Dedekind-finite cardinals and model-theoretic structures

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The candidate confirms that the work submitted is his own and that appropriate credit has been given where reference has been made to the work of others.

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Abstract

The notion of finiteness in the absence of AC has been widely studied. We consider a minimal criterion for which any class of cardinalities that satisfies it can be considered as a finiteness class. Fourteen notions of finiteness will be presented and studied in this thesis. We show how these classes relate to each other, and discuss their closure properties. Some results can be proved in ZF. Others are consistency results that can be shown by using the Fraenkel-Mostowski-model construction. Furthermore we investigate the relationship between Dedekind-finite sets and definability, and try to carry out reconstruction to recover the original structures used to construct FM-models. Later we establish a connection between tree structures and sets with their cardinalities in one of the finiteness classes, written as Δ_5.

Contents

Chapter 1

Introduction

The notion of finiteness without the Axiom of Choice (AC) has been widely studied in [Tar24], [Mos39], [Lév58], [Tru74], [Deg94], and [Gol97]. In those papers, various notions of finiteness were introduced. With AC, they all coincide, the notion of finiteness is unique and turns to be just the set ω, the set of all natural numbers, i.e. a set is finite if and only if it has a bijection with a natural number. But this is not necessarily true without AC. There could be infinite sets which in some respects behave like finite sets. All basic notations and required backgrounds are provided in Chapter 2.

In Chapter 3, we discuss the relations between notions of finiteness. First we introduce the notions of infinity which were mentioned in [Deg94], but since this thesis focusses on finiteness, we work with the dual notion, a notion of finiteness, which is also mentioned in [Her11]. We say a class of cardinals Γ is a *finiteness class* if $\omega \subseteq \Gamma$, $\aleph_0 \notin \Gamma$, and Γ is closed under $\leq$. We can see that these conditions are very fundamental and it is natural to take them as minimal criteria for a notion of finiteness.

There are many classes of finiteness that satisfy the above conditions. For instance, the class of all *weakly Dedekind-finite* cardinals, denoted by Δ_4 (as from [Tru74]), which is the class of cardinalities of sets with no countably infinite partitions, the class of *Dedekind-*

finite cardinals, denoted by Δ, which is the class of cardinalities of sets with no countably infinite subsets, and the set ω itself is also a finiteness class. It turns out that ω and Δ are the boundaries of this notion, i.e. every finiteness class lies between ω and Δ.

We gather together 14 notions of finiteness discussed in this thesis, which were mostly introduced in the references given above, and study their properties and show how they are related. We also mention notions from [Lév58] and [Deg94] which we argue should really not be allowed, on the grounds that any notion of finiteness should not include $\aleph_0$, and should be closed downwards. For instance one of Lévy's notions counts x as 'finite' provided that $x < x^2$, but by the use of some standard cardinal arithmetic in the absence of AC there is some $x > \aleph_0$ satisfying this, so this property cannot be closed downwards.

Some notions that we study in Chapter 3, for instance, the notions of *Russell-finite*, *weakly Dedekind*-finite*, and *dual Dedekind*-finite* also turn out to be notions of finiteness. We study their closure properties under various operations, e.g. closure under $+$, $\times$, and examine the relations between these classes. We also remark that there is a proper class of notions of finiteness, obtained by considering so-called MT-rank from [MT03].

The second half of Chapter 3 provides consistency results to differentiate these notions by constructing FM-models. We work in the axiom system ZFA, set theory with atoms, which was introduced by Fraenkel and Mostowski (and a later version, replacing atoms by autosingletons by Specker in [Spe57], see also [Fel71]). This is a version of ZF modified by allowing elements which are not sets, which we call *atoms* or *urelements*. A model of ZFA is called a *permutation model* or an *FM-model*. AC can be made to fail in these models so they provide a good context for studying a variety of notions of finiteness. The results in these models can be carried over to well-founded (Cohen) models of ZF by using the Jech-Sochor embedding theorem, see [JS66] or [Pin72]. In most cases the model is based on a suitable structure $\mathfrak{A}$ in the ground model, which is usually countable, and usually homogeneous, which is a strong enough condition to guarantee a rich automorphism group. Again usually the filter used to construct the FM-model is generated by finite supports. This general method is explained in [Pin72].

In Chapter 4, we investigate the relationship between Dedekind-finite sets and definability. The latter can have various senses, but it initially is taken as definable in a finite first-order language, extended to infinite first-order languages or infinitary languages later. This is inspired by some results in [Pin76], [Tru95], and [WT05]. Informally, if we drop AC, it is possible that even a set has some hidden structure, which with AC would have been destroyed. A typical case is that of amorphous sets, which as shown in [Tru95], and despite their name, can actually carry a wide variety of structures.

Those FM-models that are constructed from $\aleph_0$-categorical structures using finite supports have their set of atoms lying in Δ_4, the class of weakly Dedekind-finite cardinals, in the model. This connection was also studied in [WT05]. We will try to recover the original structures used to construct FM-models by studying possible structures that can be put on the set of atoms in the models. Our work here though based on [WT05] gives further examples and extensions.

Chapter 5 gives a start to tackling similar questions for Dedekind-finite sets which lie outside the class of weakly Dedekind-finite sets. This may involve first-order but infinite languages, or infinitary logic, inspired by Scott's Isomorphism Theorem for characterizing a countable structure by a sentence of the infinite language $\mathcal{L}_{\omega_1\omega}$. We provide a method to perform reconstruction on certain sets having countably infinite partitions with every member being weakly Dedekind-finite.

The main focus of this final chapter is on sets X whose cardinality lies in Δ_5, meaning that there is no surjection from X onto $X \cup \{*\}$ for an extra point $* \notin X$. First we give examples of trees, called *weakly* 2*-transitive*, whose cardinalities lie in Δ_5 but not in Δ_4. The main difference between the so-called '2-transitive' trees introduced by Droste in [DHM89] and the 'weakly 2-transitive' ones is that 2-transitive trees have the same ramification order throughout, but weakly 2-transitive trees need not. Using this we can form $2^{\aleph_0}$ distinct subsets of ω exhibited as the ramification orders arising in such trees, thereby providing many non-$\aleph_0$-categorical examples. Then we establish a connection between sets whose cardinalities lie in Δ_5 and a different type of tree structure. Namely $|X| \in \Delta_5$ if and only if there is no tree on a subset of X having ω levels and no leaves. We give several examples of how this works out, extending a 'pruning' lemma from [FT07] to the case in which the levels of such a tree are weakly Dedekind-finite.

Chapter 2

Preliminaries

2.1 Basic notations

Two sets have the same *size* or the same *cardinality* if there is a bijection between them. For any sets X and Y, we write $X \approx Y$ if there is a bijection between X and Y, $X \preceq Y$ if there is an injection from X into Y and $X \preceq^* Y$ if $X = \emptyset$ or there is a surjection from Y onto X. We write $|X|$ for the cardinality of X (the definition of cardinality of a set without AC will be discussed later in this chapter). Therefore $|X| = |Y|$ iff $X \approx Y$. We say $|X|$ is *less than or equal to* $|Y|$, written $|X| \leq |Y|$, if $X \preceq Y$, and we say $|X|$ is *less than* $|Y|$, written $|X| < |Y|$, if $|X| \leq |Y|$ and $|X| \neq |Y|$. We also write $|X| \leq^* |Y|$ if $X \preceq^* Y$. We write $X \subseteq Y$ for X is a subset or subclass of Y, and we use $\subset$ for *proper* subset or proper subclass, i.e. $X \subset Y$ if $X \subseteq Y$ and $X \neq Y$. We write $X \dot{\cup} Y$ for the disjoint union of X and Y and write $\dot{\bigcup} X$ for the disjoint union of a family X.

Let $<$ be a relation on a set X. For any $x, y \in X$, we write $x \parallel_< y$ if x and y are not comparable under $<$ in X, i.e. $x \neq y$, $x \not< y$ and $y \not< x$, and we may write only $x \parallel y$ if the relation is clear in the context. For any subsets A, B of X, we write $A < B$ if $x < y$ for all $x \in A$ and $y \in B$. We write $x < A$ if $\{x\} < A$. Similarly for $\parallel$.

2.2 Finite and infinite

Definition 2.2.1. A set is called *finite* if it has cardinality in ω. Otherwise it is infinite.

With the Axiom of Choice, every set can be well-ordered so we can compare their sizes. Hence $\aleph_0 \leq |X|$ for every infinite set X. Without AC this might not be true.

Definition 2.2.2. A set X is called *Dedekind-finite* if $\aleph_0 \not\leq |X|$. Otherwise it is *Dedekind-infinite*.

An equivalent definition of a Dedekind-finite set is that it is a set with no injection into a proper subset. Therefore every injection on a Dedekind-finite set must be surjective.

Clearly if $A \preceq B$, then $A \preceq^* B$. Therefore $\aleph_0 \leq |X|$ implies $\aleph_0 \leq^* |X|$ for any set X. The converse is not necessarily true without AC.

Definition 2.2.3. A set X is called *weakly Dedekind-finite* if $\aleph_0 \not\leq^* |X|$. Otherwise it is *weakly Dedekind-infinite*.

Actually it would have been better to call such a weakly Dedekind-finite set a 'strongly' Dedekind-finite set but the name is obtained from 'weakly Dedekind-infinite set' which was introduced in [Deg94]. This notion first appeared as *III-finite* in [Tar24].

With AC, all finiteness notions in this section are equivalent. We will later introduce more notions of finiteness in Chapter 3.

2.3 Homogeneous structures

Definition 2.3.1. A countable relational structure $\mathfrak{A}$ is *homogeneous* (or *ultrahomogeneous* in some textbooks) if any finite partial automorphism, that is an isomorphism between finitely generated substructures, can be extended to an automorphism.

Definition 2.3.2. The *age* of a structure $\mathfrak{A}$, written age($\mathfrak{A}$), is the class of structures (in the same language as $\mathfrak{A}$) which are isomorphic to finitely generated substructures of $\mathfrak{A}$.

Definition 2.3.3. Let $\mathcal{L}$ be a countable relational first-order language and let $\mathcal{C}$ be a class of finitely generated $\mathcal{L}$ structures.

We call $\mathcal{C}$ an *amalgamation class* if $\mathcal{C}$ satisfies the following properties.

1. *Hereditary property* (HP): If $A \in \mathcal{C}$ and B is a finitely generated substructure of A, then B is isomorphic to some structure in $\mathcal{C}$.

2. *Joint embedding property* (JEP): If $A, B \in \mathcal{C}$, then there is $C \in \mathcal{C}$ such that both A and B are embeddable in C.

3. *Amalgamation property* (AP): If $A, B, C \in \mathcal{C}$ and $e\colon A \to B$, $f\colon A \to C$ are embeddings, then there are $D \in \mathcal{C}$ and embeddings $g\colon B \to D$ and $h\colon C \to D$ such that $g \circ e = h \circ f$.

The age of any countable structure has HP and JEP but not necessarily AP.

Theorem 2.3.4 (Fraïssé [Fra53]). *Let $\mathcal{C}$ be an amalgamation class. Then there exists a unique, up to isomorphism, countable homogeneous structure $\mathfrak{A}$ whose age is $\mathcal{C}$. We call $\mathfrak{A}$ the Fraïssé limit of $\mathcal{C}$.*

Examples.

1. The Fraïssé limit of the class of finite linearly ordered sets is the rationals $\mathbb{Q}$ with the usual ordering,

2. The Fraïssé limit of the class of finite graphs is called the *random graph*,

3. The Fraïssé limit of the class of finite partially ordered sets is called the *generic partially ordered set*.

We will provide some more details for the following structures.

2.3.1 The random graph

The Fraïssé limit of the class of finite graphs is called the *Random Graph*. It was originally constructed by Rado in [Rad64], and Erdős and Rényi proved in [ER63] that it has a crucial 'randomness' property . It is most simply characterized by saying that it has a countable set of vertices Γ and for any two finite disjoint subsets U and V, there is $x \in \Gamma \setminus (U \cup V)$ joined to all members of U and to none of V. See Section 4.2.2 where we study the FM-model built from Γ.

2.3.2 The generic bipartite graph

Let $\mathcal{C}$ be the class of finite bipartite graphs with fixed parts T and B (thought of as 'top' and 'bottom'). Then $\mathcal{C}$ is an amalgamation class, and we call the Fraïssé limit of $\mathcal{C}$ the *generic bipartite graph*. We write $\mathcal{B}$ for the generic bipartite graph.

Often it is worth viewing $\mathcal{B}$ as a partial order of height 2 (in which elements on the lower level are not necessarily below those on the upper level), and therefore we will treat the edge relation as a partial order $<$.

Proposition 2.3.5. *The following are properties of* $\mathcal{B}$

- *For any finite disjoint* $U, V \subseteq T$ *there exists* $b \in B$ *such that* $b < U$ *and* $b \parallel V$.
- *For any finite disjoint* $U, V \subseteq B$ *there exists* $t \in T$ *such that* $t > U$ *and* $t \parallel V$.

Furthermore, any countable bipartite graph having these two properties is isomorphic to $\mathcal{B}$.

Note that these properties are a slightly modified version of the characterization of the random graph. More on the generic bipartite graph will be discussed in Section 3.2.1.

2.3.3 The generic partially ordered set

Let P be the generic partially ordered set, which is the Fraïssé limit of the class of all finite partially ordered sets (which is easily verified to be an amalgamation class).

Remark. If A, B, C are finite subsets of P such that $A < B$ and $C \not\leq A$ and $C \not\geq B$, then there is $x \in P$ such that $A < x < B$ and $x \parallel C$.

Furthermore, any countable partially ordered set fulfilling this condition is isomorphic to P.

The following lemma follows immediately.

Lemma 2.3.6. *Let $x \in P$ and let A, B, C be finite subsets of P such that $A < x < B$ and $x \parallel C$. There is $y \in P$ such that $y \neq x$, $y \parallel x$ and $A < y < B$ and $y \parallel C$.*

2.3.4 Henson digraphs

Definition 2.3.7. A graph $\langle G, E\rangle$ is called a *directed graph*, or abbreviated as *digraph*, if E is antisymmetric, i.e. if $\langle x, y\rangle \in E$, then $\langle y, x\rangle \notin E$.

The Fraïssé limit of the family of finite digraphs is called the *generic digraph*.

Definition 2.3.8. A digraph is called a *tournament* if there is an edge between every pair of distinct vertices.

Let $\mathcal{T}$ be a family of finite tournaments. Let $\mathrm{Forb}(\mathcal{T})$ be the family of all finite digraphs D such that D does not embed T for any $T \in \mathcal{T}$. Then it can be shown that $\mathrm{Forb}(\mathcal{T})$ is an amalgamation class. To see that $\mathrm{Forb}(\mathcal{T})$ has the amalgamation property, let $B, C \in \mathrm{Forb}(\mathcal{T})$ be such that $A = B \cap C \in \mathrm{Forb}(\mathcal{T})$. We will show that $B \cup C$, without adding new edges, lies in $\mathrm{Forb}(T)$. Suppose there is $T \in \mathrm{Forb}(\mathcal{T})$ which embeds in $B \cup C$.

Since each of B and C forbids T, $T \cap B \setminus A \neq \emptyset \neq T \cap C \setminus A$. Pick $b \in T \cap B \setminus A$ and $c \in T \cap C \setminus A$. Since T is a tournament, there is an edge between b and c, which contradicts our condition that no new edges were added to $B \cup C$. Hence T is not embeddable in $B \cup C$, and so $B \cup C \in \mathrm{Forb}(\mathcal{T})$. Therefore by the Fraïssé construction there is a generic digraph associated with it.

Definition 2.3.9. Let $\mathcal{T}$ be a family of finite tournaments not containing the 1- or 2-element tournament and such that all members are pairwise non-embeddable. We call the Fraïssé limit of $\mathrm{Forb}(\mathcal{T})$ a *Henson digraph*.

We may write $\langle D_{\mathcal{T}}, E_{\mathcal{T}} \rangle$ for a Henson digraph with age $\mathrm{Forb}(\mathcal{T})$. Note that if $\mathcal{T} = \emptyset$ then $\langle D_{\mathcal{T}}, E_{\mathcal{T}} \rangle$ is the generic digraph. If $\mathrm{Forb}(\mathcal{T})$ contains the 1-element tournament, then $D_{\mathcal{T}} = \emptyset$. And if $\mathrm{Forb}(\mathcal{T})$ contains the 2-element tournament, then $\langle D_{\mathcal{T}}, E_{\mathcal{T}} \rangle$ is the countable empty graph, so this is why these two cases are excluded.

2.4 $\aleph_0$-categorical structures

Definition 2.4.1. We say a theory T is *$\aleph_0$-categorical* if is has only one countable model up to isomorphism. A structure $\mathfrak{A}$ is an $\aleph_0$-categorical structure if $\mathrm{Th}(\mathfrak{A})$ is $\aleph_0$-categorical.

Theorem 2.4.2 (Ryll–Nardzewski). *Let T be a complete theory in a countable language. Then the following are equivalent:*

1. *T is $\aleph_0$-categorical,*
2. *every countable model $\mathcal{M}$ of T is atomic, meaning $\mathrm{tp}^{\mathcal{M}}(\overline{a})$ is isolated for all $\overline{a} \in M^n$,*
3. *each S_nT is finite, where S_nT is the set of all complete n-types p such that $p \cup T$ is satisfiable,*
4. *for some countable model $\mathfrak{B}$ of T and every $n \in \omega$, the number of orbits of the automorphism group of $\mathfrak{B}$, acting on the set of n-element subsets of $\mathrm{dom}(\mathfrak{B})$, is finite.*

The proof of the above theorem can be found in [Mar02] or [KM94], and the next theorem follows.

Theorem 2.4.3. *Every homogeneous structure over a finite relational language is $\aleph_0$-categorical.*

2.5 Set theory with atoms

2.5.1 ZFA

Set theory with atoms, denoted by ZFA, is characterised by the fact that it admits objects other than sets. More details can be found in [Jec73] and [Hal17].

Definition 2.5.1. *Atoms* or *urelements* are objects which do not have any elements and which are distinct from the empty set.

The language of ZFA is $\mathcal{L}_{\mathsf{ZFA}} = \{=, \in, \mathtt{A}\}$ where $\in$ is a binary relation symbol and $\mathtt{A}$ is a constant symbol representing the set of atoms. The axioms of ZFA are like the axioms of ZF, except the following.

Axiom of Empty Set (for ZFA):

$$\exists x\big(x \notin \mathtt{A} \land \forall z(z \notin x)\big).$$

Axiom of Extensionality (for ZFA):

$$\forall x \forall y\big((x \notin \mathtt{A} \land y \notin \mathtt{A}) \to (\forall z(z \in x \leftrightarrow z \in y) \to x = y)\big).$$

From the above two axioms, the *empty set* is unique and is denoted by $\emptyset$. A modified version of the Axiom of Foundation can be stated as follows, though it may not need changes depending on which version we are using.

Axiom of Foundation (for ZFA):

$$\forall x\big(x \neq \emptyset \wedge x \notin \mathtt{A} \to \exists y(y \in x \wedge y \cap x = \emptyset)\big).$$

Axiom of Atoms:

$$\forall x\big(x \in \mathtt{A} \leftrightarrow \big(x \neq \emptyset \wedge \neg\exists z(z \in x)\big)\big).$$

Theorem 2.5.2. Con(ZF) *implies* Con(ZFA + $\mathtt{A}$ *is infinite*).

Definition 2.5.3. For any set S and ordinal α, we define

$$\begin{aligned}
\mathcal{P}^0(S) &= S, \\
\mathcal{P}^{\alpha+1}(S) &= \mathcal{P}^\alpha(S) \cup \mathcal{P}(\mathcal{P}^\alpha(S)), \\
\mathcal{P}^\alpha(S) &= \bigcup_{\beta<\alpha} \mathcal{P}^\beta(S) \quad \text{if } \alpha \text{ is a limit ordinal.}
\end{aligned}$$

Further let $\mathcal{P}^\infty(S) = \bigcup_{\alpha\in\mathbf{ON}} \mathcal{P}^\alpha(S)$.

Theorem 2.5.4. *If $\mathcal{M}$ is a model of* ZFA *and U is the set of atoms of $\mathcal{M}$, then $\mathcal{M} = \mathcal{P}^\infty(U)$. The class $\mathcal{P}^\infty(\emptyset)$ which is a subclass of $\mathcal{M}$ is a model of* ZF.

Notation. Let $\hat{\mathbf{V}}$ denote $\mathcal{P}^\infty(\emptyset)$. We call $\hat{\mathbf{V}}$ the *kernel* or *pure part* and call members of $\hat{\mathbf{V}}$ *pure sets*.

Note that the class **ON** of ordinals is contained in $\hat{\mathbf{V}}$.

2.5.2 Fraenkel-Mostowski models

Now we will construct models for ZFA. In this section, U is the set of atoms in a model $\mathcal{M}$ of ZFA + AC.

Definition 2.5.5. Let $\mathcal{G}$ be a group of permutations of U. A set $\mathcal{F}$ of subgroups of $\mathcal{G}$ is a *normal filter* on $\mathcal{G}$ if for all subgroups H and K of $\mathcal{G}$:

(i) $\mathcal{G} \in \mathcal{F}$,

(ii) if $H \in \mathcal{F}$ and $H \leq K$, then $K \in \mathcal{F}$,

(iii) if $H \in \mathcal{F}$ and $K \in \mathcal{F}$, then $H \cap K \in \mathcal{F}$,

(iv) if $\pi \in \mathcal{G}$ and $H \in \mathcal{F}$, then $\pi H \pi^{-1} \in \mathcal{F}$,

(v) for each $a \in U$, $\{\pi \in \mathcal{G} : \pi(a) = a\} \in \mathcal{F}$.

Throughout this section, $\mathcal{G}$ is a group of permutations of U and $\mathcal{F}$ is a normal filter on $\mathcal{G}$.

Definition 2.5.6. Let $\pi \in \mathcal{G}$. Using the hierarchy of $\mathcal{P}^\alpha(U)$'s, we can define $\pi(x)$ for every x in $\boldsymbol{\mathcal{M}}$ by

$$\pi(x) = \pi[x] = \{\pi(y) : y \in x\}.$$

Remarks.

1. We sometimes write πx for $\pi(x)$.
2. It can be proved by transfinite induction that π is one-to-one, even in its extended action on the whole of $\boldsymbol{\mathcal{M}}$.

Lemma 2.5.7. *Let $\pi \in \mathcal{G}$. Then, for any x and y in $\boldsymbol{\mathcal{M}}$,*

1. *$\pi\{x, y\} = \{\pi x, \pi y\}$ and $\pi\langle x, y\rangle = \langle \pi x, \pi y\rangle$.*
2. *if f is a function, then πf is a function and $(\pi f)(\pi x) = \pi(f(x))$.*
3. *$\pi x = x$ for all $x \in \hat{\mathbf{V}}$.*

Proof. Let x and y be any elements in $\boldsymbol{\mathcal{M}}$.

1. Clearly $\pi\{x, y\} = \{\pi x, \pi y\}$ by the definition. Therefore

$$\pi\langle x, y\rangle = \pi\{\{x\}, \{x, y\}\} = \{\pi\{x\}, \pi\{x, y\}\} = \{\{\pi x\}, \{\pi x, \pi y\}\} = \langle \pi x, \pi y\rangle.$$

2. Let f be a function. By 1, $\pi f = \{\langle \pi x, \pi(f(x))\rangle : x \in \mathrm{dom}(f)\}$. To show that πf is a function, suppose $\pi x = \pi y$ for $x, y \in \mathrm{dom}(f)$. Since π is one-

to-one, $x = y$, and so $\pi(f(x)) = \pi(f(y))$. Thus πf is a function and hence $(\pi f)(\pi x) = \pi(f(x))$.

3. This can be proved straightforwardly by transfinite induction on $\hat{\mathbf{V}}$. $\square$

Definition 2.5.8. For each x in $\mathcal{M}$, we define the *setwise stabilizer* of x in $\mathcal{G}$

$$\mathcal{G}_{(x)} = \{\pi \in \mathcal{G} : \pi x = x\}.$$

Then $\mathcal{G}_{(x)}$ is a subgroup of $\mathcal{G}$. We say x is *symmetric* (with respect to $\mathcal{F}$) if $\mathcal{G}_{(x)} \in \mathcal{F}$. The notation is different in some papers or textbooks; some call this the *symmetric group* of x in $\mathcal{G}$, written $\mathrm{sym}_{\mathcal{G}}(x)$. Define a *permutation model* by

$$\mathcal{V} = \{x : x \text{ is symmetric and } x \subseteq \mathcal{V}\}.$$

Theorem 2.5.9. *The class $\mathcal{V}$ is a transitive model of* ZFA*, $\hat{\mathbf{V}} \subseteq \mathcal{V}$, and $U \in \mathcal{V}$.*

With AC, every set can be well-ordered and so it has a bijection with some ordinal, and we define its cardinality to be the least such ordinal. Defining cardinalities of sets without AC is not as straightforward.

Definition 2.5.10. The *cardinality* of a set x, denoted by $|x|$, in the model $\mathcal{V}$, is defined by

$$|x| = \mathfrak{C}(x) \cap \mathcal{P}^{\alpha}(U) \cap \mathcal{V},$$

where $\mathfrak{C}(x) = \{y \in \mathcal{V} : y \approx x\}$ and α is the least ordinal such that $\mathfrak{C}(x) \cap \mathcal{P}^{\alpha}(U) \cap \mathcal{V} \neq \emptyset$. This method is known as *Scott's trick*.

We will work in the theory ZFA + AC (for the consistency, see [Jec73]). Then we have that AC holds in the kernel $\hat{\mathbf{V}}$. By the Jech-Sochor Embedding Theorem (see [Jec73] or [JS66]), we can embed an initial segment of the permutation model into a well-founded model of ZF, so that every relation between cardinals in the permutation model also holds in the well-founded model. Hence, in order to prove that a relation between some cardinals is consistent with ZF, it is enough to find a permutation model for the statement.

Definition 2.5.11. A set I of subsets of U is a *normal ideal* if for all $E, F \subseteq U$:

(i) $\emptyset \in I$,

(ii) if $E \in I$ and $F \subseteq E$, then $F \in I$,

(iii) if $E \in I$ and $F \in I$, then $E \cup F \in I$,

(iv) if $\pi \in \mathcal{G}$ and $E \in I$, then $\pi(E) \in I$,

(v) for each $a \in U$, $\{a\} \in I$.

Remark. The set of all finite subsets of U is a normal ideal.

Definition 2.5.12. For each $E \subseteq U$, define the *pointwise stabilizer* of E under $\mathcal{G}$

$$\mathcal{G}_E = \{\pi \in \mathcal{G} : \pi(a) = a \text{ for all } a \in E\}.$$

Then $\mathcal{G}_E$ is a subgroup of $\mathcal{G}$. This is sometimes written as $\mathrm{fix}_{\mathcal{G}}(E)$.

Theorem 2.5.13. *Given a normal ideal I, then*

$$\mathcal{F} = \{H : H \text{ is a subgroup of } \mathcal{G} \text{ such that } \mathcal{G}_E \leq H \textit{ for some } E \in I\}$$

is a normal filter.

Note that given a normal ideal I, there is a corresponding normal filter $\mathcal{F}$ as defined above and we say $\mathcal{V}$ is defined from I if $\mathcal{V}$ is the permutation model defined from such $\mathcal{F}$.

Definition 2.5.14. For each x and each $E \in I$, we say that E is a *support* of x if $\mathcal{G}_E \leq \mathcal{G}_{(x)}$.

If I is the set of all finite subsets of U and $\mathcal{F}$ is the corresponding normal filter, then we say $\mathcal{F}$ is generated by finite supports.

Remarks.

1. x is symmetric iff there exists $E \in I$ such that E is a support of x. As a result, we have that
$$x \in \mathcal{V} \text{ iff } x \text{ has a support and } x \subseteq \mathcal{V}.$$

2. For each x and each $E, F \in I$, if E is a support of x and $E \subseteq F$, then F is also a support of x.

Definition 2.5.15. We call the set $\mathcal{G}(x) := \{gx : g \in \mathcal{G}\}$ the *orbit* of x under a group $\mathcal{G}$.

