Multiplicative quiver varieties and integrable particle systems

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The candidate confirms that the work submitted is his own, except where work which has formed part of jointly-authored publications has been included. The contribution of the candidate and the other authors to this work has been explicitly indicated below. The candidate confirms that appropriate credit has been given within the thesis where reference has been made to the work of others.

- Elements of Section 2.3, Section 4.2 and § 5.3.2 are based on O. Chalykh and M. Fairon, *Multiplicative quiver varieties and generalised Ruijsenaars-Schneider models*, J. Geom. Phys. **121** (2017), 413–437. I performed most preliminary computations following ideas of O. Chalykh, and I wrote the first draft.

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Abstract

The main goal of this thesis is to provide a systematic study of several integrable systems defined on complex Poisson manifolds associated to extended cyclic quivers. These spaces are particular examples of multiplicative quiver varieties of Crawley-Boevey and Shaw, for which Van den Bergh observed that they can be equipped with a Poisson bracket obtained by quasi-Hamiltonian reduction. In his approach, Van den Bergh introduced the notion of double brackets to translate the geometric quasi-Hamiltonian structure associated to these varieties directly at the level of the path algebra of the quivers. We pursue this line of thought and examine these double brackets in order to find families of algebraic elements on the path algebra of extended cyclic quivers that give rise to families of Poisson commuting functions on the corresponding multiplicative quiver varieties. This provides a way to obtain candidates for Liouville integrability, and this can be adapted to the case of degenerate integrability. For specific dimensions of these spaces, we can compute the number of functionally independent elements in each family, and conclude that we can form integrable systems. They can be written in terms of local coordinates, and be related to the trigonometric spin Ruijsenaars-Schneider system or generalisations of the latter system. As part of our construction, we also prove that their flows can be obtained by the projection method from explicit integrations performed before the quasi-Hamiltonian reduction. Another application of this work consists in describing the Poisson structure in terms of local coordinates. In particular, this allows us to prove a conjecture of Arutyunov and Frolov regarding the form of the Poisson bracket for the trigonometric spin Ruijsenaars-Schneider system.

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List of abbreviations

CM	Calogero-Moser
DG	Differential graded
GIT	Geometric Invariant Theory
KP	Kadomtsev-Petviashvili
q**KP**	q-difference Kadomtsev-Petviashvili
LHS / RHS	Left-hand side / Right-hand side
MQV	Multiplicative quiver variety
ODE	Ordinary differential equation
PDE	Partial differential equation
RS	Ruijsenaars-Schneider

Chapter 1

Introduction

1.1 Hamiltonian systems and integrability

For centuries, there has been a persistent interest in trying to understand how a mechanical system parametrised by a finite number of positions $q = (q_1, \ldots, q_n)$ could evolve over time. Based on the foundation of classical mechanics by Newton at the end of the 17$^{\text{th}}$ century, the general evolution of systems such as the two-body problem, or the problem of two fixed centres for celestial bodies could be derived. However, determining the general solution of a system for arbitrary initial positions was a harder task. To overcome this issue, two important tools were developed in the 19$^{\text{th}}$ century in the newly formulated Hamiltonian mechanics : the separation of variables by Jacobi, and the concept of integrability by Liouville. We focus on the latter notion from now on, which we call *Liouville integrability*.

Assuming that the system is governed by some potential $V(q)$, Hamilton suggested to consider a phase space parametrised by the positions q and their associated momenta $p = (p_1, \ldots, p_n)$, together with the energy function (nowadays called the Hamiltonian)

$$H(q,p) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 \, + \, V(q)\,. \tag{1.1}$$

Then the problem could be reformulated as expressing the evolution from an initial condition (q_0, p_0) in the phase space of the $2n$-uple (q,p) governed by the ordinary differential equation

$$\frac{dq_i}{dt} = \frac{\partial H(q,p)}{\partial p_i}\,, \quad \frac{dp_i}{dt} = -\frac{\partial H(q,p)}{\partial q_i}\,, \quad i = 1, \ldots, n\,. \tag{1.2}$$

We can introduce an operation $\{-,-\}$ on the functions defined on the phase space as

$$\{f,g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i} \right) , \tag{1.3}$$

which is easily seen to be antisymmetric and a biderivation. This operation also satisfies an additional property called Jacobi identity, which makes it a Poisson bracket. It is not difficult to see that we can rewrite the equations of motion (1.2) simply using the derivation $\frac{d}{dt} = \{-,H\}$, where $\{-,H\} : f \mapsto \{f,H\}$. In fact, what Liouville noticed is that given n such derivations that pairwise commute under the Poisson bracket and are independent, in other words if we have n functionally independent elements $F = (F_1, F_2, \ldots, F_n)$ with $F_1 = H$, which also satisfy $\{F_i, F_j\} = 0$ for all indices, then we can locally construct a solution to (1.2). This process, called integration by quadrature, uses only algebraic manipulations and integrations, as well as the inverse function theorem. Therefore, it does not only apply to smooth real functions as in its original formulation, but also to complex analytic functions. We refer to the functions F as a (Liouville) integrable system, and we say that H is Liouville integrable. It is worth mentioning that, to define an integrable system, H can be an arbitrary function of (q,p) not necessarily of the form (1.1). Indeed, the integration by quadrature does not require any particular form for the functions occurring in F.

From a modern point of view, what we need to define an integrable system is a space endowed locally with the operation $\{-,-\}$ given by (1.3), i.e. a manifold with a non-degenerate Poisson bracket. In fact, the definition can also be relaxed to the case of a degenerate Poisson bracket, to the case of degenerate integrability which we will encounter in this text, or to both cases. While these generalisations are almost straightforward, moving to the case of algebraic varieties is more subtle. For more on these topics, we refer to [1, 12, 114, 169].

1.2 Calogero-Moser systems

Our presentation of Liouville integrability suggests that this notion should have played an important role directly after its introduction. However, it quickly faded away at the end of the 19th century. At that time, Poincaré made the important observation that a small perturbation can break integrability, so that Liouville integrable systems are rare among mechanical systems.

As a consequence, the mathematical community lost its interest in Liouville integrability, which condemned the subject to oblivion for several decades. Fortunately, at the end of the $1960s$, the introduction of the Toda lattice [158] was a first step to revive interest in the technique of Liouville integrability[1], see [157] for further references. Another step was realised when Calogero introduced (quantum) n-body systems on the line [37, 38], which is at the basis of the present work. In the simplest case, the classical version of the system consists in taking the Hamiltonian (1.1) with rational potential

$$V(q) = \sum_{j<k} g^2 v(q_j - q_k)\,, \quad v(x) = x^{-2}\,, \tag{1.4}$$

for some coupling constant $g > 0$. Here, we can think of q as gathering the positions on the line of n particles interacting with the potential (1.4), so that each particle is characterised by its position q_i and momentum p_i, which satisfy

$$\frac{dq_i}{dt} = p_i\,, \quad \frac{dp_i}{dt} = \sum_{k\neq i} \frac{2g^2}{(q_i - q_k)^3}\,. \tag{1.5}$$

In this case, Liouville integrability for an arbitrary number of particles was established by Moser [120]. Though relatively simple in appearance, this system was on the verge of leading to intense research activities related to integrable systems. Indeed, by the time of Moser's paper publication in 1975, it was already known that we could consider the harmonic potential $v(x) = x^{-2} + \omega x^2$ [37, 38] or the trigonometric potential $v(x) = \sin(x)^{-2}$ [153, 154] in the quantum case. This led to their classical formulation, and their modification to hyperbolic or elliptic potentials, as well as generalisations related to root systems of Lie algebras as reviewed in [130]. We refer to all these systems as Calogero-Moser (or CM) systems, and we call the system with potential (1.4) the rational CM system (of type A_{n-1}).

There are three features of integrable particle systems of CM type that will play a central role in our study of Ruijsenaars-Schneider systems, which we introduce next in Section 1.3. Restricting our attention to the rational CM system, the first feature discovered by Wojciechowski is that it is degenerately integrable [172]. In general, this means that for a phase space of dimension $2n$, there exist functions $G = (G_1, \dots, G_r)$, $n \leq r \leq 2n-1$, that are functionally independent and such that the first $s = 2n - r$ functions $G_1, \dots, G_s$ Poisson commute with all the elements

[1] We confine ourselves to the classical notion of Liouville integrability, leaving historical details related to other notions of integrability for classical systems to comprehensive treatments of the subjects such as [20, 60, 61, 176].

of G. The case of Liouville integrability corresponds to $r = n$. Wojciechowski noticed that if $F = (F_1, \dots, F_n)$ is an integrable system for the rational CM case, then we can extend any of the F_i to a degenerately integrable system of $r = 2n-1$ elements, which is the maximal possible case. The second feature unveiled by Gibbons and Hermsen is that there exist spin extensions of the rational CM system [82]. This means that we can widen the phase space with initial canonical coordinates (q,p) by $2nd$ elements $(a_i^\alpha, c_i^\alpha)_{i,\alpha}$ where $i = 1, \dots, n$ corresponds to the particle label and $\alpha = 1, \dots, d$ is an index for internal degrees of freedom, called spins. For each i, we consider $(a_i^\alpha, c_i^\alpha)_\alpha$ under the constraint $\sum_\alpha a_i^\alpha c_i^\alpha = g^2$ and identify $\mathbb{R}^\times$-orbits for the action $\lambda \cdot (a_i^\alpha, c_i^\alpha)_\alpha = (\lambda a_i^\alpha, \lambda^{-1} c_i^\alpha)_\alpha$. The only new non-trivial Poisson bracket is given by $\{a_i^\alpha, c_j^\beta\} = \delta_{ij}\delta_{\alpha\beta}$, and we now look at the potential

$$V(q) = \frac{1}{2} \sum_{\substack{j,k=1 \\ k \neq j}}^{n} f_{jk} v(q_j - q_k) \,, \quad f_{jk} = \sum_{\alpha=1}^{d} a_j^\alpha c_k^\alpha \,, \quad v(x) = x^{-2} \,, \tag{1.6}$$

from which we easily get (1.4) back when $d = 1$. For this particular potential which defines what we call the rational spin CM system, Liouville integrability (and, in fact, maximal degenerate integrability) can also be shown. Finally, the third feature may be the most interesting, as it gives a geometric realisation of the phase space : it was observed by Kazhdan, Kostant and Sternberg in [95] that the phase space can be obtained by Hamiltonian (or Marsden-Weinstein) reduction. The general idea is that, beginning with a Poisson manifold M on which some Lie group G acts, we can consider a G-stable slice $N \subset M$ of codimension $\dim(G)$, such that the orbit space N/G is a manifold endowed with a Poisson bracket completely determined by the one on M. This reduction procedure has two advantages. The first one is that if a family of G-invariant functions on the bigger space M Poisson commute, then their projections after reduction will Poisson commute in N/G too. Moreover, computations with the Poisson bracket of M are usually easier, so that forming an integrable system F on N/G amounts to find Poisson commuting G-invariant functions $\tilde{F}$ in M that projects to $\frac{1}{2}\dim(N/G)$ functionally independent elements. The second one is that the vector fields defined by the functions $\tilde{F}$ (which descend to the desired vector fields associated to F) could be easier to integrate in M, so that the evolution of the system on N/G could be obtained by projecting these flows; this is the projection method. Quite amazingly, these two advantages occur when we derive the phase space of the rational CM system by Hamiltonian reduction [95].

As we mentioned, the three features that we presented were initially concerned with the rational case (and the trigonometric case for the reduction picture [95]). Though it is a very interesting subject which has attracted a lot of attention, we will skip the discussion of their possible generalisation to other potentials. Rather, let us mention that they can be adapted to the complex version of the rational CM system [170, 171], which has a very interesting consequence that motivates the present work. Namely, when Wilson described the (completed) phase space of the complex rational CM system [170] in analogy to the work of Kazhdan, Kostant and Sternberg [95], he could show that this space is endowed with a hyperkhäler structure because it is a particular example of Nakajima quiver variety [123]. This, in turn, means that the space can be defined as the moduli space of representations associated to a particular quiver, i.e. a directed graph. In his construction [123], Nakajima defined the symplectic structure on this space in terms of the matrices representing the arrows of the quiver. Therefore, it motivates the following question : *can we understand the Poisson geometry of this space directly at the level of the quiver?* This problem is at the basis of two interesting developments in modern non-commutative geometry, which are non-commutative symplectic geometry [54, 83] and non-commutative Poisson geometry [162]. In the latter case, Van den Bergh introduced a non-commutative version of Hamiltonian reduction for associative algebras, and he explicitly defined such structures associated to any quiver [162]. Thus, knowing the quiver implicitly considered by Wilson and using Van den Bergh's theory, we can partially understand the rational CM system at the level of a non-commutative algebra. This led to several attempts to enlarge our understanding of complex integrable systems of CM type using quivers. In particular, this resulted in a new simple description of the phase space for the spin version of the rational CM system [32, 118, 155, 156], but also additional generalisations [31, 43, 148, 149].

1.3 Ruijsenaars-Schneider systems

In 1986, a new class of integrable systems was introduced by Ruijsenaars and Schneider, who motivated them as a relativistic generalisation of the CM systems [144]. We refer to these systems as Ruijsenaars-Schneider (or RS) systems. As in the CM case, the potential can be rational/hyperbolic [144], or trigonometric/elliptic [140], and it can be related to different root systems as was observed by van Diejen [165, 166, 167] directly for the quantum case. To get some insight into these models, let us briefly sketch the (complex) hyperbolic case. Its phase space is associated to the positions of n particles subject to the equations of motion

$$\dot{q}_i = L_{ii}\,, \quad \dot{L}_{ii} = 2\sum_{k\neq i} L_{ik}L_{ki}\coth\left(q_i - q_k\right)\,, \tag{1.7}$$

where $L = (L_{ij})_{ij}$ is a Lax matrix for this system which is of the form

$$L_{ij} = \frac{\sinh(\gamma)}{\sinh(q_i - q_j + \gamma)}L_{jj}\,, \quad L_{jj} = e^{p_j}\sqrt{\prod_{k\neq j} f(q_k - q_j)}\,, \quad 1 \leq i,j \leq n\,.$$

Here, γ is a fixed nonzero coupling constant and f is a particular function such that (1.7) is defined by the Hamiltonian $\operatorname{tr} L$ under the Poisson bracket (1.3). (The relation between L and the original Lax matrix given in [144] can be recovered from Lemma 4.2.9.) Compared to the CM system, it is difficult to understand this model simply by looking at the equations of motion (1.7) or the Hamiltonian $\operatorname{tr} L$. Nevertheless, and quite surprisingly, systems of RS type enjoy the same kind of features that the CM systems possess.

In 1995, Krichever and Zabrodin noticed that it was possible to formulate a spin generalisation of the RS system [105], which they introduced for the elliptic potential. This is a system of n particles with coordinates $(q_i)_i$, momenta $(\dot{q}_i)_i$, and spin variables given by $(a_i^\alpha, c_i^\alpha)_{i,\alpha}$ where $i = 1,\ldots,n$ corresponds to the particle and $\alpha = 1,\ldots,d$ ranges over the number of internal degrees of freedom (spins). Introduce the functions $f_{ij} = \sum_\alpha a_i^\alpha c_j^\alpha$. Given a nonzero coupling constant γ, the equations of motion are given by

$$\ddot{q}_i = \sum_{k\neq i}[V(q_{ik}) - V(q_{ki})]\, f_{ik}f_{ki}\,, \tag{1.8a}$$

$$\dot{a}_i^\beta = -\,\lambda_i a_i^\beta + \sum_{k\neq i} V(q_{ik})\, a_k^\beta\, f_{ik}\,, \tag{1.8b}$$

$$\dot{c}_j^\beta = \lambda_i c_i^\beta - \sum_{k\neq j} V(q_{kj})\, c_k^\beta\, f_{kj}\,, \tag{1.8c}$$

where $q_{ik} := q_i - q_k$ and the elliptic potential $V(q) = \zeta(q) - \zeta(q+\gamma)$ is defined in terms of the Weierstrass zeta-function $\zeta(q)$. Of particular interest, the rational and trigonometric degenerations are given by $V^{rat}(q) = q^{-1} - (q+\gamma)^{-1}$ and[2] $V^{trig}(q) = \coth(q) - \coth(q+\gamma)$. The functions

[2] To be precise, we should write this potential $V^{hyp}(q)$ as this corresponds to the hyperbolic potential in this original real case. Since we will work in the complex setting where the trigonometric and hyperbolic potentials are equivalent, we call this potential trigonometric and stick to this terminology from now on.

$\lambda_i(t)$ can be set to zero by a suitable scaling. There are $2nd + 2n$ coordinates, but we can set the constants of motion $I_i = \dot{q}_i - \sum_\alpha a_i^\alpha c_i^\alpha$ to zero, and get a system of dimension $2nd$ after imposing a further n normalisation conditions. Following Arutyunov and Frolov [16], we interpret these conditions by considering instead the invariant spin variables $(\mathbf{a}_i^\alpha, \mathbf{c}_i^\alpha)_{i,\alpha}$, which correspond to the rescaling $\mathbf{a}_i^\alpha = (\sum_\alpha a_i^\alpha)^{-1} a_i^\alpha$ and $\mathbf{c}_i^\alpha = (\sum_\alpha a_i^\alpha) c_i^\alpha$ in the original model. For $\mathbf{f}_{ij} = \sum_\alpha \mathbf{a}_i^\alpha \mathbf{c}_j^\alpha$ obtained after normalisation of the function f_{ij}, this leads to the equations of motion

$$\dot{q}_i = \mathbf{f}_{ii} \,, \tag{1.9a}$$

$$\dot{\mathbf{a}}_i^\alpha = -\sum_{k\neq i} (\mathbf{a}_i^\alpha - \mathbf{a}_k^\alpha) \mathbf{f}_{ik} V(q_{ik}) \,, \tag{1.9b}$$

$$\dot{\mathbf{c}}_j^\alpha = \sum_{k\neq j} \left(\mathbf{c}_j^\alpha \mathbf{f}_{jk} V(q_{jk}) - \mathbf{c}_k^\alpha \mathbf{f}_{kj} V(q_{kj}) \right) , \tag{1.9c}$$

from which we see that the condition $\sum_\alpha \mathbf{a}_i^\alpha = 1$ is preserved. In the case $d = 1$, we have $\mathbf{a}_i^1 = 1$ and $\mathbf{f}_{ij} = \mathbf{c}_j^1$ for all i, j, and we recover the original RS system from (1.9a)–(1.9c). For example in the trigonometric case, we get by writing $\mathbf{c}_j^1$ as L_{jj} that (1.9a) becomes $\dot{q}_i = L_{ii}$ while (1.9c) can be written as

$$\begin{aligned}
\dot{L}_{ii} =& \sum_{k\neq i} L_{ii} L_{kk} \left[\coth(q_i - q_k) - \coth(q_i - q_k + \gamma) - \coth(q_k - q_i) + \coth(q_k - q_i + \gamma) \right] \\
=& 2 \sum_{k\neq i} \frac{\sinh(\gamma) L_{kk}}{\sinh(q_i - q_k + \gamma)} \frac{\sinh(\gamma) L_{ii}}{\sinh(q_k - q_i + \gamma)} \coth(q_i - q_k) \,,
\end{aligned}$$

so we get precisely (1.7). The Hamiltonian formulation in that case is well-known, but for the system with $d > 1$ spins and arbitrary potential, it is only given in a universal form [103]. Therefore, it is an interesting problem to find the formulation of the Poisson bracket in terms of the coordinates $(q_i, a_i^\alpha, c_i^\alpha)$ (or their normalisation). However, the problem of describing the Hamiltonian structure of the phase space of the spin RS system turns out to be hard to tackle, since it was completely solved only for the rational case [16], while for the general elliptic case there is just a partial result in the case of $n = 2$ particles [151].

The description of the complex rational case by Arutyunov and Frolov [16] was made possible by a correct understanding of the Hamiltonian reduction performed to obtain the phase space, an idea that drew on the pioneering work of Kazhdan, Kostant and Sternberg [95]. After performing the reduction, Arutyunov and Frolov were able to interpret their phase space using $2nd$ coordinates containing the positions $(q_i)_i$, together with spin variables $(\mathbf{a}_i^\alpha, \mathbf{c}_i^\alpha)_{i,\alpha}$ subject to the constraints

$\sum_\alpha \mathbf{a}_i^\alpha = 1$. Within this framework and for a fixed (complex) coupling constant γ, the Lax matrix $\mathbf{L} = (\mathbf{L}_{ij})_{ij}$ given by

$$\mathbf{L}_{ij} = \frac{\mathbf{f}_{ij}}{q_{ij} + \gamma}\,, \qquad \mathbf{f}_{ij} = \sum_{\alpha=1}^{d} \mathbf{a}_i^\alpha \mathbf{c}_j^\alpha\,, \tag{1.10}$$

naturally appeared, and allowed them to obtain explicit formulae for the Poisson bracket $\{-,-\}_A$ between all variables

$$\begin{aligned}
\{q_i, q_j\}_A =&0\,, \quad \{\mathbf{a}_i^\alpha, q_j\}_A = 0\,, \quad \{\mathbf{c}_i^\alpha, q_j\}_A = -\delta_{ij}\mathbf{c}_i^\alpha\,, &(1.11a)\\
\{\mathbf{a}_j^\gamma, \mathbf{a}_i^\alpha\}_A =&\delta_{(i\neq j)}\frac{1}{q_j - q_i}(\mathbf{a}_j^\gamma \mathbf{a}_i^\alpha + \mathbf{a}_i^\gamma \mathbf{a}_j^\alpha - \mathbf{a}_j^\gamma \mathbf{a}_j^\alpha - \mathbf{a}_i^\gamma \mathbf{a}_i^\alpha)\,, &(1.11b)\\
\{\mathbf{c}_j^\epsilon, \mathbf{a}_i^\alpha\}_A =&\delta_{\epsilon\alpha}\mathbf{L}_{ij} - \mathbf{a}_i^\alpha \mathbf{L}_{ij} + \delta_{(i\neq j)}\frac{1}{q_j - q_i}\mathbf{c}_j^\epsilon(\mathbf{a}_j^\alpha - \mathbf{a}_i^\alpha)\,, &(1.11c)\\
\{\mathbf{c}_j^\epsilon, \mathbf{c}_i^\beta\}_A =&\delta_{(i\neq j)}\frac{1}{q_j - q_i}(\mathbf{c}_j^\epsilon \mathbf{c}_i^\beta + \mathbf{c}_i^\epsilon \mathbf{c}_j^\beta) + \mathbf{c}_i^\beta \mathbf{L}_{ij} - \mathbf{c}_j^\epsilon \mathbf{L}_{ji}\,. &(1.11d)
\end{aligned}$$

To relate this phase space endowed with the Poisson bracket (1.11a)–(1.11d) to the rational spin RS system, they showed that the equations of motion for the Hamiltonian $\mathrm{tr}(\mathbf{L})$ induced by these brackets are the ones for the spin RS system in the form (1.9a)–(1.9c) with $V^{rat}(q)$, up to a factor $e^\gamma/\sinh\gamma$.

An important remark that Arutyunov and Frolov formulated is that the entries of the Lax matrices for the spin and non-spin rational RS models satisfy the same Poisson brackets defined from the same r-matrix formulation. Motivated by this relation, they introduced the spin version for the trigonometric RS model by defining the matrix $\mathbf{L} = (\mathbf{L}_{ij})_{ij}$ for

$$\mathbf{L}_{ij} = \frac{e^{q_{ij}+\gamma}}{\sinh(q_{ij}+\gamma)}\mathbf{f}_{ij}\,, \tag{1.12}$$

and they conjectured that the matrix $\mathbf{L}$ should obey the same Poisson algebra than its rational version, with the r-matrices corresponding to the spinless trigonometric case. Under this assumption, they could find the Poisson bracket for their new model in terms of the functions $(q_i, \mathbf{f}_{ij})_{ij}$, on which it takes the form

$$\{q_i, q_k\}_A =0\,, \quad \{\mathbf{f}_{ij}, q_k\}_A = -\delta_{jk}\mathbf{f}_{ij}\,, \tag{1.13a}$$

$$\begin{aligned}
\{\mathbf{f}_{ij}, \mathbf{f}_{kl}\}_A =&[\coth(q_{ik}) + \coth(q_{jl}) + \coth(q_{kj}) + \coth(q_{li})]\mathbf{f}_{ij}\mathbf{f}_{kl}\\
&+ [\coth(q_{ik}) + \coth(q_{jl}) + \coth(q_{kj}+\gamma) - \coth(q_{il}+\gamma)]\mathbf{f}_{il}\mathbf{f}_{kj}\\
&+ [\coth(q_{ki}) + \coth(q_{il}+\gamma)]\mathbf{f}_{ij}\mathbf{f}_{il} + [\coth(q_{jk}) - \coth(q_{jl}+\gamma)]\mathbf{f}_{ij}\mathbf{f}_{jl}\\
&+ [\coth(q_{ki}) - \coth(q_{kj}+\gamma)]\mathbf{f}_{kj}\mathbf{f}_{kl} + [\coth(q_{il}) + \coth(q_{lj}+\gamma)]\mathbf{f}_{lj}\mathbf{f}_{kl}\,,
\end{aligned} \tag{1.13b}$$

with the convention that a term with a vanishing denominator is omitted. However, they were unable to find what equations (1.11b)–(1.11d) would become. They attempted to replace the factors $1/q$ by $1/\coth(q)$ in (1.11b)–(1.11d), but this elementary adjustment was not successful. Indeed, as we will see in § 4.3.2, some extra terms need to be included.

As we have just explained, the reduction picture played an important role in the understanding of the rational spin RS system. To go to the trigonometric case, one has to understand the available pictures in the non-spin case, where we have two possibilities. On the one hand, there is a Poisson reduction introduced by Fock and Rosly [80], which is a discretisation of the infinite-dimensional reduction[3] of Atiyah and Bott [17]. On the other hand, we can use quasi-Hamiltonian reduction [126], an analogue of Hamiltonian reduction introduced by Alekseev, Kosmann-Schwarzbach, Malkin and Meinrenken [6, 7]. While both options are worth pursuing, we will focus on the second one for a very good reason : in his work [162], Van den Bergh did not only introduce a non-commutative version of Poisson geometry and Hamiltonian reduction, but he also introduced the non-commutative version of quasi-Hamiltonian reduction. In fact, he also showed that this theory could be applied to an arbitrary quiver, and relates to multiplicative preprojective algebras [56]. Therefore, it seems natural to consider the original quiver of Wilson, as well as the variations which give systems of rational CM type that we mentioned in Section 1.2, and try to understand their relation to the RS system. Our hope is to derive the phase space for the trigonometric spin RS system from this method, and then to prove the conjecture of Arutyunov and Frolov stating that the Poisson bracket satisfies (1.13a)–(1.13b) in this case. It will turn out to work, and we will be able to prove that the system is Liouville integrable. In fact, we will also recover the third feature of the CM system : the trigonometric spin RS system is degenerately integrable.

The relation to the spin RS system that we have just indicated is obtained by looking at an extended Jordan (or one-loop) quiver. However, it is quite satisfying to notice that our method can be adapted to other quivers, which implies that we can find new integrable particle systems. We now summarise the main results obtained in this way.

Jordan quiver extended by one arrow. We recover the completed phase space for the trigonometric RS system. We prove its Liouville integrability using non-commutative quasi-

[3]We want to restrict our attention to a finite-dimensional reduction picture, so we do not consider the possibility of studying this infinite-dimensional reduction in the present work. This alternative method can be successfully considered for the non-spin case [125].

Poisson geometry. These results have been published [41].

Jordan quiver extended by $d \geq 2$ arrows. We obtain the completed phase space for the trigonometric spin RS system. We compute the Poisson structure in local coordinates and we prove the conjecture of Arutyunov and Frolov. We show that the spin RS system is both Liouville integrable and degenerately integrable using Van den Bergh's formalism. These results have been submitted for publication [42].

Cyclic quiver on m vertices extended by several arrows. We obtain the completed phase space for a new integrable particle system, and we compute its Poisson structure in local coordinates. This allows to prove that on a suitably chosen closed submanifold, this new system restricts to the trigonometric spin RS system for $n \in \mathbb{N}^\times$ particles. This space is also the natural phase space for two other families of systems, which are $S_n \ltimes \mathbb{Z}_m^n$-invariant. All these systems are both Liouville and degenerately integrable. These results have been published when there is one additional arrow [41]; these results have been submitted for publication when there is $d \geq 2$ additional arrows pointing to the same vertex of the cyclic quiver [62].

1.4 Outline of the thesis

Chapter 2 is an introductory chapter which gathers the necessary material that we use in the rest of the present thesis. It consists of a quick overview of some well-known geometric and algebraic structures, as well as a more advanced review of the work of Van den Bergh related to double brackets [162, 163]. We particularly emphasise the notion of quasi-Hamiltonian algebras associated to quivers and their relation to multiplicative quiver varieties (or MQVs) of Crawley-Boevey and Shaw [56]. We also add some useful connections between Van den Bergh's formalism and the theory of integrable systems.

Chapter 3 is the most technical part of this thesis, where we study quasi-Hamiltonian algebras defined from the extended Jordan quiver or extended cyclic quivers. The aim of this chapter is to derive many results that will permit us to construct integrable systems and understand the Poisson structure on the corresponding MQVs. Though it can be seen as the core of the present thesis, the beauty of this chapter can only be recognised once we understand its geometric implication presented in the next two chapters; therefore we advise the reader to skip it on a first reading.

Chapter 4 deals with the MQVs corresponding to the extended Jordan quiver for specific dimensions. When the extension consists of a single arrow, we can see that the MQV is the phase space for the trigonometric RS system, and prove its Liouville integrability. When the extension consists of $d \geq 2$ arrows, we obtain the same result for the trigonometric spin RS system. In the latter case, we can show that it is both Liouville integrable and degenerately integrable. Moreover, we can explicitly obtain the Poisson brackets between local coordinates, then recover the Poisson brackets conjectured by Arutyunov and Frolov [16].

Chapter 5 deals with the MQVs corresponding to extended cyclic quivers for specific dimensions. We show that we can recover the trigonometric (spin and non-spin) RS system for particular extensions. We also obtain new generalisations of the spin RS system when we extend the cyclic quiver by an arbitrary number of arrows. Furthermore, if the cyclic quiver has m vertices, we describe new systems with $W = S_n \ltimes \mathbb{Z}_m^n$ symmetry. In all those cases, we can show that the systems are both Liouville integrable and degenerately integrable.

Chapter 6 is divided into two parts. In the first part, we describe some recent developments and new perspectives related to the models studied in this thesis. In the second part, we provide an extensive review of the different applications of double brackets since their introduction by Van den Bergh.

1.5 Conventions

We denote by $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ the sets of non-negative integers, integers, real numbers and complex numbers respectively. If we omit the zero element, we denote the corresponding sets by $\mathbb{N}^\times, \mathbb{Z}^\times, \mathbb{R}^\times, \mathbb{C}^\times$.

