varepsilon, \alpha^2, \alpha + \alpha^{-1}$. As representations over $\mathbb{R}$ the four complex irreducible representations are $2, V, 2\sigma, V$.

(5) $S^\alpha \simeq S^{\alpha^3}$ and this is a G-homeomorphism of based G-spaces.

(6) We can write $RO(G) = \mathbb{Z} \oplus \mathbb{Z}\alpha \oplus \mathbb{Z}\sigma$.

(7) The $RO(G)$-graded homotopy groups of the ring G-spectrum $X \wedge \tilde{E}\mathcal{P}$ are given by

$$\begin{aligned}\pi_\bigstar^G(X \wedge \tilde{E}\mathcal{P}) &\cong \pi_\bigstar^G(X)[\frac{1}{a_\sigma}, \frac{1}{a_\alpha}] \\ &\cong \pi_*^G(X \wedge \tilde{E}\mathcal{P}) \otimes \mathbb{Z}[a_\sigma^{\pm 1}, a_\alpha^{\pm 1}] \\ &\cong \pi_*^e(\Phi^G(X)) \otimes \mathbb{Z}[a_\sigma^{\pm 1}, a_\alpha^{\pm 1}].\end{aligned}$$

Lemma 1.9.2. *Let X be a G-spectrum. We have the following equivalences*

$$\begin{aligned}X^\Phi &= X \wedge \tilde{E}\mathcal{P} \simeq \inf_e^G(\Phi^G(X)) \wedge \tilde{E}\mathcal{P}, \\ X^\Theta &= X \wedge \tilde{E}G \simeq \inf_Q^G(\Phi^H(X)) \wedge \tilde{E}G.\end{aligned}$$

Proof. We will do the case of $G = C_2$ and for the proof we will resort to the model of orthogonal G-spectra described in Definition 1.3.6. Note that

$$\tilde{E}\mathcal{P} \simeq \tilde{E}C_2 \simeq S^{\infty\sigma}.$$

We want to show that

$$X \wedge S^{\infty\sigma} \simeq \inf_e^{C_2}(\Phi^{C_2}(X)) \wedge S^{\infty\sigma}. \quad (1.3)$$

Any C_2-spectrum Y has a canonical homotopy presentation of the form

$$Y \simeq \mathrm{hocolim}_k \Sigma^{k\rho} Y_{k\rho},$$

where ρ denotes the regular representation of C_2 and $Y_{k\rho}$ denotes the $k\rho$-th level of the spectrum Y. The geometric fixed points functor $\Phi^{C_2}(-)$ commutes with suspension and homotopy colimits so we get

$$\begin{aligned}\Phi^{C_2}(Y) &\simeq \Phi^{C_2}(\mathrm{hocolim}_k \Sigma^{-k\rho} Y_{k\rho} \wedge S^{\infty\sigma}) \\ &\simeq \mathrm{hocolim}_k \Sigma^{-k\rho} Y_{k\rho}^{C_2}\end{aligned}$$

and thus $(\Phi^{C_2}(Y))_k = Y_{k\rho}^{C_2}$. Inflating a *non-equivariant* spectrum Z to a C_2-spectrum gives at level $k\rho$

$$\mathrm{inf}_e^{C_2}(Z)_{k\rho} = Z_k \wedge S^{k\sigma}.$$

Now back to the question. At level $k\rho$, the right hand side of (1.3) looks like

$$\begin{aligned}(\mathrm{inf}_e^{C_2}(\Phi^{C_2}(X)) \wedge S^{\infty\sigma})(k\rho) &\simeq (\Phi^{C_2}(X))_k \wedge S^{k\sigma} \wedge S^{\infty\sigma} \\ &\simeq X(k\rho)^{C_2} \wedge S^{k\sigma} \wedge S^{\infty\sigma} \\ &\simeq X(k\rho)^{C_2} \wedge S^{\infty\sigma}.\end{aligned}$$

We have an obvious map to the $k\rho$-th level of the left hand side of (1.3)

$$X_{k\rho}^{C_2} \wedge S^{\infty\sigma} \to X_{k\rho} \wedge S^{\infty\sigma}$$

and this map is a weak equivalence as can been seen by checking on fixed points. □

Lemma 1.9.3 ([27, Lemma 11.2])**.** *Let X be a C_2-spectrum which is non-equivariantly connective and such that the map $X^{C_2} \to X^{hC_2}$ is a connective cover. Then the map $X^{\Phi C_2} \to X^{tC_2}$ is also a connective cover.*

The iterated Tate argument proceeds in three phases.

Phase 1**.** Consider the Tate square for the H-spectrum $\mathrm{res}_H^G(X)$

$$\begin{array}{ccc}\mathrm{res}_H^G(X) & \longrightarrow & (\mathrm{res}_H^G(X))^{\Phi} \\ \downarrow & & \downarrow \\ (\mathrm{res}_H^G(X))^h & \longrightarrow & (\mathrm{res}_H^G(X))^t.\end{array}$$

(a) *Bottom left corner:* using the H-action on $\pi_*^e(\mathrm{res}_H^G(X))$ we run the $RO(H)$-graded HFPSS for $(\mathrm{res}_H^G(X))^h$ to recover $\pi_\bigstar^H((\mathrm{res}_H^G(X))^h)$ and we remember the Q-action on $\pi_*^H((\mathrm{res}_H^G(X))^h)$.

(b) *Bottom right corner:* we invert a_σ in $\pi^H_\bigstar((\mathrm{res}^G_H(X))^h)$ to recover $\pi^H_\bigstar((\mathrm{res}^G_H(X))^t)$ and we remember the Q-action on $\pi^H_*((\mathrm{res}^G_H(X))^t)$.

(c) *Top right corner:* we use Lemma 1.9.3 to recover $\pi^H_\bigstar((\mathrm{res}^G_H(X))^\Phi)$. The homotopy of $(\mathrm{res}^G_H(X))^\Phi$ is a_σ-periodic in the sense that

$$\pi^H_\bigstar((\mathrm{res}^G_H(X))^\Phi) \cong \pi^H_*((\mathrm{res}^G_H(X))^\Phi) \otimes \mathbb{Z}[a_\sigma^\pm] \cong \pi^e_*(\Phi^H(X)) \otimes \mathbb{Z}[a_\sigma^\pm],$$

so we get $\pi^e_*(\Phi^H(X))$ and we have remembered the Q-action on this ring along the way.

Phase 2**.** Consider the Tate square for the Q-spectrum $\Phi^H(X)$

$$\begin{array}{ccc} \Phi^H(X) & \longrightarrow & (\Phi^H(X))^\Phi \\ \downarrow & & \downarrow \\ (\Phi^H(X))^h & \longrightarrow & (\Phi^H(X))^t. \end{array}$$

(a) *Bottom left corner:* from Phase 1 we know $\pi^e_*(\Phi^H(X))$ and the Q-action on it, so we run the $RO(Q)$-graded HFPSS for $(\Phi^H(X))^h$ to recover $\pi^Q_\bigstar((\Phi^H(X))^h)$.

(b) *Bottom right corner:* we invert a_σ in $\pi^Q_\bigstar((\Phi^H(X))^h)$ to recover $\pi^Q_\bigstar((\Phi^H(X))^t)$.

(c) *Top right corner:* we use Lemma 1.9.3 to recover $\pi^Q_\bigstar((\Phi^H(X))^\Phi)$. Alternatively, if the homotopy of $\Phi^G(X)$ is accessible, the $RO(Q)$-graded homotopy groups of $(\Phi^H(X))^\Phi$ are completely determined by the homotopy groups of $\Phi^G(X)$ as

$$\begin{aligned} \pi^Q_*((\Phi^H(X))^\Phi) &= \pi^Q_*(\Phi^H(X) \wedge \tilde{E}\mathcal{P}) \\ &\cong \pi^e_*((\Phi^H(X) \wedge \tilde{E}\mathcal{P})^Q) \\ &= \pi^e_*(\Phi^Q(\Phi^H(X))) \\ &\cong \pi^e_*(\Phi^G(X)) \end{aligned}$$

and

$$\begin{aligned} \pi^Q_\bigstar((\Phi^H(X))^\Phi) &= \pi^Q_\bigstar(\Phi^H(X) \wedge \tilde{E}\mathcal{P}) \\ &\cong \pi^Q_\bigstar(\Phi^H(X))[\frac{1}{a_\sigma}] \\ &\cong \pi^Q_*((\Phi^H(X))^\Phi) \otimes \mathbb{Z}[a_\sigma^\pm] \\ &\cong \pi^e_*(\Phi^G(X)) \otimes \mathbb{Z}[a_\sigma^\pm]. \end{aligned}$$

(d) *Top left corner:* the Tate square is a homotopy pullback square of rings, so we recover $\pi^Q_\bigstar(\Phi^H(X))$.

Phase 3**.** Consider the Tate square for the G-spectrum X

$$\begin{array}{ccc} X & \longrightarrow & X^{\Theta} \\ \big\downarrow & & \big\downarrow \\ X^h & \longrightarrow & X^t. \end{array}$$

(a) *Top right corner:* we can rewrite it as

$$X^{\Theta} = X \wedge \tilde{E}G \simeq \inf_Q^G(\Phi^H(X)) \wedge \tilde{E}G.$$

Observe that

$$\pi_{\bigstar}^G(X \wedge \tilde{E}G) \cong \pi_{\bigstar}^G(X)[\frac{1}{a_\alpha}] \cong \pi_{*+*\sigma}^G(X \wedge \tilde{E}G) \otimes \mathbb{Z}[a_\alpha^{\pm}]$$

and

$$\begin{aligned} \pi_{*+*\sigma}^G(X \wedge \tilde{E}G) &\cong \pi_{*+*\sigma}^G(\inf_Q^G(\Phi^H(X)) \wedge \tilde{E}G) \\ &\cong \pi_{\bigstar}^Q(\Phi^H(X)). \end{aligned}$$

The ring $\pi_{\bigstar}^Q(\Phi^H(X))$ was recovered in Phase 2, so we can compute the $RO(G)$-graded homotopy groups of X^{Θ} as

$$\pi_{\bigstar}^G(X^{\Theta}) \cong \pi_{\bigstar}^Q(\Phi^H(X)) \otimes \mathbb{Z}[a_\alpha^{\pm}].$$

(b) *Bottom left corner:* we determine the G-action on $\pi_*^e(X)$ and run the $RO(G)$-graded HFPSS for X^h to recover $\pi_{\bigstar}^G(X^h)$.

(c) *Bottom right corner:* we invert a_α in $\pi_{\bigstar}^G(X^h)$ to recover $\pi_{\bigstar}^G(X^t)$.

(d) *Top left corner:* the Tate square is a homotopy pullback square of rings, so we recover $\pi_{\bigstar}^G(X)$.

Chapter 2

Stacks

In this chapter we collect the necessary background material on stacks. Although not central to this thesis the language of stacks is required to outline the construction of the basic objects of study and we take the time and space to introduce things in sufficient detail.

We begin with a brief review of Grothendieck topologies and sites and introduce the Zariski, étale and fpqc topologies on the category of schemes in Section 2.1. In Section 2.2 we introduce stacks as categories fibred in groupoids and look at a number of examples of stacks that we will encounter later. Sheaves on stacks and the category of quasi-coherent sheaves on a stack are defined in Section 2.3 followed by a short discussion on coarse moduli spaces in Section 2.4. In Sections 2.5 and 2.6 we introduce the main examples of stacks we are interested in, the moduli stack of elliptic curves and its variations, the moduli stacks of elliptic curves with level structure, and the moduli stack of formal groups. We end the chapter with a digression on derived algebraic geometry in Section 2.7.

2.1. Grothendieck topologies

Given a scheme S we write $\mathrm{Sch}_{/S}$ for the category of S-schemes or schemes over S. There are several Grothendieck topologies that one can equip the category of schemes with and each of them restricts to a Grothendieck topology on the category of S-schemes. We are interested in the Zariski, étale and fpqc topologies which we introduce in detail below.

Recall the notions of an unramified, étale and fpqc morphism of schemes. Let R be a local ring and write $\mathfrak{m}_R$ for its maximal ideal and k_R for its residue field. A morphism of local rings $f\colon R \to S$ is a morphism of rings such that $f(\mathfrak{m}_R) \subset \mathfrak{m}_S$.

Definition 2.1.1. A morphism of local rings $f\colon R \to S$ is called

(i) *unramified* if $f(\mathfrak{m}_R)S = \mathfrak{m}_S$ and k_S is a finite separable extension of k_R,

(ii) *étale* if f is flat and unramified.

Definition 2.1.2. A morphism of schemes $f\colon X \to Y$ is called

(i) *unramified at a point* $x \in X$ (respectively *étale at a point* $x \in X$) if the morphism of stalks $f^{\#}\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is unramified (respectively étale) and

(ii) *unramified* (respectively étale) if it is unramified (respectively étale) at every point.

Definition 2.1.3. A morphism of schemes $f\colon X \to Y$ is called *fpqc* if it is faithfully flat and quasi-compact.

Next, recall the notion of a Grothendieck topology on a category.

Definition 2.1.4 ([61, Tag 03NF])**.** A *Grothendieck topology* on a category $\mathcal{C}$ is a set of families of morphisms $\{X_i \to X\}_{i\in I}$ for each object X in $\mathcal{C}$, called *coverings* of X, such that the following conditions are satisfied.

(i) If $Y \to X$ is an isomorphism, then the set $\{Y \to X\}$ is a covering.

(ii) If $\{X_i \to X\}_{i\in I}$ is a covering and $Y \to X$ is any morphism in $\mathcal{C}$, then for all $i \in I$ the fibre product $X_i \times_X Y$ exists in $\mathcal{C}$ and the family $\{X_i \times_X Y \to Y\}_{i\in I}$ is a covering.

(iii) If $\{X_i \to X\}$ is a covering and for all $i \in I$ we are given a covering $\{X_{ij} \to X_i\}_{j\in J_i}$, then the family
$$\{X_{ij} \to X\}_{i\in I, j\in J_i}$$
is a covering.

A category $\mathcal{C}$ equipped with a Grothendieck topology is called a *site*.

As mentioned above, we are interested in the Zariski, étale and fpqc Grothendieck topologies on the category of schemes. A covering of a scheme X in each of these topologies is a family $\{f_i\colon X_i \to X\}_{i\in I}$ of morphisms which are jointly surjective in the sense that
$$X = \bigcup_{i\in I} f_i(X_i)$$
and such that each f_i is

- an open immersion in the Zariski topology **zar**,
- étale in the étale topology **ét** and
- fpqc in the fpqc topology **fpqc**.

When we want to emphasize the Grothendieck topology being used we will decorate the category of schemes with a superscript and write $\mathrm{Sch}^{\mathsf{zar}}$, $\mathrm{Sch}^{\text{ét}}$ or $\mathrm{Sch}^{\mathsf{fpqc}}$ for the category of schemes equipped with the Zariski, étale and fpqc topology, respectively.

Remarks. Any Zariski covering is étale and any étale covering is fpqc, thus, in the order of increasingly finer topologies, we have the chain of inclusions

$$\textbf{zar} \subset \textbf{ét} \subset \textbf{fpqc}.$$

These topologies are examples of *subcanonical* topologies meaning that all representable presheaves on the site Sch^{τ}, where $\tau \in \{\mathsf{zar}, \text{ét}, \mathsf{fpqc}\}$, are already sheaves. The fpqc topology is the finest topology of the three and hence any presheaf satisfying the sheaf condition for the fpqc topology will be a sheaf in the étale and Zariski sites.

2.2. Categories fibred in groupoids and stacks

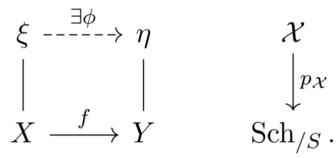
We follow the appendix of [65] and introduce stacks as categories fibred in groupoids satisfying descent. [58, Chapter 4] and [53, Chapter 1] have also been invaluable references in putting together this section.

The notion of a stack can be defined with respect to an arbitrary site $\mathcal{C}$ in the sense of Definition 2.1.4. For the purposes of this document we restrict the exposition to stacks over the site $\mathrm{Sch}_{/S}$ of S-schemes equipped with the Zariski, étale or fpqc topology.

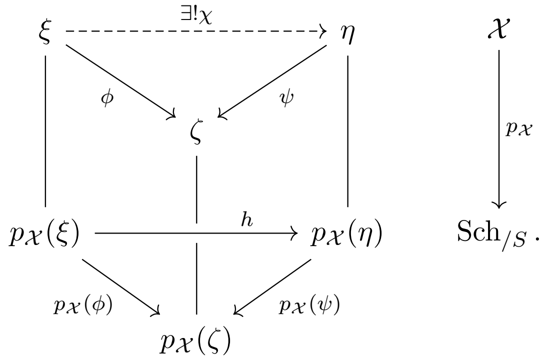
We assume that a base scheme S has been fixed and we suppress it from the terminology.

Definition 2.2.1 ([65, Definition 7.1])**.** A *category fibred in groupoids over* S is a category $\mathcal{X}$ together with a functor $p_{\mathcal{X}}\colon \mathcal{X} \to \mathrm{Sch}_{/S}$ such that the following conditions are satisfied.

(i) Given a morphism $f\colon X \to Y$ in $\mathrm{Sch}_{/S}$ and an object η in $\mathcal{X}$ such that $p_{\mathcal{X}}(\eta) = Y$ there exist a morphism $\phi\colon \xi \to \eta$ such that $p_{\mathcal{X}}(\phi) = f$



(ii) If $\phi\colon \xi \to \zeta$ and $\psi\colon \eta \to \zeta$ are morphisms in $\mathcal{X}$ and $h\colon p_{\mathcal{X}}(\xi) \to p_{\mathcal{X}}(\eta)$ is a morphism in $\mathrm{Sch}_{/S}$ such that $p_{\mathcal{X}}(\psi) \circ h = p_{\mathcal{X}}(\phi)$ there exits a unique morphism $\chi\colon \xi \to \eta$ in $\mathcal{X}$ such that $\psi \circ \chi = \phi$ and $p_{\mathcal{X}}(\chi) = h$



If $\mathcal{X}$ is a category fibred in groupoids and X is a scheme we write $\mathcal{X}(X)$ for the fibre over X, i.e. the category whose objects are objects ξ of $\mathcal{X}$ such that $p_{\mathcal{X}}(\xi) = X$ and whose morphisms are morphisms ϕ in $\mathcal{X}$ such that $p_{\mathcal{X}}(\phi) = \mathrm{Id}$.

Remarks. A couple of remarks are in order. Axiom (ii) implies that the object ξ, whose existence is postulated by (i), is unique up to a canonical isomorphism. It should be thought of as the pullback of η along f and we denote it by $f^*\eta$. Another consequence of (ii) is that a morphism ϕ in $\mathcal{X}$ is an isomorphism if and only if $p_{\mathcal{X}}(\phi)$ is. In particular, $\mathcal{X}(X)$ is a groupoid.

Definition 2.2.2. A *morphism of categories fibred in groupoids* from $\mathcal{X}$ to $\mathcal{Y}$ is a functor $f\colon \mathcal{X} \to \mathcal{Y}$ such that the diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\quad f \quad} & \mathcal{Y} \\ & {\scriptstyle p_{\mathcal{X}}} \searrow \quad \swarrow {\scriptstyle p_{\mathcal{Y}}} & \\ & \mathrm{Sch}_{/S} & \end{array}$$

commutes.

Example 2.2.3. *Let $\mathcal{X}\colon (\mathrm{Sch}_{/S})^{op} \to \mathrm{Set}$ be a contravariant functor. We can think of $\mathcal{X}$ as a category fibred in groupoids as follows. As a category $\mathcal{X}$ has objects pairs (X, ξ) where X is a scheme and ξ is an element of $\mathcal{X}(X)$. Morphisms are of the form $\phi\colon (X, \xi) \to (Y, \eta)$ where $\phi\colon X \to Y$ is a morphism of schemes such that $\mathcal{X}(\phi)(\eta) = \xi$. The functor $p_{\mathcal{X}}$ is defined as*

$$\begin{aligned} p_{\mathcal{X}}\colon \mathcal{X} &\longrightarrow \mathrm{Sch}_{/S} \\ (X, \xi) &\longmapsto X \\ \phi\colon (X, \xi) \to (Y, \eta) &\longmapsto \phi\colon X \to Y. \end{aligned}$$

Example 2.2.4. *As a follow up of the example above let X be an scheme. Then X defines a contravariant functor $\mathrm{Hom}_{\mathrm{Sch}_{/S}}(-, X)\colon (\mathrm{Sch}_{/S})^{op} \to \mathrm{Set}$ and therefore a category fibred in groupoids which we, by abuse of notation, will also denote by X.*

We now give the definition of a stack as a category fibred in groupoids satisfying descent over the site $\mathrm{Sch}^{\tau}_{/S}$ where $\tau \in \{\text{zar}, \text{ét}, \text{fpqc}\}$.

Definition 2.2.5 ([61, Tag 026B])**.** Let $\mathcal{X}$ be a category fibred in groupoids. A *descent datum* (X_i, ϕ_{ij}) in $\mathcal{X}$ relative to a covering $\{X_i \to X\}_{i \in I}$ in $\mathrm{Sch}^{\tau}_{/S}$ consists of an object $\xi_i \in \mathcal{X}(X_i)$ for each $i \in I$ and isomorphisms

$$\phi_{ij}\colon \xi_j|_{X_i \times_X X_J} \to \xi_i|_{X_i \times_X X_j}$$

in $\mathcal{X}(X_i \times_X X_j)$ for each pair $(i, j) \in I^2$ satisfying the cocycle condition. A descent datum (X_i, ϕ_{ij}) relative to a covering $\{X_i \to X\}_{i \in I}$ is called *effective* if there exists an object ξ in $\mathcal{X}(X)$ together with isomorphisms $\psi_i\colon \xi|_{X_i} \to \xi_i$ such that

$$\phi_{ij} = (\psi_i|_{X_i \times_X X_j}) \circ (\phi_j|_{X_i \times_X X_j})^{-1}.$$

Definition 2.2.6 ([65, Definition 7.3])**.** A category fibred in groupoids $\mathcal{X}$ is called a *stack* if the following conditions are satisfied.

(i) For any scheme X and any two objects ξ_1 and ξ_2 in $\mathcal{X}(X)$ the presheaf

$$\begin{aligned} \mathrm{Isom}_X(\xi_1,\xi_2)\colon (\mathrm{Sch}^{\mathcal{T}}_{/X})^{\mathrm{op}} &\longrightarrow \mathrm{Set} \\ f\colon Y \to X &\longmapsto \mathrm{Isom}_{\mathcal{X}(Y)}(f^*\xi_1, f^*\xi_2) \end{aligned}$$

is a sheaf on the site $\mathrm{Sch}^{\mathcal{T}}_{/X}$.

(ii) For any covering $\{X_i \to X\}_{i\in I}$ in $\mathrm{Sch}^{\mathcal{T}}_{/X}$ any descent datum in $\mathcal{X}$ relative to this covering is effective.

Definition 2.2.7 ([65, Definition 7.4])**.** A *morphism of stacks* is a morphism of the underlying categories fibred in groupoids.

We can form fibred products of stacks.

Definition 2.2.8 ([65, Definition 7.9])**.** Let

$$\begin{array}{ccc} & & \mathcal{X}_1 \\ & & \downarrow{\scriptstyle f_1} \\ \mathcal{X}_1 & \xrightarrow{f_2} & \mathcal{Y} \end{array}$$

be a diagram of categories fibred in groupoids. Define the *fibre product* $\mathcal{X}_1 \times_{\mathcal{Y}} \mathcal{X}_2$ as follows. The objects of $\mathcal{X}_1 \times_{\mathcal{Y}} \mathcal{X}_2$ are triples (ξ_1, ξ_2, α), where $\xi_i \in \mathcal{X}_i$ and $p_{\mathcal{X}_1}(\xi_1) = p_{\mathcal{X}_2}(\xi_2)$ and $\alpha\colon f_1(\xi_1) \to f_2(\xi_2)$ is a morphism in $\mathcal{Y}$ such that $p_{\mathcal{Y}}(\alpha) = \mathrm{Id}$. A morphism from (ξ_1, ξ_2, α) to (η_1, η_2, β) is a pair (ϕ_1, ϕ_2), where $\phi_i\colon \xi_i \to \eta_i$ is a morphism in $\mathcal{X}_i$ such that $p_{\mathcal{X}_1}(\phi_1) = p_{\mathcal{X}_2}(\phi_2)$ and $\beta \circ f_1(\phi_1) = f_2(\phi_2) \circ \alpha$. The functor $p_{\mathcal{X}_1 \times_{\mathcal{Y}} \mathcal{X}_2}$ is given by the composition

$$\mathcal{X}_1 \times_{\mathcal{Y}} \mathcal{X}_2 \longrightarrow \mathcal{X}_1 \longrightarrow \mathrm{Sch}_{/S}\,.$$

Example 2.2.9. *The category fibred in groupoids from Example 2.2.4 is a stack.*

An important example of a stack and one that we will encounter later is the quotient stack.

Definition 2.2.10. Let G be a finite group and X a scheme with a G-action. Define the *quotient stack* $[X/G]$ to be the category fibred in groupoids with objects triples (P, B, f), where $P \to B$ a principal G-bundle, $f\colon P \to X$ a G-equivariant morphisms and with morphisms cartesian diagrams of the form

$$\begin{array}{ccc} P' & \xrightarrow{g} & P \\ \downarrow & \lrcorner & \downarrow \\ B' & \longrightarrow & B \end{array}$$

compatible with the morphisms to X. The functor $p_{[X/G]}$ is defined as

$$\begin{aligned} p_{[X/G]}\colon [X/G] &\longrightarrow \mathrm{Sch}_{/S} \\ (P,B,F) &\longmapsto B \\ (f,g)\colon (E,p,e)\to(E',p',e) &\longmapsto f\colon S\to S'. \end{aligned}$$

Example 2.2.11. *Let R be a commutative ring and let $\mathbb{A}^n_R$ denote n-dimensional affine space over R. Given positive integers $a_0,\dots,a_n$, we define the* weighted projective stack *$\mathcal{P}_R(a_0,\dots,a_n)$ to be the quotient stack*

$$\mathcal{P}_R(a_0,\dots,a_n) \stackrel{\text{def}}{=} [(\mathbb{A}^{n+1}_R - \{0\})/\mathbb{G}_m].$$

The action of the multiplicative group on $\mathbb{A}^n_R$ is given as follows. The morphism of rings

$$\begin{gathered} \mathbb{Z}[t,t^{-1}]\otimes R[t_0,\dots,t_n] \leftarrow R[t_0,\dots,t_n] \\ t^{a_i}\otimes t_i \leftarrow\!\shortmid t_i \end{gathered}$$

induces a morphism of schemes

$$\mathbb{G}_m\times\mathbb{A}^{n+1}_R \to \mathbb{A}^{n+1}_R$$

and the $\mathbb{G}_m$-action is given by restricting this morphism to $\mathbb{A}^{n+1}_R - \{0\}$.

We now proceed towards the definition of a Deligne-Mumford stacks. First, we need the notion of a representable morphism.

Definition 2.2.12 ([65, Definition 7.11])**.** Let $f\colon \mathcal{X}\to\mathcal{Y}$ be a morphism of stacks. We say that f is *representable* if given any scheme X and any morphism $g\colon X\to\mathcal{Y}$ the fibre product

$$\begin{array}{ccc} X\times_{\mathcal{Y}}\mathcal{X} & \longrightarrow & \mathcal{X} \\ \big\downarrow & & \big\downarrow{\scriptstyle f} \\ X & \xrightarrow{\;g\;} & \mathcal{Y} \end{array}$$

is a scheme.

The notion of a representable morphism allows us to transfer properties of morphisms of schemes to morphisms of stacks. Let P be some property of morphisms of schemes which is stable under base change and local on the target in the sense that if $\{Y_i\to Y\}_{i\in I}$ is a covering and $f\colon X\to Y$ is a morphism of schemes then

$$f \text{ has property } P \Leftrightarrow Y_i\times_X X\to Y_i \text{ has property } P \text{ for all } i\in I.$$

Definition 2.2.13. A representable morphism of stacks $\mathcal{X} \to \mathcal{Y}$ *has property* P if given any scheme X and any morphism $X \to \mathcal{Y}$ the induced morphism of schemes $X \times_{\mathcal{Y}} \mathcal{X} \to X$ has property P.