Proposition 2.5.16. *Let $E, X \subseteq U$ where E is finite. If X is supported by E, then X is a union of orbits of U under $\mathcal{G}_E$.*

Proof. Assume X is supported by E. Let $a \in U$ be such that $\mathcal{G}_E(a) \cap X \neq \emptyset$ and let $b \in \mathcal{G}_E(a) \cap X$. Let $c \in \mathcal{G}_E(a)$. Then there are $\sigma, \tau \in \mathcal{G}_E$ such that $b = \sigma a$ and $c = \tau a$. Since $\mathcal{G}_E$ is a subgroup, $\tau\sigma^{-1} \in \mathcal{G}_E$, so as E supports X and $b \in X$, $c = \tau a = \tau\sigma^{-1} b \in X$ too. Thus $\mathcal{G}_E(a) \subseteq X$. Hence X is a union of orbits of U under $\mathcal{G}_E$. □

Examples.

1. **The Fraenkel model**: Let U be a countable set of atoms. Let $\mathcal{G}$ be the group of all permutations of U and let $\mathcal{F}$ be the filter on $\mathcal{G}$ generated by finite supports. Then the permutation model $\mathcal{N}_F$ induced from U, $\mathcal{G}$, and $\mathcal{F}$ is called the *(basic) Fraenkel model*.

2. **The Mostowski model**: Let $\langle U, < \rangle$ be a countable set of atoms isomorphic to the rationals $\langle \mathbb{Q}, <_{\mathbb{Q}} \rangle$. Let $\mathcal{G}$ be the group of order-preserving permutations of U. Let $\mathcal{F}$ be the filter on $\mathcal{G}$ generated by finite supports. The permutation model $\mathcal{N}_M$ determined by U, $\mathcal{G}$ and $\mathcal{F}$ is called the *Mostowski model*.

2.6 Infinitary languages

Definition 2.6.1. Let $\mathcal{L}$ be a language and κ be an infinite cardinal. The formulas of the infinitary logic $\mathcal{L}_{\kappa\omega}$ are defined inductively as follows.

(i) Every atomic $\mathcal{L}$-formula is a formula of $\mathcal{L}_{\kappa\omega}$.

(ii) If X is a set of formulas of $\mathcal{L}_{\kappa\omega}$ such that all of the free variables come from a fixed finite set and $|X| < \kappa$, then

$$\bigwedge_{\varphi \in X} \varphi \text{ and } \bigvee_{\varphi \in X} \varphi$$

are formulas of $\mathcal{L}_{\kappa\omega}$.

(iii) If φ is a formula of $\mathcal{L}_{\kappa\omega}$, then so are $\neg\varphi$, $\forall v\varphi$ and $\exists v\varphi$.

The following results are from [Sco65], though their proofs are not given in the original reference. The proof of Theorem 2.6.2 can be found in elementary Model Theory textbooks (e.g. see Theorem 2.4.15 in [Mar02])

Theorem 2.6.2 (Countable Isomorphism Theorem)**.** *Let $\mathfrak{A}$ be a countable $\mathcal{L}$-structure. Then there is $\varphi \in \mathcal{L}_{\omega_1\omega}$ such that $\mathfrak{B} \cong \mathfrak{A}$ if and only if $\mathfrak{B} \models \varphi$. We call φ a Scott sentence of $\mathfrak{A}$.*

The following theorem known as the Scott Countable Definability Theorem (also from [Sco65]) can be stated as follows.

Theorem 2.6.3 (Countable Definability Theorem)**.** *Let $\mathfrak{A}$ be a countable structure and let $P \subseteq A^n$ for some $n \in \omega$. Then the following are equivalent.*

1. *For any $Q \subseteq A^n$, if $\langle \mathfrak{A}, P\rangle \cong \langle \mathfrak{A}, Q\rangle$, then $P = Q$.*
2. *There is some formula $\varphi(\overline{x})$ of $\mathcal{L}_{\omega_1\omega}$ such that $\langle \mathfrak{A}, P\rangle \models \forall\overline{x}(P(\overline{x}) \leftrightarrow \varphi(\overline{x}))$.*

Proof. ($2 \Rightarrow 1$) Let φ be an $\mathcal{L}_{\omega_1\omega}$-formula satisfying 2. Let $Q \subseteq A^n$ be such that

$\langle \mathfrak{A}, P\rangle \cong \langle \mathfrak{A}, Q\rangle$. Then, by applying the isomorphism, $\langle \mathfrak{A}, Q\rangle \models \forall \overline{x}(Q(\overline{x}) \leftrightarrow \varphi(\overline{x}))$. Therefore $\overline{x} \in P \Leftrightarrow \varphi(\overline{x}) \Leftrightarrow \overline{x} \in Q$ for all $\overline{x} \in A^n$, and so $P = Q$.

$(1 \Rightarrow 2)$ Assume that for any $Q \subseteq A^n$, if $\langle \mathfrak{A}, P\rangle \cong \langle \mathfrak{A}, Q\rangle$, then $P = Q$. For each $\overline{a} \in P$, let $\varphi_{\overline{a}}(\overline{a})$ be a Scott sentence of $\langle \mathfrak{A}, \overline{a}\rangle$ and let $\varphi(\overline{x}) = \bigvee_{\overline{a} \in P} \varphi_{\overline{a}}(\overline{x})$. We will show that $\langle \mathfrak{A}, P\rangle \models \forall \overline{x}(P(\overline{x}) \leftrightarrow \varphi(\overline{x}))$.

Let $\overline{b} \in A^n$. If $\overline{b} \in P$, then $\langle \mathfrak{A}, \overline{b}\rangle \models \varphi_{\overline{b}}(\overline{b})$, so $\langle \mathfrak{A}, \overline{b}\rangle \models \varphi(\overline{b})$. Suppose $\langle \mathfrak{A}, \overline{b}\rangle \models \varphi(\overline{b})$. Then $\langle \mathfrak{A}, \overline{b}\rangle \models \varphi_{\overline{a}}(\overline{b})$ for some $\overline{a} \in P$. Since $\varphi_{\overline{a}}(\overline{a})$ is a Scott sentence of $\langle \mathfrak{A}, \overline{a}\rangle$, $\langle \mathfrak{A}, \overline{b}\rangle \cong \langle \mathfrak{A}, \overline{a}\rangle$. By 1, we have $\overline{b} = \overline{a} \in P$. $\square$

Chapter 3

Relations between notions of Dedekind-finiteness

In this chapter, we introduce the main notions of Dedekind-finiteness considered in this thesis. The object is to combine all notions previously considered in [Tar24], [Mos39], [Tru74], [Deg94], and [Gol97], and present them and some analogues as systematically as possible. The notions are presented via their corresponding 'classes', here thought of as 'notions of finiteness' (in contrast to a 'notion of infinity' studied in [Deg94]). We formulate what is understood by a 'notion of finiteness' for which a minimum requirement is that it be closed under $\leq$. For instance the notion of 'o-amorphous' is not as it stands closed under subset, so for this notion we have to including forming subsets explicitly in the definition. Other properties of the usual class of finite sets is that a finite union of finite sets is finite, related to closure under $+$ and $\times$. These will form a central theme, as well as relations between the different classes, either provably, or consistently.

3.1 Finiteness without Choice

Recall the definition of a Dedekind-finite set. A set X is Dedekind-finite if $\aleph_0 \not\leq |X|$. Otherwise it is Dedekind-infinite. Several people have studied finiteness without the Axiom of Choice. We attempt to discuss systematically most of the important definitions, see how they are related and investigate their closure properties.

A *notion of infinity* was introduced in [Deg94], and a corresponding notion of *finiteness class* was introduced in [Her11]. We will follow the latter notion since we mainly focus on finiteness in this thesis.

Definition 3.1.1. The class of cardinals Γ is a *finiteness class* if it satisfies the following.

(i) $\omega \subseteq \Gamma$,

(ii) $\aleph_0 \notin \Gamma$,

(iii) Γ is closed under $\leq$, i.e. if $|x| \in \Gamma$ and $|y| \leq |x|$, then $|y| \in \Gamma$.

It is easy to see that the set ω is a finiteness class, and with the Axiom of Choice it will be the only such class. However without AC, as we mentioned above, there is a model in which infinite Dedekind-finite sets exist, and the class of Dedekind-finite cardinals, denoted by Δ, is also a finiteness class. Furthermore we can show that these two classes are the boundaries of this notion, i.e. for any finiteness class Γ, $\omega \subseteq \Gamma \subseteq \Delta$, as shown in the following remarks, so every finiteness class is a class of Dedekind-finite cardinals.

Remarks.

1. ω is the smallest finiteness class.
2. Δ is the largest finiteness class.

Proof. It is easy to see that both ω and Δ satisfy the criteria for being finiteness classes, and ω is the smallest by the definition. To see that Δ is the largest such class, let Γ be a finiteness class and let $|X| \in \Gamma$. Since Γ is closed under $\leq$ and $\aleph_0 \notin \Gamma$, $\aleph_0 \not\leq |X|$, so $|X| \in \Delta$. □

Without AC, we shall see that it is consistent with ZF that $\omega \subset \Delta$, and also there might be other finiteness classes Γ lying between ω and Δ. First we introduce many such classes of cardinals, some of which have been extensively studied in [Deg94], [Gol97], and [Tru74]. Some notions were introduced earlier than that in [Tar24], [Mos39] and [Lév58].

Throughout this thesis, we use *finite* meaning the usual finite definition, i.e. sets having their cardinalities in ω.

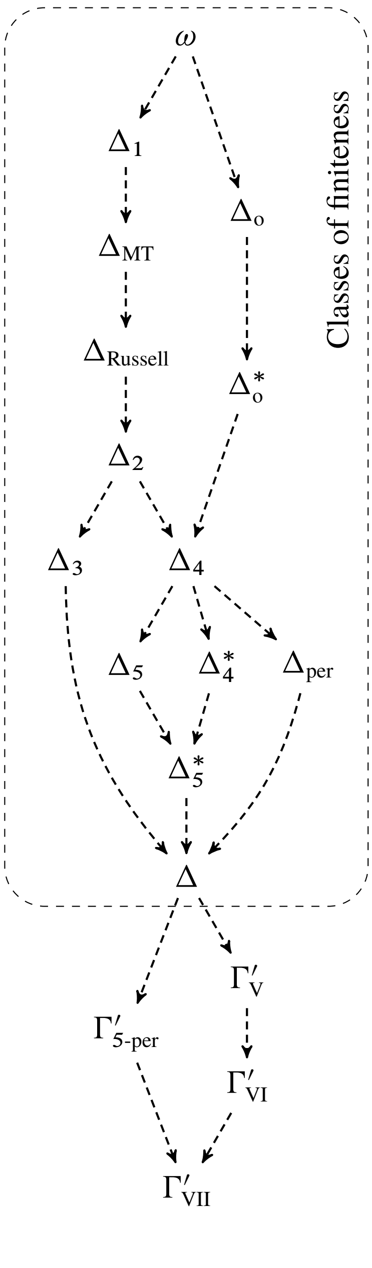
Definition 3.1.2. We introduce the following classes, of which the first fourteen are classes of Dedekind-finite cardinals, some relying on notions yet to be defined.

(i) $\omega :=$ the set of natural numbers,
(ii) $\Delta_1 := \{|x| : \text{if } x = y \dot{\cup} z, \text{then either } y \text{ or } z \text{ is finite}\}$,
(iii) $\Delta_2 := \{|x| : \text{every linearly ordered partition of } x \text{ is finite}\}$,
(iv) $\Delta_3 := \{|x| : \text{every linearly ordered subset of } x \text{ is finite}\}$,
(v) $\Delta_4 := \{|x| : x \text{ is weakly Dedekind-finite, i.e.} \ \omega \not\leq^* x\}$,
(vi) $\Delta_4^* := \{|x| : \text{there are no finite-to-one maps from a subset of } x \text{ onto } \omega\}$,
(vii) $\Delta_5 := \{|x| : x + 1 \not\leq^* x\}$,
(viii) $\Delta_5^* := \{|x| : \text{there are no non-injective finite-to-one maps from } x \text{ onto } x\}$,
(ix) $\Delta_{\mathrm{MT}} := \{|x| : x \text{ has MT-rank}\}$,
(x) $\Delta_{\mathrm{Russell}} := \{|x| : \text{every partial ordering on } x \text{ has a maximal element}\}$,
(xi) $\Delta_{\mathrm{o}} := \{|x| : x \subseteq y \text{ for some } y \text{ having a linear order so that } \langle y, <\rangle \text{ is } o\text{-amorphous}\}$,
(xii) $\Delta_{\mathrm{o}}^* := \{|x| : x \text{ has a linear order} < \text{so that } \langle x, <\rangle \text{ is weakly } o\text{-amorphous}\}$,
(xiii) $\Delta_{\mathrm{per}} := \{|x| : \text{every injection on } x \text{ has finite order}\}$,
(xiv) $\Delta := \{|x| : x \text{ is Dedekind-finite, i.e.} \ \omega \not\leq x\}$,
(xv) $\Gamma'_{\text{5-per}} = \{|x| : \text{every bijection on } x \text{ has a finite cycle}\}$,
(xvi) $\Gamma'_{\mathrm{V}} := \{|x| : 2|x| > |x|\}$,

(xvii) $\Gamma'_{\mathrm{VI}} := \{|x| : |x|^2 > |x|\}$,

(xviii) $\Gamma'_{\mathrm{VII}} := \{|x| : |x|$ is not an aleph greater than or equal to $\aleph_0\}$.

It can be shown that classes from (i)–(xiv) from the above definitions are finiteness classes, therefore they lie between ω and Δ. We can narrow down these relations further as illustrated as follows, where $\dashrightarrow$ means it is provable in ZF that there is a relation $\subseteq$ between the two classes.



Moreover we also show their closure properties under various operations, namely $+$, $\times$, union, and disjoint union, as can be defined as follows.

Definition 3.1.3. We say a finiteness class Γ is

- *closed under* $+$ if $|x|, |y| \in \Gamma$ implies $|x| + |y| \in \Gamma$,
- *closed under* $\times$ if $|x|, |y| \in \Gamma$ implies $|x| \times |y| \in \Gamma$,
- *closed under union* if $|x| \in \Gamma$ and $|y| \in \Gamma$ for all $y \in x$ implies $= |\bigcup x| \in \Gamma$,
- *closed under disjoint union* if $|x| \in \Gamma$ and $|y| \in \Gamma$ for all $y \in x$ implies $|\dot{\bigcup} x| \in \Gamma$.

More details on the relations between these classes together with their closure properties will be given in the following sections.

3.1.1 The classes Δ_1, Δ_2, Δ_3, Δ_4, and Δ_5

First we consider the classes Δ_1, Δ_2, Δ_3, Δ_4, and Δ_5. Together with ω and Δ, one can see that their definitions arise naturally from properties of finite sets. In [Lév58], Lévy introduced eight notions of finiteness, namely from *I-* to *VII-finite* together with *Ia-finite*, of which five are equivalent to five of the above seven. In particular, I-, Ia-, II-, III- and IV-finite are equivalent to definition of cardinals lying in ω, Δ_1, Δ_2, Δ_4, and Δ, respectively. The classes Δ_3 and Δ_5 were introduced in [Tru74], and in this paper, properties of all these seven classes were studied including the relations between these classes and their closure properties, along with some consistency results, as we shall recall in what follows.

Note that the latter notions from [Lév58], V-, VI-, and VII-finite, are not really interesting in this context as they fail to be finiteness classes. We will discuss them at the end of this section.

We have already remarked that ω and Δ are finiteness class. Similarly we can show that this is also true for Δ_1, Δ_2, Δ_3, Δ_4, and Δ_5, and hence they all lie between ω and Δ.

Proposition 3.1.4. *The classes Δ_1, Δ_2, Δ_3, Δ_4, and Δ_5 are finiteness classes.*

The above statements can be proved straightforwardly. Further results were shown in [Tru74] as follows.

Proposition 3.1.5. *The following relations can be proved in* ZF.

1. $\omega \subseteq \Delta_1 \subseteq \Delta_2 \subseteq \Delta_4 \subseteq \Delta_5 \subseteq \Delta$
2. $\Delta_2 \subseteq \Delta_3 \subseteq \Delta$

We note that Δ_2 is contained in both Δ_3 and Δ_4. Later we will show that it is consistent that Δ_3 and Δ_4 are incomparable and hence we do not have any relation between them. In particular, it is consistent that $\Delta_3 \not\subseteq \Delta_4$ and $\Delta_4 \not\subseteq \Delta_3$ (see Proposition 3.2.10 and Proposition 3.2.7, respectively). Furthermore some closure properties of these classes are also shown in [Tru74], which we quote here.

Theorem 3.1.6.

1. $\omega, \Delta_2, \Delta_4$ *are closed under unions,*
2. Δ_3, Δ *are closed under disjoint unions,*
3. Δ_5 *is closed under* $+$,
4. Δ_1 *is closed under* $+$ *if and only if* $\omega = \Delta_1$,
5. $\omega, \Delta_2, \Delta_3, \Delta_4, \Delta$ *are closed under* $+$ *and* $\times$.

Some classes however fail to have some of these closure properties, as follows.

Theorem 3.1.7 ([Tru74])**.**

1. *If* $\omega \neq \Delta$, *then* Δ *is not closed under unions.*
2. *If* $\omega \neq \Delta_3$, *then* Δ_3 *is not closed under unions.*
3. $\mathrm{Con}(\mathsf{ZF})$ *implies* $\mathrm{Con}(\mathsf{ZF} + \Delta_5$ *is not closed under* $\times)$.

Thomas Forster suggested another notion of finiteness: For every function $f : \omega \to X$, there is $x \in X$ such that $f^{-1}[\{x\}]$ is infinite. It turns out that this condition is also equivalent to Dedekind-finiteness.

To see this, for convenience let's call the above notion of finiteness *Forster-finite*. Thus, if X is Forster-finite, by the definition, no function $f : \omega \to X$ is an injection. Conversely, if X is not Forster-finite, then there is a function $f : \omega \to X$ such that for every $x \in X$, $f^{-1}[\{x\}]$ is finite. Since $\omega = \bigcup_{x \in \mathrm{ran}(f)} f^{-1}[\{x\}]$, $\mathrm{ran}(f)$ is infinite. Define a function $g : \min f^{-1}[\{x\}] \mapsto x$ for all $x \in \mathrm{ran}(f)$. Then g is a bijection between an infinite subset of ω and $\mathrm{ran}(f)$. Hence X is not Dedekind-finite.

The following is due to Kuratowski and the proof can be found in [Tar24] or [Her06].

Theorem 3.1.8. *For any set X, $\omega \preceq \mathcal{P}(X)$ implies $\omega \preceq^* X$.*

Corollary 3.1.9. *If $\Delta_4 = \Delta$, then $\omega = \Delta$.*

Proof. Assume $\Delta_4 = \Delta$. Let $X \notin \omega$. Then $\omega \preceq^* \mathcal{P}(X)$ since we can map the set of n-element subsets to n for each $n \in \omega$, and so $\omega \preceq \mathcal{P}(X)$. By the above theorem, $\omega \preceq^* X$, and therefore $\omega \preceq X$. Hence $X \notin \Delta$. □

Note that the definition of Δ_5 is also equivalent to the finiteness notion of *dual Dedekind-infinite* set defined in [Deg94], which can be stated as follows.

Definition 3.1.10. A set X is *dual Dedekind-finite* if there is no noninjective surjection from X onto X, i.e. every surjection on X must be injective.

It is easy to check that this is equivalent to the definition of Δ_5 given in Definition 3.1.2 and can be considered as an alternative definition.

3.1.2 Notion of MT-rank

Definition 3.1.11. A set X is *amorphous* if it is infinite and cannot be expressed as a union of two disjoint infinite sets.

We can see that the class Δ_1 is actually the class of cardinalities of amorphous or finite sets. The existence of amorphous sets is incompatible with the Axiom of Choice. This idea was extended to higher ranks in [MT03].

Definition 3.1.12. The *MT-rank* of a set X, denoted by $\mathrm{MT}(X)$, is defined as follows.

(i) $\mathrm{MT}(X) = -1$ if $X = \emptyset$,

(ii) $\mathrm{MT}(X) = \alpha$ if $\mathrm{MT}(X) \nless \alpha$ and there is $n \in \omega$ such that if $X = \dot{\bigcup}_{0\leq i\leq n} X_i$ then for some i, $\mathrm{MT}(X_i) < \alpha$.

We say a set X has MT-rank if $\mathrm{MT}(X) = \alpha$ for some ordinal α and has *MT-degree k* if k is the least n satisfying (ii).

Definition 3.1.13. Let Δ_{MT} denote the class of cardinalities of sets which have MT-rank. For any ordinal α, let $\Delta_{\mathrm{MT}_\alpha} = \{|X| \in \Delta_{\mathrm{MT}} : \mathrm{MT}(X) \leq \alpha\}$.

We can see that every non-empty $n \in \omega$ has MT-rank 0, and every amorphous set has MT-rank 1 with MT-degree 1. Hence $\Delta_{\mathrm{MT}_0} = \omega$ and Δ_{MT_1} is the class of finite sums of members of Δ_1.

Proposition 3.1.14. *The classes Δ_{MT} and $\Delta_{\mathrm{MT}_\alpha}$ are finiteness classes for all $\alpha \in$ **ON**.*

Proof. Let $\alpha \in$ **ON**. Since $\Delta_{\mathrm{MT}_0} = \omega$, $\omega \subseteq \Delta_{\mathrm{MT}_\alpha} \subseteq \Delta_{\mathrm{MT}}$. By using transfinite induction, the image of any set with MT-rank α under a function has MT-rank $\leq \alpha$. Therefore Δ_{MT} and $\Delta_{\mathrm{MT}_\alpha}$ are closed under $\leq$ and $\leq^*$.

To see that $\aleph_0 \notin \Delta_{\mathrm{MT}}$, suppose $\mathrm{MT}(\omega) = \alpha$ with MT-degree n. Then for $0 \leq i \leq n$, $X_i := \{k(n+1)+i : k \in \omega\}$ are $n+1$ infinite pairwise disjoint subsets of ω, so there is X_i such that $\mathrm{MT}(X_i) < \alpha$. But then X_i can be mapped onto ω, so $\mathrm{MT}(\omega) < \alpha$, a contradiction. Hence $\aleph_0 \notin \Delta_{\mathrm{MT}}$, and also $\aleph_0 \notin \Delta_{\mathrm{MT}_\alpha}$. □

Therefore Δ_{MT} lies between ω and Δ. Since every amorphous set has MT-rank 1 and since Δ_{MT} is closed under $\leq^*$ and $\aleph_0 \notin \Delta_{\mathrm{MT}}$, we have a stronger result $\Delta_1 \subseteq \Delta_{\mathrm{MT}} \subseteq \Delta_4$. Notice that $\Delta_{\mathrm{MT}_\alpha} \subseteq \Delta_{\mathrm{MT}_\beta}$ for all $\alpha < \beta$. One can build an FM-model containing sets having arbitrary large rank (see Section 3 in [MT03]).

Proposition 3.1.15. *For any α, it is consistent that $\Delta_{\mathrm{MT}_\beta} \subset \Delta_{\mathrm{MT}_\gamma}$ for all $\beta < \gamma \leq \alpha$.*

Therefore we have the following containments in which no relation can be reversed.

$$\Delta_{\mathrm{MT}_0} = \omega \subseteq \Delta_1 \subseteq \Delta_{\mathrm{MT}_1} \subseteq \Delta_{\mathrm{MT}_2} \subseteq \ldots \subseteq \Delta_{\mathrm{MT}_\omega} \subseteq \ldots \subseteq \Delta_{\mathrm{MT}_\alpha} \subseteq \ldots \subseteq \Delta_{\mathrm{MT}}$$

3.1.3 Russell-finiteness

The notion of *Russell-infinite* was introduced in [Deg94], and its dual notion of finiteness can be defined as follows.

Definition 3.1.16. A set X is called *Russell-finite* if it is empty, or if every partial ordering on X has a maximal element.

This is different from a *Russell-set* which is a Dedekind finite set that can be expressed as a countable union of pairwise disjoint 2-element sets (see [HT06]). A Russell-set will play some roles later.

Definition 3.1.17. Let $\Delta_{\mathrm{Russell}}$ denote the class of cardinalities of Russell-finite sets.

Now we check whether $\Delta_{\mathrm{Russell}}$ is a finiteness class and verify its closure properties.

Lemma 3.1.18. *The class $\Delta_{\mathrm{Russell}}$ is closed under $\leq$.*

Proof. Assume $|X| \in \Delta_{\text{Russell}}$ and $Y \subseteq X$ be non-empty. Let $<$ be a partial ordering on Y. Let $y \in Y$. Extend $<$ to X by letting $x < y'$ for all $x \in X \smallsetminus Y$ and $y' \geq y$ in Y. Then $<$ is a partial ordering on X and so X has a maximal element, say x_0. Since $x_0 \not< y$, $x_0 \notin X \smallsetminus Y$. Thus $x_0 \in Y$. □

Proposition 3.1.19. *The class* Δ_{Russell} *is closed under* $+$.

Proof. Let $|X|, |Y| \in \Delta_{\text{Russell}}$ where X and Y are disjoint. Let $<$ be a partial ordering on $X \cup Y$. Then $<\restriction (X \times X)$ is a partial ordering on X and so X has a maximal element, say x_0. If x_0 is a maximal element in $X \cup Y$, then we're done.

Suppose x is not a maximal element in $X \cup Y$. Then $Y' = \{y \in Y : y > x_0\}$ is not empty. Since Y is Russell finite and Δ_{Russell} is closed under $\leq$, Y' is also Russell finite and so it has a maximal element, say y_0. Since $y_0 > x_0$, y_0 is not below any member of X. Let $y \in Y$. If $y > y_0$, then $y > x_0$ and so $y \in Y'$, but y_0 is maximal in Y', a contradiction. Hence $y \not> y_0$. Therefore y_0 is maximal in $X \cup Y$. □

We could change the definition of Russell-finite by using 'minimal element' instead of 'maximal' but this doesn't make any difference. The following are some equivalent characterizations of Russell-finiteness.

Proposition 3.1.20. *The following are equivalent for non-empty* X.

1. $|X| \in \Delta_{\text{Russell}}$,
2. *in any partial ordering on* X*, every member of* X *is between a maximal and a minimal element,*
3. *every partial ordering on* X *has a bound on the length of its finite chains,*
4. *every partial ordering on* X *is well-founded.*

Proof. $(2 \Rightarrow 1)$ and $(4 \Rightarrow 1)$ are obvious, in the latter case considering the reverse ordering.

(1 ⇒ 2) Suppose $|X| \in \Delta_{\text{Russell}}$. Let $x \in X$ and $Y = \{y \in X : y \geq x\}$. Then Y contains a maximal element since Δ_{Russell} is closed under $\leq$ and it can be showed straightforwardly that it is also maximal in X.

For minimal, reverse $<$ to $>$, then we have a $>$-maximal element above x which is a $<$-minimal element below x.

(1 ⇒ 4) This follows from the fact that Δ_{Russell} is closed under $\leq$ and, from 1 applied to the reverse ordering, every element is above a minimal element.

(3 ⇒ 1) Let $<$ be a partial ordering on X. Then there is a bound on length of finite chains, say n is the least such bound. Let $\mathcal{C}$ be a chain in X with length n. Thus $\mathcal{C}$ has a maximal element, say x. Then x is also maximal in X, otherwise there is $y > x$ and $C \cup \{y\}$ has length $n + 1$, a contradiction.

(1 ⇒ 3) Let $|X| \in \Delta_{\text{Russell}}$ and $<$ be a partial ordering on X. Define recursively on n, X_n is the set of all maximal elements of $X \setminus \bigcup_{i<n} X_i$. It was shown in [Deg94] that $\Delta_{\text{Russell}} \subseteq \Delta_4$ (or see Proposition 3.1.26 for the stronger result, $\Delta_{\text{Russell}} \subseteq \Delta_2$), hence $|X| \in \Delta_4$, and so there is n such that $X_n = \emptyset$ and we have $X = \dot{\bigcup}_{i<n} X_i$. Let $\mathcal{C}$ be a chain in X. Then $|\mathcal{C} \cap X_i| \leq 1$ for all $i < n$, so $|\mathcal{C}| \leq n$. □

Proposition 3.1.21. *A disjoint Russell-finite union of Russell-finite sets is Russell-finite.*

Proof. Let $\{X_i : i \in I\}$ be a Russell-finite family of pairwise disjoint Russell-finite sets, and let $<$ be a strict partial ordering on $\dot{\bigcup}\{X_i : i \in I\}$.

