Arbitrage and derivative securities under fixed and proportional transaction costs

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1. **Abstract**

The theory of stock price models under fixed transaction costs is a relatively little explored area of modern mathematical finance. This thesis introduces a fixed transaction cost process and uses it to not only explore the world of fixed transaction costs, but also combine fixed transaction costs with proportional transaction costs (so-called combined transaction costs). We prove a no-arbitrage equivalent condition for a model under combined transaction costs by re-visiting the existing no-arbitrage conditions for proportional and fixed costs respectively, and proving each of them, by taking a first principles approach. This result can be seen as the fundamental theorem for a model under combined transaction costs.

This research on combined transaction costs also presents an extensive contribution to the analysis of European derivative securities. A distinction is made between the situation when the number of derivatives that can be traded on demand is limited compared to when it is unlimited. It will be shown that the ask and bid prices of a derivative security are unchanged by the presence of a fixed transaction cost when the derivative can be purchased in unlimited quantities. One of the main achievements here is a risk-neutral representation for the ask and bid prices of European derivative securities under combined transaction costs when the quantity of the derivatives that can be purchased on demand is restricted, as it overcomes hurdles connected to the lack of convexity which is involved in the combined cost ask price calculation algorithm.

2. **Declaration**

I declare that this thesis is a presentation of original work and I am the sole author. This work has not previously been presented for an award at this, or any other, University. All sources are acknowledged as References.

3. Acknowledgments

Firstly, I would like to thank my supervisor, Professor Tomasz Zastawniak, for the patience, kindness and wisdom that he has shown throughout my PhD. I would particularly like to express my appreciation for all the insightful advice that I have received from him during my time at the University of York, as well as the generosity with which his time was afforded to me for meetings to discuss my work. I would also like to thank my TAP panel, Dr Alexei Daletskii and Dr Alet Roux for all the excellent advice they have given me, and the Postgraduate Administrator, Nicholas Page, for providing excellent guidance through the process of completing my PhD. Finally, I would like to give special thanks to my family for their unwaivering support over all my many years of studying.

Contents

4. **Introduction**

One of the most famous results in mathematical finance is the fundamental theorem of asset pricing which asserts that in a frictionless market, the non-existence of arbitrage is equivalent to the existence of an equivalent martingale measure (EMM). This was first formalised in the paper by Harrison and Pliska, [11] and there has been a huge volume of work surrounding it ever since, e.g. see [5] and [17].

An analogous version of the fundamental theorem for models involving proportional transaction costs was first formulated by Jouini and Kallal when they proved that the non-existence of proportional cost arbitrage is equivalent to the existence of a process that fits in between the stock ask and bid price processes and admits an EMM, see [15]. This fundamental theorem for proportional transaction costs has also been shown, in a context arguably closer to our setting, by Schachermayer in [25] as well as Kabanov and Stricker in [18]. Once again Jouini and Kallal, in collaboration with Napp, showed that they were at the forefront of this area by also developing a similar result in the situation when a fixed transaction cost (but no proportional transaction cost) is in play, see [16]. They showed that the non-existence of arbitrage under fixed transaction costs is equivalent to the existence of a certain family of absolutely continuous martingale measures (AMM). This result will be referred to as the fundamental theorem for fixed transaction costs. In 2017 Lepinette and Tran [21] attempted to tackle the problem of finding a no arbitrage condition for the setting of both fixed and proportional transaction, and to the best of my knowledge is the only other (apart from this thesis) analysis of arbitrage in this setting. In that paper the absence of asymptotic arbitrage is characterised by the existence of a so-called equivalent separating probability measure. However, no link is made with risk neutral probabilities, by contrast to this thesis. Indeed it can be shown that the non-existence of an equivalent separating probability measure does not, in fact, mean that a combined cost arbitrage opportunity must be present.

In our work we will first re-prove, by a new method, a slightly re-formulated version the fundamental theorem for models under fixed transaction costs. We will prove that the non-existence of a fixed cost arbitrage opportunity is equivalent to the existence of a so called absolutely continuous martingale function (AMF), which is similar to a probability measure but is a slightly richer structure, the existence of an AMF implies the existence of an AMM and is equivalent to the existence of a certain family of AMMs. We will do this by first doing a very basic analysis of single-step sub-models and then showing that properties such as, the existence of fixed cost arbitrage or the existence of an AMF, are true in a full model if and only if they are true in in every single-step sub-model. Whilst we are doing this we will take the time to learn the fundamental concepts that are involved in models with fixed transaction costs in order to develop a good intuition for the deeper analysis that we will do in the remainder of this thesis.

Once we have examined the situation with fixed costs only and the situation with proportional costs only, we will move on to working in a setting that involves both fixed and proportional transaction costs and show that the non-existence of so called combined cost arbitrage is equivalent to the existence of a process that fits in between the stock ask and bid price processes and admits an AMF. This elegant result is the combined cost version of the fundamental theorem of asset pricing. In proving this combined cost fundamental theorem we develop a new proof of the fundamental theorem for proportional costs (in fact we prove a slightly extended version of this theorem, which will be discussed later) and then apply it to the combined cost fundamental theorem. Part of the achievement of this new result is that the proofs that we use are based on a first principles approach involving a backward induction, and this gives us a very direct understanding of why the theorem is true. An interesting point of note is that using this method of proof, it is in fact easier to prove the combined cost fundamental theorem than it is to prove the proportional cost fundamental theorem.

In the second half of this thesis we will introduce European derivative securities. The study of derivative securities in non-convex financial markets has been tackled somewhat sparingly in the past, with only three papers delving into this area, see [16], [21] and [1]. It is important to point out the work of Bank and Dolinsky [1] here, as they show that the optimal super-replication strategy under fixed transaction costs of a convex option in continuous time is a trivial buy-and-hold strategy. This work built on the results of Soner, Shreve and Cvitanic in [26], that had shown a similar result for the situation with only proportional costs in continuous time. This highlights the difficulties of a continuous time market model and motivates our choice of a discrete time approach to modeling financial markets. Having said this, there is a number of techniques that can be used to overcome the hurdles that continuous time brings, such as the work of Davis and Norman [4], who built on the work of Magill and Constantinides in [22], and showed that, in the context of maximising the utility of consumption, there must be a no trade region, where portfolio rebalancing does not take place. This area of research was also developed further in [3], [20], [9], [7] and [6].

One of the most interesting results in the theory of derivative pricing in the frictionless setting is that the ask-price of a derivative security in a frictionless viable model can always be represented as the supremum over the set of EMMs, of the expectation of the derivative payoff under each EMM (We will refer to this as the frictionless risk-neutral ask price representation theorem). This was first shown in the paper by Harrison and Pliska , see [11]. In 2008 Roux, Tokarz and Zastawniak proved the analogous version for a model under proportional transaction costs, see [24]. This was done by exploiting convex duality to link super replication strategies to certain pairs of functions. The main focus of our analysis of derivative securities will be to find an analogous version of these results in the combined cost setting, but we will also take an extensive look into the situation with fixed costs only, in order to broaden our understanding fixed costs and show many interesting results along the way, such as the fact that a derivative security in a model that does not admit arbitrage under fixed costs may not have a price that does not allow arbitrage in the extended market under fixed costs.

Under fixed costs an important distinction has to be made between the situation where a derivative can be traded in unlimited quantity compared to the situation where only a single derivative is available. It turns out that if a derivative security can be bought in unlimited quantities then its ask price is unaffected by the fixed cost process. This means that the fixed cost ask price is equal to the frictionless ask price and the combined cost ask price is equal to the proportional cost ask price. This also means that an ask price representation theorem for combined costs follows immediately from the work of Roux, Tokarz and Zastawniak [24], in the situation when a derivative can be bought in unlimited quantities. This leads us to our main problem, that is , can we find a risk-neutral representation for an ask price under combined costs when quantity of derivatives is limited? Unfortunately applying the technique used by Roux, Tokarz and Zastawniak in [24] will not work here as the fixed transaction costs removes the convexity of the sets of super-replication portfolios and thus convex duality cannot be exploited directly. Instead we will work around this by essentially removing the lack of convexity. In order to prove this risk-neutral ask price representation theorem for combined transaction costs with a limited quantity of derivatives available, we will introduce some new theoretical tools. We will extend our model to include the possibility that no trading is allowed at certain nodes. This will mean that we will need to extend many of the aforementioned results, such as the proportional cost fundamental theorem and the risk-neutral ask price representation theorem for proportional costs, which is a highly non-trivial task. It is worth noting here that the use of infinite transaction costs to model a situation where we have no trading at certain nodes has been seen before in [12]. The final part of this thesis will be the statement and proof of the risk-neutral ask price representation theorem for combined costs. This is an elegant result that we will be able to show in a relatively simple way by using the sophisticated tools that we have developed.

Overview of the structure of the thesis

This thesis can be thought of as being divided into two main parts:

Part 1: Deals with models only and does not introduce a derivative security. This is from Section 5 to Section 11.

Part 2: Analysis that is focused on derivative securities. This is from Section 12 to Section 19.

Within each part there will be three sections that set up notation at the beginning. Sections 5, 6 and 7 in part 1 and Sections 12, 13 and 14 in part 2. After these introductory sections the main sections that involve most of the key analysis will follow. The main sections will include four sections that cover the frictionless, fixed cost, proportional cost and combined cost cases respectively. At the end of the main part of this thesis there will be a concluding section, highlighting some of our main achievements and drawing attention to future directions of research, see Section 20. There will also be a glossary of definitions, conventions and notation (see Section 21), that will serve as useful point of reference for the reader. The final section of this thesis will be an appendix containing some additional results and pieces of notation, see Section 22.

5. Model set up and notation

5.1. Guide to notation.

In this thesis we will introduce a large number of mathematical objects that will be represented by various different symbols. Thus it is important to take a moment to establish some guidelines that can help us to remember the meaning of a given symbol. The following are the two main conventions that will be followed wherever possible:

1) The type of symbol will indicate the type of mathematical object according to the following rule:

• Greek alphabet $\rightarrow$ Scenarios, nodes, sets of nodes.

Examples: Set of all scenarios Ω, generic scenario ω, set of all nodes Λ, generic node λ.

Exceptions: ϵ, δ are used as real numbers for the purpose of considering a limit.

• Curly Latin letters $\rightarrow$ Sets, process of sets.

Examples: Set of times $\mathcal{T}$, Set of viable prices $\mathcal{V}$

Exceptions: $\mathcal{F}$, is used to mean a filtration (ordered collection of sets).

• Lower case Latin letters $\rightarrow$ real numbers or functions from $\mathbb{R}$ to $\mathbb{R}$.

Examples: Ask price a, bid price b, generic real number k, generic constant c, generic function f.

Exceptions: u, m, d when used as nodes.

• Upper case Latin letters $\rightarrow$ functions from Λ to $\mathbb{R}$ (processes in nearly all cases).

Examples: S, F, P, X, Y

Exceptions: T

• Bold font Latin $\rightarrow$ function to $\mathbb{R}$

There is only one example of this type: $\mathbf{Q}$

• Blackboard font Latin $\rightarrow$ can be anything.

Examples: $\mathbb{M}$, $\mathbb{P}$, $\mathbb{Q}$, $\mathbb{H}$

Please note that this rule applies to the main symbol within a cluster of symbols. For example, $a_P(D)$ will be used to represent the ask price of a derivative security D under proportional transaction costs and is a real number (a is the main symbol).

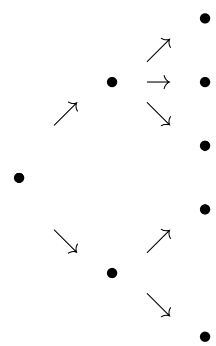
2) If we are able to use a Latin letter, we will where possible use a letter that corresponds to the name of the given mathematical object.

Examples: We will write S for stock process, F for fixed cost process and P for proportional cost process.

Exceptions: There is a large number of situations where it was simply not possible/practical to follow this rule, but none the less it is a useful guideline.

The purpose of the aforementioned guidelines is to give the reader a good first guess of the meaning of a given symbol. However an extensive glossary of definitions, conventions and notation is provided at the end of this thesis (see section 21) in order to assist the reader in finding the meaning of a symbol when in need of a memory refresh. Indeed I believe it is also well worth going over this summary a few times in order to insure that these definitions, conventions and pieces of notation are well ingrained.

5.2. **The tree structure of a model.**



In this subsection we review the function and meaning of a set of scenarios Ω and filtration $\mathcal{F}$ on Ω.

In our work, the set of scenarios Ω will be assumed to be finite and it will often be written as $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$ for some $n \in \mathbb{N}$.

Each ω_i represents a real world scenario.

The filtration $\mathcal{F}$ is used to model the way information is updated. We can deduce from $\mathcal{F}$ a time set $\mathcal{T} = \{0, 1, \ldots, T\}$, for some $T \in \mathbb{N}$, representing the times at which new information is released (we could deduce this by simply counting the number of elements of $\mathcal{F}$).

Thus we will write $\mathcal{F} = \{\mathcal{F}_t\}_{t \in \mathcal{T}}$, where $\mathcal{F}_t \subset \mathcal{F}_{t+1}$ for all $t \in \{0, 1, \ldots, T-1\}$.

$\mathcal{F}_T$ is a sigma-field and represents the collection of all possible events. An event can be thought of as a set containing a number of different scenarios (or none).

For example $\{\omega_1, \omega_3, \omega_n\}$ is an event belonging to $\mathcal{F}_T$ and has the intuitive meaning of "the event that either ω_1, ω_3 or ω_n occurs".

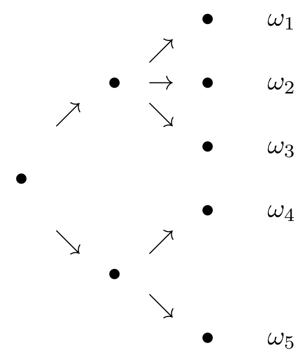
Each $\mathcal{F}_t$ represents the information available at time t, and consists of every event $\mathcal{E}$ such that it is possible to determine whether the event $\mathcal{E}$ has occurred or not at time t (no matter what happens up to and including time t).

We will soon see an example of an element of a filtration after we have seen a diagram of the tree structure.

The filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t \in \mathcal{T}}$ generates a tree structure.

The time set $\mathcal{T} = \{0, 1, \ldots, T\}$ represents the set of times when our information is updated.

Such a tree structure could look like this:



An example of an element of $\mathcal{F}_1$ is the event $\{\omega_1, \omega_2, \omega_3\}$. This is because no matter what scenario we follow, at time 1, we will be able to say either that the event $\{\omega_1, \omega_2, \omega_3\}$ definitely won't happen or that it definitely will happen.

To help gain a feel for the way in which the filtration generates the tree the following is a table that lists the atoms of $\mathcal{F}_0, \mathcal{F}_1$ and $\mathcal{F}_2$ in the example of the previous diagram.

Atoms of $\mathcal{F}_0$	Atoms of $\mathcal{F}_1$	Atoms of $\mathcal{F}_2$
Ω	$\{\omega_1, \omega_2, \omega_3\}$	$\{\omega_1\}$
	$\{\omega_4, \omega_5\}$	$\{\omega_2\}$
		$\{\omega_3\}$
		$\{\omega_4\}$
		$\{\omega_5\}$

As we have seen in the above picture a probability space and filtration can be thought of as producing a tree structure consisting of "dots and branches". We will often need to refer to the "dots" in the picture and the technical term to describe each one of them is node.

Definition 5.1. (Node)
A *node* λ in a filtration $\mathcal{F}$ with time set $\mathcal{T} = \{0, 1, \ldots, T\}$ is an atom of $\mathcal{F}_t$ for some $t \in \mathcal{T}$.

We say that such a node is a node at time t.

Definition 5.2. (Successor node)
A *successor node* μ to a node λ at time t in a filtration $\mathcal{F}$ with time set $\mathcal{T} = \{0, 1, \ldots, T\}$ is a node at time $t + 1$ such that $\mu \subset \lambda$.

Definition 5.3. (Set of successor nodes)
For any non-terminal node λ we define Succ λ to be the set of successor nodes to node λ.

Let us consider our example of a tree structure again:

$$
\begin{array}{cccccc}
 & & & & \bullet & \omega_1 \\
 & & & \nearrow & & \\
 & & \bullet & \rightarrow & \bullet & \omega_2 \\
 & \nearrow & & \searrow & & \\
 & & & & \bullet & \omega_3 \\
\bullet & & & & & \\
 & & & & \bullet & \omega_4 \\
 & \searrow & & \nearrow & & \\
 & & \bullet & & & \\
 & & & \searrow & & \\
 & & & & \bullet & \omega_5
\end{array}
$$

From this picture, the black dots represent nodes. The vertical line of nodes furthest to the right are the nodes at time 2 and the vertical line of nodes in the middle are the nodes at time 1. The node furthest to the left is the only node at time zero, it will be referred to as the root node.

We will often name the nodes in a way that is something like the way that is illustrated in the following diagram.

$$
\begin{array}{cccccc}
 & & & & uu & \omega_1 \\
 & & & \nearrow & & \\
 & & u & \rightarrow & um & \omega_2 \\
 & \nearrow & & \searrow & & \\
 & & & & ud & \omega_3 \\
\Omega & & & & & \\
 & & & & du & \omega_4 \\
 & \searrow & & \nearrow & & \\
 & & d & & & \\
 & & & \searrow & & \\
 & & & & dd & \omega_5
\end{array}
$$

It is important to remember the meaning of each node in the picture. For example $u = \{\omega_1, \omega_2, \omega_3\}$ and $uu = \{\omega_1\}$.

We have now seen all of the notation involving nodes that we will need in our definition of a model. However will see some additional notation involving nodes in Subsection 5.5.

Having examined the picture that is produced by a filtration and introduced some notation, we are now in a position to add processes to this, in order to have a full model.

5.3. **Defining a model.**

Definition 5.4. (Model)
A *model* $\mathbb{M} = (\Omega, \mathcal{F}, \mathbb{P}, S, F, P)$ consists of a set of scenarios Ω, a filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t\in\mathcal{T}}$ for some time set $\mathcal{T} = \{0, 1, \ldots, T\}$, a probability measure $\mathbb{P}$, and three processes, $S = (S_t)_{t\in\mathcal{T}}$, $F = (F_t)_{t\in\mathcal{T}}$ and $P = (P_t)_{t\in\mathcal{T}}$.

The processes S, F and P will be referred to as the stock price process, fixed transaction cost process and proportional transaction cost process respectively.

A model will have the following properties:

(1) The triple $(\Omega, \mathcal{F}_T, \mathbb{P})$ is a finite probability space.

(2) For every $\mathcal{E} \in \mathcal{F}_T$, such that $\mathcal{E} \neq \emptyset$, we have $\mathbb{P}(\mathcal{E}) > 0$.

(3) $\mathcal{F} = \{\mathcal{F}_t\}_{t\in\mathcal{T}}$ is such that $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

(4) $S = (S_t)_{t\in\mathcal{T}}$ takes values in $\mathbb{R}^+$ and is adapted to the filtration $\mathcal{F}$.

(5) $F = (F_t)_{t\in\mathcal{T}}$ takes values in $\mathbb{R}^+$ and is adapted to the filtration $\mathcal{F}$.

(6) $P = (P_t)_{t\in\mathcal{T}}$ takes values in $\mathbb{R}^\infty$ and is adapted to the filtration $\mathcal{F}$.

(7) For any non-terminal node λ there exists more than one successor node.

Note that $\mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{R}^\infty := \{0\} \cup \mathbb{R}^+ \cup \{\infty\}$. The reason for allowing P to be equal to infinity will be explained later, for now we may think of it as being like any other process.

The probability measure $\mathbb{P}$ will sometimes be referred to as the real world probability measure.

The fixed transaction cost process F represents the fixed amount that must be paid if one trades at a given time.

The proportional transaction cost process P represents the amount that must be paid per unit trade if one trades at a given time.

Convention 1. (Reference to a model)
Throughout this thesis we will construct definitions, conventions and examples without explicit mention of a model. In such situations it will be assumed that we are working in the context of a model $\mathbb{M} = (\Omega, \mathcal{F}, \mathbb{P}, S, F, P)$ with time set $\mathcal{T}$ associated with the filtration $\mathcal{F}$.

5.4. **Explanation of the model.**

As we have seen, a model consists of a probability space, filtration, time set and three processes.

We now consider the processes S, F and P and how they fit into the tree structure produced by the probability space and filtration.

Each of these processes is adapted to the filtration. This means that for any time t, the value random variables S_t, F_t and P_t is known and thus we can add them to the diagram in the following way. The numerical values given to them are for illustration purposes.

$$
\begin{array}{ccccc}
 & & & & S_2(\omega_1) = 12 \\
 & & & & F_2(\omega_1) = 2 \\
 & & & & P_2(\omega_1) = 3 \\
 & & & \nearrow & \\
 & & S_1(\omega_i) = 10 & & S_2(\omega_2) = 10 \\
 & & F_1(\omega_i) = 2 & \rightarrow & F_2(\omega_2) = 2 \\
 & & P_1(\omega_i) = 1 & & P_2(\omega_2) = 1 \\
 & \nearrow & & \searrow & \\
 & & & & S_2(\omega_3) = 8 \\
 & & & & F_2(\omega_3) = 4 \\
S_0(\omega_k) = 6 & & & & P_2(\omega_3) = 0 \\
F_0(\omega_k) = 2 & & & & \\
P_0(\omega_k) = 1 & & & & S_2(\omega_4) = 5 \\
 & & & & F_2(\omega_4) = 2 \\
 & & & & P_2(\omega_4) = 1 \\
 & \searrow & & \nearrow & \\
 & & S_1(\omega_j) = 3 & & \\
 & & F_1(\omega_j) = 1 & & \\
 & & P_1(\omega_j) = 1 & & \\
 & & & \searrow & \\
 & & & & S_2(\omega_5) = 2 \\
 & & & & F_2(\omega_5) = 3 \\
 & & & & P_2(\omega_5) = 1
\end{array}
$$

where $i = 1$, 2 or 3, $j = 4$ or 5 and $k =$ 1,2,3,4 or 5.

This diagram is the kind of picture that can be extracted from all of the models that we work with.

Note that it was a little awkward to write the part of the tree at times zero and one as there were many different scenarios that each gave the process the same value. Also the tree in the diagram is rather cluttered due to the large number of indices.

We have followed this set up as it is the most classical construct and in some circumstances will still be the most efficient form of notation. However we will now aim to introduce some notational conventions that will help us to write things in a slightly more elegant way.

Let us return to our picture of the structure of a typical model in terms of nodes and scenarios.

$$
\begin{array}{cccccc}
 & & & & uu & \omega_1 \\
 & & & \nearrow & & \\
 & & u & \rightarrow & um & \omega_2 \\
 & \nearrow & & \searrow & & \\
 & & & & ud & \omega_3 \\
\Omega & & & & & \\
 & & & & du & \omega_4 \\
 & \searrow & & \nearrow & & \\
 & & d & & & \\
 & & & \searrow & & \\
 & & & & dd & \omega_5
\end{array}
$$

Notice that any time t and scenario ω map to a single unique node.

It is also true that any node maps to a unique time, this is because of assumption (7) in definition 5.4.

Therefore an adapted process can be thought of as a function from the set of all nodes to $\mathbb{R}$.

This is summarised in the following formal convention.

Convention 2. (Notation for processes and random variables)
In our work we will interchange notation between referencing a node and referencing a time and scenario.

For example consider a node λ at time t, for some $t \in \mathcal{T}$, and scenario $\omega \in \lambda$.

Take any adapted process A.

We can write $A_t(\omega)$ to refer to the value of the process A at time t in scenario ω or we could also write $A(\lambda)$.

Occasionally we may even write $A_t(\lambda)$ (this is usually useful in situations where we are normally referring to the nodes but the emphasis on time makes the situation clearer).

In the special case when a process is predictable, for example the extended predictable process (see Definition 22.6) Y which will later be used to represent the position in stock of a portfolio strategy (this terminology will be explained later), we will sometimes write:

$Y_{t+1}(\lambda)$ which means $Y_{t+1}(\mu)$ for all nodes $\mu \in \text{Succ } \lambda$ if λ is a non-terminal node, and means $Y_{T+1}(\omega)$ for all $\omega \in \lambda$ if λ is a terminal node.

We will also sometimes write:

$Y_{+1}(\lambda)$ which means $Y_{t+1}(\lambda)$.

This means that if we wish to display our processes in a diagram using this convention then it could look like this.

$$
\begin{array}{ccccc}
 & & & & \begin{array}{c} S(uu) = 12 \\ F(uu) = 2 \\ P(uu) = 3 \end{array} \\
 & & & \nearrow & \\
 & & \begin{array}{c} S(u) = 10 \\ F(u) = 2 \\ P(u) = 1 \end{array} & \rightarrow & \begin{array}{c} S(um) = 10 \\ F(um) = 2 \\ P(um) = 1 \end{array} \\
 & \nearrow & & \searrow & \\
 & & & & \begin{array}{c} S(ud) = 8 \\ F(ud) = 4 \\ P(ud) = 0 \end{array} \\
\begin{array}{c} S(\Omega) = 6 \\ F(\Omega) = 2 \\ P(\Omega) = 1 \end{array} & & & & \\
 & & & & \begin{array}{c} S(du) = 5 \\ F(du) = 2 \\ P(du) = 1 \end{array} \\
 & \searrow & & \nearrow & \\
 & & \begin{array}{c} S(d) = 3 \\ F(d) = 1 \\ P(d) = 1 \end{array} & & \\
 & & & \searrow & \\
 & & & & \begin{array}{c} S(dd) = 2 \\ F(dd) = 3 \\ P(dd) = 1 \end{array}
\end{array}
$$

5.5. **Additional notation and terminology involving nodes.**

Definition 5.5. (Set of nodes at time t)
For any $t \in \mathcal{T}$ we define Λ_t to be the set of nodes at time t.

Definition 5.6. (Root node)
We define the term *root node* to mean the node at time 0.

Note that from the definition of a model $\mathcal{F}_0 = \{\Omega, \emptyset\}$, so Ω is the only node at time zero.

Definition 5.7. (Terminal node)
A *terminal node* is a node at time T.

Definition 5.8. (Set of all nodes)
$\Lambda := \Lambda_0 \cup \Lambda_1 \cup \ldots \cup \Lambda_T$.

Definition 5.9. (λ_t^ω)
For any $t \in \mathcal{T}$ and $\omega \in \Omega$ we will write λ_t^ω to mean the node at time t in scenario ω.

Definition 5.10. (Ancestor node)
We say that a node $\lambda \in \Lambda$ is a *ancestor node* of $\mu \in \Lambda$, if $\mu \subset \lambda$ and $\mu \neq \lambda$.

Definition 5.11. (Descendant node)
We say that a node $\mu \in \Lambda$ is a *descendant node* of $\lambda \in \Lambda$ if $\lambda \subset \mu$ and $\lambda \neq \mu$.

Definition 5.12. (Set of descendant nodes)
We define $\mathrm{desc}\lambda$ to be the set of descendant nodes to node λ.

Convention 3. (Working with more than one model)
Throughout this thesis we will sometimes need to work with more than one model. Typically we will have a model $\mathbb{M} = (\Omega, \mathcal{F}, \mathbb{P}, S, F, P)$ and a model $\mathbb{M}' = (\Omega', \mathcal{F}', \mathbb{P}', S', F', P')$. In such situations we will often write $\mathcal{T}' = \{0, \ldots, T'\}$ and Λ' to mean the set of times and set of nodes corresponding to the filtration $\mathcal{F}'$. Similarly if a model is written $\tilde{M} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{S}, \tilde{F}, \tilde{P})$ then we would write $\tilde{\mathcal{T}} = \{0, \ldots, \tilde{T}\}$ and $\tilde{\Lambda}$ for the time set and set of nodes corresponding to the filtration $\tilde{\mathcal{F}}$.

5.6. Sub-models.

Definition 5.13. (n-step sub-model at node $\lambda \in \Lambda_{t'}$)
A *n-step sub-model* at a node $\lambda \in \Lambda_{t'}$, $t' \in \mathcal{T}$ of a model $\mathbb{M} = (\Omega, \mathcal{F}, \mathbb{P}, S, F, P)$, where $t' + n \leq T$, is a model $\mathbb{M}' = (\Omega', \mathcal{F}', \mathbb{P}', S', F', P')$ such that:

- $\Omega' = \lambda$.
- $\mathcal{F}'_t = \{\mathcal{E} \in \mathcal{F}_{t'+t} : \mathcal{E} \subset \lambda\}$ for all $t \in \{0, \ldots n\}$.
- $\mathbb{P}'(\mathcal{E}) = \mathbb{P}(\mathcal{E}|\lambda)$ for all $\mathcal{E} \in \mathcal{F}'_n$.
- $S'(\mu) = S(\mu)$, $F'(\mu) = F(\mu)$ and $P'(\mu) = P(\mu)$ for all $\mu \in \Lambda'$.

Definition 5.14. (Sub-model at node $\lambda \in \Lambda$)
Let $\lambda \in \Lambda$ be such that $\lambda \in \Lambda_{t'}$ for some $t' \in \mathcal{T}$.

The sub-model at node λ is the $(T - t')-$step sub-model at node λ.

In other words it is the sub-model that starts at the node λ and ends at the terminal time.

Remark 5.15. (Sub-model at node λ)
Please note that we will frequently refer to "the sub-model at node λ" for some node λ, so it is important to remember that it has a precise meaning as given by the above definition.

5.7. Time set notation.

Definition 5.16. (Extra notation)
We will often have a time set $\mathcal{T} = \{0, \ldots, T\}$, but need to refer to time sets like $\{1, \ldots, T\}$ or $\{0, \ldots, T-1\}$, in such situations we will sometimes use the notation:

$\mathcal{T}^{-0} := \{1, \ldots, T\}$ ("$\mathcal{T}$ without 0"),

$\mathcal{T}^{-T} := \{0, \ldots, T-1\}$ ("$\mathcal{T}$ without T")

and

$\mathcal{T}^{-0,T} := \{1, \ldots, T-1\}$ ("$\mathcal{T}$ without 0 and T").

5.8. **The indicator function and process.**

Definition 5.17. (Indicator function)

$$\mathbf{1}_{\{\text{condition}\}} := \begin{cases} 1\,, & \text{if} \quad \text{condition} = \text{true}. \\ 0\,, & \text{if} \quad \text{condition} = \text{false}. \end{cases}$$

Definition 5.18. (Indicator process)
For any $\Theta \subset \Lambda$ we define $\mathbf{1}^{\Theta}$ to be an adapted process such that:

$$\mathbf{1}^{\Theta}(\lambda) := \begin{cases} 1\,, & \text{if}\, \lambda \in \Theta. \\ 0\,, & \text{if}\, \lambda \in \Lambda \setminus \Theta. \end{cases}$$

5.9. **The fixed transaction cost.**

The fixed transaction cost process $F = (F_t)_{t \in \mathcal{T}}$ consists of an adapted sequence of random variables F_t each representing the transaction cost that must be paid if a trade occurs at time t. Since the probability space is finite, the process F is bounded and we can define a constant that is an upper bound for the fixed transaction cost at any node.

Definition 5.19. (Upper bound on fixed transaction cost)
$\hat{F} := \max\limits_{\lambda \in \Lambda} F(\lambda) + 1$

(The $+1$ is there so that $\hat{F} > F(\lambda)$ for all $\lambda \in \Lambda$.)

5.10. **The proportional transaction cost.**

The proportional transaction cost process $P = (P_t)_{t \in \mathcal{T}}$ consists of a sequence of random variables P_t each representing the transaction cost that must be paid per unit of stock traded at time t.

Definition 5.20. (Ask and bid price processes of stock)
The stock ask price process S^a and stock bid price process S^b are defined as follows:

$S^a(\lambda) = S(\lambda) + P(\lambda)$ for all $\lambda \in \Lambda$.

$S^b(\lambda) = S(\lambda) - P(\lambda)$ for all $\lambda \in \Lambda$.

As it stands the only restriction on P is that the values it takes are either non-negative real numbers or ∞. Clearly this means that if $P(\lambda) \geq S(\lambda)$ then the stock bid price would be less than or equal to zero. Furthermore if $P(\lambda) = \infty$ for some node λ, then we would have $S^a(\lambda) = \infty$ and $S^b(\lambda) = -\infty$. This requires some serious justification as in the real world, one would never see a negative bid price, let alone a bid price of $-\infty$. The main reason that we allow this possibility is that it is a theoretical tool, in particular the situation when $P(\lambda) = \infty$ will be a mathematical way of saying that no trading can happen at the node λ. This theoretical tool will be necessary for us to prove the ask-price representation theorem for combined transaction costs (Theorem 19.4). This possibility of being able to say that no trading can happen at a given node also does indeed have an element of correspondence to the real world as there can be situations where stock loses liquidity and can no longer be bought or sold. It is also worth noting that having a negative bid price at a given node does not cause a problem as although it means that a trader would have to pay to sell a share of stock, the trader also still has the option of not trading at all, and a sensible investor would not sell stock at a node with negative bid price.

In this thesis we will, where possible, work with models that do not allow the possibility that $P(\lambda) \geq S(\lambda)$ as this is the most standard case. However there will be a substantial amount of results where we will need to allow the possibility. This is summarised in the following definition and convention.

Definition 5.21. (Restricted model)
We will say that a model $M = (\Omega, \mathcal{F}, \mathbb{P}, S, F, P)$ is *restricted* if and only if it is such that $P(\lambda) < S(\lambda)$ for all $\lambda \in \Lambda$. Similarly if this is the case we will say that the proportional transaction cost process P is restricted.

We will say that model is *unrestricted* if we wish to emphasise that it is not necessarily restricted.

Convention 4. (Restriction on P)
We will assume that all our definitions are applicable to unrestricted models, but we will assume that a model is restricted, unless stated otherwise for all theorems, lemmas, examples etc.

There is a one to one map between the pair of processes S and P and the pair of processes S^a and S^b (with $S^a(\lambda) \geq S^b(\lambda)$ for all λ). It will sometimes be more convenient to write a model in terms of the ask and bid stock price processes. This is shown in the following example.

Example 5.22. The model

$$
\begin{array}{ccccc}
 & & & & \begin{array}{c} S(uu)=12 \\ F(uu)=2 \\ P(uu)=3 \end{array} \\
 & & & \nearrow & \\
 & & \begin{array}{c} S(u)=10 \\ F(u)=2 \\ P(u)=1 \end{array} & \rightarrow & \begin{array}{c} S(um)=10 \\ F(um)=2 \\ P(um)=1 \end{array} \\
 & \nearrow & & \searrow & \\
\begin{array}{c} S(\Omega)=6 \\ F(\Omega)=2 \\ P(\Omega)=1 \end{array} & & & & \begin{array}{c} S(ud)=8 \\ F(ud)=4 \\ P(ud)=0 \end{array} \\
 & & & & \begin{array}{c} S(du)=5 \\ F(du)=2 \\ P(du)=1 \end{array} \\
 & \searrow & & \nearrow & \\
 & & \begin{array}{c} S(d)=3 \\ F(d)=1 \\ P(d)=1 \end{array} & & \\
 & & & \searrow & \\
 & & & & \begin{array}{c} S(dd)=2 \\ F(dd)=3 \\ P(dd)=1 \end{array}
\end{array}
$$

can equivalently be displayed as:

$$
\begin{array}{ccccc}
 & & & & \begin{array}{c} S^a(uu)=15 \\ S^b(uu)=9 \\ F(uu)=2 \end{array} \\
 & & & \nearrow & \\
 & & \begin{array}{c} S^a(u)=11 \\ S^b(u)=9 \\ F(u)=2 \end{array} & \rightarrow & \begin{array}{c} S^a(um)=11 \\ S^b(um)=9 \\ F(um)=2 \end{array} \\
 & \nearrow & & \searrow & \\
\begin{array}{c} S^a(\Omega)=7 \\ S^b(\Omega)=5 \\ F(\Omega)=2 \end{array} & & & & \begin{array}{c} S^a(ud)=8 \\ S^b(ud)=8 \\ F(ud)=4 \end{array} \\
 & & & & \begin{array}{c} S^a(du)=6 \\ S^b(du)=4 \\ F(du)=2 \end{array} \\
 & \searrow & & \nearrow & \\
 & & \begin{array}{c} S^a(d)=4 \\ S^b(d)=2 \\ F(d)=1 \end{array} & & \\
 & & & \searrow & \\
 & & & & \begin{array}{c} S^a(dd)=3 \\ S^b(dd)=1 \\ F(dd)=3 \end{array}
\end{array}
$$

Definition 5.23. (Bid-ask spread)
The bid-ask spread at a node $\lambda \in \Lambda$ is the interval $[S^b(\lambda), S^a(\lambda)]$.

5.11. Martingale measures.

Definition 5.24. (EMM)
An *equivalent martingale measure* (EMM) $\mathbb{Q}$ with respect to a process $\tilde{S}$, for which $S^b(\lambda) \leq \tilde{S}(\lambda) \leq S^a(\lambda)$ for all $\lambda \in \Lambda$, is a probability measure equivalent to the real world probability measure $\mathbb{P}$ such that the process $\tilde{S}$ is a martingale under $\mathbb{Q}$.

Definition 5.25. (AMM)
An *absolutely continuous martingale measure* (AMM) $\mathbb{Q}$ with respect to a process $\tilde{S}$, for which $S^b(\lambda) \leq \tilde{S}(\lambda) \leq S^a(\lambda)$ for all $\lambda \in \Lambda$, is a probability measure absolutely continuous with respect to the real world probability measure $\mathbb{P}$ such that the process $\tilde{S}$ is a martingale under $\mathbb{Q}$.

Convention 5. (EMM and AMM with respect to S)
In some contexts we will simply say that a probability measure $\mathbb{Q}$ is an EMM, which will mean $\mathbb{Q}$ is an EMM with respect to S.

Similarly we will sometimes say that a probability measure $\mathbb{Q}$ is an AMM, which will mean $\mathbb{Q}$ is an AMM with respect to S.

This convention is particularly useful in the situations when we are working in the frictionless or fixed cost setting.

Remark 5.26. Note that any EMM with respect to a process $\tilde{S}$ is also an AMM with respect to $\tilde{S}$.

5.12. **Conditional probability function.**

Definition 5.27. (Set of node and successor node pairs)
$\Phi = \{(\lambda, \mu) : \lambda \in \Lambda, \mu \in \mathrm{Succ}\lambda\}$.

Definition 5.28. (Conditional probability function)
A *conditional probability function* $\mathbf{Q}$ is a function from Φ to $\mathbb{R}$ such that:

(1) $\mathbf{Q}(\lambda, \mu) \in [0, 1]$ for all $(\lambda, \mu) \in \Phi$.

(2) $\sum\limits_{\mu \in \mathrm{Succ}\lambda} \mathbf{Q}(\lambda, \mu) = 1$ for all $\lambda \in \Lambda \setminus \Lambda_T$.

Remark 5.29. Each conditional probability function maps to a unique probability measure.

Definition 5.30. (probability measure corresponding to conditional probability function)
The probability measure that corresponds to conditional probability function $\mathbf{Q}$ is the probability measure $\mathbb{Q}$ such that $\mathbb{Q}(\mu) = \prod\limits_{t=0}^{T} \mathbf{Q}(\lambda_t^\omega, \lambda_{t+1}^\omega)$ for all $\mu \in \Omega_T$, $\omega \in \mu$. Note that this does not depend on ω.

Definition 5.31. (AMF)
An absolutely continuous martingale conditional probability function with respect to a process $\tilde{S}$ for which $S^b(\lambda) \leq \tilde{S}(\lambda) \leq S^a(\lambda)$ for all $\lambda \in \Lambda$ is a conditional probability function $\mathbf{Q}$ such that:

(1) $\tilde{S}(\lambda) = \sum\limits_{\mu \in \mathrm{Succ}\lambda} \mathbf{Q}(\lambda, \mu)\tilde{S}(\mu)$ for all $\lambda \in \Lambda \setminus \Lambda_T$.

(2) The probability measure corresponding to $\mathbf{Q}$ is absolutely continuous with respect to $\mathbb{P}$.

Remark 5.32. The probability function corresponding to an AMF with respect to a process $\tilde{S}$ is an AMM with respect to $\tilde{S}$.

Definition 5.33. (Expectation under an AMF)
For any AMF $\mathbf{Q}$ with corresponding AMM $\mathbb{Q}$, the symbol $\mathbb{E}_{\mathbf{Q}}$ can be understood to mean $\mathbb{E}_{\mathbb{Q}}$.

5.13. Set of martingale pairs.

Definition 5.34. ($\mathcal{P}$)
Let $\mathcal{P}$ be defined as the set of pairs $(\mathbb{Q}, \tilde{S})$ such that $\mathbb{Q}$ is an EMM with respect to $\tilde{S}$.

Definition 5.35. ($\tilde{\mathcal{P}}$)
Let $\tilde{\mathcal{P}}$ be defined as the set of pairs $(\mathbf{Q}, \tilde{S})$ such that $\mathbf{Q}$ is an AMF with respect to $\tilde{S}$.

These sets will be crucial to our work related to the fundamental theorem of asset pricing. Later we will see that the lack of proportional cost arbitrage is equivalent to the set $\mathcal{P}$ being non-empty and the lack of combined cost arbitrage is equivalent to the set $\tilde{\mathcal{P}}$ being non-empty.

6. **Portfolios and strategies**

So far we have constructed our model and introduced our most fundamental processes, S, F, P. We now introduce the portfolio strategies and trading strategies that convey the way in which an investor may interact with these processes.

6.1. **Portfolios.**

Definition 6.1. (Portfolio)
A *portfolio* is a pair (x, y) of real numbers.

For any portfolio (x, y), x will represent a position in cash and y will represent a position in stock.

6.2. **Portfolio operations.**

Definition 6.2. (Portfolio operation)
A *portfolio operation* is a function that takes a portfolio and outputs another portfolio i.e. any function $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$.

Example 6.3. (Portfolio operation)
An example of a portfolio operation would be the function $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ such that

$f(x, y) = (x + 1, y - 1)$ for all $(x, y) \in \mathbb{R}^2$.

End of example 6.3

6.3. **Self-financing portfolio operations.**

We need to define what is meant by an"acceptable portfolio operation" as we will not wish to allow trading operations such as the function $f(x, y) = (x + 1, y + 1)$ (a gain in both stock and cash). We will now consider three main types of "acceptable portfolio operation":

(1) A trade: this is where a our position in stock is increased or decreased and our position in cash is decreased or increased respectively by an amount that corresponds to the stocks value and the transaction costs that need to be paid. The dependence on how much transaction cost needs to be payed is the reason why we will need separate definitions according to whether we are considering no transaction costs, fixed costs only, proportional costs only or both fixed costs and proportional costs.

(2) A stock discard: Clearly whatever our position in stock, we should be allowed to reduce our position in stock by essentially "throwing away some shares of stock". This possibility is particularly relevant when we consider fixed transaction costs as there are situations where it is better to discard a share of stock than sell it.

(3) A cash discard: Once again it is obvious that a trader should be allowed to do this.

We will now see precise definitions of each of these. Rather than using the terminology"acceptable portfolio operation " we will use the term self-financing trading operation as this will help these definitions to correspond to the definitions of"self-financing trading strategy" that we will see later.

Definition 6.4. (Stock discard)
A *stock discard* at a node λ is a portfolio operation f such that:

$f(x, y) = (x, y - z)$

for some $z \geq 0, z \in \mathbb{R}$.

If this operation has occurred we will say that z shares of stock have been discarded.

Definition 6.5. (Cash discard)
A *cash discard* at a node λ is a portfolio operation f such that:

$f(x, y) = (x - z, y)$

for some $z \geq 0, z \in \mathbb{R}$.

If this operation has occurred we will say that z units of cash have been discarded.

Definition 6.6. (Trade)
A *frictionless self-financing trade* at a node λ is a portfolio operation f such that for some $z \in \mathbb{R}$:

$f(x, y) = (x - zS(\lambda), y + z)$ for all $(x, y) \in \mathbb{R}^2$.

A *fixed cost self-financing trade* at a node λ is a portfolio operation f such that for some $z \in \mathbb{R}$:

$f(x, y) := (x - zS(\lambda) - F(\lambda)\mathbf{1}_{z \neq 0}, y + z)$ for all $(x, y) \in \mathbb{R}^2$.

A *proportional cost self-financing trade* at a node λ is a portfolio operation f such that for some $z \in \mathbb{R}$:

$f(x, y) := (x - zS(\lambda) - |z|P(\lambda), y + z)$ for all $(x, y) \in \mathbb{R}^2$.

A *combined cost self-financing trade* at a node λ is a portfolio operation f such that for some $z \in \mathbb{R}$:

$f(x, y) := (x - zS(\lambda) - F(\lambda)\mathbf{1}_{z \neq 0} - |z|P(\lambda), y + z)$ for all $(x, y) \in \mathbb{R}^2$.

In each case we will say that z shares of stock have been traded. If $z > 0$ we will say that z shares of stock have been bought at the node λ, if $z = 0$ we will say that there was no trade and if $z < 0$ we will say that z shares of stock have been sold at the node λ.

We have now defined those portfolio operations that we deem acceptable. However although a discard may be the only way to get from one portfolio to another, any portfolio strategy that involves discards is essentially just worse off compared to a portfolio strategy that is almost identical except it does not discard. Thus we have a choice, we can deal with portfolio strategies that may involve discards or we can deal with portfolio strategies that only involve trades. In a nutshell the advantages and disadvantages of each approach can be summarised as follows:

• Allow discards: Greater flexibility when determining if a given portfolio strategy is self-financing.

• Not include discards: Simplified calculations.

In this document we will work in a world that does not allow discards.

Please note that this will have the effect of reducing the size of the set of arbitrage opportunities (we will see a precise definition of an arbitrage opportunity later) compared to if we choose to allow strategies that discard cash or stock. However the key point is that if there exists an arbitrage opportunity when discards are allowed then there exists an arbitrage opportunity without needing any discards.

Having made this decision to not allow discards we will now work towards defining what we mean by self-financing trading strategies and self-financing portfolio strategies.

6.4. **Portfolio strategies and Trading strategies.**

Typically in mathematical finance the terms trading strategy and strategy are used almost interchangeably and mean something similar to the following definition of a portfolio strategy. In our work we will make a distinction between portfolio strategies and trading strategies as defined below. We do this because it will give us extra machinery that will help us to convey mathematical concepts. For example there will be many situations in our future work where we need to give an arbitrage opportunity, in this case it will be easier to just refer to a trading strategy. There are many other examples where it will be easier to refer to a portfolio strategy.

Definition 6.7. (Portfolio strategy)
A *portfolio strategy* (X, Y) is a pair of predictable extended processes (See definition 22.6)X and Y such that $Y_0 = 0$.

For any given node λ, the pair $(X(\lambda), Y(\lambda))$ represents a position of $X(\lambda)$ units of cash and $Y(\lambda)$ units of the stock S.

Remark 6.8. (Initial position in stock equal to zero)
Please note that as stated in the above definition of a portfolio strategy we set $Y_0 = 0$, we highlight this here as it is an important point that must not be forgotten.

Definition 6.9. (Trading strategy)
A *trading strategy* Z is an adapted process of real numbers .

For each node $\lambda \in \Lambda$, $Z(\lambda)$ represents the number of shares of stock traded.

In our work we consider unrestricted models where it is possible to have $P(\lambda) = \infty$ for some node $\lambda \in \Lambda$. This means that if we want to trade at such a node then we would have to pay an infinite transaction cost. Conversely if we don't trade we shouldn't have to pay any transaction costs. This leads us to sometimes writing something like $Z(\lambda)P(\lambda)$ where $Z(\lambda) = 0$ (zero trades) and $P(\lambda) = \infty$. Since we wish to understand this to mean that we do not pay any transaction cost, we will adopt the following convention:

Convention 6. (Zero multiplied by infinity)
We will adopt the convention that zero multiplied by infinity is equal to zero.

Definition 6.10. (Trading strategy trade notation)
Take any $(x, y) \in \mathbb{R}^2$, $(x', y') \in \mathbb{R}^2$ and node $\lambda \in \Lambda$.

We will write $(x, y) \xrightarrow{Z(\lambda)} (x', y')$ if and only if

$(x', y') = (x - Z(\lambda)S(\lambda), y + Z(\lambda))$.

This can be read as: "$Z(\lambda)$ shares of stock traded in the frictionless setting moves us from the portfolio (x, y) to the portfolio(x', y') at node λ".

We will write $(x, y) \xrightarrow[F]{Z(\lambda)} (x', y')$ if and only if

$(x', y') = (x - Z(\lambda)S(\lambda) - F(\lambda)\mathbf{1}_{Z(\lambda)\neq 0}, y + Z(\lambda))$.

This can be read as: "$Z(\lambda)$ shares of stock traded in the fixed cost setting moves us from the portfolio (x, y) to the portfolio(x', y') at node λ".

We will write $(x, y) \xrightarrow[P]{Z(\lambda)} (x', y')$ if and only if

$(x', y') = (x - Z(\lambda)S(\lambda) - |Z(\lambda)|P(\lambda), y + Z(\lambda))$ and $Z(\lambda) = 0$ if $P(\lambda) \neq \infty$.

This can be read as: "$Z(\lambda)$ shares of stock traded in the proportional cost setting moves us from the portfolio (x, y) to the portfolio(x', y') at node λ".

We will write $(x, y) \xrightarrow[C]{Z(\lambda)} (x', y')$ if and only if

$(x', y') = (x - Z(\lambda)S(\lambda) - F(\lambda)\mathbf{1}_{Z(\lambda)\neq 0} - |Z(\lambda)|P(\lambda), y + Z(\lambda))$ and $Z(\lambda) = 0$ if $P(\lambda) \neq \infty$.

This can be read as: "$Z(\lambda)$ shares of stock traded in the combined cost setting moves us from the portfolio (x, y) to the portfolio(x', y') at node λ".

6.5. **Self-financing portfolio strategies.**

Definition 6.11. (Self-financing portfolio strategy)
A *frictionless self-financing portfolio strategy* is a portfolio strategy (X, Y) for which there exists a trading strategy Z such that for each node λ we have:

$$(X(\lambda), Y(\lambda)) \xrightarrow{Z(\lambda)} (X_{+1}(\lambda), Y_{+1}(\lambda)).$$

If this is the case then we will say that the portfolio strategy (X, Y) and trading strategy Z together with initial portfolio $(X_0, 0)$ *frictionless correspond* to each other. Note that this is a one to one correspondence.

A *fixed cost self-financing portfolio strategy* is a portfolio strategy (X, Y) for which there exists a trading strategy Z such that for each node λ we have:

$$(X(\lambda), Y(\lambda)) \xrightarrow[F]{Z(\lambda)} (X_{+1}(\lambda), Y_{+1}(\lambda)).$$

If this is the case then we will say that the portfolio strategy (X, Y) and trading strategy Z together with initial portfolio $(X_0, 0)$ *fixed cost correspond* to each other. Note that this is a one to one correspondence.

A *proportional cost self-financing portfolio strategy* is a portfolio strategy (X, Y) for which there exists a trading strategy Z such that for each node λ we have:

$$(X(\lambda), Y(\lambda)) \xrightarrow[P]{Z(\lambda)} (X_{+1}(\lambda), Y_{+1}(\lambda)).$$

If this is the case then we will say that the portfolio strategy (X, Y) and trading strategy Z, such that $Z(\lambda) = 0$ if $P(\lambda) = \infty$ for all $\lambda \in \Lambda$, together with initial portfolio $(X_0, 0)$ *proportional cost correspond* to each other. Note that this is a one to one correspondence.

A *combined cost self-financing portfolio strategy* is a portfolio strategy (X, Y) for which there exists a trading strategy Z such that for each node λ we have:

$$(X(\lambda), Y(\lambda)) \xrightarrow[C]{Z(\lambda)} (X_{+1}(\lambda), Y_{+1}(\lambda)).$$

If this is the case then we will say that the portfolio strategy (X, Y) and trading strategy Z, such that $Z(\lambda) = 0$ if $P(\lambda) = \infty$ for all $\lambda \in \Lambda$, together with initial portfolio $(X_0, 0)$ *combined cost correspond* to each other. Note that this is a one to one correspondence.

6.6. **Correspondence of trading strategies and portfolio strategies.**

In order to switch between trading strategies and portfolio strategies easily, we must have a good understanding of how to move from one to another directly. The lemmas in this subsection are very straightforward, but should be thoroughly understood in order to fully grasp the link.

Remark 6.12. (Definition of correspondence)
Please recall that the correspondence between trading strategies and portfolio strategies is defined precisely as part of definition 6.11.

Convention 7. (Equation involving random variables)
Throughout this thesis will sometimes write an equation involving random variables at a given time without explicitly mentioning the scenario. If this is the case then the equation may be understood to mean that it is true for any $\omega \in \Omega$. For example the equation $X_t = Y_t$ can be understood to mean $X_t(\omega) = Y_t(\omega)$ for all $\omega \in \Omega$.

Lemma 6.13. *(Portfolio strategy corresponding to a trading strategy and initial portfolio)*
Consider an unrestricted model.

Let Z be a trading strategy. Let $(x, 0)$ be an initial portfolio. Then:

The frictionless self-financing portfolio strategy (X, Y) corresponding to $(x, 0)$ and Z is given by:

$X_t = x + \sum_{i=0}^{t-1} [-Z_i S_i]$ *for all* $t \in \mathcal{T} \cup T + 1$.
$Y_t = \sum_{i=0}^{t-1} [Z_i]$ *for all* $t \in \mathcal{T} \cup T + 1$.

The fixed cost self-financing portfolio strategy (X, Y) corresponding to $(x, 0)$ and Z is given by:

$X_t = x + \sum_{i=0}^{t-1} [-Z_i S_i - F_i \boldsymbol{1}_{Z_i \neq 0}]$ *for all* $t \in \mathcal{T} \cup T + 1$.
$Y_t = \sum_{i=0}^{t-1} [Z_i]$ *for all* $t \in \mathcal{T} \cup T + 1$.

The proportional cost self-financing portfolio strategy (X, Y) corresponding to $(x, 0)$ and trading strategy Z, such that $Z(\lambda) = 0$ if $P(\lambda) = \infty$ for all $\lambda \in \Lambda$, is given by:

$X_t = x + \sum_{i=0}^{t-1} [-Z_i S_i - |Z_i| P_i]$ *for all* $t \in \mathcal{T} \cup T + 1$.
$Y_t = \sum_{i=0}^{t-1} [Z_i]$ *for all* $t \in \mathcal{T} \cup T + 1$.

The combined cost self-financing portfolio strategy (X, Y) corresponding to $(x, 0)$ and trading strategy Z, such that $Z(\lambda) = 0$ if $P(\lambda) = \infty$ for all $\lambda \in \Lambda$, is given by:

$X_t = x + \sum_{i=0}^{t-1} [-Z_i S_i - F_i \boldsymbol{1}_{Z_i \neq 0} - |Z_i| P_i]$ *for all* $t \in \mathcal{T} \cup T+1$.

$Y_t = \sum_{i=0}^{t-1} [Z_i]$ *for all* $t \in \mathcal{T} \cup T+1$.

Proof of Lemma 6.13

This follows from the definitions of self-financing portfolio strategy and trading strategy.

End of proof of Lemma 6.13

Lemma 6.14. *(Trading strategy corresponding to a portfolio strategy)*
Consider an unrestricted model.

For any frictionless self-financing portfolio strategy (X, Y)*, the trading strategy* Z *that frictionless corresponds to* (X, Y) *is given by:*

$Z(\lambda) = Y_{+1}(\lambda) - Y(\lambda)$ *for all* $\lambda \in \Lambda$.

For any fixed cost self-financing portfolio strategy (X, Y)*, the trading strategy* Z *that fixed cost corresponds to* (X, Y) *is given by:*

$Z(\lambda) = Y_{+1}(\lambda) - Y(\lambda)$ *for all* $\lambda \in \Lambda$.

For any proportional cost self-financing portfolio strategy (X, Y)*, the trading strategy* Z *that proportional cost corresponds to* (X, Y) *is given by:*

$Z(\lambda) = Y_{+1}(\lambda) - Y(\lambda)$ *for all* $\lambda \in \Lambda$.

For any combined cost self-financing portfolio strategy (X, Y)*, the trading strategy* Z *that combined cost corresponds to* (X, Y) *is given by:*

$Z(\lambda) = Y_{+1}(\lambda) - Y(\lambda)$ *for all* $\lambda \in \Lambda$.

Proof of Lemma 6.14

This follows from the definitions of self-financing portfolio strategy and trading strategy.

End of proof of Lemma 6.14

Definition 6.15. (Self-financing portfolio strategy trade at a node λ)
We will say that a self-financing portfolio strategy (X, Y) traded $Y_{+1}(\lambda) - Y(\lambda)$ shares of stock.

We will say that the self-financing portfolio strategy (X, Y) did not trade if $Y_{+1}(\lambda) - Y(\lambda) = 0$.

Here "self-financing" means self-financing in any sense (frictionless, fixed cost etc).

Example 6.16. (Trading strategy in action)
Consider a single-step model with only two successor nodes, μ_1 and μ_2, to the root node Ω.

Let the stock price process be such that $S(\Omega) = 2, S(\mu_1) = 3$ and $S(\mu_2) = 1$.

We could define a trading strategy Z as follows:

$Z(\Omega) = 1,\ Z(\mu_1) = Z(\mu_2) = -1$.

This means that at time zero we buy a share of stock and at time 1 we sell it no matter which node is reached.

One of the key advantages of trading strategies is that combined with an initial portfolio, they automatically correspond to a self-financing portfolio strategy, there is no-need to check any self-financing condition.

In this example the frictionless self-financing portfolio strategy (X, Y) that frictionless corresponds to Z and the initial portfolio $(0, 0)$ is given by:

$(X_0, Y_0) = (0, 0)$.

$(X_1, Y_1) = (-2, 1)$.

$(X_2(\mu_1), Y_2(\mu_1)) = (1, 0)$.

$(X_2(\mu_2), Y_2(\mu_2)) = (-1, 0)$.

We can then say that Z is the trading strategy that frictionless corresponds to the portfolio strategy (X, Y).

End of Example 6.16

6.7. **Frictionless value of a portfolio strategy.**

In the frictionless world the value of a frictionless self-financing portfolio strategy at a node λ is relatively unambiguous and is given by either $X(\lambda) + Y(\lambda)S(\lambda)$ or $X_{+1}(\lambda) + Y_{+1}(\lambda)S(\lambda)$ and it doesn't matter which one we choose because of the frictionless self-financing condition $X(\lambda) + Y(\lambda)S(\lambda) = X_{+1}(\lambda) + Y_{+1}(\lambda)S(\lambda)$. However in the world of transaction costs we can think of a portfolio strategy as loosing frictionless value when it trades and thus if a trade has occurred $X(\lambda) + Y(\lambda)S(\lambda) > X_{+1}(\lambda) + Y_{+1}(\lambda)S(\lambda)$. In other words the frictionless value that the portfolio strategy arrives at node λ with is greater than the frictionless value the portfolio strategy exits node λ with. Hence we will use the following useful terminology.

Definition 6.17. (Arrival value of a portfolio strategy (X, Y) at node λ)

$A^{(X,Y)}(\lambda) := X(\lambda) + Y(\lambda)S(\lambda)$ for $\lambda \in \Lambda$.

Definition 6.18. (Exit value of a portfolio strategy (X, Y) at node λ)

$E^{(X,Y)}(\lambda) := X_{+1}(\lambda) + Y_{+1}(\lambda)S(\lambda)$ for $\lambda \in \Lambda$.

Example 6.19. (Arrival value and exit value of a portfolio strategy)
Consider a model with a node $\lambda \in \Lambda$ such that $S(\lambda) = 3$ and $F(\lambda) = 1$.

Let (X, Y) be a portfolio strategy such that $(X(\lambda), Y(\lambda)) = (4, 2)$ and $(X_{+1}(\lambda), Y_{+1}(\lambda)) = (0, 3)$, in other words at this node a share of stock has been bought for the cost of 3 units and a transaction cost of 1 unit has also been paid.

The arrival value of this portfolio strategy at node λ is: $4 + 2(3) = 10$.

The exit value of this portfolio strategy at node λ is: $0 + 3(3) = 9$.

End of Example 6.19

This terminology enables us to give the following incredibly useful equivalent condition for a portfolio strategy to be self-financing.

Lemma 6.20. *(Equivalent condition for self-financing)*
Consider an unrestricted model.

A portfolio strategy (X, Y) is frictionless self-financing if and only if for any node λ we have:

$A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda)$.

A portfolio strategy (X, Y) is fixed cost self-financing if and only if for any node λ we have:

$A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda) + F(\lambda)\mathbf{1}_{\{Y_{+1}(\lambda)-Y(\lambda)\neq 0\}}$.

A portfolio strategy (X, Y) is proportional cost self-financing if and only if for any node λ we have:

$A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda) + |Y_{+1}(\lambda) - Y(\lambda)|P(\lambda)$.

A portfolio strategy (X, Y) is combined cost self-financing if and only if for any node λ we have:

$A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda) + F(\lambda)\mathbf{1}_{\{Y_{+1}(\lambda)-Y(\lambda)\neq 0\}} + |Y_{+1}(\lambda) - Y(\lambda)|P(\lambda)$.

Proof of Lemma 6.20

Combined costs

Fix a node λ.

If $P(\lambda) = \infty$ then a portfolio strategy (X, Y) is combined cost self-financing if and only if

$X_{+1}(\lambda) = X(\lambda)$ and

$Y_{+1}(\lambda) = Y(\lambda)$.

In this case it is immediate that

$A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda) + F(\lambda)\mathbf{1}_{\{Y_{+1}(\lambda)-Y(\lambda)\neq 0\}} + |Y_{+1}(\lambda) - Y(\lambda)|P(\lambda)$.