If $\mathcal{X}$ is a stack we write $\Delta_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ for the diagonal morphism.

Definition 2.2.14 ([65, Definition 7.14])**.** A *Deligne-Mumford stack* is a stack $\mathcal{X}$ such that

(i) the diagonal morphism $\Delta_{\mathcal{X}}$ is representable, quasi-compact and separated,

(ii) there exists a scheme U and an étale surjective morphism $U \to \mathcal{X}$ called an *atlas*.

Remark. The first condition above can be motivated by the following observation, see [65, Proposition 7.13] for a proof. Let $\mathcal{X}$ be a stack. Then the diagonal morphism $\Delta_{\mathcal{X}}$ is representable if and only if every morphism $X \to \mathcal{X}$ with X a scheme is representable.

Example 2.2.15. *The weighted projective stack $\mathcal{P}_R(a_0, \ldots, a_n)$ of Example 2.2.11 is a Deligne-Mumford stack if and only if all of the a_i are invertible in R. See [54, Section 2].*

We now briefly look at properties of stacks and properties of morphisms of stacks.

Definition 2.2.16. A morphism $\mathcal{X} \to \mathcal{Y}$ of Deligne-Mumford stacks is called an *open* (respectively *closed*) embedding if it is representable and has property P where P is the property of being an open (respectively closed) embedding.

Definition 2.2.17 ([61, Tag 04YM])**.** Let $\mathcal{X}$ be a Deligne-Mumford stack. A *substack* of $\mathcal{X}$ is a morphism of stacks $\mathcal{Y} \to \mathcal{X}$ which is represented by an embedding of schemes. A substack $\mathcal{Y}$ is called *open* or *closed* if the morphism $\mathcal{Y} \to \mathcal{X}$ is an open or closed embedding.

Let P be a property of schemes. We say that P is *local* in the topology $\tau \in \{\text{zar}, \text{ét}, \text{fpqc}\}$, if for every covering $\{X_i \to X\}_{i \in I}$ in $\mathrm{Sch}^{\tau}_{/S}$ we have that

$$X \text{ has property } P \Leftrightarrow X_i \text{ has property } P \text{ for all } i \in I.$$

Example 2.2.18. *Examples of properties of schemes local in the étale topology are the properties of being regular, locally noetherian, locally of finite type, quasi-compact, proper.*

Definition 2.2.19. Let P be a property of schemes local in the étale topology and $\mathcal{X}$ be a Deligne-Mumford stack. Then $\mathcal{X}$ *has property* P if there exists a surjective étale morphism $X \to \mathcal{X}$ with a X a scheme having property P.

2.3. Sheaves on stacks

In this section we review the notion of a sheaf on a Deligne-Mumford stack and introduce the category of quasi-coherent sheaves on a stack.

Definition 2.3.1 ([65, Definition 7.18])**.** Let $\mathcal{X}$ be a Deligne-Mumford stack. A *quasi-coherent sheaf* $\mathcal{F}$ on $\mathcal{X}$ consists of the following data.

(i) For each open $U \to \mathcal{X}$ a quasi-coherent sheaf $\mathcal{F}_U$ on U.

(ii) For each pair of opens U and V and a diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ & \searrow \quad \swarrow & \\ & \mathcal{X} & \end{array}$$

an isomorphism $\alpha_f \colon \mathcal{F}_U \to f^*\mathcal{F}_V$. The isomorphisms α_f are required to satisfy the cocycle condition, i.e. given atlases U, V and W and a commutative diagram

$$\begin{array}{ccccc} U & \xrightarrow{f} & V & \xrightarrow{g} & W \\ & \searrow & \downarrow & \swarrow & \\ & & \mathcal{X} & & \end{array}$$

we have

$$\alpha_{g\circ f} = \alpha_f \circ f^*\alpha_g \colon \mathcal{F}_U \xrightarrow{\cong} (g \circ f)^*\mathcal{F}_W = f^*(g^*\mathcal{F}_W).$$

A *coherent sheaf* on $\mathcal{X}$ is a quasi-coherent sheaf $\mathcal{F}$ such that all $\mathcal{F}_U$ are coherent.

Definition 2.3.2. Let $\mathcal{F}$ and $\mathcal{G}$ be quasi-coherent sheaves on a Deligne-Mumford stack $\mathcal{X}$. A *morphism* $f \colon \mathcal{F} \to \mathcal{G}$ is a collection of morphisms $f_U \colon \mathcal{F}_U \to \mathcal{G}_U$ for each atlas $U \to \mathcal{X}$ compatible with the isomorphisms α_f.

We write $\mathrm{QCoh}(\mathcal{X})$ for the category of quasi-coherent sheaves on a Deligne-Mumford stack $\mathcal{X}$. This is an abelian category with the category of coherent sheaves, denoted by $\mathrm{Coh}(\mathcal{X})$, being an abelian subcategory.

Example 2.3.3. *Let $\mathcal{X}$ be a Deligne-Mumford stack. The* structure sheaf $\mathcal{O}_{\mathcal{X}}$ *of $\mathcal{X}$ is defined by $(\mathcal{O}_{\mathcal{X}})_U = \mathcal{O}_U$ for any atlas $U \to \mathcal{X}$.*

Example 2.3.4. *Let R be a commutative ring and let $\mathbb{A}^n_R$ denote n-dimensional affine space over R. The category of quasi-coherent modules on $[\mathbb{A}^{n+1}_R/\mathbb{G}_m]$ is equivalent to the category of graded $R[t_0, \ldots, t_n]$-modules. Let m be an integer. If M is a graded module write $M[m]$ for the graded module M shifted by m, i.e. $M[m]_k = M_{m+k}$. Consider now $R[t_0, \ldots, t_n]$ as a graded module over itself, then $R[t_0, \ldots, t_n][m]$ corresponds to a line bundle on $[\mathbb{A}^{n+1}_R/\mathbb{G}_m]$ and we denote its restriction to $\mathcal{P}_R(a_0, \ldots, a_n)$ by $\mathcal{O}(m)$. See also [54, Section 2].*

Definition 2.3.5. Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a morphism of Deligne-Mumford stacks. Let $U \to \mathcal{X}$ and $V \to \mathcal{Y}$ be atlases and let $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$ and $\mathcal{G} \in \mathrm{QCoh}(\mathcal{X})$. Define

(i) the *inverse image* of $\mathcal{F}$ along f by $(f^{-1}(\mathcal{F}))_U \stackrel{\text{def}}{=} \mathcal{F}_{f(U)}$,

(ii) the *pullback* of $\mathcal{F}$ along f by $(f^*(\mathcal{F}))_U \stackrel{\text{def}}{=} \mathcal{F}_{f(U)} \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{O}_\mathcal{X}$,

(iii) the *pushforward* of $\mathcal{G}$ along f by $(f_*(\mathcal{G}))_V \stackrel{\text{def}}{=} \lim_{f(U)\to V} \mathcal{F}_V$.

2.4. Coarse moduli spaces

Definition 2.4.1. Let $\mathcal{X}$ be a Deligne-Mumford stack. A *coarse moduli space* of $\mathcal{X}$ is a scheme X together with a morphism $f\colon \mathcal{X} \to X$ such that

(i) f is initial among all morphisms from $\mathcal{X}$ to schemes,

(ii) f induces a bijection on $\pi_0(\mathcal{X}(\mathrm{Spec}(k))) \to \pi_0(X(\mathrm{Spec}(k)))$ for every algebraically closed field k, where π_0 denotes the set of isomorphism classes.

The existence of coarse moduli spaces is guaranteed for many Deligne-Mumford stacks by the following theorem.

Theorem 2.4.2 ([41], [57, Theorem 11.1.2])**.** *Let $\mathcal{X}$ be locally noetherian separated Deligne-Mumford stack. Then $\mathcal{X}$ has a coarse moduli space $f\colon \mathcal{X} \to X$. Moreover, X is separated if $\mathcal{X}$ is separated and the morphism f is proper and quasi-affine.*

2.5. The moduli stack of elliptic curves and friends

In this section we introduce the moduli stack of elliptic curves and its variations, the moduli stacks of elliptic curves with level structure. These play a central role in the definition of the ring spectra of topological modular forms in Chapter 5. We follow [55].

Definition 2.5.1. Let k be an algebraically closed field. An *elliptic curve over k* is a connected smooth proper curve E of genus 1 together with a chosen k-point $e \in E(\mathrm{Spec}(k))$.

Definition 2.5.2. Let S be a scheme. An *elliptic curve over S* is a smooth proper morphism $p\colon E \to S$ of schemes equipped with a section $e\colon S \to E$ such that every geometric fibre of E is an elliptic curve, i.e. for any morphism $f\colon \mathrm{Spec}(k) \to E$, with k an algebraically closed field, the pullback of E is an elliptic curve.

Definition 2.5.3. Let (E, p, e) and (E', p', e') be elliptic curves over schemes S and S', respectively. A *morphism of elliptic curves* is a pair

$$(f, g)\colon (E, p, e) \to (E', p', e)$$

consisting of morphisms of schemes $f\colon S \to S'$ and $g\colon E \to E'$ such that $g \circ e = e' \circ f$ and such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{g} & E' \\ \downarrow{\scriptstyle p} & \lrcorner & \downarrow{\scriptstyle p'} \\ S & \xrightarrow{f} & S' \end{array}$$

is cartesian.

We connect the definition of an elliptic curve above with the more classical notion of an elliptic curve defined by a Weierstraß equation in the following lemma.

Lemma 2.5.4 ([55, Lemma 4.3])**.** *Let R be a commutative ring and let E denote the closed subscheme of $\mathbb{P}^2_R$*

$$\mathrm{Proj}(R[x,y,z])/(y^2z + a_1xyz + a_3yz^2 = x^3 + a_2x^2z + a_4xz^2 + a_6z).$$

Then E is an elliptic curve over $\mathrm{Spec}(R)$ if the discriminant

$$\Delta = -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6$$

is invertible in R, where

$$\begin{aligned} b_2 &= a_1^2 + 4a_2 \\ b_4 &= 2a_4 + a_1a_3 \\ b_6 &= a_3^2 + 4a_6 \\ b_8 &= a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2 \end{aligned}$$

Definition 2.5.5. Define the *moduli stack of elliptic curves* $\mathcal{M}_{ell}$ to be the category fibred in groupoids over $\mathrm{Sch}_{/\,\mathrm{Spec}(\mathbb{Z})}$ with objects of the form $(S,(E,p,e))$ where S is a scheme and (E,p,e) an elliptic curve over S and with morphisms the morphisms of elliptic curves. The functor $p_{\mathcal{M}_{ell}}$ is defined as

$$\begin{aligned} p_{\mathcal{M}_{ell}}\colon \mathcal{M}_{ell} &\longrightarrow \mathrm{Sch}_{/\,\mathrm{Spec}(\mathbb{Z})} \\ (S,(E,p,e)) &\longmapsto S \\ (f,g)\colon (E,p,e) \to (E',p',e) &\longmapsto f\colon S \to S'. \end{aligned}$$

Theorem 2.5.6 ([55, Theorem 3.2])**.** *$\mathcal{M}_{ell}$ is a stack in the fpqc, and thus in the étale, topology on $\mathrm{Sch}_{/\,\mathrm{Spec}(\mathbb{Z})}$.*

In the definitions below n is always assumed to be invertible on S. For an elliptic curve E we write $E[n](S)$ for the n-torsion points of E.

Definition 2.5.7. Define the stacks $\mathcal{M}_0(n)$, $\mathcal{M}_1(n)$ and $\mathcal{M}(n)$ as the categories fibred in groupoids whose fibre at a scheme S is

$$\mathcal{M}_0(n)(S) \overset{\text{def}}{=} \{ \text{ Elliptic curves } E \text{ over } S \text{ with chosen cyclic subgroup } H \subset E \text{ of order } n \ \},$$
$$\mathcal{M}_1(n)(S) \overset{\text{def}}{=} \{ \text{ Elliptic curves } E \text{ over } S \text{ with chosen point } P \in E(S) \text{ of exact order } n \ \},$$
$$\mathcal{M}(n)(S) \overset{\text{def}}{=} \{ \text{ Elliptic curves } E \text{ over } S \text{ with chosen isomorphism } (\mathbb{Z}/n)^2 \cong E[n](S) \ \}.$$

We will also consider the compactification $\overline{\mathcal{M}}_{ell}$ of the moduli stack $\mathcal{M}_{ell}$. In the compactification we allow not only smooth elliptic curves, but also *generalized elliptic curves*. A generalized elliptic curve over an algebraically closed field is either a smooth elliptic curve or a Néron n-gon. These are obtained by gluing n copies of $\mathbb{P}^1$ and identifying (i, ∞) with $(i+1, 0)$ for all i. See [12, Section II.1] for precise definitions. The compactified versions $\overline{\mathcal{M}}_0(n)$, $\overline{\mathcal{M}}_1(n)$ and $\overline{\mathcal{M}}(n)$ are defined as the normalization of $\overline{\mathcal{M}}_{ell}$ in $\mathcal{M}_0(n)$, $\mathcal{M}_1(n)$ and $\mathcal{M}(n)$, respectively, see [12, p. IV.3].

2.6. The moduli stack of formal groups

In this section we introduce another example of a stack, the moduli stack of formal groups. This stack will make an appearance when we discuss derived stacks in the next section and will play a central role in the definition of the ring spectra of topological modular forms in Chapter 5. We follow [53, Section 2.8].

Informally, the moduli stack of formal groups $\mathcal{M}_{fg}$ is the category fibred in groupoids whose fibre at an affine scheme $\mathrm{Spec}(R)$ is the groupoid of one dimensional commutative formal groups over R and isomorphisms between them. We recall the definitions of the various ingredients below.

Definition 2.6.1. A *formal scheme* over S is a contravariant functor $(\mathrm{Sch}_{/S})^{\mathsf{op}} \to \mathrm{Set}$ that is a filtered colimit of representable functors $\mathrm{Hom}_{\mathrm{Sch}_{/S}}(-, X)$ for X an S-scheme. We write $\mathrm{fSch}_{/S}$ for the functor category of formal schemes over S and natural transformations between them.

The cartesian product in the category $\mathrm{Fun}((\mathrm{Sch}_{/S})^{\mathsf{op}}, \mathrm{Set})$ restricts to a product in the category of formal schemes over S which we denote by $\times_S$. This allows us to talk about group objects in the category $\mathrm{fSch}_{/S}$.

Definition 2.6.2. An *abstract formal group* over S is a commutative group object in $\mathrm{fSch}_{/S}$.

Example 2.6.3. *Let R be a commutative ring and $I \subset R$ be an ideal. Form the colimit*

$$\mathrm{Spf}(R) \overset{\text{def}}{=} \mathrm{colim}_n \, \mathrm{Spec}(R/I^n).$$

This is a formal scheme called the formal spectrum *of R.*

Definition 2.6.4. Let R be a commutative ring. A *commutative formal group law* over R is a power series $F \in R[[x, y]]$ such that

(i) $F(x, y) = x + y +$ terms of higher degree,

(ii) $F(x, y) = F(y, x)$,

(iii) $F(x, F(y, z)) = F(F(x, y), z)$.

Consider now the ring $R[[x]]$. The formal spectrum $\operatorname{Spf} R[[x]] = \operatorname{colim}_n \operatorname{Spec} R[x]/x^n$ is a formal scheme. Let F be a formal group law over R. Then F induces a morphism

$$\operatorname{Spf} R[[x]] \times_{\operatorname{Spec}(R)} \operatorname{Spf} R[[y]] \cong \operatorname{Spf} R[[x, y]],$$

given by sending x to y, which defines an abstract formal group over R. Here the ideal of $R[[x, y]]$ that we consider is the augmentation ideal $I = (x, y)$.

Definition 2.6.5. A 1-*dimensional commutative formal group* over S is an abstract formal group F in the sense of Definition 2.6.2 which Zariski locally is isomorphic to an abstract formal group coming from a formal group law.

Definition 2.6.6. The *moduli stack of formal groups* is the category fibred in groupoids $\mathcal{M}_{fg}$ whose fibre at an affine scheme $\operatorname{Spec}(R)$ is

$$\mathcal{M}_{fg}(\operatorname{Spec}(R)) \overset{\text{def}}{=} \{\text{Formal groups over } R \text{ and isomorphisms}\}.$$

2.7. Derived stacks

The moduli stack of elliptic curves and its variations, the moduli stacks of elliptic curves with level structure, are examples of (even periodic) derived stacks, i.e. stacks equipped with an even periodic enhancement. We review this notion below to set up the stage for the definition of the ring spectra of topological modular forms in Chapter 5. We follow [47].

Let $\mathcal{X}$ be a Deligne-Mumford stack equipped with a flat map $\mathcal{X} \to \mathcal{M}_{fg}$ to the moduli stack of formal groups and write $\mathcal{X}^{\text{aff,flat}}$ for the affine flat site of $\mathcal{X}$. The assignment

$$\begin{aligned} \mathcal{X}^{\text{aff,flat}} &\longrightarrow \{\text{homology theories}\} \\ \operatorname{Spec}(R) \to \mathcal{X} &\longmapsto MU_*(-) \otimes_{MU_*} R \end{aligned}$$

defines a presheaf of homology theories on $\mathcal{X}$ which we denote by $\mathcal{O}_{\mathcal{X}}^{\text{hom}}$.

Definition 2.7.1 ([47, Definition 2.5])**.** An *even periodic enhancement* $\mathfrak{X}$ of $\mathcal{X}$ is a sheaf $\mathcal{O}_{\mathcal{X}}^{\mathsf{top}}$ of even periodic commutative ring spectra on the affine étale site $\mathcal{X}^{\mathsf{aff,ét}}$ that extends the presheaf $\mathcal{O}_{\mathcal{X}}^{\mathsf{hom}}$ in the sense that

$$\begin{array}{ccc} & & \mathrm{CAlg}(\mathrm{Sp}) \\ & \overset{\mathcal{O}_{\mathcal{X}}^{\mathsf{top}}}{\nearrow} & \downarrow \\ \mathcal{X}^{\mathsf{aff,ét}} & \xrightarrow[\mathcal{O}_{\mathcal{X}}^{\mathsf{hom}}]{} & \{\text{homology theories}\}\,. \end{array}$$

Even periodic enhancements are examples of (even periodic) derived stacks.

Definition 2.7.2 ([47, Definition 2.6])**.** A *derived stack* is a pair $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}^{\mathsf{top}})$ where $\mathcal{X}$ is an ordinary Deligne-Mumford stack $\mathcal{X}$ and $\mathcal{O}_{\mathcal{X}}^{\mathsf{top}}$ is a sheaf of commutative ring spectra on $\mathcal{X}^{\mathsf{aff,ét}}$ together with an isomorphism $\pi_0(\mathcal{O}_{\mathcal{X}}^{\mathsf{top}}) \cong \mathcal{O}_{\mathcal{X}}$. Here $\pi_i(\mathcal{O}_{\mathcal{X}}^{\mathsf{top}})$ is the sheaf $U \mapsto \pi_i(\mathcal{O}_{\mathcal{X}}^{\mathsf{top}}(U))$. Furthermore, we require that $\pi_i(\mathcal{O}_{\mathcal{X}}^{\mathsf{top}})$ is quasi-coherent as an $\mathcal{O}_{\mathcal{X}}$-module. The derived stack $\mathfrak{X}$ is called *even periodic* if $\omega = \pi_2(\mathcal{O}_{\mathcal{X}}^{\mathsf{top}})$ is a line bundle such that

$$\pi_i(\mathcal{O}_{\mathcal{X}}^{\mathsf{top}}) = 0 \text{ for } i \text{ odd}$$

and multiplication induces isomorphisms

$$\pi_{2k}(\mathcal{O}_{\mathcal{X}}^{\mathsf{top}}) \otimes \pi_{2l}(\mathcal{O}_{\mathcal{X}}^{\mathsf{top}}) \cong \pi_{2(k+l)}(\mathcal{O}_{\mathcal{X}}^{\mathsf{top}}) \text{ for } k, l \in \mathbb{Z}.$$

Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}^{\mathsf{top}})$ be a derived stack. For a commutative ring spectrum $\mathbf{R}$ we write $\mathbf{R}\text{-mod}$ for the category of modules over $\mathbf{R}$.

Definition 2.7.3 ([47, Definition 2.9])**.** Define the category of *quasi-coherent sheaves on* $\mathfrak{X}$ as the limit

$$\mathrm{QCoh}(\mathfrak{X}) \overset{\mathsf{def}}{=} \lim_{\mathrm{Spec}(R) \to \mathcal{X}} \mathcal{O}_{\mathcal{X}}^{\mathsf{top}}(\mathrm{Spec}(R))\text{-mod},$$

where $\mathrm{Spec}(R) \to \mathcal{X}$ ranges over all étale morphisms from affine schemes into $\mathcal{X}$.

Definition 2.7.4. Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathfrak{X}$. Define the *global sections* of $\mathcal{F}$ as the limit

$$\Gamma(\mathcal{X}, \mathcal{F}) \overset{\mathsf{def}}{=} \lim_{\mathrm{Spec}(R) \to \mathcal{X} \in \mathcal{X}^{\mathsf{aff,ét}}} \mathcal{F}(\mathrm{Spec}(R)).$$

Theorem 2.7.5 ([47, Theorem 1.4])**.** *There is a categorical equivalence*

$$\mathrm{QCoh}(\mathfrak{X}) \simeq \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^{top})\text{-mod}.$$

Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathfrak{X}$ and let k be an integer. The assignment

$$\begin{aligned} \mathcal{X}^{\mathsf{aff,ét}} &\longrightarrow \mathrm{CRing} \\ \mathrm{Spec}(R) \to \mathcal{X} &\longmapsto \pi_k(\mathcal{F}(\mathrm{Spec}(R)) \end{aligned}$$

defines a quasi-coherent sheaf $\pi_i(\mathcal{F})$ on the ordinary stack $\mathcal{X}$. The homotopy groups $\pi_k(\mathcal{F})$ will be of interest to us for the following reason.

Construction 2.7.6 ([14, Chapter 5])**.** There is a spectral sequence, called the *descent spectral sequence*, with E_2 page given by

$$E_2^{st} = H^s(\mathcal{X}, \pi_t(\mathcal{F})) \Longrightarrow \pi_{t-s}(\Gamma(\mathcal{X}, \mathcal{F})).$$

In particular, taking $\mathcal{F} = \mathcal{O}_{\mathcal{X}}^{\mathsf{top}}$ and letting $\mathbf{R} = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^{\mathsf{top}})$ we get

$$E_2^{st} = H^s(\mathcal{X}, \pi_t(\omega^{\otimes \frac{t}{2}})) \Longrightarrow \pi_{t-s}(\mathbf{R}).$$

Chapter 3

Equivariant Duality

This chapter comprises the heart of the thesis. We set up an equivariant generalization of the Gorenstein duality framework introduced by Dwyer, Greenlees and Iyengar in [17] and explored non-equivariantly by Greenlees and Stojanoska in [29] and equivariantly by Greenlees and Meier in [27]. The framework allows one to view a number of dualities such as

- Poincaré duality for manifolds,
- Gorenstein duality for commutative rings,
- Benson-Carlson duality for cohomology rings of finite groups,
- Poincaré duality for groups

as instances of a single phenomenon. In the course of the chapter we introduce all of the ingredients in the following, at this stage provisional, definition of equivariant Gorenstein duality.

Definition. A map $\mathbf{R} \to \mathbf{k}$ of non-equivariantly commutative ring G-spectra is said to have *Gorenstein duality* of shift $\alpha \in RO(G)$ if there is an equivalence of $\mathbf{R}$-modules

$$\underbrace{\mathrm{Cell}_{\mathbf{k}}(\mathbf{R})}_{\mathbf{k}\text{-cellularization of } \mathbf{R}} \simeq \underbrace{\Sigma^{\alpha}}_{\text{representation suspension}} \underbrace{\mathcal{I}_{\mathbf{k}}.}_{\text{Matlis lift of } \mathbf{k}}$$

We start by discussing the equivariant generalizations of Brown-Comenetz and Anderson duality in Section 3.1. Both of these phenomena have been studied extensively in classical stable homotopy theory. Our interest in them lies in the fact that the Anderson dual of a connective non-equivariantly commutative ring G-spectrum will provide an example of a Matlis lift and will thus make sense of the right hand side in the provisional definition above. In Section 3.2 we look at the notions of cellularization and nullification in the category of modules over a ring G-spectrum. We then set up a form of Morita theory in Section 3.3 and define and characterize Matlis lifts in Section 3.4. With all ingredients at hand we define equivariant Gorenstein duality in Section 3.6. The algebraic models for cellularization and nullification and the arising local cohomology spectral sequence are examined in Sections 3.7 and 3.8. In Section 3.9 we trace the relations between Anderson and Gorenstein duality.

We extend the notational conventions introduced in Chapter 1 by adopting the following somewhat suggestive notation:

- *Non-connective* ring G-spectra will be denoted by upper-case bold letters like $\mathbf{R}$.
- *Connective* ring G-spectra will be denoted by lower-case bold letters like $\mathbf{r}$.

If $\mathbf{R}$ is a ring G-spectrum we write $\mathbf{R}\text{-mod}_G$ for the category of left $\mathbf{R}$-modules. All unspecified modules will be left modules. Given a map $f\colon \mathbf{S} \to \mathbf{R}$ of ring G-spectra we denote the restriction of scalars functor from $\mathbf{R}$-modules to $\mathbf{S}$-modules by f^*. It has both left and right adjoints

$$\mathbf{R}\text{-mod}_G \underset{f_*}{\overset{f_!}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{f^*}}}}} \mathbf{S}\text{-mod}_G$$

given by the induction $f_!(-) = - \otimes_{\mathbf{S}} \mathbf{R}$ and coinduction $f_*(-) = \mathrm{Hom}_{\mathbf{S}}(\mathbf{R}, -)$ functors, respectively. When the map f is implicit from the context we write $M{\Uparrow}_{\mathbf{S}}^{\mathbf{R}}$ for the coinduced $\mathbf{R}$-module $\mathrm{Hom}_{\mathbf{S}}(\mathbf{R}, M)$. We also often omit the restriction of scalars functor f^* from the notation.

3.1. Equivariant Brown-Comenetz and Anderson duality

In this section we look at an equivariant generalization of Brown-Comenetz and Anderson duality first introduced in [8]. Anderson duality has been studied extensively non-equivariantly, see [63], [32] and equivariantly in [59], [35] and [52].