Let L be a $<$-chain. For any $i \in I$, let $L_i = L \cap X_i$. Then L_i is a chain in X_i where $|X_i| \in \Delta_{\text{Russell}}$, so L_i must be finite for all $i \in I$. Since the X_i are pairwise disjoint, so are the L_i for all $i \in I$. Let $I' = \{i \in I : L_i \neq \emptyset\}$. Define a linear ordering $<_I$ on I' by $i_1 <_I i_2$ if $\max L_{i_1} < \max L_{i_2}$. Then we have a $<_I$-chain on I and since $|I| \in \Delta_{\text{Russell}}$, the chain is finite. Hence there are finitely many i such that $L_i \neq \emptyset$. Therefore $L = \dot{\bigcup}_{i \in I} L_i$ is a finite union of finite sets, so it is finite. □

The following corollary follows immediately.

Corollary 3.1.22. *The class Δ_{Russell} is closed under $\times$.*

It is easy to see that $n \in \Delta_{\text{Russell}}$ for all $n \in \omega$ and ω with the usual ordering has no maximal element, therefore $\aleph_0 \notin \Delta_{\text{Russell}}$. Together with Lemma 3.1.18, we have the following.

Proposition 3.1.23. *The class Δ_{Russell} is a finiteness class.*

Therefore Δ_{Russell} lies between ω and Δ. As remarked above, it was already showed in [Deg94] that every Russell-finite set is weakly Dedekind-finite, i.e. $\Delta_{\text{Russell}} \subseteq \Delta_4$. We will refine these results and find how Δ_{Russell} relates to the other finiteness classes introduced in Definition 3.1.2. First we have the next result from [MT03].

Fact 3.1.24. *If $\langle X, < \rangle$ is a non-empty partially ordered set, and X has MT-rank α, then X has minimal and maximal elements.*

Corollary 3.1.25. $\Delta_{\text{MT}} \subseteq \Delta_{\text{Russell}}$.

Proposition 3.1.26. $\Delta_{\text{Russell}} \subseteq \Delta_2$.

Proof. Suppose $|X| \notin \Delta_2$. Then there is an infinite linearly ordered partition $\langle \pi, < \rangle$ of X. If π has a $<$-maximal element, then we can construct a linear ordering on π with no maximal element. Consider the maximal conversely well-ordered final segment of $\langle \pi, < \rangle$. If it is infinite, then we can reverse the final ω^*-segment so that we have a new linear ordering on π with no maximal element. Otherwise, the segment is finite, so we can put it at the beginning of the ordering, resulting in a linear ordering on π with no maximal element. Thus we may assume that $\langle \pi, < \rangle$ has no maximal element. Define $<_X$ on X by $x <_X y$ iff $P < Q$ where $x \in P$ and $y \in Q$ and $P, Q \in \pi$. It is easy to see that $<_X$ is a partial ordering on X with no maximal element. So $|X| \notin \Delta_{\text{Russell}}$. □

Now we have the following relations.

$$\Delta_1 \subseteq \Delta_{\text{MT}} \subseteq \Delta_{\text{Russell}} \subseteq \Delta_2$$

3.1.4 More on amorphous sets

The notion of amorphous has a close link with the notion of strong minimality in model theory (see [Tru95]). Another notion which is related to amorphous is 'o-amorphous', which is related in a similar way to 'o-minimality' in model theory. First we need the following definition.

Definition 3.1.27. Let $\langle X, \leq\rangle$ be a linear ordering. A subset of X is called an *interval* if it has the form $(a, b) = \{x : a < x < b\}$ or $[a, b], [a, b), (a, b]$ – closed on one or both sides, where $a, b \in X \cup \{\pm\infty\}$. A subset Y of X is *convex* if $a < x < b$ and $a, b \in Y$ implies $x \in Y$.

Then we have the following notions.

Definition 3.1.28. We say a linear ordering $\langle X, \leq\rangle$ is *o-amorphous* if it is infinite and its only subsets (definable or not) are finite unions of intervals, and we call it *weakly o-amorphous* if its only subsets are finite unions of convex sets.

More details on o-amorphous sets can be found in [CT00]. We have every o-amorphous set is weakly o-amorphous, and Creed and Truss show ([CT00, Lemma 2.5]) that every weakly o-amorphous set is weakly Dedekind-finite. They also show that a union of two o-amorphous sets need not be o-amorphous, and provide the following easily-proved lemma.

Lemma 3.1.29 ([CT00])**.** *Let $\langle X, <\rangle$ be a linear ordering. If $X = A \cup \{x\} \cup B$ with $A < x < B$, then X is o-amorphous if and only if A and B are.*

Note that for the converse, we need an extra point x in between the two sets so that X is o-amorphous, otherwise A cannot be written as a finite union of intervals and so it is not o-amorphous. Therefore the class of cardinalities of o-amorphous sets is not a finiteness class, as we will explain after the next proof. But we can consider a slightly modified definition as stated in Definition 3.1.2, which we recall in the following.

$$\Delta_{\mathrm{o}} := \{|x| : x \subseteq y \text{ for some } y \text{ having a linear order so that } \langle y, <\rangle \text{ is } o\text{-amorphous}\}$$

$$\Delta_{\mathrm{o}}^{*} := \{|x| : x \text{ has a linear order } < \text{ so that } \langle x, <\rangle \text{ is weakly } o\text{-amorphous}\}$$

Lemma 3.1.30. $\Delta_{\mathrm{o}} \subseteq \Delta_{\mathrm{o}}^{*}$.

Proof. Let $|x| \in \Delta_{\mathrm{o}}$. Then $x \subseteq y$ for some y with a linear order $<$ such that $\langle y, <\rangle$ is o-amorphous. Hence every subset of x is a finite union of intervals whose boundaries might be in $y \setminus x$, but they are convex subsets of x. □

Proposition 3.1.31. *The classes Δ_{o} and Δ_{o}^{*} are finiteness classes.*

Proof. Clearly both Δ_{o} and Δ_{o}^{*} contain every $n \in \omega$. Next we show that $\aleph_0 \notin \Delta_{\mathrm{o}}^{*}$. Let $\langle X, <\rangle$ be a countably infinite linearly ordered set.

Case 1. There is a nonempty subset $Y \subseteq X$ with no $<$-minimum.
Since X is enumerable, Y contains a $<$-descending ω-sequence, say $\langle y_n\rangle_{n\in\omega}$ (note that this does not require AC). Then the subset $\{y_{2n} : n \in \omega\}$ of Y cannot be written as a finite union of convex sets.

Case 2. Every nonempty subset of X has a $<$-minimum.
Then $\langle X, <\rangle$ is a well-ordering and hence it contains an ω-sequence which is $<$-increasing say $\langle x_n\rangle_{n\in\omega}$, and again the set $\{x_{2n} : n \in \omega\}$ cannot be written as a finite union of convex sets.

From both cases, we can conclude that $\langle X, <\rangle$ is not weakly o-amorphous. Hence $\aleph_0 \notin \Delta_{\mathrm{o}}^{*}$, and so $\aleph_0 \notin \Delta_{\mathrm{o}}$.

The class Δ_{o} is closed under $\leq$ by its definition. For Δ_{o}^{*}, let $|X| \in \Delta_{\mathrm{o}}^{*}$ and $|Y| \leq |X|$. We may assume Y is a subset of X, and so Y inherits a linear ordering $<$ from X where

$\langle X, <\rangle$ is weakly o-amorphous. Hence every subset $Z \subseteq Y$ is a subset of X, so Z is a finite union of sets which are convex subsets of X. A convex subset of X contained in Y is a convex subset of Y, so Z is a finite union of convex subsets of Y. Thus $\langle Y, <\restriction Y \times Y\rangle$ is also weakly o-amorphous. □

Note that from the above proof, the difference between the two cases is that a convex subset of Z which is contained in Y is also a convex subset of Y, but an interval of Z which is contained in Y need not be an interval in Y, as its endpoints may have been omitted. Further as both Δ_{o} and Δ_{o}^* are finiteness classes, they lie between ω and Δ. We can refine this result further as follows.

Proposition 3.1.32. $\Delta_{\mathrm{o}}^* \subseteq \Delta_4$.

Proof. Let $|X| \in \Delta_{\mathrm{o}}^*$ and let $<$ be a linear order on X such that $\langle X, <\rangle$ is weakly o-amorphous. Suppose $|X| \notin \Delta_4$. Then X can be written as a countable disjoint union of nonempty subsets $\dot\bigcup_{n\in\omega} X_n$. Since $\langle X, <\rangle$ is weakly o-amorphous, every X_n is a finite union of convex subsets of X, say $X_n = \dot\bigcup_{i<k_n} Y_{n,i}$ for some $k_n \in \omega$ for all $n \in \omega$. Note that the $Y_{n,i}$ are pairwise comparable under $<$. Hence $\mathcal{C} := \{Y_{n,i} : i < k_n \text{ and } n \in \omega\}$ is countable and $\langle \mathcal{C}, <\rangle$ is a countable linear ordering.

Similarly to the proof of Proposition 3.1.31 where we showed that $\aleph_0 \notin \Delta_{\mathrm{o}}^*$, there is either a $<$-increasing or a $<$-decreasing ω-sequence of members of $\mathcal{C}$, the union of whose alternate subsequences cannot be written as a finite union of convex subsets of X. This contradicts $\langle X, <\rangle$ being weakly o-amorphous. Hence $|X| \in \Delta_4$. □

Therefore we have the following relations.

$$\omega \subseteq \Delta_{\mathrm{o}} \subseteq \Delta_{\mathrm{o}}^* \subseteq \Delta_4$$

3.1.5 Δ_4^* and Δ_5^*

The notion of *weakly Dedekind*-infinite* was introduced in [Gol97], which is a stronger version of weakly Dedekind-infinite set. The dual notion of finiteness is stated as follows.

Definition 3.1.33. A set X is called *weakly Dedekind*-finite* if there are no finite-to-one maps from a subset of X onto ω.

We let Δ_4^* be the class of cardinalities of weakly Dedekind*-finite sets. As we can see, the above definition is slightly different from the notion of weakly Dedekind-finite. The obvious difference is that the function is required to be finite-to-one, but in the definition we needed to quantify over all subsets to ensure that the class Δ_4^* is closed under $\leq$. It can be easily checked that $n \in \Delta_4^*$ for all $n \in \omega$ and $\aleph_0 \notin \Delta_4^*$. Therefore the following proposition follows immediately.

Proposition 3.1.34. *The class Δ_4^* is a finiteness class.*

Next we study the closure properties of Δ_4^*. Showing that Δ_4^* is closed under $+$ is straightforward.

Proposition 3.1.35. *The class Δ_4^* is closed under $+$.*

Proof. Let X and Y be sets such that $X \cap Y = \emptyset$. Suppose there is a subset $S \subseteq X \cup Y$ with a finite-to-one function $f : S \to \omega$. Then either $S \cap X$ or $S \cap Y$ has infinite image under f, so $|X| \notin \Delta_4^*$ or $|Y| \notin \Delta_4^*$. □

Proposition 3.1.36. *A disjoint weakly Dedekind*-finite union of weakly Dedekind*-finite sets is weakly Dedekind*-finite.*

Proof. Let $\{X_i : i \in I\}$ be a weakly Dedekind*-finite family of pairwise disjoint weakly Dedekind*-finite sets.

Suppose $\dot{\bigcup}\{X_i : i \in I\}$ is not weakly Dedekind*-finite. Then there is a subset $S \subseteq \dot{\bigcup}\{X_i : i \in I\}$ with a finite-to-one surjection $f\colon S \to \omega$. For each $i \in I$, let $S_i = S \cap X_i$. Then $f \restriction S_i\colon S_i \to \omega$ is also finite-to-one. Since each X_i is weakly Dedekind*-finite, $\operatorname{ran}(f \restriction S_i)$ is finite. Therefore we can associate with each i a finite subset $N_i \subseteq \omega$, where $N_i = \operatorname{ran}(f \restriction S_i)$. Furthermore S_i is finite for all $i \in I$.

Since f is finite-to-one, $f^{-1}[N]$ is finite for all finite $N \subseteq \omega$. Therefore there are finitely many i such that $S_i \subseteq f^{-1}[N]$ since all S_i are pairwise disjoint. Hence the map $g\colon i \mapsto N_i$ is finite-to-one. Since $\bigcup_{i\in I} N_i = \omega$ and all N_i are finite, there are infinitely many different N_i. Hence g maps I onto an infinite subset of $[\omega]^{<\aleph_0}$, the set of all finite subsets of ω, which is countably infinite. Hence $\operatorname{ran}(g)$ is infinite, and this contradicts weakly Dedekind*-finiteness of I. □

Corollary 3.1.37. *The class Δ_4^* is closed under $\times$.*

Both Δ_4^* and Δ_5 contain Δ_4 and are contained in Δ, but it is shown in later sections that Δ_4^* and Δ_5 are not comparable, i.e. it is consistent that $\Delta_4^* \not\subseteq \Delta_5$ and $\Delta_5 \not\subseteq \Delta_4^*$ (see Proposition 3.2.22 and Proposition 3.2.10, respectively). A stronger version of Δ_5, which has an alternative definition in Definition 3.1.10, can be obtained in the same fashion as defining Δ_4^* from Δ_4.

Definition 3.1.38. A set X is called *dual Dedekind*-finite* if there are no non-injective finite-to-one maps from X onto X, or equivalently, there are no finite-to-one maps from X onto $X \cup \{*\}$ where $* \notin X$.

Let Δ_5^* be the class of cardinalities of dual Dedekind*-finite sets. Obviously $\Delta_5 \subseteq \Delta_5^*$. Furthermore we can show that $\Delta_4^* \subseteq \Delta_5^*$. First we need the following proposition.

Proposition 3.1.39. *Let X be a set. Then $|X| \notin \Delta_5^*$ if and only if there is a subset $T \subseteq X$ carrying a finite-branching tree structure with ω levels and no leaves.*

Proof. ($\Rightarrow$) Suppose $|X| \notin \Delta_5^*$. Then there is a finite-to-one surjection $f: X \to X \cup \{*\}$ where $* \notin X$. Define L_n for $n \in \omega$ as follows. Let $L_0 = \{*\}$, $L_{n+1} = f^{-1}[L_n]$ for $n \in \omega$ and let $T = \bigcup_{n \in \omega} L_n$. Note that the L_n are finite and pairwise disjoint, and since f is a surjection, $L_n \neq \emptyset$ for all $n \in \omega$. Define $<$ on T to be the transitive closure of the relation given by $x < y$ if $y \in f^{-1}[\{x\}]$. It can be proved straightforwardly that $\langle T, < \rangle$ is a tree with no leaves with $*$ the least element, and since all L_n are finite, $\langle T, < \rangle$ is finite branching. Therefore $T \setminus \{*\}$ is our desired subset of X.

($\Leftarrow$) Suppose there is an infinite subset $T \subseteq X$ with a finite-branching tree structure satisfying the condition. Let $* \notin X$ and extend T to $T \cup \{*\}$ by letting $*$ be below all members of T. Define $f: T \to T \cup \{*\}$ by $f(x) = y$ where y is the immediate predecessor of x. Since T has no leaves and branches finitely, f is surjective and finite-to-one. We extend the domain of f to X by letting f be the identity on $X \setminus T$ so we have a finite-to-one function from X onto $X \cup \{*\}$. $\square$

To see that $\Delta_4^* \subseteq \Delta_5^*$. Suppose $|X| \notin \Delta_5^*$, then there is a subset T of X as from the above proposition, so every level of T is finite and the map given by $x \mapsto n$ where x is on level n is a finite-to-one map from a subset T of X onto ω which is also non-injective.

Proposition 3.1.40. *The class Δ_5^* is a finiteness class.*

Proof. It is easy to see that $n \in \Delta_5^*$ for all $n \in \omega$, so $\omega \subseteq \Delta_5^*$. The map $f: \omega \to \omega$ defined by $f(0) = 0$ and $f(n+1) = n$ for all $n \in \omega$ is a non-injective finite-to-one map from ω onto ω, therefore $\aleph_0 \notin \Delta_5^*$.

Let X and Y be sets such that $|Y| \leq |X|$ and suppose $|Y| \notin \Delta_5^*$. We may assume that Y is a subset of X. Then there is a non-injective finite-to-one $g: Y \to Y$. Extend g to X by letting $g(x) = x$ for all $x \in X \setminus Y$. Then g is a non-injective finite-to-one function from X onto X, and so $|X| \notin \Delta_5^*$. $\square$

Next we study the closure properties of Δ_5^* under $+$ and $\times$, which turn out to be a positive result for $+$ but not for $\times$.

Proposition 3.1.41. *The class Δ_5^* is closed under $+$.*

Proof. Let X and Y be disjoint sets. Suppose $|X| + |Y| \notin \Delta_5^*$. Then there is a subset $T \subseteq X \cup Y$ with a finite-branching tree structure satisfying the above proposition. Since T is infinite, either $T \cap X$ or $T \cap Y$ is infinite. Suppose $T_X := T \cap X$ is infinite. Then T_X is a subset of X carrying a tree structure in which every vertex branches finitely and hence it also has ω-levels. If T_X has no leaves, then $|X| \notin \Delta_5^*$ and so we're done. Suppose T_X has some leaves, say x is one such. Then the set $T_Y := \{y \in Y : y > x\}$ is a subset of Y extending x carrying a finite-branching tree structure. Since T has no leaves and x is a leaf of T_X, T_Y has no leaves. Thus T_Y is an infinite subset of Y carrying a finite-branching tree structure with ω-levels and no leaves. Hence $|Y| \notin \Delta_5^*$. $\square$

The counter example for showing that Δ_5^* is not closed under $\times$ is the same as one that is used to show the similar result for Δ_5 in [Tru74]. We will give its proof in Section 3.2 where we discuss consistency results obtaining from FM-models.

Proposition 3.1.42. *It is consistent with* ZF *that Δ_5^* is not closed under $\times$.*

Proof. See Subsection 3.2.8. $\square$

We have the following relations.

$$\begin{tikzcd}[column sep=small, row sep=small]
 & \Delta_4^* \arrow[dr, phantom, sloped, "\subsetneq"] & \\
\Delta_4 \arrow[ur, phantom, sloped, "\subsetneq"] \arrow[dr, phantom, sloped, "\subsetneq"] & & \Delta_5^* \\
 & \Delta_5 \arrow[ur, phantom, sloped, "\subsetneq"] &
\end{tikzcd}$$

3.1.6 Period-finiteness and classes beyond Δ

The notions of *period-infinite of level* (i) ($[i]$-*per-inf*) for $1 \leq i \leq 5$ were introduced in [Deg94]. We can state their dual notions of finiteness as follows.

Definition 3.1.43. A set X is

- [1]-*period-finite* if every injection on X has finite order,
- [2]-*period-finite* if every permutation of X has finite order,
- [3]-*period-finite* if every function on X has a finite cycle,
- [4]-*period-finite* if every injection on X has a finite cycle,
- [5]-*period-finite* if every permutation of X has a finite cycle.

We have the following relations

$$\begin{aligned} &\text{weakly Dedekind-finite} \Rightarrow \text{[1]-period-finite} \Rightarrow \text{[2]-period-finite} \Rightarrow \text{Dedekind-finite} \\ &\quad \Rightarrow \text{[3]-period-finite} \Rightarrow \text{[4]-period-finite} \Rightarrow \text{[5]-period-finite} \end{aligned}$$

It turns out that some of these notions actually coincide as can be shown as follows.

Proposition 3.1.44.

(i) [1]-*period-finite* $\Leftrightarrow$ [2]-*period-finite.*

(ii) *Dedekind-finite* $\Leftrightarrow$ [3]-*period-finite.*

(iii) [4]-*period-finite* $\Leftrightarrow$ [5]-*period-finite.*

Proof. It only remains to show the $\Leftarrow$ part for each statement.

(i) Since [2]-period-finite implies Dedekind-finite and a Dedekind-finite set is a set with no injection into a proper subset, every injection on a Dedekind-finite set must be surjective.

(ii) Suppose X is Dedekind-infinite. Then there is a countable subset $Y \subseteq X$, say $Y = \{x_n : n \in \omega\}$. Then the function $f : X \to X$ defined by $f(x) = x_0$ if $x \notin Y$ and $f(x_n) = x_{n+1}$ for all $n \in \omega$ has no finite cycle.

(iii) Suppose X is not [4]-period-finite. Then there is an injection $f : X \to X$ such that f has no finite cycles (including fixed points). Define a relation $\sim$ on X by $x \sim y$ iff $f^n(x) = y$ for some $n \in \mathbb{Z}$. Then $\sim$ is an equivalence relation. We call an equivalence class $[x]$ an ω-orbit if $[x] = \{f^n(y) : n \in \omega\}$ for some $y \notin \mathrm{ran}(f)$, otherwise $[x]$ is a $\mathbb{Z}$-orbit. Define g on X as follows. For any member of a $\mathbb{Z}$-orbit, g agrees with f. For an ω-orbit $\{f^n(y) : n \in \omega\}$, let

$$g(f^{2n}(y)) = f^{2n+2}(y)$$
$$g(f^{2n+1}(y)) = \begin{cases} y & \text{if } n = 0; \\ f^{2n-1}(y) & \text{if } n > 0. \end{cases}$$

Then g is a permutation on every orbit with no finite cycles and X is the disjoint union of orbits. Hence X is not [5]-period-finite. □

Note that it was mentioned in [Deg94] that there is a Dedekind-infinite set which is [3]-period-finite. This is not accurate as we have just shown that Dedekind-finiteness and [3]-period-finiteness actually coincide.

As [1]-period-finiteness is below Dedekind-finiteness, we let Δ_{per} be the class of cardinalities of [1]-period-finite sets. For [5]-period-finite which lies above Dedekind-finiteness, we let $\Gamma'_{5\text{-per}}$ be the class of cardinalities of such sets.

It can be shown that it is consistent that there is a [5]-period-finite set which is not Dedekind-finite (see Proposition 3.2.18). Hence these two notions do not coincide and since $\Delta \subseteq \Gamma'_{5\text{-per}}$, $\Gamma'_{5\text{-per}}$ is not a finiteness class.

Proposition 3.1.45. *The class Δ_{per} is a finiteness class.*

Proof. It suffices to show that Δ_{per} is closed under $\leq$. Let $|X| \in \Delta_{\text{per}}$ and $|Y| \leq |X|$. We may assume $Y \subseteq X$. Then every permutation f on Y extends to a permutation of g on X by a trivial extension. Since $|X| \in \Delta_{\text{per}}$, there is $n \in \omega$ such that $g^n = \mathrm{id}_X$. It is easy to see that $f^n = \mathrm{id}_Y$. Hence $|Y| \in \Delta_{\text{per}}$. □

Proposition 3.1.46. *The class Δ_{per} is closed under $+$.*

Proof. Let X and Y be sets such that $X \cap Y = \emptyset$. Suppose $|X| + |Y| \notin \Delta_{\text{per}}$. Then there is a permutation f on $X \cup Y$ such that f does not have finite order, i.e. $f^n \neq \mathrm{id}_{X \cup Y}$ for all $n \in \omega$.

Case 1. f has an infinite cycle. Therefore $X \cup Y$ is Dedekind-infinite. Hence either X or Y is Dedekind-infinite and so $|X| \notin \Delta_{\text{per}}$ or $|Y| \notin \Delta_{\text{per}}$.

Case 2. Every cycle of f is finite. Then there are arbitrary long finite cycles of f, i.e. $\{|C| : C \text{ is a cycle of } f\}$ is infinite.

For any cycle C under f, let $C_X = C \cap X$ and $C_Y = C \cap Y$. Then $|C| = |C_X| + |C_Y|$. Since $\{|C| : C \text{ is a cycle of } f\}$ is infinite, either $\{|C_X| : C \text{ is a cycle of } f\}$ or $\{|C_Y| : C \text{ is a cycle of } f\}$ is infinite. Suppose $\{|C_X| : C \text{ is a cycle of } f\}$ is infinite. Then we can define g on X by $g(x) = f^n(x)$ where n is the least natural number such that $f^n(x) \in X$. This is well-defined since every cycle of f is finite. It is easy to see that a cycle C' of x under g has the same cardinality as C_X where C is a cycle of x under f. Thus $|X| \notin \Delta_{\text{per}}$. □

The next 3 notions are from [Lév58], namely V-, VI-, and VII-finite, and we let Γ'_{V}, Γ'_{VI}, and Γ'_{VII} be the corresponding classes of cardinalities, respectively. None of these classes is a finiteness class as it is not closed downwards. For example, if x is an amorphous set and $y = \omega \dot\cup x$, then $|y| < 2|y|$, $|y| < |y|^2$ and $|y|$ is not an aleph, but $\aleph_0 < |y|$ and $\aleph_0$ does not belong to any finiteness class.

Note that Γ'_{VII} is the boundary of classes of cardinalities without AC, i.e. $\Gamma'_{\text{VII}} = \omega$ if and only if AC holds. Whereas showing that $\omega = \Delta$ needs a much weaker form of AC. To be precise, $\omega = \Delta$ is equivalent to '$\mathsf{W}_{\aleph_0}$: for every x, either $|x| \geq \aleph_0$ or $|x| \leq \aleph_0$', this is weaker than $\mathsf{AC}_{\aleph_0}$, the Axiom of Countable Choice. More details can be found in [Jec73, Chapter 8].

3.2 Some consistency results from FM-models

It has been shown that some inclusions stated between the classes of Dedekind-finite cardinals cannot be reversed.

For instance, since the union of two disjoint amorphous sets has an MT-rank, but is not amorphous, it is consistent that $\Delta_1 \neq \Delta_{\text{MT}}$. In fact, if $\omega \neq \Delta_1$, then $\Delta_1 \neq \Delta_{\text{MT}}$ (see [Tru74]). Now we try to show that the remaining inclusions not settled in the diagram need not hold by constructing FM-models from various structures.

3.2.1 The generic bipartite graph

Let $\mathcal{B} = \langle T \dot{\cup} B, T, B, <_{\mathcal{B}} \rangle$ be the generic bipartite graph. Let $U = \{u_a : a \in \mathcal{B}\}$ be the set of atoms and let $<$ be the partial ordering on U induced by $<_{\mathcal{B}}$, i.e. $u_b < u_t$ if $b <_{\mathcal{B}} t$. Let $\mathcal{G}$ be the group of all order-preserving permutations of U, and $\mathcal{F}$ be the filter on $\mathcal{G}$ generated by finite supports. Let $\mathcal{N}_{\mathcal{B}}$ be the permutation model determined by U, $\mathcal{G}$, and $\mathcal{F}$.

For convenience we may view U as $T \dot{\cup} B$ and we say $b < t$ if $u_b < u_t$. We will write b for a member of B and t for a member of T.

Notation.

1. For each $b \in B$, let $T_b = \{t \in T : t > b\}$ and $T_{b'} = \{t \in T : t \parallel b\}$. Then $T = T_b \dot\cup T_{b'}$ where $b < T_b$ and $b \parallel T_{b'}$.

2. For $b_1, \dots, b_n \in B$, let $T_{b_1 \dots b_i b'_{i+1} \dots b'_n}$ denote $\bigcap_{1 \le k \le i} T_{b_k} \cap \bigcap_{i+1 \le l \le n} T_{b'_l}$. We may write $T_{b_1 \dots b_i b'_{i+1} \dots b'_n}$ as $T_{\overline{b}\,\overline{b}'}$.

3. For finite $Y_1, Y_2 \subseteq B$, let T_{Y_1, Y'_2} denote $\bigcap\{T_b : b \in Y_1\} \cap \bigcap\{T_{b'} : b \in Y_2\}$.

Similarly for $t_1, \dots, t_m \in T$ and $B_{\overline{t}\,\overline{t}'}$.