The Kronecker delta function δ_{ij} (also denoted $\delta_{i,j}$ or $\delta_{(i,j)}$) takes the value $+1$ if $i = j$ or 0 otherwise. For a proposition P, we define in a similar way δ_P which takes the value $+1$ if P is true, and zero if P is false. For example, $\delta_{(i \leq j)}$ takes the value $+1$ if $i \leq j$ and 0 for $i > j$.

Consider a finite set of elements $\{a_j\}_{j \in J}$ which are totally ordered, i.e. there exists a bijective map $\rho : \{1, \dots, |J|\} \to J$ such that $a_{\rho(1)} < \dots < a_{\rho(|J|)}$. Then, the corresponding left and right products are given by

$$\overleftarrow{\prod_{j\in J}} a_j = a_{\rho(|J|)} \dots a_{\rho(2)} a_{\rho(1)} \,, \qquad \overrightarrow{\prod_{j\in J}} a_j = a_{\rho(1)} a_{\rho(2)} \dots a_{\rho(|J|)} \,.$$

Fix $d \in \mathbb{N}^\times$, $d \geq 2$, and let $J_d = \{1, \dots, d\} \subset \mathbb{N}^\times$. The ordering function on d elements is the map $o(-,-) : J_d \times J_d \to \{-1, 0, +1\}$ defined by $o(\alpha, \beta) = +1$ if $\alpha < \beta$, $o(\alpha, \beta) = 0$ if $\alpha = \beta$, and $o(\alpha, \beta) = -1$ if $\alpha > \beta$. This is a skew-symmetric map. We naturally extend the definition to any totally ordered finite set.

Chapter 2

Basic notions

2.1 Geometric formalism

In this section, any manifold M is assumed to be complex, and we write $\mathcal{O}_M$ for the sheaf of analytic functions on M. Given any sheaf $\mathcal{F}$ on M, we say that a property holds for any $f \in \mathcal{F}$ if it holds for any open subset $U \subset M$ and $f \in \mathcal{F}(U)$. We will make use of the latter short-hand notation throughout this section, as well as similar variations.

We equip the exterior algebra $\bigwedge^\bullet T_M$ with the Schouten-Nijenhuis bracket $[-,-]$ which is defined to be the Lie bracket on vector fields and is extended on multivector fields $\alpha, \beta, \gamma \in \bigwedge^\bullet T_M$ as a bilinear map of degree -1 such that

$$[\alpha, \beta] = -(-1)^{(|\alpha|-1)(|\beta|-1)}[\beta, \alpha]\,, \tag{2.1a}$$

$$[\alpha, \beta \wedge \gamma] = [\alpha, \beta] \wedge \gamma + (-1)^{(|\alpha|-1)|\beta|}\beta \wedge [\alpha, \gamma]\,, \tag{2.1b}$$

$$0 = [\alpha, [\beta, \gamma]] + (-1)^{(|\alpha|-1)(|\beta|+|\gamma|)}[\beta, [\gamma, \alpha]] + (-1)^{(|\gamma|-1)(|\alpha|+|\beta|)}[\gamma, [\alpha, \beta]]\,. \tag{2.1c}$$

2.1.1 Poisson geometry

A Poisson manifold is a manifold M endowed with a holomorphic bivector field $P \in \bigwedge^2 T_M$ such that $[P, P] = 0$ under the Schouten-Nijenhuis bracket. The map $\{-,-\} : \mathcal{O}_M^{\times 2} \to \mathcal{O}_M : (f, g) \mapsto \{f, g\} := P(df, dg)$ defines an antisymmetric $\mathbb{C}$-linear biderivation, and we can show that the condition $[P,P]=0$ is equivalent to Jacobi identity $Jac=0$, where for any $g_1,g_2,g_3\in\mathcal{O}_M$

$$Jac(g_1,g_2,g_3)=\{\{g_1,g_2\},g_3\}+\{\{g_2,g_3\},g_1\}+\{\{g_3,g_1\},g_2\}\,. \tag{2.2}$$

Indeed, $[P,P]=2\,Jac$. This implies that $\{-,-\}$ is a Poisson bracket. We will only deal with the case of a nondegenerate Poisson bracket, which means that around any $x\in M$ with local coordinates $(x_1,\dots,x_n)$, the matrix with entries $\{x_i,x_j\}$, $1\le i,j\le n=\dim M$, evaluated at x is invertible. This is equivalent to the property that M is a symplectic manifold.

We say that two functions are in *involution* if they Poisson commute, i.e. their Poisson bracket vanishes. To any $f\in\mathcal{O}_M$ we associate a vector field $V_f=\{f,-\}$ on M, and we say that vector fields of that form are *Hamiltonian*. We then remark that $[V_f,V_g]=0$ whenever f and g are in involution using (2.2). This means that if two functions are in involution, the corresponding vector fields commute, and thus their flows commute. This is central in the context of integrability that we review in § 2.1.2.

Let us mention two ways of constructing new Poisson manifolds from a reduction procedure. Consider a Lie group G with Lie algebra $\mathfrak{g}$ and a Poisson manifold M such that we have a left action $G\times M\to M$ denoted by $(g,x)\mapsto g\cdot x$. We get an action on $f\in\mathcal{O}_M$ as $(g\cdot f)(x)=f(g^{-1}\cdot x)$ for all $x\in M$. Recall that the action is said to be free if $g\cdot x=x$ implies $g=1\in G$. It is said to be proper if the map $G\times M\to M\times M$, $(g,x)\mapsto(x,g\cdot x)$ is proper, i.e. the preimage of a compact set is compact.

Recall that a morphism $\psi:M\to N$ between Poisson manifolds is Poisson if for all $f_1,f_2\in\mathcal{O}_N$, $\psi^*\{f_1,f_2\}_N=\{\psi^*f_1,\psi^*f_2\}_M$. We say that M is an invariant Poisson G-manifold if for all $g\in G$, the corresponding action is a Poisson morphism.

Proposition 2.1.1 *Assume that M is an invariant Poisson G-manifold such that the action is free and proper. Then the orbit space M/G is a Poisson manifold, and the submersion $\pi:M\to M/G$ is Poisson.*

Given any $\xi\in\mathfrak{g}$, we define

$$\xi_M(h)(x)=\frac{d}{dt}\Big|_{t=0}h\left(\exp(-t\xi)\cdot x\right)\,,\qquad\text{for all } x\in M,\ h\in\mathcal{O}(M). \tag{2.3}$$

This can be extended as a map $(-)_M:\bigwedge^\bullet\mathfrak{g}\to\bigwedge^\bullet T_M$ preserving wedge products and Schouten-Nijenhuis brackets (where we also extend the Lie bracket from $\mathfrak{g}$ to $\bigwedge^\bullet\mathfrak{g}$ using (2.1a)–(2.1c)).

Note that the element $\xi_M \in T_M$ hence obtained is not necessarily a Hamiltonian vector field. A way to overcome this issue is the existence of a moment map, which gives us in particular the tool of Hamiltonian reduction. This inspires the quasi-Hamiltonian formalism reviewed in § 2.1.3.

We say that the invariant Poisson G-manifold M is *Hamiltonian* if it is endowed with a *moment map*, that is a regular map $\mu : M \to \mathfrak{g}^*$ which is G-equivariant (i.e. $\mu(g \cdot x) = ad_g^* \mu(x)$ for ad^* the coadjoint action on $\mathfrak{g}^*$) and such that for all $\xi \in \mathfrak{g}$, $\{\mu^*(\xi), f\} = \xi_M(f)$ for all $f \in \mathcal{O}_M$.

Proposition 2.1.2 *Assume that M is Hamiltonian with moment map μ. If the action is free and proper, then, for a generic coadjoint orbit $\mathcal{O}^* \subset \mathfrak{g}^*$, the manifold $\mu^{-1}(\mathcal{O}^*)/G$ is Poisson.*

We finish this subsection by referring to [114] for a comprehensive review containing all these subjects, and [131] for an elaborated treatise on momentum maps. Both references focus on the real case, but complex versions of these results are obtained in the same way, see e.g. [1]. The results also admit natural versions for affine varieties, see [169].

2.1.2 Classical integrable systems

Definition 2.1.3 *An* integrable system *on a Poisson manifold M endowed with a nondegenerate Poisson bracket is a set of $\frac{1}{2}\dim M$ elements $(f_k)_k$, with $f_k \in \mathcal{O}_M$, that are pairwise in involution and that are functionally independent on a dense open subset of M.*

The importance of the definition comes from the fact that for any function H of the algebra generated by the $(f_k)_k$, we can implicitly compute the flows of (the Hamiltonian vector field corresponding to) H by quadrature, see e.g. [1, Section 4.2]. We say in that case that H is *(Liouville or completely) integrable*.

Example 2.1.4 *Write $\mathcal{M} = \mathfrak{gl}_n(\mathbb{C})^2 \times \mathrm{Mat}_{1\times n}(\mathbb{C}) \times \mathrm{Mat}_{n\times 1}(\mathbb{C})$ for some $n \in \mathbb{N}^\times$. This space has a nondegenerate Poisson structure given by*

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial X_{ij}} \wedge \frac{\partial}{\partial Y_{ji}} + \sum_{i=1}^{n} \frac{\partial}{\partial V_i} \wedge \frac{\partial}{\partial W_i}\,. \tag{2.4}$$

Following [170], we consider the space of matrices $(X, Y, V, W) \in \mathcal{M}$ satisfying $[X, Y] = \mathrm{Id}_n + WV$ modulo the action of $\mathrm{GL}_n(\mathbb{C})$ by $g \cdot (X, Y, V, W) = (gXg^{-1}, gYg^{-1}, Vg^{-1}, gW)$.

We denote this space by $\mathcal{C}_n$. It has a non-degenerate Poisson bracket induced by (2.4), and we can easily see that the functions $(\operatorname{tr} Y^k)_{k\in\mathbb{N}}$ are defined on $\mathcal{C}_n$ and are in involution. Write $y_1,\dots,y_n$ the eigenvalues of Y. These are independent of the $\mathrm{GL}_n(\mathbb{C})$ action, so are well-defined at each point $(X,Y,V,W)\in\mathcal{C}_n$. On the dense open subset of $\mathcal{C}_n$ where Y is diagonalisable, its eigenvalues are in fact distinct, so that the Wronskian of the functions $\frac{1}{k}\operatorname{tr} Y^k = \frac{1}{k}(y_1^k+\ldots+y_n^k)$ for $k=1,\dots,n$ is nonzero. Hence $\operatorname{tr} Y,\dots,\frac{1}{n}\operatorname{tr} Y^n$ form an integrable system in $\mathcal{C}_n$. This is the rational CM system introduced in Section 1.2. Indeed, on the dense open subset of $\mathcal{C}_n$ where X has distinct eigenvalues, we can consider a slice where $X=\operatorname{diag}(q_1,\dots,q_n)$ and

$$W^\top = -V = (1,\dots,1), \qquad Y_{ij} = \delta_{ij}p_j + \delta_{(i\neq j)}\frac{-1}{q_i-q_j}\,.$$

Then $\frac{1}{2}\operatorname{tr} Y^2$ is the (complexified) Hamiltonian for the rational Calogero-Moser system, and its Liouville integrability follows as the Hamiltonian is part of an integrable system.

The space used in Example 2.1.4 can be seen as an application of Proposition 2.1.2.

Definition 2.1.5 *A* degenerately integrable system *(also called* non-commutative integrable system *or* superintegrable system*) on a Poisson manifold M endowed with a nondegenerate Poisson bracket is a set of r elements $(f_1,\dots,f_s,f_{s+1},\dots,f_r)$ where $f_k\in\mathcal{O}_M$, with $r+s=\dim M$, such that for each $1\le i\le s$ and $1\le k\le r$ the functions f_i and f_k are in involution, and furthermore the r functions are functionally independent on a dense open subset of M.*

Remark 2.1.6 *It is important to note that this terminology is also used for a slightly different notion. In e.g. [94], this term means that for a fixed function $f\in\mathcal{O}(M)$, the algebra $\mathcal{F}(f)$ generated by the elements that Poisson commute with f is an algebra of dimension $r\ge n$ such that the kernel of the Poisson structure restricted to $\mathcal{F}$ has dimension $s=\dim M-r$. Our definition is less restrictive : in our case, we do not require that a degenerately integrable system $(f_1,\dots,f_s,f_{s+1},\dots,f_r)$ coincide with the algebra $\mathcal{F}(f_1)$.*

The proof of the integration by quadrature can be adapted to this degenerate case, and we have a notion of action-angle coordinates [124]. In particular, at a generic point we can locally integrate the flows corresponding to the Hamiltonian vector fields of $(f_1,\dots,f_s)$. We can *not* integrate locally the flows for the other functions $(f_{s+1},\dots,f_r)$ using this method.

2.1.3 Quasi-Hamiltonian manifolds

As we have seen in § 2.1.1, a possible way to obtain Poisson manifolds of smaller dimension from a given one is by the process of Hamiltonian reduction, which requires the existence of a Lie group action on a Poisson manifold, together with a moment map that takes value in the dual of the corresponding Lie algebra. An interesting question consists in generalising the construction to moment maps with value in the Lie group itself. This was studied in [7] for the symplectic case, and generalised to the Poisson case in [6]. We follow the latter reference in this section together with [106] to focus on complex manifolds. For algebraic varieties, we refer to [162, 7.12-13].

We fix a Lie group G whose Lie algebra $\mathfrak{g}$ is endowed with a non-degenerate ad-invariant bilinear form $(-,-)$. We denote by $(f_a)_a$ a basis for $\mathfrak{g}$ and $(f^a)_a$ its dual under the bilinear form. We have the structure constants $(C_{abc})_{abc}$ defined by $C_{abc} = (f_a, [f_b, f_c])$, with which we form an ad-invariant 3-form $\phi \in \bigwedge^3 \mathfrak{g}$ by

$$\phi = \frac{1}{12} \sum_{a,b,c} C_{abc} f^a \wedge f^b \wedge f^c \,. \tag{2.5}$$

Indeed, the ad-invariance follows from the identification of $\mathfrak{g}$ and $\mathfrak{g}^*$ using $(-,-)$, so that ϕ is identified with the map $\bigwedge^3 \mathfrak{g} \to \mathbb{C}$, $x \wedge y \wedge z \mapsto (x, [y,z])$ which is ad-invariant because $(-,-)$ is. The Lie group structure of G allows us to define for any $\xi \in \mathfrak{g}$ the left- and right-invariant vector fields ξ^L and ξ^R on G defined by

$$\begin{aligned} \xi^L(g)(z) &= \frac{d}{dt}\Big|_{t=0} g\left(z \cdot \exp(t\xi)\right) \,, \\ \xi^R(g)(z) &= \frac{d}{dt}\Big|_{t=0} g\left(\exp(t\xi) \cdot z\right) \,, \qquad g \in \mathcal{O}(G),\ z \in G \,. \end{aligned} \tag{2.6}$$

Assume that G acts on a manifold M. This induces for any $\xi \in \mathfrak{g}$ the infinitesimal vector field ξ_M given by (2.3), so after extension to $\bigwedge^\bullet \mathfrak{g}$, we can form the trivector field ϕ_M.

Definition 2.1.7 *We say that the G-manifold M is a* quasi-Poisson manifold *if it is equipped with a holomorphic bivector field $P \in \bigwedge^2 T_M$ which is G-invariant and satisfies $[P,P] = \phi_M$.*

The bivector field P defines an antisymmetric biderivation $\{-,-\}$ on $\mathcal{O}_M$ such that for any $g_1, g_2, g_3 \in \mathcal{O}_M$, we have for Jac defined by (2.2) that

$$Jac(g_1, g_2, g_3) = \frac{1}{2} \phi_M(g_1, g_2, g_3) \,. \tag{2.7}$$

Definition 2.1.8 *An equivariant map $\Phi : M \to G$ for the quasi-Poisson manifold (M, P) is a* multiplicative moment map *if for all $g \in \mathcal{O}_G$,*

$$\{g \circ \Phi, -\} = \frac{1}{2} \sum_a (f^a)_M \ \left((f_a^L + f_a^R)(g) \circ \Phi\right) . \tag{2.8}$$

We call the triple (M, P, Φ) a quasi-Hamiltonian manifold*. We refer to M as a quasi-Hamiltonian manifold if P and Φ are clear from the context.*

The equality (2.8) between vector fields on M means that, when taken on some $h \in \mathcal{O}_M$, the function $\{g \circ \Phi, h\}$ evaluated at $x \in M$ is equal to

$$\frac{1}{2} \sum_a [(f^a)_M(h)](x) \ [(f_a^L + f_a^R)(g)](\Phi(x)) .$$

Here, we use the dual bases $(f_a)_a$ and $(f^a)_a$ of $\mathfrak{g}$ to define the vector fields applied to h using (2.3) and the vector fields applied to g using (2.6). What is important for us is that we can obtain new examples of quasi-Hamiltonian manifolds by fusion, and that we can reduce these manifolds to the Poisson case. We review these two results now.

Theorem 2.1.9 *[6, Proposition 5.1] Let $(M, P, (\Phi_1, \Phi_2, \Psi))$ be a quasi-Hamiltonian $G \times G \times H$-manifold. Then the diagonal map $G \to G \times G$ induces a $G \times H$ action on M such that for*

$$P_{fus} = P - \frac{1}{2} \sum_a (f_a, 0)_M \wedge (0, f^a)_M , \quad \Phi_{fus} = (\Phi_1 \Phi_2, \Psi) ,$$

the triple (M, P_{fus}, Φ_{fus}) is a quasi-Hamiltonian $G \times H$-manifold.

We call that process fusion. If we interchange the role of the two copies of G, we get that the fusion yields isomorphic structures of quasi-Hamiltonian $G \times H$-manifold by [6, Proposition 5.7].

Theorem 2.1.10 *[6, Proposition 6.1] Let (M, P, Φ) be a quasi-Hamiltonian G-manifold and $\mathcal{C}$ a conjugacy class in G. Then, if the action is free and proper on $\Phi^{-1}(\mathcal{C})$, the manifold $\Phi^{-1}(\mathcal{C})/G$ inherits a Poisson structure.*

Under mild assumptions, this theorem can be extended to any G-stable submanifold as noticed in [106, Theorem 8].

2.2 Algebraic formalism

2.2.1 Quivers and path algebras

Let $Q = (Q, I)$ be a quiver with vertex set I and arrow set Q, both assumed to be finite. Define the maps $t, h : Q \to I$ that associate to every arrow a its tail and head, $t(a)$ and $h(a)$. We define a path over Q to be a finite word $\gamma = a_1 \dots a_k$ written with letters $a_i \in Q$. We take the convention that γ represents the path going through a_1, then a_2, and so on up to a_k. This means that $\gamma = 0$ if $h(a_i) \neq t(a_{i+1})$ for some i. We also consider the trivial path e_s based at the vertex $s \in I$, for each $s \in I$. The path algebra $\mathbb{C}Q$ is the $\mathbb{C}$-vector space whose basis is given by all possible paths, and on which the multiplication is defined by concatenation : if $\gamma' = a'_1 \dots a'_l$, then $\gamma\gamma'$ is the path $a_1 \dots a_k a'_1 \dots a'_l$, and this operation is extended linearly. We get that for any $s, s' \in I$, $e_s \gamma e_{s'} = \delta_{s,t(a_1)} \delta_{h(a_k),s'} \gamma$. Moreover, the unit $1 \in \mathbb{C}Q$ decomposes as $1 = \sum_{s \in I} e_s$, that is the $(e_s)_s$ form a complete set of orthogonal idempotents.

Let $\bar{Q}$ denote the double of Q, obtained by adjoining to every arrow $a \in Q$ its opposite, a^*. We get in particular, $t(a) = h(a^*)$ and $h(a) = t(a^*)$. We extend $*$ to an involution on $\bar{Q}$ by setting $(a^*)^* = a$ for all $a \in Q$. We define $\epsilon : \bar{Q} \to \{\pm 1\}$ the sign function which associates the value $+1$ to every arrow of Q and -1 to each arrow of $\bar{Q} \setminus Q$. We write $\mathbb{C}\bar{Q}$ for the path algebra of $\bar{Q}$. We view $\mathbb{C}\bar{Q}$ as a B-algebra, with $B = \oplus_{s \in I} \mathbb{C} e_s$.

Example 2.2.1 *The Jordan quiver (or one-loop quiver) Q_0 is the quiver with vertex set $I = \{0\}$ and a unique arrow $x : 0 \to 0$. Its path algebra is the polynomial algebra in one variable $\mathbb{C}[x]$.*

The tadpole quiver Q is the extension of the Jordan quiver by one arrow. It has two vertices $\{0, \infty\}$ and consists of one loop $x : 0 \to 0$ and one arrow $v : \infty \to 0$. Its double $\bar{Q}$ has two additional arrows $y = x^* : 0 \to 0$ and $w = v^* : 0 \to \infty$, see Figure 1. Its path algebra $\mathbb{C}\bar{Q}$ is defined over $B = \mathbb{C}e_0 \oplus \mathbb{C}e_\infty$.

Example 2.2.2 *Let $m \geq 2$ be an integer. The cyclic quiver Q_m is given by the vertex set $I = \mathbb{Z}_m$ and arrows $x_s : s \to s+1$ for all $s \in \mathbb{Z}_m$. Its double consists of the additional arrows $y_s = x_s^* : s+1 \to s$.*

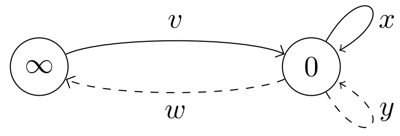

Figure 1: Double of the tadpole quiver

Next, let us introduce the notion of roots associated to a quiver. To do so, consider the Tits quadratic form associated to a quiver Q with vertex set I, which is defined by

$$q : \mathbb{Z}^I \to \mathbb{Z}\,, \quad q(\alpha) = \sum_{s\in I} \alpha_s^2 - \sum_{a\in Q} \alpha_{t(a)}\alpha_{h(a)}\,. \tag{2.9}$$

This induces a symmetric bilinear form $(-,-)$ on $\mathbb{Z}^I$ by setting $(\alpha,\beta) = q(\alpha+\beta) - q(\alpha) - q(\beta)$. For any $s \in I$, we have a corresponding element $\epsilon_s \in \mathbb{Z}^I$ such that $(\epsilon_s)_r = \delta_{sr}$. We can then form the fundamental region

$$F = \left\{\alpha \in \mathbb{N}^I \setminus \{0\} \mid (\alpha,\epsilon_s) \leq 0,\ \forall s \in I\right\} \cap \left\{\alpha \in \mathbb{N}^I \mid supp(\alpha) \text{ is connected } \right\}\,,$$

and the Weyl group W generated by the reflections

$$R_r : \mathbb{Z}^I \to \mathbb{Z}^I\,, \quad \alpha \mapsto \alpha - (\alpha,\epsilon_r)\epsilon_r\,, \qquad \text{for all } r \in I \text{ supporting no loop}\,,$$

as a subgroup of $\mathrm{Aut}(\mathbb{Z}^I)$. Then, the *real roots* are the elements in the W-orbits of $\pm\epsilon_s$ for $s \in I$ supporting no loop, while *imaginary roots* are elements in the W-orbits of $\pm\alpha$ for all $\alpha \in F$. A root is either a real root or an imaginary root. Moreover, roots are such that each root is either positive (if it belongs to $\mathbb{N}^I$) or negative (if its opposite belongs to $\mathbb{N}^I$).

Example 2.2.3 *Consider the Jordan quiver Q_0 given in Example 2.2.1. Since it consists of one vertex supporting one loop, it clearly has trivial Weyl group and no real roots. Meanwhile, we can see that its Tits form $q : \mathbb{Z} \to \mathbb{Z}$ given by (2.9) is vanishing identically. Therefore any $\alpha \in \mathbb{Z}$ satisfies $(\alpha,\epsilon_0) \leq 0$, which means that the fundamental region is $\mathbb{N}^\times$. Thus, the imaginary roots are precisely the elements of $\mathbb{Z}^\times$.*

Example 2.2.4 *Consider the cyclic quiver Q_m with $I = \mathbb{Z}_m$ given in Example 2.2.2, for some fixed integer $m \geq 2$. Using the Tits form given by (2.9), we can write that*

$$q(\alpha) = \sum_{s\in I} \alpha_s(\alpha_s - \alpha_{s+1})\,, \quad (\alpha,\epsilon_r) = 2\alpha_r - \alpha_{r+1} - \alpha_{r-1}\,,\ r,s \in I\,.$$

Introduce the element $\delta = (1,\dots,1) \in \mathbb{Z}^I$. *Clearly,* $(\delta, \epsilon_r) = 0$ *so any multiple of* δ *is an imaginary root. In fact, one can easily show that there are no other ones. If we identify* I *with* $\{0,\dots,m-1\}$, *it is a bit tedious but not complicated to show that the real roots are given by*

$$\alpha = \pm(\epsilon_i + \ldots + \epsilon_j) + k\delta\,, \quad 1 \leq i \leq j \leq m-1,\ k \in \mathbb{Z}\,.$$

For further references on quivers and their representations (that are introduced in the next section), see e.g. [49, 97]. We finish with two important constructions associated to the path algebra of a quiver. First, following Crawley-Boevey and Holland [55], we fix some $\lambda \in \mathbb{C}^I$ and denote by J_λ the two-sided ideal generated by $\left(\sum_{a\in\bar{Q}} \epsilon(a)\, aa^* - \sum_{s\in I} \lambda_s e_s\right)$ in $\mathbb{C}\bar{Q}$. The associated quotient $\Pi^\lambda(Q) = \mathbb{C}\bar{Q}/J_\lambda$ is called a *deformed preprojective algebra.*

Second, following Crawley-Boevey and Shaw [56], we consider the algebra A obtained from $\mathbb{C}\bar{Q}$ by localisation at the set of elements $(1 + aa^*)$ with $a \in \bar{Q}$. We also fix some $q \in (\mathbb{C}^\times)^I$ and a total ordering $<$ on the arrows of $\bar{Q}$. We then define the two-sided ideal J_q generated by

$$\overrightarrow{\prod_{a\in\bar{Q}}}(1 + aa^*)^{\epsilon(a)} - \sum_{s\in I} q_s e_s \ \in A\,,$$

where the elements in the product appear with respect to the ordering $<$ (so the left-most element corresponds to the smallest $a \in \bar{Q}$, and so on). The associated quotient $\Lambda^q(Q) = A/J_q$ is called a *multiplicative preprojective algebra.*

Example 2.2.5 *Consider the tadpole quiver* $\bar{Q}$ *defined in Example 2.2.1. For* $\lambda = (\lambda_0, \lambda_\infty) \in \mathbb{C}\times\mathbb{C}$, *the corresponding deformed preprojective algebra is given by*

$$\Pi^\lambda(Q) = \mathbb{C}\bar{Q}/\left\langle xy - yx - wv = \lambda_0 e_0\,,\, vw = \lambda_\infty e_\infty \right\rangle\,.$$

Denote by A *the localisation of* $\mathbb{C}\bar{Q}$ *at the elements* $1+xy, 1+yx, 1+wv, 1+vw$, *and fix the order* $x < y < v < w$. *Then, for* $q = (q_0, q_\infty) \in \mathbb{C}^\times \times \mathbb{C}^\times$, *the corresponding multiplicative preprojective algebra is given by*

$$\Lambda^q(Q) = A/\left\langle (1+xy)(1+yx)^{-1}(1+vw)(1+wv)^{-1} = q_0 e_0 + q_\infty e_\infty \right\rangle\,.$$

2.2.2 Representation spaces and GIT quotients

For a finitely generated associative algebra A over $\mathbb{C}$ and any $N \in \mathbb{N}$, the *representation space* $\operatorname{Rep}(A, N)$ is the affine scheme whose coordinate ring $\mathcal{O}(\operatorname{Rep}(A, N))$ is generated by symbols a_{ij} for $a \in A$ and $i,j = 1,\ldots,N$, such that they are linear in a and satisfy $(ab)_{ij} = \sum_k a_{ik}b_{kj}$ for any $a,b \in A$ and $1_{ij} = \delta_{ij}$. It is equivalent to see $\operatorname{Rep}(A,N)$ as parametrising algebra homomorphisms $\varrho : A \to \operatorname{Mat}_N(\mathbb{C})$, and we get that $a_{ij}(\varrho) = \varrho(a)_{ij}$ at any point $\varrho \in \operatorname{Rep}(A,N)$. It is important to remark that the definition does not depend on the chosen presentation of A, because $\operatorname{Rep}(A,N)$ represents the functor

$$\operatorname{Rep}_N^A \,:\, S \mapsto \operatorname{Hom}_{\mathbf{alg}}(A, \operatorname{Mat}_N(S))$$

from the category of finitely generated commutative algebra to the category of sets. In the spirit of [147], or more generally [57], we say that A is *smooth* (or *formally smooth* or *quasi-free*) if given any $\mathbb{C}$-algebra C and nilpotent ideal $I \subset C$, every map $\phi : A \to C/I$ can be lifted to C as $\tilde\phi : A \to C$, i.e. $\phi = \pi_I \circ \tilde\phi$ for $\pi_I : C \to C/I$ the projection map. It is equivalent to require that $\Omega_A = ker(m : A \otimes A \to A)$ is a projective A^e-module. Here, we denote by $A^e = A \otimes A^{op}$ the set of elements from $A \otimes A$ with multiplication given by $a_1a_2 = (a_1' \otimes a_1'')(a_2' \otimes a_2'') = a_1'a_2' \otimes a_2''a_1''$. Note that Ω_A can also be constructed as the module of differential 1-forms, see [57]. It was noted in [99] that if A is smooth, then all schemes $\operatorname{Rep}(A,N)$ are smooth in the usual (geometric) sense[4]. The idea is that the smoothness property in the commutative case is equivalent to the definition above for an arbitrary *commutative* $\mathbb{C}$-algebra C.

Following [162, Section 7], to any $a \in A$ we associate a matrix-valued function $\mathcal{X}(a) := (a_{ij})_{i,j=1,\ldots,N}$ on $\operatorname{Rep}(A,N)$. Then, if A is finitely generated by elements $a_1,\ldots,a_k$ subject to relations $F_l(a_1,\ldots,a_k)$ for finitely many $l = 1,\ldots,L$, we get that

$$\mathcal{O}(\operatorname{Rep}(A,N)) = \mathbb{C}[(a_1)_{ij},\ldots,(a_k)_{ij}]/\langle F_l(\mathcal{X}(a_1),\ldots,\mathcal{X}(a_k)) = 0_N \mid l = 1,\ldots,L\rangle\,.$$

In particular, if A is the free algebra on k generators, $\operatorname{Rep}(A,N)$ is just $\mathbb{A}_{\mathbb{C}}^{kN}$.