Let $\mathbf{S} \to \mathbf{R}$ be a map of commutative ring G-spectra and $T_* \to \pi_*^G(\mathbf{S}) \to \pi_*^G(\mathbf{R})$ be maps of graded commutative rings. We do not require that T_* is the homotopy of a commutative ring G-spectrum and we often take $\mathbf{S}$ to be the sphere G-spectrum S^0 or $\mathbf{R}$. Recall from Section 1.4 that given an injective abelian group I we write D_I for the duality functor

$$\begin{aligned} D_I\colon \mathrm{Mack}_G &\longrightarrow \mathrm{Mack}_G \\ \underline{M} &\longmapsto (X \mapsto \mathrm{Hom}_{\mathbb{Z}}(\underline{M}(X), I)). \end{aligned}$$

Now take an injective T_*-module J. The functor

$$\begin{aligned} \mathbf{R}\text{-mod}_G &\longrightarrow \mathrm{grAb} \\ X &\longmapsto \mathrm{Hom}_{T_*}(\pi_{-*}^G(X), J) \end{aligned}$$

is exact and therefore, by the Brown representability theorem, is represented by an $\mathbf{R}$-module $J^{\mathbf{R}} = J_{T_*}^{\mathbf{R}}$ in the sense that for any $\mathbf{R}$-module X we have

$$[X, J_{T_*}^{\mathbf{R}}]_*^{\mathbf{R},G} = \mathrm{Hom}_{T_*}(\pi_{-*}^G(X), J).$$

Definition 3.1.1. We call $J_{T_*}^{\mathbf{R}}$ the *Brown-Comenetz J-dual of* $\mathbf{R}$. More generally, given an $\mathbf{R}$-module M we define the *Brown-Comenetz J-dual of M* by the equation

$$J^M = J_{T_*}^M \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathbf{R}}(M, J_{T_*}^{\mathbf{R}}).$$

It then follows that

$$[X, J_{T_*}^M]_*^{\mathbf{R},G} = \mathrm{Hom}_{T_*}(\pi_{-*}(X \otimes_{\mathbf{R}} M), J).$$

Proposition 3.1.2. *The Brown-Comenetz dual enjoys the following properties.*

(1) The homotopy groups of $J_{T_*}^{\mathbf{R}}$ are completely determined by the definition

$$\pi_*^G(J_{T_*}^{\mathbf{R}}) = \mathrm{Hom}_{T_*}(\pi_{-*}^G(\mathbf{R}), J).$$

(2) If $\underline{M}$ is a Mackey functor and I is an injective abelian group, then

$$I_{\mathbb{Z}}^{H\underline{M}} \simeq HD_I(\underline{M}).$$

(3) Given $\mathbf{S} \to \mathbf{R}$ and $T_* \to \pi_*^G(\mathbf{S}) \to \pi_*^G(\mathbf{R})$, we have

$$J_{T_*}^{\mathbf{R}} \simeq (J_{T_*}^{\mathbf{S}})\Uparrow_{\mathbf{S}}^{\mathbf{R}}$$

(4) and

$$(J\Uparrow_{T_*}^{\pi_*^G(\mathbf{R})})_{\pi_*^G(\mathbf{R})}^{\mathbf{R}} \simeq J_{T_*}^{\mathbf{R}}.$$

Proof. Property (1) follows from the definition. To show (2) we first need to understand the homotopy Mackey functors $\underline{\pi}_*(J_{T_*}^{\mathbf{R}})$ of the Brown-Comenetz dual. Let X vary over $G/H_+ \wedge Y$ for a fixed Y. We then have

$$\begin{array}{ccc} [G/H_+ \wedge Y, J_{T_*}^{\mathbf{R}}]_*^{\mathbf{R},G} & = & \mathrm{Hom}_{T_*}(\pi_{-*}^G(G/H_+ \wedge Y), J) \\ \downarrow \cong & & \downarrow \cong \\ [Y, \mathrm{res}_H^G(J_{T_*}^{\mathbf{R}})]_*^{\mathbf{R},H} & = & \mathrm{Hom}_{T_*}(\pi_{-*}^H(Y), J) \end{array}$$

and so the value of $\underline{\pi}_*(J_{T_*}^{\mathbf{R}})$ at G/H_+ is given by

$$\underline{\pi}_*(J_{T_*}^{\mathbf{R}})(G/H_+) \stackrel{\text{def}}{=} \pi_*^H(J_{T_*}^{\mathbf{R}}) = [\mathbf{R}, \mathrm{res}_H^G(J_{T_*}^{\mathbf{R}})]_*^{\mathbf{R},H} = \mathrm{Hom}_{T_*}(\pi_{-*}^H(\mathbf{R}), J).$$

To describe the restriction and induction maps suppose that we are given subgroups $K \leq H$ of G and let $p_H^K \colon G/K_+ \to G/H_+$ denote the projection map. We can identify the restriction map from H to K from the diagram

$$\begin{array}{ccc}
[G/H_+ \wedge Y, J_{T_*}^{\mathbf{R}}]_*^{\mathbf{R},G} & = & \mathrm{Hom}_{T_*}(\pi_{-*}^G(G/H_+ \wedge Y), J) \\
\Big\downarrow \mathrm{res}_K^H = (p_H^K)^* & & \Big\downarrow \mathrm{Hom}_{T_*}((p_H^K)_*, J) \\
[G/K_+ \wedge Y, J_{T_*}^{\mathbf{R}}]_*^{\mathbf{R},G} & = & \mathrm{Hom}_{T_*}(\pi_{-*}^G(G/K_+ \wedge Y), J)
\end{array}$$

and similarly for the induction map. Part (2) is now immediate. Both the left and the right hand side have homotopy Mackey functors concentrated in degree 0 where we have

$$\begin{aligned}
\underline{\pi}_0(I_{\mathbb{Z}}^{H\underline{M}})(G/H_+) &\cong \mathrm{Hom}_{\mathbb{Z}}(\pi_0^H(H\underline{M}), I) \\
&= \mathrm{Hom}_{\mathbb{Z}}(\underline{M}(G/H_+), I) \\
&= \underline{\pi}_0(HD_I(\underline{M}))(G/H_+).
\end{aligned}$$

Properties (3) and (4) can be checked on the homotopy groups where the claims are verified by direct computations. For (3) we compute

$$\begin{aligned}
[X, J_{T_*}^{\mathbf{R}}]^{\mathbf{R},G} &= \mathrm{Hom}_{T_*}(\pi_{-*}^G(X), J) \\
&= [X, J_{T_*}^{\mathbf{S}}]^{\mathbf{S},G} \\
&= [X, \mathrm{Hom}_{\mathbf{S}}(\mathbf{R}, J_{T_*}^{\mathbf{S}})]^{\mathbf{R},G} \\
&= [X, (J_{T_*}^{\mathbf{S}})\Uparrow_{\mathbf{S}}^{\mathbf{R}}]^{\mathbf{R},G}.
\end{aligned}$$

Analogously for (4) we have

$$\begin{aligned}
[X, (J\Uparrow_{T_*}^{\pi_*^G(\mathbf{R})})_{\pi_*^G(\mathbf{R})}^{\mathbf{R}}]^{\mathbf{R},G} &= \mathrm{Hom}_{\pi_*^G(\mathbf{R})}(\pi_{-*}^G(X), J\Uparrow_{T_*}^{\pi_*^G(\mathbf{R})}) \\
&= \mathrm{Hom}_{\pi_*^G(\mathbf{R})}(\pi_{-*}^G(X), \mathrm{Hom}_{T_*}(\pi_*^G(\mathbf{R}), J)) \\
&= \mathrm{Hom}_{T_*}(\pi_{-*}^G(X), J) \\
&= [X, J_{T_*}^{\mathbf{R}}]^{\mathbf{R},G}.
\end{aligned}$$

□

Example 3.1.3. *Let A be a finite group and write HA for the associated Eilenberg-MacLane spectrum. Then the Brown-Comenetz $\mathbb{Q}/\mathbb{Z}$-dual of HA is $HA^* \simeq HA$, i.e. Eilenberg-MacLane spectra are Brown-Comenetz self-dual.*

Next we discuss the equivariant generalization of Anderson duality. This was explored by Ricka in [59] and Hill and Meier in [35]. Let again $\mathbf{S} \to \mathbf{R}$ be a map of commutative ring G-spectra, let $T_* \to \pi_*^G(\mathbf{S}) \to \pi_*^G(\mathbf{R})$ be maps of graded commutative rings and suppose that we are given an arbitrary T_*-module L of injective dimension 1 together with an injective resolution

$$0 \to L \to J_0 \to J_1 \to 0.$$

Definition 3.1.4. We define $L^{\mathbf{R}}_{T_*}$, the *Anderson L-dual of* $\mathbf{R}$, as

$$L^{\mathbf{R}} = L^{\mathbf{R}}_{T_*} \overset{\text{def}}{=} \operatorname{fib}((J_0)^{\mathbf{R}}_{T_*} \to (J_1)^{\mathbf{R}}_{T_*}).$$

Analogously to the Brown-Comenetz case, given an $\mathbf{R}$-module M we define the *Anderson L-dual* of M by the equation

$$L^M = L^M_{T_*} \overset{\text{def}}{=} \operatorname{Hom}_{\mathbf{R}}(M, L^{\mathbf{R}}_{T_*}).$$

One can show that this does not depend on the choice of the injective resolution.

Proposition 3.1.5. *The Anderson dual enjoys the following properties.*

(1) The homotopy groups of $L^{\mathbf{R}}_{T_*}$ lie in a short exact sequence of $RO(G)$-graded homotopy Mackey functors

$$0 \to \operatorname{Ext}^1_{T_*}(\underline{\pi}_{-V-1}(\mathbf{R}), L) \to \underline{\pi}_V(L^{\mathbf{R}}_{T_*}) \to \operatorname{Hom}_{T_*}(\underline{\pi}_{-V}(\mathbf{R}), L) \to 0.$$

(2) If $\underline{M}$ is a Mackey functor with $\underline{M}(G/H)$ projective for all $H \leq G$, then

$$L^{H\underline{M}}_{T_*} \simeq HD_L(\underline{M}).$$

(3) Given $\mathbf{S} \to \mathbf{R}$ and $T_* \to \pi^G_*(\mathbf{S}) \to \pi^G_*(\mathbf{R})$, we have

$$L^{\mathbf{R}}_{T_*} \simeq (L^{\mathbf{S}}_{T_*})\Uparrow^{\mathbf{R}}_{\mathbf{S}},$$

(4) and

$$(L\Uparrow^{\pi^G_*(\mathbf{R})}_{T_*})^{\mathbf{R}}_{\pi^G_*(\mathbf{R})} \simeq L^{\mathbf{R}}_{T_*}.$$

Proof. The proof is analogous to the corresponding statement about Brown-Comenetz duals. □

Comment. One reason to be interested in Anderson self-duality is that it leads to universal coefficient sequences relating homology and cohomology. Let E and X be G-spectra and let A be an abelian group. We have a short exact sequence

$$0 \to \operatorname{Ext}^1_{\mathbb{Z}}(E_{-V-1}(X), A) \to (A^E)^V(X) \to \operatorname{Hom}_{\mathbb{Z}}(E_{-V}(X), A) \to 0.$$

where V is a G-representation. If E is Anderson self-dual in the sense that A^E is equivalent to a suspension of E this sequence implies a universal coefficient theorem. See [40].

We give a number of examples, both non-equivariant and equivariant, of Anderson self-dual spectra. In all of the cases below $\mathbf{S} = S^0$ and $T_* = \mathbb{Z}$ concentrated in degree zero.

Example 3.1.6. *The ring spectra KO and KU of periodic real and complex K-theory are Anderson self-dual with shifts 2 and 4 respectively*

$$\mathbb{Z}^{KU} \simeq \Sigma^2 KU,$$
$$\mathbb{Z}^{KO} \simeq \Sigma^4 KO.$$

Example 3.1.7. *The ring spectrum Tmf of topological modular forms has been shown by Stojanoska in [63] and [62] to be Anderson self-dual with shift 21*

$$\mathbb{Z}^{\mathrm{Tmf}} \simeq \Sigma^{21}\,\mathrm{Tmf}\,.$$

Example 3.1.8. *Let $G = C_2$ and let $H\underline{\mathbb{Z}}$ be the Eilenberg-MacLane spectrum associated to the constant Mackey functor concentrated at $\mathbb{Z}$. It is shown in [27, Lemma 3.1] that*

$$\mathbb{Z}^{H\underline{\mathbb{Z}}} \simeq \Sigma^{2-2\sigma} H\underline{\mathbb{Z}}.$$

More generally, let G be a cyclic group of any order and write ε for the trivial one dimensional complex representation of G and α for a faithful one dimensional complex representation of G. Then, see [27, Remark 3.2], one has

$$\mathbb{Z}^{H\underline{\mathbb{Z}}} \simeq \Sigma^{\varepsilon-\alpha} H\underline{\mathbb{Z}}.$$

3.2. Cellularization and nullification

In this section we apply the theory of arithmetic fracture squares introduced in Section 1.2 to discuss the notions of cellularization and nullification in the category of modules over a ring G-spectrum. Our exposition is a mixture of [17], [3] and [48].

A permanent ingredient from here onwards will be a map $\mathbf{R} \to \mathbf{k}$ of ring G-spectra. We write $\boldsymbol{\mathcal{E}}$ for the endomorphism ring spectrum $\mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, \mathbf{k})$ and mod_G-$\boldsymbol{\mathcal{E}}$ for the category of right $\boldsymbol{\mathcal{E}}$-modules. The case we are mostly interested in is where $\mathbf{R}$ is connective, $\mathbf{k}$ is $H\underline{\pi}_0(\mathbf{R})$, the Eilenberg-MacLane spectrum associated to the 0-th homotopy Mackey functor of $\mathbf{R}$, and the map $\mathbf{R} \to \mathbf{k}$ is obtained by killing homotopy groups. The notation should remind the reader of the classical commutative algebra situation where one has a commutative local ring R with maximal ideal $\mathfrak{m}_R$ and residue field k_R and the map $R \to k_R$ is reduction modulo the maximal ideal.

The category $\mathbf{R}\text{-}\mathrm{mod}_G$ of modules over a ring G-spectrum is a presentable, symmetric monoidal, stable ∞-category and satisfies Hypotheses 1.2.1. Applying the theory from Section 1.2 to $\mathbf{R}\text{-}\mathrm{mod}_G$ and the $\mathbf{R}$-module $\mathbf{k}$ we get the subcategories $\mathbf{R}\text{-}\mathrm{mod}_G^{\mathbf{k}\text{-null}}$, $\mathbf{R}\text{-}\mathrm{mod}_G^{\mathbf{k}\text{-comp}}$ and $\mathbf{R}\text{-}\mathrm{mod}_G^{\mathbf{k}\text{-cell}}$ of $\mathbf{k}$-null, $\mathbf{k}$-complete and $\mathbf{k}$-cellular $\mathbf{R}$-modules, respectively. For an $\mathbf{R}$-

module M we have a $\mathbf{k}$-arithmetic fracture square

$$\begin{array}{ccc} M & \longrightarrow & (M)^\wedge_{\mathbf{k}} \\ \downarrow & & \downarrow \\ \mathrm{Null}_{\mathbf{k}}(M) & \longrightarrow & \mathrm{Null}_{\mathbf{k}}((M)^\wedge_{\mathbf{k}}) \end{array}$$

and a cofibre sequence

$$\mathrm{Cell}_{\mathbf{k}}(M) \to M \to \mathrm{Null}_{\mathbf{k}}(M) \tag{3.1}$$

separating M into a $\mathbf{k}$-cellular part and a $\mathbf{k}$-null part.

Definition 3.2.1. A non-empty full subcategory of $\mathbf{R}\text{-mod}_G$ is called *thick* if it is closed under finite colimits, retracts and desuspensions. A thick subcategory is called *localizing* if in addition it is closed under filtered colimits.

Given a collection of objects $\mathcal{C}$ in $\mathbf{R}\text{-mod}_G$ we write $\mathrm{Thick}_G(\mathcal{C})$, respectively $\mathrm{Loc}_G(\mathcal{C})$, for the smallest thick, respectively localizing, subcategory of $\mathbf{R}\text{-mod}_G$ containing the objects $G/H_+ \wedge X$ where H is a subgroup of G and X is an element of $\mathcal{C}$.

Definition 3.2.2. Let M and N be $\mathbf{R}$-modules. We say that

(i) M is *finitely built* from N if M is in $\mathrm{Thick}_G(N)$ and that

(ii) M is *built* from N if M is in $\mathrm{Loc}_G(N)$.

Definition 3.2.3. An $\mathbf{R}$-module M is called

(i) *small* if it is finitely built from $\mathbf{R}$ and

(ii) *proxy-small* if there exists an $\mathbf{R}$-module $\mathrm{K}(M)$ such that $\mathrm{K}(M)$ is finitely built from $\mathbf{R}$, M is built from $\mathrm{K}(M)$ and $\mathrm{K}(M)$ is finitely built from M . The object $\mathrm{K}(M)$ is called a *Koszul complex* associated to M.

Remark. The condition that M and $\mathrm{K}(M)$ can be built from one another implies that M and $\mathrm{K}(M)$ generate the same localizing subcategory of $\mathbf{R}\text{-mod}_G$.

Definition 3.2.4. We say that a map $\mathbf{R} \to \mathbf{k}$ of ring G-spectra is *small*, respectively *proxy-small*, if $\mathbf{k}$ is small, respectively proxy-small.

It turns out that $\mathbf{k}$-cellular $\mathbf{R}$-modules are exactly those which can be built out of $\mathbf{k}$ provided that $\mathbf{k}$ is small.

Lemma 3.2.5. *Suppose that $\mathbf{k}$ is small as an $\mathbf{R}$-module. Then an $\mathbf{R}$-module M is $\mathbf{k}$-cellular if and only if it is built from $\mathbf{k}$.*

Proof. This is a special case of [60, Theorem 3.9.3]. □

Remark. Suppose that $\mathbf{k}$ is proxy-small and write $\mathrm{K}(\mathbf{k})$ for a Koszul complex of $\mathbf{k}$. Then $\mathbf{k}$ and $\mathrm{K}(\mathbf{k})$ give rise to equivalent categories of cellular objects, i.e.

$$\mathbf{R}\text{-mod}_G^{\mathbf{k}\text{-cell}} \simeq \mathbf{R}\text{-mod}_G^{\mathrm{K}(\mathbf{k})\text{-cell}}.$$

Definition 3.2.6. An $\mathbf{R}$-module M is said to be *effectively constructible from* $\mathbf{k}$ if the natural map

$$\mathrm{eval}\colon \mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, M) \otimes_{\boldsymbol{\mathcal{E}}} \mathbf{k} \to M \tag{3.2}$$

is an equivalence.

Lemma 3.2.7 ([17, Lemma 4.4])**.** *The following are equivalent.*

(1) The map (3.2) is a $\mathbf{k}$*-equivalence.*

(2) The map (3.2) is $\mathbf{k}$*-cellularization.*

(3) $\mathrm{Cell}_{\mathbf{k}}(M)$ *is effectively constructible from* $\mathbf{k}$*.*

3.3. Morita theory

We investigate the relationship between the categories of left $\mathbf{R}$-modules and right $\boldsymbol{\mathcal{E}}$-modules where $\boldsymbol{\mathcal{E}}$ still denotes the endomorphism ring $\mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, \mathbf{k})$. Start by observing that there is an adjoint pair

$$\mathbf{R}\text{-mod}_G \underset{R}{\overset{L}{\rightleftarrows}} \mathrm{mod}_G\text{-}\boldsymbol{\mathcal{E}}, \tag{3.3}$$

where the functors are given by $L(-) = - \otimes_{\boldsymbol{\mathcal{E}}} \mathbf{k}$ and $R(-) = \mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, -)$. This adjunction restricts to an adjunction between the full subcategories of $\mathbf{k}$-cellular $\mathbf{R}$-modules and right $\boldsymbol{\mathcal{E}}$-modules. Below we show that imposing smallness conditions on $\mathbf{k}$ guarantees that the counit of this adjunction is an equivalence which, in turn, will imply that all $\mathbf{R}$-modules are effectively constructible in the sense of Definition 3.2.6.

Theorem 3.3.1. *Suppose that* $\mathbf{k}$ *is small as an* $\mathbf{R}$*-module. Then the functors* L *and* R *induce an equivalence between the* $\mathbf{k}$*-cellularization of* $\mathbf{R}\text{-mod}_G$ *and the category* $\mathrm{mod}_G\text{-}\boldsymbol{\mathcal{E}}$

$$\mathbf{R}\text{-mod}_G^{\mathbf{k}\text{-cell}} \underset{R}{\overset{L}{\rightleftarrows}} \mathrm{mod}_G\text{-}\boldsymbol{\mathcal{E}}. \tag{3.4}$$

In particular, all $\mathbf{k}$*-cellular* $\mathbf{R}$*-modules are effectively constructible from* $\mathbf{k}$*.*

Proof. This is [16, Theorem 2.1] at the level of the homotopy categories or a special case of [28, Theorem 2.7]. □

We have the following partial generalization to the proxy-small case.

Theorem 3.3.2 ([17, Theorem 4.10])**.** *Suppose* $\mathbf{k}$ *is proxy-small as an* $\mathbf{R}$*-module. Then for any* $\mathbf{R}$*-module* M *the derived counit of the adjunction (3.4)*

$$\mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, M) \otimes_{\boldsymbol{\mathcal{E}}} \mathbf{k} \to M$$

is a $\mathbf{k}$*-equivalence. In particular, all* $\mathbf{R}$*-modules are effectively constructible from* $\mathbf{k}$*.*

3.4. Matlis lifts

The next ingredient we need to discuss in order to define equivariant Gorenstein duality are Matlis lifts. We recall the relevant definitions and results from [17]. The equivariant reformulation does not contain anything original and the upshot of this section is Lemma 3.4.7 which identifies the Anderson k-dual $k^{\mathbf{r}}$ of a connective non-equivariantly commutative ring G-spectrum $\mathbf{r}$, where $k = \pi_0^G(\mathbf{r})$, as a Matlis lift of $k^{\mathbf{k}}$ for $\mathbf{k} = H\underline{\pi}_0(\mathbf{r})$. The standing assumption is still that we are given a map $\mathbf{R} \to \mathbf{k}$ of ring G-spectra.

Definition 3.4.1. An $\mathbf{R}$-module $\mathcal{I}_N$ is said to be a *Matlis lift* of a $\mathbf{k}$-module N if

(i) there is an equivalence of $\mathbf{k}$-modules $\mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, \mathcal{I}_N) \simeq N$ and

(ii) $\mathcal{I}_N$ is effectively constructible from $\mathbf{k}$, i.e. the map $\mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, \mathcal{I}_N) \otimes_{\boldsymbol{\mathcal{E}}} \mathbf{k} \to \mathcal{I}_N$ is a $\mathbf{k}$-equivalence.

Remarks. Observe that if $\mathcal{I}_N$ is a Matlis lift of N and X is an arbitrary $\mathbf{k}$-module then the restriction of scalars-coinduction adjunction equivalence

$$\mathrm{Hom}_{\mathbf{R}}(X, \mathcal{I}_N) \simeq \mathrm{Hom}_{\mathbf{k}}(X, \mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, \mathcal{I}_N))$$

implies that there is an equivalence

$$\mathrm{Hom}_{\mathbf{R}}(X, \mathcal{I}_N) \simeq \mathrm{Hom}_{\mathbf{k}}(X, N). \tag{3.5}$$

This is the crucial property of a Matlis lift. One should think of $\mathcal{I}_N$ as a lift of N to an $\mathbf{R}$-module, not the obvious one coming from the map $\mathbf{R} \to \mathbf{k}$, but one which allows for the equivalence (3.5). We will say that $\mathcal{I}_N$ has the *Matlis lifting property*. Note also that if we are given an $\mathbf{R}$-module $\mathcal{I}_N$ satisfying (i), then its $\mathbf{k}$-cellularization $\mathrm{Cell}_{\mathbf{k}}(\mathcal{I}_N)$ also satisfies (i) and so there is no loss in generality in assuming that $\mathcal{I}_N$ is $\mathbf{k}$-cellular. Requiring the stronger condition (ii) allows us to enumerate Matlis lifts below.

Example 3.4.2. *Let* R *be a commutative noetherian local ring with maximal ideal* $\mathfrak{m}$ *and residue field* k_R *and view* k_R *as an* R *module via the quotient map* $R \to k_R$*. Then the injective hull* $I(k_R)$ *of* k_R *is a Matlis lift of* k_R*. See [17, Section 7.1].*

We now show that the Matlis lifts of a $\mathbf{k}$-module N are in one-to-one correspondence with right $\boldsymbol{\mathcal{E}}$-module structures on N which extend the left $\mathbf{k}$-module structure. Note that the right multiplication action of $\mathbf{k}$ on itself gives a map $\mathbf{k}^{\mathsf{op}} \to \boldsymbol{\mathcal{E}}$.

Definition 3.4.3. An *$\mathcal{E}$-lift* of a $\mathbf{k}$-module N is a right $\mathcal{E}$-module $\tilde{N}$ such that the underlying left $\mathbf{k}$-module of $\tilde{N}$ along the map $\mathbf{k}^{\mathrm{op}} \to \mathcal{E}$ is equivalent to N. An $\mathcal{E}$-lift of N is of *Matlis type* if the coevaluation map

$$\mathrm{coeval}\colon \tilde{N} \simeq \tilde{N} \otimes_{\mathcal{E}} \mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, \mathbf{k}) \to \mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, \tilde{N} \otimes_{\mathcal{E}} \mathbf{k}) \tag{3.6}$$

is an equivalence.

Proposition 3.4.4. *There is a bijective correspondence*

$$\begin{aligned} \pi_0 \{\text{Matlis lifts } \mathcal{I}_N \text{ of } N\} &= \pi_0 \left\{\mathcal{E}\text{-lifts } \tilde{N} \text{ of } N \text{ of Matlis type}\right\} \\ \mathcal{I}_N &\longmapsto \mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, \mathcal{I}_N) \\ \tilde{N} \otimes_{\mathcal{E}} k &\longleftarrow \tilde{N}, \end{aligned}$$

where π_0 here means taking equivalence classes.

Proof. This is [17, Proposition 6.9]. Suppose $\mathcal{I}_N$ is a Matlis lift of N. Then, by definition, $N \simeq \mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, \mathcal{I}_N)$ as $\mathbf{k}$-modules and the natural right $\mathcal{E}$-action on $\mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, \mathcal{I}_N)$ provides an $\mathcal{E}$-lift of N. The map (3.6) is an equivalence since by assumption $\mathcal{I}_N$ is effectively constructible. Conversely, suppose we are given an $\mathcal{E}$-lift $\tilde{N}$ of N which is of Matlis type and let $\mathcal{I}_N = \tilde{N} \otimes_{\mathcal{E}} \mathbf{k}$. The equivalence (3.6) guarantees that $\mathcal{I}_N$ satisfies conditions (i) and (ii) of Definition 3.4.1. □

We will be interested in the case $N = \mathbf{k}$. To find a Matlis lift $\mathcal{I}_k$ then we have to

(a) define a compatible right $\mathcal{E}$-module structure on $\mathbf{k}$, i.e. an $\mathcal{E}$-lift $\tilde{\mathbf{k}}$ of $\mathbf{k}$ and

(b) show that $\tilde{\mathbf{k}}$ is of Matlis type.

The resulting Matlis lift will then be of the form $\mathcal{I}_{\mathbf{k}} \simeq \tilde{\mathbf{k}} \otimes_{\mathcal{E}} \mathbf{k}$.

We provide a couple of propositions which allow us to recognize $\mathcal{E}$-lifts of Matlis type and thus Matlis lifts.

Proposition 3.4.5. *Suppose $\mathbf{R} \to \mathbf{k}$ is small. Then any $\mathcal{E}$-lift $\tilde{N}$ of N is of Matlis type and, consequently, $\mathcal{I}_N = \tilde{N} \otimes_{\mathcal{E}} \mathbf{k}$ is a Matlis lift of N.*

Proof. By Theorem 3.3.1 the unit of the adjunction (3.4)

$$\mathrm{coeval}\colon N \to \mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, \tilde{N} \otimes_{\mathcal{E}} \mathbf{k})$$

is an equivalence. □

Proposition 3.4.6. *Suppose $\mathbf{R} \to \mathbf{k}$ is proxy-small. Then n $\mathbf{k}$-module N is of Matlis type if and only if there exists an $\mathbf{R}$-module M such that $N \simeq \mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, M)$ as right $\mathcal{E}$-modules.*

Proof. If $\tilde{N}$ is an $\boldsymbol{\mathcal{E}}$-lift of Matlis type, then $M = \tilde{N} \otimes_{\boldsymbol{\mathcal{E}}} \mathbf{k}$ has the desired property. The converse follows from Theorem 3.3.2. □

Lemma 3.4.7. *Let $\mathbf{r}$ be a connective ring G-spectrum. Let $k = \pi_0^G(\mathbf{r})$, $\mathbf{k} = H\underline{\pi}_0(\mathbf{r})$ and take $\mathbf{r} \to \mathbf{k}$ to be the map obtained by killing homotopy groups. Suppose $\underline{\pi}_0(\mathbf{r})(G/H)$ is projective over k for every $H \leq G$. Then $k^{\mathbf{r}}$ is a Matlis lift of $k^{\mathbf{k}}$ i.e.*

$$\mathrm{Hom}_{\mathbf{r}}(\mathbf{k}, k^{\mathbf{r}}) \simeq HD_k(\underline{\pi}_0(\mathbf{r})) \simeq \mathrm{Hom}_{\mathbf{k}}(\mathbf{k}, k^{\mathbf{k}}).$$

Proof. This is a corollary of Proposition 3.1.5, i.e. we have

$$\underline{\pi}_0(HD_k(\underline{\pi}_0(\mathbf{r}))) = \mathrm{Hom}_{\mathbb{Z}}(\underline{\pi}_0(\mathbf{r}), k)$$

and

$$\underline{\pi}_0(k^{\mathbf{k}}) = \mathrm{Hom}_{\mathbb{Z}}(\underline{\pi}_0(\mathbf{k}), k).$$

□

3.5. Gorenstein rings in commutative algebra

We recall the notion of a Gorenstein ring from commutative algebra. Let $(R, \mathfrak{m}, k_R)$ be a commutative local noetherian ring. The classical definition of a Gorenstein ring is that R is of finite injective dimension over itself. One can then show that if R is Gorenstein it is of injective dimension equal to its Krull dimension d and $\mathrm{Ext}_R^*(k_R, R)$ is concentrated in a single degree where it is isomorphic to k_R, that is

$$\mathrm{Ext}_R^i(k_R, R) = \begin{cases} k_R & \text{if } i = d \\ 0 & \text{otherwise.} \end{cases} \tag{3.7}$$

The real reason for considering the Gorenstein condition is the duality property it implies. Let M be an R-module and write $\Gamma_{\mathfrak{m}}(M)$ for the $\mathfrak{m}$-power torsion module of M defined as

$$\Gamma_{\mathfrak{m}}(M) \stackrel{\text{def}}{=} \{x \in M \mid \mathfrak{m}^k x = 0 \text{ for } k \gg 0\}.$$

By a theorem of Grothendieck one can identify the local cohomology of M with the right derived functors $\mathbb{R}^*\Gamma_{\mathfrak{m}}(M)$.

