Exactly Solvable Models for Topological Phases of Matter and Emergent Excitations

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The candidate confirms that the work submitted is his own and that appropriate credit has been given where reference has been made to the work of others.

 I confirm that the work submitted in this thesis is my own, except where work which has formed part of jointly authored publications has been included. The contribution of myself and the other authors to this work has been explicitly indicated below. I confirm that appropriate credit has been given within the thesis where reference has been made to the work of others.

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- I was responsible for the proposing the project, defining the model, and demonstrating the relation to the Yetter homotopy 2-type TQFT and Walker-Wang models. Martins was responsible for demonstrating the well definedness of the construction.

To my parents, Peter and Joy, my Grandparents and the fun of it.

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List of publications

[1] A. Bullivant and J. K. Pachos, *Entropic Manifestations of Topological Order in Three Dimensions*, Physical Review B, Vol. **93** 12 (APS ... 2014).

[2] A. Bullivant, M. Calçada, Z. Kádár, P. Martin and J. F. Martins, *Topological phases from higher gauge symmetry in 3+ 1 dimensions*, Physical Review B, Vol. **15** 115-118 (APS ... 2017).

[3] A. Bullivant, M. Calçada, Z. Kádár, P. Martin and J. F. Martins, *Higher lattices, discrete two-dimensional holonomy and topological phases in (3+ 1) D with higher gauge symmetry*, arXiv preprint, arXiv:1702.00868.

[4] A. Bullivant, Y. Hu and Y. Wan, *Twisted Quantum Double Model of Topological Orders with Boundaries*, arXiv preprint arXiv:1706.03611.

Abstract

Over the past 30 years experimental observations have demonstrated the existence of a variety of quantum phases of matter not admissible to a classification in terms of the Landau theory of symmetry breaking. Examples include, but are not limited to the fractional quantum Hall states and frustrated quantum magnets. Theoretical evidence supports the idea that such phases can exist in a large class of zero temperature strongly correlated condensed matter systems.

In this thesis we study a particular case of such systems called topological phases of matter. Such phases are characterised by the presence of non-local correlations which are manifest in properties such as degenerate groundstates that depend on the global topology of the system and the emergence of topological excitations. Remarkably the classification of such materials is profoundly tied to the mathematical construction of topological quantum field theories (TQFT).

In this thesis we utilise this connection to explore possible candidate Hamiltonian models for topological phases of matter. Our methodology is that of reverse engineering effective local Hamiltonians from a class of discrete TQFT's called state sums.

In chapter 5.2 we develop a construction to canonically associate to any state-sum TQFT a corresponding local Hilbert space and Hamiltonian defining a candidate model for a topological phase of matter. In chapter **??** we develop a candidate model of topological phases using ideas from higher gauge theory and higher category theory. In particular we define a Hamiltonian realisation of the Yetter Homotopy 2-type TQFT which describes a topological gauge theory, where the

gauge symmetry is given by a finite 2-group and relate a class of such models to the construction of Walker-Wang.

Building on the Hamiltonian construction for state-sum TQFT's, in the Part III of this thesis we develop an algebraic approach to understanding the topological excitations of such theories, we call tube-algebras. In chapter 10 we develop a general construction for defining tube-algebras for any unitary state-sum TQFT and describe the general features. In chapter 12 we apply this construction to the Dijkgraaf-Witten TQFT in 1+1, 2+1 and 3+1D. In chapter 13 we apply this construction to topological higher lattice gauge theories and compare the results the Dijkgraaf-Witten TQFT.

Abbreviations

TQFT	Topological quantum field theory
ss-TQFT	State-sum topological quantum field theory
TQC	Topological quantum computing
TPM	Topological phase of matter
LGT	Lattice gauge theory
TLGT	Topological lattice gauge theory
HLGT	Higher lattice gauge theory
THLGT	Topological higher lattice gauge theory
PL	Piece-wise linear
G	A group
π_1	Fundamental group
Γ	A groupoid
BG	A group presented as a groupoid
$\mathcal{G}$	A crossed module of groups
Γ^2	A 2-groupoid
$B\mathcal{G}$	A crossed module of groups presented as a 2-groupoid
Π_2	Fundamental 2-groupoid

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Chapter 1

Introduction

> Relativity unveils the origin of ordinary matter. Condensed matter come in various phases
> - Zhengang Wang

In the pursuit of comprehending the world around us we have successfully boiled down the constituents of everyday matter in terms of three principal building blocks, the electron, the proton and the neutron. Reductionist philosophy has made tremendous progress in classifying the building blocks of matter but says little about the richness of materials we see everyday. Instead the common theme of our reality appears to be that of emergence [1]. Here the material world is not described by only knowledge of the constituents but instead the admissible arrangements of such building blocks. We call such arrangements **orders**. Orders can take on many guises, such as regular orders where the constituents are arranged into repeating patterns such as in crystals or orders can be random such the distribution of molecules in a gas. To describe the spectrum of orders it is informative to define the notion of a phase of matter. Approximately, a **phase of matter** is an equivalence class of orders sharing certain physical characteristics we care about [2]. The tricky component in defining interesting phases of matter lies in the ambiguity of what equivalence class of orders to consider.

A key insight of Lev Landau [3] was that orders could be described in terms of their symmetries. From this observation he was able to develop a systematic approach to classifying orders using the principles of symmetry breaking, to define transitions between different orders and order parameters which transform non-trivially under the symmetries of the system. It was shown that such principles could be applied to a large spectrum of orders from the familiar examples of crystals and ferromagnets to superconductors. Furthermore Landau and Ginzburg [4] were able to define effective field theory descriptions describing the low energy physics of ordered systems.

In the literature it is common to suggest that for a period people thought that symmetry breaking provided a complete classification of orders in condensed matter physics, although in reality I am sure some people had doubts. Historically, the first counter example for the completeness of symmetry breaking as a classification of orders was the experimental realisation of fractional quantum Hall systems [5]. It was quickly found that such systems exhibit many different orders in the limit of zero temperature which have the same symmetry and hence could not be distinguished by symmetry breaking.

Providing an adequate generalisation of Landau theory applicable to the fractional quantum Hall effect [6, 7, 8, 9] became a new theoretical challenge for the classification of orders. The solution was the proposal of a new form of order, dubbed - **topological order** [10, 11].

Unlike symmetry breaking phases it was found that topological orders admit a characterisation in terms of the following physical properties:

- A finite energy gap between the groundstate and excited states
- The number of degenerate groundstates, which depend on the spatial topology of the system
- Non-Abelian Berry phases generated by the mapping class group of the spatial manifold, eg. modular transformations of the torus
- The existence of topological excitations, with non-trivial motion group representations generalising the bosonic/fermionic exchange statistics (anyons in 2+1D)[12, 13, 14, 15]
- When the theory is chiral, gapless edge states

The characterising properties of topological order were furthermore found to be robust in the thermodynamic limit against local perturbations which could break the symmetries of the system. This is in stark contrast to other ordered quantum phases like the Ising ferromagnet, where the degeneracy is found though symmetry breaking and weak magnetic fields which break the symmetry can be used to lift the groundstate degeneracy. Following from the robustness of the characterising features of topological order to symmetry breaking, it became apparent that such characteristics could be used to define an equivalence class of orders sharing such properties. We call this equivalence a **topological phase of matter**.

The next development in classifying topological phases of matter was the formulation of an effective field theory in the low energy/infra-red limit in analogy with Landau-Ginzburg field theories for symmetry breaking phases. This was found in a surprising place, **topological quantum field theories (TQFT)** [16, 17, 18, 19, 20]. First introduced in pursuit of a background independent theory of quantum gravity, and made axiomatic in the mathematics community [17]. TQFT's are roughly speaking quantum field theories whereby the action S is invariant under continuous deformations of space-time. The canonical example 2+1D is Chern-Simons theory where the action is given by:

$$S_{CS} = \frac{k}{2\pi} \int_M \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \tag{1.1}$$

for integer k and gauge field A. The axiomatic definition of a TQFT is given in chapter 4.

From the topological invariance of the action describing a TQFT, such theories are often much simpler to preform computations with than metric dependent field theories. In particular one can sidestep many of the perturbative issues that plague rigorous calculations in more structured field theories describing strongly correlated quantum systems. Once a topological order is identified with its effective TQFT description the characterising properties of the topological order can be directly calculated from the corresponding TQFT action.

The correspondence between topological phases of matter and TQFT reveals lots of insights into the nature of topological phases of matter. One aspect which has proved particularly fruitful in the development of fault-tolerant quantum computation [21, 2, 22, 23] is the relation between the Jones Polynomial and Chern-Simons theory, as elucidated in the work of Witten [18]. From the invariance of the action under continuous deformations it can be shown the the world-line of particles in Chern-Simons theory can be interpreted as defining knot-invariants. Such knot-invariants cannot be efficiently calculated using classical computation schemes but are efficiently simulated in topological phases of matter. It has been further shown that such invariants can be utilised to preform universal quantum computation. In this way topological phases of matter provide a promising candidate material for the physical realisation of quantum computing, where errors from the environment are shielded by the topological invariance of the theory.

1.1 Thesis Overview

Building on the effective field theory description of topological phases of matter given by a TQFT, the purpose of this thesis is to explore a set of candidate Hamiltonian models describing topological phases of matter with an emphasis on classifying the emergent topological excitations from an algebraic point of view.

With this in mind, we define the salient features of quantum many-body systems we want to describe in chapter 5.2. We then define a general framework for generating candidate Hamiltonian models for topological phases of matter using triangulated approximations of space from a given class of TQFT's called state-sums. Such Hamiltonians are defined as a sum of local, mutually commuting projection operators and are thus exactly solvable. In this construction the invariance of the underlying triangulation is emphasised. This construction generalises and includes the string-net models of Levin and Wen [24] which are defined from the Turaev-Viro TQFT and twisted quantum double models [25, 26, 27] defined from the Dijkgraaf-Witten TQFT (the Kitaev quantum double model is the untwisted example of such theories).

In chapter **??** we define a new class of Hamiltonian models generalising topological gauge theories. In particular, motivated by considerations in higher category theory we consider topological gauge theories whereby the underlying gauge group is replaced by a finite 2-group. The motivation for this section is to understand the properties of state-sum TQFTs and their Hamiltonian models which are not given by ordinary gauge groups or the Turaev-Viro/Crane-Yetter TQFT's.

In the final part of this thesis, in chapter 10 we introduce the so called tube-algebras. We argue that such algebras classify the admissible topological excitations in terms of their corresponding simple modules. In this chapter we define the general construction of such algebras applicable to any state-sum TQFT and consider some of the consequences for the theory. We then give examples of such algebras and classify the excitations. In chapter 12 we consider the Dijkgraaf-Witten TQFT in a range of space-time dimensions. In chapter 13 we apply this construction to topological higher lattice gauge theories and compare the results to the untwisted Dijkgraaf-Witten case.

Part I

Mathematical Background

Overview

In the following we review and define conventions the mathematical tools used throughout the remainder of the text. We do so in three chapters.

In chapter 2 we review two discrete constructions of topological manifolds we call cell decompositions given by triangulations and CW-complexes. These constructions will be used throughout the text to provide a mathematical model of the space/space-time of our models.

In chapter 3 we review the basic ingredients of category theory. This thesis is not concerned with the foundational topics of category theory but we will instead invoke such constructions to provide a convenient framework for describing physical systems.

In chapter 4 we will use categorical notions to define axiomatic topological quantum field theories.

Chapter 2

Cell Decompositions of Manifolds

Throughout this thesis we will be interested in topological manifolds X equipped with a **cell decomposition** as providing a suitable mathematical model of space/space-time in physical theories. By cell decomposition we mean a collection of building blocks (not so dissimilar to lego) with a set of rules which tell us how such blocks can be "glued" together to form a topological space homeomorphic to X. The two such schemes we use are **triangulations** and **CW-decompositions**. Both schemes have their advantages and relative drawbacks. In particular triangulations have the advantage of defining a finite set of building blocks, one for each dimension, called simplices, which can be considered as generalised triangles. The gluing rules (at least in low dimension) are relatively intuitive, given by identifying lower dimensional simplices. The drawback of triangulations is that for even relatively simple topological manifolds one may need to utilise a rather large number of simplices to form a homeomorphic topological space. On the other hand CW-complexes have much more freedom in the set of building blocks and hence in many cases problematic for triangulations only a few cells are needed to form a topological space homeomorphic to a given topological manifold. The drawback is that the gluing rules are often more convoluted and require defining an infinite set of data (although in practice this is not so problematic). Triangulations form a subset of CW-complexes. A the classic reference for the following material is book of Hatcher [28].

2.1 Triangulations

On our quest to define triangulations, we begin by defining the n**-simplex**:

Definition 2.1.1. An n**-simplex** $\Delta^n := [v_0 \cdots v_n]$ is the convex hull of a set of $(n+1)$ points $v_0, \cdots, v_n \in \mathbb{R}^{m \geq n}$, we refer to as **vertices**, such that all vectors $v_1 - v_0, \cdots, v_n - v_0$ are linearly independent. The orientation $\sigma(\Delta^n) := sgn(det(v_1 - v_0, \cdots, v_n - v_0))$.

The n-simplex can be seen as defining an n-dimensional version of a triangle (the 2-simplex). We will use the nomenclature: 0-simplex a vertex, 1-simplex an edge, 2-simplex a triangle, 3-simplex a tetrahedron and 4-simplex pentachords.

Definition 2.1.2. Given an n-simplex $\Delta^n = [v_0 \cdots v_n]$ the convex hull of any subset of vertices $[v_{i_0}, \cdots, v_{i_j}]$, for $0 \leq j \leq n$, is a j**-subsimplex** of Δ_n. We notate subsimplices via $\Delta^j \subseteq \Delta^n$.

Given the definition of a subsimplex we can define the notion of a **simplicial complex**:

Definition 2.1.3. A **simplicial complex** K is a set of simplices that satisfy the following two conditions:

1. Given a simplex $\Delta^n \in K$ then for any subsimplex $\Delta^j \subseteq \Delta^n$, $\Delta^j \in K$
2. The intersection of any two simplices Δ, Δ' is a single subsimplex of both simplices such that $\Delta \cap \Delta' \subseteq \Delta$ and $\Delta \cap \Delta' \subseteq \Delta'$

For many purposes it is also useful to work with a weaker notion of simplicial complex called a **Δ-complex**:

Definition 2.1.4. A **Δ-complex** is a simplicial complex whereby condition (2) is weakened such that the intersection of a pair of simplices Δ, Δ' may consist of multiple subsimplices of both simplices. In this way all simplicial complexes are Δ-complexes but not every Δ-complex is a simplicial complex.

In the text we are exclusively concerned with Δ-complexes equipped with a **branching structure**:

Definition 2.1.5. A **branching structure** is an assignment of a total ordering to the vertices of a Δ-complex. A branching structure naturally associates to each edge an orientation from the lesser to the greater ordered adjacent vertices. This condition ensures the edges on the boundary of a 2-simplex never form a cycle.

An important notion for defining triangulations of topological manifolds is:

Definition 2.1.6. The **underlying space** of a Δ-complex K is given by the union of all its simplices, treated as a topological space, denoted by $|K|$

Definition 2.1.7. Let X be a topological space. A Δ-complex K with branching structure is a **triangulation** of X if there exists a homeomorphism

$$\phi : |K| \to X. \tag{2.1}$$

Some useful constructions on Δ-complexes are:

Definition 2.1.8. The k**-skeleton** of a $\Delta-$complex K, denoted K_k is the union of all $j-$subsimplices $\Delta^j \in K$ with $j \leq k$.

Definition 2.1.9. The **boundary** of an n-dimensional Δ-complex K is an $(n-1)$-dimensional Δ-complex ∂K given by all $\Delta^{n-1} \in K$ that are the subsimplex of only a single n-simplex within K.

Definition 2.1.10. The **closure**, cl_J of a collection of simplices $J \subset K$ is given by the minimal subcomplex of K containing J.

Definition 2.1.11. The **interior**, $int(K)$ of a Δ-complex K is the set of simplices of K not contained in ∂K.

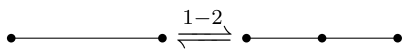
Definition 2.1.12. The **join** of two simplices $\Delta^n = [v_0 \cdots v_n], \Delta^m = [v_{n+1} \cdots v_{n+m+1}]$ is the simplex $\Delta^n \star \Delta^m = [v_0 \cdots v_{n+m+1}]$. The join $K \star J$ of two Δ-complexes K, J, is given by the union of all $\Delta \star \Delta'$, $\forall \Delta \in K$, $\forall \Delta' \in J$.

2.2 Pachner Moves

A crucial ingredient in the following constructions is that of **Pachner moves** [29]. Given an n-manifold M with a pair of PL-triangulations $\mathcal{M}, \mathcal{M}'$, the n dimensional

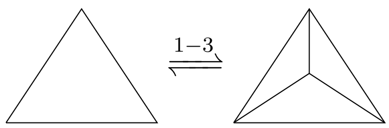
Pachner moves define a finite set of relations relating $\mathcal{M}$ to $\mathcal{M}'$ preserving the PL-structure. The moves are generated by considering the boundary of an $n+1$-simplex which defines a PL-triangulation of the n-sphere consisting of $n+2$ n-simplices. Using the hemispherical decomposition of S^n this boundary can be viewed as the gluing of two n-balls along S^{n-1}. Let Δ_l be the triangulation of the n-ball with l n-simplices and Δ_{n+2-l} the triangulation of the n-ball with $n+2-l$ n-simplices for $0 < l < n+2$ such that $\partial\Delta_l = \partial\Delta_{n+2-l}$ and $\Delta_l \cup_{\partial\Delta_l} \Delta_{n+2-l} = \partial\Delta^{n+1}$. The Pachner moves are given by replacing a region of $\mathcal{M}$ isomorphic to Δ_l with Δ_{n+2-l}. We call this move the $l-(n+2-l)$ Pachner move. The $(n+2-l)-l$ Pachner moves is naturally the inverse of the $l-(n+2-l)$ Pachner move.

Example 2.2.1. 1D Pachner move:

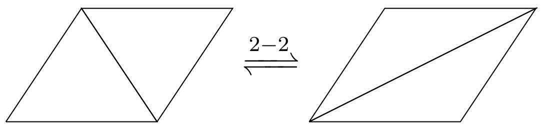


(2.2)

Example 2.2.2. 2D Pachner Moves: Tetrahedron defines two moves:

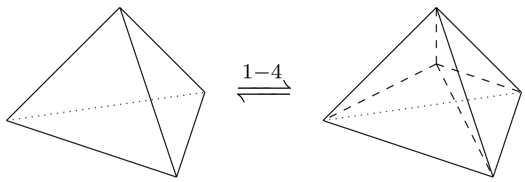


(2.3)

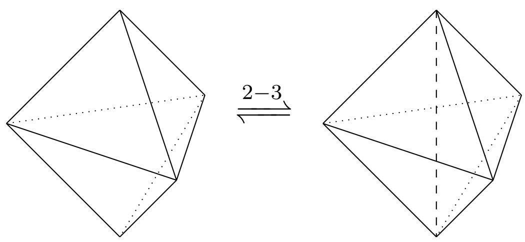


(2.4)

Example 2.2.3. 3D Pachner moves: The Pentachord defines two moves:



(2.5)



(2.6)

2.3 CW-Complexes

Δ-complexes provide a convenient methodology for generating cellulations of topological spaces with a single form of building block in each dimension. In the following we will also utilise a more general approach to formulating cellulations given by CW-complexes. Δ-complexes are naturally examples of CW-complexes but in many cases the structure of a Δ-complex is very rigid and it is often the case that a large number of simplices are required to form triangulations of a given topological manifold.

Definition 2.3.1. Given a topological manifold X, a **CW-Complex** $(X, \{\phi_a^n\}_{a \in L^n, n \in \mathbb{N}})$ is given by a collection of sets $L^0, L^1 \cdots$ for each $n \in \mathbb{N}$ and a family of continuous maps $\{\phi_a^n : D^n \to X\}_{a \in L^n}$ called **characteristic maps** satisfying the following:

1. Each characteristic map $\phi_a^n : D^n \to X$ restricts to a homeomorphism $int(D^n) \to \phi_a^n(int(D^n)) \subset X$
2. The **open cells** $c_a^n := \phi_a^n(int(D^n)) \subset X$, where $n \in \mathbb{N}$ and $a \in L^n$ form a partition of X. Ie. They are pairwise disjoint and their union is X.
3. Each $\partial(\overline{c_a^n}) := \phi_a^n(\partial(D^n)) \subset X$ is contained in the union of a finite number of open cells of dimension $< n$
4. A set $F \subset X$ is closed if, and only if, $(\phi_a^n)^{-1}(F)$ is closed in D^n, for each $n \in \mathbb{N}$ and each $a \in L^n$.

Definition 2.3.2. A **sub CW-complex** $(A, \{\phi_b^n\}_{b \in L^n, n \in \mathbb{N}})$ of a CW-complex $(X, \{\phi_a^n\}_{a \in L^n, n \in \mathbb{N}})$ is a subspace $A \subset X$ which is the union of open cells of X, such that the closure in X of each of these open cells is contained in A.

Definition 2.3.3. The n**-skeleton** X^n of a CW-complex $(X, \{\phi_a^n\}_{a \in L^n, n \in \mathbb{N}})$ is given by the union of all cells of dimension $\leq n$, with the induced topology. Note X^n is a sub CW-complex of X.

Definition 2.3.4. The **attaching map** ψ_a^n of each closed $n-$cell $\overline{c_a^n}$ is the restriction of ϕ_a^n to $\partial(D^n)$, namely:

$$\psi_a^n : \partial D^n \to \partial(\overline{c_a^n}) \subset X^{n-1} \subset X \tag{2.7}$$

The underlying topological space of the n-skeleton X^n of X is homeomorphic to the space obtained from X^{n-1} by attaching $\sqcup_{a \in L^n} D^n$ to it, along the attaching maps of the closed n-cells.

Definition 2.3.5. Given CW-complexes X and Y, a map $f : X \to Y$ is called cellular if $f(X^n) \subset Y^n$, for all $n \in \mathbb{N}$

Definition 2.3.6. If $(X, \{\phi_a^n\}_{a \in L^n, n \in \mathbb{N}})$ is a CW-complex, we call L^n the set of abstract n-cells.

Definition 2.3.7. Abstract $0, 1, 2, 3$-cells of a CW-complex will sometimes be called vertices, edges, plaquettes and blobs respectively.

Definition 2.3.8. Given two CW-complexes $(X, \{\phi_a^n\}_{a \in L^n, n \in \mathbb{N}})$ and $(Y, \{\tilde{\phi}_a^n\}_{a \in \tilde{L}^n, n \in \mathbb{N}})$. The **product CW-complex** of $X \times Y$ is given by $(X \times Y, \{\phi_a^n \times \tilde{\phi}_b^m\}_{a \in L^n, b \in \tilde{L}^m, (n,m) \in \mathbb{N}^2})$ such that the characteristic maps are given by:

$$\phi_a^n \times \tilde{\phi}_b^m : D^n \times D^m \to X \times Y \tag{2.8}$$

Chapter 3

Categories

In this thesis we often utilise the constructions of **category theory** to provide an effective description of physical theories. In the following we outline the general definitions needed for this thesis. The canonical reference for category theory is [30]. Two complimentary introductions to the subject are given in [31, 32].

Definition 3.0.1. A **category** $C = (C_0, C_1, s, t, 1, \cdot)$, is given by a pair of classes C_0, C_1 called **objects** and **morphisms** respectively, a triple of maps: **source** $s : C_1 \to C_0$, **target** $t : C_1 \to C_0$ and **unit** $1 : C_0 \to C_1$, and a **composition** $\cdot : C_1 \times_{C_0} C_1 \to C_1$, where $C_1 \times_{C_0} C_1 := \{(f, g) \in C_1 \times C_1 | t(f) = s(g) \in C_0\}$ is the class of **composable morphisms**, such that the following axioms hold:

$$s(1_x) = x = t(1_x) \tag{3.1}$$

$$s(f \cdot g) = s(f), \quad t(f \cdot g) = t(g) \tag{3.2}$$

$$1_{s(f)} \cdot f = f = f \cdot 1_{t(f)} \tag{3.3}$$

$$(f \cdot g) \cdot h = f \cdot (g \cdot h) \tag{3.4}$$

for all $x \in C_0$ and for all composable $f, g, h \in C_1$.

In the text we often utilise the **graphical notation** for categories. The reason to introduce the graphical notation is in order to highlight the **directed graph** like structure of a category.

Definition 3.0.2. A **directed graph** (V, E, σ, τ) is a pair of sets V, E called vertices and edges respectively, together with a pair of set maps $\sigma : E \to V$ and

$\tau : E \to V$. Given $e \in E$ such that $\sigma(e) = v \in V$ and $\tau(e) = v' \in V$, we denote $e \in E$ as an arrow $v \xrightarrow{e} v'$.

From this definition, categories can be visualised as a special form of directed graph whereby edges can be composed. Let C be a category and $f \in C_1$ a morphism. We notate f as follows:

$$s(f) \xrightarrow{f} t(f) \in C_1$$

Given two composable morphisms $x \xrightarrow{f} y, y \xrightarrow{g} z \in C_1$ we notate the composition via:

$$(x \xrightarrow{f} y) \cdot (y \xrightarrow{g} z) := x \xrightarrow{f} y \xrightarrow{g} z = x \xrightarrow{fg} z$$

In this way the axioms of a category can be conveniently re-expressed as follows:

$$s(x \xrightarrow{1_x} x) = x = t(x \xrightarrow{1_x} x) \tag{3.5}$$

$$s(x \xrightarrow{f} y \xrightarrow{g} z) = s(x \xrightarrow{f} y), \quad t(x \xrightarrow{f} y \xrightarrow{g} z) = t(y \xrightarrow{g} z) \tag{3.6}$$

$$s(f) \xrightarrow{1_{s(f)}f} t(f) = s(f) \xrightarrow{f} t(f) = s(f) \xrightarrow{f1_{t(f)}} t(f) \tag{3.7}$$

$$x \xrightarrow{(fg)h} w = x \xrightarrow{f(gh)} w \tag{3.8}$$

for all $x \in C_0$ and $x \xrightarrow{f} y, y \xrightarrow{g} z, z \xrightarrow{h} w \in C_1$. Note, associativity is at the heart of the unambiguous definition of such diagrams.

Example 3.0.1. Graph category: Given a directed graph $L = (V, E, \sigma, \tau)$ we can naturally associate a category. Let $L = (V, E, \sigma, \tau)$ be a directed graph, with vertex set V and edge set E. We define the category $C(L)$ as follows: Let each vertex $v \in V$ correspond to an object. For each oriented edge $e \in E$ from vertex v to v' we define a morphism $v \xrightarrow{e} v' \in C(L)_1$. To all vertices $v \in V$ we define the trivial edge $v \xrightarrow{1_v} v \in C(L)_1$. Composition of morphisms is given by all formal compositions subject to the following relations:

$$\begin{aligned} s(v \xrightarrow{1_v} v) &= v = t(v \xrightarrow{1_v} v) \\ v \xrightarrow{e} v' \xrightarrow{e'} v'' &:= v \xrightarrow{ee'} v'' \\ v \xrightarrow{e} v' \xrightarrow{1_{v'}} v' &:= v \xrightarrow{e} v' \\ v \xrightarrow{1_v} v \xrightarrow{e} v' &:= v \xrightarrow{e} v' \end{aligned} \tag{3.9}$$

Example 3.0.2. The category **Set**, is the category with sets as objects and functions as morphisms. The axioms are satisfied because composition of functions is associative and every set X admits a unique identity function $1_x : X \to X$.

We introduce the category **Set** to highlight the nature of categories. Although each object is a set, objects are regarded as possessing no substructure such that each object only carries the information of a label associated to a set. In this way we cannot ask the question of whether a certain element is contained in an object in the usual set theoretic way. Instead the information about the objects in a category is carried by it morphisms. In **Set**, each morphism $\{*\} \xrightarrow{f} X$ from a fixed one object set $\{*\}$ to a set X defines an element of X from the definition of f being a function. As such, questions about the substructure of objects are answered using the structure of morphisms instead of the structure of objects. This property outlines the ethos of category theory.

Example 3.0.3. $\mathbf{Vect}_k$: The category of all vector spaces over a fixed field k as objects and k-linear transformations as morphisms.

Example 3.0.4. **Hilb**: The category of Hilbert spaces as objects and bounded linear maps as morphisms.

3.1 Cobordism Categories

The previous examples of categories were heavily influenced by the ideas of sets and functions. General categories do not require this property. In the following we outline one such category **(n+1)Cob**, the $n+1$-dimensional cobordism category. As we will see, this category is intimately related to quantum field theory. We begin by defining $(n+1)$-dimensional **cobordisms**.

Definition 3.1.1. An $(n+1)$**-dimensional cobordism** $\Sigma \xrightarrow{M} \Sigma'$ is specified by the tuple $(M, \Sigma, \Sigma', i, i')$. Here Σ and Σ' are a pair of oriented, closed n-manifolds, M is a compact, oriented $(n+1)$-dimensional manifold and i, i' are a pair of maps

$$\Sigma \xrightarrow{i} M \xleftarrow{i'} \Sigma'.$$

Here $i : \Sigma \to M$ is an orientation preserving diffeomorphism of Σ onto $i(\Sigma) \subset \partial M$ and $i' : \Sigma' \to M$ is an orientation reversing diffeomorphism of Σ' onto $i'(\Sigma') \subset \partial M$

such that $i(\Sigma) \cup i'(\Sigma') = \partial M$ and $i(\Sigma) \cap i'(\Sigma') = \emptyset$. We refer to Σ as the **source** and Σ' as the **target**.

In the following we will consider the empty set $\emptyset$ as a closed, oriented n-manifold. In this way a closed, oriented n-manifold M can be considered as a cobordism $\emptyset \xrightarrow{M} \emptyset$. An important property of cobordisms is that they can be glued along their boundaries to form new cobordisms. Given a pair of cobordisms:

$$\Sigma \xrightarrow{i} M \xleftarrow{i'} \Sigma' \qquad \Sigma' \xrightarrow{j} N \xleftarrow{j'} \Sigma'' \tag{3.10}$$

we can form a new cobordism $\Sigma \xrightarrow{M \cup_{\Sigma'} N} \Sigma''$ with the smooth maps

$$\Sigma \xrightarrow{i} M \cup_{\Sigma'} N \xleftarrow{j'} \Sigma'' \tag{3.11}$$

using the map $i'^{-1} \circ j : \partial M \to \partial N$, where $\circ$ denotes map composition.

In order to define a category we first introduce an equivalence relation on cobordisms.

Definition 3.1.2. Let $\Sigma \xrightarrow{M} \Sigma'$ and $\Sigma \xrightarrow{M'} \Sigma'$ be a pair of $(n+1)$-dimensional cobordisms from Σ to Σ'. We then consider the two as **smooth equivalent**, if there exists an orientation preserving diffeomorphism $\psi : M \xrightarrow{\simeq} M'$ making the following diagram commute:

$$\begin{tikzcd} & M \arrow[dd, "\psi"] & \\ \Sigma \arrow[ur, "i"] \arrow[dr, "j"'] & & \Sigma' \arrow[ul, "i'"'] \arrow[dl, "j'"] \\ & M' & \end{tikzcd} \tag{3.12}$$

Definition 3.1.3. For given n, a non-negative integer, the $(n+1)$**-dimensional smooth cobordism category,** $(n+1)$**Cob**, is the category with closed, oriented n-dimensional manifolds Σ as objects. Morphisms are given as smooth equivalence classes of $(n+1)$-dimensional cobordisms. The identity morphism for an object Σ is the cobordism $\Sigma \xrightarrow{\Sigma \times I} \Sigma$. Composition is given by gluing of cobordisms.

Lemma 3.1.1. Given a pair of n-manifolds $M_1, M_2 \in (n+1)\text{Cob}_0$, if there exists a diffeomorphism $\phi : M_1 \to M_2$, this induces an isomorphism $M_1 \simeq M_2$ in $(n+1)\text{Cob}_0$.

Proof.

$$M_1 \simeq M_1 \times \{0\} \hookrightarrow M_1 \times [0,1] \hookleftarrow M_1 \times \{1\} \simeq M_1 \overset{\phi}{\simeq} M_2$$

□

There are a many variations of the smooth cobordism category. One example we will utilise in this thesis is that of the triangulated cobordism category.

Definition 3.1.4. An $(n+1)$-dimensional triangulated cobordism $\Delta(\Sigma) \xrightarrow{\Delta(M)} \Delta(\Sigma')$, is given by the tuple $(\Delta(M), \Delta(\Sigma), \Delta(\Sigma'), i, i')$. Here $\Delta(\Sigma), \Delta(\Sigma')$ are closed, oriented triangulated n-manifolds, $\Delta(M)$ is a compact triangulated $n+1$-manifold and i, i' are maps

$$\Delta(\Sigma) \xrightarrow{i} \Delta(M) \xleftarrow{i'} \Delta(\Sigma').$$

Where $i : \Delta(\Sigma) \to \Delta(M)$ is an orientation preserving embedding of $\Delta(\Sigma)$ onto $i(\Delta(\Sigma)) \subset \partial\Delta(M)$ and $i' : \Delta(\Sigma') \to \Delta(M)$ is an orientation reversing embedding of $\Delta(\Sigma')$ onto $i'(\Delta(\Sigma')) \subset \partial\Delta(M)$. Such that $i(\Delta(\Sigma)) \cup i'(\Delta(\Sigma')) = \partial(\Delta(M))$ and $i(\Delta(\Sigma)) \cap i'(\Delta(\Sigma')) = \emptyset$.

We can glue $(n+1)$-dimensional triangulated cobordisms $\Delta(\Sigma) \xrightarrow{\Delta(M)} \Delta(\Sigma')$ and $\Delta(\Sigma') \xrightarrow{\Delta(N)} \Delta(\Sigma'')$ along the boundary $\Delta(\Sigma')$ to form a new cobordism $\Delta(\Sigma) \xrightarrow{\Delta(M) \cup_{\Delta(\Sigma')} \Delta(N)} \Delta(\Sigma'')$.

Definition 3.1.5. Given a pair of triangulated $(n+1)$-dimensional cobordisms $\Delta(\Sigma) \xrightarrow{\Delta(M)} \Delta(\Sigma'), \Delta(\Sigma) \xrightarrow{\Delta(M')} \Delta(\Sigma')$, we consider them as **PL-homeomorphic equivalent** if ψ_Δ is an orientation preserving PL homeomorphism such that the following diagram commutes:

$$\begin{tikzcd} & \Delta(M) \arrow[dd, "\psi_\Delta"] & \\ \Delta(\Sigma) \arrow[ur, "i"] \arrow[dr, "j"'] & & \Delta(\Sigma') \arrow[ul, "i'"'] \arrow[dl, "j'"] \\ & \Delta(M') & \end{tikzcd} \tag{3.13}$$

Definition 3.1.6. For given $n \in \mathbb{N}$, the $(n+1)$**-dimensional triangulated cobordism category,** $(n+1)\mathbf{Cob}_\Delta$, is the category with closed, oriented n-dimensional PL triangulated manifolds $\Delta(\Sigma)$. Morphisms are given by PL-homeomorphic equivalence classes of $(n+1)$-dimensional PL triangulated cobordisms, with composition given by gluing.

3.2 Groupoids

Another example of categories we will use throughout this thesis is that of **groupoids**. For a more indepth treatment of groupoids and their relation to topology see [33].

Definition 3.2.1. Given a category C, a morphism $x \xrightarrow{f} y \in C_1$ is an **isomorphism** if there exists a two sided inverse $y \xrightarrow{f^{-1}} x \in C_1$ such that

$$
\begin{aligned}
x \xrightarrow{f} y \xrightarrow{f^{-1}} x &= x \xrightarrow{1_x} x \\
y \xrightarrow{f^{-1}} x \xrightarrow{f} y &= y \xrightarrow{1_y} y
\end{aligned}
\tag{3.14}
$$

Definition 3.2.2. A **groupoid** $\Gamma = (\Gamma_0, \Gamma_1, s, t, 1, \cdot)$ is a category where all morphisms $f \in \Gamma_1$ are isomorphisms

Example 3.2.1. Groups: The simplest examples of groupoids are given by groups. Let G be a group, then we define the groupoid $BG = (BG_0, BG_1, s, t, \cdot)$ to be the groupoid with a single object $BG_0 := *$ and morphisms $BG_1 = G$ given by elements of G such that $s(g) = t(g) = *$ for all $g \in G$. Composition of morphisms is given by composition of elements in G and the identity morphism is given by the group identity $1_* := 1_G \in G$.

Example 3.2.2. Action Groupoid: Let S denote a finite set, G a group and $\circ : G \times S \to S$ a $G-$action on the set S. We define $S//_\circ G = (S//_\circ G_0, S//_\circ G_1, s, t, \cdot)$ as the groupoid with object set $S//_\circ G_0 = S$ and morphisms $s \xrightarrow{g} g \circ s \in S//_\circ G_1$ for all $g \in G$ and $s \in S$. The identity morphism for each object $s \in S$ is given by the group identity $1_s := 1_G \in G$ and composition is inherited from composition in G when two morphisms are composable. Note the groupoid $G//_\triangleright G$ with finite group G and G-action $\triangleright$ given by conjugation, $g \xrightarrow{h} hgh^{-1} \in G//G_1$ corresponds to the quantum double $\mathcal{D}(G)$ of a finite group [34].

Example 3.2.3. Path Groupoid: A useful example not arising from groups is given by directed graphs in analogy with example 3.0.1. Let $L = (V, E, \sigma, \tau)$ be a directed graph, with vertex set V and edge set E. We define the groupoid $\Gamma(L)$ as follows: Let each vertex $v \in V$ correspond to an object. For each oriented edge $e \in E$ from vertex v to v' we define a morphism $v \xrightarrow{e} v' \in \Gamma(L)_1$. For each edge $v \xrightarrow{e} v'$ we define an edge with opposite orientation $v' \xrightarrow{e^{-1}} v := (v \xrightarrow{e} v')^{-1} \in \Gamma(L)_1$. To all vertices $v \in V$ we define the trivial edge $v \xrightarrow{1_v} v \in \Gamma(L)_1$. Composition of morphisms is given by all formal compositions subject to the following relations:

$$\begin{aligned} s(v \xrightarrow{1_v} v) =& v = t(v \xrightarrow{1_v} v) \\ v \xrightarrow{e} v' \xrightarrow{e'} & v'' = v \xrightarrow{ee'} v'' \\ v \xrightarrow{e} v' \xrightarrow{e^{-1}} & v := v \xrightarrow{1_v} v \\ v' \xrightarrow{e^{-1}} v \xrightarrow{e} & v' := v' \xrightarrow{1_v} v' \\ v \xrightarrow{e} v' \xrightarrow{1_{v'}} & v' := v \xrightarrow{e} v' \\ v \xrightarrow{1_v} v \xrightarrow{e} & v' := v \xrightarrow{e} v' \end{aligned} \tag{3.15}$$

We now introduce two important concepts in the theory of groupoids, the notion of **connected** and **stabiliser** which will use throughout the text:

Definition 3.2.3. Given a groupoid Γ, a pair of objects $a, b \in \Gamma_0$ are called **connected** if there exists $h \in \Gamma_1$ such that $s(h) = a$ and $t(h) = b$. This property defines an equivalence relation and we call the equivalence classes **connected components**. We notate the set of connected components by $\pi_0(\Gamma)$.

Definition 3.2.4. Let Γ be a groupoid and $x \in \Gamma_0$ an object, the **stabiliser** $\pi_1(x)$ is the group of morphisms

$$\pi_1(x) := \{g \in \Gamma_1 | s(g) = t(g) = x\}. \tag{3.16}$$

Proposition 3.2.1. Let Γ be a groupoid, $C \in \pi_0(\Gamma)$ a connected component and $x, y \in C$ pair of objects in C, then $\pi_1(x) \simeq \pi_1(y)$.

Proof. If x and y are elements of the same connected component $C \in \pi_0(\Gamma)$, by definition there exists $k \in \Gamma_1$ such that $s(k) = x$ and $t(k) = y$. From the existence of this morphism we can define a pair of group homomorphism

$$\begin{aligned} &\phi : \pi_1(y) \to \pi_1(x) \\ &\phi : h \mapsto khk^{-1}, \quad \forall h \in \pi_1(y) \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} &\phi^{-1} : \pi_1(x) \to \pi_1(y) \\ &\phi^{-1} : g \mapsto k^{-1}gk, \quad \forall g \in \pi_1(x) \end{aligned} \tag{3.18}$$

such that

$$\phi\phi^{-1} = 1_{\pi_1(y)} \qquad \phi^{-1}\phi = 1_{\pi_1(x)} \tag{3.19}$$

where $1_{\pi_1(y)}/1_{\pi_1(x)}$ are the $\pi_1(y)/\pi_1(x)$ identity group homomorphisms such that ϕ is a group isomorphism. □

3.2.1 Functors and Natural Equivalences

For our later purposes it will be important to compare categories. To this end we introduce the notion of **functors** and **natural transformations**.

Definition 3.2.5. Given two categories $C = (C_0, C_1, s, t, 1, \cdot)$ and $D = (D_0, D_1, s', t', 1', \cdot')$, a **functor** $F : C \to D$ is a structure preserving map between categories. A functor consists of a pair of maps $F = (F_0, F_1)$. $F_0 : C_0 \to D_0$ is a map sending objects in C to objects in D and $F_1 : C_1 \to D_1$ sending morphisms of C to morphisms in D such that, for all composable morphisms $f, g \in C_1$ and objects $x \in C_0$

$$\begin{aligned} F_1(f \cdot g) &= F_1(f) \cdot' F_1(g) \\ F_1(1_x) &= 1'_{F_0(x)} \\ s'(F_1(f)) &= F_0(s(f)) \\ t'(F_1(f)) &= F_0(t(f)) \end{aligned} \tag{3.20}$$

Example 3.2.4. Given a pair of groups G, G' with corresponding groupoids BG, BG' each functor $F : BG \to BG'$ is given by a group homomorphism $F : G \to G'$.

Definition 3.2.6. Given a pair of functors $F, \tilde{F} : C \to D$. A **natural transformation** $\eta : F \Rightarrow \tilde{F}$ is given by a map $\eta : C_0 \to D_1$ which associates to each object $x \in C_0$, a morphism $\eta_x \in D_1$ such that the following diagram **commutes**

$$\begin{tikzcd} F(x) \arrow[r, "F(a)"] \arrow[d, "\eta_x"'] & F(y) \arrow[d, "\eta_y"] \\ \tilde{F}(x) \arrow[r, "\tilde{F}(a)"'] & \tilde{F}(y) \end{tikzcd} \tag{3.21}$$

for all morphisms $x \xrightarrow{a} y \in C_1$. The requirement that the diagram commutes implies that the composition of morphisms clockwise around the diagram is equal to the composition counter clockwise,

$$\eta_x \cdot \tilde{F}(a) = F(a) \cdot \eta_y \tag{3.22}$$

Definition 3.2.7. A natural equivalence is a natural transformation $\eta : F \Rightarrow \tilde{F}$ such that there exists $\eta^{-1} : \tilde{F} \Rightarrow F$ whereby $\eta\eta^{-1} = 1_F$ and $\eta^{-1}\eta = 1_{\tilde{F}}$. Here $1_{F/\tilde{F}}$ denotes the unique identity natural transformation for the functor $F/\tilde{F}$ respectively.

Corollary 3.2.0.1. Given a pair of groupoids Γ and Γ' and functors $F, \tilde{F} : \Gamma \to \Gamma'$, any natural transformation $\eta : F \Rightarrow \tilde{F}$ is a natural equivalence.

Utilising the previous constructions we can give another example of a category.

Example 3.2.5. Functor category: Let C and D be any two categories. The functor category $[C, D]$ is the category with objects, covariant functors $F : C \to D$ and morphisms, natural transformation between such functors. This forms a category as for any functor there exists the identity natural transformation $1_F : F \Rightarrow F$ which assigns to every object $x \in C_0$ the identity morphism on $F(x)$. Furthermore composition of two natural transformations is again a natural transformation and composition is associative.

Remark 3.2.1. Given a pair of groupoids Γ, Γ', the functor category $[\Gamma, \Gamma']$ is a groupoid. This follows directly from Corollary 3.2.0.1.

3.3 Monoidal Categories

Algebraic structures in mathematics such as groups, rings and modules are based on the idea of enriching sets with suitable binary operations referred to as sums or products. The most general form of an algebraic structure is a **monoid**.

Definition 3.3.1. A **monoid** is a triple $(S, \cdot, 1)$ consisting of a set S, a binary product $\cdot : S \times S \to S$ and unit $1 \in S$ such that the following conditions hold:

1. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
2. $1 \cdot a = a = a \cdot 1$.

for all $a, b, c \in S$.

The definition of a monoidal category is designed to enrich categories in a similar vein.

Definition 3.3.2. A **monoidal category** $(C, \otimes, I, \alpha, \lambda, \rho)$, is a category C together with a functor

$$\otimes : C \times C \to C,$$

called the **monoidal product**, a distinguished object $I \in C_0$ called the monoidal unit and natural equivalences

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$

$$\lambda_A : I \otimes A \to A$$

$$\rho_A : A \otimes I \to A$$

such that the following coherence diagrams commute:

$$\begin{tikzcd}
 & (A \otimes B) \otimes (C \otimes D) \arrow[dr, "\alpha_{A,B,C\otimes D}"] & \\
((A \otimes B) \otimes C) \otimes D \arrow[ur, "\alpha_{A\otimes B,C,D}"] \arrow[d, "\alpha_{A,B,C}\otimes 1_D"'] & & A \otimes (B \otimes (C \otimes D)) \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha_{A,B\otimes C,D}"'] & & A \otimes ((B \otimes C) \otimes D) \arrow[u, "1_A \otimes \alpha_{B,C,D}"']
\end{tikzcd} \tag{3.23}$$

$$
\begin{tikzcd}
(A \otimes 1) \otimes B \arrow[rr, "\alpha_{A,1,B}"] \arrow[dr, "\rho_A \otimes 1_B"'] & & A \otimes (1 \otimes B) \arrow[dl, "1_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}
\tag{3.24}
$$

Definition 3.3.3. A braided monoidal category $(C, \otimes, I, \alpha, \lambda, \rho, \gamma)$, is a monoidal category $(C, \otimes, I, \alpha, \lambda, \rho)$, equipped with a family of isomorphisms

$$\gamma : A \otimes B \to B \otimes A$$

for each pair of objects $A, B \in C_0$ such that the following coherence diagrams commute:

$$
\begin{tikzcd}
& A \otimes (B \otimes C) \arrow[r, "\gamma_{A,B\otimes C}"] & (B \otimes C) \otimes A \arrow[dr, "\alpha_{B,C,A}"] & \\
(A \otimes B) \otimes C \arrow[ur, "\alpha_{A,B,C}"] \arrow[dr, "\gamma_{A,B} \otimes 1_C"'] & & & B \otimes (C \otimes A) \\
& (B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"'] & B \otimes (A \otimes C) \arrow[ur, "1_B \otimes \gamma_{A,C}"'] &
\end{tikzcd}
\tag{3.25}
$$

$$
\begin{tikzcd}
& (A \otimes B) \otimes C \arrow[r, "\gamma_{A\otimes B,C}"] & C \otimes (A \otimes B) \arrow[dr, "\alpha^{-1}_{C,A,B}"] & \\
A \otimes (B \otimes C) \arrow[ur, "\alpha^{-1}_{A,B,C}"] \arrow[dr, "1_A \otimes \gamma_{B,C}"'] & & & (C \otimes A) \otimes B \\
& A \otimes (C \otimes B) \arrow[r, "\alpha^{-1}_{A,C,B}"'] & (A \otimes C) \otimes B \arrow[ur, "\gamma_{A,C} \otimes 1_B"'] &
\end{tikzcd}
\tag{3.26}
$$

Definition 3.3.4. A **symmetric monoidal category**, is a braided monoidal category $(C, \otimes, I, \alpha, \lambda, \rho, \gamma)$ which satisfies the additional commutative diagram:

$$
\begin{tikzcd}
A \otimes B \arrow[rr, "1_{A\otimes B}"] \arrow[dr, "\gamma_{A,B}"'] & & A \otimes B \\
& B \otimes A \arrow[ur, "\gamma_{B,A}"'] &
\end{tikzcd}
\tag{3.27}
$$

Example 3.3.1. $\mathbf{Vect}_k$: See Example 3.0.3 for definition, is a monoidal category, with $\otimes = \otimes_k$ the ordinary tensor product over the field k of vector spaces and linear maps. It is straightforward to define natural equivalences

$$(U \otimes_k V) \otimes_k W \simeq U \otimes_k (V \otimes_k W)$$

on vector spaces which satisfy the pentagon equation and the monoidal product of linear maps makes $\otimes_k$ a functor

$$\otimes : \mathrm{Vect}_k \times \mathrm{Vect}_k \to \mathrm{Vect}_k.$$

$k \in \mathrm{Vect}_k$ is the monoidal unit, utilising

$$k \otimes_k V \simeq V \simeq V \otimes_k k$$

Furthermore Vect_k is a symmetric monoidal category. For all $U, V \in \mathrm{Vect}_{k,0}$ there are natural isomorphisms

$$\gamma_{U,V} : U \otimes_k V \to V \otimes_k U$$

which satisfy the properties of a symmetric braided monoidal category

$$\gamma_{U,V}\gamma_{V,U} = 1_{U \otimes V}.$$

Example 3.3.2. $(n+1)$**Cob**: See Definition 3.1.3, The smooth (and also triangulated) cobordism category is a symmetric monoidal category, with monoidal product given by disjoint union $\sqcup$. The monoidal unit is given by the $\emptyset$ considered as an n-dimensional manifold. It follows $\emptyset \sqcup M = M = M \sqcup \emptyset$ for all $M \in (n+1)\mathrm{Cob}_0$. By definition of disjoint union the monoidal product is associative. The canonical diffeomorphism $M \sqcup N \to N \sqcup M$ induces a natural isomorphism on all objects

$$\gamma_{M,N} : M \sqcup N \to N \sqcup M$$

satisfying the coherence diagrams for a symmetric monoidal category.

Chapter 4

Topological Quantum Field Theory

Oriented $(n+1)$D TQFT's were defined by Atiyah[17] (see also [35, 19]) as the following:

Definition 4.0.1. An oriented $(n+1)$D **Topological Quantum Field Theory** (TQFT) is a symmetric monoidal functor

$$\mathcal{Z} : (n+1)\text{Cob} \to \text{Vect}_k. \tag{4.1}$$

Despite the technical tone of the definition for a TQFT, in practice it is just a short hand expression for listing the axioms required to define a topologically invariant field theory. In particular we can unpack the definition into the following[36, 35]:

- To each closed, oriented n-manifold X, $\mathcal{Z}$ defines a finite dimensional vector space $V[X]$.
- To each closed, oriented n-manifold X, $V[\overline{X}]$ is canonically isomorphic to the dual vector space $V[X]^*$.
- Given two closed, oriented n-manifolds X and X', $V[X \sqcup X'] = V[X] \otimes V[X']$
- To the empty set $\emptyset$, considered as an $n-$manifold, $\mathcal{Z}$ assigns the 1-dimensional k-vector space $k = V[\emptyset]$.

- To each $(n{+}1)$D manifolds Y with boundary $\partial Y = \overline{X}_0 \sqcup \mathcal{X}_1$ and $\overline{X}_0 \cap \mathcal{X}_1 = \emptyset$, $\mathcal{Z}$ defines a linear map $Z[Y] : V[X_0] \to V[X_1]$.

- To closed $(n+1)$-manifolds W, viewed as a cobordism $\emptyset \xrightarrow{\mathcal{W}} \emptyset$, $\mathcal{Z}$ assigns a linear map $Z[W] : k \to k$ which is just an element of the field k which defines a diffeomorphism invariant of W.

In order to describe physically relevant TQFT's it is important to additionally require the TQFT's under consideration are **unitary**.

Definition 4.0.2. An (n+1)D **unitary TQFT** (UTQFT), is a symmetric monoidal functor,

$$\mathcal{Z} : (n+1)\text{Cob} \to \text{Hilb} \tag{4.2}$$

where Hilb is the symmetric monoidal category of Hilbert spaces and bounded linear maps (see example 3.0.4), such that for all $(n+1)$-manifolds $\mathcal{X}_0 \xrightarrow{\mathcal{Y}} \mathcal{X}_1$,

$$Z[\overline{\mathcal{Y}}] = Z[\mathcal{Y}]^\dagger : V[\mathcal{X}_1] \to V[\mathcal{X}_0] \tag{4.3}$$

where $\overline{\mathcal{Y}}$ is $\mathcal{Y}$ with orientation reversed and $\dagger$ is the adjoint operation.

4.1 Unitary State-Sum TQFT's

In the remainder of this thesis we will restrict our attention to a class of TQFT's referred to as **state-sum TQFT's** (ssTQFT). The motivation for studying ssTQFT's is the principle of locality. Inspired by relativity, the principle of locality asserts that there is no "action at a distance" but instead all influences propagate at a finite speed. In topological theories there is no metric structure and hence it makes no sense to enforce such requirements. Instead to define locality we utilise the Lagrangian formalism of field theories. It has been argued [37, 38, 39, 40, 41, 42] that TQFT's arising from local Lagrangians should be described by fully extended TQFT's [36, 37, 41, 43]. For the purposes of this thesis we will not define fully extended TQFT's but we will use the related notion of extended TQFT (see section 4.2). It is generally conjectured that ssTQFT's are in 1-1-correspondence with fully extended TQFT's [41, 44]. A proof is lacking largely due to the lack of a rigorous formulation of the most general framework for ssTQFT. The difficulties lie largely with ensuring invariants are independent of any branching structure on the triangulations. In this thesis we will side-step this open problem and give a suitable construction which captures (most) known state-sum TQFT's following the conventions of Williamson and Wang [45].

In the following the typeset W will specify a manifold while $\mathcal{W}$ will denote a triangulation with branching structure whose geometric realisation is homeomorphic to W.

Definition 4.1.1. An $(n+1)$**D-state-sum** is defined by the following collection of data:

1. A finite set L we call the **label set**
   - Generally we will decompose L in terms of subsets we refer to as the i**-label sets** L_i for $i \in \{0, \cdots, n+1\}$ such that $L = \sqcup_{0=1,..,n+1} L_i$
2. A set of **dimension functions** $d : L \to \mathbb{C}^\times$
3. The set of all **configuration maps** $s : \mathcal{K} \to L$ for each triangulated $n+1$-manifold $\mathcal{K}$, which colour each i-simplex of $\mathcal{K}$ with elements of L_i for $i \in \{1, \cdots, n+1\}$.
   - Such maps do not depend on the orientation of $\mathcal{K}$ but do depend on the choice of branching structure.
4. The **weight** $T^{\sigma(\Delta^i)}_{s(\Delta_i)} \in \mathbb{C}$ which evaluates a complex amplitude to each $n+1$-simplex $\Delta_i \in \mathcal{K}$ with colouring $s(\Delta_i)$ induced from $s : \mathcal{K} \to L$.
   - Here $\sigma \in \pm 1$ specifies the orientation of a $\Delta_i \in \mathcal{K}$

Here the idea is that the label set defines a set of "spin-like" configurations of a triangulation and the weight defines the interactions of such spins.

Given the data of an $n+1$D-state-sum we can canonically define a partition function for any oriented triangulated cobordism: Let Y be an $n+1$D cobordism with triangulation $\mathcal{Y}$ such that $\partial\mathcal{Y} = \mathcal{X}_0 \sqcup \mathcal{X}_1$, $\mathcal{X}_0 \cap \mathcal{X}_1 = \emptyset$ and $\partial(\mathcal{X}_0) = \partial(\mathcal{X}_1) = \emptyset$. We define $\mathbb{C}L$ be the complex vector space spanned by the label set L. We notate basis elements via $|l\rangle \in \mathbb{C}L$ for all $l \in L$ and equip $\mathbb{C}L$ with the orthonormal inner product

$$\langle l|l'\rangle = \delta_{l,l'} \quad \forall l, l' \in L. \tag{4.4}$$

The state-sum partition function for $\mathcal{Y}$ is given by:

$$\mathcal{Z}[\mathcal{Y}] := \sum_{\{s\}} \prod_{\Delta^{n+1}\in\mathcal{Y}} T^{\sigma(\Delta_{n+1})}_{s(\Delta_{n+1})} \prod_{\Delta\in\mathcal{Y}} d^{f(\Delta)}_{s(\Delta)} \bigotimes_{\Delta_j\in\mathcal{X}_1} |s(\Delta_j)\rangle \bigotimes_{\Delta_k\in\mathcal{X}_0} \langle s(\Delta_k)| \tag{4.5}$$

Here $f(\Delta_i) \in \{1, \frac{1}{2}\}$ where $f(\Delta_i) = 1$ if $\Delta_i \in int(\mathcal{Y})$ and $f(\Delta_i) = \frac{1}{2}$ if $\Delta_i \in \partial\mathcal{Y}$. The summation over $\{s\}$ is a complete set of configuration maps s for $\mathcal{Y}$ labelling all i-simplices for $0 < i \leq n+1$.