Lemma 3.2.1. *Every finite $E \subseteq U$ supports only finitely many subsets of U.*

In particular, if $E = \{b_1, \dots\dots b_n, t_1, \dots, t_m\}$, then there are $2^n + 2^m + n + m$ orbits of U under $\mathcal{G}_E$. Hence there are $2^{2^n+2^m+n+m}$ subsets of U supported by E.

To see this, we will show that orbits of members of $U \setminus E$ under $\mathcal{G}_E$ are either of the form $T_{\overline{b}\,\overline{b}'}$ or $B_{\overline{t},\overline{t}'}$. Since there are 2^n possible forms of $T_{\overline{b}\,\overline{b}'}$, 2^m possible forms of $B_{\overline{t}\,\overline{t}'}$ and every member of E either belongs to or does not belong to a subset of U, there are $2^{2^n+2^m+n+m}$ possible subsets of U supported by E.

Given a finite subset $E \subseteq U$, say $E = \{b_1, \dots\dots b_n, t_1, \dots, t_m\}$, the orbit of a member of $U \setminus E$ under $\mathcal{G}_E$ is of the form $T_{\overline{b}\,\overline{b}'}$ or $B_{\overline{t}\,\overline{t}'}$.

Proposition 3.2.2. *Let $E = \{b_1, \dots\dots b_n, t_1, \dots, t_m\}$. For any $x \in T \setminus E$, $\mathcal{G}_E(x)$ is of the form T_{Y_x, Y'_x} where $E \cap B = Y_x \dot\cup Y'_x$. Similarly for $x \in B \setminus E$.*

Proof. Let $x \in T \setminus E$. Let $E \cap B = E_x \dot\cup E'_x$ where $x > E_x$ and $x \parallel E'_x$. We will show that $\mathcal{G}_E(x) = T_{E_x, E'_x}$.

Let $g \in \mathcal{G}_E$. Since g preserves order and fixes E pointwise, $g(x) > E_x$ and $g(x) \parallel E'_x$. Hence $g(x) \in T_{E_x, E'_x}$.

For the converse, let $y \in T_{E_x, E'_x}$. Then $y > E_x$ and $y \parallel E'_x$. Suppose $y \neq x$. Since both x and y belong to T, $x \parallel y$. Then there is a finite partial automorphism $p\colon E \cup \{x\} \to E \cup \{y\}$ taking x to y and fixing all members of E, which can be extended to $g \in \mathcal{G}_E$. Hence $y = g(x) \in \mathcal{G}_E(x)$. $\square$

Proposition 3.2.3. *The following statements hold in $\mathcal{N}_{\mathcal{B}}$.*

1. $|U| \in \Delta_{\mathrm{Russell}}$.
2. $|U| \notin \Delta_{\mathrm{MT}}$.

Proof.

1. Let $\prec$ be a partial ordering on U supported by a finite $E \subseteq U$. Suppose $x \prec y$ and x, y lie in the same $\mathcal{G}_E$-orbit. Then $x \parallel_{<} y$ (note that here x and y are incomparable under $<$, the original partial ordering on U inherited from $<_{\mathcal{B}}$) and the finite automorphism $p\colon E \cup \{x, y\} \to E \cup \{x, y\}$ which p fixes E pointwise and swaps x and y can be extended to an automorphism $g \in \mathcal{G}_E$. Therefore $y = g(x) \prec g(y) = x$ yields a contradiction. Hence x and y are from different orbits. Since there are only finitely many orbits of $U \setminus E$ under $\mathcal{G}_E$, $\prec$ has a minimal element.

2. It suffices to show that for any $x \in T$ and finite $E \subseteq B$, $\mathcal{G}_E(x)$ does not have MT-rank in $\mathcal{N}_{\mathcal{B}}$.

   Suppose there are $x \in T$ and $E \subseteq B$ such that $\mathcal{G}_E(x)$ has MT-rank α. Without loss of generality we assume x and E are such a pair with $\mathcal{G}_E(x)$ having a least rank α. Then there is a least n such that dividing $\mathcal{G}_E(x)$ into $n + 1$ subsets, one of them must have rank $< \alpha$.

   Let $b_1, \ldots, b_n \in B \setminus E$ and let $E' = E \cup \{b_1, \ldots, b_n\}$. Then there are 2^n subsets of $\mathcal{G}_E(x)$ supported by E', and all of which are of the form $\mathcal{G}_{E'}(y)$ for some $y \in \mathcal{G}_E(x)$. Since $\mathcal{G}_E(x)$ has degree n, there is $y \in \mathcal{G}_E(x)$ such that $\mathcal{G}_{E'}(y)$ has rank $< \alpha$, which contradicts $\mathcal{G}_E(x)$ having least rank.

Since Δ_{MT} is closed under $\leq$, $|U| \notin \Delta_{\mathrm{MT}}$. □

Corollary 3.2.4. *It is consistent that* $\Delta_{\mathrm{MT}} \subset \Delta_{\mathrm{Russell}}$.

3.2.2 The generic partially ordered set

Let P be the generic partially ordered set. Let $U = \{u_p : p \in P\}$. We again abuse notion by letting $U = P$ under the induced ordering. Let $\mathcal{G}$ be the group of automorphisms of U and let $\mathcal{F}$ be the filter generated by finite supports. Let $\mathcal{N}_P$ be the permutation model determined by U, $\mathcal{G}$, and $\mathcal{F}$.

Proposition 3.2.5. $\Delta_{\mathrm{Russell}} \subset \Delta_2$ *in* $\mathcal{N}_P$.

Proof. Since U is generic partially ordered, it contains chains of all finite lengths. Therefore $|U| \notin \Delta_{\mathrm{Russell}}$.

Let $\langle \Pi, \prec \rangle$ be a linearly ordered partition of U supported by a finite set E. Let $x \notin E$, say $x \in X \setminus E$ for some $X \in \Pi$, and let $E = E_1 \dot\cup E_2 \dot\cup E_3$ where $E_1 < x < E_2$ and $E_3 \parallel_< x$. We will show that for all $z \in U$, if $E_1 < z < E_2$ and $z \parallel_< E_3$, then $z \in X$.

Let $z \in U$ satisfy the above condition, say $z \in Z \in \Pi$. The aim is to show that $Z = X$ and then we are done. If we are able to find $\pi \in \mathcal{G}_E$ such that $\pi(x) = z$ and $\pi(z) = x$, then we have $\pi(X) = Z$ and $\pi(Z) = X$, which implies $X = Z$. But since the relationship between x and z is unknown, it is not obvious whether such π exists, so we have to do this with respect to a third point y as follows.

By the method of Lemma 2.3.6, there is $y \in U$ such that $E_1 < y < E_2$ and $y \parallel_< E_3 \cup \{x, z\}$, say $y \in Y \in \Pi$. Let p be the automorphism of $E \cup \{x, y\}$ which fixes all members of E and swaps x and y. Since U is homogeneous, we can extend p to a full automorphism π on U. Then $\pi \in \mathcal{G}_E$ and so $y = \pi(x) \in \pi(X) \in \pi(\Pi) = \Pi$. Hence $\pi(X) = Y$ and similarly we can show that $\pi(Y) = X$. Suppose $X \neq Y$. Since $X, Y \in \Pi$, we may assume $X \prec Y$, but then we also have $Y = g(X) \prec g(Y) = X$ which is a contradiction. Hence $X = Y$, and similarly we can show that $Y = Z$. Therefore $X = Z$ and so $z \in X$.

Since the choice of x is arbitrary, we deduce that every z that is related to E in the same manner as x lies in the same member of the partition as x. Since E is finite, there are finitely many way to split E into 3 disjoint sets. Therefore there can only be finitely many members of Π. This shows that $|U| \in \Delta_2$. $\square$

3.2.3 Mostowski model

Recall the Mostowski model from Chapter 2. Let U be the set of atoms isomorphic to the rationals $\langle \mathbb{Q}, <_{\mathbb{Q}} \rangle$ and let $<$ be a linear ordering on U induced by $<_{\mathbb{Q}}$. Let $\mathcal{G}$ be the group of order-preserving permutations of U. Let $\mathcal{F}$ be the filter on $\mathcal{G}$ generated by finite supports. Let $\mathcal{N}_M$ be the permutation model determined by U, $\mathcal{G}$ and $\mathcal{F}$. We call $\mathcal{N}_M$ the *(ordered) Mostowski model*.

The following statement is a basic property of U in $\mathcal{N}_M$ (e.g. see Lemma 8.12 in [Hal17]).

Proposition 3.2.6. *Every finite $E \subseteq U$ supports only finitely many subsets of U.*

Actually the above statement can be strengthened to determine the exact number of subsets supported by a finite subset $E \subseteq U$. In particular, if $|E| = n$, then there are 2^{2n+1} subsets of U that are supported by E. But this is irrelevant here.

Proposition 3.2.7. *The following statements hold in $\mathcal{N}_M$.*

1. $|U| \in \Delta_4$,
2. $|U| \notin \Delta_3$.

Proof.

1. This was shown in [Tru74], but for ease of reading, we give a proof here. The result can be obtained straightforwardly from the above proposition. Suppose there is $f : U \to \omega$ with a support E. Then it can be shown that E is also a support of $f^{-1}[\{n\}]$ for all $n \in \omega$ and since E can support only finitely many subsets of U, f is not a surjection.

2. Since $\langle U, < \rangle$ is an infinite chain and the relation $<$ is in $\mathcal{N}_M$ since it has empty support, $|U| \notin \Delta_3$. $\square$

Corollary 3.2.8. *It is consistent that* $\Delta_4 \not\subseteq \Delta_3$ *and so* $\Delta_2 \subset \Delta_4$.

In Chapter 4, we will show that the set of atoms of any FM-model constructed from an $\aleph_0$-categorical structure will also lie in Δ_4 in that model.

3.2.4 $\mathbb{Q} \times \mathbb{Q}$

Let $\mathcal{Q} = \langle \mathbb{Q} \times \mathbb{Q}, <, \{P_q : q \in \mathbb{Q}\} \rangle$, where the relation $<$ on P_q is such that $\langle P_q, < \rangle \cong \langle \mathbb{Q}, <_{\mathbb{Q}} \rangle$ for all $q \in \mathbb{Q}$ and is extended to $\mathbb{Q} \times \mathbb{Q}$ by $x < y$ if $x \in P_q$ and $y \in P_r$ for some $q, r \in \mathbb{Q}$ such that $q <_{\mathbb{Q}} r$, i.e. $<$ is the lexicographic ordering on $\mathbb{Q} \times \mathbb{Q}$ where we treat P_q as $\{q\} \times \mathbb{Q}$ for all $q \in \mathbb{Q}$.

Let $U_{\mathbb{Q}^2}$ be the set of atoms indexed by $\mathcal{Q}$. Let $\mathcal{G}$ be the group of automorphisms on $U_{\mathbb{Q}^2}$ induced by $\mathrm{Aut}(\mathcal{Q})$, i.e. $\mathcal{G}$ preserves $<$ and $\langle \{P_q\}_{q \in \mathbb{Q}}, < \rangle$, and let $\mathcal{F}$ be a normal filter on $\mathcal{G}$ generated by finite supports. Let $\mathcal{N}_{\mathbb{Q}^2}$ be the corresponding FM-model.

Proposition 3.2.9. $|U_{\mathbb{Q}^2}| \in \Delta_{\mathrm{o}}^* \setminus \Delta_{\mathrm{o}}$ *in* $\mathcal{N}_{\mathbb{Q}^2}$.

Proof. First we study orbits of members of U under $\mathcal{G}_E$ for any finite $E \subseteq U$. Let $E \subseteq U$ be finite, say $E = \{a_0, a_1, \ldots, a_n\}$ where $a_0 < a_1 < \ldots < a_n$. Consider orbits of members between a_i and a_{i+1} where $a_0 \leq a_i < a_{i+1} \leq a_n$. We have 2 cases, either a_i and a_{i+1} are from the same P_q for some $q \in \mathbb{Q}$, or $a_i \in P_q$ and $a_{i+1} \in P_r$ for some different $q, r \in \mathbb{Q}$, for the latter case we also have $q <_{\mathbb{Q}} r$.

It is easy to see that if a_i and a_{i+1} are from the same P_q, then the orbits of x where $a_i < x < a_{i+1}$ is the interval (a_i, a_{i+1}). Suppose $a_i \in P_q$ and $a_{i+1} \in P_r$ for some $q, r \in \mathbb{Q}$ such that $q <_{\mathbb{Q}} r$. Then $\mathcal{G}_E$ fixes both P_q and P_r setwise. For any member x such that $a_i < x < a_{i+1}$ we have the following 3 cases , either $x \in P_q$, $x \in P_r$, or $P_q < x < P_r$. Therefore there are 3 orbits of U_E under $\mathcal{G}_E$ lying between a_i and a_{i+1}, namely $(a_i, \infty) \cap P_q$, $(-\infty, a_{i+1}) \cap P_r$, and $P_{(q,r)} := \dot{\bigcup}\{P_s : q <_{\mathbb{Q}} s <_{\mathbb{Q}} r\} = \{x \in U : P_q < x < P_r\}$.

For a_0, say $a_0 \in P_q$ for some $q \in \mathbb{Q}$, there are 2 orbits of U under $\mathcal{G}_E$ on its left, namely $(-\infty, a_0) \cap P_q$ and $P_{(-\infty,q)} := \dot{\bigcup}\{P_s : s <_{\mathbb{Q}} q\}$. Similarly for a_n which has 2 orbits of U under $\mathcal{G}_E$ on its right.

We can see that these orbits of U under $\mathcal{G}_E$ are convex subsets of U, and there are only finitely many of them for any given finite $E \subseteq U$. Therefore any subset $S \subseteq U$ is a finite union of these orbits and so it is a finite union of convex subsets of U. Hence $\langle U, < \rangle$ is weakly o-amorphous.

Now let $\prec$ be a linear ordering on U. We will show that $\langle U, \prec \rangle$ is not o-amorphous. Suppose $\prec$ is supported by E. Then, similarly to the proof of Proposition 4.2.1 where we analyzed all possible linear orderings on the set of atoms of Mostowski model, $\prec$ is also built from orbits of U under $\mathcal{G}_E$. Hence there is always an orbit of the form $P_{(q,r)}$ where q and r are either from E or $q = -\infty$ or $r = \infty$ such that $\prec$ is either $<$ or $>$ on $P_{(q,r)}$. Furthermore $P_{(q,r)}$ cannot be written as a finite union of intervals. Hence $\langle U, \prec \rangle$ is not o-amorphous. $\square$

3.2.5 ω pairs of socks

Let $U = \dot{\bigcup}_{n\in\omega} P_n$ where $P_n = \{a_n, b_n\}$ for $n \in \omega$. Let $\mathcal{G}$ be the group of permutations of U preserving each pair P_n, i.e. $\pi(P_n) = P_n$ for all $\pi \in \mathcal{G}$ and $n \in \omega$. Let $\mathcal{F}$ be the filter on $\mathcal{G}$ generated by finite supports. Let $\mathcal{N}_{\omega,2}$ be the permutation model determined by U, $\mathcal{G}$ and $\mathcal{F}$. This model is called the *second Fraenkel model* in [Jec73], but we will call it *ω pairs of socks* according to its set of atoms as this name best depicts the model. We also have modified versions of this model in later sections.

Proposition 3.2.10. *The following statements hold in $\mathcal{N}_{\omega,2}$.*

1. $|U| \in \Delta_3$,
2. $|U| \notin \Delta_4$.
3. $|U| \notin \Delta_4^*$.
4. $|U| \in \Delta_5$.

Proof.

1. Suppose there is an infinite linear ordering subset $\langle X, <\rangle$ of U with a support E. We may assume that $E = \bigcup_{n<N} P_n$ for some $N \in \omega$. Since X is infinite, there is $a_k \in X \setminus E$, where $a_k \in P_k$ for some $k > N$ and let $b_k \in P_k \setminus \{a_k\}$. Then there is $\pi \in \mathcal{G}_E$ such that $\pi(a_k) = b_k$. If $b_k \in X$, then π swaps the ordering of a_k and b_k, a contradiction, and if $b_k \notin X$, then $\pi[X] \neq X$, also a contradiction. Hence such X does not exist. Therefore $|U| \in \Delta_3$.

2. Since $\pi(P_n) = P_n$ for all $\pi \in \mathcal{G}$, the map $n \mapsto P_n$ has empty support and so $\omega \preceq^* U$, i.e. $|U| \notin \Delta_4$.

3. The function in 2 is also finite-to-one, hence $|U| \notin \Delta_4^*$.

4. Let $f{:}\, U \to U \cup \{*\}$ where $* \notin U$ be a surjection with finite support E. We may assume that $E = \bigcup_{n\leq N} P_n$ for some $N \in \omega$.

Let $m > N$. Since f is a surjection, $f^{-1}[P_m] \neq \emptyset$. Let $a \in f^{-1}[P_m]$. Suppose $a \notin P_m$. Then there is $\pi \in \mathcal{G}_E$ such that swaps members of P_m but fixes a. Thus $f\pi(a) = f(a) \neq \pi f(a)$, which contradicts $\pi f = f\pi$. Hence $a \in P_m$ so $f^{-1}[P_m] \subseteq P_m$, and since P_m is finite, $f^{-1}[P_m] = P_m$.

Thus $f[E] = E \cup \{*\}$, but E is finite, a contradiction. Therefore f does not exist and hence $|U| \in \Delta_5$. □

Corollary 3.2.11. *It is consistent that* $\Delta_5 \nsubseteq \Delta_4^*$, $\Delta_3 \nsubseteq \Delta_4$ *and so* $\Delta_2 \subset \Delta_3$.

Note that the set U is a *Russell-set* which is defined as a countable union of 2-element sets where the cartesian product of these sets is empty (a more precise definition can be found in [HT06]).

3.2.6 Rational pairs of socks

Let $U = \dot{\bigcup}_{q \in \mathbb{Q}} P_q$ where $P_q = \{u_q, v_q\}$ for $q \in \mathbb{Q}$. Let $\mathcal{G}$ be the group of permutations sending a pair to another pair and preserving the order of pairs, i.e. if $\pi(P_{q_1}) = P_{q_2}$, $\pi(P_{r_1}) = P_{r_2}$ and $q_1 <_{\mathbb{Q}} r_1$, then $q_2 <_{\mathbb{Q}} r_2$. Let $\mathcal{F}$ be the filter on $\mathcal{G}$ generated by finite supports. Let $\mathcal{N}_{\mathbb{Q},2}$ be the permutation model determined by U, $\mathcal{G}$ and $\mathcal{F}$. Then we have the following properties.

1. Each P_q is in $\mathcal{N}_{\mathbb{Q},2}$.
2. The sequence $\langle P_q : q \in \mathbb{Q} \rangle$ is in $\mathcal{N}_{\mathbb{Q},2}$.

Proposition 3.2.12. *Every finite* $E \subseteq U$ *supports only finitely many subsets of* U.

The proof of the above proposition is similar to the proof of the same statement for the Mostowski model.

Proposition 3.2.13. *The following statements hold in* $\mathcal{N}_{\mathbb{Q},2}$.

1. $|U| \notin \Delta_2$,
2. $|U| \in \Delta_3$,
3. $|U| \in \Delta_4$,
4. $|U| \notin \Delta_o^*$.

Proof.

1. Define a relation $<$ on $\{P_q : q \in \mathbb{Q}\}$ by $P_q < P_r$ if $q <_{\mathbb{Q}} r$. It is easy to see that $<$ has empty support and this is an infinite linear ordering on a partition of U. Hence $|U| \notin \Delta_2$.

2. This can be proved in a similar manner as in the Mostowski model, see the proof of Proposition 3.2.7.

3. Suppose there is a map $f : U \to \omega$ with a support E. Then we have $\pi[f^{-1}\{n\}] = f^{-1}[\{n\}]$ for all $n \in \omega$ where $\pi \in \mathcal{G}_E$. By Proposition 3.2.12, E supports only finitely many subsets of U. Therefore f is not onto, and so $|U| \in \Delta_4$.

4. It follows from 2 that there cannot be any linear ordering on U in $\mathcal{N}_{\mathbb{Q},2}$, so $|U| \notin \Delta_o^*$. □

Corollary 3.2.14. *It is consistent that*

1. $\Delta_2 \subset \Delta_3 \cap \Delta_4$,
2. $\Delta_o^* \subset \Delta_4$.

3.2.7 The circular increasing socks

Let $U = \dot{\bigcup}_{0<n\in\omega} P_n$ where $P_n = \{a_{(n,i)} : i < n\}$ for all $n \in \omega$. Let $\mathcal{G}$ be the group of permutations π of U such that for any $n \in \omega$ there exists $k_n \in \omega$ such that

$\pi(a_{(n,i)}) = a_{(n,i+k_n)}$ where addition on suffices is done mod n, and let $\mathcal{F}$ be the filter on $\mathcal{G}$ generated by finite supports. Let $\mathcal{N}_{F_{<\omega}}$ be the permutation model determined by U, $\mathcal{G}$ and $\mathcal{F}$.

Proposition 3.2.15. *The following statements hold in $\mathcal{N}_{F_{<\omega}}$.*

1. $|U| \notin \Delta_4$.
2. $|U| \in \Delta$.
3. $|U| \in \Delta_3$.
4. $|U| \in \Delta_5$.
5. $|U| \notin \Delta_{\mathrm{per}}$.

Proof.

1. By sending each member of P_n to n, we have a surjection from U onto $\omega \setminus \{0\}$. It can be easily checked that this function has empty support.

2. Suppose $|U| \notin \Delta$. Then there is an injection $f : \omega \to U$. Let E be a support of f. Since E is finite and $\mathrm{ran}(f)$ is infinite, there is $n > 1$ such that $P_n \cap E = \emptyset$ and $P_n \cap \mathrm{ran}(f) \neq \emptyset$. Let $x \in P_n \cap \mathrm{ran}(f)$. Then there is $\pi \in \mathcal{G}_E$ such that $\pi(x) \neq x$. Hence $\pi(f) \neq f$, a contradiction.

3. Let $\langle \mathcal{C}, \prec \rangle$ be a linearly ordered subset of U. Suppose $\mathcal{C}$ is infinite. If there are infinitely many n such that $|P_n \cap \mathcal{C}| = 1$, then there is an injection from an infinite subset of ω into U which contradicts $|U| \in \Delta$ from 2. Hence there are infinitely many n such that $|P_n \cap \mathcal{C}| > 1$.

   Let E be a support of $\langle \mathcal{C}, \prec \rangle$. Then there is n such that $P_n \cap E = \emptyset$ and $|P_n \cap \mathcal{C}| > 1$. If $|P_n \cap \mathcal{C}| < n$, then there are $x \in P_n \cap \mathcal{C}$ and $\pi \in \mathcal{G}_E$ such that $\pi(x) \notin P_n \cap \mathcal{C}$, and so $\pi(\mathcal{C}) \neq \mathcal{C}$. Suppose $P_n \subseteq C$. Let $P_n = \{x_1, x_2, \dots, x_n\}$ where $x_1 \prec x_2 \prec \dots \prec x_n$. Then there is $\pi \in \mathcal{G}_E$ such that $\pi(x_n) = x_1$. Hence $\pi(x_n) = x_1 \prec \pi(x_1)$. Therefore π does not preserves $\prec$.

Hence $\mathcal{C}$ must be finite and so $|U| \in \Delta_3$.

4. Let f be a surjection on U. We will show that f must be injective and therefore $|U| \in \Delta_5$. Let E be a support of f. We may assume $E = \bigcup_{i<N} P_i$ for some $N \in \omega$.

   Claim. $f[P_n] = P_n$ for all $n \geq N$.

   Let $n \geq N$. Clearly $|f[P_n]| \leq |P_n|$. Next we show that if $f(a) \in P_n$, then $a \in P_n$. Suppose $f(a) \in P_n$ but $a \notin P_n$. Then there is $\pi \in \mathcal{G}_E$ such that $\pi(a) = a$ and $\pi(f(a)) \neq f(a)$, which contradicts $\pi(f) = f$. Hence $f^{-1}[P_n] \subseteq P_n$ and so $P_n \subseteq f[P_n]$. Thus $|P_n| \leq |f[P_n]| \leq |P_n|$, and since P_n is finite, $P_n = f[P_n]$.

   This implies $f[E] = E$ and since all P_n and E are finite, f is an injection on them and hence on U.

5. Let $f \in \mathcal{G}$ be such that $k_n = 1$ for all $n > 1$. Since $f\pi = \pi f$ for all $\pi \in \mathcal{G}$, it has empty support and so we have $f \in \mathcal{N}_{F_{<\omega}}$ (note that it is not necessarily true that an FM-model contains members of the group of automorphisms used to construct it). Then for all $n \geq 1$, f^n is not the identity. □

3.2.8 The binary tree

Let U be the set of atoms isomorphic to the binary tree, say $U = \{u_\sigma : \sigma \in 2^{<\omega}\}$ with the relation $<$ on U defined by $u_\sigma < u_\rho$ if ρ extends σ. Let $\mathcal{G}$ be the group of order-preserving permutations of U and let $\mathcal{F}$ be the filter on $\mathcal{G}$ generated by finite supports. Let $\mathcal{N}_{\mathcal{T}_2}$ be the permutation model determined by U, $\mathcal{G}$, and $\mathcal{F}$.

Proposition 3.2.16. *Every permutation f on U has 'nearly' order* 1 *or* 2*, meaning that there is a finite set $E \subseteq U$ such that $f^2 = \mathrm{id}$ on $U \setminus E$.*

Proof. Let f be a permutation of U with finite support E (not assumed to preserve $<$).

We may assume that E is a finite union of levels of U until level k for some $k \in \omega$. Let $x \notin E$. Then we verify all possibilities for $f(x)$.

Let $\pi \in \mathcal{G}_E$ be such that $\pi(x) = x'$, where x' is the sibling of x, and π fixes everything not greater than x or x'. Then $\langle x, f(x)\rangle \in f$ and so $\langle \pi(x), \pi(f(x))\rangle \in \pi(f) = f$. Since f is an injection and $x \neq x' = f(x)$, $f(x) \neq \pi(f(x))$. By the choice of π, either $x \leq f(x)$ or $x' \leq f(x)$.

Suppose $f(x) > x$. Then there is $\rho \in \mathcal{G}_E$ fixing x but interchanging $f(x)$ and its sibling. Hence $\langle x, f(x)\rangle \in f$ and $\langle x, \rho(f(x))\rangle \in \rho(f) = f$, contradicting f is a function. Similarly for the case $f(x) > x'$. Hence either $f(x) = x$ or $f(x) = x'$, and therefore f has order 2 on $U \setminus E$. □

Proposition 3.2.17. *The following statements hold in $\mathcal{N}_{\mathcal{T}_2}$.*

1. $|U| \in \Delta_3$,
2. $|U| \notin \Delta_5^*$,
3. $|U| \in \Delta_{\text{per}}$.

Proof.

1. Suppose there is an infinite chain $\langle C, <\rangle$ in U with finite support E which is the union of the first k levels. Since E is finite and C is infinite, there is $u_\sigma \in C \setminus E$ such that $u_{\sigma'} \notin E$, where σ' agrees with σ except on its final entry. Let $\pi \in \mathcal{G}_E$ swap u_σ and $u_{\sigma'}$. If $u_{\sigma'} \notin C$, then π does not preserve the chain C. If $u_{\sigma'} \in C$, then u_σ and $u_{\sigma'}$ are comparable under $<$, and so π does not preserve $<$. Hence such an infinite chain doesn't belong to $\mathcal{N}_{\mathcal{T}_2}$.

2. This follows from Proposition 3.1.39.

3. Let $f: U \to U$ be a bijection in $\mathcal{N}_{\mathcal{T}_2}$ and let E be a support of f. By Proposition 3.2.16, f has at most order 2 on $U \setminus E$ and since E is finite, f has finite order on U. □

We also get results above Δ in this model.

Proposition 3.2.18. $|\omega \cup U| \in \Gamma'_{5\text{-per}}$ *in* $\mathcal{N}_{\mathcal{T}_2}$.

Proof. Let f be a bijection on $\omega \cup U$ with finite support E. Similarly to the proof of Proposition 3.2.16, we may assume that $E \cap U$ is a finite union of levels of U until some level $k \in \omega$, and we can show that f has order 2 on $U \setminus E$. Hence f has a finite cycle, so $|\omega \cup E| \in \Gamma'_{5\text{-per}}$. □

Obviously $|\omega \cup U| \notin \Delta$. We conclude consistency results holding in $\mathcal{N}_{\mathcal{T}_2}$ as follows.