If A is a B-algebra with B of the form $B = \mathbb{C}e_1 \oplus \ldots \oplus \mathbb{C}e_K$ such that the $(e_s)_s$ form a complete set of orthogonal idempotents, we can generalise the definition to a relative setting. Representation spaces are now indexed by K-tuples $\alpha = (\alpha_1,\ldots,\alpha_K) \in \mathbb{N}^K$. Given α with $\alpha_1 + \ldots + \alpha_K = N$, we embed B diagonally into $\operatorname{Mat}_N(\mathbb{C})$ so that Id_N is split into a sum of K diagonal blocks of size $\alpha_1,\ldots,\alpha_K$, representing the idempotents e_s. This means that $\mathcal{X}(e_s)$ is the s-th diagonal identity block of size α_s in Id_N. By definition, $\operatorname{Rep}_B(A,\alpha) = \operatorname{Hom}_B(A, \operatorname{Mat}_N(\mathbb{C}))$, and it can be viewed as an affine scheme in the same way as $\operatorname{Rep}(A,N)$.

[4] For an explanatory proof, combine [84, Proposition 19.1.4] and [152, Tags 00TA,00TN].

Example 2.2.6 *Consider the path algebra* $\mathbb{C}Q$ *of a quiver* Q *defined as in* § *2.2.1. The matrix* $\mathcal{X}(a)$ *representing an element* $a \in \mathbb{C}Q$ *is an* $|I| \times |I|$ *block matrix. In the case of an arrow* $a \in Q$*, we can use the idempotents to write* $a = e_{t(a)} a e_{h(a)}$*, so* a *is represented by the matrix* $\mathcal{X}(a)$ *with at most one non-zero block of size* $\alpha_{t(a)} \times \alpha_{h(a)}$ *placed in the* $t(a)$*-th block row and* $h(a)$*-th block column. Therefore, a point in* $\mathrm{Rep}_B(\mathbb{C}Q, \alpha)$ *can be viewed as a* quiver representation*, consisting of vector spaces* $\mathcal{V}_s = \mathbb{C}^{\alpha_s}$*,* $s \in I$ *and linear maps* $X_a : \mathcal{V}_{h(a)} \to \mathcal{V}_{t(a)}$ *for each* $a \in Q$*. With this interpretation, we have*

$$X_a \in \mathrm{Mat}_{\alpha_{t(a)}, \alpha_{h(a)}}(\mathbb{C}) \,, \qquad \mathrm{Rep}_B(\mathbb{C}Q, \alpha) \cong \prod_{a \in Q} \mathrm{Mat}_{\alpha_{t(a)}, \alpha_{h(a)}}(\mathbb{C}). \tag{2.10}$$

If $A = \mathbb{C}Q/J$ *for some ideal* J *generated by elements* $(\gamma_l)_l$*, then* $\mathrm{Rep}_B(A, \alpha) \subset \mathrm{Rep}_B(\mathbb{C}Q, \alpha)$ *is the subset of matrices such that* $\mathcal{X}(\gamma_l) = 0_N$ *for each* l*.*

There is a natural action of the algebraic group $\mathrm{GL}_N(\mathbb{C})$ on $\mathrm{Rep}(A, N)$ by conjugation of matrices, and in the relative case we can embed $\mathrm{GL}_\alpha := \prod_s \mathrm{GL}_{\alpha_s}(\mathbb{C})$ diagonally into $\mathrm{GL}_N(\mathbb{C})$ to get a natural GL_α action on $\mathrm{Rep}_B(A, N)$. If the latter space is a variety, the affine GIT quotient $\mathrm{Rep}_B(A, N) /\!/ \mathrm{GL}_\alpha$, whose coordinate ring is the ring of GL_α-invariant functions $\mathcal{O}(\mathrm{Rep}_B(A, N))^{\mathrm{GL}_\alpha}$, is also a variety. Note that the latter ring is finitely generated because GL_α is reductive. The GIT quotient is the orbit space $\mathrm{Rep}_B(A, N) / \mathrm{GL}_\alpha$ provided that all orbits are closed, i.e. it is a geometric quotient. As we are interested in the complex manifold structure of such varieties, it is convenient to introduce the algebraic group $G(\alpha)$ defined as $\mathrm{GL}_\alpha / \mathbb{C}^\times$, where $\mathbb{C}^\times$ denotes the subgroup $\{\prod_s \mu \, \mathrm{Id}_{\alpha_s} \mid \mu \in \mathbb{C}^\times\}$ of diagonal matrices. Indeed, this subgroup is in the stabiliser of any representation, so the action would never be free. As $G(\alpha)$ is reductive, if it acts freely on $\mathrm{Rep}_B(A, N)$ and the latter space is smooth, then all the orbits are closed and we also get that $\mathrm{Rep}_B(A, N) /\!/ \mathrm{GL}_\alpha$ is smooth. Therefore, the GIT quotient is a complex manifold is such a case.

To understand the GIT quotient from A itself, recall that we can see $\mathrm{Rep}_B(A, N)$ as the set of (relative) representations of A. Then the orbit of $\rho \in \mathrm{Rep}_B(A, N)$ is closed if and only if ρ is a semisimple representation [96]. Thus $\mathrm{Rep}_B(A, N) /\!/ G(\alpha)$ is the set of all orbits of semisimple representations. Moreover, it follows from a theorem of Le Bruyn-Procesi [107] that the coordinate ring of the GIT quotient is generated by elements of the form $\mathrm{tr}\, \mathcal{X}(a)$ for $a \in A$ [52, 53].

Example 2.2.7 *Let Π^λ be the (deformed) preprojective algebra associated to the tadpole quiver as in Example 2.2.5, with $\lambda_0 \in \mathbb{C}^\times$ and $\lambda_\infty = -n\lambda_0$. Then, the GIT quotient $\operatorname{Rep}_B(\Pi^\lambda, N)/\!/\operatorname{GL}_n$ is nothing else than the Calogero-Moser space described in Example 2.1.4 (with parameter $\lambda_0 = 1$).*

For further references on GIT quotients, we refer to [121, 89, 97].

2.3 At the crossroads : the double world

Combining Examples 2.2.1 and 2.2.7, we see a clear link between the representations of a deformed preprojective algebra for a tadpole quiver and a natural phase space for the Calogero-Moser system. Hence, it is natural to ask if the latter integrable system could be realised already at the level of the algebra Π^λ. This was first discovered by Ginzburg [83] in the context of noncommutative symplectic geometry, and a similar approach exists using the formalism of noncommutative Poisson geometry introduced by Van den Bergh [162]. We will just quickly sketch this result, as our aim is to study the corresponding noncommutative version of *quasi-Poisson* geometry and their relation to Ruijsenaars-Schneider systems. All the results in this section already appear in [162], except when explicitly stated.

2.3.1 Double brackets and associated structures

From now on, unadorned tensor products $\otimes$ are over $\mathbb{C}$ and A denotes a finitely generated associative unital $\mathbb{C}$-algebra. We often consider A as a B-algebra, i.e. with a ring homomorphism $B \to A$ for B of the form $\mathbb{C}e_1 \oplus \ldots \oplus \mathbb{C}e_K$ where the $(e_s)_s$ form a complete set of orthogonal idempotents in A.

Generalities on double brackets

We form $A\otimes A$ and denote any element $a \in A\otimes A$ as $a = a'\otimes a''$ using Sweedler's notation, when it is necessary to know the elements in each copy of the tensor product. Hence, it is possible to define $a^\circ = a''\otimes a' \in A\otimes A$. There are two A-bimodule structures on $A\otimes A$: given $b, c \in A$, the *outer* bimodule structure is given for any $a \in A \otimes A$ as $bac = ba' \otimes a''c$, while the *inner* bimodule structure is given by $b * a * c = a'c \otimes ba''$. To avoid unnecessary confusion, we always use the notation $*$ to denote the inner bimodule structure. Hence the (outer) A-bimodule structures on A and $A \otimes A$ become left A^e-modules structure, where we recall that $A^e = A \otimes A^{op}$. Following Crawley-Boevey [50], we consider the additive group

$$\mathbb{D}\mathrm{er}(A) = \mathrm{Der}(A, A \otimes A) = \{\delta \in \mathrm{Hom}_{A^e}(A, A \otimes A) \mid \delta(bc) = \delta(b)c + b\delta(c)\} \ ,$$

which becomes an A-bimodule by inheriting the inner bimodule structure on $A \otimes A$: if $\delta \in \mathbb{D}\mathrm{er}(A)$, $b_1, b_2 \in A$, then $b_1 \delta b_2 \in \mathbb{D}\mathrm{er}(A)$ is such that for any $c \in A$,

$$(b_1 \delta b_2)(c) = b_1 * \delta(c) * b_2 = \delta(c)' b_2 \otimes b_1 \delta(c)'' \,.$$

If A is a B-algebra, we consider instead $D_{A/B} = \mathbb{D}\mathrm{er}_B(A)$, the submodule of elements $\delta \in \mathbb{D}\mathrm{er}(A)$ with $\delta(B) = 0$. In this case, an important class of double derivations is given by the *gauge elements* $(E_s)_s$ indexed by the orthogonal idempotents of B, which are given by $E_s(c) = ce_s \otimes e_s - e_s \otimes e_s c$ for any $c \in A$.

Definition 2.3.1 *A* double bracket $\{\!\{-,-\}\!\}$ *on A is a $\mathbb{C}$-bilinear map $A \times A \to A \otimes A$ such that*

(D1) for any $b, c \in A$, $\{\!\{c, b\}\!\} = -\{\!\{b, c\}\!\}^\circ$ (cyclic antisymmetry);

(D2) for any $b \in A$, $\{\!\{b, -\}\!\} \in \mathbb{D}\mathrm{er}(A)$ (outer derivation property).

Due to the $\mathbb{C}$-bilinearity, we can equivalently define a double bracket as a map $A \otimes A \to A \otimes A$ satisfying *(D1)–(D2)*. Using these two conditions, one gets

(D2') for any $b_1, b_2, c \in A$, $\{\!\{b_1 b_2, c\}\!\} = b_1 * \{\!\{b_2, c\}\!\} + \{\!\{b_1, c\}\!\} * b_2$ (inner derivation property).

In other words, a double bracket is a derivation in the second argument for the outer bimodule structure, and in the first argument for the inner bimodule structure. This implies that the double bracket is completely determined by the values it takes on generators of A. In the case where A is a B-algebra, we require the double bracket to be B-bilinear, i.e. to be $\mathbb{C}$-bilinear and to vanish if one of the entries is an element of B.

More generally, given any $n \geq 2$, we endow $A^{\otimes n}$ with the obvious outer bimodule structure. For any $\varsigma \in S_n$ and $a = a_1 \otimes \ldots \otimes a_n \in A^{\otimes n}$, we let $\tau_\varsigma\, a = a_{\varsigma^{-1}(1)} \otimes \ldots \otimes a_{\varsigma^{-1}(n)}$. Then an n-bracket $\{\!\{-, \ldots, -\}\!\} : A^{\otimes n} \to A^{\otimes n}$ is a map $\mathbb{C}$-linear in each argument such that it is cyclically antisymmetric, and it is a derivation in the last argument for the outer bimodule structure :

$$\tau_{(1\dots n)} \circ \{\!\{-,\dots,-\}\!\} \circ \tau_{(1\dots n)}^{-1} = (-1)^{n+1} \{\!\{-,\dots,-\}\!\} \,,$$
$$\{\!\{a_1,\dots,a_{n-1},bc\}\!\} = \{\!\{a_1,\dots,a_{n-1},b\}\!\}\, c + b\, \{\!\{a_1,\dots,a_{n-1},c\}\!\} \,.$$

(Note that any n-bracket for n odd is commuting with the permutations $\tau_{(1\dots n)}^k$ for $k \in \mathbb{N}$.) Clearly a double bracket is a 2-bracket, and we call a 3-bracket a *triple bracket*.

Lemma 2.3.2 ([162, 2.3.1]) *Any double bracket* $\{\!\{-,-\}\!\}$ *defines an induced triple bracket* $\{\!\{-,-,-\}\!\}$*, which is given for any* $a,b,c \in A$ *by*

$$\begin{aligned} \{\!\{a,b,c\}\!\} = & \ \{\!\{a,\{\!\{b,c\}\!\}'\}\!\} \otimes \{\!\{b,c\}\!\}'' \\ & + \tau_{(123)} \{\!\{b,\{\!\{c,a\}\!\}'\}\!\} \otimes \{\!\{c,a\}\!\}'' \\ & + \tau_{(123)}^2 \{\!\{c,\{\!\{a,b\}\!\}'\}\!\} \otimes \{\!\{a,b\}\!\}'' \,. \end{aligned} \tag{2.11}$$

In the case where the induced triple bracket identically vanishes, we say that $\{\!\{-,-\}\!\}$ is a double *Poisson* bracket, and that A is a *double Poisson algebra*. We use the same terminology in the relative case.

The double Gerstenhaber algebra of polyvector fields

The Schouten-Nijenhuis bracket given by (2.1a)–(2.1c) defines a Gerstenhaber algebra structure on $\bigwedge^\bullet T_M$, which is a graded version of a Poisson bracket. Analogously, there is a graded version of a double Poisson bracket. To introduce it, fix a graded algebra D, and write $|\alpha|$ for the degree of a homogeneous element $\alpha \in D$. There is a signed S_n action defined on $D^{\otimes n}$ as follows : for a homogeneous element $\alpha = \alpha_1 \otimes \ldots \otimes \alpha_n \in D^{\otimes n}$, and for any $\varsigma \in S_n$, we define

$$\sigma_\varsigma\, \alpha = (-1)^{t(\varsigma,\alpha)} \alpha_{\varsigma^{-1}(1)} \otimes \ldots \otimes \alpha_{\varsigma^{-1}(n)} \,, \qquad \text{where}$$
$$t(\varsigma,\alpha) = \sum_{(i,j)\in I(\varsigma)} |\alpha_{\varsigma^{-1}(i)}|\,|\alpha_{\varsigma^{-1}(j)}| \,, \quad I(\varsigma) = \left\{(i,j) \mid i<j\,,\ \varsigma^{-1}(i) > \varsigma^{-1}(j)\right\}.$$

That is, $t(\varsigma,\alpha)$ counts the graded commutation of elements when applying the (unsigned) S_n action τ_ς. Then, we say that D is a *double Gerstenhaber algebra* if it is equipped with a graded bilinear map $\{\!\{-,-\}\!\} : D \times D \to D \otimes D$ such that for any $\alpha, \beta, \gamma \in D$

$$\{\!\{\alpha,\beta\}\!\} = -\,\sigma_{(12)}(-1)^{(|\alpha|-1)(|\beta|-1)}\,\{\!\{\beta,\alpha\}\!\}\ , \tag{2.12a}$$

$$\{\!\{\alpha,\beta\gamma\}\!\} = \{\!\{\alpha,\beta\}\!\}\,\gamma + (-1)^{(|\alpha|-1)|\beta|}\beta\,\{\!\{\alpha,\gamma\}\!\}\ , \tag{2.12b}$$

$$\begin{aligned} 0 =& \left\{\!\!\left\{\alpha, \{\!\{\beta,\gamma\}\!\}'\right\}\!\!\right\} \otimes \{\!\{\beta,\gamma\}\!\}'' \\ &+ (-1)^{(|\alpha|-1)(|\beta|+|\gamma|)}\sigma_{(123)}\left\{\!\!\left\{\beta, \{\!\{\gamma,\alpha\}\!\}'\right\}\!\!\right\} \otimes \{\!\{\gamma,\alpha\}\!\}' \\ &+ (-1)^{(|\gamma|-1)(|\alpha|+|\beta|)}\sigma_{(132)}\left\{\!\!\left\{\gamma, \{\!\{\alpha,\beta\}\!\}'\right\}\!\!\right\} \otimes \{\!\{\alpha,\beta\}\!\}''\ . \end{aligned} \tag{2.12c}$$

Compare (2.12a)–(2.12c) with (2.1a)–(2.1c). We now work in a relative setting, and let $D_BA := T_AD_{A/B}$ be the tensor algebra of the bimodule $D_{A/B} = \mathbb{D}\mathrm{er}_B(A)$, with elements of A in degree 0 and relative double derivations in degree $+1$. To any $\delta_1, \delta_2 \in D_{A/B}$, we associate $\{\!\{\delta_1,\delta_2\}\!\}^\sim_l = (\delta_1 \otimes 1)\delta_2 - (1 \otimes \delta_2)\delta_1$. We also let $\{\!\{\delta_1,\delta_2\}\!\}^\sim_r = -\,\{\!\{\delta_2,\delta_1\}\!\}^\sim_l$. These are B-derivations $A \to A^{\otimes 3}$, which we can see as elements in $A \otimes D_{A/B}$ and $D_{A/B} \otimes A$.

Theorem 2.3.3 ([162, 3.2]) *There is a unique structure* $\{\!\{-,-\}\!\}_{\mathrm{SN}}$ *of double Gerstenhaber algebra on* D_BA *which satisfies for any* $b, c \in A$, $\delta_1, \delta_2 \in D_{A/B}$,

$$\{\!\{b,c\}\!\}_{\mathrm{SN}} = 0\,, \quad \{\!\{\delta_1,b\}\!\}_{\mathrm{SN}} = \delta_1(b)\,, \quad \{\!\{\delta_1,\delta_2\}\!\}_{\mathrm{SN}} = \tau_{(23)}\,\{\!\{\delta_1,\delta_2\}\!\}^\sim_l + \tau_{(12)}\,\{\!\{\delta_1,\delta_2\}\!\}^\sim_r\ .$$

We call $\{\!\{-,-\}\!\}_{\mathrm{SN}}$ the *double Schouten-Nijenhuis bracket* on D_BA. The algebra D_BA is useful because it easily gives examples of n-brackets as follows.

Proposition 2.3.4 ([162, 4.1.1]) *For any* $n \in \mathbb{N}^\times$, *there is a well-defined linear map*

$$\mu : (D_BA)_n \to \{B\text{-linear } n\text{-brackets on } A\} \ :\ Q \mapsto \{\!\{-,\ldots,-\}\!\}_Q\ ,$$

which on $Q = \delta_1 \ldots \delta_n$ *is given by*

$$\{\!\{-,\ldots,-\}\!\}_Q = \sum_{i=0}^{n-1}(-1)^{(n-1)i}\tau^i_{(1\ldots n)} \circ \{\!\{-,\ldots,-\}\!\}^\sim_Q \circ \tau^{-i}_{(1\ldots n)}\ ,$$

$$\{\!\{a_1,\ldots,a_n\}\!\}^\sim_Q = \delta_n(a_n)'\delta_1(a_1)'' \otimes \delta_1(a_1)'\delta_2(a_2)'' \otimes \ldots \otimes \delta_{n-1}(a_{n-1})'\delta_n(a_n)''\,.$$

Moreover, the map μ *factors through* $D_BA/[D_BA, D_BA]$ *(for the graded commutator).*

For $n = 2$, we obtain a double bracket $\{\!\{-,-\}\!\}_{\delta_1\delta_2}$ from $\delta_1\delta_2 \in (D_BA)_2$, which is defined for any $b, c \in A$ by

$$\{\!\{b,c\}\!\}_{\delta_1\delta_2} = \delta_2(c)'\delta_1(b)'' \otimes \delta_1(b)'\delta_2(c)'' - \delta_1(c)'\delta_2(b)'' \otimes \delta_2(b)'\delta_1(c)''\,. \tag{2.13}$$

We say that an n-bracket is *differential* if it is defined from some $Q \in (D_B A)_n$ using Proposition 2.3.4. This has the following consequence : if a double bracket $\{\!\{-,-\}\!\}$ is differential for $P \in (D_B A)_2$, then the associated triple bracket $\{\!\{-,-,-\}\!\}$ given by (2.11) is differential for $\frac{1}{2}\{P,P\}_{\mathrm{SN}} \in (D_B A)_3$, where $\{-,-\}_{\mathrm{SN}} = m \circ \{\!\{-,-\}\!\}_{\mathrm{SN}}$. In the case where A is formally smooth over B (see § 2.2.2 adapting the definition of formal smoothness for B-algebras A, C) and A is both left and right flat over B, then the map μ in Proposition 2.3.4 is an isomorphism [162, 4.1.2]. In such a case, all n-brackets are differential.

Associated brackets

For any n-bracket $\{\!\{-,\dots,-\}\!\}$, we have an *associated bracket*

$$\{-,\dots,-\} = m \circ \{\!\{-,\dots,-\}\!\} : A^{\otimes n} \to A \tag{2.14}$$

obtained by multiplication (i.e. concatenation of factors). In the case of a double bracket,

$$\{b,c\} = m \circ \{\!\{b,c\}\!\} = \{\!\{b,c\}\!\}' \{\!\{b,c\}\!\}'' \,. \tag{2.15}$$

We can easily see that this operation satisfies Leibniz rule in the second argument : $\{b, c_1c_2\} = \{b,c_1\}c_2 + c_1\{b,c_2\}$ for any $b, c_1, c_2 \in A$. Now, assume that the double bracket $\{\!\{-,-\}\!\}$ is such that the bracket $m \circ \{\!\{-,-,-\}\!\}$ associated to the induced triple bracket $\{\!\{-,-,-\}\!\}$ given by (2.11) vanishes. Then the bracket $\{-,-\}$ associated to the double bracket is a *left Loday bracket*, i.e. $\{-,-\} : A \times A \to A$ is a bilinear map such that

$$\{a,\{b,c\}\} = \{\{a,b\},c\} + \{b,\{a,c\}\} \,. \tag{2.16}$$

Note that it is still a derivation in the second argument. The bracket descends to a well-defined map $A/[A,A] \times A \to A$. Moreover, we can consider that map modulo commutators, so that it yields an antisymmetric map on the vector space $A/[A,A]$. Hence $(A/[A,A], \{-,-\})$ is a Lie algebra [162, 2.4]. In fact this operation is stronger than being just a Lie bracket. In [52, 53], Crawley-Boevey introduced the notion of a H_0-*Poisson structure* on an algebra A, which is a Lie bracket $\langle -,-\rangle$ on $A/[A,A]$ such that for each $a \in A$ the map $\langle \bar{a}, -\rangle$ on $A/[A,A]$ (where $\bar{a}$ is the projection of a) is induced by a derivation $d_a : A \to A$. Then, under the above assumption, the associated bracket $\{-,-\}$ induces a H_0-Poisson structure on A.

Finally, note that associated brackets can also be defined in the graded setting. We used this fact after Proposition 2.3.4 when we introduced the bracket $\{-,-\}_{\mathrm{SN}} = m \circ \{\!\{-,-\}\!\}_{\mathrm{SN}}$ associated to the double Schouten-Nijenhuis bracket.

2.3.2 Hamiltonian and quasi-Hamiltonian algebras

From now on, we require that A is a B-algebra for $B = \oplus_{s=1}^{K} \mathbb{C} e_s$, with $K \in \mathbb{N}^{\times}$.

Hamiltonian algebras

Recall that $(A, \{\!\{-,-\}\!\})$ is a double Poisson algebra when the triple bracket given by (2.11) vanishes. In the case where the double bracket is differential and defined from an element $P \in (D_B A)_2$ by Proposition 2.3.4, this is equivalent to require $\{P,P\}_{\mathrm{SN}} = 0$ modulo $[D_B A, D_B A]$, and we say that (A,P) is a *differential* double Poisson algebra. A *moment map* $\mu \in A$ is an element $\mu = \sum_{s=1}^{K} \mu_s$ where $\mu_s \in e_s A e_s$ satisfies $\{\!\{\mu_s, c\}\!\} = E_s(c)$ for all $c \in A$. Such a triple $(A, \{\!\{-,-\}\!\}, \mu)$, or (A,P,μ), is called a *Hamiltonian algebra*.

An important class of Hamiltonian algebras is given by quivers. To state the result, we use the notations of § 2.2.1 and introduce for any $a \in \bar{Q}$ the double derivation $\frac{\partial}{\partial a} \in \mathbb{D}\mathrm{er}(\mathbb{C}\bar{Q})$ given by

$$\frac{\partial b}{\partial a} = \begin{cases} e_{t(a)} \otimes e_{h(a)} & \text{if } a = b \in \bar{Q}\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{2.17}$$

Theorem 2.3.5 ([162, 6.3.1]) *The path algebra $A = \mathbb{C}\bar{Q}$ is Hamiltonian for*

$$P = \sum_{a \in Q} \frac{\partial}{\partial a^*} \frac{\partial}{\partial a}\,, \quad \mu = \sum_{a \in Q} [a, a^*]\,.$$

Quasi-Hamiltonian algebras

Let $\{\!\{-,-\}\!\}$ be a double bracket on A. We say that it is a *double quasi-Poisson bracket* when

$$\{\!\{-,-,-\}\!\} = \frac{1}{12} \sum_{s=1}^{K} \{\!\{-,-,-\}\!\}_{E_s^3}\,. \tag{2.18}$$

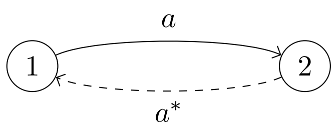

Figure 2: The simplest quiver $\bar{Q}_0$.

Here, the left-hand side is the triple bracket associated to $\{\!\{-,-\}\!\}$ by (2.11), while the right-hand side is a linear combination of the triple brackets defined by $E_s^3 \in (D_BA)_3$ using Proposition 2.3.4. A simple but tedious computation shows that for any $a,b,c \in A$

$$\begin{aligned}\{\!\{a,b,c\}\!\}_{E_s^3} =& 3\Big(ce_sa \otimes e_sb \otimes e_s - ce_sa \otimes e_s \otimes be_s - ce_s \otimes ae_sb \otimes e_s + ce_s \otimes ae_s \otimes be_s \\ & - e_sa \otimes e_sb \otimes e_sc + e_sa \otimes e_s \otimes be_sc + e_s \otimes ae_sb \otimes e_sc - e_s \otimes ae_s \otimes be_sc\Big)\,.\end{aligned}$$

If $\{\!\{-,-\}\!\}$ is differential and defined from an element $P \in (D_BA)_2$ by Proposition 2.3.4, the condition is equivalent to $\{\!\{P,P\}\!\}_{\text{SN}} = \frac{1}{6}\sum_{s=1}^K E_s^3$ modulo $[D_BA, D_BA]$, and we say that (A,P) is a *differential* double quasi-Poisson bracket. A *multiplicative moment map* (or moment map when it is clear from the context that we talk about the quasi-Poisson case) is an element $\Phi = \sum_{s=1}^K \Phi_s$ for $\Phi_s \in e_sAe_s$ such that, for all $c \in A$, we have $\{\!\{\Phi_s, c\}\!\} = \frac{1}{2}(\Phi_sE_s + E_s\Phi_s)(c)$. We can rewrite this condition as

$$\{\!\{\Phi_s, c\}\!\} = \frac{1}{2}(ce_s \otimes \Phi_s - e_s \otimes \Phi_sc + c\Phi_s \otimes e_s - \Phi_s \otimes e_sc)\,. \tag{2.19}$$

Such a triple $(A, \{\!\{-,-\}\!\}, \Phi)$, or (A,P,Φ), is called a quasi-Hamiltonian algebra.

Example 2.3.6 ([162, 6.5.1]) *Let $\bar{Q}_0$ be the quiver with two vertices $\{1,2\}$ and two arrows $a : 1 \to 2$, $a^* : 2 \to 1$, see Figure 2. Let A_0 be the algebra obtained from the path algebra $\mathbb{C}\bar{Q}_0$ by inverting the elements $(1+aa^*), (1+a^*a)$. Then, defining double derivations $\frac{\partial}{\partial a}, \frac{\partial}{\partial a^*}$ as in* (2.17)*, A_0 is quasi-Hamiltonian for*

$$P = \frac{1}{2}(1+a^*a)\frac{\partial}{\partial a}\frac{\partial}{\partial a^*} - \frac{1}{2}(1+aa^*)\frac{\partial}{\partial a^*}\frac{\partial}{\partial a}\,, \quad \Phi = (1+aa^*)(1+a^*a)^{-1}\,.$$

Remark 2.3.7 *Due to the idempotent decomposition $1 = e_1 + e_2$, $e_se_t = \delta_{st}e_t$, we can equivalently define A_0 as the algebra obtained from the path algebra $\mathbb{C}\bar{Q}_0$ by adding local inverses to $(e_1 + aa^*)$ and $(e_2 + a^*a)$. The latter means that we add to $\mathbb{C}\bar{Q}_0$ the element*

$(e_1 + aa^*)^{-1}$ *such that* $(e_1 + aa^*)^{-1}(e_1 + aa^*) = e_1 = (e_1 + aa^*)(e_1 + aa^*)^{-1}$ *and do the same for* $(e_2 + a^*a)$*. Indeed, we can use the relations* $(1 + aa^*)^{-1} = e_2 + (e_1 + aa^*)^{-1}$ *and* $(1 + a^*a)^{-1} = e_1 + (e_2 + a^*a)^{-1}$ *to go from one description to the other. In particular, note that we can write* $\Phi_1 = e_1 + aa^*$, $\Phi_2 = (e_2 + a^*a)^{-1}$.

We will need a final notion to characterise quasi-Hamiltonian algebras. We assume that A is formally smooth over B, and let Ω_A be the A^e-module generated by symbols db for $b \in A$, such that $d(b_1 b_2) = b_1.db_2 + db_1.b_2$ for any $b_1, b_2 \in A$, together with $db = 0$ if $b \in B$. This is an alternative definition of the module of non-commutative differential 1-forms (relative to B) of [57]. Slightly adapting [163], we say that the double quasi-Poisson bracket is *non-degenerate* if the map $\Omega_A \oplus (\oplus_s A E_s A) \to D_{A/B} : (a.db.c, \delta) \mapsto a \{\!\{b, -\}\!\} c + \delta$ is surjective. By [163, 8.3.1], the double bracket of Example 2.3.6 is non-degenerate.

Fusion for quasi-Hamiltonian algebras

In the Hamiltonian case, one can easily obtain Theorem 2.3.5 for arbitrary $\bar{Q}$ beginning with the particular case of the quiver $\bar{Q}_0$ given in Example 2.3.6, then obtaining the general case by 'gluing' a disjoint union of copies of $\bar{Q}_0$ to construct $\bar{Q}$. Indeed, we just need to sum the initial double Poisson brackets and moment maps together using [162, 2.5.1]. This can also easily be remarked for their geometric counterparts, with Hamiltonian manifolds. However, we have from Theorem 2.1.9 that we can not simply sum together bivectors and moment maps in the quasi-Poisson case. Hence, that theorem needs to be translated at the algebra level first.

We now recall the fusion process [162, 5.3], starting with a *differential* quasi-Hamiltonian algebra (A, P, Φ). The author has been able to show that the construction also works in the general case, which was expected by Van den Bergh. However, it would be too long to reproduce these results here, so we prefer the shorter version of [162, 5.3].