So far we have put no constraints on the weights T and in general the partition function will not give a PL-homeomorphism invariant of $\mathcal{Y}$ nor a ssTQFT.

Definition 4.1.2. An $n+1$D state-sum TQFT is a $\mathcal{Z}$ state-sum such that:

1. for each $n+1D$ Pachner relation, $\mathcal{Z}$ evaluates to the same operator for both sides of the equation
2. $\mathcal{Z}$ is independent of the vertex ordering of $int(\mathcal{Y})$

In this way the topological invariance of the theory follows exactly from a finite set of equations which the weights are required to satisfy. The difficulty in defining a ssTQFT is often lies in proving the independence of the partition function from the vertex ordering.

For $i \in \{0, 1\}$ let X_i be a closed, oriented n-manifold with triangulation $\mathcal{X}_i$ and $\mathcal{X}_i \times I$ a triangulation of $X_i \times I$ such that $\partial(\mathcal{X}_i \times I) = \mathcal{X}_i \sqcup \overline{\mathcal{X}_i}$. Given a triangulated $n+1$-dimensional cobordism $\mathcal{Y} : \mathcal{X}_0 \to \mathcal{X}_1$, the partition function $\mathcal{Z}[\mathcal{Y}]$ for a ssTQFT defines a linear map:

$$\mathcal{Z}[\mathcal{Y}] : Im(\mathcal{Z}[\mathcal{X}_0 \times I]) \to Im(\mathcal{Z}[\mathcal{X}_1 \times I]) \tag{4.6}$$

which depends only on the PL-homeomorphism class of $\mathcal{Y}$. The result follows from the relations:

$$\begin{aligned} &\mathcal{Z}[\mathcal{X}_i \times I]\mathcal{Z}[\mathcal{X}_i \times I] = \mathcal{Z}[\mathcal{X}_i \times I] \\ &\mathcal{Z}[\mathcal{Y}]\mathcal{Z}[\mathcal{X}_i \times I] = \mathcal{Z}[\mathcal{Y}] = \mathcal{Z}[\mathcal{X}_i \times I]\mathcal{Z}[\mathcal{Y}] \end{aligned} \tag{4.7}$$

which follow from PL homeomorphism invariance of the partition function.

The two primary examples of ssTQFT we will consider in the following are given by the Dijkgraaf-Witten [46] TQFT and the Yetter homotopy 2-type TQFT [47, 48].

Example 4.1.1. Dijkgraaf-Witten TQFT [46]: The $n+1$D Dijkgraaf-Witten TQFT is defined in terms of a pair (G, α^{n+1}). Here G is a finite group and $\alpha^{n+1} \in H^{n+1}(G, U(1))$ is an $n+1$-cocycle valued in $U(1)$ (see appendix A), where the action of G on $U(1)$ is given by the trivial action. The non-trivial label set $L_1 = G$ and the configuration maps are given by assigning an element $g_{ij} \in G$ to each 1-simplex $[ij]$ with vertex ordering $i < j$ given by the branching structure. For each 2-simplex $[ijk] = \Delta^2$ with $i < j < k$ we define a delta function $\delta(s([ijk])) = \delta_{g_{ij}g_{jk}, g_{ik}}$, from which we can define the weight

$$T_{s(\Delta^{n+1})}^{\sigma(\Delta^{n+1})} = \alpha^{n+1}(g_{ij}, g_{jk}, \cdots, g_{no}, g_{op})^{\sigma(\Delta^{n+1})} \prod_{\Delta^2 \in \Delta^{n+1}} \delta(s(\Delta^2)) \tag{4.8}$$

where $i < j < k < \cdots < n < o < p$ denote the $n+2$ vertices of Δ^{n+1} with branching structure induced from alphabetical ordering. The dimension functions are given by

$$d_0 = \frac{1}{|G|} \tag{4.9}$$

and are equal to 1 for all other simplices. The cocycle property

$$\delta^{n+1}\alpha^{n+1} = 1 \tag{4.10}$$

ensures the weight T defines a topologically invariant amplitude.

4.2 Extended ssTQFT's

We now extend our definition of state-sum TQFT to include $n-$manifolds with (possibly) non-empty boundary. We will make the restriction that given a $n-$manifold X with $\partial X = W$, W is a closed, oriented $(n-1)$-manifold. Furthermore in this thesis we will only consider **pinched interval cobordisms** although our musings apply more generally.

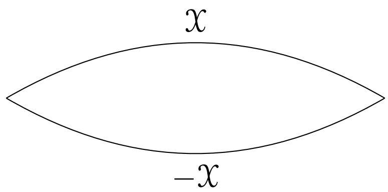
Definition 4.2.1. Let X be a compact oriented $n-$manifold with boundary $\partial X = W$, an oriented, closed $n-1$-manifold, the **pinched interval cobordism** $X \times_p [0,1]$ is given by the quotient space

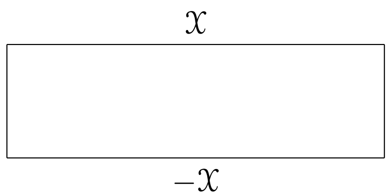
$$X \times_p [0,1] := X \times [0,1] / \sim \tag{4.11}$$

where $\sim$ is the equivalence relation

$$(w,t) \sim (w,t'), \quad \forall (w,t),(w,t') \in W \times [0,1]. \tag{4.12}$$

A consequence of this definition is that $\partial(X \times_p I) = \overline{X} \cup X$ and $\overline{X} \cap X = W$. By comparison if we instead stuck with $X \times [0,1]$ then $\partial X = \overline{X} \cup X \cup (W \times [0,1])$. In this way, if we think of $X = [0,1]$ as the 1-dimensional line element with boundary the disjoint union of points then $X \times_p [0,1]$ is a bigon whereas $X \times [0,1]$ is a rectangle, as depicted below.

$X \times_p [0,1] =$  (4.13)

$X \times [0,1] =$  (4.14)

Another consequence of this definition is that if $\partial X = \emptyset$ then $X \times_p I = X \times I$.

Definition 4.2.2. Given a triangulated pinched interval $\mathcal{Y} = \mathcal{X} \times_p [0,1]$, with $\partial \mathcal{X} = \mathcal{W}$ a **boundary condition** $\alpha \in s(\mathcal{W})$ is a choice of configuration maps $\forall \Delta^i \in \mathcal{W}$. We denote the set of all boundary conditions by $s(\mathcal{W})$.

Utilising this definition we can define the partition function with fixed boundary configuration. Let $\mathcal{Y} := \mathcal{X} \times_p I$ be as in the previous definition then we define

$Z[\mathcal{Y};\alpha]$ as follows:

$$Z[\mathcal{Y};\alpha] := \sum_{s_\alpha} \prod_{\Delta^{n+1}\in\mathcal{Y}} T^{\sigma(\Delta_{n+1})}_{s(\Delta_{n+1})} \prod_{\Delta\in\mathcal{Y}} d^{f(\Delta)}_{s(\Delta)} \bigotimes_{\Delta_j\in\mathcal{X}} |s(\Delta^j)\rangle \bigotimes_{\Delta^k\in\overline{\mathcal{X}}} \langle s(\Delta^k)| \tag{4.15}$$

$$Z[\mathcal{Y}] := \sum_{\alpha\in s(W)} Z[\mathcal{Y};\alpha] \tag{4.16}$$

Here s_α is the set of configuration maps of $\mathcal{Y} = \mathcal{X} \times_p [0,1]$ which restrict to $\alpha \in s(\mathcal{W})$ on $\mathcal{W}$ and $f \in \{0, 1, \frac{1}{2}\}$ where $f(\Delta^i) = 0$ if $\Delta^i \in \mathcal{W}$, $f(\Delta^i) = \frac{1}{2}$ if $\Delta^i \in \mathcal{X} - \mathcal{W}$ and $f(\Delta^i) = 1$ else.

Definition 4.2.3. Given compact, oriented triangulated $n-$manifold $\mathcal{X}$ with possibly non-empty boundary $\mathcal{W}$, the **state-space** with fixed boundary configuration $\alpha \in s(\mathcal{W})$ is given by

$$V[\mathcal{X};\alpha] := \mathrm{Im}\mathcal{Z}[\mathcal{X} \times_p I;\alpha] \tag{4.17}$$

such that

$$V[\mathcal{X}] = \oplus_{\alpha\in s(W)}\mathcal{Z}[\mathcal{X} \times_p I;\alpha]. \tag{4.18}$$

The state-space corresponds to the physical Hilbert space of the ssTQFT. Furthermore we note that the invariance of $\mathcal{Z}[\mathcal{X}\times_p I]$ under PL-homeomorphisms of the interior of $\mathcal{X} \times_p I$ implies

$$\mathcal{Z}[\mathcal{X} \times_p I]\mathcal{Z}[\mathcal{X} \times_p I] = \mathcal{Z}[\mathcal{X} \times_p I] \tag{4.19}$$

such that $\mathcal{Z}[\mathcal{X} \times_p I]$ is a projection operator.

4.3 Boundary Relative Triangulation Independence of $V[X, \alpha]$

Here we review the topological properties of the state-space of a ssTQFT. This section follows from the discussion of Turaev-Viro in [20] (section 2) for triangulated manfiolds without boundary.

Consider initially, X to be a closed, oriented $n-$manifold and let $Y = X \times [0,1]$. In the previous section we constructed the linear operator $\mathcal{Z}[\mathcal{Y}]$ for a triangulation $\mathcal{Y}$ of Y and when $\partial\mathcal{Y} = \overline{\mathcal{X}} \sqcup \mathcal{X}$ we defined $V[\mathcal{X}] = \mathrm{Im}\mathcal{Z}[\mathcal{Y}]$. In general we can consider triangulated cobordisms of $Y = X \times [0,1]$ of the form $\mathcal{Y}'$ with $\partial\mathcal{Y}' = \overline{\mathcal{X}} \sqcup \mathcal{X}'$ where $\mathcal{X}, \mathcal{X}'$ are two different triangulations of X. In this case $\mathcal{Z}[\mathcal{Y}'] : V[\mathcal{X}] \to V[\mathcal{X}']$ is a map from the vector space $V[\mathcal{X}]$ defined on the triangulation $\mathcal{X}$ to $V[\mathcal{X}']$ with triangulation $\mathcal{X}'$. Now using the gluing rules for cobordisms we can define two linear operators $\mathcal{Z}[\mathcal{Y}' \cup_{\mathcal{X}'} \overline{\mathcal{Y}'}] = \mathcal{Z}[\mathcal{Y}']\mathcal{Z}[\overline{\mathcal{Y}'}] : V[\mathcal{X}] \to V[\mathcal{X}]$ and $\mathcal{Z}[\overline{\mathcal{Y}'} \cup_{\mathcal{X}} \mathcal{Y}'] = \mathcal{Z}[\overline{\mathcal{Y}'}]\mathcal{Z}[\mathcal{Y}'] : V[\mathcal{X}'] \to V[\mathcal{X}']$ such that $\mathcal{Z}[\mathcal{Y}']\mathcal{Z}[\overline{\mathcal{Y}'}] = \mathcal{Z}[\mathcal{Y}]$ and $\mathcal{Z}[\overline{\mathcal{Y}'}]\mathcal{Z}[\mathcal{Y}'] = \mathcal{Z}[\mathcal{Y}'']$ and $\mathcal{Y}''$ is a triangulation of $Y = X \times [0,1]$ such that $\partial\mathcal{Y}'' = \overline{\mathcal{X}}' \sqcup \mathcal{X}'$. The equality $\mathcal{Z}[\mathcal{Y}']\mathcal{Z}[\overline{\mathcal{Y}'}] = \mathcal{Z}[\mathcal{Y}]$ follows from triangulation independence of $\mathcal{Z}$ away from the boundary and similarly for $\mathcal{Z}[\overline{\mathcal{Y}}]\mathcal{Z}[\mathcal{Y}'] = \mathcal{Z}[\mathcal{Y}'']$. These maps define an isomorphism of vector spaces such that for any two triangulations $\mathcal{X}, \mathcal{X}'$ of X, $V[\mathcal{X}] \cong V[\mathcal{X}']$. When the TQFT is unitary, this isomorphism is a unitary isomorphism by the definition $\mathcal{Z}[\overline{\mathcal{Y}}] = \mathcal{Z}[\mathcal{Y}]^{\dagger}$.

Using the same logic as in the previous paragraph we can formulate such an isomorphism between two triangulations $\mathcal{X}, \mathcal{X}'$ of X with boundary $W = \partial X$. However there is a caveat, given a triangulation $\mathcal{W}$ of W the isomorphisms only exist when $\partial\mathcal{X} = \partial\mathcal{X}' = \mathcal{W}$. This is a consequence of the definition of the pinched cobordism in our definition of an extended TQFT. In this way we only have an isomorphism class of vector spaces $V[\mathcal{X};\alpha] \cong V[\mathcal{X}';\alpha]$ where α specifies the boundary condition.

An important consequence of our definition of extended ssTQFT in the present discussion is that the dimension of $V[\mathcal{X}]$ is a topological invariant when $\partial\mathcal{X} = \emptyset$ but will depend on the choice of triangulation of $\partial\mathcal{X}$ generically. In part III we will discuss how triangulation invariance of the dimension of $V[\mathcal{X}]$ is restored.

4.4 Inner Products in $V[X]$

Given an $n+1-$manifold $Y = X \times [0,1]$ with boundary $\partial Y = \overline{X} \sqcup X$, in the category $n + 1$Cob we can consider Y in three different ways. Firstly we can consider Y as the identity map on the object X as we have made use of in the previous sections. Alternatively Y can be viewed as a map $Y : X \sqcup \overline{X} \to \emptyset$

or equivalently $Y : \overline{X} \sqcup X \to \emptyset$. From the monoidal structure of the TQFT functor the image of $X \sqcup \overline{X}$ is given by $V[X \sqcup \overline{X}] \cong V[X] \otimes V[\overline{X}]$. In this way $\mathcal{Z}[Y]$ can also be thought of as defining a map $\mathcal{Z}[Y] : V[X] \otimes V[\overline{X}] \to V[\emptyset] \cong k$ we call evaluation. Furthermore there also exists a map induced from Y such that $Y : \emptyset \to X \sqcup \overline{X}$ and $\mathcal{Z}[Y] : k \cong V[\emptyset] \to V[X \sqcup \overline{X}] \cong V[X] \otimes [\overline{X}]$ we call coevaluation. Utilising the above we can construct a canonical bilinear pairing of $V[X]$ and $V[\overline{X}]$ and an isomorphism $V[\overline{X}] \cong V[X]^*$ where $V[X]^*$ is the dual space of $V[X]$ [36].

If we restrict to unitary ssTQFT's such that $k = \mathbb{C}$, we can define a sesquilinear inner product for any two boundary preserving PL-homeomorphic triangulations $\mathcal{X}$ and $\mathcal{X}'$ of X as follows: If $a, b \in \mathbb{C}$, $|v\rangle \in \mathcal{H}[\mathcal{X}; \alpha]$ and $|w\rangle \in \mathcal{H}[\mathcal{X}'; \beta]$ and Y a triangulation of $X \times_p I$ such that $\partial\mathcal{Y} = \overline{\mathcal{X}} \cup_{\mathcal{W}} \mathcal{X}'$

$$\langle aw|bv\rangle := a^* b \langle w| Z[Y] |v\rangle \in \mathbb{C} \tag{4.20}$$

4.5 From State-Sum TQFT's to TQFT's

In this last section we review the salient properties of how the notion of a unitary state-sum TQFT can be utilised to define a unitary TQFT in the sense of the Atiyah definition 4.0.1. The principal idea is that of a **colimit** of the state spaces, see [49, 50, 51, 52] for a formal discussion. In the following we will describe the colimit in terms of the notion of a **universal cocone**.

In the following we will restrict to the case of oriented, closed n-manifolds although the results can be generalised to the case of boundaries. Let M be an oriented, closed n-manifold and $\{\mathcal{M}\}$ the set of all triangulations of M. Using Pachner moves we can define a partial ordering $\leq$ of $\{\mathcal{M}\}$, such that $\mathcal{M} \leq \mathcal{M}'$ if $\mathcal{M}$ is a refinement of $\mathcal{M}'$ by a finite number of Pachner moves. Additionally we will require $\mathcal{M} \leq \mathcal{M}$ as the trivial refinement. This poset defines a category, we call the **Pachner poset category**:

Definition 4.5.1. The **Pachner poset category** of M is the category:

- Objects: $\{\mathcal{M}\}$, all triangulations of M
- Morphisms: $\{\mathcal{M} \xrightarrow{f} \mathcal{M}'\}_{\forall \mathcal{M} \leq \mathcal{M}'}$

- Identity: Follows from relation $\mathcal{M} \leq \mathcal{M}$.

We can use the data of a unitary ssTQFT $\mathcal{Z}$ to define a functor F from the Pachner poset category to the category Hilb of Hilbert spaces and bounded linear transformations. For each triangulation $\mathcal{M}$, F defines a map $F : \mathcal{M} \mapsto V[\mathcal{M}]$, associating to each triangulation the state-space $V[\mathcal{M}]$ as defined by $Im\mathcal{Z}[\mathcal{M} \times I]$. To each partial ordering $\mathcal{M} \leq \mathcal{M}'$, F defines a unitary map defining the change of triangulation as induced from the corresponding triangulation of the cobordism $M \times I$.

We now introduce the notion of a **cocone**. A cocone $V[M_c]$ is an object in Hilb such that for all objects $V[\mathcal{M}]$ in the image of F there exists a morphism $V[\mathcal{M}] \xrightarrow{U_{\mathcal{M}}} V[M_c]$ in Hilb and for all morphisms $V[\mathcal{M}] \xrightarrow{F_f} V[\mathcal{M}']$ the following diagram commutes:

$$\begin{tikzcd} & V[M_c] & \\ V[\mathcal{M}] \arrow[ur, "U_{\mathcal{M}}"] \arrow[rr, "F_f"] & & V[\mathcal{M}'] \arrow[ul, "U_{\mathcal{M}'}"'] \end{tikzcd} \tag{4.21}$$

We define the vector space $V[M]$ as the **universal cocone**. This is to say that for all cocones $V[M_c]$ there exists a unique morphism from the cocone $V[M]$ to $V[M_c]$ such that the following diagram commutes:

$$\begin{tikzcd} & V[M_c] & \\ & V[M] \arrow[u, dotted] & \\ V[\mathcal{M}] \arrow[uur, "U_{\mathcal{M}}"] \arrow[ur, "\rho_{\mathcal{M}}"'] \arrow[rr, "F_f"] & & V[\mathcal{M}'] \arrow[uul, "U_{\mathcal{M}'}"'] \arrow[ul, "\rho_{\mathcal{M}'}"] \end{tikzcd} \tag{4.22}$$

for all $\mathcal{M} \leq \mathcal{M}'$. The unitary of the maps between such vector spaces in Hilb guarantees the existence of such a universal cone [49]. We will use the notation: $\rho_{\mathcal{M}} : V[\mathcal{M}] \to V[M]$ for all $\mathcal{M}$.

Having defined $V[M]$ we define the unitary TQFT from the unitary state-sum TQFT as the functor which assigns to all n-manifolds M, $V[M]$. To define the linear maps $Z[Y] : V[M] \to V[N]$ we first consider a triangulation of Y given by $\mathcal{M} \xrightarrow{\mathcal{Y}} \mathcal{N}$ and form $\mathcal{Z}[\mathcal{Y}]$. We can then use the unitary maps $\rho_{\mathcal{M}}, \rho_{\mathcal{N}}$ to define the triangulation independent operator via:

$$\mathcal{Z}[Y] = \rho_{\mathcal{N}} \mathcal{Z}[\mathcal{Y}] \rho_{\mathcal{M}}^{\dagger} : V[M] \to V[N]. \tag{4.23}$$

Part II

Hamiltonian Models for Topological Phases of Matter

Overview

In chapter 5 we begin by defining a suitable definition of quantum many-body system and building on this definition introduce the notion of a Hamiltonian schema, as a collection of rules which assign a quantum many-body system to a class of spatial manifolds. Using the idea of a scaling limit, we then describe a criteria for when a Hamiltonian schema defined on a discrete approximation of space admits an effective field theory described by a TQFT, we call such Hamiltonian schemas topological. From this discussion we then give a definition for a class of topological phases of matter using the language of topological Hamiltonian schemas. In section 5.2 we then introduce a recipe to canonically define an $n+1$D topological Hamiltonian schema for any triangulated approximation of n-dimensional space from the data of an $n+1$D state-sum TQFT.

In chapter 9 we construct a topological Hamiltonian schema for topological higher lattice gauge theories using ideas from higher category theory and demonstrate an equivalence between a class of such models and a class of Walker-Wang models. With this aim in mind, in chapter 6 we describe how the concepts of lattice gauge theory can be captured using the language of groupoids, functors and natural transformations. In chapter 7 we review the basic ingredients of higher category theory specialised to 2-categories which are utilised in chapter 8 to define higher lattice gauge theories with finite 2-group, generalising the construction in chapter 6.

Chapter 5

Discrete Hamiltonian Schemas for Topological Phases of Matter

The aim of this chapter is to define the notion of Hamiltonian schema and relate the concept to topological phases of matter. We begin by first giving a definition of a **quantum many-body system** [2] suitable for the following discussion:

Definition 5.0.1. A **many-body quantum system** is a triple $(\mathcal{H}, e, H)$. Here, $\mathcal{H}$ is a Hilbert space with a distinguished orthonormal basis $e = \{e_i\}$. The basis elements e_i are defined by the classical configurations of the system. H is an Hermitian operator

$$H : \mathcal{H} \to \mathcal{H}$$

we call the **Hamiltonian** and interpret the eigenvalues of H, $\{E_i\}$, where each $E_i \in \mathbb{R}$, with the energy levels of the system.

From this definition, in order to describe a many-body quantum system we first characterise the classical configurations of the physical system and then form a Hamiltonian defining the time-evolution of theory from the interactions of the classical configurations. From this data we can derive wavefunctions of the theory from the eigenvectors of H and derive the physical properties of interest in terms of Hermitian operators O_i which act on the wavefunctions.

There are two approaches to defining many-body quantum systems we call, the **fundamental** and **effective** constructions. The fundamental approach is to form a Hilbert space of all microscopic classical configurations of the quantum-many body system and define the Hamiltonian using fundamental physical principles eg. Coulombs law. For almost all systems of interest this approach is in practice impossible. For instance in quantum Hall systems there are around 10^3 electrons/μm^2 and as such 2^{1000} classical spin configurations of the system. The number of Coulombic interactions between such spin states would be orders of magnitude larger. From an analytic point of view this problem is intractable but also using techniques from numerical physics the state of the art can handle Hilbert spaces of approximate dimension 2^{70} which is still far removed from the scale required to describe such systems.

The effective construction sidesteps such difficulties by forming an educated guess about a suitable set of degrees of freedom called **effective degrees of freedom**. The effective degrees of freedom are chosen as such to provide an approximate description of the physically relevant microscopic degrees of freedom. The Hamiltonian is then defined with respect to the effective degrees of freedom by considering the most relevant interactions in the microscopic theory and treating all other interactions perturbatively [1].

In practice quantum many-body systems most naturally admit a description in terms of a geometry the theory is defined upon. Furthermore, for many systems it is interesting to compare the theory defined on a variety of geometries. To this end we define the notion Hamiltonian schema:

Definition 5.0.2. An $n+1$D **Hamiltonian schema** is a rule to define a quantum many-body system $(\mathcal{H}_M, e_M, H_{M,g})$ to a class of pairs (M, g), where M is a topological n-manifold and g is "some structure" on M. Typical examples of "some structure" include a metric or in the following a discrete structure such as a cellulation.

In many areas of physics, a discrete approximation of space or space-time provides a convenient framework for calculations. In models with a metric structure, the hope is that as the discrete structure becomes much smaller than the length scale of correlations in the system, the theory provides a concrete description

[1] Philosophically both approaches define effective descriptions but we will ignore such issues here.

of the corresponding continuum physics, eg. in lattice gauge theories [53]. In contrast, in many condensed matter systems discrete structures are physical eg. providing a description of the relative positions of atoms in a crystalline structure. We define **discrete Hamiltonian schemas** as follows:

Definition 5.0.3. An $n+1$D **Discrete Hamiltonian Schema** is an $n+1$D Hamiltonian schema associated to a class of pairs (M, g) where g is a CW-complex of M. Furthermore, the classical configurations are defined by a finite set L and configuration maps $s : g \to L$ which associate elements of L to the abstract cells of g.

Given a discrete Hamiltonian schema, a natural subset of such models is given by the concept of **locality**.

Definition 5.0.4. An $n+1$D **local Hamiltonian schema**, is an $n+1$D discrete Hamiltonian schema such that for all pairs (M, g) in the class, the Hamiltonian H_M admits a decomposition

$$H_{M,g} = -\sum_i H_{M,g;i} \tag{5.1}$$

where each $H_{M,g;i}$ is a Hermitian operator which has non-trivial action only on a sub-complex with the topology of an n-disk $D^n \subset M$.

Among the most important characterising properties of quantum many-body systems is the notion of a **gap**. The gap is defined by the difference between the two smallest energy eigenvalues of the Hamiltonian, with the eigenspace of the minimum energy eigenvalue defining the groundstate of the system. The system is called gapped if there exists a positive constant which does not depend on the geometry of system such that the gap admits a lower bound.

Definition 5.0.5. An $n+1$D discrete Hamiltonian schema is **gapped** if the eigenvalues of the corresponding Hamiltonian satisfy the following conditions: For each closed n-manifold M, let $\{g\}_{M,\leq} := \{g \leq g' \leq g'', \cdots\}$ be a partial ordering of all CW-complexes of M, where $g \leq g'$ if g' is a refinement of g and $f_M : \{g\}_{M,\leq} \to \mathbb{R}$ a positive, real, monotonically increasing function which tends to infinity in the infinite refinement limit and $\Lambda > 0$ a positive, real constant, then there exists a set of $H_{M,g}$ eigenvalues $\{E_0^1, \cdots, \}$ satisfying $|E_0^i - E_0^j| \leq e^{-f_M(g)}$ and for all other eigenvalues E_k, $|E_0^i - E^k| \geq \Lambda$.

The importance of the gap is that it protects the groundstate subspace from small perturbations of the system. Futhermore the low energy physics is dictated by the gap. Gapped systems exhibit low-energy excitations which correspond to massive excitations [54] and the correlations of the system become short-range [55, 56] in a suitable metric of the system. Conversely, gapless systems exhibit massless excitations and long-range correlations. Establishing whether a given quantum many-body systems admits a gap is in general an extremely difficult problem [57]. One class of Hamiltonian schemas which admit a gap are given as follows:

Definition 5.0.6. We call a local Hamiltonian schema **exactly solvable** if the Hamiltonian operators $\{H_{M,i}\}$ obey the relations

$$H_{M,g;i}H_{M,g;i} = H_{M,g;i}, \quad \forall i \tag{5.2}$$

$$H_{M,g;i}H_{M,g;j} = H_{M,g;j}H_{M,g;i}, \quad \forall i,j \tag{5.3}$$

for all pairs (M, g) in the class. Furthermore the lowest energy eigenspace/groundstate is given by the eigenvectors $\{|\psi\rangle\}$ of H_M such that

$$(\prod_i H_{M,g;i}) |\psi\rangle = |\psi\rangle \, . \tag{5.4}$$

Proposition 5.0.1. All exactly solvable Hamiltonian schemas are gapped.

Proof. For any exactly solvable Hamiltonian schema the Hamiltonian operators are by definition projection operators (see equation (5.2)). This property ensures the eigenvalues of $H_{M,g}$ are discrete for all pairs (M, g) in the class and the eigenvalues are either equal or $|E_i - E_j| \geq 1$ such that the Hamiltonian schema is gapped. $\square$

5.1 Discrete Hamiltonian Schemas for Topological Phases of Matter

In the introduction we defined topological phases of matter as an equivalence class of orders sharing the same topological order and stated that such phases admit an effective field theory describing the low energy/infra-red limit described by a topological quantum field theory. In this section we will formalise such ideas for a subset of topological phases which admit a description arising from discrete Hamiltonian schemas.

The first notion we need to discuss discrete Hamiltonian schemas for topological phases of matter is that of a **scaling limit** (see [50], section 2, for a formal treatment). We consider the scaling limit as the discrete Hamiltonian schema analogue of the discussion in section 4.5 relating unitary ssTQFT's to unitary TQFT's. In this way we see the scaling limit as defining a notion of continuum theory for a discrete Hamiltonian schema. We will discuss the scaling limit only of the groundstate subspace of a gapped, discrete Hamiltonian schema although the approach can be generalised to the whole Hilbert space.

Analogously to section 4.5, consider an n-manifold M and the set of all CW-complexes $\{g\}$. Again we will define a partial ordering $g \leq g'$ if g is a refinement of g' by PL-homeomorphism. Let H be an $n+1$D gapped, discrete Hamiltonian schema and $\mathcal{H}_0^{M,g}$ the groundstate subspace of H for a CW-complex g of M. We say a scaling limit for H exists, if for any pair of CW-complexes of M, $g \leq g'$ which are appropriately coarse-grained, there exists a unitary map $\phi_{g,g'} : \mathcal{H}_0^{M,g} \to \mathcal{H}_0^{M,g'}$. The motivation for this definition is the expectation that topological invariance should be an emergent rather than a microscopic symmetry of the action thus requiring a large enough set of degrees of freedom to be manifest. For such theories we can still define the colimit construction outlined 4.5 for the suitably coarse-grained CW-complexes [50]. We notate the colimit Hilbert space via $\mathcal{H}_0^M$ if it exists. Using the concept of a scaling limit we define a **topological Hamiltonian schema**:

Definition 5.1.1. An $n+1$D **topological Hamiltonian schema**, is a gapped, discrete $n+1$D Hamiltonian schema where for all n-manifolds M the scaling limit Hilbert space $\mathcal{H}_0^M$ exists and is isomorphic to $Im\mathcal{Z}[M \times I]$ for a unitary TQFT $\mathcal{Z}$.

In order to consider phases of topological Hamiltonian schemas we introduce the notion of **connectedness**:

Definition 5.1.2. Two topological Hamiltonian schemas H, H' are **connected** if there exists a homotopy of topological Hamiltonian schemas H_t for $t \in [0, 1]$ such that $H_0 = H$ and $H_1 = H'$.

Connectedness defines an equivalence relation on topological Hamiltonian schemas and we will define topological phases of matter in this language via:

Definition 5.1.3. A topological phase of matter which admits a discrete Hamiltonian schema description is an equivalence class of connected topological Hamiltonian schemas.

In the remainder of this thesis we will take the following conjecture as a truth:

Conjecture 5.1.1. Two connected topological Hamiltonian schemas have the same unitary TQFT describing the scaling limit of their groundstate subspaces.

Evidence supports the truth of this conjecture [58] although a formal proof would be a worthy research pursuit. If this conjecture is found to be false, there should exist a similar conjecture given by a stricter definition of connectedness.

One example of a stricter notion of connectedness of topological Hamiltonian schemas is given by requiring connected topological Hamtiltonian schemas additionally be **stable** [58, 2]:

Definition 5.1.4. A topological Hamiltonian schema is **stable** if

- No local operator can induce transitions between orthogonal groundstates or distinguish a pair of orthogonal groundstates
- Given any local region $A \subset M$ with the topology of an n-ball, the groundstate projector applied to A is equal to the groundstate projector applied to M restricted to the action on A.

An interesting research question would be to what extent topological Hamiltonian schemas are stable as a direct consequence of the definition.

In general it is a difficult task to ascertain whether a realistic discrete Hamiltonian schema is topological through the construction of a scaling limit. In the following we will consider one approach to this problem through the process of reverse engineering topological Hamiltonian schemas from the data of a state-sum

TQFT. The idea is to associate to a given state-sum TQFT a discrete topological Hamiltonian schema, which we will show is local, exactly solvable and whose scaling limit is given by the colimit of the state-sum TQFT. From this approach assuming the validity of conjecture 5.1.1 we would then like to classify the characteristic properties of the theory from which we can infer the properties of any connected topological Hamiltonian schema describing the same topological phase. One benefit of this approach is the explicit construction of a topological Hamiltonian schema for any local topological quantum field theory which has a realisation through a state-sum construction. Such a construction defines an effective set of degrees of freedom and an effective Hamiltonian. We can relate such a construction to realistic Hamiltonian schemas by either comparing the characterising properties of topological order or via showing a given Hamiltonian schema is connected to such models. Demonstrating the connectedness property would be useful in developing an understanding of how the effective degrees of freedom given by a state-sum TQFT can be interpreted in terms of microscopic degrees of freedom in a realistic Hamiltonian schema.

5.2 Hamiltonian Schema for Unitary State Sum TQFT's

In the following we will demonstrate how the construction of an $n+1$D unitary state sum TQFT can be canonically identified with a local, topological Hamiltonian schema defining a Hamiltonian consisting of a sum of mutually commuting local projection operators to any triangulated n-manifold $\mathcal{X}$. We call such topological Hamiltonian schemas **canonical topological Hamiltonian schemas**. Furthermore we will demonstrate how the triangulation invariance of the state-space of a ssTQFT can be implemented on the groundstate subspace of the theory. This construction also provides a complimentary view point for studying the state-space of ssTQFT's from a local operator point of view.

The first construction of a canonical topological Hamiltonian schema from a state sum TQFT was given by Levin-Wen [59, 60] who utilised the state-sum construction of the Turaev-Viro to define the model of string-net condensation.

The explicit construction of the colimit vector spaces, and the relation to the original formulation is given in [61]. In the proceeding years this approach has been utilised for a variety of unitary ssTQFT's such as the twisted quantum double models in 2+1D [25] and 3+1D [26, 62] which arise from the Dijkgraaf-Witten TQFT [46] as well as the Walker-Wang models[63, 44] arising from the Crane-Yetter-Cui TQFT [64, 65]. The canonical nature of such constructions has been known for a long time but had not been exemplified in the general setting. This section was inspired by [60] (section 7) which describes such a construction in the case of the Turaev-Viro TQFT.

5.2.1 Local Hilbert Space

Given the data of an $n+1D$ unitary ssTQFT $\mathcal{Z} := (\mathcal{Z}, T, L, d, s)$ (definition 4.1.1) we can canonically define a local Hilbert space for any compact, oriented, triangulated n-manifold $\mathcal{X}$ as follows. Let $L = \sqcup_{i=0}^{n+1} L_i$ be the label set of $\mathcal{Z}$ and $\mathbb{C}L_i$ be the complex vector space spanned by element of L_i with orthonormal inner product

$$\langle l_i | l_i' \rangle = \delta_{l_i, l_i'} \quad \forall l_i, l_i' \in L_i \tag{5.5}$$

Definition 5.2.1. The Hilbert space $\mathcal{H}[\mathcal{X}]$ of a triangulated n-manifold $\mathcal{X}$ is given by:

$$\mathcal{H}[\mathcal{X}] := \bigotimes_{i=0}^{n} \Big[\otimes_{\Delta_i \in \mathcal{X}} \mathbb{C}L_i \Big] \tag{5.6}$$

The basis elements of $\mathcal{H}[\mathcal{X}]$ are given in terms of the configurations maps such that

$$|s(\mathcal{X})\rangle := \bigotimes_{i=0}^{n} \Big[\otimes_{\Delta_i \in \mathcal{X}} |s(\Delta_i)\rangle \Big] \in \mathcal{H}[\mathcal{X}] \tag{5.7}$$

$$\langle s(\mathcal{X}) | s'(\mathcal{X}) \rangle = \delta_{s,s'}. \tag{5.8}$$

In this way we can identify each configuration of $\mathcal{X}$ with a classical configuration. The state-space $\mathcal{Z}$ assigns to $\mathcal{X}$ is a subspace of $\mathcal{H}[\mathcal{X}]$ such that:

$$V[\mathcal{X}] = \mathrm{Im}\mathcal{Z}[\mathcal{X} \times_p I] \tag{5.9}$$

or equivalently noting that $\mathcal{Z}[\mathcal{X} \times_p I]$ is a projector (see 4.19) the $+1$ eigenspace of $\mathcal{Z}[\mathcal{X} \times_p I]$:

$$V[\mathcal{X}] = \{|\psi\rangle \in \mathcal{H}[\mathcal{X}] \quad | \quad Z[\mathcal{X} \times_p I]\,|\psi\rangle = |\psi\rangle\} \subseteq \mathcal{H}[\mathcal{X}]. \tag{5.10}$$

In the following we will define the canonical Hamiltonian for $\mathcal{Z}$ such that the groundstate subspace $\mathcal{H}[\mathcal{X}]_0 := V[\mathcal{X}]$.

5.2.2 k-Local Operators

We now define the set of $k-$local operators on $\mathcal{H}(\mathcal{X})$. Let D^{n+1} be an $n+1$-ball. The boundary $\partial D^{n+1} = S^n$ has the topology of the $n-$sphere. Given an $n-$sphere S^n we can always form a decomposition in terms of two $n-$balls with opposite orientations glued along their boundaries S^{n-1}. For example: the sphere S^2 is given by gluing two 2-balls along their S^1 boundaries. We refer to this decomposition as a **hemisphere decomposition**.

Given a triangulated n-manifold $\mathcal{X}$ we define a ball to be a connected subcomplex $\mathcal{B}^n \subset \mathcal{X}$ with the topology of the $n-$ball. We refer to $\mathcal{B}^n$ as $k-$local for $k \in \mathbb{Z}$ when $k = |\Delta^0(Int(\mathcal{B}^n))|$. Generally we will be interesting in the case $k = 1$.

Definition 5.2.2. Given a $k-$local triangulated ball $\mathcal{B} \subset \mathcal{X}$, we define the $k-$local **ball operator** $H_k(\mathcal{B}, \mathcal{B}')$ in terms a triangulation of $D^{n+1} = D^n \times_p I$ with triangulation $(\mathcal{B}, \mathcal{B}')$ such that hemispherical decomposition of the boundary is given by $\overline{\mathcal{B}} \cup_{\partial\mathcal{B}} \mathcal{B}'$ and:

$$H_k(\mathcal{B}, \mathcal{B}') := \mathbb{1} \otimes Z[(\mathcal{B}, \mathcal{B}')] \otimes \mathbb{1}. \tag{5.11}$$

Here $\mathbb{1}$ are used to notate that $H_k(\mathcal{B}, \mathcal{B}')$ acts as the identity on $\mathbb{C}L_i(\Delta_i)$ where $\Delta_i \in \mathcal{X} - \mathcal{B}$. Each k-local operator defines a unitary map of Hilbert spaces

$$H_k(\mathcal{B}, \mathcal{B}') : \mathcal{H}[\mathcal{X}] \to \mathcal{H}[\mathcal{X}'] \tag{5.12}$$

where $\mathcal{X}'$ is a triangulation PL-homeomorphic to $\mathcal{X}$ given by replacing the triangulation of $\mathcal{B} \subseteq X$ with $\mathcal{B}'$.

Unitarity of the the ssTQFT directly implies:

$$
\begin{aligned}
&H_k^\dagger(\mathcal{B},\mathcal{B}') = H_{k'}(\mathcal{B}',\mathcal{B}) \\
&H_k(\mathcal{B},\mathcal{B}')H_k^\dagger(\mathcal{B},\mathcal{B}') = H_k(\mathcal{B},\mathcal{B}) \\
&H_k(\mathcal{B},\mathcal{B}')^\dagger H_k(\mathcal{B},\mathcal{B}') = H_k(\mathcal{B}',\mathcal{B}')
\end{aligned} \tag{5.13}
$$

This follows by noting $\overline{(\mathcal{B},\mathcal{B}')}$ can be identified with a triangulation $(\mathcal{B}',\mathcal{B})$ such that

$$
\begin{aligned}
H_k^\dagger(\mathcal{B},\mathcal{B}') &= \mathbb{1}\otimes Z[(\mathcal{B},\mathcal{B}')]^\dagger \otimes \mathbb{1} = \mathbb{1}\otimes \mathcal{Z}[\overline{(\mathcal{B},\mathcal{B}')}]\otimes \mathbb{1} \\
&= \mathbb{1}\otimes \mathcal{Z}[(\mathcal{B}',\mathcal{B})]\otimes \mathbb{1} = H_{k'}(\mathcal{B}',\mathcal{B})
\end{aligned} \tag{5.14}
$$

using unitarity of $\mathcal{Z}$. It then follows:

$$
\begin{aligned}
H_k(\mathcal{B},\mathcal{B}')H_k^\dagger(\mathcal{B},\mathcal{B}') &= H_k(\mathcal{B},\mathcal{B}')H_k(\mathcal{B}',\mathcal{B}) = H_k(\mathcal{B},\mathcal{B}) \\
H_k^\dagger(\mathcal{B},\mathcal{B}')H_k(\mathcal{B},\mathcal{B}') &= H_k(\mathcal{B}',\mathcal{B})H_k(\mathcal{B},\mathcal{B}') = H_k(\mathcal{B}',\mathcal{B}')
\end{aligned} \tag{5.15}
$$

where $\mathcal{Z}[(\mathcal{B},\mathcal{B}')]\mathcal{Z}[(\mathcal{B}',\mathcal{B})] = \mathcal{Z}[(\mathcal{B},\mathcal{B})]$ follows from triangulation invariance of the ssTQFT. We now give two important examples of k-local ball operators.

Example 5.2.1. k-Local state space projectors: Given a k-local ball $\mathcal{B}\subset\mathcal{X}$ and a triangulation $\mathcal{B}\times_p I$ we naturally have a k-local ball operator

$$
H_k(\mathcal{B},\mathcal{B}) : \mathcal{H}[\mathcal{X}] \to \mathcal{H}[\mathcal{X}]. \tag{5.16}
$$

Using the same logic we used to show the operators were unitary, it follows that such ball operators are necessarily idempotent:

$$
H_k(\mathcal{B},\mathcal{B})H_k(\mathcal{B},\mathcal{B}) = H_k(\mathcal{B},\mathcal{B}) \tag{5.17}
$$

and hence projection operators on $\mathcal{H}[\mathcal{X}]$

Example 5.2.2. Mutation operators: An important example of $k-$local ball operators $H_k(\mathcal{B},\mathcal{B}')$ are given by the Pachner moves (see 2.2). Each nD Pachner move is formed from an $n+1$-simplex Δ^{n+1} which forms a triangulation of an $n+1$-ball with boundary $\partial\Delta^{n+1} := \mathcal{S}^n$ a triangulation of the n-sphere S^n. Using the hemispherical decomposition of $\mathcal{S}^n$ each Pachner move defines a triangulation change from one hemisphere of $\mathcal{S}^n$ to the compliment. In this way we can define a mutation operator for each hemispherical decomposition of $\mathcal{S}^n$. Furthermore all PL-homeomorphisms of a triangulated n-manifold $\mathcal{X}$ can be constructed by composition of such operators.

5.2.3 Tent Operators

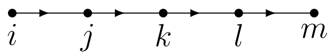
Building on the previous section we now introduce the most important example of k-local ball operators, the **tent operators**.

Definition 5.2.3. Let $\mathcal{X}$ be a triangulated n-manifold and $v_i \in int(\Delta_0(\mathcal{X}))$ a 0-simplex in the interior of $\mathcal{X}$. The closure (see definition 2.1.10) $cl_{v_i} \subseteq \mathcal{X}$ is the minimal subcomplex of X containing v_i, ie. the subcomplex generated by $\Delta_i \in \mathcal{X}$ such that $v_i \cap \Delta_i \neq \emptyset$. We can now form a triangulated $n+1$-ball $\mathcal{B}^{n+1}_{v_i}$ using the join operation $\star$ (see definition 2.1.12) such that $\mathcal{B}^{n+1}_{v_i} := cl_{v_i} \star v'_i$ where v'_i is an auxiliary vertex. Additionally to respect the branching structures we require v'_i lie in the total ordering such that $v'_i > v_i$ but $v_j > v'_i$ for all $v_j > v_i$. We define the **tent operator** as:

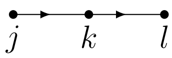
$$H_{v_i} := \mathbb{1} \otimes Z[\mathcal{B}^{n+1}_{v_i}] \otimes \mathbb{1}. \tag{5.18}$$

From the definition of cl_{v_i} this operator is a 1-local ball operator.

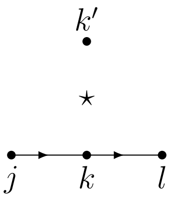
A simple example of a tent move in 1D is given by letting $\mathcal{X}$ be the interval triangulated with 5 vertices $\{j, k, l\}$

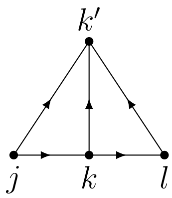
$X =$  (5.19)

then the closure cl_k of the 0-simplex k is given by

$X =$  (5.20)

and we can form the join with an auxillary 0-simplex $k < k'$ as follows:

$cl_k \star v_{k'} =$ 

$=$ 

$= [jkk'] \cup [kk'l]$ (5.21)

Theorem 5.2.1. Given any triangulated n-manifold $\mathcal{X}$, the set of tent operators $\{H_v | \forall v \in \Delta_0(int(\mathcal{X}))\}$ satisfy the following algebra:

$$\begin{aligned} H_v H_v &= H_v \quad \forall v \in \Delta_0(int(\mathcal{X})) \\ H_v H_{v'} &= H_{v'} H_v \quad \forall v, v' \in \Delta_0(int(\mathcal{X})) \end{aligned} \tag{5.22}$$

Proof. Both follow directly from the triangulation independence properties of $\mathcal{Z}$. Noting that tent operators are given by 1-local ball operators $H_1(cl_v, cl_v)$ we straight away conclude tent operators are idempotent. If $cl_v \cap cl_{v'} = \emptyset$ the operators will commute by having no common support. If $cl_v \cap cl_{v'} \neq \emptyset$ then using the $n+1$-balls defined by $cl_v \star \tilde{v}$ and $cl_{v'} \star \tilde{v}'$ we can define

$$\begin{aligned} H_{v'} H_v &= \mathbb{1} \otimes Z[cl_v \star \tilde{v} \cup_{cl_{\tilde{v}} \cap cl_{v'}} cl_{v'} \star \tilde{v}'] \otimes \mathbb{1} \\ &= \mathbb{1} \otimes Z[cl_v \star \tilde{v} \cup_{cl_{\tilde{v}'} \cap cl_v} cl_{v'} \star \tilde{v}'] \otimes \mathbb{1} = H_v H_{v'} \end{aligned} \tag{5.23}$$

where

$$Z[cl_v \star \tilde{v} \cup_{cl_{\tilde{v}} \cap cl_{v'}} cl_{v'} \star \tilde{v}'] = Z[cl_v \star \tilde{v} \cup_{cl_{\tilde{v}'} \cap cl_v} cl_{v'} \star \tilde{v}'] \tag{5.24}$$

follows by the two $n+1$D triangulations differing only by PL-homeomorphism in the interior. $\square$

5.2.4 Hamiltonian

From the previous we can always define a Hamiltonian over the Hilbert space $\mathcal{H}[\mathcal{X}]$ in terms of the tent moves.

Definition 5.2.4. Given an $n+1$D unitary ssTQFT $\mathcal{Z}$ and a triangulated n-manifold $\mathcal{X}$, we define the **canonical Hamiltonian** $H(\mathcal{Z}, \mathcal{X})$ over the Hilbert space $\mathcal{H}[\mathcal{X}]$ via:

$$H(\mathcal{Z}, \mathcal{X}) = \sum_{v \in \Delta_0(int(\mathcal{X}))} H_v. \tag{5.25}$$

In many examples there are further decompositions of the Hamiltonian available such as the Levin-Wen models [24] which define extra terms although all such models are equivalent to the canonical Hamiltonian defined here. Such Hamiltonians necessarily posses a finite gap for all choices of compact oriented n-manifold X and $n+1$D unitary ssTQFT $\mathcal{Z}$. This follows as a corollary of that fact that the tent moves are mutually commuting projection operators.

Definition 5.2.5. Given a canonical Hamiltonian $H(\mathcal{Z}, \mathcal{X})$, the **groundstate projector** $P_{\mathcal{Z},\mathcal{X}}$ is given by:

$$P_{\mathcal{Z},\mathcal{X}} = \prod_{v \in \mathcal{X}} H_v \tag{5.26}$$

Corollary 5.2.1.1.

$$P_{\mathcal{Z},\mathcal{X}} = \mathcal{Z}[\mathcal{X} \times_p I] \tag{5.27}$$

Proof. Follows by noting $\cup_{i\in\Delta_0(int(\mathcal{X}))} cl_i \star i' = \mathcal{X} \times_p I$. □

As we can apply the above construction for any triangulated n-manifold $\mathcal{X}$ using the data of $\mathcal{Z}$ this data immediately defines an exactly solvable Hamiltonian schema

$$H = (\mathcal{H}, H, s). \tag{5.28}$$

As the groundstate for such a Hamiltonian schema is given by $Im\mathcal{Z}[\mathcal{X} \times_p I]$ it immediately follows that the scaling limit of the Hamiltonian schema can be identified with the colimit of $\mathcal{Z}$ and we immediately find, (tautologically) that such Hamiltonian schemas are indeed exactly solvable topological Hamiltonian schemas.

Chapter 6

Categorical Lattice Gauge Theory

In this chapter we establish the lattice formulation of **finite gauge theories** using categorical ideas. This approach is introduced as a precursor to defining finite higher lattice gauge theory. The aim of the subsequent chapters is to define a lattice Hamiltonian schema for **topological higher lattice gauge theories**.

In gauge theories, given a Lie-group G, called the gauge group, the theory is constructed from a connection on a principle G-bundle. When the G-bundle is topologically trivial, a connection is determined by the holonomies of a 1-form field $A \in \Omega^1(M, \mathfrak{g})$ where $\mathfrak{g}$ is the Lie-algebra of G and M is the spatial manifold.

In the following we will restrict ourselves to the case where the spatial manifold M is further equipped with a discrete structure $\mathcal{M}$ in terms of a cellulation, we refer to as a **lattice**. We refer to such constructions as **lattice gauge theories**. In this formulation the parallel transport of the connection along embedded, oriented 1-submanifolds called **edges** will form the field variables of the theory by assigning an element $g_e \in G$ to each edge $e \in \mathcal{M}$. Each edge $e \in \mathcal{M}$ is equipped with an initial point, which we call source $s(e)$ and an end point which we call the target, $t(e)$. The parallel transport $g_e \in G$ defines how a charge, described by a vector $v \in V$, where V is the vector space arising from a representation ρ of G, transforms as the charge is propagated along the edge $e \in \mathcal{M}$ from $s(e)$ to $t(e)$ such that $v \mapsto \rho(g_e)v$. Since the choice of basis for V, called the **internal reference frame**, is arbitrary and not physically meaningful, it is natural to require that physically meaningful quantities in the theory be invariant under a local basis change for V called **gauge transformations**:

$$g_e \mapsto h_{s_e}^{-1} g_e h_{t_e} \tag{6.1}$$

where $h_{s_e}, h_{t_e} \in G$. The above construction has a natural description utilising ideas in category theory. In particular we will define the above ingredients of a lattice gauge theory in terms of the **lattice gauge theory groupoid** Γ_{LGT}.

In this chapter we will begin by defining an appropriate notion of a lattice suitable for lattice gauge theories. We will then utilise this construction to describe lattice gauge theories with finite gauge groups in the categorical formalism, using functors as gauge configurations and natural equivalences as gauge transformations. This construction is not new and is discussed in several articles [66, 67, 68]. Much of this material may be familiar to the reader but the purpose of this chapter is to emphasise the categorical details.

In the subsequent chapters using the categorical formulation of lattice gauge theories as the starting point we will then use ideas from **higher category theory** [31, 69, 70, 37] to introduce a generalisation of lattice gauge theory called **higher lattice gauge theory** [66, 68, 71]. The guiding principle will be to replace the gauge group G with a **gauge 2-group** $\mathcal{G}$. In this setting we will then define a field theory describing the transformation properties of charged, 1-dimensional line-like objects as they propagate along surfaces embedded in $\mathcal{M}$ and introduce a generalised notion of gauge transformations. In doing so will introduce additional field variables called 2**-parallel transports** assigned to oriented, embedded 2-manifolds. This will be done by introducing the axioms and a diagram calculus for **strict 2-groupoids**, **strict 2-functors** and **pseudo-natural transformations**. From this vantage point we will then define a Hamiltonian schema for topological higher lattice gauge theories [72, 71, 73, 47].

6.1 Lattices

In order to describe any physical theory, it is necessary to first define an appropriate mathematical model of space itself. When introducing the canonical Hamiltonian schema in chapter 5.2 we restricted to a discrete model of space defined in terms of triangulations. This has many benefits, particularly in low dimensions, by defining a single type of "building block" in each dimension and defining intuitive rules for how to compose such blocks to form spaces homeomorphic to PL-manifolds. The problem with triangulations is that there is often a need to use large numbers of simplices to triangulate even simple manifolds. To circumvent this problem, in the following we will instead invoke the technology of CW-complexes which provide us with a more general set of building blocks and gluing rules in order to model discrete spaces with fewer cellular blocks.

In the following we introduce an appropriate class of CW-complexes we call **lattices**. In subsequent sections we will build on the lattice formalism to define a categorical model of how a connection acts on the charge vector of a particle as it moves through the lattice, called the **path groupoid**. In subsequent chapters we will introduce the **path 2-groupoid** describing the analogous action of a 2-connection on the charge vector of a charged line-like object. Such constructions will play a pivotal role in our construction of gauge and higher lattice gauge theories respectively.

In the following, let $D^n := [0,1]^n$ be the n-ball with **base-point** $b_p(D^n) = (0, \cdots, 0)$ and boundary $S^n = \partial(D^{n+1})$, the n-sphere.

Definition 6.1.1. Let M be a topological manifold, with CW-complex $(M, L) := (M, \{\phi^n_a\}_{a \in L^n, n \in \mathbb{N}})$ (see definition 2.3). (M, L) is called a **lattice** for M if for each $n \in \mathbb{N}$ and $a \in L^n$:

1. A CW-decomposition Z_a of $S^{n-1} = \partial(D^n)$ is given for which the base point $b_p(D^n) = (0, \cdots, 0)$ is a closed 0-cell.

Throughout the remainder of the text we will refer to closed 0-cells as **vertices**, closed 1-cells as **edges**, closed 2-cells as **plaquettes** and closed 3-cells as **blobs**. Given a lattice (M, L), the 1-skeleton (see definition 2.3.3) can be canonically endowed with the structure of a **directed graph**. We call such a lattice with this extra data a **directed lattice** $(M, L, \to)$.

Definition 6.1.2. A **directed graph** (V, E, σ, τ) is a pair of sets V, E called vertices and edges respectively, together with a pair of set maps $\sigma : E \to V$ and $\tau : E \to V$. Given $e \in E$ such that $\sigma(e) = v$ and $\tau(e) = v'$ we denote e as an arrow $v \xrightarrow{e} v'$.

Definition 6.1.3. A **directed lattice** $(M, L, \rightarrow)$ is a lattice (M, L) where the $1-$skeleton (L^0, L^1) has the structure of a directed graph (L^0, L^1, σ, τ). This structure can be canonically induced from the characteristic maps $\phi^1_t : [0, 1] \rightarrow M$ for $t \in L^1$ as follows: Given $t \in L^1$, $\sigma(t) = \phi^1_t(0)$ and $\tau(t) = \phi^1_t(1)$.

As well as the directed lattice structure, it will be convenient for our later discussion to equip our lattice with additional orientation data for the boundaries of 2- and 3-cells.