If $P(\lambda) \neq \infty$ then a portfolio strategy (X, Y) is combined cost self-financing if and only if for some $z \in \mathbb{R}$:

$X_{+1}(\lambda) = X(\lambda) - zS(\lambda) - |z|P(\lambda) - F(\lambda)\mathbf{1}_{\{z\neq 0\}}$ and

$Y_{+1}(\lambda) = Y(\lambda) + z$.

This is true if and only if:

$X_{+1}(\lambda) = X(\lambda) - (Y_{+1}(\lambda) - Y(\lambda))S(\lambda) - |Y_{+1}(\lambda) - Y(\lambda)|P(\lambda) - F(\lambda)\mathbf{1}_{\{Y_{+1}(\lambda)-Y(\lambda)\neq 0\}}$ and

$z = Y_{+1}(\lambda) - Y(\lambda)$.

We now derive the result from the definition of $E^{(X,Y)}(\lambda)$.

$E^{(X,Y)}(\lambda) := X_{+1}(\lambda) + Y_{+1}(\lambda)S(\lambda)$

$= X(\lambda) - (Y_{+1}(\lambda) - Y(\lambda))S(\lambda) - |Y_{+1}(\lambda) - Y(\lambda)|P(\lambda) - F(\lambda)\mathbf{1}_{\{Y_{+1}(\lambda)-Y(\lambda)\neq 0\}} + Y_{+1}(\lambda)S(\lambda)$

$= X(\lambda) + Y(\lambda)S(\lambda) - |Y_{+1}(\lambda) - Y(\lambda)|P(\lambda) - F(\lambda)\mathbf{1}_{\{Y_{+1}(\lambda)-Y(\lambda)\neq 0\}}$

$= A^{(X,Y)}(\lambda) - |Y_{+1}(\lambda) - Y(\lambda)|P(\lambda) - F(\lambda)\mathbf{1}_{\{Y_{+1}(\lambda)-Y(\lambda)\neq 0\}}$

Rearranging gives:

$A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda) + F(\lambda)\mathbf{1}_{Y_{+1}(\lambda)-Y(\lambda)\neq 0} + |Y_{+1}(\lambda) - Y(\lambda)|P(\lambda)$.

The proofs for the cases of frictionless, fixed costs and proportional costs are similar.

End of proof of Lemma 6.20

This Lemma has a clear intuitive meaning, for example the combined cost part of this Lemma has the meaning: “Arrival value = Exit value + Fixed transaction cost +Proportional transaction cost”.

7. Arbitrage and strong arbitrage

7.1. Arbitrage.

Definition 7.1. (Arbitrage opportunity)
A *frictionless arbitrage opportunity* is a portfolio strategy (X, Y) such that:

(1) $(X_0, Y_0) = (0, 0)$.

(2) (X, Y) is a frictionless self-financing portfolio strategy.

(3) $X_{+1}(\lambda) \geq 0$ and $Y_{+1}(\lambda) \geq 0$ for all $\lambda \in \Lambda_T$.

(4) There exists $\lambda \in \Lambda_T$ such that $X_{+1}(\lambda) > 0$.

A *fixed cost arbitrage opportunity* is a portfolio strategy (X, Y) such that:

(1) $(X_0, Y_0) = (0, 0)$.

(2) (X, Y) is a fixed cost self-financing portfolio strategy.

(3) $X_{+1}(\lambda) \geq 0$ and $Y_{+1}(\lambda) \geq 0$ for all $\lambda \in \Lambda_T$.

(4) There exists $\lambda \in \Lambda_T$ such that $X_{+1}(\lambda) > 0$.

A *proportional cost arbitrage opportunity* is a portfolio strategy (X, Y) such that:

(1) $(X_0, Y_0) = (0, 0)$.

(2) (X, Y) is a proportional cost self-financing portfolio strategy.

(3) $X_{+1}(\lambda) \geq 0$ and $Y_{+1}(\lambda) \geq 0$ for all $\lambda \in \Lambda_T$.

(4) There exists $\lambda \in \Lambda_T$ such that $X_{+1}(\lambda) > 0$.

A *combined cost arbitrage opportunity* is a portfolio strategy (X, Y) such that:

(1) $(X_0, Y_0) = (0, 0)$.

(2) (X, Y) is a combined cost self-financing portfolio strategy.

(3) $X_{+1}(\lambda) \geq 0$ and $Y_{+1}(\lambda) \geq 0$ for all $\lambda \in \Lambda_T$.

(4) There exists $\lambda \in \Lambda_T$ such that $X_{+1}(\lambda) > 0$.

Convention 8. (Arbitrage with trading strategies)
With a slight abuse of notation we will sometimes refer to a trading strategy Z as an arbitrage opportunity. When we say this we will really mean that the portfolio strategy corresponding to the initial portfolio $(0,0)$ and the trading strategy Z is an arbitrage opportunity.

Lemma 7.2. *(Trading strategy as an arbitrage opportunity)*
Consider an unrestricted model.

A trading strategy Z is a frictionless arbitrage opportunity if and only if:

$$\sum_{t=0}^{T}[-Z_t(\omega)S_t(\omega)] \geq 0 \text{ and } \sum_{t=0}^{T}[Z_t(\omega)] \geq 0 \text{ for all } \omega \in \Omega,$$

and there exists an $\omega' \in \Omega$ such that:

$$\sum_{t=0}^{T}[-Z_t(\omega')S_t(\omega')] > 0.$$

A trading strategy Z is a fixed cost arbitrage opportunity if and only if:

$$\sum_{t=0}^{T}[-Z_t(\omega)S_t(\omega) - F_t(\omega)\mathbf{1}_{Z_t(\omega)\neq 0}] \geq 0 \text{ and } \sum_{t=0}^{T}[Z_t(\omega)] \geq 0 \text{ for all } \omega \in \Omega,$$

and there exists an $\omega' \in \Omega$ such that:

$$\sum_{t=0}^{T}[-Z_t(\omega)S_t(\omega) - F_t(\omega)\mathbf{1}_{Z_t(\omega)\neq 0}] > 0.$$

A trading strategy Z, such that $Z(\lambda) = 0$ if $P(\lambda) = \infty$ for all $\lambda \in \Lambda$, is a proportional cost arbitrage opportunity if and only if:

$$\sum_{t=0}^{T}[-Z_t(\omega)S_t(\omega) - |Z_t(\omega)|P_t(\omega)] \geq 0 \text{ and } \sum_{t=0}^{T}[Z_t(\omega)] \geq 0 \text{ for all } \omega \in \Omega,$$

and there exists an $\omega' \in \Omega$ such that:

$$\sum_{t=0}^{T}[-Z_t(\omega)S_t(\omega) - |Z_t(\omega)|P_t(\omega)] > 0.$$

A trading strategy Z, such that $Z(\lambda) = 0$ if $P(\lambda) = \infty$ for all $\lambda \in \Lambda$, is a combined cost arbitrage opportunity if and only if:

$$\sum_{t=0}^{T}[-Z_t(\omega)S_t(\omega) - F_t(\omega)\mathbf{1}_{Z_t(\omega)\neq 0} - |Z_t(\omega)|P_t(\omega)] \geq 0 \text{ and } \sum_{t=0}^{T}[Z_t(\omega)] \geq 0$$
for all $\omega \in \Omega$,

and there exists an $\omega' \in \Omega$ such that:

$$\sum_{t=0}^{T}[-Z_t(\omega)S_t(\omega) - F_t(\omega)\mathbf{1}_{Z_t(\omega)\neq 0} - |Z_t(\omega)|P_t(\omega)] > 0.$$

Proof of lemma 7.2

Follows from the definition of portfolio strategy corresponding to trading strategy, the definition of a arbitrage opportunity and Convention 8.

End of proof of lemma 7.2

7.2. Strong arbitrage.

We are about to see the definition of a strong arbitrage opportunity. The reader should be warned that this is not the most classical definition of strong arbitrage and as such it is important to be aware that it may be a slightly different notion to the notion of strong arbitrage that one would naturally have. More will be explained after we have seen the following definition.

Definition 7.3. (Strong arbitrage)
A *frictionless strong arbitrage opportunity* is a portfolio strategy (X, Y) such that for some non-terminal node $\lambda \in \Lambda$ each of the following hold:

(1) $(X(\lambda'), Y(\lambda')) = (0, 0)$ for all λ' that are not a descendant node of node λ (portfolio strategy does nothing prior to node λ or if λ is not reached).

(2) (X, Y) is a frictionless self-financing portfolio strategy.

(3) $X_{+1}(\lambda') > 0$ and $Y_{+1}(\lambda') \geq 0$ for all nodes $\lambda' \in \Lambda_T$ that are a descendant node of λ (strictly positive profit if node λ is reached).

A *fixed cost strong arbitrage opportunity* is a portfolio strategy (X, Y) such that for some non-terminal node $\lambda \in \Lambda$ each of the following hold:

(1) $(X(\lambda'), Y(\lambda')) = (0, 0)$ for all λ' that are not a descendant node of node λ (portfolio strategy does nothing prior to node λ or if λ is not reached).

(2) (X, Y) is a fixed cost self-financing portfolio strategy.

(3) $X_{+1}(\lambda') > 0$ and $Y_{+1}(\lambda') \geq 0$ for all nodes $\lambda' \in \Lambda_T$ that are a descendant node of λ (strictly positive profit if node λ is reached).

A *proportional cost strong arbitrage opportunity* is a portfolio strategy (X, Y) such that for some non-terminal node $\lambda \in \Lambda$ each of the following hold:

(1) $(X(\lambda'), Y(\lambda')) = (0, 0)$ for all λ' that are not a descendant node of node λ (portfolio strategy does nothing prior to node λ or if λ is not reached).

(2) (X, Y) is a proportional cost self-financing portfolio strategy.

(3) $X_{+1}(\lambda') > 0$ and $Y_{+1}(\lambda') \geq 0$ for all nodes $\lambda' \in \Lambda_T$ that are a descendant node of λ (strictly positive profit if node λ is reached).

A *combined cost strong arbitrage opportunity* is a portfolio strategy (X, Y) such that for some non-terminal node $\lambda \in \Lambda$ each of the following hold:

(1) $(X(\lambda'), Y(\lambda')) = (0, 0)$ for all λ' that are not a descendant node of node λ (portfolio strategy does nothing prior to node λ or if λ is not reached).

(2) (X, Y) is a combined cost self-financing portfolio strategy.

(3) $X_{+1}(\lambda') > 0$ and $Y_{+1}(\lambda') \geq 0$ for all nodes $\lambda' \in \Lambda_T$ that are a descendant node of λ (strictly positive profit if node λ is reached).

The meaning of a strong arbitrage opportunity is that it is a portfolio strategy that is guaranteed profit once it reaches a certain node and does not trade if it never reaches that given node. This is slightly different the more classical notion of a strong arbitrage opportunity which would consider a strong arbitrage opportunity to be a portfolio strategy that is guaranteed profit no matter what scenario occurs.

The reason why we will use the aforementioned definition of strong arbitrage is that it will mean that there exists fixed cost strong arbitrage in a model $\mathbb{M}$ if and only if there exists fixed cost strong arbitrage in a single-step sub-model of $\mathbb{M}$. Note that with the classical definition, the existence of fixed cost strong arbitrage in a single-step sub-model of a model $\mathbb{M}$ would not imply the existence of fixed cost strong arbitrage in $\mathbb{M}$.

7.3. **Viability of a model.**

Definition 7.4. (Viability of a model)
A model $\mathbb{M}$ is said to be *frictionless viable* if there does not exist a frictionless arbitrage opportunity in $\mathbb{M}$.

A model $\mathbb{M}$ is said to be *fixed cost viable* if there does not exist a fixed cost arbitrage opportunity in $\mathbb{M}$.

A model $\mathbb{M}$ is said to be *proportional cost viable* if there does not exist a proportional cost arbitrage opportunity in $\mathbb{M}$.

A model $\mathbb{M}$ is said to be *combined cost viable* if there does not exist a combined cost arbitrage opportunity in $\mathbb{M}$.

8. **Frictionless arbitrage**

The results of this section are well know classical results. However it is important to see them now as we will aim to develop analogues to each of them in the setting of transaction costs.

8.1. **Sub-models.**

Lemma 8.1. *(Frictionless arbitrage in a sub-model)* [2]
A model $\mathbb{M}$ is frictionless viable if and only if every single-step sub-model of $\mathbb{M}$ is frictionless viable.

Lemma 8.2. *(EMM in a sub-model)* [2]
For any model $\mathbb{M}$, there exists an EMM if and only if there exists an EMM in every single-step sub-model of $\mathbb{M}$.

8.2. **Fundamental theorem.**

Theorem 8.3. *(Frictionless fundamental theorem)* [2]
A model $\mathbb{M}$ is frictionless viable if and only if there exists an EMM with respect to S.

9. **Fixed cost arbitrage**

This section is all about fixed cost arbitrage, it is here that we will aim to answer some key questions that one could have about the conditions under which a model will admit fixed cost arbitrage. This will be heavily interwoven with results about frictionless arbitrage and we will see how the existence of fixed cost arbitrage and the existence of frictionless arbitrage are connected to each other.

A well-known result for frictionless models is the fundamental theorem of asset pricing which states that the non-existence of frictionless arbitrage in a model is equivalent to the existence of an EMM. One of the main achievements of this section is to show a similar result about fixed cost arbitrage. This result that shows an equivalence between the existence of fixed cost arbitrage to the non-existence of an AMF is very similar to a result in [16] that links the non-existence of fixed cost arbitrage to the existence of a family of AMMs. Although this result is similar to the result of Jouini, Kallal and Napp in [16], our method of proof will be substantially different and will give an excellent feel for the structure of models that admit fixed cost arbitrage and the structure of models that don't admit fixed cost arbitrage.

Our approach for showing this fundamental theorem for fixed transaction costs will be to start with the single-step model and show how concepts such as frictionless arbitrage, fixed cost arbitrage and EMM relate to each other. We will then move further towards the results in the multi-step model that we are interested in by making a connection between the multi-step model and single-step sub-models. Finally we will then be able to deduce the results that we are particularly interested in such as Theorems 9.8 and 9.10 , and we will have also proved some instructive theorems along the way.

9.1. **The single-step model.**

Let us start this subsection by studying a couple of examples in order to build some intuition about fixed transaction costs.

Example 9.1. Consider the following stock price tree with root node λ and successor nodes u and d:

$$\begin{array}{ccc} & & S(u)=3 \\ & \nearrow & S(d)=2 \\ & \nearrow & \\ S(\lambda)=1 & & \end{array}$$

The first thing to notice is that there exists a frictionless strong arbitrage opportunity, we simply choose a trading strategy Z with $Z_0 = 1$ and $Z_1 = -1$ (buy at time zero and sell at time 1).

This trading strategy is a frictionless strong arbitrage opportunity because in either scenario, the stock price goes up, so the stock will be sold at a profit. In other words if (X, Y) is the portfolio strategy that frictionless corresponds to Z and the initial portfolio $(0, 0)$, then we would have $X_2(u) > 0$, $Y_2(u) = 0$ $X_2(d) > 0$ and $Y_2(d) = 0$.

Suppose that we have a fixed transaction cost of 2 units every time we trade ($F(\lambda) = 2$ for alll $\lambda \in \Lambda$).

The trading strategy Z is not a fixed cost arbitrage opportunity. With the fixed transaction cost in play any profit that is made by the increase in the value of the stock will be counterbalanced by the fixed transaction cost. In other words if (X', Y') is a portfolio strategy that is fixed cost corresponds to $(0, 0)$ and Z then we would have $X_2'(d) = -1 < 0$ i.e. a loss if node d is reached.

However we can change this trading strategy into a fixed cost arbitrage opportunity by using an idea that we will refer to as "scaling up a trading strategy", the idea is that if we take a frictionless strong arbitrage opportunity such as trading strategy Z and multiply it by a constant $k > 0$ then the profit of the resulting trading strategy will be k times as much, no matter which scenario occurs.

Using this technique we can always turn a strong arbitrage opportunity into a fixed cost arbitrage opportunity because we can always choose a large enough constant k to counterbalance the fixed transaction cost.

In this example the trading strategy $\tilde{Z} = 10Z$ is a fixed cost arbitrage opportunity. The trading strategy $\tilde{Z}$ buys 10 shares of stock at time zero ($\tilde{Z}_0 = 10$) and sells 10 shares of stock at time 1 ($\tilde{Z}_1 = -10$). This is a fixed cost arbitrage opportunity because if node u is reached, the trading strategy will profit by 20 units due to the stock, but lose 4 units to the fixed transaction cost giving an overall profit of 16 units, and if node d is reached, the trading strategy will profit by 10 units due to the stock, but lose 4 units to the fixed transaction cost, giving an overall profit of 6 units.

The key message to take away from this example is that a frictionless strong arbitrage opportunity will not necessarily be a fixed cost arbitrage opportunity, but that same portfolio strategy can be "scaled up" to construct a fixed cost arbitrage opportunity. The consequence of this idea of "scaling up" a portfolio strategy is that the existence of frictionless strong arbitrage is equivalent to the existence of fixed cost arbitrage.

End of Example 9.1

Example 9.2. This example is very critical to understanding the difference between the existence of frictionless arbitrage and the existence of fixed cost arbitrage. Having seen the previous example a reader may now ask the questions: Why isn't the existence of frictionless arbitrage equivalent to the existence of fixed cost arbitrage? Can we not just scale up a frictionless arbitrage portfolio strategy and turn it into a fixed cost arbitrage opportunity? Answering these questions will be of great help to our understanding of fixed transaction costs and this example will explain the answer to both of them. Here is the stock price diagram for this example:

$$\begin{array}{ccc} & & S(u) = 2 \\ & \nearrow & \\ S(\lambda) = 1 & \rightarrow & S(d) = 1 \end{array}$$

Just as in the previous example this model admits frictionless arbitrage. However in this case we do not have fixed cost strong arbitrage.

The portfolio strategy Z from the previous example ($Z_0 = 1$, $Z_1 = -1$) is a frictionless arbitrage opportunity here as well. It will not lose money in scenario d but will profit in scenario u. The key difference with this example is that there is a possibility that the trading strategy will not make any profit on the stock.

Suppose once again that we have a fixed transaction cost of 2 units and to illustrate the point we will construct a trading strategy by scaling up Z by a factor of 100. Let our new portfolio strategy be $\tilde{Z} = 100Z$, so that we have $\tilde{Z}_0 = 100$ and $\tilde{Z}_1 = -100$. This trading strategy $\tilde{Z}$ will make a large profit in scenario u but in scenario d it will still break even on stock, but it will lose 4 units on transaction cost making an overall loss of 4 units. The trading strategy would make a loss in scenario ω_2 and hence is not a fixed cost arbitrage opportunity. This happens because no matter how much you scale up zero profit, you still get zero profit and in such scenarios that give zero profit the transaction cost will mean an overall loss.

Therefore we can conclude that in this example there does not exist a fixed cost arbitrage opportunity.

End of Example 9.2

We will now begin our theoretical analysis of the single-step model. We start by categorising every possible type of single-step model.

Definition 9.3. (Categorisation of the single-step model)
A single-step model $\mathbb{M}$ with root node λ and successor nodes $\mu_1, \ldots, \mu_n$ such that $S(\mu_1) \geq S(\mu_2) \geq \cdots \geq S(\mu_n)$ is said to be of:

- *type 1* if $S(\mu_1) > S_0$ and $S(\mu_n) < S_0$.
- *type 2* if $S(\mu_1) = S(\mu_n) = S_0$.
- *type 3* if either $S(\mu_1) > S_0$ and $S(\mu_n) = S_0$, or $S(\mu_1) = S_0$ and $S(\mu_n) < S_0$.
- *type 4* if $S(\mu_1) < S_0$ or $S(\mu_n) > S_0$.

$\underline{\text{Single step model of type 1}}$

$$\begin{array}{ccc} & & S(\mu_1) \\ & \nearrow & \\ S_0 & \rightarrow & \vdots \\ & \searrow & \\ & & S(\mu_n) \end{array}$$

$\underline{\text{Single step model of type 2}}$

$$S_0 \quad \rightarrow \quad S(\mu_1) = \ldots = S(\mu_n)$$

$\underline{\text{Single step model of type 3}}$

$$\begin{array}{ccc} & & S(\mu_1) \\ & \nearrow & \vdots \\ S_0 & \rightarrow & S(\mu_n) \end{array} \qquad \text{or} \qquad \begin{array}{ccc} S_0 & \rightarrow & S(\mu_1) \\ & \searrow & \vdots \\ & & S(\mu_n) \end{array}$$

$\underline{\text{Single step model of type 4}}$

$$\begin{array}{ccc} & & S(\mu_1) \\ & & \vdots \\ & \nearrow & S(\mu_n) \\ & \nearrow & \\ S_0 & & \end{array} \qquad \text{or} \qquad \begin{array}{ccc} S_0 & & \\ & \searrow & \\ & \searrow & S(\mu_1) \\ & & \vdots \\ & & S(\mu_n) \end{array}$$

Remark 9.4.
Note that every single-step model must be of either type 1,2,3 or 4.

Now that we have categorised every different possible type of single-step model, we can come to a conclusion about what types of arbitrage exist in each type of model and whether there exists a martingale measure in each type of model.

Theorem 9.5.
For any single-step model $\mathbb{M}$ *we have the following:*

(1) *There exists a fixed cost strong arbitrage opportunity if and only if* $\mathbb{M}$ *is of type 4.*

(2) *There exists a fixed cost arbitrage opportunity if and only if* $\mathbb{M}$ *is of type 4.*

(3) *There exists a frictionless strong arbitrage opportunity if and only if* $\mathbb{M}$ *is of type 4.*

(4) *There exists a frictionless arbitrage opportunity if and only if* $\mathbb{M}$ *is of either type 3 or 4.*

(5) *There exists an AMF if and only if* $\mathbb{M}$ *is of either type 1,2 or 3.*

(6) *There exists an EMM if and only if* $\mathbb{M}$ *is of either type 1 or 2.*

Proof of Theorem 9.5.

Throughout this proof I will often simply give an arbitrage trading strategy. Showing that the given trading strategy is indeed an arbitrage opportunity is relatively simple and thus will be left to the reader.

Recall that $\hat{F}$ is an upper bound on the fixed transaction cost process F (See Definition 5.19).

(1) ($\Leftarrow$)
Suppose that $S(\mu_n) > S_0$. Then consider a trading strategy Z defined by:
$Z_0 = \frac{2\hat{F}}{S(\mu_n)-S_0}$,

$Z_1 = -\frac{2\hat{F}}{S(\mu_n)-S_0}$,

is a fixed cost strong arbitrage opportunity. To see this observe that $(S(\mu_n) - S_0)$ is the minimum frictionless profit per unit stock bought. In order to get strong arbitrage we need a trading strategy that buys enough units of stock that the total profit $Z_0(S(\mu_n) - S_0)$ is greater than the maximum possible loss due to transaction cost (which is less than $2\hat{F}$).

The case when $S(\mu_1) < S_0$ involves a symmetrical argument.

($\Rightarrow$)
We show that the existence of a fixed cost strong arbitrage opportunity implies that the model

is of type 4.

For any fixed cost strong arbitrage opportunity (X, Y) we immediately have that $E_1^{(X,Y)} > 0$.

We also have that $A_0^{(X,Y)} = 0$.

Fixed cost self financing then tells us that $E_0^{(X,Y)} \leq 0$ and $A_1^{(X,Y)} > 0$.

It follows that $E_0^{(X,Y)} < A_1^{(X,Y)}$.

This means that $X_1 + Y_1 S_0 < X_1 + Y_1 S_1(\mu)$ for all $\mu \in \Lambda_1$.

This can only happen if either $S_0 < S_1$ or $S_0 > S_1$ (depending on the sign of Y_1) i.e. the model is of type 4.

(2) $(\Leftarrow)$
Since any fixed cost strong arbitrage opportunity is also a frictionless arbitrage opportunity, this implication follows from part (1).

$(\Rightarrow)$
Let (X, Y) be a fixed cost arbitrage opportunity. This means that $E^{(X,Y)}(\mu) \geq 0$ for all $\mu \in \Lambda_1$ and there exists $\mu' \in \Lambda_1$ such that $E^{(X,Y)}(\mu') > 0$.

Thus due to the fixed cost self-financing condition we also have $A^{(X,Y)}(\mu) \geq 0$ for all $\mu \in \Lambda_1$ and $A^{(X,Y)}(\mu') > 0$.

Since (X, Y) is an arbitrage portfolio strategy, $A_0^{(X,Y)} = 0$.

Observe that if (X, Y) does not trade at time zero then it is not possible to have $A^{(X,Y)}(\mu') > 0$.

Therefore fixed cost self-financing gives $A_0^{(X,Y)} \geq E_0^{(X,Y)} + F_0$ which implies that $A_0^{(X,Y)} > E_0^{(X,Y)}$.

This now means that $E_0^{(X,Y)} < A_1^{(X,Y)}$ because $E_0^{(X,Y)} < 0$ and $A_1^{(X,Y)} \geq 0$.

Therefore the model is of type 4 for the same reason as explained in part (1).

(3) The proof of this is almost identical to the proof of (1).

(4) $(\Leftarrow)$
Suppose that the model is of type 3 or 4 and that $S(\mu_n) \geq S_0$.

Observe that the trading strategy Z such that $Z_0 = 1$ and $Z_1 = -1$ is a frictionless arbitrage portfolio strategy.

The case when the model is of type 3 or 4 and $S(\mu_1) \leq S_0$ is symmetrical.

$(\Rightarrow)$
Let (X, Y) be a frictionless arbitrage opportunity. This means that $E^{(X,Y)}(\mu) \geq 0$ for all $\mu \in \Lambda_1$ and there exists $\mu' \in \Lambda_1$ such that $E^{(X,Y)}(\mu) > 0$.

Thus due to the frictionless self-financing condition we also have $A^{(X,Y)}(\mu) \geq 0$ for all $\mu \in \Lambda_1$ and $A^{(X,Y)}(\mu') > 0$.

Since (X, Y) is a frictionless arbitrage portfolio strategy, $A_0^{(X,Y)} = E_0^{(X,Y)} = 0$.

Therefore we have $E_0^{(X,Y)} \leq A^{(X,Y)}(\mu)$ for all $\mu \in \Lambda_1$ and $E_0^{(X,Y)} < A^{(X,Y)}(\mu')$.

It follows that either $S_0 \leq S_1(\mu)$ for all $\mu \in \Lambda_1$ and $S_0 < S_1(\mu')$ or $S_0 \geq S_1(\mu)$ for all $\mu \in \Lambda_1$ and $S_0 > S_1(\mu')$ (depending on the sign of Y_1). In other words the model is of type 3 or 4.

(5) $(\Leftarrow)$
We will show that if the model is of type 1,2 or 3 then we can find an AMF.

Firstly suppose that the model is of type 2 or 3. Since in either of these cases there exists an $\mu' \in \Lambda_1$ such that $S_0 = S_1(\mu')$, we can construct an AMF $\mathbf{Q}$ by simply choosing $\mathbf{Q}(\Omega, \mu') = 1$ and $\mathbf{Q}(\Omega, \mu) = 0$ for all other $\mu \in \Lambda_1$.

If the model is of type 1 then we construct an AMF $\mathbf{Q}$ by choosing $\mathbf{Q}(\Omega, \mu_1) = \frac{S_0 - S_1(\mu_n)}{S_1(\mu_1) - S_1(\mu_n)}$ and $\mathbf{Q}(\Omega, \mu_n) = \frac{S_0 - S_1(\mu_1)}{S_1(\mu_n) - S_1(\mu_1)}$ and $\mathbf{Q}(\Omega, \mu) = 0$ for all other $\mu \in \Lambda_1$.

$(\Rightarrow)$
Suppose for a contradiction that there exists an AMF $\mathbf{Q}'$ and that the model is of type 4 (i.e. not type 1,2 or 3). Using the martingale property,

$$S_0 = \mathbf{Q}'(\Omega, \mu_1)S_1(\mu_1) + \ldots + \mathbf{Q}'(\Omega, \mu_n)S_1(\mu_n).$$

We've assumed that the model is of type 4, so either $S_1(\mu_n) > S_0$ or $S_1(\mu_1) < S_0$.

If $S_1(\mu_n) > S_0$ then

$$S_0 = \mathbf{Q}'(\Omega, \mu_1)S_1(\mu_1) + \ldots + \mathbf{Q}'(\Omega, \mu_n)S_1(\mu_n) > \mathbf{Q}'(\Omega, \mu_1)S_0 + \ldots + \mathbf{Q}'(\Omega, \mu_n)S_0 = S_0$$

thus we have a contradiction (we have used the fact that $\mathbf{Q}'(\Omega, \mu_1) + \ldots + \mathbf{Q}'(\Omega, \mu_n) = 1$).

Similarly if $S_1(\mu_1) < S_0$ then

$$S_0 = \mathbf{Q}'(\Omega, \mu_1)S_1(\mu_1) + \ldots + \mathbf{Q}'(\Omega, \mu_n)S_1(\mu_n) < \mathbf{Q}'(\Omega, \mu_1)S_0 + \ldots + \mathbf{Q}'(\Omega, \mu_n)S_0 = S_0$$

and again we have a contradiction. Therefore we conclude that if there exists an AMF then the model must be of type 1,2 or 3.

(6) This follows from (4) combined with the fundamental theorem (Theorem 8.3)

End of proof of Theorem 9.5.

The next theorem is the main result of this subsection about the single-step model. It says that in a single-step model the existence of frictionless strong arbitrage, fixed cost strong arbitrage and fixed cost arbitrage are all equivalent to the non-existence of an AMF.

Theorem 9.6. *(Single-step arbitrage equivalence)*
In a single-step model the following statements are equivalent:

(1) *There exists a frictionless strong arbitrage opportunity.*

(2) *There exists a fixed cost strong arbitrage opportunity.*

(3) *There exists a fixed cost arbitrage opportunity.*

(4) *There does not exist an AMF.*

Proof of Theorem 9.6.

From Theorem 9.5 we see that each of these statements is equivalent to the model being of type 4. Thus the statements must be equivalent to each other.

End of proof of Theorem 9.6.

We have now established that in a single-step model, frictionless strong arbitrage, fixed cost arbitrage and fixed cost strong arbitrage are all equivalent to the existence of an AMF. It is now time to create some links between single-step sub-models and a full multi-step model.

9.2. **Arbitrage in single-step sub-models and in the multi-step model.**

In this section we deal with three types of arbitrage: fixed cost arbitrage, frictionless strong arbitrage and fixed cost strong arbitrage. Correspondingly this sub-section will have a three-part lemma, each part proving that the existence of the given type of arbitrage in a sub-model is equivalent to its existence in the full model.

Lemma 9.7. *(Arbitrage in the full model and arbitrage in single-step sub-models)*
Consider a model $\mathbb{M}$*:*

(1) *There exists a frictionless **strong** arbitrage opportunity in* $\mathbb{M}$ *if and only if there exists a frictionless strong arbitrage opportunity in at least one single-step sub-model of* $\mathbb{M}$*.*

(2) *There exists a fixed cost arbitrage opportunity in* $\mathbb{M}$ *if and only if there exists a fixed cost arbitrage opportunity in at least one single-step sub-model of* $\mathbb{M}$*.*

(3) *There exists a fixed cost **strong** arbitrage opportunity in* $\mathbb{M}$ *if and only if there exists a fixed cost strong arbitrage opportunity in at least one single-step sub-model of* $\mathbb{M}$*.*

Proof of Lemma 9.7.

Proof part 1($\Leftarrow$) (1),(2),(3)

Suppose that there exists a single-step sub-model $\mathbb{M}' = (\Omega', \mathcal{F}', \mathbb{P}', S', F', P')$ at node $\lambda \in \Lambda$ of model $\mathbb{M} = (\Omega, \mathcal{F}, \mathbb{P}, S, F, P)$ for which there exists a frictionless (frictionless strong, fixed cost, fixed cost strong, respectively) arbitrage opportunity Z'. Note that $\Omega' = \lambda$.

We construct a frictionless (frictionless strong, fixed cost, fixed cost strong, respectively) arbitrage trading strategy Z in the model $\mathbb{M}$:

- Do nothing at nodes that do not belong to either $\{\lambda\}$ or Succ λ:
  $Z(\mu) = 0$ if $\mu \neq \lambda$ and $\mu \notin$ Succ λ.

- If we do reach node λ then use the portfolio strategy Z':
  $Z(\lambda) = Z'(\lambda)$ and $Z(\mu) = Z'(\mu)$ for all $\mu \in$ Succ λ.

This portfolio strategy is clearly a frictionless (frictionless strong, fixed cost, fixed cost strong, respectively) arbitrage opportunity because it does nothing if the node of the sub-tree is not reached, but applies the frictionless (frictionless strong, fixed cost, fixed cost strong, respectively) arbitrage opportunity in the sub-tree once it is there.

Proof part 2 ($\Rightarrow$)

(1),(3)

We show that if there does not exist frictionless (resp. fixed cost) strong arbitrage in any single-step sub tree then there cannot exist a frictionless (resp. fixed cost) strong arbitrage opportunity in the full model.

Let us fix a non-terminal node $\lambda' \in \Lambda_{t'}$ at time $t' \in \mathcal{T}^{-T}$ and let (X, Y) be a frictionless (resp.fixed cost) self-financing portfolio strategy such that $E^{(X,Y)}(\lambda) > 0$ for all terminal descendant nodes λ to the node λ'. Note that any frictionless or fixed cost strong arbitrage opportunity must have this property for some node λ'.

Suppose that there does not exist a frictionless (resp.fixed cost) strong arbitrage opportunity in any single-step sub tree.

Our goal is to show that $A^{(X,Y)}(\lambda')$ cannot be zero and thus (X, Y) cannot be a frictionless (resp. fixed cost) strong arbitrage opportunity.

For any node λ self-financing tells us that $A^{(X,Y)}(\lambda) \geq E^{(X,Y)}(\lambda)$.

Let us write $\Lambda'_t := \{\mu \in \Lambda_t : \mu \subset \lambda'\}$, the set of nodes at time t that are either descendant nodes of λ' or are λ' itself.

For any $t \in \mathcal{T}^{-T}$ such that $t \geq t'$, we also have that $\min\limits_{\lambda \in \Lambda'_t} E^{(X,Y)}(\lambda) \geq \min\limits_{\lambda \in \Lambda'_{t+1}} A^{(X,Y)}(\lambda)$.

The reason for this is that if it were not true then for some $\tilde{t} \in \mathcal{T}$ there would exist a node $\tilde{\lambda} \in \Lambda'_{\tilde{t}}$ such that $E^{(X,Y)}(\tilde{\lambda}) < A^{(X,Y)}(\mu)$ for all $\mu \in \text{Succ}\ \tilde{\lambda}$ and this would imply that the single-step model at node $\tilde{\lambda}$ is of type 4 contradicting our assumption that there does not exist a frictionless (resp. fixed cost) strong arbitrage opportunity in any single-step sub-model.

Therefore we can deduce the following chain of inequalities:

$$\min_{\lambda \in \Lambda'_{t'}} A^{(X,Y)}(\lambda) \geq \min_{\lambda \in \Lambda'_{t'}} E^{(X,Y)}(\lambda) \geq \min_{\lambda \in \Lambda'_{t'+1}} A^{(X,Y)}(\lambda) \geq \cdots \geq \min_{\lambda \in \Lambda'_{T-1}} E^{(X,Y)}(\lambda) \geq \min_{\lambda \in \Lambda'_{T}} A^{(X,Y)}(\lambda) \geq \min_{\lambda \in \Lambda'_{T}} E^{(X,Y)}(\lambda) > 0$$

Therefore we have $A^{(X,Y)}(\lambda') \geq \min\limits_{\lambda \in \Lambda'_{t'}} A^{(X,Y)}(\lambda) > 0$.

We can conclude that if there does not exist frictionless (resp. fixed cost) strong arbitrage in any sub-model then there does not exist frictionless (resp. fixed cost) strong arbitrage in the full model.

(2)

Suppose that there does not exist a fixed cost arbitrage opportunity in any single-step sub-model and let (X, Y) be a fixed cost self-financing portfolio strategy such that $E^{(X,Y)}(\lambda) \geq 0$ for all $\lambda \in \Lambda_T$.

We will show that (X, Y) cannot be a fixed cost arbitrage opportunity.

If $Y_{+1}(\lambda) - Y(\lambda) = 0$ for all $\lambda \in \Lambda$ (portfolio strategy never trades) then clearly (X, Y) is not an arbitrage opportunity.

Let us assume that there is at least one node where the portfolio strategy does trade.

We will show that we cannot have $A_0^{(X,Y)} = 0$ and thus that this portfolio strategy cannot be a fixed cost arbitrage opportunity.

Let $t' \in \mathcal{T}$ be the smallest time at which the portfolio strategy trades in any scenario and consider a node $\lambda' \in \Lambda_{t'}$ such that $y_{+1}(\lambda') - y(\lambda') \neq 0$.

Since this is the first time at which the portfolio strategy trades, we must have $A^{(X,Y)}(\lambda) = A^{(X,Y)}(\lambda')$ for all $\lambda \in \Lambda_0 \cup \Lambda_1 \cup \cdots \cup \Lambda_{t'-1}$.

Because of the fact that $y_{+1}(\lambda') - y(\lambda') \neq 0$ the fixed cost self-financing condition becomes $A^{(X,Y)}(\lambda') = E^{(X,Y)}(\lambda') + F(\lambda')$ and thus $A^{(X,Y)}(\lambda') > E^{(X,Y)}(\lambda')$.

As in the proof of (2) and (4) we can write $\Lambda'_t := \{\mu \in \Lambda_t : \mu \subset \lambda'\}$ and deduce that:

$$\min_{\lambda \in \Lambda'_{t'}} E^{(X,Y)}(\lambda) \geq \cdots \geq \min_{\lambda \in \Lambda'_{T-1}} E^{(X,Y)}(\lambda) \geq \min_{\lambda \in \Lambda'_T} A^{(X,Y)}(\lambda) \geq \min_{\lambda \in \Lambda'_T} E^{(X,Y)}(\lambda) \geq 0$$

Therefore we have $A^{(X,Y)}(\lambda') > E^{(X,Y)}(\lambda') \geq \min_{\lambda \in \Lambda'_{t'}} E^{(X,Y)}(\lambda) \geq 0$

Since $A_0^{(X,Y)} = A^{(X,Y)}(\lambda')$ we have that $A_0^{(X,Y)} > 0$ and thus (X,Y) cannot be an arbitrage opportunity.

End of proof of Lemma 9.7.

9.3. Arbitrage in the multi-step model.

Now that we have a full understanding of arbitrage in a single-step model and we have shown how the existence of a given type of arbitrage in some single-step sub-model is equivalent to its existence in the full model, we can prove the following theorem about the existence of the various types of arbitrage in a multi-step model. In particular I think that an interesting point about this result is that, unlike with frictionless arbitrage, fixed cost arbitrage can only exist if fixed cost strong arbitrage exists.

Theorem 9.8. *(Equivalent forms of arbitrage)*
For any model $\mathbb{M}$ the following statements are equivalent:

(1) *There exists a frictionless strong arbitrage opportunity on $\mathbb{M}$.*

(2) *There exists a fixed cost strong arbitrage opportunity on $\mathbb{M}$.*

(3) *There exists a fixed cost arbitrage opportunity on $\mathbb{M}$.*

Proof of Theorem 9.8

(1) $\Leftrightarrow$ (2)

This follows from Theorem 9.6 and Lemma 9.7. The existence of frictionless strong arbitrage on $\mathbb{M}$ is equivalent to the existence of frictionless strong arbitrage in a single-step sub-model of $\mathbb{M}$ by Lemma 9.7. The existence of frictionless strong arbitrage in a single-step sub-model of $\mathbb{M}$ is equivalent to the existence of fixed cost strong arbitrage in a single-step sub-model of $\mathbb{M}$ by Theorem 9.6. The existence of fixed cost strong arbitrage in a single-step sub-model of $\mathbb{M}$ is equivalent to the existence of fixed cost strong arbitrage in $\mathbb{M}$ by Lemma 9.7.

(2) $\Leftrightarrow$ (3)

This follows from Theorem 9.6 and Lemma 9.7. The existence of fixed cost arbitrage on $\mathbb{M}$ is equivalent to the existence of fixed cost arbitrage in a single-step sub-model of $\mathbb{M}$ by Lemma 9.7. The existence of fixed cost arbitrage in a single-step sub-model of $\mathbb{M}$ is equivalent to the existence of fixed cost strong arbitrage in a single-step sub-model of $\mathbb{M}$ by Theorem 9.6. The existence of fixed cost strong arbitrage in a single-step sub-model of $\mathbb{M}$ is equivalent to the existence of fixed cost strong arbitrage in $\mathbb{M}$ by Lemma 9.7.

End of proof of Theorem 9.8

9.4. **Martingale functions: Single-step models to multi-step model.**

We have seen the connection between the existence of arbitrage in a single-step sub-model and the existence of arbitrage in the full model. In this subsection we show that there exists an AMF in every single-step sub-model if and only if there exists an AMF in the full model. This combined with Theorem 9.8 will allow us to prove a complete multi-step analogue of Theorem 9.6

Theorem 9.9. *(AMF in a sub-model)*
There exists an AMF in a model $\mathbb{M}$ *if and only if there exists an AMF in every single-step sub-model of* $\mathbb{M}$*.*

Proof of Theorem 9.9.

Proof part 1 $\underline{\Leftarrow}$

Suppose that for every single-step sub-model $\mathbb{M}^\lambda$ at node λ in the model $\mathbb{M}$ there exists an AMF $\mathbf{Q}^\lambda$ for $\mathbb{M}^\lambda$.

Recall that λ_t^ω represents the node at time $t \in \mathcal{T}$ for a scenario $\omega \in \Omega$. (See Definition 5.9)

We define a conditional probability function $\mathbf{Q}$ for $\mathbb{M}$ in the following way:

$\mathbf{Q}(\lambda, \mu) := \mathbf{Q}^\lambda(\lambda, \mu)$ for all $\mu \in \mathrm{Succ}\lambda$.

Observe that this is an AMF on $\mathbb{M}$.

Proof part 2 $\underline{\Rightarrow}$

Suppose that there exists an AMF $\mathbf{Q}$ on a model $\mathbb{M}$.

For any single-step sub-model $\mathbb{M}^\lambda$ at node λ of model $\mathbb{M}$, we can define a conditional probability function $\mathbf{Q}^\lambda$ in the following way:

$\mathbf{Q}^\lambda(\lambda, \mu) := \mathbf{Q}(\lambda, \mu)$ for all $\mu \in \mathrm{Succ}\lambda$.

Observe that this is an AMF on the single-step sub-model $\mathbb{M}^\lambda$.

End of proof of Theorem 9.9.

9.5. **Arbitrage and martingale functions in the multi-step model.**

Theorem 8.3 is a crucial result in the theory of asset pricing and finally we can now prove an analogous result relating to fixed cost arbitrage. As previously mentioned it is very similar to a result in the paper by Jouini, Kallal and Napp in [16].

Theorem 9.10. *(Fixed cost fundamental theorem)*
A model $\mathbb{M}$ is fixed cost viable if and only if there exists an AMF on $\mathbb{M}$.

Proof of Theorem 9.10.

This is a consequence of Lemma 9.7, Theorem 9.6 and Theorem 9.9.

By Lemma 9.7 the model $\mathbb{M}$ does not admit fixed cost arbitrage if and only if there does not exist fixed cost arbitrage in every single-step sub-model of $\mathbb{M}$.

By Theorem 9.6 there does not exist fixed cost arbitrage in every single-step sub-model of $\mathbb{M}$ if and only if there exists an AMF in every single-step sub-model of $\mathbb{M}$.

Finally by Theorem 9.9 there exists an AMF on every single-step sub-model of $\mathbb{M}$ if and only if there exists an AMF on $\mathbb{M}$.

End of proof of Theorem 9.10.

10. **Proportional cost arbitrage**

In this section our primary focus will be on the so called proportional cost fundamental theorem. In its pure form this result has already been shown by Jouini and Kallal in [15]. However we have bigger goals in mind with our work here, in the medium-short term, we wish to find a way of proving the proportional cost fundamental theorem that lends itself to a proof of the combined cost fundamental theorem that we will see in the next section. In the longer term, at the end of this thesis we will need this result in the theoretical setting of the case when the process P is unrestricted, in order to prove the ask price representation theorem in Section 19. The proof of this result that we will see is not only new, but arguably much simpler than the proof given by Jouini and Kallal in [15] as it starts from first principles and extensively uses induction arguments to capture the underlying reason why the theorem is true.

In Section 8 we proved everything we needed to prove by working in single-step sub-models and then moving to the multi-step model. The following example shows that under proportional costs, this approach simply won't work.

Example 10.1. (Proportional cost arbitrage in a single-step sub-model not equivalent to Proportional cost arbitrage in full model)
Consider the following stock price diagram. We look at the ask and bid price processes as they make the situation with regard to arbitrage much clearer.

$$\begin{tikzcd}
 & & \begin{matrix} S^a(uu) = 11 \\ S^b(uu) = 9 \end{matrix} \\
 & \begin{matrix} S^a(u) = 12 \\ S^b(u) = 2 \end{matrix} \arrow[ur] \arrow[dr] & \\
 & & \begin{matrix} S^a(ud) = 10 \\ S^b(ud) = 8 \end{matrix} \\
\begin{matrix} S^a(\Omega) = 6 \\ S^b(\Omega) = 4 \end{matrix} \arrow[uur] \arrow[ddr] & & \\
 & & \begin{matrix} S^a(du) = 9 \\ S^b(du) = 7 \end{matrix} \\
 & \begin{matrix} S^a(d) = 11 \\ S^b(d) = 1 \end{matrix} \arrow[ur] \arrow[dr] & \\
 & & \begin{matrix} S^a(dd) = 8 \\ S^b(dd) = 7 \end{matrix}
\end{tikzcd}$$

It is easy to observe that the full model admits proportional cost arbitrage. An investor could simply buy a single stock at time zero at the cost of 6 units and then wait until time two in order to sell the stock for at least 7 units. It is also clear that there cannot be arbitrage in any of the three single-step models as any attempt at proportional cost arbitrage in one of these single-step models would not only fail to achieve proportional cost arbitrage, but would be certain to lose money due to the huge transaction cost incurred when trading at time one.

End of example 10.1

Example 10.2. (No proportional cost arbitrage does not imply EMM (with respect to S))
Consider the following stock price diagram.

$$\begin{array}{ccc} & & S^a(u) = 10 \\ & & S^b(u) = 2 \\ & \nearrow & \\ S^a(\Omega) = 4 & & \\ S^b(\Omega) = 4 & & \\ & \searrow & \\ & & S^a(d) = 9 \\ & & S^b(d) = 1 \end{array}$$

Clearly this model does not admit proportional cost arbitrage as the transaction costs at time 1 are way to large to allow profitable trading. However, we can easily see that the process S takes the values $S(\Omega) = 4, S(u) = 6, S(d) = 5$, and thus allows frictionless arbitrage and does not admit an EMM (or even an AMF). This shows that the non-existence of proportional cost arbitrage does not imply the existence of an EMM or AMF (with respect to S).

End of example 10.2

It is in fact true that the existence of an EMM implies the non-existence of proportional cost arbitrage. This fact is included in a more general equivalent condition for the lack of proportional cost arbitrage, the so called fundamental theorem for proportional transaction costs (Theorem 10.22), which we will soon see.

We will soon show the proportional cost fundamental theorem for unrestricted models. However, we will first examine what it means for a strategy to be self-financing in an unrestricted model before doing some preparatory analysis of frictionless middle value processes.

10.1. **Proportional cost self-financing with infinite transaction costs.**

As mentioned in the introduction to this section, the theoretical situation when the process P is unrestricted is necessary for our later work. We will now explore what it means to allow the process P to be unrestricted.

10.1.1. *The meaning of a self-financing portfolio strategy.*

Recall the following definitions:6.1(Portfolio), 6.2 (Portfolio operation), 6.6 (Trade), 6.9 (Trading strategy) and 6.11 (Self-financing portfolio strategy).

Consider a node λ such that $P(\lambda) = \infty$. Any trading strategy Z must have $Z(\lambda) = 0$. It follows that for any portfolio strategy (X, Y) if $(X(\lambda), Y(\lambda)) \neq (X_{+1}(\lambda), Y_{+1}(\lambda))$ then (X, Y) is not self-financing. This is in line with our intuition about a node with an infinite transaction cost, a node with an infinite transaction cost is a node where trading is not allowed.

10.2. **Proportional cost fundamental theorem.**

10.2.1. *Frictionless middle value process.*

We now introduce the concept of a so-called frictionless middle value stock process. A frictionless middle value stock process has a very close relationship to the proportional cost fundamental theorem as it will be shown that the non-existence of proportional cost arbitrage is equivalent to the existence of a frictionless middle value stock process. Please note that in some papers the term *consistent price system* is used instead of what we will call a frictionless middle value process, see the work of Schachermayer in [25]. We use the term frictionless middle value process as I believe that in our context, it is the most self-explanatory way of naming this concept.

Definition 10.3. (Frictionless middle value stock process)
For any model $\mathbb{M} = (\Omega, \mathcal{F}, \mathbb{P}, S, F, P)$, a *frictionless middle value stock process* is a strictly positive adapted process $\tilde{S}$ such that:

$$S^b \leq \tilde{S} \leq S^a$$

and the model $\mathbb{M}' = (\Omega, \mathcal{F}, \mathbb{P}, \tilde{S}, F, P)$ does not admit frictionless arbitrage.

Remark 10.4.
Notice that since a frictionless middle value process does not admit frictionless arbitrage, there exists an EMM with respect to $\tilde{S}$.

10.2.2. *Construction of a frictionless middle value process.*

In this subsection we will show how to construct a middle value process (if it is possible to construct one) by moving backwards through the tree constructing the set of possible stock prices at a given time that could be part of a middle value process $\hat{S}$.

Definition 10.5. ($\mathcal{Q}(\lambda)$)
Let $\lambda \in \Lambda$.

We define $\mathcal{Q}(\lambda)$ to be the set of real numbers $x \in \mathbb{R}$ for which we can find a strictly positive adapted process $\tilde{S}$ such that :

$\tilde{S}(\lambda) = x$

and

$S^b(\lambda') \leq \tilde{S}(\lambda') \leq S^a(\lambda')$ for any descendant node λ' to the node λ

and the process $\tilde{S}$ does not admit frictionless arbitrage on the sub-model with root node λ.

Definition 10.6. ($\mathcal{R}(\lambda)$:"$\mathcal{Q}(\lambda)$ restricted to the bid-ask spread at node λ ")
Let $\lambda \in \Lambda$.

We set $\mathcal{R}(\lambda) := \mathcal{Q}(\lambda) \cap [S^b(\lambda), S^a(\lambda)] \cap (-\infty, \infty)$.

Remark 10.7.
Observe that the existence of a frictionless middle value process is equivalent to the set $\mathcal{R}_0$ being non-empty. Furthermore, if we have constructed $\mathcal{R}(\lambda)$ for every node λ and they are not empty, then it is easy to construct a middle value process by working forwards through the tree first choosing a $\hat{S}_0$ from $\mathcal{R}_0$ and then choosing appropriate $\hat{S}_1$s from the $\mathcal{R}_1$s etc. Note Convention 2 (e.g. $\mathcal{R}_0$ is an equivalent way of writing $\mathcal{R}(\Omega)$).

Definition 10.8. (Lower and Upper ends of the sets $\mathcal{Q}$ and $\mathcal{R}$.)
We will write $Q^b(\lambda)$ and $Q^a(\lambda)$ for the infimum and supremum of $\mathcal{Q}(\lambda)$ respectively.

We will write $R^b(\lambda)$ and $R^a(\lambda)$ for the infimum and supremum of $\mathcal{R}(\lambda)$ respectively.

We will not use these terms when $\mathcal{Q}$ and $\mathcal{R}$ are empty.

Remark 10.9. (Terminal values of the sets $\mathcal{Q}$ and $\mathcal{R}$)

Observe from the definitions of $\mathcal{Q}$ and $\mathcal{R}$ that for any terminal node $\lambda \in \Lambda_T$:

$\mathcal{Q}(\lambda) = \mathbb{R}^+$

and

$\mathcal{R}(\lambda) = [S^b(\lambda), S^a(\lambda)]$ if $P(\lambda) < S(\lambda)$.

$\mathcal{R}(\lambda) = (0, S^a(\lambda)]$ if $S(\lambda) \leq P(\lambda) < \infty$.

$\mathcal{R}(\lambda) = (0, \infty)$ if $P(\lambda) = \infty$.

Lemma 10.10. *($\mathcal{Q}(\lambda)$ is an Interval)*
For any node $\lambda \in \Lambda$, the set $\mathcal{Q}(\lambda)$ is convex and hence an interval.

Proof of Lemma 10.10.

It is enough to prove the claim for $\mathcal{Q}(\lambda)$ when λ is the root node.

Let $x', x'' \in \mathcal{Q}_0$ and let $k \in [0,1]$.

We need to show that

$x := kx' + (1-k)\,x'' \in \mathcal{Q}_0$.

Since x' and x'' belong to $\mathcal{Q}_0$, we can find strictly positive adapted processes S' and S'' such that

$S_0' = x', \quad S_0'' = x'', \qquad S_t^a \leq S_t' \leq S_t^b$ and $S_t^a \leq S_t'' \leq S_t^b \qquad \forall t \in \mathcal{T}^{-0}$,

and the stock price models S' and S'' do not allow frictionless arbitrage.

The fact that S' and S'' do not allow frictionless arbitrage means that there exist probability measures $\mathbb{Q}'$ and $\mathbb{Q}''$ such that:

(1) $\mathbb{Q}'$ is equivalent to $\mathbb{P}$

(2) S' is a strictly positive martingale under $\mathbb{Q}'$

and

(1) $\mathbb{Q}''$ is equivalent to $\mathbb{P}$

(2) S'' is a strictly positive martingale under $\mathbb{Q}''$

Set

$\mathbb{Q} := k\mathbb{Q}' + (1-k)\,\mathbb{Q}''$

and

$\tilde{S}\,(\mu) := \frac{k\mathbb{Q}'(\mu)S'(\mu)+(1-k)\mathbb{Q}''(\mu)S''(\mu)}{\mathbb{Q}(\mu)}$ for any node $\mu \in \Lambda$.

Because $\mathbb{Q}'$ and $\mathbb{Q}''$ are probability measures, so is $\mathbb{Q}$.

We also have that $\mathbb{Q}'$ and $\mathbb{Q}''$ are equivalent to $\mathbb{P}$, so $\mathbb{Q}$ is equivalent to $\mathbb{P}$.

Since $S_t^b \leq S_t' \leq S_t^a$ and $S_t^b \leq S_t'' \leq S_t^a$ for all $t \in \mathcal{T}^{-0}$, it follows that :

$S_t^b \leq \tilde{S}_t \leq S_t^a$ for all $t \in \mathcal{T}^{-0}$.

Moreover for any $t \in \mathcal{T}^{-T}$ and node $\mu \in \Lambda_t$ we have:

$$\mathbb{E}_{\mathbb{Q}}\left(\tilde{S}_{t+1}|\mu\right) = \frac{\sum\limits_{\nu\in\mathrm{Succ}(\mu)} \mathbb{Q}(\nu)\tilde{S}_{t+1}(\nu)}{\mathbb{Q}(\mu)}$$

$$= \frac{\sum\limits_{\nu\in\mathrm{Succ}(\mu)} \left(k\mathbb{Q}'(\nu)S'_{t+1}(\nu) + (1-k)\,\mathbb{Q}''(\nu)S''_{t+1}(\nu)\right)}{\mathbb{Q}(\mu)}$$

$$= \frac{k\mathbb{Q}'(\mu)\mathbb{E}_{\mathbb{Q}'}\left(S'_{t+1}|\mu\right) + (1-k)\,\mathbb{Q}''(\mu)\mathbb{E}_{\mathbb{Q}''}\left(S''_{t+1}|\mu\right)}{\mathbb{Q}(\mu)}$$

$$= \frac{k\mathbb{Q}'(\mu)S'_t(\mu) + (1-k)\,\mathbb{Q}''(\mu)S''_t(\mu)}{\mathbb{Q}(\mu)}$$

$$= \tilde{S}_t(\mu),$$

which shows that $\tilde{S}$ is a martingale under $\mathbb{Q}$.

Finally,

$\tilde{S}_0 = kS'_t + (1-k)\,S''_0 = kx' + (1-k)\,x'' = x,$

and we can conclude that indeed $x \in \mathcal{Q}_0$.

End of Proof of Lemma 10.10.

Lemma 10.11. *($\mathcal{R}(\lambda)$ is an Interval)*
For any node $\lambda \in \Lambda$, the set $\mathcal{R}(\lambda)$ is convex and hence an interval.

Proof of Lemma 10.11.

This is immediate from Lemma 10.10 and the definition of $\mathcal{R}$.

End of Proof of Lemma 10.11.

We will now do a short example in order to build some intuition about how to construct a set $\mathcal{Q}(\lambda)$ from the sets $\mathcal{R}(\mu)$, $\mu \in \mathrm{Succ}\lambda$.

Example 10.12. (Constructing $\mathcal{Q}(\lambda)$ from the $\mathcal{R}(\mu)$)
Consider a single-step binomial model $\mathbb{M}$ with root node Ω and nodes at time 1, u and d.

Suppose that $\mathcal{R}(d) = [1, 2]$ and $\mathcal{R}(u) = [3, 4]$.

Our objective is to find the set $\mathcal{Q}(\Omega)$.

Consider any real number x that lies strictly in between 1 ($= \min\{R^b(d), R^b(u)\}$) and 4 ($= \max\{R^a(d), R^a(u)\}$), lets say $x = 2$.

Set $\tilde{S}(\Omega) = 2$, $\tilde{S}(u) = 4$ and $\tilde{S}(d) = 1$.

Observe that $\tilde{S}$ does not admit frictionless arbitrage and thus 2 (or any number in between 1 and 4) belongs to $\mathcal{Q}(\Omega)$.

We will now show that 1 does not belong to $\mathcal{Q}(\Omega)$.

We will try to construct a process $\tilde{S}'$, with $\tilde{S}'(\Omega) = 1$, that has the required properties in order to understand why it is not possible.

Set $\tilde{S}'(\Omega) = 1$, we need to define $\tilde{S}'(u)$, $\tilde{S}'(d)$ in such a way that $\tilde{S}'$ does not admit frictionless arbitrage.

Since $\mathcal{R}(u) = [3, 4]$, we must choose $\tilde{S}'(u)$ to belong to this interval, lets say $\tilde{S}'(u) = 3$. This value is greater that $\tilde{S}'(\Omega) = 1$.

The only way that $\tilde{S}'$ could not admit frictionless arbitrage would be if we could set $\tilde{S}'(d)$ to be strictly less that 1 but unfortunately $\mathcal{R}(d) = [1, 2]$ so the smallest $\tilde{S}'(d)$ that we can choose is 1, and a model with $\tilde{S}'(\Omega) = 1$, $\tilde{S}'(u) = 3$ and $\tilde{S}'(d) = 1$ admits frictionless arbitrage.

We observed that $1 \notin \mathcal{Q}(\Omega)$ and by a symmetrical argument we could deduce that $4 \notin \mathcal{Q}(\Omega)$.

We can now conclude that $\mathcal{Q}(\Omega) = (1, 4)$, $Q^b = 1 = \min\limits_{\mu \in \mathrm{Succ}\lambda} R^b(\mu)$ and $Q^a = 4 = \max\limits_{\mu \in \mathrm{Succ}\lambda} R^a(\mu)$.

This example gives us the intuition we need in order to understand the following Lemma.

End of Example 10.12.

Lemma 10.13. *(Construction of $\mathcal{Q}(\lambda)$ from the $\mathcal{R}(\mu)$)*
Let $\lambda \in \Lambda$.

If for each successor node $\mu \in \mathrm{Succ}\lambda$ to node λ, $\mathcal{R}(\mu)$ is non-empty,

then:

(1) $\quad Q^a(\lambda) = \max\limits_{\mu \in \mathrm{Succ}\lambda} R^a(\mu).$

(2) $\quad Q^b(\lambda) = \min\limits_{\mu \in \mathrm{Succ}\lambda} R^b(\mu).$

Furthermore the following conditions determine whether the end points of $\mathcal{Q}(\lambda)$ belong to it.

(3)$Q^a(\lambda) \in \mathcal{Q}(\lambda)$ if and only if for all $\mu \in \mathrm{Succ}\lambda$, $Q^a(\lambda) = R^a(\mu)$ and $R^a(\mu) \in \mathcal{R}(\mu)$.

(4) $Q^b(\lambda) \in \mathcal{Q}(\lambda)$ if and only if for all $\mu \in \mathrm{Succ}\lambda$, $Q^b(\lambda) = R^b(\mu)$ and $R^b(\mu) \in \mathcal{R}(\mu)$.

If there exists a node $\mu \in$ succ λ such that $\mathcal{R}(\mu)$ is empty,

then:

(5) $\mathcal{Q}(\lambda)$ is empty.

Proof of Lemma 10.13.

It is enough to prove the claim when λ is the root node.

Let us deal with (5) first as it is trivial.

Clearly if it is not possible to construct a frictionless middle value process for any choice of $\tilde{S}(\mu)$ for some $\mu \in \text{succ}\lambda$ then it is not possible to construct a frictionless middle value process for any choice of $\tilde{S}(\lambda)$ because we would still need to choose a value for $\tilde{S}(\mu)$ such that there exists a frictionless middle value process.

We conclude that (5) is true.

Let us now deal with (1),(2),(3) and (4) .

Let us assume that $\mathcal{R}(\mu)$ is non-empty for all $\mu \in \text{Succ}\lambda$.

Let n be the number of nodes at time 1. Note that $n > 1$ due to the fact that we assume every node has more than one successor node in the definition of a model.