Corollary 3.2.19. *It is consistent that*

1. $\Delta_4 \subset \Delta_{\text{per}}$,
2. $\Delta_3, \Delta_{\text{per}} \not\subseteq \Delta_5, \Delta_5^*$,
3. $\Delta \subset \Gamma'_{5\text{-per}}$.

More on trees

First we remark that Proposition 3.2.16 does not necessarily hold in the same fashion for any n-ary tree. In particular, an FM-model built from an n-ary tree need not have a permutation on its set of atoms having 'nearly' order n. The precise value depends on the choice of structure that we put on the set of successors of each vertex. Consider the following examples for the case $n = 3$, both of which are based on a ternary tree but with different results.

Examples.

1. **Ternary tree**: Let $\mathcal{T}_3$ be a ternary tree. Construct an FM-model with the set of atoms $U_{\mathcal{T}_3}$ indexed by $\mathcal{T}_3$, $\mathcal{G}$ the group of automorphisms of $U_{\mathcal{T}_3}$ induced by $\mathrm{Aut}(\mathcal{T}_3)$. Let $\mathcal{N}_{\mathcal{T}_3}$ be the corresponding FM-model with finite supports.

Fact 3.2.20. *Every permutation on $U_{\mathcal{T}_3}$ in $\mathcal{N}_{\mathcal{T}_3}$ is 'nearly' the identity.*

Proof. Let $f : U_{\mathcal{T}_3} \to U_{\mathcal{T}_3}$ be a permutation with finite support E, which we may assume to be a finite union of levels up to level $k \in \omega$. Let $x \notin E$ and let x' and x'' be its siblings. Similarly to the proof of Proposition 3.2.16, we will verify all possibilities for $f(x)$, and by using the same argument, we ended up with 3 cases, $f(x) = x$, $f(x) = x'$, or $f(x) = x''$.

Suppose $f(x) \neq x$. Then there is $\pi \in \mathcal{G}_E$ which fixes x but swaps x' and x''. Hence $\pi f(x) \neq f(x) = f\pi(x)$, contradicting that E supports f. Thus $f(x) = x$. Similarly for x' and x''. Therefore f is the identity on $U_{\mathcal{T}_3} \setminus E$. □

2. **Circular ternary tree**: A circular ternary tree $\mathcal{T}_{C_3} = \langle T, <, R\rangle$. This is a ternary tree with an extra ternary relation R defined on each $\mathrm{succ}(x)$, the set of successors of a vertex x, so that $\langle \mathrm{succ}(x), R\rangle \cong C_3$ for all $x \in T$, so if $\mathrm{succ}(x) = \{y, y', y''\}$ we have for example $R(y, y', y'')$ and $R(y', y'', y)$ and $R(y'', y, y')$.

We construct an FM-model with the set of atoms $U_{\mathcal{T}_{C_3}}$ indexed by $\mathcal{T}_{C_3}$, and $\mathcal{G}$ the group of automorphisms of $U_{\mathcal{T}_{C_3}}$ induced by $\mathrm{Aut}(\langle T, <, R\rangle)$. Let $\mathcal{N}_{\mathcal{T}_{C_3}}$ be the corresponding FM-model with finite supports.

Fact 3.2.21. *Every permutation of $U_{\mathcal{T}_{C_3}}$ in $\mathcal{N}_{\mathcal{T}_{C_3}}$ has 'nearly' order* 3.

Proof. Similar to the proof of the ternary tree case. Let $f : U_{\mathcal{T}_{C_3}} \to U_{\mathcal{T}_{C_3}}$ be a permutation with finite support E, which we may assume to be a finite union of levels up to level $k \in \omega$. Let $x \notin E$ and let x' and x'' be its siblings. We have $f(x) = x$, $f(x) = x'$, or $f(x) = x''$.

Let $\pi \in \mathcal{G}_E$ be such that $\pi(x) = x'$. As π preserves R, we have $\pi(x') = x''$ and $\pi(x'') = x$. If $f(x) = x$, then $f(x') = f\pi(x) = \pi f(x) = \pi(x) = x'$, and also $f(x'') = x''$. If $f(x) = x'$, then $f(x') = f\pi(x) = \pi f(x) = \pi(x') = x''$, and $f(x'') = x$. Similarly if $f(x) = x''$, then we can show that $f(x') = x$ and $f(x'') = x'$.

Now let $\rho \in \mathcal{G}_E$ be such that $\rho(x) = x''$. Then $\rho(x') = x$ and $\rho(x'') = x'$. We also get the same results for $f(x)$ as in the case for π.

Therefore if $f(x) \neq x$, f either coincides with π or ρ on $\{x, x', x''\}$, and hence it has order 3 on $U_{\mathcal{T}_{C_3}} \setminus E$. □

Next we outline the idea for proving Proposition 3.1.42, taken from [Tru74]. We construct an FM-model from a binary tree but this time with an extra function f. The choice of the group of automorphisms $\mathcal{G}$ is not the same as the usual construction that we introduced earlier in the section, as instead $\mathcal{G}$ must preserve f.

Let U be the set of atoms indexed by the binary tree $\mathcal{T}_2$. Let $V, W \subseteq U$ be subsets of U defined as follows.

$$V = \{u_\sigma : \sigma \in 2^{<\omega} \text{ begins with a } 0\} \quad \text{and} \quad W = \{u_\sigma : \sigma \in 2^{<\omega} \text{ begins with a } 1\}$$

Define a function $f \colon V \times W \to V \times W \cup \{*\}$ by

$$\begin{gathered} f(\langle u_{0\alpha_1}, u_{1\beta_1}\rangle) = *, \\ f(\langle u_{0\alpha_1\alpha_2\ldots\alpha_{i+1}\alpha_{i+2}}, u_{1\beta_1\beta_2\ldots\beta_{i+1}\beta_{i+2}}\rangle) = \langle u_{0\alpha_1\alpha_2\ldots\alpha_i\beta_{i+1}}, u_{1\beta_1\beta_2\ldots\beta_i\alpha_{i+1}}\rangle, \\ \text{and} \quad f(\langle u_{\sigma_1}, u_{\sigma_2}\rangle) = \langle u_{\sigma_1}, u_{\sigma_2}\rangle \quad \text{otherwise.} \end{gathered}$$

In particular, for $u_{0^\frown\sigma_1}$ and $u_{1^\frown\sigma_2}$ with their indices having equal length > 2, f removes the last digits and swaps the second to last. It is easy to see that f is finite-to-one and surjective.

Now we want to implement f in this FM-model construction. Let $\mathcal{G}$ be the group of automorphisms on U preserving the length of the sequences, fixing V and W setwise, and further preserving f. Let $\mathcal{N}$ be the corresponding FM-model with finite supports. Then f belongs to $\mathcal{N}$ and hence we have $|V| \cdot |U| \notin \Delta_5^*$ in $\mathcal{N}$.

By the choice of $\mathcal{G}$, it preserves only length of the sequences and the function f, but not the structure of the tree as far as is apparent. Also since f swaps the second to last digits of the sequences after removed the last digits, it does not create a new function that can induce a tree structure on either V or W. We can see that both V and W are treated as increasing socks where their n^{th} pairs having size 2^n, and hence, by using the same argument as the ω pairs of socks in Proposition 3.2.10, their cardinalities lie in Δ_5 and hence Δ_5^*. For full details see [Tru74].

3.2.9 Infinitely branching tree

Let U be the set of atoms isomorphic to Baire space, say $U = \{u_\sigma : \sigma \in \omega^{<\omega}\}$ with the relation $<$ on U defined by $u_\sigma < u_\rho$ if ρ extends σ. Let $\mathcal{G}$ be the group of permutations of U preserving order and let $\mathcal{F}$ be the filter on $\mathcal{G}$ generated by finite supports. Let $\mathcal{N}_{\omega^{<\omega}}$ be the permutation model determined by U, $\mathcal{G}$, and $\mathcal{F}$.

Let E be a finite subtree of U which we assume to be closed downwards. Now we study the orbits of members of $U \setminus E$ under $\mathcal{G}_E$. Let $u_\sigma \in U \setminus E$. Then there is $u_E \in E$ such that $u_E \leq u_\sigma$, and we may assume that u_E is the greatest such. The orbit of u_σ under $\mathcal{G}_E$, written $\mathcal{G}_E(u_\sigma)$ is of the form $\{u_\rho \in U \setminus E : u_\rho \geq u_E \text{ and } |\sigma| = |\rho|\}$, i.e. the orbit of u_σ under $\mathcal{G}_E$ is the set of members of $U \setminus E$ related to E the same manner and on the same level as u_σ.

Proposition 3.2.22. *The following statements hold in $\mathcal{N}_{\omega^{<\omega}}$.*

1. $|U| \in \Delta_4^*$.
2. $|U| \in \Delta_5^*$.
3. $|U| \notin \Delta_5$.

Proof.

1. Suppose there is a finite-to-one map f from a subset $X \subseteq U$ onto ω. Let E be a support of f. Then we may assume E to be a subtree of U and E also supports X. Let $x \in X \setminus E$. Then $\mathcal{G}_E(x)$ is infinite and $f^{-1}[\{f(x)\}]$ is finite. Let $y \in \mathcal{G}_E(x) \setminus f^{-1}[\{f(x)\}]$. Then there is $\pi \in \mathcal{G}_E$ mapping x to y. Thus π does not preserve f, contradicting E supporting f.

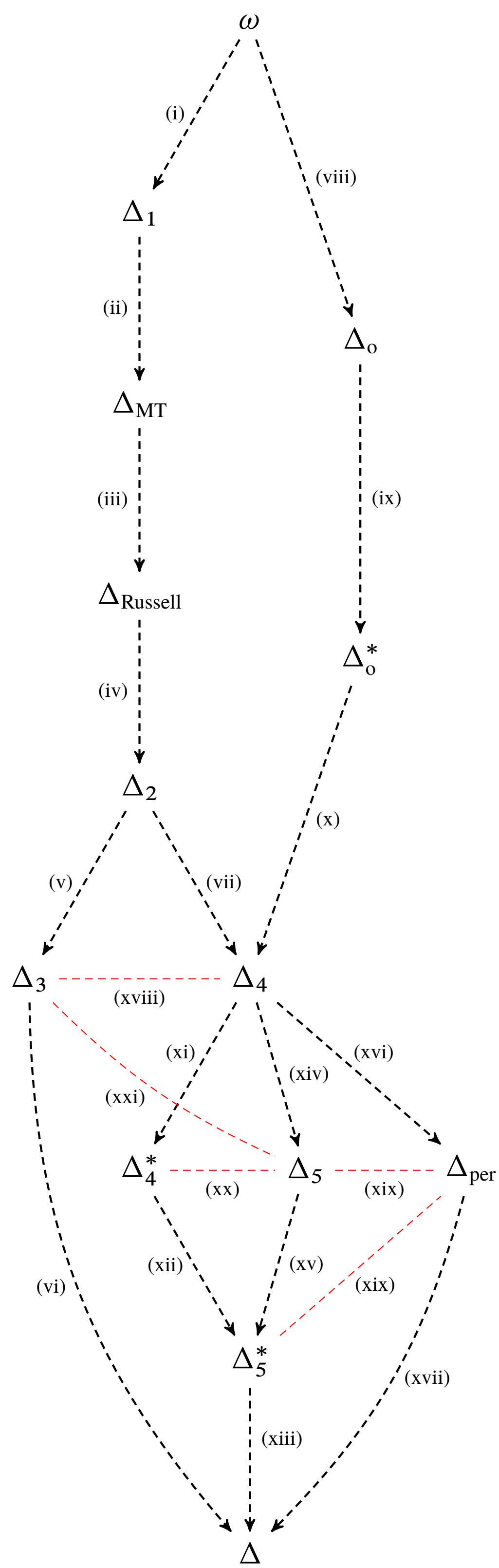
2. Follows from 1 since $\Delta_4^* \subseteq \Delta_5^*$.

3. Let $g\colon U \to U$ defined by $g(u_\emptyset) = u_\emptyset$ and $g(u_{\sigma^\frown n}) = u_\sigma$ for all $\sigma \in \omega^{<\omega}$ and $n \in \omega$. Then g is an infinite-to-one surjection on U. □

Proposition 3.2.23. *It is consistent that $\Delta_5 \subset \Delta_5^*$ and $\Delta_4^* \not\subseteq \Delta_5$.*

3.2.10 Summary

Section 3.2 shows that it is consistent with ZF that none of the relation $\subseteq$ between finiteness classes in the diagram after Definition 3.1.2 can be replaced by $=$. Furthermore we showed that in some cases classes that have no relation indicated between them are not comparable, by using various FM-models. We gather and present all these results in the following diagram.


ω
(i)
(viii)
Δ_1
(ii)
Δ_o
Δ_{MT}
(iii)
(ix)
$\Delta_{Russell}$
Δ_o^*
(iv)
Δ_2
(x)
(v)
(vii)
Δ_3
Δ_4
(xviii)
(xi)
(xvi)
(xiv)
(xxi)
Δ_4^*
Δ_5
Δ_{per}
(xx)
(xix)
(xii)
(xv)
(vi)
(xix)
Δ_5^*
(xvii)
(xiii)
Δ

The results involving the relations in dashed lines are listed as follows. First we note that $\dashrightarrow$ means there is, provable in ZF, a relation $\subseteq$ between the two classes, but - - - - means that it is consistent that two such classes are incomparable under $\subseteq$.

(i) an amorphous set X has its cardinality $|X| \in \Delta_1 \smallsetminus \omega$.

(ii) the disjoint union of two amorphous sets X and Y has $|X \dot{\cup} Y| \in \Delta_{\mathrm{MT}} \smallsetminus \Delta_1$.

(iii) $|U| \in \Delta_{\mathrm{Russell}} \smallsetminus \Delta_{\mathrm{MT}}$ in $\mathcal{N}_{\mathcal{B}}$ (Proposition 3.2.3).

(iv) $\Delta_{\mathrm{Russell}} \subset \Delta_2$ in $\mathcal{N}_P$ (Proposition 3.2.5).

(v) $\Delta_2 \subset \Delta_3$ in $\mathcal{N}_{\omega,2}$ (Corollary 3.2.11), and also in $\mathcal{N}_{\mathbb{Q},2}$ (Proposition 3.2.13).

(vi) Since $|U| \in \Delta_4 \smallsetminus \Delta_3$, $\Delta_3 \subset \Delta$ in $\mathcal{N}_M$ (Proposition 3.2.7).

(vii) $\Delta_2 \subset \Delta_4$ in $\mathcal{N}_M$ (Proposition 3.2.7), and also in $\mathcal{N}_{\mathbb{Q},2}$ (Proposition 3.2.13).

(viii) $|U| \in \Delta_{\mathrm{o}} \smallsetminus \omega$ in $\mathcal{N}_M$.

(ix) $|U_{\mathbb{Q}^2}| \in \Delta_{\mathrm{o}}^* \smallsetminus \Delta_{\mathrm{o}}^*$ in $\mathcal{N}_{\mathbb{Q}^2}$ (Proposition 3.2.9).

(x) $|U| \in \Delta_4 \smallsetminus \Delta_{\mathrm{o}}^*$ in $\mathcal{N}_{\mathbb{Q},2}$ (Proposition 3.2.13).

(xi) Since $|U| \in \Delta_4^* \smallsetminus \Delta_5$ where $\Delta_4 \subseteq \Delta_5$, $\Delta_4 \subset \Delta_4^*$ in $\mathcal{N}_{\omega^{<\omega}}$ (Proposition 3.2.22).

(xii) Since $|U| \in \Delta_5 \smallsetminus \Delta_4^*$ where $\Delta_5 \subseteq \Delta_5^*$, $\Delta_4^* \subset \Delta_5$ in $\mathcal{N}_{\omega,2}$ (Proposition 3.2.10).

(xiii) Since $|U| \in \Delta_3 \subseteq \Delta$ and $|U| \notin \Delta_5^*$, $\Delta_5^* \subset \Delta$ in $\mathcal{N}_{\mathcal{T}_2}$ (Proposition 3.2.17).

(xiv) $|U| \in \Delta_5 \smallsetminus \Delta_4$ in $\mathcal{N}_{\omega,2}$ (Proposition 3.2.10).

(xv) $|U| \in \Delta_5^* \smallsetminus \Delta_5$ in $\mathcal{N}_{\omega^{<\omega}}$ (Proposition 3.2.22).

(xvi) Since $|U| \in \Delta_{\mathrm{per}} \smallsetminus \Delta_5^*$ where $\Delta_4 \subseteq \Delta_5^*$, $\Delta_4 \subset \Delta_{\mathrm{per}}$ in $\mathcal{N}_{\mathcal{T}_2}$ (Proposition 3.2.17).

(xvii) $|U| \in \Delta_5^* \smallsetminus \Delta_{\mathrm{per}}$, so $\Delta_{\mathrm{per}} \subset \Delta$ in $\mathcal{N}_{F_{<\omega}}$ (Proposition 3.2.15).

(xviii) Δ_3 and Δ_4 are incomparable as $\Delta_3 \not\subseteq \Delta_4$ in $\mathcal{N}_{\omega,2}$ (Proposition 3.2.10) and $\Delta_4 \not\subseteq \Delta_3$ in $\mathcal{N}_M$ (Proposition 3.2.7).

(xix) Δ_{per} and Δ_5, Δ_5^* are incomparable as $\Delta_5 \not\subseteq \Delta_{\mathrm{per}}$ in $\mathcal{N}_{F_{<\omega}}$ (Proposition 3.2.15), and $\Delta_{\mathrm{per}} \not\subseteq \Delta_5^*$ in $\mathcal{N}_{\mathcal{T}_2}$ (Proposition 3.2.17).

(xx) Δ_4^* and Δ_5 are incomparable as $\Delta_4^* \not\subseteq \Delta_5$ in $\mathcal{N}_{\omega^{<\omega}}$ (Proposition 3.2.22) and $\Delta_5 \not\subseteq \Delta_4^*$ in $\mathcal{N}_{\omega,2}$ (Proposition 3.2.10).

(xxi) Δ_3 and Δ_5 are incomparable as $\Delta_3 \not\subseteq \Delta_5$ in $\mathcal{N}_{\mathcal{T}_2}$ (Proposition 3.2.17) and $\Delta_5 \not\subseteq \Delta_3$ since $\Delta_4 \not\subseteq \Delta_3$ in $\mathcal{N}_M$ (Proposition 3.2.7).

Chapter 4

Dedekind-finiteness and definability

In this chapter, we will study relationships between Dedekind-finite sets and definability. This is inspired by some results from [Pin76], [Tru95], and [WT05] where the connections between various Dedekind-finite sets with model theoretical structures are studied. For instance, there is a very close connection between weakly Dedekind-finite sets and $\aleph_0$-categorical structures. This chapter develops a theme from [WT05], going over some of the same ground but also giving more examples. We show that structures which may exist on weakly Dedekind-finite sets are definable from the original structures used in their FM-model constructions. Definability here may have various meanings, ranging from definable in a finite first-order language to an infinite first-order language or an infinitary language.

Throughout this chapter, unless otherwise stated, let $\mathfrak{A}$ be a structure and let $U_{\mathfrak{A}}$ be the set of atoms indexed by the domain of $\mathfrak{A}$, i.e. $U_{\mathfrak{A}} = \{u_a : a \in A\}$ where $A = |\mathfrak{A}|$. The group $\mathcal{G}$ is the group of automorphisms of $U_{\mathfrak{A}}$ induced by $\mathrm{Aut}(\mathfrak{A})$. Let $\mathcal{N}_{\mathfrak{A}}$ be the FM-model constructed from $\mathfrak{A}$ with finite supports. We may write $U_{\mathfrak{A}}$ and $\mathcal{N}_{\mathfrak{A}}$ as just U and $\mathcal{N}$ if the structure $\mathfrak{A}$ is clear in the context.

4.1 Weak Dedekind-finiteness and $\aleph_0$-categoricity

We first look at the following result from [WT05] in which the author studied the relations between weakly Dedekind-finite sets and $\aleph_0$-categorical structures.

Theorem 4.1.1 ([WT05])**.** *Let X be a set admitting a structure $\mathfrak{A}$ axiomatizable in a countable language and $T = \mathrm{Th}(\mathfrak{A})$. If X is weakly Dedekind-finite, then T is $\aleph_0$-categorical.*

The converse result for the set of atoms of an FM-model constructed from an $\aleph_0$-categorical structure also holds.

Proposition 4.1.2. *If $\mathfrak{A}$ is $\aleph_0$-categorical, then $U_{\mathfrak{A}}$ is weakly Dedekind-finite in $\mathcal{N}_{\mathfrak{A}}$.*

Proof. Let $f\colon U_{\mathfrak{A}} \to \omega$ be in $\mathcal{N}_{\mathfrak{A}}$, say f is supported by E. Let $\pi \in \mathcal{G}_E$. Then $\pi f = f\pi$, and so it can be proved straightforwardly that $\pi(f^{-1}[\{n\}]) = f^{-1}[\{n\}]$ for all $n \in \omega$. Since π is arbitrary, E supports $f^{-1}[\{n\}]$ for all $n \in \omega$. By Theorem 2.4.2 (the Ryll–Nardzewski theorem), there are only finitely many orbits of $U_{\mathfrak{A}}$ under $\mathcal{G}_E$. Since every subset of $U_{\mathfrak{A}}$ supported by E is a union of orbits, there are finitely many such subsets, i.e. there are only finitely many distinct $f^{-1}[\{n\}]$. Therefore f is not onto, and so $U_{\mathfrak{A}}$ is weakly Dedekind-finite. $\square$

This leads us to more questions. Given a weakly Dedekind-finite set U and a first-order structure $\mathfrak{B}$ on U, what else do we know about $\mathfrak{B}$? Furthermore, if U is the set of atoms of an FM-model constructed from some structure $\mathfrak{A}$, how are $\mathfrak{A}$ and $\mathfrak{B}$ related? These will be discussed in the next section.

4.2 Definability in permutation models

The goal of this section is to find a connection between definability and structures existing on the set of atoms. Firstly we consider the following three examples of structures constructed by the Fraïssé method and their corresponding FM-models. We will analyse relations that can be put on the set of atoms in each model.

4.2.1 The rationals: $\langle \mathbb{Q}, <_{\mathbb{Q}} \rangle$

Recall the Mostowski model $\mathcal{N}_{\mathbb{Q}}$ with the set of atoms $U_{\mathbb{Q}} = \{u_q : q \in \mathbb{Q}\}$, a linear ordering $<$ on $U_{\mathbb{Q}}$ induced from $<_{\mathbb{Q}}$ on $\mathbb{Q}$, and the group of automorphisms $\mathcal{G} = \mathrm{Aut}(U_{\mathbb{Q}}, <)$ with finite supports. We now study possible linear orderings on $U_{\mathbb{Q}}$ in the model $\mathcal{N}_{\mathbb{Q}}$.

Proposition 4.2.1. *For a finite subset $E \subseteq U_{\mathbb{Q}}$, there are finitely many linear orderings on $U_{\mathbb{Q}}$ in $\mathcal{N}_{\mathbb{Q}}$ supported by E.*

Proof. Let $E \subseteq U_{\mathbb{Q}}$ be a finite set and let $\prec$ be a strict linear ordering on $U_{\mathbb{Q}}$ in $\mathcal{N}_{\mathbb{Q}}$ supported by E, say $E = \{a_0, a_1, \ldots, a_{n-1}\}$ where $a_0 < a_1 < \ldots < a_{n-1}$.

Then there are $n+1$ orbits of $U_{\mathbb{Q}} \setminus E$ under $\mathcal{G}_E$ where each of them is an interval, either (a_i, a_{i+1}) where $0 \leq i < n-1$ or $(-\infty, a_0)$ or (a_{n-1}, ∞). Also there are n singleton orbits of E under $\mathcal{G}_E$ which are $\{a\}$ where $a \in E$.

Let I be an interval orbit, say $I = (a, b)$ where $a, b \in E \cup \{\pm\infty\}$. The claim is that $\prec$ is either $<$ or $>$ on I. It is easy to check that if there are $x, y \in I$ such that $x < y$ and $x \prec y$, then for all $z, w \in I$ such that $z < w$, we also have $z \prec w$. Therefore $\prec$ is the same as $<$ on I. Also if there are $x, y \in I$ such that $x < y$ and $x \succ y$, then for all $z, w \in I$ such that $z < w$, we have $z \succ w$, so $\prec$ is $>$ on I. We also note that I is an interval under the ordering $\prec$ as well. For if $x \prec y \prec z$ where $x, z \in I$ and $y \notin I$, then there is $\pi \in \mathcal{G}_E$ taking x to z but fixing y which violates preservation of $\prec$.

Hence there are 2^{n+1} possible values of $\prec$ on these interval orbits. And there are $2n + 1$ orbits, which can be arranged in $(2n + 1)!$ possible ways. Therefore the total possible number of $\prec$ supported by E on $U_{\mathbb{Q}}$ is $2^{2n+1}(2n + 1)!$. $\square$

4.2.2 The random graph

Let $\langle \Gamma, \sim_\Gamma \rangle$ be the random graph. Construct an FM-model by letting $U_\Gamma = \{u_\gamma : \gamma \in \Gamma\}$ be the set of atoms with the graph relation $\sim$ induced from $\sim_\Gamma$, $\mathcal{G} = \mathrm{Aut}(U_\Gamma, \sim)$, $\mathcal{F}$ be the filter on $\mathcal{G}$ generated by finite supports, and let $\mathcal{N}_\Gamma$ be the corresponding FM-model. Now we try to list all possible graph relations on U_Γ in $\mathcal{N}_\Gamma$.

Let E be a graph relation on U_Γ in $\mathcal{N}_\Gamma$. Then there is a finite $A \subseteq U_\Gamma$ supporting E. Since U_Γ is constructed from a homogeneous structure (and so it is $\aleph_0$-categorical), there are finitely many orbits of $U_\Gamma \setminus A$ under $\mathcal{G}_A$.

Note that for any orbit X of $U_\Gamma \setminus A$ under $\mathcal{G}_A$, there is a formula $\varphi(\overline{a}, x)$ where $\overline{a} \in A^{|A|}$ such that $X = \{b \in U_\Gamma : U_\Gamma \vDash \varphi(\overline{a}, b)\}$, i.e. φ tells us how the members of X relate to members of A. Therefore, for any $x, y \in X$, there is $\pi \in \mathcal{G}_A$ such that $\pi(x) = y$ since a map $A \cup \{x\} \to A \cup \{y\}$ fixing all members of A and sending x to y is a partial automorphism on U_Γ and so it can be extended to an automorphism on U_Γ.

We study the relation E by looking at all the possible forms E can take between orbits. Since every permutation in $\mathcal{G}_A$ fixes all members of A, E can be any graph relation on A. Next we investigate all possible E between orbits of $U_\Gamma \setminus A$ under $\mathcal{G}_A$ and A.

Case 1. Inter-orbits.
Let X and Y be orbits of $U_\Gamma \setminus A$ under $\mathcal{G}_A$. Consider the following statements,

(i) $\exists x \in X, \exists y \in Y, (x \sim y \wedge x E y)$,
(ii) $\exists x \in X, \exists y \in Y, (x \sim y \wedge x \not\!E y)$,
(iii) $\exists x \in X, \exists y \in Y, (x \nsim y \wedge x E y)$,

(iv) $\exists x \in X, \exists y \in Y, (x \nsim y \wedge x \not{E} y)$.

We will show that, in X, (i) $\Rightarrow \forall x \in X, \forall y \in Y, (x \sim y \rightarrow xEy)$, and similar results also hold for (ii)–(iv). With these results, we can conclude that there are 4 possibilities for E on X

- (i) + (iv) $\Rightarrow E = \sim$ between X and Y.
- (ii) + (iii) $\Rightarrow E = \overline{\sim}$ between X and Y (where $\overline{\sim}$ is the complement of $\sim$).
- (i) + (iii) $\Rightarrow E$ is complete between X and Y.
- (ii) + (iv) $\Rightarrow E$ is empty between X and Y.