Definition 6.1.4. A **dressed lattice** $(M, L, \Rightarrow)$ is a directed lattice $(M, L, \rightarrow)$ with the following additional data:

1. For each $P \in L^2$ with corresponding closed 2-cell $\overline{c^2_P}$, the boundary $\partial(\overline{c^2_P})$ is assigned an orientation from the attaching map $\psi^2_P : S^1 \rightarrow \partial(\overline{c^2_P})$.
2. For each $B \in L^3$ with corresponding closed 3-cell $\overline{c^3_B}$, the boundary $\partial(\overline{c^3_B})$ is assigned an orientation from the attaching map $\psi^3_B : S^1 \rightarrow \partial(\overline{c^3_B})$.

6.2 Path Groupoids

Building on the definition of a dressed lattice $(M, L, \Rightarrow)$ in the previous section we now introduce the **path groupoid** $\mathcal{P}(M, L)$. The intuition behind the definition of the path groupoid is to define a categorical construction for the action of a connection on the charge vector of a point particle propagating along the edges of the lattice. Variations of the path groupoid defined here exist for smooth spaces but we will not discuss them here [74, 75].

Using the fact that the charge vector corresponds to a representation (V, ρ) of the gauge group G, it is straightforward to check the following consequences: Given the parallel transport of a charge $v \in V$ along an edge e followed by an edge e' such that $t(e) = s(e')$ with gauge fields $h_e, h'_{e'} \in G$ respectively the total transformation is given by

$$v \xrightarrow{h_e} \rho(h_e)v \xrightarrow{h'_{e'}} \rho(h_e h'_{e'})v = v \xrightarrow{h_e h'_{e'}} \rho(h_e h'_{e'})v. \tag{6.2}$$

Furthermore traversing an edge e and then returning to the initial point by traversing the edge in reverse e^{-1} corresponds to the identity transformation on the charge vector

$$v \xrightarrow{h_e} \rho(h_e)v \xrightarrow{h_e^{-1}} \rho(1_G)v = v \xrightarrow{1_G} v. \tag{6.3}$$

which can alternatively be viewed as the particle traversing a path of length zero. These rules are captured in the categorical approach via the **path groupoid** described in example 3.2.3 which we reproduce here for convenience:

Definition 6.2.1. The **path groupoid** $\mathcal{P}(M, L) = (\mathcal{P}_0, \mathcal{P}_1, \sigma, \tau, 1, \cdot)$ of a dressed lattice $(M, L, \Rightarrow)$, is the groupoid with object set $\mathcal{P}_0 = L^0$, the set of vertices. The set of morphisms $\mathcal{P}_1$ are given by:

- The set of all edges $v \xrightarrow{e} v' \in L^1$
- Source, σ and target, τ are set maps induced from the directed graph structure such that $\sigma(v \xrightarrow{e} v') = v$ and $\tau(v \xrightarrow{e} v') = v'$
- For each edge $v \xrightarrow{e} v' \in L^1$, an orientation reversed edge $v' \xrightarrow{e^{-1}} v \in L^1$, such that $\sigma(v' \xrightarrow{e^{-1}} v) = \tau(v \xrightarrow{e} v') = v'$ and $\tau(v' \xrightarrow{e^{-1}} v) = \sigma(v \xrightarrow{e} v') = v$
- For each vertex $v \in \mathcal{P}_0$ a unique morphism $1_v \in \mathcal{P}(M, L)$ such that $\sigma(1_v) = v = \tau(v)$
- All formal compositions subject to the following relations:

$$\begin{aligned} v \xrightarrow{e} v' \xrightarrow{e^{-1}} v &= v \xrightarrow{1_v} v \\ v' \xrightarrow{e^{-1}} v \xrightarrow{e} v' &= v' \xrightarrow{1_{v'}} v' \\ v \xrightarrow{1_v} v \xrightarrow{e} v' = v \xrightarrow{e} v' &= v \xrightarrow{e} v' \xrightarrow{1_{v'}} v' \end{aligned} \tag{6.4}$$

6.3 Finite Lattice Gauge Theory

Using the definition of the path groupoid $\mathcal{P}(M, L)$ we can now define the data of lattice gauge theory from the categorical perspective.

Definition 6.3.1. Given a finite group G, with groupoid presentation BG (see example 3.2.1) and path groupoid $\mathcal{P}(M, L)$, a **gauge configuration** F is a functor $F : \mathcal{P}(M, L) \to BG$.

Using definition 3.2.5 we now unpack the content of definition 6.3.1. A gauge configuration $F = (F_0, F_1)$ is a pair of set maps:

$$\begin{aligned} F_0 &: \mathcal{P}(M,L)_0 \to \{*\} = BG_0 \\ F_1 &: \mathcal{P}(M,L)_1 \to G = BG_1 \end{aligned} \tag{6.5}$$

such that F_0 assigns to each vertex $v \in \mathcal{P}(M,L)_0$ the unique object $\{*\} = BG_0$ and F_1 assigns to each oriented edge $v \xrightarrow{e} v' \in \mathcal{P}(M,L)_1$ a morphism $* \xrightarrow{g_e} * \in BG_1$ where $g_e \in G$. Furthermore, functorality implies these assignments are required to satisfy the following relations:

- The identity morphism $v \xrightarrow{1_v} v \in \mathcal{P}(M,L)_1$ is assigned the group identity $1_G \in G$ for all $v \in \mathcal{P}(M,L)_0$.

$$F(v \xrightarrow{1_v} v) := * \xrightarrow{1_G} * \quad \forall v \in \mathcal{P}(M,L)_1 \tag{6.6}$$

- For all composable morphisms, $v \xrightarrow{e} v', v' \xrightarrow{f} v'' \in \mathcal{P}(M,L)_1$

$$F(v \xrightarrow{e} v' \xrightarrow{f} v'') = * \xrightarrow{g_e} * \xrightarrow{g_f} * = * \xrightarrow{g_e g_f} * = F(v \xrightarrow{ef} v'') \tag{6.7}$$

- If $F(v \xrightarrow{e} v') = * \xrightarrow{g_e} *$ then

$$F(v' \xrightarrow{e^{-1}} v) = * \xrightarrow{g_e^{-1}} * = F(v \xrightarrow{e} v')^{-1} \tag{6.8}$$

Definition 6.3.2. Given a pair of gauge configurations F and $\tilde{F}$, a **gauge transformation** η is a natural equivalence $\eta : F \to \tilde{F}$.

Using definition 3.2.6, a gauge transformation η is a set map $\eta : \mathcal{P}(M,L)_0 \to BG_1$ assigning to each vertex $v \in \mathcal{P}(M,L)_0$ a gauge transformation $* \xrightarrow{\eta_v} * \in BG_1$. For each edge $v \xrightarrow{e} v' \in \mathcal{P}(M,L)_1$ where $F(v \xrightarrow{e} v') = * \xrightarrow{g_e} *$ and $\tilde{F}(v \xrightarrow{e} v') = * \xrightarrow{\tilde{g}_e} *$ a gauge transformation $\eta : F \to \tilde{F}$ must satisfy the following commutative diagram

$$\begin{tikzcd} * \arrow[r, "g_e"] \arrow[d, "\eta_v"'] & * \arrow[d, "\eta_{v'}"] \\ * \arrow[r, "\tilde{g}_e"'] & * \end{tikzcd} \tag{6.9}$$

such that

$$\tilde{g}_e = \eta_v^{-1} g_e \eta_{v'}. \tag{6.10}$$

Definition 6.3.3. The **lattice gauge theory groupoid** $\Gamma_{LGT}(M, L; G) := [\mathcal{P}(M, L), BG]$ is the functor groupoid, with objects, gauge configurations and morphisms, gauge transformations.

6.4 Parallel Transport

Definition 6.4.1. Given a lattice groupoid $\mathcal{P}(M, L)$, a morphism $v \xrightarrow{\gamma} v'\mathcal{P}(M, L)$ and a gauge configuration $F : \mathcal{P}(M, L) \to BG$, the **parallel transport** $F(\gamma) \in G$ is the image of γ in F.

A natural consequence of this definition is that given a morphism $v \xrightarrow{\gamma} v' \in \mathcal{P}(M, L)$ and gauge configuration $F : \mathcal{P}(M, L) \to BG$, the parallel transport $F(\gamma)$ is transformed by a gauge transformation η as follows:

$$\eta : F(\gamma) \mapsto \eta_v^{-1} F(\gamma) \eta_{v'}. \tag{6.11}$$

Given a morphism $v \xrightarrow{\gamma} v' \in \mathcal{P}(M, L)$ and gauge configuration $F : \mathcal{P}(M, L) \to BG$, the parallel transport $F(\gamma)$ is transformed by a gauge transformation η as follows:

$$\eta : F(\gamma) \mapsto \eta_v^{-1} F(\gamma) \eta_{v'} \tag{6.12}$$

An important class of parallel transports for lattice gauge theories are given by **holonomies**:

Definition 6.4.2. The parallel transport $F(\gamma)$, for a morphism $v \xrightarrow{\gamma} v \in \mathcal{P}(M, L)_1$ where $s(\gamma) = t(\gamma)$ is called a **holonomy**.

6.5 Flat Gauge Configurations

So far we have constructed lattice gauge theories with no constraints on the set of admissible gauge configurations. If we wish to discuss **topological lattice gauge theories** we must first introduce the concept of flat gauge configurations.

Definition 6.5.1. Given a plaquette $P \in L^2$ and vertex $v \in \partial(P) \cap L^0$. The **quantised boundary** $\partial_v(P)$, is the morphism $v \xrightarrow{\partial_v(P)} v \in \mathcal{P}(M, L)_1$ from v to itself containing all edges in the boundary of P agreeing with the orientation of $\partial(P)$.

Definition 6.5.2. A **flat gauge configuration** F_{flat} is a gauge configuration F such that the holonomy of the quantised boundary $F(\partial_v(P)) = 1_G$ for all $P \in L^2$ and $v \in L^0$.

To see the importance of flat gauge configurations it is instructive to view them from an alternative description formulated in terms of the **fundamental groupoid**.

Definition 6.5.3. The **fundamental groupoid** $\pi_1(M, L)$ of a dressed lattice $(\mathcal{M}, L, \Rightarrow)$ is the quotient of the corresponding path groupoid $\mathcal{P}(M, L)$ by the relation

$$\partial_{b_P}(P) = v \xrightarrow{1_v} v$$

for all $P \in L^2$ where $b_P \in L^0$ is the basepoint of P (see definition 6.1.1).

This groupoid has the property that for any $x \in \Pi_1(M, L)_0$

$$\{\gamma \in \Pi_1(M, L)_1 | s(\gamma) = t(\gamma) = x\} = \pi_1(M, x) \tag{6.13}$$

where $\pi_1(M, x)$ is the fundamental group of M with base-point x. Using $\Pi_1(M, L)$ a flat gauge configuration can viewed as a functor

$$H : \Pi_1(M, L) \to \Gamma(G). \tag{6.14}$$

where for each object $x \in \Pi_1(M, L)_0$ the functor reduces to a group homomorphism

$$H : \pi_1(M, x) \to G \tag{6.15}$$

assigning a possibly non-trivial holonomy to every non-contractible 1-cycle in M.

Additionally, noting the existence of a projection functor

$$\Pi : \mathcal{P}(M, L) \to \Pi_1(M, L) \tag{6.16}$$

which acts as the identity on objects and takes morphisms $\gamma \in \mathcal{P}(M,L)_1$ to their homotopy class in $\Pi_1(M,L)$, a flat gauge configuration $F_{flat} : \mathcal{P}(M,L) \to \Gamma(G)$ can then be seen as functor for which the following diagram commutes:

$$\begin{tikzcd} \mathcal{P}(M,L) \arrow[r, "\Pi"] \arrow[rr, bend left, "F_{flat}"] & \Pi_1(M,L) \arrow[r, "H"] & \Gamma(G) \end{tikzcd} \tag{6.17}$$

In this way a flat gauge configuration can then be seen as a configuration for which the parallel transport $F(\gamma) \in G$ of a morphism γ depends only on the homotopy class of γ in M.

Chapter 7

2-Groupoids, 2-Functors and Pseudo-Natural Equivalences

In the previous chapter we used the theory of groupoids, functors and natural equivalences to define a construction of lattice gauge theories with finite gauge group. In this chapter we introduce the 2-categorical analogues of these constructions, **strict 2-groupoids**, **strict 2-functors**, **pseudo-natural equivalences** and **pseudo-modification equivalences** as a stepping stone to defining higher lattice gauge theories. In particular, the gauge group will be generalised to a **crossed module of groups**, the analogue of gauge configurations, called **2-gauge configurations** will be defined in terms of strict 2-functors and the analogue of gauge transformations, **2-gauge transformations** will be defined in terms of pseudo-natural equivalences. The definitions in this chapter are in large part a combination of the works [67, 66, 68].

7.1 Strict 2-Groupoids

In order to begin our foray into 2-groupoids we begin by defining **strict 2-categories** and an associated diagrammatic calculus:

Definition 7.1.1. A small **strict 2-category** $\mathcal{C} = (C_0, C_1, C_2, s^1, t^1, s^2, t^2, 1^1, 1^2, \cdot, \circ)$ is given by sets of, objects C_0, morphisms C_1 and 2-morphisms C_2 such that:

1. $(C_0, C_1, s^1, t^1, 1^1, \cdot)$ forms a small category

- $s^1/t^1 : C_1 \to C_0$, **source and target maps**
- $1^1 : C_0 \to C_1$, **identity map**

2. $(C_1, C_2, s^2, t^2, 1^2, \circ)$ forms a small category

   - $s^2/t^2 : C_2 \to C_1$, **source and target 2-maps**
   - $1^2 : C_1 \to C_2$, **identity 2-map**

3. $s^1(s^2(A)) = s^1(t^2(A))$ and $t^1(s^2(A)) = t^1(t^2(A))$ for all 2-morphisms $A \in C_2$. This ensures 2-morphisms can be represented by **bigons**:

$$\begin{tikzcd} s^1(s^2(A)) \arrow[r, bend left, "s^2(A)"] \arrow[r, bend right, "t^2(A)"'] \arrow[r, phantom, "\Downarrow A"] & t^1(s^2(A)) \end{tikzcd} = \begin{tikzcd} s^1(t^2(A)) \arrow[r, bend left, "s^2(A)"] \arrow[r, bend right, "t^2(A)"'] \arrow[r, phantom, "\Downarrow A"] & t^1(t^2(A)) \end{tikzcd} \tag{7.1}$$

4. **Vertical composition**, $\circ : C_2 \times_{C_1} C_2 \to C_2$, where $C_2 \times_{C_1} C_2 := \{(A, B) \in C_2 \times C_2 | t^2(A) = s^2(B)\}$, such that $\circ : (A, B) \mapsto A \circ B$. Diagrammatically we represent vertical composition as follows:

$$\begin{tikzcd} s^1(s^2(A)) \arrow[r, bend left=60, "s^2(A)"] \arrow[r] \arrow[r, bend right=60, "t^2(B)"'] \arrow[r, phantom, shift left=3ex, "\Downarrow A"] \arrow[r, phantom, shift right=3ex, "\Downarrow B"] & t^1(s^2(A)) \end{tikzcd} = \begin{tikzcd} s^1(s^2(A)) \arrow[r, bend left, "s^2(A)"] \arrow[r, bend right, "t^2(B)"'] \arrow[r, phantom, "\Downarrow A\circ B"] & t^1(s^2(A)) \end{tikzcd} \tag{7.2}$$

5. **Vertical identity**: For all $a \in C_1$ there exists $1^2_a \in C_2$, such that $s^2(1^2_a) = a = t^2(1^2_a)$ and $1^2_{s^2(A)} \circ A = A = A \circ 1^2_{t^2(A)}$ for all $A \in C_2$.

6. **Horizontal composition**: $\cdot : C_2 \times_{C_0} C_2 \to C_2$, where $C_2 \times_{C_0} C_2 := \{(A, B) \in C_2 \times C_2 | t^1(s^2(A)) = s^1(s^2(B))\}$, such that $\cdot : (A, B) \mapsto A \cdot B$.

Furthermore we require $s^2(A{\cdot}B) = s^2(A){\cdot}s^2(B)$ and $t^2(A{\cdot}B) = t^2(A){\cdot}t^2(B)$.

$$\begin{tikzcd} s^1(s^2(A)) \arrow[r, bend left, "s^2(A)"] \arrow[r, bend right, "t^2(A)"'] \arrow[r, phantom, "\Downarrow A"] & t^1(s^2(A)) \arrow[r, bend left, "s^2(B)"] \arrow[r, bend right, "t^2(B)"'] \arrow[r, phantom, "\Downarrow B"] & t^1(s^2(B)) \end{tikzcd}$$

$$= \begin{tikzcd} s^1(s^2(A \cdot B)) \arrow[r, bend left, "s^2(A\cdot B)"] \arrow[r, bend right, "t^2(A\cdot B)"'] \arrow[r, phantom, "\Downarrow A\cdot B"] & t^1(s^2(A \cdot B)) \end{tikzcd} \tag{7.3}$$

7. **Horizontal identity**: For all $a, b \in C_1$ such that $t(a) = s(b)$ (composable), $1^2(a) \cdot 1^2(b) = 1^2(a \cdot b)$

$$\begin{tikzcd} s^1(a) \arrow[r, bend left, "a"] \arrow[r, bend right, "a"'] \arrow[r, phantom, "\Downarrow 1^2_a"] & t^1(a) \arrow[r, bend left, "b"] \arrow[r, bend right, "b"'] \arrow[r, phantom, "\Downarrow 1^2_b"] & t^1(b) \end{tikzcd} = \begin{tikzcd} s^1(a \cdot b)) \arrow[r, bend left, "a\cdot b"] \arrow[r, bend right, "a\cdot b"'] \arrow[r, phantom, "\Downarrow 1^2_a\cdot 1^2_b"] & t^1(a \cdot b)) \end{tikzcd} \tag{7.4}$$

8. **Interchange law**, $(A_1 \circ B_1) \cdot (A_2 \circ B_2) = (A_1 \cdot A_2) \circ (B_1 \cdot B_2)$ for all $A_1, A_2, B_1, B_2 \in C_2$ where such compositions are defined.

$$\begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow (A_1\cdot A_2)\circ(B_1\cdot B_2)"] & \cdot \end{tikzcd} = \begin{tikzcd} \cdot \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, bend left=30, "\Downarrow A_1\cdot A_2"] \arrow[r, phantom, bend right=30, "\Downarrow B_1\cdot B_2"'] & \cdot \end{tikzcd}$$

$$= \begin{tikzcd} \cdot \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, bend left=30, "\Downarrow A_1"] \arrow[r, phantom, bend right=30, "\Downarrow B_1"'] & \cdot \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, bend left=30, "\Downarrow A_2"] \arrow[r, phantom, bend right=30, "\Downarrow B_2"'] & \cdot \end{tikzcd}$$

$$= \begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow A_1\circ B_2"] & \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow A_2\circ B_2"] & \cdot \end{tikzcd} = \begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow (A_1\circ B_1)\cdot(A_2\circ B_2)"] & \cdot \end{tikzcd} \tag{7.5}$$

Definition 7.1.2. A **2-groupoid** is a strict 2-category $\Gamma^2 = (\Gamma^2_0, \Gamma^2_1, \Gamma^2_2, s^1, t^1, s^2, t^2, 1^1, 1^2, \cdot, \circ)$ such that all morphisms Γ^2_1 and 2-morphisms Γ^2_2 are invertible. Equivalently the two categories $(\Gamma^2_0, \Gamma^2_1, s^1, t^1, \cdot)$ and $(\Gamma^2_1, \Gamma^2_2, s^2, t^2, \circ)$ are both groupoids.

In particular for each 2-morphism $A \in \Gamma_2^2$ there are two inverses, the **vertical inverse** $A^* \in \Gamma_2^2$ and the **horizontal inverse** $\overline{A} \in \Gamma_2^2$

$$
\left(\begin{tikzcd} s^1(s^2(A)) \arrow[r, bend left=50, "s^2(A)"] \arrow[r, bend right=50, "t^2(A)"'] \arrow[r, phantom, "\Downarrow A"] & t^1(s^2(A)) \end{tikzcd} \right)^* = \begin{tikzcd} s^1(s^2(A)) \arrow[r, bend left=50, "s^2(A)"] \arrow[r, bend right=50, "t^2(A)"'] \arrow[r, phantom, "\Uparrow A^*"] & t^1(s^2(A)) \end{tikzcd}
$$

$$
\overline{\begin{tikzcd} s^1(s^2(A)) \arrow[r, bend left=50, "s^2(A)"] \arrow[r, bend right=50, "t^2(A)"'] \arrow[r, phantom, "\Downarrow A"] & t^1(s^2(A)) \end{tikzcd}} = \begin{tikzcd} s^1(s^2(A)) \arrow[r, phantom, "\Downarrow \overline{A}"] & t^1(s^2(A)) \arrow[l, bend right=50, "s^2(A)^{-1}"'] \arrow[l, bend left=50, "t^2(A)^{-1}"] \end{tikzcd} \tag{7.6}
$$

such that

$$
\begin{aligned}
A \circ A^* = 1^2_{s^2(A)}, \quad & A^* \circ A = 1^2_{t^2(A)} \\
A \cdot \overline{A} = 1^2_{1_{s^1(s^2(A))}}, \quad & \overline{A} \cdot A = 1^2_{1_{t^1(s^2(A))}}
\end{aligned} \tag{7.7}
$$

In the definition of a groupoid Γ it was noted that a groupoid with one object corresponded to a categorical presentation of a group BG. In the following we will refer to a **2-group** $B\mathcal{G}$ as a 2-groupoid with a single object. For practical calculations the information of a 2-group can be concisely presented by a **crossed module** of groups $\mathcal{G}$.

7.2 Crossed Modules

Definition 7.2.1. A **crossed module** of groups $\mathcal{G} = (G, E, \partial, \triangleright)$ is a quadruple of data given by a pair of groups G and E, a pair of group homomorphisms $\partial : E \to G$ and $\triangleright : G \to Aut(E)$ such that $\triangleright : G \times E \to E$ defines a left action of G on E by automorphism. The axioms for a crossed module are given by the so called Peiffer conditions,

$$
\begin{aligned}
\partial(a \triangleright A) &= a\partial(A)a^{-1} \\
\partial(A) \triangleright B &= ABA^{-1}
\end{aligned} \tag{7.8}
$$

which hold for all $a \in G$ and $A, B \in E$.

To see the correspondence with the previous section let $\mathcal{G} = (G, E, \partial, \triangleright)$ be a crossed module. Then there exists a strict 2-groupoid $B\mathcal{G}$ such that $B\mathcal{G}_0 = *$, $B\mathcal{G}_1 = G$ and $B\mathcal{G}_2 = G \ltimes_\triangleright E$. The maps s^1 and t^1 are trivial in the sense $s^1(a) = t^1(a) = *$ for all $a \in G$. The identity 1-morphisms are given by $1^1(*) = 1_G$. The composition of 1-morphisms is given by the product in G. The 2-maps are given as follows:

$$
\begin{aligned}
s^2 :& G \ltimes_\triangleright E \to G, && (a, A) \mapsto a \\
t^2 :& G \ltimes_\triangleright E \to G, && (a, A) \mapsto \partial(A)a \\
1^2 :& G \to G \ltimes_\triangleright E, && a \mapsto (a, 1_E).
\end{aligned}
\tag{7.9}
$$

Graphically we consider the vertical compositions as follows

$$
\begin{tikzcd}
* \arrow[r, bend left=60, "a"{name=U}] \arrow[r, "b"{name=M}] \arrow[r, bend right=60, "\partial(B)b"'{name=D}] \arrow[from=U, to=M, phantom, "\Downarrow A"] \arrow[from=M, to=D, phantom, "\Downarrow B"] & *
\end{tikzcd}
=
\begin{tikzcd}
* \arrow[r, bend left=40, "a"{name=U}] \arrow[r, bend right=40, "\partial(BA)a"'{name=D}] \arrow[from=U, to=D, phantom, "\Downarrow BA"] & *
\end{tikzcd}
\tag{7.10}
$$

whenever $b = \partial(A)a$, and BA is the composition of B and A in E. Horizontal composition is given by

$$
\begin{tikzcd}
* \arrow[r, bend left=60, "a"{name=U1}] \arrow[r, bend right=60, "\partial(A)a"'{name=D1}] \arrow[from=U1, to=D1, phantom, "\Downarrow A"] & * \arrow[r, bend left=60, "b"{name=U2}] \arrow[r, bend right=60, "\partial(B)b"'{name=D2}] \arrow[from=U2, to=D2, phantom, "\Downarrow B"] & *
\end{tikzcd}
=
\begin{tikzcd}
* \arrow[r, bend left=60, "ab"{name=U}] \arrow[r, bend right=60, "\partial(A(a\triangleright B))ab"'{name=D}] \arrow[from=U, to=D, phantom, "\Downarrow A(a\triangleright B)"] & *
\end{tikzcd}
\tag{7.11}
$$

which corresponds to composition in $G \ltimes_\triangleright E$. That $\partial(A)a\partial(B)b = \partial(A(a \triangleright B))ab$ follows from the Peiffer conditions. From these definitions it is straightforward to define the vertical $(a, A)^*$ and horizontal inverse $\overline{(a, A)}$ of a 2-morphism $(a, A) \in G \ltimes_\triangleright E$ by the following:

$$
\begin{aligned}
(a, A)^* &= (\partial(A)a, A^{-1}) \\
\overline{(a, A)} &= (a^{-1}, a^{-1} \triangleright A^{-1})
\end{aligned}
\tag{7.12}
$$

7.3 2-Functors, Pseudo-Natural Transformations and Pseudo-Modifications

Now we have defined a suitable notion of 2-groupoid and 2-group we introduce the 2-categorical generalisation of functors and natural equivalences which allow us to compare strict 2-categories.

Definition 7.3.1. Let $\mathcal{C}$ and $\mathcal{D}$ be a pair of small strict 2-categories. A **strict 2-functor** $F : \mathcal{C} \to \mathcal{D}$ is a triple of maps $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$, $F_1 : \mathcal{C}_1 \to \mathcal{D}_1$ and $F_2 : \mathcal{C}_2 \to \mathcal{D}_2$ such that

1. (F_0, F_1) is a functor $(\mathcal{C}_0, \mathcal{C}_1, s^1, t^1, 1^1, \cdot) \to (\mathcal{D}_0, \mathcal{D}_1, s'^1, t'^1, 1'^1, \cdot')$
2. (F_1, F_2) is a functor $(\mathcal{C}_1, \mathcal{C}_2, s^2, t^2, 1^2, \circ) \to (\mathcal{D}_1, \mathcal{D}_2, s'^2, t'^2, 1'^2, \circ')$
3. $F_2(A \cdot B) = F_2(A) \cdot' F_2(B)$ for all $(A, B) \in \mathcal{C}_2 \times_{\mathcal{C}_0} \mathcal{C}_2$.

Definition 7.3.2. Let $F, \tilde{F} : \mathcal{C} \to \mathcal{D}$ be a pair of strict 2-functors. A **pseudo-natural transformation** $\eta : F \to \tilde{F}$ is a pair of maps $\eta_0 : \mathcal{C}_0 \to \mathcal{D}_1$ and $\eta_1 : \mathcal{C}_1 \to \mathcal{D}_2$ such that η_0 associates a morphism $\eta_x : F(x) \to \tilde{F}(x)$ to each object $x \in \mathcal{C}$ and η_1 assigns a 2-morphism

$$\begin{tikzcd} F(x) \arrow[r, "F(a)"] \arrow[d, "\eta_x"'] & F(y) \arrow[d, "\eta_y"] \arrow[dl, Rightarrow, "\eta_a"'] \\ \tilde{F}(x) \arrow[r, "\tilde{F}(a)"'] & \tilde{F}(y) \end{tikzcd} \tag{7.13}$$

such that $\eta_a : F(a) \cdot \eta_{t^1(a)} \Rightarrow \eta_{s^1(a)} \cdot \tilde{F}(a)$ to each morphism $a \in \mathcal{C}_1$. These maps are subject to the following conditions:

- For all composable morphisms $x \xrightarrow{a} y, y \xrightarrow{b} z \in \mathcal{C}_1$

$$\begin{tikzcd} F(x) \arrow[r, "F(a)"] \arrow[d, "\eta_x"'] & F(y) \arrow[r, "F(b)"] \arrow[d, "\eta_y"] \arrow[dl, Rightarrow, "\eta_a"'] & F(z) \arrow[d, "\eta_z"] \arrow[dl, Rightarrow, "\eta_b"'] \\ \tilde{F}(x) \arrow[r, "\tilde{F}(a)"'] & \tilde{F}(y) \arrow[r, "\tilde{F}(b)"'] & \tilde{F}(z) \end{tikzcd} = \begin{tikzcd} F(x) \arrow[r, "F(a)\cdot F(b)"] \arrow[d, "\eta_x"'] & F(z) \arrow[d, "\eta_z"] \arrow[dl, Rightarrow, "(1^2_{F(a)}\cdot\eta_b)\circ(\eta_a\cdot 1^2_{\tilde{F}(b)})"'] \\ \tilde{F}(x) \arrow[r, "\tilde{F}(a)\cdot\tilde{F}(b)"'] & \tilde{F}(z) \end{tikzcd} \tag{7.14}$$

- $\eta_{1^1_x} = 1^2_{\eta_x}$ for all $x \in \mathcal{C}_0$
- For any 2-morphism $a \overset{A}{\Rightarrow} b \in \mathcal{C}_2$ between morphisms $x \overset{a}{\to} y, x \overset{a}{\to} y \in \mathcal{C}_1$, the following diagram 2-commutes.

$$\begin{tikzcd}
F(x) \arrow[r, bend left, "F(a)"] \arrow[r, bend right, "F(b)"'] \arrow[r, phantom, "\Downarrow F(A)"] \arrow[d, "\eta_x"] & F(y) \arrow[d, "\eta_y"] \arrow[dl, Rightarrow, "\eta_a"'] \arrow[dl, Rightarrow, "\eta_b"] \\
\tilde{F}(x) \arrow[r, bend left, "\tilde{F}(a)"] \arrow[r, bend right, "\tilde{F}(b)"'] \arrow[r, phantom, "\Downarrow \tilde{F}(A)"] & \tilde{F}(y)
\end{tikzcd} \tag{7.15}$$

This implies

$$(F(A) \cdot 1^2_{\eta_y}) \circ \eta_b = \eta_a \circ (1^2_{\eta_x} \cdot \tilde{F}(A)) \tag{7.16}$$

The axioms for a pseudo-natural transformations guarantee that given a pair of strict 2-functors $F, \tilde{F} : \mathcal{C} \to \mathcal{D}$ and pseudo-natural transformation $\eta : F \to \tilde{F}$ the following relations hold:

$$\begin{aligned}
\eta(F(a)) \cdot \eta(F(b)) &= \eta(F(a \cdot b)) \\
\eta(F(A)) \cdot \eta(F(B)) &= \eta(F(A \cdot B)) \\
\eta(F(A)) \circ \eta(F(B)) &= \eta(F(A \circ B))
\end{aligned} \tag{7.17}$$

for all $F(a), F(b) \in \mathcal{D}_1$ and $F(A), F(B) \in \mathcal{D}_2$ whenever the composition is defined. In this way a pseudo-natural transformation of a morphism or 2-morphism depend only on the source and targets.

Definition 7.3.3. A **pseudo-natural equivalence** is a pseudo-natural transformation $\eta : F \to \tilde{F}$ such that there exists $\eta^{-1} : \tilde{F} \to F$ where $\eta\eta^{-1} = 1_F : F \to F$ the identity natural transformation for F and $\eta^{-1}\eta = 1_{\tilde{F}} : \tilde{F} \to \tilde{F}$ is the identity natural transformation for $\tilde{F}$.

For strict 2-categories there exists an extra structure which allows for the comparison of pseudo-natural transformations called **pseudo-modifications** which has no analogue in ordinary category theory. We will not need this structure to discuss higher gauge theory in the present discussion but we will return to this structure when discussing **tube algebras** for higher lattice gauge theories in chapter 13.

Definition 7.3.4. Let $\mathcal{C}, \mathcal{D}$ be a pair of small strict 2-categories, $F, \tilde{F} : \mathcal{C} \to \mathcal{D}$ a pair of strict 2-functors and $\eta, \nu : F \to \tilde{F}$ a pair of pseudo-natural transformations. A **pseudo-modification** $\mu : \eta \Rightarrow \nu$ is a map $\mathcal{C}_0 \to \mathcal{D}_2$ assigning to each object $x \in \mathcal{C}_0$ a 2-morphism $\mu_x : \eta_x \Rightarrow \nu_x$ in $\mathcal{D}$ such that

$$(F(A) \cdot \mu_y) \circ \nu_a = \eta_a \circ (\mu_x \cdot \tilde{F}(A)) \tag{7.18}$$

holds for each pair of morphisms $a, b : x \to y$ and 2-morphism $A : a \Rightarrow B$ in $\mathcal{C}$. Equivalently the following two diagram commutes:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
F(x) \arrow[r, bend left, "F(a)"] \arrow[r, bend right, "F(b)"'] \arrow[r, phantom, "\Downarrow F(A)"] \arrow[d, "\eta_x"] \arrow[d, bend right=60, "\nu_x"'] \arrow[d, phantom, "\Leftarrow \mu_x \quad"] \arrow[dr, phantom, "\Swarrow \nu_a \quad \Swarrow \eta_b"] & F(y) \arrow[d, "\eta_y"'] \arrow[d, bend left=60, "\nu_y"] \arrow[d, phantom, "\quad \mu_y \Rightarrow"] \\
\tilde{F}(x) \arrow[r, bend left, "\tilde{F}(a)"] \arrow[r, bend right, "\tilde{F}(b)"'] \arrow[r, phantom, "\Downarrow \tilde{F}(A)"] & \tilde{F}(y)
\end{tikzcd} \tag{7.19}$$

Definition 7.3.5. Given a pair of pseudo natural transformations η, ν, a **pseudo-modification equivalence** is a pseudo-modification $\mu : \eta \to \nu$ such that μ has a two sided inverse $\mu^{-1} : \nu \Rightarrow \eta$, $\mu\mu^{-1} = 1_\eta$, where 1_η is the identity pseudo-modification on η and similarly $\mu^{-1}\mu = 1_\nu$.

Corollary 7.3.0.1. Given a pair of strict 2-groupoids $\Gamma^2, \tilde{\Gamma}^2$ there exists a 2-groupoid $[\Gamma^2, \tilde{\Gamma}^2]$ with objects, strict 2-functors $F : \Gamma^2 \to \tilde{\Gamma}^2$, morphisms pseudo-natural equivalences $\eta : F \to \tilde{F}$ and 2-morphisms psuedo-modification equivalences $\mu : \eta \Rightarrow \nu$. We call this 2-groupoid the **functor 2-groupoid**.

Chapter 8

Finite Higher Lattice Gauge Theory

We now present the formalism of higher lattice gauge theory (HLGT) by analogy with conventional lattice gauge theory. We structure this section as to mirror the corresponding formalisms in lattice gauge theory. We will begin by defining a suitable 2-categorical generalisation of the path groupoid, called the **path 2-groupoid**. Building on this structure we will utilise strict 2-functors from the path 2-groupoid to a finite 2-group to define **2-gauge configurations** and pseudo-natural transformations will play the role of **2-gauge transformations**. The definitions of 2-gauge configurations and 2-gauge transformations follow from the work of Pfeiffer [66, 68].

8.1 Path 2-Groupoid

Definition 8.1.1. Let $(M, L, \Rightarrow)$ be a dressed lattice (see definition 6.1.4). There exists a small strict 2-groupoid $\mathcal{P}^2(M, L) = (\mathcal{P}_0^2, \mathcal{P}_1^2, \mathcal{P}_2^2, \sigma, \tau, s, t, 1, \mathbb{1}, \cdot, \circ)$ as follows: The groupoid $\mathcal{P}(M, L) = (\mathcal{P}_0^2, \mathcal{P}_1^2, \sigma, \tau, 1, \cdot)$ is the lattice groupoid from definition 6.2.1. The set of 2-morphisms are given as follows:

- For each $P \in L^2$ with base-point $v \in L^0$, a 2-morphism $f \in \mathcal{P}_2^2$

$$\begin{tikzcd} v \arrow[r, bend left=50, "1_v"] \arrow[r, bend right=50, "\partial(P)_v"'] \arrow[r, phantom, "\Downarrow f"] & v \end{tikzcd} \tag{8.1}$$

  such that $s(f) = 1_v$ and $t(f) = \partial(P)_v$, the quantised boundary of P (see definition 6.5.1).

- For each $P \in L^2$, a 2-morphism $f^* \in \mathcal{P}_2^2$

$$\begin{tikzcd} v \arrow[r, bend left=50, "1_v"] \arrow[r, bend right=50, "\partial(P)_v"'] \arrow[r, phantom, "\Uparrow f^*"] & v \end{tikzcd} \tag{8.2}$$

  such that $s(f^*) = t(f)$ and $t(f^*) = s(f)$.

- For each $P \in L^2$, a 2-morphism $\bar{f} \in \mathcal{P}_2^2$

$$\begin{tikzcd} v & v \arrow[l, bend right=50, "1_v"'] \arrow[l, bend left=50, "\partial(P)_v^{-1}"] \arrow[l, phantom, "\Downarrow \bar{f}"] \end{tikzcd} \tag{8.3}$$

  such that $s(\bar{f}) = 1_v$ and $t(\bar{f}) = \partial(P)_v^{-1}$.

- For each $P \in L^2$, a 2-morphism $\bar{f}^* \in \mathcal{P}_2^2$

$$\begin{tikzcd} v & v \arrow[l, bend right=50, "1_v"'] \arrow[l, bend left=50, "\partial(P)_v^{-1}"] \arrow[l, phantom, "\Uparrow \bar{f}^*"] \end{tikzcd} \tag{8.4}$$

  such that $s(\bar{f}^*) = t(f)^{-1}$ and $t(\bar{f}^*) = s(f)^{-1}$.

- For each edge $v \xrightarrow{e} v' \in L^1$, a unique 2-morphism $\mathbb{1}_e \in \mathcal{P}_2^2$

$$\begin{tikzcd} v \arrow[r, bend left=50, "e"{name=U}] \arrow[r, bend right=50, "e"'{name=D}] & v' \arrow[from=U, to=D, Rightarrow, "\mathbb{1}_e"] \end{tikzcd} \tag{8.5}$$

  such that $s(\mathbb{1}_e) = t(\mathbb{1}_e)$

- All formal vertical $\circ$ and horizontal $\cdot$ compositions of the above 2-morphisms subject to the following relations:

  1. $f \circ f^* = \mathbb{1}_{s(f)}$ and $f^* \circ f = \mathbb{1}_{t(f)}$
  2. $\mathbb{1}_{s(f)} \circ f = f = f \circ \mathbb{1}_{t(f)}$
  3. $f \cdot \overline{f} = \mathbb{1}_{1_{\sigma(s(f))}}$ and $\overline{f} \cdot f = \mathbb{1}_{1_{\tau(s(f))}}$
  4. $\mathbb{1}_{1_{\sigma(s(f))}} \cdot f = f = f \cdot \mathbb{1}_{1_{\tau(s(f))}}$
  5. $\mathbb{1}_e \cdot \mathbb{1}_{e'} = \mathbb{1}_{e \cdot e'}$
  6. Interchange law $(f_1 \cdot f_2) \circ (f_1' \cdot f_2') = (f_1 \circ f_1') \cdot (f_2 \circ f_2')$ for all composable $f_1, f_2, f_1', f_2' \in \mathcal{P}_2^2$
  7. $\mathbb{1}_{s(f)} \cdot \overline{f^*} \cdot \mathbb{1}_{t(f)} = f$
  8. $\mathbb{1}_{s(f)} \cdot \overline{f} \cdot \mathbb{1}_{t(f)} = f^*$

Definition 8.1.2. In the following, for all 2-morphisms $f \in \mathcal{P}_2^2$ we will use the notation $\overline{f}$ to notate the horizontal inverse and f^* to notate the vertical inverse as in definition 7.1.2 such that:

$$\begin{aligned} f \cdot \overline{f} &= \mathbb{1}_{1_{\sigma(s(f))}} \\ \overline{f} \cdot f &= \mathbb{1}_{1_{\tau(s(f))}} \\ f \circ f^* &= \mathbb{1}_{s(f)} \\ f^* \circ f &= \mathbb{1}_{t(f)} \\ \overline{(f^*)} &= \overline{f}^*. \end{aligned} \tag{8.6}$$

In particular utilising this notation, such 2-morphisms satisfy all relations 1-8.

The composition relations 1 − 6 guarantee $\mathcal{P}^2(M,L)$ is indeed a 2-groupoid, whereas relations 7 and 8 are of a purely geometric nature. In general, a 2-morphism $f \in \mathcal{P}^2(M,L)_2$ is given by a pair of morphisms $v \xrightarrow{\gamma} v', v \xrightarrow{\gamma'} v' \in \mathcal{P}^2(M,L)_1$ such that:

$$\begin{tikzcd} v \arrow[r, bend left=50, "\gamma"{name=U}] \arrow[r, bend right=50, "\gamma'"'{name=D}] & v' \arrow[from=U, to=D, Rightarrow, "f"] \end{tikzcd} \tag{8.7}$$

where f is a surface connecting γ and γ' which is topologically a 2-disk in (M,L). Two examples are given below:

$$\begin{tikzcd} v \arrow[loop, distance=6em, "e"] \arrow[r, phantom, "\Leftarrow f'"] & {} \end{tikzcd} \quad , \qquad \begin{tikzcd} & w \arrow[dr, "e_2"] \arrow[d, phantom, "\Downarrow f"] & \\ v \arrow[ur, "e_1"] \arrow[rr, "e_3"'] & {} & v' \end{tikzcd} \tag{8.8}$$

where $s(f') = e, t(f') = 1_v$ and $s(f) = e_1 e_2, t(f) = e_3$. Using the relation $s(f \cdot f') = s(f) \cdot s(f')$ in the definition of a strict 2-category, we can "rotate" a 2-morphism of $\mathcal{P}^2(M,L)$ by changing the source and target morphisms by composition with the identity 2-morphism. One example is the following:

$$\begin{tikzcd} & & w \arrow[dr, "e_2"] \arrow[d, phantom, "\Downarrow f"] & \\ w \arrow[r, bend left=50, "e_1^{-1}"{name=U}] \arrow[r, bend right=50, "e_1^{-1}"'{name=D}] & v \arrow[ur, "e_1"] \arrow[rr, "e_3"'] & {} & v' \arrow[from=U, to=D, Rightarrow, "\mathbb{1}_{e_1^{-1}}"] \end{tikzcd} = \begin{tikzcd} w \arrow[rr, "e_2"] \arrow[dr, "e_1^{-1}"'] \arrow[r, phantom, shift right=3ex, "\Downarrow \mathbb{1}_{e_1^{-1}} \cdot f"] & {} & v' \\ & v \arrow[ur, "e_3"'] & \end{tikzcd} \tag{8.9}$$

where $s(1_{e_1^{-1}} \cdot f) = e_2$ and $t(1_{e_1^{-1}} \cdot f) = e_1^{-1} e_3$. Similarly we can perform this operation on the right-hand side of the diagram. We call this operation **whiskering**.

Condition 7 is then be visualised on the triangle example as follows:

$$
\begin{tikzcd}
 & & w \arrow[dr, "e_2"] \arrow[d, phantom, "\Downarrow f"] & & \\
v' \arrow[r, bend left, "e_2^{-1}e_1^{-1}"] \arrow[r, bend right, "e_2^{-1}e_1^{-1}"'] \arrow[r, phantom, "\Downarrow \mathbb{1}_{e_2^{-1}e_1^{-1}}"] & w \arrow[ur, "e_1"] \arrow[rr, "e_3"'] & {} & v \arrow[r, bend left, "e_3^{-1}"] \arrow[r, bend right, "e_3^{-1}"'] \arrow[r, phantom, "\Downarrow \mathbb{1}_{e_3^{-1}}"] & v
\end{tikzcd}
$$

$$
= \begin{tikzcd}
v' \arrow[rr, "e_3^{-1}"] \arrow[dr, "e_2^{-1}"'] & {} \arrow[d, phantom, "\Downarrow \mathbb{1}_{e_2^{-1}e_1^{-1}}\cdot f\cdot e_3^{-1}"] & v \\
 & w \arrow[ur, "e_1^{-1}"'] &
\end{tikzcd}
= \begin{tikzcd}
v' \arrow[rr, "e_3^{-1}"] \arrow[dr, "e_2^{-1}"'] & {} \arrow[d, phantom, "\Downarrow f^*"] & v \\
 & w \arrow[ur, "e_1^{-1}"'] &
\end{tikzcd}
\tag{8.10}
$$

In this way relation 7 and analogously for relation 8 can be seen as ensuring there is a unique 2-morphism for each surface between a pair of morphisms with the same source and target.

Before continuing our discussion we now give two examples of composing 2-morphisms in $\mathcal{P}^2(M, L)$.

Example 8.1.1.

$$
\begin{tikzcd}
v_1 \arrow[rr, "e_{12}"] \arrow[d, "e_{14}"'] & {} \arrow[d, phantom, "\Downarrow f^{126}_{1456}"] & v_2 \arrow[r, "e_{23}"] \arrow[d, "e_{26}"] & v_3 \arrow[dd, "e_{39}"] \\
v_4 \arrow[r, "e_{45}"] \arrow[d, "e_{47}"'] \arrow[dr, phantom, "\Downarrow f^{458}_{478}"] & v_5 \arrow[r, "e_{56}"] \arrow[d, "e_{58}"] & v_6 \arrow[dr, phantom, "\Downarrow f^{5623}_{5893}"] & \\
v_7 \arrow[r, "e_{78}"'] & v_8 \arrow[rr, "e_{89}"'] & & v_9
\end{tikzcd}
\tag{8.11}
$$

Where each 2-morphism as 2-source and 2-target as follows:

$$
\begin{aligned}
e_{12}e_{26} &\xRightarrow{f^{126}_{1456}} e_{14}e_{45}e_{56} \\
e_{45}e_{48} &\xRightarrow{f^{458}_{478}} e_{47}e_{78} \\
e_{56}e_{26}^{-1}e_{23} &\xRightarrow{f^{5623}_{5893}} e_{58}e_{89}e_{39}^{-1}
\end{aligned}
\tag{8.12}
$$

Using whiskering we can define a surface:

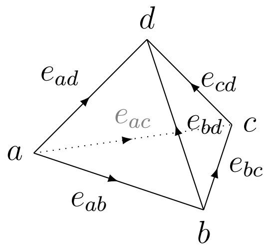
$$e_{12} \overset{f}{\Rightarrow} e_{14}e_{47}e_{78}e_{89}e_{39}^{-1}e_{23}^{-1} \tag{8.13}$$

In general there are multiple ways to express such a surface in terms of 2-morphisms. The axioms above ensure that all such compositions can be identified. One such example is depicted below:

$$\begin{aligned} f &= (f^{126}_{1456} \cdot \mathbb{1}_{e_{26}^{-1}}) \circ (\mathbb{1}_{e_{14}} \cdot f^{458}_{478} \cdot \mathbb{1}_{e_{58}^{-1}e_{56}e_{26}^{-1}}) \circ (\mathbb{1}_{e_{14}e_{47}e_{78}e_{58}^{-1}} \cdot f^{5623}_{5893} \cdot \mathbb{1}_{e_{23}^{-1}}) \\ &= (f^{126}_{1456} \cdot \mathbb{1}_{e_{26}^{-1}}) \circ (\mathbb{1}_{e_{14}e_{45}} \cdot f^{5623}_{5893} \cdot \mathbb{1}_{e_{23}^{-1}}) \circ (\mathbb{1}_{e_{14}} \cdot f^{458}_{478} \cdot \mathbb{1}_{e_{89}e_{39}^{-1}e_{23}^{-1}}) \end{aligned} \tag{8.14}$$

The two expressions for f are related via the relations defining $\mathcal{P}^2(\mathcal{M}, L)$

Example 8.1.2. Given the tetrahedron



(8.15)

with 2-morphisms

$$\begin{aligned} e_{ab}e_{bc} &\overset{f_{abc}}{\Longrightarrow} e_{ac} \\ e_{ab}e_{bd} &\overset{f_{abd}}{\Longrightarrow} e_{ad} \\ e_{ac}e_{cd} &\overset{f_{abc}}{\Longrightarrow} e_{ad} \\ e_{bc}e_{cd} &\overset{f_{abc}}{\Longrightarrow} e_{bd} \end{aligned} \tag{8.16}$$

We can cut the surface open along an edge eg. $a \xrightarrow{e_{ad}} d$ and form the surface

$e_{ad} \overset{f}{\Rightarrow} e_{ad} \in \mathcal{P}^2(M, L)_2$ as follows:

$$
\begin{tikzcd}
 & b \arrow[dr, "e_{bd}"] \arrow[dd, "e_{bc}"] \arrow[d, phantom, "\Uparrow f_{abd}" near start, shift left=6ex] & \\
a \arrow[ur, "e_{ab}"] \arrow[dr, "e_{ac}"'] \arrow[rr, bend left=90, "e_{ad}"] \arrow[rr, bend right=90, "e_{ad}"'] \arrow[r, phantom, "\Downarrow f_{abc}"] & \arrow[r, phantom, "\Uparrow f_{bcd}"] & d \\
 & c \arrow[ur, "e_{cd}"'] \arrow[u, phantom, "\Downarrow f_{acd}" near start, shift left=6ex] &
\end{tikzcd}
=
\begin{tikzcd}[column sep=huge]
a \arrow[r, bend left=70, "e_{ad}"] \arrow[r, bend left=35, "e_{ab}e_{bd}"] \arrow[r, "e_{ab}e_{bc}e_{cd}"] \arrow[r, bend right=35, "e_{ac}e_{cd}"'] \arrow[r, bend right=70, "e_{ad}"'] \arrow[r, phantom, "\Uparrow f_{abd}", shift left=7ex] \arrow[r, phantom, "\Uparrow \mathbb{1}_{e_{ab}} \cdot f_{bcd}", shift left=3ex] \arrow[r, phantom, "\Downarrow f_{abc}", shift right=3ex] \arrow[r, phantom, "\Downarrow f_{acd}", shift right=7ex] & d
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, bend left, "e_{ad}"] \arrow[r, bend right, "e_{ad}"'] \arrow[r, phantom, "\Downarrow f"] & d
\end{tikzcd}
\tag{8.17}
$$

where

$$
f = f^*_{abd} \circ (\mathbb{1}_{e_{ab}} \cdot f_{bcd})^* \circ f_{abc} \circ f_{acd}. \tag{8.18}
$$

In the diagram the first equality is given by whiskering to change the source and target morphisms and the second equality is given via vertical composition.

Definition 8.1.3. Given a dressed lattice $(M, L, \Rightarrow)$ and $B \in L^3$ with base-point $v \in L^0$, the **quantised 2-boundary** $1_v \xRightarrow{\partial^2_v(B)} 1_v \in \mathcal{P}^2(\mathcal{M}, L)_2$ is the non-trivial 2-morphism contained in $\partial(B)$ agreeing with the orientation of $\partial(B)$.

Example 8.1.3. Given the tetrahedron in example 8.1.2 with base-point $a \in \mathcal{P}^2(M, L)_0$. The quantised 2-boundary is given by:

$$
\begin{tikzcd}
a \arrow[r, bend left, "1_a"] \arrow[r, bend right, "1_a"'] \arrow[r, phantom, "{\scriptstyle \| \; f \cdot \mathbb{1}_{e_{ad}^{-1}} \; \Downarrow}"] & a
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, bend left, "e_{ad}"] \arrow[r, bend right, "e_{ad}"'] \arrow[r, phantom, "\Downarrow f"] & d \arrow[r, bend left, "e_{ad}^{-1}"] \arrow[r, bend right, "e_{ad}^{-1}"'] \arrow[r, phantom, "\Downarrow \mathbb{1}_{e_{ad}^{-1}}"] & a
\end{tikzcd}
\tag{8.19}
$$

if the surface agrees with the orientation of $\partial(\Delta_{abcd})$ and

$$
\begin{tikzcd}
a \arrow[r, bend left, "1_a"] \arrow[r, bend right, "1_a"'] \arrow[r, phantom, "{\scriptstyle \| \; (f \cdot \mathbb{1}_{e_{ad}^{-1}})^* \; \Downarrow}"] & a
\end{tikzcd}
\tag{8.20}
$$

else.

8.2 2-Gauge Configurations

Let $\mathcal{G} = (G, E, \partial, \triangleright)$ denote a finite crossed module and $B\mathcal{G}$ the presentation of $\mathcal{G}$ as a one object 2-groupoid.

Definition 8.2.1. A 2-gauge configuration F is a strict 2-functor $F : \mathcal{P}^2(M, L) \to B\mathcal{G}$.

From definition 7.3.1, a 2-gauge configuration defines a triple of set maps $F = (F_0, F_1, F_2)$ such that

$$F_0 : \mathcal{P}^2(M, L)_0 \to * \tag{8.21}$$

$$F_1 : \mathcal{P}^2(M, L)_1 \to G \tag{8.22}$$

$$F_2 : \mathcal{P}^2(M, L)_2 \to G \ltimes_\triangleright E. \tag{8.23}$$

where $F^1 = (F_0, F_1)$ defines a gauge configuration $F^1 : \mathcal{P}(M, L) \to BG$. On 2-morphisms F acts as follows:

$$F\left(\begin{tikzcd} v \arrow[r, bend left=50, "e"] \arrow[r, bend right=50, "e'"'] \arrow[r, phantom, "\Downarrow f"] & v' \end{tikzcd} \right) = \begin{tikzcd} * \arrow[r, bend left=50, "h_e"] \arrow[r, bend right=50, "\partial(H_f)h_e"'] \arrow[r, phantom, "\Downarrow H_f"] & * \end{tikzcd} \tag{8.24}$$

such that

$$t^2(H_f) \cdot s^2(H_f)^{-1} = \partial(H_f) \tag{8.25}$$

holds. Furthermore it is straightforward to verify:

$$
\begin{aligned}
&F\left(\begin{tikzcd} v \arrow[r, bend left=60, "e"] \arrow[r, bend right=60, "e"'] \arrow[r, phantom, "\Downarrow \mathbb{1}_e"] & v' \end{tikzcd}\right) = \begin{tikzcd} * \arrow[r, bend left=60, "h_e"] \arrow[r, bend right=60, "h_e"'] \arrow[r, phantom, "\Downarrow 1_E"] & * \end{tikzcd} \ , \\
&F\left(\begin{tikzcd} v \arrow[r, bend left=60, "e"] \arrow[r, bend right=60, "e'"'] \arrow[r, phantom, "\Uparrow f^*"] & v' \end{tikzcd}\right) = \begin{tikzcd} * \arrow[r, bend left=60, "h_e"] \arrow[r, bend right=60, "\partial(H_f)h_e"'] \arrow[r, phantom, "\Uparrow H_f^{-1}"] & * \end{tikzcd} \\
&F\left(\begin{tikzcd} v & v' \arrow[l, bend right=60, "e^{-1}"'] \arrow[l, bend left=60, "e'^{-1}"] \arrow[l, phantom, "\Downarrow \overline{f}"] \end{tikzcd}\right) = \begin{tikzcd} * & * \arrow[l, bend right=60, "h_e^{-1}"'] \arrow[l, bend left=60, "\partial(h_e^{-1}\triangleright H_f)h_e^{-1}"] \arrow[l, phantom, "\Downarrow h_e^{-1}\triangleright H_f^{-1}"] \end{tikzcd}
\end{aligned}
\tag{8.26}
$$

whenever

$$
F\left(\begin{tikzcd} v \arrow[r, bend left=60, "e"] \arrow[r, bend right=60, "e'"'] \arrow[r, phantom, "\Downarrow f"] & v' \end{tikzcd}\right) = \begin{tikzcd} * \arrow[r, bend left=60, "h_e"] \arrow[r, bend right=60, "\partial(H_f)h_e"'] \arrow[r, phantom, "\Downarrow H_f"] & * \end{tikzcd} \ . \tag{8.27}
$$

Finally we can verify that such functors satisfy relation 7 in definition 8.1.1 of the path 2-groupoid. If $f \in \mathcal{P}^2(M,L)_2$ and $F(f) = (h_e, H_f) \in G \ltimes_\triangleright E$ then

$$
\begin{aligned}
F(\mathbb{1}_{s^2(f)^{-1}} \cdot f \cdot \mathbb{1}_{t^2(f)^{-1}}) &= (\partial(h_e^{-1} \triangleright H_f)^{-1} h_e^{-1}, h_e^{-1} \triangleright H_f) = F(\overline{f^*}) \\
F(\mathbb{1}_{t^2(f)^{-1}} \cdot f \cdot \mathbb{1}_{s^2(f)^{-1}}) &= (\partial(h_e^{-1} \triangleright H_f)^{-1} h_e^{-1}, h_e^{-1} \triangleright H_f) = F(\overline{f^*})
\end{aligned}
\tag{8.28}
$$

8.3 2-Gauge Transformations

We now consider 2-gauge transformations. Akin to lattice gauge theory, we will consider 2-gauge transformations to be given by pseudo-natural transformations.