Theorem 3.5.1 ([31])**.** *Suppose that R is noetherian and let M be an R-module. Then*

$$H^*_{\mathfrak{m}}(R, M) \cong \mathbb{R}^*\Gamma_{\mathfrak{m}}(M).$$

Theorem 3.5.2 ([9, Theorem 3.5.8])**.** *Suppose that R is Gorenstein and let M be a finitely generated R-module. Then*

$$H^i_{\mathfrak{m}}(M) \cong \mathrm{Hom}_R(\mathrm{Ext}_R^{d-i}(M, R), I(k_R)).$$

Taking M to be R the theorem above states that all local cohomology of R is concentrated in a single degree d and there it is the injective hull of the residue field.

Corollary 3.5.3. *Let $M = R$. Then*

$$H^i_{\mathfrak{m}}(R) \cong \begin{cases} I(k_R) & \text{if } i = d \\ 0 & \text{otherwise.} \end{cases}$$

□

Remark. Note that if R is a k_R-algebra $I(k_R) = \Gamma_{\mathfrak{m}}(\mathrm{Hom}_k(R, k_R))$.

3.6. Equivariant Gorenstein duality

We are now ready to generalize the Gorenstein property of classical commutative rings to ring G-spectra and give a definition of equivariant Gorenstein duality. Before we begin, we need a reasonable substitute of the noetherian condition in the classical setting. One of the main themes of [17] is that one can restrict attention to ring spectra which are proxy-small in the sense of Definition 3.2.3. This allows one to develop a useful theory covering a wide range of examples.

Let $\mathbf{R} \to \mathbf{k}$ be a map of non-equivariantly commutative ring G-spectra. The Gorenstein condition for ring G-spectra is just a spectrum level version of the isomorphism (3.7).

Definition 3.6.1. We say that $\mathbf{R} \to \mathbf{k}$ is *Gorenstein* of shift $\alpha \in RO(G)$ if there is an equivalence of $\mathbf{R}$-modules

$$\mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, \mathbf{R}) \simeq \Sigma^{\alpha}\mathbf{k}. \tag{3.8}$$

Example 3.6.2. *We give an elementary non-equivariant example. Let $\mathbf{R} \to \mathbf{k}$ be a map of ring spectra and suppose that $\mathbf{k} \simeq \mathbf{R}/x$ for $x \in \pi_*(\mathbf{R})$. Applying $\mathrm{Hom}_{\mathbf{R}}(-, \mathbf{R})$ to the cofibre sequence*

$$\Sigma^{|x|}\mathbf{R} \to \mathbf{R} \to \mathbf{R}/x \simeq \mathbf{k}$$

we get that $\mathbf{R}$ is Gorenstein of shift $\alpha = -1 - |x|$

$$\mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, \mathbf{R}) \simeq \Sigma^{-1-|x|}\mathbf{k}.$$

This generalizes to the case where $\pi_*(\mathbf{k})$ is the quotient of $\pi_*(\mathbf{R})$ by a regular sequence $(x_1, \dots, x_n)$. Then $\mathbf{k} \simeq \mathbf{R}/x_1 \otimes_{\mathbf{R}} \cdots \otimes_{\mathbf{R}} \mathbf{R}/x_n$ and the same argument as above shows that $\mathbf{R}$ is Gorenstein of shift $\alpha = \sum_i(-1 - |x_i|)$.

If $\mathbf{R}$ is Gorenstein of shift α, then so is its $\mathbf{k}$-cellularization and therefore, without loss of generality, we can replace $\mathbf{R}$ with $\mathrm{Cell}_{\mathbf{k}}(\mathbf{R})$ in the definition above. The Gorenstein condition then says that $\mathrm{Cell}_{\mathbf{k}}(\mathbf{R})$ is, up to a shift, a Matlis lift of $\mathbf{k}$

$$\mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, \mathrm{Cell}_{\mathbf{k}}(\mathbf{R})) \simeq \Sigma^{\alpha}\mathbf{k} \simeq \Sigma^{\alpha}\,\mathrm{Hom}_{\mathbf{k}}(\mathbf{k}, \mathbf{k}).$$

One can now ask the natural question how does $\mathrm{Cell}_{\mathbf{k}}(\mathbf{R})$ compare to other Matlis lifts of $\mathbf{k}$, i.e. $\mathbf{R}$-modules $\mathcal{I}_{\mathbf{k}}$ such that

$$\mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, \mathcal{I}_{\mathbf{k}}) \simeq \mathbf{k} \simeq \mathrm{Hom}_{\mathbf{k}}(\mathbf{k}, \mathbf{k}).$$

Given a Matlis lift $\mathcal{I}_{\mathbf{k}}$ of $\mathbf{k}$ observe that we have the following equivalences of $\mathbf{R}$-modules

$$\underbrace{\mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, \mathbf{R})}_{\text{(a)}} \overset{(3.8)}{\simeq} \Sigma^{\alpha}\mathbf{k} \simeq \Sigma^{\alpha}\,\mathrm{Hom}_{\mathbf{k}}(\mathbf{k}, \mathbf{k}) \overset{(3.5)}{\simeq} \underbrace{\mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, \Sigma^{\alpha}\mathcal{I}_{\mathbf{k}})}_{\text{(b)}}.$$

The endomorphism ring $\boldsymbol{\mathcal{E}} = \mathrm{End}_{\mathbf{R}}(\mathbf{k}, \mathbf{k})$ acts on the right on (a) and on (b) equipping them with the structure of right $\boldsymbol{\mathcal{E}}$-modules. We would like to remove the $\mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, -)$ on both sides and deduce that $\mathrm{Cell}_{\mathbf{k}}(\mathbf{r}) \simeq \Sigma^{\alpha}\mathcal{I}_{\mathbf{k}}$. Provided that the equivalence above is an equivalence of right $\boldsymbol{\mathcal{E}}$-modules and $\mathbf{R} \to \mathbf{k}$ is proxy-small, the Morita theory from Section 3.3 allows us to do this.

Definition 3.6.3. We say that $\mathbf{R} \to \mathbf{k}$ is *orientably Gorenstein* of shift $\alpha \in RO(G)$ if there is an equivalence of right $\boldsymbol{\mathcal{E}}$-modules

$$\mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, \mathbf{R}) \simeq \mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, \Sigma^{\alpha}\mathcal{I}_{\mathbf{k}}).$$

Suppose that $\mathbf{R} \to \mathbf{k}$ is orientably Gorenstein of shift α and $\mathbf{k}$ is proxy-small as an $\mathbf{R}$-module so that, in particular, every $\mathbf{R}$-module has an effectively constructible $\mathbf{k}$-cellularization, i.e.

$$\mathrm{Cell}_{\mathbf{k}}(M) \simeq \mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, M) \otimes_{\boldsymbol{\mathcal{E}}} \mathbf{k}.$$

We can apply the equivalence from the Morita theory to deduce that

$$\begin{array}{cc}
\mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, \mathbf{R}) \simeq \mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, \Sigma^{\alpha}\mathcal{I}_{\mathbf{k}}) & \mathrm{mod}_G\text{-}\boldsymbol{\mathcal{E}} \\
 & \downarrow{\scriptstyle -\otimes_{\boldsymbol{\mathcal{E}}}\mathbf{k}} \\
\mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, \mathbf{R}) \otimes_{\boldsymbol{\mathcal{E}}} \mathbf{k} \simeq \mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, \Sigma^{\alpha}\mathcal{I}_{\mathbf{k}}) \otimes_{\boldsymbol{\mathcal{E}}} \mathbf{k} & \mathbf{R}\text{-}\mathrm{mod}_G \\
 & \downarrow{\scriptstyle \mathbf{k} \text{ is proxy-small}} \\
\mathrm{Cell}_{\mathbf{k}}(\mathbf{R}) \simeq \Sigma^{\alpha}\,\mathrm{Cell}_{\mathbf{k}}(\mathcal{I}_{\mathbf{k}}) & \mathbf{R}\text{-}\mathrm{mod}_G \\
 & \downarrow{\scriptstyle \mathcal{I}_{\mathbf{k}} \text{ is } \mathbf{k}\text{-cellular}} \\
\mathrm{Cell}_{\mathbf{k}}(\mathbf{R}) \simeq \Sigma^{\alpha}\mathcal{I}_{\mathbf{k}} & \mathbf{R}\text{-}\mathrm{mod}_G.
\end{array} \tag{3.9}$$

This leads us to the central definition of this document.

Definition 3.6.4. We say that $\mathbf{R} \to \mathbf{k}$ has *Gorenstein duality* of shift $\alpha \in RO(G)$ if there is an equivalence of $\mathbf{R}$-modules

$$\mathrm{Cell}_{\mathbf{k}}(\mathbf{R}) \simeq \Sigma^{\alpha}\mathcal{I}_{\mathbf{k}}. \tag{3.10}$$

3.7. Algebraic models for cellularization and nullification

In favourable situations one can construct explicit algebraic models of the cellularization and nullification of an $\mathbf{R}$-module leading to spectral sequences which allow actual computations. We show that the $\mathbf{k}$-cellularization of an $\mathbf{R}$-module M is modelled algebraically by the homotopical J-power torsion of M, also known as the stable Koszul complex of M, while the $\mathbf{k}$-nullification of M is given by the homotopical localization of M away from J. In both cases J is an appropriately chosen ideal in the $RO(G)$-graded homotopy groups of $\mathbf{R}$. In effect, we are extending the cellularization-nullification cofibre sequence (3.1) introduced in Section 3.2 to a diagram

$$\begin{array}{ccccc} \mathrm{Cell}_{\mathbf{k}}(M) & \longrightarrow & M & \longrightarrow & \mathrm{Null}_{\mathbf{k}}(M) \\ \downarrow \simeq & & \| & & \downarrow \simeq \\ \Gamma_{\mathrm{J}}(M) & \longrightarrow & M & \longrightarrow & M[J^{-1}]. \end{array}$$

In what follows M will always be an $\mathbf{R}$-module. We start by looking at an algebraic model of the $\mathbf{k}$-cellularization of M given by the stable Koszul complex of M of which we give two definitions. Suppose $J \subseteq \pi^G_\bigstar(\mathbf{R})$ is a finitely generated ideal and let $(x_1, \ldots, x_n)$ be a set of generators of J.

Definition 3.7.1. Define the *stable Koszul complex* of M with respect to the ideal J as

$$\Gamma_{\mathrm{J}}(M) \overset{\text{def}}{=} \mathrm{fib}(\mathbf{R} \to \mathbf{R}[\frac{1}{x_1}]) \otimes_{\mathbf{R}} \cdots \otimes_{\mathbf{R}} \mathrm{fib}(\mathbf{R} \to \mathbf{R}[\frac{1}{x_n}]) \otimes_{\mathbf{R}} M. \tag{3.11}$$

One can show that this depends only on the radical of the ideal J, see [25, Section 3]. There is a second definition of the stable Koszul complex which lends itself to easier manipulation in some cases. In the one generator case, for x an element of $\pi^G_\bigstar(\mathbf{R})$, it is given as

$$\kappa_{\mathbf{R}}(x) \overset{\text{def}}{=} \mathrm{colim}_l\, \Sigma^{(1-l)|x|}\mathbf{R}/x^l,$$

where the colimit is taken over the maps $\Sigma^{(1-l)|x|}\mathbf{R}/x^l \to \Sigma^{-l|x|}\mathbf{R}/x^{l+1}$ induced by the diagram of cofibre sequences

$$\begin{array}{ccccc} \Sigma^{l|x|}\mathbf{R} & \xrightarrow{x^l} & \mathbf{R} & \longrightarrow & \mathbf{R}/x^l \\ \downarrow = & & \downarrow x & & \downarrow \\ \Sigma^{l|x|}\mathbf{R} & \xrightarrow{x^{l+1}} & \Sigma^{-|x|}\mathbf{R} & \longrightarrow & \Sigma^{-|x|}\mathbf{R}/x^{l+1}. \end{array}$$

Definition 3.7.2. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a sequence of elements in $\pi^G_\bigstar(\mathbf{R})$. Define the *stable Koszul complex* of M with respect to the sequence $\mathbf{x}$ as

$$\kappa_{\mathbf{R}}(\mathbf{x}; M) \overset{\text{def}}{=} \kappa_{\mathbf{R}}(x_1) \otimes_{\mathbf{R}} \cdots \otimes_{\mathbf{R}} \kappa_{\mathbf{R}}(x_n) \otimes_{\mathbf{R}} M \tag{3.12}$$

Lemma 3.7.3. *Given* $\mathbf{x}$ *and* J *as above we have*

$$\Gamma_{\mathrm{J}}(M) \simeq \Sigma^{-(|x_1|+\cdots+|x_n|)-n}\kappa_{\mathbf{R}}(\mathbf{x}; M).$$

Proof. This is [25, Lemma 3.6] where the proof is done in the case of $M = \mathbf{R}$ and $J = (x)$. We reproduce the argument in a slightly expanded form for completeness. Observe that we have an equivalence

$$\kappa_{\mathbf{R}}(\mathbf{x}; M) \simeq \mathrm{colim}_{(l_1,\ldots,l_n)} \Sigma^{(1-l_1)|x_1|+\cdots+(1-l_n)|x_n|} M/(x_1^{l_1}, \ldots, x_n^{l_n}).$$

From here the proof of the claim is a question of rewriting

$$\begin{aligned}\Gamma_x(\mathbf{R}) &= \mathrm{fib}(\mathbf{R} \to \mathbf{R}[\frac{1}{x}]) \\ &\simeq \mathrm{fib}\,\mathrm{colim}_l(\mathbf{R} \xrightarrow{x^l} \Sigma^{-l|x|}\mathbf{R}) \\ &\simeq \mathrm{colim}_l\,\mathrm{fib}(\mathbf{R} \xrightarrow{x^l} \Sigma^{-l|x|}\mathbf{R}) \\ &\simeq \mathrm{colim}_l\,\Sigma^{-1-l|x|}\mathbf{R}/x^l \\ &\simeq \Sigma^{-|x|-1}\kappa_{\mathbf{R}}(x).\end{aligned}$$

□

The importance of the stable Koszul complex is clear from the following lemma. Write

$$\mathbf{R}/J \stackrel{\mathsf{def}}{=} \mathbf{R}/x_1 \otimes_{\mathbf{R}} \cdots \otimes_{\mathbf{R}} \mathbf{R}/x_n,$$

sometimes referred to as the *unstable Koszul complex*.

Lemma 3.7.4. *The map* $\Gamma_{\mathrm{J}}(M) \to M$ *is* $\mathbf{R}/J$*-cellularization in the category of* $\mathbf{R}$*-modules.*

Proof. This is an equivariant reformulation of [27, Proposition 3.8]. We have to show that

(a) $\Gamma_{\mathrm{J}}(M)$ is $\mathbf{R}/J$-cellular and

(b) the map $\Gamma_{\mathrm{J}}(M) \to M$ is an $\mathbf{R}/J$-equivalence.

Using the second definition of the stable Koszul complex part (a) is immediate. $\Gamma_{\mathrm{J}}(M)$ is built out of $\mathbf{R}/J$ and so, by Lemma 3.2.5, it is $\mathbf{R}/J$-cellular. To show (b), note that it is enough to prove the statement in the one generator case since $\Gamma_{\mathrm{J}}(M) = \Gamma_{x_n}(\Gamma_{(x_1,\ldots,x_{n-1})}(M))$. So suppose that $J = (x)$. We want to show that the map

$$\mathrm{Hom}_{\mathbf{R}}(\mathbf{R}/x, \Gamma_x M) \to \mathrm{Hom}_{\mathbf{R}}(\mathbf{R}/x, M)$$

is an equivalence. In fact, one can show that for arbitrary $\mathbf{R}$-modules A and B the map

$$\mathrm{Hom}_{\mathbf{R}}(A/x, \Gamma_x B) \to \mathrm{Hom}_{\mathbf{R}}(A/x, B)$$

is an equivalence. Using the cofibre sequence

$$\Gamma_x B \to B \to B[\frac{1}{x}]$$

this in turn is equivalent to showing that $\mathrm{Hom}_{\mathbf{R}}(A/x, B[\frac{1}{x}]) \simeq *$, but multiplication by x

$$\Sigma^{|x|} A \to A \to A/x$$

induces an equivalence after applying $\mathrm{Hom}_{\mathbf{R}}(-, B[\frac{1}{x}])$

$$\mathrm{Hom}_{\mathbf{R}}(A/x, B[\frac{1}{x}]) \to \mathrm{Hom}_{\mathbf{R}}(A, B[\frac{1}{x}]) \to \mathrm{Hom}_{\mathbf{R}}(\Sigma^{|x|} A, B[\frac{1}{x}]).$$

This completes the proof. □

Remark. In situations where Lemma 3.7.4 is true we will say that the $\mathbf{k}$-cellularization of an $\mathbf{R}$-module M is *algebraic*.

In Lemma 3.4.7 we have shown that the Anderson dual of an R-module M provides an example of a Matlis lift of M. In order to execute the last step in the definition of equivariant Gorenstein duality, as described in the diagram 3.9, we need to ensure that this Matlis lift is $\mathbf{R}/J$-cellular. Imposing harsh restrictions on the ideal J this holds.

Proposition 3.7.5. *Let A be an abelian group. Suppose that M is a connective $\mathbf{R}$-module and let J be the ideal*

$$J \overset{\text{def}}{=} \ker(\pi_\bigstar^G(M) \to \pi_0^G(M)).$$

Suppose that J is radically finitely generated by a regular sequence $(x_1, \cdots, x_n)$. Then the Anderson A-dual of M is $\mathbf{R}/J$-cellular.

Proof. This automatic as $\mathbf{R}/J \simeq H\underline{\pi}_0(\mathbf{R})$. See also [29, Lemma 5.1]. □

We now look at the algebraic model for the $\mathbf{k}$-nullification of an $\mathbf{R}$-module M. In the case of $M = \mathbf{R}$ this is given by a construction known as the Čech complex of $\mathbf{R}$.

Definition 3.7.6. Define the *Čech complex* of $\mathbf{R}$ by the cofibre sequence

$$\Gamma_{\mathrm{J}}(\mathbf{R}) \to \mathbf{R} \to \check{\mathrm{C}}_{\mathrm{J}}(\mathbf{R}).$$

For an $\mathbf{R}$-module M set

$$M[J^{-1}] \overset{\text{def}}{=} \check{\mathrm{C}}_{\mathrm{J}}(\mathbf{R}) \otimes_{\mathbf{R}} M.$$

We have the following corollary.

Corollary 3.7.7. *The map $M \to M[J^{-1}]$ is $\mathbf{R}/J$-nullification in the category of $\mathbf{R}$-modules.* □

Remark. We can identify the $\mathbf{R}/J$-nullification procedure as a certain Bousfield localization, namely the map $M \to M[J^{-1}]$ is right Bousfield localization in the category $\mathbf{R}\text{-mod}_G$ with respect to the $\mathbf{R}$-module $\check{C}_J(R)$. See [25, Theorem 5.1].

Provided that we are in a situation where cellularization is algebraic we can rewrite Definition 3.6.4 of equivariant Gorenstein duality as

$$\Gamma_J(\mathbf{R}) \simeq \Sigma^\alpha k^{\mathbf{R}}.$$

One should see the equivalence in this definition as the ring G-spectrum analogue of the classical Gorenstein duality statement of Corollary 3.5.3

$$H^d_{\mathfrak{m}}(R) \cong I(k_R).$$

The following diagram summarizes the various ingredients of the equivariant Gorenstein duality framework discussed so far.

$$\mathbf{R} \to \mathbf{k}$$

$\mathbf{R} \to \mathbf{k}$ is Gorenstein of shift α

$$\mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, \mathbf{R}) \simeq \mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, \Sigma^\alpha \mathcal{I}_{\mathbf{k}})$$

Morita theory provided $\mathbf{k}$ is proxy-small

$$\mathrm{Cell}_{\mathbf{k}}(\mathbf{R}) \simeq \Sigma^\alpha \,\mathrm{Cell}_{\mathbf{k}}(\mathcal{I}_{\mathbf{k}})$$

The Anderson k-dual is a $\mathbf{k}$-cellular Matlis lift

$$\mathrm{Cell}_{\mathbf{k}}(\mathbf{R}) \simeq \Sigma^\alpha k^{\mathbf{R}}$$

Cellularization is algebraic

$$\Gamma_J(\mathbf{R}) \simeq \Sigma^\alpha k^{\mathbf{R}}$$

3.8. The local cohomology spectral sequence and shift prediction

In this section we look at a local cohomology spectral sequence which arises when one has a ring G-spectrum with an algebraic cellularization. In favourable situations this spectral sequence collapses and can be used to both detect potential Gorenstein candidates and predict their expected Gorenstein shift.

Let $\mathbf{r}$ be a connective non-equivariantly commutative ring G-spectrum, let $J \subseteq \pi^G_\bigstar(\mathbf{r})$ be a finitely generated ideal and let M be an $\mathbf{r}$-module. Write $k = \pi^G_0(\mathbf{r})$, $\mathbf{k} = H\underline{\pi}_0(\mathbf{r})$ and let $\mathbf{r} \to \mathbf{k}$ be the map obtained by killing homotopy groups.

Proposition 3.8.1 ([25, Section 3])**.** *There is a spectral sequence*

$$H^*_J(\pi^G_\bigstar(M)) \Longrightarrow \pi^G_\bigstar(\Gamma_{\mathrm{J}}(M)).$$

computing the homotopy groups of $\Gamma_{\mathrm{J}}(M)$ *in terms of local cohomology.*

Corollary 3.8.2. *Suppose that* $\mathbf{r} \to \mathbf{k}$ *has Gorenstein duality of shift* α*. Then the equivalence*

$$\Gamma_{\mathrm{J}}(\mathbf{r}) \simeq \Sigma^\alpha k^{\mathbf{r}}$$

induces a local cohomology spectral sequence

$$H^*_J(\pi^G_\bigstar(\mathbf{r})) \Longrightarrow \Sigma^\alpha \pi^G_\bigstar(k^{\mathbf{r}}).$$

□

Using the local cohomology spectral sequence we can identify potential Gorenstein candidates via the following procedure. First, note that if $\mathbf{r}$ is Gorenstein, so is its rationalization. One can also show that if $\mathbf{r}$ has Gorenstein duality and the coefficient ring $\pi^e_*(\mathbf{r})$ is Cohen-Macaulay then the Hilbert series of $\pi^e_*(\mathbf{r})$ satisfies a certain functional equation. Checking whether the Hilbert series satisfies the functional equation then imposes a condition on when $\pi^e_*(\mathbf{r})$ can be Gorenstein. We discuss these steps in more detail below.

Given a prime p we write $\mathbf{r}_{(p)}$ for the p-localization of $\mathbf{r}$ at p.

Proposition 3.8.3 ([22])**.** *Suppose that* $\mathbf{r} \to \mathbf{k}$ *is Gorenstein of some shift. Then* $\mathbf{r}_{(p)} \to \mathbf{k}_{(p)}$ *is also Gorenstein of the same shift.*

Proposition 3.8.4. *Suppose that* $\pi^e_*(\mathbf{r})$ *is Gorenstein. Then* $\mathbf{r} \to \mathbf{k}$ *is also Gorenstein.*

Proof. We reproduce the argument from [19, Section 4.C]. Write $k = \pi^e_0(\mathbf{r})$. There is a spectral sequence

$$E_2^{*,*} = \mathrm{Ext}_{\pi^e_*(\mathbf{r})}(k, \pi^e_*(\mathbf{r})) \Longrightarrow \pi^e_*(\mathrm{Hom}_{\mathbf{r}}(\mathbf{k}, \mathbf{r})).$$

and this spectral sequence collapses to show that

$$\pi^e_*(\mathrm{Hom}_{\mathbf{r}}(\mathbf{k}, \mathbf{r})) = \Sigma^a k$$

for some a. The $\mathbf{r}$-module $\mathbf{k}$ is completely characterized by its homotopy so we have that $\mathrm{Hom}_{\mathbf{r}}(\mathbf{k}, \mathbf{r}) \simeq \Sigma^a \mathbf{k}$. □

There are two situations where the local cohomology spectral sequence from Corollary 3.8.2 collapses from which we deduce that the ring $\pi^e_*(\mathbf{r})$ has very special properties. If $\pi^e_*(\mathbf{r})$ is Cohen-Macaulay, the spectral sequence collapses to show that

$$H^*_J(\pi^e_*(\mathbf{r})) = \Sigma^{a+d} \pi^e_*(k^{\mathbf{r}}),$$

where d is the Krull dimension of $\pi_*^e(\mathbf{r})$. Thus the ring $\pi_*^e(\mathbf{r})$ is also Gorenstein. The spectral sequence also collapses if $\pi_*^e(\mathbf{r})$ is of Cohen-Macaulay defect 1 to give a short exact sequence

$$0 \to H_J^d(\pi_*^e(\mathbf{r})) \to \Sigma^{a+d}\pi_*^e(k^{\mathbf{r}}) \to \Sigma H_J^{d-1}(\pi_*^e(\mathbf{r})) \to 0.$$

In both cases the collapse of the spectral sequence has concrete consequences on the Hilbert series of the ring $\pi_*^e(\mathbf{r})$ which allow us to check if $\mathbf{r}$ can be Gorenstein and predict the Gorenstein shift.

Corollary 3.8.5 ([24, Theorem 6.2])**.** *Suppose that $\mathbf{r} \to \mathbf{k}$ has non-equivariant Gorenstein duality of shift a and that $\pi_*^e(\mathbf{r})$ is noetherian of Krull dimension d. Let*

$$h(t) = \sum_s \dim(\pi_s^e(\mathbf{r}))t^s$$

be the Hilbert series of $\pi_*^e(\mathbf{r})$. Then

(1) If $\pi_*^e(\mathbf{r})$ is Cohen-Macaulay it is also Gorenstein and its Hilbert series satisfies the functional equation

$$h\left(\frac{1}{t}\right) = (-1)^d t^{-d-a} h(t), \tag{3.13}$$

(2) If $\pi_*^e(\mathbf{r})$ is almost Cohen-Macaulay it is also almost Gorenstein and its Hilbert series satisfies the functional equation

$$h\left(\frac{1}{t}\right) - (-1)^d t^{-d-a} h(t) = (-1)^{d-1}(1+t)q(t), \tag{3.14}$$

where

$$q\left(\frac{1}{t}\right) = (-1)^{d-1} t^{a-d+1} q(t).$$

We summarize the Gorenstein candidate identification and shift prediction procedure in the following diagram.