Here (i) and (ii) cannot be true at the same time, and similarly for (iii) and (iv).

To see that (i) $\Rightarrow \forall x \in X, \forall y \in Y, (x \sim y \rightarrow xEy)$, suppose there are $x \in X$ and $y \in Y$ such that $x \sim y$ and xEy. Let $z \in X$ and $w \in Y$ be such that $z \sim w$. Then the map from $A \cup \{x, y\} \rightarrow A \cup \{z, w\}$ fixing all members of A, $x \mapsto z$ and $y \mapsto w$ is a partial automorphism and so it can be extended to an automorphism $\pi \in \mathcal{G}_A$. Since π preserves E, $\pi(x)E\pi(y)$, i.e. zEw.

Note that this also tells us how E behaves on X by replacing Y by X.

Case 2. Between an orbit and A.
Let X and be an orbit of $U_\Gamma \setminus A$ under $\mathcal{G}_A$. Let $a \in A$ and $x \in X$. If aEx, then aEy for all $y \in X$, since for any $y \in X$, there is $\pi \in \mathcal{G}_A$ such that $\pi(x) = y$ and so $\pi(a)E\pi(x)$, i.e. aEy. Similarly if $a \not{E} x$, then $a \not{E} y$ for all $y \in X$.

If $|A| = n$, there are 2^n orbits of $U_\Gamma \setminus A$ under $\mathcal{G}_A$. There are $2^{\frac{(n-1)n}{2}}$ possible E on A. There are 2^n possible E between A and an orbit, so there are $(2^n)^{2^n}$ possible E between A and orbits. There are $\frac{2^n(2^n+1)}{2}$ pairs of orbits (including pairing with itself), so there are $4^{\frac{2^n(2^n+1)}{2}}$ possible E between orbits supported by A. Hence there are $2^{\frac{(n-1)n}{2}+n2^n+2^n(2^n+1)}$ possible E that are supported by A.

Proposition 4.2.2. *Let E be a graph relation on U_Γ with a support A. Let X be an orbit* *of U_Γ under $\mathcal{G}_A$. Then either (X, E) is empty, complete, or* $\mathrm{Th}(X, E) = \mathrm{Th}(\Gamma, \sim_\Gamma)$.

We can obtain a similar result for any binary relation on U_Γ in $\mathcal{N}_\Gamma$. In particular, there will be no linear ordering on U_Γ in $\mathcal{N}_\Gamma$. Actually we cannot have asymmetric relations on any orbit of U_Γ. In the 'Inter-orbits' case in the above proof, we can replace Y by X to study how members in each orbit are related with respect to the relation E, and since cases (i) and (ii) cannot both hold at the same time, E cannot be asymmetric.

4.2.3 Henson digraphs

Let $\mathcal{T}$ be a family of finite tournaments such that every tournament $T \in \mathcal{T}$ has cardinality ≥ 3 and every pair of members in $\mathcal{T}$ are non-embeddable. Let $\langle D_\mathcal{T}, E_\mathcal{T}\rangle$ be the Henson digraph with age $\mathrm{Forb}(\mathcal{T})$ (see [Hen72]).

Construct an FM-model from $\langle D_\mathcal{T}, E_\mathcal{T}\rangle$. Let $U_\mathcal{T}$ be the set of atoms indexed by $D_\mathcal{T}$ with the digraph relation E induced from $E_\mathcal{T}$, $\mathcal{G} = \mathrm{Aut}(U_\mathcal{T}, E)$, $\mathcal{F}$ be the filter on $\mathcal{G}$ generated by finite supports, and let $\mathcal{N}_\mathcal{T}$ be the corresponding FM-model.

Let $\rightsquigarrow$ be a digraph relation on $U_\mathcal{T}$ in $\mathcal{N}_\mathcal{T}$ with finite support A. Let X and Y be orbits of $U_\mathcal{T} \setminus A$ under $\mathcal{G}_A$. Then we have the following possibilities.

(i) $\exists x \in X, \exists y \in Y, (x E y \wedge x \rightsquigarrow y)$,

(ii) $\exists x \in X, \exists y \in Y, (x E y \wedge y \rightsquigarrow x)$,

(iii) $\exists x \in X, \exists y \in Y, (x E y \wedge x \|_{\rightsquigarrow} y)$,

(iv) $\exists x \in X, \exists y \in Y, (y E x \wedge x \rightsquigarrow y)$,

(v) $\exists x \in X, \exists y \in Y, (y E x \wedge y \rightsquigarrow x)$,

(vi) $\exists x \in X, \exists y \in Y, (y E x \wedge x \|_{\rightsquigarrow} y)$,

(vii) $\exists x \in X, \exists y \in Y, (x \|_E y \wedge x \rightsquigarrow y)$,

(viii) $\exists x \in X, \exists y \in Y, (x \|_E y \wedge y \rightsquigarrow x)$,

(ix) $\exists x \in X, \exists y \in Y, (x \|_E y \wedge x \|_{\rightsquigarrow} y)$.

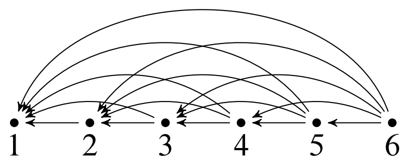
Similarly to the proof for the random graph, (i) implies $\forall x \in X, \forall y \in Y, (xEy \to x \rightsquigarrow y)$, and similarly for (ii)–(ix). Furthermore only one case from (i)–(iii) can happen at a time, similarly for (iv)–(vi) and (vii)–(ix). Therefore, there are 27 possible forms of $\rightsquigarrow$ between X and Y.

Note that if $X = Y$, i.e. considering $\rightsquigarrow$ on X, (i) and (ii) cannot both hold since we can find $\pi \in \mathcal{G}_A$ such that $\pi(x) = y$ and $\pi(y) = x$ so we cannot decide whether $x \rightsquigarrow y$ or $y \rightsquigarrow x$ and $\rightsquigarrow$ is asymmetric. Hence there are 3 possible forms of $\rightsquigarrow$ on X, either $\rightsquigarrow = E$, $\rightsquigarrow = E^{-1}$ or $\rightsquigarrow$ is empty there.

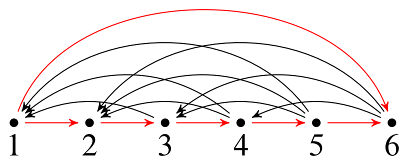
For relations between A and X, if $a \in A$, we have either $a \rightsquigarrow X$ or $X \rightsquigarrow a$ or there are no edges between a and X.

Therefore there are only finitely many possible digraph relations on U that are supported by A.

Now consider the following specific choice of $\mathcal{T}$. Let $\mathcal{T} = \{I_n : 6 \le n < \omega\}$ where L_n is the linear ordering of length n viewed as a digraph and I_n is obtained from L_n by reversing edges on adjacent vertices and between the source node and the sink node. For example, see the following diagram for the case $n = 6$ where the reversed arrows are highlighted in red.



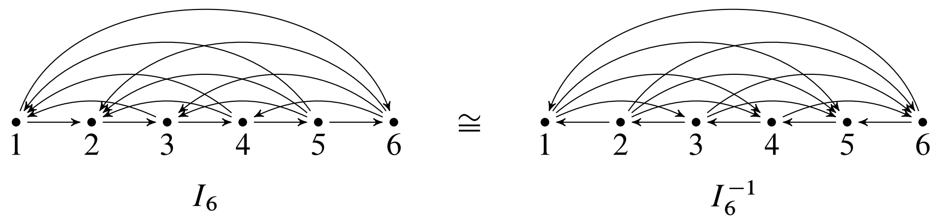
L_6



I_6

Then $\mathcal{T}$ is an infinite family of finite digraphs of cardinalities ≥ 6 such that all members are pairwise non-embeddable and have different cardinalities. More details on this construction can be found in [Che98] page 8. Furthermore, it is easy to see that $I_n \cong (I_n)^{-1}$

for every $n \in \omega$. Consider again the case $n = 6$. The function $i \mapsto 7 - i$, for all i such that $1 \leq i \leq 6$, is the desired isomorphism.



Therefore for any $I \in \mathcal{T}$, we can view I^{-1} as I (up to isomorphism) and hence we have $I^{-1} \in \mathcal{T}$ as well.

Proposition 4.2.3. *Let $\mathcal{T}_1, \mathcal{T}_2 \subseteq \mathcal{T}$ be such that $\mathcal{T}_1 \neq \mathcal{T}_2$. Let $\mathcal{N}_{\mathcal{T}_1}$ and $\mathcal{N}_{\mathcal{T}_2}$ be the FM-models constructed from the Henson digraphs $\langle D_{\mathcal{T}_1}, E_1 \rangle$ and $\langle D_{\mathcal{T}_2}, E_2 \rangle$ with the groups of automorphisms $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. Then there is a digraph relation R on $U_{\mathcal{T}_1}$ in $\mathcal{N}_{\mathcal{T}_1}$ such that $\langle U_{\mathcal{T}_1}, R \rangle \equiv \langle D_{\mathcal{T}_1}, E_1 \rangle$, but there is no such digraph relation on $U_{\mathcal{T}_2}$ in $\mathcal{N}_{\mathcal{T}_2}$.*

Proof. Let R be a digraph relation on $U_{\mathcal{T}_1}$ induced by E_1. Since the group $\mathcal{G}_1$ on $U_{\mathcal{T}_1}$ is induced by $\mathrm{Aut}(\langle D_{\mathcal{T}_1}, E_1 \rangle)$, it preserves R and hence R exists in $\mathcal{N}_{\mathcal{T}_1}$. It is easy to see that $\langle U_{\mathcal{T}_1}, R \rangle \equiv \langle D_{\mathcal{T}_1}, E_1 \rangle$.

Now we show that there is no digraph relation $\rightsquigarrow$ on $U_{\mathcal{T}_2}$ such that $\langle U_{\mathcal{T}_2}, \rightsquigarrow \rangle \equiv \langle D_{\mathcal{T}_1}, E_1 \rangle$. Without loss of generality, let $I \in \mathcal{T}_1 \setminus \mathcal{T}_2$. Let $\rightsquigarrow$ be a digraph relation on $U_{\mathcal{T}_2}$ with support A. If $\rightsquigarrow$ is the empty relation, then we're done. Suppose $\rightsquigarrow$ is not empty.

Case 1. $\rightsquigarrow$ is empty on all orbits of $U_{\mathcal{T}_2}$ under $(\mathcal{G}_2)_A$.

Then $\rightsquigarrow$ is non-empty only between orbits. Since there are only finitely many orbits, say there are k orbits, we cannot embed any tournament of size $> k$. As $\langle D_{\mathcal{T}_1}, E_1 \rangle$ admits all finite tournaments which do not embed any member of $\mathcal{T}_1$, $\langle U_{\mathcal{T}_2}, \rightsquigarrow \rangle \not\equiv \langle D_{\mathcal{T}_1}, E_1 \rangle$.

Case 2. There is an orbit X of $U_{\mathcal{T}_2}$ under $\mathcal{G}_A$ such that $\rightsquigarrow$ is non-empty on X.

As we analyzed all the possibilities of $\rightsquigarrow$ in the early part of this section, either $\rightsquigarrow\; = E_2$ or $\rightsquigarrow\; = E_2^{-1}$ on X. If $\rightsquigarrow\; = E_2$, then $\rightsquigarrow$ admits T. Suppose $\rightsquigarrow\; = E_2^{-1}$. Then $\rightsquigarrow$ admits I^{-1}. Since $I \cong I^{-1}$, E_2^{-1} also admits I. Hence $\langle U_{\mathcal{T}_2}, \rightsquigarrow\rangle \not\equiv \langle D_{\mathcal{T}_1}, E_1\rangle$.

Therefore $U_{\mathcal{T}_2}$ does not carry a digraph structure that satisfies $\mathrm{Th}(U_{\mathcal{T}_1}, E_1)$ in $\mathcal{N}_{\mathcal{T}_2}$. □

Corollary 4.2.4. *There are $2^{\aleph_0}$ first-order theories of digraphs such that for each of them, there is a corresponding weakly Dedekind-finite set. In particular, there are at least $2^{\aleph_0}$ non-equivalent (in the sense of Definition 5.3.5) weakly Dedekind-finite sets.*

Proof. Since $\mathcal{T}$ is countably infinite, this follows directly from the above proposition. □

From those three examples from Sections 4.2.1–4.2.3, we can see that a relation on $U_{\mathfrak{A}}$ in each model is related to the original relation on $U_{\mathfrak{A}}$ which is inherited from $\mathfrak{A}$ and carried over to $\mathcal{N}_{\mathfrak{A}}$ by $\mathcal{G}$.

Theorem 4.2.5. *Let $\mathfrak{A}$ be a countable structure. Then every relation on $U_{\mathfrak{A}}$ in $\mathcal{N}_{\mathfrak{A}}$ is (infinitary) definable from $\mathfrak{A}^{\mathcal{N}}$, where $\mathfrak{A}^{\mathcal{N}}$ is the structure in $\mathcal{N}_{\mathfrak{A}}$ inherited from $\mathfrak{A}$. More precisely, if $\mathfrak{A}$ is $\aleph_0$-categorical, then every relation on $U_{\mathfrak{A}}$ in $\mathcal{N}_{\mathfrak{A}}$ is definable by a first-order sentence (with finitely many parameters). Otherwise, it is definable by an $\mathcal{L}_{\omega_1\omega}$-sentence.*

First we need the following lemma, which is a basic result for any FM-model.

Lemma 4.2.6. *For all $n \in \omega$, every subset of U^n with support E is a union of orbits of U^n under $\mathcal{G}_E$.*

This is true since every subset of U^n with support E is invariant under $\mathcal{G}_E$. It remains to show that every orbit of U^n is definable for all $n \in \omega$. For $\aleph_0$-categorical structures, we have the following corollary of the Ryll-Nardzewski theorem (Theorem 2.4.2), whose proof can be found in [Eva13].

Corollary 4.2.7. *Let $\mathfrak{A}$ be an $\aleph_0$-categorical structure and let $\mathcal{G} = \mathrm{Aut}(\mathfrak{A})$.*

1. *Two n-tuples are in the same $\mathcal{G}$ orbit iff they have the same type over $\emptyset$ in $\mathfrak{A}$.*
2. *The $\emptyset$-definable subsets of A^n are unions of orbits of A^n under $\mathcal{G}$*
3. *If $X \subseteq A$ is finite, then the X-definable subsets of A^n are unions of orbits of A^n under $\mathcal{G}_X$.*

Hence, for any relation R on $U_{\mathfrak{A}}$ with finite support E, R is a union of orbits of U^n under $\mathcal{G}_E$ for some $n \in \omega$, where each orbit is E-definable, therefore R is defined by the conjunction of those formulae defining each orbit contained in R. However these methods cannot be applied for the case that $\mathfrak{A}$ is not $\aleph_0$-categorical. It is immediately true that if two tuples are in the same orbit, then they have the same type but the converse might not be true for a non-$\aleph_0$-categorical structure. There could be infinitely many orbits of U^n under $\mathcal{G}$ and a formula with finitary symbols might not able to distinguish two different orbits.

Proof of Theorem 4.2.5. Let R be a relation on $U_{\mathfrak{A}}$ with finite support E.

Case 1. $\mathfrak{A}$ is $\aleph_0$-categorical. Then every orbit of $U_{\mathfrak{A}}$ is E-definable and there are only finitely many such orbits. By Lemma 4.2.6, R is a union of orbits of $U_{\mathfrak{A}}$ under E. Therefore R is also E-definable.

Case 2. $\mathfrak{A}$ is not $\aleph_0$-categorical. Extend the language $\mathcal{L}$ to $\mathcal{L}'$ by adding every $a \in E$ as a variable symbol and let $\mathfrak{A}'$ be an $\mathcal{L}'$ structure extending $\mathfrak{A}$ with the same underlying set. Let $Q \subseteq A$. Assume $\langle \mathfrak{A}', R\rangle \cong \langle \mathfrak{A}', Q\rangle$ and let f be such an automorphism. Then $f \in \mathcal{G}_E$ and so f fixes R setwise. Hence $R = f(R) = Q$. By Theorem 2.6.3, there is an $\mathcal{L}'_{\omega_1\omega}$-formula φ defining R, where φ can be viewed as $\mathcal{L}_{\omega_1\omega}$-formula with finitely many parameters from E.

This can be applied for any function and constant on $U_{\mathfrak{A}}$. Therefore every structure on $U_{\mathfrak{A}}$ is definable by either first-order sentences or $\mathcal{L}_{\omega_1\omega}$-sentences. $\square$

4.3 Reconstruction

We have learned from the previous section that given a homogeneous structure $\mathfrak{A}$ in a finite relational language, then it is $\aleph_0$-categorical and therefore the set of atoms $U_{\mathfrak{A}}$ is weakly Dedekind-finite in the corresponding FM-model $\mathcal{N}_{\mathfrak{A}}$. In this section, we try to retrieve the original structure that is used to construct a given FM-model. Also we ask if we are given two 'different' homogeneous structures, will they give rise to different weakly Dedekind-finite sets?

Proposition 4.3.1. *Let $\mathfrak{A}$ be an $\aleph_0$-categorical structure and let $U_{\mathfrak{A}}$ be the set of atoms of the corresponding FM-model. Then for every countable $\mathfrak{B} \models \langle U_{\mathfrak{A}}, R\rangle$ where R is any n-ary relation on $U_{\mathfrak{A}}$, $\mathfrak{B}$ is $\aleph_0$-categorical.*

We can see that the above statement can be obtained immediately from Theorem 4.1.1 and Proposition 4.1.2.

Proposition 4.3.2. *Let $\mathfrak{A}$ be an $\aleph_0$-categorical structure and let E be a finite subset of $U_{\mathfrak{A}}$. Then for all $n \in \omega$ there are finitely many possible n-ary relations on $U_{\mathfrak{A}}$ in $\mathcal{N}_{\mathfrak{A}}$ supported by E.*

Proof. Let R be an n-ary relation on U with support E. We will show that R is a union of orbits of U^n under $\mathcal{G}_E$ and by the Ryll–Nardzewski Theorem (Theorem 2.4.2), there are only finitely many such orbits. Hence there are only finitely many possible R.

Let X be an orbit of U^n under $\mathcal{G}_E$. Suppose $R \cap X \neq \emptyset$. Let $\overline{a} \in R \cap X \neq \emptyset$. Then for any $\overline{b} \in X$, there is $\pi \in \mathcal{G}_E$ such that $\overline{b} = \pi(\overline{a}) \in \pi(R \cap X) = \pi(R) \cap \pi(X) = R \cap X$. Therefore $\overline{b} \in R$ and so $X \subseteq R$. □

Proposition 4.3.3. *Let $\mathfrak{A}$ be an $\aleph_0$-categorical structure. Let $\mathfrak{B} \models \mathrm{Th}(\langle U, R\rangle)$ where $\mathfrak{B}$ is countable and R is a relation on U in $\mathcal{N}_{\mathfrak{A}}$. Then there is a structure $\mathfrak{C} \cong \mathfrak{B}$ which is definable from $\mathfrak{A}$. In particular, we may take $B = A$, where A and B are the underlying sets of $\mathfrak{A}$ and $\mathfrak{B}$, respectively, and $\mathfrak{B} = \langle A, R'\rangle$ where R' is defined from R.*

Proof. Let R be a relation on U in $\mathcal{N}_{\mathfrak{A}}$. Define R' on A as follows. For any $\overline{a} = (a_1, \ldots, a_n) \in A^n$, we say $R'(a_1, \ldots, a_n)$ iff $R(u_{a_1}, \ldots, u_{a_n})$. Then, by induction on the length of φ, $\langle A, R'\rangle \vDash \varphi$ iff $\langle U, R\rangle \vDash \varphi$. Hence $\mathrm{Th}(\langle A, R'\rangle) = \mathrm{Th}(\langle U, R\rangle)$ and therefore $\langle A, R'\rangle$ is a model of $\mathrm{Th}(\langle U, R\rangle)$. By Proposition 4.1.2, U is weakly Dedekind-finite, therefore, by Theorem 4.1.1, $\mathrm{Th}(\langle U, R\rangle)$ is $\aleph_0$-categorical. Hence $\langle A, R'\rangle$ is the unique model (up to isomorphism) and this is definable from $\mathfrak{A}$. □

4.4 Above Δ_4

We now move beyond weak Dedekind-finiteness. We learned in the previous section that there is a relation between $\aleph_0$-categoricity and weak Dedekind-finiteness. Now we would like to find some property $(*)$ so that if a structure $\mathfrak{A}$ satisfies $(*)$, then $U_{\mathfrak{A}}$ is Dedekind-finite in $\mathcal{N}_{\mathfrak{A}}$. Also we want to obtain a similar result for the converse, i.e. for any given Dedekind-finite set U, what can we say about each structure $\mathfrak{A}$ that U can be equipped with?

To obtain the first result, let $\mathfrak{A}$ be a structure and let $U_{\mathfrak{A}}$ and $\mathcal{N}_{\mathfrak{A}}$ be the corresponding set of atoms and the FM-model induced from $\mathfrak{A}$ with finite supports, respectively.

If $U_{\mathfrak{A}}$ is Dedekind-infinite in $\mathcal{N}_{\mathfrak{A}}$, then there is an injection $f : \omega \to U_{\mathfrak{A}}$ in $\mathcal{N}_{\mathfrak{A}}$. Let X be a support of f. Then for any $\pi \in \mathcal{G}_X$, $\pi(f) = f$. Since $\pi(n) = n$ for all $n \in \omega$, $\pi\langle n, f(n)\rangle = \langle \pi(n), \pi(f(n))\rangle = \langle n, \pi(f(n))\rangle$, so as f is a function $\pi(f(n)) = f(n)$. Since f is an injection, all $f(n)$'s are distinct. Hence π fixes infinitely many members of $U_{\mathfrak{A}}$ for all $\pi \in \mathcal{G}_X$. Therefore if we want $U_{\mathfrak{A}}$ to be Dedekind-finite in $\mathcal{N}_{\mathfrak{A}}$, we need $\{a \in U_{\mathfrak{A}} : \pi(a) = a \text{ for all } \pi \in \mathcal{G}_X\}$ to be finite for all finite $X \subseteq U_{\mathfrak{A}}$.

Notation. For any set $X \subseteq U_{\mathfrak{A}}$, let $[X] = \{a \in U_{\mathfrak{A}} : \pi(a) = a \text{ for all } \pi \in \mathcal{G}_X\}$

A precise statement is as follows.

Proposition 4.4.1. *If $[X]$ is finite for all finite $X \subseteq U_{\mathfrak{A}}$, then $U_{\mathfrak{A}}$ is Dedekind-finite in $\mathcal{N}_{\mathfrak{A}}$.*

Let $\mathfrak{A}$ be a structure. For a formula $\varphi(x)$, we write $\varphi(\mathfrak{A})$ for the set of all $a \in A$ such that $\mathfrak{A} \models \varphi(a)$.

Definition 4.4.2.

- Let $\mathfrak{A}$ be an $\mathcal{L}$-structure and X be a subset of $|\mathfrak{A}|$. A formula $\varphi(x) \in \mathcal{L}(X)$ is called *algebraic* if $\varphi(\mathfrak{A})$ is finite.
- An element $a \in A$ is algebraic over X if it realises an algebraic $\mathcal{L}(X)$-formula. We call an element algebraic if it is algebraic over the empty set.
- The *algebraic closure* of X, written $\mathrm{acl}(X)$, is the set of all elements of $\mathfrak{A}$ algebraic over X.
- The *definable closure* of X, written $\mathrm{dcl}(X)$, is the set of all elements a such that there is an $\mathcal{L}(X)$-formula $\varphi(x)$ such that a is the unique element satisfying φ, i.e. $\{a\} = \varphi(\mathfrak{A})$.
- For a structure $\mathfrak{A}$, we say *acl is locally finite* if $\mathrm{acl}(X)$ is finite for all finite $X \subseteq A$. Similarly for *dcl is locally finite*.

It is easy to see that $\mathrm{dcl}(X) \subseteq \mathrm{acl}(X)$ but they are not necessarily equal, for example, recall the ω pairs of socks $\mathcal{S}_{\omega,2}$ with $U = \dot{\bigcup}_{n\in\omega} P_n$. We have $\mathrm{acl}(\emptyset) = U$ but $\mathrm{dcl}(\emptyset) = \emptyset$. Furthermore we have the following fact, which is a consequence of the Ryll-Nardzewski theorem.

Fact 4.4.3. *If $\mathfrak{A}$ is $\aleph_0$-categorical, then the algebraic closure of a finite set is finite. In particular $\mathfrak{A}$ is locally finite, i.e. any substructure generated by a finite subset is finite.*

Therefore

$$\aleph_0\text{-categoricity} \Rightarrow \text{acl is locally finite} \Rightarrow \text{dcl is locally finite.}$$

Proposition 4.4.4. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure and let $A \subseteq |\mathcal{M}|$ be finite. If $X \subseteq M^n$ is A-definable, then every $\mathcal{L}$-automorphism of $\mathcal{M}$ that fixes A pointwise fixes X setwise, i.e. $\pi(X) = X$ for all $\pi \in \mathcal{G}_A$.*

Proof. See Proposition 1.3.5 in [Mar02]. □

The converse of the above proposition is not necessarily true in general but it holds for $\aleph_0$-categorical structures as follows from the Ryll-Nardzewski Theorem (Theorem 2.4.2). For example, there is a rigid uncountable dense subset X of $\mathbb{R}$ (meaning that it has no non-identity automorphism) so that every $\{a\}$ is fixed by $\mathrm{Aut}(X)$ but is not $\emptyset$-definable. In the countable case, a countable model $\mathfrak{A}$ is rigid iff each element of $\mathfrak{A}$ is definable in $\mathfrak{A}$ by a formula of $\mathcal{L}_{\omega_1,\omega}$ (see [Sco65]).

Let $\mathfrak{A}$ be a structure and consider the following statement.

$$[X] \text{ is finite for all finite } X \subseteq A. \tag{$*$}$$

Notice that

$$\aleph_0\text{-categoricity} \Rightarrow (*) \Rightarrow \text{dcl locally finite.}$$

Lemma 4.4.5. *If $\mathfrak{A}$ is countable, then $[X] = \mathrm{dcl}_{\omega_1\omega}(X)$ for all finite $X \subseteq A$.*

Proof. It is easy to see that $\mathrm{dcl}_{\omega_1\omega}(X) \subseteq [X]$. Showing that every $a \in [X]$ is definable by some $\mathcal{L}_{\omega_1\omega}$-formula (with parameters from X) can be done in a similar manner as in the proof of Theorem 4.2.5 by using Theorem 2.6.3. Hence $a \in \mathrm{dcl}_{\omega_1\omega}(X)$. □

Let $\mathfrak{A}$ be a countable structure. If $\mathfrak{A}$ is $\mathrm{dcl}_{\omega_1\omega}$ locally finite, then $U_{\mathfrak{A}}$ is Dedekind-finite.

Proposition 4.4.6. *If $U_{\mathfrak{A}}$ is Dedekind-finite in $\mathcal{N}_{\mathfrak{A}}$, then $\mathrm{dcl}_{\omega_1\omega}$ is locally finite in $\mathfrak{A}$.*

Proof. Suppose $\mathrm{dcl}_{\omega_1\omega}$ is not locally finite in $\mathfrak{A}$. Then there is a finite subset $X \subseteq A$ such that $\mathrm{dcl}_{\omega_1\omega}(X)$ is infinite. Since $\mathrm{dcl}_{\omega_1\omega}(X) \subseteq A$ and A is countable, then there is a bijection $f : \omega \to \mathrm{dcl}_{\omega_1\omega}(X)$. Next we will show that f is in $\mathcal{N}_{\mathfrak{A}}$.

