Singularities of Analytic Functions and Group Representations

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Declaration

The candidate confirms that the work submitted is her own and that appropriate credit has been given where reference has been made to the work of others.

The work in chapter 3.11 of this thesis has appeared in publication as 'Numerically Stable Conditions on Rational and Essential Singularities', in Complex Variables and Elliptic Equations (2017) [2].

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Abstract

In this thesis, we demonstrate some connections between the coefficients of a Taylor series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and singularities of the function. There are many known results of this type; for example, counting the number of poles on the circle of convergence and doing convergence or overconvergence for f on any arc of holomorphy. Here, a new approach is proposed in which these kinds of results are extended by relaxing the classical conditions for singularities and convergence theorems. This is done by allowing the coefficients to be sufficiently small instead of being zero. The well-known theta function is an important example. Every point on the boundary of its domain of holomorphy is singular. This function is delivered from the *covariant transform* associated with the Heisenberg group representations. Therefore, we devote the rest of our present work to deal primarily with the *covariant transform*. We introduce three different forms of the Heisenberg group representations. The *covariant transform* allows us to construct intertwining operators related to L_2-type spaces between the representations of the Heisenberg group. The systematic usage of the covariant transform between different spaces on which the Heisenberg group representations act is another new contribution in this thesis.

To my Mom, To my Dad
To my husband and my beautiful children
I send this gift to them
with my love

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Chapter 1

Introduction

An analytic function and its expansion of the power series in the circle of convergence are at some points not identical. If Taylor series has a finite radius of convergence, the function that is represented by this series may be analytic in a larger domain. An example of this $f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$. Although the function f is analytic at every point in $\mathbb{C}$ except $z = 1$, its Taylor expansion at $z = 0$ converges only for $|z| < 1$. If a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$, has a finite radius of convergence, it is possible that a sub-series $f(z) = \sum_{n_k=0}^{\infty} a_{n_k} z^{n_k}$ can converge uniformly in a larger domain [41, §11.2]. Many of the results that have been obtained to study the convergence on a larger domain of analyticity are formulated to be based on conditions imposing coefficients to vanish [41, §11.2]. Although these results are interesting in the theoretical sense, they are not generally applicable for practical problems because, in applied science, values do not need to be strictly zero. Our generalisations of classical results represented in Chapter 2 show a certain stability in this transportation: a small variation in the Taylor coefficients preserves the function's property under consideration. In other words, a new approach is presented here that relies on estimating some coefficients of a Taylor series to be sufficiently small. This topic is a part of a broader discussion. An intriguing connection with spectral theory was discovered in [10].

The analytic *theta function* is an interesting and important example of a function with a *natural boundary* (i.e., every point on the boundary of the circle of convergence of theta function is singular). The expansion of the theta function has gaps through its Taylor coefficients. Initially, this function was introduced in the 19th century. Later, it was discovered in many different areas from number theory to quantum mechanics. A connection to the Heisenberg group representations and the group $\mathrm{SL}_2(\mathbb{R})$ was later

identified by Cartier and Mumford [10, 34]. Here, we treat the theta function through the *covariant transform*. It has been extensively studied by Grossmann, Daubechies, Perelomov and others [3, 22, 23, 37]. We will be focused on the covariant transform for the *Schrödinger group*, which is a semi-direct product of the Heisenberg group $\mathbb{H}_p^1$ and the group $\mathrm{SL}_2(\mathbb{R})$.

1.1   Outline of this Thesis

The structure of this thesis is as follows.

- In Chapter 2, we discuss some connections between the coefficients of a Taylor series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ of an analytic function f and singularities of this function. We provide some classical results [39, § 5.3, Prob.244; 41, § 11.1], which discuss the behaviour of the function f on the boundary of its domain of holomorphy. The hypotheses of these classical theorems impose that some of the Taylor coefficients vanish. We then provide our modifications of these results by eliminating the restriction of having these lacunas (gaps) through the coefficients of the Taylor expansion.

- In Chapter 3, we provide various standard notations, definitions and results from group theory and representation theory that are considered in this thesis.

- In Chapter 4, we recall fundamentals of the Heisenberg group $\mathbb{H}_p^1$ and its associated Lie algebra $\mathfrak{h}_1$. Next, we describe $\mathbb{H}_p^1$–homogeneous spaces and actions of $\mathbb{H}_p^1$ on these spaces. Further, we describe important concepts of this thesis: the Schrödinger group and the Jacobi theta function.

- In Chapter 5, we discuss the construction of induced representations of a group. In particular, we describe three forms of the Heisenberg group representations: the *left quasi-regular* representation, the *Schrödinger* representation and the *lattice* representation. Further, we define operators produced by derived representations of the Lie algebra $\mathfrak{h}_1$ of the Heisenberg group. Among these operators, the most important are called *ladder operators: creation* and *annihilation*. We discuss some properties of these operators. This leads to *vacuum* vectors, which are null solutions to the annihilation operator. In particular, we find the *vacuum* vectors of the three representations of the Heisenberg group.

- In Chapter 6, we discuss the *covariant transform* and its inverse—the *contravariant transform*. We find unitary intertwining operators between all three above Heisenberg group representations through the covariant transform and its inverse. That is, we calculate the following intertwining operators:

    i. the (pre-) Fock–Segal–Bargamann (FSB) transform and its inverse between the Schrödinger representation on $L_2(\mathbb{R})$ and the (pre-) FSB representation on $L_2(\mathbb{R}^2)$;

    ii. the Zak transform and its inverse between the Schrödinger representation on $L_2(\mathbb{R})$ and the lattice representation on $L_2(\mathbb{T}^2)$; and

    iii. the (pre-) theta transform and its inverse between the (pre-) FSB representation on $L_2(\mathbb{R}^2)$ and the lattice representation on $L_2(\mathbb{T}^2)$.

    Figure 1.1 represents the connection of these intertwining operators.

    Our contribution is the systematic usage of the covariant transform. In particular, the Zak transform and its inverse are expressed as the covariant transform between the Schrödinger and lattice representations, with the theta function being the vacuum of the latter. Similarly, expressing the pre-theta transform and its inverse throughout the same technique is also new.

    It is often preferable to deal with some spaces of analytic functions, for example, the *FSB space*, which appears in quantum mechanics. This allows us to use all theorems from complex analysis to study induced representations of $\mathbb{H}_p^1$. Such spaces can be obtained by an operation, which we refer to as *peeling*. We discuss the *peeling* of the above three representation spaces of $\mathbb{H}_p^1$.

- In Chapter 7 we conclude and discuss further work.
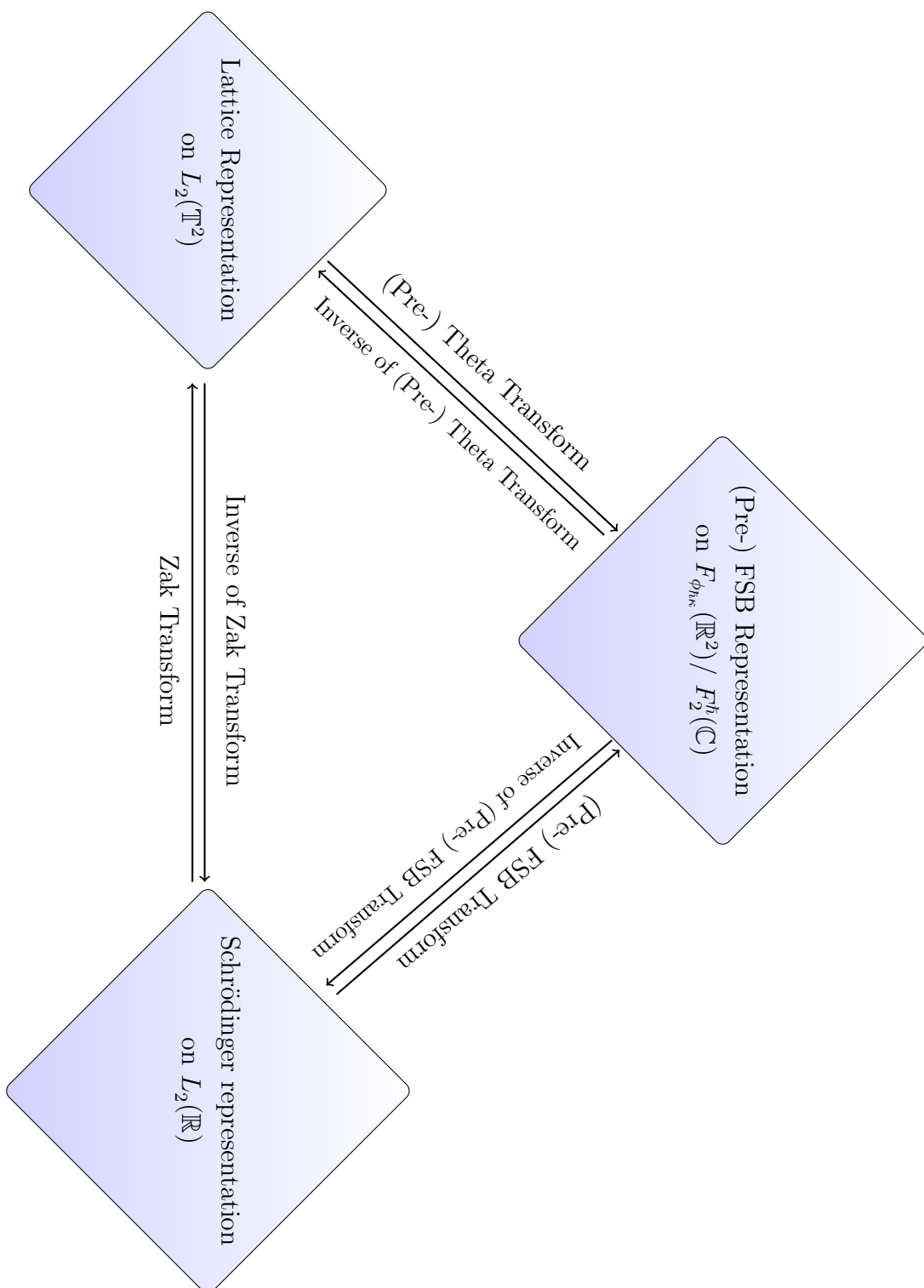
Figure 1.1: Intertwining Operators between Representations of $\mathbb{H}^1_p$

Lattice Representation
on $L_2(\mathbb{T}^2)$

(Pre-) Theta Transform

Inverse of (Pre-) Theta Transform

(Pre-) FSB Representation
on $F_{\phi_{hic}}(\mathbb{R}^2) / F^h_2(\mathbb{C})$

Inverse of Zak Transform

Zak Transform

Inverse of (Pre-) FSB Transform

(Pre-) FSB Transform

Schrödinger representation
on $L_2(\mathbb{R})$

Chapter 2

Numerically stable conditions on rational and essential singularities

It is a classical problem to analyse the boundary behaviour of an analytic function in terms of its Taylor expansion. For example, for a function f which is analytic on $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$, either f can be extended beyond the circle of convergence $\partial\mathbb{E}$, or $\partial\mathbb{E}$ is the natural boundary (i.e., every $z \in \partial\mathbb{E}$ is a singular point of f).

Many of the results that have been obtained so far (see discussion below) have hypothesises which force some of the Taylor coefficients to vanish. Although these results are interesting in the theoretical sense, they are not really applicable for practical problems because, in applied science, values may not be strictly zero. For example, in numerical methods, there is no test that can confirm that such a coefficient is exactly zero; they may only conclude that it is sufficiently close to zero.

In this chapter, we will address the following two questions for a Taylor series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ that does not necessarily have gaps.

(i) How many poles of f are on the circle of convergence?

(ii) If f is defined analytically on $\mathbb{E}$, when is $\partial\mathbb{E}$ the natural boundary for f?

We now discuss some classical results along with our generalizations, and the first example is the following.

Theorem 2.1. [39, § 5.3, Prob.244]. *Let v_n be the number of non-zero coefficients among the n coefficients $a_0, a_1, ..., a_{n-1}$. If there are only poles (and no other singularities) on the circle of convergence of the power series: $\sum_{n=0}^{\infty} a_n z^n$, the number of such poles is not smaller than $\limsup_{n\to\infty} \frac{n}{v_n}$.*

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Our first object is to replace the number v_n in this theorem by the number of coefficients that shall exceed some small values. This will constitute Theorem 2.11, which has the same conclusion as in Theorem 2.1 but in a more general setting.

We next consider the classical theorem of Ostrowski convergence for lacunary series. The definition of lacunary series used in the initial theorem is as follows.

Definition 2.2. [41, § 11.1]. An infinite power series $\sum_{v=0}^{\infty} a_v z^v$ is called a *lacunary* series if there exists an increasing sequence (m_v) of non-negative integers for $v = 0, 1, 2, \ldots$ such that $\lim_{v\to\infty}(m_{v+1} - m_v) = \infty$, and

$$a_j = 0, \quad m_v < j < m_{v+1}, \; v = 0, 1, 2, \ldots.$$

Ostrowski proved the following important property for lacunary series.

Theorem 2.3 (Ostrowski's convergence). [41, § 11.1, Thm 3]. *If $f(z) = \sum_{n=0}^{\infty} a_{m_n} z^{m_n}$ is a lacunary series with a bounded sequence of coefficients, its sequence of partial sums s_{m_n} converges uniformly on every arc of holomorphy L of f.*

Our treatment of this theorem is to relax the restrictive condition of consecutive zero coefficients by instead requiring some coefficients of the Taylor series to be sufficiently small (see Definition 2.12). More precisely, in Theorem 2.14 we relax the lacunas, whilst at the same time the conclusion of Theorem 2.3 is preserved.

Another interesting theorem is that of Ostrowski overconvergence for Ostrowski series, which is defined as follows.

Definition 2.4. [41, § 11.2]. A power series $\sum_{v=0}^{\infty} a_v z^v$ is called an *Ostrowski* series if there exists a $\delta > 0$ and two sequences (m_v) and (n_v) of non-negative integers for $v = 0, 1, 2, \ldots$ such that:

(i) $0 \leq m_0 < n_0 \leq m_1 < n_1 \leq \ldots \leq m_v < n_v \leq m_{v+1}\ldots$; and $n_v - m_v > \delta m_v$, $v = 0, 1, 2, \ldots$; and

(ii) $a_j = 0$ if $m_v < j < n_v$, $v = 0, 1, 2, \ldots.$

Let us denote by $B(r, 0)$ a disc centred at the origin with radius $r > 0$. The classical theorem of Ostrowski's overconvergence is stated as follows.

Theorem 2.5 (Ostrowski's overconvergence). [41, § 11.2, Thm 1]. *Let $f(z) = \sum_{v=0}^{\infty} a_v z^v$ be an Ostrowski series with radius of convergence $r > 0$, and let $A \subset \partial B(r, 0)$ denote the set of all the boundary points of f that are not singular. Then the sequence of sections $s_{m_k}(z) = \sum_{v=0}^{m_k} a_v z^v$ converges uniformly in a neighbourhood of $B(r, 0) \cup A$.*

We again relax the requirement for the gaps of the power series in Theorem 2.5. In Definition 2.15, we replace the consecutive zeros by adequately small coefficients instead, which allows us to prove Theorem 2.16—a generalization of Theorem 2.5.

Finally, the last classical result that interests us is Hadamard's theorem for Hadamard lacunary series that is defined as follows.

Definition 2.6. [41, § 11.2, Def 3]. An infinite power series $\sum_{v=0}^{\infty} a_v z^v$ is called a *Hadamard lacunary* series if there exists a $\delta > 0$ and an increasing sequence (m_v) of non-negative integers for $v = 0, 1, 2, \ldots$ such that

$$m_{v+1} - m_v > \delta m_v, \; v = 0, 1, 2, \ldots; \; a_j = 0, \; \text{if } m_v < j < m_{v+1}, \, a_{m_v} \neq 0.$$

Every Hadamard lacunary series is a lacunary series in the sense of Definition 2.2, and also an Ostrowski series (with $n_v = m_{v+1}$). The converse, however, is not true: for a lacunary series as in Definition 2.2, only $\lim_{v \to \infty}(m_{v+1} - m_v) = \infty$ is required, whereas for an Ostrowski series gaps need to appear only "here and there". However, the Hadamard lacunary condition requires that a gap lies between any two successive terms that actually appear.

The classical Hadamard's gap theorem is stated as follows.

Theorem 2.7 (Hadamard's gap theorem). [41, § 11.2, Thm 3.] *Every Hadamard lacunary series $\sum_{v=0}^{\infty} a_v z^v$ with radius of convergence $r > 0$ has the disc $B(r, 0)$ as a domain of holomorphy.*

Remark 2.8. The condition $a_{m_v} \neq 0$ in Definition 2.6 of a Hadamard lacunary series is necessary, otherwise we would obtain the series with zero coefficients that has trivial convergence everywhere.

Our modification of Theorem 2.7 is to once more replace the lacunas of the power series by suitably small values (see Definition 2.17 and Theorem 2.18).

2.1 Poles on the Circle of Convergence

In this section, we prove Theorem 2.11, which is a generalization of Theorem 2.1 presented in the introduction of this chapter. These results are a presentation of the relationship between coefficients of a power series and singularities of the function they present. The key to proving Theorem 2.11 will be Lemma 2.9.

Lemma 2.9. [39, § 5.3, Prob.243] *Let $\sum_{n=0}^{\infty} a_n z^n$ be the expansion into a power series of a rational function whose denominator (relative prime to the numerator) has degree q. If $A_n = \max\{|a_n|, |a_{n-1}|, ..., |a_{n-q+1}|\}$, then the radius of convergence r satisfies*

$$\lim_{n \to \infty} \sqrt[n]{A_n} = \frac{1}{r}. \tag{2.1}$$

Remark 2.10. An easy corollary of Lemma 2.9 is that for all $l \in \mathbb{Z}$, for all $\epsilon > 0$, there exists $N = N(l, \epsilon) \in \mathbb{N}$ such that $\forall n \geq N$, we have that

$$A_n > \left(\frac{1}{\rho} - \epsilon\right)^{n+l}. \tag{2.2}$$

To present our modification of Theorem 2.11, we consider the number of coefficients that shall exceed some small values instead of being non-zero in Theorem 2.1, and we obtain our new theorem as follows.

Theorem 2.11. [2] *Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series that has only poles (and no other singularities) on the circle of convergence $|z| = \rho$, as well as no other singularities inside the circle $|z| = \rho_1 > \rho$. Set $\alpha_0 = \frac{\rho}{\rho_1} < 1$. Let $\epsilon > 0$, $0 < \alpha < \alpha_0$ such that $\frac{1}{\rho_1} < \alpha \left(\frac{1}{\rho} - (\frac{1+\alpha}{\alpha})\epsilon\right)$. If v_n is the number of coefficients $|a_j| > (1 - \alpha^j)(\frac{1}{\rho} - \epsilon)^j$ among n coefficients $a_0, a_1, ..., a_{n-1}$, the number of poles on the circle of convergence is not smaller than $\limsup_{n \to \infty} \frac{n}{v_n}$.*

Proof. Let k be the number of poles counted with multiplicity on the circle of convergence. The power series in the hypotheses can be written in the form:

$$\sum_{n=0}^{\infty} a_n z^n = \left(\frac{c_1'}{(z - z_1)^{m_1}} + \frac{c_2'}{(z - z_2)^{m_2}} + ... + \frac{c_k'}{(z - z_k)^{m_p}}\right) + \sum_{n=0}^{\infty} b_n z^n, \tag{2.3}$$

such that $m_1 + \cdots + m_p = k$, where $m_j \geq 1$ are non-negative integers for $j = 1, \cdots p$. The sum in the bracket contains all singularities, which are on the boundary. We denote the expansion of the rational part

$$\left(\frac{c_1'}{(z - z_1)^{m_1}} + \frac{c_2'}{(z - z_2)^{m_2}} + ... + \frac{c_k'}{(z - z_k)^{m_p}}\right) \tag{2.4}$$

of the Taylor series $\sum_{n=0}^{\infty} a_n z^n$ by $\sum_{n=0}^{\infty} \alpha_n z^n$, which has the radius of convergence ρ. Thus, the infinite series $\sum_{n=0}^{\infty} b_n z^n$ has a bigger radius of convergence ρ_1 defined on the hypothesis, i.e.

$$\limsup_{n \to \infty} \sqrt[n]{|b_n|} = \frac{1}{\rho_1} < \alpha \left(\frac{1}{\rho} - \left(\frac{1+\alpha}{\alpha}\right)\epsilon\right). \tag{2.5}$$

Then, for $\epsilon > 0$ given in the hypothesis, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|b_n| < (\frac{1}{\rho_1} + \epsilon)^n < \alpha^n \left(\frac{1}{\rho} - \epsilon\right)^n$. As the series $\sum_{n=0}^{\infty} \alpha_n z^n$ is the expansion into a power series of the rational part of $\sum_{n=0}^{\infty} a_n z^n$, by Lemma 2.9 and Remark 2.10 (when $l = 0$ and $l = 1 - k$), we have for ϵ, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$ we have that

$$\max(|\alpha_n|, |\alpha_{n-1}|, ..., |\alpha_{n-k+1}|) > \max\left((\frac{1}{\rho} - \epsilon)^n, (\frac{1}{\rho} - \epsilon)^{n-k+1} \right).$$

Set $N = \max(N_1, N_2)$. Then, for all $n > N$, there exists at least one $\bar{n}$, $n \geq \bar{n} \geq n-k+1$ among each k consecutive elements such that

$$|\alpha_{\bar{n}}| > (\frac{1}{\rho} - \epsilon)^{\bar{n}} > \alpha^{\bar{n}}(\frac{1}{\rho} - \epsilon)^{\bar{n}} > |b_{\bar{n}}|. \tag{2.6}$$

Thus, by (2.6), when $n \geq N$, there is some $n - k + 1 \leq \bar{n} \leq n$ with

$$|a_{\bar{n}}| = |\alpha_{\bar{n}} + b_{\bar{n}}| \geq |\alpha_{\bar{n}}| - |b_{\bar{n}}| > (1 - \alpha^{\bar{n}})(\frac{1}{\rho} - \epsilon)^{\bar{n}}. \tag{2.7}$$

Consequently, $\frac{n-N}{k} \leq v_n$, where v_n is the number of coefficients $|a_j| > (1 - \alpha^j)(\frac{1}{\rho} - \epsilon)^j$ among n coefficients $a_0, a_1, ..., a_{n-1}$. Then, $\frac{n}{k} - c \leq v_n$, where $c = \frac{N}{k}$. Thus,

$$k \geq \limsup_{n \to \infty} \frac{n}{v_n}, \tag{2.8}$$

which proves the statement. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

2.2 Expansion of Analytic Functions and its convergence on the boundary

In this section, we shall generalise a result on the boundary behaviour of a power series (see Theorem 2.3), which links the extension problem for a power series with the convergence of its sequence of partial sums. This is by replacing the consecutive zero coefficients in a lacunary series (see Definition 2.2) with small values at the same places to create a quasi-lacunary series as follows.

Definition 2.12. An infinite power series $\sum_{n=0}^{\infty} a_n z^n$ is called a *quasi-lacunary* series if there exists an increasing sequence (m_v) of non-negative integers for $v = 0, 1, 2, \ldots$ such that

$$\lim_{v \to \infty} (m_{v+1} - m_v) = \infty, \quad \text{and} \quad |a_j| \leq |c_j|, \quad m_v < j < m_{v+1}, \, v = 0, 1, 2, \ldots,$$

where (c_j), $j = 0, 1, 2, \ldots$, is a $p-$ summable sequences for some $p > 1$, i.e. $\sum_{j=0}^{\infty} |c_j|^p < \infty$.

Let $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$. To give the proof of Theorem 2.14, we need the following lemma.

Lemma 2.13 (M.Riesz). [41, § 11, Lem1]. *For every arc of holomorphy $L \subset \partial\mathbb{E}$ of a power series $f(z) = \sum_{v=0}^{\infty} a_v z^v$ with radius of convergence 1, there exists a compact circular sector S with vertex at 0 such that L lies in the interior $\mathring{S}$ of S and f has a holomorphic extension $\widehat{f}$ to S. Let $z_1, z_2 \neq 0$ be the corners of S, and let w_1 and w_2 be the points of intersection of $\partial\mathbb{E}$ with $[0, z_1]$ and $[0, z_2]$, respectively. Then $|w_1| = |w_2| = 1$ and $s = |z_1| = |z_2| > 1$.*

To prove the next result, we consider the functions

$$g_n(z) = \frac{\widehat{f}(z) - s_n(z)}{z^{n+1}}(z - w_1)(z - w_2), \quad \text{where} \quad s_n(z) = \sum_{k=0}^{n} a_k z^k, \ n \in \mathbb{N}. \quad (2.9)$$

So, every function g_n is holomorphic in S. In the proof of the next theorem, S and $\widehat{f}$ can be chosen as in Lemma 2.13 and g_n as in (2.9). Denote by $(\|\widehat{f} - s_n\|)|_L$ and $\|g_n\|)|_S$ be the maxima of the absolute values of the functions $(\widehat{f} - s_n)(z)$ and $g_n(z)$ for all z in L and S, respectively. Since $|z| = 1$ for $z \in L$, the inequality

$$(\|\widehat{f} - s_n\|)|_L \leq a^{-1}(\|g_n\|)|_S, \quad \text{where} \quad a = \min_{z \in L}\{|(z - w_1)(z - w_2)|\} > 0, \quad (2.10)$$

holds for all $n \in \mathbb{N}$.

Now, we can prove our convergence theorem for a quasi-lacunary series.

Theorem 2.14 (Convergence for a quasi-lacunary series). [2] *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a quasi-lacunary series such that (a_{m_v}), where the coefficients a_{m_v} satisfy the property in Definition 2.12, is a bounded sequence of coefficients. Then the sequence of partial sums s_{m_v} converges uniformly on every arc of holomorphy L of f.*

Proof. It must be shown that $\lim_{v \to \infty}(\|\widehat{f} - s_{m_v}\|)|_L = 0$. By (2.10), it suffices to show that the sequence g_{m_v} tends locally uniformly to zero in $\mathring{S}$. Let $t \in (0, 1)$. By Vitali's theorem [41, § 7.3, Thm.2], it suffices to show that $\lim_{v \to \infty} g_{m_v}(z) = 0$ for $|z| = t$.

Setting $A = \sup |a_{m_v}|$, we have

$$(|\widehat{f}(z) - s_{m_v}(z)|)|_S = \left(\left| \sum_{k \in \cup_{n=v}^{\infty}(m_n, m_{n+1})} a_k z^k + \sum_{j=v+1}^{\infty} a_{m_j} z^{m_j} \right| \right)\Bigg|_S$$

$$\leq \sum_{k \in \cup_{n=v}^{\infty}(m_n, m_{n+1})} |a_k| t^k + \sum_{j=v+1}^{\infty} |a_{m_j}| t^{m_j}.$$

According to (2.9), we have that

$$(|g_{m_v}(z)|)|_S = \frac{(|\widehat{f}(z) - s_{m_v}(z)|)|_S}{|z^{m_v+1}|} \cdot |z - w_1||z - w_2|$$

$$\leq \frac{(|\widehat{f}(z) - s_{m_v}(z)|)|_S}{t^{m_v+1}} \cdot (1+t)^2.$$

Subsequently,

$$(|g_{m_v}(z)|)|_S \leq \sum_{k \in \cup_{n=v}^{\infty}(m_n, m_{n+1})} |a_k| t^k \frac{(1+t)^2}{t^{m_v+1}} + \sum_{j=v+1}^{\infty} |a_{m_j}| t^{m_j} \frac{(1+t)^2}{t^{m_v+1}}. \qquad (2.11)$$

The first term of the right hand side in (2.11) tends to zero. In fact, by Holder's inequality for $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\sum_{k \in \cup_{n=v}^{\infty}(m_n, m_{n+1})} |a_k| t^k \frac{(1+t)^2}{t^{m_v+1}} \leq \left(\sum_{k \in \cup_{n=v}^{\infty}(m_n, m_{n+1})} |a_k|^p \right)^{\frac{1}{p}} \left(\sum_{k \in \cup_{n=v}^{\infty}(m_n, m_{n+1})} t^{qk} \right)^{\frac{1}{q}} \frac{(1+t)^2}{t^{m_v+1}}$$

$$\leq \left(\sum_{k \in \cup_{n=v}^{\infty}(m_n, m_{n+1})} |a_k|^p \right)^{\frac{1}{p}} \left(\frac{t^{qm_v}}{1 - t^q} \right)^{\frac{1}{q}} \frac{(1+t)^2}{t^{m_v+1}}$$

$$\leq \left(\sum_{k \in \cup_{n=v}^{\infty}(m_n, m_{n+1})} |a_k|^p \right)^{\frac{1}{p}} \frac{t^{m_v}}{(1 - t^q)^{\frac{1}{q}}} \cdot \frac{(1+t)^2}{t^{m_v+1}}$$

$$= \left(\sum_{k \in \cup_{n=v}^{\infty}(m_n, m_{n+1})} |a_k|^p \right)^{\frac{1}{p}} \frac{(1+t)^2}{t(1 - t^q)^{\frac{1}{q}}} \to 0 \ as \ v \to \infty.$$

Indeed, by the hypothesis, we have $\sum_{k \in \cup_{n=v}^{\infty}(m_n, m_{n+1})} |a_k|^p < \infty$. Therefore, when $m_n \to \infty$ as $n \to \infty$, it follows that $\sum_{k \in \cup_{n=v}^{\infty}(m_n, m_{n+1})} |a_k|^p \to 0$.