Observe that, if we can define a process $\tilde{S}$ to be such that $\tilde{S}(\lambda) = q \in \mathbb{R}$ and $\tilde{S}(\mu) \in \mathcal{R}(\mu)$ for all $\mu \in \text{Succ}\lambda$, and the single-step sub-model starting at node λ does not admit frictionless arbitrage, then it is possible to define $\tilde{S}$ on the sub-model at node λ in such a way that $\tilde{S}$ is a frictionless middle value process on the sub-model at node λ. This is a consequence of the fact that the non-existence of frictionless arbitrage in a full model is equivalent to the non-existence of frictionless arbitrage in any single-step sub-model.

Claim 1

Let $q \in \mathbb{R}^+$

Then $q \in \mathcal{Q}(\lambda)$ if and only if

there exist $\mu_1 \in \text{Succ}\lambda$, $\mu_2 \in \text{Succ}\lambda$, $r_1 \in R(\mu_1)$ and $r_2 \in \mathcal{R}(\mu_2)$ such that $r_1 < q < r_2$

or

$q \in \mathcal{R}(\mu)$ for all $\mu \in \text{Succ}\lambda$.

Proof of Claim 1

We will first prove the "if" part of the claim.

Suppose there exist $\mu_1 \in \text{Succ}\lambda$, $\mu_2 \in \text{Succ}\lambda$, $r_1 \in R(\mu_1)$ and $r_2 \in \mathcal{R}(\mu_2)$ such that $r_1 < q < r_2$.

Let n be the number of nodes in Succλ and let $\mu_3, \mu_4, \ldots, \mu_n$ be the nodes belonging to Succλ that are not equal to μ_1 or μ_2.

Let $r_3, r_4, \ldots, r_n$ belong to $\mathcal{R}(\mu_3), \mathcal{R}(\mu_4), \ldots, \mathcal{R}(\mu_n)$ respectively.

Then we can set $\tilde{S}(\lambda) = q$ and $\tilde{S}(\mu_i) = r_i$ for $i = \{1, \ldots, n\}$ and observe that this means that the single-step sub-model is of type 1 (see definition 9.3) and thus does not admit frictionless arbitrage.

Therefore $q \in \mathcal{Q}(\lambda)$.

Suppose that $q \in \mathcal{R}(\mu)$ for all $\mu \in \text{Succ}\lambda$.

Then we can set $\tilde{S}(\lambda) = q$ and $\tilde{S}(\mu) = q$ for all $\mu \in \text{Succ}\lambda$ and observe that this means that the single-step sub-model is of type 2 (see definition 9.3) and thus does not admit frictionless arbitrage.

Therefore $q \in \mathcal{Q}(\lambda)$.

We now prove the "only if" part of the claim.

Suppose that there does not exist $\mu_1 \in \text{Succ}\lambda$, $\mu_2 \in \text{Succ}\lambda$, $r_1 \in R(\mu_1)$ and $r_2 \in \mathcal{R}(\mu_2)$ such that $r_1 < q < r_2$ and it is not true that $q \in \mathcal{R}(\mu)$ for all $\mu \in \text{Succ}\lambda$.

This implies that for any choice of $r_i \in \mathcal{R}(\mu_i)$ and $\mu_i \in \text{Succ}\lambda$, where $i \in \{1, \ldots, n\}$, we have either

$r_i \leq q$ for each $i \in \{1, \ldots, n\}$ and there exists $j \in \{1, \ldots, n\}$ such that $r_j < q$

or

$r_i \geq q$ for each $i \in \{1, \ldots, n\}$ and there exists $j \in \{1, \ldots, n\}$ such that $r_j > q$.

Clearly in either case if we choose $\tilde{S}(\lambda) = q$ and $\tilde{S}(\mu_i) = r_i$ for each $i \in \{1, \ldots, n\}$, then the model will admit frictionless arbitrage.

Thus we conclude that $q \notin \mathcal{Q}(\lambda)$.

End of proof of Claim 1

We now prove (1) and (2).

If $\max\limits_{\mu \in \text{Succ}\lambda} R^a(\mu) = \min\limits_{\mu \in \text{Succ}\lambda} R^b(\mu)$ then since all of the $\mathcal{R}$s are non-empty we must have $\mathcal{R}(\mu) = \{r\}$ for all $\mu \in \text{Succ}\lambda$, where $r = \max\limits_{\mu \in \text{Succ}\lambda} R^a(\mu) = \min\limits_{\mu \in \text{Succ}\lambda} R^b(\mu)$.

By Claim 1 we can conclude that $q = r$ is the only value in the set $\mathcal{Q}(\lambda)$ and thus $\max\limits_{\mu \in \text{Succ}\lambda} R^a(\mu) = \min\limits_{\mu \in \text{Succ}\lambda} R^b(\mu) = Q^a(\lambda) = Q^b(\lambda)$.

Let us now assume that $\max\limits_{\mu \in \mathrm{Succ}\lambda} R^a(\mu) \neq \min\limits_{\mu \in \mathrm{Succ}\lambda} R^b(\mu)$.

Suppose that $\max\limits_{\mu \in \mathrm{Succ}\lambda} R^a(\mu)$ and $\min\limits_{\mu \in \mathrm{Succ}\lambda} R^b(\mu)$ are not attained at the same node.

Let μ_1 be the node where the maximum is attained and let μ_2 be the node where the minimum is attained.

Take any $q \in \left(\min\limits_{\mu \in \mathrm{Succ}\lambda} R^b(\mu), \max\limits_{\mu \in \mathrm{Succ}\lambda} R^a(\mu) \right)$.

We can find $r_1 \in \mathcal{R}(\mu_1)$ and $r_2 \in \mathcal{R}(\mu_2)$ such that $r_1 < q < r_2$.

Therefore by Claim 1, we have that $q \in \mathcal{Q}(\lambda)$.

Suppose that $\max\limits_{\mu \in \mathrm{Succ}\lambda} R^a(\mu)$ and $\min\limits_{\mu \in \mathrm{Succ}\lambda} R^b(\mu)$ are attained at the same node.

Let μ_1 be the node where both the maximum and minimum are attained.

Take any $q \in \left(\min\limits_{\mu \in \mathrm{Succ}\lambda} R^b(\mu), \max\limits_{\mu \in \mathrm{Succ}\lambda} R^a(\mu) \right)$.

If there exists a node μ_2 such that there is an element r_2 of $\mathcal{R}(\mu_2)$ for which either $r_2 < q$ or $r_2 > q$ then since we can always find $r_1 \in \mathcal{R}(\mu_1)$ such that $r_1 > q$ or $r_1 < q$ it follows from Claim 1 that $q \in \mathcal{Q}(\lambda)$.

If there does not exist a node μ_2 such that there is an element r_2 of $\mathcal{R}(\mu_2)$ for which either $r_2 < q$ or $r_2 > q$, then we must have $\mathcal{R}(\mu) = \{q\}$ for all $\mu \in \mathrm{Succ}\lambda$. Therefore we can choose $r_i = q$ for $i \in \{1, \ldots, n\}$ and by Claim 1 we have $q \in \mathcal{Q}(\lambda)$.

This completes the proof of (1) and (2).

We now prove (3) and (4).

Consider $q = \max\limits_{\mu \in \mathrm{Succ}\lambda} R^a(\mu)$.

Therefore for any $i \in \{1, \ldots, n\}$ there does not exist $r \in \mathcal{R}(\mu_i)$ such that $r > q$.

Therefore by Claim 1, $q \in \mathcal{Q}(\lambda)$ if and only if there exist $r_1, \ldots, r_n$ with $r_i \in \mathcal{R}(\mu_i)$ for each $i \in \{1, \ldots, n\}$ such that:

$q = r_1 = \ldots = r_n$.

This can happen if and only if for all $\mu \in \mathrm{Succ}\lambda$, we have $Q^a(\lambda) = R^a(\mu)$ and $R^a(\mu) \in \mathcal{R}(\mu)$.

We have proved (3).

Consider $q = \min\limits_{\mu \in \mathrm{Succ}\lambda} R^b(\mu)$.

Therefore for each $i \in \{1, \ldots, n\}$, there does not exist $r \in \mathcal{R}(\mu_i)$ such that $r < q$.

Therefore by Claim 1, $q \in \mathcal{Q}(\lambda)$ if and only if there exist $r_1, \ldots, r_n$ with $r_i \in \mathcal{R}(\mu_i)$ for each

$i \in \{1, \ldots, n\}$ such that:

$q = r_1 = \ldots = r_n$.

This can happen if and only if for all $\mu \in \mathrm{Succ}\lambda$, we have $Q^b(\lambda) = R^b(\mu)$ and $R^b(\mu) \in \mathcal{R}(\mu)$.

We have proved (4).

The proof is complete.

End of Proof of Lemma 10.13.

Lemma 10.14. *(Construction of $\mathcal{R}(\lambda)$ from $\mathcal{Q}(\lambda)$)*
Let $\lambda \in \Lambda$.

If $\mathcal{Q}(\lambda)$ is non-empty and the intervals $\mathcal{Q}(\lambda)$ and $[S^b(\lambda), S^a(\lambda)]$ overlap.

Then

$R^a(\lambda) = \min\{Q^a(\lambda), S^a(\lambda)\}$

$R^b(\lambda) = \max\{Q^b(\lambda), S^b(\lambda)\}$

Furthermore the following conditions determine whether or not the end points of $\mathcal{R}(\lambda)$ belong to it:

$R^a(\lambda) \in \mathcal{R}(\lambda)$ if and only if $Q^a(\lambda) > S^a(\lambda)$ or $Q^a(\lambda) \in \mathcal{Q}(\lambda)$.

$R^b(\lambda) \in \mathcal{R}(\lambda)$ if and only if $Q^b(\lambda) < S^b(\lambda)$ or $Q^b(\lambda) \in \mathcal{Q}(\lambda)$.

If $\mathcal{Q}(\lambda)$ is empty or the intervals $\mathcal{Q}(\lambda)$ and $[S^b, S^a]$ do not overlap

then

$\mathcal{R}(\lambda)$ is empty.

Proof of Lemma 10.14.

From the definition of $\mathcal{R}$ and the fact that $\mathcal{Q}(\lambda)$ is an interval, the result follows from the basic properties of intersecting two intervals.

End of proof of Lemma 10.14.

We have now gained all the tools we need in order to be able to construct processes $\mathcal{Q}$ and $\mathcal{R}$ if it is possible to construct them. We now need understand some properties of these processes of sets.

10.2.3. *Properties of $\mathcal{Q}$ and $\mathcal{R}$.*

The following properties of $\mathcal{Q}$ and $\mathcal{R}$ are at the heart of why the proportional cost fundamental theorem is true. We will show three properties of $\mathcal{Q}$ and one property of R.

Example 10.15. (Understanding the properties of $\mathcal{Q}$ and $\mathcal{R}$)
Consider the following stock price model under proportional transaction costs:

$$
\begin{tikzcd}
 & & \begin{array}{c} S^a(uu)=15 \\ S^b(uu)=9 \end{array} \\
 & \begin{array}{c} S^a(u)=11 \\ S^b(u)=9 \end{array} \arrow[ur] \arrow[dr] & \\
 & & \begin{array}{c} S^a(ud)=8 \\ S^b(ud)=8 \end{array} \\
\begin{array}{c} S^a(\Omega)=7 \\ S^b(\Omega)=5 \end{array} \arrow[uur] \arrow[ddr] & & \\
 & & \begin{array}{c} S^a(du)=6 \\ S^b(du)=4 \end{array} \\
 & \begin{array}{c} S^a(d)=4 \\ S^b(d)=2 \end{array} \arrow[ur] \arrow[dr] & \\
 & & \begin{array}{c} S^a(dd)=3 \\ S^b(dd)=1 \end{array}
\end{tikzcd}
$$

Let us compute the sets $\mathcal{Q}$ and $\mathcal{R}$:

$\mathcal{Q}(uu)=\mathbb{R}$, $\mathcal{Q}(ud)=\mathbb{R}$, $\mathcal{Q}(du)=\mathbb{R}$, $\mathcal{Q}(dd)=\mathbb{R}$

$\mathcal{R}(uu)=[9,15]$, $\mathcal{R}(ud)=[8]$, $\mathcal{R}(du)=[4,6]$, $\mathcal{R}(dd)=[1,3]$

$\mathcal{Q}(u)=(8,15)$, $\mathcal{Q}(d)=(1,6)$

$\mathcal{R}(u)=[9,11]$, $\mathcal{R}(d)=[2,4]$

$\mathcal{Q}(\Omega)=(2,11)$

$\mathcal{R}(\Omega)=[5,7]$

Consider the path leading to node dd.

Working backwards we have $Q^b(dd)=-\infty$, $R^b(dd)=1$, $Q^b(d)=1$, $R^b(d)=2$, $Q^b(\Omega)=2$, $R^b(\Omega)=5$.

Observe that the only time this sequence increases, when going from one term to the next, is if we reach a bid price that is bigger than the current number, e.g. $S^b(d)=2>Q^b(d)=1$, so we went from $Q^b(d)=1$ to $R^b(d)=2$. This is in fact a general property and since it is always true that $Q^b(\mu)=-\infty$ for any terminal node μ, we can see that the only way we can have $Q^b(\lambda)=x$, for some node λ and real number x, is if, at some point in every sequence like the one above, there was a bid price that was greater than or equal to x.

Now consider the root node, we have $Q^b(\Omega) = 2$, notice that no matter what path the stock follows, we will at some point in the future find a bid price that is at least as big as $Q^b(\Omega) = 2$. This is a consequence of the above observation and is the idea behind Lemma 10.16.

Also notice that another way of saying the phrase "no matter what path the stock follows, we will at some point in the future find a bid price that is a least as big as $Q^b(\Omega)$" is to say "there exists a collection of disjoint descendant nodes $\mu^1, \ldots, \mu^k$ such that: $\bigcup_{i=1,\ldots,k} \mu^i = \lambda$ and $Q^b(\Omega) \leq S^b(\mu^i)$ for all $i \in \{1, \ldots, k\}$".

Again if we look at the root node, we notice that $R^b(\Omega) = 5$ and that no matter what path the stock follows, we will, either at the current time or at some point in the future, find a stock price that is at least as big as $R^b(\Omega) = 5$. This is the idea behind Lemma 10.18.

End of Example 10.15.

Lemma 10.16. *(1st property of $\mathcal{Q}$)*
Let $\lambda \in \Lambda$ and suppose that $\mathcal{Q}(\lambda)$ is non-empty .

(1) If $Q^a(\lambda) \neq \infty$ then there exists a collection of disjoint descendant nodes $\mu^1, \ldots, \mu^k$ such that:

$$\bigcup_{i=1,\ldots,k} \mu^i = \lambda$$

and

$Q^a(\lambda) \geq S^a(\mu^i)$ *for all* $i \in \{1, \ldots, k\}$.

(2) If $Q^b(\lambda) \neq 0$ then there exists a collection of disjoint descendant nodes $\mu^1, \ldots, \mu^k$ such that:

$$\bigcup_{i=1,\ldots,k} \mu^i = \lambda$$

and

$Q^b(\lambda) \leq S^b(\mu^i)$ *for all* $i \in \{1, \ldots, k\}$.

Proof of Lemma 10.16.

It is enough to prove the claim for the situation when λ is the root node.

We will prove (1) by induction.

The proof of (2) is a symmetrical argument.

Induction hypothesis $\mathbb{H}_n$ $(n \in \mathcal{T})$

For any model with n steps or less such that $\mathcal{Q}(\Omega)$ is non-empty.

We have

(1) if $Q^a(\Omega) < \infty$ then there exists a collection of disjoint descendant nodes $\mu^1, \ldots, \mu^k$ such that:

$$\bigcup_{i=1,\ldots,k} \mu^i = \Omega$$

and

$Q^a(\Omega) \geq S^a(\mu^i)$ *for all* $i \in \{1, \ldots, k\}$.

We first show that $\mathbb{H}_1$ is true.

Indeed if a model has only a single-step then for each successor node μ to the root node Ω, we have $R^a(\mu) = S^a(\mu)$. Since $Q^a(\Omega) = \max_{\mu \in \Lambda_1} R^a(\mu)$, we must have $Q^a(\Omega) \geq S^a(\mu)$ for any $\mu \in \Lambda_1$. Therefore $\mathbb{H}_1$ is true.

We now show that if $\mathbb{H}_n$ is true then $\mathbb{H}_{n+1}$ must be true.

Let us assume that $\mathbb{H}_n$ is true for some $n \in \mathbb{N}$.

Consider a model with $n + 1$ steps.

$Q^a(\Omega) = \max_{\mu \in \Lambda_1} R^a(\mu)$, so for each $\mu \in \Lambda_1$ we have that $Q^a(\Omega) \geq R^a(\mu)$.

However for each $\mu \in \Lambda_1$, $R^a(\mu) = \min\{S^a(\mu), Q^a(\mu)\}$.

It follows that $Q^a(\Omega) \geq \min\{S^a(\mu), Q^a(\mu)\}$ for all $\mu \in \Lambda_1$.

Take any $\mu \in \Lambda_1$

If $\min\{S^a(\mu), Q^a(\mu)\} = S^a(\mu)$ then immediately we have that $Q^a(\Omega) \geq S^a(\mu)$

If $\min\{S^a(\mu), Q^a(\mu)\} = Q^a(\mu)$ then we have $Q^a(\Omega) \geq Q^a(\mu)$.

Since $\mathbb{H}_n$ is assumed to be true, we can apply $\mathbb{H}_n$ to the n-step sub-model from node μ at time 1 and see that:

there exists a collection of disjoint descendant nodes $\mu^1, \ldots, \mu^k$ such that:

$$\bigcup_{i=1,\ldots,k} \mu^i = \mu$$

and

$Q^a(\mu) \geq S^a(\mu^i)$ for all $i \in \{1, \ldots, k\}$.

Thus there exists a collection of disjoint descendant nodes $\nu^1, \ldots, \nu^{\tilde{k}}$ such that:

$$\bigcup_{i=1}^{\tilde{k}} \nu^i = \Omega$$

and

$Q^a(\Omega) \geq S^a(\nu^i)$ for all $i \in \{1, \ldots, \tilde{k}\}$.

We have shown that if $\mathbb{H}_n$ is true for some $n \in \mathbb{N}$ then $\mathbb{H}_{n+1}$ must also be true.

Since $\mathbb{H}_1$ is true, $\mathbb{H}_n$ must be true for all natural numbers n.

This completes the proof of (1).

End of proof of Lemma 10.16.

Example 10.17. (About the first property of $\mathcal{Q}$)
We will now illustrate the main use of Lemma 10.16.

Suppose that for some model we have $\mathcal{Q}(\Omega) = [4, 5]$ and $[S^b(\Omega), S^a(\Omega)] = [2, 3]$.

We can now use Lemma 10.16 to deduce that there is frictionless arbitrage.

By Lemma 10.16 there exists a collection of disjoint descendant nodes $\mu^1, \ldots, \mu^k$ such that: $\bigcup_{i=1,\ldots,k} \mu^i = \Omega$ and $4 \leq S^b(\mu^i)$ for all $i \in \{1, \ldots, k\}$.

Thus we can produce arbitrage by simply buying a stock at the root node for the price 3 units, and then waiting until we reach on of the nodes $\mu^1, \ldots, \mu^k$ and selling for at least 4 units. We know that we will definitely reach one of these nodes because of the fact that $\bigcup_{i=1,\ldots,k} \mu^i = \Omega$.

End of Example 10.17.

Lemma 10.18. *(Property of $\mathcal{R}$)*
Let $\lambda \in \Lambda$ and suppose that $\mathcal{R}(\lambda)$ is non-empty.

(1) there exists a collection of disjoint nodes $\mu^1, \ldots, \mu^k$ that are either a descendant node to the node λ or λ itself and are such that:

$$\bigcup_{i=1,\ldots,k} \mu^i = \lambda$$

and

$R^a(\lambda) \geq S^a(\mu^i)$ for all $i \in \{1, \ldots, k\}$.

(2) there exists a collection of disjoint nodes $\mu^1, \ldots, \mu^k$ that are either a descendant node to the node λ or λ itself and are such that:

$$\bigcup_{i=1,\ldots,k} \mu^i = \lambda$$

and

$R^b(\lambda) \leq S^b(\mu^i)$ for all $i \in \{1, \ldots, k\}$.

Proof of Lemma 10.18.

It is sufficient to prove the claim in the case where λ is the root node.

Proof of (1).

By Lemma 10.14 we know that $R^a(\Omega) = \min\{Q^a(\Omega), S^a(\Omega)\}$.

Suppose that $R^a(\Omega) = S^a(\Omega)$.

In this case it is immediate that $\{\Omega\}$ is a collection of nodes that has the required properties.

Therefore if $R^a(\Omega) = S^a(\Omega)$ then (1) is true.

Suppose that $R^a(\Omega) = Q^a(\Omega)$.

If $R^a(\Omega) = Q^a(\Omega) = \infty$ then if follows immediately that (1) is true.

Otherwise we have $R^a(\Omega) = Q^a(\Omega) < \infty$ and Lemma 10.16 tells us that:

There exists a collection of disjoint descendant nodes $\mu^1, \ldots, \mu^k$ to the node λ such that:

$\bigcup_{i=1,\ldots,k} \mu^i = \lambda$

and

$Q^a(\Omega) \geq S^a(\mu^i)$ for all $i \in \{1, \ldots, k\}$.

Since we are supposing that $R^a(\Omega) = Q^a(\Omega)$, this implies that (1) is true.

We conclude that (1) must be true.

The proof of (2) is symetrical.

End of Proof of Lemma 10.18.

We are about to see two more properties of $\mathcal{Q}$. These properties are much more subtle than the properties that we have see so far, but they are a key point in our proof of the proportional cost fundamental theorem.

Lemma 10.19. *(Second Property of $\mathcal{Q}$)*
Let $\lambda \in \Lambda$ and suppose that $\mathcal{Q}(\lambda)$ is non-empty.

(1) If $Q^a(\lambda) \notin \mathcal{Q}(\lambda)$ and $Q^a(\lambda) < \infty$ then:

there exists a collection of disjoint descendant nodes $\mu^1, \ldots, \mu^k$ *such that* $\bigcup_{i=1,\ldots,k} \mu^i = \lambda$ *and :*

$Q^a(\lambda) \geq R^a(\mu^i)$ *for all* $i \in \{1, \ldots, k\}$ *with* $Q^a(\lambda) > R^a(\mu^i)$ *for one such* $i \in \{1, .., k\}$.

(2) If $Q^b(\lambda) \notin \mathcal{Q}(\lambda)$ *and* $Q^b(\lambda) \neq 0$ *then:*

there exists a collection of disjoint descendant nodes $\mu^1, \ldots, \mu^k$ *such that* $\bigcup_{i=1,\ldots,k} \mu^i = \lambda$ *and :*

$Q^b(\lambda) \leq R^b(\mu^i)$ *for all* $i \in \{1, \ldots, k\}$ *with* $Q^b(\lambda) < R^b(\mu^i)$ *for one such* $i \in \{1, .., k\}$.

Proof of Lemma 10.19.

It is sufficient to prove the claim with λ as the root node.

We prove (1). The proof of (2) is a symmetrical argument.

Observation 1

As a result of Lemma 10.13, the only way that we can have $Q^a(\Omega) \notin \mathcal{Q}(\Omega)$ is if :

Case 1) There exists $\mu \in \Lambda_1$ such that $Q^a(\Omega) = R^a(\mu) \notin \mathcal{R}(\mu)$
or
Case 2) There exists $\mu \in \Lambda_1$ such that $Q^a(\Omega) > R^a(\mu)$.

Induction hypothesis $\mathbb{H}_n$ $(n \in \mathcal{T})$

For any model with n *steps or less such that* $\mathcal{Q}(\Omega)$ *is non-empty, we have:*

(1) If $Q^a(\Omega) \notin \mathcal{Q}(\Omega)$ *and* $Q^a(\Omega) < \infty$ *then:*

there exists a collection of disjoint descendant nodes $\mu^1, \ldots, \mu^k$ *of the root node such that* $\bigcup_{i=1,\ldots,k} \mu^i = \Omega$ *and :*

$Q^a(\Omega) \geq R^a(\mu^i)$ *for all* $i \in \{1, \ldots, k\}$ *with* $Q^a(\Omega) > R^a(\mu_i)$ *for one such* $i \in \{1, .., k\}$

We first show that $\mathbb{H}_1$ is true.

If $Q^a(\Omega) \notin \mathcal{Q}(\Omega)$ and $Q^a(\Omega) < \infty$ then we must have $R^a(\mu) < \infty$ for all $\mu \in \Lambda_1$.

This implies that $S^a(\mu) < \infty$ for all $\mu \in \Lambda_1$ and it follows that we cannot have $R^a(\mu) \notin \mathcal{R}(\mu)$ for any terminal node $\mu \in \Lambda_1$.

It is then immediate from Observation 1 that if $\mathcal{Q}(\Omega)$ is non-empty and $Q^a(\Omega) \notin \mathcal{Q}(\Omega)$ then there must exist a node $\mu' \in \Lambda_1$ such that $Q^a(\Omega) > R^a(\mu')$.

Since $Q^a(\Omega) \geq R^a(\mu)$ for all $\mu \in \Lambda_1$ it follows that Λ_1 is a collection of nodes such that :

$\bigcup_{\mu \in \Lambda_1} \mu = \Omega$ and :

$Q^a(\Omega) \geq R^a(\mu)$ for all $\mu \in \Lambda_1$ with $Q^a(\Omega) > R^a(\mu)$ for one such $\mu \in \Lambda_1$.

Hence we conclude that $\mathbb{H}_1$ is true.

We now show that if $\mathbb{H}_n$ is true then $\mathbb{H}_{n+1}$ must be true.

Let us assume that $\mathbb{H}_n$ is true for some natural number n.

Let λ be the root node.

Consider a model with $n+1$ steps such that $\mathcal{Q}(\Omega)$ is non-empty.

From Observation 1 we know that at the root node we must have either case 1 or case 2.

Case 1 : Suppose that there exists $\mu' \in \Lambda_1$ such that $Q^a(\Omega) = R^a(\mu') \notin \mathcal{R}(\mu')$.

Lemma 10.14 tells us that for any node $\lambda' \in \Lambda$ we have:

$R^a(\lambda') \in \mathcal{R}(\lambda')$ if and only if either $Q^a(\lambda') > S^a(\lambda')$ or $Q^a(\lambda') \in \mathcal{Q}(\lambda')$.

This implies that $R^a(\mu') \notin \mathcal{R}(\mu')$ if and only if $Q^a(\mu') \leq S^a(\mu')$ and $Q^a(\mu') \notin \mathcal{Q}(\mu')$

Lemma 10.14 also tells us that $R^a(\mu') = \min\{Q^a(\mu'), S^a(\mu')\}$.

Together these imply that $R^a(\mu') = Q^a(\mu') \notin \mathcal{Q}(\mu')$.

Using the assumption that $\mathbb{H}_n$ is true we can deduce that there exists a collection of disjoint descendant nodes $\mu^1, \ldots, \mu^k$ of the node μ' such that $\bigcup_{i=1,\ldots,k} \mu^i = \mu'$ and :

$Q^a(\mu') \geq R^a(\mu^i)$ for all $i \in \{1, \ldots, k\}$ with $Q^a(\mu') > R^a(\mu^i)$ for one such $i \in \{1, .., k\}$.

We also know that $Q^a(\Omega) = R^a(\mu') = Q^a(\mu')$, so it follows that $Q^a(\Omega) \geq R^a(\mu^i)$ for all $i \in \{1, \ldots, k\}$ with $Q^a(\Omega) < R^a(\mu^i)$ for one such $i \in \{1, .., k\}$.

Therefore the set $\{\mu \in \Lambda_1 : \mu \neq \mu'\} \cup \{\mu_i : i \in \{1, \ldots, k\}\}$ is a collection of disjoint descendant nodes of the root node such that

$$\bigcup_{\mu'' \in \{\mu \in \Lambda_1 : \mu \neq \mu'\} \cup \{\mu_i : i \in \{1, \ldots, k\}\}} \mu'' = \Omega \text{ and :}$$

$Q^a(\Omega) \geq R^a(\mu'')$ for all $\mu'' \in \{\mu \in \Lambda_1 : \mu \neq \mu'\} \cup \{\mu_i : i \in \{1, \ldots, k\}\}$ with $Q^a(\Omega) > R^a(\mu'')$ for one such $\mu'' \in \{\mu \in \Lambda_1 : \mu \neq \mu'\} \cup \{\mu_i : i \in \{1, \ldots, k\}\}$.

This means that if case 1 applies then $\mathbb{H}_{n+1}$ is true.

Case 2 : Suppose that there exists a node $\mu' \in \Lambda_1$ such that $Q^a(\Omega) > R^a(\mu')$.

From Lemma 10.13 we can deduce that $Q^a(\Omega) \geq R^a(\mu)$ for all $\mu \in \Lambda_1$

This means that Λ_1 is a collection of disjoint descendant nodes such that :

$\bigcup_{\mu \in \Lambda_1} \mu = \Omega$ and :

$Q^a(\Omega) \geq R^a(\mu)$ for all $\mu \in \Lambda_1$ with $Q^a(\Omega) > R^a(\mu)$ for one such $\mu \in \Lambda_1$.

Therefore if case 2 applies then $\mathbb{H}_{n+1}$ is true.

We have shown that if $\mathbb{H}_n$ is true then if either case 1 or case 2 is applicable then $\mathbb{H}_{n+1}$ is true. Therefore since at least one of case 1 or case 2 applies, we can conclude that if $\mathbb{H}_n$ is true then $\mathbb{H}_{n+1}$ is true.

We have shown that if $\mathbb{H}_n$ is true then $\mathbb{H}_{n+1}$ must also be true. Since we have shown that $\mathbb{H}_1$ is true we can conclude that $\mathbb{H}_n$ must be true for any natural number n.

We have proved (1).

End of proof of Lemma 10.19.

Lemma 10.20. *(Third Property of $\mathcal{Q}$)*
Let $\lambda \in \Lambda$ and suppose that $\mathcal{Q}(\lambda)$ is non-empty.

(1) If $Q^a(\lambda) \notin \mathcal{Q}(\lambda)$ and $Q^a(\lambda) < \infty$ then:

there exists a collection of disjoint descendant nodes $\mu^1, \ldots, \mu^k$ such that $\bigcup_{i=1,\ldots,k} \mu^i = \lambda$ and :

$Q^a(\lambda) \geq S^a(\mu^i)$ for all $i \in \{1, \ldots, k\}$ with $Q^a(\lambda) > S^a(\mu^i)$ for one such $i \in \{1, .., k\}$.

(2) If $Q^b(\lambda) \notin \mathcal{Q}(\lambda)$ and $Q^b(\lambda) \neq 0$ then:

there exists a collection of disjoint descendant nodes $\mu^1, \ldots, \mu^k$ such that $\bigcup_{i=1,\ldots,k} \mu^i = \lambda$ and :

$Q^b(\lambda) \leq S^b(\mu^i)$ for all $i \in \{1, \ldots, k\}$ with $Q^b(\lambda) < S^b(\mu^i)$ for one such $i \in \{1, .., k\}$.

Proof of Lemma 10.20.

It is sufficient to prove the claim with λ as the root node.

Proof of (1)

Suppose that $Q^a(\Omega) \notin \mathcal{Q}(\Omega)$ and $Q^a(\Omega) < \infty$.

Lemma 10.19 tells us that:

there exists a collection of disjoint descendant nodes $\mu^1, \ldots, \mu^k$ such that $\bigcup_{i=1,\ldots,k} \mu^i = \Omega$ and :

$Q^a(\Omega) \geq R^a(\mu^i)$ for all $i \in \{1, \ldots, k\}$ with $Q^a(\Omega) > R^a(\mu^i)$ for one such $i \in \{1, .., k\}$.

Let $\tilde{i} \in \{1, \ldots, k\}$ be such that $Q^a(\Omega) > R^a(\mu^{\tilde{i}})$.

For each of these μ^i, $i \in \{1, \ldots, k\}$, Lemma 10.18 tells us that:

there exists a collection of disjoint nodes $\mu_1^i, \ldots, \mu_{k(i)}^i$ that are either descendant nodes to the node μ^i ar μ^i itself and are such that:

$$\bigcup_{j=1,\ldots,k(i)} \mu_i^j = \mu^i$$

and

$R^a(\mu^i) \geq S^a(\mu_i^j)$ for all $j \in \{1, \ldots, k(i)\}$.

It follows that the collection of nodes $\mu_1^1, \ldots, \mu_{k(1)}^1, \ldots, \mu_1^k, \ldots, \mu_{k(k)}^k$ are disjoint descendant nodes of Ω such that:

$$\bigcup_{i=1,\ldots,k} \bigcup_{j=1,\ldots,k(i)} \mu_j^i = \Omega$$

and

$Q^a(\Omega) \geq S^a(\mu_j^i)$ for all $i \in \{1, \ldots, k\}, j \in \{1, \ldots, k(i)\}$

with $Q^a(\Omega) > S^a(\mu_j^{\tilde{i}})$ for all $j \in \{1, \ldots, k(\tilde{i})\}$.

This proves that (1) is true.

End of proof of Lemma 10.20.

Example 10.21. (About the third property of $\mathcal{Q}$)
We will now illustrate the main use of Lemma 10.20.

Suppose that for some model we have $\mathcal{Q}(\Omega) = (4, 5]$ and $[S^b(\Omega), S^a(\Omega)] = [2, 4]$.

We can now use Lemma 10.20 to deduce that there is arbitrage. Note that Lemma 10.16 is insufficient for us to be able to show that there is arbitrage because it can only show that no matter what path the stock price follows there will be a bid price at least as big as 4 units (it could still be possible that all future bid prices are equal to 4).

By Lemma 10.20 there exists a collection of disjoint descendant nodes $\mu^1, \ldots, \mu^k$ such that $\bigcup_{i=1,\ldots,k} \mu^i = \Omega$ and $4 \leq S^b(\mu^i)$ for all $i \in \{1, \ldots, k\}$ with $4 < S^b(\mu^i)$ for one such $i \in \{1, .., k\}$.

Thus we can produce arbitrage by simply buying a share of stock at the root node for the price 4 units, and then waiting until we reach on of the nodes $\mu^1, \ldots, \mu^k$ and selling for at least 4 units with the possibility of selling the stock for more than 4 units. We know that we will definitely reach one of these nodes because of the fact that $\bigcup\limits_{i=1,\ldots,k} \mu^i = \Omega$

End of Example 10.21.

10.2.4. *Proportional cost fundamental theorem in full generality.*

We are now in a position to prove the proportional cost fundamental theorem. For restricted models this theorem has been proved in [15], [25] and [18], but the extension to unrestricted models is new. Also the style of this proof is different to any previous proof of the proportional cost fundamental theorem.

Theorem 10.22. *(Fundamental Theorem for proportional transaction costs)*
For any unrestricted model $\mathbb{M} = (\Omega, \mathcal{F}, \mathbb{P}, S, F, P)$, there exists a proportional cost arbitrage opportunity if and only if $\mathcal{P}$ is empty.

(See Definition 5.34 for definition of $\mathcal{P}$)

Proof of Theorem 10.22.

Observe that due to the frictionless fundamental theorem there exists a frictionless middle value process $\tilde{S}$ if and only if there exists a probability measure $\mathbb{Q}$ such that $(\mathbb{Q}, \tilde{S}) \in \mathcal{P}$.

Therefore we will prove that

For any model $\mathbb{M} = (\Omega, \mathcal{F}, \mathbb{P}, S, F, P)$, with P unrestricted, there exists a proportional cost arbitrage opportunity if and only if $\mathcal{R}(\Omega)$ is empty.

<u>Part 1</u> (If $\mathcal{P}$ is empty then there is proportional cost arbitrage)

Claim 1 If $\mathcal{R}(\Omega)$ is empty and $\mathcal{R}(\mu)$ is non-empty for all nodes $\lambda \in \Lambda_1$ then there exists a proportional cost arbitrage opportunity.

Proof of claim 1

Suppose that $\mathcal{R}(\mu)$ is non-empty for all nodes $\mu \in \Lambda_1$

This implies that $\mathcal{Q}(\Omega)$ is non-empty.

Since $\mathcal{Q}(\Omega)$ and $[S^b(\Omega), S^a(\Omega)]$ are both intervals there are only four possible ways that $\mathcal{R}(\Omega)$ can be empty:

(1) $S^a(\Omega) < Q^b(\Omega)$

(2) $S^a(\Omega) = Q^b(\Omega)$ and $Q^b(\Omega) \notin \mathcal{Q}(\Omega)$

(3) $S^b(\Omega) > Q^a(\Omega)$

(4) $S^b(\Omega) = Q^a(\Omega)$ and $Q^a(\Omega) \notin \mathcal{Q}(\Omega)$

We will show that each of these cases imply proportional cost arbitrage.

Consider **(1)**.

$S^a(\Omega) < Q^b(\Omega)$.

Note that $S^a(\Omega) > 0$ so $Q^b(\Omega) > 0$.

Lemma 10.16 tells us that :

there exists a collection of disjoint descendant nodes $\mu^1, \ldots, \mu^k$ such that:

$\bigcup_{i=1,\ldots,k} \mu^i = \Omega$

and

$Q^b(\Omega) \leq S^b(\mu^i)$ for all $i \in \{1, \ldots, k\}$

Therefore for each of these μ^i we have $S^a(\Omega) < Q^b(\Omega) \leq S^b(\mu^i)$.

The initial portfolio $(0, 0)$ combined with a trading strategy Z is a proportional cost arbitrage opportunity, where Z is defined as follows:

$Z(\Omega) = 1$ (buy a single stock at the root node)

$Z(\mu^i) = -1$ for all $i \in \{1, \ldots, k\}$. (sell a single stock when μ^i is reached)

$Z(\lambda') = 0$ for all nodes $\lambda' \in \Lambda$ such that $\lambda' \neq \Omega$ and $\lambda' \neq \mu^i$ for all $i \in \{1, \ldots, k\}$. (do nothing at any other node)

In other words we can construct arbitrage by buying a share of stock at time 0 for $S^a(\Omega)$ and then simply waiting until the node when the stock goes above $S^a(\Omega)$ is reached (which is guaranteed to happen).

Consider **(2)**.

$S^a(\Omega) = Q^b(\Omega)$ and $Q^b(\Omega) \notin \mathcal{Q}(\Omega)$

Note that $S^a(\Omega) > 0$ so $Q^b(\Omega) > 0$.

Lemma 10.20 tells us that:

there exists a collection of disjoint descendant nodes $\mu^1, \ldots, \mu^k$ such that $\bigcup_{i=1,\ldots,k} \mu^i = \Omega$ and :

$Q^b(\Omega) \leq S^b(\mu^i)$ for all $i \in \{1, \ldots, k\}$ with $Q^b(\Omega) < S^b(\mu^i)$ for one such $i \in \{1, .., k\}$.

Therefore for each of these μ^i we have $S^a(\Omega) = Q^b(\Omega) \leq S^b(\mu^i)$ and for one such $i \in \{1, .., k\}$ we have $S^a(\Omega) = Q^b(\Omega) < S^b(\mu^i)$.

Let $\tilde{i} \in \{1, .., k\}$ be such that $S^a(\Omega) = Q^b(\Omega) < S^b(\mu^{\tilde{i}})$

The initial portfolio $(0, 0)$ combined with a trading strategy Z is a proportional cost arbitrage opportunity, where Z is defined as follows:

$Z(\Omega) = 1$ (buy a single stock at the root node)

$Z(\mu^i) = -1$ for all $i \in \{1, \ldots, k\}$. (sell a single stock when μ_i is reached)

$Z(\lambda') = 0$ for all nodes $\lambda' \in \Lambda$ such that $\lambda' \neq \Omega$ and $\lambda' \neq \mu^i$ for all $i \in \{1, \ldots, k\}$. (do nothing at any other node)

Observe that it is a proportional cost arbitrage portfolio strategy because at each node μ^i we are at least selling our single stock for the same price we bought it for, so we are not making a loss in any scenario. Furthermore if the node $\mu^{\tilde{i}}$ is reached then we are able to sell the stock for a greater price than we bought it for, so we would make a strictly positive profit in any scenario where the node $\mu^{\tilde{i}}$ is reached.

Consider **(3)**.

$S^b(\lambda) > Q^a(\lambda)$.

This is a symmetrical situation to (1).

Consider **(4)**.

$S^b(\lambda) = Q^a(\lambda)$ and $Q^a(\lambda) \notin \mathcal{Q}(\lambda)$

This is a symmetrical situation to (2).

End of proof of claim 1

We will now use proof by induction.

Induction hypothesis $\mathbb{H}_n$ **(**$n \in \mathcal{T}$**)**

For any model $\mathbb{M} = (\Omega, \mathcal{F}, \mathbb{P}, S, F, P)$,*with* P *unrestricted, with* n *steps or less, there exists a proportional cost arbitrage opportunity if* $\mathcal{R}(\Omega)$ *is empty.*

We first show that $\mathbb{H}_1$ is true.

Let $\mathbb{M}_1$ be a model with only a single-step such that $\mathcal{R}(\Omega)$ is empty.

For any terminal node $\mu \in \Lambda_1$, $\mathcal{R}(\mu)$ is non-empty.

This implies that $\mathcal{Q}(\Omega)$ is non-empty.

It follows from claim 1 that there exists a proportional cost arbitrage opportunity.

We now show that if $\mathbb{H}_n$ is true then $\mathbb{H}_{n+1}$ must be true.

Suppose that $\mathbb{H}_n$ is true for some $n \in \mathbb{N}$.

Consider a model with $n+1$ steps such that $\mathcal{R}(\Omega)$ is empty.

We show that there must exist proportional cost arbitrage.

If there exists a node $\mu \in \Lambda_1$ such that $\mathcal{R}(\mu)$ is empty, then it immediately follows from $\mathbb{H}_n$ that there exists proportional cost arbitrage.

If $\mathcal{R}(\mu)$ is non-empty for all $\mu \in \Lambda_1$ then this implies that $\mathcal{Q}(\lambda)$ is non-empty and it follows from claim 1 that there exists a proportional cost arbitrage opportunity.

Therefore for any $n+1$ step model such that $\mathcal{R}(\Omega)$ is empty, there exists a proportional cost arbitrage opportunity.

We have shown that if $\mathbb{H}_n$ is true for some $n \in \mathbb{N}$ then $\mathbb{H}_{n+1}$ must also be true. Since $\mathbb{H}_1$ is true, $\mathbb{H}_n$ must be true for all natural numbers n.

We have proved that for any model $\mathbb{M}$, if $\mathcal{R}_0$ is empty then there exists a proportional cost arbitrage opportunity.

<u>Part 2</u> (If there exists proportional cost arbitrage then $\mathcal{P}$ is empty)

It is sufficient to prove that if there exists a proportional cost arbitrage opportunity then $\mathcal{R}(\Omega)$ is empty.

Let $\mathbb{M} = (\Omega, \mathcal{F}, \mathbb{P}, S, F, P)$ be a model that admits proportional cost arbitrage.

Let $\tilde{S}$ be a strictly positive adapted process such that $S^b(\lambda) \leq \tilde{S}(\lambda) \leq S^a(\lambda)$ for all $\lambda \in \tilde{\Omega}$.

Observe that the model $\tilde{\mathbb{M}} = (\Omega, \mathcal{F}, \mathbb{P}, \tilde{S}, F, \tilde{P})$, where $\tilde{P}$ is such that $\tilde{P}(\lambda) = 0$ for all $\lambda \in \Lambda$, admits proportional cost arbitrage if and only if it admits frictionless arbitrage.

Observe also that for any two models $\mathbb{M}'$ and $\mathbb{M}''$ if model $\mathbb{M}'$ has bid-ask spreads that are contained in the bid-ask spreads of $\mathbb{M}''$ then clearly if $\mathbb{M}''$ admits proportional cost arbitrage, $\mathbb{M}'$ will also admit proportional cost arbitrage. This follows from Lemma 22.7 in the Appendix.

The bid-ask spreads of $\tilde{\mathbb{M}}$ are contained in the bid-ask spreads of $\mathbb{M}$, since $\mathbb{M}$ admits proportional cost arbitrage it follows that $\tilde{\mathbb{M}}$ admits proportional cost arbitrage. Therefore $\tilde{\mathbb{M}}$ also admits frictionless arbitrage.

Therefore it is not possible to find a strictly positive adapted process $\tilde{S}$ with $S^b(\lambda) \leq \tilde{S}(\lambda) \leq S^a(\lambda)$ for all $\lambda \in \tilde{\Omega}$, such that there does not exist a frictionless arbitrage opportunity.

Therefore $\mathcal{R}(\Omega)$ is empty.

End of Proof of Theorem 10.22.

10.2.5. *Principle of two trade arbitrage under proportional costs.*

The following result is an interesting by-product of our method of proof for the combined cost fundamental theorem.

Corollary 10.23. (Principle of two trade arbitrage under proportional transaction costs)
A model $\mathbb{M}$ admits proportional cost arbitrage if and only if:

(1) there exists a non-terminal node λ and a collection of disjoint descendant nodes $\mu^1, \ldots, \mu^k$ such that $\bigcup_{i=1,\ldots,k} \mu^i = \lambda$ and either:

(1a) $S^a(\lambda) \leq S^b(\mu^i)$ for all $i \in \{1, \ldots, k\}$ with $S^a(\lambda) < S^b(\mu_i)$ for some $i \in \{1, \ldots, k\}$

or

(1b) $S^b(\lambda) \geq S^a(\mu^i)$ for all $i \in \{1, \ldots, k\}$ with $S^b(\lambda) > S^a(\mu_i)$ for some $i \in \{1, .., k\}$.

Proof of Corollary 10.23.

Follows from the proof of Theorem 10.22.

End of Proof of Corollary 10.23.

11. **Combined cost arbitrage**

In the previous section we proved the proportional cost fundamental theorem by using an induction argument. This is a new way of proving this result and now we can reap the benefits, as this method of proof can be easily adapted to the combined cost setting. In fact it turns out that using this method, it is even easier to prove the fundamental theorem for combined costs than it was to prove the proportional costs version. The reason for this is that the processes of sets, $\mathcal{Q}$ and $\mathcal{R}$ that we have seen previously, could involve either open or closed sets, and many of the technical difficulties in the proof stemmed from this. However in the proof that we will see, the analogues processes of sets, $\tilde{\mathcal{Q}}$ and $\tilde{\mathcal{R}}$, that we will soon see, will consist of only closed sets.

It is important to note that in this section, unlike the previous section, we will work with models that are such that P is restricted. This is because the combined cost fundamental theorem will not be used in any of the proofs later in this work that involve P being unrestricted.

11.1. **Combined cost fundamental theorem.**

11.1.1. *Fixed cost middle value process.*

Definition 11.1. Fixed cost middle value stock process)
For any model $\mathbb{M} = (\Omega, \mathcal{F}, \mathbb{P}, S, F, P)$, a *fixed cost middle value stock process* is a strictly positive adapted process $\hat{S}$ such that:

$$S^b \leq \tilde{S} \leq S^a$$

and the model $\mathbb{M}' = (\Omega, \mathcal{F}, \mathbb{P}, \tilde{S}, F, P)$ does not admit fixed cost arbitrage.

11.1.2. *Construction of a fixed cost middle value process.*

In this subsection we will show how to construct a fixed cost middle value process (if it is possible to construct one) by moving backwards through the tree constructing the set of possible stock prices at a given time that could be part of a fixed cost middle value process $\hat{S}$.

Definition 11.2. ($\tilde{\mathcal{Q}}(\lambda)$)
Let $\lambda \in \Lambda$.

We define $\tilde{\mathcal{Q}}(\lambda)$ to be the set of real numbers $x \in \mathbb{R}$ for which we can find a strictly positive adapted process $\tilde{S}$ such that

$\tilde{S}(\lambda) = x$

and

$S^b(\lambda') \leq \tilde{S}(\lambda') \leq S^a(\lambda')$ for any descendant node λ' to the node λ

and the process $\tilde{S}$ does not admit fixed cost arbitrage on the sub-model with root node λ.

(Recall that "the sub-model at node λ" means the sub-model that starts at node λ and continues until the terminal time).

Definition 11.3. ($\tilde{\mathcal{R}}(\lambda)$:"$\tilde{\mathcal{Q}}(\lambda)$ restricted to the bid-ask spread at node λ ")
$\tilde{\mathcal{R}}(\lambda) := \tilde{\mathcal{Q}}(\lambda) \cap [S^b(\lambda), S^a(\lambda)]$ for all $\lambda \in \Lambda$.

Remark 11.4.
Observe that the existence of a fixed cost middle value process is equivalent to the set $\tilde{\mathcal{R}}_0$ being non-empty. Furthermore, if we have constructed $\tilde{\mathcal{R}}(\lambda)$ for every node λ and they are not empty, then it is easy to construct a fixed cost middle value process by working forwards through the tree first choosing a $\hat{S}_0$ from $\tilde{\mathcal{R}}_0$ and then choosing appropriate $\hat{S}_1$s from the $\tilde{\mathcal{R}}_1$s etc.

Definition 11.5. (Lower and upper ends of the sets $\tilde{\mathcal{Q}}$ and $\tilde{\mathcal{R}}$.)
We will write $\tilde{Q}^b(\lambda)$ and $\tilde{Q}^a(\lambda)$ for the infimum and supremum of $\tilde{\mathcal{Q}}(\lambda)$ respectively.

We will write $\tilde{R}^b(\lambda)$ and $\tilde{R}^a(\lambda)$ for the infimum and supremum of $\tilde{\mathcal{R}}(\lambda)$ respectively.

We will not use these terms when $\tilde{\mathcal{Q}}$ and $\tilde{\mathcal{R}}$ are empty.

Remark 11.6. (Terminal values of the sets $\tilde{\mathcal{Q}}$ and $\tilde{\mathcal{R}}$)

Observe from the definitions of $\tilde{\mathcal{Q}}$ and $\tilde{\mathcal{R}}$ that for any terminal node $\lambda \in \Lambda_T$

$\tilde{\mathcal{Q}}(\lambda) = \mathbb{R}^+$

and

$\tilde{\mathcal{R}}(\lambda) = [S^b(\lambda), S^a(\lambda)]$.

Example 11.7. (Constructing $\tilde{\mathcal{Q}}(\lambda)$ from the $\tilde{\mathcal{R}}(\mu)$)
This is example is very similar to Example 10.12 and we will highlight some of the differences.

Consider a single-step binomial model $\mathbb{M}$ with root node Ω and nodes at time 1, u and d.

Suppose that $\tilde{\mathcal{R}}(d) = [1, 2]$ and $\tilde{\mathcal{R}}(u) = [3, 4]$

Our objective is to find the set $\tilde{\mathcal{Q}}(\Omega)$.

Consider any real number x that lies strictly in between 1 ($= \min\{R^b(d), R^b(u)\}$) and 4 ($= \max\{R^a(d), R^a(u)\}$), lets say $x = 2$.

Set $\tilde{S}(\Omega) = 2$, $\tilde{S}(u) = 4$ and $\tilde{S}(d) = 1$.

Observe that $\tilde{S}$ does not admit fixed cost arbitrage and thus 2 (or any number in between 1 and 4) belongs to $\tilde{\mathcal{Q}}(\Omega)$.

In Example 10.12 we saw that 1 does not belong to $\mathcal{Q}(\Omega)$ and 4 does not belong to $\mathcal{Q}(\Omega)$. However in this example things are different and we will see that 1 and 4 both belong to $\tilde{\mathcal{Q}}(\Omega)$.

Set $\tilde{S}(\Omega) = 1$, $\tilde{S}(u) = 4$ and $\tilde{S}(d) = 1$ and observe that this single-step model is of type 3 and thus does not admit fixed cost arbitrage.

Similarly we could set $\tilde{S}(\Omega) = 4$, $\tilde{S}(u) = 4$ and $\tilde{S}(d) = 1$ and again observe that this single-step model is of type 3 and thus does not admit fixed cost arbitrage.

It follows that $1 \in \tilde{\mathcal{Q}}(\Omega)$ and $4 \in \tilde{\mathcal{Q}}(\Omega)$.

We can now conclude that $\tilde{\mathcal{Q}}(\Omega) = [1, 4]$, $Q^b = 1 = \min_{\mu \in \mathrm{Succ}\lambda} R^b(\mu)$ and $Q^a = 4 = \max_{\mu \in \mathrm{Succ}\lambda} R^a(\mu)$.

We have shown that $\tilde{\mathcal{Q}}(\Omega)$ is a closed interval. It is in fact true that these sets $\tilde{\mathcal{Q}}$ are always closed and this is the reason why our work on fixed cost middle value processes is much simpler than our previous work on frictionless middle value processes.

End of Example 11.7.

Lemma 11.8. *(Construction of $\tilde{\mathcal{Q}}(\lambda)$ from the $\tilde{\mathcal{R}}(\mu)$*
Suppose that for each successor node μ to node λ, $\tilde{\mathcal{R}}(\mu)$ is a non-empty closed bounded interval.

Then

$$\tilde{\mathcal{Q}}(\lambda) = \Big[\min_{\mu \in \mathrm{Succ}\lambda} \tilde{R}^b(\mu), \max_{\mu \in \mathrm{Succ}\lambda} \tilde{R}^a(\mu)\Big].$$

Proof of Lemma 11.8.

Part 1 $\Big[\min_{\mu \in \mathrm{Succ}\lambda} \tilde{R}^b(\mu), \max_{\mu \in \mathrm{Succ}\lambda} \tilde{R}^a(\mu)\Big] \subset \tilde{\mathcal{Q}}(\lambda)$.

Claim 1

Consider a node $\lambda \in \Lambda \setminus \Lambda_T$ and $x \in \mathbb{R}$.

Suppose that there exists disjoint successor nodes μ_1 and μ_2 to the node λ and real numbers x_1 and x_2 such that $x_1 \in \tilde{\mathcal{R}}(\mu_1)$ and $x_2 \in \tilde{\mathcal{R}}(\mu_2)$.

Suppose also that $x_1 \leq x \leq x_2$.

Then $x \in \tilde{\mathcal{Q}}(\lambda)$.

Proof of claim 1

Since $\tilde{\mathcal{R}}(\mu)$ is non-empty for all successor nodes μ to the node λ, and $x_1 \in \tilde{\mathcal{R}}(\mu_1)$ and $x_2 \in \tilde{\mathcal{R}}(\mu_2)$, there exists a strictly positive adapted process $\tilde{S}$ such that $x_1 = \tilde{S}(\mu_1)$ and $x_2 = \tilde{S}(\mu_2)$ and $\tilde{S}$ does not admit fixed cost arbitrage on any of the sub-models starting from the successor nodes to node λ.

Take any such process $\tilde{S}$ such that $\tilde{S}(\lambda) = x$.

By assumption $x_1 \leq x \leq x_2$, so the single-step sub-model at node λ is of either type 1, type 2 or type 3 and thus does not admit fixed cost arbitrage (see definition 9.3).

It follows that this process $\tilde{S}$ will not admit fixed cost arbitrage on the single-step sub-model starting from node λ and thus that $\tilde{S}$ will not admit fixed cost arbitrage in the full model because of Lemma 9.7.

Therefore by definition we have $x \in \tilde{\mathcal{Q}}(\lambda)$.

End of proof of claim 1

Claim 2

Suppose that $x \in \left[\min_{\mu \in \mathrm{Succ}\lambda} \tilde{R}^b(\mu), \max_{\mu \in \mathrm{Succ}\lambda} \tilde{R}^a(\mu)\right]$.

Then there exists real numbers x_1 and x_2 and disjoint successor nodes μ_1 and μ_2 to the node λ such that $x_1 \in \tilde{\mathcal{R}}(\mu_1)$, $x_2 \in \tilde{\mathcal{R}}(\mu_2)$ and $x_1 \leq x \leq x_2$.

Proof of claim 2

Suppose that $\min_{\mu \in \mathrm{Succ}\lambda} \tilde{R}^b(\mu)$ and $\max_{\mu \in \mathrm{Succ}\lambda} \tilde{R}^a(\mu)$ are realised at different nodes, say nodes μ_1 and μ_2 respectively, then we set $x_1 = \tilde{R}^b(\mu_1)$ and $x_2 = \tilde{R}^a(\mu_2)$.

It follows immediately that $x_1 \leq x \leq x_2$.

Suppose that $\min_{\mu \in \mathrm{Succ}\lambda} \tilde{R}^b(\mu)$ and $\max_{\mu \in \mathrm{Succ}\lambda} \tilde{R}^a(\mu)$ are realised at the same node, call this node μ_1, then we set x_1 and x_2 according to the following rule.

If $x = \min_{\mu \in \mathrm{Succ}\lambda} \tilde{R}^b(\mu)$ then set $x_1 = \min_{\mu \in \mathrm{Succ}\lambda} \tilde{R}^b(\mu)$ and choose x_2 to be equal to any element of one of the sets belonging to $\tilde{\mathcal{R}}(\mu)$ for any successor node μ to the node λ that is not the node μ_1.

If $x = \max_{\mu \in \mathrm{Succ}\lambda} \tilde{R}^a(\mu)$ then set $x_1 = \max_{\mu \in \mathrm{Succ}\lambda} \tilde{R}^a(\mu)$ and choose x_2 to be equal to any element of one of the sets belonging to $\tilde{\mathcal{R}}(\mu)$ for any successor node μ to the node λ that is not the node μ_1.

If $\min_{\mu \in \mathrm{Succ}\lambda} \tilde{R}^b(\mu) < x < \max_{\mu \in \mathrm{Succ}\lambda} \tilde{R}^a(\mu)$ then we take any node $\mu_2 \in \mathrm{Succ}\lambda$ that is not the node μ_1 and break this situation down into two further cases:

Case 1 $R^b(\mu_2) \geq x$ then we set $x_1 = \min_{\mu \in \mathrm{Succ}\lambda} \tilde{R}^b(\mu) \in \tilde{\mathcal{R}}(\mu_1)$ and $x_2 = R^b(\mu_2) \in \tilde{\mathcal{R}}(\mu_2)$.

Case 2 $R^b(\mu_2) < x$ then we set $x_1 = \max_{\mu \in \mathrm{Succ}\lambda} \tilde{R}^a(\mu) \in \tilde{\mathcal{R}}(\mu_1)$ and $x_2 = R^b(\mu_2) \in \tilde{\mathcal{R}}(\mu_2)$.

For each of these if statements and sub-cases we have $x_1 \leq x \leq x_2$.

End of proof of claim 2

Take any $x \in \left[\min_{\mu \in \mathrm{Succ}\lambda} \tilde{R}^b(\mu), \max_{\mu \in \mathrm{Succ}\lambda} \tilde{R}^a(\mu) \right]$.

Claim 2 tells us that there exists real numbers x_1 and x_2 and disjoint successor nodes μ_1 and μ_2 to the node λ such that $x_1 \in \tilde{\mathcal{R}}(\mu_1)$, $x_2 \in \tilde{\mathcal{R}}(\mu_2)$ and $x_1 \leq x \leq x_2$.

It then follows from Claim 1 that $x \in \tilde{\mathcal{Q}}(\lambda)$.

Part 2 $\tilde{\mathcal{Q}}(\lambda) \subset \left[\min_{\mu \in \mathrm{Succ}\lambda} \tilde{R}^b(\mu), \max_{\mu \in \mathrm{Succ}\lambda} \tilde{R}^a(\mu) \right]$.

Take any $x \notin \left[\min_{\mu \in \mathrm{Succ}\lambda} \tilde{R}^b(\mu), \max_{\mu \in \mathrm{Succ}\lambda} \tilde{R}^a(\mu) \right]$.

Without loss of generality, let us assume that $x < \min_{\mu \in \mathrm{Succ}\lambda} \tilde{R}^b(\mu)$.

Then for any $\tilde{S}$ with $\tilde{S}(\lambda) = x$ and $\tilde{S}(\mu) \in \left[\min_{\mu \in \mathrm{Succ}\lambda} \tilde{R}^b(\mu), \max_{\mu \in \mathrm{Succ}\lambda} \tilde{R}^a(\mu) \right]$ we will have $\tilde{S}(\lambda) < \min_{\mu \in \mathrm{Succ}\lambda} \tilde{R}^b(\mu)$ and the single-step sub-model at node λ will be of type 4 and thus will admit fixed cost arbitrage.

Therefore $x \notin \tilde{\mathcal{Q}}(\lambda)$ and we conclude that $\tilde{\mathcal{Q}}(\lambda) \subset \left[\min_{\mu \in \mathrm{Succ}\lambda} \tilde{R}^b(\mu), \max_{\mu \in \mathrm{Succ}\lambda} \tilde{R}^a(\mu) \right]$.

End of Proof of Lemma 11.8.

Lemma 11.9. *(Construction of $\tilde{\mathcal{R}}(\lambda)$ from $\tilde{\mathcal{Q}}(\lambda)$)*
Let $\lambda \in \Lambda$.

Suppose that $\tilde{\mathcal{Q}}(\lambda)$ is a non-empty closed and bounded interval.

Then

$$\tilde{\mathcal{R}}(\lambda) = \Big[\min\{\tilde{Q}^b(\lambda), S^b(\lambda)\}, \max\{\tilde{Q}^a(\lambda), S^a(\lambda)\} \Big]$$

Proof of Lemma 11.9.

We have $\tilde{\mathcal{R}}(\lambda) := \tilde{\mathcal{Q}}(\lambda) \cap [S^b(\lambda), S^a(\lambda)]$ for any node λ. Since $\tilde{\mathcal{Q}}(\lambda)$ is an interval, the result follows from the basic properties of intersecting two intervals.

End of proof of Lemma 11.9.

Lemma 11.10. *($\tilde{\mathcal{Q}}$ and $\tilde{\mathcal{R}}$ are closed bounded intervals)*
For any non-terminal node λ, $\tilde{\mathcal{Q}}(\lambda)$ and $\tilde{\mathcal{R}}(\lambda)$ are closed bounded intervals

Proof of Lemma 11.10.