Assume that (A, P, Φ) is a quasi-Hamiltonian algebra over $B = \oplus_{s=1}^{K} \mathbb{C} e_s$, so that the $(e_s)_s$ are a complete set of orthogonal idempotents. We want to obtain an algebra from A by identifying the idempotents e_1 and e_2. (For a path algebra $\mathbb{C}\bar{Q}$, this amounts to glue vertices $1, 2$ in the underlying quiver.) To do so, recall that given algebras A, A' over B with algebra monomorphisms $i : B \to A$ and $j : B \to A'$, we define the *free algebra* $A *_B A' = T_k(A \oplus A')/I$, where I is the two-sided ideal generated by the relations $a_1 \otimes a_2 = a_1 a_2$, $a'_1 \otimes a'_2 = a'_1 a'_2$, $i(b) = j(b)$ for all $a_1, a_2 \in A$, $a'_1, a'_2 \in A$ and $b \in B$. Our first step is to construct the *extension algebra* $\bar{A}$ of A along e_1, e_2, which is given by

$$\bar{A} = A *_{\mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}\mu} (\mathrm{Mat}_2(\mathbb{C}) \oplus \mathbb{C}\mu) = A *_B \bar{B} \,, \tag{2.20}$$

where $\mu = 1 - e_1 - e_2$, and $\mathrm{Mat}_2(\mathbb{C})$ is seen as the $\mathbb{C}$-algebra generated by $e_1 = e_{11}, e_{12}, e_{21}, e_2 = e_{22}$ with $e_{st}e_{uv} = \delta_{tu}e_{sv}$. We can embed elements of A in $\bar{A}$, and any element of $D_B A$ extends to $D_{\bar{B}}\bar{A}$. This applies to $\Phi = \sum_s \Phi_s$ and P. For the second step, the *fusion algebra* A^f of A along e_1, e_2 is the algebra obtained from $\bar{A}$ by discarding elements of $e_2\bar{A} + \bar{A}e_2$, i.e.

$$A^f = \epsilon \bar{A} \epsilon \,, \qquad \text{for } \epsilon = 1 - e_2 \,. \tag{2.21}$$

We can now view A^f as a B^f-algebra, and we identify B^f with $\oplus_{s \neq 2} \mathbb{C}e_s$. In this case, an element $\bar{a} \in \bar{A}$ descends to A^f using the *trace map* Tr, given by $\mathrm{Tr}\, \bar{a} = \epsilon \bar{a} \epsilon + \epsilon e_{12} \bar{a} e_{21} \epsilon$. We also get a map $D_{\bar{B}}\bar{A} \to D_{B^f}A^f$ given by $\bar{Q} \mapsto \mathrm{Tr}\, \bar{Q} := \epsilon \bar{Q} \epsilon + \epsilon e_{12} \bar{Q} e_{21} \epsilon$. Composing with the previous map, the elements $\Phi_s \in A$ and $P \in (D_B A)_2$ define elements $\Phi_s^f \in A^f$ and $P^f \in (D_{B^f} A^f)_2$.

Theorem 2.3.8 ([162, 5.3.2]) *The B^f-algebra A^f is quasi-Hamiltonian for (P^{ff}, Φ^{ff}) given by*

$$P^{ff} = P^f - \frac{1}{2} E_1^f E_2^f \,, \qquad \Phi_1^{ff} = \Phi_1^f \Phi_2^f \,, \quad \Phi_s^{ff} = \Phi_s^f \,, \ s \neq 1, 2.$$

Note that we multiply the moment maps of the fused idempotents. Hence, the quasi-Hamiltonian structure is different if we glue idempotents in different orders. For the localised path algebra $A = \mathbb{C}\bar{Q}_{(1+aa^*)_{a \in \bar{Q}}}$ of the double $\bar{Q}$ of a quiver Q, this amounts to consider an ordering at each vertex on the arrows of $\bar{Q}$ whose tails meet at that vertex. We can now combine Example 2.3.6 and Theorem 2.3.8.

Theorem 2.3.9 ([162, 6.7.1]) *Let $\bar{Q}$ be a double quiver. For each vertex $s \in I$, consider an ordering $<_s$ on the set $T_s = \{a \in \bar{Q} \mid t(a) = s\}$. The algebra A obtained from $\mathbb{C}\bar{Q}$ by adding local inverses $(e_{t(a)} + aa^*)$ (see Remark 2.3.7) is quasi-Hamiltonian for*

$$\begin{aligned} \mathrm{P} = \frac{1}{2} \sum_{a \in \bar{Q}} \epsilon(a)(e_{h(a)} + a^* a) \frac{\partial}{\partial a} \frac{\partial}{\partial a^*} \\ - \frac{1}{2} \sum_{s \in I} \sum_{a <_s b \in T_s} \left(\frac{\partial}{\partial a^*} a^* - a \frac{\partial}{\partial a} \right) \left(\frac{\partial}{\partial b^*} b^* - b \frac{\partial}{\partial b} \right) , \end{aligned} \tag{2.22a}$$

$$\Phi = \sum_s \Phi_s \,, \quad \Phi_s = \overrightarrow{\prod_{a \in T_s}} (e_s + aa^*)^{\epsilon(a)} \,. \tag{2.22b}$$

As the fusion of non-degenerate double quasi-Poisson brackets is also non-degenerate by [163, 8.2], Theorem 2.3.9 defines a non-degenerate quasi-Hamiltonian structure on A.

Remark 2.3.10 *When the ordering $<_s$ at each vertex $s \in I$ comes from a total ordering $<$ on $\bar{Q}$, the algebra $\Lambda^q = A/(\Phi - q)$, $q \in B^\times$, is just an example of multiplicative preprojective algebra, see § 2.2.1. This algebra is the motivation behind Van den Bergh's work [162], where he successfully interpreted the element Φ as a (multiplicative) moment map. We will also call the quotient $\Lambda^q = A/(\Phi - q)$ a multiplicative preprojective algebra for any orderings $(<_s)_{s\in I}$. Indeed, we can always construct a total ordering from it by choosing a total ordering $<_I$ on I, before setting $a < b$ if $t(a) <_I t(b)$, or if $a <_s b$ when $s = t(a) = t(b)$.*

Additional results for quivers

As in Theorem 2.3.9, consider an ordering $<_s$ on the set $T_s = \{a \in \bar{Q} \mid t(a) = s\}$ for each vertex s. Write $o_s(-,-)$ for the corresponding ordering function on T_s, which we recall is defined on arrows $a, b \in T_s$ by $o_s(a,b) = +1$ if $a <_s b$, $o_s(a,b) = -1$ if $b <_s a$ and is zero otherwise. We extend this function to $\bar{Q}$ by putting $o_s(a,b) = 0$ when $a \notin T_s$ or $b \notin T_s$. We can prove the following, which explicitly gives the double bracket in terms of generators.

Proposition 2.3.11 *The biderivation of Theorem 2.3.9 takes the following form on arrows of $\bar{Q}$*

$$\{\!\{a, a\}\!\} = \frac{1}{2} o_{t(a)}(a, a^*) \left(a^2 \otimes e_{t(a)} - e_{h(a)} \otimes a^2\right) \qquad (a \in \bar{Q}), \tag{2.23}$$

$$\begin{aligned} \{\!\{a, a^*\}\!\} =& e_{h(a)} \otimes e_{t(a)} + \frac{1}{2} a^* a \otimes e_{t(a)} + \frac{1}{2} e_{h(a)} \otimes a a^* \\ &+ \frac{1}{2} o_{t(a)}(a, a^*) \left(a^* \otimes a - a \otimes a^*\right) \qquad (a \in Q), \end{aligned} \tag{2.24}$$

and for $a, b \in \bar{Q}$ such that $b \neq a, a^*$

$$\begin{aligned} \{\!\{a, b\}\!\} =& \frac{1}{2} o_{h(a)}(a^*, b)\, e_{h(a)} \otimes ab + \frac{1}{2} o_{t(a)}(a, b^*)\, ba \otimes e_{t(a)} \\ &- \frac{1}{2} o_{h(a)}(a^*, b^*)\, (b \otimes a) - \frac{1}{2} o_{t(a)}(a, b)\, (a \otimes b)\,. \end{aligned} \tag{2.25}$$

The result has interesting consequences. First, $\{\!\{a, a\}\!\} = 0$ whenever $a \in \bar{Q}$ is not a loop. Second, if $b \neq a, a^*$, then $\{\!\{a, b\}\!\} = 0$ whenever a and b do not share a common vertex. Third, we have that

(2.23)–(2.24) define all double brackets in A by use of the cyclic antisymmetry and the derivation properties.

We skip the proof of Proposition 2.3.11 for a moment, and consider a specific choice which will be of interest to us and appears as [41, Proposition 2.6]. To state this particular case, we fix for each $s \in I$ a total ordering $<_s$ on the set $M_s = \{a \in \bar{Q} \mid t(a) = s \text{ or } h(a) = s\}$. We assume that for any two arrows a, b that meet at two different vertices r, s, we have either $a <_r b, a <_s b$ or we have $b <_r a, b <_s a$. As in [62], we define the collection of all these total orderings as an *ordering* on $\bar{Q}$. We simply denote this collection as $<$, since given any arrows a, b that share at least one vertex, either $a < b$ or $b < a$.

Proposition 2.3.12 *Take an ordering in $\bar{Q}$ so that the arrows of $\bar{Q}$ are ordered in such a way that $a < a^* < b < b^*$ for any $a, b \in Q$ with $a < b$. Then one has*

$$\begin{aligned}
\{\!\{a,a\}\!\} &= \frac{1}{2}\epsilon(a)\left(a^2 \otimes e_{t(a)} - e_{h(a)} \otimes a^2\right) && (a \in \bar{Q})\,, && \text{(2.26a)}\\
\{\!\{a,a^*\}\!\} &= e_{h(a)} \otimes e_{t(a)} + \frac{1}{2}a^*a \otimes e_{t(a)} + \frac{1}{2}e_{h(a)} \otimes aa^* \\
&\quad + \frac{1}{2}(a^* \otimes a - a \otimes a^*)\delta_{h(a),t(a)} && (a \in Q)\,, && \text{(2.26b)}\\
\{\!\{a,b\}\!\} &= \frac{1}{2}e_{h(a)} \otimes ab + \frac{1}{2}ba \otimes e_{t(a)} \\
&\quad - \frac{1}{2}(b \otimes a)\delta_{h(a),h(b)} - \frac{1}{2}(a \otimes b)\delta_{t(a),t(b)} && (a,b \in \bar{Q}\,,\ a<b\,,\ b \neq a^*)\,. && \text{(2.26c)}
\end{aligned}$$

We now proceed to the proof of Proposition 2.3.11. We need the following lemma.

Lemma 2.3.13 *For $\alpha \in \bar{Q}$, define the double derivations*

$$U_\alpha = \frac{\partial}{\partial \alpha}, \qquad U_\alpha^+ = \frac{\partial}{\partial \alpha}\alpha, \qquad U_\alpha^- = \alpha\frac{\partial}{\partial \alpha}, \qquad \bar{U}_\alpha = \alpha^*\alpha\frac{\partial}{\partial \alpha}\,.$$

Then they vanish on $\beta \in \bar{Q} \setminus \{\alpha\}$ while

$$U_\alpha(\alpha) = e_{t(\alpha)} \otimes e_{h(\alpha)}, \quad U_\alpha^+(\alpha) = \alpha \otimes e_{h(\alpha)}, \quad U_\alpha^-(\alpha) = e_{t(\alpha)} \otimes \alpha, \quad \overline{U}_\alpha(\alpha) = e_{t(\alpha)} \otimes \alpha^*\alpha\,.$$

Proof. Equation (2.17) gives for any $\alpha \in \bar{Q}$,

$$\begin{aligned}
U_\alpha(\alpha) &= \left(\frac{\partial}{\partial \alpha}\right)(\alpha) = \frac{\partial \alpha}{\partial \alpha} = e_{t(\alpha)} \otimes e_{h(\alpha)}\,, \\
U_\alpha^+(\alpha) &= \left(\frac{\partial}{\partial \alpha}\alpha\right)(\alpha) = \frac{\partial \alpha}{\partial \alpha} * \alpha = e_{t(\alpha)}\alpha \otimes e_{h(\alpha)} = \alpha \otimes e_{h(\alpha)}\,, \\
U_\alpha^-(\alpha) &= \left(\alpha\frac{\partial}{\partial \alpha}\right)(\alpha) = \alpha * \frac{\partial \alpha}{\partial \alpha} = e_{t(\alpha)} \otimes \alpha e_{h(\alpha)} = e_{t(\alpha)} \otimes \alpha\,, \\
\overline{U}_\alpha(\alpha) &= \left(\alpha^*\alpha\frac{\partial}{\partial \alpha}\right)(\alpha) = \alpha^*\alpha * \frac{\partial \alpha}{\partial \alpha} = e_{t(\alpha)} \otimes \alpha^*\alpha e_{h(\alpha)} = e_{t(\alpha)} \otimes \alpha^*\alpha\,,
\end{aligned}$$

as desired. □

Proof. (Proposition 2.3.11.) We use Proposition 2.3.4 in the form of (2.13) together with its linearity. We first derive (2.23) and (2.24). To do so, consider the following terms of P

$$\begin{aligned}
\mathrm{P}_\alpha =& \frac{1}{2}\epsilon(\alpha)(1+\alpha^*\alpha)\frac{\partial}{\partial \alpha}\frac{\partial}{\partial \alpha^*} + \frac{1}{2}\epsilon(\alpha^*)(1+\alpha\alpha^*)\frac{\partial}{\partial \alpha^*}\frac{\partial}{\partial \alpha} \\
&- \frac{1}{2}\delta_{t(\alpha),h(\alpha)}\left(\frac{\partial}{\partial \alpha^*}\alpha^* - \alpha\frac{\partial}{\partial \alpha}\right)\left(\frac{\partial}{\partial \alpha}\alpha - \alpha^*\frac{\partial}{\partial \alpha^*}\right)\,,
\end{aligned}$$

for any $\alpha \in \bar{Q}$ such that either $\alpha <_{t(\alpha)} \alpha^*$ if $t(\alpha) = h(\alpha)$, or $\alpha \in Q$ otherwise. The biderivation P_α contains all the terms of P that are not trivially zero once evaluated on any two elements of the set $\{\alpha, \alpha^*\} \subset \bar{Q}$. Also, the last term is nonzero only if α is a loop. Clearly if α is not a loop, $\{\!\{\alpha, \alpha\}\!\} = \{\!\{\alpha^*, \alpha^*\}\!\} = 0$ which proves (2.23) for $a = \alpha$ in that case. Otherwise, if a is a loop we compute using (2.13) and Lemma 2.3.13 that

$$\begin{aligned}
&\{\!\{\alpha, \alpha\}\!\} = \{\!\{\alpha, \alpha\}\!\}_{\mathrm{P}_\alpha} = \{\!\{\alpha, \alpha\}\!\}_{\frac{1}{2}U_\alpha^- U_\alpha^+} \\
=&\frac{1}{2}\left(U_\alpha^+(\alpha)'U_\alpha^-(\alpha)'' \otimes U_\alpha^-(\alpha)'U_\alpha^+(\alpha)''\right) - \frac{1}{2}\left(U_\alpha^-(\alpha)'U_\alpha^+(\alpha)'' \otimes U_\alpha^+(\alpha)'U_\alpha^-(\alpha)''\right) \\
=&\frac{1}{2}\alpha^2 \otimes e_{t(\alpha)} - \frac{1}{2}e_{t(\alpha)} \otimes \alpha^2\,,
\end{aligned}$$

where we use in the last equality $e_{t(\alpha)}e_{h(\alpha)} = e_{t(\alpha)} = e_{h(\alpha)}e_{t(\alpha)}$. Similarly,

$$\begin{aligned}
&\{\!\{\alpha^*, \alpha^*\}\!\} = \{\!\{\alpha^*, \alpha^*\}\!\}_{\mathrm{P}_\alpha} = \{\!\{\alpha^*, \alpha^*\}\!\}_{\frac{1}{2}U_{\alpha^*}^+ U_{\alpha^*}^-} \\
=&\frac{1}{2}e_{t(\alpha^*)} \otimes (\alpha^*)^2 - \frac{1}{2}(\alpha^*)^2 \otimes e_{t(\alpha^*)}\,.
\end{aligned}$$

To get (2.23), we take $\alpha = a$ if $a <_{t(a)} a^*$, and $\alpha = a^*$ if $a^* <_{t(a)} a$.

Next, we compute (again with Lemma 2.3.13) since $\epsilon(\alpha^*) = -\epsilon(\alpha)$

$$\begin{aligned}
&\{\!\{\alpha,\alpha^*\}\!\} = \{\!\{\alpha,\alpha^*\}\!\}_{\mathrm{P}_\alpha} \\
=&\frac{1}{2}\epsilon(\alpha)\,\{\!\{\alpha,\alpha^*\}\!\}_{U_\alpha U_{\alpha^*}} + \frac{1}{2}\epsilon(\alpha)\,\{\!\{\alpha,\alpha^*\}\!\}_{\bar U_\alpha U_{\alpha^*}} - \frac{1}{2}\epsilon(\alpha)\,\{\!\{\alpha,\alpha^*\}\!\}_{U_{\alpha^*} U_\alpha} \\
&- \frac{1}{2}\epsilon(\alpha)\,\{\!\{\alpha,\alpha^*\}\!\}_{\bar U_{\alpha^*} U_\alpha} - \frac{1}{2}\,\{\!\{\alpha,\alpha^*\}\!\}_{U^+_{\alpha^*} U^+_\alpha} - \frac{1}{2}\,\{\!\{\alpha,\alpha^*\}\!\}_{U^-_\alpha U^-_{\alpha^*}} \\
=&\epsilon(\alpha)\, e_{h(\alpha)} \otimes e_{t(\alpha)} + \frac{1}{2}\epsilon(\alpha)\,\alpha^*\alpha \otimes e_{t(\alpha)} + \frac{1}{2}\epsilon(\alpha)\, e_{h(\alpha)} \otimes \alpha\alpha^* \\
&+ \frac{1}{2}\delta_{h(\alpha),t(\alpha)}(\alpha^* \otimes \alpha - \alpha \otimes \alpha^*)\,.
\end{aligned}$$

If a is not a loop, or if a is a loop such that $a <_{t(a)} a^*$, we take $\alpha = a$ in (2.28) to get (2.24). If a is a loop with $a^* <_{t(a)} a$, we take $\alpha = a^*$ in (2.28) to get

$$\{\!\{a^*,a\}\!\} = -e_{t(a)} \otimes e_{h(a)} - \frac{1}{2}(aa^* \otimes e_{h(a)} + e_{t(a)} \otimes a^*a) + \frac{1}{2}(a \otimes a^* - a^* \otimes a)\,.$$

Using $\{\!\{a,a^*\}\!\} = -\,\{\!\{a^*,a\}\!\}^\circ$, we find (2.24) in this last case.

Finally, to derive (2.25) we remark that P is the sum of all terms P_α as above and biderivations

$$\mathrm{P}_{\alpha,\beta} = -\,\frac{1}{2}\left(\frac{\partial}{\partial\alpha^*}\alpha^* - \alpha\frac{\partial}{\partial\alpha}\right)\left(\frac{\partial}{\partial\beta^*}\beta^* - \beta\frac{\partial}{\partial\beta}\right),$$

for all α,β, where $\alpha \neq \beta^*$, such that $t(\alpha) = t(\beta)$ with $\alpha <_{t(\alpha)} \beta$. The contribution of that biderivation is such that

$$\{\!\{\alpha,\beta\}\!\}_{\mathrm{P}_{\alpha,\beta}} = -\frac{1}{2}\,\{\!\{\alpha,\beta\}\!\}_{U^-_\alpha U^-_\beta} = -\frac{1}{2}\alpha \otimes \beta\,, \tag{2.28a}$$

$$\{\!\{\alpha^*,\beta\}\!\}_{\mathrm{P}_{\alpha,\beta}} = \frac{1}{2}\,\{\!\{\alpha^*,\beta\}\!\}_{U^+_{\alpha^*} U^-_\beta} = \frac{1}{2}e_{t(\alpha)} \otimes \alpha^*\beta\,, \tag{2.28b}$$

$$\{\!\{\alpha,\beta^*\}\!\}_{\mathrm{P}_{\alpha,\beta}} = \frac{1}{2}\,\{\!\{\alpha,\beta^*\}\!\}_{U^-_\alpha U^+_{\beta^*}} = \frac{1}{2}\beta^*\alpha \otimes e_{t(\alpha)}\,, \tag{2.28c}$$

$$\{\!\{\alpha^*,\beta^*\}\!\}_{\mathrm{P}_{\alpha,\beta}} = -\frac{1}{2}\,\{\!\{\alpha^*,\beta^*\}\!\}_{U^+_{\alpha^*} U^+_{\beta^*}} = -\frac{1}{2}\beta^* \otimes \alpha^*\,. \tag{2.28d}$$

We now have to check (2.25) for any $a,b \in \bar Q$, $b \neq a,a^*$. If $t(a) = t(b)$, either $a <_{t(a)} b$ and by (2.28a) we get a term $-\frac{1}{2}a \otimes b$ in $\{\!\{a,b\}\!\}$, or $b <_{t(a)} a$ and we get a term $-\frac{1}{2}b \otimes a$ in $\{\!\{b,a\}\!\}$, so by cyclic antisymmetry we have a term $\frac{1}{2}a \otimes b$ in $\{\!\{a,b\}\!\}$. Hence we have a term $-\frac{1}{2}o_{t(a)}(a,b)\,(a \otimes b)$ appearing in $\{\!\{a,b\}\!\}$. If $h(a) = h(b)$, either $a^* <_{h(a)} b^*$ and by (2.28d) we get a term $-\frac{1}{2}b \otimes a$ in $\{\!\{a,b\}\!\}$, or $b^* <_{h(a)} a^*$ and we get a term $-\frac{1}{2}a \otimes b$ in $\{\!\{b,a\}\!\}$. Hence we get a contributing term $-\frac{1}{2}o_{h(a)}(a^*,b^*)\,(b \otimes a)$. If $t(a) = h(b)$, either $a <_{t(a)} b^*$ and by (2.28c) we get a term $\frac{1}{2}ba \otimes e_{t(a)}$, or $b^* <_{t(a)} a$ and by (2.28b) we get a term $\frac{1}{2}e_{t(b^*)} \otimes ba$ in $\{\!\{b,a\}\!\}$. This yields $\frac{1}{2}o_{t(a)}(a,b^*)(ba \otimes e_{t(a)})$ in $\{\!\{a,b\}\!\}$. If $h(a) = t(b)$, either $a^* <_{h(a)} b$ and (2.28b) gives a term

$\frac{1}{2}e_{h(a)} \otimes ab$ for $\{\!\{a,b\}\!\}$, or $b <_{h(a)} a^*$ and (2.28c) gives a term $\frac{1}{2}ab \otimes e_{t(b)}$ for $\{\!\{b,a\}\!\}$. Thus, we have a contribution $\frac{1}{2}o_{h(a)}(a^*,b)(e_{h(a)} \otimes ab)$. Gathering the four cases gives (2.25). □

We finish by a remark on the structure of the moment map of a subquiver of $\bar{Q}$. Assume that $\bar{Q}'$ is a quiver with vertex set $I' \subset I$ and $\bar{Q}' = \{a \in \bar{Q} \,|\, t(a) \in I' \text{ and } h(a) \in I'\}$. This means that if we look at the subset of vertices I' of $\bar{Q}$ and erase all the arrows of $\bar{Q}$ which are not both starting and ending at an element of I', we recover $\bar{Q}'$. Moreover, we require that $\bar{Q}$ and $\bar{Q}'$ are endowed respectively with orderings $<, <'$ that satisfy the following conditions : whenever $a, b \in \bar{Q}'$, $a <' b$ if $a < b$ in the initial quiver $\bar{Q}$, and $a < c$ when $a \in \bar{Q}'$ but $c \in \bar{Q} \setminus \bar{Q}'$.

We construct A' as A above, and we see A' as a subalgebra of A (after adding the removed idempotents e_s for $s \in I \setminus I'$). Define elements Φ', P' by replacing $\bar{Q}$ with $\bar{Q}'$ in (2.22b) and (2.22a). Remark that we can write $\mathrm{P} = \mathrm{P}' + \mathrm{P}_{out}$ and $\Phi = (\Phi' + \sum_{s \notin I'} e_s)\Phi_{out}$ for some $\mathrm{P}_{out} \in (D_B A)_2$ and $\Phi_{out} = (\Phi_{out,s})_{s \in I}$. This statement is, in fact, a consequence of the fusion process which we have used to endow a quiver with a quasi-Hamiltonian structure [162, 6.5-6.7].

Lemma 2.3.14 *For all $b, c \in A' \subset A$, we have $\{\!\{b,c\}\!\}_{\mathrm{P}} = \{\!\{b,c\}\!\}_{\mathrm{P}'}$. In particular, for all $s \in I'$, we have $\{\!\{\Phi'_s, c\}\!\}_{\mathrm{P}} = \frac{1}{2}(\Phi'_s E_s + E_s \Phi'_s)(c)$.*

Proof. By linearity of the map in Proposition 2.3.4, we can decompose $\{\!\{-,-\}\!\}_{\mathrm{P}}$ as the sum $\{\!\{-,-\}\!\}_{\mathrm{P}'} + \{\!\{-,-\}\!\}_{\mathrm{P}_{out}}$. From (2.13), we get that $\{\!\{b,c\}\!\}_{\mathrm{P}_{out}}$ is a sum of terms of the form

$$\delta_2(c)'\delta_1(b)'' \otimes \delta_1(b)'\delta_2(c)'' - \delta_1(c)'\delta_2(b)'' \otimes \delta_2(b)'\delta_1(c)'' \,, \tag{2.29}$$

for any $b, c \in A$. By construction, P_{out} is a sum of (double) biderivations, and by inspecting (2.22a) each biderivation in P_{out} carries at least one factor $\delta_d = U_d, U_d^+, U_d^-, \bar{U}_d$ for $d \in \bar{Q} \setminus \bar{Q}'$, as defined in Lemma 2.3.13. Such a derivation δ_d vanishes on elements of A' again by Lemma 2.3.13. Therefore, if both $b, c \in A'$, all terms in (2.29) must vanish, and $\{\!\{b,c\}\!\}_{\mathrm{P}_{out}} = 0$.

Applying this to Φ'_s and $c \in A'$, $\{\!\{\Phi'_s, c\}\!\}_{\mathrm{P}} = \{\!\{\Phi'_s, c\}\!\}_{\mathrm{P}'}$. By construction, Φ' is a multiplicative moment map for $\{\!\{-,-\}\!\}_{\mathrm{P}'}$, so it satisfies (2.19). □

This previous construction can be easily generalised as follows. Assume now that $\bar{Q}' \subset \bar{Q}$ is a subquiver with the same set of vertices I. Suppose that $\bar{Q}', \bar{Q}$ are endowed with orderings $<', <$ such that whenever $a, b \in \bar{Q}'$, $a <' b$ if $a < b$ in the initial quiver $\bar{Q}$, and $a < c$ when $a \in \bar{Q}'$ but

$c \in \bar{Q} \setminus \bar{Q}'$. We can again construct A', P', Φ' as above from $\bar{Q}$, and see A' as a subalgebra of A. Note that $\Phi'_s = e_s$ if there are no arrow $a \in \bar{Q}'$ with $t(a) = s$ or $h(a) = s$. In the exact same way as Lemma 2.3.14, we can prove the next result.

Lemma 2.3.15 *For all $b, c \in A' \subset A$, we have $\{\!\{b, c\}\!\}_{\mathrm{P}} = \{\!\{b, c\}\!\}_{\mathrm{P}'}$. In particular, for all $s \in I$, with $\Phi'_s \neq e_s$, we have $\{\!\{\Phi'_s, c\}\!\}_{\mathrm{P}} = \frac{1}{2}(\Phi'_s E_s + E_s \Phi'_s)(c)$.*

More Lie brackets

Note that the triple bracket associated to a double Poisson bracket vanishes, so the associated bracket $\{-,-,-\}$ also does. A small computation shows that it is also the case for a double quasi-Poisson bracket. Hence, if $(A, \{\!\{-,-\}\!\})$ is a double (quasi-)Poisson algebra, then $\{-,-\}$ defines an H_0-Poisson structure on A by § 2.3.1.

Let $(A, \{\!\{-,-\}\!\}, \mu)$ be Hamiltonian, and put $A^\lambda = A/(\mu - \lambda)$ for some $\lambda = \sum_s \lambda_s e_s \in B$. Then the associated bracket $\{-,-\}$ descends to $A^\lambda/[A^\lambda, A^\lambda]$ and defines a Lie bracket. Similarly, if $(A, \{\!\{-,-\}\!\}, \Phi)$ is quasi-Hamiltonian and we let $A^q = A/(\Phi - q)$ for $q \in B^\times$, then $A^q/[A^q, A^q]$ also inherits an associated Lie bracket. In fact, we have again an H_0-Poisson structure on both A^λ and A^q.

2.3.3 Structures on representation spaces

We now explain how the definitions introduced in § 2.3.1 and § 2.3.2 translate to the representation spaces defined in § 2.2.2. Our presentation is mainly based on [162, Section 7].

Generalities for the rest of the section

Throughout, we continue to assume that A is a finitely generated associative algebra over $B = \oplus_{s=1}^K \mathbb{C} e_s$, with $e_s e_t = \delta_{st} e_s$, that we assume endowed with a double bracket $\{\!\{-,-\}\!\}$.

Let $I = \{1, \ldots, K\}$ and choose the dimension vector $\alpha \in \mathbb{N}^I$, setting $N = \sum_{s \in I} \alpha_s$. We consider the representation space (assumed relative to B from now on) $\mathrm{Rep}(A, \alpha)$, that we recall is such that the matrix $\mathcal{X}(e_s)$ satisfies $\mathcal{X}(e_s)_{ij} = \delta_{ij}$ if $\alpha_1 + \ldots + \alpha_{s-1} + 1 \leq i, j \leq \alpha_1 + \ldots + \alpha_s$.

I.e., $\mathcal{X}(e_s)$ is the s-th diagonal block of Id_N, and each block is of size α_s. To ease notations, denote by $R = \mathcal{O}(\mathrm{Rep}(A,\alpha))$ the coordinate ring. We have from the definition of the scheme that any element $a \in A$ induces functions $(a_{ij})_{ij}$ on $\mathrm{Rep}(A,\alpha)$, and we would like to extend this definition to arbitrary $\delta \in D_{A/B}$. For any $1 \leq i,j \leq N$, define the vector field $\delta_{ij} \in \mathrm{Der}(R)$ by

$$\delta_{ij}(b_{kl}) = \delta(b)'_{kj}\delta(b)''_{il}\,, \tag{2.30}$$

and introduce the vector field-valued matrix $\mathcal{X}(\delta)$ with (i,j) entry δ_{ij}. We call the particular disposition of indices in (2.30) the *standard index notation* as in [163]. More generally, for an element $\delta = \delta_1 \ldots \delta_n \in (D_B A)_n$ we define $\delta_{ij} \in \bigwedge_R^n \mathrm{Der}(R)$ from the matrix identity $\mathcal{X}(\delta) = \mathcal{X}(\delta_1)\ldots\mathcal{X}(\delta_n)$.