On the other hand, the second term of the right hand side in (2.11) also tends to zero. Indeed, since $\lim_{v\to\infty}(m_{v+1}-m_v)=\infty$ and $0<t<1$, then $\lim_{v\to\infty}t^{m_{v+1}-m_v}=0$,

$$\sum_{j=v+1}^{\infty}|a_{m_j}|\,t^{m_j}\frac{(1+t)^2}{t^{m_v+1}}\leq A\,\frac{t^{m_v+1}}{1-t}\cdot\frac{(1+t)^2}{t^{m_v+1}}$$

$$=A\,t^{m_{v+1}-m_v}\cdot\frac{(1+t)^2}{(1-t)t}\longrightarrow 0,$$

where, $A=\sup|a_{m_j}|$. Therefore, $\lim_{v\to\infty}(\|g_{m_v}\|)|_S=0$. Then, $\lim_{v\to\infty}(\|\widehat{f}-s_{m_v}\|)|_L=0$. Thus, the sequence of partial sums $(s_{m_v}(z))$ converges uniformly on every arc of holomorphy L of f. $\qquad\square$

2.3 Expansion of Analytic Functions and Overconvergence theorems

This section is dedicated to proving two results (namely, Theorems 2.16 and 2.18), which are further developments of Theorems 2.5 and 2.7 stated in the Introduction. These classical theorems demonstrate close relationships between overconvergence and the gaps of the power series, and once our modifications— the removal of the restrictive gap conditions— are applied we are still able to derive the same conclusion.

To show Theorem 2.16, we first need the definition of a quasi-Ostrowski series.

Definition 2.15. [2] An infinite power series $\sum_{v=0}^{\infty}a_vz^v$ is called a *quasi-Ostrowski* series if there exists a $\delta>0$, a positive decreasing sequence (c_r), $r=0,1,2,\ldots$, $c_r\searrow 0$, and two sequences (m_k) and (n_k) of non-negative integers for $k=0,1,2,\ldots$, such that:

(i) $n_k-m_k>\delta m_k$, $k=0,1,2,\ldots$, and $0\leq m_0<n_0\leq m_1<n_1\leq\ldots\leq m_k<n_k\leq m_{k+1}\ldots$; and

(ii) $|a_j|<\frac{1}{m_k^2}c_j$, $m_k<j<n_k$, $k=0,1,2\ldots$.

Recall that $B(r,0)$ denotes the open disc centred at zero and radius $r>0$. Then, we have our generalization of Theorem 2.5 as follows.

Theorem 2.16 (Overconvergence theorem for a quasi-Ostrowski series)**.** [2] *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be a quasi-Ostrowski series with radius of convergence $r>0$, and let $A\subset\partial B(r,0)$ denote the set of all the boundary points of f that are not singular. Then the*

sequence of sections $s_{m_k}(z) = \sum_{n=0}^{m_k} a_n z^n$ converges uniformly in a neighbourhood of $B(r, 0) \cup A$.

For every domain $D \subset \mathbb{C}$, let $O(\bar{D})$ denote the set of all functions that are holomorphic in an open neighbourhood of $\bar{D} = D \cup \partial D$.

Proof Theorem 2.16 . Without loss of generality, let $r = 1$ and let $c \in \partial \mathbb{E}$, where $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$. We introduce the polynomial

$$q(w) = \frac{c}{2}(w^p + w^{p+1}), \quad p \in \mathbb{N} \quad \text{and} \quad p \geq \delta^{-1},$$

(where the smaller value of p such that $p \geq \delta^{-1}$ gives the larger domain of overconvergence). Consider the function $f(q(w)) = \sum_{n=0}^{\infty} a_n q(w)^n$, which is holomorphic in $q^{-1}(\mathbb{E}) = \{w \in \mathbb{C} : |q(w)| < 1\}$. We denote by $\sum_{n=0}^{\infty} b_n w^n$ the Taylor series of $f(q(w))$ about $0 \in q^{-1}(\mathbb{E})$ and by $s_v(z)$ and $t_v(z)$ the vth partial sums of $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n w^n$, respectively. We claim that

$$|t_{(p+1)m_k}(w) - s_{m_k}(q(w))| \to 0 \quad as \quad k \to \infty \quad where \quad w \in \mathbb{E}. \tag{2.12}$$

We have $t_{(p+1)m_k}(w) = \sum_{n=0}^{(p+1)m_k} b_n w^n$, and

$$
\begin{aligned}
s_{m_k}(q(w)) &= \sum_{r=0}^{m_k} a_r q(w)^r = \sum_{r=0}^{m_k} a_r \left(\frac{c}{2}(w^p + w^{p+1})\right)^r \\
&= \sum_{r=0}^{m_k} a_r \left(\frac{c}{2}\right)^r (C_r^r w^{pr} + \ldots + C_0^r w^{(p+1)r}) \\
&= \sum_{r=0}^{m_k} a_r \left(\frac{c}{2}\right)^r \sum_{l=0}^{r} C_l^r w^{r(p+1)-l} .
\end{aligned}
$$

Each polynomial $(C_r^r w^{pr} + \ldots + C_0^r w^{(p+1)r})$ has a range of powers $pr \leq n \leq (p+1)r$. If $0 \leq r \leq m_k$, then $0 \leq n \leq (p+1)m_k$. Subsequently, $s_{m_k}(q(w))$ has a degree no greater than $(p+1)m_k$. Thus, by setting $n = r(p+1) - l$, we obtain that

$$
\begin{aligned}
s_{m_k}(q(w)) &= \sum_{n=0}^{(p+1)m_k} \sum_{r=[\frac{n}{p+1}]}^{[\frac{n}{p}]} a_r \left(\frac{c}{2}\right)^r C_{r(p+1)-n}^r w^n \\
&= \sum_{n=0}^{(p+1)m_k} d_n^{(k)} w^n,
\end{aligned}
$$

where $d_n^{(k)} = \sum_{r=[\frac{n}{p+1}]}^{[\frac{n}{p}]} a_r \left(\frac{c}{2}\right)^r C_{r(p+1)-n}^r$. This means that by Weierstrass' double series theorem [40, Ch.8, §4.2, p.250], $\sum b_n w^n$ and $\sum d_n^{(k)} w^n$ come from the series $\sum a_r q(w)^r$ by multiplying out the polynomials $q(w)^r$ and grouping the resulting series $\sum a_r(...)$ according to powers of w. Therefore, in order to prove (2.12), we need to compute the difference between $t_{(p+1)m_k}(w)$ and $s_{m_k}(q(w))$, and then show that it tends to zero uniformly. In fact, we have

$$t_{(p+1)m_k}(w) = s_{m_k}(q(w)) + (\text{ some contribution from } \sum_{r=m_k+1}^{\frac{(p+1)m_k}{p}} a_r q(w)^r),$$

because when n is from 0 to pm_k, the coefficients b_n and $d_n^{(k)}$ of the partial sums $t_{(p+1)m_k}$ and $s_{m_k}(q(w))$ are equal, respectively, i.e. $b_n = d_n^{(k)}$, for $0 \leq n \leq pm_k$. In addition, in the partial sum of $\sum_r a_r q(w)^r$ when $m_{k+1} > r \geq n_k$, by (i), each polynomial $a_r q(w)^r$, $r \geq n_k$, contains monomial $d_n^{(k)} w^n$ with $n \geq pn_k$. Since $pn_k > pm_k + p\delta m_k \geq (p+1)m_k$, $p \geq \delta^{-1}$, no such monomial contributes to $t_{(p+1)m_k}(w)$, which is a polynomial of degree no greater than $(p+1)m_k$. However, the contribution exists from each $a_r q(w)^r$ where $|a_r| < \frac{1}{m_k^2} c_r$, c_r is the notation of quasi-Ostrowski series, $m_k < r \leq \frac{(p+1)m_k}{p}$, i.e., it exists from the monomial $d_n^{(k)} w^n$ where $pm_k < n \leq (p+1)m_k < pn_k$. Since $t_{(p+1)m_k}$ has degree no greater than $(p+1)m_k$, and $b_n = d_n^{(k)}$, for $0 \leq n \leq pm_k$, the total contributions to the partial sum $t_{(p+1)m_k}$ is computed in the range of powers $pm_k < n \leq (p+1)m_k$ as follows.

$$
\begin{aligned}
|b_n - d_n^{(k)}| &= \left| \sum_{r=[\frac{n}{p+1}]}^{[\frac{n}{p}]} a_r \left(\frac{c}{2}\right)^r C_{r(p+1)-n}^r \right| \\
&\leq \sum_{r=[\frac{n}{p+1}]}^{[\frac{n}{p}]} |a_r| \frac{1}{2^r} C_{r(p+1)-n}^r, \quad where \quad pm_k < n \leq (p+1)m_k.
\end{aligned}
$$

Since $C_{r(p+1)-n}^r \leq 2^r$, then for $pm_k < n \leq (p+1)m_k$ and by using c_r the notation of

quasi-Ostrowski series, we have

$$
\begin{aligned}
|b_n - d_n^{(k)}| &\leq \sum_{r=[\frac{n}{p+1}]}^{[\frac{n}{p}]} |a_r| < \frac{1}{m_k^2} \sum_{r=[\frac{n}{p+1}]}^{[\frac{n}{p}]} c_r \\
&\leq \frac{1}{m_k^2} \sum_{r=[\frac{n}{p+1}]}^{[\frac{n}{p}]} \max_{[\frac{n}{p+1}]\leq r \leq [\frac{n}{p}]} c_r \\
&\leq \frac{1}{m_k^2} \left(\left[\frac{n}{p}\right] - \left[\frac{n}{p+1}\right] + 1 \right) c_{[\frac{n}{p+1}]} \\
&< \frac{1}{m_k^2} \left(\frac{n}{p(p+1)} + 2 \right) c_{[\frac{n}{p+1}]} .
\end{aligned}
$$

Therefore,

$$
|b_n - d_n^{(k)}| \leq
\begin{cases}
0 & \text{if} \quad 0 \leq n \leq pm_k \\
\frac{1}{m_k^2}\left(\frac{n}{p(p+1)}+2\right) c_{[\frac{n}{p+1}]} & \text{if} \quad pm_k < n \leq (p+1)m_k .
\end{cases}
$$

Consequently,

$$
\begin{aligned}
|t_{(p+1)m_k}(w) - s_{m_k}(q(w))| &= \left| \sum_{n=0}^{(p+1)m_k} (b_n - d_n^{(k)})w^n \right| = \left| \sum_{n=pm_k+1}^{(p+1)m_k} (b_n - d_n^{(k)})w^n \right| \\
&< \frac{1}{m_k^2} \sum_{n=pm_k+1}^{(p+1)m_k} \left(\frac{n}{p(p+1)} + 2 \right) c_{[\frac{n}{p+1}]} \\
&\leq \frac{1}{m_k^2} \sum_{n=pm_k+1}^{(p+1)m_k} \left(\frac{(p+1)m_k}{p(p+1)} + 2 \right) c_{[\frac{pm_k}{p+1}]} \\
&\leq \frac{1}{m_k^2} \left(\frac{m_k^2}{p} + 2m_k \right) c_{[\frac{pm_k}{p+1}]} \to 0 \quad as \quad k \to \infty ,
\end{aligned}
$$

because by hypothesis given in Definition 2.15 of a quasi-Ostrowski series, $c_{[\frac{pm_k}{p+1}]}$ decreases to zero as $k \to \infty$. Thus, (2.12) was confirmed.

We have $q^{-1}(\mathbb{E}) \supset \overline{\mathbb{E}}\backslash\{1\}$, since $|1 + w| < 2$, when $w \in \mathbb{E}\backslash\{1\}$. Hence $|q(w)| < 1$ for all $w \in \overline{\mathbb{E}}\backslash\{1\}$. Set $g(w) = f(q(w)) - s_{m_k}(q(w))$. The function $g(w) \in O(q^{-1}(\mathbb{E}))$ is thus holomorphic at every point of $\overline{\mathbb{E}}\backslash\{1\}$ for each $m_k \in \mathbb{N}$. Now, if $c \in A$, then g is also holomorphic at 1 since $q(1) = c$. Thus, the sequence of sections $(g_{(p+1)m_k})$ converges to zero in an open disk $B \supset \overline{\mathbb{E}}$. Then, by (2.12), the sequence $(s_{m_k}(z))$ now converges uniformly in $q(B)$. Since $q(B)$ is a domain containing $c = q(1)$, then $(s_{m_k}(z))$ converges uniformly in a neighbourhood of any point $c \in A$. $\square$

In the following, we provide our proof of Theorem 2.18, which is a modification of Hadamard's gap theorem (Theorem 2.7) given in the Introduction. First, however, we need to present the definition of a quasi-Hadamard lacunary series.

Definition 2.17. [2] An infinite power series $\sum_{v=0}^{\infty} a_v z^v$ is called a *quasi-Hadamard lacunary* series if there exists a $\delta > 0$, a summable positive decreasing sequence (c_r), $r = 0, 1, 2, \ldots$, $c_r \searrow 0$, and an increasing sequence (m_v) of non-negative integers for $v = 0, 1, 2, \ldots$, such that:

(i) $m_{v+1} - m_v > \delta m_v, \quad v = 0, 1, 2, \ldots$;

(ii) $|a_j| \leq \frac{1}{m_v^2} c_j, \quad \text{if} \quad m_v < j < m_{v+1}$; and

(iii) the series $\sum_{v=0}^{\infty} |a_{m_v}|$ is divergent.

Now, we present our new theorem as follows.

Theorem 2.18 (On a quasi-Hadamard lacunary series). *Every quasi-Hadamard lacunary series $\sum_{n=0}^{\infty} a_n z^n$ with radius of convergence $r > 0$ has the disc $B(r, 0)$ as a domain of holomorphy.*

Remark 2.19.   i. In Definition 2.17, condition (*iii*) that the series $\sum_{v=0}^{\infty} |a_{m_v}|$ is divergent is necessary, otherwise the previous theorem would not hold. For example, let $\sum_{n=1}^{\infty} a_n z^n = \sum_{n=1}^{\infty} \frac{1}{n^2} z^{2^n}$. Notice that conditions (i) and (ii) are satisfied, but we have $\sum_{v=1}^{\infty} |a_{m_v}| < \infty$. On the other hand, denote by $s_{m_{v+1}}$ the m_{v+1} partial sums of $\sum_{n=1}^{\infty} a_n z^n$. Then,

$$|s_{m_{v+1}}| = \left| \sum_{n=1}^{m_{v+1}} a_n z^n \right| \leq \sum_{n=1}^{m_{v+1}} |\frac{1}{n^2}|.$$

Since, $\lim_{v \to \infty} |s_{m_{v+1}}| = L < \infty$, the conclusion of Theorem 2.18 does not hold.

ii. The series of quasi-Hadamard lacunary series is a subclass of quasi-Ostrowski series.

Proof Theorem 2.18. Let $s_n(z) = \sum_{j=0}^{n} a_j z^j$ be the nth partial sums of the series $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Consider the partial sums $s_{m_k}(z)$, where the sequence of (m_k) is defined as in Definition 2.17 of a quasi-Hadamard lacunary series. For the partial sums of $s_n(z)$ we have that

$$s_n(z) = \begin{cases} s_{m_k}(z) & \text{if} \quad n = m_k, \\ s_{m_k}(z) + \sum_{j=m_k+1}^{n} a_j z^j & \text{if} \quad m_k < n < m_{k+1}, \end{cases}$$

where $|a_j| \leq \frac{1}{m_k^2} c_j$, c_j is the notation used in the quasi-Hadamard lacunary series, and $m_k < j < m_{k+1}$. Therefore,

$$
\begin{aligned}
|s_n(z) - s_{m_k}(z)| &= \left| \sum_{j=0}^{n} a_j z^j - \sum_{j=0}^{m_k} a_j z^j \right| \\
&= \left| \sum_{j=m_k+1}^{n} a_j z^j \right| \\
&\leq \sum_{j=m_k+1}^{m_{k+1}-1} |a_j| \to 0 \quad as \quad k \to \infty,
\end{aligned}
\tag{2.13}
$$

since for $m_k < j < m_{k+1}$, we have $\sum_j |a_j| \leq \frac{1}{m_k^2} \sum_j c_j < \infty$. Thus, in (2.13) $\sum_{j=m_k+1}^{m_{k+1}-1} |a_j| \to 0$ when $k \to \infty$.

Then, by (2.13), the sequence of partial sums $(s_n(z))$ converges in the same domain as $(s_{m_k}(z))$. Thus, the sequence $(s_{m_k}(z))$ diverges at every point $\zeta \notin \overline{B(r,0)}$. Hence, by Theorem 2.16 of overconvergence, all the points of $\partial B(r,0)$ are singular points of f. $\qquad \square$

Chapter 3

Preliminaries on Groups and Representation theory

In this introductory chapter, we review the elementary notions of group theory and representation theory that we will use in the remaining chapters. We mainly use the references [24, 25].

3.1 The Concept of Groups and Transformations

It is common to introduce groups in an abstract axiomatic way. To indicate the common source of this concept, we provide the definition of the complementing object—*transformation groups.*

Definition 3.1. [24, §2.1] A transformation group G is a non-void set of one-to-one mappings of a certain set X into itself with the following properties:

   i. The identity map is included in G.

   ii. If $g_1 \in G$ and $g_2 \in G$, then $g_1 g_2 \in G$.

   iii. If $g \in G$, then g^{-1} exists and belongs to G.

   Any abstract group can be viewed as a transformation group in several different ways.

Example 3.2. [24, §2.1] Let G be a group. The actions on $X = G$ can be defined as follows:

(i) The left shift

$$\tilde{\Lambda}(g) : g' \mapsto g^{-1}g'; \text{ and} \tag{3.1}$$

(ii) the right shift

$$\tilde{R}(g) : g' \mapsto g'g. \tag{3.2}$$

Note that the left action commutes with the right one; that is, for all $g, g' \in G$, we have:

$$\tilde{\Lambda}(g) \circ \tilde{R}(g') = \tilde{R}(g') \circ \tilde{\Lambda}(g). \tag{3.3}$$

For a commutative group, the left and right actions are the inverse of one-another; that is,

$$\tilde{R}(g)^{-1} = \tilde{R}(g^{-1}) = \tilde{\Lambda}(g). \tag{3.4}$$

Let V be a vector space of functions on G. The left (right) action of G into itself (3.1) can be extended naturally into a linear transformation Λ and R on V as follows:

$$\Lambda(g) : f(g') \mapsto f(g^{-1}g'), \quad f \in V; \tag{3.5}$$

and

$$R(g) : f(g') \mapsto f(g'g), \quad f \in V. \tag{3.6}$$

3.1.1 Subgroups and Homogeneous Spaces

In this section, we describe the construction of homogeneous spaces. We also express actions of a group on the homogeneous spaces through some respective parameters. To begin, let X be a set. Let G be a group acting on X by some transformations. We say that a subset $S \subset X$ is *G-invariant* if $g \cdot s \in S$ for all $g \in G$ and $s \in S$ [24, §2.1].

Now, let H be a subgroup of the group G. Let us define the space of *cosets* $X = G/H$ by the equivalence relation: $g_1 \sim g_2$ if there exists $h \in H$ such that $g_1 = g_2 h$ [24, §2.1]. The space $X = G/H$ is a *homogeneous space* under the left G-action:

$$g : g_1 H \mapsto (gg_1)H. \tag{3.7}$$

It is often convenient to have parameterisations of $X = G/H$ and express the above G-action through those parameters, as is shown below [24, §2.1]. Suppose that we have chosen a representative in each equivalence class. In other words, we have a

function (section) $\mathsf{s} : X \to G$ such that it is a right inverse to the natural projection $\mathsf{p} : G \to G/H$; that is, $\mathsf{p}(\mathsf{s}(x)) = x$ for all $x \in X$.

The set G can be identified with the direct product $G \sim G/H \times H$; that is, for any $g \in G$, we have $g = \mathsf{s}(\mathsf{p}(g))h$, for some $h \in H$ depending on g. Indeed, from the definition of s and p, the point $\mathsf{s}(\mathsf{p}(g))$ belongs to the same class of the point as g; that is, $\mathsf{s}(\mathsf{p}(g)) \sim g$. Then, any $g \in G$ has a unique decomposition of the form

$$g = s(x)h, \tag{3.8}$$

where $x = \mathsf{p}(g) \in X$ and $h \in H$. We define a map r associated to s through the following identities:

$$x = \mathsf{p}(g), \quad h = r(g) := \mathsf{s}(x)^{-1}g. \tag{3.9}$$

The G-action on x in terms of these parameterisations (the maps s and p) is as follows:

$$g : x \mapsto g \cdot x = \mathsf{p}(g * \mathsf{s}(x)), \tag{3.10}$$

where $*$ is the multiplication on G. This is illustrated by the following commutative diagram:

$$
\begin{array}{ccc}
G & \xrightarrow{\,g*\,} & G \\
{\scriptstyle s}\big\Updownarrow{\scriptstyle p} & & {\scriptstyle s}\big\Updownarrow{\scriptstyle p} \\
X & \xrightarrow{\,g\cdot\,} & X
\end{array}
\tag{3.11}
$$

3.1.2 Lie Groups

Groups can have some additional analytical structures. For example, they can be a topological space with a corresponding notion of limit and respective continuity. To perform some analysis on the groups, we need *Lie groups*, defined as follows.

Definition 3.3. [25, App.III, §1.1] A *Lie group* is a smooth manifold G endowed with a multiplication law that is a smooth map $G \times G \to G$ satisfying the usual group axioms.

Consider the particular case in which G is a smooth manifold and at the same time a subgroup of $GL(n, \mathbb{R})$ of $n \times n$ invertible matrices whose entries are in $\mathbb{R}$. Such a group is usually called a *matrix Lie group* [25, App.III, §1.1]. A matrix Lie group is *locally compact* [8, Ch.3,§3.1] in the natural topology; that is, there exists a compact neighbourhood of every point. Henceforth, a group G will mean a locally compact matrix group.

Example 3.4. The following are non-commutative matrix Lie groups:

(i) The group $\mathrm{SL}_2(\mathbb{R})$ [29] is a set of 2×2 matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with real entries $a, b, c, d \in \mathbb{R}$, and the determinant $\det A = ad - bc$ is equal to 1. The group law coincides with the matrix multiplication:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix}.$$

The identity is the unit matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and the inverse of A is:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

(ii) Let $SL_3(\mathbb{R})$ be the 3×3 matrices with the unit determinant [19, Ch.1,§2]. We define a matrix $SL_3(s, x, y) \in SL_3(\mathbb{R})$, for $(s, x, y) \in \mathbb{R}^3$, by

$$SL_3(s, x, y) = \begin{pmatrix} 1 & x & s \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.12}$$

It is easy to verify that

$$SL_3(s, x, y) \, SL_3(s', x', y') = SL_3(s + s' + xy', x + x', y + y'). \tag{3.13}$$

Thus, the set

$$\{ SL_3(s, x, y) \in SL_3(\mathbb{R}) : \ (s, x, y) \in \mathbb{R}^3 \} \tag{3.14}$$

with the group law (3.13) forms a subgroup of $GL(3, \mathbb{R})$. The matrix multiplication (3.13) in the coordinates $(s, x, y) \in \mathbb{R}^3$ can be reduced to the following group law:

$$(s, x, y) \, (s', x', y') = (s + s' + xy', x + x', y + y'). \tag{3.15}$$

The triple real numbers (s, x, y) with the group multiplication (3.15) are called the *polarised Heisenberg group* and denoted by $\mathbb{H}_p^1$ [18]. The identity is $(0, 0, 0)$, and the inverse of (s, x, y) is $(s, x, y)^{-1} = (-s + xy, -x, -y)$.

The group $\mathrm{SL}_2(\mathbb{R})$ and the polarised Heisenberg group are two examples of matrix Lie groups, which will be the main objects in the reminder of this thesis. Note that every closed subgroup H of a Lie group G is itself a Lie group. Moreover, the respective homogeneous space G/H is a smooth manifold on which G acts by smooth transformations (see [25, App.III, §1.3,Thm.4]).

3.1.3 Lie Algebras

The theory of Lie groups is closely related to the respective infinitesimal object—a Lie algebra—given as follows.

Definition 3.5. [25, App.III, §1.2] Let $\mathfrak{g}$ be a vector space over some field F (in our thesis, we shall use only fields of real and complex numbers) endowed with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the following axioms: for all X, Y, and Z in $\mathfrak{g}$, we have

   i. bilinearity: $[aX + bY, Z] = a[X, Z] + b[Y, Z]$, for all scalars a, b in F;

   ii. alternating on $\mathfrak{g}$: $[X, X] = 0$; and

   iii. Jacobi identity: $[X, [Y, Z]] + [Y, [X, Z]] + [Z, [X, Y]] = 0$.

The map $[\cdot,\cdot]$ is called the *commutator*.

An important class of Lie algebras is formed by *matrix Lie algebras* [25, App.III, §1.2], which are subspaces of $M_n(F)$ (a Lie algebra of $n \times n$ matrices over the field F of real or complex numbers) and are closed with respect to the ordinary matrix commutator

$$[X, Y] = XY - YX. \tag{3.16}$$

This class is universal because of Ado's Theorem (any Lie algebra is isomorphic to a matrix Lie algebra [25, App.III, §1.2]).

For every matrix Lie group G there is an associated matrix Lie algebra $\mathfrak{g}$. An important relation between them is the *exponential map* [25, App.III, §1.2]:

$$\begin{aligned} \exp : \mathfrak{g} &\to G : \\ X &\mapsto \exp X = \mathrm{e}^X. \end{aligned} \tag{3.17}$$

The expansion map for a matrix Lie group is defined by the following Taylor series:

$$\exp(X) = \sum_{k \geq 0} \frac{(X)^k}{k!}. \tag{3.18}$$

The Lie algebra $\mathfrak{g}$ can be realised in several different ways (see [25, App.III, §1.2]). We we will mention two of them in Subsections 3.1.4 and 3.1.5.

3.1.4 One-parameter Subgroups and Lie Algebras

Let G be a matrix Lie group and $\mathfrak{g}$ be its associated matrix Lie algebra. For the first realisation of the matrix Lie algebra, we consider a *one-dimensional continuous subgroup* $x(t)$ of G as a group homomorphism of $x : (\mathbb{R}, +) \to G$ [25, App.III, §1.2,Thm.2]. It follows from the well-known property of the exponent that for any element $X \in \mathfrak{g}$, the curve $x(t) = exptX$, $t \in \mathbb{R}$ is a *one parameter subgroup* of G that satisfies:

$$x(t)\,x(s) = x(t+s), \quad x(0) = I, \quad x'(t)|_{t=0} = X, \quad \text{for all } t, s \in \mathbb{R}, \tag{3.19}$$

where I is the identity matrix in G. An example of this will be considered for the polarised Heisenberg group in Subsection 4.3.

3.1.5 Invariant Vector Fields and Lie Algebras

The second realisation of the matrix Lie algebra $\mathfrak{g}$ [25, App.III, §1.2] is identified through the left (right) *invariant vector fields* on a group G; that is, first-order differential operators $\tilde{X}$ defined at every point of G and invariant under the left (right) shifts:

$$\tilde{X}\Lambda = \Lambda\tilde{X} \qquad (\tilde{X}R = R\tilde{X}).$$

The Lie bracket of two left (right) invariant vector fields is also a left (right) invariant vector field. We will give an example of this for the Lie algebra of the polarised Heisenberg group in Subsection 4.3.

3.2 Representations of Groups

Groups act on other sets by means of transformations. Among various group actions there is an important subclass called *linear representations*.

Definition 3.6. [24, §7.1] A *representation* in the wide sense means a homomorphism of a group G:

$$\rho(g_1 g_2) = \rho(g_1)\rho(g_2), \quad \text{for all } g_1, g_2 \in G, \tag{3.20}$$

into the group of one-to-one mappings of a certain set $\mathcal{H}$ onto itself. The representation ρ is called *linear* if $\mathcal{H}$ is a linear space and the mappings $\rho(g)$ are linear operators. The space $\mathcal{H}$ is called the *representation space*.

If e is the identity of G and $g \in G$, we have $\rho(g^{-1}) = \rho(g)^{-1}$ and $\rho(e) = I$, where I is the identity operator. Let ρ be a representation of a Lie group G on a Hilbert space $\mathcal{H}$. A *strong continuity* of ρ means that for any vector $u \in \mathcal{H}$ and for any convergent sequence $(g_n) \to g \in G$, we have [25, App.V, §1.1]

$$||\rho(g_n)u - \rho(g)u|| \to 0.$$

More details on other types of continuous representation can be found in [24, §7.1; 25, App.V, §1.1]. In this thesis, we agree that the term 'representation' of Lie groups always means a 'linear strongly continuous representation'. Furthermore, the majority of representations will be from the following important class, which has many additional properties:

Definition 3.7. [24, §7.3] A representation ρ of a Lie group G in a Hilbert space $\mathcal{H}$ is called a *unitary representation* if $\rho(g)$ is a unitary operator for all $g \in G$.