If for some time $t \in \mathcal{T}$ we have that $\tilde{\mathcal{R}}(\mu)$ is a closed bounded interval for all nodes $\mu \in \Lambda_t$ then it follows from Lemma 11.8 and Lemma 11.9 that $\tilde{\mathcal{R}}(\lambda)$ and $\tilde{\mathcal{Q}}(\lambda)$ are closed bounded intervals for all nodes $\lambda \in \Lambda_{t-1}$

Since $\tilde{\mathcal{R}}(\mu)$ is a closed bounded interval for all nodes $\mu \in \Lambda_T$ it follows by induction that $\tilde{\mathcal{Q}}(\lambda)$ and $\tilde{\mathcal{R}}(\lambda)$ are closed bounded intervals for all non-terminal nodes λ.

End of proof of Lemma 11.10.

11.1.3. *Properties of $\tilde{\mathcal{Q}}$.*

In our earlier section about the properties of $\mathcal{Q}$ and $\mathcal{R}$, we needed three properties of $\mathcal{Q}$ and one property of $\mathcal{R}$ in order to have what we needed for the proportional cost fundamental theorem. In this section we only need one lemma because we don't have to account for the fact that the ends of the sets $\mathcal{Q}$ and $\mathcal{R}$ may not belong to the sets themselves. The idea behind the following lemma is very similar to the idea behind Lemma 10.16, and thus it is worth taking a look at Example 10.15 in order to understand how it works.

Lemma 11.11. *(Property of $\tilde{\mathcal{Q}}$)*
Suppose that $\tilde{\mathcal{Q}}(\lambda)$ is non-empty for some node $\lambda \in \Lambda$.

Then

(1) there exists a collection of disjoint descendant nodes $\mu^1, \dots, \mu^k$ to the node λ such that

$$\bigcup_{i=1}^{k} \mu^i = \lambda$$

and

$\tilde{Q}^a(\lambda) \geq S^a(\mu^i)$ *for all* $i \in \{1, \dots, k\}$

(2) there exists a collection of disjoint descendant nodes $\mu^1, \dots, \mu^k$ to the node λ such that

$$\bigcup_{i=1}^{k} \mu^i = \lambda$$

and

$\tilde{Q}^b(\lambda) \leq S^b(\mu^i)$ *for all* $i \in \{1, \dots, k\}$

Proof of Lemma 11.11.

It is enough to prove the claim for the situation when λ is the root node.

We prove (1) by induction.

Induction hypothesis $\mathbb{H}_n$ $(n \in \mathcal{T})$

For any model with n steps or less such that $\tilde{\mathcal{Q}}(\Omega)$ is non-empty,

We have

(1) there exists a collection of disjoint descendant nodes $\mu^1, \dots, \mu^k$ to the root node such that

$$\bigcup_{i=1}^{k} \mu^i = \Omega$$

and

$\tilde{Q}^a(\Omega) \geq S^a(\mu^i)$ *for all* $i \in \{1, \dots, k\}$.

We first show that $\mathbb{H}_1$ is true.

Indeed if a model has only a single-step then for each successor node μ to the root node λ, we have $\tilde{R}^a(\mu) = S^a(\mu)$. Since $\tilde{Q}^a(\Omega) = \max\limits_{\mu \in \Lambda_1} \tilde{R}^a(\mu)$, we must have $\tilde{Q}^a(\Omega) \geq S^a(\mu)$ for all $\mu \in \Lambda_1$.

Therefore $\mathbb{H}_1$ is true.

We now show that if $\mathbb{H}_n$ is true then $\mathbb{H}_{n+1}$ must be true.

Let us assume that $\mathbb{H}_n$ is true for some $n \in \mathbb{N}$.

Consider a model with $n+1$ steps.

$\tilde{Q}^a(\Omega) = \max_{\mu \in \Lambda_1} \tilde{R}^a(\mu)$, so for each $\mu \in \Lambda_1$ we have that $\tilde{Q}^a(\Omega) \geq \tilde{R}^a(\mu)$.

However for each $\mu \in \Lambda_1$, $\tilde{R}^a(\mu) = \min\{S^a(\mu), \tilde{Q}^a(\mu)\}$.

It follows that $\tilde{Q}^a(\Omega) \geq \min\{S^a(\mu), \tilde{Q}^a(\mu)\}$ for all $\mu \in \Lambda_1$.

Take any $\mu \in \Lambda_1$.

If $\min\{S^a(\mu), \tilde{Q}^a(\mu)\} = S^a(\mu)$ then immediately we have that $\tilde{Q}^a(\Omega) \geq S^a(\mu)$.

If $\min\{S^a(\mu), \tilde{Q}^a(\mu)\} = \tilde{Q}^a(\mu)$ then we have $\tilde{Q}^a(\Omega) \geq \tilde{Q}^a(\mu)$.

Since $\mathbb{H}_n$ is assumed to be true, we can apply $\mathbb{H}_n$ to the n-step sub-model from node μ at time 1 and see that

there exists a collection of disjoint descendant nodes $\mu^1, \ldots, \mu^k$ to the node μ such that

$\bigcup_{i=1}^{k} \mu^i = \mu$

and

$\tilde{Q}^a(\mu) \geq S^a(\mu^i)$ for all $i \in \{1, \ldots, k\}$.

Thus there exists a collection of disjoint descendant nodes $\nu^1, \ldots, \nu^{\tilde{k}}$ such that:

$\bigcup_{i=1}^{\tilde{k}} \nu^i = \Omega$

and

$\tilde{Q}^a(\Omega) \geq S^a(\nu^i)$ for all $i \in \{1, \ldots, \tilde{k}\}$.

We have shown that if $\mathbb{H}_n$ is true for some $n \in \mathbb{N}$ then $\mathbb{H}_{n+1}$ must also be true. Since $\mathbb{H}_1$ is true, $\mathbb{H}_n$ must be true for all natural numbers n.

This completes the proof of (1).

The proof of (2) is a completely symmetrical argument.

End of Proof of Lemma 11.11.

11.1.4. *Combined cost fundamental theorem.*

We are now in position to state and prove the fundamental theorem for combined transaction costs. This elegant result is the culmination of the technique that we have developed of the the course of this section and the previous section.

Theorem 11.12. *(Fundamental theorem for combined transaction costs)*
For any model $\mathbb{M} = (\Omega, \mathcal{F}, \mathbb{P}, S, F, P)$, there exists a combined cost arbitrage opportunity if and only if $\tilde{\mathcal{P}}$ is empty.

Proof of Theorem 11.12.

Observe that due to Theorem 9.10, which tells us that a model admits fixed cost arbitrage if and only if it admits an AMF, we can deduce that $\tilde{\mathcal{P}}$ is empty if and only $R(\Omega)$ is empty.

Therefore we will prove that for any model $\mathbb{M} = (\Omega, \mathcal{F}, \mathbb{P}, S, F, P)$, there exists a combined cost arbitrage opportunity if and only if $R(\Omega)$ is empty.

<u>Part 1</u> (If $R(\Omega)$ is empty then there exists a combined cost arbitrage opportunity)

Let λ be the root node.

Claim 1 If $\tilde{\mathcal{R}}(\Omega)$ is empty and $\tilde{\mathcal{R}}(\mu)$ is non-empty for all nodes $\mu \in \Lambda_1$, then there must exist a combined cost arbitrage opportunity.

Proof of claim 1

Suppose that $\tilde{\mathcal{R}}(\mu)$ is non-empty for all nodes $\mu \in \Lambda_1$.

By Lemma 11.8 and Lemma 11.10 this implies that $\tilde{\mathcal{Q}}(\Omega)$ is non-empty and

$$\tilde{\mathcal{Q}}(\Omega) = \Big[\min_{\mu \in \mathrm{Succ}\lambda} \tilde{R}^b(\mu), \max_{\mu \in \mathrm{Succ}\lambda} \tilde{R}^a(\mu) \Big].$$

Suppose that $\tilde{\mathcal{R}}(\Omega)$ is empty.

There are only two ways that this can happen:

(1) $S^a(\Omega) < \tilde{Q}^b(\Omega)$

(2) $S^b(\Omega) > \tilde{Q}^a(\Omega)$

Consider (1).

By Lemma 11.11 there exists a collection of disjoint descendant nodes $\mu^1, \ldots, \mu^k$ to the root node Ω such that

$$\bigcup_{i=1,\ldots,k} \mu^i = \Omega$$

and

$\tilde{Q}^b(\Omega) \leq S^b(\mu^i)$ for all $i \in \{1, \ldots, k\}$

so
$S^a(\Omega) \leq S^b(\mu^i)$ for all $i \in \{1, \ldots, k\}$.

The following trading strategy Z is a combined cost arbitrage opportunity.

$Z(\Omega) = \frac{(T+1)\hat{F}}{\min\limits_{\mu \in \{\mu_1 \cdots \mu_n\}} S^b(\mu) - S^a(\Omega)}$
$Z(\mu) = -\frac{(T+1)\hat{F}}{\min\limits_{\mu \in \{\mu_1 \cdots \mu_n\}} S^b(\mu) - S^a(\Omega)}$ for all $\mu \in \{\mu_1 \ldots \mu_n\}$.
$Z(\lambda') = 0$ for all other nodes $\lambda' \in \Lambda$.

This is a combined cost arbitrage opportunity because in every scenario it will make a profit of at least $Z(\Omega) \min\limits_{\mu \in \{\mu_1 \cdots \mu_n\}} S^b(\mu) - S^a(\Omega)$ due to the difference in the ask price at time zero and the bid price that the stock is sold at, and this will be equal to $(T+1)\hat{F}$ which is greater than the maximum possible fixed transaction cost that the trading strategy will have to pay.

If $S^b(\Omega) > \tilde{Q}^a(\Omega)$ then the combined cost arbitrage opportunity is symmetrical to the combined cost arbitrage opportunity that we have just seen.

End of proof of claim 1

Induction hypothesis $\mathbb{H}_n$ $(n \in \mathcal{T})$

For any model $\mathbb{M} = (\Omega, \mathcal{F}, \mathbb{P}, S, F, P)$ *with* n *steps or less, if* $\tilde{\mathcal{R}}(\Omega)$ *is empty then there exists a combined cost arbitrage opportunity.*

We first show that $\mathbb{H}_1$ is true.

In a single-step model $\tilde{\mathcal{R}}(\mu)$ is non-empty for all nodes $\mu \in \Lambda_1$.

It then follows immediately from Claim 1 that if $\tilde{\mathcal{R}}(\Omega)$ is empty then there exists a combined cost arbitrage opportunity.

Therefore $\mathbb{H}_1$ is true.

We now show that $\mathbb{H}_n$ is true then $\mathbb{H}_{n+1}$ must be true.

Suppose that $\mathbb{H}_n$ is true for some $n \in \mathbb{N}$

Let $\mathbb{M}$ be a model with $n+1$ steps that does not admit proportional cost arbitrage.

This implies that for any node $\mu \in \Lambda_1$ the sub-model with root node μ will not admit combined cost arbitrage and thus by our assumption that $\mathbb{H}_n$ is true, $R(\mu)$ is non-empty for all nodes $\mu \in \Lambda_1$.

Since $\mathbb{M}$ does not admit combined cost arbitrage Claim 1 combined with the fact that $R(\mu)$ is non-empty for all nodes $\mu \in \Lambda_1$ implies that $\tilde{\mathcal{R}}(\Omega)$ is non-empty.

Therefore $\mathbb{H}_{n+1}$ must be true.

We have shown that if $\mathbb{H}_n$ is true for some $n \in \mathbb{N}$ then $\mathbb{H}_{n+1}$ must also be true.

Since $\mathbb{H}_1$ is true we conclude that $\mathbb{H}_n$ must be true for all natural numbers.

<u>Part 2</u> (If there exists a combined cost arbitrage opportunity then $R(\Omega)$ is empty)

Let $\mathbb{M} = (\Omega, \mathcal{F}, \mathbb{P}, S, F, P)$ be a model that admits combined cost arbitrage.

Let $\tilde{S}$ be a strictly positive adapted process such that $S^b(\lambda) \leq \tilde{S}(\lambda) \leq S^a(\lambda)$ for all $\lambda \in \tilde{\Omega}$.

Observe that the model $\tilde{\mathbb{M}} = (\Omega, \mathcal{F}, \mathbb{P}, \tilde{S}, F, \tilde{P})$, where $\tilde{P}$ is such that $\tilde{P}(\lambda) = 0$ for all $\lambda \in \Lambda$, admits combined cost arbitrage if and only if it admits fixed cost arbitrage.

Observe also that for any two models $\mathbb{M}'$ and $\mathbb{M}''$ if model $\mathbb{M}'$ has bid-ask spreads that are contained in the bid-ask spreads of $\mathbb{M}''$ then clearly if $\mathbb{M}''$ admits combined cost arbitrage, $\mathbb{M}'$ will also admit combined cost arbitrage. This is as a consequence of Lemma 22.7 in the Appendix.

The bid-ask spreads of $\tilde{\mathbb{M}}$ are contained in the bid-ask spreads of $\mathbb{M}$, since $\mathbb{M}$ admits combined cost arbitrage it follows that $\tilde{\mathbb{M}}$ admits combined cost arbitrage. Therefore $\tilde{\mathbb{M}}$ also admits fixed cost arbitrage.

Therefore it is not possible to find a strictly positive process $\tilde{S}$ with $S^b(\lambda) \leq \tilde{S}(\lambda) \leq S^a(\lambda)$ for all $\lambda \in \tilde{\Omega}$, such that there does not exist a fixed cost arbitrage opportunity.

Therefore $\tilde{\mathcal{R}}(\Omega)$ is empty.

End of Proof of Theorem 11.12.

12. **European derivative securities**

We now begin our investigation into derivative securities. Our main goal in this half of the thesis is to find risk-neutral representations of the ask and bid prices of derivatives under fixed and proportional transaction costs. In this section we provide the groundwork for our investigations by formulating what we mean by a derivative, as well as showing how strategies will relate to derivatives. We begin with the definition of a derivative.

12.1. **Derivative security definition.**

Definition 12.1. (Derivative security (European))
A *derivative security* D defined in a model $\mathbb{M}$ is an $\mathcal{F}_T$ measurable random variable $D = (D^0, D^1)$, representing the portfolio which the derivative security buyer will receive at time T.

Definition 12.2. (Derivative security initial price)
An *initial price*, d_0 of a derivative security D is a real number. It represents the amount that the buyer of the derivative security must pay at time zero in order to purchase the derivative security.

Example 12.3. (Payoff of a European forward contract)
Let us look at an example of a typical payoff. A European forward contract with forward price k is a contract that guarantees that the holder will receive a unit of stock at the price k.

Consider a forward contract D, with a forward price of 5 units.

This means that the payoff, (D^0, D^1), of the derivative security D at every terminal node would be the portfolio $(-5, 1)$ because, at time T, the holder of the contract would receive one share of stock in exchange for 5 units of cash.

Example 12.4. (Payoff of a European Call Option)
Consider a single-step model with only two terminal nodes, u and d. Let us set $S(\Omega) = 3, S(u) = 5$ and $S(d) = 1$. We will also set $F(\lambda) = 1$ and $P(\lambda) = 0$ for all $\lambda \in \Lambda$.

Consider a call option D that gives the holder the option the right to receive a single share of stock in exchange for 3 units of cash.

In this case the payoff portfolios of the derivative D would be as follows:

$(D^0(u), D^1(u)) = (-3, 1)$ (This is because a rational investor would definitely exercise at this node).

$(D^0(d), D^1(d)) = (0, 0)$ (This is because a rational investor would never exercise the option at this node).

12.2. **Derivative super-replication portfolio strategies.**

Definition 12.5. (Derivative super-replication portfolio strategy)
A *frictionless super-replication portfolio strategy* (with respect to D) is a portfolio strategy (X, Y) for which there exists a trading strategy Z such that for each node λ we have:

$(X(\lambda), Y(\lambda)) \xrightarrow{Z(\lambda)} (X_{+1}(\lambda), Y_{+1}(\lambda))$ if λ is not a terminal node.

$(X(\lambda) - D^0, Y(\lambda) - D^1) \xrightarrow{Z(\lambda)} (X_{+1}(\lambda), Y_{+1}(\lambda)),\ X_{+1}(\lambda) \geq 0$ and $Y_{+1}(\lambda) \geq 0$ if λ is a terminal node.

If this is the case then we will say that the portfolio strategy (X, Y) and trading strategy Z together with initial portfolio $(X_0, 0)$ *frictionless super-replication correspond* to each other (with respect to D). Note that this is a one to one correspondence.

A *fixed cost super-replication portfolio strategy* (with respect to D) is a portfolio strategy (X, Y) for which there exists a trading strategy Z such that for each node λ we have:

$(X(\lambda), Y(\lambda)) \xrightarrow[F]{Z(\lambda)} (X_{+1}(\lambda), Y_{+1}(\lambda))$ if λ is not a terminal node.

$(X(\lambda) - D^0, Y(\lambda) - D^1) \xrightarrow[F]{Z(\lambda)} (X_{+1}(\lambda), Y_{+1}(\lambda)),\ X_{+1}(\lambda) \geq 0$ and $Y_{+1}(\lambda) \geq 0$ if λ is a terminal node.

If this is the case then we will say that the portfolio strategy (X, Y) and trading strategy Z together with initial portfolio $(X_0, 0)$ *fixed cost super-replication correspond* to each other (with respect to D). Note that this is a one to one correspondence.

A *proportional cost super-replication portfolio strategy* (with respect to D) is a portfolio strategy (X, Y) for which there exists a trading strategy Z such that for each node λ we have:

$(X(\lambda), Y(\lambda)) \xrightarrow[P]{Z(\lambda)} (X_{+1}(\lambda), Y_{+1}(\lambda))$ if λ is not a terminal node.

$(X(\lambda) - D^0, Y(\lambda) - D^1) \xrightarrow[P]{Z(\lambda)} (X_{+1}(\lambda), Y_{+1}(\lambda)),\ X_{+1}(\lambda) \geq 0$ and $Y_{+1}(\lambda) \geq 0$ if λ is a terminal node.

If this is the case then we will say that the portfolio strategy (X, Y) and trading strategy Z, such that $Z(\lambda) = 0$ if $P(\lambda) = \infty$ for all $\lambda \in \Lambda$, together with initial portfolio $(X_0, 0)$ *proportional cost super-replication correspond* to each other (with respect to D). Note that this is a one to one correspondence.

A *combined cost super-replication portfolio strategy* (with respect to D) is a portfolio strategy (X, Y) for which there exists a trading strategy Z such that for each node λ we have:

$(X(\lambda), Y(\lambda)) \xrightarrow[C]{Z(\lambda)} (X_{+1}(\lambda), Y_{+1}(\lambda))$ if λ is not a terminal node.

$(X(\lambda) - D^0, Y(\lambda) - D^1) \xrightarrow[C]{Z(\lambda)} (X_{+1}(\lambda), Y_{+1}(\lambda)),\ X_{+1}(\lambda) \geq 0$ and $Y_{+1}(\lambda) \geq 0$ if λ is a terminal node.

If this is the case then we will say that the portfolio strategy (X, Y) and trading strategy Z, such that $Z(\lambda) = 0$ if $P(\lambda) = \infty$ for all $\lambda \in \Lambda$, together with initial portfolio $(X_0, 0)$ *combined cost super-replication correspond* to each other(with respect to D). Note that this is a one to one correspondence.

Definition 12.6. (Set of super-replication portfolio strategies for D)
$\mathrm{Srep}(D) :=$ The set of frictionless derivative super-replication portfolio strategies (with respect to D).

$\mathrm{Srep}_F(D) :=$ The set of fixed cost derivative super-replication portfolio strategies (with respect to D).

$\mathrm{Srep}_P(D) :=$ The set of proportional cost derivative super-replication portfolio strategies (with respect to D).

$\mathrm{Srep}_C(D) :=$ The set of combined cost derivative super-replication portfolio strategies (with respect to D).

If it is necessary to specify the model in which the portfolio strategies must be super-replication portfolio strategies, then the model will be denoted as a superscript e.g. $\mathrm{Srep}^{\mathbb{M}}(D)$.

Definition 12.7. (Super-replicability)
We will say that a derivative security is *frictionless super-replicable* if and only if there exists a frictionless super-replication portfolio strategy.

We will say that a derivative security is *fixed cost super-replicable* if and only if there exists a fixed cost super-replication portfolio strategy.

We will say that a derivative security is *proportional cost super-replicable* if and only if there exists a proportional cost super-replication portfolio strategy.

We will say that a derivative security is *combined cost super-replicable* if and only if there exists a combined cost super-replication portfolio strategy.

Remark 12.8. (Super-replicability)
Every derivative in a restricted model is frictionless, fixed cost, proportional cost and combined cost super-replicable. It is also true that every derivative security in an unrestricted model is frictionless and fixed cost super-replicable. However it is not necessarily true that a derivative security in an unrestricted model is proportional cost or combined cost super-replicable. We will see more on this later.

12.3. **Derivative self-financing portfolio strategies.**

Definition 12.9. (Derivative self-financing portfolio strategy)
A *frictionless derivative self-financing portfolio strategy* (with respect to D and d_0) is a portfolio strategy (X, Y) for which there exists a trading strategy Z such that for each node λ we have:

$(X(\lambda) + d_0, Y(\lambda)) \xrightarrow{Z(\lambda)} (X_{+1}(\lambda), Y_{+1}(\lambda))$ if λ is the root node.

$(X(\lambda), Y(\lambda)) \xrightarrow{Z(\lambda)} (X_{+1}(\lambda), Y_{+1}(\lambda))$ if λ is not the root node nor a terminal node.

$(X(\lambda) - D^0, Y(\lambda) - D^1) \xrightarrow{Z(\lambda)} (X_{+1}(\lambda), Y_{+1}(\lambda))$, $X_{+1}(\lambda) \geq 0$ and $Y_{+1}(\lambda) \geq 0$ if λ is a terminal node.

If this is the case then we will say that the portfolio strategy (X, Y) and trading strategy Z together with initial portfolio $(X_0, 0)$ *frictionless self-financing correspond* to each other (with respect to D and d_0). Note that this is a one-to-one correspondence.

A *fixed cost derivative self-financing portfolio strategy* (with respect to D and d_0) is a portfolio strategy (X, Y) for which there exists a trading strategy Z such that for each node λ we have:

$(X(\lambda) + d_0, Y(\lambda)) \xrightarrow[F]{Z(\lambda)} (X_{+1}(\lambda), Y_{+1}(\lambda))$ if λ is the root node.

$(X(\lambda), Y(\lambda)) \xrightarrow[F]{Z(\lambda)} (X_{+1}(\lambda), Y_{+1}(\lambda))$ if λ is not the root node nor a terminal node.

$(X(\lambda) - D^0, Y(\lambda) - D^1) \xrightarrow[F]{Z(\lambda)} (X_{+1}(\lambda), Y_{+1}(\lambda))$, $X_{+1}(\lambda) \geq 0$ and $Y_{+1}(\lambda) \geq 0$ if λ is a terminal node.

If this is the case then we will say that the portfolio strategy (X, Y) and trading strategy Z together with initial portfolio $(X_0, 0)$ *fixed cost self-financing correspond* to each other(with respect to D and d_0). Note that this is a one-to-one correspondence.

A *proportional cost derivative self-financing portfolio strategy* (with respect to D and d_0) is a portfolio strategy (X, Y) for which there exists a trading strategy Z such that for each node λ we have:

$(X(\lambda) + d_0, Y(\lambda)) \xrightarrow[P]{Z(\lambda)} (X_{+1}(\lambda), Y_{+1}(\lambda))$ if λ is the root node.

$(X(\lambda), Y(\lambda)) \xrightarrow[P]{Z(\lambda)} (X_{+1}(\lambda), Y_{+1}(\lambda))$ if λ is not the root node nor a terminal node. $(X(\lambda) - D^0, Y(\lambda) - D^1) \xrightarrow[P]{Z(\lambda)} (X_{+1}(\lambda), Y_{+1}(\lambda))$, $X_{+1}(\lambda) \geq 0$ and $Y_{+1}(\lambda) \geq 0$ if λ is a terminal node.

If this is the case then we will say that the portfolio strategy (X, Y) and trading strategy Z, such that $Z(\lambda) = 0$ if $P(\lambda) = \infty$ for all $\lambda \in \Lambda$, together with initial portfolio $(X_0, 0)$ *proportional cost self-financing correspond* to each other(with respect to D and d_0). Note that this is a one-to-one correspondence.

A *combined cost derivative self-financing portfolio strategy* (with respect to D and d_0) is a portfolio strategy (X, Y) for which there exists a trading strategy Z such that for each node λ we have:
$(X(\lambda) + d_0, Y(\lambda)) \xrightarrow[C]{Z(\lambda)} (X_{+1}(\lambda), Y_{+1}(\lambda))$ if λ is the root node.

$(X(\lambda), Y(\lambda)) \xrightarrow[C]{Z(\lambda)} (X_{+1}(\lambda), Y_{+1}(\lambda))$ if λ is not the root node nor a terminal node.

$(X(\lambda) - D^0, Y(\lambda) - D^1) \xrightarrow[C]{Z(\lambda)} (X_{+1}(\lambda), Y_{+1}(\lambda))$, $X_{+1}(\lambda) \geq 0$ and $Y_{+1}(\lambda) \geq 0$ if λ is a terminal node.

If this is the case then we will say that the portfolio strategy (X, Y) and trading strategy Z, such that $Z(\lambda) = 0$ if $P(\lambda) = \infty$ for all $\lambda \in \Lambda$, together with initial portfolio $(X_0, 0)$ *combined cost self-financing correspond* to each other(with respect to D and d_0). Note that this is a one-to-one correspondence.

12.4. The portfolio strategy corresponding to a trading strategy.

Lemma 12.10. *(Portfolio strategy super-replication corresponding to a trading strategy) Consider an unrestricted model.*

Let $x \in \mathbb{R}$.

The frictionless super-replication portfolio strategy (X, Y) that corresponds to initial portfolio $(x, 0)$ and trading strategy Z (with respect to D) is given by:

$$X_t = x + \sum_{i=0}^{t} \Big[- Z_i S_i \Big] - D^0 \boldsymbol{1}_{\{t=T+1\}}$$
$$Y_t = \sum_{i=0}^{t} [Z_i] - D^1 \boldsymbol{1}_{\{t=T+1\}}$$
for all $t \in \mathcal{T} \cup \{T+1\}$.

The fixed cost super-replication portfolio strategy (X, Y) that corresponds to initial portfolio $(x, 0)$ and trading strategy Z (with respect to D) is given by:

$$X_t = x + \sum_{i=0}^{t} \Big[- Z_i S_i - F_i \boldsymbol{1}_{\{Z_i \neq 0\}} \Big] - D^0 \boldsymbol{1}_{\{t=T+1\}}$$
$$Y_t = \sum_{i=0}^{t} [Z_i] - D^1 \boldsymbol{1}_{\{t=T+1\}}$$
for all $t \in \mathcal{T} \cup \{T+1\}$.

The proportional cost super-replication portfolio strategy (X, Y) that corresponds to initial portfolio $(x, 0)$ and trading strategy Z, such that $Z(\lambda) = 0$ if $P(\lambda) = \infty$ for all $\lambda \in \Lambda$, (with respect to D) is given by:

$$X_t = x + \sum_{i=0}^{t} \Big[- Z_i S_i - |Z_i| P_i \Big] - D^0 \boldsymbol{1}_{\{t=T+1\}}$$

$$Y_t = \sum_{i=0}^{t} [Z_i] - D^1 \boldsymbol{1}_{\{t=T+1\}}$$

for all $t \in \mathcal{T} \cup \{T+1\}$.

The combined cost super-replication portfolio strategy (X, Y) *that corresponds to initial portfolio* $(x, 0)$ *and trading strategy* Z*, such that* $Z(\lambda) = 0$ *if* $P(\lambda) = \infty$ *for all* $\lambda \in \Lambda$*, (with respect to* D*) is given by:*

$$X_t = x + \sum_{i=0}^{t} \Big[- Z_i S_i - F_i \boldsymbol{1}_{\{Z_i \neq 0\}} - |Z_i| P_i \Big] - D^0 \boldsymbol{1}_{\{t=T+1\}}$$

$$Y_t = \sum_{i=0}^{t} [Z_i] - D^1 \boldsymbol{1}_{\{t=T+1\}}$$

for all $t \in \mathcal{T} \cup \{T+1\}$.

Proof of Lemma 12.10

Follows from the definition of a derivative super-replication portfolio strategy.

End of proof of Lemma 12.10

Lemma 12.11. *(Portfolio strategy that self-financing corresponding to trading strategy and initial portfolio)*
Consider an unrestricted model.

Let Z be a trading strategy. Let $x \in \mathbb{R}$.

The frictionless derivative self-financing portfolio strategy (X, Y) that corresponds to initial portfolio $(x, 0)$ and trading strategy Z (with respect to D and d_0) is given by:

$$X_t = x + d_0 \mathbf{1}_{\{t \neq 0\}} + \sum_{i=0}^{t} \Big[- Z_i S_i \Big] - D^0 \mathbf{1}_{\{t=T+1\}}$$
$$Y_t = \sum_{i=0}^{t} [Z_i] - D^1 \mathbf{1}_{\{t=T+1\}}$$
for all $t \in \mathcal{T} \cup \{T+1\}$.

The fixed cost derivative self-financing portfolio strategy (X, Y) that corresponds to initial portfolio $(x, 0)$ and trading strategy Z (with respect to D and d_0) is given by:

$$X_t = x + d_0 \mathbf{1}_{\{t \neq 0\}} + \sum_{i=0}^{t} \Big[- Z_i S_i - F_i \mathbf{1}_{\{Z_i \neq 0\}} \Big] - D^0 \mathbf{1}_{\{t=T+1\}}$$
$$Y_t = \sum_{i=0}^{t} [Z_i] - D^1 \mathbf{1}_{\{t=T+1\}}$$
for all $t \in \mathcal{T} \cup \{T+1\}$.

The proportional cost derivative self-financing portfolio strategy (X, Y) that corresponds to initial portfolio $(x, 0)$ and trading strategy Z, such that $Z(\lambda) = 0$ if $P(\lambda) = \infty$ for all $\lambda \in \Lambda$, (with respect to D and d_0) is given by:

$$X_t = x + d_0 \mathbf{1}_{\{t \neq 0\}} + \sum_{i=0}^{t} \Big[- Z_i S_i - |Z_i| P_i \Big] - D^0 \mathbf{1}_{\{t=T+1\}}$$
$$Y_t = \sum_{i=0}^{t} [Z_i] - D^1 \mathbf{1}_{\{t=T+1\}}$$
for all $t \in \mathcal{T} \cup \{T+1\}$.

The combined cost derivative self-financing portfolio strategy (X, Y) that corresponds to initial portfolio $(x, 0)$ and trading strategy Z, such that $Z(\lambda) = 0$ if $P(\lambda) = \infty$ for all $\lambda \in \Lambda$, (with respect to D and d_0) is given by:

$$X_t = x + d_0 \mathbf{1}_{\{t \neq 0\}} + \sum_{i=0}^{t} \Big[- Z_i S_i - F_i \mathbf{1}_{\{Z_i \neq 0\}} - |Z_i| P_i \Big] - D^0 \mathbf{1}_{\{t=T+1\}}$$
$$Y_t = \sum_{i=0}^{t} [Z_i] - D^1 \mathbf{1}_{\{t=T+1\}}$$
for all $t \in \mathcal{T} \cup \{T+1\}$.

Proof of Lemma 12.11

Follows from the definition of a derivative super-replication portfolio strategy.

End of proof of Lemma 12.11

12.5. **The trading strategy corresponding to a portfolio strategy.**

Lemma 12.12. *(Trading strategy super-replication corresponding to a portfolio strategy) Consider an unrestricted model.*

For any frictionless super-replication strategy (X, Y)*, the trading strategy* Z *that frictionless super-replication corresponds to* (X, Y) *(with respect to derivative* D*) is given by:*

$Z(\lambda) = Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda)$ *for all* $\lambda \in \Lambda_T$.

$Z(\lambda) = Y_{+1}(\lambda) - Y(\lambda)$ *for all* $\lambda \in \Lambda \setminus \Lambda_T$.

For any fixed cost super-replication strategy (X, Y)*, the trading strategy* Z *that fixed cost super-replication corresponds to* (X, Y) *(with respect to derivative* D*) is given by:*

$Z(\lambda) = Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda)$ *for all* $\lambda \in \Lambda_T$.

$Z(\lambda) = Y_{+1}(\lambda) - Y(\lambda)$ *for all* $\lambda \in \Lambda \setminus \Lambda_T$.

For any proportional cost super-replication strategy (X, Y)*, the trading strategy* Z *that proportional cost super-replication corresponds to* (X, Y) *(with respect to derivative* D*) is given by:*

$Z(\lambda) = Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda)$ *for all* $\lambda \in \Lambda_T$.

$Z(\lambda) = Y_{+1}(\lambda) - Y(\lambda)$ *for all* $\lambda \in \Lambda \setminus \Lambda_T$.

For any combined cost super-replication strategy (X, Y)*, the trading strategy* Z *that combined cost super-replication corresponds to* (X, Y) *(with respect to derivative* D*) is given by:*

$Z(\lambda) = Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda)$ *for all* $\lambda \in \Lambda_T$.

$Z(\lambda) = Y_{+1}(\lambda) - Y(\lambda)$ *for all* $\lambda \in \Lambda \setminus \Lambda_T$.

Proof of Lemma 12.12

Follows from the definition of a correspondence between a super-replication portfolio strategy and a trading strategy.

End of proof of Lemma 12.12

Lemma 12.13. *(Trading strategy self-financing corresponding to a portfolio strategy)*
Consider an unrestricted model.

For any frictionless self-financing strategy (X, Y), the trading strategy Z that frictionless self-financing corresponds to (X, Y) (with respect to derivative D) is given by:

$Z(\lambda) = Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda)$ *for all* $\lambda \in \Lambda_T$.

$Z(\lambda) = Y_{+1}(\lambda) - Y(\lambda)$ *for all* $\lambda \in \Lambda \setminus \Lambda_T$.

For any fixed cost self-financing strategy (X, Y), the trading strategy Z that fixed cost self-financing corresponds to (X, Y) (with respect to derivative D) is given by:

$Z(\lambda) = Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda)$ *for all* $\lambda \in \Lambda_T$.

$Z(\lambda) = Y_{+1}(\lambda) - Y(\lambda)$ *for all* $\lambda \in \Lambda \setminus \Lambda_T$.

For any proportional cost self-financing strategy (X, Y), the trading strategy Z that proportional cost self-financing corresponds to (X, Y) (with respect to derivative D) is given by:

$Z(\lambda) = Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda)$ *for all* $\lambda \in \Lambda_T$.

$Z(\lambda) = Y_{+1}(\lambda) - Y(\lambda)$ *for all* $\lambda \in \Lambda \setminus \Lambda_T$.

For any combined cost self-financing strategy (X, Y), the trading strategy Z that combined cost self-financing corresponds to (X, Y) (with respect to derivative D) is given by:

$Z(\lambda) = Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda)$ *for all* $\lambda \in \Lambda_T$.

$Z(\lambda) = Y_{+1}(\lambda) - Y(\lambda)$ *for all* $\lambda \in \Lambda \setminus \Lambda_T$.

Proof of Lemma 12.13

Follows from the definition of a correspondence between a derivative self-financing portfolio strategy and a trading strategy.

End of proof of Lemma 12.13

Definition 12.14. (Number of trades process of a strategy)
For any portfolio strategy (X, Y) and derivative D, we define the real valued process $Z^{(X,Y)}$ as follows:

$Z^{(X,Y)}(\lambda) = Y_{+1}(\lambda) - Y(\lambda)$ if λ is a non-terminal node.

$Z^{(X,Y)}(\lambda) = Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda)$ if λ is a terminal node.

In the rare cases when there is more than one derivative security it will be noted which derivative this process refers to.

12.6. **Sets of portfolios from which super-replication is possible.**

The following definition is admittedly slightly convoluted, but there is a simple idea at its heart. Consider a portfolio (x, y), node λ and derivative D. For the purpose of explanation we will work in the frictionless setting for this discussion. Our aim is to answer the question "If we arrive at node λ with the portfolio (x, y), is it possible to trade in a self-financing way and cover the derivative payoff ?". We are about to formally define the set $\mathcal{Z}^{(x,y)}(\lambda)$. This set will describe the answer to the aforementioned question. If $\mathcal{Z}^{(x,y)}(\lambda)$ is empty then it means that it is not possible to trade in a self-financing way and still cover the derivative payoff. If $\mathcal{Z}^{(x,y)}(\lambda)$ is not empty then it means that it is possible to trade in a self-financing way and cover the derivative payoff. The equations that we are about to see are best understood by comparing them to the equations in Lemma 12.10. Essentially the equations are almost a way of saying that "there exists a portfolio strategy in the sub-model at node λ with initial portfolio (x, y) that super-replicates the derivative on the sub-model". The only reason that this is technically not what we are saying, is that, by definition, a super-replication portfolio strategy starts with a position of zero shares of stock. Hence the reason for the added complication to the following definition.

Definition 12.15. (Set of strategies enabling a portfolio to super-replicate)
Let $\lambda \in \Lambda$ and let D be a derivative security in a model $\mathbb{M}$.

Let $\mathbb{M}^\lambda$ be the sub-model at node λ.

We denote by S^λ, F^λ, P^λ, Ω^λ and $\mathcal{T}^\lambda$ the stock process, fixed transaction cost process, proportional transaction cost process, set of scenarios and set of times, in sub-model $\mathbb{M}^\lambda$ respectively.

For any $(x, y) \in \mathbb{R}^2$ we define $\mathcal{Z}^{(x,y)}(\lambda)$ to be the set of trading strategies Z in M^λ such that:

$$x - D^0(\omega) + \sum_{t\in\mathcal{T}^\lambda} -S^\lambda_t(\omega)Z_t(\omega) \geq 0 \text{ and } y + \sum_{t\in\mathcal{T}^\lambda} Z_t(\omega) - D^1(\omega) \geq 0 \ ,$$

for all $\omega \in \Omega^\lambda$.

For any $(x, y) \in \mathbb{R}^2$ we define $\mathcal{Z}_F^{(x,y)}(\lambda)$ to be the set of trading strategies Z in M^λ such that:

$$x - D^0(\omega) + \sum_{t\in\mathcal{T}^\lambda} \Big[- S^\lambda_t(\omega)Z_t(\omega) - F^\lambda_t \mathbf{1}_{\{Z_t\neq 0\}}\Big] \geq 0 \text{ and}$$

$$y - D^1(\omega) + \sum_{t\in\mathcal{T}^\lambda} Z_t(\omega) \geq 0 \ , \text{ for all } \omega \in \Omega^\lambda.$$

For any $(x, y) \in \mathbb{R}^2$ we define $\mathcal{Z}_P^{(x,y)}(\lambda)$ to be the set of trading strategies Z in M^λ such that:

$$x - D^0(\omega) + \sum_{t\in\mathcal{T}^\lambda} \Big[- S^\lambda_t(\omega)Z_t(\omega) - |Z_t|P^\lambda_t\Big] \geq 0 \text{ and}$$

$$y - D^1(\omega) + \sum_{t\in\mathcal{T}^\lambda} Z_t(\omega) \geq 0 \ , \text{ for all } \omega \in \Omega^\lambda.$$

For any $(x, y) \in \mathbb{R}^2$ we define $\mathcal{Z}_C^{(x,y)}(\lambda)$ to be the set of trading strategies Z in M^λ such that:

$$x - D^0(\omega) + \sum_{t\in\mathcal{T}^\lambda} \Big[- S^\lambda_t(\omega)Z_t(\omega) - F^\lambda_t \mathbf{1}_{\{Z_t\neq 0\}} - |Z_t|P^\lambda_t\Big] \geq 0 \text{ and}$$

$$y - D^1(\omega) + \sum_{t\in\mathcal{T}^\lambda} Z_t(\omega) \geq 0 \ , \text{ for all } \omega \in \Omega^\lambda.$$

We are now in a good position to introduce the terminology that we have been aiming for in this subsection.

Definition 12.16. (Arrival set of portfolios that can super-replicate the derivative)
We define the arrival set of portfolios that can super-replicate the derivative as follows:

$\mathcal{A}^D(\lambda) := \{(x,y) \in \mathbb{R}^2 : \mathcal{Z}^{(x,y)}(\lambda) \neq \emptyset\}.$

$\mathcal{A}^D_F(\lambda) := \{(x,y) \in \mathbb{R}^2 : \mathcal{Z}^{(x,y)}_F(\lambda) \neq \emptyset\}.$

$\mathcal{A}^D_P(\lambda) := \{(x,y) \in \mathbb{R}^2 : \mathcal{Z}^{(x,y)}_P(\lambda) \neq \emptyset\}.$

$\mathcal{A}^D_C(\lambda) := \{(x,y) \in \mathbb{R}^2 : \mathcal{Z}^{(x,y)}_C(\lambda) \neq \emptyset\}.$

Definition 12.17. (Exit set of portfolios that can super-replicate the derivative)
We define the exit set of portfolios that can super-replicate the derivative as follows:

$\mathcal{E}^D(\lambda) := \bigcap_{\mu \in \mathrm{Succ}\lambda} \mathcal{A}^D(\mu)$ if λ is a non-terminal node.
$\mathcal{E}^D(\lambda) := \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ if λ is a terminal node.

$\mathcal{E}^D_F(\lambda) := \bigcap_{\mu \in \mathrm{Succ}\lambda} \mathcal{A}^D_F(\mu)$ if λ is a non-terminal node.
$\mathcal{E}^D_F(\lambda) := \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ if λ is a terminal node.

$\mathcal{E}^D_P(\lambda) := \bigcap_{\mu \in \mathrm{Succ}\lambda} \mathcal{A}^D_P(\mu)$ if λ is a non-terminal node.
$\mathcal{E}^D_P(\lambda) := \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ if λ is a terminal node.

$\mathcal{E}^D_C(\lambda) := \bigcap_{\mu \in \mathrm{Succ}\lambda} \mathcal{A}^D_C(\mu)$ if λ is a non-terminal node.
$\mathcal{E}^D_C(\lambda) := \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ if λ is a terminal node.

Definition 12.18. (Linear slice)
We will say that a set in $\mathbb{R}^2$ is a *linear slice* if it can be written as $\{(x,y) : y \geq mx + c, b \leq x \leq a\}$, for some $c, m \in \mathbb{R}$, $a \in \mathbb{R} \cup \{\infty\}$, $b \in \mathbb{R} \cup \{-\infty\}$, with $a \leq b$.

Definition 12.19. (Piecewise linear hull)
We will say that a set in $\mathbb{R}^2$ is a *piecewise linear hull* if it can be written as the union of finitely many linear slices.

Remark 12.20. Note that any linear slice is a closed set and thus any piecewise linear hull is also a closed set.

Lemma 12.21. *(Arrival and exit sets are piecewise linear hulls)*
For any unrestricted model $\mathbb{M}$ and proportional cost super-replicable derivative D we have the following:

If $\mathbb{M}$ is frictionless viable then:

(1) For any node $\lambda \in \Lambda$, we have that $\mathcal{A}^D(\lambda)$ and $\mathcal{E}^D(\lambda)$ are piecewise linear hulls.

(2) For any node $\lambda \in \Lambda$, we have that $\mathcal{A}^D_F(\lambda)$ and $\mathcal{E}^D_F(\lambda)$ are piecewise linear hulls.

If $\mathbb{M}$ is proportional cost viable then:

(3) For any node $\lambda \in \Lambda$, we have that $\mathcal{A}^D_P(\lambda)$ and $\mathcal{E}^D_P(\lambda)$ are piecewise linear hulls.

(4) For any node $\lambda \in \Lambda$, we have that $\mathcal{A}^D_C(\lambda)$ and $\mathcal{E}^D_C(\lambda)$ are piecewise linear hulls.

Proof of Lemma 15.4

We prove (4). The proofs of $(1) - (3)$ are similar, but much more straightforward.

The sets $\mathcal{A}^D_C(\lambda)$ and $\mathcal{E}^D_C(\lambda)$ are non-empty for any node $\lambda \in \Lambda$. To see this observe that the point $(\max\limits_{\mu \in \Lambda_T} D^0(\mu), \max\limits_{\mu \in \Lambda_T} D^1(\mu))$ belongs to $\mathcal{A}^D_C(\lambda)$ and $\mathcal{E}^D_C(\lambda)$ for any node $\lambda \in \Lambda$.

Observe that the sets $\mathcal{A}^D_C(\lambda)$ and $\mathcal{E}^D_C(\lambda)$ are connected sets. To see this, notice that for any portfolio (x, y) belonging to either of these sets, the point $(x + b, y + a)$ belongs to the set for any $a, b \in \mathbb{R}^+$. This means that for any pair of points (x_1, y_1) and (x_2, y_2) that belong to either $\mathcal{A}^D_C(\lambda)$ or $\mathcal{E}^D_C(\lambda)$, we can draw a straight line from (x_1, y_1) to $(\min\{x_1, x_2\}, \max\{y_1, y_2\})$ and another straight line form $(\min\{x_1, x_2\}, \max\{y_1, y_2\})$ to (x_2, y_2), and the combination of these two lines will connect (x_1, y_1) to (x_2, y_2) and lie entirely in the set.

We first show that the non-existence of proportional cost arbitrage implies that for all $\lambda \in \Lambda$, $\mathcal{A}^D_C(\lambda) \neq \mathbb{R}^2$ and $\mathcal{E}^D_C(\lambda) \neq \mathbb{R}^2$.

Suppose that $\mathcal{A}^D_C(\lambda) = \mathbb{R}^2$ for some node $\lambda \in \Lambda$.

Let $m := \min\limits_{\lambda \in \Lambda_T} \min\{D^0(\lambda), D^1(\lambda)\}$, the minimum possible frictionless cash or stock position of the derivative payoff.

Let $\mathbb{M}^\lambda$ be the sub-model at node λ.

We denote by S^λ, F^λ, P^λ, Ω^λ and $\mathcal{T}^\lambda$ the stock process, fixed transaction cost process, proportional transaction cost process, set of scenarios and set of times, in sub-model $\mathbb{M}^\lambda$ respectively.

Consider the initial portfolio $(m-1, m-1)$. From the definition of $\mathcal{A}_C^D(\lambda)$ there exists a trading strategy Z in the model $\mathbb{M}^\lambda$ such that:

$$m-1-D^0(\omega)+\sum_{t\in\mathcal{T}^\lambda}\Big[-S_t^\lambda(\omega)Z_t(\omega)-F_t^\lambda(\omega)\mathbf{1}_{\{Z_t\neq 0\}}-|Z_t(\omega)|P_t^\lambda(\omega)\Big]\geq 0 \text{ and}$$
$$m-1-D^1(\omega)+\sum_{t\in\mathcal{T}^\lambda}Z_t(\omega)\geq 0\text{ , for all }\omega\in\Omega^\lambda.$$

This implies that:

$$\sum_{t\in\mathcal{T}^\lambda}\Big[-S_t^\lambda(\omega)Z_t(\omega)-F_t^\lambda(\omega)\mathbf{1}_{\{Z_t\neq 0\}}-|Z_t(\omega)|P_t^\lambda(\omega)\Big]\geq -(m-1-D^0(\omega))>0 \text{ and}$$
$$\sum_{t\in\mathcal{T}^\lambda}Z_t(\omega)\geq -(m-1-D^1(\omega))>0\text{ , for all }\omega\in\Omega^\lambda.$$

By Lemma 7.2 this implies that Z is a combined cost arbitrage opportunity and also a proportional cost arbitrage opportunity in the model $\mathbb{M}^\lambda$. Thus there would exist a proportional cost arbitrage opportunity in $\mathbb{M}$, contradicting the fact that $\mathbb{M}$ is proportional cost viable.

Therefore we cannot have $\mathcal{A}_C^D(\lambda)=\mathbb{R}^2$ for any node $\lambda\in\Lambda$.

Suppose that $\mathcal{E}_C^D(\lambda)=\mathbb{R}^2$ for some node $\lambda\in\Lambda$. From the definition of $\mathcal{E}_C^D$ this is only possible if λ is a non-terminal node. Therefore this would imply that $\mathcal{A}_C^D(\mu)=\mathbb{R}^2$ for all $\mu\in\text{Succ}\lambda$ and since we have already deduced that this cannot be the case, we conclude that $\mathcal{E}_C^D(\lambda)\neq\mathbb{R}^2$ for all $\lambda\in\Lambda$.

Observe that for any terminal node μ, $\mathcal{E}_C^D(\mu)$ is a piecewise linear hull.

We show that $\mathcal{A}_C^D(\lambda)$ and $\mathcal{E}_C^D(\lambda)$ are piecewise linear hulls for all $\lambda\in\Lambda$ by working backwards through the tree.

Observe that the intersection of finitely many piecewise linear hulls is either empty or a piecewise linear hull.

By definition, for any non-terminal node $\lambda\in\Lambda$, we have $\mathcal{E}_C^D(\lambda)=\bigcap_{\mu\in\text{Succ}\lambda}\mathcal{A}_C^D(\mu)$.

It follows that for any non-terminal node λ such that $\mathcal{A}_C^D(\mu)$ is a piecewise linear hull for all $\mu\in\text{Succ}\lambda$, we have that $\mathcal{E}_C^D(\lambda)$ is a piecewise linear hull.

Fix a node $\lambda\in\Lambda$ and suppose that $\mathcal{E}_C^D(\lambda)$ is a piecewise linear hull.

We will show that $\mathcal{A}_C^D(\lambda)$ is a piecewise linear hull.

$\mathcal{A}_C^D(\lambda)=\{(x,y)\in\mathbb{R}^2:\text{there exists a combined cost trade } f \text{ such that } f(x,y)\in\mathcal{E}_C^D(\lambda)\}$.

We will write $\mathcal{E}_C^D(\lambda)+F(\lambda)$ to mean $\{(x,y):(x-F(\lambda),y)\in\mathcal{E}_C^D(\lambda)\}$.

Note that since $\mathcal{E}_C^D(\lambda)$ is a piecewise linear hull, we have that $\mathcal{E}_C^D(\lambda)+F(\lambda)$ is a piecewise linear hull.

We have that

$$\mathcal{A}_C^D(\lambda)=\mathcal{E}_C^D(\lambda)\cup\{(x,y)\in\mathbb{R}^2:\text{there exists a combined cost trade } f \text{ such that } f(x,y)\in\mathcal{E}_C^D(\lambda)\}$$

$= \mathcal{E}_C^D(\lambda) \cup \{(x,y) \in \mathbb{R}^2$: there exists a proportional cost trade f such that $f(x,y) \in \mathcal{E}_C^D(\lambda) + F(\lambda)\}$

$= \mathcal{E}_C^D(\lambda) \cup \{(x,y) \in \mathbb{R}^2$: there exists a $z \in \mathbb{R}$ such that $(x - zS(\lambda) - |z|P(\lambda), y+z) \in \mathcal{E}_C^D(\lambda) + F(\lambda)\}$.

$= \mathcal{E}_C^D(\lambda) \cup \{(x,y) \in \mathbb{R}^2$: there exists a $z \in \mathbb{R}$ such that $(x - zS^a(\lambda)\mathbf{1}_{z>0} - zS^b(\lambda)\mathbf{1}_{z<0}, y+z) \in \mathcal{E}_C^D(\lambda) + F(\lambda)\}$.

Let $\mathcal{V}(\mathcal{E}_C^D(\lambda) + F(\lambda))$ be the set of vertices of the set $\mathcal{E}_C^D(\lambda) + F(\lambda)$.

Let $n(\lambda)$ be the number of vertices of the set $\mathcal{E}_C^D(\lambda) + F(\lambda)$.

Suppose that $\mathcal{V}(\mathcal{E}_C^D(\lambda) + F(\lambda))$ is non-empty.

Let $\hat{\mathcal{V}}(\mathcal{E}_C^D(\lambda) + F(\lambda)) := \{(x,y)$: for some non-negative constant z, $(x-z,y) \in \mathcal{V}(\mathcal{E}_C^D(\lambda) + F(\lambda))\}$, "The set of points vertically above the vertices".

We will write $\mathbb{V}_i(\lambda)$ to mean the ith vertex of the set $\mathcal{E}_C^D(\lambda) + F(\lambda)$ for any $i \in \{1, \ldots, n(\lambda)\}$.

Let $\hat{\mathcal{V}}_i(\mathcal{E}_C^D(\lambda) + F(\lambda)) := \{(x,y)$: for some non-negative constant z, $(x-z,y) = \mathbb{V}_i(\mathcal{E}_C^D(\lambda) + F(\lambda))\}$ for any $i \in \{1, \ldots, n(\lambda)\}$, "The set of points vertically above the ith vertex".

Then

$\mathcal{A}_C^D(\lambda) = \mathcal{E}_C^D(\lambda) \bigcup \{(x,y) \in \mathbb{R}^2$: there exists a $z \in \mathbb{R}$ such that $(x - zS^a(\lambda)\mathbf{1}_{z>0} - zS^b(\lambda)\mathbf{1}_{z<0}, y+z) \in \hat{\mathcal{V}}(\mathcal{E}_C^D(\lambda) + F(\lambda))\}$.

$= \mathcal{E}_C^D(\lambda) \bigcup_{i \in \{1,\ldots,n(\lambda)\}} \{(x,y) \in \mathbb{R}^2$: there exists a $z \in \mathbb{R}$ such that $(x - zS^a(\lambda)\mathbf{1}_{z>0} - zS^b(\lambda)\mathbf{1}_{z<0}, y+z) \in \hat{\mathcal{V}}_i(\mathcal{E}_C^D(\lambda) + F(\lambda))\}$.

Notice that each $\{(x,y) \in \mathbb{R}^2$: there exists a $z \in \mathbb{R}$ such that $(x - zS^a(\lambda)\mathbf{1}_{z>0} - zS^b(\lambda)\mathbf{1}_{z<0}, y+z) \in \hat{\mathcal{V}}_i(\mathcal{E}_C^D(\lambda) + F(\lambda))\}$ is a piecewise linear hull because it is the set of points above the pair of straight lines with gradients S^a and S^b respectively, that meet at the vertex.

We have expressed $\mathcal{A}_C^D(\lambda)$ as a finite union of piecewise linear hulls, it follows that $\mathcal{A}_C^D(\lambda)$ is a piecewise linear hull.

Suppose that $\mathcal{V}(\mathcal{E}_C^D(\lambda) + F(\lambda))$ is empty.

This can only happen if $\mathcal{E}_C^D(\lambda) + F(\lambda)$ is a half plane.

Therefore $\{(x,y) \in \mathbb{R}^2$: there exists a $z \in \mathbb{R}$ such that $(x - zS^a(\lambda)\mathbf{1}_{z>0} - zS^b(\lambda)\mathbf{1}_{z<0}, y+z) \in \mathcal{E}_C^D(\lambda) + F(\lambda)\} = \mathcal{E}_C^D(\lambda) + F(\lambda)$.

This is because if there existed a pair $(x,y) \in \mathbb{R}^2$ that did not belong to $\mathcal{E}_C^D(\lambda) + F(\lambda)$, and a real number $z \in \mathbb{R}$ such that $(x - zS^a(\lambda)\mathbf{1}_{z>0} - zS^b(\lambda)\mathbf{1}_{z<0}, y+z) \in \mathcal{E}_C^D(\lambda) + F(\lambda)$ then since $\mathcal{E}_C^D(\lambda) + F(\lambda)$ is a half plane this would imply that the set $\{(x,y) \in \mathbb{R}^2$: there exists a $z \in \mathbb{R}$ such that $(x - zS(\lambda) - |z|P(\lambda), y+z) \in \mathcal{E}_C^D(\lambda) + F(\lambda)\}$ is the whole of $\mathbb{R}^2$ and thus that there exists proportional cost arbitrage in the model.

Since for any terminal node μ we have $\mathcal{E}_C^D(\mu) = \{(x,y) \in \mathbb{R}^2 : x \geq 0 \text{ and } y \geq 0\}$ which is a piecewise linear hull, it follows by induction that $\mathcal{A}_C^D(\lambda)$ and $\mathcal{E}_C^D(\lambda)$ are piecewise linear hulls, for all $\lambda \in \Lambda$.

This completes the proof.

End of proof of Lemma 15.4

12.7. **Equivalent conditions for self-financing and super-replication.**

Lemma 12.22. *(Equivalent condition for derivative super-replication)*
Consider an unrestricted model.

A portfolio strategy (X,Y) is a frictionless super-replication portfolio strategy (with respect to derivative D) if and only if for any node λ we have:

$A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda)$ *if λ is not a terminal node.*

$A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda) + D^0(\lambda) + D^1(\lambda)S(\lambda)$, $X_{+1}(\lambda) \geq 0$ *and* $Y_{+1}(\lambda) \geq 0$ *if λ is a terminal node.*

A portfolio strategy (X,Y) is a fixed cost super-replication portfolio strategy (with respect to derivative D) if and only if for any node λ we have:

$A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda) + F(\lambda)\boldsymbol{1}_{Y_{+1}(\lambda)-Y(\lambda)\neq 0}$ *if λ is not a terminal node.*

$A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda) + D^0(\lambda) + D^1(\lambda)S(\lambda) + F(\lambda)\boldsymbol{1}_{\{Y_{+1}(\lambda)-Y(\lambda)+D^1(\lambda)\neq 0\}}$, $X_{+1}(\lambda) \geq 0$ *and* $Y_{+1}(\lambda) \geq 0$ *if λ is a terminal node.*

A portfolio strategy (X,Y) is a proportional cost super-replication portfolio strategy (with respect to derivative D) if and only if for any node λ we have:

$A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda) + |Y_{+1}(\lambda) - Y(\lambda)|P(\lambda)$ *if λ is not a terminal node.*

$A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda) + D^0(\lambda) + D^1(\lambda)S(\lambda) + |Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda)|P(\lambda)$, $X_{+1}(\lambda) \geq 0$ *and* $Y_{+1}(\lambda) \geq 0$ *if λ is a terminal node.*

A portfolio strategy (X,Y) is a combined cost super-replication portfolio strategy (with respect to derivative D) if and only if for any node λ we have:

$A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda) + F(\lambda)\boldsymbol{1}_{Y_{+1}(\lambda)-Y(\lambda)\neq 0} + |Y_{+1}(\lambda) - Y(\lambda)|P(\lambda)$ *if λ is not a terminal node.*

$A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda) + D^0(\lambda) + D^1(\lambda)S(\lambda) + F(\lambda)\boldsymbol{1}_{\{Y_{+1}(\lambda)-Y(\lambda)+D^1(\lambda)\neq 0\}} + |Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda)|P(\lambda)$, $X_{+1}(\lambda) \geq 0$ *and* $Y_{+1}(\lambda) \geq 0$ *if λ is a terminal node.*

Proof of Lemma 12.22

Combined costs

Fix a node λ.

If $P(\lambda) = \infty$ then a portfolio strategy (X, Y) is a combined cost super-replication portfolio if and only if:

$X_{+1}(\lambda) = X(\lambda)$ and $Y_{+1}(\lambda) = Y(\lambda)$ if λ is not a terminal node.

$X_{+1}(\lambda) = X(\lambda) - D^0(\lambda)$ and $Y_{+1}(\lambda) = Y(\lambda) - D^1(\lambda)$ if λ is a terminal node.

The result follows immediately.

If $P(\lambda) \neq \infty$ then we consider the following cases:

Case 1 λ is the root node

In this case the derivative super-replication condition is identical to the standard self-financing condition. This equivalence has already been shown in Lemma 6.20.

Case 2 λ is neither a terminal node nor the root node

In this case the derivative super-replication condition is identical to the standard self-financing condition. This equivalence has already been shown in Lemma 6.20.

Case 3 λ is a terminal node

Portfolio strategy (X, Y) is a combined cost super-replication portfolio strategy if and only if at every node $\lambda \in \Lambda$ there is a real number $z \in \mathbb{R}$ such that :

$X_{+1}(\lambda) = X(\lambda) - D^0(\lambda) - (z)S(\lambda) - |z|P(\lambda) - F(\lambda)\mathbf{1}_{\{z\neq 0\}}$,

$Y_{+1}(\lambda) = Y(\lambda) - D^1(\lambda) + z$,

and $X_{+1}(\lambda) \geq 0$, $Y_{+1}(\lambda) \geq 0$.

This is true if and only if:

$X_{+1}(\lambda) = X(\lambda) - D^0(\lambda) - (Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda))S(\lambda) - |Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda)|P(\lambda) - F(\lambda)\mathbf{1}_{\{Y_{+1}(\lambda)-Y(\lambda)+D^1(\lambda)\neq 0\}}$,

$z = Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda)$,

and $X_{+1}(\lambda) \geq 0$ and $Y_{+1}(\lambda) \geq 0$.

We now derive the equality $A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda) + D^0(\lambda) + D^1(\lambda)S(\lambda) + F(\lambda)\mathbf{1}_{\{Y_{+1}(\lambda)-Y(\lambda)+D^1(\lambda)\neq 0\}} + |Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda)|P(\lambda)$

from the definition of $E^{(X,Y)}(\lambda)$

$E^{(X,Y)}(\lambda) := X_{+1}(\lambda) + Y_{+1}(\lambda)S(\lambda)$

$= X(\lambda) - D^0(\lambda) - (Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda))S(\lambda) - |Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda)|P(\lambda) - F(\lambda)\mathbf{1}_{\{Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda) \neq 0\}} + Y_{+1}(\lambda)S(\lambda)$

$= X(\lambda) - D^0(\lambda) - (-Y(\lambda) + D^1(\lambda))S(\lambda) - |Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda)|P(\lambda) - F(\lambda)\mathbf{1}_{\{Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda) \neq 0\}}$

$= X(\lambda) + Y(\lambda)S(\lambda) - D^0(\lambda) - D^1(\lambda)S(\lambda) - |Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda)|P(\lambda) - F(\lambda)\mathbf{1}_{\{Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda) \neq 0\}}$

$= A^{(X,Y)}(\lambda) - D^0(\lambda) - D^1(\lambda)S(\lambda) - |Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda)|P(\lambda) - F(\lambda)\mathbf{1}_{\{Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda) \neq 0\}}$

Rearranging gives:

$A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda) + D^0(\lambda) + D^1(\lambda)S(\lambda) + |Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda)|P(\lambda) + F(\lambda)\mathbf{1}_{\{Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda) \neq 0\}}.$

The proofs for the cases of frictionless, fixed costs and proportional costs are similar.