Definition 8.3.1. Given a pair of 2-gauge configurations $F, \tilde{F} : \mathcal{P}^2(M,L) \to B\mathcal{G}$, a 2-gauge transformation is given by a pseudo-natural transformation $\eta : F \Rightarrow \tilde{F}$.

From definition 7.3.2, a 2-gauge transformation consists of a pair of set maps

$$
\begin{aligned}
\eta_0 &: \mathcal{P}^2(M,L) \to G \\
\eta_1 &: \mathcal{P}^2(M,L) \to G \ltimes_{\triangleright} E
\end{aligned}
\tag{8.29}
$$

For each 1-morphism $v \xrightarrow{e} v' \in \Gamma^2(M,L)_1$, where $F(v \xrightarrow{e} v') = * \xrightarrow{h_e} *$, $\tilde{F}(v \xrightarrow{e} v') = * \xrightarrow{\tilde{h}_e} * \in \Gamma(\mathcal{G})$ a 2-gauge transformation consists of $\eta_v, \eta_{v'} \in G$ and $(h_e\eta_{v'}, \eta_e) \in G \ltimes_{\triangleright} E$ such that the following diagram commutes:

$$
\begin{tikzcd}
* \arrow[r, "h_e"] \arrow[d, "\eta_v"'] & * \arrow[d, "\eta_{v'}"] \arrow[dl, Rightarrow, "\eta_e"'] \\
* \arrow[r, "\tilde{h}_e"'] & *
\end{tikzcd}
\tag{8.30}
$$

or equivalently

$$
\tilde{h}_e = \eta_v^{-1}\partial(\eta_e)h_e\eta_{v'}. \tag{8.31}
$$

To consider the action on a 2-morphism

$$
F\left(
\begin{tikzcd}
v \arrow[r, bend left=50, "e"] \arrow[r, bend right=50, "e'"'] \arrow[r, phantom, "\Downarrow f"] & v'
\end{tikzcd}
\right) =
\begin{tikzcd}
* \arrow[r, bend left=50, "h_e"] \arrow[r, bend right=50, "\partial(H_f)h_e"'] \arrow[r, phantom, "\Downarrow H_f"] & *
\end{tikzcd}
$$

$$
\tilde{F}\left(
\begin{tikzcd}
v \arrow[r, bend left=50, "e"] \arrow[r, bend right=50, "e'"'] \arrow[r, phantom, "\Downarrow f"] & v'
\end{tikzcd}
\right) =
\begin{tikzcd}
* \arrow[r, bend left=50, "\tilde{h}_e"] \arrow[r, bend right=50, "\partial(\tilde{H}_f)\tilde{h}_e"'] \arrow[r, phantom, "\Downarrow \tilde{H}_f"] & *
\end{tikzcd}
\tag{8.32}
$$

we utilise the tin can axiom:

$$
\begin{tikzcd}
* \arrow[r, bend left, "h_e"] \arrow[r, bend right, "\partial(H_f)h_e"'] \arrow[r, phantom, "\Downarrow H_f"] \arrow[d, "\eta_v"'] & * \arrow[d, "\eta_{v'}"] \arrow[dl, Rightarrow, shift left=2, "\eta_{e'}"] \arrow[dl, Rightarrow, shift right=2, "\eta_e"'] \\
* \arrow[r, bend left, "\tilde{h}_e"] \arrow[r, bend right, "\partial(\tilde{H}_f)\tilde{h}_e"'] \arrow[r, phantom, "\Downarrow \tilde{H}_f"] & *
\end{tikzcd}
\tag{8.33}
$$

such that:

$$\begin{aligned}\tilde{H}_f &= \eta_v^{-1} \triangleright [\eta_{e'} H_f \eta_e^{-1}] \\ \tilde{h_e} &= \eta^{-1}\partial(\eta_e) h_e \eta.\end{aligned} \tag{8.34}$$

8.4 2-Parallel Transport

In this section we define the notion of **2-holonomy** generalising the notion of holonomy in lattice gauge theory.

Definition 8.4.1. Given a 2-gauge configuration F and a 2-morphism $S \in \mathcal{P}^2(M,L)_2$, a **2-holonomy** is given by $F(S) = (h,H) \in G \ltimes E$.

Definition 8.4.2. Given a 2-holonomy $F(S) = (h,H) \in G \ltimes E$ of the 2-morphism $S \in \mathcal{P}^2(M,L)_2$, the **surface holonomy**, $H^2(F(S))$ is given by:

$$\begin{aligned} H^2 &: F(S) \to E \\ H^2 &: (h,H) \mapsto H^2(h,H) = H \in E \end{aligned} \tag{8.35}$$

Similarly we define the 2-source $s(h,H) = h \in G$ as the **source holonomy** and the 2-target $t(h,H) = \partial(H)h \in G$ the **target holonomy**.

Remark 8.4.1. Given a 2-holonomy $F(S) = (h,H) \in G \ltimes_\triangleright E$ of a 2-morphism $S \in \mathcal{P}^2(M,L)_2$ such that $s^2(S) = t^2(S)$, the relation in equation (8.25) requires the surface holonomy $H \in ker\partial \subseteq E$ is restricted to take values in the Abelian subgroup $ker\partial \subseteq E$.

Lemma 8.4.1. Given a 2-configuration $F : \mathcal{P}^2(M,L) \to B\mathcal{G}$ and 2-morphism $\gamma \overset{S}{\Rightarrow} \gamma \in \mathcal{P}^2(M,L)_2$ with holonomy $F(S) = (h,H) \in G \ltimes_\triangleright ker\partial$. The 2-holonomy transforms under a 2-gauge transformation $\eta : F \to \tilde{F}$ via:

$$\begin{aligned} \eta &: F(S) \to \tilde{F}(S) \\ \eta &: (h,H) \mapsto (\eta_{\sigma(\gamma)}\partial(\eta_\gamma)h\eta_{\tau(\gamma)}, \eta_{\sigma(\gamma)} \triangleright H) \end{aligned} \tag{8.36}$$

where $\eta_{\sigma(\gamma)}, \eta_{\tau(\gamma)} \in G$ and $\eta_\gamma \in E$.

Proof. Follows from definition of 2-gauge transformation and that $ker\partial \subseteq E$ is an Abelian subgroup of E. $\square$

Remark 8.4.2. In particular this implies that the 2-holonomy $F(\partial_v^2(B)) = (1_G, H) \in G \ltimes_{\triangleright} ker\partial$ of a quantised 2-boundary $\partial_v^2(B)$ transforms as

$$\eta : (1_G, H) \to (1_G, \eta_v \triangleright H). \tag{8.37}$$

Definition 8.4.3. A **2-flat gauge configuration** F_{2-flat}, is a 2-gauge configuration $F_{2-flat} : \mathcal{P}^2(M, L) \to \mathcal{G}$ such that for all $B \in L^3$, $F_{2-flat}(\partial_v^2(B)) = (1_G, 1_E)$.

8.5 Fundamental 2-Groupoid

Akin to the fundamental groupoid, given a lattice 2-groupoid we can also define the **fundamental 2-groupoid**.

Definition 8.5.1. The **fundamental 2-groupoid** $\Pi_2(M, L)$ of a dressed lattice $(M, L, \Rightarrow)$ is the quotient of the path 2-groupoid $\mathcal{P}^2(M, L)$ under the relation

$$\partial_v(B) = \mathbb{1}_{1_v} \tag{8.38}$$

for all $B \in L^3$ with basepoint $b_p(B) = v$.

There exists a strict 2-functor

$$P^2 : \mathcal{P}^2(M, L) \to \Pi_2(M, L) \tag{8.39}$$

which acts as the identity on objects and morphisms, and each surface $f \in \mathcal{P}^2(M, L)$ is sent to its homotopy class in M.

Using the 2-functor P^2 we can define a 2-flat gauge configuration F_{2-flat} akin to a flat gauge configuration:

$$\begin{tikzcd} \Gamma^2(M, L) \arrow[r, "P^2"] \arrow[rr, bend left, "F_{2-flat}"] & \Pi_2(M, L) \arrow[r, "H"] & \Gamma(\mathcal{G}) \end{tikzcd} \tag{8.40}$$

such that the above diagram commutes.

Chapter 9

Hamiltonian Schema for Higher Lattice Gauge Theory

In the following: let M be a topological manifold with lattice decomposition (M, L) and $\mathcal{G} = (G, E, \partial, \triangleright)$ a finite crossed module of groups. From this data we will construct an exactly solvable topological Hamiltonian schema $(\mathcal{H}(M, L; \mathcal{G}), \{e_i\}, H(M, L; \mathcal{G}))$ whereby the Hamiltonian $H(M, L; \mathcal{G})$ is a sum of local, mutually commuting projection operators. We will then demonstrate that the groundstates are given by the state space of the Yetter homotopy 2-type TQFT [47].

9.1 Hilbert Space

Given a topological manifold M with dressed lattice decomposition $(M, L, \Rightarrow)$ and finite crossed module $\mathcal{G}$, let

$$\Theta := \{F : \mathcal{P}^2(M, L) \to B\mathcal{G}\} \tag{9.1}$$

be the set of all 2-gauge configurations F. We define the Hilbert space $\mathcal{H}(M, L)$ to be

$$\mathcal{H}(M, L; \mathcal{G}) := \mathbb{C}\Theta, \tag{9.2}$$

the complex vector space spanned by 2-gauge configurations. We notate the basis elements by:

$$|F\rangle \in \mathcal{H}(M, L; \mathcal{G}), \quad \forall F \in \Theta \tag{9.3}$$

Additionally we equip $\mathcal{H}(M,L)$ with an orthonormal inner product such that

$$\langle F|F'\rangle = \delta_{F,F'} \quad \forall F, F' \in \Theta. \tag{9.4}$$

In this way we identify the classical states with 2-gauge configurations.

9.2 Gauge Operators

In the following we will define a set of operators acting on $\mathcal{H}(M,L;\mathcal{G})$ we call **gauge spikes** which are induced from the 2-gauge transformations of higher lattice gauge theory.

In definition 8.3.1 we defined 2-gauge transformations as pseudo-natural transformations between 2-gauge configurations. The data of a 2-gauge transformation can be specified using the directed graph $(M,L,\rightarrow)$ of the lattice (M,L) and the crossed module $\mathcal{G}=(G,E,\partial,\triangleright)$. In this way a 2-gauge transformation can be indexed by an element of the set:

$$\mathcal{T}(M,L;\mathcal{G}) = G^{|L^0|} \times E^{|L^1|}. \tag{9.5}$$

To specify elements we assign an enumeration to each vertex $v \in L^0$ and edge $e \in L^1$ in $(M,L,\rightarrow)$ such that

$$(g_{v_1},\cdots,g_{v_{|L^0|}};H_{e_1},\cdots,H_{e_{|L^1|}}) \in \mathcal{T}(M,L;\mathcal{G}). \tag{9.6}$$

From the definition of a 2-gauge transformation it is straightforward to endow $\mathcal{T}(M,L;\mathcal{G})$ with a group structure, where the product is given by

$$\begin{aligned}&(g_{v_1},\cdots,g_{v_{|L^0|}};H_{e_1},\cdots,H_{e_{|L^1|}})(g'_{v_1},\cdots,g'_{v_{|L^0|}};H'_{e_1},\cdots,H'_{e_{|L^1|}}\\ &= (g_{v_1}g'_{v_1},\cdots,g_{v_{|L^0|}}g'_{v_{|L^0|}};(g_{s(e_1)}\triangleright H'_{e_1})H_{e_1},\cdots,(g_{s(e_{|L^1|})}\triangleright H'_{e_{|L^1|}})H_{e_{|L^1|}}).\end{aligned} \tag{9.7}$$

The identity is given by

$$1_{\mathcal{T}(M,L;\mathcal{G})} = (1_G,\cdots,1_G;1_E\cdots,1_E). \tag{9.8}$$

with inverse

$$(g_{v_1},\cdots,g_{v_{|L^0|}};H_{e_1},\cdots,H_{e_{|L^1|}})^{-1} = (g_{v_1}^{-1},\cdots,g_{v_{|L^0|}}^{-1};g_{s(e_1)}^{-1}\triangleright H_{e_1}^{-1},\cdots,g_{s(e_{|L^1|})}^{-1}\triangleright H_{e_{|L^1|}}^{-1}). \tag{9.9}$$

Utilising the group $\mathcal{T}(M, L; \mathcal{G})$ of 2-gauge transformations, we now define **vertex gauge spikes** and **edge gauge spikes**.

Definition 9.2.1. A **vertex gauge spike** $A_v^g \in \mathcal{T}(M, L; \mathcal{G})$, for vertex $v \in L^0$ and $g \in G$ is a 2-gauge transformation $A_v^g := (g_{v_1}, \cdots, g_{v_{|L^0|}}; H_{e_1}, \cdots, H_{e_{|L^1|}})$ where $g_v = g$, $g_{v'} = 1_G$ for all $v' \in L^0$ where $v' \neq v$ and $H_e = 1_E$ for all $e \in L^1$.

Definition 9.2.2. An **edge gauge spike** $A_e^H \in \mathcal{T}(M, L; \mathcal{G})$, for edge $e \in L^1$ and $H \in E$ is a 2-gauge transformation $A_e^H := (g_{v_1}, \cdots, g_{v_{|L^0|}}; H_{e_1}, \cdots, H_{e_{|L^1|}})$ where $H_e = H$, $H_{e'} = 1_E$ for all $e' \in L^1$ where $e' \neq e$ and $g_v = 1_G$ for all $v \in L^0$.

Utilising the group multiplication structure of $\mathcal{T}(M, L; \mathcal{G})$ it is straightforward to prove the following two lemmas:

Lemma 9.2.1. All gauge transformations $\eta \in \mathcal{T}(M, L; \mathcal{G})$ can be constructed as a product of vertex and edge gauge spikes.

Lemma 9.2.2. The vertex and edge gauge spikes satisfy the following relations:

$$\begin{aligned}
A_v^g A_v^{g'} &= A_v^{gg'} & &\forall v \in L^0, \forall g, g' \in G \\
A_e^H A_e^{H'} &= A_e^{HH'} & &\forall e \in L^1, \forall H, H' \in E \\
A_v^g A_{v'}^{g'} &= A_{v'}^{g'} A_{v'}^g & &\forall v \neq v' \in L^0, \forall g, g' \in G \\
A_e^H A_{e'}^{H'} &= A_{e'}^{H'} A_e^H & &\forall e \neq e' \in L^1, \forall H, H' \in E \\
A_v^g A_e^H &= A_e^H A_v^g & &\forall e \in L^1, \forall v \neq s(e) \in L^0, \forall g \in G, \forall H \in E \\
A_{s(e)}^g A_e^{g^{-1} \triangleright H} &= A_e^H A_{s(e)}^g & &\forall e \in L^1, \forall g \in G, \forall H \in E
\end{aligned} \tag{9.10}$$

Given the above construction we can define an action of the vertex and edge gauge spikes on $\mathcal{H}(M, L; \mathcal{G})$ as follows: Given a pair of 2-gauge configurations $F, F' \in \Theta(M, L; \mathcal{G})$, if $\eta \in \mathcal{T}(M, L; \mathcal{G})$ defines a 2-gauge transformation $\eta : F \to F'$ then we notate $F' := \eta \cdot F$. From the group structure of $\mathcal{T}(M, L; \mathcal{G})$ it follows that if $F' = \eta \cdot F$ then $F = \eta^{-1} \cdot F'$. Using this notation we define a **2-gauge operator** as follows:

Definition 9.2.3. Given a 2-gauge transformation $\eta \in \mathcal{T}(M, L; \mathcal{G})$, the **2-gauge operator** $\hat{\eta}$ is a linear map

$$\hat{\eta} : \mathcal{H}(M, L; \mathcal{G}) \to \mathcal{H}(M, L; \mathcal{G}) \tag{9.11}$$

such that

$$\hat{\eta} := \sum_{F \in \Theta(M,L;\mathcal{G})} |\eta \cdot F\rangle \langle F| \,. \tag{9.12}$$

Definition 9.2.4. Given a vertex gauge spike $A_v^g \in \mathcal{T}(M,L;\mathcal{G})$, the **vertex gauge spike operator** is the gauge operator $\hat{A}_v^g$.

Definition 9.2.5. Given an edge gauge spike $A_e^H \in \mathcal{T}(M,L;\mathcal{G})$, the **edge gauge spike operator** is the gauge operator $\hat{A}_e^H$.

From these definitions we now introduce the vertex and edge gauge projectors, by symmetrising over the vertex and edge gauge operators as follows:

Definition 9.2.6. The **vertex gauge projector** $\hat{A}_v := \frac{1}{|G|}\sum_{g \in G} \hat{A}_v^g$

Definition 9.2.7. The **edge gauge projector** $\hat{A}_e := \frac{1}{|E|}\sum_{H \in E} \hat{A}_e^H$

Lemma 9.2.3. The vertex and edge gauge projectors satisfy the following relations:

$$\begin{aligned}
\hat{A}_v\hat{A}_v &= \hat{A}_v && \forall v \in L^0 \\
\hat{A}_e\hat{A}_e &= \hat{A}_e && \forall e \in L^1 \\
\hat{A}_v\hat{A}_{v'} &= \hat{A}_{v'}\hat{A}_v && \forall v, v' \in L^0 \\
\hat{A}_e\hat{A}_{e'} &= \hat{A}_{e'}\hat{A}_e && \forall e, e' \in L^1 \\
\hat{A}_v\hat{A}_e &= \hat{A}_e\hat{A}_v && \forall v \in L^0, \forall e \in L^1
\end{aligned} \tag{9.13}$$

Proof. Follows from definition and lemma 9.2.2 □

9.3 2-Holonomy Operator

We now introduce the 2-holonomy operator. This operator is a self adjoint projection operator. For $K \in ker(\partial) \subseteq E$ and blob $b \in L^3$, with basepoint $v \in L^0$ we define:

$$B_b^K : \mathcal{H}(M,L;\mathcal{G}) \to \mathcal{H}(M,L;\mathcal{G}) \tag{9.14}$$

such that

$$B_b^K |F\rangle = |F\rangle\, \delta_{H^2(F(\partial_v^2(B))),K} \tag{9.15}$$

where $H^2(\partial_v^2(b))$ is the 2-holonomy of the quantised 2-boundary of b (definition 8.1.3).

Lemma 9.3.1. The 2-holonomy projectors satisfy the following relations:

$$B_b^K B_b^{K'} = B_b^K \delta_{K,K'}, \quad \forall K, K' \in ker\partial, \forall b \in L^3 \tag{9.16}$$
$$B_b^K B_{b'}^{K'} = B_b^K B_{b'}^{K'}, \quad \forall K, K' \in ker\partial, \forall b,' \in L^3 \tag{9.17}$$

Definition 9.3.1. 2-holonomy 2-flatness projector

$$B_b := B_b^{1_E} \tag{9.18}$$

In particular the 2-holonomy, 2-flatness projector doesn't depend on the basepoint of b as any redefinition of basepoint will give the same result.

It follows from the previous that the 2-holonomy projectors satisfy the following relations with the vertex and edge gauge operators:

Lemma 9.3.2. mixed relations:

$$\hat{A}^g_{b_p(b)} B_b^K = B_b^{g \triangleright K} \hat{A}^g_v, \quad \forall g \in G, \forall K \in ker\partial, \forall e \in L^1 \tag{9.19}$$
$$\hat{A}^g_v B_b^K = B_b^{g \triangleright K} \hat{A}^g_v, \quad \forall g \in G, \forall K \in ker\partial, \forall v \in L^0, \forall e \in L^1 \tag{9.20}$$
$$\hat{A}^H_e B_b^K = B_b^K \hat{A}^H_e, \quad \forall H \in E, \forall K \in ker\partial, \forall e \in L^2, \forall b \in L^3 \tag{9.21}$$

where $b_p(b)$ is the basepoint of $b \in L^3$.

9.4 Hamiltonian

Now we have defined the Hilbert space and local operators, we define the topological higher lattice gauge theory Hamiltonian.

Definition 9.4.1. Topological Higher Lattice Gauge Theory Hamiltonian

$$H(M, L; \mathcal{G}) := -\sum_{v \in int(L^0(M))} \hat{A}_v - \sum_{e \in int(L^1(M))} \hat{A}_e - \sum_{b \in int(L^3(M))} B_b \tag{9.22}$$

This Hamiltonian is exactly solvable, as all operators are local, in the sense of having non-trivial action only on disks with the topology on an n-ball and all operators are mutually commuting projectors following from lemmas 9.2.3, 9.3.1 and 9.3.2. From these relations the groundstate projector is given by:

Definition 9.4.2. Groundstate projector

$$P(M,L;\mathcal{G}) := \prod_{v\in int(L^0(M))} \hat{A}_v \prod_{e\in int(L^1(M))} \hat{A}_e \prod_{b\in int(L^3(M))} B_b \tag{9.23}$$

and the groundstate subspace is given by

$$\mathcal{H}(M,L;\mathcal{G})_0 := \{|F\rangle \in \mathcal{H}(M,L;\mathcal{G}) | P(M,L;\mathcal{G})\,|F\rangle = |F\rangle\}. \tag{9.24}$$

In the following it will be useful to expand the expression of the groundstate projector in terms of 2-gauge transformation operators such that:

$$P(M,L;\mathcal{G}) = \frac{1}{|G|^{|int(L^0(M))|}|E|^{|int(L^1(M))|}} \sum_{\eta\in\tilde{\mathcal{T}}(M,L;\mathcal{G})} \hat{\eta} \prod_{b\in L^3} B_b. \tag{9.25}$$

where $\tilde{\mathcal{T}} \subseteq \mathcal{T}(M, L : \mathcal{G})$ is the subgroup of 2-gauge transformations where each

$$(g_{v_1}, \cdots, g_{v_{|L^0(M)|}}; H_{e_1}, \cdots, H_{e_{|L^1(M)|}}) \in \mathcal{T}(M,L;\mathcal{G}) \tag{9.26}$$

is an element of $\tilde{\mathcal{T}}(M,L;\mathcal{G})$ if and only if $g_{v_i} = 1_G$ if $v_i \notin int(L^0(M))$ and $H_{e_j} = 1_E$ if $e_j \notin int(L^1(M))$. For a closed manifold M, $\mathcal{T}(M,L;\mathcal{G}) = \tilde{\mathcal{T}}(M,L;\mathcal{G})$.

9.5 Relation to Yetter TQFT

We now demonstrate that the groundstate subspace of the topological higher lattice gauge theory Hamiltonian schema correspond to the state space defined by the Yetter homotopy 2-type TQFT. In the following we will restrict to the case of closed spatial manifolds. In chapter 13 we will discuss a class of spatial manifolds with boundary.

We begin by first defining the Yetter TQFT for lattices.

Definition 9.5.1. Let Y be an $n+1$ dimensional cobordism with boundary $\partial Y = \overline{M_1} \sqcup M_2$, with $\partial(M_1) = \partial(M_2) = \emptyset$, $(Y, L, \Rrightarrow)$ a dressed lattice decomposition of Y and Θ_{2-flat} the set of all 2-flat configurations $F_{2-flat} : \Pi_2(M, L) \to B\mathcal{G}$. The Yetter TQFT is then the state-sum TQFT $\mathcal{Z}^{\mathcal{G}}_{Yetter}$:

$$\begin{aligned}&\mathcal{Z}^{\mathcal{G}}_{Yetter}(Y, L)\\ &:= \frac{|E|^{|L^0(Y)|-\frac{1}{2}|L^0(\partial(Y))|-|L^1(Y)|+\frac{1}{2}|L^1(\partial(Y))|}}{|G|^{|L^0(Y)|-\frac{1}{2}|L^0(\partial(Y))|}} \sum_{F\in\Theta_{2-flat}} |F(M_2, L_2)\rangle \langle F(M_1, L_1)|\end{aligned} \tag{9.27}$$

where $F(M_i, L_i)$ is the restriction of the functor $F : \Pi_2(Y, L) \to B\mathcal{G}$ to $F(M_i, L_i) : \Pi_2(M_i, L_i) \to B\mathcal{G}$.

In particular the Yetter TQFT is invariant under PL-homeomorphic transformations of the 2-lattice of the interior of (Y, L).

Let $(M, L, \Rrightarrow)$ be a dressed lattice for a closed n-manifold M, the statespace for $(M, L, \Rrightarrow)$ is given by the vector space $V^{\mathcal{G}}_{Yetter}(M, L) = Im(\mathcal{Z}^{\mathcal{G}}_{Yetter}(M \times I, L \times I))$, where:

Definition 9.5.2. Given a lattice (M, L) of a closed topological manifold M, the **cylinder lattice** $(M \times I, L \times I)$ is given as follows: Let $([0, 1], L_I)$ be the lattice decomposition of the interval $[0, 1]$ with two 0-cells and a single 1-cell. Then $(M \times I, L \times I)$ is the lattice given by the product CW-complex $(M, L) \times ([0, 1], L_I)$.

We now demonstrate a useful relation, relating 2-flat gauge configurations of a cylinder lattice and 2-gauge transformations:

Lemma 9.5.1. Let M be a closed topological manifold, with dressed lattice $(M, L, \Rrightarrow)$. Letting $F_{2-flat,0} : \Gamma^2(M, L) \to \mathcal{G}$ define a 2-flat gauge configuration and $\eta : F_{2-flat,0} \to F_{2-flat,1}$ a 2-gauge transformation. There is a one to one correspondence between pairs $(F_{2-flat,0}, \eta)$ and 2-flat gauge configurations of the cylinder lattice $(M \times I, L \times I)$.

Proof. Let $F_{2-flat,0}, F_{2-flat,1} : \Pi_2(M, L) \to B\mathcal{G}$ be a pair of 2-flat gauge configurations such that there exists a 2-gauge transformation $\eta : F_{2-flat,0} \to F_{2-flat,1}$, defined by $\eta = (g_{v_1}, \cdots, g_{v_{|L^0(M)|}}; H_{e_1}, \cdots, H_{e_{|L^2(M)|}}) \in \mathcal{T}(M, L; \mathcal{G})$. Further, let $(M \times I, L \times I)$ be the cylinder lattice of (M, L) and $F_{2-flat} : \Gamma^2(M \times I, L \times I) \to \mathcal{G}$

a 2-flat gauge configuration such that on the subcomplex $(M \times 0, L \times 0)$, F_{2-flat} restricts to $F_{2-flat,0} : \Gamma^2(M \times 0, L \times 0) \to \mathcal{G}$. The 2-flat gauge configuration F_{2-flat} is then defined using η by functorially assigning $g_{v_i} \in G$ to each edge $v_i \times I \in (M \times I, L \times I)$ and $H_{e_j} \in E$ to each plaquette $e_j \times I \in (M \times I, L \times I)$:

$$F_{2-flat} : \left(v_i \times 0 \xrightarrow{v_i \times [0,1]} v_i \times 1\right) \mapsto \left(* \xrightarrow{g_{v_i}} *\right)$$

$$F_{2-flat} : \begin{tikzcd} s(e_j) \times 0 \arrow[r, "e_j \times 0"] \arrow[d, "s(e_j)\times I"'] & t(e_j) \times 0 \arrow[d, "t(e_j)\times I"] \arrow[dl, Rightarrow, "e_j \times I"] \\ s(e_j) \times 1 \arrow[r, "e_j \times 1"'] & t(e_j) \times 1 \end{tikzcd} \mapsto \begin{tikzcd} * \arrow[r, "F_{2-flat,0}(e_j \times 0)"] \arrow[d, "g_{s(e_j)}"'] & * \arrow[d, "g_{t(e_j)}"] \arrow[dl, Rightarrow, "H_{e_j}"] \\ * \arrow[r, "F_{2-flat,1}(e_j \times 1)"'] & * \end{tikzcd} \tag{9.28}$$

The requirement of 2-flatness is then imposed by requiring the boundary of each blob $p \times [0,1] \in (M \times I, L \times I)$ defined from the plaquette $p \in (M, L)$ form a 2-commutative diagram. This requirement uniquely specifies the 2-gauge configuration of $(M \times 1, L \times 1)$ be given by $F_{2-flat,1} : \Pi_2(M \times 1, L \times 1) \to B\mathcal{G}$. Such 2-gauge configurations are formally equivalent to the definition of a pseudo-natural transformation, $\eta : F_{2-flat,0} \to F_{2-flat,1}$. This follows as requiring 2-flatness is equivalent to the commutativity of the tin-can axiom (8.33) defining a pseudo-natural transformation such that both are in one to one correspondence. □

Theorem 9.5.2. The groundstate projector $P(M, L; \mathcal{G}) = \mathcal{Z}^{\mathcal{G}}_{Yetter}(M \times I; L \times I)$, for a closed n-manifold with dressed lattice $(M, L, \Rightarrow)$.

Proof. We first consider the normalisation factors in equation 9.5.1 for the cylinder lattice $(M \times I, L \times I)$, which are given by:

$$|G|^{-|L^0(M)|}|E|^{-|L^1(M)|}. \tag{9.29}$$

This follows as $(M \times I, L \times I)$ has no internal vertices, and all vertices on the boundary are given by the two copies of the vertices L^0 from (M, L). Further, there are $2|L^0|$ edges on the boundary as induced from the two copies of (M, L) and there are $|L^0|$ internal edges occurring from the product of the vertices with the interval. Applying these rules we find the previous normalisation constant.

Now utilising lemma 9.5.1 we can rewrite the state-sum as follows:

$$
\begin{aligned}
&\mathcal{Z}^{\mathcal{G}}_{Yetter}(M \times I; L \times I) \\
&= \frac{1}{|G|^{|L^0(M)|}|E|^{|L^1(M)|}} \sum_{F_{2-flat}:\Pi_2(M\times I, L\times I)\to B\mathcal{G}} |F_{2-flat,1}(M,L)\rangle \langle F_{2-flat,0}(M,L)| \\
&= \frac{1}{|G|^{|L^0(M)|}|E|^{|L^1(M)|}} \sum_{\eta\in\mathcal{T}(M,L;\mathcal{G})} \sum_{H_{2-flat}:\Pi_2(M,L)\to B\mathcal{G}} |\eta \cdot H_{2-flat}(M,L)\rangle \langle H_{2-flat}(M,L)| \\
&= \frac{1}{|G|^{|L^0(M)|}|E|^{|L^1(M)|}} \sum_{\eta\in\mathcal{T}(M,L;\mathcal{G})} \sum_{H_{2-flat}:\Gamma^2(M,L)\to B\mathcal{G}} (\prod_{b\in L^3} B_b) |\eta \cdot H(M,L)\rangle \langle H(M,L)| \\
&= \frac{1}{|G|^{|L^0(M)|}|E|^{|L^1(M)|}} (\prod_{b\in L^3} B_b) \sum_{\eta} \hat{\eta} \\
&= P(M, L; \mathcal{G}).
\end{aligned}
\tag{9.30}
$$

Between the first and second lines we apply the definition of $\mathcal{Z}^{\mathcal{G}}_{Yetter}(M\times I, L\times I)$ where for $i \in \{0,1\}$, $F_{2-flat,i}(M,L)$ is the restriction of F_{2-flat} to $(M \times i, L \times i)$. Between the second and third line we directly apply lemma 9.5.1. Between lines three and four we use the relation:

$$
(\prod_{b\in L^3} B_b) |H(M,L)\rangle = \begin{cases} |H(M,L)\rangle, & \text{if } H : \Gamma^2(M,L) \to B\mathcal{G} \text{ is } 2-flat \\ 0, & \text{else} \end{cases}
\tag{9.31}
$$

which follows from the definition 9.3.1 of B_p and the definition of a 2-flat 2-gauge configuration combined with the fact that 2-gauge transformations preserve 2-flatness. Between lines four and five we apply definition 9.12 for $\hat{\eta}$. □

This relation of the groundstate subspace to the state space of the Yetter TQFT demonstrates the existence of a colimit of the groundstate and hence the Hamiltonian schema for topological higher lattice gauge theory defines an exactly solvable topological Hamiltonian schema.

9.6 Groundstate subspace

In the following we construct the groundstate subspace for the topological higher lattice gauge theory Hamiltonian schema for a closed topological n-manifold. We will discuss the groundstate subspace for a class of manifolds with boundary and the topological excitations in chapter 13. In the following we will use the language of groupoids, in particular the notions of connect component (definition 3.2.3), stabiliser subgroup (definition 3.2.4) and proposition 3.2.1.

We begin by making some observations about the groundstate subspace using the form of the groundstate projector $P(M, L; \mathcal{G})$. First of all, 2-gauge configurations which are not 2-flat are in the kernel of the groundstate projector. In this way a 2-gauge configuration is in the groundstate subspace if and only if it is 2-flat. Secondly given an arbitrary element of $\mathcal{H}(M, L; \mathcal{G})$

$$|\psi\rangle = \sum_{F \in \Theta(M,L;\mathcal{G})} \lambda_F |F\rangle \tag{9.32}$$

the gauge vertex and edge operators imply, if there exists $\eta \in \mathcal{T}(M, L; \mathcal{G})$ such that $F' = \eta \cdot F$ then $\lambda_F = \lambda_{F'}$ if the state is in the groundstate subspace.

Let $[\Pi_2(M, L), B\mathcal{G}]$ be the strict 2-functor 2-groupoid with objects 2-flat gauge configurations of $\mathcal{P}^2(M, L)$, morphisms 2-gauge transformations and 2-morphisms pseudo-modification equivalences (see corollary 7.3.0.1). Furthermore let $\Gamma_{THLGT}(M, L; \mathcal{G})$ be the underlying groupoid of $[\Pi_2(M, L), B\mathcal{G}]$ given by forgetting the 2-morphisms. In the following we will show the data of $\Gamma_{THLGT}(M, L; \mathcal{G})$ can be used to define the groundstate subspace $\mathcal{H}[M, L; \mathcal{G}]_0$.

The first observation is that the 2-flat subspace of $\mathcal{H}(M, L; \mathcal{G})$ is given by the Hilbert space

$$\mathcal{H}[M, L; \mathcal{G}]_{2-flat} := \mathbb{C}\Gamma_{THLGT}(M, L; \mathcal{G})_0 \subseteq \mathcal{H}[M, L; \mathcal{G}]. \tag{9.33}$$

and

$$\mathcal{H}[M, L; \mathcal{G}]_0 \subseteq \mathcal{H}[M, L; \mathcal{G}]_{2-flat}. \tag{9.34}$$

Let $\pi_0(\Gamma_{THLGT}(M, L; \mathcal{G})) = \{C_i\}_{i \in \{1, \cdots, |\pi_0(\Gamma_{THLGT}(M,L;\mathcal{G}))|\}}$ be the set of connected components of $\Gamma_{THLGT}(M, L; \mathcal{G})$. Given a connected component C_i the stabiliser subgroup of any pair of objects $x, y \in C_i$ are isomorphic $\pi_1(x) \cong \pi_1(y)$ (proposition 3.2.1) such that we can define $|\pi_1(C)|$ from any representative element. For any connected component C_i and representative element $F_i \in C_i$ we can define the normalised vector:

$$|C_i\rangle := \frac{1}{\sqrt{|G|^{|L^0|}|E|^{|L^1|}|\pi_1(C_i)|}} \sum_{\eta \in \mathcal{T}(M,L;\mathcal{G})} |\eta \cdot F_i\rangle \tag{9.35}$$

In particular, such vectors are independent of the choice of representative element by redefinition of the 2-gauge transformation η. It is straightforward to verify:

$$P(M,L;\mathcal{G})\,|C_i\rangle = |C_i\rangle \tag{9.36}$$

and

$$\langle C_j|C_i\rangle = \delta_{i,j} \tag{9.37}$$

for all $i,j \in \{1,\cdots,|\pi_0(\Gamma_{THLGT}(M,L;\mathcal{G}))|\}$. The second equality follows as by definition a gauge transformation $\eta \in \mathcal{T}(M,L;\mathcal{G})$ cannot change the connected component of a 2-flat 2-gauge configuration. In this way we see that

$$\mathcal{H}(M,L;\mathcal{G})_0 = \mathbb{C}\{|C_i\rangle\}_{i\in\{1,\cdots,|\pi_0(\Gamma_{THLGT}(M,L;\mathcal{G}))|\}}. \tag{9.38}$$

We can verify this basis is complete for the groundstate subspace using the groundstate projector:

$$\begin{aligned}
dim\mathcal{H}(M,L;\mathcal{G})_0 &= TrP(M,L;\mathcal{G}) \\
&= \frac{1}{|G|^{|L^0|}|E|^{|L^1|}} \sum_{\eta\in\mathcal{T}(M,L;\mathcal{G})} \sum_{F\in\Gamma_{THLGT}(M,L;\mathcal{G})_0} \delta_{\eta\cdot F,F} \\
&= \sum_{F\in\Gamma_{THLGT}(M,L;\mathcal{G})_0} \frac{|\pi_1(F)|}{|G|^{|L^0|}|E|^{|L^1|}} \\
&= \sum_{C_i\in\pi_0(\Gamma_{THLGT}(M,L;\mathcal{G})_0)} \frac{|\pi_1(C_i)||C_i|}{|G|^{|L^0|}|E|^{|L^1|}} \\
&= |\pi_0(\Gamma_{THLGT}(M,L;\mathcal{G})_0)|.
\end{aligned} \tag{9.39}$$

where we used the relation $|\pi_1(C_i)||C_i| = |G|^{|L^0|}|E|^{|L^1|}$ for all C_i which follows from the orbit stabiliser theorem for a group with action on a set.

It is known that there is a natural bijection between elements of $\pi_0(M,L;\mathcal{G})$ and homotopy classes of maps $M \to B_{\mathcal{G}}$, where $B_{\mathcal{G}}$ is the classifying space of the crossed module $\mathcal{G}$ as explained in [76, 77]. In this way we can alternatively view the groundstate subspace as the complex vector space spanned by homotopy classes of such maps. In particular this result also demonstrates the independence of the groundstate degeneracy from the choice of lattice without refering to the Yetter homotopy 2-type TQFT.

9.7 Relation to Walker Wang Models

In this section we discuss the relation between the Walker-Wang model [63] and the Hamiltonian schema for topological higher lattice gauge theory. In particular we outline a duality map between our model with the finite crossed module $\mathcal{E} = (\mathbf{1}_E, E, \partial, \triangleright)$, where $\partial : E \to 1_E$ and $\triangleright$ is the identity and the Walker-Wang model based on the symmetric fusion category $Rep(E)$, where E is any finite Abelian group.

9.7.1 Walker-Wang Model

To begin, we briefly outline the Walker-Wang model. The Walker-Wang model is a 3+1D model of string-net condensation with groundstates proposed to describe time-reversal invariant topological phases of matter in the bulk and chiral anyon theories on the boundary [78]. Such models correspond to a topological Hamiltonian schema with scaling limit corresponding to the colimit of the Crane-Yetter-Kauffman state-sum TQFT [64].

The Walker-Wang model is specified by two pieces of input data, a unitary braided fusion category (UBFC) $\mathcal{C}$ and a cubulation C of a 3-manifold M^3. In the following we will define the generic model on a trivalent graph defined from the 1-skeleton C^1 of C where vertices are canonically resolved to trivalent vertices see fig 9.1. We will then restrict the input to a symmetric braided fusion category $rep(E)$ and remove the vertex resolution condition. We will make the assumption that the cubulation of the manifold is simple: namely all faces have 4-edges and each vertex is 6-valent[1].

The Walker-Wang model is defined on the trivalent cubic graph given by the resolved 1-skeleton C^1 of C (see fig 9.2) with directed edges. The Hilbert space has an orthonormal basis given by all colourings of the directed edges of C^1 by labels from $\mathcal{L} = \{1, a, b, c, \cdots\}$, with orthonormal inner product on colourings. For each edge label $a \in \mathcal{L}$ there is a conjugate label $a^* \in \mathcal{L}$ which satisfies the

[1]Every 3-manifold has a presentation in terms of a cubulation, in other words in terms of a partition into 3-dimensional cubes, which only intersect along a common face. However in some cases the vertices of a cubulation may not be six-valent. For some manifolds these features are not avoidable; see [79].

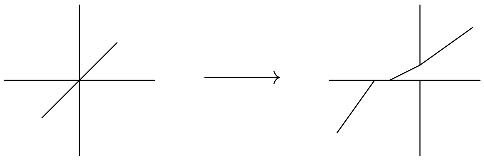


Figure 9.1: Resolution of 6-valent vertex to a trivalent vertex.

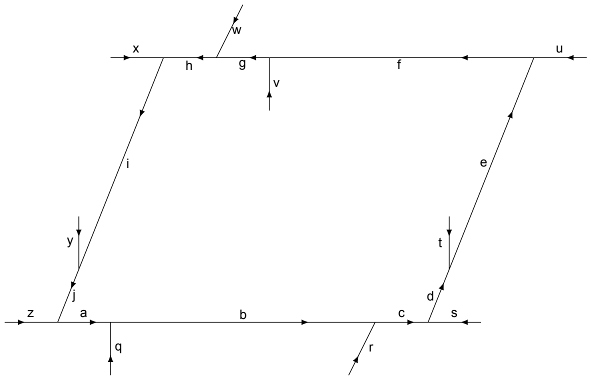


Figure 9.2: Trivalent plaquette with oriented edges for Walker-Wang model.

relation $a^{**} = a$. We define the states such that reversing the direction of an edge and conjugating the edge label gives the same state of the Hilbert space as the original configuration. The label set $\mathcal{L}$ has a unique element $1 \in \mathcal{L}$ we call the vacuum which satisfies the relation $1 = 1^*$.

To specify the Hamiltonian we introduce the fusion algebra of the label set [80, 81, 21, 22]. A fusion rule is an associative, commutative product of labels such that for $a, b, c \in \mathcal{L}$, $a \otimes b = \sum_c N_{ab}^c c$. Here $N_{ab}^c \in \mathbb{Z}^+$ is a non-negative integer called the fusion multiplicity. In the following we will restrict to the case of "multiplicity free" which is the restriction $N_{ab}^c \in \{0, 1\}\ \forall a, b, c \in \mathcal{L}$. The fusion multiplicities satisfy the following relations

$$N_{ab}^c = N_{ba}^c \tag{9.40}$$

$$N_{ab}^1 = \delta_{ab^*} \tag{9.41}$$

$$N_{a1}^b = \delta_{ab} \tag{9.42}$$

$$\sum_{x \in \mathcal{L}} N_{ab}^x N_{xc}^d = \sum_{x \in \mathcal{L}} N_{ax}^d N_{cd}^x. \tag{9.43}$$

Given the label set and fusion algebra we define $d : \mathcal{L} \to \mathbb{C}$ such that $\forall a \in \mathcal{L}$,

$d : a \mapsto d_a$ and $d_{a^*} = d_a$. We will refer to d_a as the quantum dimension of the label a and $D = \sqrt{\sum_{a\in\mathcal{L}} d_a^2}$ as the total quantum dimension. The quantum dimensions are required to satisfy

$$d_a d_b = \sum_c N_{ab}^c d_c. \tag{9.44}$$

Additionally we define $\alpha_i = sgn(d_i) \in \{\pm 1\}$ which satisfies

$$\alpha_i \alpha_j \alpha_k = 1 \tag{9.45}$$

if $N_{ij}^{k^*} = 1$.

Given the fusion algebra and quantum dimensions we define the $6j$-symbols which enforce the associativity of fusion of processes. The $6j$-symbols are a map $F : \mathcal{L}^6 \to \mathbf{C}$ which satisfy the following relations

$$F^{ijm}_{j^*i^*1} = \frac{v_m}{v_i v_j} N^{m^*}_{ij} \tag{9.46}$$

$$F^{ijm}_{kln} = F^{klm^*}_{jin^*} = F^{jim}_{lkn^*} = F^{mij}_{nk^*l^*}\frac{v_m v_n}{v_j v_l} = \overline{F^{j^*i^*m^*}_{l^*k^*n}} \tag{9.47}$$

$$\sum_n F^{mlq}_{kp^*n} F^{jip}_{mns^*} F^{js^*n}_{lkr^*} = F^{jip}_{q^*kr} F^{riq^*}_{mls^*} \tag{9.48}$$

$$\sum_n F^{mlq}_{kp^*n} F^{l^*m^*i^*}_{pk^*n} = \delta_{iq}\delta_{mlq}\delta_{k^*ip} \tag{9.49}$$

where $v_a = \sqrt{d_a}$.

The final piece of data required to define the Walker-Wang model is the braiding relations or R-matrices. The R-matrices are a map $R : \mathcal{L}^3 \to \mathbf{C}$ which are required to satisfy the Hexagon equations which ensure the compatibility of braiding and fusion. The Hexagon equations are as follows

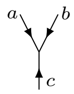
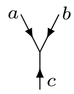
$$\begin{aligned} \sum_g F^{cad^*}_{be^*g} R^e_{gc} F^{abg^*}_{ce^*f} &= R^d_{ac} F^{acd^*}_{be^*f} R^f_{bc} \\ \sum_g F^{e^*bd}_{cag} R^e_{ad} F^{e^*ag}_{bcf} &= R^d_{ac} F^{e^*bd}_{acf} R^f_{ab}. \end{aligned} \tag{9.50}$$

The data $(\mathcal{L}, N, d, F, R)$ forms a UBFC. Examples of solutions to the above data are representations of a finite group or a quantum group (see for example, [80] for a list of examples).

Using the above data we can write down the Walker-Wang Hamiltonian. The Hamiltonian is of the following form

$$H = -\sum_{v\in C^0} A_v - \sum_{p\in C^2} B_p \tag{9.51}$$

where C^0 is the vertex set of C and C^2 is the set of 2-cells, we call plaquettes. The plaquettes are defined with reference to the original square faces of C before the vertex resolution. The term A_v is the vertex operator and acts on the 3-edges adjacent to a vertex. We define the action of A_v on states as follows

$A_v \Big|$  $\Big\rangle = \delta(abc) \Big|$  $\Big\rangle$ (9.52)

where $\delta(abc) = 1$ if $N_{ab}^{c*} \geq 1$ and $\delta(abc) = 0$ else.

The plaquette operator B_p takes a more complicated form in terms of the $6j$-symbols and R-matrices. Using Fig 9.2 as the basis, B_p has the following form

$$\begin{aligned} B_p^n = &\sum_{a',b',c',d',e',f',g',h',i',j'} R^{d}_{t^*e}\overline{R^{d'}_{t^*e'}}R^{f'}_{v^*g'}\overline{R^{f}_{v^*g}} \\ &F^{qb^*a}_{n^*a'b'^*}F^{rc^*b}_{n^*b'c'^*}F^{sd^*c}_{n^*c'd'^*}F^{te^*d}_{n^*d'e'^*}F^{uf^*e}_{n^*e'f'^*} \\ &F^{vg^*f}_{n^*f'g'^*}F^{wh^*g}_{n^*g'h'^*}F^{xi^*h}_{n^*h'i'^*}F^{y^*j^*i}_{n^*i'j'^*}F^{z^*a^*j}_{n^*j'a'^*} \\ &\times \left|a',b',c',d',e',f',g',h',i',j'\right\rangle \left\langle a,b,c,d,e,f,g,h,i,j\right| \end{aligned} \tag{9.53}$$

$$B_p = \sum_{n\in\mathcal{L}} \frac{d_n}{D^2} B_p^n. \tag{9.54}$$

We define the inner product of such states by

$$\langle a,b,c,\cdots|a',b',c',\cdots\rangle = \delta_{aa'}\delta_{bb'}\delta_{cc'}\cdots. \tag{9.55}$$

9.7.2 The Symmetric Braided Fusion Category $Rep(E)$

In the following we will be interested in the UBFC $Rep(E)$, where $(E,+)$ is a finite Abelian group, given as follows:

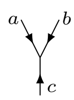
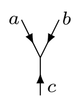
The label set of $Rep(E)$ is given by elements of E, with the vacuum label 1 given by the identity element $0 \in E$ and $a^* = [-a]_N$. The quantum dimension

$d_a = 1$ for all $a \in E$ and $D^2 = |E|$. The fusion multiplicities are multiplicity free with $N_{ab}^c = \delta_{a+b,c}$ such that the fusion rules are given by the group composition rules (we use $+$ for the group composition as E is an Abelian group) and $a \otimes b = [a+b]_N$ for all $a, b \in E$. We list the data of $Rep(E)$ below.

$$\begin{aligned}
\mathcal{L} &= \text{underlying set of } E \\
a \otimes b &= a + b \\
a^* &= -a \\
d_a &= 1 \qquad \forall a \in \mathcal{L} \\
D &= |E| \\
N_{ab}^c &= \delta_{a+b,c} \\
F_{kln}^{ijm} &= \delta_{i+j,-m}\delta_{k+l,m}\delta_{l+i,-n}\delta_{j+k,n} \\
R_{ij}^k &= \delta_{i+j,k}
\end{aligned} \tag{9.56}$$

9.7.3 Walker-Wang Models for $Rep(E)$

Utilising $Rep(E)$ as defined in the previous section as the input data of the Walker-Wang model we may write the terms of the Hamiltonian as follows. The vertex operator acts on basis elements as

$A_v \Big|$  $\Big\rangle = \delta_{a+b+c,0} \Big|$  $\Big\rangle$ (9.57)

which energetically penalises configurations of labels around vertices which do not fuse to the identity object.

To define the plaquette operator we first choose an orientation of the plaquette (although the action of B_p is independent of the choice taken). In the following we choose an anti-clockwise convention and define $p^{\pm}$ as the set of edges with direction parallel/anti-parallel to the choice of orientation. We may then write the plaquette operator for $n \in E$ as follows

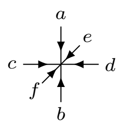
$$B_p^n = \Big(\prod_{v \in p} A_v\Big) \prod_{e \in p^+} \Sigma_e^n \prod_{e' \in p^-} \Sigma_{e'}^{-n} \tag{9.58}$$

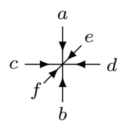
where Σ_e^n acts on the label l of edge e such that $\Sigma_e^n : l \mapsto l + n$. The operators Σ_e^n commute for all edges and $\Sigma_e^n \Sigma_e^m = \Sigma_e^{n+m}$. The operator B_p in the Hamiltonian is then equal to

$$B_p = \frac{1}{|E|} \sum_{n \in E} B_p^n. \tag{9.59}$$

As such an operator symmetrises over all group elements the action on basis states is independent of orientation convention for the plaquette.

As the model based for $Rep(E)$ does not have any strict dependency on the trivalent lattice we may equally well define the model on a cubic lattice without changing the dynamics of the model. Under such a transformation the vertex operator becomes:

$A_v \Bigg|$ 

$\Bigg\rangle = \delta_{a+b+c+d+e+f,0} \Bigg|$ 

$\Bigg\rangle$ (9.60)

while the plaquette operator takes the same form with the trivalent vertex operators replaced with the 6-valent counterpart.

9.7.4 Topological Higher Lattice Gauge Theory for $\mathcal{E}$

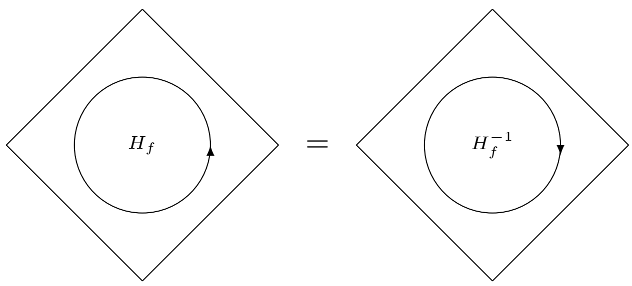
In the following we begin by discussing the THLGT Hamiltonian schema with crossed module $\mathcal{E} = (1_E, E, \partial, \triangleright)$ where E is a finite Abelian group, $\partial : E \to 1_E$ is the group homomorphism which takes all elements of E to the trivial group given by the identity of E and the group action $\triangleright$ is trivial, acting as the identity map. We will first describe the general features of such a model demonstrating how much of the previous discussion can be simplified for such models. We will then show that for a 3D lattice with the 1-skeleton of the dual lattice a directed graph this theory reproduces the Walker-Wang model with input category $Rep(E)$.

We begin by discussing 2-gauge configurations. The edge 2-gauge configurations will be trivial as a direct consequence of $G = 1_E$. In this way all edges are assigned the trivial group. From the previous discussion, 2-gauge configurations of plaquettes are given by assignments of the group E. Such assignments we defined with respect to a source and target path given by boundary relative homotopic paths in (M, L). In the following this data can be vastly simplified.

Given a 2-morphism $f \in \mathcal{P}^2(M, L)_2$, a 2-gauge configuration F is given by a strict 2-functor defining the following map:

$$F:\ \begin{tikzcd} v \arrow[rr, bend left, "s(f)"] \arrow[rr, bend right, "t(f)"'] \arrow[rr, phantom, "\Downarrow f"] & & v' \end{tikzcd} \ \mapsto\ \begin{tikzcd} * \arrow[rr, bend left, "1_E"] \arrow[rr, bend right, "1_E"'] \arrow[rr, phantom, "\Downarrow H_f"] & & * \end{tikzcd} \tag{9.61}$$

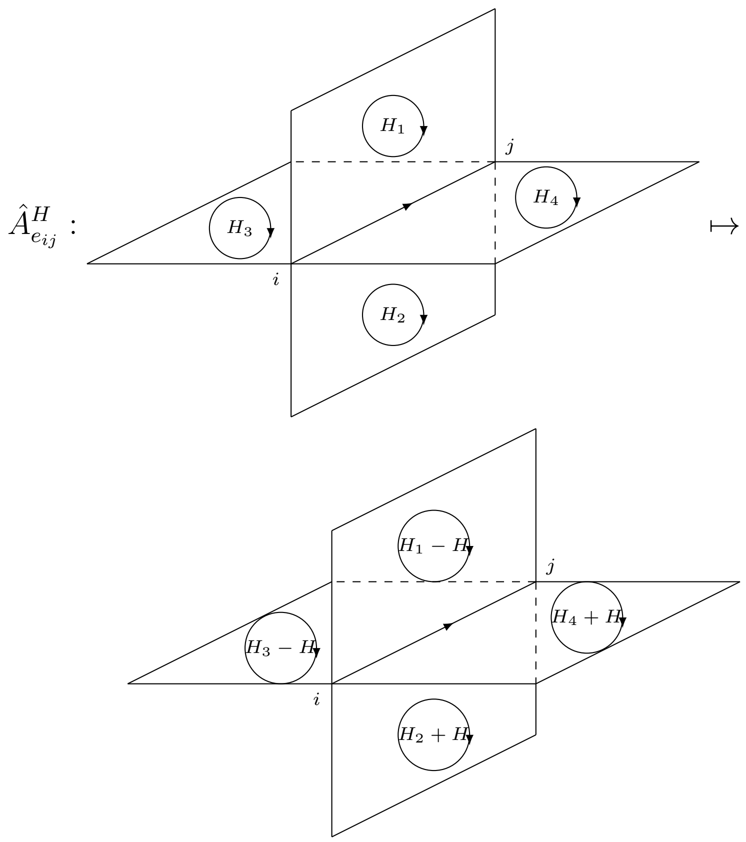
Changes of the source and target morphisms $s(f), t(f) \in \mathcal{P}^2(M, L)_1$ were defined using the operation of whiskering. This was preformed by horizontal composition with the identity 2-morphism of the morphism which changed the source and target morphism. For the crossed module $\mathcal{E}$, by definition such operations act trivially on the face labels. In this way we can neglect the exact source and target morphisms and instead consider the path $s(f)t^{-1}(f) \in \mathcal{P}^2(M, L)_1$ defining an orientation to the boundary of the 2-morphism. Whiskering does not change the orientation of this circle. We can visualise this orientation by assigning an oriented circle on each plaquette as follows:



(9.62)

The operations of taking the horizontal and vertical inverse of the morphism both have the same action given by changing the orientation assigned to a plaquette which by functoriality correspond to taking the inverse of the group element assigned to the plaquette. In this way we can define a 2-gauge configuration by first assigning an oriented circle to each plaquette of the lattice and assigning an element $H_f \in E$ to each face and changing the orientation of such plaquettes corresponds to taking the inverse of the group element assignment. From functorality we identify these two 2-gauge configurations as both are defined by the same strict 2-functor.