The representation theory is much simpler if ρ is unitary [24, §7.3]. If the dimensionality of $\mathcal{H}$ is infinite, then ρ is an *infinite-dimensional* representation. There is a natural equivalent relation on the set of all representations of a group, which is defined by an *intertwining property*.

Definition 3.8. [24, §7.2] Let ρ_1 and ρ_2 be two representations of a Lie group G in spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. An operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ is called an *intertwining operator* between ρ_1 and ρ_2 if for every $g \in G$

$$U\rho_1(g) = \rho_2(g)U, \tag{3.21}$$

that is, the following diagram commutes

$$\begin{array}{ccc} \mathcal{H}_1 & \xrightarrow{U} & \mathcal{H}_2 \\ \downarrow{\rho_1(g)} & & \downarrow{\rho_2(g)} \\ \mathcal{H}_1 & \xrightarrow{U} & \mathcal{H}_2 \end{array} \tag{3.22}$$

Furthermore, unitary representations ρ_1 and ρ_2 are *unitary equivalent representations* if and only if there is a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ intertwining ρ_1 and ρ_2. In this case, we write $\rho_1 \sim \rho_2$.

3.3   Decomposition of Representations

It is a standard mathematical approach to study representations through their decomposition into smaller blocks. To be able to do this, we have to clarify some relevant notions.

Definition 3.9. [24, §7] Let ρ be a linear representation of a Lie group G in a Hilbert space $\mathcal{H}$. A linear subspace $L \subset \mathcal{H}$ is an *invariant subspace* for ρ if for any $x \in L$ and $g \in G$ the vector $\rho(g)x$ again belongs to L.

There are always two trivial invariant subspaces: the null space and the entire $\mathcal{H}$. All others are *non-trivial invariant subspaces*.

Let ρ be a linear representation of a Lie group G in a Hilbert space $\mathcal{H}$. If there are only two trivial invariant subspaces, then ρ is an *irreducible representation*. Otherwise for any non-trivial invariant and irreducible subspace $L \subset \mathcal{H}$, we can define the *restriction of the representation ρ on L* and obtain a *subrepresentation* of ρ acting on L (see [24, §7]). If there is a non-trivial subrepresentation, then the representation is called *reducible* (see [24, §7]).

The following type of representations is easier to study.

Definition 3.10. [24, §7] A linear representation ρ of a Lie group G on a space $\mathcal{H}$ is called *decomposable* if there are two non-trivial invariant subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$.

An important property of the unitary representation is that any representation is either irreducible or decomposable. The irreducibility of a representation is often established by Schur's lemma.

Lemma 3.11. (Classical Schur's lemma)[4, Lem. 4.3.1] *Let ρ be a strongly continuous unitary irreducible representation of a Lie group G on a Hilbert space $\mathcal{H}$. If $U : \mathcal{H} \to \mathcal{H}$ is a linear bounded operator and U commutes with $\rho(g)$, for all $g \in G$, then $U = c\,I$, for some $c \in \mathbb{C}$.*

A *character* χ of a commutative locally compact group G is a continuous function $\chi : G \to \mathbb{C}$ [8, Ch.5,§5.1], which satisfies

$$|\chi(g)| = 1, \quad \chi(g_1 g_2) = \chi(g_1)\,\chi(g_2), \quad \text{for all } g_1, g_2 \in G, \tag{3.23}$$

that is, a character is a (one-dimensional) continuous irreducible unitary representation of G.

Example 3.12. [8, Ch.5,§5.1] For $G = \mathbb{R}^n$, every character χ has the form:

$$\chi(x) = \exp\left(2\pi i(y_1 x_1 + y_2 x_2 + \cdots + y_n x_n)\right) = \exp\left(2\pi i \langle y, x \rangle\right), \quad \text{with } y, x \in \mathbb{R}^n.$$

$$(3.24)$$

Chapter 4

Preliminaries on the Heisenberg Group

In this chapter we provide some basic definitions of and results on the Heisenberg group and its Lie algebra. Next, we give an account of continuous and non-commutative subgroups H of $\mathbb{H}_p^1$. We also describe the respective homogeneous spaces $X = \mathbb{H}_p^1/H$ of $\mathbb{H}_p^1$ and actions of the Heisenberg group on X. Then, we briefly discuss the Schrödinger group. In the last section, we give an introduction of the Jacobi theta function. We mainly use the references [19, 32] (see also [17, 18, 25]).

4.1 The Heisenberg Group

Let $n \geq 1$ be an integer. For two real n-vectors x, $y \in \mathbb{R}^n$, we write xy for their inner product [19, Ch.1,§2]:

$$xy = x_1y_1 + x_2y_2 + \cdots + x_ny_n, \quad \text{where } x = (x_1, x_2, \ldots, x_n),\ y = (y_1, y_2, \ldots, y_n). \quad (4.1)$$

Similarly for complex vectors z, $w \in \mathbb{C}^n$, we define:

$$z\bar{w} = z_1\bar{w}_1 + z_2\bar{w}_2 + \cdots + z_n\bar{w}_n, \quad \text{where } z = (z_1, z_2, \ldots, z_n),\ w = (w_1, w_2, \ldots, w_n). \quad (4.2)$$

Definition 4.1. [19, Ch1.,§2] The *symplectic form* ω on $\mathbb{R}^{2n}$ is a function of two vectors given by

$$\omega(x, y; x', y') = xy' - x'y, \qquad \text{where } (x, y), (x', y') \in \mathbb{R}^{2n}. \qquad (4.3)$$

The symplectic form ω possesses the following properties:

i. ω is anti-symmetric: $\omega(x, y; x', y') = -\omega(x', y'; x, y)$.

ii. ω is bilinear:

$$\omega(x, y; \alpha x', \alpha y') = \alpha \omega(x, y; x', y') \quad \text{for all } \alpha \in \mathbb{R}^n,$$
$$\omega(x, y; x' + x'', y' + x'') = \omega(x, y; x', y') + \omega(x, y; x'', y'').$$

iii. Let $z = x + \mathrm{i}y$ and $w = x' + \mathrm{i}y'$. Then, ω can be expressed through the complex inner product (4.2) as $\omega(x, y; x', y') = -\Im(z\bar{w})$, where $\Im$ is the imaginary part of a complex number.

Definition 4.2. [19, Ch1.,§2] An element of the n-dimensional *Heisenberg group* $\mathbb{H}^n$ is $(s, x, y) \in \mathbb{R}^{2n+1}$, where $s \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$. The group law on $\mathbb{H}^n$ is given as follows:

$$(s, x, y) \cdot (s', x', y') = (s + s' + \tfrac{1}{2}\omega(x, y; x', y'), x + x', y + y'), \tag{4.4}$$

where ω is the symplectic form.

The identity is $(0, 0, 0)$, and $(s, x, y)^{-1} = (-s, -x, -y)$.

Let us introduce complexified coordinates (s, z) on $\mathbb{H}^1$ with $z = x + \mathrm{i}y$. As

$$\omega(x, y; x', y') = -\Im(z\bar{z}') = \Im(z'\bar{z}),$$

the group law can be written as:

$$(s, z) \cdot (s', z') = (s + s' + \tfrac{1}{2}\Im(z'\bar{z}), z + z').$$

For the sake of simplicity, in this thesis we will work with the one-dimensional Heisenberg group $\mathbb{H}^1$. The group law on $\mathbb{H}^1$ can be expressed in an equivalent form of the polarised Heisenberg group law $\mathbb{H}^1_p$ (3.15) [19, § 1.2]:

$$(s, x, y) \cdot (s', x', y') = (s + s' + xy', x + x', y + y'). \tag{4.5}$$

The map

$$\psi : \mathbb{H}^1 \longrightarrow \mathbb{H}^1_p$$
$$(s, x, y) \mapsto (s + \frac{1}{2}xy, x, y) \tag{4.6}$$

is an isomorphism of the Heisenberg group $\mathbb{H}^1$ to its polarised form $\mathbb{H}^1_p$. We observe that [19, Ch1.,§2]

$$Z = \{(s,0,0) \in \mathbb{H}^1_p : \ s \in \mathbb{R}\} \tag{4.7}$$

is the *centre* of $\mathbb{H}^1_p$, that is $(s,0,0)$, which commutes with any other element of $\mathbb{H}^1_p$. In this thesis, we only consider the polarised Heisenberg group $\mathbb{H}^1_p$, and we simply call it the Heisenberg group.

4.2   Automorphisms of the Heisenberg Group

A group automorphism is its homomorphism to itself. Among all of the automorphisms of the Heisenberg group we are interested in those connected to symplectic transformations. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of the symplectic group $\mathrm{Sp}(2) \sim \mathrm{SL}_2(\mathbb{R})$ with real entries $a,b,c,d \in \mathbb{R}$, where the determinant $\det A = ad - bc$ is equal to 1.

i. The following transformation ϕ_A [19, Ch.1] is an automorphism of $\mathbb{H}^1$:

$$\phi_A : \mathbb{H}^1 \longrightarrow \mathbb{H}^1$$
$$(s,x,y) \mapsto (s, A(x,y)) \tag{4.8}$$
$$= (s, \tilde{x}, \tilde{y})$$

where

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{4.9}$$

The map $A : (x,y) \mapsto (\tilde{x}, \tilde{y})$ is called a *symplectic map*.

ii. An automorphism $\phi^p_A : \mathbb{H}^1_p \to \mathbb{H}^1_p$ in polarised form is calculated as follows. Recall the homorphism $\psi : \mathbb{H}^1 \to \mathbb{H}^1_p$ (4.6)

$$\psi : (s,x,y) \mapsto (s + \frac{1}{2}xy, x, y). \tag{4.10}$$

The inverse map $\psi^{-1} : \mathbb{H}^1_p \to \mathbb{H}^1$ is:

$$\psi^{-1} : (s,x,y) \mapsto (s - \frac{1}{2}xy, x, y). \tag{4.11}$$

To define the automorphism of $\mathbb{H}^1_p$, we use the following commutative diagram:

$$\begin{array}{ccc} \mathbb{H}^1 & \xrightarrow{\phi_A} & \mathbb{H}^1 \\ \psi^{-1} \uparrow & & \downarrow \psi \\ \mathbb{H}^1_p & \xrightarrow{\phi^p_A} & \mathbb{H}^1_p \end{array} \tag{4.12}$$

where ϕ_A is the automorphism of $\mathbb{H}^1$ (4.8). Then:

$$
\begin{aligned}
\phi_A^p(s, x, y) &= \psi \circ \phi_A \circ \psi^{-1}(s, x, y) \\
&= \psi \circ \phi_A\left(s - \frac{1}{2}xy, x, y\right) \\
&= \psi\left(s - \frac{1}{2}xy, \tilde{x}, \tilde{y}\right) \\
&= \left(s + \frac{1}{2}(\tilde{x}\tilde{y} - xy), \tilde{x}, \tilde{y}\right) \\
&= \left(s + \frac{1}{2}(ac\,x^2 + 2bc\,xy + bd\,y^2), \tilde{x}, \tilde{y}\right),
\end{aligned}
\tag{4.13}
$$

where

$$
\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.
\tag{4.14}
$$

4.3 The Lie Algebra of the Heisenberg Group

The Lie algebra of the Heisenberg group is denoted by $\mathfrak{h}_1$ and is also called the *Weyl algebra*. The Lie algebra $\mathfrak{h}_1$ is a three-dimensional real vector space $\mathbb{R}^3$ with basis vectors

$$
S = (1, 0, 0), \quad X = (0, 1, 0), \quad Y = (0, 0, 1),
\tag{4.15}
$$

satisfying the following commutation relations:

$$
[X, Y] = S, \quad [X, S] = [Y, S] = 0.
\tag{4.16}
$$

In order to identify the Lie algebra corresponding to $\mathfrak{h}_1$, it is convenient to use a matrix representation. Given $(s, x, y) \in \mathfrak{h}_1$, we define the $m(s, x, y) \in M_3(\mathbb{R})$ by [19, Ch.1,§2]

$$
m(s, x, y) = \begin{pmatrix} 0 & x & s - \frac{1}{2}xy \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix},
\tag{4.17}
$$

which forms the 3×3 matrix Lie algebra. In fact, the matrix $SL_3(s, x, y)$ from (3.12) is created by the exponential map $SL_3(s, x, y) = \exp(m(s, x, y))$:

$$
\begin{pmatrix} 1 & x & s \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \exp \begin{pmatrix} 0 & x & s - \frac{1}{2}xy \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}.
\tag{4.18}
$$

Thus, we link $\mathfrak{h}_1$ with the one-parameter subgroups of the Heisenberg group $\mathbb{H}_p^1$ (see Subsection 3.1.3):

$$s(t) = (t, 0, 0), \quad x(t) = (0, t, 0), \quad \text{and} \quad y(t) = (0, 0, t). \tag{4.19}$$

Also, there is a realisation of $\mathfrak{h}_1$ as the collection of invariant vector fields. Let f be a differentiable function on $\mathbb{H}_p^1$. The left and the right shifts on f are

$$\Lambda(g) : f(g') \mapsto f(g^{-1}g') \quad \text{and} \quad R(g) : f(g') \mapsto f(g'g) \quad g, g' \in \mathbb{H}_p^1. \tag{4.20}$$

We calculate the invariant vector fields on the Heisenberg group $\mathbb{H}_p^1$ through a connection to the above one-parameter subgroups. The left invariant vector fields corresponding to the subgroups (4.19) are obtained through the differentiation of the right actions of these subgroups:

$$S^l = \partial_s, \quad X^l = \partial_x, \quad Y^l = x\,\partial_s + \partial_y. \tag{4.21}$$

Similarly, the right-invariant vector fields are obtained by the derivation of the left actions of the subgroups (4.19):

$$S^r = -\partial_s, \quad X^r = -y\,\partial_s - \partial_x, \quad Y^r = -\partial_y. \tag{4.22}$$

The left (right) invariant vector fields form a bases of the Lie algebra $\mathfrak{h}_1$ (see [25, App.III, §1.3]). As expected, they satisfy the Heisenberg *commutator relation*

$$[X^{l(r)}, Y^{l(r)}] = S^{l(r)}. \tag{4.23}$$

4.4   Continuous Subgroups of $\mathbb{H}_p^1$ and Homogeneous Spaces

In this section, we describe $\mathbb{H}_p^1$–homogeneous spaces X and actions of $\mathbb{H}_p^1$ on each space. The main references of this section are [19, Ch.1; 32].

4.4.1   1D Continuous Subgroups and 2D Homogeneous Spaces

Let us consider a one-dimensional continuous subgroup of $\mathbb{H}_p^1$:

$$Z = \{(s, 0, 0) \in \mathbb{H}_p^1 : \ s \in \mathbb{R}\}, \tag{4.24}$$

which is the centre (4.7) of $\mathbb{H}_p^1$.

By (3.8), the space $\mathbb{H}_p^1/Z$ can be identified with $\mathbb{R}^2$ through the decomposition $(s, x, y) = (0, x, y)(s, 0, 0)$ for each $(s, x, y) \in \mathbb{H}_p^1$.

Next, we describe the actions of $\mathbb{H}_p^1$ on the homogeneous space $\mathbb{H}_p^1/Z$. See Section 3.1.1 for the background theory and definitions of maps $\mathsf{p} : \mathbb{H}_p^1 \to X$ and $\mathsf{s} : X \to \mathbb{H}_p^1$, where $X = \mathbb{H}_p^1/H$ is the respective homogeneous space for a subgroup $H \subset \mathbb{H}_p^1$. We use the following parametrisations:

$$\begin{aligned} \mathsf{p} &: (s', x', y') \mapsto (x', y'), \\ \mathsf{s} &: (x', y') \mapsto (0, x', y'). \end{aligned} \tag{4.25}$$

Let $g = (s, x, y) \in \mathbb{H}_p^1$. By (3.10), the action of g on $\tilde{x} = (x', y') \in X = \mathbb{H}_p^1/Z$ is calculated as follows.

$$\begin{aligned} g^{-1} \cdot \tilde{x} &= \mathsf{p}(g^{-1} * \mathsf{s}(\tilde{x})) \\ &= \mathsf{p}((s, x, y)^{-1} * (0, x', y')) \\ &= \mathsf{p}((-s + xy, -x, -y) * (0, x', y')) \\ &= (x' - x, y' - y). \end{aligned} \tag{4.26}$$

4.4.2 A 2D Continuous Subgroup and a 1D Homogeneous Space

In this subsection, we consider a two-dimensional continuous subgroup [32] of $\mathbb{H}_p^1$ given as follows:

$$H_x' = \{(s, 0, y) \in \mathbb{H}_p^1 : s, y \in \mathbb{R}\}. \tag{4.27}$$

The homogeneous space $X = \mathbb{H}_p^1/H_x'$ is parametrised by $\mathbb{R}$ because, by (3.8), X can be identified through the decomposition $(s, x, y) = (0, x, 0)(s - xy, 0, y)$ for each $(s, x, y) \in \mathbb{H}_p^1$. Using the maps $\mathsf{p} : (s', x', y') \mapsto x'$ and $\mathsf{s} : x' \mapsto (0, x', 0)$, by (3.10), the action of $g = (s, x, y) \in \mathbb{H}_p^1$ on $x' \in \mathbb{H}_p^1/H_x' = \mathbb{R}$ is

$$\begin{aligned} g^{-1} \cdot x' &= \mathsf{p}(g^{-1} * \mathsf{s}(x')) \\ &= x' - x \end{aligned} \tag{4.28}$$

4.5 A Non-commutative Discontinuous Subgroup and a Homogeneous Space

In this section, we consider a *non-commutative discontinuous* subgroup H_d of $\mathbb{H}_p^1$ and the respective homogenous space $X = \mathbb{H}_p^1/H_d$. The main references for this section are [10, §11, §12; 34, §3, p.8].

Definition 4.3. [12, §1] Let $v = (v_1, v_2)$, $w = (w_1, w_2)$ be two non-zero linearly independent vectors in $\mathbb{R}^2$. The set of vectors

$$
\begin{aligned}
\mathbf{\Gamma} &= \mathbb{Z}v + \mathbb{Z}w \\
&= \left\{ m_1 v + m_2 w \in \mathbb{R}^2 : m_1, m_2 \in \mathbb{Z} \right\}
\end{aligned}
\tag{4.29}
$$

forms a *lattice* in $\mathbb{R}^2$.

The lattice Γ can be represented by parallelograms formed by two vectors v and w as in Figure 4.1.
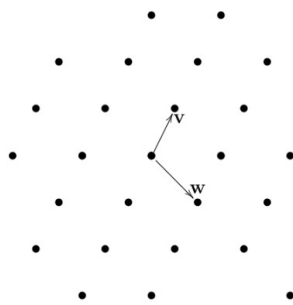

Figure 4.1: Parallelogram Lattice in $\mathbb{R}^2$ generated by v and w.

This Γ can be transformed to the lattice of squares (see Figure 4.2) as follows. Suppose that Γ is defined by two complex vectors $v = v_1 + iv_2$ and $w = w_1 + iw_2$, which spans a parallelogram of the unit area, that is, $v_1 w_2 - v_2 w_1 = 1$. Then, the matrix $A = \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix} \in SL_2(\mathbb{C})$. The inverse of it is $A^{-1} = \begin{pmatrix} w_2 & -w_1 \\ -v_2 & v_1 \end{pmatrix}$. The linear transformation of $\mathbb{R}^2$ defined by the latter matrix maps points of the lattice spanned by v and w to the standard lattice $\mathbb{Z}^2$ (see [36, Ch.7]).
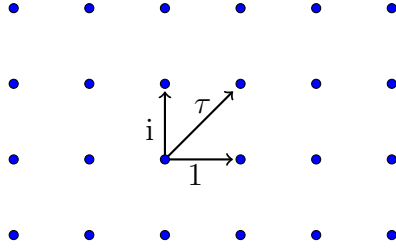
Figure 4.2: Square Lattice in $\mathbb{R}^2$ generated by 1 and i

Denote by $\mathbb{C}_+$ the upper half-plane. A non-commutative discontinuous subgroup H_τ, $\tau \in \mathbb{C}_+$, of the Heisenberg group $\mathbb{H}_p^1$ (see [10, §11, §12; 34, §3, p.8]) is defined by:

$$H_\tau = \left\{ (s,n,k)_\tau := (s, n + \tau k) : (n,k) \in \mathbb{Z}^2, s \in \mathbb{R} \right\}, \tag{4.30}$$

where $\Gamma = \{ n + \tau k : n,k \in \mathbb{Z} \}$ is the lattice of parallelograms generated by 1 and τ. As the parallelogram lattice Γ can be mapped into a square lattice Γ', we consider Γ' to define the following non-commutative discontinuous subgroup H_d:

$$H_d = \{ (s,n,k) = (s, n + \mathrm{i}k) : (n,k) \in \mathbf{\Gamma'}, s \in \mathbb{R} \}. \tag{4.31}$$

That is, H_τ for $\tau = \mathrm{i}$. The subgroup H_d has a *discrete subgroup* H_d^* (see [10, §1.1; 34, §3, p.8]):

$$H_d^* = \{ (t,n,k) = (m, n + \mathrm{i}k) : (n,k) \in \mathbf{\Gamma'}, t \in \mathbb{Z} \}. \tag{4.32}$$

We have $H_d^* \subset H_d \subset \mathbb{H}_p^1$. In this thesis, for the sake of simplicity, we only consider the subgroup H_d. We consider a topology on H_d by embedding into $\mathbb{H}_p^1$.

By (3.8), the homogeneous space $X = \mathbb{H}_p^1/H_d$ can be identified with the torus

$$\mathbb{T}^2 = \{ (u,v) : u,v \in [0,1) \} \tag{4.33}$$

through the following decomposition

$$(s,x,y) = (0, \{x\}, \{y\})(s - \{x\}[y], [x], [y]), \tag{4.34}$$

where $[x] \in \mathbb{Z}$ and $\{x\} \in [0,1)$ denote the *integer* and *fractional* parts of x, respectively, that is, $[x] + \{x\} = x \in \mathbb{R}$. Let $(s,x,y) \in \mathbb{H}_p^1$ and $\tilde{x} = (u,v) \in \mathbb{H}_p^1/H_d$. By (3.10), we calculate the $\mathbb{H}_p^1$-action on $\mathbb{H}_p^1/H_d = \mathbb{T}^2$ as follows. For

$$\begin{aligned} \mathsf{p} &: (s,x,y) \mapsto (\{x\}, \{y\}), \\ \mathsf{s} &: (u,v) \mapsto (0,u,v), \end{aligned} \tag{4.35}$$

we have

$$
\begin{aligned}
g^{-1} \cdot \tilde{x} &= \mathsf{p}(g^{-1} * \mathsf{s}(x)) \\
&= \mathsf{p}((s, x, y)^{-1} * (0, u, v)) \\
&= \mathsf{p}\left(-s + xy - xv, u - x, v - y\right) \\
&= (\{u - x\}, \{v - y\}).
\end{aligned}
\tag{4.36}
$$

This action is called a 'periodic shift'.

4.6   Schrödinger Group

Since the group law on the Heisenberg group is defined using the symplectic form ω, any transformation of $\mathbb{H}^1$ that preserves ω can be used to define a group automorphism. In general, whenever we have a group acting by automorphisms on another group, we can consider their *semi-direct product*.

Definition 4.4. [42, §7] Let N and K be groups, and let $\phi : K \longrightarrow \mathrm{Aut}(N)$ (we write $\phi_k := \phi(k)$, which is an automorphism corresponding to K acting on N). The semi-direct product of N by K denoted by $N \rtimes K$ is the set of ordered pairs $\{(n, k) : n \in N, \ k \in K\}$ together with the binary operation defined by

$$
(n_1, k_1) * (n_2, k_2) = (n_1 \phi_{k_1}(n_2), k_1 k_2). \tag{4.37}
$$

If $N = \mathbb{H}^1$ and $K = \mathrm{Sp}(2)$, we can build the semi-direct product $\tilde{G} = \mathbb{H}^1 \rtimes \mathrm{Sp}(2)$ [17, 32], where $\mathrm{Sp}(2)$ is the symplectic group. Since the symplectic group $\mathrm{Sp}(2)$ is isomorphic to $\mathrm{SL}_2(\mathbb{R})$ [32], the semi-direct product can be written as:

$$
\tilde{G} = \mathbb{H}^1 \rtimes \mathrm{SL}_2(\mathbb{R}). \tag{4.38}
$$

This group is called the *Schrödinger group* or *Jacobi group*. Consider two matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ of the group $\mathrm{SL}_2(\mathbb{R})$ (see Example 3.4 (i)). Let $g = (s, x, y), g' = (s', x', y') \in \mathbb{H}^1$. By (4.37), the group law of $\tilde{G} = \mathbb{H}^1 \rtimes \mathrm{SL}_2(\mathbb{R})$ is

$$
\begin{aligned}
(g, A) * (g', A') &= (g \cdot \phi_A(g'), AA') \\
&= ((s, x, y) \cdot (s', \tilde{x}', \tilde{y}'), AA') \\
&= (s + s' + \frac{1}{2}(x\tilde{y}' - \tilde{x}'y), x + \tilde{x}', y + \tilde{y}'), AA'),
\end{aligned}
\tag{4.39}
$$

where the action $\phi_A(g')$ is defined in (4.8) and

$$\begin{pmatrix} \tilde{x}' \\ \tilde{y}' \end{pmatrix} = \begin{pmatrix} ax' + by' \\ cx' + dy' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}. \tag{4.40}$$

Now, we give the construction of the Schrödinger group in polarised form.

i. The semi-direct product of $\mathbb{H}_p^1$ by the group $\mathrm{SL}_2(\mathbb{R})$ is

$$G = \mathbb{H}_p^1 \rtimes \mathrm{SL}_2(\mathbb{R}). \tag{4.41}$$

Similar to (4.39), the group law of $G = \mathbb{H}_p^1 \rtimes \mathrm{SL}_2(\mathbb{R})$ is given by:

$$(g, A) * (g', A') = (g \cdot \phi_A^p(g'), AA'), \tag{4.42}$$

where, the action $\phi_A^p(g')$ is defined in (4.13) by:

$$\phi_A^p(g') = (s' + \frac{1}{2}(ac\, x'^2 + x'y'\,(ad + bc) + bd\, y'^2 - x'y'), \tilde{x}', \tilde{y}'), \tag{4.43}$$

and $\begin{pmatrix} \tilde{x}' \\ \tilde{y}' \end{pmatrix} = \begin{pmatrix} ax' + by' \\ cx' + dy' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$. We call G the *polarised Schrödinger group*.

ii. The Heisenberg group is a subgroup of the polarised Schrödinger group because

$$\mathbb{H}_p^1 = \mathbb{H}_p^1 \rtimes \{I\} \subset \mathbb{H}_p^1 \rtimes \mathrm{SL}_2(\mathbb{R}). \tag{4.44}$$

In this thesis, we only consider the polarised Schrödinger group and call it the Schrödinger group.

4.7   Introduction of the Jacobi Theta Functions

The object connecting two parts of this thesis is the analytic function $\Theta(z, \tau)$ in two variables called the *Jacobi theta function* or simply the *theta function*. We mainly use [34] as a reference for this section. Other additional sources are [8, 10, 14, 19, 43].

Definition 4.5. [34] Let $z \in \mathbb{C}$ and $\tau \in \mathbb{C}_+$, where $\mathbb{C}_+$ is the upper half-plane. The Jacobi theta function is defined by:

$$\Theta(z, \tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau} e^{2\pi i n z}. \tag{4.45}$$

Let Z be the centre of the Heisenberg group $\mathbb{H}_p^1$. If $H = Z \times \mathrm{SL}_2(\mathbb{Z})$ is a subgroup of the Schrödinger group $\mathbb{H}_p^1 \rtimes \mathrm{SL}_2(\mathbb{R})$ (4.41), the domain $X = \mathbb{C} \times \mathbb{C}_+$, on which Θ is defined, is a homogeneous space obtained as follows [34]:

$$X = \mathbb{H}_p^1 \rtimes \mathrm{SL}_2(\mathbb{R})/Z \times \mathrm{SL}_2(\mathbb{Z}). \tag{4.46}$$

We now take a closer look at Θ as a function of z, with τ fixed, by recording its basic structural properties.