End of proof of Lemma 12.22

Lemma 12.23. *(Equivalent condition for derivative self-financing)*
Consider an unrestricted model.

A portfolio strategy (X, Y) is frictionless derivative self-financing portfolio strategy (with respect to D with initial price d_0) if and only if for any node λ we have:

$A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda) - d_0$ *if λ is the root node.*

$A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda)$ *if λ is neither the root node nor a terminal node.*

$A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda) + D^0(\lambda) + D^1(\lambda)S(\lambda)$ *if λ is a terminal node.*

A portfolio strategy (X, Y) is fixed cost derivative self-financing portfolio strategy (with respect to D with initial price d_0) if and only if for any node λ we have:

$A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda) - d_0 + F(\lambda)\mathbf{1}_{Y_{+1}(\lambda) - Y(\lambda) \neq 0}$ *if λ is the root node.*

$A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda) + F(\lambda)\mathbf{1}_{Y_{+1}(\lambda) - Y(\lambda) \neq 0}$ *if λ is neither the root node nor a terminal node.*

$A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda) + D^0(\lambda) + D^1(\lambda)S(\lambda) + F(\lambda)\mathbf{1}_{\{Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda) \neq 0\}}$ *if λ is a terminal node.*

A portfolio strategy (X, Y) is proportional cost derivative self-financing portfolio strategy (with respect to D with initial price d_0) if and only if for any node λ we have:

$A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda) - d_0 + |Y_{+1}(\lambda) - Y(\lambda)|P(\lambda)$ *if λ is the root node.*

$A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda) + |Y_{+1}(\lambda) - Y(\lambda)|P(\lambda)$ *if λ is neither the root node nor a terminal node.*

$A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda) + D^0(\lambda) + D^1(\lambda)S(\lambda) + |Y_{+1}(\lambda) - Y(\lambda) + D^1|P(\lambda)$ *if λ is a terminal node.*

A portfolio strategy (X, Y) is combined cost derivative self-financing portfolio strategy (with respect to

D with initial price d_0) if and only if for any node λ we have:

$A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda) - d_0 + F(\lambda)\boldsymbol{1}_{Y_{+1}(\lambda)-Y(\lambda)\neq 0} + |Y_{+1}(\lambda) - Y(\lambda)|P(\lambda)$ *if λ is the root node.*

$A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda) + F(\lambda)\boldsymbol{1}_{Y_{+1}(\lambda)-Y(\lambda)\neq 0} + |Y_{+1}(\lambda) - Y(\lambda)|P(\lambda)$ *if λ is neither the root node nor a terminal node.*

$A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda) + D^0(\lambda) + D^1(\lambda)S(\lambda) + F(\lambda)\boldsymbol{1}_{\{Y_{+1}(\lambda)-Y(\lambda)+D^1(\lambda)\neq 0\}} + |Y_{+1}(\lambda) - Y(\lambda) + D^1|P(\lambda)$ *if λ is a terminal node.*

Proof of Lemma 12.23

Combined costs

Fix a node λ

If $P(\lambda) = \infty$ then a portfolio strategy (X, Y) is a combined cost super-replication portfolio if and only if:

$X_{+1}(\lambda) = X(\lambda) + d_0$ and $Y_{+1}(\lambda) = Y(\lambda)$ if λ is the root node.

$X_{+1}(\lambda) = X(\lambda)$ and $Y_{+1}(\lambda) = Y(\lambda)$ if λ is not a terminal node nor the root node.

$X_{+1}(\lambda) = X(\lambda) - D^0(\lambda)$ and $Y_{+1}(\lambda) = Y(\lambda) - D^1(\lambda)$ if λ is a terminal node.

The result follows immediately.

If $P(\lambda) \neq \infty$ then we consider the following cases:

Case 1 λ is the root node

Portfolio strategy (X, Y) is combined cost self-financing if and only if for some $z \in \mathbb{R}$:

$X_{+1}(\lambda) = X(\lambda) + d_0 - (z)S(\lambda) - |z|P(\lambda) - F(\lambda)\mathbf{1}_{\{z\neq 0\}}$ and

$Y_{+1}(\lambda) = Y(\lambda) + z.$

This is true if and only if:

$X_{+1}(\lambda) = X(\lambda) + d_0 - (Y_{+1}(\lambda) - Y(\lambda))S(\lambda) - |Y_{+1}(\lambda) - Y(\lambda)|P(\lambda) - F(\lambda)\mathbf{1}_{\{Y_{+1}(\lambda)-Y(\lambda)\neq 0\}}$ and

$z = Y_{+1}(\lambda) - Y(\lambda).$

We now derive the result from the definition of $E^{(X,Y)}(\lambda)$

$E^{(X,Y)}(\lambda) := X_{+1}(\lambda) + Y_{+1}(\lambda)S(\lambda)$

$= X(\lambda) + d_0 - (Y_{+1}(\lambda) - Y(\lambda))S(\lambda) - |Y_{+1}(\lambda) - Y(\lambda)|P(\lambda) - F(\lambda)\mathbf{1}_{\{Y_{+1}(\lambda)-Y(\lambda)\neq 0\}} + Y_{+1}(\lambda)S(\lambda)$

$= X(\lambda) + d_0 + Y(\lambda)S(\lambda) - |Y_{+1}(\lambda) - Y(\lambda)|P(\lambda) - F(\lambda)\mathbf{1}_{\{Y_{+1}(\lambda)-Y(\lambda)\neq 0\}}$

$= A^{(X,Y)}(\lambda) + d_0 - |Y_{+1}(\lambda) - Y(\lambda)|P(\lambda) - F(\lambda)\mathbf{1}_{\{Y_{+1}(\lambda)-Y(\lambda)\neq 0\}}$

Rearranging gives:

$A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda) + F(\lambda)\mathbf{1}_{Y_{+1}(\lambda)-Y(\lambda)\neq 0} + |Y_{+1}(\lambda) - Y(\lambda)|P(\lambda) - d_0.$

Case 2 λ is neither a terminal node nor the root node

In this case the derivative self-financing condition is identical to the standard self-financing condition. This equivalence has already been shown in Lemma 6.20.

Case 3 λ is a terminal node

Portfolio strategy (X, Y) is combined cost self-financing if and only if for some $z \in \mathbb{R}$:

$X_{+1}(\lambda) = X(\lambda) - D^0(\lambda) - (z)S(\lambda) - |z|P(\lambda) - F(\lambda)\mathbf{1}_{\{z\neq 0\}}$ and

$Y_{+1}(\lambda) = Y(\lambda) - D^1(\lambda) + z.$

This is true if and only if:

$X_{+1}(\lambda) = X(\lambda) - D^0(\lambda) - (Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda))S(\lambda) - |Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda)|P(\lambda) - F(\lambda)\mathbf{1}_{\{Y_{+1}(\lambda)-Y(\lambda)+D^1(\lambda)\neq 0\}}$

and

$z = Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda).$

We now derive the result from the definition of $E^{(X,Y)}(\lambda)$

$E^{(X,Y)}(\lambda) := X_{+1}(\lambda) + Y_{+1}(\lambda)S(\lambda)$

$= X(\lambda) - D^0(\lambda) - (Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda))S(\lambda) - |Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda)|P(\lambda) - F(\lambda)\mathbf{1}_{\{Y_{+1}(\lambda)-Y(\lambda)+D^1(\lambda)\neq 0\}} +$
$Y_{+1}(\lambda)S(\lambda)$

$= X(\lambda) - D^0(\lambda) - (-Y(\lambda) + D^1(\lambda))S(\lambda) - |Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda)|P(\lambda) - F(\lambda)\mathbf{1}_{\{Y_{+1}(\lambda)-Y(\lambda)+D^1(\lambda)\neq 0\}}$

$= X(\lambda) + Y(\lambda)S(\lambda) - D^0(\lambda) - D^1(\lambda)S(\lambda) - |Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda)|P(\lambda) - F(\lambda)\mathbf{1}_{\{Y_{+1}(\lambda)-Y(\lambda)+D^1(\lambda)\neq 0\}}$

$= A^{(X,Y)}(\lambda) - D^0(\lambda) - D^1(\lambda)S(\lambda) - |Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda)|P(\lambda) - F(\lambda)\mathbf{1}_{\{Y_{+1}(\lambda)-Y(\lambda)+D^1(\lambda)\neq 0\}}$

Rearranging gives:

$A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda) + D^0(\lambda) + D^1(\lambda)S(\lambda) + |Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda)|P(\lambda) + F(\lambda)\mathbf{1}_{\{Y_{+1}(\lambda)-Y(\lambda)+D^1(\lambda)\neq 0\}}.$

The proofs for the cases of frictionless, fixed costs and proportional costs are similar.

End of proof of Lemma 12.23

13. **Extended arbitrage**

In this section we will explore the notion of an extended arbitrage opportunity and show how it can be represented in terms of trading strategies as well as the more standard representation in terms of portfolio strategies.

13.1. **Extended arbitrage opportunities.**

Definition 13.1. (Extended arbitrage opportunity)
A *frictionless extended arbitrage opportunity* on a derivative security D with initial price d_0 consists of a portfolio strategy (X, Y) and integer n (representing the quantity of derivative security D sold for the initial derivative security price d_0), such that each of the following conditions holds:

(1) $(X_0, Y_0) = (0, 0)$.

(2) (X, Y) is frictionless self-financing with respect to derivative nD with initial price nd_0.

(3) For all $\lambda \in \Lambda_T$ we have $X_{+1}(\lambda) \geq 0$ and $Y_{+1}(\lambda) \geq 0$.

(4) There exists at least one $\lambda \in \Lambda_T$ such that $X_{+1}(\lambda) > 0$.

We will sometimes refer to such a portfolio strategy as a *frictionless extended arbitrage opportunity with n derivative securities*. If $n > 0$ we may specify that this is a frictionless extended arbitrage opportunity for the option seller. Similarly if $n < 0$ then we may specify that this is a frictionless extended arbitrage opportunity for the option buyer.

A *fixed cost extended arbitrage opportunity* on a derivative security D with initial price d_0 consists of a portfolio strategy (X, Y) and integer n (representing the quantity of derivative security D sold for the initial derivative security price d_0), such that each of the following conditions holds:

(1) $(X_0, Y_0) = (0, 0)$.

(2) (X, Y) is fixed cost self-financing with respect to derivative nD with initial price nd_0.

(3) For all $\lambda \in \Lambda_T$ we have $X_{+1}(\lambda) \geq 0$ and $Y_{+1}(\lambda) \geq 0$.

(4) There exists at least one $\lambda \in \Lambda_T$ such that $X_{+1}(\lambda) > 0$.

We will sometimes refer to such a portfolio strategy as a *fixed cost extended arbitrage opportunity with n derivative securities*. If $n > 0$ we may specify that this is a fixed cost extended arbitrage opportunity for the option seller. Similarly if $n < 0$ then we may specify that this is a fixed cost extended arbitrage opportunity for the option buyer.

A *proportional cost extended arbitrage opportunity* on a derivative security D with initial price d_0 consists of a portfolio strategy (X, Y) and integer n (representing the quantity of derivative security D sold for the initial derivative security price d_0), such that each of the following conditions holds:

(1) $(X_0, Y_0) = (0, 0)$.

(2) (X, Y) is proportional cost self-financing with respect to derivative nD with initial price nd_0.

(3) For all $\lambda \in \Lambda_T$ we have $X_{+1}(\lambda) \geq 0$ and $Y_{+1}(\lambda) \geq 0$.

(4) There exists at least one $\lambda \in \Lambda_T$ such that $X_{+1}(\lambda) > 0$.

We will sometimes refer to such a portfolio strategy as a *proportional cost extended arbitrage opportunity with n derivative securities.* If $n > 0$ we may specify that this is a proportional cost extended arbitrage opportunity for the option seller. Similarly if $n < 0$ then we may specify that this is a proportional cost extended arbitrage opportunity for the option buyer.

A *combined cost extended arbitrage opportunity* on a derivative security D with initial price d_0 consists of a portfolio strategy (X, Y) and integer n (representing the quantity of derivative security D sold for the initial derivative security price d_0), such that each of the following conditions holds:

(1) $(X_0, Y_0) = (0, 0)$.

(2) (X, Y) is combined cost self-financing with respect to derivative nD with initial price nd_0.

(3) For all $\lambda \in \Lambda_T$ we have $X_{+1}(\lambda) \geq 0$ and $Y_{+1}(\lambda) \geq 0$.

(4) There exists at least one $\lambda \in \Lambda_T$ such that $X_{+1}(\lambda) > 0$.

We will sometimes refer to such a portfolio strategy as a *combined cost extended arbitrage opportunity with n derivative securities.* If $n > 0$ we may specify that this is a combined cost extended arbitrage opportunity for the option seller. Similarly if $n < 0$ then we may specify that this is a combined cost extended arbitrage opportunity for the option buyer.

Convention 9. (Trading strategy arbitrage opportunity)
With a slight abuse of notation we will say that a trading strategy Z and integer n is an extended arbitrage opportunity (with respect to derivative D with initial price d_0) if and only if the portfolio strategy that self-financing corresponds to trading strategy Z and initial portfolio $(0, 0)$ with respect to derivative nD and initial price nd_0 is an extended arbitrage opportunity.

13.2. **Trading strategy as an extended arbitrage opportunity.**

Lemma 13.2. *(Trading strategy as an extended arbitrage opportunity)*
Consider an unrestricted model.

A trading strategy Z and integer n is a frictionless extended arbitrage opportunity (with respect to derivative D with initial price d_0) if and only if:

$$nd_0 + \sum_{t=0}^{T} \Big[-Z_t(\omega) S_t(\omega) \Big] - nD^0(\omega) \geq 0$$

and $\sum_{t=0}^{T}[Z_t(\omega)] - nD^1(\omega) \geq 0$ *for all* $\omega \in \Omega$.

and there exists $\omega' \in \Omega$ *such that:*

$$nd_0 + \sum_{t=0}^{T}\Big[-Z_t(\omega')S_t(\omega')\Big] - nD^0(\omega') > 0$$

A trading strategy Z *and integer* n *is a fixed cost extended arbitrage opportunity (with respect to derivative* D *with initial price* d_0*) if and only if:*

$$nd_0 + \sum_{t=0}^{T}\Big[-Z_t(\omega)S_t(\omega) - F_t(\omega)\boldsymbol{1}_{\{Z_t(\omega)\neq 0\}}\Big] - nD^0(\omega) \geq 0$$

and $\sum_{t=0}^{T}[Z_t(\omega)] - nD^1(\omega) \geq 0$ *for all* $\omega \in \Omega$.

and there exists $\omega' \in \Omega$ *such that:*

$$nd_0 + \sum_{t=0}^{T}\Big[-Z_t(\omega')S_t(\omega') - F_t(\omega')\boldsymbol{1}_{\{Z_t(\omega')\neq 0\}}\Big] - nD^0(\omega') > 0$$

A trading strategy Z *and integer* n *is a proportional cost extended arbitrage opportunity (with respect to derivative* D *with initial price* d_0*) if and only if:*

$$nd_0 + \sum_{t=0}^{T}\Big[-Z_t(\omega)S_t(\omega) - |Z_t(\omega)|P_t(\omega)\Big] - nD^0(\omega) \geq 0$$

and $\sum_{t=0}^{T}[Z_t(\omega)] - nD^1(\omega) \geq 0$ *for all* $\omega \in \Omega$.

and there exists $\omega' \in \Omega$ *such that:*

$$nd_0 + \sum_{t=0}^{T}\Big[-Z_t(\omega')S_t(\omega') - |Z_t(\omega')|P_t(\omega')\Big] - nD^0(\omega') > 0$$

A trading strategy Z *and integer* n *is a combined cost extended arbitrage opportunity (with respect to derivative* D *with initial price* d_0*) if and only if:*

$$nd_0 + \sum_{t=0}^{T}\Big[-Z_t(\omega)S_t(\omega) - F_t(\omega)\boldsymbol{1}_{\{Z_t(\omega)\neq 0\}} - |Z_t(\omega)|P_t(\omega)\Big] - nD^0(\omega) \geq 0$$

and $\sum_{t=0}^{T}[Z_t(\omega)] - nD^1(\omega) \geq 0$ *for all* $\omega \in \Omega$.

and there exists $\omega' \in \Omega$ *such that:*

$$nd_0 + \sum_{t=0}^{T}\Big[-Z_t(\omega')S_t(\omega') - F_t(\omega')\boldsymbol{1}_{\{Z_t(\omega')\neq 0\}} - |Z_t(\omega')|P_t(\omega')\Big] - nD^0(\omega') > 0$$

Proof of Lemma 13.2

Follows from the definition of an extended arbitrage opportunity.

End of proof of Lemma 13.2

14. **Derivative pricing**

In this section our goal is to define what we mean by ask and bid prices of derivatives. As we will see there will be different types of ask and bid price according to whether we consider the availability of the derivative to be restricted or not. The distinction between limited availability and unlimited availability is necessary as the mathematics behind pricing a derivative under limited availability is much simpler and substantially different to pricing a derivative with unlimited availability.

14.1. **Unlimited availability.**

14.1.1. *Viable price (unlimited availability).*

Definition 14.1. (Viable price)
An initial price d_0 is said to be a *frictionless viable price* (with respect to derivative D) if there does not exist a frictionless extended arbitrage opportunity (with respect to derivative D and initial price d_0).

An initial price d_0 is said to be a *fixed cost viable price* (with respect to derivative D) if there does not exist a fixed cost extended arbitrage opportunity (with respect to derivative D and initial price d_0).

An initial price d_0 is said to be a *proportional cost viable price* (with respect to derivative D) if there does not exist a proportional cost extended arbitrage opportunity (with respect to derivative D and initial price d_0).

An initial price d_0 is said to be a *combined cost viable price* (with respect to derivative D) if there does not exist a combined cost extended arbitrage opportunity (with respect to derivative D and initial price d_0).

14.1.2. *Set of viable prices (unlimited availability).*

Definition 14.2. (Notation for set of arbitrage free prices)

$\mathcal{V}(D) := \{d_0 \in \mathbb{R} : d_0 \text{ is a frictionless viable price}\}$

$\mathcal{V}_F(D) := \{d_0 \in \mathbb{R} : d_0 \text{ is a fixed cost viable price }\}$

$\mathcal{V}_P(D) := \{d_0 \in \mathbb{R} : d_0 \text{ is a proportional cost viable price}\}$

$\mathcal{V}_C(D) := \{d_0 \in \mathbb{R} : d_0 \text{ is a combined cost viable price }\}$

14.1.3. *Ask and bid price (unlimited availability).*

Definition 14.3. (Ask price of a derivative security)
The *frictionless ask price* $a(D)$ *of a derivative security* D is the supremum of initial prices d_0 such that there does not exist a frictionless extended arbitrage opportunity.

$a(D) := \sup \mathcal{V}(D)$.

The *fixed cost ask price* $a_F(D)$ *of a derivative security* D is the supremum of initial prices d_0 such that there does not exist a fixed cost extended arbitrage opportunity.

$a_F(D) := \sup \mathcal{V}_F(D)$.

The *proportional cost ask price* $a_P(D)$ *of a derivative security* D is the supremum of initial prices d_0 such that there does not exist a proportional cost extended arbitrage opportunity.

$a_P(D) := \sup \mathcal{V}_P(D)$.

The *combined cost ask price* $a_C(D)$ *of a derivative security* D is the supremum of initial prices d_0 such that there does not exist a combined cost extended arbitrage opportunity.

$a_C(D) := \sup \mathcal{V}_C(D)$.

Definition 14.4. (Bid price of a derivative security)
The *frictionless bid price* $b(D)$ *of a derivative security* D is the infimum of initial prices d_0 such that there does not exist a frictionless extended arbitrage opportunity.

$b(D) := \inf \mathcal{V}(D)$

The *fixed cost bid price* $b_F(D)$ *of a derivative security* D is the infimum of initial prices d_0 such that there does not exist a fixed cost extended arbitrage opportunity.

$b_F(D) := \inf \mathcal{V}_F(D)$

The *proportional cost bid price* $a_P(D)$ *of a derivative security* D is the infimum of initial prices d_0 such that there does not exist a proportional cost extended arbitrage opportunity.

$b_P(D) := \inf \mathcal{V}_P(D)$

The *combined cost bid price* $b_C(D)$ *of a derivative security* D is the infimum of initial prices d_0 such that there does not exist a combined cost extended arbitrage opportunity.

$b_C(D) := \inf \mathcal{V}_C(D)$

14.2. **Limited availability.**

14.2.1. *Viable price (limited availability).*

Definition 14.5. (Viable price with n derivatives available)
An initial price d_0 is said to be a *frictionless viable price with n derivatives available* (with respect to D) if there does not exist a frictionless extended arbitrage opportunity with j derivative securities (with respect to D and d_0) for any $j \in \mathbb{Z}$ such that $-n \leq j \leq n$.

An initial price d_0 is said to be a *fixed cost viable price with n derivatives available* (with respect to D) if there does not exist a fixed cost extended arbitrage opportunity with j derivative securities (with respect to D and d_0) for any $j \in \mathbb{Z}$ such that $-n \leq j \leq n$.

An initial price d_0 is said to be a *proportional cost viable price with n derivatives available* (with respect to D) if there does not exist a proportional cost extended arbitrage opportunity with j derivative securities (with respect to D and d_0) for any $j \in \mathbb{Z}$ such that $-n \leq j \leq n$.

An initial price d_0 is said to be a *combined cost viable price with n derivatives available* (with respect to D) if there does not exist a combined cost extended arbitrage opportunity with j derivative securities (with respect to D and d_0) for any $j \in \mathbb{Z}$ such that $-n \leq j \leq n$.

14.2.2. *Set of viable prices (limited availability).*

Definition 14.6. (Set of arbitrage free prices (limited availability))

Let $n \in \mathbb{N}$.

$\mathcal{V}^n(D) := \{d_0 : d_0 \text{ is a frictionless viable price with } n \text{ derivatives}\}$

$\mathcal{V}^n_F(D) := \{d_0 : d_0 \text{ is a fixed cost viable price with } n \text{ derivatives}\}$

$\mathcal{V}^n_P(D) := \{d_0 : d_0 \text{ is a proportional cost viable price with } n \text{ derivatives }\}$

$\mathcal{V}^n_C(D) := \{d_0 : d_0 \text{ is a combined cost viable price with } n \text{ derivatives}\}$

14.2.3. *Ask and bid price (limited availability).*

Definition 14.7. (Ask price of a derivative security (limited availability))
The *frictionless ask price with $n \in \mathbb{N}$ derivatives available $a^n(D)$ of a derivative security D* is the the supremum of prices d_0 such that there does not exist a frictionless extended arbitrage opportunity with j derivative securities for any $j \in \mathbb{Z}$ such that $-n \leq j \leq n$.

$a^n(D) := \sup \mathcal{V}^n(D)$.

The *fixed cost ask price with $n \in \mathbb{N}$ derivatives available $a^n_F(D)$ of a derivative security D* is the the supremum of prices d_0 such that there does not exist a fixed cost extended arbitrage opportunity with j derivative securities for any $j \in \mathbb{Z}$ such that $-n \leq j \leq n$.

$a^n_F(D) := \sup \mathcal{V}^n_F(D)$.

The *proportional cost ask price with $n \in \mathbb{N}$ derivatives available $a^n_P(D)$ of a derivative security D* is the the supremum of prices d_0 such that there does not exist a proportional cost extended arbitrage opportunity with j derivative securities for any $j \in \mathbb{Z}$ such that $-n \leq j \leq n$.

$a^n_P(D) := \sup \mathcal{V}^n_P(D)$.

The *combined cost ask price with $n \in \mathbb{N}$ derivatives available $a^n_C(D)$ of a derivative security D* is the the supremum of prices d_0 such that there does not exist a combined cost extended arbitrage opportunity with j derivative securities for any $j \in \mathbb{Z}$ such that $-n \leq j \leq n$.

$a^n_C(D) := \sup \mathcal{V}^n_C(D)$.

Definition 14.8. (Bid price of a derivative security (limited availability)
The *frictionless bid price with n derivatives available $b^n(D)$ of a derivative security D* is the the infimum of prices d_0 such that there does not exist a frictionless extended arbitrage opportunity with j derivative securities for any $j \in \mathbb{Z}$ such that $-n \leq j \leq n$.

$b^n(D) := \inf \mathcal{V}^n(D)$.

The *fixed cost bid price with n derivatives available $b^n_F(D)$ of a derivative security D* is the the infimum of prices d_0 such that there does not exist a fixed cost extended arbitrage opportunity with j derivative securities for any $j \in \mathbb{Z}$ such that $-n \leq j \leq n$.

$b^n_F(D) := \inf \mathcal{V}^n_F(D)$.

The *proportional cost bid price with n derivatives available $b^n_P(D)$ of a derivative security D* is the the infimum of prices d_0 such that there does not exist a proportional cost extended arbitrage opportunity with j derivative securities for any $j \in \mathbb{Z}$ such that $-n \leq j \leq n$.

$b^n_P(D) := \inf \mathcal{V}^n_P(D)$.

The *combined cost bid price with n derivatives available $b^n_C(D)$ of a derivative security D* is the the infimum of prices d_0 such that there does not exist a combined cost extended arbitrage opportunity with j derivative securities for any $j \in \mathbb{Z}$ such that $-n \leq j \leq n$.

$b^n_C(D) := \inf \mathcal{V}^n_C(D)$.

15. **Universal properties of ask and bid prices**

As we have already mentioned, our goal is to find representations of ask and bid prices under fixed and proportional transaction costs. However, in our current situation we have a lot of different terms to focus on in order to fully achieve this goal. For example we have ask price, bid price, ask price with one derivative available, ask price with n derivatives available. In this section our main aim will be to narrow our focus by finding relationships between some of these prices so that we can restrict our attention to a subset of them.

15.1. **Relationship between ask price and bid price.**

The following theorem, about the relationship between the ask price of a derivative security D and the bid price of the derivative security $-D$, is a well known result for derivative securities in the frictionless case and under proportional transaction costs, so it is very reassuring to know that it also holds under both fixed transaction costs and combined transaction costs. It is particularly useful because it means that any results about the ask price can immediately produce a corresponding result about the bid price.

Theorem 15.1. *(Relationship between ask price and bid price)*
Let $n \in \mathbb{N}$.

For any derivative security $D = (D^0, D^1)$ *in an unrestricted model* $\mathbb{M}$ *we have the following:*

If $\mathbb{M}$ *is frictionless viable then*

(1) $a(D) = -b(-D)$.

(2) $a_F(D) = -b_F(-D)$.

(3) $a^n(D) = -b^n(-D)$.

(4) $a^n_F(D) = -b^n_F(-D)$.

If $\mathbb{M}$ *is proportional cost viable then*

(5) $a_P(D) = -b_P(-D)$.

(6) $a_C(D) = -b_C(-D)$.

(7) $a^n_P(D) = -b^n_P(-D)$.

(8) $a^n_C(D) = -b^n_C(-D)$.

Where $-D = (-D^0, -D^1)$.

Proof of Theorem 15.1.

(1)

Suppose that for a derivative security D with initial price d_0 there exists a frictionless extended arbitrage opportunity on D consisting of portfolio strategy (X, Y) and integer n'.

From the definition of a frictionless extended arbitrage opportunity we can see that for a derivative security $-D$ with initial price $-d_0$ and payoff portfolio $-D$, the combination of portfolio strategy (X, Y) with integer $-n'$ will be a frictionless extended arbitrage opportunity on $-D$. To see this take a look at the definition of a frictionless extended arbitrage opportunity as well as the frictionless derivative self-financing equivalent condition, Lemma 12.23, and observe that the conditions stated are unchanged by moving from n', d_0 and D to $-n', -d_0$ and $-D$. We still get the following equations:

$A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda) - n'd_0$ if λ is the root node.

$A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda)$ if λ is neither the root node nor a terminal node.

$A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda) + n'D^0(\lambda) + n'D^1(\lambda)S(\lambda)$ if λ is a terminal node.

This means that any element of $\mathbb{R} \setminus \mathcal{V}(D)$ (the set of initial prices such that there exists a frictionless extended arbitrage opportunity on D) is an element of $\mathbb{R} \setminus \mathcal{V}(-D)$ multiplied by -1 and also any element of $\mathbb{R} \setminus \mathcal{V}(-D)$ is an element of $\mathbb{R} \setminus \mathcal{V}(D)$ multiplied by -1.

Therefore the set $\mathbb{R} \setminus \mathcal{V}(D)$ is the set $\mathbb{R} \setminus \mathcal{V}(-D)$ reflected about 0 and we must have $\sup \mathcal{V}(D) = -\inf \mathcal{V}(-D)$ which by definition is equivalent to $a(D) = -b(-D)$.

(2)-(8)

Almost identical to (1).

End of proof of Theorem 15.1.

15.2. **Representation of ask price with n derivatives available.**

Theorem 15.2. *(Representation of ask price with n derivatives available)*
For any derivative security D in an unrestricted model $\mathbb{M}$ and any $n \in \mathbb{N}$ we have the following:

If $\mathbb{M}$ is frictionless viable then

(1) $a^n(D) = \frac{1}{n}a^1(nD)$

(2) $a^n_F(D) = \frac{1}{n}a^1_F(nD)$.

If $\mathbb{M}$ is proportional cost viable then

(3) $a^n_P(D) = \frac{1}{n}a^1_P(nD)$

(4) $a^n_C(D) = \frac{1}{n}a^1_C(nD)$.

Where nD is the derivative security such that $nD = (nD^0, nD^1)$.

Proof of Theorem 15.2.

We prove (4), (1)-(3) are almost identical.

Suppose for a contradiction that $a^n_C(D) > \frac{1}{n}a^1_C(nD)$.

Let initial price d_0 be such that $a^n_C(D) > d_0 > \frac{1}{n}a^1_C(nD)$

This implies that initial price d_0 is such that there exists a combined cost extended arbitrage opportunity with one derivative available with respect to derivative nD and initial price nd_0. Furthermore this extended arbitrage opportunity must be an extended arbitrage opportunity for the seller (See appendix lemma 22.9).

Let this extended arbitrage opportunity be represented by trading strategy Z with one derivative security sold.

By Lemma 13.2, this means that:

$$nd_0 + \sum_{i=0}^{T}\Big[-Z_i(\omega)S_i(\omega) - F_i(\omega)\mathbf{1}_{\{Z_i(\omega)\neq 0\}} - |Z_i(\omega)|P_i(\omega)\Big] - nD^0(\omega) \geq 0$$

and $\sum\limits_{i=0}^{T}[Z_i(\omega)] - nD^1(\omega) \geq 0$ for all $\omega \in \Omega$,

and there exists $\omega' \in \Omega$ such that:

$$nd_0 + \sum_{i=0}^{T}\Big[-Z_i(\omega')S_i(\omega') - F_i(\omega')\mathbf{1}_{\{Z_i(\omega')\neq 0\}} - |Z_i(\omega')|P_i(\omega')\Big] - nD^0(\omega') > 0.$$

By Lemma 13.2, this means that the trading strategy Z combined with selling n derivatives is a combined cost extended arbitrage opportunity with respect to derivative D and initial price d_0.

The existence of a combined cost extended arbitrage opportunity for the seller with respect to D

and d_0 such that $d_0 < a^n_C(D)$ contradicts the fact that $a^n_C(D)$ is the supremum of initial prices for which there does not exist a combined cost extended arbitrage opportunity. Note that Lemma 22.9 is relevant in this deduction.

Therefore $a^n_C(D) \leq \frac{1}{n} a^1_C(nD)$.

Suppose that $a^n_C(D) < \frac{1}{n} a^1_C(nD)$.

Let initial price d_0 be such that $a^n_C(D) < d_0 < \frac{1}{n} a^1_C(nD)$

This implies that initial price d_0 is such that there exists a combined cost extended arbitrage opportunity with n derivatives available with respect to derivative D and initial price d_0. Furthermore this extended arbitrage opportunity must be an extended arbitrage opportunity for the seller (See appendix lemma 22.9).

Let this extended arbitrage opportunity be represented by trading strategy Z with n derivative securities sold (We can assume that there exists an extended arbitrage opportunity with exactly n derivatives sold because of Lemma 22.10).

By lemma 13.2, this means that:

$$nd_0 + \sum_{i=0}^{T} \Big[- Z_i(\omega) S_i(\omega) - F_i(\omega) \mathbf{1}_{\{Z_i(\omega) \neq 0\}} - |Z_i(\omega)| P_i(\omega) \Big] - nD^0(\omega) \geq 0$$

and $\sum_{i=0}^{T} [Z_i(\omega)] - nD^1(\omega) \geq 0$ for all $\omega \in \Omega$.

and there exists $\omega' \in \Omega$ such that:

$$nd_0 + \sum_{i=0}^{T} \Big[- Z_i(\omega') S_i(\omega') - F_i(\omega') \mathbf{1}_{\{Z_i(\omega') \neq 0\}} - |Z_i(\omega')| P_i(\omega') \Big] - nD^0(\omega') > 0$$

By Lemma 13.2, this means that the trading strategy Z combined with selling a single derivative is a combined cost extended arbitrage opportunity with respect to derivative nD and initial price nd_0.

The existence of a combined cost extended arbitrage opportunity for the seller with respect to nD and nd_0 such that $nd_0 < a^n_C(D)$ contradicts the fact that $a^n_C(D)$ is the supremum of initial prices for which there does not exist a combined cost extended arbitrage opportunity. Note that Lemma 22.9 is relevant in this deduction.

Therefore $a^n_C(D) \geq \frac{1}{n} a^1_C(nD)$.

We can conclude that $a^n_C(D) = \frac{1}{n} a^1_C(nD)$.

End of proof of Theorem 15.2.

15.3. **Ask price written as the cheapest super-replication strategy.**

Theorem 15.3. *(Ask price represented as cheapest super-replication strategy)*
Consider an unrestricted model $\mathbb{M}$.

If $\mathbb{M}$ is frictionless viable then:

(1) $a^1(D) = \inf\{X_0 : (X, Y) \in \mathrm{Srep}(D)\}$

and

(2) $a^1_F(D) = \inf\{X_0 : (X, Y) \in \mathrm{Srep}_F(D)\}$.

If $\mathbb{M}$ is proportional cost viable then:

(3) $a^1_P(D) = \inf\{X_0 : (X, Y) \in \mathrm{Srep}_P(D)\}$

and

(4) $a^1_C(D) = \inf\{X_0 : (X, Y) \in \mathrm{Srep}_C(D)\}$.

(See definitions 14.7 and 12.6.)

Proof of Theorem 15.3.

We will prove (4). The proofs of (1),(2) and (3) are almost identical.

Part 1 Show that $a^1_C(D) \leq \inf\{X_0 : (X, Y) \in \mathrm{Srep}_C(D)\}$.

Suppose for a contradiction that $a^1_C(D) > \inf\{X_0 : (X, Y) \in \mathrm{Srep}_C(D)\}$.

This implies that D is combined cost super-replicable in $\mathbb{M}$.

Let $d \in \mathbb{R}$ be such that:

$a^1_C(D) > d > \inf\{X_0 : (X, Y) \in \mathrm{Srep}_C(D)\}$.

This implies that there exists a portfolio strategy (X, Y) that combined cost super-replicates D such that $X_0 < d$.

Let (X', Y') be a portfolio strategy such that $X'(\Omega) = 0$, $X'_{+1}(\lambda) = X_{+1}(\lambda) + d - X_0$ and $Y'_{+1}(\lambda) = Y_{+1}(\lambda)$ for all $\lambda \in \Lambda \setminus \Omega$.

We will show that the portfolio strategy (X', Y') combined with selling n lots of the derivative security is a combined cost extended arbitrage opportunity.

Since the position in stock of this strategy is equal to that of (X, Y) at every time step, and the position in cash is shifted by a constant amount, we only need to check that at time zero $A^{(X',Y')}(\Omega) = E^{(X',Y')}(\Omega) + |Y'_{+1}(\Omega)|P(\Omega) + F(\Omega)\mathbf{1}_{\{Y_{+1}(\Omega)\neq 0\}} - d$ in order to know that portfolio strategy (X', Y') is combined cost self-financing with respect to derivative D with initial price d. See Lemma 12.23.

$$
\begin{aligned}
A^{(X',Y')}(\Omega) &= A^{(X,Y)}(\Omega) - X_0 \\
&= E^{(X,Y)}(\Omega) + |Y'_{+1}(\Omega)|P(\Omega) + F(\Omega)\mathbf{1}_{\{Y_{+1}(\Omega)\neq 0\}} - X_0 \\
&= E^{(X',Y')}(\Omega) + X_0 - d + |Y'_{+1}(\Omega)|P(\Omega) + F(\Omega)\mathbf{1}_{\{Y'_{+1}(\Omega)\neq 0\}} - X_0 \\
&= E^{(X',Y')}(\Omega) + |Y'_{+1}(\Omega)|P(\Omega) + F(\Omega)\mathbf{1}_{\{Y'_{+1}(\Omega)\neq 0\}} - d
\end{aligned}
$$

Therefore (X', Y') is combined cost self-financing with respect to derivative D with initial price d.

We also have that $X'_{+1}(\lambda) = X_{+1}(\lambda) + d - X_0 > 0$ and $Y'_{+1}(\lambda) = Y_{+1}(\lambda) \geq 0$ for all $\lambda \in \Lambda_T$.

We can conclude that (X', Y') combined with selling a single derivative is a combined cost extended arbitrage opportunity with respect to derivative D with initial price d.

This contradicts the fact that $a^1_C(D)$ is the supremum of prices for which there does not exist a combined cost extended arbitrage opportunity.

Therefore we must have

$a^1_C(D) \leq \inf\{X_0 : (X, Y) \in \mathrm{Srep}_C(D)\}$.

Part 2 Show that $a^1_C(D) \geq \inf\{X_0 : (X, Y) \in \mathrm{Srep}_C(D)\}$.

Suppose for a contradiction that $a^1_C(D) < \inf\{X_0 : (X, Y) \in \mathrm{Srep}_C(D)\}$.

Let d be an initial price of the derivative such that:

$a^1_C(D) < d < \inf\{X_0 : (X, Y) \in \mathrm{Srep}_C(D)\}$.

Since d is greater than $a^1_C(D)$ there must exist a combined cost extended arbitrage opportunity with respect to derivative D and initial price d consisting of a portfolio strategy (X, Y) and integer 1 (Lemma 22.9 tells us that the extended arbitrage opportunity must involve selling the derivative) .

From the definition of a combined cost extended arbitrage opportunity with a single derivative security sold we see that (X, Y) must be a combined cost self-financing portfolio strategy with respect to derivative D with initial price d.

This means that (X, Y) is such that $0 \geq X_1 + Y_1 S_0 - d$.

Let (X', Y') be a portfolio strategy such that $X'_0 = d$, $X'_{+1}(\lambda) = X_{+1}(\lambda) + d$ and $Y'_{+1}(\lambda) = Y_{+1}(\lambda)$ for all $\lambda \in \Lambda \setminus \Omega$.

We show that (X', Y') is a super-replication portfolio strategy for D.

$$
\begin{aligned}
A^{(X',Y')}(\Omega) &= A^{(X,Y)}(\Omega) + d \\
&= E^{(X,Y)}(\Omega) + |Y_{+1}(\Omega)|P(\Omega) + F(\Omega)\mathbf{1}_{\{Y_{+1}(\Omega)\neq 0\}} + d \\
&= E^{(X',Y')}(\Omega) + d + |Y'_{+1}(\Omega)|P(\Omega) + F(\Omega)\mathbf{1}_{\{Y'_{+1}(\Omega)\neq 0\}} + d \\
&= E^{(X',Y')}(\Omega) + |Y'_{+1}(\Omega)|P(\Omega) + F(\Omega)\mathbf{1}_{\{Y'_{+1}(\Omega)\neq 0\}}
\end{aligned}
$$

We also have that $X'_{+1}(\lambda) = X_{+1}(\lambda) + d > 0$ and $Y'_{+1}(\lambda) = Y_{+1}(\lambda) \geq 0$ for all $\lambda \in \Lambda_T$.

Thus the portfolio strategy (X', Y') is a combined cost super-replication portfolio strategy for D with initial portfolio $(d, 0)$.

Hence we have found a portfolio strategy (X', Y') that is a combined cost supper replication portfolio strategy for D such that X'_0 is less than $\inf\{X_0 : (X, Y) \in \mathrm{Srep}_C(D)\}$, which gives us a contradiction.

Therefore we cannot have $a^1_C(D) < \inf\{X_0 : (X, Y) \in \mathrm{Srep}_C(D)\}$.

We conclude that $a^1_C(D) = \inf\{X_0 : (X, Y) \in \mathrm{Srep}_C(D)\}$.

End of proof of Theorem 15.3.

15.4. **Realisation of the cheapest super-replication strategy.**

Lemma 15.4. *(Realisation of cheapest super-replication portfolio strategy)*
For any unrestricted model $\mathbb{M}$ and proportional cost super-replicable derivative D we have the following:

If $\mathbb{M}$ is frictionless viable then:

(1) There exists a portfolio strategy $(X', Y') \in \mathrm{Srep}(D)$ such that $X'_0 = \inf\{X_0 : (X,Y) \in \mathrm{Srep}(D)\}$.

(2) There exists a portfolio strategy $(X', Y') \in \mathrm{Srep}_F(D)$ such that $X'_0 = \inf\{X_0 : (X,Y) \in \mathrm{Srep}_F(D)\}$.

If $\mathbb{M}$ is proportional cost viable then:

(3) There exists a portfolio strategy $(X', Y') \in \mathrm{Srep}_P(D)$ such that $X'_0 = \inf\{X_0 : (X,Y) \in \mathrm{Srep}_P(D)\}$.

(4) There exists a portfolio strategy $(X', Y') \in \mathrm{Srep}_C(D)$ such that $X'_0 = \inf\{X_0 : (X,Y) \in \mathrm{Srep}_C(D)\}$.

Proof of Lemma 15.4

We prove (4). The proofs of $(1) - (3)$ are similar.

It is sufficient for us to prove that $\{X_0 : (X,Y) \in \mathrm{Srep}_C(D)\}$ is a non-empty closed set.

Recall the definitions of $\mathcal{A}^D_C$ and $\mathcal{E}^D_C$, Definitions 12.16 and 12.17.

Observe that $\{X_0 : (X,Y) \in \mathrm{Srep}_C(D)\} = \mathcal{A}^D_C(\Omega) \cap \{(x,0) : x \in \mathbb{R}\}$ and is non-empty due to the assumption that D is super-replicable.

We have shown in Lemma 12.21 that the sets $\mathcal{A}^D_C(\lambda)$ is a linear hull for all $\lambda \in \Lambda$, so $\mathcal{A}^D_C(\Omega)$ is a piecewise linear hull.

Since a piecewise linear hull is a closed set, it follows that $\mathcal{A}^D_C(\Omega)$ is a closed set.

Therefore $\{X_0 : (X,Y) \in \mathrm{Srep}_C(D)\} = \mathcal{A}^D_C(\Omega) \cap \{(x,0) : x \in \mathbb{R}\}$ is a closed set.

This completes the proof.

End of proof of Lemma 15.4

15.5. **Conclusion of universal properties of ask and bid prices.**

We started this section by finding a relationship between the ask price and the bid price of derivative securities under both limited and unlimited availability. This means that we can now focus our attention on the ask price of a derivative security knowing that any properties that we deduce can be directly applied to the derivative security bid price using this relationship. Furthermore we have shown that the ask price of a derivative security with $n \in \mathbb{N}$ derivatives available is equal to the ask price of another derivative security with only a single derivative available. This means that under limited availability we can restrict our attention to analysing the situation with only a single derivative available.

We have also shown that the ask price with a single derivative available can be expressed as the cheapest super-replication strategy. This is an extremely important representation of ask price with limited availability as it lends itself to an algorithm for finding the ask price under limited availability.

16. **Derivative pricing in the frictionless setting**

We will now have a very brief look at some frictionless results that will serve as a guide for the results that we will be aiming for in the transaction cost setting.

16.1. **Existence of a frictionless viable price.**

Lemma 16.1. *(Existence of frictionless viable price in a frictionless viable model)*
For any frictionless viable model $\mathbb{M}$ with a derivative security D there exists a frictionless viable price.

Proof of Lemma 16.1.

This is a very well known result, but it is often not explicitly stated. Hence we will simply indicate how it follows from other well known results.

Since the model is frictionless viable it follows from the fundamental theorem that there must exist an EMM $\mathbb{Q}$.

From theorem 16.2 it follows that $d_0 = \mathbb{E}_{\mathbb{Q}}(D^0 + D^1 S_T)$ is a viable price for derivative security D.

End of proof of Lemma 16.1.

16.2. **Representation of ask price.**

Theorem 16.2. *(Frictionless viable price expressed as an expectation)* [5]
In a frictionless viable model $\mathbb{M}$ with derivative security D, an initial price d_0 is a frictionless viable price of D if and only if there exists an EMM $\mathbb{Q}$ such that d_0 can be expressed as:

$$d_0 = \mathbb{E}_{\mathbb{Q}}(D^0 + D^1 S_T).$$

Theorem 16.3. *(Frictionless ask price representation theorem)* [5]
For any frictionless viable model $\mathbb{M}$ with derivative D, we have:

$$a(D) = \sup_{\{\text{EMM } \mathbb{Q}\}} \mathbb{E}_{\mathbb{Q}}(D^0_T + D^1_T S_T)$$

17. **Derivative pricing under fixed costs**

In this section we turn our attention to pricing a derivative under fixed costs. We will be particularly interested in developing analogous results to the lemmas and theorems that we saw in the previous section about frictionless arbitrage. A key point of note in this section is that we have to be very careful about what model we are dealing with, as we will see, it turns out that a model that is fixed cost viable but not frictionless viable is in fact very difficult to deal with indeed.

17.1. **Fixed cost viable price in a fixed cost viable model.**

17.1.1. *Unlimited availability.*

If a model is frictionless viable then we know that any derivative security has at least one frictionless viable price. However an interesting quirk of a model under fixed transaction costs is that it is not necessarily true that if the model is fixed cost viable then a derivative security has a fixed cost viable price. Here is an example of a derivative with no fixed cost viable price.

Example 17.1. (A derivative security with no fixed cost fair price)
Consider the following diagram of a stock process S and derivative security D.

$$
\begin{array}{ccccccccc}
 & & & & 6 & & (1,0) & & \text{Scenario} \quad \omega_1 \\
 & & & \nearrow & & & & & \\
 & & 5 & \rightarrow & 5 & & (1,0) & & \text{Scenario} \quad \omega_2 \\
 & & & & & & & & \\
 & & & & 2 & & (0,0) & & \text{Scenario} \quad \omega_3 \\
 & \nearrow & & \nearrow & & & & & \\
1 & \rightarrow & 1 & \rightarrow & 1 & & (0,0) & & \text{Scenario} \quad \omega_4
\end{array}
$$

We learned from Section 9 that any model $\mathbb{M}$ with such a stock price tree will admit frictionless arbitrage but will not admit fixed cost arbitrage.

We now show the very interesting result that despite the fact that there is no fixed cost arbitrage in the model, no matter what initial price d_0 is chosen for the derivative security $D = (D^0, D^1)$, the model will admit fixed cost extended arbitrage.

The following proof that there exists fixed cost strong arbitrage no matter what d_0 is chosen, will work for any fixed cost process F, but for simplicity we will assume that $F(\lambda) = 1$ for all $\lambda \in \Lambda$.

Our first guess at a viable price for D would be $d_0 = 0$ as this is the expectation of $D^0 + D^1 S_T$ under an AMM $\mathbb{Q}$.

However in this case the creation of a fixed cost extended arbitrage opportunity is very easy, we just buy a single derivative security. This will cost us nothing initially and at time 2 the payoff will be greater than zero in every scenario and in scenario ω_1 or ω_2 the payoff will be positive.

Furthermore if d_0 were less that zero then the same portfolio strategy, of buying a single derivative security, would work even better as we would make a profit in every scenario.

If $d_0 \geq 1$ then we can achieve fixed cost arbitrage by simply selling the derivative.

Suppose that $0 < d_0 < 1$.

We construct a trading strategy Z and integer n that constitutes a fixed cost extended arbitrage opportunity d :

$n = -\lceil \frac{2}{d_0} \rceil$ (The smallest whole number of derivative securities that we have to trade in order to receive at least 2 units of cash at time zero (2 units of cash is the transaction cost of creating and liquidating the following trading strategy).

$Z_0 = 100$ (something as large as we want)

$Z_1 = -100$

$Z_2 = 0$

This is a fixed cost arbitrage opportunity because in scenarios ω_3 and ω_4 the trading strategy will lose 2 units to the fixed transaction cost, but will profit by at least 2 units from the derivative security, giving an overall non-negative profit in scenarios ω_3 and ω_4. In scenarios ω_1 and ω_2 the trading strategy will lose at most 2 units due to the derivative but will gain at least 400 units due to the stock.

We conclude that this derivative security when defined in this model does not have a fixed cost viable price.

Let us consider why this happened. The first thing to understand is that the only reason that we do not have fixed cost arbitrage on the model is that in order to purchase a trading strategy, an investor must pay the transaction cost and this means that if the stock price stays the same the investor will not be able to recover the money lost on the transaction cost, which is what they would need to do in order to have a fixed cost arbitrage opportunity.

One way that a derivative security can help to create arbitrage is by effectively covering the (relatively small) transaction cost by profiting in the situations where the stock price stays the same, while in the situations where the value of the trading strategy goes up, the profits of the portfolio strategy can far out way any losses made on the derivative security.

Indeed the reason why choosing an initial price as the expectation of $D^0 + D^1 S_T$ under a AMM is a good choice in this example is because it forced the scenario in which the stock price stays the same to correspond with the situation in which the position in the derivative security does not make a profit, if we buy a trading strategy as well as a derivative security we will have no way of recovering the lost transaction cost in scenario ω_4.

Nonetheless, in this case, fixed cost extended arbitrage is still not prevented because of the fact that this price allows fixed cost extended arbitrage that involves only the derivative security.

End of Example 17.1

It is also possible for a derivative security in a model that is not frictionless viable to have a fixed cost viable price. Clearly if a derivative security has a constant payoff consisting only of cash, then the derivative security will have a viable price equal to that amount of cash.

Example 17.2.

$$
\begin{array}{ccccccc}
 & & 2 & & (1,0) & \text{Scenario} & \omega_1 \\
 & \nearrow & & & & & \\
1 & \rightarrow & 1 & & (1,0) & \text{Scenario} & \omega_2
\end{array}
$$

In this example the price $d_0 = 1$ is clearly a fixed cost viable price.

End of Example 17.2

We would now like to find out under what conditions there exists a fixed cost viable price.

In example 17.1 the only reason that choosing the initial price to be the expectation of $D^0 + D^1 S_T$ under an AMM didn't work was that if that was the price then there existed and extended fixed cost arbitrage opportunity that involved only the derivative security.

Thus the natural question arises: If choosing d_0 to be the expectation of $D^0 + D^1 S_T$ under an AMM does not allow arbitrage with just the derivative security, is it possible for it to still admit a fixed cost extended arbitrage opportunity? The next example answers this question.

Example 17.3. (Another derivative security with no fixed cost viable price)
Consider the following diagram of a stock process S and derivative security D. We also set $F_0 = F_1 = F_2 = 1$.

$$
\begin{array}{ccccccccc}
 & & & & 6 & & (-1,0) & \text{Scenario} & \omega_1 \\
 & & & \nearrow & & & & & \\
 & & 5 & \rightarrow & 5 & & (4,0) & \text{Scenario} & \omega_2 \\
 & & & & 2 & & (0,0) & \text{Scenario} & \omega_3 \\
 & \nearrow & & \nearrow & & & & & \\
1 & \rightarrow & 1 & \rightarrow & 1 & & (0,0) & \text{Scenario} & \omega_4
\end{array}
$$

We can immediately see that the expectation of $D^0 + D^1 S_T$ under the AMM will be equal to 0.

Consider initial price $d_0 = 0$. We cannot create a fixed cost extended arbitrage opportunity by just buying and selling the cash and derivative security D as there will always be a possibility that we make a loss on the derivative security.

However we can construct a fixed cost extended arbitrage opportunity consisting of trading strategy Z and integer n as follows:

Buy a single derivative security at time 0, i.e. $n = -1$;

$Z_0 = 0$

$Z_1(\omega_1) = Z_1(\omega_2) = 100$

$Z_1(\omega_3) = Z_1(\omega_4) = 0$

$Z_2(\omega_1) = Z_2(\omega_2) = -100$

$Z_2(\omega_3) = Z_2(\omega_4) = 0$

At time 2 we will have zero profit/loss in scenarios ω_3 and ω_4, but we will have a profit of $4 - 2 = 2$ in scenario ω_2 and a profit of $100 - 1 - 2 = 97$ in scenario ω_1.

This shows that even though setting the initial price of the derivative security as the expectation of $(D^0 + D^1 S_T)$ under an AMM did not allow a fixed cost extended arbitrage opportunity with just the derivative there was still a fixed cost extended arbitrage opportunity.

End of Example 17.3

17.1.2. *Limited availability.*

Example 17.4. (Existence of a viable price with limited availability)

Consider the stock price model from Example 17.1 with a slightly different derivative:

$$\begin{array}{cccccccc}
 & & & & 6 & (6,0) & \text{Scenario} & \omega_1 \\
 & & & \nearrow & & & & \\
 & & 5 & \rightarrow & 5 & (6,0) & \text{Scenario} & \omega_2 \\
 & & & & 2 & (0,0) & \text{Scenario} & \omega_3 \\
 & \nearrow & & \nearrow & & & & \\
1 & \rightarrow & 1 & \rightarrow & 1 & (0,0) & \text{Scenario} & \omega_4
\end{array}$$

Once again we will assume that $F(\lambda) = 1$ for all $\lambda \in \Lambda$.

We now consider the set of fixed cost viable prices with only one derivative available $\mathcal{V}_F^1(D)$.

We would hope that due to the restriction on the number of derivatives there would exist a fixed cost viable price with limited availability.

Let us consider some possible prices.

If $d_0 \geq 6$ we can achieve fixed cost arbitrage by just selling the derivative at time zero and not trading any shares of stock.

If $d_0 \leq 0$ we can achieve fixed cost arbitrage by just buying the derivative at time zero and not trading any shares of stock.

If $2 \leq d_0 < 6$ we can achieve fixed cost arbitrage by applying the following trading strategy Z as well as selling a single derivative security:

$Z_0 = 2$

$Z_1 = 0$

$Z_2 = -2$

Suppose that $0 < d_0 < 2$.

It turns out that when the price is in this range there does not exist a fixed cost arbitrage opportunity.

To see this we consider our options for producing arbitrage.

If we buy the derivative then we will make a loss on the derivative in scenario ω_4 and we will have no possible way of recovering this loss as the stock price does not change in scenario ω_4.

If we sell the derivative then we will make a profit on the derivative in scenarios ω_3 and ω_4 but this profit will not be as large as the transaction cost of 2 units if we buy and sell stock along the path ω_4 or ω_3.

Therefore there cannot be a fixed cost extended arbitrage opportunity that sells the derivative and trades at either the root node or the node $\{\omega_3, \omega_4\}$.

Furthermore a portfolio strategy that sells the derivative at a price in this range will make a loss in scenario ω_2. The only possible way that we could possibly create a fixed cost extended arbitrage opportunity is if we were to trade at the node $\{\omega_1, \omega_2\}$ but if this happens then there will definitely be a big overall loss in scenario ω_2.

Therefore there cannot exist a fixed cost extended arbitrage opportunity if the price is in the range $(0, 2)$.

End of Example 17.4

Conclusion

I believe that the introduction of a limit on the availability of a derivative ensures the existence of at least one viable price (although this remains unproven). However these models that are fixed cost viable but are not frictionless viable are anything but "well behaved" and have little practical value. Therefore we will leave this as a place for further research for those whom are interested.

In the rest of our investigations we will work in a frictionless viable model when dealing with fixed transaction costs and a proportional cost viable model when dealing with combined transaction costs.

17.2. **Fixed cost viable price in a frictionless viable model.**

17.2.1. *Unlimited availability.*

An interesting point that highlights a subtlety of fixed transaction costs is that under proportional transaction costs or in the frictionless setting many authors (e.g. see [24]) would use a different definition of ask price, namely:

"$a(D) = \inf\{X_0 : (X, Y) \in \text{Srep}(D)\}$"

and

"$a_P(D) = \inf\{X_0 : (X, Y) \in \text{Srep}_P(D)\}$".

Indeed in such a setting this would be equivalent to defining it as the supremum of initial prices that do not allow extended arbitrage.

However under fixed transaction costs these two ways of defining the fixed cost ask price are not equivalent.

It can be true that

$a_F(D) \neq \inf\{X_0 : (X, Y) \in \text{Srep}_C(D)\}$.

This can be seen in the following example.

Example 17.5. $(a_F(D) \neq \inf\{X_0 : (X, Y) \in \text{Srep}_F(D)\})$
Consider the following diagram of a stock process S and derivative security D.

We will set $F_0 = F_1 = 1$.

$$\begin{array}{ccl} & & 3 \quad (-1,1) \quad \text{Scenario} \quad \omega_1 \\ & \nearrow & \\ 2 & & \\ & \searrow & \\ & & 1 \quad (-1,2) \quad \text{Scenario} \quad \omega_2 \end{array}$$

In this example the fixed cost ask price is $d_0 = 1.5$ (we can calculate this easily because as we will see later in Theorem 17.7 it is equal to the frictionless ask price), but the least cost supper replicating portfolio strategy is the portfolio strategy (X, Y) with $(X_0, Y_0) = (X_1, Y_1) = (3, 0)$ which has an initial value of 3 (we will show this later in Example 17.14). The reason for this difference is that the cheapest portfolio strategy that super-replicates D is potentially more expensive than the cost per derivative security of purchasing a portfolio strategy that super-replicates more that one derivative security.

End of example 17.5

Our next result, Lemma 17.6 is about the existence of a fixed cost viable price. We note this here as many of the theorems further on in this section will assume the existence of a fixed cost viable price. This may seem trivial but as we saw in the previous subsection, the existence of a viable price is not something we can always take for granted.

Theorem 17.6. *(Existence of fixed cost viable price in a frictionless viable model)*
For any frictionless viable model $\mathbb{M}$ with a derivative security D there exists a fixed cost viable price.

Proof of Theorem 17.6.

From Lemma 16.1 we know that there exists a frictionless viable price d_0. Since there is no frictionless extended arbitrage for this price d_0, there cannot be a fixed cost extended arbitrage opportunity for this price either, so the price d_0 is a fixed cost viable price.

End of proof of Theorem 17.6.

The following theorem effectively closes the question of how to compute the fixed cost ask and bid prices of a derivative security with unlimited quantities of a derivative available in a frictionless viable model. It shows that if the price strays away from what is a frictionless viable price then a fixed cost arbitrage opportunity may be constructed by purchasing a frictionless extended arbitrage opportunity in large enough quantities to cancel out the total transaction cost incurred by the portfolio strategy.

Theorem 17.7. *(Equivalence of frictionless price and fixed cost price)*
For any derivative security D in a frictionless viable model $\mathbb{M}$, we have :

$a(D) = a_F(D)$.

Proof of Theorem 17.7.

Any fixed cost extended arbitrage opportunity is also a frictionless extended arbitrage opportunity, so the set

$\mathcal{V}(D) = \{d_0 : \text{there does not exist a frictionless extended arbitrage opportunity}\}$

is contained in the set

$\mathcal{V}_F(D) = \{d_0 : \text{there does not exist a fixed cost extended arbitrage opportunity}\}$

This implies that the supremum of the $\mathcal{V}_F(D)$ must be greater than or equal to the supremum of $\mathcal{V}(D)$ and the infimum of $\mathcal{V}_F(D)$must be less than or equal to the infimum of $\mathcal{V}(D)$.

It follows immediately that $b_F(D) \leq b(D)$ and $a(D) \leq a_F(D)$.

We now show that we cannot have $a(D) < a_F(D)$.

Suppose for a contradiction that $a(D) < a_F(D)$.

This implies that for any derivative security price d_0 such that $a(D) < d_0 < a_F(D)$, there exists a frictionless extended arbitrage opportunity for the derivative seller, but there does not exist a fixed cost extended arbitrage opportunity for the derivative seller. See Lemma 22.9.

Consider initial prices d and d' such that $a(D) < d < d' < a_F(D)$, since they are greater than $a(D)$ there must exist a frictionless extended arbitrage opportunity for the seller if the derivative security price is equal to either of them.

Let a trading strategy Z and integer $n > 0$ (n must be > 0 by Lemma 22.9), be a frictionless extended arbitrage opportunity with respect to derivative D and initial price d.

Therefore by Lemma 13.2, we have:

$$nd + \sum_{t=0}^{T} \Big[- Z_t(\omega) S_t(\omega) \Big] - nD^0(\omega) \geq 0$$

and $\sum_{t=0}^{T} [Z_t(\omega)] - nD^1(\omega) \geq 0$ for all $\omega \in \Omega$.

and there exists $\omega' \in \Omega$ such that:

$$nd + \sum_{t=0}^{T} \Big[- Z_t(\omega') S_t(\omega') \Big] - nD^0(\omega') > 0$$

It follows that if we replace d by d' we get:

$$nd' + \sum_{t=0}^{T} \Big[- Z_t(\omega) S_t(\omega) \Big] - nD^0(\omega) \geq n(d' - d) > 0$$

and $\sum_{t=0}^{T} [Z_t(\omega)] - nD^1(\omega) \geq 0$ for all $\omega \in \Omega$.