For each $a \in \mathrm{dcl}_{\omega_1\omega}(X)$ there is an $L_{\omega_1\omega}$-formula φ_a with parameters from X such that a is the unique point such that $(\mathfrak{A}, X \cup \{a\}) \vDash \varphi_a(a)$. Hence for any $\pi \in \mathcal{G}_X$, $\pi(a) = a$. Since $\pi(n) = n$ for all $n \in \omega$, we have $\pi(f) = f$. Thus $f \in \mathcal{N}_{\mathfrak{A}}$. Thus $U_{\mathfrak{A}}$ is Dedekind-infinite in $\mathcal{N}_{\mathfrak{A}}$. $\square$

Chapter 5

Beyond Δ_4

In this chapter we try to perform reconstruction on sets that lie outside Δ_4, weakly Dedekind-infinite sets. First we investigate FM-model constructions in which their set of atoms are not weakly Dedekind-infinite. Chapter 4 shows us there is a connection between $\aleph_0$-categorical structures and FM-models with their set of atoms weakly Dedekind-finite. Hence to find FM-models with their sets of atoms lying above Δ_4 we should start with non-$\aleph_0$-categorical structures, for example, structures with a countably infinite partition, which we will study in the first section.

The rest of this chapter studies trees in this context. In the first case, we consider trees whose branches are densely ordered, in particular the so-called 'weakly 2-transitive trees' from [DHM89], which we show give rise to $2^{\aleph_0}$ essentially distinct members of $\Delta_5 \setminus \Delta_4$. The main part of this final section however considers well-founded trees of height ω, which have a close connection with sets which lie outside Δ_5. We concentrate on trees of this kind that are balanced, where all points on any particular level behave in the same way, and show how an arbitrary tree of height ω and no leaves can be suitably 'pruned' to give a balanced tree (subject to some conditions).

5.1 Sets with countably infinite partitions

Let $\mathfrak{A}$ be a structure with its domain A having a countably infinite partition $\Pi = \{A_i : i \in \omega\}$ whose group of automorphisms $\mathrm{Aut}(\mathfrak{A})$ preserves each member of Π setwise, i.e. $\pi[A_i] = [A_i]$ for all $\pi \in \mathrm{Aut}(\mathfrak{A})$ and $i \in \omega$. Now construct an FM-model $\mathcal{N}_{\mathfrak{A}}$ from $\mathfrak{A}$ with the set of atoms $U_{\mathfrak{A}}$ induced by $\mathfrak{A}$ and the group of automorphisms $\mathcal{G}$ on $U_{\mathfrak{A}}$ induced by $\mathrm{Aut}(\mathfrak{A})$ with finite supports. Since each member of Π is fixed setwise by $\mathrm{Aut}(\mathfrak{A})$, its corresponding set lies in $\mathcal{N}_{\mathfrak{A}}$ and so $|U_{\mathfrak{A}}| \notin \Delta_4$ in $\mathcal{N}_{\mathfrak{A}}$.

Since we are studying Dedekind-finite sets, the choice of $\mathfrak{A}$ must be made carefully so that there will not be infinitely many singleton orbits under its automorphism group (to be precise we want $\mathfrak{A}$ to have $\mathrm{dcl}_{\omega_1,\omega}$ locally finite), otherwise $U_{\mathfrak{A}}$ will be Dedekind-infinite in $\mathcal{N}_{\mathfrak{A}}$, as in the following example.

Example. Let $\{\mathfrak{A}_i : i \in \omega\}$ be a family of countable structures $\mathfrak{A}_i$ such that $\mathfrak{A}_i \cong \langle [0,1) \cap \mathbb{Q}, <_{\mathbb{Q}} \rangle$ for all $i \in \omega$ and the A_i, the domains of $\mathfrak{A}_i$, are pairwise disjoint. Let $A = \dot{\bigcup}_{i \in \omega} A_i$ and $\mathfrak{A} = \langle A, \{A_i\}_{i\in\omega}, \{<_i\}_{i\in\omega} \rangle$ where A_i is a unary relation and $<_i$ is the relation on A_i from $\mathfrak{A}_i$ for all $i \in \omega$. Note that for all $i \in \omega$, $\mathfrak{A}_i$ is $\aleph_0$-categorical but not homogeneous.

Construct an FM-model model from the above $\mathfrak{A}$ as follows. Let $U_{\mathfrak{A}}$ be the set of atoms induced from $\mathfrak{A}$, say $U_{\mathfrak{A}} = \{u_a : a \in A\}$. We write U_i for $\{u_a : a \in A_i\}$. Let $\mathcal{G}$ be the group of automorphisms on $U_{\mathfrak{A}}$ induced from $\mathrm{Aut}(\mathfrak{A})$, and let $\mathcal{N}_{\mathfrak{A}}$ be the corresponding FM-model with finite supports.

Since $\mathcal{G}$ fixes each member of Π, the sequence $\langle U_i : i \in \omega \rangle$ has empty support, and so it is in the model $\mathcal{N}_{\mathfrak{A}}$. Therefore $|U_{\mathfrak{A}}| \notin \Delta_4$ in $\mathcal{N}_{\mathfrak{A}}$. Furthermore we can define a function $f : i \mapsto u_i$ where u_i is the $<_i$-least member of U_i for all $i \in \omega$. This function also has empty support and so it lies in $\mathcal{N}_{\mathfrak{A}}$. Hence $|U_{\mathfrak{A}}| \notin \Delta$ in $\mathcal{N}_{\mathfrak{A}}$.

The structure $\mathfrak{A}$ in the above example is not suitable for our study. Even if each member of the partition carries an $\aleph_0$-categorical structure, its corresponding set of atoms may turn out to be Dedekind-infinite in the FM-model. So having each member of the partition $\aleph_0$-categorical is not sufficient for making the cardinality of the set of atoms lie in Δ in the FM-model.

This can be fixed by requiring that the group of automorphisms acts transitively on all but finitely many members of the partition. The next example is a modified version of the ω pairs of socks, changing the size of each member of the partition.

Example. Recall the ω pairs of socks $\mathcal{S}_{\omega,2} = \langle \dot{\bigcup}_{i\in\omega} P_i, < \rangle$ where each $|P_i| = 2$ for all $i \in \omega$ and $<$ is defined on $\dot{\bigcup}_{i\in\omega} P_i$ by $P_i < P_j$ if $i < j$ for all $i, j \in \omega$. Instead of having each $|P_i| = 2$ for all $i \in \omega$, we can alter $|P_i|$ to be other cardinals. Start with the finite case where each P_i is finite and carries no structure (we will deal with the case that $|P_i|$ is weakly Dedekind-finite later). Let $n \in \omega$ and let $\mathcal{S}_{\omega,n} = \langle \dot{\bigcup}_{i\in\omega} P_i, < \rangle$ where $|P_i| = n$ for all $i \in \omega$ and let the relation $<$ be defined on $\dot{\bigcup}_{i\in\omega} P_i$ by $P_i < P_j$ if $i < j$. Then $\mathrm{Aut}(\mathcal{S}_{\omega,n}) \restriction P_i \cong \mathrm{Sym}(P_i)$ for all $i \in \omega$.

Obviously $|U_{\mathcal{S}_{\omega,n}}| \notin \Delta_4$ in $\mathcal{N}_{\mathcal{S}_{\omega,n}}$, but this time we have $|U_{\mathcal{S}_{\omega,n}}| \in \Delta$ in $\mathcal{N}_{\mathcal{S}_{\omega,n}}$ provided that $n \geq 2$ which can be verified in a similar way as ω pairs of socks (see Proposition 3.2.10).

Notice that we assumed that no P_i carries any structure. This requirement is not necessary in general for an FM-model construction. For instance the Mostowski model $\mathcal{N}_{\mathbb{Q}}$, which is constructed by letting the set of atoms $U_{\mathbb{Q}}$ be induced from the structure on rationals with its usual ordering $\langle \mathbb{Q}, <_{\mathbb{Q}} \rangle$ and letting the group $\mathcal{G}$ to be the group of automorphisms on $U_{\mathbb{Q}}$ induced from $\mathrm{Aut}(\langle \mathbb{Q}, <_{\mathbb{Q}} \rangle)$. We can see that the group $\mathrm{Aut}(\langle \mathbb{Q}, <_{\mathbb{Q}} \rangle)$ is a subgroup of $\mathrm{Sym}(\mathbb{Q})$, the symmetric group on $\mathbb{Q}$, or even can be considered as a subgroup of the symmetric group on ω, $\mathrm{Sym}(\omega)$, if $\mathbb{Q}$ is enumerated as $\{q_n : n \in \omega\}$. Instead of constructing the model $\mathcal{N}_{\mathbb{Q}}$ from $\langle \mathbb{Q}, <_{\mathbb{Q}} \rangle$, we can do it by letting U be induced by ω, as in the Fraenkel model $\mathcal{N}_\omega$, but choose the group of automorphisms $\mathcal{G}$ on U be induced by $\mathrm{Aut}(\langle \mathbb{Q}, <_{\mathbb{Q}} \rangle)$ as a subgroup of $\mathrm{Sym}(\omega)$, via the enumeration.

To summarize the above paragraph, the above FM-model construction with the set of atoms induced from $\langle \dot{\bigcup}_{i\in\omega} P_i, <\rangle$ can be varied by the choice of the group $\mathcal{G}$.

5.1.1 Reconstruction for sets with a countably infinite partition

Now we try to do a reconstruction for weakly Dedekind-infinite sets by dividing it into countably infinite parts as we discussed in the previous section.

Let $\mathcal{N}$ be an FM-model with the set of atoms U such that $|U| \notin \Delta_4$ in $\mathcal{N}$. Let $\mathcal{L}$ be a finite relational language. Then we extend $\mathcal{L}$ by adding countably infinitely many relational symbols. Let $\mathcal{L}' = \mathcal{L} \cup \{P_i\}_{i\in\omega}$ where each P_i is a unary relation symbol, and let $\mathfrak{A}$ be an $\mathcal{L}'$ structure on U in $\mathcal{N}$. As $|U| \notin \Delta_4$, U has a countably infinite partition, and we interpret these relational symbols P_i to correlate to each member of the partition. Then we have

$$\mathfrak{A} \vDash \bigwedge_{i\in\omega} \exists x\, (P_i(x)) \wedge \forall x \left(\bigvee_{i\in\omega} P_i(x) \wedge \bigwedge_{i\neq j} \neg \left(P_i(x) \wedge P_j(x)\right) \right).$$

Note that the above sentence is an $\mathcal{L}'_{\omega_1\omega}$-sentence. This allows us to study properties of each member of the partition as we can let $U_i = \{a \in U : \mathfrak{A} \vDash P_i(a)\}$. For example, let $n, i \in \omega$ and let $\varphi_n(P_i)$ be the following sentence.

$$\varphi_n(P_i) \equiv \exists x_0 \exists x_1 \ldots \exists x_{n-1} \left(\bigwedge_{k\neq l} x_k \neq x_l \wedge \bigwedge_{k<n} P_i(x_k) \wedge \forall x \left(P_i(x) \rightarrow \bigvee_{k<n} x = x_k \right) \right)$$

Then if we have $\mathfrak{A} \vDash \varphi_n(P_i)$, we know that $|U_i| = n$. This gives us a rough idea that the structure we used to construct this model might be similar to $\mathcal{S}_{\omega,n}$ that we have discussed before.

In the case that every member of a partition is weakly Dedekind-finite, we can use a result from the previous chapter where we performed the reconstruction on weakly Dedekind-finite sets. We perform reconstruction on this weakly Dedekind-infinite set by considering possible structures that can be put on each member of a partition in $\mathcal{N}$. Therefore, the following result on reconstruction of weakly Dedekind-infinite sets that we would like is as follows.

Conjecture 5.1.1. *Let $\mathcal{N}$ be an FM-model with the set of atoms U. Let $\mathfrak{A}$ be an $\mathcal{L}$-structure on U in $\mathcal{N}$, where $\mathcal{L}$ is a finite relational language. If there is a countable partition Π of U such that every member of Π lies in Δ_4, then there is a unique minimal countable model $\mathfrak{B}$ in the ground model $\mathcal{M}$ such that* $\mathrm{Th}(\mathfrak{B}) = \mathrm{Th}(\mathfrak{A})$.

The idea of the proof should be as follows.

Expand the language $\mathcal{L}$ to $\mathcal{L}'$ by adding countably infinitely many unary relation symbols $\{P_i : i \in \omega\}$. Then $\mathcal{L}'$ is still a countable language. Let $\mathfrak{A}'$ be the $\mathcal{L}'$-structure extending $\mathfrak{A}$. Enumerate Π as $\{U_i : i \in \omega\}$. For any $i \in \omega$, we say $\mathfrak{A}' \models P_i(a)$ if $a \in U_i$.

Let $\mathfrak{A}_n$ be the restriction of $\mathfrak{A}'$ to the structure on $\bigcup_{i<n} U_i$. Since each U_i is weakly Dedekind-finite for all $i \in \omega$, U_n is also weakly Dedekind-finite and so $\mathrm{Th}(\mathfrak{A}_n)$ is $\aleph_0$-categorical for all $n \in \omega$. Then by using results from Chapter 4, there is a unique countable $\mathcal{L}'$-structure $\mathfrak{B}_n$ such that $\mathrm{Th}(\mathfrak{A}_n) = \mathrm{Th}(\mathfrak{B}_n)$. We wish to show that $\mathfrak{B}_n$ is a substructure of $\mathfrak{B}_{n+1}$ and form $\mathfrak{B} := \bigcup_{n\in\omega} \mathfrak{B}_n$. This would be the desired structure. Some details still remain to be verified.

We make some further remarks about the above in the special case of ω pairs of socks. We may describe this in (at least) two different languages. Most commonly, $A = \dot{\bigcup}_{i\in\omega} P_i$, $|P_i| = 2$, and here language is infinite first-order. Note that this structure is homogeneous. We can build the FM-model $\mathcal{N}_{\omega,2}$ and $|U_{\omega,2}| \in \Delta_5 \smallsetminus \Delta_4$. If we try to recover the structure $\langle A, \{P_i : i \in \omega\}\rangle$, then we expect to obtain the original structure, but notice this is *not* unique model of its theory because it is not $\aleph_0$-categorical (there are infinitely many 1-types). What are the other models? The type $\{\neg P_i : i \in \omega\}$ is realized, but all these 'infinite' points are unrelated. A reconstruction result would say that we find a unique 'minimal' countable model.

An alternative way of axiomatizing ω pairs of socks is via a partial order. But now non-standard models are very different since any 'infinite' point must lie in a copy of $\mathbb{Z} \times \{0, 1\}$ under $\langle m, i\rangle < \langle n, j\rangle$ iff $m < n$. However the unique 'minimal' model is essentially the same as before.

Note that we have just considered the case where each member of the partition is weakly Dedekind-finite. Unfortunately it is not necessarily true that every weakly Dedekind-infinite set can be written as a countably union of weakly Dedekind-finite sets, see an example in the next section.

5.1.2 Quasi-amorphous sets

First we introduce quasi-amorphous sets, which were defined and studied in [CT01].

Definition 5.1.2. A set X is called *quasi-amorphous* if

(i) X is uncountable,
(ii) every subset of X is either countable or co-countable (complement of countable),
(iii) every uncountable subset contains a countably infinite subset.

An FM-model containing a quasi-amorphous set can be constructed in the same manner as the construction of the Fraenkel-model, apart from the set of atoms, which now will be indexed by ω_1, and the choice of supports which are now allowed to be countably infinite. A more detailed construction is as follows.

Let U_{ω_1} be the set of atoms indexed by ω_1, i.e. $U_{\omega_1} = \{u_\alpha : \alpha < \omega_1\}$. Let $\mathcal{G}$ be the symmetric group on U and let $\mathcal{F}$ be the filter on $\mathcal{G}$ generated by countable supports. Let $\mathcal{N}_{\omega_1}$ be the corresponding FM-model.

Fact 5.1.3. *The set of atoms U_{ω_1} is quasi-amorphous in $\mathcal{N}_{\omega_1}$.*

Proof. We check that U_{ω_1} satisfies the three properties for being quasi-amorphous.

(i) Since the model $\mathcal{N}_{\omega_1} \subseteq \mathcal{M}$, where $\mathcal{M}$ is the ground model, and U_{ω_1} is uncountable in $\mathcal{M}$, U_{ω_1} is also uncountable in $\mathcal{N}_{\omega_1}$.

(ii) This part is similar to the proof that every subset of the set of atoms of the Fraenkel model is either finite or co-finite.

Let X be a subset of U_{ω_1}. Then there is a countable subset E of U_{ω_1} supporting X. We will show that either $X \subseteq E$ or $U_{\omega_1} \setminus X \subseteq E$, hence X is either countable or co-countable.

Suppose $X \nsubseteq E$. Then there is $x \in X \setminus E$. Let $y \in U_{\omega_1} \setminus E$. Then there is $\pi \in \mathcal{G}_E$ such that $\pi(x) = y$. Hence $y = \pi(x) \in \pi(X) = X$. Therefore $U_{\omega_1} \setminus E \subseteq X$, i.e. $U_{\omega_1} \setminus X \subseteq E$.

(iii) The fact that every uncountable subset X of U_{ω_1} contains a countably infinite subset follows from the fact that every countably infinite subset of U_{ω_1} is supported by itself, therefore they all lie in $\mathcal{N}_{\omega_1}$. □

By property (iii) of quasi-amorphous sets, we have the following fact.

Fact 5.1.4. $|U_{\omega_1}| \notin \Delta$ *in* $\mathcal{N}_{\omega_1}$.

Quasi-amorphous family of pairs of socks

Now we will modify this construction so that the set of atoms U is Dedekind-finite but weakly Dedekind-infinite, i.e. $|U| \in \Delta \setminus \Delta_4$, but so that every countably infinite partition of U contains a weakly Dedekind-infinite set.

The idea is to construct a quasi-amorphous family of pairs of socks, where the group of automorphisms $\mathcal{G}$ can permute the pairs. Let $U_{\omega_1,2} = \dot{\bigcup}_{\alpha<\omega_1} P_\alpha$ where $P_\alpha = \{u_\alpha, v_\alpha\}$ are pairwise disjoint and let $\Pi = \{P_\alpha : \alpha < \omega_1\}$. Let $\mathcal{G}$ be the group of automorphisms of $U_{\omega_1,2}$ preserving Π setwise, i.e. $\mathcal{G} = \{\pi \in \mathrm{Sym}(U_{\omega_1,2}) : \pi(\Pi) = \Pi\}$. Let $I = \{E \cup \Lambda : E \subseteq U_{\omega_1,2}$ is finite and $\Lambda \subseteq \Pi$ is countable$\}$. It is easy to see that I is a normal ideal. Let $\mathcal{F}$ be the filter generated by $\{\mathcal{G}_{E\cup\Lambda} : E \cup \Lambda \in I\}$. Then $\mathcal{F}$ is a normal filter of $\mathcal{G}$. Let $\mathcal{N}_{\omega_1,2}$ be the corresponding FM-model.

Proposition 5.1.5. $|U_{\omega_1,2}| \in \Delta \smallsetminus \Delta_4$ *in* $\mathcal{N}_{\omega_1,2}$.

Proof. First we show that $|U| \notin \Delta_4$. By the construction of $\mathcal{N}_{\omega_1,2}$, every countable subset $\Lambda \subseteq \Pi$ supports itself and hence it stays countable in $\mathcal{N}_{\omega_1,2}$. Thus given any countably infinite $\Lambda \subseteq \Pi$, we have $\omega \preceq \Lambda \subseteq \mathcal{P}(U)$ and so $\omega \preceq^* U$.

Showing $|U| \in \Delta$ is done by a standard argument. Let $f : \omega \to U$ be a function supported by $E \cup \Lambda$ for some finite $E \subseteq U$ and countable $\Lambda \subseteq \Pi$. Suppose $\mathrm{ran}(f)$ is infinite. Since E is finite, there is $x \in \mathrm{ran}(f) \smallsetminus E$, say $x = f(n)$ for some $n \in \omega$. Let $\pi \in \mathcal{G}_{E\cup\Lambda}$ be such that $\pi(x) \neq x$. Then $\pi(n) = n$ and so $\langle n, \pi(x)\rangle = \langle \pi(n), \pi(x)\rangle \in \pi(f) = f$. Since $\langle n, x\rangle$ is also in f, this contradicts f is a function. Thus $\mathrm{ran}(f)$ is finite and so f is not injective. $\square$

Proposition 5.1.6. *The set of atoms $U_{\omega_1,2}$ cannot be written as a countable union of weakly Dedekind-finite sets in $\mathcal{N}_{\omega_1,2}$, i.e. every countable partition of $U_{\omega_1,2}$ in $\mathcal{N}_{\omega_1,2}$ must contain a weakly Dedekind-infinite set.*

Proof. Let Θ be a countable partition of $U_{\omega_1,2}$ in $\mathcal{N}_{\omega_1,2}$ with support $E \cup \Lambda$, where $E \subseteq U_{\omega_1,2}$ is finite and $\Lambda \subseteq \Pi$ is countable. We may assume that E is a finite union of some P_α's. Then for all $\pi \in \mathcal{G}_{E\cup\Lambda}$ and $X \in \Theta$, $\pi(X) = X$. Since $E \cup \Lambda$ is countable, there are uncountably many $\gamma < \alpha$ such that $P_\gamma \cap E = \emptyset$ and $P_\gamma \notin \Lambda$, let γ_1 and γ_2 be any such γ. Let $Y \in \Theta$ be such that $P_{\gamma_1} \cap Y \neq \emptyset$, say $u_{\gamma_1} \in P_{\gamma_1} \cap Y$. Let π

interchange the two members of P_{γ_1} and fix all other points. Then $\pi \in \mathcal{G}_{E\cup\Lambda}$ so it fixes Y. Hence $v_{\gamma_1} = \pi(u_{\gamma_1}) \in \pi[P_{\gamma_1} \cap Y] = P_{\gamma_1} \cap Y$, so $P_{\gamma_1} \subseteq Y$. Furthermore there will be $\rho \in \mathcal{G}_{E\cup\Lambda}$ such that $\rho[P_{\gamma_1}] = P_{\gamma_2}$. Hence $P_{\gamma_2} = \rho[P_{\gamma_1}] \subseteq \rho[Y] = Y$. Since γ_1 and γ_2 are arbitrary, Y contains uncountably many P_γ. In fact it contains all but countably many P_γ.

Let Γ be a countably infinite set of γ's such that $P_\gamma \subseteq Y$. Then $f\colon P_\gamma \mapsto \gamma$ maps a subset of Y onto a countably infinite set, and is supported by Γ, hence it lies in $\mathcal{N}_{\omega_1,2}$. Therefore Y is weakly Dedekind-infinite in $\mathcal{N}_{\omega_1,2}$. □

Corollary 5.1.7. *It is consistent that there is a Dedekind-infinite set that cannot be written as a countable union of weakly Dedekind-finite sets.*

Therefore the method introduced in Conjecture 5.1.1 cannot be applied for all weakly Dedekind-infinite sets. So far we can therefore only perform reconstruction for weakly Dedekind-infinite sets that can be written as a countable union of weakly Dedekind-finite sets.

5.2 Weakly 2-transitive trees and Δ_5

Definition 5.2.1. A poset $\langle T, \leq\rangle$ is called a *tree* if the following four conditions are satisfied:

(i) for each $a \in T$, the set $\{x \in T : x \leq a\}$ is linearly ordered,
(ii) for all $a, b \in T$ there is $c \in T$ with $c \leq a$ and $c \leq b$,
(iii) there are $a, b \in T$ with $a \not\leq b$ and $b \not\leq a$, i.e. $a \parallel b$,
(iv) $\langle T, \leq\rangle$ contains an infinite chain.

Definition 5.2.2. Let T be a tree. We say T is

- *k-homogeneous* if every isomorphism between k-element subsets extends to an automorphism of T,
- *k-transitive* if whenever two k-subsets of T are isomorphic, then there is an automorphism of T taking one to the other,
- *weakly k-transitive* if any isomorphism between chains of T of length k extends to an automorphism of T.

Remark. Let T be a tree. Then T^+ is the smallest tree which contains T and is a *meet-semilattice* (every nonempty finite subset has a meet or a greatest lower bound). It can be proved that T^+ exists and is unique up to isomorphism (see [Dro85]).

Definition 5.2.3. The set of *ramification points* of a tree T is defined by

$$\mathrm{ram}(T) = \{a \in T^+ : a = \inf\{b, c\} \text{ for some } b, c \in T \text{ with } b \parallel c\}.$$

Remark. $T^+ = T \cup \mathrm{ram}(T)$.

Definition 5.2.4. Let $x \in T$. A relation E on $\{y \in T : y > x\}$ defined by $y_1 E y_2$ if there is $z \in T^+$ such that $x < z \leq \{y_1, y_2\}$ is an equivalence relation. We call the equivalence classes of E *cones* at x.

1. If $a \in \mathrm{ram}(T)$, we let $C(a)$ be the set of all cones at a.
2. If $a \in \mathrm{ram}(T)$, we say that a is *special ramification point* of T if a has a cone which has a smallest element, that is, if a is *covered* in T^+ by some $b \in T$ (meaning that there is no point c with $a < c < b$).

   Let $\mathrm{ram_s}(T)$ denote the set of all special ramification points of T. If $a \in \mathrm{ram_s}(T)$, we let $C_s(a)$ (respectively $C_n(a)$) denote the set of all cones at a with (respectively without) a smallest element.

3. For each finite or infinite cardinal $k \geq 2$, let

$$\mathrm{ram}_k(T) = \{a \in \mathrm{ram}(T) \smallsetminus \mathrm{ram}_s(T) : |C(a)| = k\}, \text{ and}$$
$$\mathrm{ram}_\infty(T) = \{a \in \mathrm{ram}(T) \smallsetminus \mathrm{ram}_s(T) : C(a) \text{ is infinite}\}.$$

It has been shown in [DHM89] that if $\mathcal{T}$ has infinitely many distinct ramification orders, then it is not $\aleph_0$-categorical, so in the FM-model built from such $\mathcal{T}$, $|U_\mathcal{T}| \notin \Delta_4$, and this can happen in $2^{\aleph_0}$ ways. Hence we have the following theorem.

Theorem 5.2.5 ([DHM89]). *There are $2^{\aleph_0}$ pairwise non-isomorphic countable weakly 2-transitive trees.*

Let $\mathcal{T}$ be a countable weakly 2-transitive tree. For notational ease, we enlarge $\mathcal{T}$ by adjoining a minimum point $-\infty$ (though strictly speaking, $-\infty \notin \mathcal{T}$). For any $X \subseteq T$, let $\bigwedge X = \{z \in T : z = x \wedge y \text{ for some } x, y \in X \cup \{-\infty\}\}$. Recall that $[X]$ denotes the set of all members of T which are fixed by $\mathrm{Aut}(\mathcal{T})_X$. It is easy to see that $-\infty \in [X]$ for all $X \subseteq T$.

Lemma 5.2.6. *For every finite $X \subseteq T$, $\bigwedge X = [X]$.*

Proof. ($\subseteq$) Let $x \in \bigwedge X$. Then $x = y \wedge z$ for some $y, z \in X \cup \{-\infty\}$. If $\pi \in \mathrm{Aut}(\mathcal{T})_X$, then π fixes y and z and since π is an automorphism, π also fixes x. Hence $x \in [X]$.

($\supseteq$) Suppose $x \notin \bigwedge X$. Then $x > a$ for some $a \in \bigwedge X$. Let a be the greatest such element. Consider the following two cases. If for every $b \in \bigwedge X$, $b \not> x$, then let $y > x$, so $y \not< b$ for all $b \in \bigwedge X$. If there exists $b \in \bigwedge X$ such that $b > x$, then let y be such that $a < x < y < b$. In both cases, y has the same relation as x to all members of $\bigwedge X$, and we can find $\pi \in \mathrm{Aut}(\mathcal{T})_X$ such that $\pi(x) = y$. Thus $x \notin [X]$. ☐

Since the two notations above coincide, we can write $\bigwedge X$ as $[X]$ and consider them as the same thing. Now consider an FM-model induced by a weakly 2-transitive tree. Let $U_\mathcal{T}$

be the set of atoms indexed by a weakly 2-transitive tree $\mathcal{T} = \langle T, \leq \rangle$. Let $\mathcal{G}$ be the group of automorphisms of $U_{\mathcal{T}}$ induced by $\mathrm{Aut}(\mathcal{T})$, and let $\mathcal{N}_{\mathcal{T}}$ be the corresponding FM-model with finite supports.