Proposition 4.6. [34] *Let* $z \in \mathbb{C}$ *and* $\tau \in \mathbb{C}_+$. *The theta function* Θ *satisfies the following properties:*

(i) $\Theta(z + 1, \tau) = \Theta(z, \tau)$ *(periodic).*

(ii) $\Theta(z + \tau, \tau) = \mathrm{e}^{-\pi \mathrm{i}\tau} \mathrm{e}^{-2\pi \mathrm{i}z} \Theta(z, \tau)$ *(quasi-periodic).*

The above two periodicities together imply the following:

$$\Theta(z + a\tau + b, \tau) = \mathrm{e}^{-\pi \mathrm{i}a^2\tau - 2\pi \mathrm{i}az}\Theta(z, \tau), \quad where \quad a, b \in \mathbb{Z}. \tag{4.47}$$

By Proposition 4.6, the theta function Θ has *double quasi-periodic* behaviour with respect to the lattice Γ (4.29) generated by 1 and τ. That is, it is *periodic* with respect to $z \mapsto z + b$, $b \in \mathbb{Z}$ and *quasi-periodic* with respect to $z \mapsto z + a\tau$, $a \in \mathbb{Z}$. Therefore, for a fixed $\tau \in \mathbb{C}_+$, there are two possibilities to study the theta function (see [34]):

  i. Θ is a double quasi-periodic function on $\mathbb{R}^2$.

 ii. Θ is a function on the square $\mathbb{T}^2$.

Taking into account both variables of Θ, we consider it as function on the homogeneous space

$$(z, \tau) \in X = \mathbb{H}_p^1 \rtimes \mathrm{SL}_2(\mathbb{R})/H_d \rtimes \mathrm{SL}_2(\mathbb{Z}) = \mathbb{T}^2 \times \mathbb{C}_+, \tag{4.48}$$

where H_d is the non-commutative discontinuous subgroup defined in (4.32) of $\mathbb{H}_p^1$.

Chapter 5

Representations of the Heisenberg Group and Ladder Operators

The main purpose of this chapter is to describe the construction of induced representations of the Heisenberg group $\mathbb{H}_p^1$ on Hilbert spaces of the following forms:

i. the *left quasi-regular* representation induced from a character of the centre Z;

ii. the *Schrödinger* representation induced from a character of H_x'; and

iii. the *lattice* representation induced from a character of the non-commutative subgroup H_d.

Furthermore, we discuss *ladder operators: creation* and *annihilation* operators produced by derived representations of the Lie algebra $\mathfrak{h}_1$ of the Heisenberg group.

5.1 Left Regular Representations of $\mathbb{H}_p^1$

In the present section, we extend the action of $\mathbb{H}_p^1$ on itself by left shift to a linear representation

$$\Lambda(g) : F(g') \mapsto F(g^{-1}g'), \qquad g, g' \in \mathbb{H}_p^1 \tag{5.1}$$

on a certain linear space of functions on $\mathbb{H}_p^1$ [8, §1.1]. The main references of this section are [18; 24, §13; 25, App.V,§ 2; 32].

The Lebesgue measure $dg = ds\,dx\,dy$ on $\mathbb{H}_p^1 \sim \mathbb{R}^3$ is invariant under the left shift (3.1). This measure is also invariant under right shifts. Thus, $\mathbb{H}_p^1$ is *unimodular*. The action (5.1) on the Hilbert space $L_2(\mathbb{H}_p^1, dg)$ of square integrable functions

on $\mathbb{H}_p^1$ is unitary [25, App.V, §2] and is called the *left regular* representation. Let $\chi_\hbar(s,0,0) = e^{2\pi i \hbar s}$ be the character of the centre Z of $\mathbb{H}_p^1$ [8, Ch.5,§5.1]. Let $L_2^\chi(\mathbb{H}_p^1)$ be the space of functions on $\mathbb{H}_p^1$ having the properties [28]:

$$F(gh) = \bar{\chi}_\hbar(h)F(g), \qquad \text{for all } g \in \mathbb{H}_p^1, \ h \in Z \tag{5.2}$$

and

$$\int_{\mathbb{R}^2} |F(0,x,y)|^2 \, dx \, dy < \infty. \tag{5.3}$$

We call the property (5.2) the *H-covariance property*. The space $L_2^\chi(\mathbb{H}_p^1)$ is invariant under the left $\mathbb{H}_p^1$-shifts (5.1) because the left and right shifts commute.

5.1.1 Induced Representations of $\mathbb{H}_p^1$ on Homogeneous Spaces

In this section, we discuss a particular case of induced representations in the sense of Mackey, where the induction is performed from a character of a subgroup H of a group G [24, §13.2]. A detailed consideration of this topic can be found in [24, §13.2; 25, App.V, §2; 32] and Subsection 3.1.1.

Let H be a subgroup of the Heisenberg group $\mathbb{H}_p^1$ and $\mathbb{H}_p^1/H$ be the respective homogeneous space. Consider a continuous section $\mathsf{s} : \mathbb{H}_p^1/H \to \mathbb{H}_p^1$, which is a right inverse of the natural projection $\mathsf{p} : \mathbb{H}_p^1 \to \mathbb{H}_p^1/H$. Any element $g \in \mathbb{H}_p^1$ can be uniquely decomposed as $g = \mathsf{s}(\mathsf{p}(g))\mathsf{r}(g)$, where the map $\mathsf{r} : G \to H$ is defined by the previous identity (see Subsection 3.1.1 and [24, §13.2]):

$$\mathsf{r}(g) = \mathsf{s}(\mathsf{p}(g))^{-1}g. \tag{5.4}$$

Let χ be a character of H. Let $L_2(\mathbb{H}_p^1/H)$ be a space of square integrable functions on the homogeneous space $\mathbb{H}_p^1/H$, which will be a subset of Euclidean space with the Lebesgue measure. We define a *lifting* $\mathcal{L}_\chi : L_2(\mathbb{H}_p^1/H) \to L_2^\chi(\mathbb{H}_p^1)$ [32] as follows:

$$\begin{aligned} [\mathcal{L}_\chi f](g) &= \bar{\chi}(\mathsf{r}(g))f(\mathsf{p}(g)), \qquad \text{where } f \in L_2(\mathbb{H}_p^1/H) \\ &=: F(g). \end{aligned} \tag{5.5}$$

The image space of the lifting $\mathcal{L}_\chi$ satisfies the *H-covariance property* (5.2). Indeed, by (5.5), if $F \in L_2^\chi(\mathbb{H}_p^1)$, we have

$$F(gh) = \bar{\chi}(\mathsf{r}(gh))f(\mathsf{p}(gh)). \tag{5.6}$$

Now, as $g \in \mathbb{H}_p^1$, we have $g = \mathsf{s}(\mathsf{p}(g))\mathsf{r}(g)$; therefore,

$$
\begin{aligned}
gh &= \mathsf{s}(\mathsf{p}(gh))\mathsf{r}(gh) \\
&= \mathsf{s}(\mathsf{p}(g))\mathsf{r}(gh) \\
&= g\,(\mathsf{r}(g))^{-1}\mathsf{r}(gh).
\end{aligned}
$$

Hence, $h = (\mathsf{r}(g))^{-1}\mathsf{r}(gh)$, that is, $\mathsf{r}(g)h = \mathsf{r}(gh)$. Thus, the identity (5.6) becomes

$$
\begin{aligned}
F(gh) &= \bar{\chi}(h)\bar{\chi}(\mathsf{r}(g))f(\mathsf{p}(g)) \\
&= \bar{\chi}(h)F(g).
\end{aligned}
$$

This shows that any lifting function $\mathcal{L}_\chi f = F$ possesses the H-covariance property. The space $L_2^\chi(\mathbb{H}_p^1)$ is not invariant under right $\mathbb{H}_p^1$-shifts

$$
R(g): F(g') \mapsto F(g'g), \quad g', g \in \mathbb{H}_p^1 \tag{5.7}
$$

because the H-covariance property (5.2) is not preserved. Now, we define the *pulling* $\mathcal{P}: L_2^\chi(\mathbb{H}_p^1) \to L_2(\mathbb{H}_p^1/H)$ by $[\mathcal{P}F](x) = F(\mathsf{s}(x))$ [32]. The induced representation ρ_χ on $L_2(\mathbb{H}_p^1/H)$ for the character χ on H is generated by the following formula [32]:

$$
\rho_\chi(g) = \mathcal{P} \circ \Lambda(g) \circ \mathcal{L}_\chi. \tag{5.8}
$$

This can be represented by the following commutative diagram.

$$
\begin{array}{ccc}
L_2^\chi(\mathbb{H}_p^1) & \xrightarrow{\Lambda(g)} & L_2^\chi(\mathbb{H}_p^1) \\
\mathcal{L}_\chi \Big\uparrow\Big\downarrow \mathcal{P} & & \mathcal{L}_\chi \Big\uparrow\Big\downarrow \mathcal{P} \\
L_2(\mathbb{H}_p^1/H) & \xrightarrow{\rho_\chi(g)} & L_2(\mathbb{H}_p^1/H)
\end{array}
$$

Figure 5.1: Induced representation from a character of a subgroup

Thus, the formula of the induced representation on the homogeneous space $\mathbb{H}_p^1/H$ (5.8) is

$$
[\rho_\chi(g)f](x) = \bar{\chi}(\mathsf{r}(g^{-1} * \mathsf{s}(x)))f(g^{-1} \cdot x). \tag{5.9}
$$

5.1.2 Induction from Continuous Subgroups

In this section, we use formula (5.9)

$$[\rho_\chi(g)f](x) = \bar{\chi}(\mathsf{r}(g^{-1} * \mathsf{s}(x)))f(g^{-1} \cdot x) \tag{5.10}$$

to construct induced representations from the continuous subgroups Z and H'_x obtained in Subsections 4.4.1 and 4.4.2, respectively. Let $\hbar \neq 0$ be a real number, which can be associated with the *Planck constant* [9].

i. In the following we construct a representation induced from the character $\chi_\hbar(s, 0, 0) = e^{2\pi i \hbar s}$ of the centre Z. Consider the natural projection and section maps defined in (4.25):

$$\begin{aligned} \mathsf{p} &: (s', x', y') \mapsto (x', y'), \\ \mathsf{s} &: (x', y') \mapsto (0, x', y'). \end{aligned} \tag{5.11}$$

Let $g = (s, x, y) \in \mathbb{H}^1_p$. The map $\mathsf{r} : \mathbb{H}^1_p \to Z$ is

$$\begin{aligned} \mathsf{r}(g) &= \mathsf{s}(\mathsf{p}(g))^{-1} g \\ &= (0, x, y)^{-1} g \\ &= (xy, -x, -y)\,(s, x, y) \\ &= (s, 0, 0). \end{aligned}$$

For $\tilde{x} = (x', y') \in \mathbb{R}^2 = \mathbb{H}^1_p/Z$, we have

$$\begin{aligned} \mathsf{r}(g^{-1} * \mathsf{s}(\tilde{x})) &= \mathsf{r}((-s + xy, -x, -y)\,(0, x', y')) \\ &= \mathsf{r}(-s + xy - xy', x' - x, y' - y) \\ &= (-s - x(y' - y), 0, 0). \end{aligned} \tag{5.12}$$

Recall the $\mathbb{H}^1_p$-action on $\mathbb{R}^2$ (4.26)

$$g^{-1} \cdot \tilde{x} = (x' - x, y' - y). \tag{5.13}$$

For $f \in L_2(\mathbb{R}^2)$, substituting the calculations from (5.12) and (5.13) into the general induced representation formula (5.10) implies

$$[\Lambda_\hbar(s, x, y)f](x', y') = e^{2\pi i \hbar(s + x(y' - y))} f(x' - x, y' - y). \tag{5.14}$$

This representation is equivalent to a restriction of the left regular representation to the space of functions having a Z-covariance property (5.2). We call this representation the *left quasi-regular representation*. This is reducible [28], and we will decompose it into irreducible components (see Subsection 6.4.1).

ii. Let $H = H'_x = \{(s, 0, y) \in \mathbb{H}^1_p, \, s \in \mathbb{R}, \, y \in \mathbb{R}^1\}$. We construct an induced representation from a character $\chi_\hbar(s, 0, 0) = e^{2\pi i \hbar s}$ of H'_x as follows. Consider the projection and section maps $p : (s', x', y') \mapsto x'$ and $s : x' \mapsto (0, x', 0)$ defined in Subsection 4.4.2. The map $r : \mathbb{H}^1_p \to H'_x$ (5.4) for $g = (s, x, y) \in \mathbb{H}^1_p$ is

$$
\begin{aligned}
r(g) &= s(p(g))^{-1} g \\
&= (0, x, 0)^{-1} g \\
&= (0, -x, 0)(s, x, y) \\
&= (s - xy, 0, y).
\end{aligned}
$$

For $\tilde{x} = t \in \mathbb{R} = \mathbb{H}^1_p / H'_x$, we have

$$
\begin{aligned}
r(g^{-1} * s(\tilde{x})) &= r((-s + xy, -x, -y)(0, t, 0)) \\
&= r(-s + xy, t - x, -y) \\
&= (-s + ty, 0, -y).
\end{aligned} \tag{5.15}
$$

The $\mathbb{H}^1_p$-action on $\mathbb{R}$ defined in (4.28) is

$$
g^{-1} \cdot t = t - x. \tag{5.16}
$$

Therefore, for $f \in L_2(\mathbb{R})$, if we substitute our calculations from (5.15) and (5.16) into the general formula (5.9) of the induced representation, we obtain

$$
[\rho_\hbar(s, x, y) f](t) = e^{2\pi i \hbar(s - ty)} f(t - x). \tag{5.17}
$$

This is a unitary irreducible representation on $L_2(\mathbb{R})$ called the *Schrödinger representation* [19, CH.1].

We will see later that all these representations (and many others) with the same value $\hbar$ are unitarily equivalent by the Stone–von Neumann theorem given in Section 5.2. In this thesis, for the sake of simplicity, we only consider the positive Planck constant $\hbar > 0$.

5.1.3 Induction from a Non-commutative Subgroup

In this section, we construct an induced representation on $L_2(\mathbb{T}^2)$ from a character $\chi_m(s', n, k) = \mathrm{e}^{2\pi m i s'}$, $m \in \mathbb{Z}\backslash\{0\}$, of the subgroup (4.32)

$$H_d = \{(s, n, k) := (s, n + \mathrm{i}k) : (n, k) \in \mathbf{\Gamma}', s \in \mathbb{R}\}.$$

Since this subgroup is not commutative, we need to consider only characters with kernels containing the subgroup's commutator consisting of elements $(m, 0, 0) \in \mathbb{H}_p^1$ with $m \in \mathbb{Z}$; therefore, $m \in \mathbb{Z}$ as above. For the sake of simplicity, we only consider $m \in \mathbb{N}$, which can be associated with a positive *Planck constant* [9].

For $F \in L_2^{\chi_m}(\mathbb{H}_p^1)$, the H-covariance property $F(gh) = \bar{\chi}_m(h)F(g)$ (5.2) with $g = (s, x, y)$ and $h = (-xk, n, k) \in H_d$ implies

$$F(s, x + n, y + k) = \mathrm{e}^{2\pi m \mathrm{i} x k}F(s, x, y). \tag{5.18}$$

Thus, the space $L_2^{\chi_m}(\mathbb{H}_p^1)$ consists of functions F, which are double quasi-periodic (periodic in x and quasi-periodic in y). Consider the natural projection and section maps defined in (4.35):

$$\begin{aligned}
\mathsf{p} &: (s, x, y) \mapsto (\{x\}, \{y\}), \\
\mathsf{s} &: (u, v) \mapsto (0, u, v).
\end{aligned} \tag{5.19}$$

We calculate the map $\mathsf{r} : \mathbb{H}_p^1 \to H_d$ (5.4) as follows:

$$\begin{aligned}
\mathsf{r}(g) &= \mathsf{s}(\mathsf{p}(g))^{-1}\, g \\
&= \mathsf{s}(\{x\}, \{y\})^{-1}\, (s, x, y) \\
&= (\{x\}\{y\}, -\{x\}, -\{y\})\, (s, x, y) \\
&= (s - \{x\}[y], [x], [y]).
\end{aligned} \tag{5.20}$$

Since $m \in \mathbb{N}$, we have

$$\chi_m(\mathsf{r}(s, x, y)) = \mathrm{e}^{2\pi m \mathrm{i}(s - \{x\}[y])} = e^{2\pi m \mathrm{i}(s - (x - [x])[y])} = \mathrm{e}^{2\pi m \mathrm{i}(s - x[y])}.$$

The $\mathbb{H}_p^1$-action on $\mathbb{H}_p^1/H_d = \mathbb{T}^2$ calculated in (4.36) is

$$g^{-1} \cdot \tilde{x} = (\{u - x\}, \{v - y\}). \tag{5.21}$$

Thus, for a function $f \in L_2(\mathbb{T}^2)$, the representation induced from the character χ_m of H_d is

$$
\begin{aligned}
[\rho_m(s,x,y)f](u,v) &= \bar{\chi}_m(\mathsf{r}(g^{-1} * \mathsf{s}(x)))f(g^{-1} \cdot x) \\
&= \bar{\chi}_m(-s - x(v-y) - \{u-x\}[v-y], [u-x], [v-y])f(g^{-1} \cdot x) \\
&= \mathrm{e}^{2\pi m\mathrm{i}(s + x(v-y) + (u-x)[v-y])}f(\{u-x\}, \{v-y\}).
\end{aligned}
$$

(5.22)

Remark 5.1. [35, §8.1] We have two possibilities to treat the function $f \in L_2(\mathbb{T}^2)$:

    i. f is a double quasi-periodic function on $\mathbb{R}^2$; and

    ii. f is a function on the torus $\mathbb{T}^2 = \{(u,v) : u, v \in [0,1)\}$ (4.33).

If f is considered as a double quasi-periodic function on $\mathbb{R}^2$, by Remark 5.1, the representation (5.22) becomes

$$
[\rho_m(s,x,y)f](u,v) = \mathrm{e}^{2\pi m\mathrm{i}(s + x(v-y))}f(u-x, v-y).
$$

(5.23)

The representation ρ_m is unitary irreducible on $L_2(\mathbb{T}^2)$ and is called the *lattice* representation [10].

5.2   Stone–von Neumann Theorem

So far we have constructed several families of strongly continuous unitary infinite-dimensional irreducible representations of the Heisenberg group $\mathbb{H}_p^1$ parametrised by $\hbar \in \mathbb{R} \backslash \{0\}$. The theorem of Stone–von Neumann holds that any two irreducible unitary representations of $\mathbb{H}_p^1$ with the same $\hbar$ are equivalent.

Theorem 5.2 (Stone–von Neumann). [19, Ch.1,§5] *Let ρ be a strongly continuous unitary representation of $\mathbb{H}^n$ on a Hilbert space H, such that $\rho(s,0,0) = e^{2\mathrm{i}\pi\hbar s}I$ for a non-zero real $\hbar$. Then $H = \oplus H_\alpha$, where the H_α's are mutually orthogonal subspaces of H, each invariant under ρ, such that that the restriction $\rho|_{H_\alpha}$ is unitarily equivalent to the Schrödinger representation $\rho_\hbar$ for each α. In particular, if ρ is irreducible then ρ is equivalent to $\rho_\hbar$.*

5.3 Ladder Operators

If ρ is a representation of a Lie group G acting on a Hilbert space $\mathcal{H}$, it is useful to pass to its derived representations $\mathrm{d}\rho$ of the Lie algebra $\mathfrak{g}$ (cf. [25, Ch.2]). For the Heisenberg group $\mathbb{H}_p^1$, the important operators produced by derived representations are called *ladder operators*. The historic origin of ladder operators is the study of spectrum of quantum harmonic oscillator (cf. [9, § II.3]). Thereafter, mathematicians realised that these operators are also useful for the representations theory of the Heisenberg group (cf. [25, Ch.2]). Our main references are [9, §II.3; 25, §2]. Additional useful sources are [17; 19, Ch.1; 20, § 2.5; 21, § 2.2; 38, § 1.3].

To begin, let ρ be a representation of $\mathbb{H}_p^1$ on a Hilbert space $\mathcal{H}$. Let

$$S = (1, 0, 0), \quad X = (0, 1, 0), \quad Y = (0, 0, 1) \tag{5.24}$$

be the basis (5.24) of the Lie algebra $\mathfrak{h}_1$ of the Heisenberg group $\mathbb{H}_p^1$. Consider the derived representations of $\mathfrak{h}_1$:

$$\mathrm{d}\rho^S = \frac{\mathrm{d}}{\mathrm{d}t}\rho(\mathrm{e}^{tS})|_{t=0}, \quad \mathrm{d}\rho^X = \frac{\mathrm{d}}{\mathrm{d}t}\rho(\mathrm{e}^{tX})|_{t=0}, \quad \mathrm{d}\rho^Y = \frac{\mathrm{d}}{\mathrm{d}t}\rho(\mathrm{e}^{tY})|_{t=0}. \tag{5.25}$$

To simplify the above expressions, we denote $\tilde{S} = \mathrm{d}\rho^S$, $\tilde{X} = \mathrm{d}\rho^X$ and $\tilde{Y} = \mathrm{d}\rho^Y$. The commutation relation between $\tilde{X}$ and $\tilde{Y}$ is

$$[\tilde{X}, \tilde{Y}] = \tilde{S}.$$

The set $\{\tilde{X}, \tilde{Y}, \tilde{S}\}$ spans representations of the Lie algebra $\mathfrak{h}_1$ of the Heisenberg group $\mathbb{H}_p^1$. If ρ is irreducible, $\tilde{S}$ is a multiple of the identity operator I, that is, $\tilde{S} = -\mathrm{i}\hbar I$, where $\hbar > 0$ a positive real number (cf. [9, § II.3; 25, §2]). Now, we provide the definition of the ladder operators.

Definition 5.3. [9, § II.3] Let $\kappa > 0$ be some fixed number. In the above notations, the ladder operators a^+, a^- are defined as follows:

$$a^- = \frac{1}{\sqrt{2\hbar\kappa}}(\kappa(\mathrm{i}\tilde{X}) + \mathrm{i}(\mathrm{i}\tilde{Y})), \quad a^+ = \frac{1}{\sqrt{2\hbar\kappa}}(\kappa(\mathrm{i}\tilde{X}) - \mathrm{i}(\mathrm{i}\tilde{Y})), \quad \hbar > 0, \tag{5.26}$$

with the commutator

$$[a^-, a^+] = a^- a^+ - a^+ a^- = I. \tag{5.27}$$

The operator a^+ (respectively, a^-) is known as the *creation* (respectively, *annihilation*) operator. These names are borrowed from quantum mechanics.

In this thesis, we fix the parameter $\kappa > 0$ of the ladder operators $a^{\pm}$ [5]. Since ρ is unitary, $\tilde{X}$ and $\tilde{Y}$ are *skew-adjoint*, that is, $\tilde{X}^* = -\tilde{X}$ and $\tilde{Y}^* = -\tilde{Y}$. Thus, $i\tilde{X}$ and $i\tilde{Y}$ are *self-adjoint*. Consequently, the creation and annihilation operators are *adjoint* of each other, that is, $(a^-)^* = a^+$ on $\mathcal{H}$.

5.4   Number Operator and Ladder Operators

Let ρ be a unitary irreducible representation of the Heisenberg group $\mathbb{H}_p^1$ on a Hilbert space $\mathcal{H}$. Consider the creation and annihilation operators a^+ and a^- (5.26) of ρ of the Lie algebra $\mathfrak{h}_1$ of the Heisenberg group, respectively. We define the *number operator* N to be $N = a^+a^-$ [9, § II.3]. As a consequence of (5.27), there are commutation relations between N and $a^{\pm}$ given by:

$$[N, a^-] = -a^-, \tag{5.28}$$

$$[N, a^+] = a^+. \tag{5.29}$$

Due to these commutators, the spectral decomposition [17] of the operator N is rooted in the following notion.

Definition 5.4. [25, Ch.2, §2.6] In the above notations, a vector $\phi_0 \in \mathcal{H}$ is called a *vacuum vector* if it is a null solution of the annihilation operator (i.e., $a^-\phi_0 = 0$).

The main properties of the number operator N and the ladder operators $a^{\pm}$ follow from the commutation relations of (5.28) and (5.29).

i. If ϕ_0 is a vacuum of the irreducible representation ρ, then, for a fixed κ, ϕ_0 is unique up to scalar multiplication (see [25, Ch.2, §2.6]).

ii. For $n \geq 0$, the vectors

$$\phi_n = \frac{1}{\sqrt{n!}}(a^+)^n \phi_0 \tag{5.30}$$

are the *eigenvectors* of N with the *eigenvalues* n, that is, $N\phi_n = n\phi_n$. Furthermore, we have

$$a^+\phi_n = \sqrt{n+1}\,\phi_{n+1}, \quad a^-\phi_n = \sqrt{n}\,\phi_{n-1}. \tag{5.31}$$

If ρ is the Schrödinger representation $\rho_\hbar$, the vectors in (5.30) are the celebrated *Hermite functions* (see [25, Ch.2, §2.6]).

    iii. The vectors $\{\phi_n\}_{n \geq 0}$ (5.30) form an orthonormal basis. In fact, assume a Hilbert subspace H_1 of $\mathcal{H}$ spanned by $\{\phi_n\}_{n \geq 0}$. By (5.31), the subspace H_1 is invariant under both operators $a^{\pm}$. Since ρ is irreducible, H_1 must coincide with the whole space (see [25, Ch.2, §2.6]).

The name of the ladder operators $a^{\pm}$ is explained by the following diagram, which visualises relations (5.31):

$$0 \xleftarrow[a^-]{} \phi_0 \underset{a^-}{\overset{a^+}{\rightleftarrows}} \phi_1 \underset{a^-}{\overset{a^+}{\rightleftarrows}} \phi_2 \underset{a^-}{\overset{a^+}{\rightleftarrows}} \phi_3 \underset{a^-}{\overset{a^+}{\rightleftarrows}} \cdots$$

Figure 5.2: Ladder operators

    The following subsections are devoted to calculating the vacuums for the left quasi-regular, the Schrödinger and the lattice representations of $\mathbb{H}_p^1$.

    The Gaussian function is a crucial element in the theory of the Heisenberg group, and we will repeatedly use its properties [17]. The main feature of this function is that the Gaussian represents vacuums for the representations of $\mathbb{H}_p^1$, which will be discussed in detail in Subsections 5.4.1, 5.4.2 and 5.4.3.

5.4.1 Vacuum of the Left Quasi-Regular Representation

In this subsection, we look for vacuums of the left quasi-regular representation (5.14):

$$[\Lambda_\hbar(s, x, y)f](x', y') = e^{2\pi i \hbar(s + x(y' - y))} f(x' - x, y' - y). \tag{5.32}$$

To begin, we calculate the following derived representations. Let $h = 2\pi\hbar > 0$. The derived representations of $\Lambda_\hbar$ of $\mathfrak{h}_1$ are

$$d\Lambda_\hbar^X = 2\pi i \hbar y - \partial_x, \quad d\Lambda_\hbar^Y = -\partial_y,$$

and the annihilation operator $a_{\Lambda_\hbar}^-$ is

$$a_{\Lambda_\hbar}^- = d\Lambda_\hbar^{\kappa X - iY} = 2\pi i \hbar \kappa y - (\kappa \partial_x - i \partial_y). \tag{5.33}$$

The ladder operators $a_{\Lambda_\hbar}^{\pm}$ act on the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ of smooth rapidly decreasing functions, which is dense in $L_2(\mathbb{R}^2)$ (see [25, § 2.3]).

It is useful to write the above differential operators in terms of a complex variable z [33]. Let us consider $z = \sqrt{\frac{h}{2\kappa}}(x + i\kappa y)$, $x, y \in \mathbb{R}$. Then, we have

$$\partial_z = \frac{1}{\sqrt{2h\kappa}}(\kappa\partial_x - i\partial_y), \quad \partial_{\bar{z}} = \frac{1}{\sqrt{2h\kappa}}(\kappa\partial_x + i\partial_y). \tag{5.34}$$

To calculate the vacuums of $\Lambda_\hbar$ annihilated by $a_{\Lambda_\hbar}^-$, we solve the partial differential equation $a_{\Lambda_\hbar}^- \phi = 0$ (5.33), for a fixed $\kappa > 0$, using the method of characteristics [1]. The solutions are the Gaussian-type functions

$$\begin{aligned}
\phi_{\hbar\kappa}(z, \bar{z}) &= e^{-\psi(\bar{z})} \cdot e^{\frac{1}{4}(z-\bar{z})^2} \\
&= e^{-\psi(x-i\kappa y)}e^{-\frac{h\kappa}{2}y^2} \in \mathcal{S}(\mathbb{R}^2),
\end{aligned} \tag{5.35}$$

where ψ is an arbitrary smooth function depends on $\bar{z}$. Therefore, we obtain infinitely many vacuums up to scalars annihilated by $a_{\Lambda_\hbar}^-$.