$\mathbf{r}$ is Gorenstein

Proposition 3.8.3

↪ $\mathbf{r}_{(0)}$ is Gorenstein

orientability condition

↪ $\mathbf{r}_{(0)}$ has Gorenstein duality

Corollary 3.8.5

↪ $\pi_*^e(\mathbf{r}_{(0)})$ is (almost) Gorenstein

Corollary 3.8.5

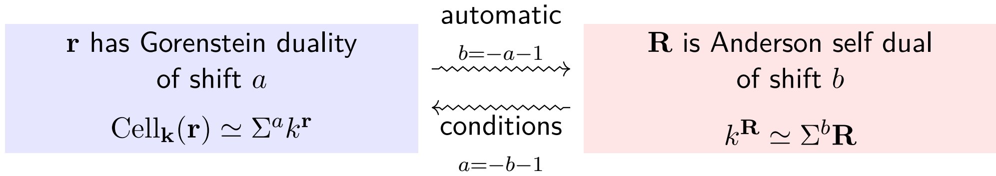
↪ $h(t)$ satisfies (3.13) or (3.13)

3.9. The Anderson-Gorenstein duality duumvirate

In this last section we make explicit a connection between Gorenstein duality for a connective ring spectrum and Anderson self-duality for the its nullification. Let $\mathbf{r}$ be a commutative ring spectrum, $k = \pi_0(\mathbf{r})$ and let $\mathbf{k}$ and $\mathbf{R}$ denote the commutative ring spectra

$$\mathbf{k} \stackrel{\text{def}}{=} Hk \text{ and}$$
$$\mathbf{R} \stackrel{\text{def}}{=} \mathrm{Null}_{\mathbf{k}}(\mathbf{r}).$$

Finally, let $\mathbf{r} \to \mathbf{k}$ be the map obtained by killing all higher homotopy groups. We illustrate schematically the relationship between Anderson and Gorenstein duality in the following diagram.



First, we record the fact that Gorenstein duality for $\mathbf{r} \to \mathbf{k}$ automatically implies Anderson self-duality for $\mathbf{R}$.

Theorem 3.9.1. *Suppose that $\mathbf{r} \to \mathbf{k}$ has non-equivariant Gorenstein duality of shift a. Then the ring spectrum $\mathbf{R}$ is Anderson self-dual of shift $b = -a - 1$, i.e. there is an equivalence of $\mathbf{R}$-modules*

$$k^{\mathbf{R}} \simeq \Sigma^b \mathbf{R}.$$

Proof. This is [29, Proposition 4.1]. The statement of the theorem follows immediately if one shows that the map

$$\varepsilon\colon \mathrm{Cell}_{\mathbf{k}}(\mathbf{r}) \to \mathbf{r}$$

is self-dual in the sense that applying $\mathrm{Hom}_{\mathbf{r}}(-, k^{\mathbf{r}})$ to ε we get the a-th desuspension of ε. By assumption $\mathbf{r} \to \mathbf{k}$ has Gorenstein duality of shift a, i.e. there is an equivalence of $\mathbf{r}$-modules $\mathrm{Cell}_{\mathbf{k}}(\mathbf{r}) \simeq \Sigma^a k^{\mathbf{r}}$. Thus we may use $\mathrm{Hom}_{\mathbf{r}}(-, \Sigma^{-a}\mathrm{Cell}_{\mathbf{k}}(\mathbf{r}))$ as the dualization. The map ε then dualizes to

$$\varepsilon^*\colon \Sigma^{-a}\mathrm{Cell}_{\mathbf{k}}(\mathbf{r}) \simeq \mathrm{Hom}_{\mathbf{r}}(\mathbf{r}, \Sigma^{-a}\mathrm{Cell}_{\mathbf{k}}(\mathbf{r})) \to \mathrm{Hom}_{\mathbf{r}}(\mathrm{Cell}_{\mathbf{k}}(\mathbf{r}), \Sigma^{-a}\mathrm{Cell}_{\mathbf{k}}(\mathbf{r})) \simeq \Sigma^{-a}\mathbf{r}.$$

This has the universal property of $\mathbf{k}$-cellularization and therefore is the suspension of ε. □

Next we show that under addition connectivity and projectivity assumptions on the homotopy groups of $\mathbf{r}$, Anderson self-duality for $\mathbf{R}$ implies Gorenstein duality for $\mathbf{r} \to \mathbf{k}$.

Theorem 3.9.2. *Suppose* $\mathbf{R}$ *is Anderson self-dual of shift* $b = -a - 1$ *with* $a \leq -2$. *If* $\pi_i(\mathbf{r}) = 0$ *for* $i \geq a + 1$ *and* $\pi_a(\mathbf{r})$ *is projective over* k, *then* $\mathbf{r} \to \mathbf{k}$ *has non-equivariant Gorenstein duality of shift* $a = -b - 1$, *i.e. there is an equivalence of* $\mathbf{r}$*-modules*

$$\mathrm{Cell}_{\mathbf{k}}(\mathbf{r}) \simeq \Sigma^a k^{\mathbf{r}}.$$

Proof. This is [29, Lemma 4.3]. By assumption $\mathbf{R}$ is Anderson self-dual of shift b so we have an equivalence of $\mathbf{R}$-modules

$$k^{\mathbf{R}} \simeq \Sigma^b \mathbf{R}.$$

Now apply $\mathrm{Hom}_{\mathbf{r}}(-, k^{\mathbf{r}})$ to the cofibre sequence

$$\mathrm{Cell}_{\mathbf{k}}(\mathbf{r}) \to \mathbf{r} \to \mathbf{R} \tag{3.15}$$

to obtain

$$\mathrm{Hom}_{\mathbf{r}}(\mathrm{Cell}_{\mathbf{k}}(\mathbf{r}), k^{\mathbf{r}}) \leftarrow k^{\mathbf{r}} \leftarrow \mathrm{Hom}_{\mathbf{r}}(\mathbf{R}, k^{\mathbf{r}}).$$

Recall from Section 3.1 that by definition the Anderson k-dual of the $\mathbf{r}$-module $\mathbf{R}$ is given by $k^{\mathbf{R}} = \mathrm{Hom}_{\mathbf{r}}(\mathbf{R}, k^{\mathbf{r}})$. Thus the rightmost term in the sequence above is $k^{\mathbf{R}} \simeq \Sigma^b \mathbf{R}$. Suspending this sequence $a = -b - 1$ times we get

$$\Sigma^a \mathrm{Hom}_{\mathbf{r}}(\mathrm{Cell}_{\mathbf{k}}(\mathbf{r}), k^{\mathbf{r}}) \leftarrow \Sigma^a k^{\mathbf{r}} \leftarrow \Sigma^{-1} \mathbf{R}$$

and taking a mapping cone we get the cofibre sequence

$$\Sigma^a k^{\mathbf{r}} \to \Sigma^a \mathrm{Hom}_{\mathbf{r}}(\mathrm{Cell}_{\mathbf{k}}(\mathbf{r}), k^{\mathbf{r}}) \to \mathbf{R}. \tag{3.16}$$

We would like to show that this is equivalent to the original sequence (3.15). We start by arguing that the middle term is equivalent to $\mathbf{r}$. From the hypothesis it follows that $\Sigma^a \mathrm{Hom}_{\mathbf{r}}(\mathrm{Cell}_{\mathbf{k}}(\mathbf{r}), k^{\mathbf{r}})$ is connective. Indeed, its homotopy groups sit in a short exact sequence

$$\begin{aligned} 0 &\to \mathrm{Ext}^1_k(\pi_{-i+a-1}(\mathrm{Cell}_{\mathbf{k}}(\mathbf{r})), k) \\ &\to \pi_i(\Sigma^a \mathrm{Hom}_{\mathbf{r}}(\mathrm{Cell}_{\mathbf{k}}(\mathbf{r}), k^{\mathbf{r}})) \\ &\to \mathrm{Hom}_k(\pi_{-i+a}(\mathrm{Cell}_{\mathbf{k}}(\mathbf{r})), k) \to 0. \end{aligned}$$

By assumption for $i \leq -1$ both the Ext and Hom terms vanish and we have

$$\mathbf{r} \simeq \mathbf{R}[a+2, \infty) \simeq \Sigma^a \mathrm{Hom}_{\mathbf{r}}(\mathrm{Cell}_{\mathbf{k}}(\mathbf{r}), k^{\mathbf{r}}))[a+2, \infty] \simeq \Sigma^a \mathrm{Hom}_{\mathbf{r}}(\mathrm{Cell}_{\mathbf{k}}(\mathbf{r}), k^{\mathbf{r}}).$$

Thus the middle term of the sequence is $\mathbf{r}$. It remains to check that the map from it to $\mathbf{R}$ has the universal property that the map $\mathbf{r} \to \mathrm{Null}_{\mathbf{k}}(\mathbf{r}) \simeq \mathbf{R}$ has. This is immediate as the fibre $\Sigma^{-b-1} k^{\mathbf{r}}$ of this map is $\mathbf{k}$-cellular and we conclude that $\mathrm{Cell}_{\mathbf{k}}(\mathbf{r}) \simeq \Sigma^a k^{\mathbf{r}}$. □

Chapter 4

Modular Forms

In this chapter we recall the arithmetic notion of a modular form and we collect a number of computational results necessary for Chapter 5 and Chapter 6. We start in Section 4.1 by giving two definitions of modular forms, one analytic following [13] and one algebro-geometric following [56]. In Section 4.2 we review the dimension formulas for computing the dimension of the space of weight k modular forms with respect to a congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$. In Section 4.3 we sketch two methods that can be used to identify the natural $(\mathbb{Z}/n)^{\times}$-action on the graded ring of modular forms with respect to the congruence subgroup $\Gamma_1(n)$.

4.1. Classical modular forms with level structure

Let $\mathbb{H}$ denote the complex upper half-plane. The group $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on $\mathbb{H}$ via Möbius transformations, given $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathbb{H}$

$$\gamma(z) = \frac{az+b}{cz+d}.$$

Definition 4.1.1. Let f be a complex valued function on the upper half-plane, let $k \in \mathbb{Z}$ and let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Define a new function $f|[\gamma]_k \colon \mathbb{H} \to \mathbb{C}$ by

$$f|[\gamma]_k(z) \overset{\text{def}}{=} \det(\gamma)^{k-1}(cz+d)^{-k} f(\gamma(z)).$$

This is sometimes known as the *slash operator*.

Definition 4.1.2. Let n be a positive integer. The subgroup

$$\Gamma(n) \overset{\text{def}}{=} \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod n \right\},$$

of $\mathrm{SL}_2(\mathbb{Z})$ is called the *principal congruence subgroup of level* n.

Definition 4.1.3. A subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ is called a *congruence subgroup* if it contains $\Gamma(n)$ for some positive integer n.

Example 4.1.4. *The groups*

$$\Gamma_0(n) \overset{def}{=} \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod n \right\},$$
$$\Gamma_1(n) \overset{def}{=} \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod n \right\}$$

are congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$*. Clearly*

$$\Gamma(n) \subset \Gamma_1(n) \subset \Gamma_0(n) \subset \mathrm{SL}_2(\mathbb{Z}).$$

For the rest of this section let Γ denote one of the subgroups $\Gamma(n)$, $\Gamma_1(n)$ or $\Gamma_0(n)$ and k be a non-negative integer.

Definition 4.1.5. A *holomorphic modular form of weight* k *with respect to* Γ is a function $f\colon \mathbb{H} \to \mathbb{C}$ satisfying the following conditions.

(i) f is holomorphic on $\mathbb{H}$,

(ii) $f|[\gamma]_k = f$ for all $\gamma \in \Gamma$,

(iii) f is holomorphic at the cusps.

We will write $\mathrm{mf}_k(\Gamma;\mathbb{C})$ for the collection of all weight k holomorphic modular forms with respect to Γ. One can show that this is a finite dimensional vector space over $\mathbb{C}$. Pointwise multiplication of a weight k and a weight l modular forms produces a weight $k+l$ modular form and we will write $\mathrm{mf}_*(\Gamma;\mathbb{C})$ for the graded commutative ring

$$\mathrm{mf}_*(\Gamma;\mathbb{C}) = \bigoplus_{k\geq 0} \mathrm{mf}_k(\Gamma;\mathbb{C}).$$

If we instead require f to be meromorphic at the cusps we get the notion of a *meromorphic modular form of weight* k *with respect to* Γ. We will write $\mathrm{MF}_k(\Gamma;\mathbb{C})$ for the collection of all weight k meromorphic modular forms with respect to Γ and $\mathrm{MF}_*(\Gamma;\mathbb{C})$ for the corresponding graded ring. Given a subring $R \subset \mathbb{C}$ one can further consider $\mathrm{mf}_*(\Gamma;R)$ and $\mathrm{MF}_*(\Gamma;R)$, the subrings of holomorphic, respectively meromorphic, modular forms which have a q-expansion with coefficients in R.

In Chapter 5 we will study the Hilbert series of the graded rings $\mathrm{mf}_*(\Gamma;\mathbb{C})$ and it will be important that these rings enjoy certain finiteness properties.

Proposition 4.1.6 ([51, Theorem 5.14])**.** *The rings* $\mathrm{mf}_*(\Gamma;\mathbb{C})$ *are Cohen-Macaulay.*

There is another definition of modular forms of a more geometric flavour. Given an elliptic curve $p\colon E \to S$ let $\omega_E = p_*\Omega^1_{E/S}$. This is known to be a line bundle on S, see [12, p. II 1.6]. Define a line bundle ω_{ell} on $\mathcal{M}_{ell}$ as follows. Given a map $s\colon S \to \mathcal{M}_{ell}$ from a scheme

S to the moduli stack of elliptic curves let $p\colon E \to S$ be the corresponding elliptic curve. We associate with (S, s) the line bundle ω_E.

Let R be a $\mathbb{Z}[\frac{1}{n}]$-algebra and write $\mathcal{M}(\Gamma)_R$ for the pullback

$$\mathcal{M}(\Gamma)_R = \mathcal{M}(\Gamma) \times_{\mathrm{Spec}(\mathbb{Z}[\frac{1}{n}])} \mathrm{Spec}(R).$$

Denote the projection $\overline{\mathcal{M}}(\Gamma) \to \overline{\mathcal{M}}_{ell,R}$ by f and use the same name for its restriction $\mathcal{M}(\Gamma)_R \to \mathcal{M}_{ell,R}$. Let ω denote the pullback $f^*\omega_{ell}$. We can define holomorphic modular forms of weight k over R as sections of $\omega^{\otimes k}$

$$\mathrm{mf}_k(\Gamma; R) \overset{\text{def}}{=} H^0(\mathcal{M}(\Gamma)_R, \omega^{\otimes k}).$$

The line bundle ω_{ell} extends to the compactified stack $\overline{\mathcal{M}}(\Gamma)$ and we can also define meromorphic modular forms of weight k over R as

$$\mathrm{MF}_k(\Gamma; R) \overset{\text{def}}{=} H^0(\overline{\mathcal{M}}(\Gamma)_R, \omega^{\otimes k}).$$

4.2. Dimension formulas

Given a congruence subgroup Γ one can prove a general formula for the dimension of the weight k piece of $\mathrm{mf}_*(\Gamma; \mathbb{C})$. We include the relevant results following [13]. The dimension formulas will be used in Chapter 5 where we compute the Hilbert series of the graded rings $\mathrm{mf}_*(\Gamma; \mathbb{C})$ for various congruence subgroups Γ.

Definition 4.2.1. Define the *modular curve* $Y(\Gamma)$ as the quotient space

$$Y(\Gamma) \overset{\text{def}}{=} \Gamma/\mathbb{H}.$$

This is a Riemann surface which can be compactified and we will write $\overline{Y}(\Gamma)$ for its compactification.

The complex points of the stacks $\mathcal{M}(\Gamma)$ and $\overline{\mathcal{M}}(\Gamma)$ for orbifolds and we have equivalences

$$\mathcal{M}(\Gamma)_{\mathbb{C}} \simeq Y(\Gamma) \text{ and } \overline{\mathcal{M}}(\Gamma)_{\mathbb{C}} \simeq \overline{Y}(\Gamma).$$

The dimension formulas are derived by treating the modular curve $Y(\Gamma)$ as an orbifold, describing the subspace $\mathrm{mf}_k(\Gamma; \mathbb{C})$ in terms of divisors and then applying a suitable version of the Riemann-Roch theorem. Let g denote the genus of the modular curve $Y(\Gamma)$ and let ε_2 and ε_3 denote the number of elliptic points of $Y(\Gamma)$ with period 2 and period 3, respectively. Finally, let ε_∞ denote the number of cusps of $Y(\Gamma)$.

Theorem 4.2.2 ([13, Theorem 3.5.1]). *Let k be an even integer. Then*

$$\dim_{\mathbb{C}}(\mathrm{mf}_k(\Gamma;\mathbb{C})) = \begin{cases} (k-1)(g-1) + \left[\frac{k}{4}\right]\varepsilon_2 + \left[\frac{k}{3}\right]\varepsilon_3 + \frac{k}{2}\varepsilon_\infty & \text{if } k \geq 2 \\ 1 & \text{if } k = 2 \\ 0 & \text{if } k \leq 0. \end{cases}$$

A similar formula exist for the odd case. Note that if k is odd and Γ contains the negative identity matrix, then $\mathrm{mf}_k(\Gamma;\mathbb{C}) = 0$. Let $\varepsilon_\infty^{\text{reg}}$ and $\varepsilon_\infty^{\text{irr}}$ denote the number of regular and irregular cusps of $Y(\Gamma)$, respectively.

Theorem 4.2.3 ([13, Theorem 3.6.1]). *Let k be an odd integer. Then*

$$\dim_{\mathbb{C}}(\mathrm{mf}_k(\Gamma;\mathbb{C})) = \begin{cases} (k-1)(g-1) + \left[\frac{k}{3}\right]\varepsilon_3 + \frac{k}{2}\varepsilon_\infty^{\text{reg}} + \frac{k-1}{2}\varepsilon_\infty^{\text{irr}} & \text{if } k \geq 3 \\ 0 & \text{if } k < 0. \end{cases}$$

4.3. The Tate normal form and Eisenstein methods

Let us consider the case of modular forms with respect to the congruence subgroup $\Gamma_1(n)$. Observe that there are two, a priori different, $(\mathbb{Z}/n)^\times$-actions on $\mathrm{mf}_*(\Gamma_1(n);\mathbb{C})$. If we take the classical definition of modular forms, then there is a $(\mathbb{Z}/n)^\times$-action on $\mathrm{mf}_*(\Gamma_1(n);\mathbb{C})$ defined as follows. Let $d \in (\mathbb{Z}/n)^\times$ and let σ_d be any element of $\mathrm{SL}_2(\mathbb{Z})$ such that

$$\sigma_d \equiv \begin{pmatrix} \overline{d} & 0 \\ 0 & d \end{pmatrix} \bmod n,$$

where $\overline{d}$ is the multiplicative inverse of $d \bmod n$. Then d acts on $\mathrm{mf}_k(\Gamma_1(n);\mathbb{C})$ by sending f to $f|[\sigma_d]_k$. The action depends only on d and not on the choice of the matrix σ_d. If we instead take the algebro-geometric definition, then by functoriality there is a $(\mathbb{Z}/n)^\times$-action on $\mathrm{mf}_*(\Gamma_1(n);\mathbb{C})$ induced by the $(\mathbb{Z}/n)^\times$-action on the torsion points of exact order n on the moduli stack $\mathcal{M}_1(n)$. Luckily, one can show that two actions coincide. The theorem below suggests the direction of the argument and the full details can be found [56].

Recall that elliptic curves over $\mathbb{C}$ can be seen as quotients $\mathbb{C}/\Lambda_\tau$ with Λ_τ a lattice in $\mathbb{C}$ of the form $\mathbb{Z} \oplus \tau\mathbb{Z}$ and $\tau \in \mathbb{H}$.

Theorem 4.3.1 ([13, Theorem 1.5.1]). *There is a bijective correspondence between isomorphism classes of pairs (E_τ, P), where E_τ is an elliptic curve over $\mathbb{C}$ and P is a point of order n, and points on the modular curve $Y(\Gamma_1(n))$ given by*

$$\begin{aligned} \pi_0\{E_\tau = \mathbb{C}/\Lambda_\tau, \tau \in \mathbb{H}\} &= \pi_0\{\Gamma_1(n)\tau \mid \tau \in \mathbb{H}\} \\ E_\tau &\longleftrightarrow \Gamma_1(n)\tau. \end{aligned}$$

We now give a short description of two methods that can be used to determine the $(\mathbb{Z}/n)^\times$-action on $\mathrm{mf}_*(\Gamma_1(n);\mathbb{C})$.

4.3.1. The Tate normal form method

This method was outlined in [45, Proposition 3.2] and [6, Theorem 1.1.1]. Let $p\colon E \to S$ be an elliptic curve over a scheme S and let P be a chosen point on E of exact order n. Zariski locally, the elliptic curve is given by a Weierstraß equation. We can use a change of coordinates to move the point P to $(0,0)$ in such a way that the tangent line at P has zero slope. The elliptic curve will then have a Weierstraß equation of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2.$$

This is known as the *homogeneous Tate normal form* of (E, P). To illustrate the argument suppose that n is odd and that we want to determine the action of the element $[2] \in (\mathbb{Z}/n)^\times$. To understand this action we need to take the elliptic E with the chosen point $P = (0,0)$ and use a transformation to move the point $[2]P$ to coordinates $(0,0)$ while preserving the shape of the Weierstraß equation. The new coefficients will tell us how $[2]$ acts on the ring $\mathrm{mf}_*(\Gamma_1(n);\mathbb{C})$. If 2 is a primitive root modulo n, i.e. a generator of $(\mathbb{Z}/n)^\times$ this determines the full action. Otherwise, we will need the action of other elements as well.

4.3.2. The Eisenstein method

The space $\mathrm{mf}_k(\Gamma_1(n);\mathbb{C})$ naturally decomposes into its subspaces of cusp forms, which we will denote by $\mathrm{cf}_k(\Gamma_1(n);\mathbb{C})$, and the corresponding quotient space

$$\mathrm{ef}_k(\Gamma_1(n);\mathbb{C}) \overset{\text{def}}{=} \mathrm{mf}_k(\Gamma_1(n);\mathbb{C})/\,\mathrm{cf}_k(\Gamma_1(n);\mathbb{C}),$$

the *Eisenstein subspace*. Using [13, Theorem 4.1.8] one can write an explicit basis for the Eisenstein subspace in terms of modified Eisenstein series on which the $(\mathbb{Z}/n)^\times$-action is described explicitly by the multiplication with the respective character. Provided that we are in a situation where no cusp exist forms this determines the $(\mathbb{Z}/n)^\times$-action on $\mathrm{mf}_k(\Gamma_1(n);\mathbb{C})$.

Chapter 5

Topological Modular Forms

In this chapter we switch gears from number theory to topology and we introduce the main objects of study in the thesis, the commutative ring spectra of topological modular forms with level structure.

The theory of topological modular forms is a fairly new and active research area in algebraic topology with rich applications in topology and intricate connections to number theory. The spectrum of topological modular forms is a generalized Eilenberg-Steenrod cohomology theory that is in a precise sense the union of all elliptic cohomology theories arising from elliptic curves. It is a homotopy theoretic refinement of the classical ring of integral modular forms. A brief and rather sketchy definition of the spectra TMF, Tmf and tmf, following [14], is provided below.

Recall that we write $\mathcal{M}_{fg}$ for the moduli stack of formal groups classifying all 1-dimensional formal groups and their isomorphisms. Thus maps $\mathrm{Spec}(R) \to \mathcal{M}_{fg}$ with R a commutative ring are in one-to-one correspondence with formal groups over R. Every complex oriented cohomology theory gives rise to a formal group law and there is a universal one, complex bordism theory MU. The resulting formal group law is the universal formal group law, meaning that homomorphisms of rings $MU_* \to R$ are in one-to-one correspondence with formal group laws over R. Conversely, given a homomorphism of rings $MU_* \to R$ that classifies a formal group law over a ring R, the Landweber exact functor theorem gives an algebraic criterion that ensures the functor $X \mapsto MU_*(X) \otimes_{MU_*} R$, where X is a space, satisfies exactness and thus yields a homology theory. The Landweber condition can also be expressed in the language of stacks. A formal group law classified by a map $MU_* \to R$ is Landweber exact if and only if the corresponding map $\mathrm{Spec}(R) \to \mathcal{M}_{fg}$ is flat. We will be interested in even periodic cohomology theories, so we will use the periodic version of complex bordism

$$MP = \vee_{n \in \mathbb{Z}} \Sigma^{2n} MU.$$

There is a map $\mathcal{M}_{ell} \to \mathcal{M}_{fg}$ sending an elliptic curve to its completion at the identity and this map is flat, see [14, Theorem 3.12]. Therefore, given any flat map $\mathrm{Spec}(R) \to \mathcal{M}_{ell}$ classifying an elliptic curve E over a ring R we get a flat map $\mathrm{Spec}(R) \to \mathcal{M}_{fg}$ and thus, by the stacky version of the Landweber exact functor theorem, a homology theory. Write

$\mathcal{O}^{\mathsf{hom}}$ for the resulting presheaf on the affine flat site of $\mathcal{M}_{ell}$

$$\begin{aligned} \mathcal{O}^{\mathsf{hom}} \colon \mathcal{M}_{ell}^{\mathsf{aff,flat}} &\longrightarrow \{\text{homology theories}\} \\ \operatorname{Spec}(R) \to \mathcal{M}_{ell} &\longmapsto MP_*(-) \otimes_{MP_0} R. \end{aligned}$$

We would like to define a single universal elliptic homology theory and the standard way to build such an object from a presheaf is to take global sections. Unfortunately, the site of affine schemes over the moduli stack $\mathcal{M}_{ell}$ has no initial object and therefore no notion of global sections. We can try to define the universal elliptic homology theory as the limit $\lim_{U \in \mathcal{U}} \mathcal{O}^{\mathsf{hom}}(U)$, where $\mathcal{U}$ is a cover of $\mathcal{M}_{ell}$. However, the category of homology theories is not complete and this limit does not exists. The way to remedy this is to recall that by Brown's representability theorem homology theories are represented by spectra. The category of spectra is better behaved than the category of homology theories, in particular, it has limits and colimits. If we can show that the presheaf $\mathcal{O}^{\mathsf{hom}}$ rigidifies to a sheaf of spectra, we can build our desired universal object by taking a homotopy limit in the category of spectra. This is the content of the following theorem due to Goerss-Hopkins-Miller and Lurie.

Theorem 5.0.1 (Goerss-Hopkins-Miller and Lurie)**.** *The moduli stack of elliptic curves admits an even periodic enhancement, i.e. there is a sheaf of commutative ring spectra* $\mathcal{O}^{top} \colon \mathcal{M}_{ell}^{\mathsf{aff,\acute{e}t}} \longrightarrow \operatorname{CAlg}(\operatorname{Sp})$ *on the affine étale site of* $\mathcal{M}_{ell}$ *extending the presheaf* $\mathcal{O}^{hom}$*. Furthermore, this sheaf extends to the compactification* $\overline{\mathcal{M}}_{ell}$*.*

We can now define the commutative ring spectra of topological modular forms, that come in the following three variants characterized, among other things, by their peculiar capitalization.

$$\begin{aligned} \mathrm{TMF} &\overset{\mathsf{def}}{=} \mathcal{O}^{\mathsf{top}}(\mathcal{M}_{ell}), \\ \mathrm{Tmf} &\overset{\mathsf{def}}{=} \mathcal{O}^{\mathsf{top}}(\overline{\mathcal{M}}_{ell}), \\ \mathrm{tmf} &\overset{\mathsf{def}}{=} \mathrm{Tmf}\langle 0 \rangle. \end{aligned}$$

The name *topological modular forms* comes from the fact that the homotopy groups of these spectra are closely connected to the classical rings of modular forms. The ring $\pi_*(\mathrm{TMF})$ is rationally isomorphic to the ring of weakly holomorphic integral modular forms

$$\mathrm{mf}_*(\mathrm{SL}_2(\mathbb{Z};\mathbb{Z})) \cong \mathbb{Z}[c_4, c_6, \Delta^{\pm 1}]/(c_4^3 - c_6^2 - 1728\Delta),$$

and $\pi_*(\mathrm{tmf})$ is rationally isomorphic to the subring of integral modular forms

$$\mathrm{MF}_*(\mathrm{SL}_2(\mathbb{Z});\mathbb{Z}) \cong \mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - 1728\Delta).$$

Here the elements c_4, c_6 and Δ have degrees $8, 12$ and 24 respectively.