It is interesting to note that, for any $a \in A$, $\{\!\{a,-\}\!\} \in D_{A/B}$ defines a vector field by $\{\!\{a,-\}\!\}_{ij}(b_{kl}) = \{\!\{a,b\}\!\}'_{kj}\,\{\!\{a,b\}\!\}''_{il}$. This motivates the following result.

Proposition 2.3.16 ([162, 7.5.1, 7.8]) *There is a unique antisymmetric biderivation $\{-,-\} : R \times R \to R$ such that for all $a,b \in A$ and $1 \leq i,j,k,l \leq N$,*

$$\{a_{ij}, b_{kl}\} = \{\!\{a,b\}\!\}'_{kj}\;\{\!\{a,b\}\!\}''_{il}\;. \tag{2.31}$$

Moreover, if $\{\!\{-,-\}\!\}$ is differential and defined from an element $P \in (D_B A)_2$ by Proposition 2.3.4, then the biderivation (2.31) is defined by the bivector field $\mathrm{tr}(\mathcal{X}(P))$. In the latter case, we denote the map by $\{-,-\}_P$.

Allowing graded components, the following is expected.

Proposition 2.3.17 ([162, 7.6]) *Let $P,Q \in D_B A$. Then*

$$[P_{ij}, Q_{kl}] = (\{\!\{P,Q\}\!\}'_{\mathrm{SN}})_{kj}(\{\!\{P,Q\}\!\}''_{\mathrm{SN}})_{il}\,.$$

I.e. the (geometric) Schouten-Nijenhuis bracket on $\mathrm{Rep}(A,\alpha)$ is determined by the double Schouten-Nijenhuis bracket on A.

Finally, recall that the algebraic group $\mathrm{GL}_\alpha := \prod_{s=1}^K \mathrm{GL}_{\alpha_s}(\mathbb{C})$ acts by conjugation on $\mathrm{Rep}(A,\alpha)$, and we have[5] $g \cdot \mathcal{X}(a) = g^{-1}\mathcal{X}(a)g$. We obtain an induced action of the Lie algebra

[5] Here we follow [162, Sect. 7], but we will use later the action $g \cdot \mathcal{X}(a) := g\mathcal{X}(a)g^{-1}$ when working on multiplicative quiver varieties. This slight change can be easily forgotten as it will just amount to act with the inverse element from the algebraic group.

$\mathfrak{g}_\alpha := \prod_{s=1}^K \mathfrak{gl}_{\alpha_s}(\mathbb{C})$ given by $\xi \cdot \mathcal{X}(a) = [\mathcal{X}(a), \xi]$ for any $\xi \in \mathfrak{g}_\alpha$. Moreover, the gauge elements $(E_s)_s$ correspond to the action of $\mathfrak{g}_\alpha$ as follows.

Proposition 2.3.18 ([162, 7.9.1]) *Fix $s \in I$ and take $i, j \in \{1, \dots, N\}$ such that $\alpha_1 + \ldots + \alpha_{s-1} + 1 \leq i, j \leq \alpha_1 + \ldots + \alpha_s$. Let $F_{ij} \in \mathfrak{g}_\alpha$ be the elementary matrix which is $+1$ at (i, j) entry and zero otherwise. Then the derivation $E_{s,ji}$ is such that $E_{s,ji}(h) = F_{ij} \cdot h$ for any $h \in R$.*

The Hamiltonian case

Let us adopt the notation $a_{ij,kl,uv} := a'_{ij} a''_{kl} a'''_{uv} \in R$ for an element $a = a' \otimes a'' \otimes a''' \in A^{\otimes 3}$. As in [162, 7.5.2], we can compute using (2.31) that for any $a, b, c \in A$,

$$Jac(a_{ij}, b_{kl}, c_{uv}) = \{\!\{a, b, c\}\!\}_{uj,il,kv} - \{\!\{a, c, b\}\!\}_{kj,iv,ul} \,, \tag{2.32}$$

where $\{\!\{-,-,-\}\!\}$ is the triple bracket defined by $\{\!\{-,-\}\!\}$ and Jac is defined by (2.2).

Since (2.32) vanishes for a double Poisson bracket, we can use the general results introduced in the previous subsection and the smoothness criterion of § 2.2.2 to derive the following result.

Theorem 2.3.19 ([162, 7.11.1]) *Assume that $(A, \{\!\{-,-\}\!\}, \mu)$ is a Hamiltonian algebra. We have that $\operatorname{Rep}(A, \alpha)$ is a GL_α-scheme with Poisson bracket $\{-,-\}$ determined from $\{\!\{-,-\}\!\}$ by (2.31), and the matrix-valued function $\mathcal{X}(\mu) : \operatorname{Rep}(A, \alpha) \to \mathfrak{g}_\alpha$ is a moment map. Furthermore, if A is smooth, $\operatorname{Rep}(A, \alpha)$ admits a structure of a Hamiltonian $G(\alpha)$-Poisson manifold.*

Here, we identified $\mathfrak{g}_\alpha$ and its dual through the trace pairing. Note also that we pass to $G(\alpha) = \mathrm{GL}_\alpha / \mathbb{C}^\times$, as the action is never free otherwise.

Example 2.3.20 *Theorem 2.3.5 yields that any double quiver $\bar{Q}$ induces that $\operatorname{Rep}(A, \alpha)$ (in the relative setting) is a Poisson variety, and admits a moment map. Consider the tadpole quiver of Example 2.2.1. Then we have a Hamiltonian manifold structure on the representation space described in Example 2.2.7, which precisely gives the reduced symplectic space described in Example 2.1.4.*

This example is precisely the motivation for the study of quasi-Hamiltonian algebras that we will do : we will try to understand spaces that could be carrying integrable systems from a coordinate-free point of view.

Example 2.3.21 *Consider a double quiver $\bar{Q}$, and dimension vectors $\alpha, \mathbf{w} \in \mathbb{N}^I$. We extend $\bar{Q}$ by adjoining to I one vertex, denoted ∞, and by adding $\mathbf{w}_s$ arrows $\infty \to s$ for each $s \in I$. Write $\bar{Q}^{\mathbf{w}}$ the quiver hence obtain. Moreover, we extend α to $\widetilde{\alpha}$ by adding to it $\alpha_\infty = 1$. Again, Theorem 2.3.5 gives that $\mathrm{Rep}(\bar{Q}^{\mathbf{w}}, \widetilde{\alpha})$ is a $G_{\widetilde{\alpha}}$-variety with a Poisson bracket and momentum map. Then $\mu^{-1}(0)//G(\widetilde{\alpha})$ is an example of Nakajima quiver variety [123] for 0 stability parameter, as explained in [51].*

The latter example justifies the study by Yamakawa of the multiplicative analogues of this construction [174], and in particular the introduction of such *framing*, i.e. one extra vertex and several arrows pointing towards the initial vertices in $\bar{Q}$. We return to this procedure at the end of this section.

The quasi-Hamiltonian case

We choose as before the algebraic group GL_α with Lie algebra $\mathfrak{g}_\alpha$. Again, we consider the trace pairing on $\mathfrak{g}_\alpha$, and we note that the dual basis to $(F_{ij})_{ij}$ (see Proposition 2.3.18 for the definition of F_{ij} and which indices are considered to form the basis) is $(F_{ji})_{ij}$.

Proposition 2.3.22 ([162, 7.12.1]) *The trivector field on $\mathrm{Rep}(A, \alpha)$ induced by $\phi \in \bigwedge^3 \mathfrak{g}_\alpha$ defined in (2.5) is given by $\frac{1}{6} \sum_s \mathrm{tr}\, \mathcal{X}(E_s^3)$.*

We now extend the result of [162, 7.13] to the non-differential case, namely double quasi-Poisson algebras define quasi-Poisson structures on their representation spaces. It was already noticed by Van den Bergh to be possible, though not proved in [162]. The author believes this slightly longer proof is worth being written done, and we go through it now.

Recalling relation (2.32), we remark that a similar statement can be proved in the following form.

Lemma 2.3.23 *Assume that $Q \in (D_B A)_3$. Then the following equality holds for any $a, b, c \in A$*

$$\mathrm{tr}\, \mathcal{X}(Q)(a_{ij}, b_{kl}, c_{uv}) = (\{\!\{a, b, c\}\!\}_Q)_{uj,il,kv} - (\{\!\{a, c, b\}\!\}_Q)_{kj,iv,ul}\,, \tag{2.33}$$

In fact, this result implies (2.32) in the case where $\{\!\!\{-,-\}\!\!\}$ is differential. Indeed, the double bracket is defined by an element $P \in (D_B A)_2$, so the triple bracket $\{\!\!\{-,-,-\}\!\!\}$ is defined by an element of $(D_B A)_3$, viz. $\frac{1}{2}\{P,P\}_{\mathrm{SN}}$ by [162, 4.2.3]. Meanwhile, $\{-,-\}$ is defined by $\mathrm{tr}\,\mathcal{X}(P)$ using Proposition 2.3.16, so Jac is defined by the trivector $\frac{1}{2}[\mathrm{tr}\,\mathcal{X}(P), \mathrm{tr}\,\mathcal{X}(P)]$. Summing over $i=j$, $k=l$ in Proposition 2.3.17 thus yield that Jac is defined by $\frac{1}{2}\,\mathrm{tr}\,\mathcal{X}(\{P,P\}_{\mathrm{SN}})$. It remains to take $Q = \frac{1}{2}\{P,P\}_{\mathrm{SN}}$ in (2.33) to get (2.32).

Proof. (Lemma 2.3.23.) Write $Q = \delta^1\delta^2\delta^3$. From Proposition 2.3.4 we can get

$$\begin{aligned}
\{\!\!\{a,b,c\}\!\!\}_Q =& \delta^3(c)'\delta^1(a)'' \otimes \delta^1(a)'\delta^2(b)'' \otimes \delta^2(b)'\delta^3(c)'' \\
&+ \delta^2(c)'\delta^3(a)'' \otimes \delta^3(a)'\delta^1(b)'' \otimes \delta^1(b)'\delta^2(c)'' \\
&+ \delta^1(c)'\delta^2(a)'' \otimes \delta^2(a)'\delta^3(b)'' \otimes \delta^3(b)'\delta^1(c)''\,,
\end{aligned}$$

and similarly by swapping b and c

$$\begin{aligned}
\{\!\!\{a,c,b\}\!\!\}_Q =& \delta^3(b)'\delta^1(a)'' \otimes \delta^1(a)'\delta^2(c)'' \otimes \delta^2(c)'\delta^3(b)'' \\
&+ \delta^2(b)'\delta^3(a)'' \otimes \delta^3(a)'\delta^1(c)'' \otimes \delta^1(c)'\delta^2(b)'' \\
&+ \delta^1(b)'\delta^2(a)'' \otimes \delta^2(a)'\delta^3(c)'' \otimes \delta^3(c)'\delta^1(b)''\,.
\end{aligned}$$

Now, recall that the matrix-valued vector field $(\delta_{ij})_{ij}$ representing an element $\delta \in (D_B A)_1$ satisfies $\delta_{ab}(c_{uv}) = \delta(c)'_{ub}\delta(c)'_{av}$ in standard index notation. We compute

$$\begin{aligned}
\mathrm{Tr}(Q)(a_{ij}, b_{kl}, c_{uv}) =& \sum_{rst} (\delta^1_{rs} \wedge \delta^2_{st} \wedge \delta^3_{tr})(a_{ij}, b_{kl}, c_{uv}) \\
=& \sum_{rst} \delta^1(a)'_{is}\delta^1(a)''_{rj}\ \delta^2(b)'_{kt}\delta^2(b)''_{sl}\ \delta^3(c)'_{ur}\delta^3(c)''_{tv} \\
&+ \sum_{rst} \delta^3(a)'_{ir}\delta^3(a)''_{tj}\ \delta^1(b)'_{ks}\delta^1(b)''_{rl}\ \delta^2(c)'_{ut}\delta^2(c)''_{sv} \\
&+ \sum_{rst} \delta^2(a)'_{it}\delta^2(a)''_{sj}\ \delta^3(b)'_{kr}\delta^3(b)''_{tl}\ \delta^1(c)'_{us}\delta^1(c)''_{rv} \\
&- \sum_{rst} \delta^1(a)'_{is}\delta^1(a)''_{rj}\ \delta^3(b)'_{kr}\delta^3(b)''_{tl}\ \delta^2(c)'_{ut}\delta^2(c)''_{sv} \\
&- \sum_{rst} \delta^3(a)'_{ir}\delta^3(a)''_{tj}\ \delta^2(b)'_{kt}\delta^2(b)''_{sl}\ \delta^1(c)'_{us}\delta^1(c)''_{rv} \\
&- \sum_{rst} \delta^2(a)'_{it}\delta^2(a)''_{sj}\ \delta^1(b)'_{ks}\delta^1(b)''_{rl}\ \delta^3(c)'_{ur}\delta^3(c)''_{tv}\,,
\end{aligned}$$

which we rewrite after summation as

$$\begin{aligned}\mathrm{Tr}(Q)(a_{ij},b_{kl},c_{uv}) = \; & (\delta^3(c)'\delta^1(a)'')_{uj}(\delta^1(a)'\delta^2(b)'')_{il}(\delta^2(b)'\delta^3(c)'')_{kv} \\ & + (\delta^2(c)'\delta^3(a)'')_{uj}(\delta^3(a)'\delta^1(b)'')_{il}(\delta^1(b)'\delta^2(c)'')_{kv} \\ & + (\delta^1(c)'\delta^2(a)'')_{uj}(\delta^2(a)'\delta^3(b)'')_{il}(\delta^3(b)'\delta^1(c)'')_{kv} \\ & - (\delta^3(b)'\delta^1(a)'')_{kj}(\delta^1(a)'\delta^2(c)'')_{iv}(\delta^2(c)'\delta^3(b)'')_{ul} \\ & - (\delta^2(b)'\delta^3(a)'')_{kj}(\delta^3(a)'\delta^1(c)'')_{iv}(\delta^1(c)'\delta^2(b)'')_{ul} \\ & - (\delta^1(b)'\delta^2(a)'')_{kj}(\delta^2(a)'\delta^3(c)'')_{iv}(\delta^3(c)'\delta^1(b)'')_{ul} \,.\end{aligned}$$

We can conclude from our choice of notation. □

The following is the general version of [162, 7.13.2].

Theorem 2.3.24 *Assume that $(A, \{\!\{-,-\}\!\}, \Phi)$ is a quasi-Hamiltonian algebra. We have that $\mathrm{Rep}(A,\alpha)$ is a GL_α-scheme with quasi-Poisson bracket $\{-,-\}$ determined from $\{\!\{-,-\}\!\}$ by (2.31), and the matrix-valued function $\mathcal{X}(\Phi) : \mathrm{Rep}(A,\alpha) \to \mathrm{GL}_\alpha$ is a multiplicative moment map.*

Proof. We only need to show (2.7) on generators of the coordinate ring R, so fix $a, b, c \in A$. We remark that by Proposition 2.3.22 the 3-vector field $\phi_{\mathrm{Rep}(A,\alpha)}$ is given by $\frac{1}{6}\sum_s \mathrm{tr}(E_s^3)$, hence we can write

$$\frac{1}{2}\phi_{\mathrm{Rep}(A,\alpha)}(a_{ij},b_{kl},c_{uv}) = \frac{1}{12}\sum_s \mathrm{tr}(E_s^3)(a_{ij},b_{kl},c_{uv})\,.$$

Using Lemma 2.3.23, this is the same as

$$\left(\frac{1}{12}\sum_s \{\!\{a,b,c\}\!\}_{E_s^3}\right)_{uj,il,kv} - \left(\frac{1}{12}\sum_s \{\!\{a,c,b\}\!\}_{E_s^3}\right)_{kj,iv,ul}$$

But then, since the double bracket is quasi-Poisson we get by definition

$$\frac{1}{2}\phi_{\mathrm{Rep}(A,\alpha)}(a_{ij},b_{kl},c_{uv}) = (\{\!\{a,b,c\}\!\})_{uj,il,kv} - (\{\!\{a,c,b\}\!\})_{kj,iv,ul}\,,$$

which is nothing else but $Jac(a_{ij},b_{kl},c_{uv})$ by (2.32).

Finally, to prove that equality (2.8) holds, it is enough to reproduce the proof of [162, 7.13.2] which does not use the assumption that the quasi-Poisson algebra is differential. □

Corollary 2.3.25 *Under the assumptions of Theorem 2.3.24 and provided that A is smooth, the space $\mathrm{Rep}(A,\alpha)$ admits a structure of a quasi-Hamiltonian $G(\alpha)$-manifold.*

Again, we replaced GL_α by $G(\alpha)$ as we always have a copy of $\mathbb{C}^\times$ in the stabiliser of any point otherwise.

Fix a quiver $\bar{Q}$. Applying Theorem 2.3.24 to the structure obtained in Theorem 2.3.9 endows the multiplicative quiver varieties of Crawley-Boevey and Shaw [56] with a Poisson structure. We now turn to the study of these spaces.

Multiplicative Quiver Varieties

We follow [41, 2.6], and refer to [56] and [174] for details about multiplicative quiver varieties. Let Q be a quiver, $\bar{Q}$ its double, and fix $\alpha \in \mathbb{N}^I$ a dimension vector. We identify $\mathrm{Rep}(\mathbb{C}\bar{Q}, \alpha)$ with representations of $\bar{Q}$ as in Example 2.2.6, i.e. we attach the vector space $\mathcal{V}_s = \mathbb{C}^{\alpha_s}$ at each vertex $s \in I$, and a point is determined by the data $(X_a)_a$ such that $X_a \in \mathrm{Mat}_{\alpha(t(a))\times\alpha(h(a))}(\mathbb{C})$ for all arrows $a \in \bar{Q}$. We can use the entries of these matrices as coordinates. We then consider the subspace $\mathrm{Rep}(A, \alpha)$ which is such that for all $a \in \bar{Q}$, $\det(\mathrm{Id}_{\mathcal{V}_{t(a)}} + X_a X_{a^*}) \neq 0$. Hence $\mathrm{Rep}(A, \alpha)$ is a smooth variety. Then Corollary 2.3.25 and Theorem 2.3.9 endow $\mathrm{Rep}(A, \alpha)$ with a structure of quasi-Poisson $G(\alpha)$-manifold. Using the identification we have made, together with Proposition 2.3.12 and (2.31), we get the following characterisation of the quasi-Poisson bracket on $\mathrm{Rep}(A, \alpha)$, that we denote $\{-,-\}_{\mathrm{P}}$.

Proposition 2.3.26 *Take an ordering in $\bar{Q}$ so that the arrows of $\bar{Q}$ are ordered in such a way that $a < a^* < b < b^*$ for any $a, b \in Q$ with $a < b$. Then the quasi-Poisson bracket on $\mathrm{Rep}(A, \alpha)$ is completely determined by*

$$\{(X_a)_{ij}, (X_a)_{kl}\}_{\mathrm{P}} = \frac{1}{2}\epsilon(a)\left((X_a^2)_{kj}\delta_{il} - \delta_{kj}(X_a^2)_{il}\right) \qquad (a \in \bar{Q}), \tag{2.34a}$$

$$\begin{aligned}\{(X_a)_{ij}, (X_{a^*})_{kl}\}_{\mathrm{P}} = &\,\delta_{kj}\delta_{il} + \frac{1}{2}(X_{a^*}X_a)_{kj}\delta_{il} + \frac{1}{2}\delta_{kj}(X_aX_{a^*})_{il} \\ &+ \frac{1}{2}\delta_{h(a),t(a)}\left((X_{a^*})_{kj}(X_a)_{il} - (X_a)_{kj}(X_{a^*})_{il}\right) \qquad (a \in Q),\end{aligned} \tag{2.34b}$$

while if $a, b \in \bar{Q}$ are such that $a < b$ but $b \neq a^*$,

$$\begin{aligned}\{(X_a)_{ij}, (X_b)_{kl}\}_{\mathrm{P}} = &\,\frac{1}{2}\delta_{h(a),t(b)}\delta_{kj}(X_aX_b)_{il} + \frac{1}{2}\delta_{h(b),t(a)}\delta_{il}(X_bX_a)_{kj} \\ &- \frac{1}{2}\delta_{h(a),h(b)}(X_b)_{kj}(X_a)_{il} - \frac{1}{2}\delta_{t(a),t(b)}(X_a)_{kj}(X_b)_{il}\,.\end{aligned} \tag{2.35}$$

When writing $(X_a)_{ij}$ above, we always assume $1 \leq i \leq \alpha_{t(a)}$, $1 \leq j \leq \alpha_{h(a)}$.

Fix some $q = (q_s)_s \in (\mathbb{C}^\times)^I$. Assuming an ordering is taken, a level set of the moment map $\{\mathcal{X}(\Phi) = \prod_s q_s \operatorname{Id}_{\mathcal{V}_s}\}$ is nothing else than $\operatorname{Rep}(\Lambda^q, \alpha)$, the moduli space of representations (of fixed dimension α) of the multiplicative preprojective algebra Λ^q associated to Q. Hence $\operatorname{Rep}(\Lambda^q, \alpha)$ is a closed affine subvariety. This space is easily seen to be empty whenever we have $q^\alpha := \prod_s q_s^{\alpha_s} \neq 1$ [56, Lemma 1.5].

Isomorphism classes of representations correspond to orbits under the group $G(\alpha)$, and the semi-simple representations correspond to closed orbits as recalled in § 2.2.2. Thus, the points in the affine variety $\mathcal{S}_{\alpha,q} := \operatorname{Rep}(\Lambda^q, \alpha)//G(\alpha)$ correspond to semi-simple representations of Λ^q of dimension α, and $\mathcal{S}_{\alpha,q}$ is called a *multiplicative quiver variety*, that we abbreviate MQV from now on. Note that the space $\mathcal{S}_{\alpha,q}$ is a Poisson variety. Indeed, the quasi-Poisson bracket of Proposition 2.3.26 descends to a Poisson bracket on $\mathcal{O}(\mathcal{S}_{\alpha,q}) = \mathcal{O}(\operatorname{Rep}(\Lambda^q, \alpha))^{G(\alpha)}$, by restriction to $G(\alpha)$-invariant functions [162, Proposition 1.7]. We are particularly interested in the case where the GIT quotient is a geometric quotient, so that we can hope to have a Poisson manifold structure after reduction. Restricting to the case where all representations in $\operatorname{Rep}(\Lambda^q, \alpha)$ are simple, we find the next result based on [56, Theorem 1.10].

Theorem 2.3.27 *[41, Theorem 2.8] Let $p(\alpha) = 1 - q(\alpha)$, where q is the Tits form (2.9). Suppose that $\operatorname{Rep}(\Lambda^q, \alpha)$ is non-empty and all representations in $\operatorname{Rep}(\Lambda^q, \alpha)$ are simple. Then α is a positive root of Q and $\operatorname{Rep}(\Lambda^q, \alpha)$ is a smooth affine variety of dimension $g + 2p(\alpha)$, with $g = \dim G(\alpha) = \sum_{s\in I} \alpha_s^2 - 1$. The group $G(\alpha)$ acts freely on $\operatorname{Rep}(\Lambda^q, \alpha)$, so $\mathcal{S}_{\alpha,q} = \operatorname{Rep}(\Lambda^q, \alpha)/G(\alpha)$ is a Poisson manifold of dimension $2p(\alpha)$, obtained by quasi-Hamiltonian reduction.*

As we observed that the double bracket that induces $\{-,-\}_{\mathrm{P}}$ is defined by a non-degenerate biderivation P (see [163, Sect. 8]), we get in fact from [163, Proposition 5.2] and [6, Theorem 10.3] that $\mathcal{S}_{\alpha,q}$ is a symplectic manifold when any representation in $\operatorname{Rep}(\Lambda^q, \alpha)$ is simple.

Finally, note that the Poisson bracket on the affine variety $\mathcal{S}_{\alpha,q}$ (that we only assume to be non-empty) can be directly understood inside $\Lambda^q/[\Lambda^q, \Lambda^q]$. Indeed, the coordinate ring $\mathcal{O}(\operatorname{Rep}(\Lambda^q, \alpha))^{G(\alpha)}$ is generated by elements of the form $\operatorname{tr} \mathcal{X}(\gamma)$ for $\gamma \in \Lambda^q$, which amounts to know γ projected to $\Lambda^q/[\Lambda^q, \Lambda^q]$. Remarking that the ideal generated by $\Phi - q$ is a Lie ideal for the Lie bracket on $A/[A, A]$ defined by the associated bracket $\{-,-\} = m \circ \{\!\{-,-\}\!\}$, it suffices to take traces in (2.31) to obtain

$$\{\operatorname{tr}\mathcal{X}(\gamma), \operatorname{tr}\mathcal{X}(\gamma')\}_P = \operatorname{tr}\mathcal{X}\left(\{\!\{\gamma,\gamma'\}\!\}'\,\{\!\{\gamma,\gamma'\}\!\}''\right) = \operatorname{tr}\mathcal{X}(\{\gamma,\gamma'\}), \tag{2.36}$$

where we see $\{-,-\}$ as a bilinear map on $\Lambda^q/[\Lambda^q,\Lambda^q]$. Hence, the Poisson bracket of traces of matrices representing two paths *is* the trace of the matrix representing the associated bracket applied to these paths.

Multiplicative Quiver Varieties obtained by framing

The examples of MQVs that we study in this thesis come from a particular class of quivers obtained by framing (see [174, Section 6.2] with trivial stability parameter), which we now define. Let Q be an arbitrary quiver with vertex set I. A *framing* of Q is a quiver $\widetilde{Q}$ whose set of vertices is given by $\widetilde{I} = I \cup \{\infty\}$, and whose set of arrows is given by the ones of Q together with additional arrows $\{v_{s,\beta} : \infty \to s \mid 1 \leq \beta \leq d_s\}$ to each $s \in I$ with fixed $d_s \in \mathbb{N}$. As an example, we can consider the tadpole quiver from Example 2.2.1, which is the simplest framing of a Jordan quiver. Given arbitrary $\alpha \in \mathbb{N}^I$ and $q \in (\mathbb{C}^\times)^I$, we extend them from I to $\widetilde{I}$ by putting $\alpha_\infty = 1$ and $q_\infty = q^{-\alpha} = \prod_{s\in I} q_s^{-\alpha_s}$, so that we are considering

$$\widetilde{\alpha} = (1,\alpha)\,, \qquad \widetilde{q} = q^{-\alpha}e_\infty + \sum_{s\in I} q_s e_s\,. \tag{2.37}$$

Note that $\widetilde{q}^{\widetilde{\alpha}} = 1$. We define the representation space $\operatorname{Rep}(\Lambda^{\widetilde{q}},\widetilde{\alpha})$ associated to the multiplicative preprojective algebra of $\widetilde{Q}$ with parameter $\widetilde{q}$, and construct the quotient

$$\widetilde{\mathcal{M}}_{\alpha,q}(Q) := \operatorname{Rep}(\Lambda^{\widetilde{q}},\widetilde{\alpha})//G(\widetilde{\alpha})\,, \quad \text{where } G(\widetilde{\alpha}) \cong \prod_{s\in I}\operatorname{GL}_{\alpha_s} = \operatorname{GL}_\alpha\,,$$

which is a MQV. In order to have a symplectic manifold structure, we say that $q = \sum_{s\in I} q_s e_s$ is *regular* if $q^\alpha \neq 1$ for any root α of the quiver Q. We have the following result, which is a multiplicative analogue of [123, Theorem 2.8], [25, Proposition 3].

Proposition 2.3.28 *[41, Proposition 2.9] Choose an arbitrary framing $\widetilde{Q}$ of Q and let $\widetilde{\alpha}$ and $\widetilde{q}$ be defined as above. If q is regular, then every module of dimension $\widetilde{\alpha}$ over the multiplicative preprojective algebra $\Lambda^{\widetilde{q}}$ is simple. Hence, the group GL_α acts freely on $\operatorname{Rep}(\Lambda^{\widetilde{q}},\widetilde{\alpha})$ and the MQV $\widetilde{\mathcal{M}}_{\alpha,q}(Q)$ is smooth.*

It follows that if q is regular and $\widetilde{\mathcal{M}}_{\alpha,q}(Q) \neq \emptyset$, then $\widetilde{\alpha} = (1,\alpha)$ is a positive root of $\widetilde{Q}$ and $\widetilde{\mathcal{M}}_{\alpha,q}(Q)$ is a smooth affine variety of dimension $2p(\widetilde{\alpha})$. Moreover, it is endowed with a symplectic form.

2.3.4 Relation to integrability

Assume that $(A, \{\!\{-,-\}\!\})$ is a double (quasi-)Poisson algebra over $B = \sum_{s=1}^{K} \mathbb{C}e_s$. In case A admits a moment map μ (resp. multiplicative moment map Φ), we denote by A^q_{red} the algebra $A/(\mu - q)$ for $q \in B$ (resp. $A/(\Phi - q)$ for $q \in B^\times$). We write $\{-,-\}$ for the bracket associated to $\{\!\{-,-\}\!\}$ on A defined by (2.15).

Definition 2.3.29 *Two elements $b, c \in A$ are* in involution *if $\{b,c\} \in [A,A]$. An element $b \in A$ is* involutive *if $\{b^k, b^l\} \in [A,A]$ for all $k,l \in \mathbb{N}$.*

To get an example of involutive element, we remark the following result that appears in different forms throughout Chapter 3.