In particular, there is a special vacuum of (5.35) such that it is annihilated by two operators $a_{\Lambda_\hbar}^-$ and $a_{R_\hbar}^-$, where $a_{R_\hbar}^-$ is the derived representation from the *right regular* action $R_\hbar$ given by:

$$[R_\hbar(g)f](x', y') = e^{-2\pi i\hbar(s+x'y)}f(x'+x, y'+y). \tag{5.36}$$

Therefore, the corresponding derived representations of $\mathfrak{h}_1$ are:

$$dR_\hbar^X = \partial_x, \quad dR_\hbar^Y = -2\pi i\hbar x + \partial_y, \tag{5.37}$$

and the operator $a_{R_\hbar}^- =: \mathcal{L}^{\kappa X + iY}$ is:

$$\mathcal{L}^{\kappa X + iY} = 2\pi\hbar x + (\kappa\partial_x + i\partial_y). \tag{5.38}$$

We call $\mathcal{L}^{\kappa X + iY}$ the *right ladder* operator, which acts on $\mathcal{S}(\mathbb{R}^2)$. Now, the partial differential equation

$$\mathcal{L}^{\kappa X + iY}\phi_{\hbar\kappa} = \sqrt{2h\kappa}\left(\frac{1}{2}(z + \bar{z}) + \partial_{\bar{z}}\right)\phi_{\hbar\kappa} = 0$$

holds if and only if $\psi(\bar{z})$ of the vacuums (5.35) is $\psi(\bar{z}) = \frac{1}{2}\bar{z}^2 + c$, for a constant $c \in \mathbb{C}$. Let $c_0 \in \mathbb{C}$ be a non-zero arbitrary constant. The unique vacuum annihilated by both $a_{\Lambda_\hbar}^-$ and $\mathcal{L}^{\kappa X + iY}$ is as follows:

$$\begin{aligned}
\phi_{0,\hbar\kappa}(z, \bar{z}) &= c_0\, e^{-\frac{1}{2}\bar{z}^2 + \frac{1}{4}(z-\bar{z})^2 + c} \\
&= c_0\, e^{\frac{1}{4}(z^2 - \bar{z}^2 - 2z\bar{z}) + c}.
\end{aligned} \tag{5.39}$$

The special rôle of this vacuum will be revealed in Subsections 6.4.1 and 6.5.1.

5.4.2   Vacuum of the Schrödinger Representation

To calculate the vacuum of the Schrödinger representation, we follow a similar method to the one listed Subsection 5.4.1. The formula of the Schrödinger representation (5.17) is

$$[\rho_\hbar(s,x,y)f](t) = e^{2\pi i\hbar(s-ty)} f(t-x). \tag{5.40}$$

The derived representations of $\rho_\hbar$ of $\mathfrak{h}_1$ are

$$d\rho_\hbar^X = -\partial_x, \quad d\rho_\hbar^Y = -2\pi i\hbar x,$$

and the annihilation operator $a_{\rho_\hbar}^-$ is:

$$a_{\rho_\hbar}^- = d\rho_\hbar^{\kappa X - iY} = -2\pi\hbar t - \kappa\partial_t. \tag{5.41}$$

The operators $a_{\rho_\hbar}^\pm$ act on $\mathcal{S}(\mathbb{R}) \subset L_2(\mathbb{R})$ (cf. [25, Ch.2, §2.3]). After simple calculation for $a_{\rho_\hbar}^- \phi_{\hbar\kappa} = 0$, for a fixed $\kappa > 0$, we conclude that the unique vacuum $\phi_{\hbar\kappa}$ up to scalars is the Gaussian function [17]

$$\phi_{\hbar\kappa}(t) = e^{-\frac{\pi\hbar}{\kappa}t^2} \in \mathcal{S}(\mathbb{R}) \subset L_2(\mathbb{R}). \tag{5.42}$$

5.4.3   Vacuum of the Lattice Representation

An interesting observation is that the vacuum for the lattice representation is to be the well-known theta function (up to a scalar multiplication) (4.45). To begin, the derived representations of ρ_m of $\mathfrak{h}_1$ are

$$d\rho_m^X = 2\pi imy - \partial_x, \quad d\rho_m^Y = -\partial_y,$$

and the annihilation operator $a_{\rho_m}^-$ of the lattice representation (5.23) is

$$a_{\rho_m}^- = d\rho_m^{\kappa X - iY} = 2\pi im\kappa y - (\kappa\partial_x - i\partial_y). \tag{5.43}$$

The ladder operators $a_{\rho_m}^\pm$ act on $\mathcal{S}(\mathbb{T}^2) \subset L_2(\mathbb{T}^2)$ [25, Ch.2, §2.3].

Let $h = 2\pi m$. For $\omega = \sqrt{\frac{h}{2\kappa}}(\kappa y + ix)$ and $\omega' = \sqrt{\frac{h}{2\kappa}}(\kappa v + iu)$ of $\mathbb{C}$, the formula of the lattice representation

$$[\rho_m(s,x,y)f](u,v) = e^{2\pi mi(s+x(v-y))} f(u-x,v-y) \tag{5.44}$$

in terms of the variables ω and ω' after simplification is as follows:

$$[\rho_m(s,\omega)f](\omega',\bar{\omega}') := \mathrm{e}^{2\pi m \mathrm{i}s}\, \mathrm{e}^{\frac{1}{2}(\omega-\bar{\omega})(\omega'+\bar{\omega}')+\frac{1}{2}(\bar{\omega}^2-\omega^2)} f(\omega'-\omega,\bar{\omega}'-\bar{\omega}). \qquad (5.45)$$

The annihilation operator $a^-_{\rho_m}$ (5.43) in terms of the variables ω and ω' is

$$\begin{aligned}
a^-_{\rho_m} &= \mathrm{d}\rho_m^{\kappa X - \mathrm{i}Y} = 2\pi \mathrm{i}m\kappa y - (\kappa\partial_x - \mathrm{i}\partial_y) \\
&= \sqrt{2h\kappa}\,\, \mathrm{i}\,\left(\frac{1}{2}(\omega+\bar{\omega}) + \partial_{\bar{\omega}}\right).
\end{aligned} \qquad (5.46)$$

One can directly check that, for a fixed $\kappa > 0$, up to scalars, the function

$$\begin{aligned}
\Phi_{m\kappa}(u,v) &= \mathrm{e}^{-\frac{\pi m}{\kappa}(u^2-2\mathrm{i}\kappa uv)}\Theta_{m\kappa}\left(\sqrt{\frac{h}{2\kappa}}((\kappa v+\mathrm{i}u),\mathrm{i})\right) \\
&= \mathrm{e}^{\frac{1}{4}(3\omega^2-\bar{\omega}^2-2\omega\bar{\omega})}\,\Theta_{m\kappa}(\omega,\mathrm{i}) \\
&= \Phi_{m\kappa}(\omega,\bar{\omega})
\end{aligned} \qquad (5.47)$$

is a null solution of $a^-_{\rho_m}$ (5.46), where $\Theta_{m\kappa}(\omega,\mathrm{i})$ is the analytic theta function (4.5) in $\omega = \sqrt{\frac{h}{2\kappa}}(\kappa y + \mathrm{i}x) \in \mathbb{C}$. We will present our calculation of $\Phi_{m\kappa}(\omega,\bar{\omega})$ (5.47) in Remark 6.42 after introducing the *covariant transform*.

Chapter 6

Covariant Transform and Analyticity

In this chapter, we present our constructions based on group representations of:

i. the pre-Fock–Segal–Bargamann (FSB) transform and its inverse;

ii. the Zak transform and its inverse;

iii. the theta function;

iv. the pre-theta transform and its inverse; and

v. the Fock-Segal-Bargmann (FSB) space of analytic functions.

To perform these, we use the *covariant transform*. The original contributions are the interpretations of the *Zak transform* and the *pre-theta transform* with their inversions through the covariant transform. Thereafter, the theta function is completely defined in terms of group representations.

Much of the the materials in this chapter are scattered through the literature, but our presentation is more systematic in the usage of group representations. The main sources for this chapter are [4, Ch.8; 9; 17; 25; 29; 31; 32].

6.1   Covariant Transform

Consider a representation ρ of a group G on a Hilbert space $\mathcal{H}$. A map $\mathcal{W}$ from $\mathcal{H} \otimes \mathcal{H}$ to a space $L(G)$ of functions over G is defined by a *matrix coefficient* [29] via a pair of

vectors $f, \phi \in \mathcal{H}$ as follows:

$$\mathcal{W}(f, \phi)(g) = \langle f, \rho(g)\phi \rangle, \quad g \in G. \tag{6.1}$$

If ρ is unitary, we obtain the following identity:

$$\mathcal{W}(f, \phi)(g) = \langle \rho(g^{-1})f, \phi \rangle = \langle f, \rho(g)\phi \rangle, \quad g \in G. \tag{6.2}$$

Moreover, if ρ is a strongly continuous unitary representation of a Lie group G on $\mathcal{H}$, it is clear from the Schwartz's inequality that $\mathcal{W}(u, v)$ is a continuous bounded function on G. Providing a more detailed description of the space $L(G)$ is an important and challenging task of harmonic analysis (see [13, 18]). If a non-zero vector ϕ is fixed, we have a map from $\mathcal{H}$ to $L(G)$. In this thesis, the non-zero fixed vector ϕ is called a *fiducial vector* (is also known as *aka vacuum vector, ground state, mother wavelet,* etc.).

Definition 6.1. [29] Let ρ be a representation of a group G on a Hilbert space $\mathcal{H}$. If $\phi \in \mathcal{H}$ is a fiducial vector, the linear map $\mathcal{W}_\phi : \mathcal{H} \to L(G)$:

$$\begin{aligned} \mathcal{W}_\phi : f &\mapsto \mathcal{W}(f, \phi) \\ &= \langle f, \rho(g)\phi \rangle, \quad g \in G \end{aligned} \tag{6.3}$$

is called the covariant transform.

The main property of the covariant transform is given in Proposition 6.2.

Proposition 6.2. [29] *Let ρ be a unitary representation of a group G on a Hilbert space $\mathcal{H}$. The covariant transform $\mathcal{W}_\phi$ intertwines the representation ρ on $\mathcal{H}$ and the left regular action Λ of G on $L(G)$:*

$$\mathcal{W}_\phi(\rho(g)f) = \Lambda(g) \circ \mathcal{W}_\phi(f) \quad \forall g \in G, \tag{6.4}$$

where $\Lambda(g) : f(g') \mapsto f(g^{-1}g')$, $g, g' \in G$ (5.1).

Proof. We have that

$$\begin{aligned} [\mathcal{W}_\phi(\rho(g)f)](g') &= \langle \rho(g)f, \rho(g')\phi \rangle \\ &= \langle \rho(g'^{-1})\rho(g)f, \phi \rangle \\ &= \langle \rho(g^{-1}g')^{-1}f, \phi \rangle \\ &= \langle f, \rho(g^{-1}g')\phi \rangle \\ &= [\Lambda(g) \circ \mathcal{W}_\phi(f)](g'). \end{aligned} \tag{6.5}$$

$\square$

To discover some preferable fiducial vectors, we use a particular case of a result from [29, § 5]. Let G be a Lie group and ρ be its representation in a Hilbert space $\mathcal{H}$. Let $[\mathcal{W}_\phi f](g) = \langle f, \rho(g)\phi \rangle$ be the covariant transform defined by a fiducial vector $\phi \in H$. Then, the covariant transform intertwines right shifts on the group G with the associated action ρ on fiducial vectors:

$$R(g) \circ \mathcal{W}_\phi = \mathcal{W}_{\rho(g)\phi}. \tag{6.6}$$

There are many interesting applications of this simple observation [5, 6, 29, 30, 32].

Proposition 6.3. *Let G be a Lie group with a Lie algebra $\mathfrak{g}$ and ρ be a representation of G on a Hilbert space $L_2(\mathbb{R}^n)$. We denote the derived representation of ρ by $\mathrm{d}\rho^X$. Let ϕ be a fiducial vector in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ such that $(\sum_{j=1}^n a_j \mathrm{d}\rho^{X_j})\phi = 0$, for some $a_j \in \mathbb{C}$. Then, the covariant transform $\mathcal{W}_\phi$ obeys the condition:*

$$\left(\sum_{j=1}^n \bar{a}_j \mathrm{d}R^{X_j} \right) \tilde{f} = 0, \quad \tilde{f} = \mathcal{W}_\phi(f), \text{ for any } f \in L_2(\mathbb{R}^n), \tag{6.7}$$

with $\mathrm{d}R$ being the derived form of the right regular representation of G.

Proof. Let $f \in L_2(\mathbb{R}^n)$. Since $R(g) \circ \mathcal{W}_\phi(f) = \mathcal{W}_{\rho(g)\phi}(f)$ (6.6), then

$$\left(\sum_{j=1}^n \bar{a}_j \mathrm{d}R^{X_j} \right) \circ \mathcal{W}_\phi(f) = \mathcal{W}_{(\sum_{j=1}^n a_j \mathrm{d}\rho^{X_j}(g))\phi}(f). \tag{6.8}$$

As $(\sum_{j=1}^n a_j \mathrm{d}\rho^{X_j})\phi = 0$, we thus have $\left(\sum_{j=1}^n \bar{a}_j \mathrm{d}R^{X_j} \right) \tilde{f} = 0$, which proves the statement. $\square$

Consider the Heisenberg group $G = \mathbb{H}_p^1$. Let $\rho_\hbar$ be the Schrödinger representation. For $f, \phi \in L_2(\mathbb{R})$, the corresponding matrix coefficient of $\rho_\hbar$ at (f, ϕ) is:

$$\begin{aligned} \mathcal{W}(f, \phi)(x, y) &= \langle f, \rho_\hbar(0, x, y)\phi \rangle \\ &= \int_{\mathbb{R}} e^{2\pi i \hbar y t} f(t) \bar{\phi}(t - x) \, \mathrm{d}t. \end{aligned} \tag{6.9}$$

The map $\mathcal{W}(f, \phi)$ (6.9) is called the *Fourier-Wigner transform* [19, Ch.1, §4].

Proposition 6.4. [19, Ch.1, §4] *Let $\rho_\hbar$ be the Schrödinger representation of $\mathbb{H}_p^1$. The Fourier-Wigner transform $\mathcal{W}(f, \phi)(x, y) = \langle f, \rho_\hbar(0, x, y)\phi \rangle$ is a linear map of the spaces*

$$\begin{aligned} \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R}) &\to \mathcal{S}(\mathbb{R}^2), \\ L_2(\mathbb{R}) \times L_2(\mathbb{R}) &\to L_2(\mathbb{R}^2). \end{aligned} \tag{6.10}$$

Moreover, $\mathcal{W} : L_2(\mathbb{R}) \times L_2(\mathbb{R}) \to L_2(\mathbb{R}^2)$ is 'sesqui-unitary'; that is, for all f_1, g_1, f_2, g_2 of $L_2(\mathbb{R})$, we have

$$\langle \mathcal{W}(f_1, \phi_1), \mathcal{W}(f_2, \phi_2) \rangle_{L_2(\mathbb{R}^2)} = \langle f_1, f_2 \rangle_{L_2(\mathbb{R})} \overline{\langle \phi_1, \phi_2 \rangle}_{L_2(\mathbb{R})}. \tag{6.11}$$

The sesqui-unitary property (6.11) of the Fourier-Wigner transform can be generalised for any equivalent representation ρ of $\mathbb{H}_p^1$ as follows.

Proposition 6.5. *Let ρ be a strongly continuous unitary irreducible representation of the Heisenberg group $\mathbb{H}_p^1$ on a Hilbert space $\mathcal{H}$. Then, the matrix coefficient $\mathcal{W}(f, \phi)(x, y) = \langle f, \rho(0, x, y)\phi \rangle$ (6.2) is a linear map of the spaces*

$$\mathcal{H} \times \mathcal{H} \to L_2(\mathbb{R}^2). \tag{6.12}$$

Moreover, for all $f_1, \phi_1, f_2, \phi_2 \in \mathcal{H}$, the matrix coefficient $\mathcal{W}$ (6.2) is sesqui-unitary on $\mathcal{H} \times \mathcal{H} \to L_2(\mathbb{R}^2)$, that is,

$$\langle \mathcal{W}(f_1, \phi_1), \mathcal{W}(f_2, \phi_2) \rangle_{L_2(\mathbb{R}^2)} = \langle f_1, f_2 \rangle_{\mathcal{H}} \overline{\langle \phi_1, \phi_2 \rangle}_{\mathcal{H}}. \tag{6.13}$$

Proof. By the Stone–von Neumann Theorem 5.2, two unitary irreducible representations of $\mathbb{H}_p^1$, for the same Planck constant $\hbar$, are equivalent. Thus, there is a unitary operator $U : \mathcal{H} \to L_2(\mathbb{R})$ such that it intertwines the Schrödinger representation $\rho_\hbar$ and the representation ρ, for the same Planck constant $\hbar$, that is, $\rho = U^{-1}\rho_\hbar U$. Then, for $f, \phi \in \mathcal{H}$, we have

$$\begin{aligned}
\mathcal{W}(f, \phi)(x, y) = \langle f, \rho(0, x, y)\phi \rangle &= \langle f, U^{-1}\rho_\hbar(0, x, y) U\phi \rangle \\
&= \langle Uf, \rho_\hbar(0, x, y) U\phi \rangle \\
&= \langle \tilde{f}, \rho_\hbar(0, x, y) \tilde{\phi} \rangle = \mathcal{W}_1(\tilde{f}, \tilde{\phi})(x, y).
\end{aligned} \tag{6.14}$$

By Proposition 6.4, the Fourier-Wigner transform $\mathcal{W}_1$ is a linear map of the spaces of $L_2(\mathbb{R}) \times L_2(\mathbb{R}) \to L_2(\mathbb{R}^2)$ (6.10) and sesqui-unitary. Thus, according to the identity (6.14), the matrix coefficient $\mathcal{W}(f, \phi)$ of the equivalent representation ρ is a linear map of $\mathcal{H} \times \mathcal{H} \to L_2(\mathbb{R}^2)$ and sesqui-unitary. $\qquad\square$

The sesqui-unitary property (6.14) for a unitary irreducible representation ρ of $\mathbb{H}_p^1$ is known as *square integrability modulo the centre Z* [19, Ch.1,§4].

6.2 Induced Covariant Transform on $\mathbb{H}_p^1$

In this section, we study the covariant transform that produces functions on a homogeneous space rather than the entire group.

Definition 6.6. [29] Let ρ be a representation of $\mathbb{H}_p^1$ on a Hilbert space $\mathcal{H}$ and H be a closed subgroup of $\mathbb{H}_p^1$. Let $X = \mathbb{H}_p^1/H$ be a homogeneous space. Let $\phi_0 \in \mathcal{H}$ be a fiducial vector such that

$$\rho(h)\phi_0 = \chi(h)\phi_0, \quad \text{for all } h \in H, \tag{6.15}$$

for some character χ of H. The *induced covariant transform* $\mathcal{W}_{\phi_0}^{\rho}$ is a map from the Hilbert space $\mathcal{H}$ to a space $W(X)$ of functions on $X = \mathbb{H}_p^1/H$ given as follows:

$$\begin{aligned}
\mathcal{W}_{\phi_0}^{\rho} : f \mapsto &\tilde{f}(\mathsf{s}(x)) \\
&= \langle f, \rho(\mathsf{s}(x))\phi_0 \rangle, \qquad x \in X,
\end{aligned} \tag{6.16}$$

where s is a continuous section from X to $\mathbb{H}_p^1$.

Note that the map $f \mapsto \tilde{f}(\mathsf{s}(x))$ (6.16) intertwines ρ on $\mathcal{H}$ with a representation ρ_χ on $W(X)$ induced by the character χ of the subgroup H. Indeed,

$$\begin{aligned}
\rho_\chi(g) \circ [\mathcal{W}_{\phi_0}^{\rho}(f)](x) &= \bar{\chi}(\mathsf{r}(g^{-1} * \mathsf{s}(x)))[\mathcal{W}_{\phi_0}^{\rho}(f))](g^{-1}.x) \\
&= \bar{\chi}(\mathsf{r}(g^{-1} * \mathsf{s}(x))) \langle f, \rho(\mathsf{s}(g^{-1}.x))\phi_0 \rangle \\
&= \langle f, \rho(\mathsf{s}(g^{-1}.x)) \chi(\mathsf{r}(g^{-1} * \mathsf{s}(x)))\phi_0 \rangle \\
&= \langle f, \rho(\mathsf{s}(g^{-1}.x)) \rho(\mathsf{r}(g^{-1} * \mathsf{s}(x)))\phi_0 \rangle \\
&= \langle f, \rho(\mathsf{s}(g^{-1}.x)\, \mathsf{r}(g^{-1} * \mathsf{s}(x)))\phi_0 \rangle.
\end{aligned} \tag{6.17}$$

The last action coincides with the induced representation (5.9). Alternatively, this can be seen from the fact that any function of the image of the induced covariant transform (6.16) has the H-covariance property $\tilde{f}(gh) = \bar{\chi}(h)\tilde{f}(g)$ [29]. Indeed, for all $g \in G$ and $h \in H$, since $\rho(h)\phi_0 = \chi(h)\phi_0$ (6.15), we have

$$\begin{aligned}
\tilde{f}(gh) &= \langle f, \rho(gh)\phi_0 \rangle \\
&= \langle f, \rho(g)\rho(h)\phi_0 \rangle \\
&= \langle f, \rho(g)\chi(h)\phi_0 \rangle \\
&= \bar{\chi}(h) \langle f, \rho(g)\phi_0 \rangle \\
&= \bar{\chi}(h)\tilde{f}(g).
\end{aligned} \tag{6.18}$$

In Section 5.1.1, we described three forms of the representations of $\mathbb{H}_p^1$: the left quasi-regular, the Schrödinger and the lattice representations. By the Stone–von Neumann Theorem 5.2, these representations for the same Plank constant $\hbar$ shall be intertwined by unitary operators. The exact form of intertwining operators can be found by the induced covariant transform. The following subsections will provide the all three possible intertwining operators:

    i. between the Schrödinger representation and the left quasi-regular representation;

    ii. between the Schrödinger representation and the lattice representation; and

    iii. between the left quasi-regular representation and the lattice representation.

6.2.1   The (Pre-) Fock-Segal-Bargmann Transform

Let $[\rho_\hbar(s, x, y)f](t) = \mathrm{e}^{2\pi\mathrm{i}\hbar(s-ty)} f(t - x)$ be the Schrödinger representation (5.17). We look for the induced covariant transform $\mathcal{W}_\phi^{\rho_\hbar} : L_2(\mathbb{R}) \to L_2(\mathbb{R}^2)$, which intertwines the Schrödinger representation and the left quasi-regular representation restricted to an irreducible component of $L_2(\mathbb{R}^2)$. In fact, for the character $\chi_\hbar(s, 0, 0) = \mathrm{e}^{2\pi\mathrm{i}\hbar s}$ of the centre $Z = \{(s, 0, 0) \in \mathbb{H}_p^1 : \; s \in \mathbb{R}\}$, any vector $\phi \in L_2(\mathbb{R})$ satisfies (6.15)

$$\rho_\hbar(s, 0, 0)\phi = \chi_\hbar(s, 0, 0)\phi, \quad \text{for all } (s, 0, 0) \in Z.$$

Thus, for all $f \in L_2(\mathbb{R})$, the induced covariant transform $\mathcal{W}_\phi^{\rho_\hbar}$ for any fiducial vector $\phi \in L_2(\mathbb{R})$ is:

$$[\mathcal{W}_\phi^{\rho_\hbar}(f)](x, y) = \int_\mathbb{R} f(t) \, \mathrm{e}^{2\pi\mathrm{i}\hbar ty} \, \phi(t - x) \, \mathrm{d}t. \tag{6.19}$$

The main properties of $\mathcal{W}_\phi^{\rho_\hbar}$ follow from the general properties of the covariant transform.

Corollary 6.7. *Let $\phi \in L_2(\mathbb{R})$ be a fiducial vector such that $\|\phi\| = 1$. The covariant transform $\mathcal{W}_\phi^{\rho_\hbar} : L_2(\mathbb{R}) \to L_2(\mathbb{R}^2)$ is a unitary intertwining operator between the Schrödinger representation $\rho_\hbar$ on $L_2(\mathbb{R})$ and the left quasi-regular representation $\Lambda_\hbar$ restricted on the image space*

$$F_\phi(\mathbb{R}^2) := \{\mathcal{W}_\phi^{\rho_\hbar}(f) : f \in L_2(\mathbb{R})\}. \tag{6.20}$$

In particular, $\Lambda_\hbar$ is an irreducible representation on $F_\phi(\mathbb{R}^2)$.

Since, by Proposition 6.3, there is a special component which is annihilated by the right ladder $\mathcal{L}^{\kappa X + \mathrm{i}Y}$ (5.38), we specify ϕ to the corresponding fiducial vector. That is, we consider the vacuum of the Schrödinger representation (up to normalisation) $\phi_{\hbar\kappa}(t) = 2^{1/4}\mathrm{e}^{-\frac{\pi\hbar}{\kappa}t^2}$ (5.42). The corresponding induced covariant transform (6.3) with the measure renormalised by the factor $(\frac{\hbar}{\kappa})^{1/2}$ is given by:

$$
\begin{aligned}
\tilde{f}(x, y) := [\mathcal{W}^{\rho_\hbar}_{\phi_{\hbar\kappa}}(f)](x, y) &= \langle f, \rho_\hbar(0, x, y)\phi_{\hbar\kappa}\rangle \\
&= \left(\frac{\hbar}{\kappa}\right)^{1/2} 2^{1/4} \int_{\mathbb{R}} f(t)\,\mathrm{e}^{2\pi\mathrm{i}\hbar ty}\,\mathrm{e}^{-\frac{\pi\hbar}{\kappa}(t-x)^2}\,\mathrm{d}t.
\end{aligned}
\tag{6.21}
$$

This is called the *pre-FSB transform* [35, §4.2] from $L_2(\mathbb{R})$ into

$$
F_{\phi_{\hbar\kappa}}(\mathbb{R}^2) = \{\tilde{f} : \tilde{f} = \mathcal{W}^{\rho_\hbar}_{\phi_{\hbar\kappa}}(f),\ f \in L_2(\mathbb{R})\}.
\tag{6.22}
$$

The image space $F_{\phi_{\hbar\kappa}}(\mathbb{R}^2)$ is a subspace of square integrable functions on $\mathbb{R}^2$, called the *pre-FSB space*. The left quasi-regular representation $\Lambda_\hbar$ restricted on the pre-FSB space $F_{\phi_{\hbar\kappa}}(\mathbb{R}^2)$ is called the *pre-FSB representation*. The prefix "pre-" is removed by a unitary operator—the *peeling*, which will produce the *FSB space* of analytic functions on $\mathbb{C}$ in Subsection 6.5.1.

6.2.2 The Zak Transform

In this subsection, we derive the Zak transform as a special case of the covariant transform. This will allow us to recover results from [8, Ch 9; 10; 19, Ch 4]. To begin, we give the standard definition of the Zak transform and some related properties.

Definition 6.8. [35, §8.1] Let $f \in \mathcal{S}(\mathbb{R})$ be a function of the Schwartz space of smooth rapidly decreasing functions on $\mathbb{R}$. For $m \in \mathbb{N}$, the Zak transform on $(u, v) \in \mathbb{R}^2$ is defined as follows:

$$
[\tilde{\mathcal{Z}}f](u, v) = \sum_{n \in \mathbb{Z}} f(u + n)\,\mathrm{e}^{2\pi\mathrm{i}mnv}.
\tag{6.23}
$$

The function $[\tilde{\mathcal{Z}}f](u, v)$ is quasi-periodic in u and periodic in v.

In the most fundamental situation, it is considered $m = 1$ in (6.23) [35, §8.1]. Since $f \in \mathcal{S}(\mathbb{R})$ is a smooth rapidly decreasing function, the Zak transform of f is well-defined through a convergent series. For convenience, we define an associated transform.

Definition 6.9. Let $f \in \mathcal{S}(\mathbb{R})$ be a function of the Schwartz space of smooth rapidly decreasing functions on $\mathbb{R}$. For $m \in \mathbb{N}$, the *co-Zak transform* is defined as follows:

$$[\mathcal{Z}f](u,v) = e^{2\pi i muv}[\tilde{\mathcal{Z}}f](u,v), \tag{6.24}$$

where $[\tilde{\mathcal{Z}}f](u,v)$ is the Zak transform defined as above.

We denote the space of functions $\tilde{f}(u,v)$, which are quasi-periodic in u, periodic in v and square integrable by $\tilde{L}_2(\mathbb{T}^2)$. The factor $e^{2\pi i muv}$ in (6.24) swaps the periodicity property on the variables (u,v) due to the following useful lemma.