Proposition 5.2.7. *$|U_{\mathcal{T}}| \in \Delta_5$ in $\mathcal{N}_{\mathcal{T}}$.*

Proof. Let $f : U_{\mathcal{T}} \to U_{\mathcal{T}} \cup \{*\}$, where $* \notin U_{\mathcal{T}}$, be surjective in $\mathcal{N}_{\mathcal{T}}$ with finite support X. We assume that X contains $-\infty$.

We remark that for any x, y, if $f(x) = y$, then $y \in [X \cup \{x\}]$. For since X supports f, $\pi(f) = f$ for all $\pi \in \mathcal{G}_X$. Then $\langle x, y \rangle \in f$ and so $\langle \pi x, \pi y \rangle \in \pi(f) = f$ for all $\pi \in \mathcal{G}_X$. Hence if $\pi y \neq y$, then $\pi x \neq x$ since f is a function. If $y \notin [X \cup \{x\}]$, then there is $\pi \in \mathcal{G}_{X \cup \{x\}} \subseteq \mathcal{G}_X$ such that $\pi(y) \neq y$ but $\pi(x) = x$, which contradicts $f(x) = y$. Hence $y \in [X \cup \{x\}]$. Also note that $[X \cup \{x\}] = [X] \cup \{x\} \cup \{x \wedge a : a \in X\}$.

Let $y \notin [X]$ and let x be such that $f(x) = y$. Then $y \in [X \cup \{x\}] \setminus [X] \subseteq \{x\} \cup \{x \wedge a : a \in X\}$.

Case 1. $y \not\leq a$ for all $a \in [X]$.
Then $y \notin \{x \wedge a : a \in X\}$. Hence $y \in \{x\}$, i.e. $y = x$ and so $f(y) = y$.

Case 2. $y \leq b$ for some $b \in [X]$.
If $x \in [X]$, then $[X] = [X \cup \{x\}] \ni y$, which contradicts $y \notin [X]$. Hence $x \notin [X]$. Suppose $x \not\leq a$ for all $a \in [X]$. Then by case 1, we can show that $f(x) = x$ and since $f(x) = y$, $x = y \leq b$ where $b \in [X]$, a contradiction.

Therefore $x \leq c$ for some $c \in [X]$. Then for any $a \in X$, either $x \wedge a = x$ or $x \wedge a \in [X]$. Thus $[X \cup \{x\}] = [X] \cup \{x\} \cup \{x \wedge a : a \in X\} = [X] \cup \{x\}$. Since $y \notin [X]$, $y = x$. Hence $f(y) = y$.

From both cases, we have $f(y) = y$ for all $y \notin [X]$. Hence $f[[X]] = [X] \cup \{*\}$ which is a contradiction since f is surjective but $[X]$ is finite. Therefore such f does not exist in $\mathcal{N}_{\mathcal{T}}$ and so $|U_{\mathcal{T}}| \in \Delta_5$ in $\mathcal{N}_{\mathcal{T}}$. $\square$

Hence we get an example of sets with their cardinalities lying in $\Delta_5 \setminus \Delta_4$. Notice that these weakly 2-transitive trees have densely ordered branches, whereas the binary tree has ω levels, hence all branches are finite or of length ω. There is an intimate relationship between tree structures of this kind, and sets with cardinality not lying in Δ_5, explained in the next section. We shall give a wide variety of examples of tree structures of this kind.

5.3 Trees and beyond Δ_5

As the method we used in the first section cannot be applied to every weakly Dedekind-infinite set, we may need more information from the structure, other than the fact that it can be divided into countably infinite sets. First consider the following example.

Example. The binary tree vs. ω pairs of socks

Consider the binary tree $\mathcal{T}_2 = \langle 2^{<\omega}, \subseteq \rangle$ and the ω pairs of socks $\mathcal{S}_{\omega,2} = \langle \dot{\bigcup}_{n\in\omega} P_n, < \rangle$. We can see that these two structures share some similar properties, for instance every member has two immediate successors, and both structures have ω levels where each level is finite. A major difference is that two members of the socks structure might share the same immediate successors, i.e. even though there is a linear ordering on the partition $\{P_i : i \in \omega\}$, there is no specific link between members of each consecutive pair of socks, whereas each member of the tree has its own immediate successors; more specifically the ordering on $\mathcal{T}$ is semi-linear (the downward closure of every member is well-ordered) but this is not true for the socks $\mathcal{S}_{\omega,2}$.

Both the binary tree and the ω pairs of socks have their corresponding set of atoms not lying in Δ_4 in the corresponding FM-models, but the binary tree ends up above Δ_5, unlike the pairs of socks. In particular, $|U_{\mathcal{S}_{\omega,2}}| \in \Delta_5 \setminus \Delta_4$ but $|U_{\mathcal{T}_2}| \in \Delta \setminus \Delta_5$ in their corresponding FM-models.

As the above example shows there is a link between sets carrying tree structures and Δ_5 as we have already seen in Proposition 3.1.39 (and later in Proposition 5.3.1). In this section, FM-models constructed from tree structures which have ω levels will be studied.

Proposition 5.3.1. *Let X be a set. Then $|X| \notin \Delta_5$ if and only if there is a subset $T \subseteq X$ carrying a tree structure with ω levels and no leaves.*

Proof. The proof is the same as the proof of Proposition 3.1.39 just dropping the 'finiteness' parts. $\square$

Corollary 5.3.2. *Let $\mathcal{M}$ be a model for $\mathsf{ZFA} + \mathsf{AC}$ with set of atoms $U_\mathcal{T}$ where $\mathcal{T}$ is a tree structure on $U_\mathcal{T}$ with ω levels and no leaves. Let $\mathcal{G}$ be any group of permutations of $U_\mathcal{T}$ which preserves the tree structure and let $\mathcal{N}_\mathcal{T}$ be the FM-model built from $U_\mathcal{T}$ and $\mathcal{G}$ using finite supports. Then $|U_\mathcal{T}| \notin \Delta_5$ in $\mathcal{N}_\mathcal{T}$.*

Proof. Since $\mathcal{G}$ preserves the tree structure, $U_\mathcal{T}$ carries the tree structure inherited from $\mathcal{T}$ in $\mathcal{N}_\mathcal{T}$. Therefore, as in the proof of Proposition 5.3.1, there is a surjective $f : U_\mathcal{T} \to U_\mathcal{T} \cup \{*\}$, where $* \notin U_\mathcal{T}$. The function f then has empty support since $\mathcal{G}$ preserves the tree structure. Hence $f \in \mathcal{N}_\mathcal{T}$ and so $|U_\mathcal{T}| \notin \Delta_5$. $\square$

Therefore, to study FM-models in which the set of atoms does not lie in Δ_5, we can start by studying tree structures with ω levels instead.

The general scenario is as follows. We start with a tree $\mathcal{T}$ with ω levels and no leaves in a model $\mathcal{M}$ of $\mathsf{ZFA} + \mathsf{AC}$ for which there is a sequence $\langle \mathfrak{A}_n : n \in \omega \rangle$ of non-empty finite or countable $\aleph_0$-categorical structures such that if x lies in the n^{th} level of $\mathcal{T}$, $\operatorname{succ}(x)$ is identified with $\operatorname{dom} \mathfrak{A}_n$. For the group $\mathcal{G}$ of automorphisms of $\mathcal{T}$ we take the group of tree automorphisms f which for each x in the n^{th} level also induce an isomorphism from $\operatorname{succ}(x)$ to $\operatorname{succ}(f(x))$, where they are both view as copies of $\mathfrak{A}_n$ under the chosen indexation. In what comes next, we shall consider such cases in which $\operatorname{succ}(x)$ is finite or more generally $\aleph_0$-categorical.

5.3.1 Balanced trees

To study sets not lying in Δ_5 by using FM-models constructions, in view of what we have just shown, we are naturally led to consider possible tree structures with ω levels and no leaves, and it is more convenient to insist that the group of automorphisms act transitively on each level of the tree. This is captured by the idea of 'balanced tree', which was introduced in [FT07], and can be extended as follows.

Definition 5.3.3. A tree $\mathcal{T}$ is called a *balanced tree* if the sets of immediate successors of each vertex x, denoted by $\mathrm{succ}(x)$, in the same level are equivalent.

In the presence of AC, 'equivalent' just means 'isomorphic' but in the absence of AC, we mean that any first-order structure which the first can be equipped with is elementarily equivalent to some first-order structure which can be put on the other (this will be defined and discussed later for the case $\mathrm{succ}(x)$ is infinite).

For example, if $\mathcal{T}$ is finitely branching, then $\mathcal{T}$ is balanced if every vertex in the same level has the same degree, i.e. $|\,\mathrm{succ}(x)|$ is constant for all x in the same level. The case that $\mathcal{T}$ branches infinitely is more complicated so we will deal with this in the later part of this section.

Note that balanced finitely branching trees were used to study versions of König's Lemma in [FT07], and our work here can be viewed as generalizing that.

We remark that if a tree is balanced, then its group of automorphisms acts transitively on each level of the tree. We now show how an arbitrary tree with ω levels and no leaves can be pruned to a form a balanced one, starting with the finitely branching case.

Theorem 5.3.4. *For any tree $\mathcal{T} = \langle T, < \rangle$ with ω levels and no leaves where each level is finite, there is a balanced subtree T^* fulfilling the same things.*

The following method appears to work easily in this case. The idea is to equalize the branching degree of every vertex in each level. On level 1, prune branches on level 2 corresponding to any vertices on level 1 whose degree is not equal to the minimum, so that all vertices on level 1 have same degree. Repeat this on level $2, 3, \ldots$ and so on. Notice that each level is fixed beyond a finite stage, so this method terminates. Also, at all stages, the pruned tree still has no leaves and ω levels and this is also true for the subtree eventually formed after ω steps. However this method appears to use the Axiom of Choice, since we may have to choose which branches to remove infinitely many times. Indeed we now show by means of an FM-model that this use of AC is unavoidable.

Example. Let $\mathcal{T}_2$ be the binary tree and $\mathcal{T}_3$ be the ternary tree, i.e. $\mathcal{T}_2 = \{2^{<\omega}, \subseteq\}$ and $\mathcal{T}_3 = \{3^{<\omega}, \subseteq\}$. Let $\mathcal{T}_{2+3}$ be the tree constructed by joining $\mathcal{T}_2$ and $\mathcal{T}_3$ and adding a minimum point $*$. By the above method, we keep removing branches from vertices in each level from the $\mathcal{T}_3$-side so that every vertex has degree 2, and at the end we should end up with having the subtree $\mathcal{T}^* \subseteq \mathcal{T}$ isomorphic to the binary tree $\mathcal{T}_2$. To see that this cannot be done without AC, construct the FM-model $\mathcal{N}_{\mathcal{T}_{2+3}}$ from $\mathcal{T}_{2+3}$ by the usual construction with finite supports, and let $U_{\mathcal{T}_{2+3}}$ be the set of atoms in the model. We write $U_{\mathcal{T}_2}$, $U_{\mathcal{T}_3}$, and u_*, to correspond to $\mathcal{T}_2$, $\mathcal{T}_3$ and $*$, respectively. Suppose there is $U_{\mathcal{T}^*}$, the set of atoms corresponding to $\mathcal{T}^*$, which lies in $\mathcal{N}_{\mathcal{T}_{2+3}}$ with finite support E containing the root. We can see that $U_{\mathcal{T}^*} = U_{\mathcal{T}_2} \cup U_{\mathcal{T}_2^*} \cup \{u_*\}$, where $U_{\mathcal{T}_2^*} \subseteq U_{\mathcal{T}_3}$. Let x be a maximal member of $E \cap (U_{\mathcal{T}_3} \cup \{u_*\})$. Then there are y, z in $U_{\mathcal{T}_2^*}$ extending x on the same level. But then we can find $\pi \in \mathcal{G}_E$ such that $\pi[\{y, z\}] = \{y, w\}$ where w is another vertex extending x in $U_{\mathcal{T}_3}$ that gets removed in the pruning process. Hence $\pi(U_{\mathcal{T}_2^*}) \neq U_{\mathcal{T}_2^*}$, i.e. the subset $U_{\mathcal{T}_2^*}$ does not exist in the model $\mathcal{N}_{\mathcal{T}_{2+3}}$.

Therefore pruning a tree must be done more carefully. We will show that there is another method to prune finite branching trees without using the Axiom of Choice as follows.

Proof of Theorem 5.3.4. Let S be a sequence of natural numbers such that every number occurs infinitely often, say $S = \langle k_n \rangle_{n \in \omega}$. We will construct a decreasing sequence of subtrees of T such that for each n, T_{n+1} is pruned on level k_n so every member has the same degree. Let L_n be the n^{th} level of T.

First let $T_0 = T$. For $n \in \omega$, suppose T_n has been constructed. Let X_n be the set of members of height k_n in T_n with minimum degree, i.e. $X_n = \{x \in L_{k_n} \cap T_n : x$ is of least degree$\}$. Let $T_{n+1} = \bigcup_{x \in X_n} \{y \in T : y \leq x \text{ or } y > x\}$. It can be easily checked that T_{n+1} is a subtree of T_n and has no leaves. Let $T^* = \bigcap_{n \in \omega} T_n$. It remains to show that T^* is balanced and contains no leaves.

Let $x \in T^*$. Then $\mathrm{succ}(x) \cap T^* = \bigcap_{n \in \omega} (\mathrm{succ}(x) \cap T_n)$, where $\langle \mathrm{succ}(x) \cap T_n \rangle_{n \in \omega}$ is a decreasing chain and $\mathrm{succ}(x) \cap T_n$ is finite and non-empty for all $n \in \omega$. Hence $\bigcap_{n \in \omega} (\mathrm{succ}(x) \cap T_n) \neq \emptyset$, so x is not a leaf.

For each n, let $I_n = \{i \in \omega : k_i = n\}$. Then $\langle X_i \rangle_{i \in I_n}$ is a decreasing sequence of subsets of L_n. Note that I_n is infinite since we chose S so that every n occurs infinitely often. Since L_n is finite, the sequence X_i must terminate at some least i_n. Hence the level n^{th} of T^*, $L_n \cap T^*$, must equal X_{i_n}, where each member of X_{i_n} has the same degree. Therefore T^* is balanced. □

With the method introduced in the previous proof, pruning the tree $\mathcal{T}_{2+3}$ leaves only the $\mathcal{T}_2$ part and in addition the minimum vertex $*$. This pruning method can also make a finite branching tree whose corresponding set of atoms is Dedekind-infinite give rise to a pruned subtree which is Dedekind-finite. For instance, there is a finite branching tree $\mathcal{T}$ such that $|U_{\mathcal{T}}| \notin \Delta$ in $\mathcal{N}_{\mathcal{T}}$, but having a balanced $\mathcal{T}' \subseteq \mathcal{T}$ such that $|U_{\mathcal{T}'}| \in \Delta$ in $\mathcal{N}_{\mathcal{T}}$. We give such an example here.

Example. We construct a tree $\mathcal{T}$ as a countable disjoint union of binary trees with addition relations on their roots.

Let $\{\mathcal{B}_i : i \in \omega\}$ be a countable family of pairwise disjoint binary trees, say $\mathcal{B}_i = \langle B_i, <_i \rangle \cong \mathcal{T}_2$ for all $i \in \omega$. Let b_i be the root of $\mathcal{B}_i$. Let $T = \dot{\bigcup}_{i \in \omega} B_i$, partially ordered by $<$ the union of all the $<_i$ and the relation on $\{b_i : i \in \omega\}$ induced from the usual ordering on ω. It is easy to see that $\mathcal{T}$ is finite branching without leaves, and every vertex in each B_i has its level in $\mathcal{T}$ increased by i from what it is in $\mathcal{B}_i$. Furthermore there is a unique vertex in each level with ramification order 3. Thus the set $R := \{x \in T : |\operatorname{succ}(x)| = 3\}$ is countable.

Construct an FM-model $\mathcal{N}_\mathcal{T}$ from $\mathcal{T}$ with set of atoms $U_\mathcal{T}$ indexed by $\mathcal{T}$, $\mathcal{G}$ induced by $\operatorname{Aut}(\mathcal{T})$, and finite supports. It is easy to see that $\mathcal{G}$ fixes every member of $\{u_x : x \in R\}$ so it lies in and is countable in $\mathcal{N}_\mathcal{T}$. Thus $|U_\mathcal{T}| \notin \Delta$ in $\mathcal{N}_\mathcal{T}$. Next we prune $\mathcal{T}$ according to the method introduced in Theorem 5.3.4. This gives us a finite branching subtree $\mathcal{T}' = \mathcal{B}_0 \cong \mathcal{T}_2$, which by the method of Section 3.2.8 one shows that $|U_{\mathcal{T}'}| \in \Delta$ in $\mathcal{N}_\mathcal{T}$.

Now we will discuss the case in which $\operatorname{succ}(x)$ is infinite for some x. We already have given some discussion of this above We always assume that T and $\operatorname{succ}(x)$ are countably infinite for all $x \in T$, e.g. let $T = \omega^{<\omega}$ be equipped with the relation $\subseteq$. Construct the FM-model $\mathcal{N}_\mathcal{T}$ by the usual construction so that the set of atoms $U_\mathcal{T}$ carries the original tree structure $\mathcal{T} := \langle T, \subseteq \rangle$, and furthermore we will have $\operatorname{succ}(u_x)$, the corresponding set of atoms of $\operatorname{succ}(x)$ (which we may write only as $\operatorname{succ}(x)$ in the future proof), being weakly Dedekind-finite in $\mathcal{N}_\mathcal{T}$ for all $x \in T$. In short, $U_\mathcal{T}$ admits a tree structure with each vertex branching weakly Dedekind-finitely.

Since each vertex branches weakly Dedekind-finitely, we can no longer use the method in the proof of the finite branching case because we still have no method to distinguish two weakly Dedekind-finite sets. Consider the following equivalence relation between two sets.

Definition 5.3.5. For any sets A and B, we write $A \equiv B$ if for any first-order structure $\mathfrak{A}$ that can be put onto A in a countable language, there exists a first-order structure $\mathfrak{B}$ on B over the same language such that $\mathfrak{A} \equiv \mathfrak{B}$.

Now we will try to make sense of the definition of balanced tree in the case that $\mathcal{T}$ branches weakly Dedekind-finitely. It seems reasonable to say that two weakly Dedekind-finite sets, that are constructed from countable sets, are ‘the same’ if they are equivalent under the relation $\equiv$ defined above.

Suppose we have two weakly Dedekind-finite sets X and Y such that $X \not\equiv Y$. Then there must be some structure $\mathfrak{A}$ that we can put on X and a sentence φ which is true in $\mathfrak{A}$, but not in any first-order structure $\mathfrak{B}$ on Y. Since the set of sentences is countable, we can choose such a least sentence φ such that, under the same interpretation, φ distinguishes X and Y, i.e. without loss of generality, φ holds in a structure on X but not in any structure that can be put on Y, which we write $X < Y$.

This method can be applied to every pair of weakly Dedekind-finite sets, but then if there are infinitely many pairs of such sets we still need to choose which language will be used to distinguish sets from each pair. Since we are dealing with trees having ω levels, it suffices to fix a language $\mathcal{L} = \{\mathcal{R}, \mathcal{F}, \mathcal{C}\}$ where the set of relation symbols $\mathcal{R}$, the set of function symbols $\mathcal{F}$, and the set of constant symbols $\mathcal{C}$ are all countable. Hence the set of $\mathcal{L}$-sentences is countable, and we enumerate them as $\{\varphi_n : n \in \omega\}$. Now let $\mathcal{X}$ be a weakly Dedekind-finite family of weakly Dedekind-finite sets. The method will successively cut down $\mathcal{X}$ to nonempty subfamilies $\mathcal{Y}_n$. This give a descending sequence of $\mathcal{Y}_n$'s, and since $\mathcal{X}$ is weakly Dedekind-finite, this sequence terminates at some $n \in \omega$.

These remarks lead to a version of Theorem 5.3.4 for the case that the tree $\mathcal{T}$ has each level weakly Dedekind-finite.

Theorem 5.3.6. *For any tree $\mathcal{T} = \langle T, < \rangle$ with ω levels and no leaves where each level is weakly Dedekind-finite, there is a balanced subtree T^* fulfilling the same properties.*

Proof. Modify the proof of the finite branching case in Theorem 5.3.4 as follows.

If all members of $L_{k_n} \cap T_n$ are $\equiv$-equivalent, then $X_n = L_{k_n} \cap T_n$ is unchanged. Otherwise, choose the least formula φ such that some but not all members of $L_{k_n} \cap T_n$ satisfy φ and let $X_n = \{x \in L_{k_n} \cap T_n : x \models \varphi\}$. This ensures that for each k, $\{X_n : k_n = k\}$ decreases and because L_k is weakly Dedekind-finite, this terminates. The rest of the proof is similar. $\square$

Therefore, given an FM-model $\mathcal{N}$ with the set of atoms U equipped with an ω-level tree structure where each level is (infinite) weakly Dedekind-finite, we may assume that U is balanced, and at the n^{th} level we can associate with it a unique countable structure $\mathfrak{A}_n$. Let $\mathcal{T}$ be a tree of height ω where each vertex branches according to those structures, i.e. if x is in level n, then $\operatorname{succ}(x) \cong \mathfrak{A}_n$. Then the FM-model $\mathcal{N}_{\mathcal{T}}$ constructed from $\mathcal{T}$ by the construction given above has the set of atoms $U_{\mathcal{T}}$ equivalent to U.

The following are example of balanced trees.

5.3.2 $\operatorname{succ}(x)$ is finite

Let $\sigma = \langle \sigma_k : k \in \omega \rangle$ be a sequence of positive integers greater than 1 and let $\mathcal{T}_\sigma$ be a tree in which all elements on the k^{th} level have exactly σ_k immediate successors, i.e. if x is on the k^{th} level, then $|\operatorname{succ}(x)| = \sigma_k$. This construction is due to [FT07]. Then $\mathcal{T}_\sigma$ is balanced, and there are $2^{\aleph_0}$ different trees that can be constructed by this method. One can see that all balanced finite branching trees are isomorphic to one of these trees.

Construct an FM-model from $\mathcal{T}_\sigma$ by letting $U_{\mathcal{T}_\sigma}$ be the set of atoms indexed by $\mathcal{T}_\sigma$, $\mathcal{G}$ be the group of automorphisms of $U_{\mathcal{T}_\sigma}$ induced by $\operatorname{Aut}(\mathcal{T}_\sigma)$, and $\mathcal{N}_{\mathcal{T}_\sigma}$ be the corresponding FM-model with finite supports.

We illustrate how this notion can be described in $\mathcal{L}_{\omega_1\omega}$-sentences. First we give examples of first-order sentences telling properties of each member x of $U_{\mathcal{T}_\sigma}$ as follows.

- x is a root of $U_{\mathcal{T}_\sigma}$:
$$\varphi_0(x) \equiv \forall y(x \leq y).$$
- x has level $n > 0$:
$$\begin{aligned}\varphi_n(x) \equiv \exists y_0 \exists y_1 \ldots \exists y_{n-1} \Bigg(& y_0 < y_1 < \ldots < y_{n-1} < x \\ & \wedge \forall z_0 \forall z_1 \ldots \forall z_{n-1} \Bigg(z_0 < z_1 < \ldots < z_{n-1} < x \to \bigwedge_{i<n} (z_i = y_i) \Bigg) \Bigg).\end{aligned}$$

- x has ramification order σ_n:

$$\chi_{\sigma_n}(x) \equiv \exists y_0 \exists y_1 \ldots \exists y_{\sigma_n - 1} \left(\bigwedge_{i \neq j} (y_i \neq y_j) \wedge \bigwedge_{i < \sigma_n} (x < y_i) \right. \\ \left. \wedge \forall z \left(x < z \to \bigvee_{i < \sigma_n} (y_i \leq z) \right) \right).$$

Hence we have

$$\langle U_{\mathcal{T}_\sigma}, < \rangle \vDash \forall x \left((\forall y (x \leq y) \wedge \chi_{\sigma_0}(x)) \vee \bigvee_{n \in \omega} \varphi_n(x) \wedge \bigvee_{n \in \omega} (\varphi_n(x) \to \chi_{\sigma_n}(x)) \right).$$

Furthermore, for any set X, if there is a partial ordering $<$ on X such that $\langle X, < \rangle$ satisfies the above $\mathcal{L}_{\omega_1 \omega}$-sentence, then we know that $\langle X, < \rangle$ carries the tree structure T_σ.

5.3.3 No structure on $\mathrm{succ}(x)$

This construction is the same as in Section 3.2.9. Let $\mathcal{T} = \langle \omega^{<\omega}, \subseteq \rangle$. Then we can see that for each $x \in \omega^{<\omega}$, $\mathrm{succ}(x)$ is infinite. We put no structure on $\mathrm{succ}(x)$, therefore $\mathrm{Aut}(T) \upharpoonright \mathrm{succ}(x) \cong \mathrm{Sym}(\mathrm{succ}(x))$ for all x.

Construct an FM-model $\mathcal{N}_\mathcal{T}$ from $\mathcal{T}$ with the set of atoms $U_\mathcal{T}$ induced from $\mathcal{T}$. Then we have $|U_\mathcal{T}| \notin \Delta_5$. By the same argument as the proof for the similar result for the binary tree in Proposition 3.2.17, we have $|U_\mathcal{T}| \in \Delta_3$ in $\mathcal{N}_\mathcal{T}$.

Proposition 5.3.7. *Let L_n be the n^{th} level of $U_\mathcal{T}$. Then $\mathrm{MT}(L_n) = n$ for all $n \in \omega$.*

Proof. Prove by induction on n, using the fact that an MT-rank n union of amorphous sets have MT-rank $n + 1$. $\square$

5.3.4 $\mathrm{succ}(x) \cong \langle \mathbb{Q}, <_{\mathbb{Q}} \rangle$

Now we consider the case that $\mathrm{succ}(x)$ carries some structure, starting from a simple one, the rationals $\langle \mathbb{Q}, <_{\mathbb{Q}} \rangle$.

Let $\mathcal{T}_{\mathbb{Q}}$ be a tree such that each vertex x branches into $\mathbb{Q}$ successors. We may view T as $\mathbb{Q}^{<\omega}$ equipped with two binary relations $\leq_T$ and $<_{\mathbb{Q}}$ which are defined as follows. The relation $\leq_T$ is the usual tree ordering on T, say $x \leq_T y$ if $x \subseteq y$, while the relation $<_{\mathbb{Q}}$ focuses on each $\mathrm{succ}(x)$, say $y <_{\mathbb{Q}} z$ if $y = x^\frown r$ and $z = x^\frown s$ for some $x \in \mathbb{Q}^{<\omega}$, $r, s \in \mathbb{Q}$ and $r < s$. We can see that $\langle \mathrm{succ}(x), <_{\mathbb{Q}} \rangle \cong \langle \mathbb{Q}, <_{\mathbb{Q}} \rangle$. Let $\mathcal{T}_{\mathbb{Q}} = \langle \mathbb{Q}^{<\omega}, \leq_T, <_{\mathbb{Q}} \rangle$. Construct an FM-model by letting the set of atoms $U_{\mathcal{T}_{\mathbb{Q}}}$ be induced from $\mathcal{T}_{\mathbb{Q}}$, $\mathcal{G}$ the group of automorphisms of $U_{\mathcal{T}}$ be induced from $\mathrm{Aut}(\mathcal{T}_{\mathbb{Q}})$, and $\mathcal{N}_{\mathcal{T}_{\mathbb{Q}}}$ the corresponding FM-model be constructed by finite supports. We will refer to a member $u_q \in U_{\mathcal{T}_{\mathbb{Q}}}$ as only q and $U_{\mathcal{T}_{\mathbb{Q}}}$ as only U.

We can see that $\mathrm{succ}(0)$ is the same (equivalent) as the set of atoms $U_{\mathbb{Q}}$ of the Mostowski model, i.e. it is o-amorphous. Hence we have the following proposition.

Proposition 5.3.8. $|U_{\mathcal{T}_{\mathbb{Q}}}| \notin \Delta_3$ *in* $\mathcal{N}_{\mathcal{T}_{\mathbb{Q}}}$.

However the higher levels are weakly o-amorphous. For example, at level 2, L_2 is the same as the FM-model that was built from $\mathbb{Q} \times \mathbb{Q}$, which is weakly o-amorphous, but not o-amorphous, see Section 3.2.4.

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