5.1. Topological modular forms with level structure

The above constructions have been extended to elliptic curves with level structure. Let Γ be one of $\Gamma_1(n)$, $\Gamma_0(n)$ or $\Gamma(n)$. The stacks $\mathcal{M}(\Gamma)$ are étale over $\mathcal{M}_{ell}$ and we can define

$$\mathrm{TMF}(\Gamma) \overset{\text{def}}{=} \mathcal{O}^{\text{top}}(\mathcal{M}(\Gamma)).$$

In contrast, the stacks $\overline{\mathcal{M}}(\Gamma)$ are in general not étale over $\overline{\mathcal{M}}_{ell}$. As a remedy, Hill and Lawson extended the sheaf $\mathcal{O}^{\text{top}}$ to the log-étale site of the compactified moduli stack $\overline{\mathcal{M}}_{ell}$, see [36]. The maps $\overline{\mathcal{M}}(\Gamma) \to \overline{\mathcal{M}}_{ell}$ are log-étale and thus we can define

$$\mathrm{Tmf}(\Gamma) \overset{\text{def}}{=} \mathcal{O}^{\text{top}}(\overline{\mathcal{M}}(\Gamma)),$$
$$\mathrm{tmf}(\Gamma) \overset{\text{def}}{=} \mathrm{Tmf}(\Gamma)\langle 0\rangle.$$

Notation 5.1.1. When $\Gamma = \Gamma_1(n)$ we will use the notation $\mathrm{TMF}_1(n), \mathrm{Tmf}_1(n)$ and $\mathrm{tmf}_1(n)$ for $\mathrm{TMF}(\Gamma_1(n)), \mathrm{Tmf}(\Gamma_1(n))$ and $\mathrm{tmf}(\Gamma_1(n))$, respectively. The appropriate changes in decoration will be made when Γ is $\Gamma_0(n)$ or $\Gamma(n)$.

Remark. As noted in the discussion after [35, Definition 4.1], the definition of $\mathrm{tmf}(\Gamma)$ above should be considered appropriate for $n \geq 2$ if $\mathrm{tmf}(\Gamma)$ is even and $\pi_*^e(\mathrm{tmf}(\Gamma))$ is isomorphic to the ring $\mathrm{mf}_*(\Gamma;\mathbb{Z})$ of integral holomorphic modular forms with respect to Γ. The second condition is always true, but in general, there can be a non-trivial $\pi_1^e(\mathrm{tmf}(\Gamma))$. The first case of this occurring in the case of $\mathrm{Tmf}_1(n)$ is when $n = 23$ as shown in [10]. The significance of the number 23 will become even greater in Section 5.4, where we look for candidates for Gorenstein duality and Anderson self-duality among the ring spectra $\mathrm{tmf}_1(n)$ and $\mathrm{Tmf}_1(n)$, respectively.

5.2. The descent spectral sequence

Recall the descent spectral sequence construction from Section 2.7 of Chapter 2. Specializing to the derived stacks $\overline{\mathcal{M}}(\Gamma)$ the spectral sequence takes the form

$$E_2^{st} = H^s(\overline{\mathcal{M}}(\Gamma), \omega^{\otimes t/2}) \Longrightarrow \pi_{t-s}(\mathrm{Tmf}(\Gamma)).$$

where $\omega = \pi_2(\mathcal{O}^{\text{top}})$. The stacks $\overline{\mathcal{M}}(\Gamma)$ are of cohomological dimension 1 and the spectral sequence collapses at its E_2 page so we obtain

$$\pi_t(\mathrm{Tmf}(\Gamma)) \cong \begin{cases} H^0(\overline{\mathcal{M}}(\Gamma), \omega^{\otimes \frac{t}{2}}) & \text{for } t \text{ even} \\ H^1(\overline{\mathcal{M}}(\Gamma), \omega^{\otimes \frac{t+1}{2}}) & \text{for } t \text{ odd.} \end{cases}$$

5.3. Equivariance

We now concentrate on the moduli stacks $\mathcal{M}_1(n)$ and their compactifications $\overline{\mathcal{M}}_1(n)$. Recall that $\mathcal{M}_1(n)$ classifies elliptic curves E together with a chosen point P in E of exact order n. Sending (E,P) to (E,kP) for $[k] \in (\mathbb{Z}/n)^\times$ defines an action of $(\mathbb{Z}/n)^\times$ on $\mathcal{M}_1(n)$ that extends to an action on the compactified stack $\overline{\mathcal{M}}_1(n)$. By functoriality, this induces an action on the global sections $\mathrm{Tmf}_1(n) = \mathcal{O}^{\mathsf{top}}(\overline{\mathcal{M}}_1(n))$. Write $G = (\mathbb{Z}/n)^\times$ for brevity. The sheaf $\mathcal{O}^{\mathsf{top}}$ is a sheaf on the affine étale site of $\overline{\mathcal{M}}_1(n)$, but we can also view it as a sheaf on the G-equivariant affine étale site $\overline{\mathcal{M}}_1(n)_G^{\mathsf{aff,ét}}$, where coverings are families

$$\{U_i \to \overline{\mathcal{M}}_1(n)\}_{i\in I}$$

of G-equivariant étale morphisms $U_i \to \overline{\mathcal{M}}_1(n)$ with U_i affine. Evaluating the sheaf $\mathcal{O}^{\mathsf{top}}$ on such a covering would land us in the category of commutative ring spectra with a G-action which can be though of as the functor category $F(BG_+, \mathrm{CAlg}(\mathrm{Sp}))$. We can then take the cofree construction of the resulting ring spectrum and get a commutative ring G-spectrum. Abusing notation we shall use $\mathcal{O}^{\mathsf{top}}$ to denote the composition

$$\overline{\mathcal{M}}_1(n)_G^{\mathsf{aff,ét}} \xrightarrow{\mathcal{O}^{\mathsf{top}}} F(BG_+, \mathrm{CAlg}(\mathrm{Sp})) \xrightarrow{F(EG_+,-)} \mathrm{CAlg}(\mathrm{Sp}_G).$$

Remark. It is not clear that one gets a strict $(\mathbb{Z}/n)^\times$-action in the ∞-categorical framework. Meier argues in [52, Example 6.12] that one does get a strict $(\mathbb{Z}/2)^\times$-action on $\mathrm{Tmf}_1(n)$ which suffices for the examples of level 4 and level 6 that we study. Alternatively one could work instead with a model category framework.

In the rest of this section we examine a number of a priori unexpected equivariant properties of the sheaf $\mathcal{O}^{\mathsf{top}}$ and the ring G-spectra $\mathrm{TMF}_1(n)$, $\mathrm{Tmf}_1(n)$ and $\mathrm{tmf}_1(n)$.

Proposition 5.3.1. *Let n be square-free integer. If $U \to \overline{\mathcal{M}}_1(n)$ is a G-equivariant affine open in $\overline{\mathcal{M}}_1(n)_G^{\mathsf{aff,ét}}$, the Tate spectrum of $\mathcal{O}^{\mathsf{top}}(U)$ vanishes, that is $\mathcal{O}^{\mathsf{top}}(U)^{tG} \simeq *$.*

Proof. This is a special case of [47, Theorem 5.10]. To apply the theorem we have to check that the map $U/G \to \mathcal{M}_{fg}$ is tame. This follows because the map $\overline{\mathcal{M}}_{ell} \to \mathcal{M}_{fg}$ is representable by [47, Theorem 7.2]. The map $\overline{\mathcal{M}}_0(n) \to \mathcal{M}_{ell}$ is representable provided that n is square-free, see [11, p. 4.1.1]. Finally, $U/G \to \overline{\mathcal{M}}_0(n)$ is also representable as $\overline{\mathcal{M}}_1(n) \to \overline{\mathcal{M}}_0(n)$ is a G-Galois covering and U/G sits as an open immersion inside $\overline{\mathcal{M}}_1(n)/G$. □

Proposition 5.3.2. *Let $n \geq 2$. The map $\mathrm{Tmf}_0(n) \to \mathrm{Tmf}_1(n)$ induces an equivalence*

$$\mathrm{Tmf}_1(n)^{hG} \simeq \mathrm{Tmf}_0(n).$$

Proof. This was first shown by Hill and Lawson in [36, Theorem 6.1]. An alternative proof is given by Meier in [52, Proposition A.4]. □

For $n \geq 3$ the ring spectra $\mathrm{TMF}_1(n)$, $\mathrm{Tmf}_1(n)$ and $\mathrm{tmf}_1(n)$ have a natural C_2-action as described in [52, Example 6.12]. One can then ask whether any of these ring C_2-spectra are strongly even in the sense of Definition 1.6.1.

Proposition 5.3.3 ([52, Example 6.12])**.** $\mathrm{TMF}_1(n)$ *is strongly even as a ring C_2-spectrum.*

The current state of affairs for the connective versions of $\mathrm{Tmf}_1(n)$ is less satisfying with the number 23 making yet another appearance.

Proposition 5.3.4 ([52, Example 6.13])**.** $\mathrm{tmf}_1(n)$ *is strongly even as a ring C_2-spectrum for $2 < n < 23$.*

Another peculiar feature of the ring G-spectra of topological modular forms with $\Gamma_1(n)$ structure is that we can reconstruct, both equivariantly and non-equivariantly, the non-connective spectrum $\mathrm{Tmf}_1(n)$ from the Čech complex of its connective cover $\mathrm{tmf}_1(n)$. Let J denote the ideal

$$J \overset{\mathsf{def}}{=} \ker(\pi_\bigstar^G(\mathrm{tmf}_1(n)) \to \pi_0^G(\mathrm{tmf}_1(n))$$

and suppose that J is radically finitely generated by a regular sequence $\mathbf{x} = (x_1, \ldots, x_n)$.

Proposition 5.3.5. *Let f be a non-constant homogeneous polynomial in $(x_1, \ldots, x_n)$. Then the map*

$$\mathrm{tmf}_1(n)[f^{-1}] \to \mathrm{Tmf}_1(n)[f^{-1}]$$

is an equivalence of G-spectra.

Proof. The proof goes as [35, Lemma 4.21]. □

Proposition 5.3.6. *Let f be a non-constant homogeneous polynomial in $(x_1, \ldots, x_n)$ and write $D(f)$ for the nonvanishing locus of the underlying element $f \in H^0(\overline{\mathcal{M}}_1(n), w^*)$. Then there is an equivalence of G-spectra*

$$\mathrm{Tmf}_1(n)[f^{-1}] \simeq \mathcal{O}^{top}(D(f)).$$

Proof. The proof goes as [35, Lemma 4.22]. □

Proposition 5.3.7. *There is an equivalence of G-spectra*

$$\mathrm{Tmf}_1(n) \simeq \check{\mathrm{C}}_{\mathrm{J}}(\mathrm{tmf}_1(n)).$$

Proof. This is an equivariant version of [29, Lemma 5.2]. Let us abbreviate $\mathbf{r} = \mathrm{tmf}_1(n)$ and $\mathbf{R} = \mathrm{Tmf}_1(n)$. We assume that we are in a situation where the connective cover $\mathbf{r}$ can be constructed as the naive truncation of $\mathbf{R}$ and so we have a map $l \colon \mathbf{r} \to \mathbf{R}$ inducing a monomorphism on homotopy groups. We will show that $\mathbf{R}$ has the universal property that $\check{\mathrm{C}}_{\mathrm{J}}(\mathbf{r})$ enjoys. First recall that by Propositions 5.3.5 and 5.3.6 given $x \in \pi_\bigstar^G(\mathbf{R})$ we have an equivalence

$$\mathbf{R}[1/x] \simeq \mathbf{r}[1/x].$$

Let

$$\mathbf{r}/\mathbf{x} = \mathbf{r}/x_1 \otimes_{\mathbf{r}} \mathbf{r}/x_2 \otimes_{\mathbf{r}} \cdots \otimes_{\mathbf{r}} \mathbf{r}/x_n$$

denote the unstable Koszul complex of $\mathbf{r}$ with respect to the set of radical generators $\mathbf{x}$ of J. We have to show that $\mathbf{R}$ admits no $\mathbf{r}$-module maps from $\mathbf{r}/\mathbf{x}$, i.e.

$$[\mathbf{r}/\mathbf{x}, \mathbf{R}]_*^{\mathbf{r},G} = 0.$$

The stack $\overline{\mathcal{M}}_1(n)$ admits a finite open cover by substacks of the form $\overline{\mathcal{M}}_1(n)[\frac{1}{y}]$ for $y \in J$ whose intersections are of the same form. By part (2) of Proposition 5.3.1 there is an equivalence

$$\mathbf{R}[1/y] \simeq \mathcal{O}^{\mathsf{top}}(\overline{\mathcal{M}}_1(n)[1/y]).$$

We have assumed that

$$H^0(\overline{\mathcal{M}}_1(n)[\frac{1}{y}], \omega^{\otimes * /2}) = \pi_*^e(\mathbf{r}[\frac{1}{y}]).$$

Moreover, since y acts nilpotently on $\mathbf{r}/\mathbf{x}$ we have

$$[\mathbf{r}/\mathbf{x}, \mathbf{T}[1/y]]_*^{\mathbf{r},G} = 0.$$

Finally, $\mathbf{R}$ is built from the spectra $\mathbf{R}[\frac{1}{y}]$ and so

$$[\mathbf{r}/\mathbf{x}, \mathbf{R}]_*^{\mathbf{r},G} = 0$$

□

5.4. Gorenstein candidates

We now come to the main question subject to investigation in this thesis. Recall that the spectra $\mathrm{tmf}_1(n)$ can be constructed as genuine ring $(\mathbb{Z}/n)^\times$- spectra and as such they are legitimate candidates for equivariant Gorenstein duality. We ask the following question.

Question Does $\mathrm{tmf}_1(n)$ exhibit *equivariant* Gorenstein duality?

Naturally, one first looks at the non-equivariant situation. A surprising fact is that there is a *finite* list of levels n for which $\mathrm{tmf}_1(n)$ exhibits the Gorenstein duality phenomenon *non-equivariantly*. In [52] Meier determined a finite list of cases where the ring spectrum $\mathrm{Tmf}_1(n)$ is Anderson self-dual. Combining his results with [29] we see that the non-equivariant Gorenstein duality picture for $\mathrm{tmf}_1(n)$ is as follows.

Theorem 5.4.1. *The ring spectrum* $\mathrm{tmf}_1(n)$ *has non-equivariant Gorenstein duality if and only if* $n \in \{1, \ldots, 8, 11, 14, 15\}$ *or potentially* $n = 23$ *with non-equivariant Gorenstein duality shifts* a *as follows.*

n	1	2	3	4	5	6	7	8	11	14	15	23
a	-22	-14	-10	-8	-6	-6	-4	-4	-2	-2	-2	0

Proof. This is [52, Theorem 5.14] combined with [29, Lemma 4.3]. □

One can alternatively identify the list of potential Gorenstein candidates by studying the Hilbert series of the rationalization of the non-equivariant homotopy groups of $\mathrm{tmf}_1(n)$ as described in Section 3.8 of Chapter 3. We go through this argument below to produce a list of levels n identical to the one in Theorem 5.4.1.

First, we describe the general form of the Hilbert series of the graded ring $\mathrm{mf}_*(\Gamma_1(n);\mathbb{C})$. This is a direct computation using the dimension formulas from Section 4.2 of Chapter 4. Let g be the genus of $\mathcal{M}(\Gamma_1(n))_{\mathbb{C}}$, let ε_∞ be the number of cusps of $\mathcal{M}(\Gamma_1(n))_{\mathbb{C}}$ and write m for the dimension of the space $\mathrm{mf}_1(\Gamma_1(n);\mathbb{C})$ of weight 1 modular forms.

Lemma 5.4.2. *The Hilbert series of the graded ring* $\mathrm{mf}_*(\Gamma_1(n);\mathbb{C})$ *are given by*

$$h(t) = \begin{cases} \frac{1}{(1-t^4)(1-t^6} & \text{if } n=1 \\ \frac{1}{(1-t^2)(1-t^4)} & \text{if } n=2 \\ \frac{1}{(1-t)(1-t^3)} & \text{if } n=3 \\ \frac{1}{(1-t)(1-t^2)} & \text{if } n=4 \end{cases}$$

and for $n > 4$ *by*

$$h(t) = \frac{1+(m-2)t+(-2m+g+\varepsilon_\infty)t^2+(m-\frac{\varepsilon_\infty}{2})t^3}{(1-t)^2}.$$

□

Now suppose $\mathrm{tmf}_1(n)$ has non-equivariant Gorenstein duality of some shift. The corresponding ring $\mathrm{mf}_*(\Gamma_1(n);\mathbb{C})$ is Cohen-Macaulay by Proposition 4.1.6 and therefore, by Corollary 3.8.5, its Hilbert series $h(t)$ will satisfy one of the Gorenstein type functional equations (3.13) or (3.14). From the shape of $h(t)$ we see that this can happen if and only if the polynomial in the numerator is palindromic, which is the case for $n = 1,2,3,4$ and in general if

$$\begin{aligned} 1 &= m - \frac{\varepsilon_\infty}{2} \\ m-2 &= -2m+g+\varepsilon_\infty. \end{aligned}$$

There is an explicit formula for the dimension of the space of weight 1 modular forms *only when* $\varepsilon_\infty > 2g-2$ in which case $m = \frac{\varepsilon_\infty}{2}$, see [13, Theorem 3.5.1]. We identify a range of levels n where this is case using the following lemma by Mitankin.

Lemma 5.4.3. *There exists an integer* n_0*, such that for* $n > n_0$ *we have* $\varepsilon_\infty \leq 2g-2$*.*

Proof. We can see this by looking at the difference

$$\begin{aligned}
\varepsilon_\infty - 2g - 2 &= \varepsilon_\infty - 2 - \frac{d_n}{6n} + \varepsilon_\infty + 2 \\
&= 2\varepsilon_\infty - \frac{d_n}{6n} \\
&= \sum_{d|n} \phi(d)\phi(\frac{n}{d}) - \frac{n^2}{12}\prod_{p|n}\left(1 - \frac{1}{p^2}\right) \\
&\leq n\sum_{d|n} 1 - \frac{n^2}{12}\prod_{p|n}\left(1 - \frac{1}{p^2}\right).
\end{aligned}$$

It is well-known that $\sum_{d|n} 1 = O_\varepsilon(n^\varepsilon)$. On the other hand, we have

$$\prod_{p|n}\left(1 - \frac{1}{p^2}\right) \geq \prod_{p}\left(1 - \frac{1}{p^2}\right) = 6/\pi^2.$$

Thus

$$n\sum_{d|n} 1 - \frac{n^2}{12}\prod_{p|n}\left(1 - \frac{1}{p^2}\right) \leq 0$$

for n big enough. This proves the result. □

Suppose $4 < n \leq n_0$. The Hilbert series of $\mathrm{mf}_*(\Gamma_1(n);\mathbb{C})$ then takes the form

$$h(t) = \frac{1 + (\frac{\varepsilon_\infty}{2} - 2)t + gt^2}{(1-t)^2}.$$

We see that the numerator can be palindromic if and only if $g = 0, 1$. The modular curve $\mathcal{M}_1(n)_\mathbb{C}$ is of genus 0 or 1 for a finite number of values of n and so by direct computation we obtain the table below. In the last column we display the predicted non-equivariant Gorenstein shift. The values of g and ε_∞ are computed using the formulae in [13, Figure 3.3, Figure 3.4]. The genus information can also be found in [39, A029937].

The detect the case of $n = 23$ we argue as follows. Recall from Section 4.3 of Chapter 4 that the space of weight 1 modular forms decomposes into its subspace of cusp forms and the corresponding quotient space of Eisenstein forms

$$\mathrm{mf}_1(\Gamma_1(n);\mathbb{C}) = \mathrm{ef}_1(\Gamma_1(n);\mathbb{C}) \oplus \mathrm{cf}_1(\Gamma_1(n);\mathbb{C}).$$

The dimension of the Eisenstein subspace can be calculated using the dimension formulas, see [13, Chapter 4], and is given by

$$\dim_\mathbb{C}(\mathrm{ef}_1(\Gamma_1(n);\mathbb{C})) = \frac{\varepsilon_\infty}{2}.$$

For $n = 23$ we get that $\varepsilon_\infty = 22$ and therefore $\dim_{\mathbb{C}}(\mathrm{ef}_1(\Gamma_1(23); \mathbb{C})) = 11$. By [10] the dimension of the space $\mathrm{cf}_1(\Gamma_1(23))$ is 1 and we obtain

$$m = \dim_{\mathbb{C}}(\Gamma_1(23); \mathbb{C}) = 12.$$

Plugging this value of m into the general expression of the Hilbert series from Lemma 5.4.2 we recover the case $n = 23$.

Table 5.1.: Hilbert series of $\mathrm{mf}_*(\Gamma_1(n); \mathbb{C})$ for $3 \leq n \leq n_0$, $n = 23$ and $g = 0, 1$

n	g	ε_∞	$h(t)$	predicted shift
1	0	1	$\frac{1}{(1-t^4)(1-t^6)}$	-22
2	0	2	$\frac{1}{(1-t^2)(1-t^4)}$	-14
3	0	2	$\frac{1}{(1-t)(1-t^3)}$	-10
4	0	3	$\frac{1}{(1-t)(1-t^2)}$	-8
5	0	4	$\frac{1}{(1-t)^2}$	-6
6	0	4	$\frac{1}{(1-t)^2}$	-6
7	0	6	$\frac{1+t}{(1-t)^2}$	-4
8	0	6	$\frac{1+t}{(1-t)^2}$	-4
11	1	10	$\frac{1+3t+t^2}{(1-t)^2}$	-2
14	1	12	$\frac{1+4t+t^2}{(1-t)^2}$	-2
15	1	16	$\frac{1+6t+t^2}{(1-t)^2}$	-2
23	12	22	$\frac{1+10t+10t^2+t^3}{(1-t)^2}$	0

It would be useful to have the picture in the case of $\Gamma_0(n)$ as well. Let g denote the genus of $\mathcal{M}(\Gamma_0(n))_{\mathbb{C}}$ and let ε_2 and ε_3 denote the number of elliptic points of $\mathcal{M}(\Gamma_0(n))_{\mathbb{C}}$ with period 2 and period 3, respectively. Finally, let ε_∞ be the number of cusps of $\mathcal{M}(\Gamma_0(n))_{\mathbb{C}}$.

Lemma 5.4.4. *The Hilbert series of the graded ring* $\mathrm{mf}_*(\Gamma_0(n); \mathbb{C})$ *for* $n > 2$ *is of form*

$$h(T) = \frac{1 + (g + e_\infty)T + (3g + e_2 + e_3 + 2e_\infty - 4)T^2}{(1 - T^2)(1 - T^3)} + \\ + \frac{(4g + e_2 + 2e_3 + 2e_\infty - 5)T^3 + (3g + e_2 + e_3 + e_\infty - 3)T^4 + gT^5}{(1 - T^2)(1 - T^3)},$$

where $T = t^2$. □

Again, we see that the numerator can be palindromic if and only if $g = 0$ or 1. The two cases are resolved by the table below. The values of g, ε_2, ε_3 and ε_∞ are computed using the formulae in [13, Figure 3.3, Figure 3.4] and can also be found in [39, A001617], [39, A000089], [39, A000086] and [39, A001616] respectively.

Table 5.2.: Hilbert series of $\mathrm{mf}_*(\Gamma_0(n);\mathbb{C})$ for $n > 2$ and $g = 0, 1$

n	g	ε_2	ε_3	ε_∞	$h(T)$	predicted shift
3	0	0	1	2	$\frac{1+T^2}{(1-T)(1-T^3)}$	-10
4	0	0	0	3	$\frac{1}{(1-T)^2}$	-10
5	0	2	0	2	$\frac{1+T^2}{(1-T)(1-T^2)}$	-6
6	0	0	0	4	$\frac{1+T}{(1-T)^2}$	-6
7	0	0	2	2	$\frac{1+2T^2+T^3}{(1-T)(1-T^3)}$	-6
8	0	0	0	4	$\frac{1+T}{(1-T)^2}$	-6
9	0	0	0	4	$\frac{1+T}{(1-T)^2}$	-6
11	1	0	0	2	$\frac{1+T^2}{(1-T)^2}$	-2
14	1	0	0	4	$\frac{(1+T)^2}{(1-T)^2}$	-2
15	1	0	0	4	$\frac{(1+T)^2}{(1-T)^2}$	-2
20	1	0	0	6	$\frac{1+4T+T^2}{(1-T)^2}$	-2
24	1	0	0	8	$\frac{1+6T+T^2}{(1-T)^2}$	-2
27	1	0	0	6	$\frac{1+4T+T^2}{(1-T)^2}$	-2
32	1	0	0	8	$\frac{1+6T+T^2}{(1-T)^2}$	-2
36	1	0	0	12	$\frac{1+10T+T^2}{(1-T)^2}$	-2

We now move to the equivariant picture. Equivariant Gorenstein duality implies non-equivariant Gorenstein duality and therefore the list of $\mathrm{tmf}_1(n)$'s that *could* exhibit the duality equivariantly is the one from Theorem 5.4.1. Levels 1 and 2 are not interesting equivariantly as the group acting is the trivial group. The first equivariantly non-trivial cases are levels $3, 4$ and 6 where the group acting is C_2. The case of $n = 3$ was computed by Greenlees and Meier in [27] and we give computations for $n = 4$ and $n = 6$ in Chapter 6. In fact, we can look at the C_2-equivariant picture for all levels n. Recall again that for $n \geq 3$ the ring spectrum $\mathrm{tmf}_1(n)$ has a natural C_2-action as described in [52, Example 6.12]. One can ask whether any of the $\mathrm{tmf}_1(n)$'s in the finite list above exhibit C_2-equivariant Gorenstein duality. In [52] Meier refines his non-equivariant Anderson self-duality statements for $\mathrm{Tmf}_1(n)$ to C_2-equivariant statements. Combined with an equivariant version of Theorem 3.9.2 his results answer the C_2-equivariant Gorenstein duality question for $n = 3, 5, 7, 11, 15$. We summarize the above discussion in the following theorem.

Theorem 5.4.5. *The ring C_2-spectrum $\mathrm{tmf}_1(n)$ has equivariant Gorenstein duality if and only if $n \in \{3, 5, 7, 11, 15\}$ or potentially $n \in \{4, 6, 8, 14, 23\}$ with equivariant Gorenstein duality shifts δ as follows.*

n	3	4	5	6	7	8	11	14	15	23
δ	$-8-2\sigma$	$-7-\sigma$	-5	-6	$-5+\sigma$	$?$	$-4+2\sigma$	$?$	$-4+2\sigma$	$?$

Proof. For $n = 3$ this is [27]. For $n = 3, 5, 7, 11, 15$ this is [52, Section 6.3]. The cases of $n = 4$ and $n = 6$ are contained in the Chapter 6. □

Studying equivariantly examples for higher n quickly becomes difficult due to the complexity of the $RO(G)$-graded computations. Progress on the first non C_2-case of level 5 is reported in Chapter 7.

5.5. The Anderson-Gorenstein-Serre duality triumvirate

We now look at yet another duality phenomenon this time coming from algebraic geometry, namely Serre duality. In [52] Meier shows that there is an intricate connection between the presence of Serre duality on the moduli stacks $\overline{\mathcal{M}}_1(n)$ and Anderson self-duality of the corresponding ring spectra $\mathrm{Tmf}_1(n)$ arising as the global sections of the sheaf $\mathcal{O}^{\mathrm{top}}$ on $\overline{\mathcal{M}}_1(n)$. Combining his results with the relation between Anderson and Gorenstein duality discussed in Section 3.9 of Chapter 3 we formulate a context where all three duality phenomena are interlaced.

Let $\mathbb{Z} = \mathbb{Z}[\frac{1}{n}]$ and write f_n for the projection $\overline{\mathcal{M}}_1(n) \to \overline{\mathcal{M}}_{ell}$. Let ω denote the pullback to

$\overline{\mathcal{M}}_1(n)$ of the line bundle ω_{ell} on $\overline{\mathcal{M}}_{ell}$.

$$\begin{array}{ccc} \omega = f_n^*\omega_{ell} & & \omega_{ell} \cong \pi_2(\mathcal{O}^{top}) \\ \Big| & & \Big| \\ \overline{\mathcal{M}}_1(n) & \xrightarrow{f_n} & \overline{\mathcal{M}}_{ell} \end{array}$$

First, we record a proposition which determines the general shape of the stacks $\overline{\mathcal{M}}_1(n)$.

Lemma 5.5.1 ([52, Proposition 2.4])**.** *$\overline{\mathcal{M}}_1(n)$ is a weighted projective stack for $1 \leq n \leq 4$ and represented by a projective scheme over $\mathbb{Z}$ for $n \geq 5$.*

We have the following two immediate consequences.