Lemma 6.10. *Let $\tilde{f}(u,v) \in \tilde{L}_2(\mathbb{T}^2)$ be a function, which is quasi-periodic in u and periodic in v. Then, $f(u,v) = e^{2\pi i muv}\tilde{f}(u,v) \in L_2(\mathbb{T}^2)$ is periodic in u and quasi-periodic in v. Moreover, the map*

$$\begin{aligned} U : \tilde{f}(u,v) &\mapsto e^{2\pi i muv}\tilde{f}(u,v) \\ &= f(u,v) \end{aligned} \tag{6.25}$$

is a unitary intertwining operator between the lattice representation (5.23)

$$[\rho_m(s,x,y)f](u,v) = e^{2\pi m i(s+x(v-y))}f(u-x,v-y) \tag{6.26}$$

on $L_2(\mathbb{T}^2)$ and the corresponding lattice representation on $\tilde{L}_2(\mathbb{T}^2)$ given by

$$[\tilde{\rho}_m(s,x,y)\tilde{f}] = e^{2\pi m i(s-uy)}f(u-x,v-y). \tag{6.27}$$

Proof. Let $n,k \in \mathbb{Z}$. Since $\tilde{f}(u,v)$ is quasi-periodic in u and periodic in v, then

$$\tilde{f}(u+n,v) = e^{-2\pi m i nv}\tilde{f}(u,v), \quad \text{and} \quad \tilde{f}(u,v+k) = \tilde{f}(u,v). \tag{6.28}$$

Therefore, we have

$$\begin{aligned} f(u+n,v) &= e^{2\pi i m(u+n)v}\tilde{f}(u+n,v) \\ &= e^{2\pi i m(u+n)v}e^{-2\pi i mnv}\tilde{f}(u,v) \\ &= e^{2\pi i muv}\tilde{f}(u,v) \\ &= f(u,v). \end{aligned} \tag{6.29}$$

Thus, $f(u,v)$ is periodic in u. Now,

$$\begin{aligned} f(u,v+k) &= e^{2\pi i mu(v+k)}\tilde{f}(u,v+k) \\ &= e^{2\pi i muk}e^{2\pi i muv}\tilde{f}(u,v) \\ &= e^{2\pi i muk}f(u,v). \end{aligned} \tag{6.30}$$

Hence, $f(u, v)$ is quasi-periodic in v. The map U is obviously a unitary operator:

$$U : \tilde{L}_2(\mathbb{T}^2, \mathrm{d}u\,\mathrm{d}v) \longrightarrow L_2(\mathbb{T}^2, \mathrm{d}u\,\mathrm{d}v). \tag{6.31}$$

Furthermore, U intertwines ρ_m on $L_2(\mathbb{T}^2)$ and the corresponding lattice representation $\tilde{\rho}_m := U^{-1} \circ \rho_m \circ U$ on $\tilde{L}_2(\mathbb{T}^2)$. The formula of $\tilde{\rho}_m := U^{-1} \circ \rho_m \circ U$ after simplification is as follows:

$$\begin{aligned}
[\mathrm{e}^{-2\pi\mathrm{i}muv}\rho_m(s, x, y)\mathrm{e}^{2\pi\mathrm{i}muv}\tilde{f}](u, v) &= \mathrm{e}^{2\pi m\mathrm{i}(s-uy)}\tilde{f}(u - x, v - y) \\
&= [\tilde{\rho}_m(s, x, y)\tilde{f}](u, v),
\end{aligned} \tag{6.32}$$

which proves the statement. $\qquad\square$

Remark 6.11.   i. Since the lattice representation (5.23)

$$[\rho_m(s, x, y)f](u, v) = \mathrm{e}^{2\pi m\mathrm{i}(s+x(v-y))} f(u - x, v - y) \tag{6.33}$$

is irreducible on $L_2(\mathbb{T}^2)$, the vacuum

$$\Phi_{m\kappa}(u, v) = \mathrm{e}^{-\frac{\pi m}{\kappa}(u^2 - 2\mathrm{i}\kappa uv)}\Theta_{m\kappa}(\sqrt{\frac{h}{2\kappa}}((\kappa v + \mathrm{i}u), \mathrm{i})$$

of the representation ρ_m obtained in (5.47) is unique up to scalar multiplication (see Section 5.4 in (iii)).

ii. By Lemma 6.10, if we multiply $\Phi_{m\kappa}(u, v)$ by $\mathrm{e}^{-2\pi\mathrm{i}muv}$, we obtain a function $\mathrm{e}^{-\frac{\pi m}{\kappa}u^2}\Theta_{m\kappa}(\sqrt{\frac{h}{2\kappa}}(\kappa v + \mathrm{i}u), \mathrm{i})$ in the corresponding space $\tilde{L}_2(\mathbb{T}^2)$ that has the same double quasi-periodic property but in the opposite way. Since $U^{-1} : L_2(\mathbb{T}^2) \to \tilde{L}_2(\mathbb{T}^2)$ of (6.25) is a unitary intertwining operator between ρ_m and the corresponding lattice representation $\tilde{\rho}_m$ (6.27), the function

$$\tilde{\Phi}_{m\kappa}(u, v) = \mathrm{e}^{-\frac{\pi m}{\kappa}u^2}\Theta_{m\kappa}(\sqrt{\frac{h}{2\kappa}}(\kappa v + \mathrm{i}u), \mathrm{i}) \tag{6.34}$$

is the corresponding unique vacuum up to scalar multiplication annihilated by

$$a_{\tilde{\rho}_m}^- = \mathrm{d}\tilde{\rho}_m^{\kappa X - \mathrm{i}Y} = -2\pi mu - (\kappa\partial_u - \mathrm{i}\partial_y). \tag{6.35}$$

Let $[\rho_\hbar(s, x, y)f](t) = \mathrm{e}^{2\pi\mathrm{i}\hbar(s-ty)} f(t - x)$ be the Schrödinger representation and χ_m be the character of the non-commutative subgroup $H_d = \{(s, n, k) : \ s \in \mathbb{R}, n, k \in \mathbb{Z}\}$ of $\mathbb{H}_p^1$. To calculate the induced covariant transform

$$\mathcal{W}_{\phi_0}^{\rho_\hbar} : L_2(\mathbb{R}) \to L_2(\mathbb{T}^2), \tag{6.36}$$

we first need a fiducial vector ϕ_0 satisfying (6.15), that is, for all $(s, n, k) \in H_d$, we have

$$e^{2\pi i\hbar s}e^{-2\pi i\hbar tk}\,\phi_0(t-n) = e^{2\pi ims}\phi_0(t). \tag{6.37}$$

The left- and the right-hand sides of (6.37) are equal if and only if

  i. $\hbar = m$; and

  ii. the function ϕ_0 is a periodic function and $supp(\phi_0) \subseteq \mathbb{Z}$. This implies that $supp(\phi_0) = \mathbb{Z}$.

Thus, the only vector ϕ_0 satisfying (6.37) would be the *Dirac comb* distribution, that is, $\phi_0(t) = \sum_{n\in\mathbb{Z}} \delta(t-n)$, which is is a periodic distribution constructed from the *Dirac delta* $\delta(t)$ [11].

Remark 6.12. In Definition 6.1, the covariant transform $\mathcal{W}_{\phi_0}$ is defined for a fiducial vector being a vector of a Hilbert space. In [15, 16, 27, 28], the covariant transform $\mathcal{W}_{\phi_0}$ was treated in Banach spaces. A further generalisation to *Frechet spaces* would allow to use the Dirac comb as a fiducial vector (mother wavelet). However, in this thesis, we do not consider the covariant transform in such generality. Therefore, we will only heuristically proceed with the Dirac comb distribution as a regular function. A posteriori justification of obtained results can be made by direct arguments.

Let $s : \mathbb{T}^2 \to \mathbb{H}_p^1 : (u, v) \mapsto (0, u, v)$ be the continuous section defined in (4.35). For $f \in \mathcal{S}(\mathbb{R}) \subset L_2(\mathbb{R})$, we calculate the induced covariant transform $[\mathcal{W}_{\phi_0}^{\rho_\hbar}(f)](u, v) = \langle f, \rho_\hbar(s(\tilde{x}))\phi_0 \rangle$, $\tilde{x} \in \mathbb{T}^2$, as follows:

$$
\begin{aligned}
[\mathcal{W}_{\phi_0}^{\rho_\hbar}(f)](u, v) &= \langle f, \rho_\hbar(0, u, v)\phi_0 \rangle \\
&= \int_{\mathbb{R}} f(t)\, e^{2\pi i\hbar tv}\, \bar{\phi}_0(t-u)\, \mathrm{d}t \\
&= \int_{\mathbb{R}} f(t)\, e^{2\pi imtv} \sum_{n\in\mathbb{Z}} \delta(t-(u+n))\, \mathrm{d}t, \quad \hbar = m \\
&= \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} f(t)\, e^{2\pi imtv}\delta(t-(u+n))\, \mathrm{d}t \\
&= \sum_{n\in\mathbb{Z}} f(u+n)\, e^{2\pi imv(u+n)} \\
&= e^{2\pi imuv} \sum_{n\in\mathbb{Z}} f(u+n)\, e^{2\pi imvn}.
\end{aligned}
\tag{6.38}
$$

This is the co-Zak transform (6.24), which possesses the following properties.

Corollary 6.13. *For $f \in L_2(\mathbb{R})$, let*

$$[\mathcal{W}_{\phi_0}^{\rho_\hbar}(f)](u,v) = \mathrm{e}^{2\pi i m u v} \sum_{n \in \mathbb{Z}} f(u+n) \, \mathrm{e}^{2\pi i m n v} = [\mathcal{Z}f](u,v)$$

be the co-Zak transform. Then, we have the following properties:

   i. The operator $\mathcal{Z}: L_2(\mathbb{R}) \to L_2(\mathbb{T}^2)$ is unitary.

   ii. The operator $\mathcal{Z}: L_2(\mathbb{R}) \to L_2(\mathbb{T}^2)$ intertwines the Schrödinger representation $\rho_\hbar$ and the lattice representation ρ_m. That is, $\rho_m \circ \mathcal{Z} = \mathcal{Z} \circ \rho_\hbar$, for $\hbar = m$.

   iii. The image space of $[\mathcal{Z}f](u,v)$ consists of functions $\tilde{f}(u,v)$ that have the double-quasi-periodic property on $\mathbb{R}^2$.

The statements i–iii were proved by direct arguments in many references, such as [7, Ch.I; 19, Ch.1,§10]. If a theory of the covariant transform for the Dirac comb distribution were at our disposal, this would imply the following properties of the Zak transform:

   i. the first property is parallel to the sesqui-unitarity (6.13) of the covariant transform. That is, for $f \in L_2(\mathbb{R})$, we have that

$$||\mathcal{Z}f||_{L_2(\mathbb{T}^2)}^2 = ||f||_{L_2(\mathbb{R})}^2, \tag{6.39}$$

   where $|| \cdot ||$ denotes the norm and the subscript indicates the Hilbert space in which the function lies.

   ii. Since $\mathcal{W}_{\phi_0}^{\rho_\hbar} = \mathcal{Z}$, the second property corresponds exactly to the intertwining property in (6.17), i.e. for $\hbar = m$, we have

$$\rho_m \circ \mathcal{W}_{\phi_0}^{\rho_\hbar} = \mathcal{W}_{\phi_0}^{\rho_\hbar} \circ \rho_\hbar. \tag{6.40}$$

   iii. The image space of the induced covariant transform $\tilde{f} = \mathcal{W}_{\phi_0}^{\rho_\hbar}(f)$ has the H-covariance property (6.18) $\tilde{f}(gh) = \bar{\chi}(h)\tilde{f}(g)$, for all $g \in G$ and $h \in H$. For the subgroup $H_d = \{(s,n,k) = (s, n+ik) : (n,k) \in \mathbf{\Gamma}', s \in \mathbb{R}\}$ (4.32), it is exactly the double-quasi-periodic property (5.18):

$$\tilde{f}(u+n, v+k) = \mathrm{e}^{2\pi m i u k} \tilde{f}(u,v). \tag{6.41}$$

This indicates that the covariant transform is worth to be extended for singular fiducial vectors.

Remark 6.14. In Subsection 5.4.3, we showed that $\Phi_{m\kappa}(\omega, \bar{\omega})$ (5.47) is the vacuum of the lattice representation ρ_m. The same vacuum can be obtained from scratch by the following consideration. By Corollary 6.13, the Schrödinger representation $\rho_\hbar$ (5.17) and the lattice representation ρ_m are intertwined by the co-Zak transform $\mathcal{Z} = \mathrm{e}^{2\pi\mathrm{i}muv}\tilde{\mathcal{Z}}$ for the same Planck constant $\hbar = m$. Therefore, $\mathcal{Z} = \mathrm{e}^{2\pi\mathrm{i}muv}\tilde{\mathcal{Z}}$ intertwines the derived representations $a^-_{\rho_\hbar} = \mathrm{d}\rho_\hbar^{\kappa X - \mathrm{i}Y}$ and $a^-_{\rho_m} = \mathrm{d}\rho_m^{\kappa X - \mathrm{i}Y}$:

$$\mathcal{Z} \circ a^-_{\rho_\hbar} = a^-_{\rho_m} \circ \mathcal{Z}. \tag{6.42}$$

Since $\phi_{\hbar\kappa}(t) = \mathrm{e}^{-\frac{\pi\hbar}{\kappa}t^2}$ is the vacuum annihilated by $a^-_{\rho_\hbar}$ (see Subsection 5.4.2), the function $[\mathcal{Z}\phi_{\hbar\kappa}]$ would be annihilated by $a^-_{\rho_m}$ as well. We calculate $[\mathcal{Z}\phi_{\hbar\kappa}]$ as follows (cf. [19, Ch 4]):

$$
\begin{aligned}
\Phi_{m\kappa}(u, v) &:= [\mathcal{Z}\phi_{\hbar\kappa}](u, v) = \mathrm{e}^{2\pi\mathrm{i}muv}[\tilde{\mathcal{Z}}\phi_{\hbar\kappa}](u, v) \\
&= \mathrm{e}^{2\pi\mathrm{i}muv} \sum_{n \in \mathbb{Z}} \phi_{\hbar\kappa}(u + n)\mathrm{e}^{2\pi\mathrm{i}mnv} \\
&= \mathrm{e}^{2\pi\mathrm{i}muv} \sum_{n \in \mathbb{Z}} \mathrm{e}^{-\frac{\pi m}{\kappa}(u+n)^2}\mathrm{e}^{2\pi\mathrm{i}mnv} \\
&= \mathrm{e}^{-\frac{\pi m}{\kappa}(u^2 - 2\mathrm{i}\kappa uv)} \sum_{n \in \mathbb{Z}} \mathrm{e}^{-\frac{\pi m}{\kappa}n^2}\, \mathrm{e}^{\frac{2\pi\mathrm{i}m}{\kappa}n(\kappa v + \mathrm{i}u)} \\
&= \mathrm{e}^{\frac{1}{4}(3\omega^2 - \bar{\omega}^2 - 2\omega'\bar{\omega})} \sum_{n \in \mathbb{Z}} \mathrm{e}^{-\frac{\pi m}{\kappa}n^2}\, \mathrm{e}^{2\pi\mathrm{i}n\sqrt{\frac{m}{\pi\kappa}}\omega} \\
&= \mathrm{e}^{\frac{1}{4}(3\omega^2 - \bar{\omega}^2 - 2\omega\bar{\omega})}\, \Theta_{m\kappa}(\omega, \mathrm{i}) \\
&=: \Phi_{m\kappa}(\omega, \bar{\omega}),
\end{aligned}
\tag{6.43}
$$

where $\omega = \sqrt{\frac{h}{2\kappa}}(\kappa v + \mathrm{i}u) \in \mathbb{C}$, $h = 2\pi m$ and $\Theta_{m\kappa}$ is the theta function (see Section 4.7).

6.2.3 The (Pre-) Theta Transform

In the present subsection, we look for an intertwining operator

$$\mathcal{W}_\Theta^{\rho_m} : L_2(\mathbb{T}^2) \to L_2(\mathbb{R}^2) \tag{6.44}$$

between the lattice representation and the left quasi-regular representation restricted to an irreducible component of $L_2(\mathbb{R}^2)$. Although the formula of the left quasi-regular representation (5.14)

$$[\Lambda_\hbar(s,x,y)f](x',y') = e^{2\pi i\hbar(s+x(y'-y))} f(x'-x, y'-y) \qquad (6.45)$$

is similar to the lattice representation's formula (5.23)

$$[\rho_m(s,x,y)f](u,v) = e^{2\pi m i(s+x(v-y))} f(u-x, v-y), \qquad (6.46)$$

they are rather different, e.g., they act on different spaces $L_2(\mathbb{R}^2)$ and $L_2(\mathbb{T}^2)$, respectively.

Let $\chi_\hbar(s,0,0) = e^{2\pi i\hbar s}$ be the character of the centre $Z = \{(s,0,0) : s \in \mathbb{R}\}$ of $\mathbb{H}_p^1$. To calculate the induced covariant transform $\mathcal{W}_\Theta^{\rho_m}$, we need to find a fiducial vector ϕ_0 satisfying (6.15). As was already mentioned (see Subsection 6.2.1), any vector $\phi_0 \in L_2(\mathbb{T}^2)$ satisfies that $\rho_m(s,0,0)\phi_0 = \chi_\hbar(s,0,0)\phi_0$, for all $(s,0,0) \in Z$ and $\hbar = m$.

As in the case of the pre-FSB transform (see Subsection 6.2.1) and by Proposition 6.3, if we specify ϕ_0 to be the vacuum $\Phi_{m\kappa}$ obtained in (5.47) of the lattice representation ρ_m, the image space of $\mathcal{W}_\Theta^{\rho_m}(f)$, $f \in L_2(\mathbb{T}^2)$ is annihilated by the right ladder $\mathcal{L}^{\kappa X + iY}$ (5.38). Thus, we set

$$\phi_0(u,v) = \Phi_{m\kappa}(u,v) = e^{-\frac{\pi m}{\kappa}(u^2 - 2i\kappa uv)} \Theta_{m\kappa}(\sqrt{\frac{h}{2\kappa}}((\kappa v + iu), i).$$

Let $\mathsf{s} : \mathbb{R}^2 \to \mathbb{H}_p^1 : (x,y) \mapsto (0,x,y)$ (4.35) be the continuous section. For $\hbar = m$, the induced covariant transform $\mathcal{W}_\Theta^{\rho_m} : L_2(\mathbb{T}^2) \to L_2(\mathbb{R}^2)$ is calculated as follows:

$$\begin{aligned}
[\mathcal{W}_\Theta^{\rho_m}(f)](x,y) &= \langle f, \rho_m(\mathsf{s}(\tilde{x}))\phi_0 \rangle, \quad \tilde{x} \in \mathbb{T}^2 \\
&= \int_{\mathbb{T}^2} f(u,v)\, e^{-2\pi i m x(v-y)}\, \bar{\phi}_0(u-x, v-y)\, du\, dv \\
&= \int_{\mathbb{T}^2} f(u,v) e^{-2\pi i m u(v-y) - \frac{\pi m}{\kappa}(u-x)^2} \qquad (6.47) \\
&\qquad \times \bar{\Theta}_{m\kappa}\left(\sqrt{\frac{h}{2\kappa}}(\kappa(v-y) + i(u-x)), i \right) du\, dv.
\end{aligned}$$

After simplifications, (6.47) becomes

$$[\mathcal{W}_\Theta^{\rho_m}(f)](x,y) = \int_{\mathbb{T}^2} f(u,v) \, \mathrm{e}^{-\frac{\pi m}{\kappa}(u^2+2\mathrm{i}\kappa uv)-\frac{\pi m}{\kappa}(x^2-2u(x+\mathrm{i}\kappa y))}$$
$$\times \bar{\Theta}_{m\kappa}\left(\sqrt{\frac{h}{2\kappa}}(\kappa(v-y)+\mathrm{i}(u-x)),\mathrm{i}\right) \mathrm{d}u\,\mathrm{d}v \qquad (6.48)$$
$$= \tilde{f}(x,y).$$

Thus, we obtain the induced covariant transform $\mathcal{W}_\Theta^{\rho_m}$ from $L_2(\mathbb{T}^2)$ into $L_2(\mathbb{R}^2)$. We call $\mathcal{W}_\Theta^{\rho_m}$ the *pre-theta transform*. The image space of the pre-theta transform

$$F_\Theta(\mathbb{R}^2) = \{\tilde{f} : \tilde{f} = \mathcal{W}_\Theta^{\rho_m}(f), \ f \in L_2(\mathbb{T}^2)\} \qquad (6.49)$$

consists of null solutions of $\mathcal{L}^{\kappa X+\mathrm{i}Y}$. The prefix "pre-" is removed when the operator $\mathcal{L}^{\kappa X+\mathrm{i}Y}$ is translated into the analyticity condition $\partial_{\bar{z}}$ by a unitary peeling operator (see Subsection 6.5.1).

The general properties of the covariant transform $\mathcal{W}_\Theta^{\rho_m}$ yield corresponding properties of the pre-theta transform.

Corollary 6.15. *The pre-theta transform* $\mathcal{W}_\Theta^{\rho_m} : L_2(\mathbb{T}^2) \to L_2(\mathbb{R}^2)$ *given by*

$$[\mathcal{W}_\Theta^{\rho_m}(f)](x,y) = \int_{\mathbb{T}^2} f(u,v) \, \mathrm{e}^{-\frac{\pi m}{\kappa}(u^2+2\mathrm{i}\kappa uv)-\frac{\pi m}{\kappa}(x^2-2u(x+\mathrm{i}\kappa y))}$$
$$\times \bar{\Theta}_{m\kappa}\left(\sqrt{\frac{h}{2\kappa}}(\kappa(v-y)+\mathrm{i}(u-x)),\mathrm{i}\right) \mathrm{d}u\,\mathrm{d}v \qquad (6.50)$$

is a unitary intertwining operator between the lattice representation ρ_m *on* $L_2(\mathbb{T}^2)$ *and the left quasi-regular representation* $\Lambda_\hbar$ *on* $F_\Theta(\mathbb{R}^2)$.

6.3 Contravariant Transform on $\mathbb{H}_p^1$

The goal of the present section is to introduce the *contravariant transform* $\mathcal{M}_\psi$ with some basic properties [29]. Examples of $\mathcal{M}_\psi$ are also given in Subsections 6.3.1, 6.3.2 and 6.3.3. In Section 6.2, we study the covariant transform $\mathcal{W}_\phi$ (6.70), which is a map from a Hilbert space $\mathcal{H}$ to a space $W(X)$ of functions on a homogeneous space X. The contravariant transform $\mathcal{M}_\psi$ is the *adjoint* of $\mathcal{W}_\phi$ and sends a function f on $L_1(X)$ to a vector on $\mathcal{H}$.

Definition 6.16. [29] Let ρ be a representation of the Heisenberg group $\mathbb{H}_p^1$ on a Hilbert space $\mathcal{H}$ and H be a closed subgroup of $\mathbb{H}_p^1$. Let $X = \mathbb{H}_p^1/H$ be a homogeneous space, which will be a subset of Euclidean space with the Lebesgue measure. The contravariant transform $\mathcal{M}_\psi$ for a *reconstruction vector* $\psi \in \mathcal{H}$ is a map $\mathcal{M}_\psi^\rho : L_1(X) \to \mathcal{H}$ given by

$$
\begin{aligned}
\mathcal{M}_\psi^\rho : \ \tilde{\nu} &\mapsto \mathcal{M}_\psi(\tilde{\nu}) \\
&= \int_X \tilde{\nu}(x) \ \psi_x \ \mathrm{d}\mu(x) \\
&= \int_X \tilde{\nu}(x) \ \rho(\mathsf{s}(x)) \ \mathrm{d}\mu(x) \ \psi,
\end{aligned}
\tag{6.51}
$$

where $\psi_x = \rho(\mathsf{s}(x))\psi$ and s is a continuous section from $X = \mathbb{H}_p^1/H$ to $\mathbb{H}_p^1$ (see Section 3.1.1).

Let ρ be a strongly continuous unitary irreducible representation of $\mathbb{H}_p^1$ on a Hilbert space $\mathcal{H}$. For non-orthogonal fiducial and reconstruction vectors, the contravariant transform $\mathcal{M}_\psi$ and the covariant transform $\mathcal{W}_\phi$ (6.3) are adjoints

$$
\langle \mathcal{M}_\psi \, \tilde{\nu}, u \rangle_{\mathcal{H}} = \langle \tilde{\nu}, \mathcal{W}_\phi \, u \rangle_{L_2(\mathbb{R}^2)},
\tag{6.52}
$$

where $u \in \mathcal{H}$ (see [4, Ch.8,§8.1]). To see this, we note that the contravariant transform $\mathcal{M}_\psi$ intertwines the left regular representation Λ (5.1) on $\mathbb{H}_p^1$ and ρ on $\mathcal{H}$ [29]:

$$
\mathcal{M}_\psi \circ \Lambda(g) = \rho(g) \circ \mathcal{M}_\psi.
\tag{6.53}
$$

Combining with $\mathcal{W}_\phi \circ \rho(g) = \Lambda(g) \circ \mathcal{W}_\phi$ (6.4), we see that the composition $\mathcal{M}_\psi \circ \mathcal{W}_\phi$ of the covariant and contravariant transform intertwines ρ with itself. That is,

$$
(\mathcal{M}_\psi \circ \mathcal{W}_\phi) \circ \rho(g) = \rho(g) \circ (\mathcal{M}_\psi \circ \mathcal{W}_\phi).
\tag{6.54}
$$

Therefore, Schur's lemma 3.11 implies that

$$
\mathcal{M}_\psi \circ \mathcal{W}_\phi = c \, I,
\tag{6.55}
$$

for some constant $c \in \mathbb{C}$. Alternatively, the sesqui-unitary property (6.13) implies that $\mathcal{M}_\psi \circ \mathcal{W}_\phi = \langle \psi, \phi \rangle I$. Indeed, if $f, g \in \mathcal{H}$, we have

$$
\begin{aligned}
\langle \mathcal{M}_\psi \circ \mathcal{W}_\phi f, g \rangle &= \langle \mathcal{W}_\phi f, \mathcal{W}_\psi g \rangle \\
&= \langle f, g \rangle \overline{\langle \phi, \psi \rangle} \\
&= \langle \langle \psi, \phi \rangle f, g \rangle.
\end{aligned}
\tag{6.56}
$$

Therefore, for non-orthogonal vectors ψ and ϕ, we obtain $\langle \psi, \phi \rangle = c \neq 0$. Thus, $\mathcal{M}_\psi \circ \mathcal{W}_\phi$ is a scalar multiple of the identity operator I. Thus, $c^{-1}\mathcal{M}_\psi$ is the inverse operator of $\mathcal{W}_\phi$.

6.3.1 The Inverse of the (Pre-) FSB Transform

Let $[\rho_\hbar(s, x, y)f](t) = e^{2\pi i \hbar(s-ty)} f(t-x)$ be the Schrödinger representation (5.17). The contravariant transform $\mathcal{M}_\psi^{\rho_\hbar} : L_2(\mathbb{R}^2) \to L_2(\mathbb{R})$ associated with a vector $\psi \in L_2(\mathbb{R})$ provides the reconstruction formula [33]:

$$\mathcal{M}_\psi^{\rho_\hbar} : f \mapsto \int_{\mathbb{R}^2} f(x, y) \, \rho_\hbar(0, x, y) \, \psi \, \mathrm{d}x \, \mathrm{d}y. \tag{6.57}$$

In particular, if the reconstruction vector is the vacuum $\psi_{\hbar\kappa}(t) = 2^{1/4} e^{-\frac{\pi\hbar}{\kappa} t^2}$ of the Schrödinger representation, we obtain that

$$\begin{aligned}
\mathcal{M}_\psi^{\rho_\hbar} : f \mapsto &2^{1/4} \int_{\mathbb{R}^2} f(x, y) \, e^{-2\pi i \hbar ty} \, e^{-\frac{\pi\hbar}{\kappa}(t-x)^2} \, \mathrm{d}x \, \mathrm{d}y \\
= &2^{1/4} e^{-\frac{\pi\hbar}{\kappa} t^2} \int_{\mathbb{R}^2} f(x, y) \, e^{-\frac{\pi\hbar}{\kappa}(x^2 - 2t(x - i\kappa y))} \, \mathrm{d}x \, \mathrm{d}y.
\end{aligned} \tag{6.58}$$

This is known as the *inverse of the pre-FSB transform* [35, §4.2].

6.3.2 The Inverse of the Zak Transform

In Subsection 6.2.2, we derived the co-Zak transform $\mathcal{Z} : L_2(\mathbb{R}) \to L_2(\mathbb{T}^2)$ (6.38) through the induced covariant transform $\mathcal{W}_{\phi_0}^{\rho_\hbar}$. Now, we calculate its inverse using the contravariant transform. To begin, we provide the standard definition of the inverse of the Zak transform.

Definition 6.17. [35, §8.1] Let $\tilde{L}_2(\mathbb{T}^2)$ be a space of square integrable functions $\tilde{g}(x, v)$ that are quasi-periodic in x and periodic in v. Let $\tilde{g} = \tilde{\mathcal{Z}}f$ be the Zak transform of $f \in \mathcal{S}(\mathbb{R}) \subset L_2(\mathbb{R})$. The function f can be reconstructed using the following formula:

$$\begin{aligned}
\tilde{\mathcal{Z}}^{-1} : &\tilde{L}_2(\mathbb{T}^2) \to L_2(\mathbb{R}) \\
&[\tilde{\mathcal{Z}}^{-1}\tilde{g}](x) = \int_{\mathbb{T}} \tilde{g}(x, v) e^{-2\pi i mnv} \, \mathrm{d}v, \quad n \in \mathbb{Z}, \ m \in \mathbb{N}.
\end{aligned} \tag{6.59}$$

The operator $\tilde{\mathcal{Z}}^{-1}$ is called *the inverse of the Zak transform*.