We now consider the action of 2-gauge transformations. The vertex gauge operator has trivial action following from $G = 1_E$. In this way we only need consider edge gauge operators. To define such operators we need only consider the direct graph structure of the 1-skeleton of the lattice and the orientation of the plaquettes defined previously. Given an oriented edge $e_{ij} \in \mathcal{P}^2(M, L)_1$, the edge gauge operator $\hat{A}^H_{e_{ij}}$ has non-trivial action only on plaquettes adjacent to e_{ij}. Such an operator has trivial action on the 2-gauge configuration of all edges. To adjacent plaquettes the action is given by $\hat{A}^H_{e_{ij}} : H_f \mapsto H_f + H^\epsilon$ where $\epsilon \in \{\pm 1\}$. The value of ϵ is inferred from whether the orientation of the edge adjacent to the plaquette f is parallel or anti-parallel to the orientation of the plaquette such that $\epsilon = +1$ if the edge is parallel and $\epsilon = -1$ if the edge is anti-parallel. An example of this action is given as follows:



(9.63)

From this definition we can define the edge gauge projector immediately via $\hat{A}_{e_{ij}} := \frac{1}{|E|} \sum_{H \in E} \hat{A}^H_{e_{ij}}$. In this way the edge gauge operator is independent of the orientation of the edge and only depends on the relative orientation of the plaquettes adjacent to e_{ij}.

We finally discuss the 2-flatness operator. In order to calculate the 2-flatness operator we first choose a prospective for defining the orientation of the plaquettes either from inside or outside the blob (3-cell). The operator will be independent of which perspective is chosen. Picking a perspective, say from outside of the blob, we define $\eta_f \in \{\pm 1\}$ such that $\eta = 1$ if the orientation of the plaquette is clockwise from the chosen prospective and $\eta = -1$ if the orientation is anti-clockwise. In this way we define the 2-flatness operator via:

$$\hat{B}_b = \delta_{\sum_{f \in \partial b} H_f^{\eta_f}, 1_E}. \tag{9.64}$$

where the summation is over plaquettes f in the boundary of the blob b.

Following from the discussion the Hamiltonian can be described for a lattice (M, L) is defined in terms of two non-trivial terms:

$$H(M, L; \mathcal{E}) = -\sum_{b \in (M,L)_3} \hat{B}_b - \sum_{e \in (M,L)_1} \hat{A}_e. \tag{9.65}$$

In the subsequent section we will show how this Hamiltonian schema corresponds to the Walker-Wang model for $Rep(E)$ when the 1-skeleton of the lattice forms a directed graph.

9.7.5 $3D$ THLGT Model on the Dual Lattice

After defining the THLGT Hamiltonian schema for crossed module $\mathcal{E}$ we now define the model on a dual lattice. We define dualisation by a map which takes the n-cells of a cellular decomposition of a d-manifold to the $(d-n)$-cells of the dual cellulation. In the following we will consider the spatial dimension to be 3 and that our lattice is a cubulation such that the dual cell decomposition is also a cubulation, such a restriction is for ease of presentation and the arguments follow straightforwardly outside of such a restriction whenever the 1-skeleton of the dual is a graph. In this case the cubes (3-cells) are taken to vertices (0-cells) of the new cellulation, square faces (2-cells) are taken to edges (1-cells) and edges (1-cells) are taken to faces (2-cells). In this way we can canonically map the

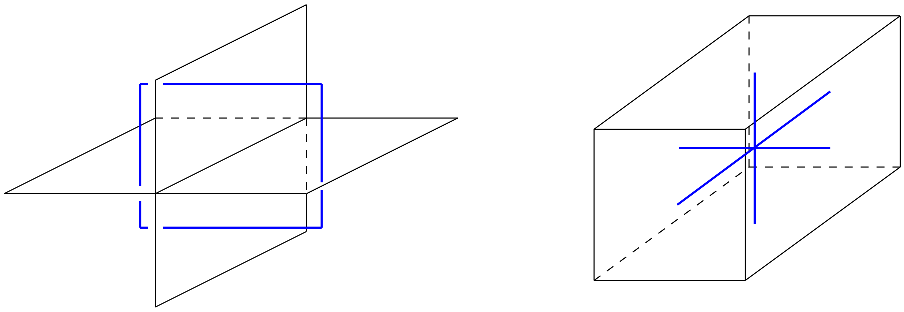
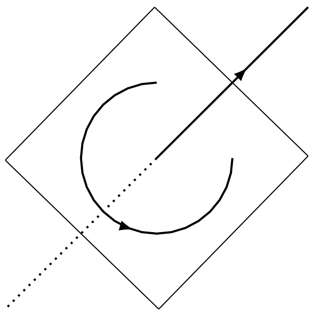


Figure 9.3: Examples of the dual of a cubic lattice. The edges of the original lattice are black and the dual edges blue.

THLGT model with degrees of freedom on faces to a dual lattice where the face labels are now on edges. Examples are shown in figure 9.3 where black edges are of the original lattice and blue are dual.

Utilising the duality map discussed previously the direction of dual edges are inherited from the orientation of faces. The direction is defined by the right hand rule, such that if the fingers of your right hand points in the direction of the orientation arrow of the plaquette the thumb gives the direction of the dual edge, eg.



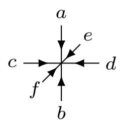
(9.66)

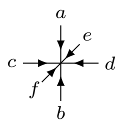
We now consider the edge gauge operators on the dual lattice. Using the orientation of the edge in the original lattice we can define an orientation to the dual plaquette using the right hand rule, such that if the thumb of your right hand points in the direction of the edge, the plaquette is oriented with respect to the direction the fingers point in. Letting $\tilde{e}$ be an edge of the dual lattice with group element $H_{\tilde{e}}$ assigned, we define $\Sigma_{\tilde{e}}^{H} : H_{\tilde{e}} \mapsto H_{\tilde{e}} + H$, with trivial action on all other edges. Using this convention we can describe the edge gauge projector via:

$$\hat{A}_{\tilde{p}} := \frac{1}{|E|} \sum_{H \in E} \prod_{\tilde{e}^{+} \in \tilde{p}} \Sigma_{\tilde{e}}^{H} \prod_{\tilde{e}^{-1} \in \tilde{p}} \Sigma_{\tilde{e}}^{-H} \tag{9.67}$$

where $\tilde{e}^{+}$ is the set of edges in the dual plaquette $\tilde{p}$ with orientation parallel to $\tilde{p}$ and $\tilde{e}^{-1}$ is the set of edges with anti-parallel orientation. It is straightforward to verify such conventions define the same action as the edge gauge operator on the original lattice.

Similarly for a vertex $\tilde{v}$ of the dual lattice we can define the 2-flatness projector on the dual lattice as follows:

$\hat{B}_{\tilde{v}} \Big|$ 

$\Big\rangle = \delta_{a+b+c+d+e+f,0} \Big|$ 

$\Big\rangle$ (9.68)

Thus we see that the 2-flatness condition on the cubic cells becomes a vertex condition on the dual lattice.

9.7.6 Comparison of Models

Using the discussion outlined in the previous sections we now compare the THLGT model with input $\mathcal{E}$ and the Walker-Wang model with input $Rep(E)$. Both models are defined on a cubic lattice Γ with a local Hilbert space defined by $\mathcal{H} = \otimes_{e \in \Gamma} \mathbf{C}^{|E|}$ with edge labels indexed by the group E. The Hamiltonian for the THLGT and Walker-Wang models can respectively be written as follows:

$$H_{Yetter}(\mathcal{E}) = -\sum_{v \in \Gamma} \hat{B}_v - \sum_{p \in \Gamma} \Big(\frac{1}{|E|} \sum_{h \in E} \prod_{e^{+} \in p} \Sigma_e^{h} \prod_{e^{-} \in p} \Sigma_e^{-h}\Big) \tag{9.69}$$

$$H_{WW}(Rep(E)) = -\sum_{v \in \Gamma} A_v - \sum_{p \in \Gamma} \Big(\frac{1}{|E|} \sum_{h \in E} \prod_{e^{+} \in p} \Sigma_e^{h} \prod_{e^{-} \in p} \Sigma_e^{-h}\Big)\Big(\prod_{v \in p} A_v\Big) \tag{9.70}$$

where in equation (9.69) we have substituted equations (9.67) and (9.68) into equation (9.65). Noting that we can identify $A_v = \hat{B}_v$ in the two models the only difference in the definition of the two Hamiltonians is the second term which acts on plaquettes of the lattice. This difference is immaterial as the only distinguishing feature of the term $(\prod_{v \in p} A_v)$ is to increase the energy penalty for configurations which do not satisfy the vertex constraint to twice the energy cost of creating plaquette defects. From such a point of view the two Hamiltonians have the same ground-state configurations and the excitations will have the same measurable properties such as braid statistics but the energy cost will be increased for the creation of vertex violations in the Walker-Wang model in comparison to the energy cost in the THLGT model.

Furthermore this relation between the two models implies that we can consider the groundstates of the Walker-Wang model with $Rep(E)$ as corresponding to homotopy classes of maps form the spatial 3-manifold M to the classifying space $B_{\mathcal{E}}$.

Part III

Quasi-Particles and Tube Algebras

Overview

In chapter 5 we demonstrated that the state spaces of state-sum TQFT's admit a description in terms of exactly solvable Hamiltonian schemas. In particular they admit Hamiltonians consisting of local, mutually commuting projection operators. In the following we will expand on this description to develop an algebraic approach to understanding topological excitations arising in such theories. See appendix B for a brief introduction to finite dimensional algebras, modules and related constructions utilised in the following.

In previous studies topological excitations have been successfully described in terms of so called ribbon operators which generate quasi particles on their boundaries while commuting with the Hamiltonian along their bulk [24, 27, 82, 83, 84]. However such ribbon operators or their higher dimensional analogues can be extremely difficult to define for general topological Hamiltonian schemas. With this in mind, in this section we will utilise an alternative approach, exploiting the length scale invariance of such theories.

In particular we will introduce the so called tube-algebra which generalises the construction of Ocneanu [85] for the Turaev-Viro TQFT and later discussed for picture TQFT's in the notes of Kevin Walker [38]. We will argue that the simple modules of the tube algebra correspond to the **irreducible topological excitations** in a state-sum TQFT. A reformulation of the Ocneanu tube algebra in terms of the string-net construction appears in [86] and the case of 2+1D topological finite gauge theories is discussed in [84].

In chapter 10 we introduce the construction of tube algebras as applicable to any state-sum TQFT and the describe the salient features of the tube algebra. In particular we prove that such tube algebras are finite dimensional, associative, $*$-algebras and as a corollary are semisimple. Subsequently, we will show that although our construction of tube algebras will depend on a triangulation of the spatial manifold, by considering Morita equivalence classes of tube-algebras, triangulation independence can be restored. We will also describe the centre of the tube algebra and consider the relation to minimum entropy states [87].

In chapter 11 we consider and classify the algebraic properties and representation theory of a class of tube algebras we call **twisted groupoid-like**. The results of this section will be directly applied in the subsequent discussion of the Dijkgraaf-Witten and THLGT tube algebras.

In chapter 12 we apply the tube algebra construction to the Dijkgraaf-Witten TQFT in 1+1D, 2+1D and 3+1D. The results in 1+1D and 2+1D confirm the work of others, in particular giving an interpretation of the results of the twisted quantum double in [34]. The new component in the Dijkgraaf-Witten case is the classification of point and loop-like excitations in 3+1D.

In chapter 13 we apply the tube algebra construction to the case of topological higher gauge theories. In this case we define the tube algebra canonically using the functor 2-groupoid between the fundamental 2-groupoid and a finite crossed module. We study examples of this constructions in 1+1D, 2+1D, 3+1D mirroring the discussion of the Dijkgraaf-Witten TQFT and discuss the correspondence.

Chapter 10

Tube Algebras

Given a gapped quantum many-body system with translational symmetry, the ground state subspace necessarily has a uniform energy density E_0 across the spatial manifold. For such systems we define excitations to be local regions of the spatial manifold with higher energy density, $E_0 + \Delta E$ for finite $\Delta E > 0$.

Excitations naturally admit a classification into two classes: **local** and **topological**.

Definition 10.0.1. A **local excitation** in a gapped quantum many body quantum system is an excitation which can be created and annihilated via local operators.

Examples of local excitations include bit flips in qudit models. Conversely

Definition 10.0.2. A **topological excitation** in a gapped many body quantum system is an excitation which *cannot* be created or annihilated by any finite set of local operators.

A general excitation in a gapped many body quantum system will be a **composite** of local and topological excitations and as such we introduce the looser notion of

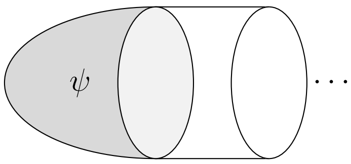
Definition 10.0.3. The **topological type** of an excitation in a gapped many body quantum system is the equivalence class of topological excitations which differ by a local excitation.

We now relate the above discussion to the canonical Hamiltonian formalism for unitary ssTQFT's. Let M be a closed spatial manifold with triangulation $\mathcal{M}$ such that $\partial\mathcal{M} = \emptyset$. In section 5.2.4 the canonical Hamiltonian for a unitary ssTQFT $\mathcal{Z}$ on spatial manifold $\mathcal{M}$ was defined as:

$$H(\mathcal{Z}, \mathcal{M}) = -\sum_{i \in \Delta^0(\mathcal{M})} H_i. \tag{10.1}$$

The spatial geometry of the theory is fully encoded in the set of local projection operators $\{H_i\}$ acting in a local neighbourhood $cl_i \subset \mathcal{M}$ of the vertex $i \in \Delta^0(\mathcal{M})$. From the projection property the operators H_i assign the eigenvalue $E_i = +1$ to ground state configurations in the local neighbourhood of $i \in \Delta^0(\mathcal{M})$ and $E_j = 0$ to excited states. In this way the canonical Hamiltonian naturally gives us an approach to understanding a notion of spatial location of excitations in the theory in terms of the triangulation choice $\mathcal{M}$.

For simplicity we will first consider classifying the topological excitation type of point-particle excitations in $2+1$D theories before describing the general picture. Given a point-particle excitation ψ, in a local neighbourhood of ψ the projection operators signal the presence of an excitation. If the local neighbourhood is much smaller than the global topological features of $\mathcal{M}$, this local neighbourhood will have the topology of a 2-disk D^2. In the following we will identify ψ as both the excitation and the local neighbourhood. Without loss of generality from triangulation invariance of the theory away from the excitation we can always find a triangulation and isomorphic Hilbert space such that this is the case. We visualise this local piece of $\mathcal{M}$ as follows:



(10.2)

Here the grey region correspond to the excited region with local energy density $E_0 + \Delta E$ and the white regions correspond to the regions with ground state local energy density E_0.

We now want to understand the topological excitation type of the grey region. To do so we use the following physically inspired assumptions:

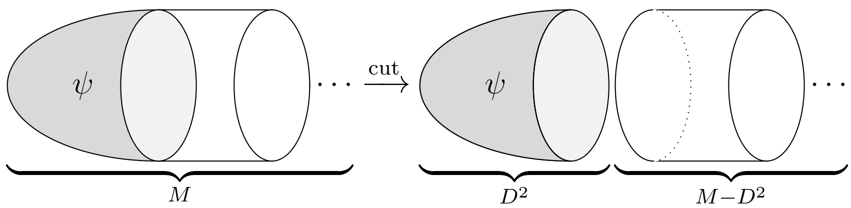
1. No local operator acting on the interior of ψ can change the topological excitation type of ψ.

2. Entanglement between the excited and groundstate regions characterises the excitation

3. Topological excitations should be **scale invariant**, in the sense that they are measurable at all length scales.

Such assumptions are not necessarily independent but it is useful to phrase them in such a way. Assumption 1. follows from definition 10.0.3 of topological excitation type. Assumption 2. is not fully independent of 1. The entanglement in topological phases of matter should only depend on a local neighbourhood of the boundary between the two regions due to the gapped and local structure of the Hamiltonian. Furthermore the entanglement for such systems describes a quantity which is invariant under local unitary operations which occur on the compliment of the entanglement cut and the groundstate regions of the manifold should be invariant under local unitary operators. Assumption 3. follows as the physical theory lacks a metric and hence their is no notion of length scale.

To consider the entanglement between the two regions we implement a **cut** along the boundary between the two regions:



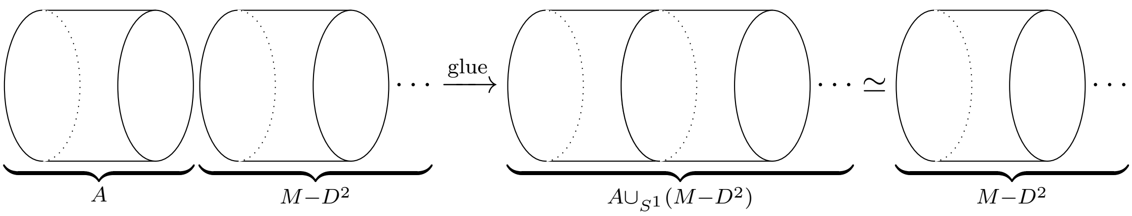
(10.3)

In particular we require such cuts to be reversible. In terms of the triangulation, the cuts we consider split the boundary degrees of freedom so that both sides of the cut carry a copy of the boundary and boundary configuration with opposite orientation. The entanglement between the two regions naturally defines a non-trivial **boundary condition** in terms of the admissible configurations of the boundary shared between the excited and the ground state regions. In this way we can alternatively view the excitation as either being described by entanglement or equivalently a boundary condition on $\mathcal{M} - \psi$.

Building upon our assumption that topological excitation types can be described in terms of the entanglement between the ground state and excited regions we can redefine the problem of classifying topological excitation types with the classification of boundary conditions. So far we have made no assumptions about the nature of such boundary conditions. We now use assumption 3 in a stronger form.

- If the topological excitation type is scale invariant, the physical properties should be invariant under the process of **gluing** more space around the boundary which doesn't change the topology of $\mathcal{M} - \psi$.

In particular noting that $\partial(M - D^2) = S^1$, such that $\partial(\mathcal{M} - \psi)$ defines a triangulation $\mathcal{S}^1$ of the circle, we can glue a triangulation $\mathcal{A}$ of $A = S^1 \times I$ to $\mathcal{M} - \psi$ without changing the topology. This is visualised below:



(10.4)

where $\simeq$ denotes a PL-homeomorphism. In terms of the groundstate subspace, the canonical Hamiltonian associates to the triangulations, we can consider gluing as defining a map:

$$\mathcal{H}[\mathcal{A}]_0 \otimes \mathcal{H}[\mathcal{M} - \psi]_0 \xrightarrow{\text{glue}} \mathcal{H}[\mathcal{A} \cup_{\mathcal{S}^1} (\mathcal{M} - \psi)] \xrightarrow{\mathcal{Z}[GL]} \mathcal{H}[\mathcal{M} - \psi]_0 \tag{10.5}$$

here $\mathcal{Z}[GL]$ is the linear transformation of the Hilbert spaces given by the mutation operators associated to the PL-homeomorphism $\mathcal{A} \cup_{\mathcal{S}^1} (\mathcal{M} - \psi) \simeq \mathcal{M} - \psi$.

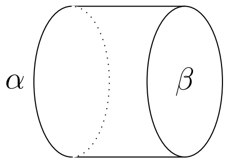
In the case $\mathcal{M} - \psi = \mathcal{A}$ (ie. the presence of two particles on the sphere) it is straightforward to see this map defines an algebra on $\mathcal{H}[A]$:

$$\mathcal{H}[\mathcal{A}]_0 \otimes \mathcal{H}[\mathcal{A}]_0 \xrightarrow{\text{glue}} \mathcal{H}[\mathcal{A} \cup_{S^1} \mathcal{A}] \xrightarrow{\mathcal{Z}[GL]} \mathcal{H}[\mathcal{A}]_0 \tag{10.6}$$

where the product $\circ$ is given by gluing followed by applying $\mathcal{Z}[GL]$. We call $(\mathcal{H}[\mathcal{A}]_0, \circ)$ the $\mathcal{S}^1$**-tube algebra**. Using this observation we can naturally identify

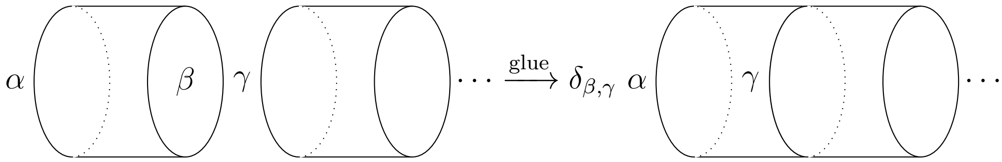
$\mathcal{H}[\mathcal{M}-\psi]_0$ as defining a **module** over the algebra $(\mathcal{H}[\mathcal{A}]_0, \circ)$ with the action been given by gluing then applying $\mathcal{Z}[GL]$.

A natural requirement for the gluing to be well defined is that we can only glue manifolds along identified boundary configurations. Following the conventions of section 4.2, let $\mathcal{H}[\mathcal{A};\alpha,\beta]_0$ be the Hilbert space over $\mathcal{A}$ with fixed field configurations $s(\mathcal{S}^1 \times 0) = \alpha$ and $s(\mathcal{S}^1 \times 1) = \beta$ of $\partial\mathcal{A}$:

$\mathcal{H}[\mathcal{A};\alpha,\beta]_0 :=$ 

(10.7)

Given $\mathcal{H}[\mathcal{M}-\psi;\gamma]_0$ we require

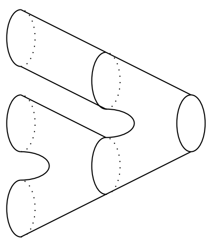


(10.8)

In this way we see that the boundary configuration dictated by the entanglement of a particle-excitation on $\mathcal{M}-\psi$ forms $\mathcal{H}[\mathcal{M}-\psi]_0$ submodules of the tube algebra. Utilising this correspondence, in the following we will identify excitations with modules of the tube algebra. Furthermore, we will identify:

- reducible modules with composite topological excitation types
- simple modules with irreducible excitations

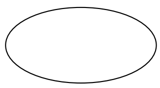
So far we discussed one topological particle excitation. In general we will want to consider a manifold M with multiple particle excitations. In such cases we will consider $M - \sqcup_i^n D^2$. Defining an orientation to each boundary we can consider this space as a cobordism $\sqcup_i^m D^2 \xrightarrow{M-\sqcup_i^n D^2} \sqcup_j^{n-m} D^2$. For example, for $n = 4$ and $m = 3$:


(10.9)

In this way $\mathcal{H}[\mathcal{M} - \sqcup_i \psi_i]_0$ can be considered as a $(\otimes_{i=1}^{m} \mathcal{H}[\mathcal{A}]_0, \otimes_{j=1}^{n-m} \mathcal{H}[\mathcal{A}]_0)$-bimodule. We will consider each such bimodule as a configuration of n-excitations on $\mathcal{M}$. In general the topology of $M - \sqcup_i D^2$ will play a role in determining the set of n-particle configurations.

In the previous discussion we have outlined the features of particle excitations in 2+1D and argued that such excitations on a manifold M can be classified in terms of bimodules of the tube algebra. This argument can be straightforwardly applied to excitations with topologies different to the point and in arbitrary dimensions as follows: Given an excitation ψ on a triangulated manifold $\mathcal{M}$, with local neighbourhood $\mathcal{N}$ in $\mathcal{M}$. The tube algebra, we will call the $\partial\mathcal{N}$-tube algebra, will be given by considering the Hilbert spaces over $\mathcal{H}[\partial\mathcal{N} \times I]_0$ and algebra product defined in the same manner as the previous example. Furthermore, we can consider multiple excitation states analogously to the $2 + 1$D case by considering bimodules associated to $\mathcal{M} - \sqcup_i \mathcal{N}_i$ where we can additionally allow the topology of each $\mathcal{N}_i$ to differ.

One particular example we will consider in the following is that of loops in $3 + 1$D theories. Loops are excitations with the topology of the circle S^1. In 3-dimensional space the local neighbourhood of a loop is given by the solid tori $D^2 \times S^1$



$$\xrightarrow{\text{local neighbourhood}} D^2 \times S^1 \tag{10.10}$$

Using $\partial(D^2 \times S^1) = S^1 \times S^1 = T^2$ we find the corresponding tube-algebra is given by the groundstate subspace of a triangulation of $T^2 \times I$. The classification of loop-like excitations in $3 + 1$D is of primary interest because such excitations are expected to provide non-trivial motion group representations ie. the loop-braid group [88, 89], generalising the braid group of point particles in $2 + 1$D or the necklace group [90].

The previous discussion was rather informal with regard to defining the tube algebras and the gluing procedure. In the subsequent section will define such a construction and demonstrate that the tube algebras are finite, associative algebras.

10.1 Tubes and Tube Algebras

In the following we formalise the previous discussion for unitary $n+1$D state sum TQFT $\mathcal{Z}$. We begin by defining **tubes**.

Definition 10.1.1. Given a closed, oriented $(n-1)$-manifold W with triangulation $\mathcal{W}$, we define the $\mathcal{W}$-**Tube** to be the triangulated $n-$manifold $\mathcal{W}_{tube} := \mathcal{W} \times [0,1]$.

By definition $\partial \mathcal{W}_{tube} = \overline{\mathcal{W}} \sqcup \mathcal{W}$, where we identify $\overline{\mathcal{W}} = \mathcal{W} \times 0$ and $\mathcal{W}$ with $\mathcal{W} \times 1$.

Using the canonical Hamiltonian schema for a state-sum TQFT $\mathcal{Z}$ we can canonically assign a Hilbert space $\mathcal{H}[\mathcal{W}_{tube}]$ to the $\mathcal{W}$-tube. Using the the ground state projector $P_{\mathcal{Z},\mathcal{W}_{tube}}$ or equivalently the pinched cobordism $\mathcal{W}_{tube} \times_p I$ we define the ground state subspace $\mathcal{H}[\mathcal{W}_{tube}]_0 \subseteq \mathcal{H}[\mathcal{W}_{tube}]$ via:

$$\mathcal{H}[\mathcal{W}_{tube}]_0 = \mathrm{Im} P_{\mathcal{Z},\mathcal{W}_{tube}} = \mathrm{Im}\mathcal{Z}[\mathcal{W}_{tube} \times_p I]. \tag{10.11}$$

From section 4.2, due to the presence of boundaries, $\mathcal{H}[\mathcal{W}_{tube}]$ admits a bi-grading:

$$\mathcal{H}[\mathcal{W}_{tube}] = \bigoplus_{\alpha,\beta \in s(\mathcal{W})} \mathcal{H}[\mathcal{W}_{tube}; \alpha, \beta] \tag{10.12}$$

which further restricts to the ground state subspaces

$$\mathcal{H}[\mathcal{W}_{tube}]_0 = \bigoplus_{\alpha,\beta \in s(\mathcal{W})} \mathcal{H}[\mathcal{W}_{tube}; \alpha, \beta]_0. \tag{10.13}$$

In particular we will use the convention that $\alpha \in s(\mathcal{W})$ is the configuration restricted to $\mathcal{W} \times 0$ and $\beta \in s(\mathcal{W})$ is the configuration restricted to $\mathcal{W} \times 1$.

As discussed in the previous section we can define a gluing procedure on such Hilbert spaces when the boundary configurations are identified. To do so we first introduce some notation: Let

$$\mathcal{H}[\mathcal{W}_{tube}] \otimes_{\mathcal{W}} \mathcal{H}[\mathcal{W}_{tube}] \subseteq \mathcal{H}[\mathcal{W}_{tube}] \otimes \mathcal{H}[\mathcal{W}_{tube}] \tag{10.14}$$

where

$$\mathcal{H}[\mathcal{W}_{tube}] \otimes_{\mathcal{W}} \mathcal{H}[\mathcal{W}_{tube}] := \bigoplus_{\alpha,\beta,\gamma \in s(\mathcal{W})} \mathcal{H}[\mathcal{W}_{tube}; \alpha, \beta] \otimes \mathcal{H}[\mathcal{W}_{tube}; \beta, \gamma]. \tag{10.15}$$

We now define the projector $\mathbb{P}$:

$$\mathbb{P} : \mathcal{H}[\mathcal{W}_{tube}] \otimes \mathcal{H}[\mathcal{W}_{tube}] \to \mathcal{H}[\mathcal{W}_{tube}] \otimes_{\mathcal{W}} \mathcal{H}[\mathcal{W}_{tube}] \tag{10.16}$$

by the action

$$\mathbb{P} : |v_{\alpha,\beta}\rangle \otimes |v_{\beta',\gamma}\rangle \mapsto |v_{\alpha,\beta}\rangle \otimes |v_{\beta',\gamma}\rangle\, \delta_{\beta,\beta'} \tag{10.17}$$

on basis elements $|v_{\alpha,\beta}\rangle \in \mathcal{H}[\mathcal{W}_{tube}; \alpha; \beta]$ and $|v_{\beta',\gamma}\rangle \in \mathcal{H}[\mathcal{W}_{tube}; \beta', \gamma]$ which can then be extended linearly to the entirety of $\mathcal{H}[\mathcal{W}_{tube}] \otimes \mathcal{H}[\mathcal{W}_{tube}]$.

We identify elements of $\mathcal{H}[\mathcal{W}_{tube}] \otimes_{\mathcal{W}} \mathcal{H}[\mathcal{W}_{tube}]$ with elements of $\mathcal{H}[\mathcal{W}_{tube} \cup_{\mathcal{W}} \mathcal{W}_{tube}]$ where $\mathcal{W}_{tube} \cup_{\mathcal{W}} \mathcal{W}_{tube}$ is the triangulation given by gluing two $\mathcal{W}$-tubes along $\mathcal{W}$. This identification follows from the definition of $\mathcal{H}[\mathcal{W}_{tube}]$ in section 5.2.1 in terms of tensor factors associated to the simplices of $\mathcal{W}_{tube}$. In general given $|v_{\alpha,\beta}\rangle \in \mathcal{H}[\mathcal{W}_{tube}; \alpha, \beta]_0$ and $|v_{\beta,\gamma}\rangle \in \mathcal{H}[\mathcal{W}_{tube}; \beta, \gamma]_0$

$$|v_{\alpha,\beta}\rangle \otimes |v_{\beta,\gamma}\rangle \notin \mathcal{H}[\mathcal{W}_{tube} \cup_{\mathcal{W}} \mathcal{W}_{tube}; \alpha, \gamma]_0. \tag{10.18}$$

This is because no projection operators have been applied to a local neighbourhood around the gluing.

Now given the Hilbert space $\mathcal{H}[\mathcal{W}_{tube} \cup_{\mathcal{W}} \mathcal{W}_{tube}; \alpha, \gamma]$ we can use the pinched cobordism $GL_W := (W \times I) \times_p I$ to define a triangulated cobordism $GL_{\mathcal{W}}$ such that the boundary is given by: $\partial GL_{\mathcal{W}} := (\overline{\mathcal{W}_{tube} \cup_{\mathcal{W}} \mathcal{W}_{tube}}) \cup_{\overline{\mathcal{W}} \sqcup \mathcal{W}} \mathcal{W}_{tube}$. In this way

$$Z[GL_{\mathcal{W}}] : \mathcal{H}[\mathcal{W}_{tube} \cup_{\mathcal{W}} \mathcal{W}_{tube}] \to \mathcal{H}[\mathcal{W}_{tube}]_0. \tag{10.19}$$

The reason the map is into $\mathcal{H}[\mathcal{W}_{tube}]_0$ not $\mathcal{H}[\mathcal{W}_{tube}]$ follows from:

$$Z[\mathcal{W}_{tube} \times_p I]Z[GL_{\mathcal{W}}] = Z[GL_{\mathcal{W}}] = Z[GL_{\mathcal{W}}]Z[\mathcal{W}_{tube} \cup_{\mathcal{W}} \mathcal{W}_{tube} \times_p I] \tag{10.20}$$

as required by triangulation invariance of $\mathcal{Z}$.

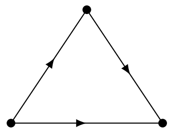
Example 10.1.1. Let us clarify our discussion with some intuition about what is happening here. Imagine $W := *$ is a single point, with triangulation $\mathcal{P}$ as a single 0-simplex Δ^0. A triangulation of $\mathcal{P}$-tube can be given as the 1-simplex Δ^1.

$$\mathcal{P}_{tube} := \bullet\!\!\longrightarrow\!\!\!-\!\!\!-\!\!\bullet \tag{10.21}$$

and similarly

$$\mathcal{P}_{tube} \cup_{\mathcal{P}} \mathcal{P}_{tube} :=$$ (10.22)

In this way we can consider $GL_{\mathcal{P}}$ as a pinched cobordism $GL_{\mathcal{P}} : \mathcal{P}_{tube} \cup_{\mathcal{P}} \mathcal{P}_{tube} \to \mathcal{P}_{tube}$ which is simply given by the triangle:

$$GL_{\mathcal{P}} =$$  (10.23)

Remark 10.1.1. Notice in the previous example we can define the triangulation $GL_{\mathcal{P}} = \mathcal{P}_{tube} \star \mathcal{P}$ in terms of the join operation (see 2.1.12).

Using the above we can now define the tube algebra product $\circ$ on $\mathcal{H}[\mathcal{W}_{tube}]_0$ in terms of the sequence of operations:

$$\mathcal{H}[\mathcal{W}_{tube}]_0 \otimes \mathcal{H}[\mathcal{W}_{tube}]_0 \xrightarrow{\mathbb{P}} \mathcal{H}[\mathcal{W}_{tube} \cup_{\mathcal{W}} \mathcal{W}_{tube}] \xrightarrow{\mathcal{Z}[GL_{\mathcal{W}}]} \mathcal{H}[\mathcal{W}_{tube}]_0 \tag{10.24}$$

such that

$$\circ := \mathcal{Z}[GL_{\mathcal{W}}]\mathbb{P} : \mathcal{H}[\mathcal{W}_{tube}]_0 \otimes \mathcal{H}[\mathcal{W}_{tube}]_0 \to \mathcal{H}[\mathcal{W}_{tube}]_0 \tag{10.25}$$

For vectors we can define the structure coefficients for the algebra via:

$$\circ : |v_{\alpha,\beta}\rangle \otimes |v_{\beta',\gamma}\rangle \mapsto \delta_{\beta,\beta'} \sum_{\{w_{\alpha,\gamma}\}} \mathcal{Z}[GL_{\mathcal{W}}]^{w_{\alpha,\gamma}}_{v_{\alpha,\beta}\otimes v_{\beta,\gamma}} |w_{\alpha,\gamma}\rangle \tag{10.26}$$

where the summation $\{|w_{\alpha,\gamma}\rangle\}$ is over the complete set of basis vectors for $\mathcal{H}[\mathcal{W}_{tube}; \alpha, \gamma]$ and

$$\mathcal{Z}[GL_{\mathcal{W}}]^{w_{\alpha,\gamma}}_{v_{\alpha,\beta}\otimes v_{\beta,\gamma}} := \langle w_{\alpha,\gamma}| \, \mathcal{Z}[GL_{\mathcal{W}}] \, |v_{\alpha,\beta}\rangle \otimes |v_{\beta,\gamma}\rangle \in \mathbb{C}. \tag{10.27}$$

Definition 10.1.2. Given an n+1D unitary ssTQFT $\mathcal{Z}$ and a closed, oriented, triangulated $n-1$-manifold $\mathcal{W}$, the **$\mathcal{W}$-tube algebra** is the $\mathbb{C}$-algebra on $\mathcal{H}[\mathcal{W}_{tube}]_0$ with product $\circ = Z[GL_{\mathcal{W}}]\mathbb{P}$.

Corollary 10.1.0.1. For all $n+1$D unitary ssTQFT $\mathcal{Z}$ the $\mathcal{W}$-tube algebra is a finite dimensional algebra.

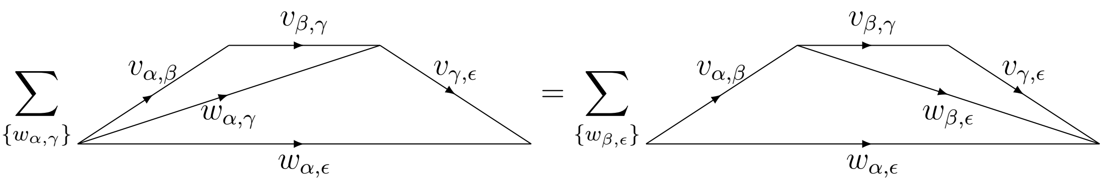
Proof. Follows directly from the finite dimensionality of $\mathcal{H}[\mathcal{W}_{tube}]_0$. $\square$

Proposition 10.1.1. For all $n+1$D unitary ssTQFT $\mathcal{Z}$ the $\mathcal{W}$-tube algebra is associative.

Proof. Follows directly from triangulation invariance of $\mathcal{Z}$ for $n+1$-manifolds. Let $|u_{\alpha,\beta}\rangle, |v_{\beta,\gamma}\rangle, |w_{\gamma,\epsilon}\rangle \in \mathcal{H}[\mathcal{W}_{tube}]_0$. To demonstrate associativity we require:

$$\sum_{\{w_{\alpha,\gamma}\}} \Big(\langle w_{\alpha,\epsilon}| \mathcal{Z}[GL_{\mathcal{W}}] |w_{\alpha,\gamma}\rangle \otimes |v_{\gamma,\epsilon}\rangle \Big)\Big(\langle w_{\alpha,\gamma}| \mathcal{Z}[GL_{W}] |v_{\alpha,\beta}\rangle \otimes |v_{\beta,\gamma}\rangle \Big)$$
$$= \sum_{\{w_{\beta,\epsilon}\}} \Big(\langle w_{\alpha,\epsilon}| \mathcal{Z}[GL_{\mathcal{W}}] |v_{\alpha,\beta}\rangle \otimes |w_{\beta,\epsilon}\rangle \Big)\Big(\langle w_{\beta,\epsilon}| \mathcal{Z}[GL_{W}] |v_{\beta,\gamma}\rangle \otimes |v_{\gamma,\epsilon}\rangle \Big). \quad (10.28)$$

Both of these expressions can be straightforwardly interpreted as matrix elements of a pinched cobordism of $\mathcal{W}_{tube} \times_p I$. Each expression corresponds to a different but PL-homeomorphic triangulation of $\mathcal{W}_{tube} \times_p$ where the boundary of both given by $\overline{\mathcal{W}_{tube} \cup_{\mathcal{W}} \mathcal{W}_{tube} \cup_{\mathcal{W}} \mathcal{W}_{tube}} \cup \mathcal{W}_{tube}$. We can visualise this by drawing each $\mathcal{W}_{tube}$ as a line segment:



(10.29)

As both expressions include summations over complete bases for the bulk edge colouring both matrix elements are independent of their triangulations and hence the above matrix elements are equal from the definition of $\mathcal{Z}$. $\square$

In any quantum theory, a unitary operator U is called a symmetry of the Hamiltonian H if $U^{\dagger}HU = H$. Given a symmetry U, each eigenspace of H can be further decomposed into eigenspaces of U which form a representation of U. It is common to call the eigenvalues of U **good quantum numbers** [2]. In analogy, it is straightforward to verify

$$P(\mathcal{Z}, W_{tube})(|v\rangle \circ |w\rangle) = (P(\mathcal{Z}, W_{tube})\,|v\rangle \circ (P(\mathcal{Z}, W_{tube})\,|w\rangle) \quad (10.30)$$

ie. applying the groundstate projector $P(\mathcal{Z}, W_{tube})$ before or after taking the tube algebra product gives the same result for all $|v\rangle, |w\rangle \in \mathcal{H}[W_{tube}]_0$. From this stand point we can consider the tube-algebra product as a generalised symmetry of the groundstate and call the module labels good quantum numbers.

10.2 $*$-Algebras and Semisimplicity

In this section we demonstrate that the tube algebra admits the additional structure of being a $*$-algebra. As a corollary we will show that the tube algebra is semisimple.

We begin by recalling the definition of a $*$-algebra and $*$-representation, see eg. [91, 92] for an accessible introduction.

Definition 10.2.1. Let A be a complex algebra. A is a $*$-algebra if it additionally admits a map $* : A \to A$, we notate via $* : a \mapsto a^*$ for all $a \in A$, satisfying the following properties:

- $(a^*)^* = a, \quad \forall a \in A$
- $(ab)^* = b^* a^*, \quad \forall a, b \in A$
- $(a + b)^* = a^* + b^*, \quad \forall a, b \in A$
- $(\lambda a)^* = \overline{\lambda} a^*, \quad \forall \lambda \in \mathbb{C}, \forall a \in A.$

Here $\overline{\lambda}$ denotes the complex conjugate of $\lambda \in \mathbb{C}$.

Definition 10.2.2. Given a $*$-algebra A, a $*$-representation of A on a Hilbert space $\mathcal{H}$ is a map $\pi : A \to B(\mathcal{H})$, where $B(\mathcal{H})$ is the set of bounded linear operators on $\mathcal{H}$ such that:

- π is linear
- π is a homomorphism $\pi(ab) = \pi(a)\pi(b)$ $\forall a, b \in A$
- $\langle \pi(a)v|w\rangle = \langle v|\pi(a^*)w\rangle$ $\forall v, w \in \mathcal{H}, \forall a \in A.$

From the definition of a $*$-representation we can immediately infer the following two results:

Proposition 10.2.1. If $\mathcal{K} \subset \mathcal{H}$ is an invariant subspace of a $*$-representation $(\pi, \mathcal{H})$, then so is the orthogonal complement $\mathcal{K}^\perp$.

Proof. If $v \in \mathcal{K}$ and $w \in \mathcal{K}^{\perp}$, then

$$\langle \pi(a)v, w\rangle = 0 = \langle v, \pi(a^*)w\rangle \quad \forall a \in A \tag{10.31}$$

which implies $\pi(a^*)w \in \mathcal{K}^{\perp}$ for all $a \in A$ hence $\mathcal{K}^{\perp} \subset \mathcal{H}$ is an invariant subspace. $\square$

Corollary 10.2.0.1. Any finite dimensional $*$-algebra A is semisimple.

Following from corollary 10.2.0.1, if we establish that the tube algebra for a unitary state-sum TQFT defines a $*$-algebra and the corresponding Hilbert space $\mathcal{H}[\mathcal{W}_{tube}]_0$ is the regular $*$-representation, it directly follows that the tube algebra is semisimple. This will be the focus of the following discussion.

Given a closed, oriented, triangulated $n-1$-manifold $\mathcal{W}$ we defined the tube $\mathcal{W}_{tube} := \mathcal{W} \times I$. In the following we will additionally notate the orientation reversal of $\mathcal{W}_{tube}$ via $\overline{\mathcal{W}_{tube}}$. Let $|g\rangle \in \mathcal{H}[\mathcal{W}_{tube}]$ be a basis element of $\mathcal{H}[\mathcal{W}_{tube}]$ and thus a configuration of $\mathcal{W}_{tube}$, noting that configurations do not depend on orientation of the manifold (see section 4.1.1) we can canonically associate to $|g\rangle$ a basis element $|g^*\rangle \in \mathcal{H}[\overline{\mathcal{W}_{tube}}]$, given by the same labelling of the simplices as defined by $|g\rangle$. Given this correspondence we define the following maps:

$$\begin{aligned} * :&\mathcal{H}[\mathcal{W}_{tube}] \to \mathcal{H}[\overline{\mathcal{W}_{tube}}] \\ * :&\lambda\,|g\rangle \mapsto (\lambda\,|g\rangle)^* := \overline{\lambda}\,|g^*\rangle \end{aligned} \tag{10.32}$$

and

$$\begin{aligned} * :&\mathcal{H}[\overline{\mathcal{W}_{tube}}] \to \mathcal{H}[\mathcal{W}_{tube}] \\ * :&\lambda\,|g^*\rangle \mapsto (\lambda\,|g^*\rangle)^* := \overline{\lambda}\,|g\rangle \end{aligned} \tag{10.33}$$

for any basis elements $|g\rangle \in \mathcal{H}[\mathcal{W}_{tube}]$ and $|g^*\rangle \in \mathcal{H}[\overline{\mathcal{W}_{tube}}]$ from which we extend linearly to the whole of $\mathcal{H}[\mathcal{W}_{tube}]$ and $\mathcal{H}[\overline{\mathcal{W}_{tube}}]$ respectively.

We now consider the groundstate subspaces. Considering $\overline{\mathcal{W}_{tube}} \times_p I$, this triangulation can be seen as a cobordism in two ways:

$$\overline{\mathcal{W}_{tube}} \times_p I : \overline{\mathcal{W}_{tube}} \to \overline{\mathcal{W}_{tube}} \tag{10.34}$$

$$\overline{\mathcal{W}_{tube}} \times_p I : \mathcal{W}_{tube} \to \mathcal{W}_{tube}. \tag{10.35}$$

Using this correspondence it follows the matrix elements satisfy the relation:

$$\langle h^*| \, \mathcal{Z}[\overline{\mathcal{W}_{tube}} \times_p I] \, |g^*\rangle = \langle g| \, \mathcal{Z}[\overline{\mathcal{W}_{tube}} \times_p I] \, |h\rangle \tag{10.36}$$

for any pair of basis elements $|g\rangle, |h\rangle \in \mathcal{H}[\mathcal{W}_{tube}]$. Additionally, taking the orientation reversal of $\overline{\mathcal{W}_{tube}} \times_p I$ we find a cobordism $\overline{\overline{\mathcal{W}_{tube}} \times_p I} : \mathcal{W}_{tube} \to \mathcal{W}_{tube}$ such that $\mathcal{Z}[\overline{\overline{\mathcal{W}_{tube}} \times_p I}] = \mathcal{Z}[\mathcal{W}_{tube} \times_p I]$. From this identification and equation (10.36), for a unitary ssTQFT it follows:

$$\langle h^*| \, \mathcal{Z}[\overline{\mathcal{W}_{tube}} \times_p I] \, |g^*\rangle = \overline{\langle h| \, \mathcal{Z}[\mathcal{W}_{tube} \times_p I] \, |g\rangle} \tag{10.37}$$

for any pair of basis elements $|g\rangle, |h\rangle \in \mathcal{H}[\mathcal{W}_{tube}]$. From equation (10.37) it directly follows:

$$|v\rangle^* \in \mathcal{H}[\overline{\mathcal{W}_{tube}}]_0, \quad \forall \, |v\rangle \in \mathcal{H}[\mathcal{W}_{tube}]_0 \tag{10.38}$$

$$|v\rangle \in \mathcal{H}[\mathcal{W}_{tube}]_0, \quad \forall \, |v\rangle^* \in \mathcal{H}[\overline{\mathcal{W}_{tube}}]_0. \tag{10.39}$$

Using the results above the $*$-map can be lifted to an involution on $\mathcal{H}[\mathcal{W}_{tube}]_0$ by noting for all $|g^*\rangle \in \mathcal{H}[\overline{\mathcal{W}_{tube}}]_0$ we can can define a triangulated pinched interval cobordism $\mathcal{O} : \overline{\mathcal{W}_{tube}} \to \mathcal{W}_{tube}$ forming an isomorphism $\mathcal{H}[\overline{\mathcal{W}_{tube}}]_0 \cong \mathcal{H}[\mathcal{W}_{tube}]_0$. The corresponding operator $\mathcal{Z}[\mathcal{O}] : \mathcal{H}[\overline{\mathcal{W}_{tube}}]_0 \to \mathcal{H}[\mathcal{W}_{tube}]_0$ can be expressed using the gluing cobordism matrix elements via:

$$\mathcal{Z}[\mathcal{O}] = \sum_{\{|g\rangle\},\{|g'\rangle\},\{|h\rangle\},\{|k\rangle\}} \mathcal{Z}^k_{g\otimes h} \mathcal{Z}^h_{g'\otimes k} \, |g'\rangle \, \langle g^*| \tag{10.40}$$

where the summation is over $\{|i\rangle\}$ for $i \in \{g, g', h, k\}$ is a summation over a complete, orthonormal basis for $\mathcal{H}[\mathcal{W}_{tube}]_0$. This operator obeys the relations:

$$\mathcal{Z}[\mathcal{O}]\mathcal{Z}[\mathcal{O}]^\dagger = \mathcal{Z}[\mathcal{W}_{tube} \times_p I], \quad \mathcal{Z}[\mathcal{O}]^\dagger \mathcal{Z}[\mathcal{O}] = \mathcal{Z}[\overline{\mathcal{W}_{tube}} \times_p I] \tag{10.41}$$

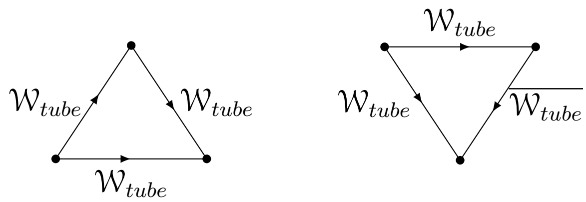
ensuring it defines an isomorphism between the two Hilbert spaces. Composing the star relation previously with $\mathcal{Z}[\mathcal{O}]$ defines an involution $* : \mathcal{H}[\mathcal{W}_{tube}]_0 \to \mathcal{H}[\mathcal{W}_{tube}]_0$.

In order to for the above involution to define a $*$-structure on the tube algebra we additionally need to verify the relation $(|g\rangle \, |h\rangle)^* = |h^*\rangle \, |g^*\rangle$ for all $|g\rangle, |h\rangle \in$

$\mathcal{H}[\mathcal{W}_{tube}]_0$. To show this relation we begin by making the following observations: Given the triangulation $GL_{\mathcal{W}}$ from the previous section, we can consider the same triangulation defining the following pinched interval cobordisms:

$$\begin{aligned}
GL_{\mathcal{W}} &: \mathcal{W}_{tube} \cup \mathcal{W}_{tube} \to \mathcal{W}_{tube} \\
GL_{\mathcal{W}} &: \mathcal{W}_{tube} \to \mathcal{W}_{tube} \cup \overline{\mathcal{W}_{tube}} \\
GL_{\mathcal{W}} &: \overline{\mathcal{W}_{tube}} \cup \mathcal{W}_{tube} \to \mathcal{W}_{tube} \\
GL_{\mathcal{W}} &: \overline{\mathcal{W}_{tube}} \to \overline{\mathcal{W}_{tube}} \cup \overline{\mathcal{W}_{tube}} \\
GL_{\mathcal{W}} &: \mathcal{W}_{tube} \to \overline{\mathcal{W}_{tube}} \cup \mathcal{W}_{tube} \\
GL_{\mathcal{W}} &: \mathcal{W}_{tube} \cup \overline{\mathcal{W}_{tube}} \to \overline{\mathcal{W}_{tube}}.
\end{aligned} \tag{10.42}$$

These relations can be visualised by considering $\mathcal{W}_{tube}$ by the line element such that $GL_{\mathcal{W}}$ is given by a triangle and considering the rotations by $\frac{\pi}{3}$ changing the source and targets of $GL_{\mathcal{W}}$. For example the first two relations can be visualised via:



(10.43)

where the cobordism is defined from the top to the bottom of the triangle. Defining matrix elements of $\mathcal{Z}[GL_{\mathcal{W}}]$ such that the subscript defines the configuration of the source and the superscript the target configuration we find the following relations:

$$\mathcal{Z}[GL_{\mathcal{W}}]^{k}_{g\otimes h} = \mathcal{Z}[GL_{\mathcal{W}}]^{k\otimes h^*}_{g} = \mathcal{Z}[GL_{\mathcal{W}}]^{h^*}_{k^*\otimes g} = \mathcal{Z}[GL_{\mathcal{W}}]^{h^*\otimes g^*}_{k^*} = \mathcal{Z}[GL_{\mathcal{W}}]^{g^*}_{h\otimes k^*} \tag{10.44}$$

for all $|g\rangle, |h\rangle, |k\rangle \in \mathcal{H}[\mathcal{W}_{tube}]$. Additionally we can exchange the source and target configurations in the matrix element by taking the complex conjugate eg:

$$\mathcal{Z}[GL_{\mathcal{W}}]^{k}_{g\otimes h} = \overline{\mathcal{Z}[GL_{\mathcal{W}}]^{g\otimes h}_{k}} = \mathcal{Z}[\overline{GL_{\mathcal{W}}}]^{g\otimes h}_{k}. \tag{10.45}$$

which follows as a direct consequence of unitarity of the ssTQFT by the relation $\mathcal{Z}[\mathcal{Y}]^\dagger = \mathcal{Z}[\overline{\mathcal{Y}}]$ for any triangulated $n+1$-cobordism $\mathcal{Y}$.

Using these relations we can verify:

$$
\begin{aligned}
(|g\rangle\,|h\rangle)^* = (\sum_{\{|k\rangle\}} \mathcal{Z}[GL_{\mathcal{W}}]^{k}_{g\otimes h}\,|k\rangle)^* = \sum_{\{|k\rangle\}} \overline{\mathcal{Z}[GL_{\mathcal{W}}]^{k}_{g\otimes h}}\,|k^*\rangle \\
\sum_{\{|k\rangle\}} \mathcal{Z}[GL_{\mathcal{W}}]^{g\otimes h}_{k}\,|k^*\rangle = \sum_{\{|k\rangle\}} \mathcal{Z}[GL_{\mathcal{W}}]^{k^*}_{h^*\otimes g^*}\,|k^*\rangle = |h^*\rangle\,|g^*\rangle
\end{aligned}
\tag{10.46}
$$

for all $|g\rangle\,, |h\rangle \in \mathcal{H}[\mathcal{W}_{tube}]_0$. From this relation and the previous we have established that the tube algebra define the $*$-algebra structure of definition 10.2.1.

To conclude that the tube algebra is semisimple we need to verify the inner product on $\mathcal{H}[\mathcal{W}_{tube}]_0$ obeys the relation:

$$
\langle k|gh\rangle = \langle kh^*|g\rangle \tag{10.47}
$$

for all $|g\rangle\,, |h\rangle\,, |k\rangle \in \mathcal{H}[\mathcal{W}_{tube}]_0$. Using the inner product defined in section 4.4 and unitarity it directly follows:

$$
\langle k|gh\rangle = \mathcal{Z}[GL_{\mathcal{W}}]^{k}_{g\otimes h} = \mathcal{Z}[GL_{\mathcal{W}}]^{k\otimes h^*}_{g} = \langle kh^*|g\rangle\,. \tag{10.48}
$$

Semisimplicity of the tube algebra has the consequence that for all unitary state-sum TQFT's the number of irreducible topological excitations is finite. Additionally, any composite topological excitation type can be understood in terms of direct sums of irreducible topological excitation types.

10.3 Morita Equivalence

In the previous section we defined the $\mathcal{W}$-tube algebra for a fixed triangulation $\mathcal{W}$. In general given a closed, oriented $n-1$-manifold W, there is no canonical choice for a triangulation $\mathcal{W}$, instead we usually find ourselves making a choice which serves only to simplify computations. The problem of such a freedom is that in general there is no unique tube algebra over W but instead a class of algebras, one for each choice of triangulation. This problem is particularly worrying in our formalism as we wish to classify topological excitation types which we are assuming have properties invariant under changes of length scale and we would expect to be invariant under a choice of triangulation $\mathcal{W}$. It is immediate that the class of W algebras cannot be given by the equivalence of isomorphism.

This is because the dimension of the tube algebra increases with the number of boundary configurations which is proportional to the number of simplices in the triangulation of the boundary. Instead the appropriate equivalence class of algebras is given by **Morita equivalence**, a good reference is [93].

Theorem 10.3.1. Two algebras A and B are Morita equivalent if and only if there exists an $A-B$-bimodule P and a $B-A$-bimodule Q such that $P \otimes_B Q \simeq A$ and $Q \otimes_A P \simeq B$.