Therefore the trading strategy Z and integer n is an frictionless extended arbitrage opportunity with respect to derivative D and initial price d'.

The reason that we have considered both the price d and the price d' is that now we have shown that when the price is d' we have a frictionless extended arbitrage opportunity with a strict inequality on the final position in cash. This means that it can be thought of as a"frictionless extended strong arbitrage opportunity".

In order to construct a fixed cost extended arbitrage opportunity we will effectively "scale up" this portfolio strategy, and because it can now be thought of as a "frictionless strong extended arbitrage opportunity", when the initial option price is d', the transaction cost will be overridden in every scenario.

The profit in each scenario is at least $n(d' - d)$ and the total fixed transaction cost paid in each scenario will be less than $(T + 1)\hat{F}$. (See definition of $\hat{F}$, Definition 5.19.)

Therefore in order to "scale up" this portfolio strategy by a large enough amount we need an integer n' (representing the scale factor) that for which $n' > n$ and such that:

$$n'n(d' - d) \geq (T + 1)\hat{F}$$

Therefore we set $n' = \frac{(T+1)\hat{F}}{n(d'-d)}$ (note that this is strictly positive).

We can deduce that:

$$n'nd' + \sum_{t=0}^{T} \Big[-n'Z_t(\omega)S_t(\omega)\Big] - n'nD^0(\omega) \geq (T+1)\hat{F} > 0$$

and $\sum_{t=0}^{T} [n'Z_t(\omega)] - n'nD^1(\omega) \geq 0$ for all $\omega \in \Omega$.

This implies that:

$$n'nd' + \sum_{t=0}^{T} \Big[-n'Z_t(\omega)S_t(\omega) - F_t(\omega)\mathbf{1}_{Z_t(\omega)\neq 0}\Big] - n'nD^0(\omega) > 0$$

and $\sum_{t=0}^{T} [n'Z_t(\omega)] - n'nD^1(\omega) \geq 0$ for all $\omega \in \Omega$.

(Because $(T+1)\hat{F} > \sum_{t=0}^{T} F_t(\omega)\mathbf{1}_{Z_t(\omega)\neq 0}$)

By Lemma 13.2 the trading strategy $n'Z$ and integer $n'n$ is a fixed cost extended arbitrage opportunity with respect to derivative D with initial price d'. Since $n'n$ is positive, this is an extended arbitrage opportunity for the seller.

This means that for any initial price d_0 such that $a(D) < d_0 < a_F(D)$ there exists a fixed cost arbitrage opportunity for the seller. This is a contradiction since by Lemma 22.9, any initial price less than $a_F(D)$ should be either a fixed cost viable price or should allow only a fixed cost extended arbitrage opportunity for the buyer.

Therefore $a(D) \geq a_F(D)$.

We can conclude that $a(D) = a_F(D)$.

End of proof of Theorem 17.7.

Note that Theorem 17.7 is a special case of Theorem 19.1 which we will see later.
Our goal is now to show a similar result to Theorem 16.2 about a fixed cost viable price. However this proves to be more tricky than one would expect. As we will see the "if" part of the theorem has an analogous version for the fixed cost viable price, but the "only if" part does not.

Theorem 17.8. *(Fixed cost viable price expressed as an expectation)*
In a frictionless viable model $\mathbb{M}$ with derivative security D ,an initial price d_0 of derivative security D is a fixed cost viable price of D if there exists an EMM $\mathbb{Q}$ such that

$$d_0 = \mathbb{E}_{\mathbb{Q}}(D^0 + D^1 S_T).$$

Proof of Theorem 17.8.

From Theorem 16.2 we know that $d_0 = \mathbb{E}_{\mathbb{Q}}(D^0 + D^1 S_T)$ is a frictionless viable price, therefore it is a fixed cost viable price.

End of proof of Theorem 17.8.

Notice that so far we have shown that the fixed cost ask price is equal to the frictionless ask price and the fixed cost bid price is equal to the frictionless bid price, but we have not proved anything about whether or not d_0 is a fixed cost viable price when it is equal to either the fixed cost ask price or fixed cost bid price. We would like to be able to prove that any fixed cost viable price can be expressed as an expectation under an EMM, but unfortunately this is simply false as our next example shows.

Example 17.9.

Consider the following diagram of a stock process S and derivative security D. We will set $F_0 = F_1 = 1$.

$$\begin{array}{ccllll}
 & & 4 & (0,1) & \text{node} & u \\
 & \nearrow & 3 & (1,0) & \text{node} & m \\
 & \nearrow & & & & \\
2 & & & & & \\
 & \searrow & & & & \\
 & & 1 & (0,1) & \text{node} & d
\end{array}$$

Let us first calculate the derivative security frictionless payoffs:

$D^0(u) + D^1(u)S(u) = 0 + (1)(4) = 4.$

$D^0(m) + D^1(m)S(m) = 1 + (0)(3) = 1.$

$D^0(d) + D^1(d)S(d) = 0 + (1)(1) = 1.$

Now let us consider the possible EMMs in this model. By inspection we can see that for any EMM $\mathbb{Q}$, $\mathbb{E}_{\mathbb{Q}}(D^0 + D^1 S_T)$ must be greater than 1 as this is the minimum frictionless payoff.

We can also see that for any $\epsilon > 0$ we can always find a EMM $\mathbb{Q}$ such that $\mathbb{Q}(u)$ is less than ϵ, which means that $\mathbb{E}_{\mathbb{Q}}(D^0 + D^1 S_T)$ can get arbitrarily close to 1 over the set of EMMs.

Thus the frictionless bid price and fixed cost bid price are equal to 1.

If $d_0 = 1$ then clearly we can create a frictionless extended arbitrage opportunity by buying a single derivative security. However we will now prove that we cannot create a fixed cost extended arbitrage opportunity:

Suppose that we try to create a fixed cost arbitrage opportunity without buying or selling any socks (using the derivative only).

If our portfolio strategy involves buying the derivative security then in scenario d we will make a loss because we will only receive a share of stock at time one and since we are not trading we will not be able to convert it into cash. Thus we would make a loss of one unit overall.

If our portfolio strategy involves selling the derivative security then clearly we would make a huge loss in scenario u.

Therefore any fixed cost arbitrage opportunity must involve buying or selling stock.

Suppose that our attempt at a fixed cost arbitrage opportunity involves buying stock at time zero.

Then no matter whether we buy or sell the derivative security we will make a loss on both the stock and the derivative security in scenario d.

Therefore any fixed cost arbitrage opportunity must involve selling stock at time zero.

Suppose that we sell some stock at time zero.

Then if scenario m occurs we will make a loss on the stock, and no matter whether we buy or sell the derivative security, we will gain nothing from the derivative security.

Therefore there cannot be a fixed cost extended arbitrage opportunity involving selling stock.

Thus we can conclude that there does not exist a fixed cost extended arbitrage opportunity when $d_0 = 1$.

Furthermore there does not exist an EMM $\mathbb{Q}$ for which $\mathbb{E}_{\mathbb{Q}}(D^0 + D^1 S_T) = 1$.

In summary we have found a fixed cost viable price that cannot be expressed as the expectation of the frictionless payoff under an EMM $\mathbb{Q}$. Notice that this fixed cost viable price can be expressed as the expectation of the frictionless payoff under an AMM (the AMM $\mathbb{Q}$ with $\mathbb{Q}(u) = 0$).

In our quest to find a result similar to Theorem 16.2, we may now suspect that the expectation of the frictionless payoff under any AMM is a fixed cost viable price. However the next example shows that this is not the case either.

End of Example 17.9

Example 17.10. Consider the following diagram of a stock process S and derivative security D. We will also set $F_0 = F_1 = 1$.

$$
\begin{array}{ccllll}
 & & 4 & (0,1) & \text{Scenario} & u \\
 & \nearrow & 3 & (1,0) & \text{Scenario} & m \\
 & \nearrow & & & & \\
2 & & & & & \\
 & \searrow & & & & \\
 & & 1 & (1,0) & \text{Scenario} & d
\end{array}
$$

The only difference between this and the previous example (Example 17.9) is that now $D(d)$ is equal to $(1,0)$ instead of $(0,1)$.

All of the frictionless payoffs are the same as the previous example and $d_0 = 1$ is still equal to the expectation of the frictionless payoff under an AMM.

Now though we have a fixed cost extended arbitrage opportunity if $d_0 = 1$.

This fixed cost extended arbitrage consists of simply buying one derivative security at time 0. Then at time 1 we will break even in scenarios d and m, but will make a profit in scenario u.

End of Example 17.10

Unfortunately for now, our ideas for a result similar to Theorem 16.2 have been quashed. However there is still scope for determining the conditions under which the expectation of the frictionless payoff under an AMM is a fixed cost viable price, thus potentially giving a result under some additional assumptions.

Theorem 17.11. *(Fixed cost viable price expressed as an expectation under an AMM)*
Let $\mathbb{M}$ be a frictionless viable model with derivative security D defined on $\mathbb{M}$. For any fixed cost viable price d_0, there exists an AMM $\mathbb{Q}$ such that:

$$d_0 = \mathbb{E}_{\mathbb{Q}}(D^0 + D^1 S_T).$$

Proof of Theorem 17.11.

Let the number of possible scenarios be N and let $\Omega = \{\omega_1, \ldots, \omega_N\}$.

Define $Q^A := \{(\mathbb{Q}(w_1), \ldots, \mathbb{Q}(w_N)) : \mathbb{Q} \text{ is an AMM } \}$

and $Q^E := \{(\mathbb{Q}(w_1), \ldots, \mathbb{Q}(w_N)) : \mathbb{Q} \text{ is an EMM } \}$.

Take any constant N-dimensional vector v.

Define $U^A := \{q \cdot v : q \in Q^A\}$ and $U^E := \{q \cdot v : q \in Q^E\}$

The set Q^A is a compact set.

We now show that U^A is also compact (we will show that for every convergent sequence in U^A the limit is also in U^A).

Observe that U^A is bounded.

Take any convergent sequence $(u_n)_{n\in\mathbb{N}}$ in U^A with limit u (i.e $u_n \to u$).

We can write each u_n as $q_n \cdot v$ for some sequence $(q_n)_{n\in\mathbb{N}}$ in Q^A (so we can write $q_n \cdot v \to u$).

Because Q^A is compact we can find a subsequence of $(q_n)_{n\in\mathbb{N}}$ (lets call it $(q_{n_k})_{k\in\mathbb{N}}$ for some increasing sequence of natural numbers n_k with $k \in \mathbb{N}$) with a limit $q \in Q^A$.

Therefore $q_{n_k} \cdot v \to q \cdot v$.

Moreover, $q \cdot v \in U^A$ because $q \in Q^A$.

Since $q_{n_k} \cdot v$ is a subsequence of $q_n \cdot v$ it must have the same limit, and thus $u = q \cdot v \in U^A$.

We now know that the set $U^A = \{q \cdot v : q \in Q^A\}$ is compact.

Set $v = (D^0(\omega_1) + D^1(\omega_1)S_T(\omega_1), \ldots, D^0(\omega_n) + D^1(\omega_n)S_T(\omega_n))$.

Then clearly $(a(D), b(D)) \subset U^E \subset [a(D), b(D)]$, see Theorem 16.2.

Observe that $U^E \subset U^A$. We can now see that since U^A is a closed set and $U^A \supset U^E \supset (a(D), b(D))$, we must have $U^A \supset [a(D), b(D)] = [a_F(D), b_F(D)]$.

Therefore we can conclude that any fixed cost viable price is an element of U^A i.e. it can be expressed as an expectation of the frictionless derivative security payoff under an AMM.

End of proof of Theorem 17.11.

Example 17.12. (False result about expectation under AMM)

Conjecture

Suppose that $\min_{\lambda \in \Lambda_T} (D^0(\lambda) + D^1(\lambda)S_T(\lambda)) < d_0 < \max_{\lambda \in \Lambda_T} (D^0(\lambda) + D^1(\lambda)S_T(\lambda))$
(This means that we cannot achieve arbitrage using only the derivative security and cash).

Then for a given price d_0 the following statements are equivalent:

- There exists an AMM $\mathbb{Q}$ such that $d_0 = \mathbb{E}_{\mathbb{Q}}(D^0 + D^1 S_T)$
- d_0 is a fixed cost viable price.

Counter example to conjecture.

Consider the following model with $F_0 = F_1 = F_2 = 1$:

$$\begin{array}{ccccccc}
 & & & & 5 & (100,0) & \omega_1 \\
 & & & \nearrow & & & \\
 & & 4 & & & & \\
 & \nearrow & & \searrow & & & \\
 & & & & 3 & (100,0) & \omega_2 \\
3 & \to & 3 & \to & 3 & (5,0) & \omega_3 \\
 & & & & 10 & (4,0) & \omega_4 \\
 & \searrow & & \nearrow & & & \\
 & & 2 & & & & \\
 & & & \searrow & & & \\
 & & & & 1 & (100,0) & \omega_5
\end{array}$$

The probability measure $\mathbb{Q}$ with $\mathbb{Q}(\omega_3) = 1$ is obviously an AMM.

The expectation of the frictionless payoff under this AMM is $\mathbb{E}_{\mathbb{Q}}(D^0 + D^1 S_T) = 5$ and $\min_{\lambda \in \Lambda_T}(D^0(\lambda) + D^1(\lambda)S_T(\lambda)) = 4 < 5 < 100 = \max_{\lambda \in \Lambda_T}(D^0(\lambda) + D^1(\lambda)S_T(\lambda))$.

We now show that $d_0 = 5$ is not a viable price.

This is a consequence of the ability to re-balance our position in stock, without this possibility I believe that the result would hold(e.g in 1 step models).

Arbitrage portfolio strategy:

- At time 0 buy a single derivative security.
- At time 1 do nothing at nodes $\{\omega_1, \omega_2\}$ and $\{\omega_3\}$, but if node $\{\omega_4, \omega_5\}$ is reached then buy a single stock.
- At time 2 we liquidate all positions and find that we have profit in all scenarios.

End of Example 17.12.

Theorem 17.13. *(Range of Expectation under AMMs)*
$\{\mathbb{E}_{\mathbb{Q}}(D^0 + D^1 S_T) : \mathbb{Q} \text{ is an AMM}\} = [b(D), a(D)] = [b_F(D), a_F(D)]$

Proof of Theorem 17.13.

From Theorem 17.11 we know that

$[a(D), b(D)] = [a_F(D), b_F(D)] \subset \{\mathbb{E}_{\mathbb{Q}}(D^0 + D^1 S_T) : \mathbb{Q} \text{ is an AMM}\}$

We need to prove that

$\{\mathbb{E}_{\mathbb{Q}}(D^0 + D^1 S_T) : \mathbb{Q} \text{ is an AMM}\} \subset [a(D), b(D)] = [a_F(D), b_F(D)]$

Notice that $[a(D), b(D)] = \{\mathbb{E}_{\mathbb{Q}}(D^0 + D^1 S_T) : \mathbb{Q} \text{ is an EMM}\}$.

Observe that for any AMM $\mathbb{Q}$ and $\epsilon > 0$ it is possible to find an EMM $\mathbb{Q}'$ such that

$|\mathbb{E}_{\mathbb{Q}'}(D^0 + D^1 S_T) - \mathbb{E}_{\mathbb{Q}}(D^0 + D^1 S_T)| < \epsilon$.

It follows that the closure of the set $\{\mathbb{E}_{\mathbb{Q}}(D^0 + D^1 S_T) : \mathbb{Q} \text{ is an AMM}\}$ is contained in the closure of the set $\{\mathbb{E}_{\mathbb{Q}}(D^0 + D^1 S_T) : \mathbb{Q} \text{ is an EMM}\} = [a(D), b(D)]$

End of proof of Theorem 17.13.

17.2.2. *Limited availability.*

In this subsection we will have a brief glance at the situation with fixed costs only with a derivative under limited availability. Although one of our main goals here is to find a risk-neutral representation for the ask price of a derivative with limited availability, the reason why we will not go into great depth now, is that this result is a special case of the combined cost risk-neutral representation theorem that we will see later.

We will now do an example where we compute $a_F^1(D)$.

Although the example we are about to see involves only a single step model and the available quantity of derivatives is one, it displays all the techniques that we would need to be able to price $a_F^k(D)$ in a model with more steps for any $k \in \mathbb{N}$.

Example 17.14. (Computation of $a_F^1(D)$)

Consider the following diagram of a stock process S and derivative security D.

We will set $F_0 = F_1 = 1$.

$$\begin{array}{ccccc} & & 3 & (-1,1) & \text{Node} \quad u \\ & \nearrow & & & \\ 2 & & & & \\ & \searrow & & & \\ & & 1 & (-1,2) & \text{Node} \quad d \end{array}$$

Theorem 15.3 tells us that:

$a_F^1(D) = \inf\{X_0 : (X, Y) \in \mathrm{Srep}_F(D)\}$.

Recall the definitions of $\mathcal{A}_F^D$ and $\mathcal{E}_F^D$, Definitions 12.16 and 12.17.

We will now compute $a_F^1(D)$.

We will do this by first finding the regions $\mathcal{A}_F^D(u)$ and $\mathcal{A}_F^D(d)$ and then working backwards deducing the region $\mathcal{E}_F^D(\Omega)$ before finally deducing the region $\mathcal{A}_F^D(\Omega)$.

Once we have found $\mathcal{A}_F^D(\Omega)$ the ask price $a_F(D)$ will simply be the y coordinate of the lowest point on the y axis that is in the region $\mathcal{A}_F^D(\Omega)$.

Step 1 (Finding $\mathcal{A}_F^D(u)$ and $\mathcal{A}_F^D(d)$)

We start by finding the functions whose epigraphs are the piecewise linear hulls $\mathcal{A}_F^D(u)$ and $\mathcal{A}_F^D(d)$ (Note that the epigraph of a function f is the set $\{(y, x) : y \geq f(x)\}$). This is shown on the graph below:

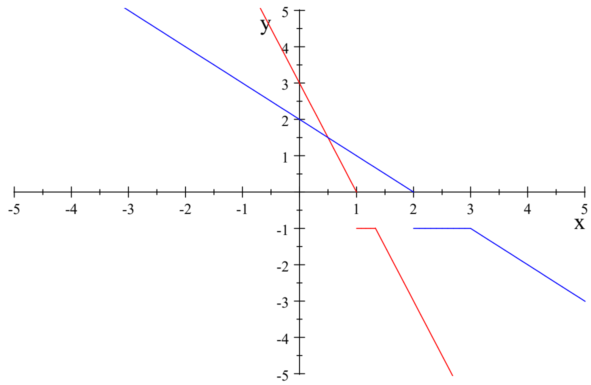


FIGURE 17.1. Functions whose epigraphs are $\mathcal{A}_F^D(u)$ and $\mathcal{A}_F^D(d)$.

Red line:

$$\begin{cases} y = -3x + 3, & \text{if } x < 1. \\ y = -1\,, & \text{if } 1 \leq x \leq \frac{4}{3}. \\ y = -3x + 3\,, & \text{if } \frac{4}{3} < x. \end{cases}$$

this can also be written as:

$$\begin{cases} y = -3x + 3, & \text{if } x < 1. \\ y = \min\{-1, -3x + 3\}\,, & \text{if } 1 \leq x. \end{cases}$$

The red line represents the function whose epigraph is the piecewise linear hull $\mathcal{A}_F^D(u)$.

The equation $y = -3x + 3$ comes from taking a line of gradient -3 (3 being the stock price at node u), intersecting it with the point (-1,1)(the derivative security payoff portfolio) and then raising this line up by one in order to take into account the transaction cost.

Any point above this line represents a portfolio that we can trade in a fixed cost self financing way into a portfolio that dominates (x coord $\geq$ -1, y coord $\geq$ 1) the portfolio (-1,1).

However in order to find the region that includes all portfolios that can be converted in a fixed cost self financing way into a portfolio that dominates the portfolio $(-1, 1)$, we need to also include any point to the right of $(-1, 1)$.

This is why we take the minimum between -1 and $y = -3x + 3$ for $x \geq 1$.

Blue line:

$$\begin{cases} y = -x + 2, & \text{if } x < 2. \\ y = -1\,, & \text{if } 2 \leq x \leq 3. \\ y = -x + 2\,, & \text{if } 3 < x. \end{cases}$$

this can also be written as:

$$\begin{cases} y = -x + 2, & \text{if } x < 2. \\ y = \min\{-1, -x + 2\}\,, & \text{if } 2 \leq x. \end{cases}$$

The blue line represents the function whose epigraph is the piecewise linear hull $\mathcal{A}_F^D(d)$. These equations are deduced by exactly the same reasoning we used to find the boundary of the region $\mathcal{A}_F^D(u)$.

Step 2 (Intersecting $\mathcal{A}_F^D(u)$ and $\mathcal{A}_F^D(d)$)

We have found the sets of portfolios the we can have at each of the terminal nodes that would allow a given portfolio to fixed cost super-replicate the derivative security payoff. We now need to find the set of portfolios that we can exit time zero with , that will allow us to fixed cost super-replicate the derivative security payoff no matter what scenario occurs (i.e. we need to find $\mathcal{E}_F^D(\Omega)$). This set of portfolios is the intersection of the region above the blue line and the region above the red line, so the function whose epigraph is the piecewise linear hull $\mathcal{E}_F^D(\Omega)$, will be the maximum of the above functions. This is shown on the graph below:

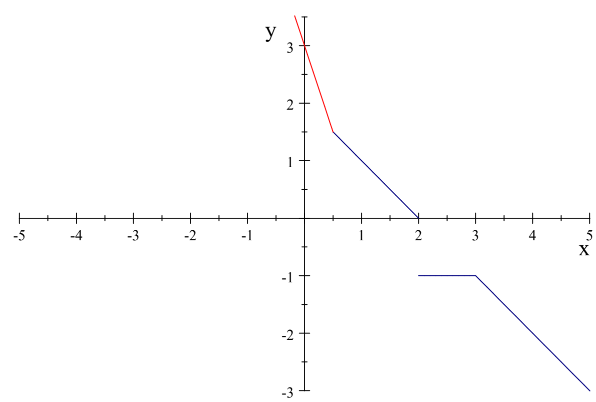


FIGURE 17.2. Function whose epigraph is the piecewise linear hull $\mathcal{E}_F^D(\Omega)$.

Step 3 (Finding the set $\mathcal{A}_F^D(\Omega)$)

If we did not have friction we would now simply draw a line of gradient -2 (because 2 is the stock price at time zero) that just touches the region $\mathcal{E}_F^D(\Omega)$ but does not enter it, and any point above this line would represent a portfolio that we can trade into a portfolio in $\mathcal{E}_F^D(\Omega)$. Thus since doing a fixed cost trade into $\mathcal{E}_F^D(\Omega)$ is equivalent to doing a frictionless trade into the region one unit above $\mathcal{E}_F^D(\Omega)$, we can find the set we are looking for by simply drawing $\mathcal{E}_F^D(\Omega)$ raised by one step and then drawing a line of gradient -2 that just touches but does not enter our new region. First we will draw the above region raised by one unit, this is shown in green in the graph below:

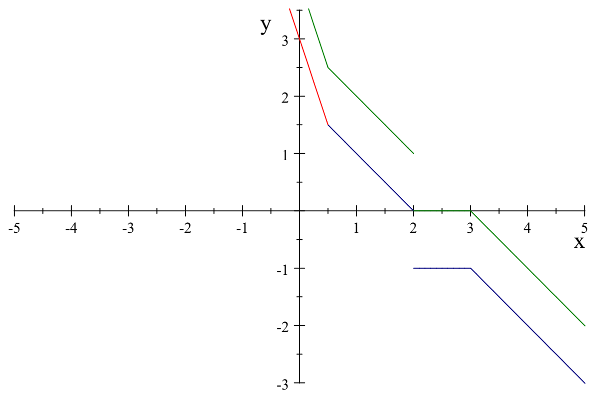


FIGURE 17.3. $\mathcal{E}_F^D(\Omega)$ raised by one unit.

We now draw a line of gradient -2 that just touches this new region. This is the black line in the graph below:

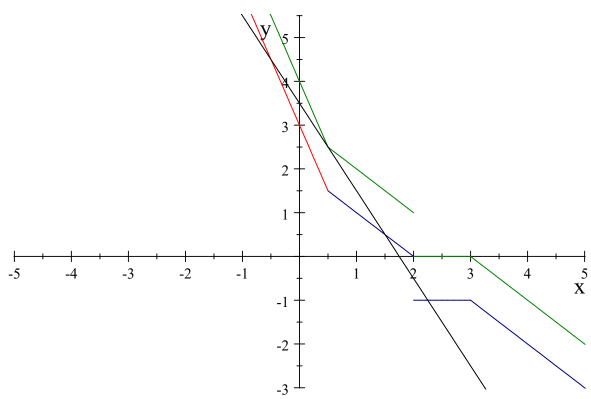


FIGURE 17.4. Re-balancing line drawn.

The region above the black line is the set of portfolios at time 0 that can be traded into the region of portfolios that super-replicate the derivative security payoff no matter what the scenario.

<u>**Step** 4</u> (Intersecting the set of portfolios that we can trade for a portfolio that we can exit time 1 with and the set of portfolios that are already acceptable)

Finally we can find the boundary of the set of all portfolios that can be part of a fixed cost self-financing trading strategy that super-replicates the derivative security payoff by taking the maximum of the blue, red and black lines. This function whose epigraph is the piecewise linear hull $\mathcal{A}_F^D(\Omega)$ is shown below:

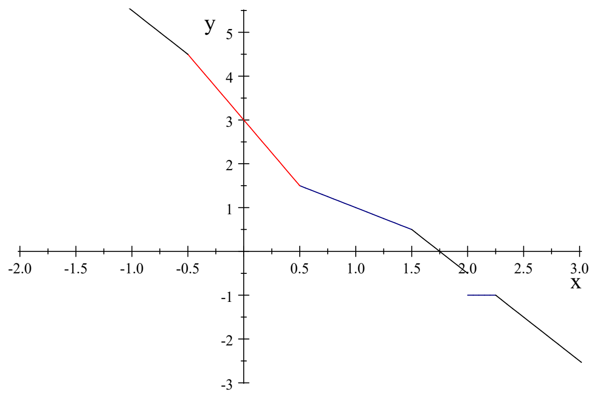


FIGURE 17.5. Function whose epigraph is the piecewise linear hull $\mathcal{A}_F^D(\Omega)$.

Step 5 (Finding the lowest initial value of all portfolios at time zero that can be part of a super-replicating portfolio strategy)

In order to find the portfolio strategy with the cheapest initial value we simply look at the lowest point on the y-axis that is in this region. This is the point (3,0) and the initial value of this portfolio is 3. Therefore $\alpha_1(D) = 3$.

We can also see from the graphs that the portfolio (3,0) is contained in all of the regions $\mathcal{A}_F^D(\Omega), \mathcal{E}_F^D(\Omega), \mathcal{A}_F^D(u)$ and $\mathcal{A}_F^D(\Omega)$, so we can say that the trading strategy (X, Y) defined by $(X_0, Y_0) = (3, 0)$ and $(X_1, Y_1) = (3, 0)$ is a trading strategy that realises optimal super-replication.

End of Example 17.14

The graphs that we have just seen are a good illustration of the nature of pricing a derivative security in a model under fixed transaction costs. There is an interesting point to be made here in relation to a model under proportional transaction costs. A pricing algorithm similar to the one described above can be used to price a derivative security under proportional transaction costs (see [24]), and such an algorithm will produce regions with boundaries that are all convex functions. In that setting one can then consider the convex dual of these convex functions and these dual functions lend themselves easily to making a connection with risk-neutral probabilities. The point to be made here is that an analogous transformation to the dual functions is impossible under fixed transaction costs as the fixed transaction cost causes that the functions that we get to be neither convex nor concave.

18. **Derivative pricing under proportional transaction costs**

In this section our main focus will be on the proportional cost ask price representation theorem (Theorem 18.9). This result was shown by Roux, Tokarz and Zastawniak in [24] and applies to restricted models only. Our aim will be to extend this theorem to unrestricted models, so that we can use this result in our work on the combined cost ask price representation theorem in Section 19. First we must investigate the super-replicability of derivatives, as the introduction of the possibility of an infinite proportional transaction cost can lead to situations where a derivative is not super-replicable.

18.1. **Super-replicability.**

An interesting side effect of the introduction of unrestricted models, is that there can now be situations where it may not be possible to super-replicate a derivative. In our later work it will be crucial that we take this into account and thus we need an equivalent condition for proportional cost super-replicability.

Lemma 18.1. *(Proportional cost super-replicability condition)*
Let $\mathbb{M}$ be an unrestricted proportional cost viable model and let D be a derivative security in $\mathbb{M}$.

D is proportional cost super-replicable if and only if:

For any $\omega \in \Omega$ such that $D^1(\omega) > 0$ there exists a $t \in \mathcal{T}$ such that $P_t(\omega) < \infty$.

Proof of Lemma 18.1

Suppose that there exists $\omega' \in \Omega$ such that $D^1(\omega') > 0$ but $P_t(\omega') = \infty$ for all $t \in \mathcal{T}$.

This means that for any portfolio strategy (X, Y) that proportional cost super-replication corresponds to a trading strategy Z and initial portfolio $(X_0, 0)$, we would have $Y_t(\omega') = 0$ for all $t \in \mathcal{T}$ and $Y_{T+1}(\omega') = -D^1(\omega') < 0$.

Thus since for any proportional cost super-replication portfolio strategy there must exist a trading strategy and initial portfolio that proportional cost super-replication corresponds to it, there cannot exist a proportional cost super-replication portfolio strategy. Hence D is not super-replicable

Suppose that for any $\omega \in \Omega$ such that $D^1(\omega) > 0$ there exists a $t \in \mathcal{T}$ such that $P_t(\omega) < \infty$.

Another way of saying this is that for any node $\mu \in \Lambda_T$ such that $D^1(\mu) > 0$ there exists a node $\lambda \in \Lambda$ that is either an ascendant node to node μ or μ itself, such that $P(\lambda) < \infty$.

Let $\mathcal{D}^+$ be the set of nodes λ such that there exists a node $\mu \in \Lambda_T$ with $D^1(\mu) > 0$ that is either a descendant node to node λ or λ itself.

Let a trading strategy Z be defined as follows:

$Z(\lambda) = \max_{\lambda' \in \Lambda_T} (D^1(\lambda'))$ if $\lambda \in \mathcal{D}^+$ and $P(\lambda) < \infty$.

$Z(\lambda) = 0$ otherwise.

The portfolio strategy that proportional cost corresponds to this trading strategy along with the initial portfolio $(\max_{\lambda \in \Lambda_T} D^0(\lambda) + T \max_{\lambda' \in \Lambda_T} D^1(\lambda') \max_{\lambda \notin \Theta} [S^a(\lambda)], 0)$ is a proportional cost super-replication portfolio strategy.

This completes the proof.

End of proof of Lemma 18.1

18.2. Ask price indifference to availability under proportional costs.

The following result is fairly obvious, but since we have highlighted the need for a distinction between unlimited availability and limited availability for a derivative under fixed costs, it makes sense to double check that the proportional cost setting is unaffected by the availability of the derivative.

Theorem 18.2. *(Proportional cost ask price unaffected by availability)*
Let $n \in \mathbb{N}$

In a proportional cost viable unrestricted model $\mathbb{M}$ *with derivative security* D *we have:*

$a_P(D) = a_P^1(D)$.

Proof of Theorem 18.2.

Clearly if there exists a proportional cost extended arbitrage opportunity with 1 derivative available then this extended arbitrage opportunity is also an extended arbitrage opportunity with unlimited availability.

Suppose that there exists a trading strategy Z and integer n that is a proportional cost extended arbitrage opportunity, where n is such that $|n| > 1$.

Observe that the trading strategy $Z' = \frac{1}{|n|} Z$ and integer $\frac{n}{|n|}$ is a proportional cost extended arbitrage opportunity.

It follows that $\mathcal{V}_P(D) = \mathcal{V}_P^1(D)$ and thus that $a_P(D) = a_P^1(D)$.

End of proof of Theorem 18.2.

18.3. **Bounded number of trades of super-replication strategies.**

The results in this section are purposefully designed for use in the proofs of the next section where we extend the proportional cost ask price representation theorem to unrestricted models. Whilst they have very limited value in their own right, they still highlight some useful restrictions on super-replication portfolio strategies.

18.3.1. *Bounding $A^{(X,Y)}$ and $E^{(X,Y)}$ in the frictionless setting.*

Lemma 18.3. *(Minimum possible $A^{(X,Y)}(\lambda)$ and $E^{(X,Y)}(\lambda)$ for D)*

Let D be a derivative security in a frictionless viable unrestricted model $\mathbb{M}$.

Then the constant $c \in \mathbb{R}$ defined by

$c := \min_{\lambda \in \Lambda_T} \{D^0(\lambda) + D^1(\lambda)S_T(\lambda)\} - 1$ *is such that*

(1) *For every portfolio strategy $(X,Y) \in \mathrm{Srep}(D)$ and node $\lambda \in \Lambda$ we have $A^{(X,Y)}(\lambda) > c$.*

(2) *For every portfolio strategy $(X,Y) \in \mathrm{Srep}(D)$ and non-terminal node $\lambda \in \Lambda$ we have $E^{(X,Y)}(\lambda) > c$.*

Proof of Lemma 18.3.

Part 1

We show that (1) is true.

Suppose for a contradiction that (1) is false.

Let portfolio strategy (X,Y) and node λ be such that $A^{(X,Y)}(\lambda) \leq c$ and $(X,Y) \in \mathrm{Srep}(D)$.

Case 1: $\lambda \notin \Lambda_T$ (λ is not a terminal node)

Since $(X,Y) \in \mathrm{Srep}(D)$, by Lemma 12.22 we have:

$A_T^{(X,Y)}(\mu) = E_T^{(X,Y)}(\mu) + D^0(\mu) + D^1(\mu)S_T(\mu)$ with $X_{+1}(\mu) \geq 0$ and $Y_{+1}(\mu) \geq 0$ for all $\mu \in \Lambda_T$.

This implies that $A_T^{(X,Y)}(\mu) \geq D^0(\mu) + D^1(\mu)S_T(\mu)$ for all $\mu \in \Lambda_T$

Therefore we have:

$A^{(X,Y)}(\lambda) \leq c = \min_{\mu' \in \Lambda_T} \{D^0(\mu') + D^1(\mu')S_T(\mu')\} - 1 < A_T^{(X,Y)}(\mu)$ for all $\mu \in \Lambda_T$.

This implies that there exists frictionless arbitrage in the model, contradicting the assumption that the model was frictionless viable (to see this apply Lemma 22.3 to the sub-model at node λ).

Case 2: $\lambda \in \Lambda_T$ (λ is a terminal node)

It is immediately obvious from the equivalent condition of frictionless super-replication (Lemma 12.22) that if $A_T^{(X,Y)}(\lambda) \leq c$ then (X, Y) does not super-replicate D.

We can conclude that since both case 1 and case 2 lead to a contradiction, (1) must be true.

Part 2

We show that (2) is true.

For any frictionless self-financing portfolio strategy (X, Y) we have that for any non-terminal node λ , there exists a successor node μ to the node λ, such that:

$E_t^{(X,Y)}(\lambda) \geq A_{t+1}^{(X,Y)}(\mu)$ because otherwise there would exist a frictionless arbitrage opportunity (to see this observe that $E_t^{(X,Y)}(\lambda) = A_t^{(X,Y)}(\lambda)$ due to the self-financing condition and take a look at Lemma 22.1).

Therefore, combining this with part 1, we have:

$c < A_{t+1}^{(X,Y)}(\mu) \leq E_t^{(X,Y)}(\lambda)$ for any portfolio strategy $(X, Y) \in \mathrm{Srep}(D)$ and non-terminal node λ.

End of proof of Lemma 18.3.

Lemma 18.4. *($E^{(X,Y)}$ and $A^{(X,Y)}$ bounded above at every node for a given X_0)*

Let D be a derivative security in a frictionless viable unrestricted model $\mathbb{M}$.

Let $x_0 \in \mathbb{R}$.

Then:

There exists a constant $c \in \mathbb{R}$ such that for any portfolio strategy $(X,Y) \in \operatorname{Srep}(D)$ with $X_0 = x_0$ we have:

$E^{(X,Y)}(\lambda) < c$ and $A^{(X,Y)}(\lambda) < c$

for all $\lambda \in \Lambda$.

Proof of Lemma 18.4.

We first use induction to prove that there exists a constant $c \in \mathbb{R}$ such that for any $(X,Y) \in \operatorname{Srep}(D)$ with $X_0 = x_0$, we have $A^{(X,Y)}(\lambda) < c$ for all $\lambda \in \Lambda$.

Induction hypothesis $\mathbb{H}_n$ $(n \in \mathcal{T})$

There exists a constant $c \in \mathbb{R}$ such that for any portfolio strategy $(X,Y) \in \operatorname{Srep}(D)$ with $X_0 = x_0$ we have:

$A^{(X,Y)}(\lambda) < c$

for all $\lambda \in \bigcup\limits_{i=0}^{n} \Lambda_i$.

We first show that $\mathbb{H}_0$ is true.

Set $c := x + 1$.

It follows immediately that :

$A^{(X,Y)}(\Omega) = X_0 < c = X_0 + 1.$

Therefore property (1) of $\mathbb{H}_0$ holds.

We now show that if $\mathbb{H}_n$ is true for some $n \in \mathcal{T}^{-T}$ then $\mathbb{H}_{n+1}$ must also be true.

Let $n \in \mathcal{T}^{-T}$.

Let c_n be a constant satisfying $\mathbb{H}_n$.

Due to the frictionless self-financing condition this also gives us $A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda) < c_n$ for any node $\lambda \in \bigcup_{t\in\{0,\ldots,n\}} \Lambda_t$ and portfolio strategy $(X,Y) \in \text{Srep}(D)$ with $X_0 = x_0$.

We will construct a constant c_{n+1} that satisfies $\mathbb{H}_{n+1}$.

In order to do this we will first construct a constant c^{λ}_{n+1} for each node $\lambda \in \Lambda_n$ such that for any successor node $\mu \in \text{Succ } \lambda$ we have:

$A^{(X,Y)}(\mu) < c^{\lambda}_{n+1}$ for all $(X,Y) \in \text{Srep}(D)$ with $X_0 = x_0$

Consider any node $\lambda \in \Lambda_n$ and portfolio strategy $(X,Y) \in \text{Srep}(D)$.

Case 1 $\max_{\mu\in\text{Succ}\lambda} S(\mu) = \min_{\mu\in\text{Succ}\lambda} S(\mu)$

This implies that $\max_{\mu\in\text{Succ}\lambda} S(\mu) = \min_{\mu\in\text{Succ}\lambda} S(\mu) = S(\lambda)$, or else we would have a clear case of frictionless arbitrage, see Theorem 9.5.

Since $S(\mu) = S(\lambda)$ for all $\mu \in \text{Succ } \lambda$, we must also have $E^{(X,Y)}(\lambda) = A^{(X,Y)}(\mu)$ for all $\mu \in \text{Succ } \lambda$.

In view of this, we set $c^{\lambda}_{n+1} = c_n$.

This means that for any node $\lambda \in \Lambda_n$ that is such that $\max_{\mu\in\text{Succ}\lambda} S(\mu) = \min_{\mu\in\text{Succ}\lambda} S(\mu)$ we have:

$A^{(X,Y)}(\mu) = E^{(X,Y)}(\lambda) = A^{(X,Y)}(\lambda) < c^{\lambda}_{n+1}$ for all successor nodes μ to λ.

Case 2 $\max_{\mu\in\text{Succ}\lambda} S(\mu) > \min_{\mu\in\text{Succ}\lambda} S(\mu)$

Suppose that $Y_{+1}(\lambda) = 0$.

It is immediate that $A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda) = A^{(X,Y)}(\mu)$ for all $\mu \in \text{Succ}\lambda$.

This implies that $A^{(X,Y)}(\mu) \le c_n$.

Suppose (X,Y) is such that $Y_{+1}(\lambda) > 0$.

Then:

$$\max_{\mu\in\text{Succ}\lambda}\{A^{(X,Y)}(\mu)\} - E^{(X,Y)}(\lambda) = Y_{+1}(\lambda)[\max_{\mu\in\text{Succ}\lambda}\{S(\mu)\} - S(\lambda)]$$

and

$$\min_{\mu\in\text{Succ}\lambda}\{A^{(X,Y)}(\mu)\} - E^{(X,Y)}(\lambda) = Y_{+1}(\lambda)[\min_{\mu\in\text{Succ}\lambda}\{S(\mu)\} - S(\lambda)].$$

This implies that:

$$\max_{\mu\in\text{Succ}\lambda}\{A^{(X,Y)}(\mu)\} - E^{(X,Y)}(\lambda) = \frac{\max_{\mu\in\text{Succ}\lambda}\{S(\mu)\}-S(\lambda)}{\min_{\mu\in\text{Succ}\lambda}\{S(\mu)\}-S(\lambda)}\Big[\min_{\mu\in\text{Succ}\lambda}\{A^{(X,Y)}(\mu)\} - E^{(X,Y)}(\lambda)\Big].$$

Adding $E^{(X,Y)}(\lambda)$ to both sides gives us:

$$\max_{\mu\in\mathrm{Succ}\lambda}\{A^{(X,Y)}(\mu)\} = \frac{\max\limits_{\mu\in\mathrm{Succ}\lambda}\{S(\mu)\}-S(\lambda)}{\min\limits_{\mu\in\mathrm{Succ}\lambda}\{S(\mu)\}-S(\lambda)}\Big[\min_{\mu\in\mathrm{Succ}\lambda}\{A^{(X,Y)}(\mu)\} - E^{(X,Y)}(\lambda)\Big] + E^{(X,Y)}(\lambda)$$

Observe that $\frac{\max\limits_{\mu\in\mathrm{Succ}\lambda}\{S(\mu)\}-S(\lambda)}{\min\limits_{\mu\in\mathrm{Succ}\lambda}\{S(\mu)\}-S(\lambda)}$ is negative because if it were not then there would exist frictionless arbitrage in the single-step sub-model at node λ. It follows that we can replace $\min\limits_{\mu\in\mathrm{Succ}\lambda}\{A^{(X,Y)}(\mu)\}$ by a lower bound for $A^{(X,Y)}$ and get:

$$\max_{\mu\in\mathrm{Succ}\lambda}\{A^{(X,Y)}(\mu)\} = \frac{\max\limits_{\mu\in\mathrm{Succ}\lambda}\{S(\mu)\}-S(\lambda)}{\min\limits_{\mu\in\mathrm{Succ}\lambda}\{S(\mu)\}-S(\lambda)}\Big[\tilde{c} - E^{(X,Y)}(\lambda)\Big] + E^{(X,Y)}(\lambda),$$

where $\tilde{c}$ is the lower bound for $A^{(X,Y)}$ referred to in Lemma 18.3.

By using the induction hypothesis $\mathbb{H}_n$ combined with the frictionless self-financing condition we can also now replace $E^{(X,Y)}(\lambda)$ by an upper bound for $E^{(X,Y)}$, c_n, and get:

$$\max_{\mu\in\mathrm{Succ}\lambda}\{A^{(X,Y)}(\mu)\} \leq \frac{\max\limits_{\mu\in\mathrm{Succ}\lambda}\{S(\mu)\}-S(\lambda)}{\min\limits_{\mu\in\mathrm{Succ}\lambda}\{S(\mu)\}-S(\lambda)}\Big[\tilde{c} - c_n\Big] + c_n.$$

Also note that $\min\limits_{\mu\in\mathrm{Succ}\lambda}\{S(\mu)\} - S(\lambda)$ must be less than zero because otherwise the single-step sub-model at node λ would be of type 3 or 4 (see Definition 9.3) and there would exist frictionless arbitrage in the model.

Suppose that (X,Y) is such that $Y_{+1}(\lambda) < 0$.

In a similar fashion to the case when $Y_{+1}(\lambda) > 0$ we can show that:

$$\max_{\mu\in\mathrm{Succ}\lambda}\{A^{(X,Y)}(\mu)\} \leq \frac{\min\limits_{\mu\in\mathrm{Succ}\lambda}\{S(\mu)\}-S(\lambda)}{\max\limits_{\mu\in\mathrm{Succ}\lambda}\{S(\mu)\}-S(\lambda)}\Big[\tilde{c} - c_n\Big] + c_n.$$

Therefore if λ is such that $\max\limits_{\mu\in\mathrm{Succ}\lambda} S(\mu) > \min\limits_{\mu\in\mathrm{Succ}\lambda} S(\mu)$ then we define:

$$c^{\lambda}_{n+1} := \max\left\{c_n, \frac{\max\limits_{\mu\in\mathrm{Succ}\lambda}\{S(\mu)\}-S(\lambda)}{\min\limits_{\mu\in\mathrm{Succ}\lambda}\{S(\mu)\}-S(\lambda)}\Big[\tilde{c} - c_n\Big] + c_n, \frac{\min\limits_{\mu\in\mathrm{Succ}\lambda}\{S(\mu)\}-S(\lambda)}{\max\limits_{\mu\in\mathrm{Succ}\lambda}\{S(\mu)\}-S(\lambda)}\Big[\tilde{c} - c_n\Big] + c_n\right\}.$$

This means that for any node $\lambda \in \Lambda_n$ that is such that $\max\limits_{\mu\in\mathrm{Succ}\lambda} S(\mu) > \min\limits_{\mu\in\mathrm{Succ}\lambda} S(\mu)$ we have:

$A^{(X,Y)}(\mu) < c^{\lambda}_{n+1}$ for all successor nodes μ to λ.

We can now define:

$$c_{n+1} := \max\left\{c_n, \max_{\lambda\in\Lambda_n}\{c^{\lambda}_{n+1}\}\right\}.$$

From the way we have constructed c_{n+1} we know that:

$A^{(X,Y)}(\lambda) < c_{n+1}$ for any node λ belonging to $\bigcup\limits_{t \in \{0,\ldots,n+1\}} \Lambda_t$.

We have constructed a constant c_{n+1} that satisfies the requirements of $\mathbb{H}_{n+1}$.

Therefore $\mathbb{H}_{n+1}$ is true.

We have shown that $\mathbb{H}_0$ is true.

We have also shown that if $\mathbb{H}_n$ is true for some $n \in \mathcal{T}^{-T}$ then $\mathbb{H}_{n+1}$ must also be true.

Therefore, by induction, we can conclude that $\mathbb{H}_n$ is true for any $n \in \mathcal{T}$.

Let c be a constant such that for any node $\lambda \in \Lambda$ and portfolio strategy $(X, Y) \in \text{Srep}(D)$ with $X_0 = x_0$, we have $A^{(X,Y)}(\lambda) < c$.

We now show that the exit values of a portfolio strategy are also bounded above by a constant.

Consider a node $\lambda \in \Lambda$.

If $\lambda \in \Lambda \setminus \Lambda_T$ then it is immediate from the frictionless self-financing condition that the exit value of portfolio strategy (X, Y) at node λ is bounded above by c.

If $\lambda \in \Lambda_T$ then the frictionless super-replication condition gives us $A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda) + D^0(\lambda) + D^1(\lambda)S(\lambda)$, see Lemma 12.22.

Therefore $E^{(X,Y)}(\lambda) < c - \min\limits_{\lambda' \in \Lambda_T} \{D^0(\lambda') + D^1(\lambda')S(\lambda')\}$.

Finally we define:

$\hat{c} := c + |\min\limits_{\lambda' \in \Lambda_T} \{D^0(\lambda') + D^1(\lambda')S(\lambda')\}|$, so that $A^{(X,Y)}(\lambda) < \hat{c}$ and $E^{(X,Y)}(\lambda) < \hat{c}$ for any node $\lambda \in \Lambda$ and strategy (X, Y) with $X_0 = x_0$.

This completes the proof.

Note that from the induction step we can see that a full construction of this constant $\hat{c}$ would involve only the stock process S and the derivative security D.

End of proof of Lemma 18.4.

18.3.2. *Upper bounds on trading under proportional costs.*

Lemma 18.5. *(Upper bound on proportional cost trading in a frictionless viable model)*

Let $\mathbb{M}$ be a frictionless viable unrestricted model.

Let $\lambda \in \Lambda$.

Let $P(\lambda) > 0$.

Let $x_0 \in \mathbb{R}$.

Then there exists a constant $c \in \mathbb{R}$ such that for any portfolio strategy $(X, Y) \in \mathrm{Srep}_P(D)$ with $X_0 = x_0$ we have :

$|Z^{(X,Y)}(\lambda)| < c.$

(To see the definition of $Z^{(X,Y)}(\lambda)$ see Definition 12.14)

Proof of Lemma 18.5.

The number of shares of stock traded by a super-replication portfolio strategy (X, Y) at node λ, $Z^{(X,Y)}(\lambda)$ is equal to $Y_{+1}(\lambda) - Y(\lambda)$ if λ is not a terminal node . If λ is a terminal node then $Z^{(X,Y)}(\lambda)$ is equal to $Y_{+1}(\lambda) - Y(\lambda) + D^1_T(\lambda)$.

If $P(\lambda) = \infty$ then we have that $Z^{(X,Y)}(\lambda) = 0$ and thus the result follows immediately.

Suppose that $P(\lambda) \neq \infty$

Case 1 $\lambda \notin \Lambda_T$

Since the stock must be traded in a proportional cost self-financing way, we have:

$A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda) + P(\lambda)|Y_{+1}(\lambda) - Y(\lambda)|$ (see Lemma 6.20).

This implies that $|Y_{+1}(\lambda) - Y(\lambda)| = \frac{A^{(X,Y)}(\lambda) - E^{(X,Y)}(\lambda)}{P(\lambda)} \leq \frac{\hat{c}-\tilde{c}}{P(\lambda)}$,

where $\hat{c}$ is the constant referred to in Lemma 18.4 and $\tilde{c}$ is the constant referred to in Lemma 18.3

Therefore we define:

$c := \frac{\hat{c}-\tilde{c}}{P(\lambda)}$.

By construction we have that $|Z^{(X,Y)}(\lambda)| = |Y_{+1}(\lambda) - Y(\lambda)| < c$.

Case 2 $\lambda \in \Lambda_T$

Since (X, Y) proportional cost super-replicates D, we have:

$A^{(X,Y)}(\lambda) = E^{(X,Y)}(\lambda) + D^0(\lambda) + D^1(\lambda)S(\lambda) + P(\lambda)|Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda)|$ (See Lemma 12.22).

This implies that $|Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda)| = \frac{A^{(X,Y)}(\lambda) - E^{(X,Y)}(\lambda) - (D^0(\lambda) + D^1(\lambda)S(\lambda))}{P(\lambda)}$
$\leq \frac{\hat{c} - \tilde{c} - (D^0(\lambda) + D^1(\lambda)S(\lambda))}{P(\lambda)}$,

where $\hat{c}$ is the constant referred to in Lemma 18.4 and $\tilde{c}$ is the constant referred to in Lemma 18.3.

Therefore we define:

$c = \frac{\hat{c} - \tilde{c} - (D^0(\lambda) + D^1(\lambda)S(\lambda))}{P(\lambda)}$.

By construction we have that $|Z^{(X,Y)}(\lambda)| = |Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda)| < c$.

End of proof of Lemma 18.5.

Lemma 18.6. *(Upper bound on proportional cost sales at a node in a proportional cost viable model)*

Let $\mathbb{M} = (\Omega, \mathcal{F}, \mathbb{P}, S, F, P)$ be a proportional cost viable unrestricted model.

Let $\lambda \in \Lambda$.

Let $x_0 \in \mathbb{R}$.

Let $\tilde{S}$ be a frictionless middle value process. (see Definition 10.3)

Suppose that $S^b(\lambda) < \tilde{S}(\lambda)$.

Then there exists a constant $c \in \mathbb{R}$ such that for any portfolio strategy $(X, Y) \in \mathrm{Srep}_P(D)$ with $X_0 = x_0$ we have:

$-Z^{(X,Y)}(\lambda) < c$.

Proof of Lemma 18.6.

Let $\tilde{\mathbb{M}} = (\Omega, \mathcal{F}, \mathbb{P}, \tilde{S}, F, \tilde{P})$ be a model such that:

$\tilde{S}^b(\lambda) = S^b(\lambda)$ and

$\tilde{P}(\lambda') = 0$ for all $\lambda' \neq \lambda$, $\lambda' \in \Lambda$

Note that because of the fact that $S^b(\lambda) < \tilde{S}(\lambda)$ we have that $\tilde{P}(\lambda) > 0$.

Case 1 $\lambda \notin \Lambda_T$

The model $\tilde{\mathbb{M}}$ and node λ satisfy the conditions of Lemma 18.5, so there exists a constant c such that $|Y_{+1}(\lambda) - Y(\lambda)| < c$, for any $(X, Y) \in \mathrm{Srep}^{\tilde{\mathbb{M}}}(D)$ with $X_0 = x_0$.

For any portfolio strategy $(X, Y) \in \mathrm{Srep}_P^{\mathbb{M}}(D)$ with initial value X_0 and number of trades at node λ, $Y_{+1}(\lambda) - Y(\lambda) < 0$ it is possible to find a portfolio strategy $(\tilde{X}, \tilde{Y}) \in \mathrm{Srep}_P^{\tilde{\mathbb{M}}}(D)$ with initial value

$\tilde{X}_0 = X_0$ and number of trades at node λ, $\tilde{Y}_{+1}(\lambda) - \tilde{Y}(\lambda) = Y_{+1}(\lambda) - Y(\lambda) < 0$ because the bid-ask spreads of $\tilde{\mathbb{M}}$ are contained in the bid-ask spreads of $\mathbb{M}$ at every node, except λ where the bid prices are identical. See Lemma 22.8.

It follows that for every $(X,Y) \in \mathrm{Srep}^{\mathbb{M}}(D)$ with $X_0 = x_0$, the constant c is such that $-(Y_{+1}(\lambda) - Y(\lambda)) < c$

Case 2 $\lambda \in \Lambda_T$

The model $\tilde{\mathbb{M}}$ and node λ satisfy the conditions of Lemma 18.5, so there exists a constant c such that $|Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda)| < c$.

For any portfolio strategy $(X,Y) \in \mathrm{Srep}_P^{\mathbb{M}}(D)$ with initial value X_0 and number of trades at node λ, $Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda) < 0$ it is possible to find a portfolio strategy $(\tilde{X}, \tilde{Y}) \in \mathrm{Srep}_P^{\tilde{\mathbb{M}}}(D)$ with initial value $\tilde{X}_0 = X_0$ and number of trades at node λ, $\tilde{Y}_{+1}(\lambda) - \tilde{Y}(\lambda) + D^1(\lambda) = Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda) < 0$ because the bid-ask spreads of $\tilde{\mathbb{M}}$ are contained in the bid-ask spreads of $\mathbb{M}$ at every node, except λ where the bid prices are identical. See Lemma 22.8.

It follows that for every $(X,Y) \in \mathrm{Srep}^{\mathbb{M}}(D)$ with $X_0 = x_0$, the constant c is such that $-(Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda)) < c$.

End of proof of Lemma 18.6.

Lemma 18.7. *(Upper bound on number of proportional cost buys at a node in a proportional cost viable model)*

Let $\mathbb{M} = (\Omega, \mathcal{F}, \mathbb{P}, S, F, P)$ *be a proportional cost viable unrestricted model.*

Let $\lambda \in \Lambda$.

Let $x_0 \in \mathbb{R}$.

Let $\tilde{S}$ *be a frictionless middle value process. (see Definition 10.3)*

Suppose that $S_a(\lambda) > \tilde{S}(\lambda)$.

Then there exists a constant $c \in \mathbb{R}$ *such that for any portfolio strategy* $(X,Y) \in \mathrm{Srep}_P(D)$ *with* $X_0 = x_0$ *we have:*

$Z^{(X,Y)}(\lambda) < c$.

Proof of Lemma 18.7.

Let $\tilde{\mathbb{M}} = (\Omega, \mathcal{F}, \mathbb{P}, \tilde{S}, F, \tilde{P})$ be a model such that:

$\tilde{S}^a(\lambda) = S^a(\lambda)$ and

$\tilde{P}(\lambda') = 0$ for $\lambda' \neq \lambda$, $\lambda' \in \Lambda$

Note that because of the fact that $S^a(\lambda) > \tilde{S}(\lambda)$ we have that $\tilde{P}(\lambda) > 0$.

Case 1 $\lambda \notin \Lambda_T$

The model $\tilde{\mathbb{M}}$ and node λ satisfy the conditions of Lemma 18.5, so there exists a constant c such that $|Y_{+1}(\lambda) - Y(\lambda)| < c$ for any $(X, Y) \in \mathrm{Srep}_P^{\tilde{\mathbb{M}}}(D)$ with $X_0 = x_0$.

For any portfolio strategy $(X, Y) \in \mathrm{Srep}_P^{\mathbb{M}}(D)$ with initial value X_0 and number of trades at node λ, $Y_{+1}(\lambda) - Y(\lambda) > 0$ it is possible to find a portfolio strategy $(\tilde{X}, \tilde{Y}) \in \mathrm{Srep}_P^{\tilde{\mathbb{M}}}(D)$ with initial value $\tilde{X}_0 = X_0$ and number of trades at node λ, $\tilde{Y}_{+1}(\lambda) - \tilde{Y}(\lambda) = Y_{+1}(\lambda) - Y(\lambda) > 0\}$ because the bid-ask spreads of $\tilde{\mathbb{M}}$ are contained in the bid-ask spreads of $\mathbb{M}$ at every node, except λ where the bid prices are identical. See Lemma 22.8.

It follows that for every $(X, Y) \in \mathrm{Srep}_P^{\mathbb{M}}(D)$, the constant c is such that $(Y_{+1}(\lambda) - Y(\lambda)) < c$.

Case 2 $\lambda \in \Lambda_T$

The model $\tilde{\mathbb{M}}$ and node λ satisfy the conditions of Lemma 18.5, so there exists a constant c such that $|Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda)| < c$ for any $(X, Y) \in \mathrm{Srep}_P^{\tilde{\mathbb{M}}}(D)$ with $X_0 = x_0$.

For any portfolio strategy $(X, Y) \in \mathrm{Srep}_P^{\mathbb{M}}(D)$ with initial value X_0 and number of trades at node λ, $Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda) > 0$ it is possible to find a portfolio strategy $(\tilde{X}, \tilde{Y}) \in \mathrm{Srep}_P^{\tilde{\mathbb{M}}}(D)$ with initial value $\tilde{X}_0 = X_0$ and number of trades at node λ, $\tilde{Y}_{+1}(\lambda) - \tilde{Y}(\lambda) + D^1(\lambda) = Y_{+1}(\lambda) - Y(\lambda) + D^1(\lambda) > 0\}$ because the bid-ask spreads of $\tilde{\mathbb{M}}$ are contained in the bid-ask spreads of $\mathbb{M}$ at every node, except λ where the bid prices are identical. See Lemma 22.8.

It follows that for any $(X, Y) \in \mathrm{Srep}_P^{\tilde{\mathbb{M}}}(D)$ with $X_0 = x_0$, the constant c is such that $-(Y_{+1}(\lambda) - Y(\lambda) + D^1) < c$.

End of proof of Lemma 18.7.

Lemma 18.8. *(Upper bound on proportional cost trading in a proportional cost viable model)*
Let $\mathbb{M}$ be a proportional cost viable unrestricted model.

Let $\Theta \subset \Lambda$.

Let $x_0 \in \mathbb{R}$

Suppose that $\mathbb{M}$ is such that:

there exists a frictionless middle value process $\tilde{S}$ for which $S^b(\lambda) < \tilde{S}(\lambda) < S^a(\lambda)$ for all $\lambda \in \Theta$.

Then there exists a constant c such that for any node $\lambda \in \Theta$ and portfolio strategy $(X, Y) \in \mathrm{Srep}_P(D)$

with $X_0 = x_0$*, we have*

$|Z^{(X,Y)}(\lambda)| < c.$

Proof of Lemma 18.8.

From Lemma 18.6 we know that for any node $\lambda \in \Theta$ there exists a constant $\breve{c}^{\lambda} \in \mathbb{R}$ such that for any portfolio strategy $(X,Y) \in \mathrm{Srep}_P(D)$ with $X_0 = x_0$ we have:

$-Z^{(X,Y)}(\lambda) < \breve{c}^{\lambda}.$

We also know form Lemma 18.7 that for any node $\lambda \in \Theta$ there exists a constant $\hat{c}^{\lambda} \in \mathbb{R}$ such that for any portfolio strategy $(X,Y) \in \mathrm{Srep}_P(D)$ with $X_0 = x_0$ we have:

$Z^{(X,Y)}(\lambda) < \hat{c}^{\lambda}.$

Therefore for any node $\lambda \in \Theta$ and portfolio strategy $(X,Y) \in \mathrm{Srep}_p(D)$ we have:

$|Z^{(X,Y)}(\lambda)| < \max\{|\hat{c}^{\lambda}|, |\breve{c}^{\lambda}|\}.$

Set $c^{\lambda} = \max\{\hat{c}^{\lambda}, \breve{c}^{\lambda}\}$

Then for any node $\lambda \in \Theta$ and portfolio strategy $(X,Y) \in \mathrm{Srep}_P(D)$ with $X_0 = x_0$, we have:

$|Z^{(X,Y)}(\lambda)| < c^{\lambda}.$

Finally we define $c =: \max\limits_{\lambda \in \Theta} c^{\lambda}$ and we have:

For any node $\lambda \in \Theta$ and portfolio strategy $(X,Y) \in \mathrm{Srep}_P(D)$ with $X_0 = x_0$:

$|Z^{(X,Y)}(\lambda)| < c.$

End of proof of Lemma 18.8.

18.4. **Extended proportional cost ask price representation theorem.**

Our goal in this section is to extend the proportional cost ask price representation theorem to include unrestricted models. We will do this in three steps. First we will extend the proportional cost ask price representation theorem to include the possibility that $S(\lambda) = P(\lambda)$ for some node $\lambda \in \Lambda$. We will then extend this extended version to include the possibility that $S(\lambda) < P(\lambda) < \infty$ for some node $\lambda \in \Lambda$. Finally we will make another extension to include the possibility that $P(\lambda) = \infty$ for some node $\lambda \in \Lambda$.