Let $[\rho_\hbar(s, x, y)f](t) = e^{2\pi i \hbar(s-ty)} f(t-x)$ be the Schrödinger representation and $\chi_\hbar = e^{2\pi i \hbar s}$ be the character of the subgroup $H_x' = \{(s, 0, y) : \ s, y \in \mathbb{R}\}$ (4.27). We look for the contravariant transform $\mathcal{M}_{\psi_0}^{\rho_\hbar} : L_2(\mathbb{T}^2) \to L_2(\mathbb{R})$. To calculate it, we need

to find a reconstruction vector ψ_0 satisfying the condition in (6.15), that is, for all $h \in H'_x = \{(s, 0, y) : s, y \in \mathbb{R}\}$, we want

$$e^{2\pi i h(s - ty)}\psi_0(t) = e^{2\pi i hs}\psi_0(t). \tag{6.60}$$

Thus, the left-hand side of (6.60) is equal to the right-hand side if and only if ψ_0 is supported at $\{0\}$. Further analysis shows that the reconstruction vector ψ_0 satisfying the condition (6.60) is the Dirac delta distribution $\delta(t)$ [11]. As discussed in Remark 6.12, we manipulate heuristically using the distribution $\delta(t)$ as a regular function.

Let $\tilde{x} = (u, v) \in X = \mathbb{T}^2 = \mathbb{H}^1_p / H_d$, where H_d is the non-commutative subgroup (4.32). Consider the section map $\mathsf{s} : \mathbb{T}^2 \to \mathbb{H}^1_p : (u, v) \mapsto (0, u, v)$ defined in (4.35). For $g \in L_2(\mathbb{T}^2)$ and $\hbar = m$, we have that

$$
\begin{aligned}
\mathcal{M}^{\rho_\hbar}_{\psi_0} : g &\mapsto \int_{\mathbb{T}} \int_{\mathbb{T}} g(u, v)\, \rho_\hbar(\mathsf{s}(\tilde{x}))\, \psi_0(t)\, \mathrm{d}u\, \mathrm{d}v \\
&= \int_0^1 \int_0^1 g(u, v)\, \rho_\hbar(0, u, v)\, \psi_0(t)\, \mathrm{d}u\, \mathrm{d}v \\
&= \int_0^1 \int_0^1 g(u, v)\, e^{-2\pi i m t v} \delta(t - u)\, \mathrm{d}u\, \mathrm{d}v \\
&= \int_0^1 g(t, v)\, e^{-2\pi i m t v}\, \mathrm{d}v \\
&= \int_0^1 \tilde{g}(t, v)\, \mathrm{d}v.
\end{aligned}
\tag{6.61}
$$

Since $g(t, v)$ is contained in the space $L_2(\mathbb{T}^2)$ of square integrable functions that are periodic in t and quasi-periodic in v, by our Lemma 6.10, multiplying $g(t, v)$ by $e^{-2\pi i m t v}$ produces a function that has the same double quasi-periodicity property of $g(t, v)$ but in the opposite way. In other words, $\tilde{g}(t, v) = g(t, v) \cdot e^{-2\pi i m t v} \in \tilde{L}_2(\mathbb{T}^2)$ is quasi-periodic in t and periodic in v and square integrable. Moreover, since $t \in \mathbb{R} \approx [0, 1] \times \mathbb{Z}$, then $t = x + n$, for some $x \in [0, 1]$ and $n \in \mathbb{Z}$. Therefore, for $t = x + n$, (6.61) becomes

$$
\begin{aligned}
\int_0^1 \tilde{g}(x + n, v)\, \mathrm{d}v &= \int_0^1 \tilde{g}(x, v)\, e^{-2\pi i m n v}\, \mathrm{d}v \\
&= [\mathcal{M}^{\rho_\hbar}_{\psi_0} g](x) = [\mathcal{M}^{\rho_\hbar}_{\psi_0}(e^{2\pi i m x v}\, \tilde{g})](x).
\end{aligned}
\tag{6.62}
$$

Thus, $\mathcal{M}^{\rho_\hbar}_{\psi_0}$ is the inverse of the induced covariant transform $\mathcal{W}^{\rho_\hbar}_{\phi_0}$ (6.38) from $L_2(\mathbb{T}^2)$ into $L_2(\mathbb{R})$.

Corollary 6.18. *Let $\tilde{g}(x,v) = g(x,v) \cdot e^{-2\pi i m x v}$ such that $g \in L_2(\mathbb{T}^2)$. The contravariant transform* (6.62),

$$
\begin{aligned}
[\mathcal{M}_{\psi_0}^{\rho_\hbar}(e^{2\pi i m x v}\ \tilde{g})](x) &= \int_0^1 \tilde{g}(x,v)\ e^{-2\pi i m n v}\ \mathrm{d}v \\
&= [\tilde{\mathcal{Z}}^{-1}\tilde{g}](x),
\end{aligned}
\tag{6.63}
$$

is the inverse of the Zak transform. For $g \in L_2(\mathbb{T}^2)$, one can write the inverse of the co-Zak transform $\mathcal{Z}^{-1}g = \tilde{\mathcal{Z}}^{-1}\ e^{-2\pi i m u v}g$.

6.3.3   The Inverse of the (Pre-) Theta Transform

In Subsection 6.2.3, we found the pre-theta transform from $L_2(\mathbb{T}^2)$ into $L_2(\mathbb{R}^2)$. Now, we calculate the *inverse of the pre-theta transform*

$$
\mathcal{M}_{\Theta}^{\rho_m} : L_2(\mathbb{R}^2) \to L_2(\mathbb{T}^2),
\tag{6.64}
$$

where $[\rho_m(s,x,y)f](u,v) = e^{2\pi i m(s+x(v-y))}f(u-x,v-y)$ is the lattice representation.

Let $\mathsf{s} : \mathbb{R}^2 \to \mathbb{H}_p^1 : (x,y) \mapsto (0,x,y)$ (4.25) be the continuous section. For $\hbar = m$ and $f \in L_2(\mathbb{R}^2)$, the contravariant transform $\mathcal{M}_\psi^{\rho_m}$ associated with a reconstruction vector $\psi \in L_2(\mathbb{T}^2)$ from Definition 6.16 is

$$
\mathcal{M}_\psi^{\rho_m} : f \mapsto \int_{\mathbb{R}^2} f(x,y)\ \rho_m(0,x,y)\ \mathrm{d}x\,\mathrm{d}y\ \psi(u,v).
\tag{6.65}
$$

In particular, if we set the reconstruction vector ψ by the lattice representation's vacuum (5.47)

$$
\Phi_{m\kappa}(u,v) = e^{-\frac{\pi m}{\kappa}(u^2 - 2i\kappa u v)}\Theta_{m\kappa}(\sqrt{\frac{h}{2\kappa}}((\kappa v + iu), i),
\tag{6.66}
$$

the integral transformation $\mathcal{M}_\psi^{\rho_m}$ (6.65) after simplification becomes

$$
\begin{aligned}
\mathcal{M}_{\psi_{\hbar\kappa}}^{\rho_m}(f) = e^{-\frac{\pi m}{\kappa}u^2 - 2i\kappa u v} \int_{\mathbb{T}^2} f(x,y)\ e^{-\frac{\pi m}{\kappa}(x^2 - 2u(x - i\kappa y))} \\
\times \Theta_{m\kappa}\left(\sqrt{\frac{h}{2\kappa}}(\kappa(v-y) + i(u-x), i)\right)\ \mathrm{d}x\,\mathrm{d}y.
\end{aligned}
\tag{6.67}
$$

Thus, we obtain the inverse operator of $\mathcal{W}_\Theta^{\rho_m}$ (6.48). We call the transformation $\mathcal{M}_\Theta^{\rho_m}$ (6.67) the inverse of the pre-theta transform.

6.4 Decomposing Representations of $\mathbb{H}_p^1$ into Irreducible Components

It is very common in mathematics to split complex structures into simpler pieces. Thus, the aim of this section is to separate a strongly continuous unitary reducible representation of the Heisenberg group $\mathbb{H}_p^1$ into irreducible components. For decomposing general representations of a Lie group G, readers can look at [24, § 8; 25, App.IV,§2.5].

Consider $G = \mathbb{H}_p^1$. In Section 6.1, we defined the covariant transform $\mathcal{W}_\phi$ (6.3) by fixing the second variable of the matrix coefficient as a fiducial vector ϕ. If we fix the first vector in the matrix coefficient (6.2) rather than the second one, we call this operator the *co-covariant transform* $\tilde{\mathcal{W}}_\phi$.

Definition 6.19. [29] Let ρ be a representation of the Heisenberg group $\mathbb{H}_p^1$ on a Hilbert space $\mathcal{H}$. Let ϕ be a fiducial vector in $\mathcal{H}$. The co-covariant transform $\tilde{\mathcal{W}}_\phi$ is given by

$$
\begin{aligned}
\tilde{\mathcal{W}}_\phi : \mathcal{H} &\to L(\mathbb{H}_p^1) \\
f &\mapsto \overline{\langle \rho(g)f, \phi \rangle} \\
&= \langle \phi, \rho(g)f \rangle.
\end{aligned}
\tag{6.68}
$$

If ρ is unitary, then we have

$$
\begin{aligned}
\tilde{\mathcal{W}}_\phi : \mathcal{H} &\to L(\mathbb{H}_p^1) \\
f &\mapsto \overline{\langle f, \rho(g^{-1})\phi \rangle} \\
&= \langle \phi, \rho(g)f \rangle.
\end{aligned}
\tag{6.69}
$$

Let H be a closed subgroup of $\mathbb{H}_p^1$ and $X = \mathbb{H}_p^1/H$ be the respective homogeneous space, which will be a subset of Euclidean space with the Lebesgue measure. Similarly to the covariant transform $\mathcal{W}_\phi$, we define $\tilde{\mathcal{W}}_\phi$ for $X = \mathbb{H}_p^1/H$ by

$$
\begin{aligned}
\tilde{\mathcal{W}}_\phi : \mathcal{H} &\to W(X) \\
f &\mapsto \overline{\langle f, \rho(\mathsf{s}(x)^{-1})\phi \rangle} \\
&= \langle \phi, \rho(\mathsf{s}(x))f \rangle = \tilde{\phi}_f(x), \quad x \in X = \mathbb{H}_p^1/H,
\end{aligned}
\tag{6.70}
$$

where s is any continuous section from X to $\mathbb{H}_p^1$ and $W(X)$ is the space of functions on the homogeneous space X.

Since the Schrödinger (5.17) and the lattice (5.23) representations are already irreducible, we only need to decompose the left quasi-regular representation $\Lambda_\hbar$ (5.14) as shown in Subsection 6.4.1.

6.4.1 Decomposing the Left Quasi-Regular Representation

In this subsection, we decompose the left quasi-regular representation $\Lambda_\hbar$ (5.14) into irreducible components on $L_2(\mathbb{R}^2)$. To begin, we provide the following result, which is an immediate consequence of Proposition 6.3.

Corollary 6.20. *Let $G = \mathbb{H}_p^1$ be the Heisenberg group and*

$$[\rho_\hbar(s, x, y)f](t) = \mathrm{e}^{2\pi\mathrm{i}\hbar(s-ty)} f(t - x) \tag{6.71}$$

be the Schrödinger representation. Consider the vacuum $\phi_{\hbar\kappa} \in L_2(\mathbb{R})$ of $\rho_\hbar$ as a fiducial vector of the co-covariant transform $\tilde{\mathcal{W}}_{\phi_{\hbar\kappa}}^{\rho_\hbar}$. Then, the image space

$$\{\tilde{\phi}_f = \tilde{\mathcal{W}}_{\phi_{\hbar\kappa}}(f) : \ f \in L_2(\mathbb{R})\} \tag{6.72}$$

of the co-covariant transform $\tilde{\mathcal{W}}_{\phi_{\hbar\kappa}}^{\rho_\hbar} : L_2(\mathbb{R}) \to L_2(\mathbb{R}^2)$ is a linear subspace of $L_2(\mathbb{R}^2)$ consisting of all vacuum vectors of the form $\tilde{\phi}_f$ of the left quasi-regular representation $\Lambda_\hbar$ (5.14).

Proof. By Proposition 6.4, the co-covariant transform $\tilde{\mathcal{W}}_{\phi_{\hbar\kappa}}$ is a linear map from $L_2(\mathbb{R})$ to $L_2(\mathbb{R}^2)$. According to Proposition 6.2, we have

$$\Lambda_\hbar(g) \circ \tilde{\mathcal{W}}_{\phi_{\hbar\kappa}} = \tilde{\mathcal{W}}_{\rho_\hbar(g)\phi_{\hbar\kappa}}, \quad g \in \mathbb{H}_p^1. \tag{6.73}$$

To show that $\tilde{\phi}_f$ is a vacuum for each function $f \in L_2(\mathbb{R})$, we need to verify that $\tilde{\phi}_f$ is annihilated by $a_{\Lambda_\hbar}^- = \mathrm{d}\Lambda_\hbar^{\kappa X - \mathrm{i}Y}$ (5.33). By (6.73), we have that

$$\mathrm{d}\Lambda_\hbar^{\kappa X - \mathrm{i}Y} \circ \tilde{\mathcal{W}}_{\phi_{\hbar\kappa}} = \tilde{\mathcal{W}}_{\mathrm{d}\rho_\hbar^{\kappa X - \mathrm{i}Y} \phi_{\hbar\kappa}}. \tag{6.74}$$

Hence,

$$\begin{aligned}
\mathrm{d}\Lambda_\hbar^{\kappa X} \langle \phi_{\hbar\kappa}, \rho_\hbar(\mathsf{s}(x))f \rangle &= \langle \phi_{\hbar\kappa}, \mathrm{d}\rho_\hbar^{-\kappa X} \rho_\hbar(\mathsf{s}(x))f \rangle \\
&= \langle \mathrm{d}\rho_\hbar^{\kappa X} \phi_{\hbar\kappa}, \rho_\hbar(\mathsf{s}(x))f \rangle
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{d}\Lambda_\hbar^{-\mathrm{i}Y} \langle \phi_{\hbar\kappa}, \rho_\hbar(\mathsf{s}(x))f \rangle &= \langle \phi_{\hbar\kappa}, \mathrm{d}\rho_\hbar^{\mathrm{i}Y} \rho_\hbar(\mathsf{s}(x))f \rangle \\
&= \langle \mathrm{d}\rho_\hbar^{-\mathrm{i}Y} \phi_{\hbar\kappa}, \rho_\hbar(\mathsf{s}(x))f \rangle.
\end{aligned}$$

Since $\phi_{\hbar\kappa}$ is the Schrödinger representation's vacuum obtained in Subsection 5.4.2, it is annihilated by $a^-_{\rho_\hbar} = \mathrm{d}\rho^{\kappa X - \mathrm{i} Y}_\hbar$. Then,

$$\mathrm{d}\Lambda^{\kappa X - \mathrm{i} Y}_\hbar \langle \phi_0, \rho_\hbar(\mathsf{s}(x)) f \rangle = \langle \mathrm{d}\rho^{\kappa X - \mathrm{i} Y}_\hbar \phi_{\hbar\kappa}, \rho_\hbar(\mathsf{s}(x)) f \rangle = 0, \tag{6.75}$$

i.e., $a^-_{\Lambda_\hbar} \tilde{\phi}_f = 0$. Therefore, for all $f \in L_2(\mathbb{R})$, the image space of the co-covariant transform is a space of all vacuums $\tilde{\phi}_f$ of $\Lambda_\hbar$. $\qquad\square$

We now turn to decomposing the space $L_2(\mathbb{R}^2)$ into a direct sum of invariant irreducible subspaces $L_2(\mathbb{R}^2) = \oplus_{n \in \mathbb{N}} H_n$ under the left quasi-regular representation $\Lambda_\hbar$. Consider the vacuum $\phi_{0,\hbar\kappa}(z, \bar{z}) = \mathrm{e}^{\frac{1}{4}(z^2 - \bar{z}^2 - 2z\bar{z}) + c}$ of $\Lambda_\hbar$ obtained in (5.39), which is the unique vector (up to normalisation) in $L_2(\mathbb{R}^2)$ annihilating by both operators $a^-_{\Lambda_\hbar}$ (5.33) and $\mathcal{L}^{\kappa X + \mathrm{i} Y}$ (5.38). Set $\phi_{00} := \phi_{0,\hbar\kappa}$, then all vectors $\phi_{0j} = (a^+_{R_\hbar})^j \phi_{00}$ are vacuum due to the commutation of the left and right actions:

$$\begin{aligned} a^-_{\Lambda_\hbar} \phi_{0n} &= a^-_{\Lambda_\hbar} (a^+_{R_\hbar})^n \phi_{00} \\ &= (a^+_{R_\hbar})^n a^-_{\Lambda_\hbar} \phi_{00} = 0. \end{aligned} \tag{6.76}$$

For each vacuum ϕ_{0j}, the collection of vectors $\phi_{ij} = (a^+_{\Lambda_\hbar})^i \phi_{0j}$ form an orthonormal basis of an irreducible component of the left quasi-regular representation with the respective ladder structure (cf. (5.31))

$$a^+_{\Lambda_\hbar} \phi_{ij} = \sqrt{i+1}\ \phi_{i+1,j}, \quad a^-_{\Lambda_\hbar} \phi_{ij} = \sqrt{i}\ \phi_{i-1,j}. \tag{6.77}$$

This is complemented by the respective action of the right derived representation

$$a^+_{R_\hbar} \phi_{ij} = \sqrt{j+1}\ \phi_{i,j+1}, \qquad a^-_{R_\hbar} \phi_{ij} = \sqrt{j}\ \phi_{i,j-1}. \tag{6.78}$$

Two actions—the left and the right—jointly create the two-dimensional lattice structure shown in Fig. 6.1. The collection of all vectors ϕ_{ij} is an orthonormal basis of $L_2(\mathbb{R}^2)$ [44]. In greater detail, the first row represents an orthonormal basis of the image space of the co-covariant transform vacuums $\{\tilde{\phi}_f = \tilde{\mathcal{W}}_{\phi_{\hbar\kappa}}(f) : f \in L_2(\mathbb{R})\}$ (6.72). This implies that the operator $a^-_{\Lambda_\hbar}$ annihilates the whole space generated by the first horizontal ladder in Figure 6.1.

Due to the commutativity of the left and right actions, the first column is an orthonormal basis of the covariant transform image $\mathcal{W}_{\phi_{\hbar\kappa}}(f)$ (the pre-FSB component $H_0 := F_{\phi_{\hbar\kappa}} = \{\tilde{f} : \tilde{f} = \mathcal{W}_{\phi_{\hbar\kappa}}(f), f \in L_2(\mathbb{R})\}$ (6.22)) annihilated by

$$\mathcal{L}^{\kappa X + \mathrm{i} Y} \phi_{j0} = 0, \quad \forall, \phi_{j0} \in H_0, \ \ j \in \mathbb{N}. \tag{6.79}$$
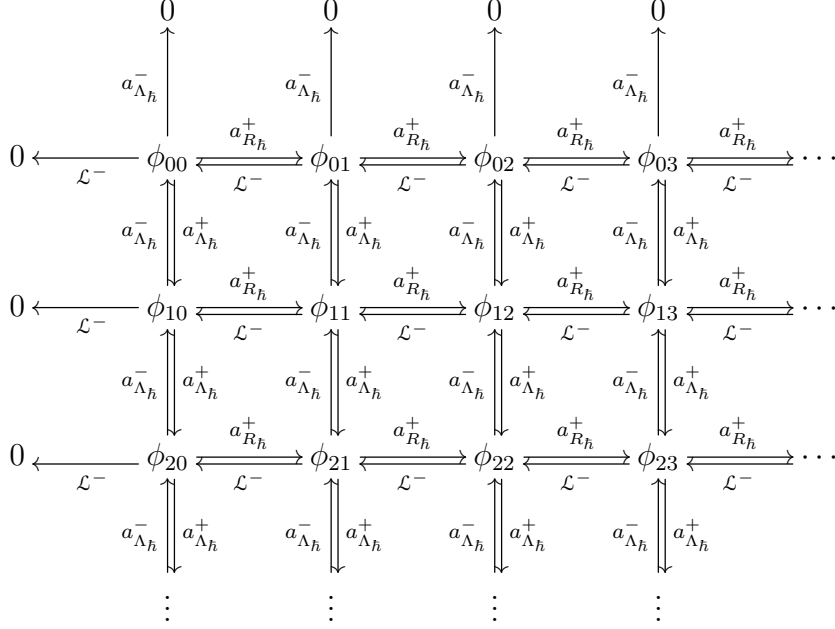
Figure 6.1: Decomposing the left quasi-regular representation

Since the $\mathcal{W}_{\phi_{\hbar\kappa}}(f)$ is a unitary operator between the invariant irreducible space $L_2(\mathbb{R})$ and the image space H_0, the component H_0 is invariant and irreducible with respect to $\Lambda_\hbar$ (see Corollary 6.7). The other vertical ladders also generate corresponding invariant irreducible components H_n ($n \in \mathbb{N}$). This is because for each $n \in \mathbb{N}$, there is a covariant transform $\mathcal{W}_{\phi_{0n}}^{\Lambda_\hbar} : H_n \to L_2(\mathbb{R})$ such that its fiducial vector is taken to be the vacuum ϕ_{0n} of $\Lambda_\hbar$. Then, $\mathcal{W}_{\phi_{0n}}^{\Lambda_\hbar}(f)$ intertwines the left quasi-regular representation $\Lambda_\hbar$ restricted on H_n ($n \in \mathbb{N}$) and the Schrödinger representation $\rho_\hbar$ on $L_2(\mathbb{R})$, which is already invariant and irreducible.

Now, we decompose the space $L_2(\mathbb{R}^2)$ into a direct sum of the invariant irreducible subspaces

$$L_2(\mathbb{R}^2) = \oplus_{n=0}^\infty H_n, \tag{6.80}$$

where each H_n is generated by the n-th unique vertical ladder, i.e.,

$$H_n = \overline{span\{\phi_{0n}, \phi_{1n}, \cdots\}}. \tag{6.81}$$

Each component H_n, for $n = 0, 1, \cdots$, possesses the following properties for all $j \in \mathbb{N}$:

$$(\mathcal{L}^{\kappa X + \mathrm{i}Y})^m \phi_{jn} = 0, \ \forall m \geq n \qquad \text{and} \qquad (\mathcal{L}^{\kappa X + \mathrm{i}Y})^{n-1} \phi_{jn} \neq 0. \tag{6.82}$$

If $n = 0$, we have the pre-FSB space H_0. If $n > 0$, the components H_n are called the n-th *true pre-poly-analytic subspace* [44]. Fig. 6.1 visualises a decomposition of $L_2(\mathbb{R}^2)$ into irreducible subspaces of the left quasi-regular representation. The described decomposition is not unique, but it is distinguished among infinitely many others through its connections to the poly-analytic functions.

6.5 Peeling Representations of $\mathbb{H}_p^1$ and Analyticity

Since the annihilation operator provides a useful characterisation of an irreducible component of a representation ρ we are interested in expressing it in the most transparent form. The peeling of the representation ρ is useful because it simplifies the corresponding annihilation operator to be a linear combination of derivatives only. Therefore, it also simplifies the structure of the eigenvectors ϕ_n (5.30) forming an orthonormal basis of the initial irreducible space $\mathcal{H}$. Sometimes, say for the pre-FSB representation, we are able to peel an irreducible representation to a space of analytic functions. In this situation, one can use all complex analysis theorems to study the induced representations of $\mathbb{H}_p^1$.

Definition 6.21. [33] Let G be a group and ρ be a representation on a space $L_2(X, \mathrm{d}\mu(x))$ of square integrable functions on some set X with an appropriated measure $\mathrm{d}\mu(x)$. A peeling ε_d is an invertible operator of multiplication defined by a function $d(x)$ on X:

$$\varepsilon_d : f(x) \mapsto e^{d(x)} f(x). \tag{6.83}$$

The operator ε_d is unitary for suitably related measures:

$$\varepsilon_d : L_2(X, \mathrm{d}\mu(x)) \to L_2(X, \mathrm{d}\nu(x)) \tag{6.84}$$

such that $\mathrm{d}\nu(x) = e^{-2\Re d(x)}\mathrm{d}\mu(x)$.

In this thesis, all considered peelings use smooth $d(x)$ on a domain X in a Euclidean space. We will discuss the choice of $d(x)$ for the pre-FSB, Schrödinger and lattice representations in Subsections 6.5.1, 6.5.2 and 6.5.3, respectively.

6.5.1 Peeling the (Pre-) FSB Representation

Let $\Lambda_\hbar$ be the pre-FSB representation (5.14), which acts irreducibly on the pre-FSB space $F_{\phi_{\hbar\kappa}}$ (6.22). Consider the variables $z, \bar{z} \in \mathbb{C}$, where $z = \sqrt{\frac{h}{2\kappa}}(x + i\kappa y)$ and $h = 2\pi\hbar > 0$. In this subsection, we peel the representation $\Lambda_\hbar$ into the corresponding one $\tilde{\Lambda}_\hbar$ that acts on the FSB space of analytic functions. Moreover, we want to simultaneously simplify two 'orthogonal' sets of ladder operators from Fig. 6.1. To perform this, we look for a peeling operator satisfying the following conditions:

i. The peeling defined by $e^{d(z,\bar{z})}$ shall intertwine the right annihilation operator $\mathcal{L}^{\kappa X + iY} = 2\pi\hbar x + (\kappa\partial_x + i\partial_y)$ and the Cauchy–Riemann operators $\partial_{\bar{z}} = \kappa\partial_x + i\partial_y$:

$$\partial_{\bar{z}} \, e^{d(z,\bar{z})} f(z, \bar{z}) = e^{d(z,\bar{z})} \mathcal{L}^{\kappa X + iY} f(z, \bar{z}). \tag{6.85}$$

A simple differential equation for (6.85) implies that $d(z, \bar{z}) = \tilde{\psi}(z) + \frac{1}{4}(z + \bar{z})^2$, where $\tilde{\psi}$ is an arbitrary smooth function of z alone.

ii. The same peeling shall intertwine the left annihilation operator $a^-_{\Lambda_\hbar} = 2\pi i\hbar\kappa y - (\kappa\partial_x - i\partial_y)$ with (a multiple of) the complex derivative $\partial_z = (\kappa\partial_x - i\partial_y)$. This fixes $\tilde{\psi}(z) = -\frac{1}{2}z^2 - c$ and the peeling operator becomes

$$\begin{aligned}
\varepsilon_d \cdot I = e^{d(z,\bar{z})} \cdot I &= e^{\frac{h}{4\kappa}(x^2 + \kappa^2 y^2 - 2i\kappa xy) - c} \cdot I \\
&= e^{-\frac{1}{2}z^2 + \frac{1}{4}(z+\bar{z})^2 - c} \cdot I \\
&= e^{\frac{1}{4}(\bar{z}^2 - z^2 + 2z\bar{z}) - c} \cdot I.
\end{aligned} \tag{6.86}$$

The consequence of the above conditions is that the peeling maps the vacuum ϕ_{00}, which is killed by both the left and right annihilation operators to the function identically equal to $c_0 \in \mathbb{C}$, $c_0 \neq 0$, which is killed by both ∂_z and $\partial_{\bar{z}}$. To find the corresponding representation on the space $F_2^\hbar$ of analytic functions after peeling, we consider its composition with the covariant transform:

$$\begin{aligned}
F(x, y) &= \left(\frac{\hbar}{\kappa}\right)^{1/2} e^{\frac{h}{4\kappa}(x^2 + \kappa^2 y^2 - 2i\kappa xy) - c} \int_{\mathbb{R}} f(t) \, e^{2\pi i\hbar ty} \, e^{-\frac{\pi\hbar}{\kappa}(t-x)^2} \, dt \\
&= \left(\frac{\hbar}{\kappa}\right)^{1/2} \int_{\mathbb{R}} f(t) \, e^{-\frac{\pi\hbar}{\kappa}t^2} \, e^{\frac{2\pi\hbar}{\kappa}t(x + i\kappa y)} \, e^{-\frac{\pi\hbar}{2\kappa}(x^2 - \kappa^2 y^2 + 2i\kappa xy)} \, dt \\
&= \left(\frac{\hbar}{\kappa}\right)^{1/2} \int_{\mathbb{R}} f(t) \, e^{-\frac{h}{2\kappa}t^2 + \sqrt{\frac{2h}{\kappa}} \, tz - \frac{1}{2}z^2} \, dt =: F(z),
\end{aligned} \tag{6.87}$$

where F is an analytic function. Indeed, the function $F(z) = \mathrm{e}^{d(z,\bar{z})}\tilde{f}(z)$ (6.87) satisfies the Cauchy-Riemann equation

$$\partial_{\bar{z}}F(z) = (\kappa\partial_x + \mathrm{i}\partial_y)F(x,y) = 0.$$