The heart of this theorem lies in the definition of the tensor product of bimodules over the algebra itself rather than over a field as is often the case in physics. Such a tensor product (among others) additionally requires the constraint:

$$\begin{aligned} P \cdot b \otimes_B Q &= P \otimes_B b \cdot Q \quad \forall b \in B \\ Q \cdot a \otimes_A P &= Q \otimes_A a \cdot P \quad \forall a \in A \end{aligned} \tag{10.49}$$

Theorem 10.3.2. Given a closed, oriented $n-1$-manifold W and a pair of triangulations $\mathcal{W}, \mathcal{W}'$ of W, let ${}_{\mathcal{W}}Q_{\mathcal{W}'}$ denote a triangulation of $W \times I$ such that $W \times 0$ has triangulation $\mathcal{W}$ and $W \times 1$ has triangulation $\mathcal{W}'$. Similarly we define ${}_{\mathcal{W}'}Q_{\mathcal{W}}$ by the conditions $W \times 0$ has triangulation $\mathcal{W}'$ and W has triangulation $\mathcal{W}$. Then:

$$\begin{aligned} \mathcal{H}[{}_{\mathcal{W}}Q_{\mathcal{W}'}]_0 \otimes_{\mathcal{W}'-\mathrm{tube}} \mathcal{H}[{}_{\mathcal{W}'}Q_{\mathcal{W}}]_0 &\simeq \mathcal{H}[\mathcal{W}_{tube}]_0 \\ \mathcal{H}[{}_{\mathcal{W}'}Q_{\mathcal{W}}]_0 \otimes_{\mathcal{W}-\mathrm{tube}} \mathcal{H}[{}_{W}Q_{\mathcal{W}'}]_0 &\simeq \mathcal{H}[\mathcal{W}'_{tube}]_0 \end{aligned} \tag{10.50}$$

Where $\otimes_{\mathcal{W}'-\mathrm{tube}}/\otimes_{\mathcal{W}-\mathrm{tube}}$ is the tensor product over the $\mathcal{W}'-\mathrm{tube}/\mathcal{W}-\mathrm{tube}$ algebra.

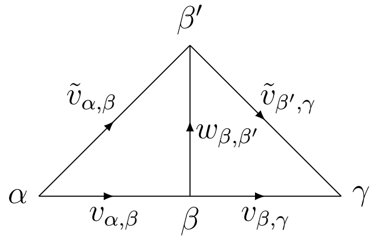
Proof. Given $|v_{\alpha,\beta}\rangle, |v_{\beta,\gamma}\rangle \in \mathcal{H}[\mathcal{W}_{tube}]_0$ we can naturally make the identification $|v_{\alpha,\beta}\rangle \otimes |v_{\beta,\gamma}\rangle \in \mathcal{H}[\mathcal{W}_{tube} \cup_{\mathcal{W}} \mathcal{W}_{tube}]$. In order for such elements to be in the groundstate subspace $\mathcal{H}[\mathcal{W}_{tube} \cup_{\mathcal{W}} \mathcal{W}_{tube}]_0$ they additionally have to satisfy the constraint:

$$\prod_{i \in \Delta^0(\mathcal{W} \times 1)} H_i \, |v_{\alpha,\beta}\rangle \otimes |v_{\beta,\gamma}\rangle = |v_{\alpha,\beta}\rangle \otimes |v_{\beta,\gamma}\rangle \,. \tag{10.51}$$

as by definition they are already groundstates away from the boundary. Using the matrix elements of $\mathcal{Z}[GL_{\mathcal{W}}]$ we can write down such an operator via:

$$\prod_{i\in\Delta^0(\mathcal{W}\times 1)} H_i = \sum_{\substack{w_{\beta,\beta'}\\ v_{\alpha,\beta},\tilde{v}_{\alpha,\beta'}\\ v_{\beta,\gamma},\tilde{v}_{\beta',\gamma}}} \mathcal{Z}[GL_{\mathcal{W}}]^{\tilde{v}_{\alpha,\beta'}}_{v_{\alpha,\beta}\otimes w_{\beta,\beta'}} \overline{\mathcal{Z}[GL_{\mathcal{W}}]^{v_{\beta,\gamma}}_{w_{\beta,\beta'}\otimes\tilde{v}_{\beta',\gamma}}} \left|\tilde{v}_{\alpha,\beta'}\right\rangle \left\langle v_{\alpha,\beta}\right| \otimes \left|\tilde{v}_{\beta',\gamma}\right\rangle \left\langle v_{\beta,\gamma}\right| \tag{10.52}$$

Intuitively this expression can be seen by considering $\mathcal{W}_{tube}$ as line segments and considering the pinched cobordism as below:



(10.53)

Looking at the matrix elements we see they can be interpreted as the structure coefficients of the tube algebra such that:

$$\begin{aligned} \sum_{\tilde{v}_{\alpha,\beta'}} \mathcal{Z}[GL_{\mathcal{W}}]^{\tilde{v}_{\alpha,\beta'}}_{v_{\alpha,\beta}\otimes w_{\beta,\beta'}} \left|\tilde{v}_{\alpha,\beta'}\right\rangle &= \left|v_{\alpha,\beta}\right\rangle \left|w_{\beta,\beta'}\right\rangle \\ \sum_{v_{\beta,\gamma}} \mathcal{Z}[GL_{\mathcal{W}}]^{v_{\beta,\gamma}}_{w_{\beta,\beta'}\otimes\tilde{v}_{\beta',\gamma}} \left|v_{\beta,\gamma}\right\rangle &= \left|w_{\beta,\beta'}\right\rangle \left|\tilde{v}_{\beta',\gamma}\right\rangle \end{aligned} \tag{10.54}$$

for all $\left|w_{\beta,\beta'}\right\rangle \in \mathcal{H}[\mathcal{W}_{tube};\beta,\beta']$. In this way we can rewrite the groundstate projector on the boundary in terms of the algebra product:

$$\prod_{i\in\Delta^0(\mathcal{W}\times 1)} H_i = \sum_{\substack{w_{\beta,\beta'}\\ v_{\alpha,\beta}\\ \tilde{v}_{\beta',\gamma}}} \left[\left|v_{\alpha,\beta}\right\rangle \left|w_{\beta,\beta'}\right\rangle\right] \left\langle v_{\alpha,\beta}\right| \otimes \left|\tilde{v}_{\beta'\gamma}\right\rangle \left[\left\langle w_{\beta',\beta}\right| \left\langle\tilde{v}_{\beta',\gamma}\right|\right]. \tag{10.55}$$

The summation over $\left|v_{\beta,\gamma}\right\rangle$ and $\left|\tilde{v}_{\alpha,\beta'}\right\rangle$ are no longer needed as they are fixed by the other elements. Having written down the groundstate projector on the boundary in this form we see that this defines a projector exactly into the subspace

$$\mathcal{H}[\mathcal{W}_{tube}]_0 \otimes_{\mathcal{W}-\text{tube}} \mathcal{H}[\mathcal{W}_{tube}]_0 = \mathcal{H}[\mathcal{W}_{tube} \cup_{\mathcal{W}} \mathcal{W}_{tube}]_0. \tag{10.56}$$

Additionally because this subspace satisfies (10.51) and the groundstate projector away from the boundary we can define a mutation operator such that:

$$\mathcal{H}[\mathcal{W}_{tube}]_0 \otimes_{\mathcal{W}-\text{tube}} \mathcal{H}[\mathcal{W}_{tube}]_0 \cong \mathcal{H}[\mathcal{W}_{tube}]_0 \tag{10.57}$$

Without loss of generality we can assume ${}_{\mathcal{W}}Q_{\mathcal{W}'}$ has the form of $\mathcal{W}_{tube}/\mathcal{W}'_{tube}$ in a local neighbourhood of each boundary and similarly for ${}_{\mathcal{W}'}Q_{\mathcal{W}}$ using a mutation operator. In this way the same result can be applied to the case of gluing bimodules along a boundary. □

Corollary 10.3.2.1. Given a closed, oriented $n-1$-manifold W, for any pair of triangulations $\mathcal{W}, \mathcal{W}'$ of W, $\mathcal{W}$-tube is Morita equivalent to $\mathcal{W}'$-tube.

This corollary is outlined in [86] however no proof existed in the literature to the authors knowledge, although the result was widely believed.

Lemma 10.3.3. The modules of Morita equivalent algebras are in one-one correspondence

From this lemma (see [93] for a proof) we can take any choice of triangulation of $\mathcal{W}$ and classify the topological excitation types knowing that such modules will be in one-one correspondence for any other choice of triangulation.

In section 4.3 it was noted that for a spatial manifold X with triangulation $\mathcal{X}$, the dimension of the state-space $\mathcal{H}[\mathcal{X}]_0$ defined a triangulation independent and hence topological invariant for the manifold X when $\partial X = \emptyset$ but when $\partial X \neq \emptyset$ the dimension depended on the choice of triangulation of ∂X. Using the discussion of Mortia equivalence and the semisimplicity of the tube algebra, given a spatial manifold X with non-empty boundary we can decompose $\mathcal{H}[\mathcal{X}]_0$ in terms of simple bimodules of the boundary tube algebras. In particular semi-simplicity implies there are finite such simple bimodules. As any other choice of boundary triangulations defines a Mortia equivalent tube algebra for the boundaries and the number of simple bimodules for are in one-one correspondence it follows that the number of simple bimodules of the boundary tube algebras given by $\mathcal{H}[\mathcal{X}]_0$ defines a quantity invariant under mutation of the triangulation in the bulk and boundary and hence defines a topologically invariant quantity to X.

10.4 Centre of the Tube Algebra

Given a closed, oriented $n-1$-manifold W with triangulation $\mathcal{W}$ we will now show the centre $Z[\mathcal{W}-\text{tube}]$ corresponds $\mathcal{H}[\mathcal{W}\times S^1]_0$.

Let $\mathcal{W}_{S^1}$ be a triangulation of $M\times S^1$ induced from $\mathcal{W}_{tube}$ by identifying $\mathcal{W}\times 0=\mathcal{W}\times 1$, with associated Hilbert space $\mathcal{H}[\mathcal{W}_{S^1}]$. There is an immediate observation that:

$$\mathcal{H}[\mathcal{W}_{S^1}]=\oplus_{\alpha\in s(\mathcal{W})}\mathcal{H}[\mathcal{W}_{tube};\alpha,\alpha] \tag{10.58}$$

by considering $\mathcal{H}[\mathcal{W}_{S^1}]$ as the subspace of $\mathcal{H}[\mathcal{W}_{tube}]$ such that both boundary configurations are identified. Additionally we find

$$\mathcal{H}[\mathcal{W}_{S^1}]_0\subseteq\oplus_{\alpha\in s(\mathcal{W})}\mathcal{H}[\mathcal{W}_{tube};\alpha,\alpha]_0. \tag{10.59}$$

The reason that $\mathcal{H}[\mathcal{W}_{S^1}]_0$ is subspace and not equal to $\oplus_{\alpha\in s(\mathcal{W})}\mathcal{H}[\mathcal{W}_{tube};\alpha,\alpha]_0$ is that a state in $|v_{\alpha,\alpha}\rangle\in\oplus_{\alpha\in s(\mathcal{W})}\mathcal{H}[\mathcal{W}_{tube};\alpha,\alpha]_0$ may not satisfy

$$\Big(\prod_{i\in\Delta^0(\mathcal{W}\times 0)}H_i\Big)\,|v_{\alpha,\alpha}\rangle=|v_{\alpha,\alpha}\rangle \tag{10.60}$$

even if it is a ground state in the compliment and hence is not necessarily an element of $\mathcal{H}[\mathcal{W}_{S^1}]_0$.

Theorem 10.4.1.

$$\mathcal{H}[\mathcal{W}_{S^1}]_0=Z(\mathcal{W}-\text{tube}) \tag{10.61}$$

Where $Z(\mathcal{W}-\text{tube})$ is the center of the $\mathcal{W}$-tube algebra:

$$Z(\mathcal{W}-\text{tube})=\{|v\rangle\in\mathcal{H}[\mathcal{W}_{tube}]_0\quad|\quad|w\rangle\,|v\rangle=|v\rangle\,|w\rangle\;\forall\,|w\rangle\in\mathcal{H}[\mathcal{W}_{tube}]\} \tag{10.62}$$

Proof. Follows directly from theorem 10.3.2. □

As the tube algebra is finite and semisimple it follows from simple representation theoretic arguments (see appendix B.3) that there is a one-one correspondence with simple modules and a basis for the center. In particular this implies a one-one correspondence with groundstates of W_{S^1} and simple modules of W-tube such that:

Corollary 10.4.1.1. For any tube algebra $\mathcal{W}$-tube

$$dim\mathcal{H}[W_{S^1}]_0 = \text{number of simple modules of } W\text{-tube} \tag{10.63}$$

One example of this result is the observation the number of simple topological excitations in 2+1D is in one-one correspondence with the groundstate degeneracy of the torus T^2.

Furthermore we can gain additional insight onto the centre of tube algebras by thinking of the Morita equivalence class of tube algebras.

Lemma 10.4.2. Given a pair of Morita equivalent algebras A and B, $Z(A) \simeq Z(B)$.

As such for a pair of triangulations W, W' of $\mathcal{W}$ we can consider the isomorphism

$$\mathcal{H}[W_{S^1}]_0 \simeq \mathcal{H}[W'_{S^1}]_0 \tag{10.64}$$

arising from PL-homeomorphisms between W_{S^1} and W'_{S^1} or from the Mortia equivalence of W-tube and W'-tube.

Chapter 11

Twisted Groupoid-Like Tube Algebras

In the following discussion our prototypical example of tube algebras will be given by a set of algebras we call **twisted groupoid-like algebras**.

The structure of this chapter is to first introduce twisted representations of finite groups and their character theory. Building on the theory of twisted group representations we will then introduce twisted groupoid algebras and show that the representation of such algebras can be constructed in terms of twisted group representations. We then define the notion of twisted groupoid-like algebras and classify the simple modules in terms of the representation theory of twisted groupoid algebras. In the final sections of this chapter we will consider the generic features of tube algebras given by twisted groupoid-like algebras.

11.1 Twisted Representations of Finite Groups

In the section we provide some key results about twisted representations of finite groups (sometimes referred to as projective representations) which we will generalise in the subsequent section to the case of twisted representations of finite groupoids. A key reference for the theory of twisted group algebras is [94].

Definition 11.1.1. Let G be a finite group and $\beta \in H^2(G, U(1))$ a normalised

2-cocycle, such that $\beta : G \times G \to U(1)$ is a $U(1)-$valued function satisfying:

$$\begin{aligned}
&\frac{\beta(g,h)\beta(gh,k)}{\beta(g,hk)\beta(h,k)} = 1 \quad \forall g,h,k \in G \\
&\beta(g,e) = \beta(e,g) = 1 \quad \forall g \in G \\
&\beta(g,g^{-1}) = \beta(g^{-1},g) \quad \forall g \in G.
\end{aligned} \tag{11.1}$$

A $\beta-$**twisted representation** (ρ, V) of G for a vector space V, is a homomorphism

$$\rho : G \to \mathrm{End}(V)$$

satisfying:

$$\begin{aligned}
\rho(g)\rho(h) &= \beta(g,h)\rho(gh) \quad \forall g,h \in G \\
\rho(e) &= \mathbb{1}
\end{aligned} \tag{11.2}$$

In the limiting case that $\beta(g,h) = 1$ for all $g,h \in G$, ρ reduces to an ordinary representation of G. Analogously to the an ordinary group representation a $\beta-$twisted representation can alternatively be viewed as a representation of the **twisted group algebra** ${}^{\beta}\mathbb{C}G$.

Definition 11.1.2. Let G be a finite group and $\beta \in H^2(G, U(1))$ a normalised 2-cocycle, the twisted group algebra ${}^{\beta}\mathbb{C}G$ is the $\mathbb{C}-$algebra $\mathbb{C}\{|g\rangle\}_{\forall g \in G}$ with multiplication:

$$|g\rangle\,|h\rangle = \beta(g,h)\,|gh\rangle\,.$$

The $2-$cocycle condition ensures ${}^{\beta}\mathbb{C}G$ is associative:

$$|g\rangle\,(|h\rangle\,|k\rangle) = \beta(g,hk)\beta(h,k)\,|ghk\rangle = \beta(g,h)\beta(gh,k)\,|ghk\rangle = (|g\rangle\,|h\rangle)\,|k\rangle\,. \tag{11.3}$$

Akin to the case for ordinary group representations it can be shown ${}^{\beta}\mathbb{C}G$ is a semi-simple algebra such that every representation can be written as the direct sum of irreducible representations. Additionally given a $\beta-$twisted representation of G, (ρ, V) with any inner product, the representation is unitarisable, such that there always exists a new inner product whereby

$$< v, w >=< \rho(g)v, \rho(g)w > \quad \forall v,w \in V, \quad \forall g \in G. \tag{11.4}$$

This implies the existence of a presentation of $\rho(g)$ as a unitary matrix for all ρ and $g \in G$ such that $\rho(g)^\dagger\rho(g) = 1_V = \rho(g)\rho(g)^\dagger$.

Let $\{(\rho_i, V_i)\}$ denote the set of unitary irreducible representations of ${}^\beta\mathbb{C}G$ up to isomorphism and $D^{\rho_i}(g)$ the matrix presentation of $\rho_i(g)$, the representation matrices satisfy the following conditions:

$$\begin{aligned}
&\sum_n D^{\rho_i}_{mn}(g)D^{\rho_i}_{no}(h) = \beta(g,h)D^{\rho_i}_{mo}(gh)\\
&\overline{D^{\rho_i}_{mn}(g)} = \frac{1}{\beta(g,g^{-1})}D^{\rho_i}_{nm}(g^{-1})\\
&\frac{1}{|G|}\sum_{g\in G} D^{\rho_i}_{mn}(g)\overline{D^{\rho_j}_{m'n'}(g)} = \frac{\delta_{\rho_i,\rho_j}}{d_{\rho_i}}\delta_{m,m'}\delta_{n,n'}\\
&\frac{1}{|G|}\sum_{\{\rho_i\}}\sum_{m,n} d_{\rho_i}D^{\rho_i}_{mn}(g)\overline{D^{\rho_i}_{mn}(g')} = \delta_{g',g'}
\end{aligned} \tag{11.5}$$

for all $g, h \in G$ and d_{ρ_i} :=dim(V_i).

11.1.1 Character Theory

It is well known in the case of finite groups that representations of the group algebra $\mathbb{C}G$ are classified up to equivalence by their characters. The character $\chi^\rho(g)$ of a group element $g \in G$ in the representation ρ is given by the trace of the matrix $\chi^\rho(g) := Tr\rho(g)$. For ordinary group representations it is known that the characters are invariant under the action of conjugation such that $\chi^\rho(g) = \chi^\rho(h^{-1}gh)$ for all $g, h \in G$. In the case of the twisted group algebra ${}^\beta\mathbb{C}G$ an analogous statement holds however the characters are instead invariant under the following conjugation relation.

Lemma 11.1.1. For $\rho : G \to End(V)$ a $\beta-$twisted representation of G, for all $g, h \in G$ the following conjugation holds:

$$\chi^\rho(h^{-1}gh) = \frac{\beta(h, h^{-1}gh)}{\beta(g,h)}\chi^\rho(g) \tag{11.6}$$

Proof. From the definition of ρ

$$\rho(h^{-1}gh) = \beta^{-1}(g,h)\beta^{-1}(h^{-1},gh)\rho(h^{-1})\rho(g)\rho(h) \tag{11.7}$$

Applying

$$\begin{aligned}\rho(h)\rho(h)^{-1} = \mathbb{1} = \rho(hh^{-1}) = \beta(h,h^{-1})^{-1}\rho(h)\rho(h^{-1})\\ \implies \beta(h,h^{-1})\rho(h)^{-1} = \rho(h^{-1})\end{aligned} \tag{11.8}$$

we find

$$\rho(h^{-1}gh) = \frac{\beta(h,h^{-1})}{\beta(g,h)\beta(h^{-1},gh)}\rho(h)^{-1}\rho(g)\rho(h). \tag{11.9}$$

Finally using the 2-cocycle condition arising from the triple $(h^{-1},h,h^{-1}gh)$

$$\frac{\beta(h^{-1},h)}{\beta(h^{-1},gh)} = \beta(h,h^{-1}gh) \tag{11.10}$$

such that

$$\rho(h^{-1}gh) = \frac{\beta(h,h^{-1}gh)}{\beta(g,h)}\rho(h)^{-1}\rho(g)\rho(h) \tag{11.11}$$

and taking the trace on both sides gives the desired result. □

We call characters which satisfy the above relation **β-twisted characters**. β-twisted characters satisfy the following conditions which follow from equation (11.5):

$$\begin{aligned}\frac{1}{|G|}\sum_{g\in G}\chi_{\rho_i}(g)\overline{\chi_{\rho_j}(g)} = \delta_{\rho_i,\rho_j}\\ \sum_{\{\rho_i\}}\chi_{\rho_i}(g)\overline{\chi_{\rho_i}(h)} = \begin{cases}\frac{|G|}{|C^A|} & \text{if } g,h\in C^A\\ 0 & \text{else}\end{cases}\end{aligned} \tag{11.12}$$

where C^A is a conjugacy class of G. It has been shown that representations of the finite $\beta-$twisted group are classified up to isomorphism by such characters [94].

11.2 Twisted Representations of Finite Groupoids

Following from the previous section we now introduce the twisted representation theory of finite groupoids (see section 3.2) building on the results of the previous section. The discussion largely follows from the beautifully written paper of Simon Willerton [95] which sparked the authors interest in twisted groupoid algebras.

Definition 11.2.1. Let Γ be a finite groupoid and $\Gamma_1 \times_c \Gamma_1 = \{(k,h) \in \Gamma_1 \times \Gamma_1 | t(k) = s(h)\}$ the space of composable morphisms in Γ_1 such that:

$$a \xrightarrow{\ k\ } b \xrightarrow{\ h\ } c \ = \ a \xrightarrow{\ kh\ } c \tag{11.13}$$

a **normalised groupoid 2-cocycle** $\beta \in H^2(\Gamma, U(1))$ is $U(1)$ valued function $\beta : \Gamma_1 \times_c \Gamma_1 \to U(1)$ such that the following hold:

$$\begin{aligned}
&\beta_{s(g)}(g, 1_{t(g)}) = \beta_{s(g)}(1_{s(g)}, g) = 1\\
&\frac{\beta_{s(h)}(h,g)\beta_{s(k)}(k,hg)}{\beta_{s(k)}(kh,g)\beta_{s(k)}(k,h)} = 1\\
&\beta_{s(g)}(g, g^{-1}) = \beta_{t(g)}(g^{-1}, g).
\end{aligned} \tag{11.14}$$

for all $g, h, k \in \Gamma_1$ where $t(g) = s(h)$ and $t(h) = s(k)$.

Using the definition of the normalised groupoid 2−cocycle β we can define the **twisted groupoid algebra** ${}^{\beta}\mathbb{C}\Gamma$.

Definition 11.2.2. Let Γ be a finite groupoid Γ and $\beta \in H^2(\Gamma, U(1))$ a normalised groupoid 2-cocyle, the **twisted groupoid algebra** ${}^{\beta}\mathbb{C}\Gamma$, is the $\mathbb{C}-$algebra $\mathbb{C}\{|g\rangle\}_{g\in\Gamma_1}$ with multiplication:

$$|g\rangle\,|h\rangle = \beta_{s(g)}(g,h)\,|gh\rangle\,\delta_{t(g),s(h)}. \tag{11.15}$$

Analogously to the twisted group algebra ${}^{\beta}\mathbb{C}G$, the groupoid 2−cocycle condition ensures ${}^{\beta}\mathbb{C}\Gamma$ is an associative algebra.

Definition 11.2.3. Let Γ be a finite groupoid Γ and $\beta \in H^2(\Gamma, U(1))$ a normalised groupoid 2-cocyle, a **β-twisted representation of** Γ is a representation (ρ, V) for a vector space V of ${}^{\beta}\mathbb{C}\Gamma$. Requiring $\rho : {}^{\beta}\mathbb{C}\Gamma \to \mathrm{End}(V)$ to define a homomorphism implies:

$$\rho(g)\rho(h) = \beta_{s(g)}(g,h)\rho(gh)\delta_{s(h),t(g)} \quad \forall g, h \in \Gamma_1. \tag{11.16}$$

It has been shown that the finite twisted groupoid algebra is a semi-simple algebra [95] with the proof mirroring the analogous proof for group algebras.

In order to construct representations of ${}^{\beta}\mathbb{C}\Gamma$ we will use the definitions of connected component $\pi_0\Gamma$ of Γ and the stabiliser group $\pi_1(x)$ of an object $x \in \Gamma_0$, see definitions 3.2.3, 3.2.4 respectively. Using the set of connected components, the groupoid algebra can be decomposed into sub-algebras indexed by connected components of the groupoid such that ${}^{\beta}\mathbb{C}\Gamma = \oplus_{C\in\pi_0(\Gamma)}{}^{\beta}\mathbb{C}\Gamma_C$, where ${}^{\beta}\mathbb{C}\Gamma_C := \mathbb{C}\{g \in \Gamma_1 | s(g) \in C\}$. This follows from the fact: for two disjoint connected components $C, C' \in \pi_0(\Gamma)$, $|g\rangle\,|h\rangle = 0$ for all $|g\rangle \in {}^{\beta}\mathbb{C}\Gamma_C$ and $|h\rangle \in {}^{\beta}\mathbb{C}\Gamma_{C'}$. In this way we can find all representations of ${}^{\beta}\mathbb{C}\Gamma$ as direct sums of representations of ${}^{\beta}\mathbb{C}\Gamma_C$. We can now form representations of ${}^{\beta}\mathbb{C}\Gamma_C$ as follows:

Let $\{c_1, \cdots, c_{|C|}\} = C \in \pi_0(\Gamma)$ index the objects of the connected component C. For each $c_a \in C$ we define a single morphism $c_a \xrightarrow{q_a} c_1 \in \Gamma_C$ with the requirement $q_{c_1} = 1_{c_1}$ is the identity morphism of $c_1 \in C$. Let $\pi_1(c_1)$ be the stabiliser group of c_1 and $\beta_{c_1} \in H^2(\pi_1(c_1), U(1))$ the restriction of the groupoid $2-$cocycle β to the group $2-$cocycle over $\pi_1(c_1)$. Defining $R : \pi_1(c_1) \to \mathrm{End}(W)$ to be a unitary $\beta_{c_1}-$twisted group representation of $\pi_1(c_1)$. Letting $V = \mathbb{C}C \otimes W$ we can define the matrix presentation for a representation $F_{C,R} : {}^{\beta}\mathbb{C}\Gamma_C \to \mathrm{End}(V)$ as

$$\mathcal{D}^{C,R}_{am,bn}(k) = \delta_{s(k),c_a}\delta_{t(k),c_b}\frac{\beta_{c_1}(q_a^{-1}, k)}{\beta_{c_1}(q_a^{-1}kq_b, q_b^{-1})}D^R_{m,n}(q_a^{-1}kq_b). \tag{11.17}$$

Here the indices $a, b \in \{1, \cdots, |C|\}$ and $m, n \in \{1, \cdots, \dim(V)\}$. It is straightforward but cumbersome to verify such a matrix is a ${}^{\beta}\mathbb{C}\Gamma$ homomorphism, such that:

$$\sum_{b,n} \mathcal{D}^{C,R}_{am,bn}(k)\mathcal{D}^{C,R}_{bn,co}(k') = \beta_{s(k)}(k, k')\mathcal{D}^{C,R}_{am,co}(kk') \tag{11.18}$$

using the groupoid $2-$cocycle relation. Furthermore if the representation R is an irreducible $\beta_{c_1}-$twisted representation of $\pi_1(c_1)$ it follows that $F_{C,R}$ is an irreducible representation of ${}^{\beta}\mathbb{C}\Gamma$. It follows from lemma 3.2.1 the representations are unitarily equivalent for any choice of $c_i \in C \in \pi_0(\Gamma)$ in the definition of $\pi_1(c_i)$ and non-canonical choice of morphisms $q_i \in \Gamma_1$.

The representations acts on $\mathbb{C}C \otimes W$ $=\mathrm{span}_{\mathbb{C}}\{|c_i, v_n\rangle\,|i \in \{1, \cdots, |C|\}, n \in \{1, \cdots, |d_R|\}\}$ as follows:

$$\sum_{b,n} \mathcal{D}^{C,R}_{am,bn}(k)\,|c_i, v_o\rangle = \frac{\beta_{c_1}(q_a^{-1}, k)}{\beta_{c_1}(q_a^{-1}kq_i, q_i^{-1})}\,|c_i, D^R_{mo}(q_a^{-1}kq_i)v_o\rangle\,\delta_{s(k),c_a}\delta_{t(k),c_i} \tag{11.19}$$

Unitarity of the representation R gives rise to unitarity of $F_{C,R}$ such that the representations satisfy the following conjugation relation:

$$\overline{\mathcal{D}^{C,R}_{am,bn}(k)} = \frac{1}{\beta_{s(k)}(k,k^{-1})}\mathcal{D}^{C,R}_{bn,am}(k^{-1}). \tag{11.20}$$

Definition 11.2.4. In the following we will use the notation:

$$(F_{C,R}, V) := (C,R) \tag{11.21}$$

and denote the set of irreducible representations of ${}^{\beta}\mathbb{C}\Gamma$ up to isomorphism via:

$$\{(C,R)\}. \tag{11.22}$$

The dimension of the represention (C,R) can be conveniently expressed as follows

$$\dim(C,R) = d_R|C| := d_{C,R} \tag{11.23}$$

Definition 11.2.5. Let Γ be a finite groupoid and $x \in \Gamma_0$ an object, we define the set of morphisms with source x via:

$$M(x) := \{g \in \Gamma_1 | s(g) = x\} \tag{11.24}$$

Proposition 11.2.1. Let Γ be a finite groupoid and $C \in \pi_0(\Gamma)$ a connected component, for all $x, y \in C$, $|M(x)| = |M(y)| := |M(C)|$.

Proof. Follows analogously to prop 3.2.1. □

Using the previous definition we will now express two useful relations for twisted groupoid representations

$$\sum_{\{C,R\}}\sum_{\substack{a,n\\b,m}} d_{C,R}\mathcal{D}^{C,R}_{am,bn}(k)\overline{\mathcal{D}^{C,R}_{am,bn}(k')} = \delta_{k,k'}|M_{s(k)}| \tag{11.25}$$

$$\sum_{k\in\Gamma_1}\mathcal{D}^{C,R}_{am,bn}(k)\overline{\mathcal{D}^{C',R'}_{a'm',b'n'}(k)} = \frac{|M_C|}{d_{C,R}}\delta_{C,C'}\delta_{R,R'}\delta_{a,a'}\delta_{b,b'}\delta_{m,m'}\delta_{n,n'} \tag{11.26}$$

These relations follow directly from the definition of $\mathcal{D}^{C,R}_{am,bn}(k)$ and the corresponding relations for twisted group representations.

Definition 11.2.6. In the following given an irreducible representation (C,R) we will often use the notation I, J, K to denote the pair (a,m) where $a \in \{1, \cdots, |C|\}$ and $m \in \{1, \cdots, |R|\}$.

11.2.1 Character Theory

Analogously to twisted group representations, we can associate to twisted groupoid representations a character theory where:

$$\chi^{C,R}(k) := \sum_{a,m} \mathcal{D}^{C,R}_{am,am}(k) = \delta_{s(k),t(k)}\delta_{c_a,s(k)} \frac{\beta_{c_1}(q_a^{-1}, k)}{\beta_{c_1}(q_a^{-1} k q_a, q_a^{-1})} \chi^R(q_a^{-1} k q_a) \quad (11.27)$$

for all $k \in \Gamma_1$. Such characters satisfy a conjugation type relation generalising that of lemma 11.1.1 such that for all $k, x \in \Gamma_1$ where $s(x) = s(k) = t(k)$:

$$\chi^{C,R}(x^{-1} k x) = \frac{\beta_{s(k)}(k, x)}{\beta_{s(k)}(x, x^{-1} k x)} \chi^{C,R}(k). \quad (11.28)$$

The proof follows identically to lemma 11.1.1 while taking into account the source and target maps. A useful consequence of this result is that:

Proposition 11.2.2. For all representations (C, R) of the twisted groupoid algebra ${}^{\beta}\mathbb{C}\Gamma$, if there exists $k, x \in \Gamma_1$ such that $s(k) = t(k) = s(x)$ and $x^{-1} k x = k$, then $\chi^{C,R}(k) = 0$ if $\beta_{s(k)}(k, x) \neq \beta_{s(k)}(x, k)$.

Proof. Follows directly from conjugation property for such a pair $k, x \in \Gamma_1$:

$$\chi^{C,R}(k) = \frac{\beta_{s(k)}(k, x)}{\beta_{s(k)}(x, k)} \chi^{C,R}(k).$$

□

11.3 Twisted Groupoid-Like Algebras

In the following chapters it will be useful to define **twisted groupoid-like algebras** and their properties in terms of twisted groupoid algebras.

Definition 11.3.1. Given a twisted groupoid algebra ${}^{\beta}\mathbb{C}\Gamma$, there exists a corresponding **twisted groupoid-like algebra** ${}^{\beta}\mathbb{C}\widetilde{\Gamma}$ given by the $\mathbb{C}$-algebra over the vector space $\mathbb{C}\Gamma_1 := \{|g\rangle\}_{g \in \Gamma_1}$ with algebra product:

$$|g\rangle\,|h\rangle = \frac{\beta_{s(g)}(g, h)}{\sqrt{|M(s(g))|}}\,|gh\rangle\,\delta\_t(g), s(h) \quad \forall g, h \in \Gamma_1. \quad (11.29)$$

For all $|g\rangle \in \mathbb{C}\Gamma_1$ there exists a $*$-structure (see 10.2.1) given by:

$$|g\rangle^* := \frac{1}{\beta_{s(g)}(g, g^{-1})} |g^{-1}\rangle \in \mathbb{C}\Gamma_1 \tag{11.30}$$

such that,

$$\begin{aligned} |g\rangle^* |g\rangle &= \frac{1}{\sqrt{|M(s(g))|}} |1_{s(g)}\rangle \\ |g\rangle |g\rangle^* &= \frac{1}{\sqrt{|M(s(g))|}} |1_{t(g)}\rangle \end{aligned} \tag{11.31}$$

for all $g \in \Gamma_1$.

Remark 11.3.1. From the definition it follows that ${}^{\beta}\mathbb{C}\Gamma \cong {}^{\beta}\mathbb{C}\tilde{\Gamma}$. The existence of such an isomorphism guarantees that twisted groupoid-like algebras are semisimple.

In the next section we will utilise the representation matrices of twisted groupoid algebras to find a basis for twisted groupoid-like algebras to defining the simple modules.

11.4 Canonical Basis for Twisted Groupoid-Like Algebras

Following from the semisimplicity of twisted groupoid-like algebras in this chapter we construct an isomorphism between ${}^{\beta}\mathbb{C}\tilde{\Gamma}$ and the direct sum of irreducible β-twisted representations of Γ. We call this basis the **canonical basis** of ${}^{\beta}\mathbb{C}\tilde{\Gamma}$ following from example B.3. We will show in subsequent chapters that this basis is intimately related to simple excitations in a variety of state-sum TQFT's.

Let $\mathbb{C}\{|g\rangle\}_{g\in\Gamma_1}$ be the regular module of ${}^{\beta}\mathbb{C}\tilde{\Gamma}$ equipped with the inner product

$$\langle g|h\rangle = \delta_{g,h} \quad \forall g, h \in \Gamma_1. \tag{11.32}$$

We define a ${}^{\beta}\mathbb{C}\tilde{\Gamma}$-module isomorphism by the relations

$$\begin{aligned} |C, R; I, I'\rangle &:= \sqrt{\frac{d_{C,R}}{|M(C)|}} \sum_{g\in\Gamma_1} \overline{\mathcal{D}^{C,R}_{II'}(g)} |g\rangle \\ |g\rangle \ &:= \frac{1}{\sqrt{|M(s(g))|}} \sum_{\{(C,R)\}} \sqrt{d_{C,R}} \sum_{I,I'} \mathcal{D}^{C,R}_{II'}(g) |C, R; I, I'\rangle \end{aligned} \tag{11.33}$$

Definition 11.4.1. We will denote $\{|C,R;I,I'\rangle\} := \{|C,R;I,I'\rangle\}_{\forall\{C,R\},\forall I,I'\in\{1,\cdots,d_{C,R}\}}$ the complete set of such basis states.

Proposition 11.4.1. The basis $\{|C,R;I,I'\rangle\}$ is orthonormal, such that:

$$\langle C',R';J,J'|C,R;I,I'\rangle = \delta_{C,C'}\delta_{R,R'}\delta_{J,I}\delta_{J',I'} \tag{11.34}$$

Proof. Both follow directly from definition and equations (11.25) and (11.26):

$$\begin{aligned}
&\langle C',R';J,J'|C,R;I,I'\rangle \\
=&\sqrt{\frac{d_{(C,R)}d_{(C',R')}}{|M(C)||M(C')|}}\sum_{g\in\Gamma}\overline{\mathcal{D}^{C,R}_{II'}(g)}\mathcal{D}^{C',R'}_{JJ'}(g) \\
&\sqrt{\frac{d_{(C,R)}d_{(C',R')}}{|M(C)||M(C')|}}\frac{|M(C)|}{d_{(C,R)}}\delta_{C,C'}\delta_{R,R'}\delta_{I,J}\delta_{I',J'} \\
&= \delta_{C,C'}\delta_{R,R'}\delta_{I,J}\delta_{I',J'}
\end{aligned} \tag{11.35}$$

where we used equation (11.26) between lines 2 and 3. □

Proposition 11.4.2. The basis $\{|C,R;I,I'\rangle\}$ is a complete basis for ${}^{\beta}\mathbb{C}\Gamma$

Proof. To prove this statement we verify $|\{|C,R;I,I'\rangle\}| = |\Gamma_1|$ as follows:

$$\begin{aligned}
&\sum_{\{(C,R)\}}\sum_{I,I'}\langle C,R;I,I'|C,R;I,I'\rangle = \frac{d_{(C,R)}}{|M(C)|}\sum_{\{(C,R)\}}\sum_{I,I'}\sum_{g\in\Gamma}\overline{\mathcal{D}^{C,R}_{II'}(g)}\mathcal{D}^{C,R}_{II'}(g) \\
&= \sum_{g\in\Gamma}1 = |\Gamma_1|.
\end{aligned} \tag{11.36}$$

□

Proposition 11.4.3. The algebra product in $\mathbb{C}\{|C,R;I,I'\rangle\}$ is given by:

$$|C,R;I,I'\rangle\,|C',R';J,J'\rangle = \frac{1}{\sqrt{d_{(C,R)}}}\,|C,R;I,J'\rangle\,\delta_{I',J}\delta_{C,C'}\delta_{R,R'} \tag{11.37}$$

Proof.

$$\begin{aligned}
&|C,R;I,I'\rangle\,|C',R';J,J'\rangle \\
&= \sqrt{\frac{d_{(C,R)}d_{(C',R')}}{|M(C)||M(C')|}} \sum_{g,h\in\Gamma_1} \overline{\mathcal{D}^{C,R}_{II'}(g)\mathcal{D}^{C',R'}_{JJ'}(h)} \frac{\beta_{s(g)}(g,h)}{\sqrt{|M(s(g))|}}\,|gh\rangle \\
&= \sqrt{\frac{d_{(C,R)}d_{(C',R')}}{|M(C)||M(C')|}} \sum_{g,h\in\Gamma_1} \overline{\mathcal{D}^{C,R}_{II'}(g)\mathcal{D}^{C',R'}_{JJ'}(h)} \\
&\frac{\beta_{s(g)}(g,h)}{|M_{s(g)}|} \sum_{\{(\tilde{C},\tilde{R})\}} \sqrt{d_{C,R}} \sum_{K,K'} \mathcal{D}^{\tilde{C},\tilde{R}}_{KK'}(gh)\,|C,R;K,K'\rangle \\
&= \sqrt{\frac{d_{(C,R)}d_{(C',R')}}{|M(C')||M(C)|}} \frac{1}{|M(s(g))|} \sum_{\{(\tilde{C},\tilde{R})\}} \sum_{K,K',\tilde{K}} \sum_{g,h\in\Gamma_1} \\
&\overline{\mathcal{D}^{C,R}_{II'}(g)}\mathcal{D}^{\tilde{C},\tilde{R}}_{K\tilde{K}}(g)\overline{\mathcal{D}^{C',R'}_{JJ'}(h)}\mathcal{D}^{\tilde{C},\tilde{R}}_{\tilde{K}K'}(h)\,|C,R;K,K'\rangle \\
&= \frac{1}{\sqrt{d_{(C,R)}}}\,|C,R;I,J'\rangle\,\delta_{C,C'}\delta_{R,R'}\delta_{I',J}
\end{aligned} \tag{11.38}$$

where we used the definition of $|gh\rangle$ from equation (11.33) between lines 2 and 3 and the orthogonality condition from equation (11.26) between lines 4 and 5. □

By comparison with the discussion in example B.3 this new basis can be identified with the canonical basis for a matrix algebra and this basis transformation defines the isomorphism between ${}^{\beta}\mathbb{C}\Gamma$ and the direct sum of matrix algebras. The numerical constant $\frac{1}{\sqrt{d_{C,R}}}$ in the product is purely an artefact of use choosing our new basis to be orthonormal with respect to the inner product on $\mathbb{C}\Gamma$.

Definition 11.4.2. In the following we will call the basis $\{|C,R;I,I'\rangle\}$ the **canonical basis**.

A useful corollary of the product in equation (11.4.3) is:

$$\begin{aligned}
|g\rangle\;\;|C,R;I,I'\rangle &= \frac{1}{\sqrt{|M(C)|}} \sum_{J} \mathcal{D}^{C,R}_{IJ}(g)\,|C,R;J,I'\rangle \quad \forall g\in\Gamma_1 \\
|C,R;I,I'\rangle\,|g\rangle &= \frac{1}{\sqrt{|M(C)|}} \sum_{J'} \mathcal{D}^{C,R}_{J'I'}(g)\,|C,R;I,J'\rangle \quad \forall g\in\Gamma_1
\end{aligned} \tag{11.39}$$

which follows from equation (11.4.3) and the definition of $|g\rangle$ in 11.33.

11.4.1 Central Basis

Utilising the canonical basis we can straightforwardly define a basis for the centre of the twisted groupoid-like algebra. Given ${}^{\beta}\mathbb{C}\tilde{\Gamma}$, the central subalgebra $Z({}^{\beta}\mathbb{C}\tilde{\Gamma})$ is defined by

$$Z({}^{\beta}\mathbb{C}\tilde{\Gamma}) = \{|a\rangle \in {}^{\beta}\mathbb{C}\tilde{\Gamma}|\, |a\rangle\, |h\rangle = |h\rangle\, |a\rangle\, \forall\, |h\rangle \in \Gamma_1\} \tag{11.40}$$

From the semi-simplicity of ${}^{\beta}\mathbb{C}\tilde{\Gamma}$ and example B.3 it directly follows:

$$\dim Z({}^{\beta}\mathbb{C}\tilde{\Gamma}) = \text{number of simple } {}^{\beta}\mathbb{C}\tilde{\Gamma}\text{-modules up to isomorphism} \tag{11.41}$$

Using the canonical basis in the previous section we can define an orthonormal basis for $Z({}^{\beta}\mathbb{C}\Gamma)$ as follows:

$$|\chi^{C,R}\rangle = \frac{1}{\sqrt{d_{C,R}}} \sum_{I} |C,R;I,I\rangle = \frac{1}{\sqrt{|M_C|}} \sum_{k\in\Gamma_1} \overline{\chi^{C,R}(k)}\, |k\rangle \tag{11.42}$$

such that

$$Z({}^{\beta}\mathbb{C}\Gamma) = \mathbb{C}\{|\chi^{C,R}\rangle\}_{\{(C,R)\}} \tag{11.43}$$

That the basis is orthonormal follows directly from equation (11.4.1). It is straightforward to verify such elements are indeed central such that: $\forall\, |g\rangle \in {}^{\beta}\mathbb{C}\tilde{\Gamma}$

$$\begin{aligned} &|C,R\rangle\, |g\rangle = \frac{1}{\sqrt{d_{C,R}}} \sum_{I} |C,R;II\rangle\, |g\rangle \\ &= \frac{1}{\sqrt{d_{C,R}|M(C)|}} \sum_{I,J} |C,R;IJ\rangle\, \mathcal{D}^{C,R}_{JI}(g) = \frac{1}{\sqrt{d_{C,R}|M(C)|}} \sum_{I,J} |C,R;JI\rangle\, \mathcal{D}^{C,R}_{IJ}(g) \\ &= |g\rangle\, |C,R\rangle \end{aligned} \tag{11.44}$$

We now relate the previous constructions back to a class of tube algebras given by twisted groupoid-like algebras giving a physical interpretation to the tube algebra in analogy with gauge theories.

11.5 Twisted Groupoid-Like Tube Algebras

We now discuss twisted groupoid-like algebras in the context of tube algebras.

Definition 11.5.1. Given an oriented, closed $n-1$-manifold with triangulation $\mathcal{W}$. The $\mathcal{W}$-tube algebra is twisted groupoid-like if $(\mathcal{H}[\mathcal{W}_{tube}]_0, \circ)$ admits a complete orthonormal basis such that $\mathcal{H}[\mathcal{W}_{tube}]_0 \cong \mathbb{C}\Gamma$ and the algebra product matrix elements are given by:

$$\mathcal{Z}[GL_{\mathcal{W}}] = \sum_{g,h\in\Gamma} \frac{\beta_{s(g)}(g,h)}{\sqrt{|M(s(g))|}} \,|gh\rangle \;\; (\langle g| \;\otimes \langle h| \;)\delta_{t(g),s(h)}. \tag{11.45}$$

where $\beta \in H^2(\Gamma, U(1))$.

A consequence of this definition, using equation (10.40), is that the $*$-structure on $\mathcal{H}[\mathcal{W}_{tube}]_0$ is given by:

$$|g\rangle^* = \frac{1}{\beta_{s(g)}(g,g^{-1})}\,|g^{-1}\rangle \tag{11.46}$$

for all basis elements $|g\rangle \in \mathcal{H}[\mathcal{W}_{tube}]_0$.

From this relation, using the groundstate basis relation,

$$\mathcal{Z}[\mathcal{W}_{tube} \times_p I]\,|g\rangle \; = |g\rangle \;\; , \quad \forall\, |g\rangle \;\in \mathcal{H}[W_{tube}]_0 \tag{11.47}$$

it is immediate that the canonical basis $\{|C,R;I,I'\rangle\}$ for a twisted groupoid-like, $\mathcal{W}$-tube algebra defines a complete, orthonormal basis for $\mathcal{H}[\mathcal{W}_{tube}]_0$:

$$\mathcal{Z}[\mathcal{W}_{tube} \times_p I]\,|C,R;I,I'\rangle = |C,R;I,I'\rangle\,, \quad \forall\, |C,R;I,I'\rangle \in \{|C,R;I,I'\rangle\} \tag{11.48}$$

using propositions 11.4.1 and 11.4.2. Furthermore, using the central canonical basis $\{|\chi^{C,R}\rangle\}$ we find a complete, orthonormal basis of $\mathcal{H}[\mathcal{W}_{S^1}]_0$, where

$$\mathcal{Z}[\mathcal{W}_{S^1} \times I] = \sum_{g,h\in\Gamma} \frac{1}{|M(s(g))|} \frac{\beta_{s(g)}(g,h)}{\beta_{s(h)}(h,h^{-1}gh)}\,|h^{-1}gh\rangle \;\; \langle g| \;\; \delta_{s(g),t(g)}\delta_{s(g),s(h)} \tag{11.49}$$

such that:

$$\begin{aligned} \mathcal{Z}[\mathcal{W}_{S^1} \times I]\,|\chi^{C,R}\rangle = |\chi^{C,R}\rangle\,, \quad \forall\, |\chi^{C,R}\rangle \in \{|\chi^{C,R}\rangle\} \\ |\{|\chi^{C,R}\rangle\}| = dim\mathcal{H}[\mathcal{W}_{S^1}]_0. \end{aligned} \tag{11.50}$$

Given an $n-1$-manifold W with triangulation $\mathcal{W}$. If the $\mathcal{W}$-tube algebra is a twisted groupoid-like algebra, from Morita equivalence of tube algebras for W we have a one-one correspondence between modules and we can consider the pairs (C, R) defining a twisted groupoid representation, as defining quantum numbers for excitations of any triangulation of W. As the algebra product commutes with the Hamiltonian we can see that such quantum numbers are preserved in the groundstate adding to the validity that they are useful quantities.

In analogy with the quantum double of a finite group [34] which represents a groupoid algebra, we will now discuss a physical interpretation of the pair (C, R) defining a representation. In the following we will call C defining a connected component of objects a flux-like quantum number and the representation R of $Z(C)$ a charge-like quantum number. The flux-like quantum numbers admit an interpretation as the equivalence class of configurations on the boundary related by the length scale invariance of the system. In this way a generic configuration of the boundary will be an element of the vector space $\mathbb{C}C$. A measurement of the boundary will project elements of this vector space to a basis element $c_i \in C$ defining a classical configuration of the boundary. The notion of R as a charge-like quantum number also makes sense from this stand-point. Given a flux configuration $c_i \in C$, gluing a tube can only permute the configuration to another element of C. The charge R then tells us how a classical configurations transforms under the action of adding more space around the boundary. In particular, the representation decomposes $\mathbb{C}C$ in terms of the symmetries of the boundary configuration which map $c_i \in C$ to itself under the addition of more space around the boundary.

Using this interpretation we can consider each element of the canonical basis $\{|C, R; I, I'\rangle\}$ as defining a pair of excitations $\mathcal{W}_{tube}$ localised at each boundary. In particular, taking the vector space $\mathbb{C}C \otimes V_R$ as the internal vector space of each excitation located on the boundaries, we interpret such states as corresponding to well defined flux and charge states of $I = (a, m) \in \mathbb{C}C \otimes V_R$ on $W \times 0$ and $I' = (b, n) \in \mathbb{C}C \otimes V_R$ on $\mathcal{W} \times 1$.

From the above constructions once we establish that a given $\mathcal{W}-$tube algebra is twisted groupoid-like we can define a complete orthonormal basis for $\mathcal{W}_{tube}$ and $\mathcal{W}_{S^1}$ and define the quantum numbers of the simple topological excitation types via the pair (C, R). The internal vector space of the (C, R) excitation can then be defined by $\mathbb{C}C \otimes V_R$.

11.6 Minimum Entropy States

When outlining the physical reasoning behind the tube algebra we made assumptions about the entanglement between the excitation and the groundstate of the system. In the following we will argue that the center of twisted groupoid-like tube algebras correspond to the minimally entangled states of $\mathcal{H}[\mathcal{W}_{S^1}]_0$ when the entanglement cut is taken along the codimension 1-submanifold $\mathcal{W} \sqcup \mathcal{W}$ cutting $\mathcal{W}_{S^1}$ into two disjoint tubes $\mathcal{W}_{tube} \sqcup \mathcal{W}_{tube}$. An example of such a cut is cutting the torus into two cylinders by cutting along two disjoint circles S^1.

Assuming $\mathcal{W}$-tube is twisted groupoid-like we have an orthonormal canonical basis for $\mathcal{W}-$ tube given by:

$$\{|C, R; I, J\rangle\} \tag{11.51}$$

for each simple module (C, R) with dimension $d_{C,R}$ and $I, J \in \{1, \cdots, d_R\}$ an orthonormal basis for (C, R). In this basis the algebra product is given by:

$$|R; I, I'\rangle\, |R'; J, J'\rangle = \frac{1}{\sqrt{d_{(C,R)}}}\, |R; I, J'\rangle\, \delta_{R,R'}\delta_{J,I'}. \tag{11.52}$$

Using the correspondence $Z(W - \text{tube}) = \mathcal{H}[W_{S^1}]_0$, we can define normalised basis elements for $\mathcal{H}[W_{S^1}]_0$ via:

$$|\chi^{C,R}\rangle := \frac{1}{\sqrt{d_{(C,R)}}} \sum_I |R; I, I\rangle \tag{11.53}$$

Using the relation from theorem 10.3.2:

$$\mathcal{H}[W_{tube}]_0 \simeq \mathcal{H}[W_{tube}]_0 \otimes_{W-tube} \mathcal{H}[W_{tube}]_0 \tag{11.54}$$

we can identify:

$$|C, R; I, I'\rangle = \frac{1}{\sqrt{d_{(C,R)}}} \sum_J |C, R; I, J\rangle \otimes |C, R; J, I'\rangle\,. \tag{11.55}$$

and define a Schmidt decomposition of the states $|\chi^{C,R}\rangle$:

$$|\chi^{C,R}\rangle := \frac{1}{\sqrt{d_{C,R}}} \sum_I |R; I, I\rangle = \frac{1}{d_R} \sum_{I,J} |R; I, J\rangle \otimes |R; J, I\rangle \tag{11.56}$$

This decomposition is naturally associated to the decomposition of W_{S^1} into the disjoint union of two copies of W_{tube}.

We can now define a general state of $\mathcal{H}[W_{S^1}]$ via:

$$|\psi\rangle := \sum_{C,R,I} \frac{\alpha_{C,R,I}}{\sqrt{d_R}} |R; I, I\rangle = \sum_{C,R,I} \frac{\alpha_{C,R,I,I}}{d_R} \sum_J |C, R; I, J\rangle \otimes |C, R; J, I\rangle \quad (11.57)$$

which is normalised:

$$\sum_{\{C,R,I\}} |\alpha_{C,R,I}|^2 = 1. \quad (11.58)$$

We now write down the density matrix:

$$\rho_{AB} = \sum_{C,R,I} \sum_{C',R',J} \frac{\alpha_{C,R,I}\alpha^*_{C',R',J}}{d_{(C,R)}d_{(C',R')}} \sum_{K,L} |C, R; I, K\rangle \langle C', R'; J, L| \otimes |C, R; K, I\rangle \langle C', R'; L, J| \quad (11.59)$$

and corresponding reduced density matrix into the first tensor component:

$$\rho_A = \sum_{C,R,I,K} \frac{|\alpha_{C,R,I}|^2}{d^2_{(C,R)}} |C, R; I, K\rangle \langle C, R; I, K| \quad (11.60)$$

As the basis is diagonal we can form the Von-Neumann entropy via:

$$S_A = -\sum_{C,R,I} \frac{|\alpha_{C,R,I}|^2}{d_{(C,R)}} \log \frac{|\alpha_{C,R,I}|^2}{d^2_{(C,R)}}. \quad (11.61)$$

Taking variations of S_A with respect to each $|\alpha_{C,R,I}|^2$

$$\begin{aligned} \frac{dS_A}{d(|\alpha_{C,R,I}|^2)} &= -\log \frac{|\alpha_{C,R,I}|^2}{d^2_{(C,R)}} + 1 \\ \frac{d^2 S_A}{d(|\alpha_{C,R,I}|^2)^2} &= -\frac{d^2_{(C,R)}}{|\alpha_{C,R,I}|^2} \end{aligned} \quad (11.62)$$

We find a maxima when all $|\alpha_{C,R,I}|^2$ are equal and a minima exactly when only one term $|\alpha_{C,R,I}|^2 = 1$ and all others vanish. In this way we conclude the minimal entropy states of $\mathcal{H}[W_{S^1}]_0$ are exactly the central canonical basis elements.

This result validates and expands the conjecture in [87] there exists a one-one correspondence between minimal entropy states of the torus and simple topological excitation types when the tube algebra is twisted groupoid-like. This result can be directly applied to any $\mathcal{W}$-tube algebra as the relations follow directly from semisimplicity of the tube algebra.

Chapter 12

State Sum Tube Algebras For Dijkgraaf-Witten Theory

In this chapter we apply the construction of tube algebras for unitary state-sum TQFT's to the example of the Dijkgraaf-Witten TQFT [46]. For an overview of the Dijkgraaf-Witten state-sum TQFT see example 4.1.1. In this chapter we will discuss point-particle like topological excitation types arising in the 1+1D and 2+1D and both point-particle and loop like topological excitation types in 3+1D. In particular we will find that all examples studied in this chapter provide examples of twisted groupoid-like tube algebras discussed in the previous chapter.

12.1 State Sum Tube Algebra For 1+1D Dijkgraaf-Witten Theory

We begin by giving the simplest non-trivial example of the Dijkgraaf-Witten tube algebra. In 1+1D there is a unique choice for the boundary manifold $\mathcal{W}$ given by the 0-dimensional point P. Taking the point triangulated as a 0-simplex we can define $\mathcal{P}_{tube} = \Delta^1$. Any other triangulation will give rise to an isomorphic vector space and isomorphic algebras and so we are free to choose the simplest triangulation without loss of generality.