18.4.1. *The existing theorem.*

Theorem 18.9. *(Proportional cost ask price representation theorem)* [24]

For any proportional cost viable model $\mathbb{M}$ such that $P(\lambda) < S(\lambda)$ for all $\lambda \in \Lambda$, the proportional cost ask price of a derivative security D is given by:

$$a_P(D) = \sup_{(Q,\tilde{S})\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1\tilde{S}_T).$$

(For definition of $\mathcal{P}$ see Definition 5.34)

18.4.2. *Extending the theorem to include bid prices equal to zero.*

Theorem 18.10. *(Theorem 18.9 extended to include $P = S$)*

For any proportional cost viable model $\mathbb{M}$ with $P \leq S$ for all $\lambda \in \Lambda$, the proportional cost ask price of a derivative security D is given by:

$$a_P(D) = \sup_{(Q,\tilde{S})\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1\tilde{S}_T).$$

Proof of Theorem 18.10.

Let $\mathbb{M} = (\Omega, \mathcal{F}, \mathbb{P}, S, F, P)$ be a proportional cost viable model such that $S^b(\lambda) = 0$ if and only if $\lambda \in \Theta$ for some arbitrary non-empty collection of nodes $\Theta \subset \Lambda$.

Let $\tilde{S}$ be a frictionless middle value process with respect to model $\mathbb{M}$.

We know that such a process $\tilde{S}$ exists because of Theorem 10.22.

We will say that $\epsilon > 0$ is sufficiently small if and only if $\epsilon < \min_{\lambda\in\Lambda} \tilde{S}(\lambda)$.

For any sufficiently small $\epsilon > 0$ we will define the following.

Let $\mathbb{M}^\epsilon = (\Omega, \mathcal{F}, \mathbb{P}, S^\epsilon, F, P^\epsilon)$ be a model such that:

If $\lambda \in \Theta$ then

$S^{\epsilon b}(\lambda) = \epsilon$

and

$S^{\epsilon a}(\lambda) = S^a(\lambda)$ (here $S^{\epsilon a}$ and $S^{\epsilon b}$ refer to the stock ask and bid price process in model $\mathbb{M}^\epsilon$.)

If $\lambda \notin \Theta$ then

$S^{\epsilon b}(\lambda) = S^b(\lambda)$

and

$S^{\epsilon a}(\lambda) = S^a(\lambda)$

Note that ϵ being sufficiently small implies that the model $\mathbb{M}^\epsilon$ will not admit proportional cost arbitrage (because $\tilde{S}$ is a frictionless middle value process with respect to $\mathbb{M}^\epsilon$).

Let $i := \inf\{X_0 : (X, Y) \in \mathrm{Srep}_P^{\mathbb{M}}(D)\}$ (proportional cost ask price of D in model $\mathbb{M}$).

Let $i^\epsilon := \inf\{X_0 : (X, Y) \in \mathrm{Srep}_P^{\mathbb{M}^\epsilon}(D)\}$ (proportional cost ask price of D in model $\mathbb{M}^\epsilon$).

Let $\mathcal{P}^\epsilon$ be defined as the set of pairs $(\mathbb{Q}, \tilde{S})$ such that $\mathbb{Q}$ is equivalent to $\mathbb{P}$, $\tilde{S}$ is a strictly positive martingale under $\mathbb{Q}$ and $\tilde{S}(\lambda) \in [S^{\epsilon b}(\lambda), S^{\epsilon a}(\lambda)]$ for all $\lambda \in \Lambda$.

We have created the framework that will allow us to complete this proof.

We will now separate the proof into three parts:

Part 1: Show that $\lim\limits_{\epsilon \to 0} i^\epsilon = i$.

Part 2: Show that $\lim\limits_{\epsilon \to 0} \sup\limits_{(Q,\tilde{S}) \in \mathcal{P}^\epsilon} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1 \tilde{S}_T) = \sup\limits_{(Q,\tilde{S}) \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1 \tilde{S}_T)$.

part 3: Conclusion.

<u>Part 1</u>

Show that $\lim\limits_{\epsilon \to 0} i^\epsilon = i$.

Observe that $[S^{\epsilon b}(\lambda), S^{\epsilon a}(\lambda)] \subset [S^b(\lambda), S^a(\lambda)]$ for all $\lambda \in \Lambda$ and sufficiently small $\epsilon > 0$.

Therefore for any sufficiently small $\epsilon > 0$ and portfolio strategy (X, Y) belonging to $\mathrm{Srep}_P^{\mathbb{M}}(D)$ we can find a portfolio strategy (X', Y') belonging to $\mathrm{Srep}_P^{\mathbb{M}^\epsilon}(D)$ with the same initial portfolio as (X, Y) such that $X'_{+1}(\lambda) \geq X_{+1}(\lambda)$ for all $\lambda \in \Lambda_T$ and $Y'_{+1}(\lambda) \geq Y_{+1}(\lambda)$ for all $\lambda \in \Lambda_T$.

This implies that $i^\epsilon \leq i$ for any sufficiently small $\epsilon > 0$.

Taking the limit as epsilon tends to zero gives us $\lim\limits_{\epsilon \to 0} i^\epsilon \leq i$.

Note that the limit $\lim\limits_{\epsilon \to 0} i^\epsilon$ exists because i^ϵ does not decrease as ϵ increases and it is bounded above by i.

We now aim to show that $\lim\limits_{\epsilon \to 0} i^\epsilon \geq i$.

In order to do this we will use our previous work on bounding the number of trades of super-replication strategies at certain nodes.

Let $f^\epsilon : \mathbb{R} \mapsto \mathbb{R}$ be defined as follows:

$f^\epsilon(z) := \inf\{c \in \mathbb{R} : -Z^{(X,Y)}(\lambda) < c \quad \forall (X,Y) \in \mathrm{Srep}_P^{M^\epsilon}, \lambda \in \Theta : X_0 = z\}$ for all $z \in \mathbb{R}$.

$f^\epsilon(z)$ can be thought of as the smallest upper bound on the number of trades of any strategy belonging to $\mathrm{Srep}_P^{M^\epsilon}$ with initial value z.

Let us take a moment to check that $f^\epsilon(z) < \infty$ for all $z \in \mathbb{R}$.

From Lemma 18.6 we know that for any node $\lambda \in \Theta$ there exists a constant $c^\lambda \in \mathbb{R}$ such that for any portfolio strategy $(X,Y) \in \mathrm{Srep}_P^{\mathbb{M}^\epsilon}(D)$ with $X_0 = z$ we have that $-Z^{(X,Y)}(\lambda) < c^\lambda$.

Therefore we can define $c := \max\limits_{\lambda \in \Theta} c^\lambda$ and we have that:

For any portfolio strategy $(X,Y) \in \mathrm{Srep}_P^{\mathbb{M}^\epsilon}(D)$ with $X_0 = z$ we have $-Z^{(X,Y)}(\lambda) < c$.

Therefore $f^\epsilon(z) < c (< \infty)$.

• Claim 1 : f^ϵ is a non-decreasing function for all sufficiently small epsilon.

Take any $z' \in \mathbb{R}$ and $z \in \mathbb{R}$ with $z' > z$.

Consider a strategy $(X,Y) \in \mathrm{Srep}_P^{\mathbb{M}^\epsilon}$.

Observe that the strategy that proportional cost super-replication corresponding to initial portfolio $(z', 0)$ and trading strategy $Z^{(X,Y)}$ also belongs to $(X,Y) \in \mathrm{Srep}_P^{\mathbb{M}^\epsilon}$.

It follows that for any sufficiently small ϵ we have $f^\epsilon(z') \geq f^\epsilon(z)$.

• Claim 2 : For any $z \in \mathbb{R}$, $f^\epsilon(z)$ does not increase as ϵ decreases.

Observe that for any sufficiently small $\tilde{\epsilon}$ and ϵ with $\tilde{\epsilon} > \epsilon$, we have $f^{\tilde{\epsilon}}(z) \leq f^\epsilon(z)$ because the bid-ask spreads of $\mathbb{M}^{\tilde{\epsilon}}$ are contained in the bid-ask spreads of $\mathbb{M}^\epsilon$. To see this, remember that f^ϵ and $f^{\tilde{\epsilon}}$ are the smallest upper bounds on the number of trades in $\mathbb{M}^\epsilon$ and $\mathbb{M}^{\tilde{\epsilon}}$ respectively, and note that for any portfolio strategy belonging to $\mathrm{Srep}_P^{\mathbb{M}^\epsilon}$ there exists a portfolio strategy belonging to $\mathrm{Srep}_P^{\mathbb{M}^{\tilde{\epsilon}}}$ that has the same initial value and the same proportional cost super-replication corresponding trading strategy.

We have now proved Claim 1 and Claim 2, and thus we are in a position to use the functions f^ϵ for sufficiently small ϵ.

From lemma 15.4 we know that for each ϵ there exists a portfolio strategy realising the initial value i^ϵ. Let us call such a portfolio strategy $(X^\epsilon, Y^\epsilon) \in \mathrm{Srep}_P^{\mathbb{M}^\epsilon}(D)$ and let Z^ϵ be the trading strategy that proportional cost super-replication corresponds to (X^ϵ, Y^ϵ) in model $\mathbb{M}^\epsilon$. Note that the initial portfolio of (X^ϵ, Y^ϵ) is $(i^\epsilon, 0)$.

For each portfolio strategy (X^ϵ, Y^ϵ) we can construct a portfolio strategy $(\bar{X}^\epsilon, \bar{Y}^\epsilon)$ that proportional cost super-replication corresponds to trading strategy Z^ϵ and initial portfolio $(i^\epsilon + f^\epsilon(i)T\epsilon, 0)$ in model

$\mathbb{M}$.

We check that $(\bar{X}^\epsilon, \bar{Y}^\epsilon)$ is a super replication portfolio strategy in $\mathbb{M}$, i.e. that $\bar{X}^\epsilon_{T+1}$ and $\bar{Y}^\epsilon_{T+1}$ are non-negative.

Note that since both portfolio strategies are generated by the same trading strategy and the only difference between $\mathbb{M}$ and $\mathbb{M}^\epsilon$ is the difference in bid price at nodes belonging to Θ. The only difference between the two portfolio strategies is the difference in their initial values and the difference in the cost in selling at a node belonging to Θ.

In what follows we will use the following notation:

For any real number z we will write z^- to mean $\max\{-z, 0\}$.

$\bar{X}^\epsilon_{T+1}(\omega) - X^\epsilon_{T+1}(\omega) = i^\epsilon + T\epsilon f^\epsilon(i) - i^\epsilon - \sum_{t=0}^{T} \epsilon(Z^\epsilon_t(\omega)^-)\mathbf{1}_{\lambda^\omega_t \in \Theta}$ (see Lemma 12.11)

$= T\epsilon f^\epsilon(i) - \sum_{t=0}^{T} \epsilon(Z^{(X^\epsilon, Y^\epsilon)}_t(\omega)^-)\mathbf{1}_{\lambda^\omega_t \in \Theta}$

$\geq T\epsilon f^\epsilon(i) - \sum_{t=0}^{T} \epsilon f^\epsilon(i^\epsilon)\mathbf{1}_{\lambda^\omega_t \in \Theta}$

$\geq T\epsilon f^\epsilon(i) - \sum_{t=0}^{T} \epsilon f^\epsilon(i^\epsilon)$

$= T\epsilon f^\epsilon(i) - T\epsilon f^\epsilon(i^\epsilon)$

$\geq T\epsilon f^\epsilon(i) - T\epsilon f^\epsilon(i)$ (this is because $i^\epsilon \leq i$ and f is a non-decreasing function)

$= 0$

for all $\omega \in \Omega$.

Therefore $\bar{X}^\epsilon_{T+1}(\omega) \geq X^\epsilon_{T+1}(\omega) \geq 0$ for all $\omega \in \Omega$.

Clearly the final number of shares of stock is the same for both portfolio strategies as they both start with zero shares of stock and are associated with the same trading strategy.

$\bar{Y}^\epsilon_{T+1}(\omega) = Y^\epsilon_{T+1}(\omega) \geq 0$ for all $\omega \in \Omega$.

We can now conclude that $(\bar{X}^\epsilon, \bar{Y}^\epsilon) \in \mathrm{Srep}^{\mathbb{M}}(D)$.

We also know that $\bar{X}^\epsilon_0 = i^\epsilon + f^\epsilon(i)T\epsilon$

Since i is the infimum of the initial values of portfolio strategies $\in \mathrm{Srep}^{\mathbb{M}}_P(D)$, this means that $i^\epsilon + f^\epsilon(i)T\epsilon \geq i$ for any sufficiently small $\epsilon > 0$.

Taking the limit as epsilon tends to zero of both sides of this equation gives us:

$\lim_{\epsilon \to 0} i^\epsilon + \lim_{\epsilon \to 0}(f^\epsilon(i)T\epsilon) \geq \lim_{\epsilon \to 0} i$

$\Rightarrow \lim_{\epsilon \to 0} i^\epsilon \geq i$

We have $\lim_{\epsilon \to 0} i^\epsilon \geq i$ and $\lim_{\epsilon \to 0} i^\epsilon \leq i$ and therefore we can conclude that:

$\lim_{\epsilon \to 0} i^\epsilon = i$.

Part 2

Show that $\lim_{\epsilon \to 0} \sup_{(\mathbb{Q},\tilde{S}) \in \mathcal{P}^\epsilon} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1 \tilde{S}_T) = \sup_{(\mathbb{Q},\tilde{S}) \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1 \tilde{S}_T)$.

For any $(\mathbb{Q}, \tilde{S}) \in \mathcal{P}^\epsilon$, we have $(\mathbb{Q}, \tilde{S}) \in \mathcal{P}$, so $\mathcal{P}^\epsilon \subset \mathcal{P}$.

This implies that:

$\sup_{(\mathbb{Q},\tilde{S}) \in \mathcal{P}^\epsilon} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1 \tilde{S}_T) \leq \sup_{(\mathbb{Q},\tilde{S}) \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1 \tilde{S}_T)$ for all sufficiently small $\epsilon > 0$.

Hence:

$\lim_{\epsilon \to 0} \sup_{(\mathbb{Q},\tilde{S}) \in \mathcal{P}^\epsilon} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1 \tilde{S}_T) \leq \sup_{(\mathbb{Q},\tilde{S}) \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1 \tilde{S}_T)$.

Note that we know that the limit on the left hand side exists because $\sup_{(\mathbb{Q},\tilde{S}) \in \mathcal{P}^\epsilon} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1 \tilde{S}_T)$ does not decrease as ϵ decreases and is bounded above by $\sup_{(\mathbb{Q},\tilde{S}) \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1 \tilde{S}_T)$.

Take any $(\mathbb{Q}, \tilde{S}) \in \mathcal{P}$, since $\tilde{S}$ is strictly positive we can always find an epsilon such that $(\mathbb{Q}, \tilde{S}) \in \mathcal{P}^\epsilon$.

Thus $\mathcal{P} \subset \bigcup_{\text{Sufficiently small } \epsilon} \mathcal{P}^\epsilon$

This implies that:

$\lim_{\epsilon \to 0} \sup_{(\mathbb{Q},\tilde{S}) \in \mathcal{P}^\epsilon} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1 \tilde{S}_T) \geq \sup_{(\mathbb{Q},\tilde{S}) \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1 \tilde{S}_T)$ for all sufficiently small $\epsilon > 0$.

Therefore:

$\lim_{\epsilon \to 0} \sup_{(\mathbb{Q},\tilde{S}) \in \mathcal{P}^\epsilon} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1 \tilde{S}_T) = \sup_{(\mathbb{Q},\tilde{S}) \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1 \tilde{S}_T)$

Part 3

Conclusion.

From Theorem 18.9 we know that for any sufficiently small $\epsilon > 0$ we have

$i^\epsilon = \sup_{(\mathbb{Q},\tilde{S}) \in \mathcal{P}^\epsilon} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1 \tilde{S}_T)$.

Taking the limit as epsilon tends to zero on both sides gives:

$\lim_{\epsilon \to 0} i^\epsilon = \lim_{\epsilon \to 0} \sup_{(\mathbb{Q},\tilde{S}) \in \mathcal{P}^\epsilon} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1 \tilde{S}_T)$

From parts 1 and 2 we know that this is equivalent to:

$i = \sup_{(\mathbb{Q},\tilde{S}) \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1 \tilde{S}_T)$

That is to say:

$$a_P(D) = \sup_{(\mathbb{Q},\tilde{S})\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1\tilde{S}_T).$$

Finally we note that since Θ is an arbitrary set of nodes, this result is true for any model $\mathbb{M}$ that has $P(\lambda) = S(\lambda)$ at certain nodes $\lambda \in \Theta$.

End of proof of Theorem 18.10.

18.4.3. *Extending the theorem to include negative bid prices.*

Theorem 18.11. *(Theorem 18.10 extended to include $P_t > S_t$ (but not including $P_t = \infty$))*

For any proportional cost viable model $\mathbb{M}$, with finite P, the proportional cost ask price of a derivative security D is given by:

$$a_P(D) = \sup_{(Q,\tilde{S})\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1\tilde{S}_T).$$

Proof of Theorem 18.11.

Let $\mathbb{M} = (\Omega, \mathcal{F}, \mathbb{P}, S, F, P)$ be a proportional cost viable model such that $S^b(\lambda) < 0$ if and only if $\lambda \in \Theta$ for some collection of nodes $\Theta \subset \Lambda$.

Let $\mathbb{M}^0 = (\Omega, \mathcal{F}, \mathbb{P}, S^0, F, P^0)$ be a proportional cost viable model where S^0 and P^0 are such that:

${S^0}^b(\lambda) = 0$ if and only if $\lambda \in \Theta$ and ${S^0}^b(\lambda) = S^b(\lambda)$ if and only if $\lambda \notin \Theta$.

${S^0}^a(\lambda) = S^a(\lambda)$ for all λ.

Note that $\mathbb{M}^0$ does not admit proportional cost arbitrage either due to Theorem 10.22.

Observe that for any proportional cost self-financing portfolio strategy (X, Y) in $\mathbb{M}$, that sells stock at a node belonging to Θ we can construct a portfolio strategy (X', Y') that is identical to (X, Y) except that it doesn't sell sell stock at any node belonging to Θ, that is also proportional cost self-financing in $\mathbb{M}$ (because selling on Θ was loosing money with no gain). The portfolio strategy (X', Y') would then also be proportional cost self-financing on $\mathbb{M}^0$ (because the ask prices of the two models are identical everywhere and the bid prices are identical at nodes not belonging to Θ).

It follows that:

$$a_P^{\mathbb{M}}(D) = \inf\{X_0 : (X,Y) \in \mathrm{Srep}_P^{\mathbb{M}}(D)\} = \inf\{X_0 : (X,Y) \in \mathrm{Srep}_P^{\mathbb{M}^0}(D)\}$$

$$= \sup_{(Q,\tilde{S})\in\mathcal{P}^{\mathbb{M}^0}} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1\tilde{S}_T).$$

$$= \sup_{(Q,\tilde{S})\in\mathcal{P}^{\mathbb{M}}} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1\tilde{S}_T).$$

This completes the proof.

(note that the final equality holds because the definition of $\mathcal{P}$ specifies that S must be a strictly positive process, so the fact that some of the bid-ask spreads involve negative bid prices is the same as if the bid price were zero on the nodes where it is negative).

End of proof of Theorem 18.11.

18.4.4. *Extending the theorem to include infinite transaction costs.*

Theorem 18.12. *(Theorem 18.11 extended to include $P_t > S_t$ (including $P_t = \infty$))*

For any proportional cost viable unrestricted model $\mathbb{M}$, the proportional cost ask price of a derivative security D is given by:

$$a_P(D) = \sup_{(Q,\tilde{S})\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1\tilde{S}_T).$$

Proof of Theorem 18.12.

Let $\mathbb{M} = (\Omega, \mathcal{F}, \mathbb{P}, S, F, P)$ be a proportional cost viable unrestricted model.

Let $\Theta \subset \Lambda$ be the collection of nodes $\lambda \in \Lambda$ such that $P(\lambda) = \infty$ if and only if $\lambda \in \Theta$.

Since this model does not admit proportional cost arbitrage we know that there must exist a frictionless middle value process $\tilde{S}$.

Let $\hat{s}^a = \max_{\lambda\notin\Theta}[S^a(\lambda)]$ "the maximum finite ask price in $\mathbb{M}$".

We will prove the claim by considering this model as the limit of a sequence of models with finite transaction costs.

We will say that δ is sufficiently large if and only if $\delta > \hat{s}^a$ and $\delta > \tilde{S}(\lambda)$ for all $\lambda \in \Lambda$.

<u>Construction of $\mathbb{M}^\delta$ for sufficiently large δ</u>

For any sufficiently large δ we will define a model $\mathbb{M}^\delta$ in the following way.

Note that δ being sufficiently large will mean that $\mathbb{M}^\delta$ will not admit proportional cost arbitrage due to the proportional cost fundamental theorem (Theorem 10.22).

Let $\mathbb{M}^\delta = (\Omega, \mathcal{F}, \mathbb{P}, S^\delta, F, P^\delta)$ be a model where S^δ and P^δ are such that:

${S^\delta}^a(\lambda) = \delta$ if $\lambda \in \Theta$ and ${S^\delta}^a(\lambda) = S^a(\lambda)$ if $\lambda \notin \Theta$.

${S^\delta}^b(\lambda) = 0$ if $\lambda \in \Theta$ and ${S^\delta}^b(\lambda) = S^b(\lambda)$ if $\lambda \notin \Theta$.

The following terminology will also be used for any sufficiently large δ.

Let $\mathcal{P}_\delta$ be defined as the set of pairs $(\mathbb{Q}, \tilde{S})$ such that $\mathbb{Q}$ is equivalent to $\mathbb{P}$, $\tilde{S}$ is a strictly positive martingale under $\mathbb{Q}$ and $\tilde{S}(\lambda) \in [{S^\delta}^b(\lambda), {S^\delta}^a(\lambda)]$ for all $\lambda \in \Lambda$.

We will write $a_P^{\mathbb{M}}(D)$ to mean the proportional cost ask price of D in model $\mathbb{M}$ with unlimited availability and similarly we will write $a_P^{\mathbb{M}^\delta}(D)$ to mean the proportional cost ask price of D in model $\mathbb{M}^\delta$ with unlimited availability.

We have created the framework that will allow us to complete this proof.

Please note that now that we have P unrestricted in $\mathbb{M}$ it is possible that for a given derivative security D there does not exist a super-replication portfolio strategy. Thus, the situation where a derivative D is super-replicable and the case where it is not super-replicable will need to be considered separately in certain situations.

The following is an outline of this proof.

• **Part 1**: Show that $a_P^{\mathbb{M}}(D) = \lim_{\delta\to\infty} a_P^{\mathbb{M}^\delta}(D)$.

—Part 1a: Case when D is proportional cost super-replicable in $\mathbb{M}$.

—Part 1b: Case when D is not proportional cost super-replicable in $\mathbb{M}$.

• **Part 2**: Show that $\lim_{\delta\to\infty} \sup_{(Q,\tilde{S})\in\mathcal{P}_\delta} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1\tilde{S}_T) = \sup_{(Q,\tilde{S})\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1\tilde{S}_T)$.

• **Part 3**: Conclusion.

Part 1a

Show that $a_P^{\mathbb{M}}(D) = \lim_{\delta\to\infty} a_P^{\mathbb{M}^\delta}(D)$ when D is proportional cost super-replicable in $\mathbb{M}$.

From Theorems 15.3 and 18.2 we know that:

$a_P^{\mathbb{M}}(D) = \inf\{X_0 : (X,Y) \in \mathrm{Srep}_P^{\mathbb{M}}(D)\}$

and

$a_P^{\mathbb{M}^\delta}(D) = \inf\{X_0 : (X,Y) \in \mathrm{Srep}_P^{\mathbb{M}^\delta}(D)\}$

for all sufficiently large δ.

We will now aim to construct a subset of $\mathrm{Srep}_P^{\mathbb{M}^\delta}(D)$ called $\mathrm{Srep3}_P^{\mathbb{M}^\delta}(D)$ that consists of portfolio strategies in $\mathrm{Srep}_P^{\mathbb{M}^\delta}(D)$ that have three particular properties, and are such that for any portfolio strategy belonging to $\mathrm{Srep}_P^{\mathbb{M}^\delta}(D)$ but not belonging to $\mathrm{Srep3}_P^{\mathbb{M}^\delta}(D)$ there exists a portfolio strategy that does belong to $\mathrm{Srep3}_P^{\mathbb{M}^\delta}(D)$ that has the same initial portfolio. Essentially $\mathrm{Srep3}_P^{\mathbb{M}^\delta}(D)$ can be thought of as the set of portfolio strategies belonging to $\mathrm{Srep}_P^{\mathbb{M}^\delta}(D)$ that do not involve poor trading choices such as choosing to sell stock when the bid price is negative.

We first construct the sets $\mathrm{Srep1}_P^{\mathbb{M}^\delta}(D)$ and $\mathrm{Srep2}_P^{\mathbb{M}^\delta}(D)$ as stepping stones towards constructing $\mathrm{Srep3}_P^{\mathbb{M}^\delta}(D)$.

<u>Construction of set $\mathrm{Srep1}_P^{\mathbb{M}^\delta}(D)$</u>

For any sufficiently large δ we define $\mathrm{Srep1}_P^{\mathbb{M}^\delta}(D)$ to be the set of portfolio strategies $(X,Y) \in \mathrm{Srep}_P^{\mathbb{M}^\delta}(D)$ such that $Z^{(X,Y)}(\lambda) \geq 0$ for all $\lambda \in \Theta$.

For future reference, we will refer to this property of strategies in $\mathrm{Srep1}_P^{\mathbb{M}^\delta}(D)$ as follows:

$\star$ Property 1) $Z^{(X,Y)}(\lambda) \geq 0$ for all $\lambda \in \Theta$.

• Claim 1 $\{X_0 : (X,Y) \in \text{Srep}_P^{\mathbb{M}^\delta}(D)\} = \{X_0 : (X,Y) \in \text{Srep1}_P^{\mathbb{M}^\delta}(D)\}$.

Since $\text{Srep1}_P^{\mathbb{M}^\delta}(D) \subset \text{Srep}_P^{\mathbb{M}^\delta}(D)$ it immediately follows that

$\{X_0 : (X,Y) \in \text{Srep1}_P^{\mathbb{M}^\delta}(D)\} \subset \{X_0 : (X,Y) \in \text{Srep}_P^{\mathbb{M}^\delta}(D)\}$.

Let $(X^1, Y^1) \in \text{Srep}_P^{\mathbb{M}^\delta}(D)$

Let Z^1 be the trading strategy that proportional cost super-replication corresponds to (X^1, Y^1) in the model $\mathbb{M}^\delta$.

Let Θ^- be the set of nodes $\lambda \in \Lambda$ such that $Z^1(\lambda) < 0$ and $\lambda \in \Theta$.

We define a trading strategy $\hat{Z}^1$ as follows:

$\hat{Z}^1(\lambda) = 0$ if $\lambda \in \Theta^-$ (No trade instead of paying to sell)

$\hat{Z}^1(\lambda) = Z^1(\lambda)$ if $\lambda \notin \Theta^-$

Let $(\hat{X}^1, \hat{Y}^1)$ be the portfolio strategy that proportional cost super-replication corresponds to $\hat{Z}^1$ and the initial portfolio $(X_0^1, 0)$.

Observe that $\hat{X}_{T+1}^1 \geq X_{T+1}^1$ and $\hat{X}_{T+1}^1 \geq X_{T+1}^1$.

Therefore $\{X_0 : (X,Y) \in \text{Srep}_P^{\mathbb{M}^\delta}(D)\} \subset \{X_0 : (X,Y) \in \text{Srep1}_P^{\mathbb{M}^\delta}(D)\}$.

This completes the proof of Claim 1.

Construction of set $\text{Srep2}_P^{\mathbb{M}^\delta}(D)$

For any sufficiently large δ we define $\text{Srep2}_P^{\mathbb{M}^\delta}(D)$ to be the set of portfolio strategies (X,Y) such that $(X,Y) \in \text{Srep1}_P^{\mathbb{M}^\delta}(D)$ and $Z^{(X,Y)}(\lambda) = 0$ for any non terminal node $\lambda \in \Theta$.

We will refer to this property of strategies in $\text{Srep2}_P^{\mathbb{M}^\delta}(D)$ as follows:

$\star$ Property 2) $Z^{(X,Y)}(\lambda) = 0$ for any non terminal node $\lambda \in \Theta$.

• Claim 2 $\{X_0 : (X,Y) \in \text{Srep1}_P^{\mathbb{M}^\delta}(D)\} = \{X_0 : (X,Y) \in \text{Srep2}_P^{\mathbb{M}^\delta}(D)\}$.

Since $\text{Srep2}_P^{\mathbb{M}^\delta}(D) \subset \text{Srep1}_P^{\mathbb{M}^\delta}(D)$ it immediately follows that

$\{X_0 : (X,Y) \in \text{Srep2}_P^{\mathbb{M}^\delta}(D)\} \subset \{X_0 : (X,Y) \in \text{Srep1}_P^{\mathbb{M}^\delta}(D)\}$.

Let $(X^2, Y^2) \in \text{Srep1}_P^{\mathbb{M}^\delta}(D)$

Let Z^2 be the trading strategy that proportional cost supper-replication corresponds to (X^2, Y^2) in $\mathbb{M}^\delta$.

Let Θ^+ be the set of non-terminal nodes λ such that $Z^{(X^2,Y^2)}(\lambda) > 0$ and $\lambda \in \Theta$.

In order to understand the following portfolio strategy, recall that for any node belonging to Θ the stock ask price is equal to δ which is the largest possible ask price in $\mathbb{M}^\delta$. This means that if at a given non-terminal node the ask price is equal to δ, there is no point in purchasing a share of stock at this moment since the price can only go down.

Define a new trading strategy $\hat{Z}^2$ as follows:

We postpone buying stock at the price δ until the terminal time:

$\hat{Z}^2(\lambda) = 0$ if $\lambda \in \Theta^+$.

At the terminal time we buy the same amount of stock that we would have bought anyway, plus all the stock that we postponed purchasing earlier:

$\hat{Z}^2(\lambda) = Z^2(\lambda) + \sum_{t=0}^{T} Z^2(\lambda_t^\omega)\mathbf{1}_{\lambda_t^\omega \in \Theta^+}$ if $\lambda \in \Lambda_T$,
where ω is any scenario belonging to λ.

At the remaining non-terminal nodes we follow the same trading strategy:

$\hat{Z}^2(\lambda) = Z^2(\lambda)$ if $\lambda \in \Lambda \setminus (\Theta^+ \cup \Lambda_T)$

Let $(\hat{X}^2, \hat{Y}^2)$ be the portfolio strategy that proportional cost corresponds to trading strategy $\hat{Z}^2$ and initial portfolio $(X_0^2, 0)$.

Observe that $\hat{Y}_T^2 \geq Y_T^2$ and $\hat{X}_T^2 \geq X_T^2$.

It follows that $(\hat{X}^2, \hat{Y}^2)$ belongs to $\mathrm{Srep2}_P^{\mathbb{M}^\delta}(D)$.

Therefore for any portfolio strategy $(X^2, Y^2) \in \mathrm{Srep1}_P^{\mathbb{M}^\delta}(D)$ there exists a portfolio strategy $(\hat{X}^2, \hat{Y}^2) \in \mathrm{Srep2}_P^{\mathbb{M}^\delta}(D)$ with the same initial value.

We must have $\{X_0 : (X,Y) \in \mathrm{Srep1}_P^{\mathbb{M}^\delta}(D)\} \subset \{X_0 : (X,Y) \in \mathrm{Srep2}_P^{\mathbb{M}^\delta}(D)\}$.

This completes the proof of claim 2.

Construction of the set $\mathrm{Srep3}_P^{\mathbb{M}^\delta}(D)$

The motivation behind $\mathrm{Srep3}_P^{\mathbb{M}^\delta}(D)$ has it's roots in the super-replication condition. Thus we will first examine the super-replication condition.

From Lemma 18.1, we know that the assumption that D is proportional cost super-replicable is equivalent to:

Sup rep condition 1) For any $\omega \in \Omega$ such that $D^1(\omega) > 0$ there exists a $t \in \mathcal{T}$ such that $P_t(\omega) < \infty$.

We will now take some additional liberties with notation in order to continue to express this condition in terms of nodes instead of scenarios whilst keeping the meaning of the condition clear.

For any $\mu \in \Lambda_T$, adapted process A and time $t \in \mathcal{T}$, we will write $A_t(\mu)$ to mean $A_t(\omega)$ for all $\omega \in \mu$.

For any $\mu \in \Lambda_T$ and time $t \in \mathcal{T}$, we will write λ_t^μ to mean the ascendant node to node μ at time t.

In our context we can now write Sup rep condition 1 as :

For any $\mu \in \Lambda_T$ such that $D^1(\mu) > 0$, there exists a $t \in \mathcal{T}$ such that $\lambda_t^\mu \notin \Theta$.

Let Γ be the set of nodes $\mu \in \Lambda_T$ such that we have $\lambda_t^\mu \in \Theta$ for all $t \in \mathcal{T}$.

For any sufficiently large δ, we define $\text{Srep3}_P^{\mathbb{M}^\delta}(D)$ be the set of portfolio strategies (X, Y) such that $(X, Y) \in \text{Srep2}_P^{\mathbb{M}^\delta}(D)$ and for any node $\mu \in \Gamma$, we have that $Z_t^{(X,Y)}(\mu) = 0$ for all $t \in \mathcal{T}$.

We will refer to this property of strategies belonging to $\text{Srep3}_P^{\mathbb{M}^\delta}(D)$ as follows:
$\star$ Property 3) For any node $\mu \in \Gamma$, we have that $Z_t^{(X,Y)}(\mu) = 0$ for all $t \in \mathcal{T}$.

• Claim 3 $\{X_0 : (X, Y) \in \text{Srep2}_P^{\mathbb{M}^\delta}(D)\} = \{X_0 : (X, Y) \in \text{Srep3}_P^{\mathbb{M}^\delta}(D)\}$.

Since $\text{Srep3}_P^{\mathbb{M}^\delta}(D) \subset \text{Srep2}_P^{\mathbb{M}^\delta}(D)$ it immediately follows that

$\{X_0 : (X, Y) \in \text{Srep3}_P^{\mathbb{M}^\delta}(D)\} \subset \{X_0 : (X, Y) \in \text{Srep2}_P^{\mathbb{M}^\delta}(D)\}$.

Let (X^3, Y^3) be a portfolio strategy belonging to $\text{Srep2}_P^{\mathbb{M}^\delta}(D)$ and let Z^3 be a trading strategy that proportional cost super-replication corresponds to (X^3, Y^3) in $\mathbb{M}^\delta$.

Let $\bar{\Theta}$ be the set of nodes $\mu \in \Gamma$ such that $D^1(\mu) \leq 0$ and $Z^{(X^3,Y^3)}(\mu) > 0$.

Let trading strategy $\hat{Z}^3$ be defined as follows

$\hat{Z}^3(\lambda) = 0$ if $\lambda \in \bar{\Theta}$

$\hat{Z}^3(\lambda) = Z^3(\lambda)$ if $\lambda \in \Lambda \setminus \bar{\Theta}$.

Let $(\hat{X}^3, \hat{Y}^3)$ be the portfolio strategy that proportional cost super-replication corresponds to $\hat{Z}^3$ with initial portfolio $(X_0^3, 0)$.

Observe that $(\hat{X}^3, \hat{Y}^3)$ belongs to $\text{Srep3}_P^{\mathbb{M}^\delta}(D)$.

This is because, due to the fact that the portfolio strategy $(\hat{X}^3, \hat{Y}^3) \in \text{Srep2}_P^{\mathbb{M}^\delta}(D)$, we must have $(\hat{X}^3(\mu), \hat{Y}^3(\mu)) = (\hat{X}_0^3, 0)$ for any node $\mu \in \Gamma$, and since we must have $D^1(\mu) \leq 0$ (due to Sup rep condition 1), this means that buying stock at this node will not help the portfolio strategy to super-replicate D.

Therefore for any portfolio strategy $(X^3, Y^3) \in \text{Srep2}_P^{\mathbb{M}^\delta}(D)$ there exists a portfolio strategy $(\hat{X}^3, \hat{Y}^3) \in \text{Srep3}_P^{\mathbb{M}^\delta}(D)$ with the same initial value.

We must have $\{X_0 : (X, Y) \in \text{Srep2}_P^{\mathbb{M}^\delta}(D)\} \subset \{X_0 : (X, Y) \in \text{Srep3}_P^{\mathbb{M}^\delta}(D)\}$.

This completes the proof of Claim 3.

<u>Construction of a function g^δ for each sufficiently large δ</u>

Let $g^\delta(z) := \inf\{k \in \mathbb{R} : |Z^{(X,Y)}(\lambda)| \leq k \quad \forall \lambda \in \Theta, (X,Y) \in \text{Srep3}_P^{\mathbb{M}^\delta} \text{ with } X_0 = z\}$

for all sufficiently large δ and $z \in \mathbb{R}$.

$g^\delta(z)$ can be thought of as being the smallest upper bound on the number of trades of any strategy belonging to $\text{Srep3}_P^{\mathbb{M}^\delta}$ with initial value z.

First let us check that this infimum produces a finite number.

By Lemma 18.8 we know that for any δ there exists a constant c^δ such that for any node $\lambda \in \Theta$ and portfolio strategy $(X,Y) \in \text{Srep}_P^{\mathbb{M}^\delta}(D)$ with $X_0 = z$, we have $|Z^{(X,Y)}(\lambda)| < c^\delta$.

Therefore $g^\delta(z) \in \mathbb{R}$ for all sufficiently large δ and $z \in \mathbb{R}$, since $|Z^{(X,Y)}(\lambda)|$ is bounded below by zero and is bounded above by c^δ.

• Claim 4 : g^δ is a non-decreasing function

Take any $z' \in \mathbb{R}$ and $z \in \mathbb{R}$ with $z' > z$.

Consider a strategy $(X,Y) \in \text{Srep3}_P^{\mathbb{M}^\delta}$.

Observe that the strategy that proportional cost super-replication corresponds to initial portfolio $(z', 0)$ and trading strategy $Z^{(X,Y)}$ also belongs to $\text{Srep3}_P^{\mathbb{M}^\delta}$.

It follows that for any sufficiently large δ we have $g^\delta(z') \geq g^\delta(z)$.

• Claim 5 For any $z \in \mathbb{R}$, $g^\delta(z)$ does not increase as δ increases.

Observe that for any sufficiently large $\tilde{\delta}$ and δ with $\tilde{\delta} < \delta$, we have $g^{\tilde{\delta}}(z) \geq g^\delta(z)$ because the bid-ask spreads of $\mathbb{M}^{\tilde{\delta}}$ are contained in the bid-ask spreads of $\mathbb{M}^\delta$. To see this, remember that g^δ and $g^{\tilde{\delta}}$ are the smallest upper bounds on the number of trades in $\mathbb{M}^\delta$ and $\mathbb{M}^{\tilde{\delta}}$ respectively, and note that for any portfolio strategy belonging to $\text{Srep3}_P^{\mathbb{M}^\delta}$ there exists a portfolio strategy belonging to $\text{Srep3}_P^{\mathbb{M}^{\tilde{\delta}}}$ that has the same initial value and the same proportional cost super-replication corresponding trading strategy.

• Claim 6 $\lim\limits_{\delta\to\infty} g^\delta(z) = 0$ for all $z \in \mathbb{R}$.

Let us fix $z \in \mathbb{R}$

Let us also fix sufficiently large $\tilde{\delta}$ and consider any sufficiently large δ such that $\tilde{\delta} < \delta$.

If $g^\delta(z) = 0$ then it is immediate that $\lim\limits_{\delta\to\infty} g^\delta(z) = 0$ because $g^\delta(z)$ is non-negative for all sufficiently large δ and does not increase as δ increases.

Suppose that $g^\delta(z) > 0$.

For any $\epsilon > 0$ there must exist a portfolio strategy $(X^\epsilon, Y^\epsilon) \in \text{Srep3}_P^{\mathbb{M}^\delta}(D)$ with $X_0 = z$ and node $\lambda^\epsilon \in \Theta$ such that $g^\delta(z) - \epsilon \leq |Z^{(X^\epsilon,Y^\epsilon)}(\lambda^\epsilon)| \leq g^\delta(z)$.

This is because if it were not true then there would exist an ϵ such that for all portfolio strategies $(X,Y) \in \text{Srep3}_P^{\mathbb{M}^\delta}(D)$ and nodes $\lambda \in \Theta$ we would have $g^\delta(z) - \epsilon > |Z^{(X,Y)}(\lambda)|$, but this would imply that $g^\delta(z) - \epsilon$ is an upper bound on the number of trades of any portfolio strategy belonging to $\text{Srep3}_P^{\mathbb{M}^\delta}(D)$ with initial value z, contradicting the fact that $g^\delta(z)$ is the infimum of such upper bounds.

Let Z^ϵ be the trading strategy that proportional cost super-replication corresponds to (X^ϵ, Y^ϵ) in the model $\mathbb{M}^\delta$.

Further, let us define a trading strategy $\tilde{Z}^\epsilon$ as follows:

$\tilde{Z}^\epsilon(\lambda) := Z^\epsilon(\lambda)$ if $\lambda \notin \Theta$.

$\tilde{Z}^\epsilon(\lambda) := Z^\epsilon(\lambda)\frac{\delta}{\tilde{\delta}}$ if $\lambda \in \Theta$.

Let $(\tilde{X}^\epsilon, \tilde{Y}^\epsilon)$ be the portfolio strategy that proportional cost super-replication corresponds to $\tilde{Z}^\epsilon$ and initial portfolio $(X_0^\epsilon, 0)$ in $\mathbb{M}^{\tilde{\delta}}$.

Note that since (X^ϵ, Y^ϵ) satisfies property 1), property 2) and property 3), so does $(\tilde{X}^\epsilon, \tilde{Y}^\epsilon)$.

Let us consider an expression for the final position in cash of portfolio strategy (X^ϵ, Y^ϵ) for any scenario $\omega \in \Omega$.

$$
\begin{aligned}
X_{T+1}^\epsilon(\omega) &= X_0^\epsilon(\omega) - D^0(\omega) + \sum_{t=0}^{T}[-S_t^\delta(\omega)Z_t^\epsilon(\omega) - |Z_t^\epsilon(\omega)|P_t^\delta(\omega)] \\
&= X_0^\epsilon(\omega) - D^0(\omega) + \sum_{t=0}^{T}[-S_t^\delta(\omega)Z_t^\epsilon(\omega) - |Z_t^\epsilon(\omega)|P_t^\delta(\omega)]\mathbf{1}_{\lambda_t^\omega \in \Theta} \\
&\quad + \sum_{t=0}^{T}[-S_t^\delta(\omega)Z_t^\epsilon(\omega) - |Z_t^\epsilon(\omega)|P_t^\delta(\omega)]\mathbf{1}_{\lambda_t^\omega \notin \Theta} \\
&= X_0^\epsilon(\omega) - D^0(\omega) + [-S_T^\delta(\omega)Z_T^\epsilon(\omega) - |Z_T^\epsilon(\omega)|P_T^\delta(\omega)]\mathbf{1}_{\lambda_T^\omega \in \Theta} \\
&\quad + \sum_{t=0}^{T}[-S_t^\delta(\omega)Z_t^\epsilon(\omega) - |Z_t^\epsilon(\omega)|P_t^\delta(\omega)]\mathbf{1}_{\lambda_t^\omega \notin \Theta} \\
&\quad \text{(because of Property 2)} \\
&= X_0^\epsilon(\omega) - D^0(\omega) - [Z_T^\epsilon(\omega)\delta]\mathbf{1}_{\lambda_T^\omega \in \Theta} \\
&\quad + \sum_{t=0}^{T}[-S_t^\delta(\omega)Z_t^\epsilon(\omega) - |Z_t^\epsilon(\omega)|P_t^\delta(\omega)]\mathbf{1}_{\lambda_t^\omega \notin \Theta} \\
&\quad \text{(because of property 1)} \\
&= X_0^\epsilon(\omega) - D^0(\omega) - [Z_T^\epsilon(\omega)\delta]\mathbf{1}_{\lambda_T^\omega \in \Theta} \\
&\quad + \sum_{t=0}^{T}[-S_t^{\tilde{\delta}}(\omega)Z_t^\epsilon(\omega) - |Z_t^\epsilon(\omega)|P_t^{\tilde{\delta}}(\omega)]\mathbf{1}_{\lambda_t^\omega \notin \Theta} \\
&\quad \text{(because } \mathbb{M}^{\tilde{\delta}} \text{ and } \mathbb{M}^\delta \text{ are identical on nodes not belonging to } \Theta\text{).} \\
&= X_0^\epsilon(\omega) - D^0(\omega) - [Z_T^\epsilon(\omega)\tfrac{\delta}{\delta'}\delta']\mathbf{1}_{\lambda_T^\omega \in \Theta}
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{t=0}^{T} [-S_t^{\tilde{\delta}}(\omega) Z_t^{\epsilon}(\omega) - |Z_t^{\epsilon}(\omega)| P_t^{\tilde{\delta}}(\omega)] \mathbf{1}_{\lambda_t^{\omega} \notin \Theta} \\
&= X_0^{\epsilon}(\omega) - D^0(\omega) - [\tilde{Z}_T^{\epsilon}(\omega)\delta'] \mathbf{1}_{\lambda_T^{\omega} \in \Theta} \\
&\quad + \sum_{t=0}^{T} [-S_t^{\tilde{\delta}}(\omega) \tilde{Z}_t^{\epsilon}(\omega) - |\tilde{Z}_t^{\epsilon}(\omega)| P_t^{\tilde{\delta}}(\omega)] \mathbf{1}_{\lambda_t^{\omega} \notin \Theta} \\
&= \tilde{X}_0^{\epsilon}(\omega) - D^0(\omega) + \sum_{t=0}^{T} [-S_t^{\tilde{\delta}}(\omega) \tilde{Z}_t^{\epsilon}(\omega) - |\tilde{Z}_t^{\epsilon}(\omega)| P_t^{\tilde{\delta}}(\omega)] \\
&= \tilde{X}_{T+1}^{\epsilon}(\omega).
\end{aligned}
$$

Therefore if $X_{T+1}^{\epsilon} \geq 0$ then $\tilde{X}_{T+1}^{\epsilon} \geq 0$.

Let us now consider an expression for the final position in stock of portfolio strategy $(X^{\epsilon}, Y^{\epsilon})$.

$$
\begin{aligned}
Y_{T+1}^{\epsilon}(\omega) &= -D^1(\omega) + \sum_{t=0}^{T} Z_t^{\epsilon}(\omega) \\
&= -D^1 + \sum_{t=0}^{T} Z_t^{\epsilon}(\omega) \mathbf{1}_{\lambda_t^{\omega} \notin \Theta} \\
&\quad + \sum_{t=0}^{T} Z_t^{\epsilon}(\omega) \mathbf{1}_{\lambda_t^{\omega} \in \Theta} \\
&\leq -D^1(\omega) + \sum_{t=0}^{T} Z_t^{\epsilon}(\omega) \mathbf{1}_{\lambda_t^{\omega} \notin \Theta} \\
&\quad + \sum_{t=0}^{T} Z_t^{\epsilon}(\omega) \tfrac{\delta}{\tilde{\delta}} \mathbf{1}_{\lambda_t^{\omega} \in \Theta} \\
&= -D^1(\omega) + \sum_{t=0}^{T} \tilde{Z}_t^{\epsilon}(\omega) \mathbf{1}_{\lambda_t^{\omega} \notin \Theta} \\
&\quad + \sum_{t=0}^{T} \tilde{Z}_t^{\epsilon}(\omega) \mathbf{1}_{\lambda_t^{\omega} \in \Theta} \\
&= -D^1(\omega) + \sum_{t=0}^{T} \tilde{Z}_t^{\epsilon}(\omega) \\
&= \tilde{Y}_{T+1}^{\epsilon}(\omega).
\end{aligned}
$$

Therefore if $Y_{T+1}^{\epsilon} \geq 0$ then $\tilde{Y}_{T+1}^{\epsilon} \geq 0$.

We have shown that if $(X^{\epsilon}, Y^{\epsilon}) \in \mathrm{Srep3}_P^{\mathbb{M}^{\delta}}(D)$ then $(\tilde{X}^{\epsilon}, \tilde{Y}^{\epsilon}) \in \mathrm{Srep3}_P^{\mathbb{M}^{\tilde{\delta}}}(D)$.

Consider the number of trades made by portfolio strategy $(\tilde{X}^{\epsilon}, \tilde{Y}^{\epsilon})$ at the node λ^{ϵ}.

$\tilde{Z}^{\epsilon}(\lambda^{\epsilon}) = \frac{\delta}{\tilde{\delta}} Z^{\epsilon}(\lambda^{\epsilon}) = \frac{\delta}{\tilde{\delta}} |Z^{(X^{\epsilon}, Y^{\epsilon})}(\lambda)| \geq \frac{\delta}{\tilde{\delta}} g^{\delta}(z) - \frac{\delta}{\tilde{\delta}} \epsilon$

This implies that $g^{\tilde{\delta}}(z) \geq \frac{\delta}{\tilde{\delta}} g^{\delta}(z) - \frac{\delta}{\tilde{\delta}} \epsilon$.

Taking the limit as ϵ tends to zero of both sides give us: $g^{\tilde{\delta}}(z) \geq \frac{\delta}{\tilde{\delta}} g^{\delta}(z)$

This implies that $\frac{\tilde{\delta}}{\delta} g^{\tilde{\delta}}(z) \geq g^{\delta}(z) \geq 0$.

Therefore we can take the limit as δ tends to infinity and get:

$\lim_{\delta\to\infty} g^\delta(z) = 0$ for any $z \in \mathbb{R}$.

We have constructed a function g^δ for each sufficiently large δ and shown it to have certain properties that we will use later.

Using the sets $\mathrm{Srep3}_P^{\mathbb{M}^\delta}(D)$ and the functions g^δ to show that $a_P^{\mathbb{M}}(D) = \lim_{\delta\to\infty} a_P^{\mathbb{M}^\delta}(D)$

In order to keep our notation clear and compact we will write:

$i := \inf\{X_0 : (X,Y) \in \mathrm{Srep}_P^{\mathbb{M}}(D)\} = a_P^{\mathbb{M}}(D)$ (Ask price of D in model $\mathbb{M}$)

and

$i^\delta := \inf\{X_0 : (X,Y) \in \mathrm{Srep}_P^{\mathbb{M}^\delta}(D)\} = a_P^{\mathbb{M}^\delta}(D)$ (Ask price of D in model $\mathbb{M}^\delta$)

Since for any sufficiently large δ, the bid-ask spreads of $\mathbb{M}^\delta$ are contained in the bid-ask spreads of $\mathbb{M}$ it follows immediately that $i^\delta \le i$ and thus $\lim_{\delta\to\infty} i^\delta \le i$. Note that the limit $\lim_{\delta\to\infty} i^\delta$ exists because i^δ does not decrease as δ increases, and is bounded above by i.

We now just need to show that $\lim_{\delta\to\infty} i^\delta \ge i$.

As a result of Claim 1, Claim 2 and Claim 3 we know that :

$i^\delta = \inf\{X_0 : (X,Y) \in \mathrm{Srep3}_P^{\mathbb{M}^\delta}(D)\}$

Let $(X^\delta, Y^\delta) \in \mathrm{Srep3}_P^{\mathbb{M}^\delta}(D)$ be a portfolio strategy that realises i^δ.

(We know that such a strategy exists because of Lemma 15.4 and the fact that we have already proved that for any portfolio strategy belonging to $\mathrm{Srep}_P^{\mathbb{M}^\delta}(D)$ there exists a portfolio strategy belonging to $\mathrm{Srep3}_P^{\mathbb{M}^\delta}(D)$ with the same initial value).

Let Z^δ be a trading strategy that proportional cost super-replication corresponds to the portfolio strategy (X^δ, Y^δ) in the model $\mathbb{M}^\delta$.

We will now construct a portfolio strategy $(\tilde{X}^\delta, \tilde{Y}^\delta)$ by giving its associated initial portfolio $(\tilde{X}_0^\delta, 0)$ and trading strategy $\tilde{Z}^\delta$ that proportional cost correspond to it in the model $\mathbb{M}$.

Let the initial portfolio of $(\tilde{X}^\delta, \tilde{Y}^\delta)$ be $(\tilde{X}_0^\delta, 0) = (i^\delta + g^\delta(i)\hat{s}^a, 0)$, where $\hat{s}^a = \max_{\lambda\notin\Theta}[S^a(\lambda)]$.

Note that since g^δ is an increasing function, for any sufficiently large δ, we have $g^\delta(i) \ge g^\delta(i^\delta)$.

Recall that Γ is the set of nodes $\mu \in \Lambda_T$ such that we have $\lambda_t^\mu \in \Theta$ for all $t \in \mathcal{T}$.

Let Δ be defined as a collection of disjoint nodes that do not belong to Θ such that for any terminal node μ that does not belong to Γ, we have either $\mu \in \Delta$ or μ has an ascendant node that belongs to Δ.

We will write $\Delta = \{\lambda_1, \ldots, \lambda_n\}$ for some $n \in \mathbb{N}$.

We define a trading strategy $\tilde{Z}^\delta$ as follows:

$\tilde{Z}^\delta(\lambda) = 0$ if $\lambda \in \Theta$.

$\tilde{Z}^\delta(\lambda) = Z^\delta(\lambda) + g^\delta(i)$ if $\lambda \in \Delta$.

$\tilde{Z}^\delta(\lambda) = Z^\delta(\lambda)$ if $\lambda \notin \Theta$ and $\lambda \notin \Delta$.

Observe that properties 1, 2 and 3 also apply to this portfolio strategy $(\tilde{X}^\delta, \tilde{Y}^\delta)$.

We now check that $(\tilde{X}^\delta, \tilde{Y}^\delta)$ proportional cost super-replicates D.

For any ω such that $D^1(\omega) \leq 0$ and $\lambda_t^\omega \in \Theta$ for all $t \in \mathcal{T}$ we have:

$$
\begin{aligned}
X_{T+1}^\delta(\omega) &= X_0^\delta(\omega) - D^0(\omega) + \sum_{t=0}^{T}[-S_t^\delta(\omega)Z_t^\delta(\omega) - |Z_t^\delta(\omega)|P_t^\delta(\omega)] \\
&= X_0^\delta(\omega) - D^0(\omega) + \sum_{t=0}^{T}[-S_t^\delta(\omega)Z_t^\delta(\omega) - |Z_t^\delta(\omega)|P_t^\delta(\omega)]\mathbf{1}_{\lambda_t^\omega \in \Theta} \\
&\quad + \sum_{t=0}^{T}[-S_t^\delta(\omega)Z_t^\delta(\omega) - |Z_t^\delta(\omega)|P_t^\delta(\omega)]\mathbf{1}_{\lambda_t^\omega \notin \Theta} \\
&= X_0^\delta(\omega) - D^0(\omega) + \sum_{t=0}^{T}[-S_t^\delta(\omega)Z_t^\delta(\omega) - |Z_t^\delta(\omega)|P_t^\delta(\omega)]\mathbf{1}_{\lambda_t^\omega \in \Theta} \\
&= X_0^\delta(\omega) - D^0(\omega) \\
&\qquad \text{(because of property 3).} \\
&= i^\delta - D^0(\omega) \\
&\leq i^\delta + g^\delta(i)\hat{s}^a - D^0(\omega) \\
&= \tilde{X}_0^\delta(\omega) - D^0(\omega) \\
&= \tilde{X}_0^\delta(\omega) - D^0(\omega) + \sum_{t=0}^{T}[-S_t(\omega)\tilde{Z}_t^\delta(\omega) - |\tilde{Z}_t^\delta(\omega)|P_t(\omega)] \\
&\qquad \text{(because of } Z_t(\omega) = 0 \text{ whenever } \lambda_t^\omega \in \Theta). \\
&= \tilde{X}_{T+1}^\delta(\omega)
\end{aligned}
$$

For any ω that is not such that $D^1(\omega) \leq 0$ and $\lambda_t^\omega \in \Theta$ for all $t \in \mathcal{T}$ we have the following:

(Note that for such an ω we will, with a slight abuse of notation, write λ_Δ^ω to mean the node belonging to Δ that contains ω. There can only be one such node because the nodes contained in Δ are disjoint.)

$$
\begin{aligned}
X_{T+1}^\delta(\omega) &= X_0^\delta(\omega) - D^0(\omega) + \sum_{t=0}^{T}[-S_t^\delta(\omega)Z_t^\delta(\omega) - |Z_t^\delta(\omega)|P_t^\delta(\omega)] \\
&= X_0^\delta(\omega) - D^0(\omega) + \sum_{t=0}^{T}[-S_t^\delta(\omega)Z_t^\delta(\omega) - |Z_t^\delta(\omega)|P_t^\delta(\omega)]\mathbf{1}_{\lambda_t^\omega \in \Theta} \\
&\quad + \sum_{t=0}^{T}[-S_t^\delta(\omega)Z_t^\delta(\omega) - |Z_t^\delta(\omega)|P_t^\delta(\omega)]\mathbf{1}_{\lambda_t^\omega \notin \Theta}
\end{aligned}
$$

$$= X_0^\delta(\omega) - D^0(\omega) + [-S_T^\delta(\omega)Z_T^\delta(\omega) - |Z_T^\delta(\omega)|P_T^\delta(\omega)]\mathbf{1}_{\lambda_T^\omega \in \Theta}$$

$$+ \sum_{t=0}^{T}[-S_t^\delta(\omega)Z_t^\delta(\omega) - |Z_t^\delta(\omega)|P_t^\delta(\omega)]\mathbf{1}_{\lambda_t^\omega \notin \Theta}$$

(because of Property 2)

$$= X_0^\delta(\omega) - D^0(\omega) + [-Z_T^\epsilon(\omega)\delta]\mathbf{1}_{\lambda_T^\omega \in \Theta}$$

$$+ \sum_{t=0}^{T}[-S_t^\delta(\omega)Z_t^\delta(\omega) - |Z_t^\delta(\omega)|P_t^\delta(\omega)]\mathbf{1}_{\lambda_t^\omega \notin \Theta}$$

(because of Property 1).

$$= X_0^\delta(\omega) - D^0(\omega) + [-Z_T^\delta(\omega)\delta]\mathbf{1}_{\lambda_T^\omega \in \Theta}$$

$$+ \sum_{t=0}^{T}[-S_t(\omega)Z_t^\delta(\omega) - |Z_t^\delta(\omega)|P_t(\omega)]\mathbf{1}_{\lambda_t^\omega \notin \Theta}$$

(because $\mathbb{M}$ and $\mathbb{M}^\delta$ are identical on nodes not belonging to Θ).

$$= X_0^\delta(\omega) - D^0(\omega) + [-Z_T^\delta(\omega)\delta]\mathbf{1}_{\lambda_T^\omega \in \Theta}$$

$$+ \sum_{t=0}^{T}[-S_t(\omega)Z_t^\delta(\omega) - |Z_t^\delta(\omega)|P_t(\omega)]\mathbf{1}_{\lambda_t^\omega \notin \Theta}\mathbf{1}_{\lambda_t^\omega \notin \Delta}$$

$$+ \sum_{t=0}^{T}[-S_t(\omega)Z_t^\delta(\omega) - |Z_t^\delta(\omega)|P_t(\omega)]\mathbf{1}_{\lambda_t^\omega \notin \Theta}\mathbf{1}_{\lambda_t^\omega \in \Delta}$$

$$= X_0^\delta(\omega) - D^0(\omega) + [-Z_T^\delta(\omega)\delta]\mathbf{1}_{\lambda_T^\omega \in \Theta}$$

$$+ \sum_{t=0}^{T}[-S_t(\omega)Z_t^\delta(\omega) - |Z_t^\delta(\omega)|P_t(\omega)]\mathbf{1}_{\lambda_t^\omega \notin \Theta}\mathbf{1}_{\lambda_t^\omega \notin \Delta}$$

$$-S(\lambda_\Delta^\omega)Z^\delta(\lambda_\Delta^\omega) - |Z^\delta(\lambda_\Delta^\omega)|P(\lambda_\Delta^\omega)$$

(See definition of Δ).

$$= X_0^\delta(\omega) - D^0(\omega) + [-Z_T^\delta(\omega)\delta]\mathbf{1}_{\lambda_T^\omega \in \Theta}$$

$$+ \sum_{t=0}^{T}[-S_t(\omega)Z_t^\delta(\omega) - |Z_t^\delta(\omega)|P_t(\omega)]\mathbf{1}_{\lambda_t^\omega \notin \Theta}\mathbf{1}_{\lambda_t^\omega \notin \Delta}$$

$$-S(\lambda_\Delta^\omega)(\tilde{Z}^\delta(\lambda_\Delta^\omega) - g^\delta(i)) - |\tilde{Z}^\delta(\lambda_\Delta^\omega) - g^\delta(i)|P(\lambda_\Delta^\omega)$$

(See definition of Δ and definition of $\tilde{Z}$).

$$\leq X_0^\delta(\omega) - D^0(\omega) + [-Z_T^\delta(\omega)\delta]\mathbf{1}_{\lambda_T^\omega \in \Theta}$$

$$+ \sum_{t=0}^{T}[-S_t(\omega)Z_t^\delta(\omega) - |Z_t^\delta(\omega)|P_t(\omega)]\mathbf{1}_{\lambda_t^\omega \notin \Theta}\mathbf{1}_{\lambda_t^\omega \notin \Delta}$$

$$+g^\delta(i)S^a(\lambda_\Delta^\omega) - S(\lambda_\Delta^\omega)\tilde{Z}^\delta(\lambda_\Delta^\omega) - |\tilde{Z}^\delta(\lambda_\Delta^\omega)|P(\lambda_\Delta^\omega)$$

(This is because $-|x-y| \leq |y| - |x|$ for all $x, y \in \mathbb{R}$).

$$= X_0^\delta(\omega) - D^0 + [-Z_T^\delta(\omega)\delta]\mathbf{1}_{\lambda_T^\omega \in \Theta}$$

$$+\sum_{t=0}^{T}[-S_t(\omega)\tilde{Z}_t^{\delta}(\omega)-|\tilde{Z}_t^{\delta}(\omega)|P_t(\omega)]\mathbf{1}_{\lambda_t^{\omega}\notin\Theta}\mathbf{1}_{\lambda_t^{\omega}\notin\Delta}$$

$$+g^{\delta}(i)S^{a}(\lambda_{\Delta}^{\omega})-S(\lambda_{\Delta}^{\omega})\tilde{Z}^{\delta}(\lambda_{\Delta}^{\omega})-|\tilde{Z}^{\delta}(\lambda_{\Delta}^{\omega})|P(\lambda_{\Delta}^{\omega})$$

(because Z and $\tilde{Z}$ are identical at nodes not belonging to Θ).

$$=X_0^{\delta}(\omega)-D^0(\omega)+[-Z_T^{\delta}(\omega)\delta]\mathbf{1}_{\lambda_T^{\omega}\in\Theta}+g^{\delta}(i)S^{a}(\lambda_{\Delta}^{\omega})$$

$$+\sum_{t=0}^{T}[-S_t(\omega)\tilde{Z}_t^{\delta}(\omega)-|\tilde{Z}_t^{\delta}(\omega)|P_t(\omega)]\mathbf{1}_{\lambda_t^{\omega}\notin\Theta}$$

$$\leq X_0^{\delta}(\omega)-D^0(\omega)+g^{\delta}(i)S^{a}(\lambda_{\Delta}^{\omega})$$

$$+\sum_{t=0}^{T}[-S_t(\omega)\tilde{Z}_t^{\delta}(\omega)-|\tilde{Z}_t^{\delta}(\omega)|P_t(\omega)]\mathbf{1}_{\lambda_t^{\omega}\notin\Theta}$$

(Property 1 ensures that $[-Z_T^{\delta}(\omega)\delta]\mathbf{1}_{\lambda_T^{\omega}\in\Theta}$ is non-positive).

$$\leq X_0^{\delta}(\omega)-D^0(\omega)+g^{\delta}(i)S^{a}(\lambda_{\Delta}^{\omega})$$

$$+\sum_{t=0}^{T}[-S_t(\omega)\tilde{Z}_t^{\delta}(\omega)-|\tilde{Z}_t^{\delta}(\omega)|P_t(\omega)]$$

(because $\tilde{Z}$ is zero on nodes belonging to Θ).

$$\leq X_0^{\delta}(\omega)-D^0(\omega)+g^{\delta}(i)\hat{s}^{a}$$

$$+\sum_{t=0}^{T}[-S_t(\omega)\tilde{Z}_t^{\delta}(\omega)-|\tilde{Z}_t^{\delta}(\omega)|P_t(\omega)]$$

$$=i^{\delta}-D^0(\omega)+g^{\delta}(i)\hat{s}^{a}$$

$$+\sum_{t=0}^{T}[-S_t(\omega)\tilde{Z}_t^{\delta}(\omega)-|\tilde{Z}_t^{\delta}(\omega)|P_t(\omega)]$$

$$=\tilde{X}_0(\omega)-D^0(\omega)+\sum_{t=0}^{T}[-S_t(\omega)\tilde{Z}_t^{\delta}(\omega)-|\tilde{Z}_t^{\delta}(\omega)|P_t(\omega)]$$

$$=\tilde{X}_{T+1}(\omega)$$

(because $(\tilde{X},\tilde{Y})$ is generated by $(\tilde{X}_0,0)$ and $\tilde{Z}$ in the model $\mathbb{M}$).