(i) $\overline{\mathcal{M}}_1(n)$ has cohomological dimension 1. In other words, for any $\mathcal{F} \in \mathrm{QCoh}(\overline{\mathcal{M}}_1(n))$ we have
$$H^i(\overline{\mathcal{M}}_1(n), \mathcal{F}) = 0 \text{ for } i \geq 1.$$

(ii) $\overline{\mathcal{M}}_1(n)$ has Serre duality in the sense that there exists a dualizing sheaf $\Omega^1_{\overline{\mathcal{M}}_1(n)}$ on $\overline{\mathcal{M}}_1(n)$ such that for any $\mathcal{F} \in \mathrm{QCoh}(\overline{\mathcal{M}}_1(n))$ we have
$$H^0(\overline{\mathcal{M}}_1(n), \mathcal{F}) \cong H^1(\overline{\mathcal{M}}_1(n), \mathcal{F}^\vee \otimes \Omega^1_{\overline{\mathcal{M}}_1(n)})^\vee.$$

Next, we quote another result by Meier which says that for a finite number of levels n the dualizing sheaf on the stack $\overline{\mathcal{M}}_1(n)$ is of a very particular form, namely a tensor power of the line bundle ω. (Un)surprisingly, this finite list of levels n coincides with the list of Theorem 5.4.1.

Lemma 5.5.2 ([52, Proposition 5.7])**.** *We have that $\Omega^1_{\overline{\mathcal{M}}_1(n)} \cong \omega^{\otimes i}$ if and only if $n \in \{1, \ldots, 8, 11, 14, 15, 23\}$.*

Last, we show that whenever $\Omega^1_{\overline{\mathcal{M}}_1(n)}$ is a tensor power of ω, the ring spectrum $\mathrm{Tmf}_1(n)$ is Anderson self-dual.

Theorem 5.5.3. *We have $\mathbb{Z}^{\mathrm{Tmf}_1(n)} \simeq \Sigma^b \mathrm{Tmf}_1(n)$ as $\mathrm{Tmf}_1(n)$-modules if and only if b is odd and $\Omega^1_{\overline{\mathcal{M}}_1(n)} \cong \omega^{\otimes -m}$ with $m = \frac{b-1}{2}$.*

Proof. The case of $n = 1$ is contained in [63] and [62]. For $n \geq 2$ this is [52, Proposition 5.12]. Let us write $\mathbf{R} = \mathrm{Tmf}_1(n)$ for brevity. By Serre duality the pairing

$$H^0(\overline{\mathcal{M}}_1(n), \omega^{\otimes -i-m}) \otimes H^1(\overline{\mathcal{M}}_1(n), \omega^{\otimes i}) \to H^1(\overline{\mathcal{M}}_1(n), \omega^{\otimes -m}) \cong \mathbb{Z}$$

is perfect; note that all occurring groups are finitely generated $\mathbb{Z}$-modules and torsion free by [52, Lemma 5.9] and this implies that $\pi_*(\mathbf{R})$ is torsion free as well. Let D be a generator of $H^1(\overline{\mathcal{M}}_1(n), \omega^{\otimes -m}) \cong \pi_{-2m-1}(\mathbf{R}) \cong \mathbb{Z}$. We have an isomorphism

$$\phi\colon \pi_i(\mathbb{Z}^{\mathbf{R}}) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Ab}}(\pi_{-i}(\mathbf{R}), \mathbb{Z}).$$

Write δ for the element in $\pi_{2m+1}(\mathbb{Z}^{\mathbf{R}})$ with the property that $\phi(\delta)(D) = 1$. The element δ corresponds to a $\mathbf{R}$-linear map $\hat{\delta}\colon \Sigma^{2m+1}\mathbf{R} \to \mathbb{Z}^{\mathbf{R}}$. We obtain the following commutative diagram

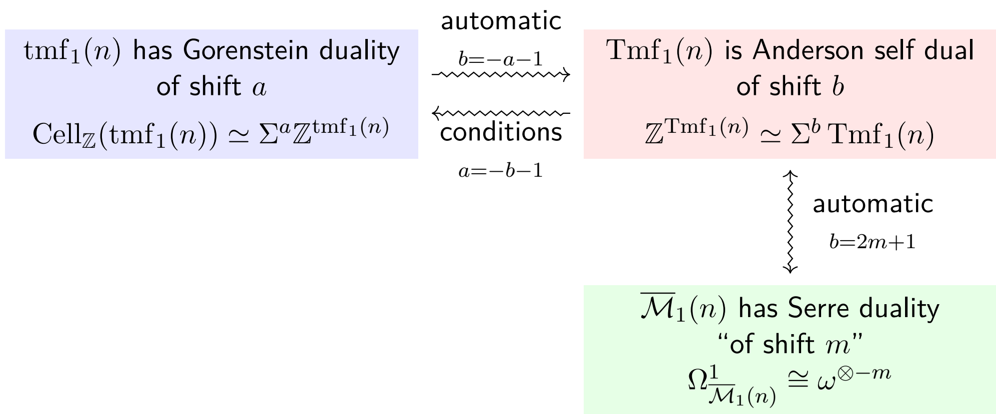
$$\begin{array}{ccccc}
\pi_{i-2m-1}(\mathbf{R}) \otimes \pi_{-i}(\mathbf{R}) & \xrightarrow{\hat{\delta}_* \otimes \mathrm{Id}} & \pi_i(\mathbb{Z}^{\mathbf{R}}) \otimes \pi_{-i}(\mathbf{R}) & \xrightarrow[\cong]{\phi \otimes \mathrm{Id}} & \mathrm{Hom}_{\mathrm{Ab}}(\pi_{-i}(\mathbf{R}), \mathbb{Z}) \otimes \pi_{-i}(\mathbf{R}) \\
\downarrow & & & & \downarrow \mathrm{ev} \\
\pi_{-2m-1}(\mathbf{R}) & & \xrightarrow[\cong]{\phi(\delta)} & & \mathbb{Z}.
\end{array}$$

The left vertical map is a perfect pairing by Serre duality and the right vertical map is a perfect pairing by definition. It follows that the map $\hat{\delta}$ induces an isomorphism

$$\hat{\delta}_*\colon \pi_{i-2m-1}(\mathbf{R}) \to \pi_i(\mathbb{Z}^{\mathbf{R}})$$

for all i and therefore $\hat{\delta}$ is an equivalence of $\mathbf{R}$-modules. □

We summarize the Anderson-Gorenstein-Serre duality interlace in the following diagram.



Chapter 6

Toy Examples

In this chapter we put to work the framework we have set and study two toy examples of ring G-spectra that could exhibit equivariant Gorenstein duality. We look at the ring C_2-spectra of connective topological forms with level 4 and level 6 structure, respectively. These were the natural candidates to look at after Greenlees and Meier studied the ring C_2-spectrum of topological modular forms with level 3 structure in [27]. We summarize the overall attack strategy in the following recipe.

Strategy. Let $G = (\mathbb{Z}/n)^\times$. To study equivariant Gorenstein duality for $\mathrm{tmf}_1(n)$ we proceed as follows.

(a) Identify the moduli stack $\overline{\mathcal{M}}_1(n)$. In the cases of interest $\overline{\mathcal{M}}_1(n)$ is either a weighted projective stack or a $\mathbb{P}^1$.

(b) Calculate the non-equivariant homotopy groups of $\mathrm{Tmf}_1(n)$ via the descent spectral sequence.

(c) Deduce the non-equivariant homotopy groups of $\mathrm{tmf}_1(n)$. In the cases of interest, with the exception of $n = 23$, $\mathrm{tmf}_1(n)$ can be constructed as the naive truncation of $\mathrm{Tmf}_1(n)$ and so $\pi^e_*(\mathrm{tmf}_1(n))$ can be readily obtained from $\pi^e_*(Tmf_1(n))$.

(d) Identify the G-action on $\pi^e_*(\mathrm{tmf}_1(n))$ via either the Tate normal form or the Eisenstein method.

(e) Calculate the $RO(G)$-graded homotopy groups of $\mathrm{tmf}_1(n)$ via a Tate diagram argument or the slice spectral sequence.

(f) Provided that $\pi^G_\bigstar(\mathrm{tmf}_1(n))$ is Gorenstein, try to obtain a Gorenstein duality statement by checking the orientability condition.

(g) Descend to $\mathrm{tmf}_0(n)$.

6.1. Topological modular forms with level 4 structure

Write $\mathcal{M}_1(4)$ for the moduli stack of elliptic curves with a chosen point of exact order 4 and $\overline{\mathcal{M}}_1(4)$ for its Deligne-Mumford compactification. For the sake of brevity we will write $\mathbb{Z}$ for $\mathbb{Z}[\frac{1}{2}]$.

Lemma 6.1.1 ([51, Example 2.1])**.** *The moduli stack $\overline{\mathcal{M}}_1(4)$ is a weighted projective stack*

$$\overline{\mathcal{M}}_1(4) \simeq \mathcal{P}_{\mathbb{Z}}(1,2).$$

Lemma 6.1.2. *The non-equivariant homotopy groups of $\mathrm{tmf}_1(4)$ are given by*

$$\pi^e_*(\mathrm{tmf}_1(4)) \cong \mathbb{Z}[a_1, a_2],$$

where $|a_i| = 2i$.

Proof. We run the descent spectral sequence for $\mathrm{Tmf}_1(4)$. The E_2-page is given by

$$E_2^{st} = H^s(\overline{\mathcal{M}}_1(4), \omega^{\otimes \frac{t}{2}})$$

and the spectral sequence converges to $\pi^e_{t-s}(\mathrm{Tmf}_1(4))$. The moduli stack $\overline{\mathcal{M}}_1(4)$ has cohomological dimension 1 and so the spectral sequence collapses at the E_2-page. Furthermore $\overline{\mathcal{M}}_1(4)$ has Serre duality with a dualizing sheaf $\Omega^1_{\overline{\mathcal{M}}_1(4)} \cong \omega^{\otimes -3}$ in the sense that

$$H^0(\overline{\mathcal{M}}_1(4), \mathcal{F}) \cong H^1(\overline{\mathcal{M}}_1(4), \mathcal{F}^\vee \otimes \omega^{\otimes -3})^\vee,$$

for any quasi-coherent sheaf $\mathcal{F}$ on $\overline{\mathcal{M}}_1(4)$, see [52]. Thus we can write

$$\pi^e_t(\mathrm{Tmf}_1(4)) \cong \begin{cases} H^0(\overline{\mathcal{M}}_1(4), \omega^{\otimes \frac{t}{2}}) & \text{for } t \text{ even} \\ H^1(\overline{\mathcal{M}}_1(4), \omega^{\otimes \frac{t+1}{2}}) \cong H^0(\overline{\mathcal{M}}_1(4), \omega^{\otimes \frac{-t-7}{2}})^\vee & \text{for } t \text{ odd.} \end{cases}$$

There are certain structural features that can be observed in the homotopy groups of $\mathrm{Tmf}_1(4)$ as a consequence of the presence of Serre duality on the moduli stack. By Riemann-Roch $H^0(\overline{\mathcal{M}}_1(4), \omega^{\otimes -i}) = 0$ for $i > 0$ and so

$$\pi^e_{2i}(\mathrm{Tmf}_1(4)) \cong H^0(\overline{\mathcal{M}}_1(4), \omega^{\otimes i}) = 0,$$

for $i < 0$, i.e. all negative homotopy groups of even degree vanish. Furthermore

$$\pi^e_{2i+1}(\mathrm{Tmf}_1(4)) \cong H^0(\overline{\mathcal{M}}_1(4), \omega^{\otimes -i-4})^\vee = 0,$$

for $i > -4$, i.e. all positive homotopy groups of odd degree also vanish. We depict the structure of $\pi_*^e(\mathrm{Tmf}_1(4))$ in the picture below.

H^1	0	□	0	□	0	□	0	0	0	0	0	0												
H^0													□	0	□	0	□	0	□	0	□	0	□	0
	-12		-10		-8		-6		-4		-2		0		2		4		6		8		10	

The sheaf ω can be identified with $\mathcal{O}(1)_{\mathcal{P}_{\mathbb{Z}}(1,2)}$ and using [54, Proposition 2.5] we see that

$$H^s(\mathcal{P}_{\mathbb{Z}}(1,2), \omega^{\otimes *}) = \begin{cases} \mathbb{Z}[a_1, a_2] & \text{for } s = 0 \\ \mathbb{Z}[a_1, a_2]^\vee & \text{for } s = 1 \\ 0 & \text{for } s \geq 2 \end{cases}$$

or

$$\pi_*^e(\mathrm{Tmf}_1(4)) \cong \mathbb{Z}[a_1, a_2] \oplus \Sigma^{-7}\mathbb{Z}[a_1, a_2]^\vee.$$

Due to the gap between $\pi_{\geq 0}^e(\mathrm{Tmf}_1(4))$ and $\pi_{<0}^e(\mathrm{Tmf}_1(4))$ we can obtain the connective cover $\mathrm{tmf}_1(4)$ as the naive truncation of $\mathrm{Tmf}_1(4)$ and deduce

$$\pi_*^e(\mathrm{tmf}_1(4)) \cong \mathbb{Z}[a_1, a_2].$$

□

Lemma 6.1.3. *C_2 acts on $\pi_*^e(\mathrm{tmf}_1(4))$ by fixing a_2 and sending a_1 to $-a_1$.*

Proof. This is analogous to the level 3 case and follows immediately from the Tate normal form method. □

Lemma 6.1.4. *The $RO(C_2)$-graded equivariant homotopy groups of $\mathrm{tmf}_1(4)$ are given by*

$$\pi_\bigstar^{C_2}(\mathrm{tmf}_1(4)) \cong \mathbb{Z}[\bar{a}_2, \bar{b}, u^{\pm 1}],$$

where $|\bar{a}_2| = 2 + 2\sigma$, $|\bar{b}| = 1 + \sigma$ and $|u| = 2 - 2\sigma$.

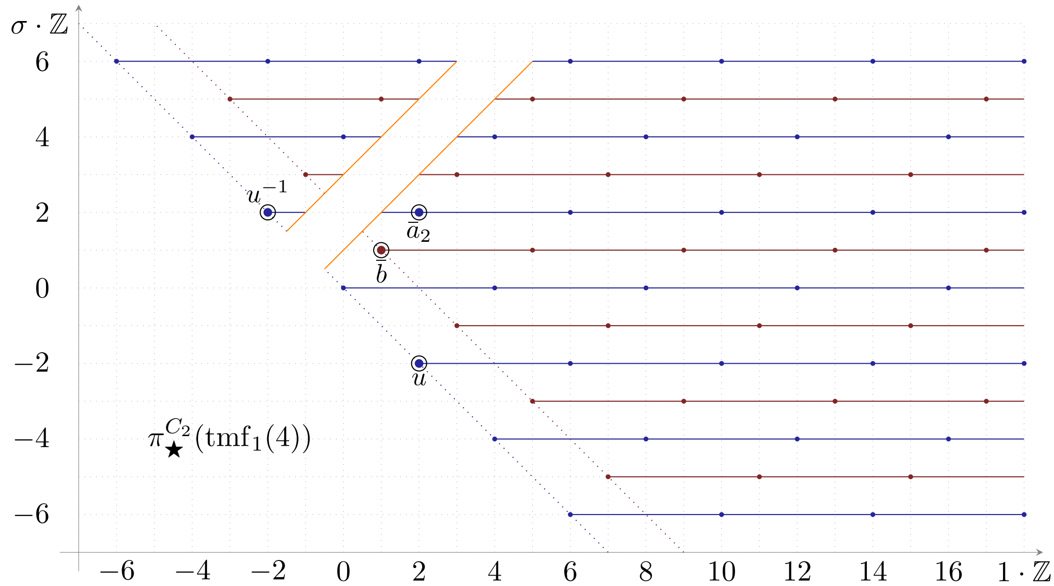
Proof. We run the $RO(C_2)$-graded homotopy fixed points spectral sequence form $\mathrm{tmf}_1(4)$. Its E_2-page is given by

$$E_2^{pq}(r) = H^q(C_2, \pi_{p+q}^e(\mathrm{tmf}_1(4) \wedge S^{-r\sigma})) \Longrightarrow \pi_{p+r\sigma}^{C_2}(\mathrm{tmf}_1(4)^h).$$

As the group order is invertible in the coefficient module only the H^0 entries are non-zero and thus we get

$$\pi_*^e(\mathrm{tmf}_1(4) \wedge S^{-r\sigma})^{C_2} \cong \begin{cases} \Sigma^{-r}\pi_*^e(\mathrm{tmf}_1(4))^{C_2} & \text{for } t \text{ even} \\ \Sigma^{-r}\Sigma^{-2}\pi_*^e(\mathrm{tmf}_1(4))^{C_2} & \text{for } t \text{ odd.} \end{cases}$$

We display this information in the picture below.



Blue lines indicate copies of the ring $\mathbb{Z}[a_1^2, a_2]$ and red lines indicate copies of the same ring shifted by -2. Writing this out we get

$$\pi_\bigstar^{C_2}(\mathrm{tmf}_1(4)) \cong \mathbb{Z}[\bar{a}_2, \bar{b}, u^{\pm 1}],$$

where $|\bar{a}_2| = 2 + 2\sigma$, $|\bar{b}| = 1 + \sigma$ and $|u| = 2 - 2\sigma$ as claimed. $\square$

Corollary 6.1.5. $\mathrm{tmf}_1(4)$ *is Gorenstein of shift* $\alpha = -5 - 3\sigma$

$$\mathrm{Hom}_{\mathrm{tmf}_1(4)}(H\underline{\mathbb{Z}}, \mathrm{tmf}_1(4)) \simeq \Sigma^{-5-3\sigma} H\underline{\mathbb{Z}}.$$

Proof. This follows from the description of the ring $\pi_\bigstar^{C_2}(\mathrm{tmf}_1(4))$ obtained above with Gorenstein shift calculated as $\alpha = -(|\bar{a}_2| + |\bar{b}|) - 2 = -5 - 3\sigma$. $\square$

From the shape of the $RO(C_2)$-graded coefficients the following is immediate.

Corollary 6.1.6. $\mathrm{tmf}_1(4)$ *is strongly even.* $\square$

We next want to move on to Gorenstein duality, so let us collect the necessary ingredients.

(i) By Corollary 6.1.5 $\mathrm{tmf}_1(4)$ is Gorenstein of shift $\alpha = -5 - 3\sigma$

$$\mathrm{Hom}_{\mathrm{tmf}_1(4)}(H\underline{\mathbb{Z}}, \mathrm{tmf}_1(4)) \simeq \Sigma^{-5-3\sigma} H\underline{\mathbb{Z}}.$$

(ii) By Lemma 3.4.7 the Anderson $\mathbb{Z}$-dual of $\mathrm{tmf}_1(4)$ is a Matlis lift of $\mathbb{Z}^{H\underline{\mathbb{Z}}}$

$$\mathrm{Hom}_{\mathrm{tmf}_1(4)}(H\underline{\mathbb{Z}}, \mathbb{Z}^{\mathrm{tmf}_1(4)}) \simeq \mathrm{Hom}_{H\underline{\mathbb{Z}}}(H\underline{\mathbb{Z}}, \mathbb{Z}^{H\underline{\mathbb{Z}}}).$$

(iii) By Example 3.1.8 the Anderson $\mathbb{Z}$-dual of $H\underline{\mathbb{Z}}$ is given by

$$\mathbb{Z}^{H\underline{\mathbb{Z}}} \simeq \Sigma^{2-2\sigma} H\underline{\mathbb{Z}}.$$

Combining (i), (ii) and (iii) we get the following equivalences

$$\mathrm{Hom}_{\mathrm{tmf}_1(4)}(H\underline{\mathbb{Z}}, \mathrm{tmf}_1(4)) \simeq \Sigma^{-5-3\sigma} H\underline{\mathbb{Z}} \simeq \mathrm{Hom}_{\mathrm{tmf}_1(4)}(H\underline{\mathbb{Z}}, \Sigma^{-7-\sigma}\mathbb{Z}^{\mathrm{tmf}_1(4)}).$$

We would like to remove the $\mathrm{Hom}_{\mathrm{tmf}_1(4)}(H\underline{\mathbb{Z}}, -)$ on both sides from this equivalence. Recalling the discussion from Section 3.6 of Chapter 3 we can do this provided the right action of

$$\mathcal{E} = \mathrm{End}_{\mathrm{tmf}_1(4)}(H\underline{\mathbb{Z}}, H\underline{\mathbb{Z}})$$

on both sides of the equivalence is the same. At the moment, this is a conjecture.

Conjecture 6.1.7. *There is a unique $\mathcal{E}$-module structure on $H\underline{\mathbb{Z}}$.*

Corollary 6.1.8. *Provided the conjecture holds, we have the following corollaries.*

(1) $\mathrm{tmf}_1(4)$ *has equivariant Gorenstein duality of shift* $-7-\sigma$

$$\mathrm{Cell}_{H\underline{\mathbb{Z}}}(\mathrm{tmf}_1(4)) \simeq \Sigma^{-7-\sigma}\mathbb{Z}^{\mathrm{tmf}_1(4)}.$$

(2) $\mathrm{Tmf}_1(4)$ *is Anderson self-dual of shift* $6+\sigma$

$$\mathbb{Z}^{\mathrm{Tmf}_1(4)} \simeq \Sigma^{6+\sigma}\,\mathrm{Tmf}_1(4).$$

□

We can descend to $\mathrm{tmf}_0(4)$ and deduce that it is also Gorenstein.

Lemma 6.1.9. $\mathrm{tmf}_0(4)$ *is Gorenstein of shift* $a = -10$.

Proof. This follows directly once we observe the right things, namely that

(1) the order of C_2 is invertible in $\pi_*^e(\mathrm{tmf}_1(4))$ so the HFPSS collapses (its E_2-page is concentrated in the 0-th column) giving $\pi_*^e(\mathrm{tmf}_1(4))^{C_2} \cong \pi_*^{C_2}(\mathrm{tmf}_1(4)^h)$,

(2) $\mathrm{tmf}_1(4)$ is cofree, i.e. $\mathrm{tmf}_1(4)^h \simeq \mathrm{tmf}_1(4)$.

Combining these two facts we get

$$\begin{aligned}\pi_* \operatorname{tmf}_0(4) &\cong \pi_*^e(\operatorname{tmf}_1(4)^{C_2}) \\ &\cong \pi_*^{C_2}(\operatorname{tmf}_1(4)) \\ &\cong \pi_*^e(\operatorname{tmf}_1(4)^{C_2} \\ &\cong \mathbb{Z}[a_1^2, a_2],\end{aligned}$$

where $|a_1^2| = 4$ and $|a_2| = 4$. This is Gorenstein of shift $a = -(4+4) - 2 = -10$ which agrees with the shift predicted by the computation of the Hilbert series of $\operatorname{mf}_*(\Gamma_1(4); \mathbb{C})$ in Section 5.4 of Chapter 5. □

6.2. Topological modular forms with level 6 structure

The calculations for the case $n = 6$ proceed analogously. Write $\mathcal{M}_1(6)$ for the moduli stack of elliptic curves with a chosen point of exact order 6 and $\overline{\mathcal{M}}_1(6)$ for its Deligne-Mumford compactification. The group acting on the stack is $(\mathbb{Z}/6)^\times \cong C_2$. For the rest of this section we write $\mathbb{Z}$ for $\mathbb{Z}[\frac{1}{6}]$.

Lemma 6.2.1 ([51, Example 2.5])**.** *The moduli stack $\overline{\mathcal{M}}_1(6)$ is a projective line*

$$\overline{\mathcal{M}}_1(6) \simeq \mathbb{P}^1_{\mathbb{Z}}.$$

Lemma 6.2.2. *The non-equivariant homotopy groups of $\operatorname{tmf}_1(6))$ are given by*

$$\pi_*^e(\operatorname{tmf}_1(6)) \cong \mathbb{Z}[x, y],$$

where $|x| = |y| = 2$.

Proof. The proof is analogous to the level 4 case. □

Lemma 6.2.3. *C_2 acts on $\pi_*^e(\operatorname{tmf}_1(6))$ by sending x to $-x$ and y to $-y$.*

Proof. Only the Eisenstein method is tractable here. We know that $\operatorname{mf}_1(6)_*$ is generated by two modular forms in degree 1 and that there are no cusp forms of weight 1. We write a basis of the weight-1 Eisenstein subspace $\operatorname{mf}_1(6)_1 = \mathcal{E}_1(6)_1$ in terms of modified Eisenstein series. Let $A_{6,1}$ be the set of triples $(\{\psi, \phi\}, t)$ such that ψ and ϕ are primitive Dirichlet characters modulo u and v with $(\psi\phi)(-1) = -1$ and $tuv \mid 6$. Then the set

$$\left\{ E^{\psi,\phi,t} \mid (\{\psi, \phi\}, t) \in A_{6,1} \right\}$$

gives a basis for $\mathcal{E}_1(6)_1$. There are no non-trivial primitive Dirichlet characters modulo 2 and 6 so the only possible values for $(\{u, v\}, t)$ are $(\{3, 1\}, 1)$ and $(\{3, 1\}, 2)$. Thus a basis of

$\mathrm{mf}_1(6)_1$ is given by the modified Eisenstein series $E^{\psi_3,\phi_1,1}$ and $E^{\psi_3,\phi_1,3}$ and the C_2-action on the basis elements is through the character ψ_3 (ϕ_1 is the trivial character). □

Lemma 6.2.4. *The $RO(C_2)$-graded equivariant homotopy groups of $\mathrm{tmf}_1(6)$ are given by*

$$\pi_\bigstar^{C_2}(\mathrm{tmf}_1(6)) \cong \mathbb{Z}[\bar{v}, \bar{w}, , u^{\pm 1}],$$

where $|\bar{v}| = |\bar{w}| = 1 + \sigma$ and $|u| = 2 - 2\sigma$.

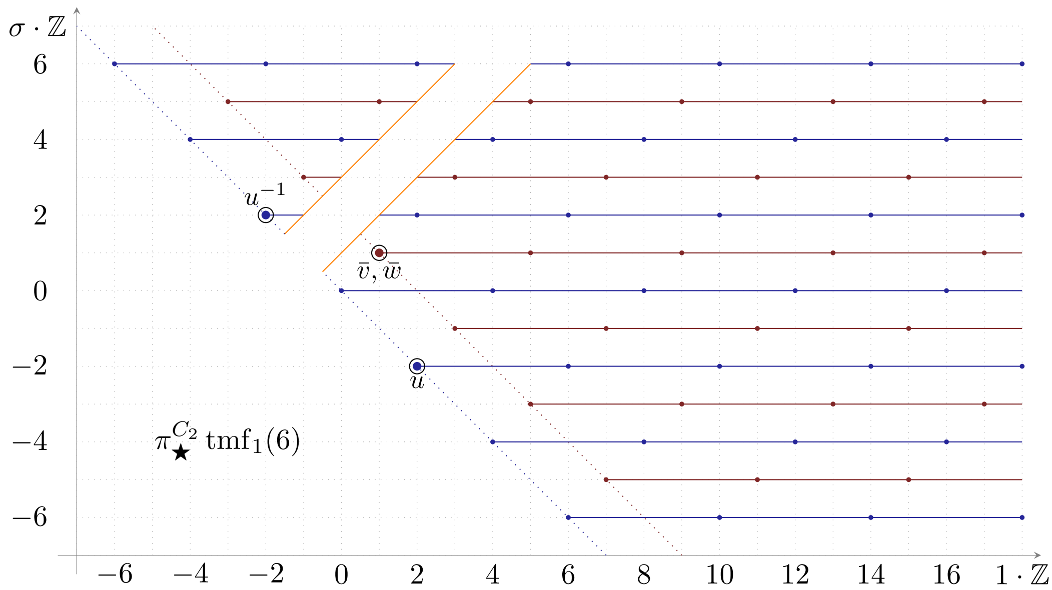
Proof. We run the $RO(C_2)$-graded homotopy fixed points spectral sequence for $\mathrm{tmf}_1(6)$. The E_2-page is given by

$$E_2^{pq}(r) = H^q(C_2, \pi^e_{p+q}(\mathrm{tmf}_1(6) \wedge S^{-r\sigma})) \Rightarrow \pi^{C_2}_{p+r\sigma}(\mathrm{tmf}_1(6)^h).$$

As the group order is invertible in the coefficient module only the H^0 entries are non-zero and thus we get

$$\pi^e_*(\mathrm{tmf}_1(6) \wedge S^{-r\sigma})^{C_2} \cong \begin{cases} \Sigma^{-r}\pi^e_*(\mathrm{tmf}_1(6))^{C_2} & \text{for } t \text{ even} \\ \Sigma^{-r}(\pi^e_*(\mathrm{tmf}_1(6)) \otimes \delta)^{C_2} & \text{for } t \text{ odd.} \end{cases}$$

We display this information in the picture below.