$\mathcal{P}_{tube} :=$ (12.1)

Let G be a finite group and $\beta \in H^2(G, U(1))$ a 2-cocycle. The 1+1D Dijkgraaf-Witten TQFT assigns elements $g \in G$ to oriented 1-simplices. The Hilbert space $\mathcal{H}[\mathcal{P}_{tube}]_0$ is given by:

$$\mathcal{H}[\mathcal{P}_{tube}]_0 = \mathbb{C}\{|\bullet\xrightarrow{\quad g \quad}\bullet\rangle\} \quad \forall g \in G \tag{12.2}$$

equipped with the inner product:

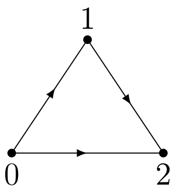
$$\langle\bullet\xrightarrow{\quad g \quad}\bullet|\bullet\xrightarrow{\quad h \quad}\bullet\rangle = \delta_{g,h}. \tag{12.3}$$

There is a trivial bigrading as there is a unique configuration of the point $\mathcal{P}$.

Utilising the previous discussion we can now define the algebra product on $\mathcal{H}[\mathcal{P}_{tube}]_0$ as follows:

$$\begin{aligned} &|\overset{0}{\bullet}\xrightarrow{\quad g \quad}\overset{1}{\bullet}\rangle \quad |\overset{1}{\bullet}\xrightarrow{\quad h \quad}\overset{2}{\bullet}\rangle \xrightarrow{glue} |\overset{0}{\bullet}\xrightarrow{\quad g \quad}\overset{1}{\bullet}\xrightarrow{\quad h \quad}\overset{2}{\bullet}\rangle \\ &\xmapsto{\mathcal{Z}[GL_{\mathcal{P}}]} \frac{\beta(g,h)}{\sqrt{|G|}}|\overset{0}{\bullet}\xrightarrow{\quad gh \quad}\overset{2}{\bullet}\rangle \end{aligned} \tag{12.4}$$

Here

$\mathcal{Z}[GL_{\mathcal{P}}] = \mathcal{Z}[$  $] = \frac{1}{\sqrt{|G|}} \sum_{g,h \in G} \beta(g,h)\, |\bullet\xrightarrow{gh}\rangle\, \langle\xrightarrow{g}\bullet\xrightarrow{h}|$ (12.5)

Summarising the algebra product:

$$|\bullet\xrightarrow{\quad g \quad}\bullet\rangle\, |\bullet\xrightarrow{\quad h \quad}\bullet\rangle = \frac{\beta(g,h)}{\sqrt{|G|}}\, |\bullet\xrightarrow{\quad gh \quad}\bullet\rangle \tag{12.6}$$

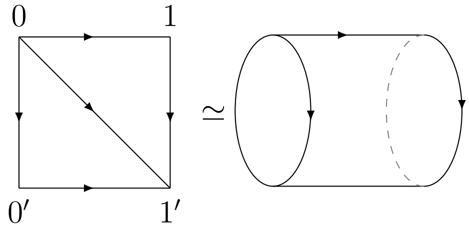
we can directly identify the 1+1D Dijkgraaf-Witten tube algebra with the twisted groupoid-like algebra ${}^{\beta}\mathbb{C}\widetilde{BG}$. In this way we see that simple point-particle topological excitation types in the 1+1D Dijkgraaf-Witten TQFT can be identified with β-twisted representations of the group G, defining point-like topological excitations carrying a charge-like quantum number and no flux-like quantum number.

Additionally using the results from the previous chapter, we can define the canonical basis for $\mathcal{P}$-tube which gives an orthonormal basis for $\mathcal{H}[\mathcal{P}_{tube}]_0$ which diagonalises the algebra product and we can interpret this space as defining the state space of $\mathcal{P}_{tube}$ with a simple excitation type localised on each boundary. Furthermore the central basis for this groupoid-like algebra can be identified with $\mathcal{H}[\mathcal{P}_{S^1}]_0$.

12.2 State Sum Tube Algebra for 2+1D Dijkgraaf-Witten

We now give the 2+1D $\mathcal{W}-$tube algebra. In 2+1D there is a unique choice of the boundary manifold W given by the circle S^1. Taking $\mathcal{S}^1$ as a triangulation of the circle with a single edge and vertex, $\mathcal{S}^1_{tube} = \mathcal{S}^1 \times [0,1]$ can be defined in terms of the following triangulation:

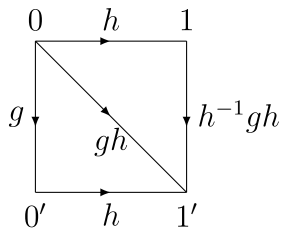
$\mathcal{S}^1_{tube} :=$



(12.7)

with the identification of vertices $0 = 0'$, $1 = 1'$ and edges $[01] = [0'1']$.

Let G be a finite group and $\alpha \in H^3(G, U(1))$ a normalised 3-cocyle. The Dijkgraaf-Witten TQFT assigns elements $g \in G$ to oriented 1-simplices. The Hilbert space $\mathcal{H}[S^1_{tube}]_0$ is spanned by basis elements



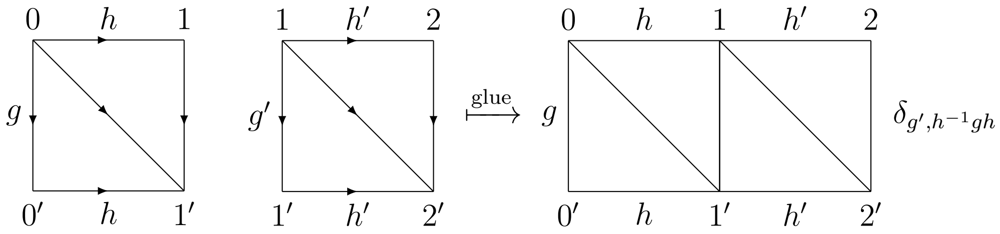
$:= |g \xrightarrow{h}\rangle$ (12.8)

such that

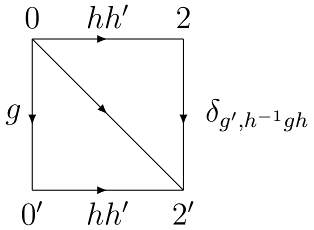
$$\mathcal{H}[S^1_{tube}]_0 = \mathbb{C}\{|g \xrightarrow{h}\rangle\}_{\forall g,h \in G} \tag{12.9}$$

.

We define a gluing of basis elements as follows:



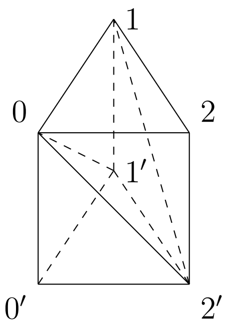
$\xmapsto{Z[GL_{\mathcal{S}^1}]} \frac{\beta_g(h,h')}{\sqrt{|G|}}$



(12.10)

This can be thought as $GL_{\mathcal{S}^1} = GL_{\mathcal{P}} \times S^1$, with the identification $[012] = [0'1'2']$, or as $\mathcal{S}^1_{tube} \star \mathcal{S}^1$ as in conjecture **??**.

$GL_{\mathcal{S}^1} :=$



$$= [00'1'2']^{+1} \cup [011'2']^{-1} \cup [0122']^{+1} \tag{12.11}$$

Such that

$$Z[GL_{\mathcal{S}^1}] = \sum_{g,h,h' \in G} \frac{\beta_g(h,h')}{\sqrt{|G|}} \, |g \xrightarrow{hh'}\rangle \, (\langle g \xrightarrow{h}| \otimes \langle h^{-1}gh \xrightarrow{h'}|) \tag{12.12}$$

where

$$\frac{\alpha(g,h,h')\alpha(h,h',h'^{-1}h^{-1}ghh')}{\alpha(h,h^{-1}gh,h')} := \beta_g(h,h'). \tag{12.13}$$

It is straightforward to demonstrate $\beta \in H^2(G//G, U(1))$ defines a normalised groupoid 2-cocycle by application of the 3−cocyle condition:

$$\begin{aligned} \frac{\beta_{k^{-1}xk}(h,g)\beta_x(k,hg)}{\beta_x(kh,g)\beta_x(k,h)} &= 1 \quad \forall x,k,h,g \in G \\ \beta_e(k,h) = \beta_k(e,h) &= \beta_k(h,e) \quad \forall k,h \in G \\ \beta_g(h,h^{-1}) &= \beta_{h^{-1}gh}(h^{-1},h) \quad \forall g,h \in G. \end{aligned} \tag{12.14}$$

It immediately follows that the $2+1D$ Dijkgraaf-Witten $\mathcal{S}^1$-tube algebra defines a twisted action groupoid-like algebra ${}^{\beta}\mathbb{C}(\widetilde{G//G})$, where the group action is given by conjugation $h \triangleright g := h^{-1}gh$. The algebra ${}^{\beta}\mathbb{C}(G//G)$ was first discussed by Roche et al [34] as the twisted quantum double algebra and was shown to define a quasi-Hopf algebra. We summarise the algebra product below:

$$|g \xrightarrow{h}\rangle\, |g' \xrightarrow{h'}\rangle = \frac{\beta_g(h,h')}{\sqrt{|G|}}\, |g \xrightarrow{hh'}\rangle\, \delta_{g',h^{-1}gh}. \tag{12.15}$$

12.2.1 Representation Theory of Twisted Quantum Double

The representation theory of the twisted quantum double follows directly from the discussion of section 11.2. Each irreducible representation is specified by a pair (C,R) where $C \in \pi_0(G//G)$ is a connected component and R is a $\beta-$twisted representation of the group $\pi_1(C)$. The connected components $\pi_0(G//G)$ are given by conjugacy classes of the group such that given $g \in G$ the connected component $C(g) = \{h | h = x^{-1}gx \quad \forall x \in G\}$. To construct the irreducible representations (C,R), let $c_1 \in C = \{c_1, \cdots, c_{|C|}\}$ be a representative element of C and for each $c_a \in C$ we define a morphism $c_a \xrightarrow{q_a} c_1$ where $c_1 \xrightarrow{q_1} c_1 := c_1 \xrightarrow{1_{c_1}} c_1$. Then let R be a β_{c_1}-twisted irreducible representation of the group $\pi_1(c_1) = \{h \in G | hc_1 = c_1 h\}$. The components of the representations can be written as follows:

$$\mathcal{D}^{C,R}_{am,bn}(|g \xrightarrow{h}\rangle) = \delta_{g,c_a}\delta_{c_b,h^{-1}c_a h}\frac{\beta_{c_1}(q_a^{-1},h)}{\beta_{c_1}(q_a^{-1}hq_b,q_b^{-1})} D^R_{m,n}(q_a^{-1}hq_b). \tag{12.16}$$

Where D^R is the matrix of the β-twisted $\pi_1(c_1)$ representation R. It is straightforward to verify $\{\mathcal{D}^{C,R}\}_{C,R}$ satisfy the relations of equations (11.25) and (11.26) with $|M_x| = |G|$ for all $x \in G$.

We can now define the canonical basis for the cylinder (see section 11.4) via the relation:

$$am \overset{C,R}{\bullet\!\!\longrightarrow\!\!\bullet} bn = \sqrt{\frac{d_{C,R}}{|G|}} \sum_{g,h} \overline{\mathcal{D}^{C,R}_{am,bn}(g \xrightarrow{h})}\, |g \xrightarrow{h}\rangle \tag{12.17}$$

and verify

$$\mathcal{H}[\mathcal{S}^1_{tube}]_0 = \mathbb{C}\{\ am \overset{C,R}{\bullet\!\!\longrightarrow\!\!\bullet} bn\ \}_{\forall (C,R), \forall a,b\in\{1,\cdots,|C|, \forall m,n \in \{1,\cdots,d_R\}} \tag{12.18}$$

Furthermore it follows the algebra product for the canonical basis is given by:

$$am \overset{C,R}{\bullet\!\!\longrightarrow\!\!\bullet} bn \ \star\ a'm' \overset{C',R'}{\bullet\!\!\longrightarrow\!\!\bullet} b'n' = \frac{1}{\sqrt{d_{C,R}}}\ am \overset{C,R}{\bullet\!\!\longrightarrow\!\!\bullet} b'n'\ \delta_{C,C'}\delta_{R,R'}\delta_{b,a'}\delta_{n,m'}. \tag{12.19}$$

In this way we for each simple representation (C,R) we find a simple bimodule of the $\mathcal{S}^1_{tube}$ algebra as:

$$\mathbb{C}\{\ am \overset{C,R}{\bullet\!\!\longrightarrow\!\!\bullet} bn\ \}_{\forall a,b\in\{1,\cdots,|C|, \forall m,n \in \{1,\cdots,d_R\}}. \tag{12.20}$$

Simplicity follows as any subspace fails to be a bimodule.

From this discussion we can identify the irreducible point-like topological excitation types in the 2+1D Dijkgraaf-Witten TQFT as consisting of composite flux-charge particles, where the flux-like quantum number is a conjugacy class $C \subset G$ and the charge is an irreducible representation of centraliser subgroup $Z(C) \subseteq G$ of C.

12.3 3+1D Dijkgraaf-Witten TQFT

We now describe the $3+1$D Dijkgraaf-Witten tube algebra. The main difference between the $3+1$D tube algebra and the lower dimensional analogues is the presence of excitations with different topologies. In particular such excitations admit a classification in terms of the boundary of their local neighbourhoods, which form closed surfaces. It is well known that up to diffeomorphism all closed surfaces are described in terms of their genus $g \geq 0 \in \mathbb{Z}_0^+$. In this way for any closed, compact surface W^g with genus g and triangulation $\mathcal{W}^g$ we can consider the tube algebra $\mathcal{H}[\mathcal{W}^g_{tube}]_0$. In the following we will consider the simplest two examples: $W^0 = S^2$ defining particle like excitations, $W^1 = T^2 = S^1 \times S^1$ describing loop-like excitations.

Before we describe specific examples we will first define a canonical way to define a triangulation of $W^g \times I$ and GL_{W^g} for any triangulation $\mathcal{W}^g$ of W^g. We begin by considering a triangulation $\mathcal{W}^g \times I$ in terms of $\mathcal{W}^g$. Let $a < b$ be a pair of ordered labels and $i < j < k$ an ordered triple of labels, we define a triangulation of $D^2 \times I$ as follows:

$$I^{\pm}_{[ab]}([ijk]) :=[a_i a_j a_k b_k]^{\pm} \cup [a_i a_j b_j b_k]^{\mp} \cup [a_i b_i b_j b_k]^{\pm} \tag{12.21}$$

where the $\pm$ superscript represents the orientation of the 3-simplex. The ordering of the vertices are induced from the orderings $a < b$ and $i < j < k$ by the relations:

$$a_* < b_{*'} \tag{12.22}$$

for any pair of labels $*, *'$ independent of the ordering and

$$a_x < a_y, \quad b_x < b_y \tag{12.23}$$

for any pair of labels $x < y$. Utilising this notation we can define a triangulation $\mathcal{W}^g \times I$ of $W^g \times I$ via:

$$\mathcal{W}^g \times [ab] := \bigcup_{[ijk] \in \mathcal{W}^g} I^{\sigma([ijk])}_{[ab]}([ijk]) \tag{12.24}$$

where $\sigma([ijk]) \in \pm$ is the orientation of the 2-simplex $[ijk] \in \mathcal{W}^g$.

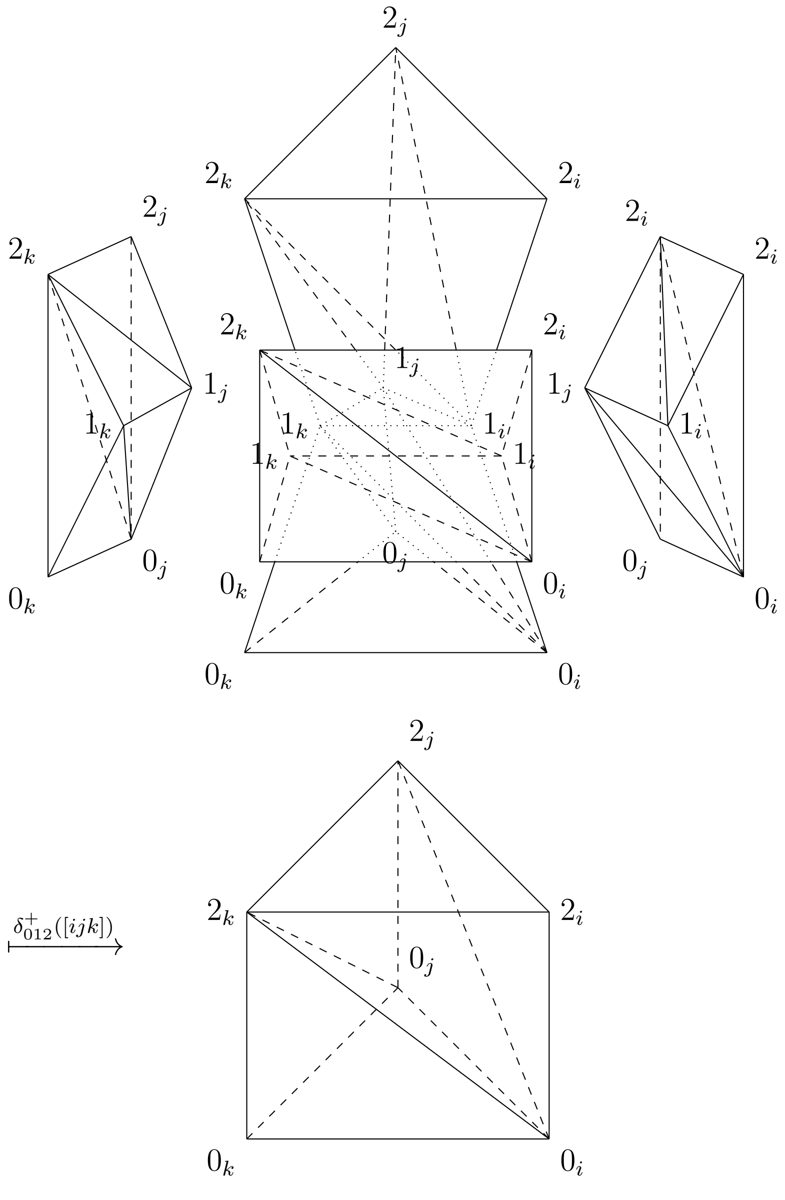
Building on this notation we can define the triangulated pinched cobordism

$$GL_{\mathcal{W}^g} : \mathcal{W}^g \times [01] \cup \mathcal{W}^g \times [12] \to \mathcal{W}^g \times [02] \tag{12.25}$$

as follows: Let,

$$\begin{aligned}\delta^{\pm}_{[abc]}([ijk]) :=& [a_i b_i b_j c_j c_k]^{\pm} \cup [a_i b_i b_j b_k c_k]^{\mp} \cup [a_i a_j b_j b_k c_k]^{\pm} \\ & \cup [a_i b_i c_i c_j c_k]^{\mp} \cup [a_i a_j b_j c_j c_k]^{\pm} \cup [a_i a_j a_k b_k c_k]^{\mp}\end{aligned} \tag{12.26}$$

with vertices ordered by the rules in equations (12.22) and (12.23). A visualisation of this triangulation is depicted via:



(12.27)

where $\delta^+_{012}([ijk])$ can be thought of as the series of Pachner moves relating the left hand side to the right. Using this definition it is straightforward to check using the boundary map that a triangulation of GL_{W^g} can be given by:

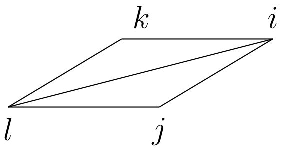
$$GL_{\mathcal{W}^g} := \bigcup_{[ijk]\in\mathcal{W}^g} \delta^{\sigma([ijk])}_{[012]}([ijk]) \tag{12.28}$$

such that $GL_{\mathcal{W}^g}$ defines the triangulated cobordism in equation (12.25).

12.3.1 S^2-Tube Algebra

We now consider the tube algebra $(\mathcal{H}[\mathcal{W}^0_{tube}]_0, \circ)$. This tube algebra arises from considering point-like excitations in $3+1$D. This can be seen by noting the local neighbourhood of a point in 3-dimensional space is given by the 3-disk D^3 which has boundary, the sphere $S^2 = W^0$.

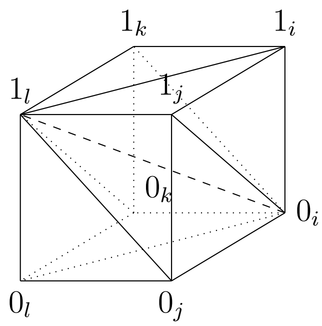
Making a choice of triangulation of the sphere $\mathcal{W}^0$

$\mathcal{W}^0 :=$  (12.29)

with the identifications:

$$\begin{aligned} &jk \\ [ij] &= [ik] \\ [jl] &= [kl] \end{aligned} \tag{12.30}$$

We define the triangulation $\mathcal{W}^0 \times I$, with identifications induced from $\mathcal{W}^0$ as follows:

 (12.31)

For each $a, b, c, g, h \in G$ we define a configuration $|(g,h) \xrightarrow{a,b,c}\rangle \in s(\mathcal{W}^0 \times I)$

via the assignments:

$$\begin{aligned}
&[0_i 1_i] = a \\
&[0_j 1_j] = [0_k 1_k] = b \\
&[0_l 1_l] = c \\
&[0_i 0_j] = [0_i 0_k] = g \\
&[0_j 0_l] = [0_k 0_l] = h \\
&[0_i 0_l] = gh \\
&[0_i 1_j] = [0_i 1_k] = gb \\
&[0_j 1_l] = [0_l 1_l] = hc \\
&[1_i 1_j] = [1_i 1_k] = a^{-1}gb \\
&[1_j 1_l] = [1_k 1_l] = b^{-1}hc \\
&[1_i 1_l] = a^{-1}ghc
\end{aligned} \tag{12.32}$$

such that:

$$\mathcal{H}[\mathcal{W}^0_{tube}]_0 = \mathbb{C}\{|(g,h) \xrightarrow{a,b,c}\rangle\}_{\forall a,b,c,g,h \in G}. \tag{12.33}$$

Using $GL_{\mathcal{W}^0} = \delta^-_{[012]}([ijl]) \cup \delta^+([ikl])$ with the induced identifications we can write down $\mathcal{Z}[GL_{\mathcal{W}^0}]$ as follows:

$$\mathcal{Z}[GL_{\mathcal{W}^0}] = \frac{1}{|G|^{\frac{3}{2}}} \sum_{g,h,a,b,c,a',b',c'} |(g,h) \xrightarrow{aa',bb',cc}\rangle \left(\langle (g,h) \xrightarrow{a,b,c} | \otimes \langle (a^{-1}gb, b^{-1}hc) \xrightarrow{a',b',c'} |\right) \tag{12.34}$$

Notice that the 3-cocycle terms cancel out from the final expression, this follows from the chosen triangulation of the sphere being given by two copies of D^2 with opposite orientation glued along their boundary. Using $\mathcal{Z}[GL_{\mathcal{W}^0}]$ the algebra product is given as:

$$|(g,h) \xrightarrow{a,b,c}\rangle \, |(g',h') \xrightarrow{a',b',c'}\rangle = \frac{1}{|G|^{\frac{3}{2}}} \, |g,h; aa', bb', cc'\rangle \, \delta_{g', a^{-1}gb} \delta_{h', b^{-1}hc} \tag{12.35}$$

Which defines a groupoid-like algebra $\mathbb{C}\widetilde{G^2//G^3}$, were $G^2//G^3$ is the action groupoid with objects pairs $(g,h) \in G^2$ and morphisms $(a,b,c) \in G^3$, with action given by $(a,b,c) \triangleright (g,h) = (a^{-1}gb, b^{-1}hc)$.

Representation Theory

The representation theory of $\widetilde{G^2//G^3}$ is rather straightforward. We define the representation in terms of the action groupoid $G^2//G^3$. From the definition of the action, the groupoid $G^2//G^3$ has a single connected component given by the object set $G \times G$. Given a representative object $(g, h) \in G \times G$ in the connected component the stabiliser subgroup is given by:

$$\pi_1(g, h) \cong G. \tag{12.36}$$

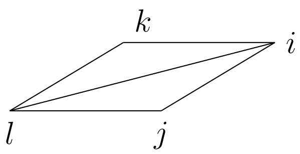
This can be seen by taking the subgroup $\pi_1(g, h) := \{(a, g^{-1}ag, h^{-1}g^{-1}agh)\}_{\forall a \in G} \subset G^3$.

In this way we can identify irreducible point-like topological excitation types in $3 + 1$D Dijkgraaf-Witten TQFT with irreducible representations of the group G. As would be expected the charges do not carry flux quantum numbers as all gauge configurations of the boundary are identified.

12.3.2 T^2-Tube Algebra

We now consider the second simplest example in $3+1$D, the $W^1 = T^2$ tube algebra. As mention briefly in the previous discussion this algebra classifies loop-like topological excitation types in $3 + 1$D. This can be seen by considering a excitation with the topology of the circle S^1 embedded in a 3-manifold and noticing the boundary of the local neighbourhood has topology T^2.

We begin by defining a triangulation $\mathcal{W}^1$ of the torus:

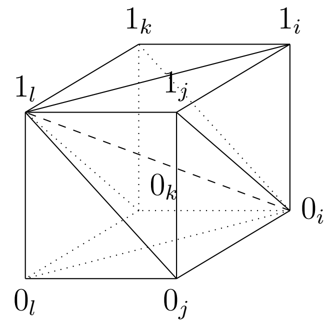
$\mathcal{W}^1 :=$ 

(12.37)

with the identifications

$$\begin{aligned} i &= j = k = l \\ [ij] &= [kl] \\ [ik] &= [jl] \end{aligned} \tag{12.38}$$

From this triangulation we can define $\mathcal{W}^1 \times I$:



(12.39)

with identifications induced from those of $\mathcal{W}^1$.

Let $G \times_c G := \{(g,h) \in G \times G | gh = hg\}$. For $(g,h) \in G \times_c G$ and $a \in G$ we can define configurations $|(g,h) \xrightarrow{a}\rangle \in s(\Sigma^1 \times I)$ through the assigments:

$$\begin{aligned}
&[0_i 0_j] = [0_k 0_l] = g \\
&[0_i 0_k] = [0_j 0_l] = h \\
&[0_i 0_l] = gh = hg \\
&[0_i 1_i] = [0_j 1_j] = [0_k 1_k] = [0_l 1_l] = a \\
&[0_i 1_j] = [0_k 1_l] = ga \\
&[0_i 1_k] = [0_j 1_l] = ha \\
&[1_i 1_j] = [1_k 1_l] = a^{-1} g a \\
&[1_i 1_k] = [1_j 1_l] = a^{-1} h a \\
&[1_i 1_l] = a^{-1} g h a = a^{-1} h g a
\end{aligned} \tag{12.40}$$

such that

$$\mathcal{H}[\mathcal{W}^1_{tube}]_0 = \mathbb{C}\{|(g,h) \xrightarrow{a}\rangle\}_{\forall (g,h) \in G \times_c G, a \in G} \tag{12.41}$$

Using $GL_{\mathcal{W}^1} = \delta^-_{[012]}(ijl)\delta^+_{[012]}(ikl)$ with the induced identifications we can evaluate

$$\mathcal{Z}[GL_{\mathcal{W}^1}] = \frac{1}{\sqrt{|G|}} \sum_{(g,h) \in G \times_c G} \sum_{a,a' \in G} \beta_{g,h}(a,a') \, |(g,h) \xrightarrow{aa'}\rangle \left(\langle (g,h) \xrightarrow{a}| \otimes \langle (a^{-1} g a, a^{-1} h a) \xrightarrow{a'}|\right) \tag{12.42}$$

Where

$$\beta_{h,g}(a,a') := \frac{\alpha(a,a',a'^{-1}a^{-1}gaa',a'^{-1}a^{-1}haa')\alpha(a,a^{-1}ha,a',a'^{-1}a^{-1}gaa')}{\alpha(a,a',a'^{-1}a^{-1}haa',a'^{-1}a^{-1}gaa')\alpha(h,a,a',a'^{-1}a^{-1}gaa')} \frac{\alpha(a,a^{-1}ga,a^{-1}ha,a')\alpha(h,a,a^{-1}ga,a')}{\alpha(a,a^{-1}ga,a',a'^{-1}a^{-1}gaa')\alpha(a,a^{-1}ha,a^{-1}ga,a')} \frac{\alpha(g,a,a',a'^{-1}a^{-1}haa')\alpha(g,h,a,a')}{\alpha(g,a,a^{-1}ha,a')\alpha(h,g,a,a')}. \tag{12.43}$$

It is straightforward but tedious to verify $\beta \in H^2(G \times_c G//G, U(1))$ is a normalised groupoid 2-cocyle using the 4-cocycle relations, such that:

$$\frac{\beta_{x^{-1}hx,x^{-1}gx}(y,z)\beta_{h,g}(x,yz)}{\beta_{h,g}(xy,z)\beta_{h,g}(x,y)} = 1 \tag{12.44}$$

From these expression we can write down the $\mathcal{W}^1$-tube algebra $(\mathcal{H}[\mathcal{W}^1_{tube}], \circ)$ product:

$$|(g,h) \xrightarrow{a}\rangle\, |(g',h') \xrightarrow{a'}\rangle = \frac{\beta_{g,h}(a,a')}{\sqrt{|G|}}\, |(g,h) \xrightarrow{aa'}\rangle\, \delta_{a^{-1}ga,g'}\delta_{a^{-1}ha,h'} \tag{12.45}$$

In this way we can see the $\mathcal{W}^1$-tube algebra defines a twisted groupoid-like algebra ${}^{\beta}\mathbb{C}G \,\widetilde{\times_c}\, G//G$ where the groupoid $G\times_c G//G$ is an action groupoid with objects, elements of $G \times_c G$ and morphisms, elements of G with the action given by simultaneous conjugation $a\triangleright(g,h) = (a^{-1}ga, a^{-1}ha)$. We will refer to the groupoid algebra ${}^{\beta}\mathbb{C}G \times_c G//G$ as the **twisted quantum triple**. We will elucidate the naming in section 12.3.4.

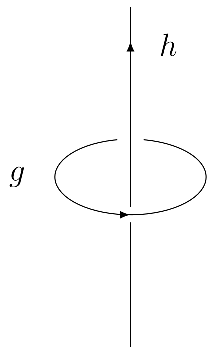
12.3.3 Representation Theory of Twisted Quantum Triple

Using the twisted quantum triple ${}^{\beta}\mathbb{C}G \,\widetilde{\times_c}\, G//G$ it is straightforward to define the irreducible representations of ${}^{\beta}\mathbb{C}G \times_c G//G$.

The connected components $\pi_0(G\times_c G//G)$ are given by the orbits of $G\times_c G$ under simultaneous conjugation. Given an orbit $C \in \pi_0(G \times_c G//G)$ and a representative element $(g,h) \in C \subseteq G\times_c G$ the stabiliser $\pi_1(g,h) = \{a \in G | ag = ga, ah = ha\} = Z(g) \cap Z(h)$ is given by the joint stabiliser of the pair $(g,h) \in G\times_c G$.

In this way irreducible loop-like topological excitation types carry both flux-like quantum numbers associated to $C \in \pi_0(G \times_c G//G)$ and charges given by the joint stabiliser of $(g, h) \in C$.

There is a natural interpretation of the flux-like quantum numbers as a loop threaded by an infinite line. In this way we consider the pairs $(g, h) \in G \times_c G$ to correspond to a loop carrying a flux of $g \in G$ threaded by an external flux of type $h \in G$. As we measure fluxes in topological gauge theories by transporting charges particles along closed paths, the holonomies of such paths are homotopy invariants. For the fluxes to be well defined under homotopy changes of the particles path we are naturally lead to the constraint that the two fluxes satisfy $[g, h] = 1_G$.



(12.46)

Similar results have been obtained in the untwisted case by assuming the correspondence between groundstates of the three torus and simple loop-like topological excitation types, which naturally leads to a similar interpretation of the flux quantum numbers[96, 97, 97, 98].

Additionally in such studies the authors investigated the action of the mapping class group on the 3-torus. The mapping class group is given by $SL(3, \mathbb{Z})$ and contains the torus mapping class group $SL(2, \mathbb{Z})$ as a subgroup. $SL(3, \mathbb{Z})$ has two generators S and T. In particular the T matrix is interpreted as the generalisation of the Dehn twist of $SL(2, \mathbb{Z})$ which is known to be related to anyonic spin in 2+1D. By this relation it has been argued that an orthonormal basis for T^3 for which the T matrix is diagonal should correspond to a groundstate with a well defined loop excitation threading the 3-torus with the diagonal elements defining phase factors which can be interpreted as a notion of spin, involving a framed loop turning itself inside out. Such a basis is often called the fusion or quasiparticle basis [26, 96]. From section 11.4.1 we can immediately identify the central basis of the $\mathcal{W}^1$-tube algebra with the groundstate basis $\mathcal{H}[\mathcal{W}^1_{S^1}]_0$ which fully agrees with the quasi-particle basis defined in [26, 98]. In this way we can identify a spin-like phase factor to our loop-like excitations arising from the tube algebra approach in terms of the T matrix.

12.3.4 Comultiplication Structure

In the previous section we referred to the $\mathcal{W}^1$-tube algebra as the twisted quantum triple $T(G)$. In the following we support this naming by demonstrating that the twisted quantum triple admits a comultiplication and antipode structure analogously to the twisted quantum double.

Using section the quasi-Hopf algebra structure of the twisted quantum double of a finite group [34] as inspiration we can analogously form a comultiplication algebra homomorphism $\Delta : T(G) \to T(G) \otimes T(G)$ for the twisted quantum triple algebra as follows:

$$\Delta(|(g,h) \xrightarrow{a}\rangle) = \sum_{g_1 g_2 = g} \gamma_{a,h}(g_1,g_2)\,|(g_1,h) \xrightarrow{a}\rangle \otimes |(g_2,h) \xrightarrow{a}\rangle \tag{12.47}$$

Here, $[g_i, h] = 1_G$ and

$$\begin{aligned} \gamma_h(g_1,g_2,g_3) &:= \frac{\alpha(g_1,g_2,g_3,h)\alpha(g_1,h,g_2,g_3)}{\alpha(g_1,g_2,h,g_3)\alpha(h,g_1,g_2,g_3)} \\ \gamma_{\eta,h}(g_1,g_2) &:= \frac{\gamma_h(g_1,g_2,\eta)\gamma_h(\eta,\eta^{-1}g_1\eta,\eta^{-1}g_2\eta)}{\gamma_h(g_1,\eta,\eta^{-1}g_2\eta)} \end{aligned} \tag{12.48}$$

It can be verified $\gamma_h(g_1,g_2,g_3)$ defines a 3-cocycle $\gamma_h \in H^3(Z(h),U(1))$.

Using the 4-cocycle conditions, we can verify the following properties of Δ:
Δ is Quasi-coassociative

$$(\Delta \otimes 1)\Delta((g,h) \xrightarrow{a}) = \phi_h (1 \otimes \Delta)\Delta((g,h) \xrightarrow{a})\phi_h^{-1} \tag{12.49}$$

with associator

$$\phi_h := \sum_{g_1,g_2,g_3} \gamma_h(g_1,g_2,g_3)\,|(g_1,h) \xrightarrow{1_G}\rangle \otimes |(g_2,h) \xrightarrow{1_G}\rangle \otimes |(g_3,h) \xrightarrow{1_G}\rangle\,. \tag{12.50}$$

for all $(g,h) \xrightarrow{a} \in T(G)$. This follows from the identity:

$$\gamma_h^{-1}(g_1,g_2,g_3)\gamma_{\eta,h}(g_1,g_2)\gamma_{\eta,h}(g_1g_2,g_3)\gamma_h(\eta^{-1}g_1\eta,\eta^{-1}g_2\eta,\eta^{-1}g_3\eta) = \gamma_{\eta,h}(g_2,g_3)\gamma_{\eta,h}(g_1,g_2g_3) \tag{12.51}$$

Δ is an algebra homomorphism:

$$\Delta((g,h)\xrightarrow{a})\Delta((g',h')\xrightarrow{a'}) = \Delta((g,h)\xrightarrow{aa'})\delta_{g',a^{-1}ga}\delta_{h',a^{-1}ha} \tag{12.52}$$

for all $(g,h)\xrightarrow{a}, (g',h')\xrightarrow{a'} \in T(G)$, which follows from the identity:

$$\beta_{g,h_1h_2}(x,y)\gamma_{xy,g}(h_1,h_2) = \beta_{g,h_1}(x,y)\beta_{g,h_2}(x,y)\gamma_{x,g}(h_1,h_2)\gamma_{y,x^{-1}gx}(x^{-1}h_1x,x^{-1}h_2x) \tag{12.53}$$

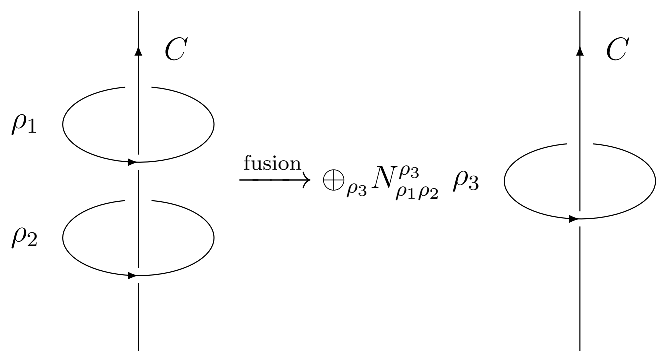
Using the comultiplication map and the semisimplicity of $T(G)$ we can naturally define the tensor product of representations:

$$\rho_1 \otimes \rho_2 = \oplus_{\rho_3} N^{\rho_3}_{\rho_1,\rho_2}\rho_3 \tag{12.54}$$

We interpret this tensor decomposition as the fusion of loop-like excitations, analogously to the particle fusion in the twisted quantum double, where the fusion coefficients:

$$\begin{aligned} N^{\rho_3}_{\rho_1,\rho_2} &:= \frac{1}{|G|}\sum_{\substack{h,a\in G\\ g\in Z(h)}} \mathrm{Tr}(\mathcal{D}^{\rho_1\otimes\rho_2}(\Delta((g,h)\xrightarrow{a}))\overline{\mathcal{D}^{\rho_3}((g,h)\xrightarrow{a})}) \\ &= \frac{1}{|G|}\sum_{g_1,g_2,h}\gamma_{a,h}(g_1,g_2)\chi^{\rho_1}((g_1,h)\xrightarrow{a}))\chi^{\rho_2}((g_2,h)\xrightarrow{a}))\overline{\chi^{\rho_3}((g_1g_2,h)\xrightarrow{a}))}. \end{aligned} \tag{12.55}$$

We visualise this process of fusion as follows:



(12.56)

Where the value of the threaded flux $C \subseteq G$ applies a constraint on the loop-like topological excitation types admissible.

Aside from the comulitplication structure $T(G)$ also admits an involution $S : T(G) \to T(G)$:

$$S((g,h) \to a) = \frac{1}{\beta_{h,g^{-1}}(a,a^{-1})\gamma_{a,h}(g,g^{-1})}(a^{-1}g^{-1}a, a^{-1}ha) \xrightarrow{a^{-1}} \tag{12.57}$$

defining an algebra anti-homomorphism

$$S(a \cdot b) = S(b) \cdot S(a) \tag{12.58}$$

Again using the twisted quantum double as inspiration we expect this map to be related to dual representations of the twisted quantum double such that:

$$\mathcal{D}^{\rho^*}_{II'}((g,h) \to a) := \mathcal{D}^{\rho}_{I'I}(S((g,h) \to a)) \tag{12.59}$$

where ρ^* is the dual representation of ρ.

Although this structure has many of the hall marks of a quasi-Hopf algebra [34] akin to the twisted quantum double. This is not the case. It turns out in practice that there do not exist conunit maps which are also algebra homomorphisms. In the case that the 4-cocycle is trivial, this algebra becomes a weak Hopf algebra. In my review of the subject there does not seem to be an existing definition of this structure although the correct description should be related to Hopf algebras which are simultaneously weak in the sense that the counit is not an algebra homomorphism but also quasi as dictated by the non-trivial cocycles. We leave such subtleties to further research.

Chapter 13

Tube Algebras for Topological Higher Lattice Gauge Theories

In this chapter we will consider the general case of tube algebras in topological higher lattice gauge theories. In particular we will begin by outlining some properties of 2-groupoids we will utilise in the following before describing the simplest tube algebra, the 1+1D theory and comparing the results to the Dijkgraaf-Witten theory. Building on this example we will then introduce the general formulation of the tube algebra in $n+1$D before giving examples in 2+1D and 3+1D mirroring the discussion with the Dijkgraaf-Witten theory. In particular we will find that the untwisted Dijkgraaf-Witten theory forms an example of the topological higher lattice gauge theories.

The general recipe for constructing the topological higher lattice gauge theory tube algebra for gauge 2-group $B\mathcal{G}$ defined by the crossed module $\mathcal{G} = (G, E, \partial, \triangleright)$ and lattice W, is given by first constructing the functor 2-groupoid $[\Pi_2(W), B\mathcal{G}]$ (see corollary 7.3.0.1) and demonstrating that there canonically exists an ordinary groupoid $\mathbb{P}[\Pi_2(W), B\mathcal{G}]$, whose corresponding groupoid-like algebra defines the W-tube algebra.

13.1 Properties of 2-Groupoids

Before we discuss the general construction of the tube algebra for topological higher lattice gauge theories, it is informative to define some general properties of 2-groupoids.

Definition 13.1.1. Let Γ^2 be a strict 2-groupoid and $x, y \in \Gamma_0^2$ a pair of objects, we call x and y **connected** if there exists a morphism $x \xrightarrow{k} y \in \Gamma_1^2$. This property defines an equivalence relation and we call the equivalence classes **connected components**. We notate the set of connected components by $\pi_0(\Gamma^2)$.

Note the similarity with definition 3.2.3 for the connected component of a groupoid. Analogously we can define an equivalence relation on the set of morphisms in a strict 2-groupoid as follows:

Definition 13.1.2. Let Γ^2 be a strict 2-groupoid and $g, g' \in \Gamma_1^2$ a pair of morphisms, we call g and g' **2-connected** if there exists a 2-morphism $g \overset{A}{\Rightarrow} g' \in \Gamma_2^2$. This property defines an equivalence relation and we call the equivalence classes **2-connected components**. We notate the set of 2-connected components by $\pi_0^1(\Gamma^2)$.

Definition 13.1.3. Let Γ^2 be a strict 2-groupoid and $g \in \Gamma_1^2$ a morphism, we define the set of 2-morphisms with 2-source g via:

$$M^2(g) := \{A \in \Gamma_2^2 | s^2(A) = g\}$$

Proposition 13.1.1. Let $C_i \in \pi_0(\Gamma^2)$ be a connected component of a strict 2-groupoid Γ^2 and $x, y \in C_i$. Given a pair of morphisms $g, h \in \Gamma_1^2$ such that $s^1(g) = x$ and $s^1(h) = y$, then there exists a bijection between $M^2(g)$ and $M^2(h)$.

Proof. From the definition of a connected component, if $x, y \in C_i$ there exists $x \xrightarrow{k} y \in \Gamma_1^2$, such that we can define a pair of functions:

$$\begin{aligned} &\phi : M(h) \to M(g) \\ &\phi : B \mapsto 1_k^2 \cdot B \cdot 1_{h^{-1}kg}^2, \quad \forall B \in M^2(h) \end{aligned} \tag{13.1}$$

and

$$\begin{aligned} &\phi^{-1} : M(g) \to M(h) \\ &\phi^{-1} : A \mapsto 1_{k^{-1}}^2 \cdot A \cdot 1_{g^{-1}kh}^2, \quad \forall A \in M^2(g). \end{aligned} \tag{13.2}$$

Using the identities $1^2_g \cdot 1^2_h = 1^2_{gh}$ and $1^2_{s^1(s^2(A))} \cdot A = A \cdot 1^2_{t^1(s^2(A))}$ for all $g, h \in \Gamma^2_1$ and $A \in \Gamma^2_2$ in the definition of Γ^2 if follows ϕ and ϕ^{-1} satisfy:

$$\phi\phi^{-1} = 1_{M^2(h)} \qquad \phi^{-1}\phi = 1_{M^2(g)} \tag{13.3}$$

where $1_{M^2(h)}/1_{M^2(g)}$ is the identity function on $M^2(h)/M^2(g)$ such that ϕ defines a bijection. □

From this proposition it is immediate that for all $g \in \Gamma^2_1$, $|M^2(g)|$ depends only on the connected component of $s^1(g)$ and hence we will often use the notation $M^2(C_i) := |M^2(g)|$ whenever $s(g) \in C_i \in \pi_0(\Gamma^2)$.

Definition 13.1.4. Let Γ^2 be a strict 2-groupoid and $g \in \Gamma^2_1$ a morphism. The **2-stabiliser** of $g \in \Gamma^2_1$ is the group:

$$\pi^2_1(g) := \{A \in \Gamma^2_2 | s^2(A) = t^2(A) = g\} \tag{13.4}$$

with group product given by vertical composition.

Proposition 13.1.2. Let Γ^2 be a strict 2-groupoid and $C_i \in \pi_0(\Gamma^2)$ a connected component. For all $g, h \in \Gamma^2_1$ such that $s^1(g), s^1(h) \in C_i$, $\pi^2_1(g) \cong \pi^2_1(h)$.

Proof. Follows from the proof of proposition 13.1.1 by noting that the functions ϕ and ϕ^{-1} defined in equations (13.1) and (13.2) applied to $\pi^2_1(h)$ and $\pi^2_1(g)$ respectively define a group isomorphism by application of the interchange law $(A_1 \cdot A_2) \circ (B_1 \cdot B_2) = (A_1 \circ B_1) \cdot (A_2 \circ B_2)$ for all composable $A_1, A_2, B_1, B_2 \in \Gamma^2_2$ and $1^2_g \circ 1^2_g = 1^2_g$ for all $g \in \Gamma^2_1$. □

From this proposition, similarly to $M^2(g)$ we will often use the notation $|\pi^2_1(g)| := |\pi^2_1(C_i)|$ whenever $s^1(g) \in C_i \in \pi_0(\Gamma^2)$.

Definition 13.1.5. Given a strict 2-groupoid Γ^2 and morphism $g \in \Gamma^2_1$, the **2-orbit** is defined as follows:

$$Orb^2(g) = \{h \in \Gamma^2_1 | \exists A \in \Gamma^2_2, s^2(A) = g, t^2(A) = h\} \tag{13.5}$$

Proposition 13.1.3. Given a strict 2-groupoid Γ^2, for all morphisms $g \in \Gamma^2_1$,

$$|M^2(g)| = |\pi^2_1(g)||Orb^2(g)|$$

Proof. Follows from orbit-stabiliser theorem. ◻

This result combined with the previous propositions imply $|Orb^2(C_i)| := |Orb^2(g)|$ depends only on the connected component of $s^1(g) \in C_i \in \pi_0(\Gamma^2)$ for all $g \in \Gamma_1^2$ and

$$|Orb^2(C_i)| = \frac{|M^2(C_i)|}{|\pi_1^2(C_i)|}. \tag{13.6}$$

We now make our last definition of this section:

Definition 13.1.6. Given a strict 2-groupoid Γ^2 there exists a strict groupoid $\mathbb{P}\Gamma^2$, we call the **underlying groupoid** such that:

- Objects:
$$\mathbb{P}\Gamma_0^2 := \Gamma_0^2$$
- Morphisms:
$$\mathbb{P}\Gamma_1^2 := \Gamma_1^2/\sim$$
where $\sim$ is the 2-connected equivalence class.
- Composition: induced from composition of morphisms in Γ_1^2 by noting for all $g, g', h, h' \in \Gamma_1^2$ such that $g \sim g'$, $h \sim h'$ and $t(g) = t(g') = s(h) = s(h')$, $gh \sim g'h'$ follows from horizontal composition in Γ_1^2.

$$\begin{tikzcd} x \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, "g'"'] \arrow[r, phantom, "\Downarrow A"] & y \arrow[r, bend left=50, "h"] \arrow[r, bend right=50, "h'"'] \arrow[r, phantom, "\Downarrow B'"] & z \arrow[r, phantom, "="] & x \arrow[r, bend left=50, "gg'"] \arrow[r, bend right=50, "hh'"'] \arrow[r, phantom, "\Downarrow A\cdot B"] & z \end{tikzcd} \tag{13.7}$$

It is straightforward to show $\mathbb{P}\Gamma^2$ is indeed a groupoid. Further more using the previous propositions it follows that for all $x \in C_i \in \pi_0(\mathbb{P}\Gamma^2) = \pi_0(\Gamma^2)$,

$$|M(x)|_{\mathbb{P}\Gamma^2} := |M(C_i)|_{\mathbb{P}\Gamma^2} = \frac{|M(C_i)|_{\Gamma^2}}{|Orb^2(C_i)|_{\Gamma^2}}. \tag{13.8}$$

Here the subscripts are to clarify which groupoid/2-groupoid we are evaluating the set in.

13.2 State Sum Tube Algebra for 1+1D Higher Lattice Gauge Theory

The simplest example of the higher lattice gauge theory tube algebra is the $1{+}1D$ example. In this case we consider our tubes to be given by a lattice decomposition of the interval $\mathcal{P}_{tube} := [0, 1]$.

We begin by describing the fundamental 2-groupoid (see definition 8.5.1) of the dressed lattice $(*, v)$ of a a single vertex v. The fundamental 2-groupoid $\Pi_2(*, v)$ is the trivial strict 2-groupoid consisting of:

- Single object $v \in \Pi_2(*, v)_0$
- Identity morphism $v \xrightarrow{1_v} v \in \Pi_2(*, v)_1$
- Identity 2-morphism $\begin{tikzcd} v \arrow[r, bend left=50, "1_v"{name=U}] \arrow[r, bend right=50, "1_v"'{name=D}] & v \arrow[Rightarrow, from=U, to=D, "\mathbb{1}_{1_v}"] \end{tikzcd} \in \Pi_2(*, v)_2$.

Using the definition of the fundamental 2-groupoid $\Pi_2(*, v)$ we can define the functor 2-groupoid $[\Pi_2(*, v), B\mathcal{G}]$ as follows:

- Objects: strict 2-functors $F : \Pi_2(*, v) \to \mathcal{G}$. Such functors assign to the vertex v the single object of $B\mathcal{G}$.
- Morphisms: pseudo-natural transformations $\{g : F \to F\}_{\forall g \in G}$
- 2-Morphisms are given by pseudo-natural transformations $\{\eta : g \Rightarrow \partial(\eta)g\}_{\forall \eta \in E}$

Now using this data we wish to define the P-tube algebra $(\mathcal{H}[P_{tube}], \circ)$. From lemma 9.5.1 we can identify configurations $s(P_{tube})$ with the morphism set of $[\Pi_2(*, v), B\mathcal{G}]$.

$$s(P_{tube}) := \{|\bullet \xrightarrow{\ \ g\ \ } \bullet\rangle\}_{\forall g \in G} \tag{13.9}$$

In order to define the ground state subspace $\mathcal{H}[P_{tube}]_0$ we need to define the subspace of $\mathbb{C}s(P_{tube})$ which is invariant under the ground state projector $P(P_{tube}; B\mathcal{G})$.

To this end we endow $\mathbb{C}s(P_{tube})$ with an orthonormal inner product:

$$\langle \bullet\xrightarrow{\quad g' \quad}\bullet | \bullet\xrightarrow{\quad g \quad}\bullet \rangle = \delta_{g,g'}. \tag{13.10}$$

From this relation we can define the ground state projector as:

$$P(P_{tube};\mathcal{G}) = \frac{1}{|E|}\sum_{\substack{g\in G\\ H\in E}} |\bullet\xrightarrow{\quad \partial(H)g \quad}\bullet\rangle\langle\bullet\xrightarrow{\quad g \quad}\bullet| \tag{13.11}$$

In this way we can immediately define the dimension of $\mathcal{H}[P_{tube}]_0$:

$$dim\mathcal{H}[P_{tube}]_0 = TrP(P_{tube};B\mathcal{G}) = \frac{1}{|E|}\sum_{\substack{g\in G\\ H\in E}} \delta_{g,\partial(H)g} = \frac{|G||ker\partial|}{|E|} = \frac{|G|}{|Im\partial|} \tag{13.12}$$

where in the last identity we used $|E| = |ker\partial||Im\partial|$ which follows from ∂ defining a homomorphism $\partial : E \to G$. The ground state degeneracy is an integer by noting $Im\partial$ is a normal subgroup of G.

Let $\pi_0^1([\Pi_2(*,v),B\mathcal{G}]) = \{\mathcal{C}_1,\cdots,\mathcal{C}_{|\pi_0^1([\Pi_2(*,v),B\mathcal{G}])|}\}$ be the set of 2-connected components of $[\Pi_2(*,v),B\mathcal{G}]$. It is straightforward to verify $|\mathcal{C}| = \frac{|G|}{|Im\partial|}$. Furthermore given a representative element $g \in C_i$ we can make the identification $g(Im\partial) = \mathcal{C}_i$. In this way, noting $Im\partial$ is a normal subgroup of G we can naturally associate to each 2-connected component $\mathcal{C}_i$ an element $i \in G_\partial := G/Im\partial$. Each connected 2-component $\mathcal{C}_i$ defines an orthonormal basis element of $\mathcal{H}[P_{tube}]_0$ by taking a representative element $g \in \mathcal{C}_i$ and symmeterising over the edge gauge operator such that:

$$\begin{aligned} |\xrightarrow{i}\rangle &:= \frac{1}{\sqrt{|E||ker\partial|}}\sum_{H\in E} |\bullet\xrightarrow{\quad \partial(H)g \quad}\bullet\rangle\,. \\ \langle\xrightarrow{i'} | \xrightarrow{i}\rangle &= \delta_{i,i'} \end{aligned} \tag{13.13}$$

and we conclude:

$$\mathcal{H}[P_{tube}]_0 = \mathbb{C}\{|\xrightarrow{i}\rangle\}_{i\in G_\partial}. \tag{13.14}$$

We can now define the gluing operator on elements of $\mathbb{C}s(P_{tube})$ before linearising to elements of $\mathcal{H}[P_{tube}]_0$:

$$
\begin{aligned}
&|\, \overset{0}{\bullet} \xrightarrow{\quad g \quad} \overset{1}{\bullet} \,\rangle \;\; |\, \overset{1}{\bullet} \xrightarrow{\quad h \quad} \overset{2}{\bullet} \,\rangle \xrightarrow{glue} |\, \overset{0}{\bullet} \xrightarrow{\quad g \quad} \overset{1}{\bullet} \xrightarrow{\quad h \quad} \overset{2}{\bullet} \,\rangle \\
&\xmapsto{\mathcal{Z}[GL]_P} \frac{1}{\sqrt{|G||E|}} \sum_{H\in E} |\, \overset{0}{\bullet} \xrightarrow{\partial(H)gh} \overset{2}{\bullet} \,\rangle
\end{aligned}
\tag{13.15}
$$

where

$$
\mathcal{Z}[GL_P] = \mathcal{Z}\left[\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd}\right] = \frac{1}{\sqrt{|G||E|}} \sum_{\substack{g,h\in G \\ H\in E}} |\, \bullet \xrightarrow{\partial(H)gh} \bullet \,\rangle (\langle \bullet \xrightarrow{g} \bullet | \otimes \langle \bullet \xrightarrow{h} \bullet |)
\tag{13.16}
$$

In this way we see the product acts by composition of the basis elements of $\mathbb{C}s(P_{tube})$ in G followed by applying the ground state projector. On basis elements of $\mathcal{H}[P_{tube}]_0$ the product is given by:

$$
\begin{aligned}
&|\xrightarrow{i}\rangle\, |\xrightarrow{j}\rangle = \Big(\frac{1}{\sqrt{|E||ker\partial|}} \sum_{F\in E} |\bullet \xrightarrow{\quad \partial(F)g \quad} \bullet\rangle\Big)\Big(\frac{1}{\sqrt{|E||ker\partial|}} \sum_{H\in E} |\bullet \xrightarrow{\quad \partial(H)h \quad} \bullet\rangle\Big) \\
&= \frac{1}{|E|^2|ker\partial|\sqrt{|G|}} \sum_{\eta,F,H\in E} |\, \bullet \xrightarrow{\partial(\eta F g \triangleright H)gh} \bullet \,\rangle = \frac{1}{\sqrt{|E||ker\partial|}} \sqrt{\frac{|Im\partial|}{|G|}} \sum_{\eta\in E} |\bullet \xrightarrow{\quad \partial(\eta)gh \quad} \bullet\rangle \\
&= \sqrt{\frac{|Im\partial|}{|G|}}\, |\xrightarrow{ij}\rangle
\end{aligned}
\tag{13.17}
$$

where in the last equality we identify $ij \in G_\partial$ as the group product of $i, j \in G_\partial$ in G_∂. In this way the P-tube algebra for the higher lattice gauge theory is given by the groupoid-like algebra $\widetilde{BG_\partial}$. This can also be seen as the groupoid-like algebra $\mathbb{CP}[\widetilde{\Pi_2(*, v)}, B\mathcal{G}] = \mathbb{CP}\widetilde{B\mathcal{G}}$.