Therefore if $X_{T+1}^{\delta}\geq 0$ then $\tilde{X}_{T+1}^{\delta}\geq 0$.

For any ω such that $D^1(\omega)\leq 0$ and $\lambda_t^{\omega}\in\Theta$ for all $t\in\mathcal{T}$ we have:

$$Y_{T+1}^{\delta}(\omega)=D^1(\omega)+\sum_{t=0}^{T}Z_t^{\delta}(\omega)$$

$$=D^1(\omega)$$

$$=D^1(\omega)+\sum_{t=0}^{T}\tilde{Z}_t^{\delta}(\omega)$$

$$=\tilde{Y}_{T+1}^{\delta}(\omega)$$

For any ω that is not such that $D^1(\omega) \leq 0$ and $\lambda_t^\omega \in \Theta$ for all $t \in \mathcal{T}$ we have:

$$\begin{aligned}
Y_{T+1}^\delta(\omega) &= D^1(\omega) + \sum_{t=0}^{T} Z_t^\delta(\omega) \\
&= D^1(\omega) + \sum_{t=0}^{T} Z_t^\delta(\omega)\mathbf{1}_{\lambda_t^\omega \notin \Theta} + \sum_{t=0}^{T} Z_t^\delta(\omega)\mathbf{1}_{\lambda_t^\omega \in \Theta} \\
&= D^1(\omega) + \sum_{t=0}^{T} Z_t^\delta(\omega)\mathbf{1}_{\lambda_t^\omega \notin \Theta} + Z_T^\delta(\omega)\mathbf{1}_{\lambda_T^\omega \in \Theta} \\
&\leq D^1(\omega) + \sum_{t=0}^{T} Z_t^\delta(\omega)\mathbf{1}_{\lambda_t^\omega \notin \Theta} + g^\delta(i) \\
&= D^1(\omega) + \sum_{t=0}^{T} \tilde{Z}_t^\delta(\omega)\mathbf{1}_{\lambda_t^\omega \notin \Theta} \\
&= D^1(\omega) + \sum_{t=0}^{T} \tilde{Z}_t^\delta(\omega) \\
&= \tilde{Y}_{T+1}^\delta.
\end{aligned}$$

Therefore if $Y_{T+1}^\delta \geq 0$ then $\tilde{Y}_{T+1}^\delta \geq 0$.

it follows that if $(X^\delta, Y^\delta) \in \mathrm{Srep3}_P^{\mathbb{M}^\delta}(D)$ then $(\tilde{X}^\delta, \tilde{Y}^\delta) \in \mathrm{Srep}_P^{\mathbb{M}}(D)$.

The initial portfolio of $(\tilde{X}^\delta, \tilde{Y}^\delta)$ is $(i^\delta + g^\delta(i)\hat{s}^a, 0)$.

Therefore $i \leq i^\delta + g^\delta(i)\hat{s}^a$.

This implies that $\lim_{\delta\to\infty} i \leq \lim_{\delta\to\infty} i^\delta + \lim_{\delta\to\infty} g^\delta(i)\hat{s}^a$.

Since we have shown that $\lim_{\delta\to\infty} g^\delta(i) = 0$ it follows that $i \leq \lim_{\delta\to\infty} i^\delta$.

We now know that $i \geq \lim_{\delta\to\infty} i^\delta$ and $i \leq \lim_{\delta\to\infty} i^\delta$, and therefore we can conclude that:

$i = \lim_{\delta\to\infty} i^\delta$.

Part 1b

Show that $a_P^{\mathbb{M}}(D) = \lim_{\delta\to\infty} a_P^{\mathbb{M}^\delta}(D)$ when D is not proportional cost super-replicable in $\mathbb{M}$.

First let us observe that since D is not super-replicable, $\mathrm{Srep}_P^{\mathbb{M}}(D)$ is empty, so $a_P^{\mathbb{M}}(D) = \inf\{X_0 : (X, Y) \in \mathrm{Srep}_P^{\mathbb{M}}(D)\} = \infty$.

In order to complete this part of the proof, we now just need to show that $\lim_{\delta\to\infty} a_P^{\mathbb{M}^\delta}(D) = \infty$.

Since D is not super-replicable there exists a scenario $\omega' \in \Omega$ such that $D^1(\omega') > 0$ and $P_t(\omega') = \infty$ for all $t \in \mathcal{T}$.

This means that ω' is such that $D^1(\omega') > 0$ and $P_t^\delta(\omega') = \delta$ for all $t \in \mathcal{T}$.

Suppose that the stock price follows the path ω'. Any super-replication portfolio strategy (X, Y) must have a high enough initial value to be able to buy $D^1(\omega')$ shares of stock at some point along the path ω' and still have a positive position in cash at the end ($X_{T+1} \geq 0$). Since it is not possible to achieve a net gain in cash from the stock, the minimum amount of cash you would need in order to buy this amount of stock for, if the stock price follows the path ω', would be $\delta D^1(\omega')$. It follows that for any such (X, Y), we have $X_0 \geq \delta D^1(\omega')$.

Therefore $a_P^{\mathbb{M}^\delta}(D) = \inf\{X_0 : (X, Y) \in \mathrm{Srep}_P^{\mathbb{M}^\delta}(D)\} \geq \delta D^1(\omega')$.

Taking the limit as $\delta \to \infty$ gives:

$$\lim_{\delta \to \infty} a_P^{\mathbb{M}^\delta}(D) = \infty.$$

Thus we can conclude that

$$a_P^{\mathbb{M}}(D) = \lim_{\delta \to \infty} a_P^{\mathbb{M}^\delta}(D) \; (= \infty).$$

Part 2

Show that $\lim\limits_{\delta \to \infty} \sup\limits_{(Q, \tilde{S}) \in \mathcal{P}_\delta} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1 \tilde{S}_T) = \sup\limits_{(Q, \tilde{S}) \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1 \tilde{S}_T)$.

For any $(Q, \tilde{S}) \in \mathcal{P}_\delta$, we have $(Q, \tilde{S}) \in \mathcal{P}$, so $\mathcal{P}_\delta \subset \mathcal{P}$.

This implies that:

$$\sup_{(Q, \tilde{S}) \in \mathcal{P}_\delta} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1 \tilde{S}_T) \leq \sup_{(Q, \tilde{S}) \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1 \tilde{S}_T) \text{ for all } \delta.$$

Hence:

$$\lim_{\delta \to \infty} \sup_{(Q, \tilde{S}) \in \mathcal{P}_\delta} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1 \tilde{S}_T) \leq \sup_{(Q, \tilde{S}) \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1 \tilde{S}_T).$$

Take any $(Q, \tilde{S}) \in \mathcal{P}$, since $\tilde{S}$ is strictly positive we can always find an δ such that $(Q, \tilde{S}) \in \mathcal{P}_\delta$.

Therefore $\mathcal{P} \subset \bigcup\limits_{\text{Sufficiently large } \delta} \mathcal{P}^\delta$.

This implies that:

$$\lim_{\delta \to \infty} \sup_{(Q, \tilde{S}) \in \mathcal{P}_\delta} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1 \tilde{S}_T) \geq \sup_{(Q, \tilde{S}) \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1 \tilde{S}_T)$$

Therefore:

$$\lim_{\delta \to \infty} \sup_{(Q, \tilde{S}) \in \mathcal{P}_\delta} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1 \tilde{S}_T) = \sup_{(Q, \tilde{S}) \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1 \tilde{S}_T)$$

Part 3

Conclusion.

From Theorem 18.11, we know that:

$$a_P^{\mathbb{M}^\delta}(D) = \sup_{(Q,\tilde{S})\in\mathcal{P}_\delta} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1\tilde{S}_T).$$

Taking the limit as δ tends to infinity on both sides gives:

$$\lim_{\delta\to\infty} a_P^{\mathbb{M}^\delta}(D) = \lim_{\delta\to\infty} \sup_{(Q,\tilde{S})\in\mathcal{P}_\delta} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1\tilde{S}_T)$$

From parts 1 and 2 we know that this is equivalent to:

$$a_P^{\mathbb{M}}(D) = \sup_{(Q,\tilde{S})\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}(D^0 + D^1\tilde{S}_T).$$

Finally we note that since Θ is an arbitrary set of nodes, this result is true for any model $\mathbb{M}$.

End of proof of Theorem 18.12.

19. **Derivative pricing under combined costs**

We will now explore ask price representations of derivatives under combined costs. As it turns out, the fixed transaction cost process has no effect on the ask price of a derivative security with unlimited availability and thus, as we will see, the combined cost ask price is equal to the proportional cost ask price under unlimited availability. The main achievement of this section is to formulate and prove an ask price representation theorem for a derivative under combined transaction costs with limited availability, as this will require some sophisticated mathematical tools. Fortunately, we have already paved the way for this proof, as the work we did in the previous section, in order to extend the proportional cost ask price representation theorem will play a key role in the deductions that are made.

19.1. **Restrictions on viability.**

In this section we we will work only with models that are proportional cost viable. The reason for this is that as seen previously (see Example 17.1) if a model is fixed cost viable but not frictionless viable then a derivative security may not have a fixed cost viable price and the model becomes of little practical significance. The relationship between the fixed cost setting and the frictionless setting can be thought of as a special case of the relationship between the combined cost setting and the proportional cost setting, because in any model with $P(\lambda) = 0$ for all $\lambda \in \Lambda$ frictionless arbitrage is equivalent to proportional cost arbitrage and fixed cost arbitrage is equivalent to combined cost arbitrage.

19.2. **Unlimited availability.**

The following theorem is a more general version of Theorem 17.7.

Theorem 19.1. *(Equivalence of proportional cost price and combined cost price)*
For any derivative security D in a proportional cost viable model $\mathbb{M}$, we have that :

$a_P(D) = a_C(D)$.

Proof of Theorem 19.1.

Any combined cost extended arbitrage opportunity with respect to D and d_0 is also a proportional cost extended arbitrage opportunity with respect to D and d_0, so the set $\mathcal{V}_P(D)$ is contained in the set $\mathcal{V}_C(D)$

This implies that the supremum of the $\mathcal{V}_C(D)$ must be greater than or equal to the supremum of $\mathcal{V}_P(D)$ and the infimum of $\mathcal{V}_C(D)$must be less than or equal to the infimum of $\mathcal{V}_P(D)$.

It follows immediately that $b_C(D) \leq b_P(D)$ and $a_P(D) \leq a_C(D)$.

We now show that we cannot have $a_P(D) < a_C(D)$.

Suppose for a contradiction that $a_P(D) < a_C(D)$.

This implies that for any real number d_0 such that $a_P(D) < d_0 < a_C(D)$, there exists a proportional cost extended arbitrage opportunity with respect to D and d_0 for the derivative seller, but there does not exist a combined cost extended arbitrage opportunity for the derivative seller. See Lemma 22.9.

Consider real numbers d and d' (representing initial prices of the derivative) such that $a_P(D) < d < d' < a_C(D)$, since they are greater than $a_P(D)$ there must exist a proportional cost extended arbitrage opportunity for the seller if the derivative security price is equal to either of them.

Let a trading strategy Z and integer $n > 0$ (n must be > 0 by Lemma 22.9), be a proportional cost extended arbitrage opportunity with respect to derivative D and initial price d.

Therefore by Lemma 13.2, we have:

$$nd + \sum_{t=0}^{T}\Big[-Z_t(\omega)S_t(\omega) - |Z_t(\omega)|P_t(\omega)\Big] - nD_T^0(\omega) \geq 0$$

and $\sum\limits_{t=0}^{T}[Z_t(\omega)] - nD_T^1(\omega) \geq 0$ for all $\omega \in \Omega$.

and there exists $\omega' \in \Omega$ such that:

$$nd + \sum_{t=0}^{T}\Big[-Z_t(\omega')S_t(\omega') - |Z_t(\omega')|P_t(\omega')\Big] - nD_T^0(\omega') > 0$$

It follows that if we replace d by d' we get:

$$nd' + \sum_{t=0}^{T}\Big[-Z_t(\omega)S_t(\omega) - |Z_t(\omega)|P_t(\omega)\Big] - nD_T^0(\omega) \geq n(d'-d) > 0$$

and $\sum\limits_{t=0}^{T}[Z_t(\omega)] - nD_T^1(\omega) \geq 0$ for all $\omega \in \Omega$.

Therefore the trading strategy Z and integer n is an proportional cost extended arbitrage opportunity with respect to derivative D and initial price d'.

The reason that we have considered both the price d and tthe price d' is that now we have shown that when the price is d' we have a proportional cost extended arbitrage opportunity with a strict inequality on the final position in cash. This means that it can be thought of as a"proportional cost extended strong arbitrage opportunity".

In order to construct a combined cost extended arbitrage opportunity we will effectively "scale up" this portfolio strategy, and because it can now be thought of as a "proportional cost strong extended arbitrage opportunity", when the initial option price is d', the transaction cost will be overridden in every scenario.

The profit in each scenario is at least $n(d'-d)$ and the total fixed transaction cost paid in each scenario will be less than $(T+1)\hat{F}$. (See definition of $\hat{F}$, Definition 5.19.)

Therefore in order to "scale up" this portfolio strategy by a large enough amount we need an integer n' (representing the scale factor) for which $n' > n$ and such that:

$$n'n(d'-d) \geq (T+1)\hat{F}$$

Therefore we set $n' = \frac{(T+1)\hat{F}}{n(d'-d)}$ (note that this is strictly positive).

We can deduce that:

$$n'nd' + \sum_{t=0}^{T} \Big[- n'Z_t(\omega)S_t(\omega) - |n'Z_t(\omega)|P_t(\omega) \Big] - n'nD_T^0(\omega) \geq (T+1)\hat{F} > 0$$

and $\sum_{t=0}^{T} [n'Z_t(\omega)] - n'nD_T^1(\omega) \geq 0$ for all $\omega \in \Omega$.

This implies that:

$$n'nd' + \sum_{t=0}^{T} \Big[- n'Z_t(\omega)S_t(\omega) - |n'Z_t(\omega)|P_t(\omega) - F_t(\omega)\mathbf{1}_{Z_t(\omega)\neq 0} \Big] - n'nD_T^0(\omega) > 0$$

and $\sum_{t=0}^{T} [n'Z_t(\omega)] - n'nD_T^1(\omega) \geq 0$ for all $\omega \in \Omega$.

(Because $(T+1)\hat{F} > \sum_{t=0}^{T} F_t(\omega)\mathbf{1}_{Z_t(\omega)\neq 0}$)

By Lemma 13.2 the trading strategy $n'Z$ and integer $n'n$ is a combined cost extended arbitrage opportunity with respect to derivative D with initial price d'. Since $n'n$ is positive, this is an extended arbitrage opportunity for the seller.

This means that for any initial price d_0 such that $a_P(D) < d_0 < a_C(D)$ there exists a combined cost arbitrage opportunity for the seller. This is a contradiction since by Lemma 22.9, any initial price less than $a_C(D)$ should be either a combined cost viable price or should allow only a combined cost extended arbitrage opportunity for the buyer.

Therefore $a_P(D) \geq a_C(D)$.

We can conclude that $a_P(D) = a_C(D)$.

End of proof of Theorem 19.1.

19.3. **Limited availability.**

19.3.1. *Local definitions for this subsection.*

Definition 19.2. (Adjusted stock price model)
For any model $\mathbb{M}$ and collection of nodes $\Theta \subset \Lambda$ we define the following:

Let $\mathbb{M}^{\Theta} = (\Omega, \mathcal{F}, \mathbb{P}, S, F, P^{\Theta})$,

where P^{Θ} is such that:

If $\lambda \in \Theta$ then $P^{\Theta}(\lambda) = P(\lambda)$.

If $\lambda \notin \Theta$ then $P^{\Theta} = \infty$.

$\mathbb{M}^{\Theta}$ can be interpreted to mean the model $\mathbb{M}$ but with no proportional (or combined) cost trading at any node not belonging to Θ.

Definition 19.3. ($\mathcal{P}^{\Theta}$)
Let $\Theta \subset \Lambda$.

We will write $\mathcal{P}^{\Theta}$ to mean the set of pairs $(\mathbb{Q}, \tilde{S})$ such that $\mathbb{Q}$ is an EMM with respect to $\tilde{S}$ in the model $\mathbb{M}^{\Theta}$.

See definition 5.34.

19.3.2. *The combined cost ask price risk-neutral representation theorem.*

Theorem 19.4. *(Combined cost ask price representation theorem)*

For any proportional cost viable model $\mathbb{M}$, with derivative security D, the combined cost ask price of D with one derivative security available is given by:

$$a_C^1(D) = \min_{\Theta \subset \Lambda} \sup_{(Q,\tilde{S}) \in \mathcal{P}^\Theta} \mathbb{E}_{\mathbb{Q}}(D_T^0 + D_T^1 \tilde{S}_T + \sum_{t \in \mathcal{T}} F_t \boldsymbol{1}_t^\Theta).$$

Proof of Theorem 19.4.

Let $\mathbb{M} = (\Omega, \mathcal{F}, \mathbb{P}, S, F, P)$ (note that P is restricted) .

Let D be a derivative security with $D = (D^0, D^1)$.

Recall that for any $\Theta \subset \Lambda$ we define $\mathbb{M}^\Theta := (\Omega, \mathcal{F}, \mathbb{P}, S, F, P^\Theta)$ where P^Θ is such that:

If $\lambda \in \Theta$ then $P^\Theta(\lambda) = P(\lambda)$.

If $\lambda \notin \Theta$ then $P^\Theta = \infty$.

(Please note that here Θ can be thought of as the set of nodes where we can trade, this is in contrast to a similar situation in Theorem 18.12 where Θ can be thought of as the set of nodes where no trading happens).

Let D^Θ be a derivative security such that $D^\Theta = (D^0 + \sum_{t \in \mathcal{T}} F_t \mathbf{1}_t^\Theta, D^1)$.

We will write $a_P^{\mathbb{M}^\Theta}(D)$ to mean the proportional cost ask price of D in model $\mathbb{M}^\Theta$ with unlimited availability. Note that under proportional costs restricting the availability does not affect the ask price of a derivative, but we specify unlimited availability in order to be precise.

For any portfolio strategy (X, Y), we will write $Z^{\Theta(X,Y)}$ to mean the number of trades process with respect to derivative D^Θ, see definition 12.14.

The following is an overview of the chain of equalities that this proof will follow, each of these equalities needs to be shown:

$$\begin{aligned}
a_C^1(D) &\overset{\mathbf{(1)}}{=} \inf\{X_0 : (X,Y) \in \mathrm{Srep}_C^{\mathbb{M}}(D)\} \\
&\overset{\mathbf{(2)}}{=} \min_{\Theta \subset \Lambda} \inf\{X_0 : (X,Y) \in \mathrm{Srep}_C^{\mathbb{M}}(D) \text{ and } Z^{(X,Y)}(\lambda) \neq 0 \Leftrightarrow \lambda \in \Theta\ \} \\
&\overset{\mathbf{(3)}}{=} \min_{\Theta \subset \Lambda} \inf\{X_0 : (X,Y) \in \mathrm{Srep}_P^{\mathbb{M}}(D^\Theta) \text{ and } Z^{\Theta(X,Y)}(\lambda) \neq 0 \Leftrightarrow \lambda \in \Theta\} \\
&\overset{\mathbf{(4)}}{=} \min_{\Theta \subset \Lambda} \inf\{X_0 : (X,Y) \in \mathrm{Srep}_P^{\mathbb{M}^\Theta}(D^\Theta)\} \\
&\overset{\mathbf{(5)}}{=} \min_{\Theta \subset \Lambda} \{a_P^{\mathbb{M}^\Theta}(D^\Theta)\}
\end{aligned}$$

$$\overset{(6)}{=} \min_{\Theta\subset\Lambda} \sup_{(Q,\tilde{S})\in\mathcal{P}^{\Theta}} \mathbb{E}_{\mathbb{Q}}((D^0_T + \sum_{t\in\mathcal{T}} F_t \mathbf{1}^{\Theta}_t) + D^1_T \tilde{S}_T).$$

We will now prove each of these 6 equalities in turn:

Part 1 Proof of **(1)**

$a^1_C(D) = \inf\{X_0 : (X,Y) \in \mathrm{Srep}^{\mathbb{M}}_C(D)\}$.

See Theorem 15.3.

Part 2 Proof of **(2)**

$\inf\{X_0 : (X,Y) \in \mathrm{Srep}^{\mathbb{M}}_C(D)\}$

$= \min\limits_{\Theta\subset\Lambda} \inf\{X_0 : (X,Y) \in \mathrm{Srep}^{\mathbb{M}}_C(D)$ and $Z^{(X,Y)}(\lambda) \neq 0 \Leftrightarrow \lambda \in \Theta\ \}$.

Let (X', Y') be a portfolio strategy realising $\inf\{X_0 : (X,Y) \in \mathrm{Srep}^{\mathbb{M}}_C(D)\}$
(we know that such a portfolio strategy exists because of Lemma 15.4).

We can associate with this portfolio strategy (X', Y') a collection of nodes $\Theta' \in \Lambda$ consisting of the nodes λ at which $Z^{(X',Y')}(\lambda) \neq 0$.

For this choice of Θ' we have

$\inf\{X_0 : (X,Y) \in \mathrm{Srep}^{\mathbb{M}}_C(D)$ and $Z^{(X,Y)}(\lambda) \neq 0$ if and only if $\lambda \in \Theta'\ \}$

$= X'_0$

$= \inf\{X_0 : (X,Y) \in \mathrm{Srep}^{\mathbb{M}}_C(D)\}$.

Therefore

$\inf\{X_0 : (X,Y) \in \mathrm{Srep}^{\mathbb{M}}_C(D)\} \geq \min\limits_{\Theta\subset\Lambda} \inf\{X_0 : (X,Y) \in \mathrm{Srep}^{\mathbb{M}}_C(D)$ and $Z^{(X,Y)}(\lambda) \neq 0 \Leftrightarrow \lambda \in \Theta$ $\}$.

Since $\{X_0 : (X,Y) \in \mathrm{Srep}^{\mathbb{M}}_C(D)$ and $Z^{(X,Y)}(\lambda) \neq 0 \Leftrightarrow \lambda \in \Theta\ \} \subset \{X_0 : (X,Y) \in \mathrm{Srep}^{\mathbb{M}}_C(D)\}$ for any Θ it follows that

$\inf\{X_0 : (X,Y) \in \mathrm{Srep}^{\mathbb{M}}_C(D)\} \leq \min\limits_{\Theta\subset\Lambda} \inf\{X_0 : (X,Y) \in \mathrm{Srep}^{\mathbb{M}}_C(D)$ and $Z^{(X,Y)}(\lambda) \neq 0 \Leftrightarrow \lambda \in \Theta$ $\}$.

And thus we can conclude that:

$\inf\{X_0 : (X,Y) \in \mathrm{Srep}_C^{\mathbb{M}}(D)\} = \min\limits_{\Theta\subset\Lambda} \inf\{X_0 : (X,Y) \in \mathrm{Srep}_C^{\mathbb{M}}(D)$ and $Z^{(X,Y)}(\lambda) \neq 0 \Leftrightarrow \lambda \in \Theta$ $\}$.

Part 3 Proof of **(3)**

$\min\limits_{\Theta\subset\Lambda} \inf\{X_0 : (X,Y) \in \mathrm{Srep}_C^{\mathbb{M}}(D)$ and $Z^{(X,Y)}(\lambda) \neq 0 \Leftrightarrow \lambda \in \Theta$ $\}$

$= \min\limits_{\Theta\subset\Lambda} \inf\{X_0 : (X,Y) \in \mathrm{Srep}_P^{\mathbb{M}}(D^\Theta)$ and $Z^{\Theta(X,Y)}(\lambda) \neq 0 \Leftrightarrow \lambda \in \Theta\}$.

Take any $\Theta \in \Lambda$.

We show that

$\inf\{X_0 : (X,Y) \in \mathrm{Srep}_C^{\mathbb{M}}(D)$ and $Z^{(X,Y)}(\lambda) \neq 0 \Leftrightarrow \lambda \in \Theta$ $\}$

$= \inf\{X_0 : (X,Y) \in \mathrm{Srep}_P^{\mathbb{M}}(D^\Theta)$ and $Z^{\Theta(X,Y)}(\lambda) \neq 0 \Leftrightarrow \lambda \in \Theta\}$

Let Z be a trading strategy that is such that $Z(\lambda) \neq 0$ if and only if $\lambda \in \Theta$.

Let $(x, 0)$ be an initial portfolio.

Let (X^C, Y^C) be a portfolio strategy that proportional cost super-replication corresponds to Z and $(x, 0)$ with respect to derivative D.

Let (X^P, Y^P) be a portfolio strategy that combined cost super-replication corresponds to Z and $(x, 0)$ with respect to derivative D^Θ.

We show that $(X^C_{T+1}, Y^C_{T+1}) = (X^P_{T+1}, Y^P_{T+1})$.

$$\begin{aligned}
X^C_{T+1} &= x + \sum_{t=0}^{T} \Big[- Z_t S_t - F_t \mathbf{1}_{Z_t \neq 0} - |Z_t| P_t \Big] - D^0_T \\
&= x + \sum_{t=0}^{T} \Big[- Z_t S_t - |Z_t| P_t \Big] - D^0_T - \sum_{t=0}^{T} F_t \mathbf{1}_{Z_t \neq 0} \\
&= x + \sum_{t=0}^{T} \Big[- Z_t S_t - |Z_t| P_t \Big] - D^0_T - \sum_{t=0}^{T} F_t \mathbf{1}^\Theta_t \\
&= X^P_{T+1}.
\end{aligned}$$

$$\begin{aligned}
Y^C_{T+1} &= Y_0 + \sum_{t=0}^{T} [Z_t] - D^1_T \\
&= Y^P_{T+1}
\end{aligned}$$

This means that for any portfolio strategy (X^C, Y^C) that combined cost super-replicates D and trades at a node λ if and only if $\lambda \in \Theta$, there exists a portfolio strategy (X^P, Y^P) that has the same initial portfolio and proportional cost super-replicates D^Θ and trades at a node λ if and only if $\lambda \in \Theta$.

The reverse is also true, i.e. for any portfolio strategy (X^P, Y^P) that proportional cost super-replicates D^Θ and trades at a node λ if and only if $\lambda \in \Theta$, there exists a portfolio strategy (X^C, Y^C) that has the same initial portfolio and combined cost super-replicates D and trades at a node λ if and only if $\lambda \in \Theta$.

Therefore

$\inf\{X_0 : (X,Y) \in \mathrm{Srep}_C^{\mathbb{M}}(D) \text{ and } Z^{(X,Y)}(\lambda) \neq 0 \text{ if and only if } \lambda \in \Theta\ \}$

$= \inf\{X_0 : (X,Y) \in \mathrm{Srep}_P^{\mathbb{M}}(D^\Theta) \text{ and } Z^{\Theta(X,Y)}(\lambda) \neq 0 \text{ if and only if } \lambda \in \Theta\}$

It follows that **(3)** is true.

Part 4 Proof of **(4)**

$\min\limits_{\Theta \subset \Lambda} \inf\{X_0 : (X,Y) \in \mathrm{Srep}_P^{\mathbb{M}}(D^\Theta) \text{ and } Z^{\Theta(X,Y)}(\lambda) \neq 0 \Leftrightarrow \lambda \in \Theta\}$
$= \min\limits_{\Theta \subset \Lambda} \inf\{X_0 : (X,Y) \in \mathrm{Srep}_P^{\mathbb{M}^\Theta}(D^\Theta)\}.$

We show that:

$\inf\{X_0 : (X,Y) \in \mathrm{Srep}_P^{\mathbb{M}}(D^\Theta) \text{ and } Z^{\Theta(X,Y)}(\lambda) \neq 0 \Leftrightarrow \lambda \in \Theta\}.$

$= \inf\{X_0 : (X,Y) \in \mathrm{Srep}_P^{\mathbb{M}^\Theta}(D^\Theta)\}$

for any $\Theta \subset \Lambda$.

Take any $\Theta \subset \Lambda$.

Observe that if a portfolio strategy (X,Y) belongs to $\mathrm{Srep}_P^{\mathbb{M}^\Theta}(D^\Theta)$ then $Z^{\Theta(X,Y)}(\lambda) \neq 0$ implies that $\lambda \in \Theta$.

Therefore:

$\inf\{X_0 : (X,Y) \in \mathrm{Srep}_P^{\mathbb{M}^\Theta}(D^\Theta)\}$

$= \inf\{X_0 : (X,Y) \in \mathrm{Srep}_P^{\mathbb{M}^\Theta}(D^\Theta) \text{ and } Z^{\Theta(X,Y)}(\lambda) \neq 0 \Rightarrow \lambda \in \Theta\}.$

Consider a strategy $(\tilde{X},\tilde{Y}) \in \mathrm{Srep}_P^{\mathbb{M}^\Theta}(D^\Theta)$ that is such that $Z^{\Theta(\tilde{X},\tilde{Y})}(\lambda') = 0$ for some $\lambda' \in \Theta$.

Consider any $\epsilon > 0$.

We define a trading strategy $\tilde{Z}^\epsilon$ as follows:

$\tilde{Z}^\epsilon(\lambda) = \frac{\epsilon}{S^\Theta(\lambda') + P^\Theta(\lambda')}$ if $\lambda = \lambda'$.

$\tilde{Z}^\epsilon(\lambda) = Z^{\Theta(\tilde{X},\tilde{Y})}(\lambda)$ if $\lambda \in \Lambda$ and $\lambda \neq \lambda'$.

Let $(\tilde{X}^\epsilon,\tilde{Y}^\epsilon)$ be the portfolio strategy that proportional cost super-replication corresponds to the trading strategy $\tilde{Z}^\epsilon$ and initial portfolio $(\tilde{X}_0 + \epsilon, 0)$.

Observe that $(\tilde{X}^\epsilon,\tilde{Y}^\epsilon)$ belongs to $\mathrm{Srep}_P^{\mathbb{M}^\Theta}(D^\Theta)$ because it has just enough additional cash at to buy

$\frac{\epsilon}{S^\Theta(\lambda')+P^\Theta(\lambda')}$ shares of stock at node λ', but trades in exactly the same way as $Z^{\Theta(\tilde{X},\tilde{Y})}$ everywhere else.

We have shown that for any strategy $(\tilde{X},\tilde{Y}) \in \mathrm{Srep}_P^{\mathbb{M}^\Theta}(D^\Theta)$ that is such that $Z^{\Theta(\tilde{X},\tilde{Y})}(\lambda') = 0$ for some $\lambda' \in \Theta$ we can find another strategy, $(\tilde{X}^\epsilon,\tilde{Y}^\epsilon) \in \mathrm{Srep}_P^{\mathbb{M}^\Theta}(D^\Theta)$ and $\epsilon > 0$, with initial position in cash $\tilde{X}_0 + \epsilon$, that is such that $Z^{\Theta(\tilde{X}^\epsilon,\tilde{Y}^\epsilon)}(\lambda') \neq 0$.

Since we can make ϵ arbitrarily small, it follows that:

$\inf\{X_0 : (X,Y) \in \mathrm{Srep}_P^{\mathbb{M}^\Theta}(D^\Theta) \text{ and } Z^{\Theta(X,Y)}(\lambda) \neq 0 \Rightarrow \lambda \in \Theta\}$

$= \inf\{X_0 : (X,Y) \in \mathrm{Srep}_P^{\mathbb{M}^\Theta}(D^\Theta) \text{ and } Z^{\Theta(X,Y)}(\lambda) \neq 0 \Leftrightarrow \lambda \in \Theta\}$.

The models $\mathbb{M}$ and $\mathbb{M}^\Theta$ are identical on nodes that belong to Θ, so we have that:

$\inf\{X_0 : (X,Y) \in \mathrm{Srep}_P^{\mathbb{M}^\Theta}(D^\Theta) \text{ and } Z^{\Theta(X,Y)}(\lambda) \neq 0 \Leftrightarrow \lambda \in \Theta\}$.

$= \inf\{X_0 : (X,Y) \in \mathrm{Srep}_P^{\mathbb{M}}(D^\Theta) \text{ and } Z^{\Theta(X,Y)}(\lambda) \neq 0 \Leftrightarrow \lambda \in \Theta\}$.

This completes the proof that:

$\inf\{X_0 : (X,Y) \in \mathrm{Srep}_P^{\mathbb{M}}(D^\Theta) \text{ and } Z^{\Theta(X,Y)}(\lambda) \neq 0 \Leftrightarrow \lambda \in \Theta\}$.

$= \inf\{X_0 : (X,Y) \in \mathrm{Srep}_P^{\mathbb{M}^\Theta}(D^\Theta)\}$

for any $\Theta \subset \Lambda$.

It follows immediately that:

$\min\limits_{\Theta\subset\Lambda} \inf\{X_0 : (X,Y) \in \mathrm{Srep}_P^{\mathbb{M}}(D^\Theta) \text{ and } Z^{\Theta(X,Y)}(\lambda) \neq 0 \Leftrightarrow \lambda \in \Theta\}$

$= \min\limits_{\Theta\subset\Lambda} \inf\{X_0 : (X,Y) \in \mathrm{Srep}_P^{\mathbb{M}^\Theta}(D^\Theta)\}$.

<u>Part 5</u> Proof of **(5)**

$\min\limits_{\Theta\subset\Lambda} \inf\{X_0 : (X,Y) \in \mathrm{Srep}_P^{\mathbb{M}^\Theta}(D^\Theta)\} = \min\limits_{\Theta\subset\Lambda}\{a_P^{\mathbb{M}^\Theta}(D^\Theta)\}$.

This follows from Theorem 15.3.

<u>Part 6</u> Proof of **(6)**

$\min\limits_{\Theta\subset\Lambda}\{a_P^{\mathbb{M}^\Theta}(D^\Theta)\} = \min\limits_{\Theta\subset\Lambda} \sup\limits_{(Q,\tilde{S})\in\mathcal{P}^\Theta} \mathbb{E}_{\mathbb{Q}}((D_T^0 + \sum\limits_{t\in\mathcal{T}} F_t \mathbf{1}_t^\Theta) + D_T^1 \tilde{S}_T)$.

This follows from Theorem 18.12.

End of proof of Theorem 19.4.

20. **Conclusions**

The analysis that has been conducted in this thesis takes the most fundamental situation (single risky-asset, discrete time, European derivative) and applies a common sense approach in order to derive a number of advanced results. This means that although many of these proofs are only done in this fundamental setting, they strike at the heart of why the results are true and thus lend themselves to extension into a larger framework. An instructive example of this, is our new proof of the proportional cost fundamental theorem, that immediately allowed us to adapt the proof in order to prove the combined cost fundamental theorem. Interestingly it turned out that the proof of the combined cost fundamental theorem by this method was even simpler than the proof of the proportional cost fundamental theorem. A clear next step for research relating to the fundamental theorem, would be to prove a analogous result, for a model with multiple risky assets under combined transaction costs.

One of the most interesting results that we have derived in this thesis is the risk-neutral representation for the ask price of a derivative under combined costs with limited availability as it highlights an imaginative way of getting around the problems connected to the lack of convexity in markets with a fixed transaction cost. This idea could potentially now be used in other contexts. In 2008 Roux and Zastawniak produced groundbreaking work on the theory of American options, see [23]. In particular they found a risk-neutral representation for the ask price of an American derivative security under proportional transaction costs. An interesting direction of study would be to prove a similar result for American derivative securities in the combined cost setting.

Over the course of this thesis we have constructed a model set up that lends itself to the study of combined transaction costs and used it to produce a number of beautiful results, with the combined cost fundamental theorem and ask price representation theorem being highlights. I hope that the reader has enjoyed the journey of exploration that we have been through together, and wish that it will inspire many more fascinating results in the future.

21. **Glossary of definitions, conventions and notation**

Table of definitions part 1

Definition	Concepts introduced	Symbols introduced
5.1	Node	
5.2	Successor Node	
5.4	Model	$\mathbb{M}, S, F, P, \mathbb{R}^+, \mathbb{R}^\infty$
5.5	Set of nodes at time t	Λ_t
5.6	Root node	Ω
5.7	Terminal node	
5.8	Set of all nodes	Λ
5.3	Set of successor nodes	$\text{Succ}\lambda$
5.9	Node at time t in scenario ω	λ_t^ω
5.10	Ancestor node	
5.11	Descendant node	
5.12	Set of Descendant nodes	$\text{Desc}\lambda$
5.13	n-step sub-model at a node λ	
5.14	Sub-model at node λ	
5.16	Time set notation	$\mathcal{T}^{-0}, \mathcal{T}^{-T}, \mathcal{T}^{-0,T}$
5.17	Indicator function	$\mathbf{1}_{\text{condition}}$
5.18	Indicator process	$\mathbf{1}^\Theta$
5.19	Upper bound on fixed transaction cost	$\hat{F}$
5.20	Stock ask and bid price processes	S^a, S^b
5.21	Restricted model, unrestricted model	
5.23	Bid-ask spread	
5.24	Equivalent martingale measure with respect to a process $\tilde{S}$ (EMM)	
5.25	Absolutely continuous martingale measure with respect to a process $\tilde{S}$ (AMM)	
5.27	Set of node and successor node pairs	Φ
5.28	Conditional probability function	$\mathbf{Q}$
5.30	Probability measure corresponding to a conditional probability function	
5.31	Absolutely continuous martingale conditional probability function (AMF)	
5.33	Expectation under an AMF	$\mathbb{E}_{\mathbf{Q}}$
5.34	Set of EMM pairs	$\mathcal{P}$
5.35	Set of AMM pairs	$\tilde{\mathcal{P}}$
6.1	Portfolio	
6.2	Portfolio operation	
6.4	Stock discard	
6.5	Cash discard	

Table of definitions part 2

Definition	Concepts introduced	Symbols introduced
6.6	Trade portfolio operation	
6.7	Portfolio strategy	(X,Y)
6.9	Trading strategy	Z
6.10	Trade notation	$\xrightarrow{Z(\lambda)}, \xrightarrow[F]{Z(\lambda)}, \xrightarrow[P]{Z(\lambda)}, \xrightarrow[C]{Z(\lambda)}$
6.11	Self-financing portfolio strategy, correspondence between trading strategies and self-financing portfolio strategies	
6.15	Terminology for when a portfolio strategy is deemed to have traded or not	
6.17	Arrival value of a portfolio strategy	$A^{(X,Y)}$
6.18	Exist value of a portfolio strategy	$E^{(X,Y)}$
7.1	Arbitrage opportunity	
7.3	Strong arbitrage opportunity	
7.4	Viable model	
9.3	Catogorisation of single-step models	
10.3	Frictionless middle value process	
10.5	Process of sets of real numbers such that it is possible to construct a frictionless middle value process	$\mathcal{Q}$
10.6	$\mathcal{R} := \mathcal{Q} \cap [S^b, S^a]$	$\mathcal{R}$
10.8	Lower and upper ends of the sets $\mathcal{Q}$ and $\mathcal{R}$	Q^a, Q^b, R^a, R^b
11.1	Fixed cost middle value stock process	
11.2	Process of sets of real numbers such that it is possible to construct a fixed cost middle value process	$\tilde{\mathcal{Q}}, \tilde{\mathcal{R}}$
11.3	$\tilde{\mathcal{R}} := \tilde{\mathcal{Q}} \cap [S^b, S^a]$	$\tilde{\mathcal{R}}$
11.5	Lower and upper ends of the sets $\tilde{\mathcal{Q}}$ and $\tilde{\mathcal{R}}$	$\tilde{Q}^a, \tilde{Q}^b, \tilde{R}^a, \tilde{R}^b$
12.1	European derivative security	D
12.2	Initial price	d_0
12.5	Derivative super-replication portfolio strategy, correspondence between trading strategies and derivative super-replication portfolio strategies	
12.6	Set of Super-replication portfolio strategies for D	$\mathrm{Srep}(D), \mathrm{Srep}_F(D),$ $\mathrm{Srep}_P(D), \mathrm{Srep}_C(D)$
12.7	Super-replicability	
12.9	Derivative self-financing portfolio strategy, correspondence between trading strategies and derivative self-financing portfolio strategies	
12.14	Number of trades process of a portfolio strategy	$Z^{(X,Y)}$
12.15	Set of strategies enabling a portfolio to super-replicate	$\mathcal{Z}^{(x,y)}, \mathcal{Z}_F^{(x,y)}, \mathcal{Z}_P^{(x,y)}, \mathcal{Z}_C^{(x,y)}$
12.16	Arrival set	$\mathcal{A}^D, \mathcal{A}_F^D, \mathcal{A}_P^D, \mathcal{A}_C^D$
12.17	Exit set	$\mathcal{E}^D, \mathcal{E}_F^D, \mathcal{E}_P^D, \mathcal{E}_C^D$
12.18	Linear slice	
12.19	Piecewise linear hull	

Table of definitions part 3

Definition	Concepts introduced	Symbols introduced
13.1	Extended arbitrage opportunity, extended arbitrage opportunity with n derivatives, extended arbitrage opportunity for the seller vs extended arbitrage opportunity for the buyer	
14.3	Ask price of a derivative security	$a(D), a_F(D),$ $a_P(D), a_C(D)$
14.4	Bid price of a derivative security	$b(D), b_F(D),$ $b_P(D), b_C(D)$
14.5	Viable price of a derivative security with limited availability	
14.7	Ask price of a derivative security with limited availability	$a^n(D), a^n_F(D),$ $a^n_P(D), a^n_C(D)$
14.8	Bid price of a derivative security with limited availability	$b^n(D), b^n_F(D),$ $b^n_P(D), b^n_C(D)$
19.2	Adjusted stock price model	$\mathbb{M}^\Theta$
19.3	Set of martingale pairs on an adjusted stock price model	$\mathcal{P}^\Theta$
22.4	Adapted process	
22.5	Predictable process	
22.6	Extended predictable process	

Table of conventions

Convention	Description
1	Reference to a model
2	Notation for processes and random variables
3	Convention for notation when there is more than one model
4	Restricted P vs Unrestricted P
5	EMM and AMM with respect to S
6	Zero multiplied by infinity
7	Equations involving random variables
8	Reference to a trading strategy as an arbitrage opportunity
9	Reference to a trading strategy as an extended arbitrage opportunity

22. **Appendix**

22.1. **Additional results about frictionless arbitrage.**

Lemma 22.1.
Let (X, Y) be a frictionless (resp fixed cost, proportional cost, combined cost) self-financing portfolio strategy.

Suppose that for some non-terminal node $\lambda \in \Lambda$ we have that $A^{(X,Y)}(\lambda) < A^{(X,Y)}(\mu)$ for all $\mu \in \mathrm{Succ}\lambda$.

Then there exists a frictionless arbitrage opportunity in the single-step sub-model with root node λ.

Proof of Lemma 22.1

The frictionless (resp fixed cost, proportional cost, combined cost) self-financing condition tells us that $E^{(X,Y)}(\lambda) \leq A^{(X,Y)}(\lambda)$

Therefore it follows that $E^{(X,Y)}(\lambda) < A^{(X,Y)}(\mu)$ for all $\mu \in \mathrm{Succ}\lambda$.

This implies that $Y(\mu)S(\lambda) < Y(\mu)S(\mu)$ for all $\mu \in \mathrm{Succ}\lambda$.

Therefore either $S(\lambda) < S(\mu)$ for all $\mu \in \mathrm{Succ}\lambda$ or $S(\lambda) > S(\mu)$ for all $\mu \in \mathrm{Succ}\lambda$

This means that the single-step model at node λ is of type 4 and thus admits frictionless arbitrage.

End of proof of Lemma 22.1

Lemma 22.2.
Let (X, Y) be a frictionless (resp fixed cost, proportional cost, combined cost) self-financing portfolio strategy.

Let $t \in \mathcal{T}^{-T}$

Suppose that $\min\limits_{\omega\in\Omega} A_t^{(X,Y)}(\omega) < \min\limits_{\omega\in\Omega} A_{t+1}^{(X,Y)}(\omega)$

Then there exists a frictionless arbitrage opportunity.

Proof of Lemma 22.2

Since $\min\limits_{\omega\in\Omega} A_t^{(X,Y)}(\omega) < \min\limits_{\omega\in\Omega} A_{t+1}^{(X,Y)}(\omega)$, there must exist a node $\lambda \in \Lambda_t$ such that:

$A^{(X,Y)}(\lambda) < A^{(X,Y)}(\mu)$ for all $\mu \in \mathrm{Succ}\lambda$.

It follows from Lemma 22.1 that there exists a frictionless arbitrage opportunity.

End of proof of Lemma 22.2

Lemma 22.3.
Let (X,Y) be a frictionless (resp fixed cost, proportional cost, combined cost) self-financing portfolio strategy.

Suppose that $\min_{\omega\in\Omega} A_0^{(X,Y)}(\omega) < \min_{\omega\in\Omega} A_T^{(X,Y)}(\omega)$.

Then there exists a frictionless arbitrage opportunity.

Proof of Lemma 22.3

If $\min_{\omega\in\Omega} A_t^{(X,Y)}(\omega) \geq \min_{\omega\in\Omega} A_{t+1}^{(X,Y)}(\omega)$ for all $t \in \mathcal{T}^{-T}$ then we would have :
$\min_{\omega\in\Omega} A_0^{(X,Y)}(\omega) \geq \min_{\omega\in\Omega} A_T^{(X,Y)}(\omega)$.
Therefore there must exist a $t \in \mathcal{T}^{-T}$ such that $\min_{\omega\in\Omega} A_t^{(X,Y)}(\omega) < \min_{\omega\in\Omega} A_{t+1}^{(X,Y)}(\omega)$.

It follows from Lemma 22.2 that there exists a frictionless arbitrage opportunity.

End of proof of Lemma 22.3

22.2. **Processes.**

Definition 22.4. (Adapted process)
An *adapted process* A on a filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t\in\mathcal{T}}$ and set of possible outcomes Ω is a sequence of random variables $A_0, A_1, \ldots, A_T$ such that A_t is $\mathcal{F}_t$ measurable for all $t \in \mathcal{T}$.

Definition 22.5. (Predictable process)
A *predictable process* A on a filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t\in\mathcal{T}}$ and set of possible outcomes Ω is a sequence of random variables $A_0, A_1, \ldots, A_T$ such that A_t is $\mathcal{F}_{t-1}$ measurable for all $t \in \mathcal{T}^{-0}$ and A_0 is $\mathcal{F}_0$ measurable.

Definition 22.6. (Extended predictable process)
An *extended predictable process* A on a filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t\in\mathcal{T}}$ and set of possible outcomes Ω is a sequence of random variables $A_0, A_1, \ldots, A_T, A_{T+1}$ such that A_t is $\mathcal{F}_{t-1}$ measurable for all $t \in \mathcal{T}^{-0} \cup \{T+1\}$ and A_0 is $\mathcal{F}_0$ measurable.

22.3. **Bid-ask spreads.**

Lemma 22.7.
Let $\mathbb{M} = (\Omega, \mathcal{F}, \mathbb{P}, S, F, P)$ and $\tilde{\mathbb{M}} = (\Omega, \mathcal{F}, \mathbb{P}, \tilde{S}, F, \tilde{P})$ be unrestricted proportional cost viable models.

Suppose that the bid-ask spreads of $\mathbb{M}$ are contained in the bid-ask spreads of $\tilde{\mathbb{M}}$.

For any proportional cost self-financing portfolio strategy $(\tilde{X}, \tilde{Y})$ there exists a proportional cost self-financing portfolio strategy (X, Y) such that $X_0 = \tilde{X}_0$, $X_{T+1} \geq \tilde{X}_{T+1}$ and $Y_{T+1} \geq \tilde{Y}_{T+1}$.

For any combined cost self-financing portfolio strategy $(\tilde{X}, \tilde{Y})$ there exists a combined cost self-financing portfolio strategy (X, Y) such that $X_0 = \tilde{X}_0$, $X_{T+1} \geq \tilde{X}_{T+1}$ and $Y_{T+1} \geq \tilde{Y}_{T+1}$.

Proof of Lemma 22.7

We prove the proportional cost case.

Let $(\tilde{X}, \tilde{Y})$ be a proportional cost self-financing strategy that proportional cost corresponds to trading strategy $\tilde{Z}$ in model $\tilde{\mathbb{M}}$.

Let (X, Y) be the proportional cost self-financing strategy that proportional cost corresponds to trading strategy $\tilde{Z}$ and initial portfolio $(\tilde{X}_0, 0)$ in the model $\mathbb{M}$.

Observe that (X, Y) is such that $X_0 = \tilde{X}_0$, $X_{T+1} \geq \tilde{X}_{T+1}$ and $Y_{T+1} \geq \tilde{Y}_{T+1}$.

The proof for the combined cost case is an identical argument.

End of proof of Lemma 22.7

Lemma 22.8.
Let $\mathbb{M} = (\Omega, \mathcal{F}, \mathbb{P}, S, F, P)$ and $\tilde{\mathbb{M}} = (\Omega, \mathcal{F}, \mathbb{P}, \tilde{S}, F, \tilde{P})$ be unrestricted proportional cost viable models.

Suppose that the bid-ask spreads of $\mathbb{M}$ are contained in the bid-ask spreads of $\tilde{\mathbb{M}}$.

Then for any portfolio strategy $(\tilde{X}, \tilde{Y}) \in \mathrm{Srep}_P^{\tilde{\mathbb{M}}}$ there exists a portfolio strategy $(X, Y) \in \mathrm{Srep}_P^{\mathbb{M}}(D)$ such that $X_0 = \tilde{X}_0$, $X_{T+1} \geq \tilde{X}_{T+1}$, $Y_{T+1} \geq \tilde{Y}_{T+1}$ and

$Z^{(X,Y)}(\lambda) = Z^{(\tilde{X},\tilde{Y})}(\lambda)$ for all $\lambda \in \Lambda$.

Proof of Lemma 22.8

Let $(\tilde{X}, \tilde{Y}) \in \mathrm{Srep}_P^{\tilde{\mathbb{M}}}$.

Let Z be the trading strategy that proportional cost super-replication corresponds to $(\tilde{X}, \tilde{Y})$ in the model $\tilde{\mathbb{M}}$.

This means that:

$$\tilde{X}_T = \tilde{X}_0 + \sum_{t=0}^{T} \Big[- Z_t \tilde{S}_t - |Z_t| \tilde{P}_t \Big] - D^0.$$

$$\tilde{Y}_T = \sum_{t=0}^{T} [Z_t] - D^1.$$

(see Lemma 12.10).

Let (X, Y) be the portfolio strategy that proportional cost corresponds to Z with the initial portfolio $(\tilde{X}_0, 0)$ in the model $\tilde{\mathbb{M}}$.

This means that:

$$X_T = \tilde{X}_0 + \sum_{t=0}^{T} \Big[- Z_t S_t - |Z_t| P_t \Big] - D^0.$$

$$Y_T = \sum_{t=0}^{T} [Z_t] - D^1.$$

Observe that $X_{T+1} \geq \tilde{X}_{T+1}$ and $Y_{T+1} \geq \tilde{Y}_{T+1}$.

Since (X, Y) and $(\tilde{X}, \tilde{Y})$ correspond to the same trading strategy Z it is immediate that $Z^{(X,Y)}(\lambda) = Z^{(\tilde{X},\tilde{Y})}(\lambda)$ for all $\lambda \in \Lambda$.

End of proof of Lemma 22.8

22.4. **Types of extended arbitrage.**

22.4.1. *Buyers arbitrage vs Sellers arbitrage.*

Lemma 22.9.
For any derivative security D in an unrestricted model $\mathbb{M}$ we have the following:

If $\mathbb{M}$ is frictionless viable then:

(1) If $\mathcal{V}(D)$ is non-empty and $d_0 > a(D)$ then for any frictionless extended arbitrage opportunity consisting of portfolio strategy (X, Y) and integer n, we must have $n > 0$ (i.e. it is an arbitrage opportunity for the seller).

(2) If $\mathcal{V}_F(D)$ is non-empty and $d_0 > a_F(D)$ then for any fixed cost extended arbitrage opportunity consisting of portfolio strategy (X, Y) and integer n, we must have $n > 0$ (i.e. it is an arbitrage opportunity for the seller).

(3) If $\mathcal{V}(D)$ is non-empty and $d_0 < a(D)$ then for any frictionless extended arbitrage opportunity consisting of portfolio strategy (X, Y) and integer n, we must have $n < 0$.

(4) If $\mathcal{V}_F(D)$ is non-empty and $d_0 < a_F(D)$ then for any fixed cost extended arbitrage opportunity consisting of portfolio strategy (X, Y) and integer n, we must have $n < 0$.

If $\mathbb{M}$ is proportional cost viable then:

(5) If $\mathcal{V}_P(D)$ is non-empty and $d_0 > a_P(D)$ then for any proportional cost extended arbitrage opportunity consisting of portfolio strategy (X, Y) and integer n, we must have $n > 0$.

(6) If $\mathcal{V}_C(D)$ is non-empty and $d_0 > a_C(D)$ then for any combined cost extended arbitrage opportunity consisting of portfolio strategy (X, Y) and integer n, we must have $n > 0$.

(7) If $\mathcal{V}_P(D)$ is non-empty and $d_0 < a_P(D)$ then for any proportional cost extended arbitrage opportunity consisting of portfolio strategy (X, Y) and integer n, we must have $n < 0$.

(8) If $\mathcal{V}_C(D)$ is non-empty and $d_0 < a_C(D)$ then for any combined cost extended arbitrage opportunity consisting of portfolio strategy (X, Y) and integer n, we must have $n < 0$.

(In other words, if the derivative price is greater than the ask price then there is an extended arbitrage opportunity for the seller and if the derivative price is less than the bid price then there is an extended arbitrage opportunity for the buyer.)

Proof of Lemma 22.9.

Proof of (1)

Suppose that $d_0 > a(D)$ and there exists a frictionless extended arbitrage opportunity consisting of trading strategy (X, Y) and integer $n < 0$ ($n = 0$ would contradict the assumption that $\mathbb{M}$ is frictionless viable).

Notice that the only time that d_0 appears in the conditions for this to be a frictionless extended arbitrage opportunity is the self-financing condition at the root node which reads as follows:

$A^{(X,Y)}(\Omega) \geq E^{(X,Y)}(\Omega) - nd_0$ (See lemma 12.23).

Because of the assumption that $n \leq 0$, as d_0 decreases the RHS decreases and the equation will still hold.

This means that for any initial price $d_0' \leq d_0$ (including $d_0' \leq a(D)$) there will still exist a frictionless extended arbitrage opportunity.

This contradicts the assumption that there exists a frictionless viable price and the fact that $a(D)$ is the supremum of frictionless viable prices.

Therefore if $d_0 > a(D)$ and there exists a frictionless extended arbitrage opportunity then it must be a frictionless extended arbitrage opportunity for the seller.

This completes the proof of (1).

The proofs of (2),(5) and (6) are almost identical.

The proofs of (3),(4),(7) and (8) follow from a symmetrical argument to the proofs of (1),(2),(5) and (6) respectively.

End of proof of Lemma 22.9.

22.4.2. *Scaling up an extended arbitrage opportunity.*

Lemma 22.10.
Let $n \in \mathbb{N}$

For any derivative security D we have the following:

(1) If there exists a frictionless extended arbitrage opportunity with $n' \in \mathbb{Z}$ derivatives where $0 < n' \leq n$, then there exists a frictionless extended arbitrage opportunity with precisely n derivative securities.

(2) If there exists a fixed cost extended arbitrage opportunity with $n' \in \mathbb{Z}$ derivatives where $0 < n' \leq n$, then there exists a fixed cost extended arbitrage opportunity with precisely n derivative securities.

(3) If there exists a proportional cost extended arbitrage opportunity with $n' \in \mathbb{Z}$

derivatives where $0 < n' \leq n$, then there exists a proportional cost extended arbitrage opportunity with precisely n derivative securities.

(4) If there exists a combined cost extended arbitrage opportunity with $n' \in \mathbb{Z}$ derivatives where $0 < n' \leq n$, then there exists a combined cost extended arbitrage opportunity with precisely n derivative securities.

(5) If there exists a frictionless extended arbitrage opportunity with $n' \in \mathbb{Z}$ derivatives where $0 > n' \geq -n$, then there exists a frictionless extended arbitrage opportunity with precisely $-n$ derivative securities.

(6) If there exists a fixed cost extended arbitrage opportunity with $n' \in \mathbb{Z}$ derivatives where $0 > n' \geq -n$, then there exists a fixed cost extended arbitrage opportunity with precisely n derivative securities.

(7) If there exists a proportional cost extended arbitrage opportunity with $n' \in \mathbb{Z}$ derivatives where $0 > n' \geq -n$, then there exists a proportional cost extended arbitrage opportunity with precisely n derivative securities.

(8) If there exists a combined cost extended arbitrage opportunity with $n' \in \mathbb{Z}$ derivatives where $0 > n' \geq -n$, then there exists a combined cost extended arbitrage opportunity with precisely n derivative securities.

Proof of Lemma 22.10.

Proof of (1)

Let $n \in \mathbb{N}$ and suppose that there exists a frictionless extended arbitrage opportunity consisting of trading strategy Z and n' derivatives, for some $n' \in \mathbb{Z}$ such that $0 < n' \leq n$.

Let Z' be a process such that $Z'(\lambda) = \frac{n}{n'} Z(\lambda)$ for all $\lambda \in \Lambda$.

Observe that this trading strategy Z' combined with purchasing n lots of derivative security D is an extended arbitrage opportunity. See lemma 13.2.

The best way to see this is to think of the example of having a fixed cost extended arbitrage opportunity with one derivative security, and observe that if we purchase the associated trading strategy twice then this could combine with purchasing two derivative securities to give a fixed cost extended arbitrage opportunity with two derivative securities. The factor $\frac{n}{n'}$ corresponds to the factor $\frac{2}{1} = 2$ in the example of going from one derivative security to two.

The proofs of (2),(3) and (4) are almost identical to the proof of (1).

The proofs of (5),(6),(7) and (8) are symmetrical to the proofs of (1),(2),(3) and (4) respectively.

End of proof of Lemma 22.10.

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