Lemma 2.3.30 *If $b \in A$ satisfies $\{\!\{b,b\}\!\} = \sum_{t\in J}(b^t \otimes a_t - a_t \otimes b^t)$ where $J \subset \mathbb{N}$ is a finite set and $a_t \in A$ for all $t \in J$, then b is involutive.*

Proof. We use the derivation properties *(D2)–(D2')* to find for $k,l \in \mathbb{N}$

$$\begin{aligned}\left\{\!\!\left\{b^k, b^l\right\}\!\!\right\} =& \sum_{\sigma=1}^{k}\sum_{\tau=1}^{l} b^{\sigma-1} * b^{\tau-1}\,\{\!\{b,b\}\!\}\,b^{l-\tau} * b^{k-\sigma} \\ =& \sum_{\sigma=1}^{k}\sum_{\tau=1}^{l}\sum_{t\in J}\left(b^{k+\tau-\sigma+t-1} \otimes b^{\sigma-1}a_t b^{l-\tau} - b^{\tau-1}a_t b^{k-\sigma} \otimes b^{l+\sigma-\tau+t-1}\right).\end{aligned}$$

Applying the multiplication map as in (2.15) yields

$$\begin{aligned}\{b^k, b^l\} =& k\sum_{\tau=1}^{l}\sum_{t\in J}(b^{k+\tau+t-2}a_t b^{l-\tau} - b^{\tau-1}a_t b^{l+k-\tau+t-1}) \\ =& k\sum_{\tau=1}^{l}\sum_{t\in J}\left[b^{k+t-1}, b^{\tau-1}a_t b^{l-\tau}\right],\end{aligned}$$

finishing the proof since $[A,A]$ is the vector space of all commutators. □

Fix a dimension vector α for the rest of this section. Recall that the space $\mathrm{Rep}(A,\alpha)//G(\alpha)$ inherits a Poisson bracket from the (quasi-)Poisson bracket $\{-,-\}$ defined in Proposition 2.3.16. The same is true for $\mathcal{M}_{red} := \mathrm{Rep}(A^q_{red},\alpha)//G(\alpha)$ if A admits a (multiplicative) moment map. The next two statements hold for $\mathrm{Rep}(A^q_{red},\alpha)//G(\alpha)$ when defined.

Proposition 2.3.31 *If $b,c \in A$ are in involution, then $\mathrm{tr}\,\mathcal{X}(b)$ and $\mathrm{tr}\,\mathcal{X}(c)$ are in involution on $\mathrm{Rep}(A,\alpha)//G(\alpha)$ and on $\mathrm{Rep}(A^q_{red},\alpha)//G(\alpha)$ (when defined).*

Proof. These are clearly elements of the coordinate ring. Looking at them as functions on $\mathrm{Rep}(A,\alpha)$, we have by (2.31)

$$\{\mathrm{tr}\,\mathcal{X}(b),\mathrm{tr}\,\mathcal{X}(c)\} = \sum_{i,j}\{b_{ii},c_{jj}\} = \sum_{i,j} \{\!\!\{b,c\}\!\!\}'_{ji}\,\{\!\!\{b,c\}\!\!\}''_{ij} = \mathrm{tr}\, m\circ\{\!\!\{b,c\}\!\!\}\ .$$

This last term vanishes by Definition 2.3.29 since we take the trace of a commutator. □

Proposition 2.3.32 *If $b \in A$ is involutive, the symmetric functions of the eigenvalues of $\mathcal{X}(b)$ are in involution on $\mathrm{Rep}(A,\alpha)//G(\alpha)$.*

We now assume that $\mathcal{M}_{red}$ is defined and is smooth, so that we look at this space as a complex manifold. Furthermore, we restrict to the case where the Poisson bracket is non-degenerate. For example, in the quasi-Hamiltonian case, this is the case if the double quasi-Poisson bracket is non-degenerate, see § 2.3.2. We get criteria for integrability by a simple combination of Definition 2.1.3 and Propositions 2.3.31–2.3.32.

Corollary 2.3.33 *Let $b_1,\dots,b_n \in A$ be pairwise in involution with $\dim\mathcal{M}_{red} = 2n$. If $\mathrm{tr}\,\mathcal{X}(b_1),\dots,\mathrm{tr}\,\mathcal{X}(b_n)$ are functionally independent on a dense open subset of $\mathcal{M}_{red}$, then they form an integrable system.*

Corollary 2.3.34 *Let $b \in A$ be involutive. Assume that there exists $J \subset \mathbb{N}^\times$, $|J| = \frac{1}{2}\dim\mathcal{M}_{red}$, such that $(\mathrm{tr}\,\mathcal{X}(b^j))_{j\in J}$ are functionally independent on a dense open subset of $\mathcal{M}_{red}$. Then they form an integrable system.*

There is an obvious reason why we do not take the first n powers of $\mathcal{X}(b)$: we could have $b \notin \oplus_s e_s A e_s$, so that $\mathrm{tr}\,\mathcal{X}(b) = 0$. Such examples appears in § 3.2.1.

We now turn to the case of degenerate integrability. We continue with notations as above.

Definition 2.3.35 *Let $b, c \in A$. We say that c is in* strong involution *with b if $\{b, c\} = 0$. We say that b and c are in* total involution *if $\{b, c\} = 0 = \{c, b\}$. An element $b \in A$ is* strongly involutive *if for all $k, l \in \mathbb{N}$, b^k is in strong involution with b^l.*

Note that there exists elements in strong involution but not in total involution : as we can see in Lemma 3.1.7 (with notations therein), $\{u^k, w_\alpha v_\beta u^l\} = 0$ but going through the proof of that statement we can remark that $\{w_\alpha v_\beta u^l, u^k\} \neq 0$. We can also remark that, in the definition of a strongly involutive element, we can equivalently require all powers to be in total involution.

Lemma 2.3.36 *If $c_1, c_2 \in A$ are in strong involution with an element $b \in A$, then $\{b, c_1 c_2\} = 0$.*

Proof. The bracket $\{\!\{-,-\}\!\}$ has the derivation property *(D2)* in its second variable, so that we have $\{b, c_1 c_2\} = c_1\{b, c_2\} + \{b, c_1\}c_2$. □

For $b \in A$, let us form the subalgebra generated by B and some fixed elements in strong involution with b. Bringing Lemma 2.3.36 together with the B-linearity of the double bracket, we can see that any element of that subalgebra is also in strong involution with b.

Definition 2.3.37 *Let $b \in A$ be a strongly involutive element. An* involutive chain *with respect to b is given by algebra inclusions $\mathbb{C}[b] \subset I(b) \subset A$, where $\mathbb{C}[b] \subset A$ denotes the subalgebra generated by b and the subalgebra B, while $I(b) \subset A$ is a subalgebra generated by b, the subalgebra B, and some elements that are in strong involution with b^k for all $k \in \mathbb{N}$.*

Note that without the assumption that b is strongly involutive, the first inclusion could be false. Moreover, an element in strong involution only with b is not necessarily in strong involution with any b^k, $k \geq 2$, since there is no derivation property in the first variable for the associated bracket (2.14).

We can now look at the structure induced on representation spaces. Let $\mathcal{O}_{I(b)}$ be the subring of $\mathbb{C}[\mathrm{Rep}(A, \alpha)]$ generated by elements $\{\mathrm{tr}\,\mathcal{X}(c)^k \mid c \in I(b), k \in \mathbb{N}\}$. Let $\hat{\mathcal{O}}_{I(b)}$ be the Poisson subalgebra generated by $\mathcal{O}_{I(b)}$ in the (quasi-)Poisson algebra $\mathbb{C}[\mathrm{Rep}(A, \alpha)]$.

Proposition 2.3.38 *The centre $\mathcal{Z}(\hat{\mathcal{O}}_{I(b)})$ contains $\mathbb{C}[\mathrm{tr}\,\mathcal{X}(b)^k \mid k \in \mathbb{N}]$.*

Proof. Assume that $c \in I(z)$. Then, as in the proof of Proposition 2.3.31, $\{\operatorname{tr} \mathcal{X}(b)^k, \operatorname{tr} \mathcal{X}(c)\} = \operatorname{tr} \mathcal{X}(\{z^k, c\})$ for all k, which is zero by assumption on c. This implies $\{\operatorname{tr} \mathcal{X}(b)^k, \mathcal{O}_{I(b)}\} = 0$ by Leibniz rule.

By definition, an element $\gamma \in \hat{\mathcal{O}}_{I(b)}$ is a sum of terms which are obtained from a finite number of elements $g_1, \dots, g_l \in \mathcal{O}_{I(b)}$, by multiplication or taking Poisson brackets. Using the linearity of the bracket, we just need to show that it holds by induction on k. If $k = 2$, either $\gamma = g_1 g_2$ in which case $\{\operatorname{tr} \mathcal{X}(b)^k, \gamma\} = 0$ by the first part of the proof, or $\gamma = \{g_1, g_2\}$. Now, recall that the (quasi-)Poisson bracket $\{-,-\}$ satisfies Jacobi identity on functions of the form $\operatorname{tr}(c)$. In particular, it implies that the bracket is Poisson on $\mathcal{O}_{I(b)}$, hence

$$\{\operatorname{tr} \mathcal{X}(b)^k, \{g_1, g_2\}\} = \{\{\operatorname{tr} \mathcal{X}(b)^k, g_1\}, g_2\} + \{g_1, \{\operatorname{tr} \mathcal{X}(b)^k, g_2\}\} = 0\,.$$

A similar argument works to show the induction step. □

We find in this way a criterion for degenerate integrability by combining Definition 2.1.5 and Proposition 2.3.38.

Corollary 2.3.39 *Assume that $z \in A$ is strongly involutive. Let $b_1, \dots, b_n \in A$, with $n = \dim \mathcal{M}_{red} - m$, be in strong involution with all $(z^k)_{k\in\mathbb{N}}$. If for some $J \subset \mathbb{N}$ with $|J| = m$, the functions $(\operatorname{tr} \mathcal{X}(z^j))_{j\in J}$ and $\operatorname{tr} \mathcal{X}(b_1), \dots, \operatorname{tr} \mathcal{X}(b_n)$ are functionally independent on a dense open subset, then they form a degenerately integrable system.*

Our understanding of (degenerately) integrable systems in Chapter 4 and Chapter 5 will be mostly based on a correct choice of local coordinates where one of the functions given in Corollary 2.3.33 or Corollary 2.3.39 has an interesting form, see e.g. Example 2.1.4 which is based on an underlying Hamiltonian algebra by Example 2.3.20. Meanwhile, recall that it was noticed by Kazhdan, Kostant and Sternberg [95] that flows of integrable systems on a phase space obtained by Hamiltonian reduction could be easier to integrate on the initial space, and that it was the case for the Calogero-Moser system. With the notation of Example 2.1.4, the flow under $H_k = \frac{1}{k} \operatorname{tr} Y^k$ in $\mathcal{C}_n$ can be obtained from the flow defined by H_k in $\mathcal{M}$, where it takes the linear form $X(t) = X(0) + tY(0)^{k-1}$, while $Y(t), V(t)$ and $W(t)$ are constant. Indeed, it suffices to look at a slice around the point at time t where X is diagonalisable (which is possible generically). In that way, we can obtain the values of $(q_i(t), p_i(t))_i$ up to permutations. Therefore, if we identify a (degenerately) integrable system on $\mathcal{M}_{red}$, we can try to see if the equations of motion for $G(\alpha)$-invariant functions on $\operatorname{Rep}(A,\alpha)$ that descend to the (degenerately) integrable system can be explicitly integrated on $\operatorname{Rep}(A,\alpha)$.

Following this line of thought, assume that we want to integrate in $\operatorname{Rep}(A,\alpha)$ the flow associated to a function $\operatorname{tr}\mathcal{X}(b)$. We need to know the equations of motion for a set of generators of the coordinate ring, which are entries of some matrices $\mathcal{X}(a_1),\ldots,\mathcal{X}(a_k)$ if $a_1,\ldots,a_k$ generate A over B. Hence, we just need to analyse the k matrix-valued equations of motion $\{\operatorname{tr}\mathcal{X}(b),\mathcal{X}(a_i)\}$, which by (2.31) are nothing else than

$$\{\operatorname{tr}\mathcal{X}(b),\mathcal{X}(a_i)\}=\mathcal{X}(\{b,a_i\})\,, \tag{2.38}$$

where the bracket on the right-hand side is the associated bracket in A. Hence, given a family of elements in involution $b_1,\ldots,b_n$ and generators $a_1,\ldots,a_k$ of A, the discussion motivates the computation of the associated brackets $\{b_j,a_i\}$ for $1\le j\le n$ and $1\le i\le k$. A careful reader can notice that two points need to be clarified :

- Is it true that the flows are constrained to the subspace $\operatorname{Rep}(A^q_{red},\alpha)$?
- Do the flows defined for $b,b'\in A$ by (2.38) descend to the same flows on $\mathcal{M}_{red}$ if the elements b,b' induce the same function on $\mathcal{M}_{red}$?

We discuss the quasi-Hamiltonian case where $A^q_{red}=A/(\Phi-q)$, as the Hamiltonian case is shown along those lines and is easier.

To answer the first question, remark that since $\Phi=\sum_s\Phi_s$ is a multiplicative moment map, we have by (2.19)

$$\{b,\Phi_s\}=-m\circ\{\!\{\Phi_s,b\}\!\}^\circ=-\frac{1}{2}m\circ(\Phi_s\otimes be_s-\Phi_sb\otimes e_s+e_s\otimes b\Phi_s-e_sb\otimes\Phi_s)=0\,.$$

Hence, by B-linearity, $\{b,\Phi-q\}=0$.

To answer the second question, we obviously have by (2.38) that $\{\operatorname{tr}\mathcal{X}(b),\mathcal{X}(a_i)\}=\{\operatorname{tr}\mathcal{X}(b'),\mathcal{X}(a_i)\}$ if $b-b'\in[A,A]$. Hence, it remains to show that the evolution of any $\mathcal{X}(a_i)$ under the flow associated to $\{\operatorname{tr}\mathcal{X}(\Phi-q),-\}$ is constant in $\mathcal{M}_{red}$. Using again (2.19), we have for any $a\in A$

$$\{\Phi-q,a\}=\frac{1}{2}\sum_s m\circ(ae_s\otimes\Phi_s-e_s\otimes\Phi_sa+a\Phi_s\otimes e_s-\Phi_s\otimes e_sa)=a\Phi-\Phi a\,.$$

Thus, we get for the flow associated to $\operatorname{tr}\mathcal{X}(\Phi - q)$ that $\mathcal{X}(a)(t) = e^{-t\Phi}\mathcal{X}(a)e^{t\Phi}$ on $\operatorname{Rep}(A^q_{red}, \alpha)$. This means that such flows evolve along an orbit in $\operatorname{Rep}(A^q_{red}, \alpha)$, so that the induced dynamics in $\mathcal{M}_{red}$ is trivial.

Remark 2.3.40 *We could be interested in the right-hand sides of* (2.38) *only, and interpret it as systems of ordinary differential equations on a non-commutative algebra. We will* not *treat that problem at all. However the interested reader can take a considerable step, though trivial, in that direction by looking at the matrix flows that we derive as being of the form $\mathcal{X}(a(t) = ...)$. For further developments on the subject, see [14, 119, 127] and references therein. See also [15] for a slight modification of double brackets to understand Kontsevich system introduced in [173].*

Chapter 3

Quasi-Hamiltonian algebras defined from cyclic quivers

For the whole chapter, we heavily rely on the conventions introduced at the end of Chapter 1 and the constructions introduced in Chapter 2.

3.1 Jordan quiver

Fix $d \in \mathbb{N}^\times$. In this section, we look at a general framing of a Jordan quiver by d arrows, which corresponds in the simplest case $d = 1$ to the tadpole quiver from Example 2.2.1. To be precise, let Q be the quiver with vertices $\{0, \infty\}$, and arrows $x : 0 \to 0$ and $v_\alpha : \infty \to 0$ for $1 \leq \alpha \leq d$. The double $\bar{Q}$ of Q consists of the additional arrows $y = x^* : 0 \to 0$ and $w_\alpha = v_\alpha^* : 0 \to \infty$ for $1 \leq \alpha \leq d$. We write $\mathbb{C}\bar{Q}$ for the corresponding path algebra, and A for the algebra obtained by (locally) inverting the elements $e_{t(a)} + aa^*$ for all $a \in \bar{Q}$. These are B-algebras for $B = \mathbb{C}e_0 \oplus \mathbb{C}e_\infty$. We consider the following ordering on the vertices of $\bar{Q}$ from now on

$$\begin{aligned} \text{at } \infty : \quad & v_1 < w_1 < \ldots < v_d < w_d \,, \\ \text{at } 0 : \quad & x < y < v_1 < w_1 < \ldots < v_d < w_d \,. \end{aligned} \tag{3.1}$$

Note that it is induced by the *total* order $x < y < v_1 < w_1 < \ldots < v_d < w_d$ on $\bar{Q}$.

Remark 3.1.1 *For the remainder of this section, the Greek indices $\alpha, \beta, \gamma, \epsilon$ are always assumed to take value in the set $\{1, \ldots, d\}$.*

3.1.1 Quasi-Hamiltonian formalism

The algebra $A = \mathbb{C}\bar{Q}_{(e_{t(a)}+aa^*)}$ is quasi-Hamiltonian by Theorem 2.3.9. To understand this structure, we first describe the double quasi-Poisson bracket $\{\!\{-,-\}\!\}$ induced by P, which is well-defined on $\mathbb{C}\bar{Q}$. From Proposition 2.3.12, we have that

$$\{\!\{x,x\}\!\} = \frac{1}{2}\left(x^2 \otimes e_0 - e_0 \otimes x^2\right) , \quad \{\!\{y,y\}\!\} = -\frac{1}{2}\left(y^2 \otimes e_0 - e_0 \otimes y^2\right) , \tag{3.2a}$$

$$\{\!\{x,y\}\!\} = e_0 \otimes e_0 + \frac{1}{2}yx \otimes e_0 + \frac{1}{2}e_0 \otimes xy + \frac{1}{2}(y \otimes x - x \otimes y) , \tag{3.2b}$$

$$\{\!\{x,w_\alpha\}\!\} = \frac{1}{2}e_0 \otimes xw_\alpha - \frac{1}{2}x \otimes w_\alpha , \quad \{\!\{x,v_\alpha\}\!\} = \frac{1}{2}v_\alpha x \otimes e_0 - \frac{1}{2}v_\alpha \otimes x , \tag{3.2c}$$

$$\{\!\{y,w_\alpha\}\!\} = \frac{1}{2}e_0 \otimes yw_\alpha - \frac{1}{2}y \otimes w_\alpha , \quad \{\!\{y,v_\alpha\}\!\} = \frac{1}{2}v_\alpha y \otimes e_0 - \frac{1}{2}v_\alpha \otimes y , \tag{3.2d}$$

while between framing arrows it takes the form

$$\{\!\{v_\alpha,v_\beta\}\!\} = -\frac{1}{2}\,o(\alpha,\beta)\,(v_\beta \otimes v_\alpha + v_\alpha \otimes v_\beta) , \tag{3.3a}$$

$$\{\!\{w_\alpha,w_\beta\}\!\} = -\frac{1}{2}\,o(\alpha,\beta)\,(w_\beta \otimes w_\alpha + w_\alpha \otimes w_\beta) , \tag{3.3b}$$

$$\begin{aligned}\{\!\{v_\alpha,w_\beta\}\!\} = &\delta_{\alpha\beta}\left(e_0 \otimes e_\infty + \frac{1}{2}w_\alpha v_\alpha \otimes e_\infty + \frac{1}{2}e_0 \otimes v_\alpha w_\alpha\right) \\ &+ \frac{1}{2}\,o(\alpha,\beta)\,(e_0 \otimes v_\alpha w_\beta + w_\beta v_\alpha \otimes e_\infty) ,\end{aligned} \tag{3.3c}$$

where $o(-,-)$ is the ordering function on d elements defined in Section 1.5. To derive (3.3a), note that Proposition 2.3.12 gives for $\alpha < \beta$ that $\{\!\{v_\alpha, v_\beta\}\!\} = -\frac{1}{2}(v_\beta \otimes v_\alpha + v_\alpha \otimes v_\beta)$. This is because $v_\alpha < v_\beta$, and their heads and tails coincide. We then find (3.3a) by cyclic antisymmetry of the double bracket. Identities (3.3b) and (3.3c) are obtained in the same way. Next, we can write the moment map in A with respect to this quasi-Poisson bracket using Theorem 2.3.9, and it is given by $\Phi = \Phi_0 + \Phi_\infty$ where

$$\Phi_0 = (e_0 + xy)(e_0 + yx)^{-1} \prod_{\alpha=1,\ldots,d}^{\longrightarrow} (e_0 + w_\alpha v_\alpha)^{-1} \in e_0 A e_0 , \tag{3.4a}$$

$$\Phi_\infty = \prod_{\alpha=1,\ldots,d}^{\longrightarrow} (e_\infty + v_\alpha w_\alpha) \in e_\infty A e_\infty . \tag{3.4b}$$

For $q_0, q_\infty \in \mathbb{C}^\times$, we set $\widetilde{q} = q_0 e_0 + q_\infty e_\infty$ and define $\Lambda^{\widetilde{q}} = A/(\Phi - \widetilde{q})$. Since our ordering comes from a total ordering, $\Lambda^{\widetilde{q}}$ is a multiplicative preprojective algebra associated to Q with parameter $\widetilde{q}$ as defined in [56].

3.1.2 An interesting localisation

Consider the algebra $A' = A_x$ obtained by formally adjoining an element $x^{-1} = e_0 x^{-1} e_0$ such that $xx^{-1} = e_0 = x^{-1}x$. As $(e_0 + xy)^{-1} \in A$, both $z := y + x^{-1}$ and z^{-1} lie in A'. The double bracket descends to A', and it takes the following form with z

$$\{\!\{z, z\}\!\} = -\frac{1}{2}\left(z^2 \otimes e_0 - e_0 \otimes z^2\right) , \tag{3.5a}$$

$$\{\!\{x, z\}\!\} = \frac{1}{2} zx \otimes e_0 + \frac{1}{2} e_0 \otimes xz + \frac{1}{2}(z \otimes x - x \otimes z) , \tag{3.5b}$$

$$\{\!\{z, w_\alpha\}\!\} = \frac{1}{2} e_0 \otimes z w_\alpha - \frac{1}{2} z \otimes w_\alpha , \quad \{\!\{z, v_\alpha\}\!\} = \frac{1}{2} v_\alpha z \otimes e_0 - \frac{1}{2} v_\alpha \otimes z , \tag{3.5c}$$

To show this, we use the B-linearity of the double bracket and the derivation rules *(D2)–(D2')*. For example, $\{\!\{x, z\}\!\} = \{\!\{x, y\}\!\} + \{\!\{x, x^{-1}\}\!\}$ and the first term is given by (3.2b). For the second, we have for any $a \in A$

$$0 = \{\!\{a, e_0\}\!\} = \{\!\{a, xx^{-1}\}\!\} = \{\!\{a, x\}\!\} x^{-1} + x \{\!\{a, x^{-1}\}\!\} ,$$

so that $\{\!\{x, x^{-1}\}\!\} = -x^{-1} \{\!\{x, x\}\!\} x^{-1}$, which is $\frac{1}{2}(x^{-1} \otimes x - x \otimes x^{-1})$ by (3.2a). The algebra A' also inherits the moment map. Thus $(A, \{\!\{-,-\}\!\}, \Phi)$ is quasi-Hamiltonian for the double bracket given above and the moment map $\Phi = \Phi_0 + \Phi_\infty$ defined by (3.4a)–(3.4b). In particular, we can rewrite (3.4a) as

$$\Phi_0 = xzx^{-1}z^{-1} \prod_{\alpha=1,\dots,d}^{\longrightarrow} (e_0 + w_\alpha v_\alpha)^{-1} . \tag{3.6}$$

The localisation of the algebra $\Lambda^{\widetilde{q}}$ is now defined as $A'/(\Phi - \widetilde{q})$, which can be understood as the algebra A' with the relations

$$xzx^{-1}z^{-1} = q_0 \prod_{\alpha=1,\dots,d}^{\longleftarrow} (e_0 + w_\alpha v_\alpha) , \qquad \prod_{\alpha=1,\dots,d}^{\longrightarrow} (e_\infty + v_\alpha w_\alpha) = q_\infty e_\infty .$$

It seems natural to choose a way to rewrite the products appearing in these relations as sums. In our case, we will see in Chapter 4 that we only care about the first identity, which leads us to the introduction of *spin elements* a'_α, c'_α in A', defined as follows :

$$a'_\alpha = w_\alpha , \quad c'_1 = v_1 z , \quad c'_\alpha = v_\alpha (e_0 + w_{\alpha-1} v_{\alpha-1}) \dots (e_0 + w_1 v_1) z . \tag{3.7}$$

Note that we can define the c'_α inductively using

$$c'_\alpha = \sum_{\lambda=1}^{\alpha-1} v_\alpha w_\lambda c'_\lambda + v_\alpha z\,. \tag{3.8}$$

With this choice, the first relation is equivalent to $xzx^{-1} = q_0 z + q_0 \sum_\alpha a'_\alpha c'_\alpha$.

We are interested in the double brackets between the elements $(x, z, a'_\alpha, c'_\alpha)$, and the only ones that we do not have are those involving c'_α. The proofs of the next few results are postponed to § 3.3.1.

Proposition 3.1.2 *For any $\alpha, \beta = 1, \dots, d$,*

$$\{\!\{x, c'_\alpha\}\!\} = \frac{1}{2} c'_\alpha x \otimes e_0 + \frac{1}{2} c'_\alpha \otimes x\,, \quad \{\!\{z, c'_\alpha\}\!\} = -\frac{1}{2} c'_\alpha z \otimes e_0 + \frac{1}{2} c'_\alpha \otimes z \tag{3.9a}$$

$$\begin{aligned} \{\!\{a'_\alpha, c'_\beta\}\!\} =& -\frac{1}{2} c'_\beta a'_\alpha \otimes e_0 + \frac{1}{2}\left(o(\alpha,\beta) - \delta_{\alpha\beta}\right) e_\infty \otimes a'_\alpha c'_\beta \\ & - \delta_{\alpha\beta}\left(e_\infty \otimes z + \sum_{\lambda=1}^{\beta-1} e_\infty \otimes a'_\lambda c'_\lambda \right), \end{aligned} \tag{3.9b}$$

where the last sum is omitted for $\beta = 1$.

Lemma 3.1.3 *For any $\alpha, \beta = 1, \dots, d$,*

$$\{\!\{v_\alpha, c'_\beta\}\!\} = \frac{1}{2} c'_\beta \otimes v_\alpha - \frac{1}{2}\left(o(\alpha,\beta) + \delta_{\alpha\beta}\right) v_\alpha \otimes c'_\beta\,. \tag{3.10}$$

Proposition 3.1.4 *For any $\alpha, \beta = 1, \dots, d$,*

$$\{\!\{c'_\alpha, c'_\beta\}\!\} = \frac{1}{2} o(\alpha,\beta)\left(c'_\beta \otimes c'_\alpha - c'_\alpha \otimes c'_\beta\right)\,. \tag{3.11}$$

Note that in the case $d = 1$ we can simply write

$$\{\!\{a'_1, c'_1\}\!\} = -\frac{1}{2} c'_1 a'_1 \otimes e_0 - \frac{1}{2} e_\infty \otimes a'_1 c'_1 - e_\infty \otimes z\,, \quad \{\!\{c'_1, c'_1\}\!\} = 0\,,$$

instead of (3.9b) and (3.11).

3.1.3 Associated brackets

Recall that the bracket $\{-,-\}$ associated to $\{\!\{-,-\}\!\}$ by (2.15) defines a left Loday bracket on A, which descends to a Lie bracket on $A/[A,A]$, see § 2.3.2. Moreover, since the left

Loday bracket is a derivation in the second variable, it descends to an H_0-Poisson structure on $A/[A,A]$. Similarly, this holds for A' and $A'/[A',A']$. This structure defines a Poisson bracket on corresponding varieties, which is obtained by quasi-Hamiltonian reduction by Theorem 2.3.24. Furthermore, the latter Poisson bracket is completely characterised by the associated bracket using (2.36). This motivates the results that we gather now, and their possible relation to integrability as in § 2.3.4.

General results

Now, we derive several identities involving the associated bracket (both as a Loday bracket on A or a Lie bracket on $A/[A,A]$), whose importance will be made precise in Chapter 4. All proofs are postponed to § 3.3.2.

Lemma 3.1.5 *For any $k,l \geq 1$, we get in $A'/[A',A']$*

$$\{x^k, x^l\} = 0\,, \quad \{x^k, a'_\alpha c'_\beta x^l\} = k\, a'_\alpha c'_\beta x^{k+l}\,, \tag{3.12a}$$

$$\begin{aligned}
\{a'_\gamma c'_\epsilon x^k, a'_\alpha c'_\beta x^l\} =& \frac{1}{2}\left(\sum_{r=1}^{k} - \sum_{r=1}^{l}\right)\left(a'_\alpha c'_\beta x^r a'_\gamma c'_\epsilon x^{k+l-r} + a'_\alpha c'_\beta x^{k+l-r} a'_\gamma c'_\epsilon x^r\right)\\
&+ \frac{1}{2} o(\alpha,\gamma)(a'_\gamma c'_\epsilon x^k a'_\alpha c'_\beta x^l + a'_\alpha c'_\epsilon x^k a'_\gamma c'_\beta x^l)\\
&+ \frac{1}{2} o(\epsilon,\beta)(a'_\alpha c'_\beta x^k a'_\gamma c'_\epsilon x^l - a'_\alpha c'_\epsilon x^k a'_\gamma c'_\beta x^l)\\
&+ \frac{1}{2}[o(\epsilon,\alpha) + \delta_{\alpha\epsilon}]\, a'_\alpha c'_\epsilon x^k a'_\gamma c'_\beta x^l - \frac{1}{2}[o(\beta,\gamma) + \delta_{\beta\gamma}]\, a'_\alpha c'_\epsilon x^k a'_\gamma c'_\beta x^l\\
&+ \delta_{\alpha\epsilon}\left(z x^k + \sum_{\lambda=1}^{\epsilon-1} a'_\lambda c'_\lambda x^k\right) a'_\gamma c'_\beta x^l\\
&- \delta_{\beta\gamma}\, a'_\alpha c'_\epsilon x^k \left(z x^l + \sum_{\mu=1}^{\beta-1} a'_\mu c'_\mu x^l\right)\,.
\end{aligned} \tag{3.12b}$$

In fact, (3.12a) holds in A' for the left Loday bracket $\{-,-\}$. A similar statement is true for the different variants of Lemma 3.1.5 that we state in the remainder of this subsection and in § 3.2.3. (We will not write it explicitly to keep those results as simple as possible.)

As a first variant of Lemma 3.1.5, it seems natural to try to write the analogue of this result when x is replaced by z.