At this point we compare this result to the $1+1$D Dijkgraaf-Witten P-tube algebra ${}^{\beta}\mathbb{C}\widetilde{BG}$ (see section 12.1). This algebra is the same as that arising in the Dijkgraaf-Witten TQFT when the group is given as G_∂ and we choose the 2-cocycle $\beta \in H^2(G, U(1))$ to be trivial, ie. $\beta(g, h) = 1$ for all $g, h \in G$. As such in analogy with gauge theories we associate to each irreducible representation ρ_i of G_∂ a point-like topological excitation type with charge ρ_i.

The interesting observation is that when $ker\partial \neq E$ the gauge symmetry of the edges G is broken down to the normal subgroup to G_∂. In this way it seems natural that the presence of $ker\partial \neq E$ corresponds to a confinement mechanism in the theory. This makes sense in terms of the 2-gauge configurations defined by:

$$F : \Pi_2(\mathcal{M}) \to B\mathcal{G}. \tag{13.18}$$

for some lattice approximation of space $\mathcal{M}$. In general when $\ker\partial \neq E$ the colouring of edges around a plaquette is not defined by a flat connection:

$$F_1 : \pi_1(\mathcal{M}) \to BG \tag{13.19}$$

but instead the 1-holonomy around a plaquette is given by $\partial(H)$ where H is the 2-holonomy of the face and as such does not give rise to a homotopy invariant holonomy. In such a case taking the quotient of a holonomy $G/Im\partial$ does return a holonomy which only depends on the homotopy class of the path. Hence particles are not strictly topological if they are charged under the full edge gauge group G. The relation between higher gauge symmetry and confinement is discussed in [100] although such ideas are still in need of further development.

13.3 State Sum Tube Algebra for Higher Lattice Gauge Theory

In this section we outline the general case for the higher lattice gauge theory tube algebra and discuss examples in the subsequent sections. The main result used in this section is lemma 9.5.1 which relates 2-gauge transformations of the fundamental 2-groupoid $\Pi_2(M, L)$ to 2-gauge configurations of the 2-groupoid $\Pi_2(M \times I, L \times I)$. This section is structured to closely follow the previous 1+1D example to provide intuition of the calculation.

In the following we will consider the $n+1$D Yetter TQFT and fixed finite crossed module $\mathcal{G} = (G, E, \partial, \triangleright)$. Additionally let W be a closed, compact, oriented $n-1$-manifold with dressed lattice decomposition $\mathcal{W} := (W, L, \Rightarrow)$ and corresponding lattice 2-groupoid $\Gamma^2(\mathcal{W}) := \Gamma^2(W, L)$ and fundamental 2-groupoid $\Pi_2(\mathcal{W}) := \Pi_2(W, L)$. For the following we will also assume without loss of generality there is an enumeration of the vertices and edges of $\mathcal{W}$.

The data required to define the $\mathcal{W}$-tube algebra is given by defining the functor 2-groupoid $[\Pi_2(\mathcal{W}), B\mathcal{G}]$ (see corollary 7.3.0.1). This 2-groupoid is defined by:

- Objects: 2-flat 2-gauge configurations (equivalently strict 2-functors)

$$\{F : \Pi_2(\mathcal{W}) \to B\mathcal{G}\} = [\Pi_2(\mathcal{W}), B\mathcal{G}]_0$$

  In particular the set of 2-flat 2-gauge configurations, can be specified by a subset of $G^{|L^0|} \times E^{|L^1|}$.

- Morphisms: 2-gauge transformations (equivalently pseudo-natural transformations)

$$\{F \xrightarrow{\eta} \eta \cdot F\} = [\Pi_2(\mathcal{W}), B\mathcal{G}]_1$$

  Each 2-gauge transformation, is specified by a 2-gauge configuration F defining the source and an element $\eta \in G^{|L^0|} \times E^{|L^1|}$ (see section 9.2). The target object is uniquely specified by this pair, in this way we will often not specify the target in diagrams. We use the notation $\eta \cdot F$ for the target object to highlight that 2-gauge transformations are given by a group action on the set of 2-gauge configurations.

- Composition: the composition of 2-gauge transformations was defined in 9.2:

$$F \xrightarrow{\eta} \eta \cdot F \xrightarrow{\eta'} \eta\eta' \cdot F = F \xrightarrow{\eta\eta'} \eta\eta' \cdot F$$

where for $\eta = (g_1, \cdots, g_{|L|^0}; H_1, \cdots, H_{|L|^0}), \eta' = (g'_1, \cdots, g'_{|L|^0}; H'_1, \cdots, H'_{|L|^0}) \in G^{|L^0|} \times E^{|L^1|}$,

$$\eta\eta' := (g_1 g'_1, \cdots, g_{|L|^0} g'_{|L|^0}; (g_{s(e_1)} \triangleright H'_1)H_1, \cdots, (g_{s(e_{|L^1|})} \triangleright H'_{|L^1|})H_{|L^1|}) \in G^{|L^0|} \times E^{|L^1|}$$

- 2-Morphisms: pseudo-modification equivalences

$$\left\{\begin{tikzcd}[column sep=huge] F \arrow[r, bend left=50, "\eta"] \arrow[r, bend right=50, "\mu\circ\eta"'] \arrow[r, phantom, "\Downarrow\mu"] & F' \end{tikzcd}\right\} = [\Pi_2(\mathcal{W}), \mathcal{G}]_2$$

Each pseudo-modification equivalence is specified by a 2-gauge transformation $F \xrightarrow{\eta}$ defining the 2-source and an element $\mu \in E^{|L^0|}$. This pair uniquely specifies the 2-target morphism and similarly as for morphisms we will often not specify the target in diagrams. We will use the notation $\mu \circ \eta$ for the target morphism of μ to highlight that pseudo-modifications are given by a group action on the set of morphisms. In general a pseudo-modification equivalence acts on $F \xrightarrow{\eta}$ where $\eta = (g_1, \cdots, g_{|L|^0}; H_1, \cdots, H_{|L|^0}) \in G^{|L^0|} \times E^{|L^1|}$ and $(\mu_1, \cdots, \mu_{|L^0|}) \in E^{|L^0|}$ via:

$$\mu : (g_1, \cdots, g_{|L|^0}; H_1, \cdots, H_{|L|^0}) \mapsto (\partial(\mu_1)g_1, \cdots, \partial(\mu_{|L^0|})g_{|L|^0}; \tilde{H}_1, \cdots, \tilde{H}_{|L|^0}).$$

In general $\tilde{H}_i$ will depend on μ, η and F. We can write down a general expression by requiring the following diagram is 2-commutative for each edge $v_i \xrightarrow{e_{ij}} v_j \in L^1$:

$$\begin{tikzcd}[row sep=huge, column sep=huge]
* \arrow[r, bend left, "g_i"] \arrow[r, bend right, "\partial(\mu_i)g_i"'] \arrow[r, phantom, "\Downarrow\mu_i"] \arrow[d, "F(e_{ij})"'] \arrow[dr, phantom, "H_{ij}\Nearrow\ \Nearrow\tilde{H}_{ij}"] & * \arrow[d, "\tilde{F}(e_{ij})"] \\
* \arrow[r, bend left, "g_j"] \arrow[r, bend right, "\partial(\mu_j)g_j"'] \arrow[r, phantom, "\Downarrow\mu_j"] & *
\end{tikzcd} \tag{13.20}$$

which implies

$$\tilde{H}_{ij} = \mu_i H_{ij}(F(e_i) \triangleright \mu_j^{-1}). \tag{13.21}$$

Note pseudo-modifications do not change the source and targets of a 2-gauge transformations and hence in terms of the group actions $(\mu \circ \eta) \cdot F = \eta \cdot F$ for all $\mu \in E^{|L^0|}$.

- Vertical and horizontal composition:

$$\begin{tikzcd} F \arrow[r, bend left=60, "\eta"] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3ex, "\Downarrow\mu"] \arrow[r, phantom, shift right=3ex, "\Downarrow\mu'"] & {} \end{tikzcd} = \begin{tikzcd} F \arrow[r, bend left=40, "{(g,H)}"] \arrow[r, bend right=40] \arrow[r, phantom, "\Downarrow\mu'\mu"] & {} \end{tikzcd}$$

$$\begin{tikzcd} F \arrow[r, bend left=60, "\eta"] \arrow[r, bend right=60] \arrow[r, phantom, "\Downarrow\mu"] & {} \arrow[r, bend left=60, "\eta'"] \arrow[r, bend right=60] \arrow[r, phantom, "\Downarrow\mu'"] & {} \end{tikzcd} = \begin{tikzcd} F \arrow[r, bend left=60, "\eta\eta'"] \arrow[r, bend right=60] \arrow[r, phantom, "\Downarrow\mu(\eta\triangleright\mu')"] & {} \end{tikzcd} \tag{13.22}$$

whenever defined. Here $\mu'\mu \in E^{|L^0|}$ is just the group composition of μ' and μ in $E^{|L^0|}$ and $\eta \triangleright \mu$ is defined for $\eta = (g_1, \cdots, g_{|L|^0}; H_1, \cdots, H_{|L|^0}) \in G^{|L^0|} \times E^{|L^1|}$ and $\mu = (\mu_1, \cdots, \mu_{|L^0|}) \in E^{|L^0|}$ in terms of the action $\triangleright : G \to Aut(E)$ defined by $\mathcal{G}$ via:

$$\eta \triangleright \mu := (g_1 \triangleright \mu_1, \cdots, g_{|L^0|} \triangleright \mu_{|L^0|}) \in |E|^{|L^0|}.$$

Utilising the data defined by $[\Pi_2(\mathcal{W}), \mathcal{G}]$ we can now define the $\mathcal{W}$-tube algebra. This first step is to define the 2-flat 2-gauge configurations of $\mathcal{W}_{tube} = (W \times I, L \times I)$. Following from lemma 9.5.1 this data is captured by $[\Pi_2(\mathcal{W}), \mathcal{G}]_1$. In this way we will define the vector space

$$\mathbb{C}[\Pi_2(\mathcal{W}), \mathcal{G}]_1 \tag{13.23}$$

with orthonormal inner product

$$\langle F' \xrightarrow{\eta'} | F \xrightarrow{\eta} \rangle = \delta_{F,F'} \delta_{\eta,\eta'}. \tag{13.24}$$

for all $F \xrightarrow{\eta}, F' \xrightarrow{\eta'} \in [\Pi_2(\mathcal{W}), B\mathcal{G}]_1$.

The next step is to define the ground state projector $P(\mathcal{W}_{tube}; B\mathcal{G})$. To this end we introduce the following lemma:

Lemma 13.3.1. Given a closed, compact, oriented manifold W with lattice decomposition $\mathcal{W} := (W, L)$ and corresponding lattice decompositon $\mathcal{W}_{tube} := (W \times I, L \times I)$. Given a 2-flat 2-gauge configuration $F : \Pi_2(\mathcal{W} \times I) \to B\mathcal{G}$, the set of 2-gauge transformations which restrict to the identity on $\mathcal{W} \times 0$ and $\mathcal{W} \times I$ are in one-to-one correspondence with elements $[\Pi_2(\mathcal{W}), B\mathcal{G}]_2$.

Proof. Follows from definition. For each vertex $v_i \in L^0(\mathcal{W})$ there exists an edge $= v_i \times I \in L^1(\mathcal{W} \times I)$. A 2-gauge transformation which restricts to the identity on $\mathcal{W} \times 0$ and $\mathcal{W} \times 1$ is given by the product of edge gauge spikes for each edge $v_i \times I$. Defining $\hat{A}^{\mu_i}_{v_i \times I}$ for each such vertex v_i, we see this defines an element $\mu = (\mu_1, \cdots, \mu_{|L^0|}) \in E^{|L^0|}$. Taking the definition of a 2-gauge transformation for each edge $v_i \times I$ and requiring the 2-gauge transformation variables for the source $v_i \times 0$ and target $v \times 1$ vertices, $\eta_{v_i \times 0} = 1_G, \eta_{v_i \times 1} = 1_G \in G$ as in the definition of $\hat{A}^{\mu_i}_{v_i \times I}$ the 2-commutative diagrams for the 2-gauge transformation reduce to the 2-commutative diagram in equation (13.20) defining a pseudo-modification equivalence in $[\Pi_2(\mathcal{W}), B\mathcal{G}]_1$. $\square$

Using this correspondence we can now define the ground state projector $P(\mathcal{W}_{tube}; B\mathcal{G})$ in terms of pseudo-natural equivalences:

$$P(\mathcal{W}_{tube}; B\mathcal{G}) = \frac{1}{|E|^{|L^0|}} \sum_{\mu \in [\Pi_2(\mathcal{W}), \mathcal{G}]} |t^2(\mu)\rangle \langle s^2(\mu)| \tag{13.25}$$

or equivalently in terms of the correspond group action on morphisms:

$$P(\mathcal{W}_{tube}; B\mathcal{G}) = \frac{1}{|E|^{|L^0|}} \sum_{\mu \in E^{|L^0|}} \sum_{\eta \in [\Pi_2(\mathcal{W}), B\mathcal{G}]} |\mu \circ \eta\rangle \langle \eta| \tag{13.26}$$

We can now define the dimension of $\mathcal{H}[\mathcal{W}_{tube}; B\mathcal{G}]_0$ as follows:

$$\begin{aligned} dim\mathcal{H}[\mathcal{W}_{tube}]_0 = TrP(\mathcal{W}_{tube}; B\mathcal{G}) &= \frac{1}{|E|^{|L^0|}} \sum_{\mu \in E^{|L^0|}} \sum_{\eta \in [\Pi_2(\mathcal{W}), B\mathcal{G}]_1} \delta_{\eta, \mu\circ\eta} \\ = \frac{1}{|E|^{|L^0|}} \sum_{\eta \in [\Pi_2(\mathcal{W}), \mathcal{G}]_1} |\pi_1^2(\eta)| &= \sum_{C_i \in \pi_0([\Pi_2(\mathcal{W}), B\mathcal{G}]} |C_i||M(C_i)|_{\mathbb{P}[\Pi_2(\mathcal{W}), B\mathcal{G}]} \\ = |\mathbb{P}[\Pi_2(\mathcal{W}), B\mathcal{G}]_1| & \end{aligned} \tag{13.27}$$

Here $\mathbb{P}[\Pi_2(\mathcal{W}), B\mathcal{G}]$ is the underlying groupoid of $[\Pi_2(\mathcal{W}), B\mathcal{G}]$ (see definition 13.1.6) and $|\mathbb{P}[\Pi_2(\mathcal{W}), B\mathcal{G}]_1|$ is the number of morphisms of $\mathbb{P}[\Pi_2(\mathcal{W}), B\mathcal{G}]$. This follows directly from the results outlined in section 13.1.

We now define a basis for $\mathcal{H}[\mathcal{W}_{tube}; B\mathcal{G}]_0$. From the definition of $\mathbb{P}[\Pi_2(\mathcal{W}), B\mathcal{G}]$ we can canonically associate to each morphism $\bar{\eta} \in \mathbb{P}[\Pi_2(\mathcal{W}), B\mathcal{G}]_1$ a corresponding 2-orbit of morphisms $D_i \subseteq [\Pi_2(\mathcal{W}), B\mathcal{G}]_1$. Given any representative $\eta \in D_i$ we define the vector:

$$|\bar{\eta}\rangle := \frac{1}{\sqrt{|E|^{|L^0|}|\pi_1^2(s^1(\eta))|_{[\Pi_2(\mathcal{W}),B\mathcal{G}]}}} \sum_{\mu \in E^{|L^0|}} |\mu \circ \eta\rangle \tag{13.28}$$

in terms of the vectors $|\eta\rangle \in \mathbb{C}[\Pi_2(\mathcal{W}), B\mathcal{G}]_1$. Such states are independent of the choice of representative element. It follows for all $\bar{\eta}, \bar{\eta}' \in \mathbb{P}[\Pi_2(\mathcal{W}), B\mathcal{G}]_1$ the canonically associated vectors $|\bar{\eta}\rangle, |\bar{\eta}\rangle$ are orthonormal:

$$\langle \bar{\eta} | \bar{\eta}' \rangle = \delta_{\bar{\eta},\bar{\eta}'}. \tag{13.29}$$

and satisfy

$$P(\mathcal{W}_{tube}; \gamma)\, |\bar{\eta}\rangle = |\bar{\eta}\rangle \tag{13.30}$$

such that we can make the identification:

$$\mathcal{H}[\mathcal{W}_{tube}; B\mathcal{G}]_0 = \mathbb{C}\{|\bar{\eta}\rangle\}_{\forall \bar{\eta} \in \mathbb{P}[\Pi_2(\mathcal{W}),B\mathcal{G}]_1} \subseteq \mathbb{C}[\Pi_2(\mathcal{W}), B\mathcal{G}]_1. \tag{13.31}$$

We now turn our attention to defining the tube algebra product. To do so, we need to evaluate the matrix elements of

$$\mathcal{Z}[GL_{\mathcal{W}}] : \mathcal{H}[\mathcal{W}_{tube}]_0 \otimes_{\mathcal{W}} \mathcal{H}[\mathcal{W}_{tube}]_0 \to \mathcal{H}[\mathcal{W}_{tube}]_0.$$

This task is vastly simpler compared to the Dijkgraaf-Witten theory, where we had to specify the whole triangulation. Here we will instead look at the matrix elements locally and infer the matrix elements using the fact that $\mathcal{Z}[GL_{\mathcal{W}}]$ depends only on the 3-skeleton of the lattice $GL_{\mathcal{W}}$ and is independent of the exact choice of lattice. In this way we will construct a canonical lattice for $GL_{\mathcal{W}}$ and describe 2-flat 2-gauge configurations from which we can read off matrix elements.

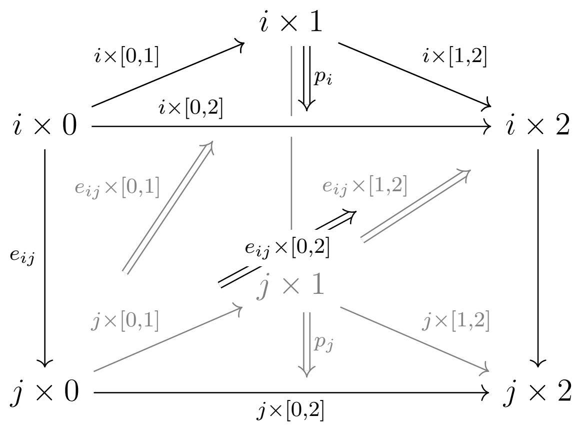
From the definition of $GL_{\mathcal{W}} : \mathcal{W}_{tube} \cup_{\mathcal{W}} \mathcal{W}_{tube} \to \mathcal{W}_{tube}$ we immediately determine the boundary lattice of $GL_{\mathcal{W}}$. In the following we will notate this pinched cobordism as follows:

$$GL_{\mathcal{W}} : \mathcal{W} \times [0,1] \cup_{\mathcal{W}\times 1} \mathcal{W} \times [1,2] \to \mathcal{W} \times [0,2].$$

From this data we will make the choice that the boundary of $GL_{\mathcal{W}}$ completely determines the 0- and 1-skeleton of $GL_{\mathcal{W}}$. To define the 3-skeleton will now introduce $|L^0|$ additional 2-cells and $|L^1|$ 3-cells. The additional 2-cells are given by: for each vertex i of $\mathcal{W}$ we have three edges $i \times [0,1], i \times [1,2], i \times [0,2]$ in the boundary of $GL_{\mathcal{W}}$ defining a triangle. To each such triple of edges we will add a triangular plaquette p_i.

$$\begin{tikzcd} & i \times 1 \arrow[dr, "i\times[1,2]"] \arrow[d, phantom, "p_i"] & \\ i \times 0 \arrow[ur, "i\times[0,1]"] \arrow[rr, "i\times[0,2]"'] & {} & i \times 2 \end{tikzcd} \tag{13.32}$$

The additional 3-cells of $GL_{\mathcal{W}}$ are as follows: for each edge $i \xrightarrow{e_{ij}} j$ in $\mathcal{W}$ we add a 3-cell to the interior of the prism



(13.33)

All i-cells for $i > 3$ will not contribute to the matrix elements of $\mathcal{Z}[GL_{\mathcal{W}}]$ and hence we can circumvent defining such cells.

We now define a 2-flat 2-gauge connection of $GL_{\mathcal{W}}$ in terms of a 2-flat 2-gauge connection specified by an element of $\mathcal{H}[\mathcal{W}_{tube}]_0 \otimes_{\mathcal{W}} \mathcal{H}[\mathcal{W}_{tube}]_0$. This data immediately defines a 2-flat 2-gauge configuration of $\mathcal{W} \times [0,1] \cup_{\mathcal{W}\times 1} \mathcal{W} \times [1,2] \subset GL_{\mathcal{W}}$. In order to uniquely specify the 2-flat 2-gauge configuration of the interior of $\mathcal{W} \times [0,2]$ we only need to specify an element $\mu_i \in E$ to each plaquette p_i:

$$\begin{tikzcd} & i \times 1 \arrow[dr, "g_i'"] \arrow[d, phantom, "\Downarrow\mu_i"] & \\ i \times 0 \arrow[ur, "g_i"] \arrow[rr, "\partial(\mu_i) g_i g_i'"'] & {} & i \times 2 \end{tikzcd} \tag{13.34}$$

In this way, requiring that assignment of group elements defines a 2-gauge configuration uniquely specifies the 2-gauge configuration of the edge $i \times [0,2]$ for each vertex i in $\mathcal{W}$. Requiring that each 3-cell is 2-flat or equivalently the boundary is 2-commutative uniquely specifies the plaquettes $e_{ij} \times [0,2]$ as follows:

$$\begin{tikzcd}
& i\times 1 \arrow[dr, "g'_i"] \arrow[dd, dash] \arrow[d, phantom, "\Downarrow \mu_i"] & \\
i\times 0 \arrow[ur, "g_i"] \arrow[rr, "\partial(\mu_i) g_i g'_i"] \arrow[dd, "F(e)"'] \arrow[dr, phantom, "\Uparrow H_e"] & \mu_i(g_i\triangleright H'_e)H_e(F(e)\triangleright\mu_j^{-1}) & i\times 2 \arrow[dd] \arrow[dl, phantom, "\Uparrow H'_e"] \\
& j\times 1 \arrow[dr, "g'_j"] \arrow[d, phantom, "\Downarrow \mu_j"] & \\
j\times 0 \arrow[ur, "g_j"] \arrow[rr, "\partial(\mu_j) g_j g'_j"'] & & j\times 2
\end{tikzcd} \tag{13.35}$$

Applying these rules to th whole of $GL_{\mathcal{W}}$ and comparing with the composition of morphisms in $[\Pi_2(\mathcal{W}), B\mathcal{G}]$ we can immediately write down the matrix elements of $\mathcal{Z}[GL_{\mathcal{W}}]$:

$$\mathcal{Z}[GL_{\mathcal{W}}] = \frac{1}{|G|^{\frac{|L^0|}{2}}|E|^{|L^0|+\frac{|L^1|}{2}}} \sum_{\mu \in E^{|L^0|}} \sum_{\eta,\eta' \in [\Pi_2(\mathcal{W}),B\mathcal{G}]_1} |\mu \circ \eta\eta'\rangle\, (\langle \eta| \otimes \langle \eta'|)\delta_{t^1(\eta),s^1(\eta')} \tag{13.36}$$

In terms of the basis elements of $\mathcal{H}[\mathcal{W}_{tube}]_0$ the product becomes:

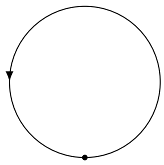
$$\begin{aligned}
&|\bar{\eta}\rangle\,|\bar{\eta}'\rangle = \\
&\frac{1}{|G|^{\frac{|L^0|}{2}}|E|^{2|L^0|+\frac{|L^1|}{2}}}\frac{1}{|\pi_1^2(C_i)|} \sum_{\mu,\mu_1,\mu_2 \in E^{|L^0|}} |\mu \circ [(\mu_1 \circ \eta_1)(\mu_2 \circ \eta_2)]\rangle\, \delta_{t^1(\eta_1),s^1(\eta_2)} \\
&= \frac{1}{|G|^{\frac{|L^0|}{2}}|E|^{2|L^0|+\frac{|L^1|}{2}}}\frac{1}{|\pi_1^2(C_i)|} \sum_{\mu,\mu_1,\mu_2 \in E^{|L^0|}} |\mu\mu_1(\eta_1 \triangleright \mu_2) \circ \eta_1\eta_2\rangle\, \delta_{t^1(\eta_1),s^1(\eta_2)} \\
&= \frac{1}{|G|^{\frac{|L^0|}{2}}|E|^{\frac{|L^1|}{2}}}\frac{1}{|\pi_1^2(C_i)|} \sum_{\mu \in E^{|L^0|}} |\mu \circ \eta_1\eta_2\rangle\, \delta_{t^1(\eta_1),s^1(\eta_2)} \\
&= \sqrt{\frac{|Orb^2(C_i)|}{|G|^{|L^0|}|E|^{|L^1|}}}\frac{1}{\sqrt{|E|^{|L^0|}|\pi_1^2(C_i)|}} \sum_{\mu \in E^{|L^0|}} |\mu \circ \eta_1\eta_2\rangle\, \delta_{t^1(\eta_1),s^1(\eta_2)} \\
&= \frac{1}{\sqrt{|M(C_i)|_{\mathbb{P}[\Pi_2(\mathcal{W}),B\mathcal{G}]}}}\, |\bar{\eta}\bar{\eta}'\rangle\, \delta_{t^1(\bar{\eta}),s^1(\bar{\eta}')}
\end{aligned} \tag{13.37}$$

In this way we see that the $\mathcal{W}$-tube algebra defines a groupoid-like algebra given by the groupoid $\mathbb{P}[\Pi_2(\mathcal{W}), \mathcal{G}]$ induced from the 2-groupoid $[\Pi_2(\mathcal{W}), \mathcal{G}]$ by sending morphisms to their equivalence class under the relation of being 2-connected.

13.4 State Sum Tube Algebra for 2+1D Higher Lattice Gauge Theory

After discussing the general case, we turn our attention to the 2+1D example. As in the Dijkgraaf-Witten case the boundary manifold necessarily has the topology of S^1. In the following we will utilise the simplest dressed lattice decomposition of S^1 consisting of a unique oriented edge and unique vertex:

$$\mathcal{W} = (S^1, L) := \tag{13.38}$$



The fundamental 2-groupoid $\Pi_2(\mathcal{W})$ is given as follows:

- Objects: unique vertex, $v = \Pi_2(\mathcal{W})_0$
- Morphisms: one non-trivial morphism

$$v \xrightarrow{e} v \in \Pi_2(\mathcal{W})_1$$

- 2-Morphisms: identity 2-morphisms:

$$\left\{ \begin{tikzcd} v \arrow[r, bend left, "1_v"] \arrow[r, bend right, "1_v"'] \arrow[r, phantom, "\Downarrow \mathbb{1}_{1_v}"] & v \end{tikzcd} , \begin{tikzcd} v \arrow[r, bend left, "e"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow \mathbb{1}_{e}"] & v \end{tikzcd} \right\} = \Gamma^2(S^1, L)_2 \tag{13.39}$$

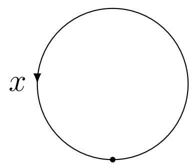
Given $\Pi_2(\mathcal{W})$ we can define the functor 2-groupoid $[\Pi_2(\mathcal{W}), \mathcal{G}]$ as follows:

- Objects: strict 2-functors

$$\{x : \Pi_2(\mathcal{W}) \to B\mathcal{G}\}_{\forall x \in G} = [\Pi_2(\mathcal{W}), B\mathcal{G}]_0$$

given by assigning $x \in G$ to the unique edge of $\mathcal{W}$

$$x(\Pi_2(\mathcal{W})) :=$$



- Morphisms: pseudo-natural transformations

$$\{x \xrightarrow{(g,H)}\}_{\forall x, g\in G, \forall H\in E} = [\Pi_2(\mathcal{W}), B\mathcal{G}]_1$$

The source and target maps are given by:

$$\begin{aligned} s^1(x \xrightarrow{(g,H)}) &= x \\ t^1(x \xrightarrow{(g,H)}) &= g^{-1}\partial(H)xg \end{aligned} \tag{13.40}$$

- Composition:

$$x \xrightarrow{(g,H)} g^{-1}\partial(H)xg \xrightarrow{(g',H')} = x \xrightarrow{(gg',(g\triangleright H')H)}$$

- 2-Morphisms: pseudo-modifications

$$\left\{ \begin{tikzcd} x \arrow[r, bend left=50, "{(g,H)}"] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow\eta"] & {} \end{tikzcd} \right\}_{\forall x, g\in G, H,\eta\in E} = [\Pi_2(\mathcal{W}), B\mathcal{G}]_2$$

The 2-source and 2-target maps are given by:

$$\begin{aligned} s^2\left(\begin{tikzcd} x \arrow[r, bend left=50, "{(g,H)}"] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow\eta"] & {} \end{tikzcd} \right) &= x \xrightarrow{(g,H)} \\ t^2\left(\begin{tikzcd} x \arrow[r, bend left=50, "{(g,H)}"] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow\eta"] & {} \end{tikzcd} \right) &= x \xrightarrow{(\partial(\eta)g, \eta H(x\triangleright\eta^{-1}))} \end{aligned} \tag{13.41}$$

- Vertical and horizontal composition:

$$
\begin{tikzcd}[column sep=large]
x \arrow[r, bend left=60, "{(g,H)}", ""{name=U, below}] \arrow[r, ""{name=M, above}, ""{name=M2, below}] \arrow[r, bend right=60, ""{name=D, above}] \arrow[Rightarrow, from=U, to=M, "\eta"] \arrow[Rightarrow, from=M2, to=D, "\eta'"] & {}
\end{tikzcd}
=
\begin{tikzcd}[column sep=large]
x \arrow[r, bend left=40, "{(g,H)}", ""{name=A, below}] \arrow[r, bend right=40, ""{name=B, above}] \arrow[Rightarrow, from=A, to=B, "\eta'\eta"] & {}
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=large]
x \arrow[r, bend left=60, "{(g,H)}", ""{name=A, below}] \arrow[r, bend right=60, ""{name=B, above}] \arrow[Rightarrow, from=A, to=B, "\eta"] & {} \arrow[r, bend left=60, "{(g',H')}", ""{name=C, below}] \arrow[r, bend right=60, ""{name=D, above}] \arrow[Rightarrow, from=C, to=D, "\eta'"] & {}
\end{tikzcd}
=
\begin{tikzcd}[column sep=large]
x \arrow[r, bend left=60, "{(gg',(g\rhd H')H)}", ""{name=E, below}] \arrow[r, bend right=60, ""{name=F, above}] \arrow[Rightarrow, from=E, to=F, "\eta(g\rhd\eta')"] & {}
\end{tikzcd}
\tag{13.42}
$$

whenever defined.

We now have defined $[\Pi_2(\mathcal{W}), B\mathcal{G}]$, from the discussion in the previous section we can immediately identify the $\mathcal{W}$-tube algebra product:

$$
|\bar{\eta}\rangle\,|\bar{\eta}'\rangle = \frac{1}{\sqrt{|M(s(\bar{\eta}))|_{\mathbb{P}[\Pi_2(\mathcal{W}),B\mathcal{G}]}}}\,|\bar{\eta}\bar{\eta}'\rangle\,\delta_{t(\bar{\eta}),s(\bar{\eta}')} \tag{13.43}
$$

for all $\bar{\eta}, \bar{\eta}' \in \mathbb{P}[\Pi_2(\mathcal{W}), B\mathcal{G}]_1$, where $|\bar{\eta}\rangle$ are defined in equation (13.28).

One immediate observation is that this algebra reduces to the untwisted Dijkgraaf-Witten $\mathcal{S}^1$-algebra when $E = 1_G$ (see section 12.2). From the algebra we see that fluxes associated to point-like topological excitation types are given by orbits of G under the action

$$
x \xrightarrow{(g,H)} g^{-1}\partial(H)xg, \quad \forall (g,H) \in G \times E. \tag{13.44}
$$

and the charges are given by irreducible representations of stabilisers of a representative of the orbit after taking the quotient by the equivalence $\sim$ of being 2-connected.

$$
\{(g,H) \in G \times E/\sim | x = g^{-1}\partial(H)xg\} \tag{13.45}
$$

Additionally the central subalgebra here defines an orthonormal basis for the torus T^2, and the dimension of this subalgebra gives the dimension of the torus groundstate degeneracy.

Example 13.4.1. We now give an example: Let $\mathcal{G} := (Z_2, Z_3, \partial, \triangleright)$ be a crossed module with $Z_3 = (\{0,1,2\}, +)$ and $Z_2 := (\{1,-1\}, \times)$, $\partial(a) = 1$, and $x \triangleright a = a^x$ for all $x \in Z_2$ and $a \in Z_3$. We now define $\mathbb{P}[\Pi_2(\mathcal{W}), B\mathcal{G}]$, there are two connected components given by the elements of Z_2. For the connected component 1, $|M(1)| = 6$, the pseudo-modifications such that no morphisms are identified and the simple modules are given by irreducible representations of $Z_2 \ltimes_{\triangleright} Z_3$. In the connected component -1, $|M(-1)| = 2$. The pseudo-modification identifies morphisms $-1 \xrightarrow{x,a} \sim -1 \xrightarrow{x,b}$ for all $a, b \in Z_3$. In this way the simple modules for the connected component -1 are given by irreducible representations of Z_2. We also see the groundstate degeneracy on the torus for such a theory is 5.

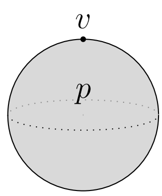
13.5 State Sum Tube Algebra for 3+1D Higher Lattice Gauge Theory

13.5.1 S^2-Tube Algebra

We now consider the $\mathcal{W}$-tube algebra for higher lattice gauge theory associated to the sphere S^2. We begin by defining a lattice for S^2. To this end we define the simplest directed lattice $\mathcal{W} := (S^2, L^0 = v, L^2 = p\}, \Rightarrow)$ corresponding to a single vertex v and single plaquette p. The plaquette is defined by defining the attaching map

$$\psi_p^2 := \partial D^2 \to (0,0)$$

which identifies all points of the boundary with the basepoint of D^2. This is visualised as follows:



(13.46)

From $\mathcal{W}$ we can define the fundamental 2-groupoid $\Pi_2(\mathcal{W})$ as follows:

- Objects: $\Pi_2(\mathcal{W})_0 = v$
- Morphisms: $\Pi_2(\mathcal{W})_1 = v \xrightarrow{1_v} v$, the unique identity morphism on v
- 2-Morphisms: $\Pi_2(\mathcal{W})_2 := \{$ $\begin{tikzcd} v \arrow[rr, bend left, "1_v"] \arrow[rr, bend right, "1_v"'] \arrow[rr, phantom, "\Downarrow F"] & & v \end{tikzcd}$, $\begin{tikzcd} v \arrow[rr, bend left, "1_v"] \arrow[rr, bend right, "1_v"'] \arrow[rr, phantom, "\Downarrow 1^2_{1_v}"] & & v \end{tikzcd}$ $\}$

Given the lattice 2-groupoid $\Pi_2(\mathcal{W})$ we define $[\Pi_2(\mathcal{W}), B\mathcal{G}]$ as follows:

- Objects: 2-flat 2-gauge configurations

  $$\{A : \Pi_2(\mathcal{W}) \to B\mathcal{G}\}_{\forall A \in ker\partial}$$

  Which for each element $A \in ker\partial$, $A(F) = A \in ker\partial$. We are restricted to $A \in ker\partial \subseteq E$ due to the source and target of F coinciding with the identity morphism 1_v.

- Morphisms: 2-gauge transformations

$$\{A \xrightarrow{g} g \triangleright A\}_{\forall g \in G}$$

Each 2-gauge transformation changes the 2-holonomy of F by the action $\triangleright : G \to Aut(\partial(H))$.

- 2-Morphisms: Pseudo-modifications

$$\begin{tikzcd} \{\ A \arrow[rr, bend left=50, "g"] \arrow[rr, bend right=50, "\partial(H)g"'] \arrow[rr, phantom, "\Downarrow H"] & & g \triangleright A\ \}_{\forall H \in E} \end{tikzcd}$$

From the axioms of a crossed module we can verify $\partial(E)g \triangleright A = E(g \triangleright A)E^{-1} = g \triangleright A$ where the last equality follows from $g \triangleright ker\partial = ker\partial$ for all $g \in G$.

From the previous discussion we can straight away identify the groupoid $\mathbb{P}[\Pi_2(\mathcal{W}), B\mathcal{G}] := ker\partial // G_\partial$ where $G_\partial = G/Im\partial$, as the action groupoid with objects $A \in ker\partial$ and action of $a \in G_\partial$ on $ker\partial$ defined by the crossed module action $\triangleright$.

In this way we can immediately deduce that the irreducible point-like topological excitations in $3+1$D topological higher lattice gauge theories are given by orbits of $ker\partial$ under the action of G and irreducible representations of the stabiliser of the orbit under the action of G_∂ induced from the action of G. We interpret such orbits of $ker\partial$ as the 2-flux associated to a point-particle in the sense it is only measurable to a 2-charged string traversing a sphere around the point and not a point-particle. We interpret the irreducible representation as an ordinary charge associated to the particle as it is measurable by the Aharonov-Bohm phase[101] with a flux loop. In the case $E = 1_G$ the W-tube algebra reduced to the Dijkraaf-Witten S^2-tube algebra where the point-particle like topological excitations types are classified by irreducible representations of G (see 12.3.1).

13.5.2 T^2-Tube Algebra

We now introduce the T^2 Tube Algebra. Let $T^2 = S^1 \times S^1$ be given a directed lattice $\mathcal{W} = (T^2, L^0 = v, L^1 = \{e, m\}, L^2 = p)$ given as follows:

$$\begin{tikzcd} v \arrow[r, "e"] \arrow[d, "m"'] \arrow[dr, phantom, "p"] & v \arrow[d, "m"] \\ v \arrow[r, "e"'] & v \end{tikzcd} \tag{13.47}$$

where all vertices are identified and the top, bottom and left, right edges respectively are identified.

From this data we can define the fundamental 2-groupoid $\Pi_2(\mathcal{W})$ in terms of the following generating morphisms:

- Objects: $v = \Pi_2(\mathcal{W})_0$
- Morphisms: $v \xrightarrow{e} v, v \xrightarrow{m} v \in \Pi_2(\mathcal{W})_1$
- 2-morphism: $\begin{tikzcd} v \arrow[r, bend left=50, "em"{name=U}] \arrow[r, bend right=50, "me"'{name=D}] & v \arrow[Rightarrow, from=U, to=D, "F"] \end{tikzcd} \in \Pi_2(\mathcal{W})_2 :=$

From this data we can define the functor 2-groupoid, $[\Pi_2(\mathcal{W})_0, B\mathcal{G}]$:

- Objects: 2-gauge configurations

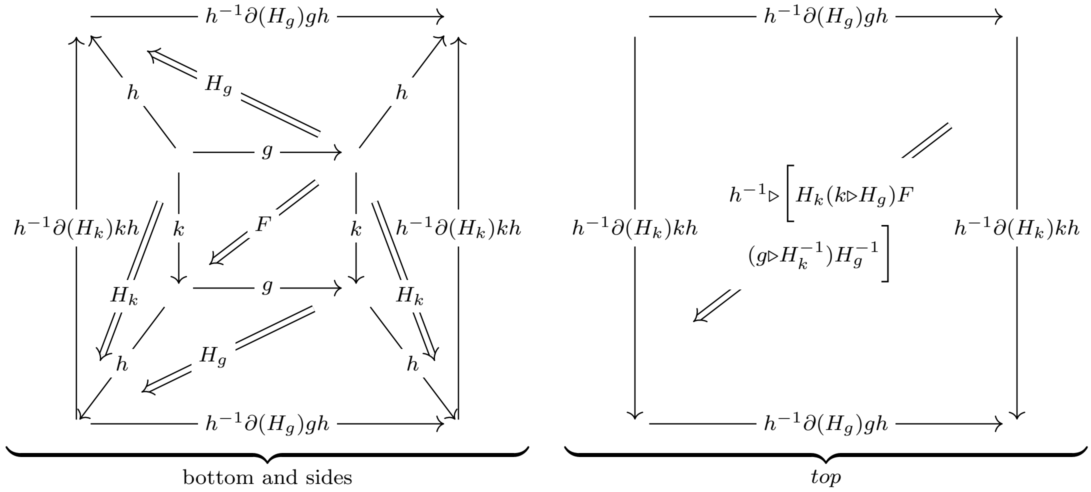
$$\{(g, h; H) : \mathcal{P}^2(\mathcal{W}) \to B\mathcal{G}\}_{\forall (g,h,H) \in G^2 \times_c E}$$

  where $G^2 \times_c E := \{(g, h, H) \in G \times G \times E | \partial(H) = ghg^{-1}h^{-1}\}$

- Morphisms: 2-gauge transformations

$$\{(g, k, F) \xrightarrow{(h, H_g, H_k)} (h^{-1}\partial(H_g)gh, h^{-1}\partial(H_k)kh, h^{-1}\triangleright \Big[H_k(k\triangleright H_g)F(g\triangleright H_k^{-1})H_g^{-1}\Big])\}$$

These can be visualised via:



(13.48)

- 2-Morphisms: pseudo-modifications

$$\Gamma^2(W;\mathcal{G})_2 := \{ \begin{tikzcd}(g,k,F) \arrow[r, bend left=50, "{(k,H_g,H_k)}"] \arrow[r, bend right=50, "{\mu\circ(k,H_g,H_k)}"'] \arrow[r, phantom, "\Downarrow\mu"] & {} \end{tikzcd} \}_{\forall \mu \in E}$$

where

$$\mu \circ (k, H_g, H_k) := (\partial(\mu)k, \mu H_g(g \triangleright \mu^{-1}), \mu H_k(k \triangleright \mu^{-1}))$$

From this data we immediately define the W-tube algebra.

There are a lot of subtleties in this tube algebra and we postpone a full interpretation of all details to future work. The algebra is suggestive that there are 1-fluxes associated to each non-contractible cycle of the torus given by $g, h \in G$ and a 2-flux whose image $\partial(F) = [g, h]$ specifies the degree to which the two elements are not required to commute, generalising the torus Dijkgraaf-Witten tube algebra, where we required $[g, h] = 1_G$ (see section 12.3.2). It is immediate that in the case $E = 1_G$ we recover the untwisted Dijkgraaf-Witten T^2 tube algebra. That $F \neq ker\partial$, we cannot interpret $F \in E$ as the 2-flux associated to a surface diffeomorphic to the sphere and instead a more thorough treatment is required to understand such observables in the theory. It appears that the gauge transformation variables $H_k, H_g \in E$ classify a 2-charge degree of freedom associated to 2-charged strings wrapping around the non-trivial cycles of the torus. The variable $h \in G$ corresponds to an ordinary charge degree of freedom. Moving forward it is important to understand how the transformation properties of all such degrees of freedom depend on each other. We do not know a consistent interpretation of the flux degrees of freedom akin to the Dijkgraaf-Witten theory and so postpone an interpretation of the tube algebra for future research.

Appendix A

Group Cohomology

In this chapter we review the basis ingredients of the group cohomology used in the definition of the Dijkgraaf-Witten TQFT. Let G be a finite group and M a G-**module**

Definition A.0.1. A G-**module**, M is an Abelian group M enriched with a G-action $\triangleright : G \times M \to M$ such that:

$$\begin{aligned} g \triangleright (ab) &= (g \triangleright a)(g \triangleright b), \quad \forall g \in G, \forall a, b \in M \\ gh \triangleright a &= g \triangleright (h \triangleright a), \quad \forall g, h \in G, \forall a \in M \end{aligned} \tag{A.1}$$

A function of the form

$$\begin{aligned} c^n &: G^n \to M \\ (g_1, \cdots, g_n) &\mapsto c^n(g_1, \cdots, g_n) \end{aligned} \tag{A.2}$$

is called an **n-cochain**. The set of all n-cochains is denoted by $C^n(G, M)$ and forms an Abelian group

$$c(g_1, \cdots, g_n)c'(g_1, \cdots, g_n) = cc'(g_1, \ldots, g_n)$$

with multiplication given by the structure of M. There exists a natural map

$$\delta^n : C^n(G, M) \to C^{n+1}(G, M) \tag{A.3}$$

called the **coboundary operator**, where

$$(\delta^n c^n)(g_1, \cdots, g_{n+1}) := [g_1 \triangleright c^n(g_2, \cdots, g_{n+1})] c^n(g_1, \cdots, g_n)^{(-1)^{n+1}} \prod_{i=1}^{n} c^n(g_1, \cdots, g_{i-1}, g_i g_{i+1}, \cdots, g_n)^{(-1)^i}. \tag{A.4}$$

We call n-cochains which satisfy

$$\delta^n c^n = 1 \tag{A.5}$$

n-cocycles and denote the subgroup of n-cocyles via $Z^n(G, M) \subseteq C^n(G, M)$. Given an n-cochain $c^n \in C^n(G, M)$ such that

$$c^n = \delta^{n-1} c^{n-1} \tag{A.6}$$

we call such n-cochains **n-coboundaries** and denote the subgroup of such n-cochains $B^n(G, M)$.

From these definitions we define the equivalence class of n-cocycles related by an n-coboundary via:

$$H^n(G, M) := \frac{Z^n(G, M)}{B^n(G, M)} = \frac{ker(\delta^{n+1})}{Im(\delta^{n+1})} \tag{A.7}$$

which we call the n-th **cohomology group**. We call an n-cocycle **trivial** if it is in the equivalence class of the unit $1 \in M$. An n-cocycle $c^n \in Z^n(G, M)$ is called **normalised** if

$$c^n(g_1, \cdots, g_n) = 1_M \tag{A.8}$$

whenever there exists an $i \in \{1, \cdots, n\}$ such that $g_i = 1_G \in G$. A natural consequence of this definition is that for all n-cocycles there exists a normalised n-cocycle in the equivalence class under n-coboundaries.

Appendix B

Finite Dimensional Algebras

In this chapter we review and define conventions for finite dimensional algebras which compliment the discussion in the main text. Roughly speaking an algebra is a vector space enriched by defining a "multiplication" of vectors. We formalise this structure as follows:

Definition B.0.1. A $k-$algebra over the field k, is a $k-$vector space A together with two bilinear maps

$$\begin{aligned} m &: A \otimes A \to A \\ \eta &: k \otimes A \to A \end{aligned} \tag{B.1}$$

We denote these maps in components as $m(a,b) \mapsto ab$ and $\eta(r,a) \mapsto ra$ for all $a, b \in A$ and $r \in k$. Additionally we require these maps to respect the following axioms:

$$\begin{aligned} a(bc) = (ab)c \quad & \forall a, b, c \in A \\ 1_k a = a = a 1_k \quad & \forall a \in A. \end{aligned} \tag{B.2}$$

Definition B.0.2. A $k-$algebra A is finite dimensional when the underlying vector space is finite dimensional.

Additionally we will make use of the notion of a sub-algebra $A' \subset A$:

Definition B.0.3. Given a $k-$algebra A. An A sub-algebra $A' \subset A$ is a sub-vector space $A' \subset A$ with bilinear maps m, η induced from A which are closed in

A' such that

$$\begin{aligned} a'b' \in A' \quad & \forall a', b' \in A' \\ ra' \in A' \quad & \forall r \in k, \quad \forall a' \in A'. \end{aligned} \tag{B.3}$$

An important example of of an A sub-algebra is the central sub-algebra $Z(A)$:

Definition B.0.4. Given a $k-$Algebra A. The central sub-algebra

$$Z(A) := \{a \in A | ab = ba \quad \forall b \in A\} \tag{B.4}$$

it is straightforward to verify this is indeed an A sub-algebra.

B.1 Modules

An important object in the study of algebras is given by modules. Modules are defined to provide a natural generalisation of the notion of vector spaces. In the abstract formulation, a $k-$vector space V can be thought of as an Abelian group $(V, +)$ which is enriched by an action of the field $\alpha : k \otimes V \to V$ called scalar multiplication satisfying certain axioms. Utilising this point of view an $A-$module is defined analogously such that the field k can generalised to be an algebra A:

Definition B.1.1. Let A be a $k-$algebra. A left $A-$module is an Abelian group $(M, +)$ together with a linear map $\alpha : A \otimes M \to M$, denoted by $\alpha(a, m) \mapsto am$ such that

$$\begin{aligned} (ab)n = a(bn) \quad & \forall a, b \in A, \quad \forall n \in M \\ 1_k n = n \quad & \forall n \in M \\ (a + b)n = an + bn \quad & \forall a, b \in A, \quad \forall n \in M \\ a(n + m) = an + am \quad & \forall a \in A, \quad \forall m, n \in M \end{aligned} \tag{B.5}$$

Similarly we can define a right $A-$module:

Definition B.1.2. Let A be a $k-$algebra. A right $A-$module is an Abelian group $(M, +)$ together with a linear map $\beta : M \otimes A \to M$, denoted by $\beta(m, a) \mapsto ma$ such that

$$\begin{aligned} n(ab) &= (na)b \quad \forall a, b \in A, \quad \forall n \in M \\ n &= n1_k \quad \forall n \in M \\ n(a+b) &= na + nb \quad \forall a, b \in A, \quad \forall n \in M \\ (n+m)a &= na + ma \quad \forall a \in A, \quad \forall m, n \in M. \end{aligned} \tag{B.6}$$

Furthermore in the subsequent discussion we will be interested in so called bimodules:

Definition B.1.3. Let A, A' be a pair of $k-$algebras. An $(A - A')$bimodule B is an Abelian group $(B, +)$ such that B is simultaneously a left $A-$module, a right $A'-$module and

$$(an)b = a(nb) \quad \forall a, b \in A, \quad \forall n \in B. \tag{B.7}$$

As for the case of vector spaces, it is often useful to define sub-modules:

Definition B.1.4. Given a left $A-$module M and an Abelian subgroup $M' \subseteq M$. M' defines a left submodule of M if

$$an' \in M' \quad \forall a \in A, \quad \forall n' \in M' \tag{B.8}$$

and similarly for right sub-modules where the left action is replaced by a right action and bimodules. The following definitions are the same for left/right/bi modules and so we choose to drop the prefix:

Definition B.1.5. An $A-$module M is called simple if the only submodules of M are the trivial group 0 and M itself.

Definition B.1.6. An $A-$module M is called semi-simple if M is the direct sum of simple modules M_i such that $M = \oplus_i M_i$.

Definition B.1.7. Given an algebra A, A itself forms an $A-$module which we call the regular module.

Definition B.1.8. An algebra A is called semi-simple when the regular module is semi-simple.

An important tool in the study of algebras is given by representations which characterise how an algebra A acts on a module M:

Definition B.1.9. Given an algebra A and a left/right $A-$module M, a left/right representation of A is a pair (ρ, M) where ρ is an algebra homomorphism $\rho : A \to End(M)$.

Definition B.1.10. A representation (R, M) is called irreducible if M is a simple module.

A consequence of this definition is that if A is a semi-simple algebra, then any representation can be constructed by the direct sum of irreducible representations.

B.2 Example: Matrix Algebras

Let kM_n be the $k-$algebra of $n \times n$ complex matrices with the algebra product given by matrix multiplication. The canonical basis for kM_n is given by $\{e_{ab}\}_{a,b\in\{1,\cdots,n\}}$ where e_{ab} corresponds to the matrix with entry $(a, b) = 1$ and zero's elsewhere. In this basis the algebra product is given by

$$e_{ab}e_{b'c} = e_{ac}\delta_{b,b'}. \tag{B.9}$$

Given the definition of kM_n it is straightforward to find the left simple modules. For each $c \in \{1, \cdots, n\}$ let L_c be the n dimensional $k-$vector space $L_c := \mathbb{C}\{e_{ic}\}_{i\in 1,\cdots,n}$. It is straightforward to show each L_c defines a left kM_n module

$$AL_c = L_c \tag{B.10}$$

Furthermore such modules are simple as any sub-vector space $\tilde{L}_c \subset L_c$ fails to be a left kM_n module. Noting that the regular module $kM_n = \oplus_{c\in\{1\cdots,n\}}L_c$ and the isomorphism $L_c \simeq L_d$ for all $c, d \in \{1, \cdots, n\}$ we see that kM_n has one left simple module upto isomorphism of dimension n which occurs in the regular representation $n-$times.

Utilising the simple module L_c of kM_n in the canonical basis the associated representation $\rho : kM_n \to \mathrm{End}(L_c) = kM_n$ is irreducible due to the simplicity of L_c and ρ is the algebra homomorphism given by $\rho(e_{ab}) = e_{ab}$ for all $e_{ab} \in kM_n$.

Similarly we can define the n simple right modules $R_c = k\{e_{ci}\}_{i\in 1,\cdots,n}$ for $c \in \{1, \cdots, n\}$ which are all isomorphic. There is one simple kM_n bimodule given by kM_n itself where the irreducible representation $\rho : kM_n \to \mathrm{End}(kM_n)$ is given by $\rho(e_{ab}) = e_{ab}$ for all $e_{ab} \in kM_n$.

In the canonical basis the identity element of kM_n is given by $1_n = \sum_a e_{aa}$ and the central sub-algebra $Z(kM_n) = \{a \in M_n | ab = ba, \quad \forall b \in kM_n\}$ is the 1-dimensional sub-algebra given by $k1_n$.

B.3 Example: Semi-Simple Algebras

An important result we state without proof is that:

Theorem B.3.1. A semi-simple $k-$algebra A is isomorphic to the direct sum of matrix algebras kM_n, such that

$$A \simeq kM_{n_1} \oplus \cdots \oplus kM_{n_m} \tag{B.11}$$

From theorem (B.3.1) we can generalise the previous example of matrix algebras to the case of any semi-simple algebra. If A is any semi-simple algebra there exists an isomorphism such that $A \simeq M = kM_{n_1} \oplus \cdots \oplus kM_{n_m}$. From the existence of such an isomorphism it follows that there exists a canonical basis for A such that

$$e^i_{ab} e^j_{b'c} = e^i_{ac} \delta_{i,j} \delta_{b,b'} \tag{B.12}$$

for $i, j \in \{1, \cdots, m\}$, $a, b \in \{1, \cdots, n_i\}$ and $b', c \in \{1, \cdots, n_j\}$. Furthermore A has m simple left/right modules upto isomorphism of dimension $n_1, \cdots, n_m$ respectively and the regular module of A is a direct sum of such left/right simple modules with each module of dimension n_i occurring n_i times in the decomposition. Each simple left/right module is the simple left/right module L^i/R^i

associated to M_{n_i} for $i \in \{1, \cdots, m\}$ and the representation $\rho : A \to \mathrm{End}(L^i/R^i)$ is given by $\mathrm{End}(M) = M$. From the previous it follows that we have the sum of squares rule

$$\dim(A) = \sum_{i=1}^{m} n_i^2. \tag{B.13}$$

Furthermore there are m simple $A-$bimodules given by kM_i for $i \in \{1, \cdots, m\}$.

A consequence of semi-simplicity is that the central sub-algebra of A, $Z(A) \simeq k1_{n_1} \oplus \cdots \oplus k1_{n_m}$ and $\dim Z(A) = m =$ number of simple left/right modules upto isomorphism=number of simple bimodules.

From the previous we see that defining an isomorphism from the basis of A to the semi-direct product of matrix algebras in the canonical basis gives rise to all the information about the simple modules and the thus irreducible representations.

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