The integral (6.87) is known as the FSB transform. The image $F_2^{\hbar}$ of the FSB transforms is called the FSB space. It is a closed subspace of

$$L_2(\mathbb{R}^2, \mathrm{e}^{-\frac{\hbar}{2\kappa}(x^2+\kappa^2y^2)+2c}\mathrm{d}x\,\mathrm{d}y) = L_2(\mathbb{C}, \mathrm{e}^{-|z|^2+2c}\mathrm{d}z\,\mathrm{d}\bar{z}), \qquad (6.88)$$

and often only values $\hbar = 1$, $\kappa = 1$ and $c = 0$ are used [44]. Now, we calculate the corresponding representation $\tilde{\Lambda}_{\hbar}$:

$$\begin{aligned}
\mathrm{e}^{d(z',\bar{z}')}\Lambda_{\hbar}(g)\mathrm{e}^{-d(z',\bar{z}')}F(z') &= \mathrm{e}^{\frac{1}{4}(\bar{z}'^2-z'^2+2z'\bar{z}')}\Lambda_{\hbar}(s,z)\mathrm{e}^{-\frac{1}{4}(\bar{z}'^2-z'^2+2z'\bar{z}')}F(z') \\
&= \mathrm{e}^{\hbar\mathrm{i}s-\frac{1}{2}(z^2-\bar{z}^2)-\frac{1}{4}(\bar{z}^2-z^2+2z\bar{z})+\bar{z}z'}\ F(z'-z) \\
&= \mathrm{e}^{\hbar\mathrm{i}s+\frac{1}{4}(\bar{z}^2-z^2-2z\bar{z})+\bar{z}z'}F(z'-z) \\
&=: [\tilde{\Lambda}_{\hbar}(s,z)F](z'),
\end{aligned} \qquad (6.89)$$

for any analytic function F and $g \in \mathbb{H}_p^1$. The action

$$[\tilde{\Lambda}_{\hbar}(s,z)F](z') = \mathrm{e}^{\hbar\mathrm{i}s+\frac{1}{4}(\bar{z}^2-z^2-2z\bar{z})+\bar{z}z'}F(z'-z) \qquad (6.90)$$

is called the FSB representation. It agrees with the previously imposed annihilation operator:

$$a_{\tilde{\Lambda}_{\hbar}}^- = \mathrm{d}\tilde{\Lambda}_{\hbar}^{\kappa X-\mathrm{i}Y} = -\sqrt{2\hbar\kappa}\ \partial_z, \qquad (6.91)$$

which annihilates the vacuum $\tilde{\phi}_{\hbar\kappa}(z) = c_0 \in \mathbb{C}$. Furthermore, the intertwining property implies:

Corollary 6.22. *The action $\tilde{\Lambda}_{\hbar} = \varepsilon_d \circ \Lambda_{\hbar}(g) \circ \varepsilon_d^{-1}$ (6.90) is a unitary irreducible representation of $\mathbb{H}_p^1$ in the FSB space $F_2^{\hbar}$. Two such actions, $\tilde{\Lambda}_{\hbar}$ and $\tilde{\Lambda}_{\hbar'}$, are not equivalent for $\hbar \neq \hbar'$.*

Similar to the pre-FSB transform, we calculate the composition of the pre-theta

transform $\tilde{f}$ (6.48) and the peeling $\mathrm{e}^{\frac{h}{4\kappa}(x^2+\kappa^2 y^2-2\mathrm{i}\kappa xy)-c}$ (6.86), $h = 2\pi m$, as follows:

$$
\begin{aligned}
\tilde{F}(x,y) &= \mathrm{e}^{\frac{h}{4\kappa}(x^2+\kappa^2 y^2-2\mathrm{i}\kappa xy)-c}\,\tilde{f}(x,y) \\
&= \int_{\mathbb{T}^2} f(u,v)\,\mathrm{e}^{-\frac{\pi m}{\kappa}(u^2+2\mathrm{i}\kappa uv)-\frac{\pi m}{2\kappa}(x^2-\kappa^2 y^2+2\mathrm{i}\kappa xy)+\frac{2\pi m}{\kappa}u(x+\mathrm{i}\kappa y)-c} \\
&\qquad\qquad\qquad\qquad \times \bar{\Theta}_{m\kappa}\left(\sqrt{\frac{h}{2\kappa}}(\kappa(v-y)+\mathrm{i}(u-x),\mathrm{i}\right)\mathrm{d}u\,\mathrm{d}v \\
&= \int_{\mathbb{T}^2} f(u,v)\,\mathrm{e}^{-\frac{\pi m}{\kappa}(u^2+2\mathrm{i}\kappa uv)-\frac{1}{2}z^2+2\sqrt{\frac{\pi m}{\kappa}}uz-c}\,\bar{\Theta}_{m\kappa}(-\mathrm{i}(\bar{z}'-\bar{z}),\mathrm{i})\,\mathrm{d}u\,\mathrm{d}v \\
&= \tilde{F}(z),
\end{aligned}
$$

(6.92)

where $z = \sqrt{\frac{h}{2\kappa}}(x+\mathrm{i}\kappa y) \in \mathbb{C}$. It was considered in Subsection 5.4.3 that the theta function $\Theta_{m\kappa}(\omega,\mathrm{i})$ is analytic in $\omega = \sqrt{\frac{h}{2\kappa}}(\kappa y+\mathrm{i}x) = \mathrm{i}\bar{z}$. Thus, $\bar{\Phi}(\bar{z}) = \bar{\Theta}_{m\kappa}(\mathrm{i}\bar{z},\mathrm{i})$ is analytic in z. Therefore, the function $\tilde{F}(z) = \mathrm{e}^{d(z,\bar{z})}\tilde{f}(z,\bar{z})$ (6.92) satisfies the Cauchy-Riemann equation

$$
\partial_{\bar{z}}F(z) = (\kappa\partial_x+\mathrm{i}\partial_y)F(x,y) = 0. \tag{6.93}
$$

We call $\tilde{F}(x,y)$ the *theta transform*. Note that in both cases of peeling the pre-FSB and pre-theta spaces, we obtained the well-known space of analytic functions, which can be found in many references, such as [8, 19, 35]. Similar to the decomposition of the left quasi-regular representation on $L_2(\mathbb{R}^2)$, we decompose FSB representation into irreducible components

$$
L_2(\mathbb{C}, \mathrm{e}^{-2\Re(d_{n,h}(z,\bar{z}))}\mathrm{d}z\,\mathrm{d}\bar{z}) = \oplus_{n=0}^{\infty}F_2^{n,h}(\mathbb{C}),
$$

where $F_2^{n,h}(\mathbb{C})$ are the corresponding invariant irreducible subspaces having the following property:

$$
F_2^{n,h}:\quad \partial_{\bar{z}}^m f = 0,\ \forall m \geq n,\quad \partial_{\bar{z}}^{n-1}f \neq 0. \tag{6.94}
$$

The spaces $F_2^{n,h}(\mathbb{C})$ are called the *n-th true poly-analytic space* [44]. In particular, if $n = 0$, we deliver to the FSB space of analytic functions.

6.5.2 Peeling the Schrödinger Representation

In this subsection, we peel the Schrödinger representation $\rho_{\hbar}$ (5.17) so that the corresponding annihilation operator in the corresponding space will be only the derivative

∂_t. As in the case of peeling the pre-FSB representation, it is a matter of simplifying the eigenfunction (vacuum) of the initial component.

Let $c \in \mathbb{C}$ be a non-zero arbitrary constant. In Subsection 5.4.2, we obtained the vacuum $\phi_{\hbar\kappa}(t) = c\,e^{-\frac{\pi\hbar}{\kappa}t^2} \in L_2(\mathbb{R})$ of the annihilation operator $a^-_{\rho_\hbar} = d\rho_\hbar^{\kappa X - iY} = -2\pi\hbar t - \kappa\partial_t$ (5.41). We choose a peeling operator ε_d defined by $d(t)$ such that

$$\begin{aligned}
\varepsilon_d : \phi_{\hbar\kappa} &\mapsto e^{d(t)}\phi_{\hbar\kappa} \\
c\,e^{-\frac{\pi\hbar}{\kappa}t^2} &\mapsto \tilde{\phi} = c.
\end{aligned} \tag{6.95}$$

Thus,

$$\varepsilon_d \cdot I = e^{\frac{\pi\hbar}{\kappa}t^2} \cdot I. \tag{6.96}$$

Then, ε_d transforms the Hermite functions $H_n(t)e^{-\frac{\pi\hbar}{\kappa}t^2}$ from the initial representation space of $\rho_\hbar$, to the corresponding *Hermite polynomials* $H_n(t)$, where the equivalent representation $e^{d(t)} \circ \rho_\hbar \circ e^{-d(t)}$ acts. The operator ε_d is unitary:

$$\varepsilon_d : L_2(\mathbb{R}, dt) \longrightarrow L_2(\mathbb{R}, e^{-2\frac{\pi\hbar}{\kappa}t^2}dt). \tag{6.97}$$

The system of chains in Figure 6.2 represents the peeling transformation. The cor-

$$
\begin{array}{ccc}
H_0(t)e^{-\frac{\pi\hbar}{\kappa}t^2} & \xrightarrow{\varepsilon_d} & H_0(t) \\
a^-\Big\updownarrow a^+ & & \tilde{a}^-\Big\updownarrow \tilde{a}^+ \\
H_1(t)e^{-\frac{\pi\hbar}{\kappa}t^2} & & H_1(t) \\
a^-\Big\updownarrow a^+ & & \tilde{a}^-\Big\updownarrow \tilde{a}^+ \\
H_2(t)e^{-\frac{\pi\hbar}{\kappa}t^2} & & H_2(t) \\
a^-\Big\updownarrow a^+ & & \tilde{a}^-\Big\updownarrow a^+ \\
H_1(t)e^{-\frac{\pi\hbar}{\kappa}t^2} & & H_1(t) \\
a^-\Big\updownarrow a^+ & & \tilde{a}^-\Big\updownarrow \tilde{a}^+ \\
\vdots & & \vdots
\end{array}
$$

Figure 6.2: Transforming Hermite functions to Hermite polynomials

responding representation $\tilde{\rho}_\hbar$ acting on $F \in L_2(\mathbb{R}, e^{-2\frac{\pi\hbar}{\kappa}t^2}dt)$ is calculated as follows:

$$[\mathrm{e}^{d(t)} \circ \rho_\hbar(s,x,y) \circ \mathrm{e}^{-d(t)}F](t) = \mathrm{e}^{\frac{\pi\hbar}{\kappa}t^2}\mathrm{e}^{2\pi\mathrm{i}\hbar(s-ty)}\mathrm{e}^{-\frac{\pi\hbar}{\kappa}(t-x)^2}F(t-x)$$
$$= \mathrm{e}^{2\pi\mathrm{i}\hbar s}\mathrm{e}^{-\frac{\pi\hbar}{\kappa}(x^2-2xt+2\mathrm{i}\kappa yt)}F(t-x)$$
$$= \mathrm{e}^{2\pi\mathrm{i}\hbar s}\mathrm{e}^{-\frac{\pi\hbar}{\kappa}(x^2-2t(x-\mathrm{i}\kappa y))}F(t-x)$$
$$=: [\tilde{\rho}_\hbar(s,x,y)F](t). \tag{6.98}$$

Consequently, the corresponding derived representations of the Lie algebra $\mathfrak{h}_1$ are

$$\mathrm{d}\tilde{\rho}_\hbar^X = 2\pi\hbar t - \partial_t, \quad \mathrm{d}\tilde{\rho}_\hbar^Y = -2\pi\mathrm{i}\hbar t \tag{6.99}$$

and the annihilation operator $a_{\tilde{\rho}_\hbar}^-$ is

$$a_{\tilde{\rho}_\hbar}^- = \mathrm{d}\tilde{\rho}_\hbar^{\kappa X - \mathrm{i}Y} = -\partial_t, \tag{6.100}$$

which annihilates the vacuum $\tilde{\phi}(t) = c$.

6.5.3 Peeling the Lattice Representation

The purpose of peeling the lattice representation ρ_m is similar to that of peeling the Schrödinger representation $\rho_\hbar$. That means we will simplify the initial vacuum (5.47)

$$\Phi(\omega, \bar{\omega}) = c\,\mathrm{e}^{\frac{1}{4}(3\omega^2 - \bar{\omega}^2 - 2\omega\bar{\omega})}\,\Theta_{m\kappa}(\omega, \mathrm{i}), \qquad \omega = \sqrt{\frac{h}{2\kappa}}(\kappa v + \mathrm{i}u) \in \mathbb{C}, \tag{6.101}$$

of ρ_m calculated in Subsection 5.4.3, for any non-zero $c \in \mathbb{C}$. We choose a peeling operator ε_d defined by $d(\omega, \bar{\omega})$ such that

$$\varepsilon_d : \Phi(\omega, \bar{\omega}) \mapsto \mathrm{e}^{d(\omega, \bar{\omega})}\Phi(\omega, \bar{\omega})$$
$$= c\,\Theta_{m\kappa}(\omega, \mathrm{i}). \tag{6.102}$$

Thus,

$$\varepsilon_d \cdot I = \mathrm{e}^{d(\omega, \bar{\omega})} \cdot I$$
$$= \mathrm{e}^{-\frac{1}{4}(3\omega^2 - \bar{\omega}^2 - 2\omega\bar{\omega})} \cdot I. \tag{6.103}$$

Then, ε_d is a unitary operator:

$$\varepsilon_d : L_2(\mathbb{T}^2, \mathrm{d}\omega\,\mathrm{d}\bar{\omega}) \longrightarrow \hat{L}_2(\mathbb{T}^2, \mathrm{e}^{-2\Re d(\omega, \bar{\omega})}\mathrm{d}\omega\,\mathrm{d}\bar{\omega})$$
$$\longrightarrow \hat{L}_2(\mathbb{T}^2, \mathrm{e}^{-\frac{2\pi m}{\kappa}u^2}\mathrm{d}u\,\mathrm{d}v) =: \tilde{H}. \tag{6.104}$$

Furthermore, for all $F \in \tilde{H}$, the corresponding irreducible lattice representation $\tilde{\rho}_m$ acting on $\tilde{H}$ is

$$
[\mathrm{e}^{d(\omega', \bar{\omega}')} \rho_m(s, \omega) \mathrm{e}^{-d(\omega', \bar{\omega}')} F](\omega', \bar{\omega}') = \mathrm{e}^{2\pi m \mathrm{i} s} \, \mathrm{e}^{\frac{1}{4}(\omega - \bar{\omega})^2 + \omega(\bar{\omega}' - \omega')} F(\omega' - \omega, \bar{\omega}' - \bar{\omega})
$$
$$
=: [\tilde{\rho}_m(s, \omega) F](\omega', \bar{\omega}'). \tag{6.105}
$$

Therefore, the corresponding annihilation operator is simply

$$
a_{\tilde{\rho}_m}^- = \mathrm{d}\tilde{\rho}_m^{\kappa X - \mathrm{i}Y} = \mathrm{i}\sqrt{2h\kappa}\, \partial_{\bar{\omega}}, \tag{6.106}
$$

which annihilates the theta function $\tilde{\Phi} =: c\,\Theta_{m\kappa}(\omega, \mathrm{i})$.

Chapter 7

Epilogue

7.1 Conclusion and Further Work

Taylor coefficients can be viewed as one of the fundamental examples of the covariant transform [26, 29] of analytic functions. Many classical results of harmonic analysis describe how the properties of functions are "transported" by the covariant transform [28, 30]. The results presented in Chapter 2 show a certain stability in this transportation: a small variation in the Taylor coefficients preserves certain properties of the function under consideration.

Our work in the remaining chapters provides a detailed construction of equivalent representations of the (polarised) Heisenberg group $\mathbb{H}_p^1$. This is summarised in Table 7.1. Furthermore, we have also found the corresponding vacuums and irreducible components. Using the covariant transform in a systematic way is a new approach to produce:

  i. the Zak transform and its inverse;

 ii. the theta function as a vacuum of the Zak transform; and

iii. the pre-theta transform and its inverse.

Furthermore, we used the peeling operator to obtain the FSB space of analytic functions. In Figure 1.1, we have shown how our work on the intertwining operators relates these representations of Heisenberg group $\mathbb{H}_p^1$.

The main objective of the subsequent research will be a proper extension of the covariant transform for distributions. This shall be done to cover all aspects of the

Zak transform and its inverse. Furthermore, our research will make a closer connection between singularities of a complex analytic function f and the representations theory of the Schrödinger group $G = \mathbb{H}_p^1 \rtimes \mathrm{SL}_2(\mathbb{R})$. We shall pay close attention to the theta function Θ, which has the real line as a natural boundary. Some recent research about the natural boundaries for power series can be found in [10].

Representations of the Heisenberg group			
Representation	Schrödinger ρ_h	FSB Λ_h	Lattice ρ_m
Subgroup H	$H'_x = \{(s,0,y) \in \mathbb{H}^1_p, s \in \mathbb{R}, y \in \mathbb{R}^1\}$ (4.27)	$Z = \{(s,0,0) \in \mathbb{H}^1_p, s \in \mathbb{R}\}$ (4.24)	$H_d = \{(s,n,k) : (n,k) \in \Gamma', s \in \mathbb{R}\}$ (4.32)
Character $\bar\chi$	$\bar\chi_h(s,0,y) = e^{-2\pi i\hbar s}$	$\bar\chi_h(s,0,0) = e^{-2\pi i\hbar s}$	$\bar\chi_m(s,n,k) = e^{-2\pi m i s}$
Homogeneous space $X = G/H$	$X = \mathbb{R}$	$X = \mathbb{R}^2$	$X = \mathbb{T}^2$
Formula of representation on $L_2(X)$	$[\rho_h(s,x,y)f](x') = e^{2\pi i\hbar(s-x'y)} f(x'-x)$ (5.17)	$[\Lambda_h(s,x,y)f](x',y') = e^{2\pi i\hbar(s+x(y'-y))} f(x'-x, y'-y)$ (5.14)	$[\rho_m(s,x,y)f](u,v) = e^{2\pi m i(s+x(v-y))} f(u-x, v-y)$ (5.23)
Vacuum	$\phi_{\hbar\kappa}(t) = e^{-\frac{\pi\hbar}{\kappa}t^2}$ (5.42)	$\phi_{\hbar\kappa}(z, \bar z) = e^{\frac{1}{4}(z^2 - \bar z^2 - 2z\bar z)+c}$ (5.39)	$\Phi(\omega, \bar\omega) = e^{\frac{1}{4}(3\omega^2 - \bar\omega^2 - 2\omega\bar\omega)} \Theta_{m\kappa}(\omega, i)$ (5.47)
Peeling operator	$e^{d(t)} \cdot I = e^{\frac{\pi\hbar}{\kappa}t^2} \cdot I$ (6.96)	$e^{d(z',\bar z')} = e^{\frac{1}{4}(\bar z'^2 - z'^2 + 2z'\bar z')-c}$ (6.86)	$e^{-\frac{1}{4}(3\omega^2 - \bar\omega^2 - 2\omega\bar\omega)}$ (6.103)
Corresponding representation after peeling	$\tilde\rho_h(s,x,y)F](t) = e^{2\pi i\hbar s}e^{-\frac{\pi\hbar}{\kappa}(x^2 - 2t(x-i\kappa y))} F(t-x)$ (6.98)	$[\tilde\Lambda_h(s,z)F](z') = e^{\hbar i s + \frac{1}{4}(\bar z^2 - z^2 - 2z\bar z)+\bar z z'} F(z'-z)$ (6.90)	$[\tilde\rho_m(0,\omega)F](\omega', \bar\omega') = e^{\frac{1}{4}(\omega^2 - 2\omega\bar\omega)+\omega(\bar\omega'-\omega')} F(\omega'-\omega, \bar\omega'-\bar\omega)$ (6.105)
Irreducible component	$L_2(\mathbb{R}, e^{-2\frac{\pi\hbar}{\kappa}t^2}dt)$ (6.97)	$F_2^{\hbar}(\mathbb{C}, e^{-|z|^2+2c}dz\,d\bar z)$ (6.88)	$L_2(\mathbb{T}^2, e^{-\frac{\hbar}{\kappa}u^2}du\,dv)$ (6.104)
Vacuum after peeling	$\tilde\phi(t) = c$ (6.100)	$\tilde\phi(z) = c_0$ (6.91)	$\tilde\Phi(\omega, \bar\omega) = c\Theta_{m\kappa}(\omega, i)$ (6.102)

Table 7.1: A Summary of Information about Representations of $\mathbb{H}^1_p$

Bibliography

[1] M. B. Abbott, *An introduction to the method of characteristics*, American Elsevier, New York, 1966. MR0201100 ↑51

[2] A. Alameer, *Numerically stable conditions on rational and essential singularities*, Complex Variables and Elliptic equations **63** (2017), no. 5, 640–651. arXiv:1604.06654. ↑i, 8, 10, 12, 16

[3] S. T. Ali, J.-P. Antoine, and J.-P. Gazeau, *Coherent states, wavelets, and their generalizations*, Second, Theoretical and Mathematical Physics, Springer, New York, 2014. MR3154614 ↑2

[4] _____, *Coherent states, wavelets, and their generalizations*, Second, Theoretical and Mathematical Physics, Springer, New York, 2014. MR3154614 ↑26, 55, 69

[5] F. Almalki and V. V. Kisil, *Geometric dynamics of a harmonic oscillator, arbitrary minimal uncertainty states and the smallest step 3 nilpotent Lie group*, J. Phys. A: Math. Theor **52** (2019), 025301. arXiv:1805.01399. ↑49, 57

[6] _____, *Solving the Schrödinger equation by reduction to a first-order differential operator through a coherent states transform* (2019). arXiv:1903.03554 . ↑57

[7] L. Auslander and R. Tolimieri, *Abelian harmonic analysis, theta functions and function algebras on a nilmanifold*, Lecture Notes in Mathematics, Vol. 436, Springer-Verlag, Berlin-New York, 1975. MR0414785 ↑65

[8] R. Berndt, *Representations of linear groups*, Vieweg, Wiesbaden, 2007. An introduction based on examples from physics and number theory. MR2340988 ↑21, 26, 27, 38, 41, 42, 61, 80

[9] A. Bohm, *Quantum mechanics: foundations and applications*, Third, Texts and Monographs in Physics, Springer-Verlag, New York, 1993. With the collaboration of Mark Loewe. MR1226227 ↑44, 46, 48, 49, 55

[10] P. Cartier, *Quantum mechanical commutation relations and theta functions*, Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965), 1966, pp. 361–383. MR0216825 ↑1, 2, 35, 36, 38, 47, 61, 86

[11] A. Córdoba, *Dirac combs*, Letters in Mathematical Physics **17** (Apr. 1989), 191–196. ↑64, 71

[12] I. Dolgachev, *Lectures on modular forms*, Apuntes, 1997. ↑35

[13] M. Duflo and C. C. Moore, *On the regular representation of a nonunimodular locally compact group*, J. Functional Analysis **21** (1976), no. 2, 209–243. MR52#14145 ↑56

[14] M. Eichler and D. Zagier, *The theory of Jacobi forms*, Progress in Mathematics, vol. 55, Birkhäuser Boston, Inc., Boston, MA, 1985. MR781735 ↑38

[15] H. G. Feichtinger and K. H. Gröchenig, *Banach spaces related to integrable group representations and their atomic decompositions. I*, J. Funct. Anal. **86** (1989), no. 2, 307–340. MR1021139 (91g:43011) ↑64

[16] _____, *Banach spaces related to integrable group representations and their atomic decompositions. II*, Monatsh. Math. **108** (1989), no. 2-3, 129–148. MR1026614 (91g:43012) ↑64

[17] V. Fischer, *The Heisenberg oscillator*, Bull. Math. Sci. **2** (2012), no. 1, 181–191. MR2942676 ↑29, 37, 48, 49, 50, 52, 55

[18] V. Fischer and M. Ruzhansky, *Lower bounds for operators on graded Lie groups*, C. R. Math. Acad. Sci. Paris **351** (2013), no. 1-2, 13–18. MR3019754 ↑22, 29, 41, 56

[19] G. B. Folland, *Harmonic analysis in phase space*, Annals of Mathematics Studies, vol. 122, Princeton University Press, Princeton, NJ, 1989. ↑22, 29, 30, 31, 32, 33, 38, 45, 47, 48, 57, 58, 61, 65, 66, 80

[20] J.-P. Gazeau, SL(2, **R**)-*coherent states and integrable systems in classical and quantum physics*, Quantization, coherent states, and complex structures (Białowieża, 1994), 1995, pp. 147–158. MR1403156 ↑48

[21] _____, *Coherent States in Quantum Physics*, Wiley-VCH Verlag, 2009. ↑48

[22] A. Grossmann, J. Morlet, and T. Paul, *Transforms associated to square integrable group representations. I. General results*, J. Math. Phys. **26** (1985), no. 10, 2473–2479. MR803788 ↑2

[23] _____, *Transforms associated to square integrable group representations. II. Examples*, Ann. Inst. H. Poincaré Phys. Théor. **45** (1986), no. 3, 293–309. MR868528 ↑2

[24] A. A. Kirillov, *Elements of the theory of representations*, Springer-Verlag, Berlin, 1976. Translated from the Russian by Edwin Hewitt, Grundlehren der Mathematischen Wissenschaften, Band 220. ↑19, 20, 24, 25, 26, 41, 42, 73

[25] _____, *Lectures on the orbit method*, Graduate Studies in Mathematics, vol. 64, American Mathematical Society, Providence, RI, 2004. MR2069175 ↑19, 21, 23, 24, 25, 29, 33, 41, 42, 48, 49, 50, 52, 55, 73

[26] V. V. Kisil, *Analysis in* **R**^{1,1} *or the principal function theory*, Complex Variables Theory Appl. **40** (1999), no. 2, 93–118. arXiv:funct-an/9712003. MR1744876 (2000k:30078) ↑85

[27] _____, *Relative convolutions. I. Properties and applications*, Adv. Math. **147** (1999), no. 1, 35–73. arXiv:funct-an/9410001,. MR1725814 (2001h:22012) ↑64

[28] _____, *Wavelets in Banach spaces*, Acta Appl. Math. **59** (1999), no. 1, 79–109. arXiv:math/9807141, On-line. ↑42, 45, 64, 85

[29] _____, *Erlangen programme at large: an Overview*, Advances in applied analysis, 2012, pp. 1–94. arXiv:1106.1686. ↑22, 55, 56, 57, 59, 68, 69, 73, 85

[30] _____ , *The real and complex techniques in harmonic analysis from the point of view of covariant transform*, Eurasian Math. J. **5** (2014), 95–121. ↑57, 85

[31] _____ , *Uncertainty and analyticity*, Current trends in analysis and its applications, 2015, pp. 583–590 (English). arXiv:1312.4583. ↑55

[32] _____ , *Symmetry, geometry and quantization with hypercomplex numbers*, Geometry, integrability and quantization XVIII, 2017, pp. 11–76. arXiv:1611.05650. MR3616912 ↑29, 33, 34, 37, 41, 42, 43, 55, 57

[33] _____ , *Toeplitz operators on the Fock–Segal–Bargmann space and two-sided convolutions on the Heisenberg group* (2019). (in preparation). ↑51, 70, 77

[34] D. Mumford, *Tata lectures on theta. I*, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2007. With the collaboration of C. Musili, M. Nori, E. Previato and M. Stillman, Reprint of the 1983 edition. MR2352717 ↑2, 35, 36, 38, 39

[35] Y. A. Neretin, *Lectures on Gaussian integral operators and classical groups*, EMS Series of Lectures in Mathematics, European Mathematical Society (EMS), Zürich, 2011. MR2790054 ↑47, 61, 70, 80

[36] C. D. Olds, A. Lax, and G. P. Davidoff, *The geometry of numbers*, Anneli Lax New Mathematical Library, vol. 41, Mathematical Association of America, Washington, DC, 2000. Appendix I by Peter D. Lax. MR1817689 ↑35

[37] A. Perelomov, *Generalized coherent states and their applications*, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1986. MR87m:22035 ↑2

[38] _____ , *Generalized coherent states and their applications*, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1986. MR858831 ↑48

[39] G. Pólya and G. Szegó, *Problems and theorems in analysis. I*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 193, Springer-Verlag, Berlin-New York, 1978. Series, integral calculus, theory of functions, Translated from the German by D. Aeppli, Corrected printing of the revised translation of the fourth German edition. MR580154 ↑2, 5, 8

[40] R. Remmert, *Theory of complex functions*, Graduate Texts in Mathematics, vol. 122, Springer-Verlag, New York, 1991. Translated from the second German edition by Robert B. Burckel, Readings in Mathematics. MR1084167 ↑14

[41] _____ , *Classical topics in complex function theory*, Vol. 172, Springer Science & Business Media, 2013. ↑1, 2, 6, 7, 10

[42] J. J. Rotman, *An introduction to the theory of groups*, Fourth, Graduate Texts in Mathematics, vol. 148, Springer-Verlag, New York, 1995. MR1307623 ↑37

[43] E. M. Stein and R. Shakarchi, *Complex analysis*, Princeton Lectures in Analysis, II, Princeton University Press, Princeton, NJ, 2003. MR1976398 ↑38

[44] N. L. Vasilevski, *On the structure of Bergman and poly-Bergman spaces*, Integral Equations Operator Theory **33** (1999), no. 4, 471–488. ↑75, 77, 79, 80

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