Lemma 3.1.6 *For any $k,l\geq 1$, we get in $A'/[A',A']$*

$$\begin{aligned}
\{z^k,z^l\} =&0\,,\quad \{z^k,a'_\alpha c'_\beta z^l\}=0\,, &\text{(3.13a)}\\
\{a'_\gamma c'_\epsilon z^k, a'_\alpha c'_\beta z^l\} =&\frac{1}{2}\left(\sum_{r=1}^{k}+\sum_{r=1}^{l}\right)\left(a'_\alpha c'_\beta z^r a'_\gamma c'_\epsilon z^{k+l-r} - a'_\alpha c'_\beta z^{k+l-r} a'_\gamma c'_\epsilon z^r\right)\\
&+ a'_\alpha c'_\beta a'_\gamma c'_\epsilon z^{k+l} + \frac{1}{2}o(\alpha,\gamma)(a'_\gamma c'_\epsilon z^k a'_\alpha c'_\beta z^l + a'_\alpha c'_\epsilon z^k a'_\gamma c'_\beta z^l)\\
&- a'_\alpha c'_\beta z^{k+l} a'_\gamma c'_\epsilon + \frac{1}{2}o(\epsilon,\beta)(a'_\alpha c'_\beta z^k a'_\gamma c'_\epsilon z^l - a'_\alpha c'_\epsilon z^k a'_\gamma c'_\beta z^l)\\
&+\frac{1}{2}[o(\epsilon,\alpha)+\delta_{\alpha\epsilon}]\, a'_\alpha c'_\epsilon z^k a'_\gamma c'_\beta z^l - \frac{1}{2}[o(\beta,\gamma)+\delta_{\beta\gamma}]\, a'_\alpha c'_\epsilon z^k a'_\gamma c'_\beta z^l\\
&+\delta_{\alpha\epsilon}\left(a'_\gamma c'_\beta z^{k+l+1} + \sum_{\lambda=1}^{\epsilon-1} a'_\lambda c'_\lambda z^k a'_\gamma c'_\beta z^l\right)\\
&-\delta_{\beta\gamma}\left(a'_\alpha c'_\epsilon z^{k+l+1} + \sum_{\mu=1}^{\beta-1} a'_\alpha c'_\epsilon z^k a'_\mu c'_\mu z^l\right)\,. &\text{(3.13b)}
\end{aligned}$$

Moreover, (3.13a) *holds for any $k,l\in\mathbb{Z}$.*

This time, we can see that $\mathbb{C}[z^{\pm 1}]/\left[\mathbb{C}[z^{\pm 1}],\mathbb{C}[z^{\pm 1}]\right]$ is an abelian subalgebra of $A'/[A',A']$, whose Lie bracket with any element in the infinite set $\{a'_\alpha c'_\beta z^k \mid k\in\mathbb{Z},\ 1\leq \alpha,\beta\leq d\}$ vanishes.

A modification of those two results consists in using the elements $(v_\alpha,w_\alpha)_\alpha$ that originally appear in the definition of A (or A'). Moreover, it allows us to work in greater generalities. Consider $u\in\{x,y,z,e_0+xy\}$, and assume that we are in A if $u\neq z$, and in A' if $u=z$. We have that $\epsilon(x)=+1$, $\epsilon(y)=-1$, and we choose to set $\epsilon(z)=-1$, $\epsilon(e_0+xy)=+1$. Then we can write $\{\!\{u,u\}\!\}=\frac{1}{2}\epsilon(u)[u^2\otimes e_0 - e_0\otimes u^2]$. Furthermore, we have in each case

$$\{\!\{u,w_\alpha\}\!\}=\frac{1}{2}e_0\otimes uw_\alpha - \frac{1}{2}u\otimes w_\alpha\,,\quad \{\!\{u,v_\alpha\}\!\}=\frac{1}{2}v_\alpha u\otimes e_0 - \frac{1}{2}v_\alpha\otimes u\,. \tag{3.14}$$

Indeed, all these double brackets are given in that form in § 3.1.1 and § 3.1.2, or they can be computed for $u=e_0+xy$. Consider the elements $w_\alpha v_\beta u^l$ for any $l\in\mathbb{N}$ and $\alpha,\beta=1,\dots,d$. In the case $u=z$, we have for all α that $w_\alpha v_1 z^l = a'_\alpha c'_1 z^{l-1}$.

Lemma 3.1.7 *For any $k,l \geq 1$, we have in $A/[A,A]$ (or in $A'/[A',A']$ when $u=z$) that*

$$\{u^k, u^l\} = 0\,, \quad \{u^k, w_\alpha v_\beta u^l\} = 0\,, \tag{3.15a}$$

$$\begin{aligned}
\{w_\gamma v_\epsilon u^k, w_\alpha v_\beta u^l\} =& \frac{1}{2}\left[o(\gamma,\beta) + o(\epsilon,\alpha) - o(\epsilon,\beta) - o(\gamma,\alpha)\right] w_\alpha v_\epsilon u^k\, w_\gamma v_\beta u^l \\
&+ \frac{1}{2} o(\gamma,\beta)\; w_\alpha v_\beta\, w_\gamma v_\epsilon u^{k+l} + \frac{1}{2} o(\epsilon,\alpha)\; w_\alpha v_\beta u^{k+l}\, w_\gamma v_\epsilon \\
&- \frac{1}{2} o(\epsilon,\beta)\; w_\alpha v_\beta u^k\, w_\gamma v_\epsilon u^l - \frac{1}{2} o(\gamma,\alpha)\; w_\alpha v_\beta u^l\, w_\gamma v_\epsilon u^k \\
&- \delta_{\gamma\beta}\left[w_\alpha v_\epsilon u^{k+l} + \frac{1}{2} w_\alpha v_\beta\, w_\gamma v_\epsilon u^{k+l} + \frac{1}{2} w_\alpha v_\epsilon u^k\, w_\gamma v_\beta u^l\right] \\
&+ \delta_{\alpha\epsilon}\left[w_\gamma v_\beta u^{k+l} + \frac{1}{2} w_\alpha v_\beta u^{k+l}\, w_\gamma v_\epsilon + \frac{1}{2} w_\alpha v_\epsilon u^k\, w_\gamma v_\beta u^l\right] \\
&+ \frac{1}{2}\epsilon(u)\left[\sum_{\tau=1}^{k-1} w_\alpha v_\beta u^{k+l-\tau} w_\gamma v_\epsilon u^\tau + \sum_{\sigma=1}^{l} w_\alpha v_\beta u^{k+\sigma} w_\gamma v_\epsilon u^{l-\sigma}\right] \\
&- \frac{1}{2}\epsilon(u)\left[\sum_{\sigma=1}^{l-1} w_\alpha v_\beta u^{\sigma} w_\gamma v_\epsilon u^{k+l-\sigma} + \sum_{\tau=1}^{k} w_\alpha v_\beta u^{k-\tau} w_\gamma v_\epsilon u^{l+\tau}\right]
\end{aligned} \tag{3.15b}$$

The next result follows from Definitions 2.3.35, 2.3.37 and the remark after Lemma 3.1.5.

Corollary 3.1.8 *The element u is strongly involutive. Furthermore, if $I(u)$ is the subalgebra of A (or A') generated by B, u and elements $w_\alpha v_\beta u^k$, we get an involutive chain $\mathbb{C}[u] \subset I(u) \subset A$ (or A').*

Embeddings of quivers

We now consider two constructions of families of elements in involution, which are associated to subquivers of $\bar{Q}$. Proofs for all non-trivial results are provided in § 3.3.3.

First, let $\bar{Q}_0$ be the quiver with vertices $\{0,\infty\}$ and arrows $x, y : 0 \to 0$. In terms of the construction described before Lemma 2.3.14, this is the subquiver $\bar{Q}'$ supported at $I' = \{0\}$ to which we add the vertex ∞ as a disconnected component. Next, let $\bar{Q}_1$ be obtained from $\bar{Q}_0$ by adding arrows $v_1 : \infty \to 0$ and $w_1 : 0 \to \infty$. Similarly, we can consider $\bar{Q}_1, \ldots, \bar{Q}_d$, where $\bar{Q}_\alpha$ is obtained from the previous quiver $\bar{Q}_{\alpha-1}$ by adding v_α, w_α. Clearly, $\bar{Q}_d = \bar{Q}$. We can take on each subquiver $\bar{Q}_\alpha$ the ordering obtained by restricting (3.1) to the arrows of $\bar{Q}_\alpha$. This gives a quasi-Hamiltonian structure on A_α, the algebra $\mathbb{C}\bar{Q}_\alpha$ localised at the elements $e_{t(a)} + aa^*$ for

$a \in \bar{Q}_\alpha$. This yields in turn a chain of quasi-Hamiltonian algebras

$$A_0 \subset A_1 \subset A_2 \subset \ldots \subset A_d = A\,, \tag{3.16}$$

which can be localised at x. In particular, note that the moment map $\Phi^{(\alpha)}$ of A_α is defined in any A_β, $\beta \geq \alpha$. Moreover, it can be obtained inductively as

$$\Phi^{(0)} = (e_0 + xy)(e_0 + yx)^{-1} + e_\infty, \quad \Phi^{(\alpha)} = \Phi^{(\alpha-1)}(1 + w_\alpha v_\alpha)^{-1}(1 + v_\alpha w_\alpha)\,. \tag{3.17}$$

We begin by only considering the embedding $A_0 \subset A$. Let $\phi = \Phi_0 = (e_0 + xy)(e_0 + yx)^{-1}$, or $xzx^{-1}z^{-1}$ when we localise A at x. We also assume that there is a formal inverse in A to $u \in \{x, y, z, e_0 + xy\}$, so we work in the localised algebra if u is not already invertible.

Proposition 3.1.9 *Let $U_{+,\eta} = u(1 + \eta\phi)$, $U_{-,\eta} = u(1 + \eta\phi^{-1})$, for arbitrary $\eta \in \mathbb{C}$ playing the role of a spectral parameter. Then, if $\epsilon(u) = -1$,*

$$\{U^K_{+,\eta}, U^L_{+,\eta'}\} = 0 \quad \textit{mod}\ [A, A], \qquad \textit{for any}\ \eta, \eta' \in \mathbb{C}\,. \tag{3.18}$$

If $\epsilon(u) = +1$,

$$\{U^K_{-,\eta}, U^L_{-,\eta'}\} = 0 \quad \text{mod}\ [A, A], \qquad \text{for any}\ \eta, \eta' \in \mathbb{C}\,. \tag{3.19}$$

The result also holds for $u = x + y^{-1}$, $\epsilon(x + y^{-1}) = +1$, if we decide to localise at y. We get the following by definition 2.3.29.

Corollary 3.1.10 *Assume $\epsilon(u) = -1$ and for all $K \in \mathbb{N}$, develop $U^K_{+,\eta} = \sum_{k=0}^K u_{K,k}\eta^k$. Then any two elements in the set $\{u_{K,k} \mid K \in \mathbb{N},\ 0 \leq k \leq K\}$ are in involution. The same holds for $\epsilon(u) = +1$ with $U_{-,\eta}$ instead.*

Proof. We use Proposition 3.1.9 to get all brackets in the discussion below. First, taking $\eta = \eta' = 0$, we get that $\{u_{K,0}, u_{L,0}\} = 0$ for all $K, L \in \mathbb{N}$ (though we already knew it by Lemma 3.1.7). This gives by linearity, taking only $\eta' = 0$, $\{U^K_{+,\eta} - u_{K,0}, u_{L,0}\} = 0$. Since $\frac{1}{\eta}(U^K_{+,\eta} - u^K) = u_{K,1} + o(\eta)$, we get from our previous computation that $\{u_{K,1}, u_{L,0}\} = 0$ by dividing by η then taking $\eta = 0$. Repeating the argument, $\{u_{K,k}, u_{L,0}\} = 0$ for all $k \leq K$ and all L. Repeating the argument by developing in terms of η', we can conclude. □

If we consider the full chain given in (3.16), we can get a different result. To state it, we restrict to the cases $u \in \{y, z\}$, and we define the element $u_{(\alpha)} = \Phi^{(\alpha)}u$ for each $0 \leq \alpha \leq d$.

Proposition 3.1.11 *For any $K, L \in \mathbb{N}$, $0 \leq \alpha, \beta \leq d$, the elements $u^K_{(\alpha)}, u^L_{(\beta)}$ are in involution.*

Note that we can write $y_{(0)} = (e_0+xy)y(e_0+xy)^{-1}$ and $z_{(0)} = xzx^{-1}$, hence Proposition 3.1.11 gives other families of elements that commute with any y^K in the Lie algebra $A/[A, A]$ (or z^K in $A'/[A'.A']$) for any d.

Computations for the Loday bracket

The translation of Proposition 3.1.9 on representation spaces is that for $\epsilon(u) = -1$ the functions $(\operatorname{tr}\mathcal{X}(U^K_{+,\eta}))_{K\in\mathbb{N}}$ Poisson commute after quasi-Hamiltonian reduction, as we will see in Chapter 4. It is also true for $(\operatorname{tr}\mathcal{X}(U^K_{-,\eta}))_{K\in\mathbb{N}}$ with $\epsilon(u) = +1$. By definition of the Poisson bracket, we have in fact that such functions are commuting under the *quasi-Poisson* bracket, i.e. on the representation spaces *before* reduction. Hence the flows defined by the corresponding vector fields $\{\operatorname{tr}\mathcal{X}(U^K_{+,\eta}), -\}_{\mathrm{P}}$ (again, before reduction and it is similar for the other families) commute. If we can integrate the flows before reduction, it suffices to project them to get the flows on the reduced space. This section deals with the derivation of the defining vector fields before reduction, see also the end of § 2.3.4 as we will make an extensive use of (2.38).

Lemma 3.1.12 *Write $U_\eta = z(1 + \eta\phi)$ with $\phi = xzx^{-1}z^{-1}$. The left Loday bracket $\{-, -\} : A' \times A' \to A'$ satisfies for any $K \in \mathbb{N}^\times$*

$$\frac{1}{K}\{U^K_\eta, x\} = -\,\eta\phi U^{K-1}_\eta zx - xU^{K-1}_\eta z\,, \quad \frac{1}{K}\{U^K_\eta, z\} = -zU^{K-1}_\eta z + U^{K-1}_\eta z^2\,,$$
$$\frac{1}{K}\{U^K_\eta, v_\beta\} = 0\,, \qquad \frac{1}{K}\{U^K_\eta, w_\beta\} = 0\,.$$

Let A_y be the algebra A localised at y in the same way as we obtained A' from A.

Lemma 3.1.13 *Write $\bar{U}_\eta = y(1 + \eta\phi)$ with $\phi = (e_0 + xy)(e_0 + yx)^{-1}$. The left Loday bracket $\{-, -\} : A_y \times A_y \to A_y$ satisfies for any $K \in \mathbb{N}^\times$*

$$\frac{1}{K}\{\bar{U}^K_\eta, x\} = -\,\bar{U}^{K-1}_\eta - x\bar{U}^{K-1}_\eta y - \eta\phi\bar{U}^{K-1}_\eta(e_0 + yx)\,,$$
$$\frac{1}{K}\{\bar{U}^K_\eta, y\} = -\,y\bar{U}^{K-1}_\eta y + \bar{U}^{K-1}_\eta y^2\,, \qquad \frac{1}{K}\{\bar{U}^K_\eta, v_\beta\} = 0\,, \qquad \frac{1}{K}\{\bar{U}^K_\eta, w_\beta\} = 0\,.$$

Lemma 3.1.14 *Write $\hat{U}_\eta = x(1+\eta\phi^{-1})$ with $\phi = xzx^{-1}z^{-1}$. The left Loday bracket $\{-,-\} : A' \times A' \to A'$ satisfies for any $K \in \mathbb{N}^\times$*

$$\begin{aligned}
\frac{1}{K}\{\hat{U}_\eta^K, x\} =& x\hat{U}_\eta^{K-1}x - \hat{U}_\eta^{K-1}x^2 \,, \\
\frac{1}{K}\{\hat{U}_\eta^K, z\} =& z\hat{U}_\eta^{K-1}x + \eta\phi^{-1}\hat{U}_\eta^{K-1}xz \,, \\
\frac{1}{K}\{\hat{U}_\eta^K, v_\beta\} =& 0 \,, \qquad \frac{1}{K}\{\hat{U}_\eta^K, w_\beta\} = 0 \,.
\end{aligned}$$

Note the similarity of Lemmae 3.1.12 and 3.1.14 if we swap x and z.

Lemma 3.1.15 *Write $\tilde{U}_\eta = (e_0+xy)(1+\eta\phi^{-1})$ with $\phi = (e_0+xy)(e_0+yx)^{-1}$. The left Loday bracket $\{-,-\} : A \times A \to A$ satisfies for any $K \in \mathbb{N}^\times$*

$$\begin{aligned}
\frac{1}{K}\{\tilde{U}_\eta^K, x\} =& -\tilde{U}_\eta^{K-1}(e_0+xy)x - \eta x\phi^{-1}\tilde{U}_\eta^{K-1}(e_0+xy) \,, \\
\frac{1}{K}\{\tilde{U}_\eta^K, (e_0+xy)\} =& (e_0+xy)\tilde{U}_\eta^{K-1}(e_0+xy) - \tilde{U}_\eta^{K-1}(e_0+xy)^2 \,, \\
\frac{1}{K}\{\tilde{U}_\eta^K, v_\beta\} =& 0 \,, \qquad \frac{1}{K}\{\tilde{U}_\eta^K, w_\beta\} = 0 \,.
\end{aligned}$$

All proofs can be found in § 3.3.4. Next, we can state similar results for the elements given in Proposition 3.1.11, see also § 3.3.4 for the proofs.

Lemma 3.1.16 *Write $z_{(\alpha)} = \Phi^{(\alpha)}z$ with $\Phi^{(\alpha)}$ given in (3.17) for $0 \leq \alpha \leq d$. The left Loday bracket $\{-,-\} : A' \times A' \to A'$ satisfies for any $K \in \mathbb{N}^\times$*

$$\begin{aligned}
&\frac{1}{K}\{z_{(\alpha)}^K, x\} = -z_{(\alpha)}^K x \,, \quad \frac{1}{K}\{z_{(\alpha)}^K, z\} = 0 \,, \\
&\frac{1}{K}\{z_{(\alpha)}^K, v_\beta\} = v_\beta z z_{(\alpha)}^{K-1}\Phi^{(\alpha)} \,, \quad \frac{1}{K}\{z_{(\alpha)}^K, w_\beta\} = -zz_{(\alpha)}^{K-1}\Phi^{(\alpha)}w_\beta \,, \quad \beta \leq \alpha \,, \\
&\frac{1}{K}\{z_{(\alpha)}^K, v_\beta\} = 0 \,, \quad \frac{1}{K}\{z_{(\alpha)}^K, w_\beta\} = 0 \,, \quad \beta > \alpha \,.
\end{aligned}$$

Moreover, $\{z_{(\alpha)}^K, z_{(\alpha)}\} = \{z_{(\alpha)}^K, \Phi^{(\alpha)}\} = 0$.

Lemma 3.1.17 *Write $y_{(\alpha)} = \Phi^{(\alpha)}y$ with $\Phi^{(\alpha)}$ given in (3.17) for $0 \leq \alpha \leq d$. The left Loday bracket $\{-,-\} : A \times A \to A$ satisfies for any $K \in \mathbb{N}^\times$*

$$\begin{aligned}
&\frac{1}{K}\{y_{(\alpha)}^K, x\} = -y_{(\alpha)}^{K-1}\Phi^{(\alpha)} - y_{(\alpha)}^K x \,, \quad \frac{1}{K}\{y_{(\alpha)}^K, y\} = 0 \,, \\
&\frac{1}{K}\{y_{(\alpha)}^K, v_\beta\} = v_\beta y y_{(\alpha)}^{K-1}\Phi^{(\alpha)} \,, \quad \frac{1}{K}\{y_{(\alpha)}^K, w_\beta\} = -yy_{(\alpha)}^{K-1}\Phi^{(\alpha)}w_\beta \,, \quad \beta \leq \alpha \,, \\
&\frac{1}{K}\{y_{(\alpha)}^K, v_\beta\} = 0 \,, \quad \frac{1}{K}\{y_{(\alpha)}^K, w_\beta\} = 0 \,, \quad \beta > \alpha \,.
\end{aligned}$$

Moreover, $\{y_{(\alpha)}^K, y_{(\alpha)}\} = \{y_{(\alpha)}^K, \Phi^{(\alpha)}\} = 0$.

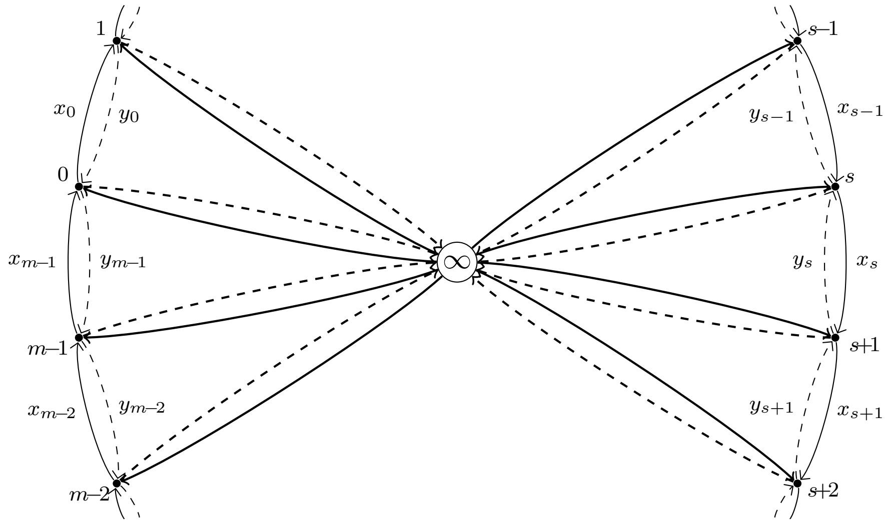

Figure 3: The double of a cyclic quiver on m vertices with general framing. The thick arrow $\infty \to s$ represents the d_s elements $(v_{s,\alpha})_\alpha$, while the thick dashed arrow $s \to \infty$ represents their d_s doubles $(w_{s,\alpha})_\alpha$.

3.2 Cyclic quivers

Fix an integer $m \geq 2$ and let $I = \mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$. When we consider I as a set, we identify it with $\{0, \dots, m-1\}$ by sending an element $s \in I$ to its representative in $\{0, \dots, m-1\}$. Moreover, fix $\mathbf{d} = (d_0, \dots, d_{m-1}) \in \mathbb{N}^I$ such that $|\mathbf{d}| = \sum_{s \in I} d_s \geq 1$. Without loss of generality, we simply assume that $d_0 \geq 1$ while $d_s \in \mathbb{N}$ for $s \in I \setminus \{0\}$.

We look at a quasi-Hamiltonian structure associated to (the path algebra of the double of) a cyclic quiver on m arrows, see Example 2.2.2, with framing corresponding to $\mathbf{d}$. We explicitly define the quiver $\bar{Q}_\mathbf{d}$ in the following way. Let $Q_\mathbf{d}$ be the quiver with vertex set $\widetilde{I} = I \cup \{\infty\}$, and whose edge set consists, for all $s \in I$, of $d_s + 1$ arrows given by $x_s : s \to s+1$ and $v_{s,\alpha} : \infty \to s$ with $\alpha = 1, \dots, d_s$. There is no arrow $\infty \to s$ when $d_s = 0$. The double $\bar{Q}_\mathbf{d}$ of $Q_\mathbf{d}$ is then given by the same vertex set and $2m + 2|\mathbf{d}|$ arrows given by the ones above together with $y_s = x_s^* : s+1 \to s$, $w_{s,\alpha} = v_{s,\alpha}^* : s \to \infty$ for all $1 \leq \alpha \leq d_s$ and $s \in I$, see Figure 3.

We write $\mathbb{C}\bar{Q}_\mathbf{d}$ for the path algebra of $\bar{Q}_\mathbf{d}$, and we let A denote the algebra obtained by inverting the elements $e_{t(a)} + aa^*$ for all $a \in \bar{Q}_\mathbf{d}$. These are B-algebras for $B = \oplus_{s \in I} \mathbb{C}e_s \oplus \mathbb{C}e_\infty$. We consider the ordering $<$ on the vertices of $\bar{Q}_{\mathbf{d}}$ given by

$$\text{at } 0: \qquad x_0 < y_0 < x_{m-1} < y_{m-1} < v_{0,1} < w_{0,1} < \ldots < v_{0,d_0} < w_{0,d_0} \tag{3.20a}$$

$$\text{at } s: \qquad x_s < y_s < x_{s-1} < y_{s-1} < v_{s,1} < w_{s,1} < \ldots < v_{s,d_s} < w_{s,d_s} \tag{3.20b}$$

$$\begin{aligned}\text{at } \infty: \qquad & v_{0,1} < w_{0,1} < \ldots < v_{0,d_0} < w_{0,d_0} < v_{1,1} < w_{1,1} < \ldots \\ & \ldots < v_{m-1,1} < w_{m-1,1} < \ldots < v_{m-1,d_{m-1}} < w_{m-1,d_{m-1}}\end{aligned} \tag{3.20c}$$

Here, we omit the elements $v_{s,1}, \ldots, w_{s,d_s}$ in the ordering at a vertex when $d_s = 0$. The ordering just defined *is not* induced by a total order on $\bar{Q}_{\mathbf{d}}$.

Remark 3.2.1 *For the remainder of this section, we adopt several conventions, some of which have already been noted.*

The indices r, s, p, q range in I, and when we write $o(r, s)$ we mean $o_m(r', s')$, the value of the ordering function on m elements (defined in Section 1.5) evaluated on the pair (r', s') which is the representative of (r, s) under the identification of $I \times I$ and $\{0, \ldots, m-1\}^{\times 2}$ as sets.

When we consider a couple (s, α), for example as index of $v_{s,\alpha}$, we assume that $s \in I$ as we have just explained and α ranges over the set $\{1, \ldots, d_s\}$. We omit such couples when $d_s = 0$. We call those pairs (s, α) the admissible spins (or spin indices). *The greek indices $\alpha, \beta, \gamma, \epsilon$ corresponding to such couples follow the same convention.*

The notation $\delta_{rs} o(\alpha, \beta)$ corresponding to admissible spins $(s, \alpha), (r, \beta)$ takes the value 0 if $r \neq s$, while if $r = s$ it denotes $o_{d_s}(\alpha, \beta)$, the ordering function on d_s elements evaluated on the pair (α, β), which is well-defined since by convention $\alpha, \beta \in \{1, \ldots, d_s\}$. In the latter case if $d_s = 0$, this identically vanishes.

3.2.1 Quasi-Hamiltonian formalism

The algebra $A = \mathbb{C}\bar{Q}_{(e_{t(a)}+aa^*)}$ is quasi-Hamiltonian by Theorem 2.3.9, and we can characterise its double quasi-Poisson bracket using Proposition 2.3.12. On the arrows forming the cycle, we have that

$$\{\!\{x_r,x_s\}\!\} =\frac{1}{2}\delta_{(s,r-1)}\ x_{r-1}x_r\otimes e_r\ -\ \frac{1}{2}\delta_{(s,r+1)}\ e_{r+1}\otimes x_rx_{r+1}\,, \tag{3.21a}$$

$$\{\!\{y_r,y_s\}\!\} =\frac{1}{2}\delta_{(s,r-1)}\ e_r\otimes y_ry_{r-1}\ -\ \frac{1}{2}\delta_{(s,r+1)}\ y_{r+1}y_r\otimes e_{r+1}\,, \tag{3.21b}$$

$$\begin{aligned}\{\!\{x_r,y_s\}\!\} &=\delta_{sr}\left(e_{r+1}\otimes e_r+\frac{1}{2}y_rx_r\otimes e_r+\frac{1}{2}e_{r+1}\otimes x_ry_r\right)\\ &\quad -\frac{1}{2}\delta_{(s,r-1)}\ x_r\otimes y_{r-1}\ +\frac{1}{2}\delta_{(s,r+1)}\ y_{r+1}\otimes x_r\ .\end{aligned} \tag{3.21c}$$

For these arrows with the framing arrows, we can find

$$\{\!\{x_r,w_{s,\alpha}\}\!\} =\frac{1}{2}\delta_{(s,r+1)}\,e_{r+1}\otimes x_rw_{r+1,\alpha}-\frac{1}{2}\delta_{rs}\,x_r\otimes w_{r,\alpha}\,, \tag{3.22a}$$

$$\{\!\{x_r,v_{s,\alpha}\}\!\} =\frac{1}{2}\delta_{rs}\,v_{r,\alpha}x_r\otimes e_r-\frac{1}{2}\delta_{(s,r+1)}\,v_{r+1,\alpha}\otimes x_r\,, \tag{3.22b}$$

$$\{\!\{y_r,w_{s,\alpha}\}\!\} =\frac{1}{2}\delta_{rs}\,e_r\otimes y_rw_{r,\alpha}-\frac{1}{2}\delta_{(s,r+1)}\,y_r\otimes w_{r+1,\alpha}\,, \tag{3.22c}$$

$$\{\!\{y_r,v_{s,\alpha}\}\!\} =\frac{1}{2}\delta_{(s,r+1)}v_{r+1,\alpha}y_r\otimes e_{r+1}-\frac{1}{2}\delta_{rs}v_{r,\alpha}\otimes y_r\,. \tag{3.22d}$$

Finally, between the framing arrows, we get

$$\begin{aligned}\{\!\{v_{s,\alpha},v_{r,\beta}\}\!\} =& -\frac{1}{2}\,o(s,r)v_{s,\alpha}\otimes v_{r,\beta}\\ &-\frac{1}{2}\,\delta_{sr}o(\alpha,\beta)\left(v_{r,\beta}\otimes v_{s,\alpha}+v_{s,\alpha}\otimes v_{r,\beta}\right)\,,\end{aligned} \tag{3.23a}$$

$$\begin{aligned}\{\!\{w_{s,\alpha},w_{r,\beta}\}\!\} =& -\frac{1}{2}\,o(s,r)w_{r,\beta}\otimes w_{s,\alpha}\\ &-\frac{1}{2}\,\delta_{sr}o(\alpha,\beta)\left(w_{r,\beta}\otimes w_{s,\alpha}+w_{s,\alpha}\otimes w_{r,\beta}\right)\,,\end{aligned} \tag{3.23b}$$

$$\begin{aligned}\{\!\{v_{s,\alpha},w_{r,\beta}\}\!\} =&\frac{1}{2}o(s,r)\,w_{r,\beta}v_{s,\alpha}\otimes e_\infty\\ &+\delta_{sr}\delta_{\alpha\beta}\left(e_s\otimes e_\infty+\frac{1}{2}w_{r,\beta}v_{s,\alpha}\otimes e_\infty+\frac{1}{2}e_s\otimes v_{s,\alpha}w_{r,\beta}\right)\\ &+\frac{1}{2}\,\delta_{sr}o(\alpha,\beta)\left(e_s\otimes v_{s,\alpha}w_{r,\beta}+w_{r,\beta}v_{s,\alpha}\otimes e_\infty\right)\,,\end{aligned} \tag{3.23c}$$

All double brackets can be easily obtained. We limit ourselves to show (3.23a), leaving the rest to the reader. To evaluate $\{\!\{v_{s,\alpha},v_{r,\beta}\}\!\}$, if $s=r$, we just have to remark that it is the same computation that leads to (3.3a). So it remains the case $s\neq r$, where we only have $t(v_{s,\alpha})=\infty=t(v_{r,\beta})$. If $s<r$, we get by Proposition 2.3.12 that $\{\!\{v_{s,\alpha},v_{r,\beta}\}\!\}=-\frac{1}{2}v_{s,\alpha}\otimes v_{r,\beta}$. If $s>r$, we have similarly $\{\!\{v_{r,\beta},v_{s,\alpha}\}\!\}=-\frac{1}{2}v_{r,\beta}\otimes v_{s,\alpha}$, so that by cyclic antisymmetry $\{\!\{v_{s,\alpha},v_{r,\beta}\}\!\}=+\frac{1}{2}v_{s,\alpha}\otimes v_{r,\beta}$. Considering all cases at once, we have precisely (3.23a). It will be useful to remark that we can write from (3.23c) (note the change of indices!)

$$\begin{aligned}\{\!\{w_{s,\alpha}, v_{r,\beta}\}\!\} =&\frac{1}{2}o(s,r)e_\infty \otimes w_{s,\alpha}v_{r,\beta} \;-\delta_{sr}\delta_{\alpha\beta}e_\infty \otimes e_s \\ &+\frac{1}{2}\delta_{sr}[o(\alpha,\beta)-\delta_{\alpha\beta}](e_\infty \otimes w_{s,\alpha}v_{r,\beta} + v_{r,\beta}w_{s,\alpha}\otimes e_s)\,.\end{aligned} \tag{3.24}$$

In the form (3.21a)–(3.22d), the equations involving the arrows x_s, y_s do not seem to be a natural generalisation of (3.2a)–(3.2d). To exhibit their similarities, introduce the elements $x = \sum_s x_s$, $y = \sum_s y_s$, so that $x \in \oplus_s e_s A e_{s+1}$ and $y \in \oplus_s e_s A e_{s-1}$. Then we can write (3.22a)–(3.22d) as

$$\begin{aligned}\{\!\{x, w_{s,\alpha}\}\!\} &= \frac{1}{2}\,e_s \otimes x w_{s,\alpha} - \frac{1}{2}\,e_s x \otimes w_{s,\alpha}\,, \quad \{\!\{x, v_{s,\alpha}\}\!\} = \frac{1}{2}\,v_{s,\alpha}x \otimes e_s - \frac{1}{2}\,v_{s,\alpha}\otimes x e_s\,,\\ \{\!\{y, w_{s,\alpha}\}\!\} &= \frac{1}{2}\,e_s \otimes y w_{s,\alpha} - \frac{1}{2}\,e_s y \otimes w_{s,\alpha}\,, \quad \{\!\{y, v_{s,\alpha}\}\!\} = \frac{1}{2}\,v_{s,\alpha}y \otimes e_s - \frac{1}{2}\,v_{s,\alpha}\otimes y e_s\,.\end{aligned}$$

which is similar to (3.2c)–(3.2d). Furthermore, set $F_a = \sum_{s\in I} e_{s+a}\otimes e_s$ for any $a \in \mathbb{Z}$. We have for example

$$F_1 = \sum_{s\in I} e_{i+1}\otimes e_i\,, \quad F_{-1} = \sum_{s\in I} e_{s-1}\otimes e_s = \sum_{s\in I} e_s \otimes e_{s+1}\,.$$

Then, we can show that

$$\{\!\{x,x\}\!\} = \frac{1}{2}\left(x^2F_1 - F_1x^2\right)\,, \quad \{\!\{y,y\}\!\} = -\frac{1}{2}\left(y^2F_{-1} - F_{-1}y^2\right) \tag{3.25a}$$

$$\{\!\{x,y\}\!\} = F_1 + \frac{1}{2}\left(yxF_1 + F_1xy - xF_1y + yF_1x\right)\,, \tag{3.25b}$$

which contains in particular (3.2a) and (3.2b) if we extend these double brackets to the case $m = 1$, where then $I = \{0\}$ and $F_a = e_0 \otimes e_0$ for all $a \in \mathbb{Z}$. Let us prove that the first identity holds, the other cases being similar. We have by (3.21a)

$$\begin{aligned}\{\!\{x,x\}\!\} &= \sum_{r,s}\{\!\{x_r,x_s\}\!\} = \frac{1}{2}\sum_r x_{r-1}x_r \otimes e_r - \frac{1}{2}\sum_r e_{r+1}\otimes x_r x_{r+1}\\ &= \frac{1}{2}x^2\sum_r e_{r+1}\otimes e_r - \frac{1}{2}\sum_r e_{r+1}\otimes e_r x^2 = \frac{1}{2}x^2F_1 - \frac{1}{2}F_1x^2\,,\end{aligned}$$

where we used that $xe_{r+1} = x_r e_{r+1} = x_r$ and $e_r x = e_r x_r = x_r$.