Here, blue lines indicate copies of $\mathbb{Z}[x^2, y^2] \oplus xy\mathbb{Z}[x^2, y^2]$ and red lines indicate copies of $x\mathbb{Z}[x^2, y^2] \oplus y\mathbb{Z}[x^2, y^2]$. Writing this out we get

$$\pi_\bigstar^{C_2}(\mathrm{tmf}_1(6)) \cong \mathbb{Z}[\bar{v}, \bar{w}, u^{\pm 1}],$$

where $|\bar{v}| = |\bar{w}| = 1 + \sigma$ and $|u| = 2 - 2\sigma$ as claimed. □

Corollary 6.2.5. $\mathrm{tmf}_1(6)$ *is Gorenstein of shift* $\alpha = -4 - 2\sigma$

$$\mathrm{Hom}_{\mathrm{tmf}_1(6)}(H\underline{\mathbb{Z}}, \mathrm{tmf}_1(6)) \simeq \Sigma^{-2-2\sigma} H\underline{\mathbb{Z}}.$$

Proof. This follows from the description of the ring $\pi_\bigstar^{C_2}(\mathrm{tmf}_1(6))$ obtained above with Gorenstein shift calculated as by $\alpha = -(|\bar{v}| + |\bar{w}|) - 2 = -4 - 2\sigma$. □

From the shape of the $RO(C_2)$-graded coefficients the following is immediate.

Corollary 6.2.6. $\mathrm{tmf}_1(6)$ *is strongly even.* □

To obtain Gorenstein duality we need to remove the $\mathrm{Hom}_{\mathrm{tmf}_1(6)}(H\underline{\mathbb{Z}}, -)$ from both sides of the equivalence

$$\mathrm{Hom}_{\mathrm{tmf}_1(6)}(H\underline{\mathbb{Z}}, \mathrm{tmf}_1(6)) \simeq \Sigma^{-4-2\sigma} H\underline{\mathbb{Z}} \simeq \mathrm{Hom}_{\mathrm{tmf}_1(6)}(H\underline{\mathbb{Z}}, \Sigma^{-6}\mathbb{Z}^{\mathrm{tmf}_1(6)}).$$

We can do this provided that the right action of

$$\mathcal{E} = \mathrm{Hom}_{\mathrm{tmf}_1(6)}(H\underline{\mathbb{Z}}, H\underline{\mathbb{Z}})$$

action on both sides of the equivalence is the same. This is again a conjecture.

Conjecture 6.2.7. *There is a unique right* $\mathcal{E}$*-module structure on* $H\underline{\mathbb{Z}}$*.*

Corollary 6.2.8. *Provided the conjecture holds, we have the following corollaries.*

(1) $\mathrm{tmf}_1(6)$ *has equivariant Gorenstein duality of shift* -6

$$\mathrm{Cell}_{H\underline{\mathbb{Z}}}(\mathrm{tmf}_1(6)) \simeq \Sigma^{-6}\mathbb{Z}^{\mathrm{tmf}_1(6)}.$$

(2) $\mathrm{Tmf}_1(6)$ *is Anderson self-dual of shift* 5

$$\mathbb{Z}^{\mathrm{Tmf}_1(6)} \simeq \Sigma^5 \, \mathrm{Tmf}_1(6).$$

□

We can descend to $\mathrm{tmf}_0(6)$ and show that it is also Gorenstein.

Lemma 6.2.9. $\mathrm{tmf}_0(6)$ *is Gorenstein of shift* -6*.*

Proof. Again, the HFPSS collapses as the order of C_2 is invertible in $\pi_*^e(\mathrm{tmf}_1(6))$ and $\mathrm{tmf}_1(6)$ is cofree, so we get

$$\begin{aligned}\pi_*(\mathrm{tmf}_0(6)) &\cong \pi_*^e(\mathrm{tmf}_1(6)^{C_2})\\ &\cong \pi_*^{C_2}(\mathrm{tmf}_1(6))\\ &\cong \pi_*^e(\mathrm{tmf}_1(6))^{C_2}\\ &\cong \mathbb{Z}[x^2, xy, y^2]\\ &\cong \mathbb{Z}[X, Y, Z]/_{(XY=Z^2)},\end{aligned}$$

where $|X| = |Y| = |Z| = 4$. This is Gorenstein of shift $a = -(|X|+|Y|+|Z|)+2+8 = -6$ which agrees with the shift predicted by the computation of the Hilbert of $\mathrm{mf}_*(\Gamma_1(6); \mathbb{C})$ in Section 5.4 of Chapter 5. □

Chapter 7

Future Directions

In this final chapter we record two directions in which the author tried to further pursue the ideas discussed so far.

The first is in line of reconstructing the Gorenstein duality picture for another entry from the list in Theorem A. The natural candidate in order of increasing level and increasing group size is the ring C_4-spectrum $\mathrm{tmf}_1(5)$. We review the state of affairs on this matter in Section 7.1 below and report on the progress made so far.

The second direction which was partially explored is the computation of the (non-)equivariant Picard groups of the ring spectra of topological modular forms present in Theorem A. We review this in Section 7.2.

7.1. Topological modular forms with level 5 structure

Topological modular forms with a level 5 structure have been studied previously in the literature. We summarize the relevant for us results below.

Let $\mathcal{M}_1(5)$ denote the moduli stack of elliptic curves with a chosen point of exact order 5 and write $\overline{\mathcal{M}}_1(5)$ for its Deligne-Mumford compactification. Let $\mathbb{Z} = \mathbb{Z}[\frac{1}{5}]$.

Lemma 7.1.1. *The moduli stack $\overline{\mathcal{M}}_1(5)$ is a projective line, i.e.*

$$\overline{\mathcal{M}}_1(5) \simeq \mathbb{P}^1_{\mathbb{Z}} = \mathrm{Proj}(\mathbb{Z}[x, y]).$$

Proof. See [6] or [51, Example 2.5] for an alternative argument. □

Lemma 7.1.2 ([6])**.** *The non-equivariant homotopy groups of $\mathrm{tmf}_1(5)$ are given by*

$$\pi^e_*(\mathrm{tmf}_1(5)) \cong \mathbb{Z}[x, y],$$

where $|x| = |y| = 2$.

Lemma 7.1.3. *C_4 acts on $\pi^e_*(\mathrm{tmf}_1(5))$ by sending x to y and y to $-x$.*

Proof. This was first done in [6, Lemma 2.1.2]. One can give an alternative proof via the Eisenstein method which is also tractable here. □

Behrens and Ormsby have compute the $\mathbb{Z}$-graded equivariant homotopy groups of $\mathrm{TMF}_1(5)$ in [6]. The iterated Tate argument described in Section 1.9 was specifically designed to attack the level 5 computations. It is the hope of the author that using the results of Behrens and Ormsby one can bootstrap the iterated Tate argument and recover the full $RO(C_4)$-graded homotopy type of $\mathrm{tmf}_1(5)$ which is a necessary ingredient in the recipe one uses to check for Gorenstein duality. The author has made partial progress with this strategy in recovering the full $RO(C_2)$-graded homotopy groups of $\mathrm{tmf}_1(5)$.

An alternative approach for attacking the level 5 computations emerged from discussions with Meier and Zeng. In [34] Hill, Hopkins and Ravenel define a ring C_4-spectrum, denoted there by $k_{[2]}$, by norming up the real bordism spectrum $MU_{\mathbb{R}}$ to a C_4-spectrum and then killing a range of C_2-equivariant homotopy elements. There are strong indications that the spectrum $k_{[2]}$ is secretly $\mathrm{tmf}_1(5)$ undercover. The non-equivariant and C_2-equivariant groups of the spectra $\mathrm{tmf}_1(5)$ and $k_{[2]}$ fully coincide and there are hints that the same is true C_4-equivariantly. If one can show that this is indeed the case, the slice spectral sequence computations done in [34] will allow one to predict the potential Anderson self-duality shift of $\mathrm{Tmf}_1(5)$, by arguments analogous to the level 3 computations of Hill and Meier in [35], and from there the potential Gorenstein duality shift of $\mathrm{tmf}_1(5)$. This is a work in progress.

7.2. Equivariant Picard groups

The ring spectra of topological modular forms with level structure are in addition commutative rings. This allows one to consider symmetric monoidal ∞-categories of modules over them which give rise to well-behaved invariants of algebraic or algebro-geometric type. One such invariant is the Picard group and its equivariant generalization.

Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a symmetric monoidal category. Following [49], define $\mathrm{Pic}(\mathcal{C})$, the *Picard group* of $\mathcal{C}$, to be the group of isomorphism classes of invertible objects in $\mathcal{C}$, i.e. objects $X \in \mathcal{C}$ such that there exits an object $Y \in \mathcal{C}$ such that $X \otimes Y = \mathbf{1}$. If $(\mathcal{C}, \otimes, \mathbf{1})$ is a symmetric monoidal ∞-category there is a more fundamental invariant of $\mathcal{C}$ which remembers all isomorphisms and higher isomorphisms. Define $\mathcal{P}\mathrm{ic}(\mathcal{C})$, the *Picard ∞-groupoid* of $\mathcal{C}$, to be the ∞-groupoid of invertible objects and equivalences between them. Clearly we have $\pi_0(\mathcal{P}\mathrm{ic}(\mathcal{C})) \cong \mathrm{Pic}(\mathrm{Ho}(\mathcal{C}))$. As $\mathcal{C}$ is a symmetric monoidal ∞-category, $\mathcal{P}\mathrm{ic}(\mathcal{C})$ inherits the structure of a group-like E_∞-space and thus, by a result of Boardman-Vogt and May, there is a connective spectrum $\mathfrak{pic}(\mathcal{C})$ with $\Omega^\infty \mathfrak{pic}(\mathcal{C}) \simeq \mathcal{P}\mathrm{ic}(\mathcal{C})$. Given a ring spectrum $\mathbf{R}$ we can now define the *Picard group* of $\mathbf{R}$ to be $\mathrm{Pic}(\mathrm{Ho}(\mathbf{R}\text{-}\mathrm{mod}))$ and the *Picard space* $\mathcal{P}\mathrm{ic}(\mathbf{R})$ to be $\mathcal{P}\mathrm{ic}(\mathbf{R}\text{-}\mathrm{mod})$, where $\mathbf{R}\text{-}\mathrm{mod}$ denotes the ∞-category of $\mathbf{R}$-modules. If $\mathbf{R}$ is in addition commutative, then the category $\mathbf{R}\text{-}\mathrm{mod}$ is even a symmetric monoidal ∞-category and we define the *Picard spectrum* $\mathfrak{pic}(\mathbf{R})$ to be $\mathfrak{pic}(\mathbf{R}\text{-}\mathrm{mod})$.

One can add an equivariant dash to this. Let G be a finite group, $\mathbf{R}$ be a commutative ring G-spectrum and write $\mathbf{R}\text{-mod}_G$ for the category of equivariant $\mathbf{R}$-modules. We have the following proposition, see [35, Section 6].

Proposition 7.2.1. *If $\mathbf{R}$ is cofree and $\mathbf{S} \to \mathbf{R}$ is a faithful G-Galois extension in the sense of Rognes then there is a monoidal equivalence $\mathrm{Ho}(\mathbf{S}\text{-mod}) \simeq \mathrm{Ho}(\mathbf{R}\text{-mod}_G)$ and therefore $\mathrm{Pic}(\mathbf{S}) \cong \mathrm{Pic}(\mathbf{R}\text{-mod}_G)$.*

Following [35] we denote the group $\mathrm{Pic}(\mathbf{R}\text{-mod}_G)$ of invertible equivariant $\mathbf{R}$-modules by $\mathrm{Pic}_G(\mathbf{R})$ and refer to it as the *equivariant Picard group* of $\mathbf{R}$. This group has been computed for the ring spectrum of topological modular forms with level 3 structure in [35].

The author would like to complete the duality story of the $\mathrm{tmf}_1(n)$ spectra with a computation of their corresponding Picard groups. The non-equivariant Picard groups for levels 4 and 6 have been computed by the author and the equivariant computations are a work in progress. If sufficient progress is made in the reconstruction of the level 5 picture, one can hope to also look into the, much harder, computation of the Picard group for level 5.

Appendix A

Dualities for topological modular forms

n	$\overline{\mathcal{M}}_1(n)$	$h(t)$	G	$\pi_*^e(\mathrm{tmf}_1(n))$	G-action	$\Omega^1_{\overline{\mathcal{M}}_1(n)/\mathbb{Z}[\frac{1}{n}]}$	a	b	α	β	δ
1	$\mathcal{P}_{\mathbb{Z}[\frac{1}{6}]}(4,6)$	$\frac{1}{(1-t^4)(1-t^6)}$	e	$\mathbb{Z}[\frac{1}{6}][\overset{8}{c_4}, \overset{12}{c_6}]$	trivial	$\omega_{ell}^{\otimes -10}$	-22	21	n/a	n/a	n/a
2	$\mathcal{P}_{\mathbb{Z}[\frac{1}{2}]}(2,4)$	$\frac{1}{(1-t^2)(1-t^4)}$	e	$\mathbb{Z}[\frac{1}{2}][\overset{4}{b_2}, \overset{8}{b_4}]$	trivial	$\omega^{\otimes -6}$	-14	13	n/a	n/a	n/a
3	$\mathcal{P}_{\mathbb{Z}[\frac{1}{3}]}(1,3)$	$\frac{1}{(1-t)(1-t^3)}$	C_2	$\mathbb{Z}[\frac{1}{3}][\overset{2}{a_1}, \overset{6}{a_3}]$	$a_1 \mapsto -a_1$ $a_3 \mapsto -a_3$	$\omega^{\otimes -4}$	-10	9	$-6-4\sigma$	$7+2\sigma$	$-8-2\sigma$
4	$\mathcal{P}_{\mathbb{Z}[\frac{1}{2}]}(1,2)$	$\frac{1}{(1-t)(1-t^2)}$	C_2	$\mathbb{Z}[\frac{1}{2}][\overset{2}{a_1}, \overset{4}{a_2}]$	$a_1 \mapsto -a_1$ $a_2 \mapsto a_2$	$\omega^{\otimes -3}$	-8	7	$-5-3\sigma$	$6+\sigma$	$-7-\sigma$
5	$\mathbb{P}^1_{\mathbb{Z}[\frac{1}{5}]}$	$\frac{1}{(1-t)^2}$	C_4	$\mathbb{Z}[\frac{1}{5}][\overset{2}{x}, \overset{2}{y}]$	$x \mapsto y$ $y \mapsto -x$	$\omega^{\otimes -2}$	-6	5			
6	$\mathbb{P}^1_{\mathbb{Z}[\frac{1}{6}]}$	$\frac{1}{(1-t)^2}$	C_2	$\mathbb{Z}[\frac{1}{6}][\overset{2}{x}, \overset{2}{y}]$	$x \mapsto -x$ $y \mapsto -y$	$\omega^{\otimes -2}$	-6	5	$-4-2\sigma$	5	-6
7	$\mathbb{P}^1_{\mathbb{Z}[\frac{1}{7}]}$	$\frac{1+t}{(1-t)^2}$	C_6	$\mathbb{Z}[\frac{1}{7}][\overset{2}{x}, \overset{2}{y}, \overset{2}{z}]/(xy+yz+zx)$	$x \mapsto -z$ $y \mapsto -x$ $z \mapsto -y$	$\omega^{\otimes -1}$	-4	3			
8	$\mathbb{P}^1_{\mathbb{Z}[\frac{1}{2}]}$	$\frac{1+t}{(1-t)^2}$	$C_2 \times C_2$	3 generators of degree 2		$\omega^{\otimes -1}$	-4	3			
11		$\frac{1+3t+t^2}{(1-t)^2}$	C_{10}	5 generators of degree 2			-2	1			
14		$\frac{1+4t+t^2}{(1-t)^2}$	C_6	6 generators of degree 2			-2	1			
15		$\frac{1+6t+t^2}{(1-t)^2}$	$C_2 \times C_4$	8 generators of degree 2			-2	1			
23		$\frac{1+10t+10t^2+t^3}{(1-t)^2}$	C_{22}			ω	0	-1			

Glossary

Bibliography

[1] D. W. Anderson. "Universal coefficient theorems for K-theory". Mimeographed notes. Univ. California, Berkeley, Calif., 1969.

[2] Andrew Baker and Birgit Richter. "Invertible modules for commutative $\mathbb{S}$-algebras with residue fields". In: *Manuscripta Math.* 118.1 (2005), pp. 99–119.

[3] Tobias Barthel, Drew Heard, and Gabriel Valenzuela. "Local duality for structured ring spectra". In: *J. Pure Appl. Algebra* 222.2 (2018), pp. 433–463.

[4] Clark Barwick, Saul Glasman, and Jay Shah. "Spectral Mackey functors and equivariant algebraic K-theory, II". In: *Tunis. J. Math.* 2.1 (2020), pp. 97–146.

[5] Hyman Bass. "On the ubiquity of Gorenstein rings". In: *Math. Z.* 82 (1963), pp. 8–28.

[6] Mark Behrens and Kyle Ormsby. "On the homotopy of $Q(3)$ and $Q(5)$ at the prime 2". In: *Algebr. Geom. Topol.* 16.5 (2016), pp. 2459–2534.

[7] Andrew J. Blumberg and Michael A. Hill. "Operadic multiplications in equivariant spectra, norms, and transfers". In: *Adv. Math.* 285 (2015), pp. 658–708.

[8] Edgar H. Brown Jr. and Michael Comenetz. "Pontrjagin duality for generalized homology and cohomology theories". In: *Amer. J. Math.* 98.1 (1976), pp. 1–27.

[9] Winfried Bruns and Jürgen Herzog. *Cohen-Macaulay rings*. Vol. 39. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1993, pp. xii+403.

[10] Kevin Buzzard. "Computing weight one modular forms over $\mathbb{C}$ and $\overline{\mathbb{F}}_p$". In: *Computations with modular forms*. Vol. 6. Contrib. Math. Comput. Sci. Springer, Cham, 2014, pp. 129–146.

[11] Brian Conrad. "Arithmetic moduli of generalized elliptic curves". In: *J. Inst. Math. Jussieu* 6.2 (2007), pp. 209–278.

[12] P. Deligne and M. Rapoport. "Les schémas de modules de courbes elliptiques". In: (1973), 143–316. Lecture Notes in Math., Vol. 349.

[13] Fred Diamond and Jerry Shurman. *A first course in modular forms*. Vol. 228. Graduate Texts in Mathematics. Springer-Verlag, New York, 2005, pp. xvi+436.

[14] Christopher L. Douglas et al., eds. *Topological modular forms*. Vol. 201. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2014, pp. xxxii+318.

[15] Andreas W. M. Dress. "Contributions to the theory of induced representations". In: (1973), 183–240. Lecture Notes in Math., Vol. 342.

[16] W. G. Dwyer and J. P. C. Greenlees. "Complete modules and torsion modules". In: *Amer. J. Math.* 124.1 (2002), pp. 199–220.

[17] W. G. Dwyer, J. P. C. Greenlees, and S. Iyengar. "Duality in algebra and topology". In: *Adv. Math.* 200.2 (2006), pp. 357–402.

[18] W. G. Dwyer, J. P. C. Greenlees, and S. B. Iyengar. "Gross-Hopkins duality and the Gorenstein condition". In: *J. K-Theory* 8.1 (2011), pp. 107–133.

[19] J. P. C. Greenlees. "Ausoni-Bökstedt duality for topological Hochschild homology". In: *J. Pure Appl. Algebra* 220.4 (2016), pp. 1382–1402.

[20] J. P. C. Greenlees. "First steps in brave new commutative algebra". In: *Interactions between homotopy theory and algebra*. Vol. 436. Contemp. Math. Amer. Math. Soc., Providence, RI, 2007, pp. 239–275.

[21] J. P. C. Greenlees. "Four approaches to cohomology theories with reality". In: *An alpine bouquet of algebraic topology*. Vol. 708. Contemp. Math. Amer. Math. Soc., Providence, RI, 2018, pp. 139–156.

[22] J. P. C. Greenlees. "Homotopy invariant commutative algebra over fields". In: *Building bridges between algebra and topology*. Adv. Courses Math. CRM Barcelona. Birkhäuser/Springer, Cham, 2018, pp. 103–169.

[23] J. P. C. Greenlees. "Spectra for commutative algebraists". In: *Interactions between homotopy theory and algebra*. Vol. 436. Contemp. Math. Amer. Math. Soc., Providence, RI, 2007, pp. 149–173.

[24] J. P. C. Greenlees and G. Lyubeznik. "Rings with a local cohomology theorem and applications to cohomology rings of groups". In: *J. Pure Appl. Algebra* 149.3 (2000), pp. 267–285.

[25] J. P. C. Greenlees and J. P. May. "Completions in algebra and topology". In: *Handbook of algebraic topology*. North-Holland, Amsterdam, 1995, pp. 255–276.

[26] J. P. C. Greenlees and J. P. May. "Generalized Tate cohomology". In: *Mem. Amer. Math. Soc.* 113.543 (1995), pp. viii+178.

[27] J. P. C. Greenlees and Lennart Meier. "Gorenstein duality for real spectra". In: *Algebr. Geom. Topol.* 17.6 (2017), pp. 3547–3619.

[28] J. P. C. Greenlees and B. Shipley. "The cellularization principle for Quillen adjunctions". In: *Homology Homotopy Appl.* 15.2 (2013), pp. 173–184.

[29] J. P. C. Greenlees and V. Stojanoska. "Anderson and Gorenstein duality". In: *Geometric and topological aspects of the representation theory of finite groups*. Vol. 242. Springer Proc. Math. Stat. Springer, Cham, 2018, pp. 105–130.

[30] Moritz Groth. "A short course on ∞-categories". In: *arXiv e-prints* (July 2010). arXiv: 1007.2925.

[31] Robin Hartshorne. *Local cohomology*. Vol. 1961. A seminar given by A. Grothendieck, Harvard University, Fall. Springer-Verlag, Berlin-New York, 1967, pp. vi+106.

[32] Drew Heard and Vesna Stojanoska. “K-theory, reality, and duality”. In: *J. K-Theory* 14.3 (2014), pp. 526–555.

[33] M. A. Hill, M. J. Hopkins, and D. C. Ravenel. “On the nonexistence of elements of Kervaire invariant one”. In: *Ann. of Math. (2)* 184.1 (2016), pp. 1–262.

[34] Michael A. Hill, Michael J. Hopkins, and Douglas C. Ravenel. “The slice spectral sequence for the C_4 analog of real K-theory”. In: *Forum Math.* 29.2 (2017), pp. 383–447.

[35] Michael A. Hill and Lennart Meier. “The C_2-spectrum $\mathrm{T}mf_1(3)$ and its invertible modules”. In: *Algebr. Geom. Topol.* 17.4 (2017), pp. 1953–2011.

[36] Michael Hill and Tyler Lawson. “Topological modular forms with level structure”. In: *Invent. Math.* 203.2 (2016), pp. 359–416.

[37] Philip S. Hirschhorn. *Model categories and their localizations*. Vol. 99. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2003, pp. xvi+457.

[38] Po Hu and Igor Kriz. “Real-oriented homotopy theory and an analogue of the Adams-Novikov spectral sequence”. In: *Topology* 40.2 (2001), pp. 317–399.

[39] OEIS Foundation Inc. *The On-Line Encyclopedia of Integer Sequences*. 2019. URL: https://oeis.org.

[40] Paul C. Kainen. “Universal coefficient theorems for generalized homology and stable cohomotopy”. In: *Pacific J. Math.* 37 (1971), pp. 397–407.

[41] Seán Keel and Shigefumi Mori. “Quotients by groupoids”. In: *Ann. of Math. (2)* 145.1 (1997), pp. 193–213.

[42] L. Gaunce Lewis Jr. “The $RO(G)$-graded equivariant ordinary cohomology of complex projective spaces with linear $\mathbf{Z}/p$ actions”. In: *Algebraic topology and transformation groups (Göttingen, 1987)*. Vol. 1361. Lecture Notes in Math. Springer, Berlin, 1988, pp. 53–122.

[43] Jacob Lurie. *Higher Algebra*. 2017. URL: http://www.math.harvard.edu/~lurie/papers/HA.pdf.

[44] Jacob Lurie. *Higher topos theory*. Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xviii+925.

[45] Mark Mahowald and Charles Rezk. “Topological modular forms of level 3”. In: *Pure Appl. Math. Q.* 5.2, Special Issue: In honor of Friedrich Hirzebruch. Part 1 (2009), pp. 853–872.

[46] M. A. Mandell and J. P. May. “Equivariant orthogonal spectra and S-modules”. In: *Mem. Amer. Math. Soc.* 159.755 (2002), pp. x+108.

[47] Akhil Mathew and Lennart Meier. “Affineness and chromatic homotopy theory”. In: *J. Topol.* 8.2 (2015), pp. 476–528.

[48] Akhil Mathew, Niko Naumann, and Justin Noel. “Nilpotence and descent in equivariant stable homotopy theory”. In: *Adv. Math.* 305 (2017), pp. 994–1084.

[49] Akhil Mathew and Vesna Stojanoska. “The Picard group of topological modular forms via descent theory”. In: *Geom. Topol.* 20.6 (2016), pp. 3133–3217.

[50] J. P. May. *Equivariant homotopy and cohomology theory*. Vol. 91. CBMS Regional Conference Series in Mathematics. Amer. Math. Soc., Providence, RI, 1996, pp. xiv+366.

[51] Lennart Meier. “Additive decompositions for rings of modular forms”. In: *arXiv e-prints* (Oct. 2017). arXiv: 1710.03461.

[52] Lennart Meier. “Topological modular forms with level structure: decompositions and duality”. In: *arXiv e-prints* (June 2018). arXiv: 1806.06709.

[53] Lennart Meier. “United Elliptic Cohomology”. PhD thesis. University of Bonn, 2012.

[54] Lennart Meier. “Vector bundles on the moduli stack of elliptic curves”. In: *J. Algebra* 428 (2015), pp. 425–456.

[55] Lennart Meier and Viktoriya Ozornova. *Moduli stack of elliptic curves*. 2017. URL: http://www.staff.science.uu.nl/~meier007/Mell.pdf.

[56] Lennart Meier and Viktoriya Ozornova. “Rings of modular forms and a splitting of $TMF_0(7)$”. In: *arXiv e-prints* (Dec. 2018). arXiv: 1812.04425.

[57] Martin Olsson. *Algebraic spaces and stacks*. Vol. 62. American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2016, pp. xi+298. URL: https://doi.org/10.1090/coll/062.

[58] Rory Potter. “Derived Categories of Surfaces and Group Actions”. PhD thesis. University of Sheffield, 2018.

[59] Nicolas Ricka. “Equivariant Anderson duality and Mackey functor duality”. In: *Glasg. Math. J.* 58.3 (2016), pp. 649–676.

[60] Stefan Schwede and Brooke Shipley. “Stable model categories are categories of modules”. In: *Topology* 42.1 (2003), pp. 103–153.

[61] The Stacks Project Authors. *Stacks Project*. 2019. URL: https://stacks.math.columbia.edu.

[62] Vesna Stojanoska. “Calculating descent for 2-primary topological modular forms”. In: *An alpine expedition through algebraic topology*. Vol. 617. Contemp. Math. Amer. Math. Soc., Providence, RI, 2014, pp. 241–258.

[63] Vesna Stojanoska. “Duality for topological modular forms”. In: *Doc. Math.* 17 (2012), pp. 271–311.

[64] Angelo Vistoli. “Grothendieck topologies, fibered categories and descent theory”. In: *Fundamental algebraic geometry*. Vol. 123. Math. Surveys Monogr. Amer. Math. Soc., Providence, RI, 2005, pp. 1–104.

[65] Angelo Vistoli. “Intersection theory on algebraic stacks and on their moduli spaces”. In: *Invent. Math.* 97.3 (1989), pp. 613–670.