All these double brackets define the quasi-Poisson structure on A. To fully understand the quasi-Hamiltonian structure, remark that the moment map is given by $\Phi = \sum_s \Phi_s + \Phi_\infty$, where

$$\Phi_s =(e_s + x_s y_s)(e_s + y_{s-1}x_{s-1})^{-1}\prod_{1\leqslant\alpha\leqslant d_s}^{\longrightarrow}(e_s + w_{s,\alpha}v_{s,\alpha})^{-1} \in e_s A e_s\,, \tag{3.26a}$$

$$\Phi_\infty = \prod_{1\leqslant s\leqslant m-1}^{\longrightarrow}\;\prod_{1\leqslant\alpha\leqslant d_s}^{\longrightarrow}(e_\infty + v_{s,\alpha}w_{s,\alpha}) \in e_\infty A e_\infty\,, \tag{3.26b}$$

by direct application of Theorem 2.3.9. For $\mathbf{q} = (q_0, \dots, q_{m-1}) \in (\mathbb{C}^\times)^I$ and $q_\infty \in \mathbb{C}^\times$, we can set $\widetilde{q} = \sum_s q_s e_s + q_\infty e_\infty$ and form the multiplicative preprojective algebra[6] $\Lambda^{\widetilde{q}} = A/(\Phi - \widetilde{q})$.

3.2.2 Additional localisation

Motivated by the localisation for the Jordan quiver and subsequent developments, see § 3.1.2, we consider the algebra $A' = A_x$ obtained by inverting x. To define the latter, we add elements $x_s^{-1} = e_{s+1} x_s^{-1} e_s$ to A such that $x_s x_s^{-1} = e_s = x_s^{-1} x_s$ for all $s \in I$. Then $x^{-1} := \sum_s x_s^{-1}$ satisfies $xx^{-1} = 1 - e_\infty = x^{-1}x$. We have that $z_s := y_s + x_s^{-1} \in A'$ and $z_s^{-1} \in A'$ since $z_s^{-1} = (e_s + x_s y_s)^{-1} x_s$. We form the element $z = y + x^{-1} = \sum_s z_s$ which is readily seen to be such that $z \in \oplus_s e_{s+1} A' e_s$. We can compute double brackets with z and get

$$\{\!\{z, z\}\!\} = -\frac{1}{2}\left(z^2 F_{-1} - F_{-1} z^2\right) \tag{3.27a}$$

$$\{\!\{x, z\}\!\} = \frac{1}{2}\left(zxF_1 + F_1 xz - xF_1 z + zF_1 x\right) , \tag{3.27b}$$

$$\{\!\{z, x\}\!\} = -\frac{1}{2}\left(xzF_{-1} + F_{-1} zx - zF_{-1} x + xF_{-1} z\right) , \tag{3.27c}$$

where the last identity is obtained by cyclic antisymmetry together with the following straightforward result.

Lemma 3.2.2 *Assume that $a \in \oplus_s e_s A' e_{s+r}$, $b \in \oplus_s e_s A' e_{s-r}$ for some $r \in \mathbb{Z}$. We have that $(baF_r)^\circ = F_{-r} ba$ and $(aF_r b)^\circ = bF_{-r} a$.*

It is also easy to show that

$$\begin{aligned} \{\!\{z, w_{s,\alpha}\}\!\} &= \frac{1}{2} e_s \otimes z w_{s,\alpha} - \frac{1}{2} e_s z \otimes w_{s,\alpha} , \\ \{\!\{z, v_{s,\alpha}\}\!\} &= \frac{1}{2} v_{s,\alpha} z \otimes e_s - \frac{1}{2} v_{s,\alpha} \otimes z e_s . \end{aligned} \tag{3.28}$$

We choose to introduce the following *spin variables* in A' :

$$a'_{s,\alpha} = w_{s,\alpha} , \quad c'_{s,1} = v_{s,1} z , \quad c'_{s,\alpha} = v_{s,\alpha}(e_s + w_{s,\alpha-1} v_{s,\alpha-1}) \dots (e_s + w_{s,1} v_{s,1}) z , \tag{3.29}$$

such that the elements $c'_{s,\alpha}$ admit an alternative inductive definition as

$$c'_{s,\alpha} = \sum_{\lambda=1}^{\alpha-1} v_{s,\alpha} w_{s,\lambda} c'_{s,\lambda} + v_{s,\alpha} z . \tag{3.30}$$

[6] Recall that we can recover a total order to match the definition given in [56], see Remark 2.3.10.

It is important to remark that $a'_{s,\alpha} = e_s a'_{s,\alpha} e_\infty$ and $c'_{s,\alpha} = e_\infty c'_{s,\alpha} e_{s-1}$. This is due to the fact that $v_{s,\alpha} z = v_{s,\alpha} z_{s-1}$. The reason for this mysterious choice of elements comes from the fact that, in A', the identity $\Phi_s = q_s e_s$ with Φ_s given by (3.26a) for some $q_s \in \mathbb{C}^\times$ is equivalent to

$$x_s z_s x_{s-1}^{-1} = q_s z_{s-1} + q_s \sum_{\alpha=1}^{d_s} a'_{s,\alpha} c'_{s,\alpha} \,.$$

We can now formulate the double quasi-Poisson bracket evaluated on the elements $(x, z, a'_{s,\alpha}, c'_{s,\alpha})$. Only the double brackets involving $c'_{s,\alpha}$ are not already known, and we compute them in § 3.3.5.

Proposition 3.2.3 *For any $r, s \in I$, $\alpha = 1, \dots, d_s$, $\gamma = 1, \dots, d_r$,*

$$\begin{aligned}
\{\!\{x, a'_{s,\alpha}\}\!\} &= \frac{1}{2} e_s \otimes x a'_{s,\alpha} - \frac{1}{2} e_s x \otimes a'_{s,\alpha} \,, \\
\{\!\{z, a'_{s,\alpha}\}\!\} &= \frac{1}{2} e_s \otimes z a'_{s,\alpha} - \frac{1}{2} e_s z \otimes a'_{s,\alpha} \,, \qquad (3.31a) \\
\{\!\{x, c'_{s,\alpha}\}\!\} &= \frac{1}{2} c'_{s,\alpha} x \otimes e_{s-1} + \frac{1}{2} c'_{s,\alpha} \otimes x e_{s-1} \,, \\
\{\!\{z, c'_{s,\alpha}\}\!\} &= -\frac{1}{2} c'_{s,\alpha} z \otimes e_{s-1} + \frac{1}{2} c'_{s,\alpha} \otimes z e_{s-1} \,, \qquad (3.31b) \\
\{\!\{a'_{s,\alpha}, a'_{r,\gamma}\}\!\} &= -\frac{1}{2} o(s,r)\, a'_{r,\gamma} \otimes a'_{s,\alpha} - \frac{1}{2} \delta_{sr} o(\alpha,\gamma) \left(a'_{r,\gamma} \otimes a'_{s,\alpha} + a'_{s,\alpha} \otimes a'_{r,\gamma}\right) . \qquad (3.31c)
\end{aligned}$$

Proposition 3.2.4 *For any $r, s \in I$, $\alpha = 1, \dots, d_s$, $\beta = 1, \dots, d_r$,*

$$\begin{aligned}
\{\!\{a'_{s,\alpha}, c'_{r,\beta}\}\!\} =& \frac{1}{2} o(s,r)\, e_\infty \otimes a'_{s,\alpha} c'_{r,\beta} - \frac{1}{2} \delta_{(r,s+1)}\, c'_{r,\beta} a'_{s,\alpha} \otimes e_s \\
&+ \frac{1}{2} \delta_{sr} \left(o(\alpha,\beta) - \delta_{\alpha\beta}\right) e_\infty \otimes a'_{s,\alpha} c'_{r,\beta} \qquad (3.32) \\
&- \delta_{sr} \delta_{\alpha\beta} \left(e_\infty \otimes e_s z + \sum_{\lambda=1}^{\beta-1} e_\infty \otimes a'_{s,\lambda} c'_{s,\lambda} \right) ,
\end{aligned}$$

where the last sum is omitted for $\beta = 1$.

Lemma 3.2.5 *For any $r, s \in I$, $\alpha = 1, \dots, d_s$, $\beta = 1, \dots, d_r$,*

$$\begin{aligned}
\{\!\{v_{s,\alpha}, c'_{r,\beta}\}\!\} =& -\frac{1}{2} o(s,r)\, v_{s,\alpha} \otimes c'_{r,\beta} + \frac{1}{2} \delta_{(r,s+1)} c'_{r,\beta} \otimes v_{s,\alpha} \\
&- \frac{1}{2} \delta_{sr} \left(o(\alpha,\beta) + \delta_{\alpha\beta}\right) v_{s,\alpha} \otimes c'_{r,\beta} \,. \qquad (3.33)
\end{aligned}$$

Proposition 3.2.6 *For any $r, s \in I$, $\alpha = 1, \dots, d_s$, $\beta = 1, \dots, d_r$,*

$$\{\!\{c'_{s,\alpha}, c'_{r,\beta}\}\!\} = -\frac{1}{2} o(s,r)\, c'_{s,\alpha} \otimes c'_{r,\beta} + \frac{1}{2} \delta_{sr} o(\alpha,\beta) \left(c'_{r,\beta} \otimes c'_{s,\alpha} - c'_{s,\alpha} \otimes c'_{r,\beta}\right) . \qquad (3.34)$$

It is interesting to compare these double brackets with those for a framed Jordan quiver obtained in § 3.1.1 and § 3.1.2.

3.2.3 Associated brackets

For the rest of this subsection, we write $k \underset{m}{\equiv} l$ to denote that k is congruent to l modulo m, for given $k, l \in \mathbb{Z}$. This notation is also well-defined on elements of $I = \mathbb{Z}_m$.

We now translate most of the results from § 3.1.3 to cyclic quivers. We will only keep the results needed for our study of the corresponding MQVs in Chapter 5. All proofs can be found in § 3.3.6, § 3.3.7 and § 3.3.8.

General results

Lemma 3.2.7 *For any* $r, s, p, q \in I$, $\alpha = 1, \dots, d_s$, $\beta = 1, \dots, d_r$, $\gamma = 1, \dots, d_p$, $\epsilon = 1, \dots, d_q$, *and* $k, l \geq 1$, *we have in* $A'/[A', A']$

$$\{x^k, x^l\} = 0\,, \quad \{x^k, a'_{s,\alpha}c'_{r,\beta}x^l\} = k\, a'_{s,\alpha}c'_{r,\beta}x^{k+l}\,, \tag{3.35}$$

$$\begin{aligned}
\{a'_{p,\gamma}c'_{q,\epsilon}x^k, a'_{s,\alpha}c'_{r,\beta}x^l\} =& \frac{1}{2}\sum_{v=1}^{k}\left(a'_{s,\alpha}c'_{r,\beta}x^v a'_{p,\gamma}c'_{q,\epsilon}x^{k+l-v} + a'_{s,\alpha}c'_{r,\beta}x^{k+l-v}a'_{p,\gamma}c'_{q,\epsilon}x^v\right) \\
&- \frac{1}{2}\sum_{v=1}^{l}\left(a'_{s,\alpha}c'_{r,\beta}x^v a'_{p,\gamma}c'_{q,\epsilon}x^{k+l-v} + a'_{s,\alpha}c'_{r,\beta}x^{k+l-v}a'_{p,\gamma}c'_{q,\epsilon}x^v\right) \\
&+ \frac{1}{2}[o(p,r) - o(p,s) + o(q,s) - o(q,r)]\, a'_{s,\alpha}c'_{q,\epsilon}x^k a'_{p,\gamma}c'_{r,\beta}x^l \\
&+ \frac{1}{2}\delta_{ps}o(\alpha,\gamma)\left(a'_{p,\gamma}c'_{q,\epsilon}x^k a'_{s,\alpha}c'_{r,\beta}x^l + a'_{s,\alpha}c'_{q,\epsilon}x^k a'_{p,\gamma}c'_{r,\beta}x^l\right) \\
&+ \frac{1}{2}\delta_{qr}o(\epsilon,\beta)\left(a'_{s,\alpha}c'_{r,\beta}x^k a'_{p,\gamma}c'_{q,\epsilon}x^l - a'_{s,\alpha}c'_{q,\epsilon}x^k a'_{p,\gamma}c'_{r,\beta}x^l\right) \\
&+ \frac{1}{2}\delta_{qs}[o(\epsilon,\alpha) + \delta_{\alpha\epsilon}]\, a'_{s,\alpha}c'_{q,\epsilon}x^k a'_{p,\gamma}c'_{r,\beta}x^l \\
&- \frac{1}{2}\delta_{pr}[o(\beta,\gamma) + \delta_{\beta\gamma}]\, a'_{s,\alpha}c'_{q,\epsilon}x^k a'_{p,\gamma}c'_{r,\beta}x^l \\
&+ \delta_{qs}\delta_{\alpha\epsilon}\left(z + \sum_{\lambda=1}^{\epsilon-1}a'_{s,\lambda}c'_{s,\lambda}\right)x^k a'_{p,\gamma}c'_{r,\beta}x^l \\
&- \delta_{pr}\delta_{\beta\gamma}\; a'_{s,\alpha}c'_{q,\epsilon}x^k\left(z + \sum_{\mu=1}^{\beta-1}a'_{p,\mu}c'_{p,\mu}\right)x^l\,.
\end{aligned} \tag{3.36}$$

In particular, in order for the elements on which we evaluate the bracket to be nonzero, we need $k \underset{m}{\equiv} 0$ *for* x^k*, while* $l \underset{m}{\equiv} s-(r-1)$ *for* $a'_{s,\alpha}c'_{r,\beta}x^l$*, and* $k \underset{m}{\equiv} p-(q-1)$ *for* $a'_{p,\gamma}c'_{q,\epsilon}x^k$*.*

Write $1_I = 1 - e_\infty = \sum_s e_s$, and consider $u \in \{x, y, z, 1_I + xy\}$. We assume that we work in A if $u \neq z$ and in A' otherwise. We set $\epsilon(x) = \epsilon(1_I + xy) = +1$, $\epsilon(y) = \epsilon(z) = -1$ to extend the map $\epsilon : \bar{Q}_{\mathbf{d}} \to \{\pm 1\}$. We also set $\theta(u) = \epsilon(u)$ if $u = x, y, z$ while $\theta(u) = 0$ when $u = 1_I + xy$, so that $u \in \oplus_s e_s A e_{s+\theta(u)}$. Then, it is easy to check that we can write in all cases

$$\{\!\{u, u\}\!\} = \frac{1}{2}\epsilon(u)[u^2 F_{\theta(u)} - F_{\theta(u)}u^2]\,, \tag{3.37a}$$

$$\{\!\{u, w_{s,\alpha}\}\!\} = \frac{1}{2}\, e_s \otimes u w_{s,\alpha} - \frac{1}{2}\, e_s u \otimes w_{s,\alpha}\,,$$

$$\{\!\{u, v_{s,\alpha}\}\!\} = \frac{1}{2}\, v_{s,\alpha} u \otimes e_s - \frac{1}{2}\, v_{s,\alpha} \otimes u e_s\,. \tag{3.37b}$$

We can prove the following result.

Lemma 3.2.8 *For any* $k, l \in \mathbb{N}$, $r, s \in I$, $\alpha = 1, \ldots, d_s$, $\beta = 1, \ldots, d_r$,

$$\{u^k, u^l\} = 0\,, \quad \{u^k, w_{s,\alpha}v_{r,\beta}u^l\} = 0\,, \tag{3.38}$$

and for $p, q \in I$, $\gamma = 1, \ldots, d_p$, $\epsilon = 1, \ldots, d_q$,

$$\begin{aligned}
&\{w_{p,\gamma}v_{q,\epsilon}u^k, w_{s,\alpha}v_{r,\beta}u^l\} \\
=&\frac{1}{2}\,[o(p,r) + o(q,s) - o(p,s) - o(q,r)]\, w_{s,\alpha}v_{q,\epsilon}u^k\, w_{p,\gamma}v_{r,\beta}u^l \\
&+ \frac{1}{2}\delta_{ps}o(\alpha,\gamma)(w_{s,\alpha}v_{q,\epsilon}u^k w_{p,\gamma}v_{r,\beta}u^l + w_{p,\gamma}v_{q,\epsilon}u^k w_{s,\alpha}v_{r,\beta}u^l) \\
&+ \frac{1}{2}\delta_{qr}o(\beta,\epsilon)(w_{s,\alpha}v_{q,\epsilon}u^k w_{p,\gamma}v_{r,\beta}u^l + w_{s,\alpha}v_{r,\beta}u^k w_{p,\gamma}v_{q,\epsilon}u^l) \\
&+ \frac{1}{2}\delta_{qs}[o(\epsilon,\alpha) + \delta_{\epsilon\alpha}](w_{s,\alpha}v_{q,\epsilon}u^k w_{p,\gamma}v_{r,\beta}u^l + w_{s,\alpha}v_{r,\beta}u^{k+l}w_{p,\gamma}v_{q,\epsilon}) \\
&- \frac{1}{2}\delta_{pr}[o(\beta,\gamma) + \delta_{\beta\gamma}](w_{s,\alpha}v_{q,\epsilon}u^k w_{p,\gamma}v_{r,\beta}u^l + w_{s,\alpha}v_{r,\beta}w_{p,\gamma}v_{q,\epsilon}u^{k+l}) \\
&+ \delta_{qs}\delta_{\epsilon\alpha}\, w_{p,\gamma}v_{r,\beta}u^{k+l} - \delta_{pr}\delta_{\gamma\beta}\, w_{s,\alpha}v_{q,\epsilon}u^{k+l} \\
&+ \frac{1}{2}\epsilon(u)\,\left[\sum_{\tau=1}^{k-1} w_{s,\alpha}v_{r,\beta}u^{k+l-\tau}w_{p,\gamma}v_{q,\epsilon}u^\tau + \sum_{\tau=1}^{l} w_{s,\alpha}v_{r,\beta}u^{k+\tau}w_{p,\gamma}v_{q,\epsilon}u^{l-\tau}\right] \\
&- \frac{1}{2}\epsilon(u)\,\left[\sum_{\sigma=1}^{l-1} w_{s,\alpha}v_{r,\beta}u^\tau w_{p,\gamma}v_{q,\epsilon}u^{k+l-\tau} + \sum_{\tau=1}^{k} w_{s,\alpha}v_{r,\beta}u^{k-\tau}w_{p,\gamma}v_{q,\epsilon}u^{l+\tau}\right]\,.
\end{aligned} \tag{3.39}$$

These identities hold directly in A (or A' for $u = z$). If we work modulo commutators, the elements on which we evaluate the bracket are nonzero only if $k\theta(u) \underset{m}{\equiv} 0$ for u^k, while $l\theta(u) \underset{m}{\equiv}$

$s-r$ for $w_{s,\alpha}v_{r,\beta}u^l$. This follows from the decomposition $u \in \oplus_s e_s A e_{s+\theta(u)}$. Remark also that when $p=q=r=s$, (3.39) can be written in the form (3.15b). Indeed, this is obvious for $u = 1_I + xy$, while in the other three cases we have that $l = l_0 m, k = k_0 m$ so we replace u^m by u and in the last sums only the terms $\tau = \tau_0 m$ have a nonzero contribution.

An immediate consequence of Lemma 3.2.8 and Definitions 2.3.35, 2.3.37 is as follows.

Corollary 3.2.9 *The element u is strongly involutive. Furthermore, if $I(u)$ is the subalgebra of A (or A') generated by B, u and elements $w_{s,\alpha}v_{r,\beta}u^l$, we get an involutive chain $\mathbb{C}[u] \subset I(u) \subset A$ (or A').*

In the particular case where $u = 1_I + xy$, we have that $u \in \oplus_s e_s A e_s$ and we can get a slightly different result.

Lemma 3.2.10 *Fix $u = 1_I + xy$. For any $k,l \in \mathbb{N}$, $r,s,t \in I$, $\alpha = 1,\dots,d_s$, $\beta = 1,\dots,d_r$,*

$$\{(e_s u e_s)^k, (e_r u e_r)^l\} = 0\,, \quad \{(e_t u e_t)^k, w_{s,\alpha}v_{r,\beta}u^l\} = 0\,. \tag{3.40}$$

In particular, for any $t \in I$, the element $e_t u e_t$ is strongly involutive. Furthermore, if $I_t(u)$ is the subalgebra of A generated by B and the elements $e_s u e_s$, $w_{s,\alpha}v_{r,\beta}u^l$ for all possible indices, we get an involutive chain $\mathbb{C}[e_t u e_t] \subset I_t(u) \subset A$.

Embeddings of quivers

Remark 3.2.11 *Recall that the admissible spin indices (s,α) introduced in Remark 3.2.1 are such that $s \in I$, and $1 \leq \alpha \leq d_s$ (where we omit such terms for $d_s = 0$). We put a total order on these couples by setting $(s,\alpha) < (r,\beta)$ whenever $s < r$ as elements of $\{0,\dots,m-1\}$, or when $\alpha < \beta$ if $s = r$. We also denote by $\bar{s} \in I$ the element such that $(\bar{s}, d_{\bar{s}})$ is maximal with respect to this ordering. Equivalently, $\bar{s}$ is the highest number in $\{0,\dots,m-1\}$ satisfying $d_{\bar{s}} \neq 0$.*

Let $\mathrm{Ord}(\mathbf{d}) = \{(s,\alpha) \mid s \in I,\ 1 \leq \alpha \leq d_s\}$be the set of admissible spin indices, which is a totally ordered set by the above construction. If we consider $\{1,\dots,|\mathbf{d}|\}$ with its natural total order, there is a unique map $\rho : \{1,\dots,|\mathbf{d}|\} \to \mathrm{Ord}(\mathbf{d})$ preserving total orders on both sets. It satisfies $\rho(1) = (0,1)$ and $\rho(|\mathbf{d}|) = (\bar{s}, d_{\bar{s}})$.

First, let $\bar{Q}_0$ be the cyclic quiver obtained by removing the framing vertices $v_{s,\alpha}, w_{s,\alpha}$ from $\bar{Q}_{\mathbf{d}}$. In terms of the construction described before Lemma 2.3.14, this is the subquiver $\bar{Q}'$ supported at I to which we add the vertex ∞ as a disconnected component. Next, let $\bar{Q}_1$ be obtained from $\bar{Q}_0$ by adding arrows $v_{\rho(1)} = v_{0,1} : \infty \to 0$ and $w_{\rho(1)} = w_{0,1} : 0 \to \infty$. Similarly, we can consider $\bar{Q}_1, \bar{Q}_2, \ldots, \bar{Q}_{|\mathbf{d}|}$, where $\bar{Q}_j$ is obtained from the previous quiver $\bar{Q}_{j-1}$ by adding $v_{\rho(j)}, w_{\rho(j)}$. Clearly, $\bar{Q}_{|\mathbf{d}|}$ is our original quiver $\bar{Q}_{\mathbf{d}}$.

We can take on each subquiver $\bar{Q}_j$, $j \in \{0\} \cup \{1, \ldots, |\mathbf{d}|\}$, the ordering obtained by restricting (3.20a)–(3.20c) to the arrows of $\bar{Q}_j$. This gives a quasi-Hamiltonian structure on A_j, the algebra $\mathbb{C}\bar{Q}_j$ localised at the elements $e_{t(a)} + aa^*$ for $a \in \bar{Q}_j$. This yields in turn a chain of quasi-Hamiltonian algebras

$$A_0 \subset A_1 \subset A_2 \subset \ldots \subset A_{|\mathbf{d}|} = A\,, \tag{3.41}$$

which can be localised at x. In particular, note that for any $j, j' \in \{0\} \cup \{1, \ldots, |\mathbf{d}|\}$, $j \leq j'$, the moment map $\Phi^{(j)}$ of A_j is defined in $A_{j'}$. Moreover, it can be obtained inductively as

$$\begin{aligned} \Phi^{(0)} =& \sum_{s\in I}(e_s + x_s y_s)(e_s + y_{s-1}x_{s-1})^{-1} + e_\infty, \\ \Phi^{(j)} =& \Phi^{(j-1)}(1 + w_{\rho(j)}v_{\rho(j)})^{-1}(1 + v_{\rho(j)}w_{\rho(j)})\,. \end{aligned} \tag{3.42}$$

We begin by considering the simplest embedding $A_0 \subset A$. We define $\phi = \sum_s \phi_s$ by setting $\phi_s = \Phi_s^{(0)}$ for any $s \in I$. That is $\phi_s = (e_s + x_s y_s)(e_s + y_{s-1}x_{s-1})^{-1}$ or $\phi_s = x_s z_s x_{s-1}^{-1} z_{s-1}^{-1}$ when we localise A at x. For the next proposition, we assume that we work in A localised at $u \in \{x, y, z, 1_I + xy\}$.

Proposition 3.2.12 *Let $U_{+,\eta} = u(1 + \eta\phi)$, $U_{-,\eta} = u(1 + \eta\phi^{-1})$, for arbitrary $\eta \in \mathbb{C}$ playing the role of a spectral parameter. Let $K, L \in \mathbb{N}^\times$. Then, if $\epsilon(u) = -1$,*

$$\{U^K_{+,\eta}, U^L_{+,\eta'}\} = 0 \quad \textit{mod } [A, A], \qquad \textit{for any } \eta, \eta' \in \mathbb{C}\,. \tag{3.43}$$

If $\epsilon(u) = +1$,

$$\{U^K_{-,\eta}, U^L_{-,\eta'}\} = 0 \quad \text{mod } [A, A], \qquad \text{for any } \eta, \eta' \in \mathbb{C}\,. \tag{3.44}$$

Note that when u is not $1_I + xy$ and K is not divisible by m, then $U^K_{+,\eta} = 0$ mod $[A, A]$ and the result is trivial. Indeed, $U^K_{+,\eta} \in \oplus_s e_s A e_{s+K\theta(u)}$. This is also true for the other functions $U^K_{-,\eta}$ in Proposition 3.2.12. The next result is proved as Corollary 3.1.10.

Corollary 3.2.13 *Assume $\epsilon(u) = -1$ and for all $K \in \mathbb{N}$, develop $U_{+,\eta}^K = \sum_{k=0}^K u_{K,k}\eta^k$. Then any two elements in the set $\{u_{K,k} \mid K \in \mathbb{N},\, 0 \leq k \leq K\}$ are in involution. The same holds for $\epsilon(u) = +1$ with $U_{-,\eta}$ instead.*

Considering this time the full chain (3.41), we can get a different result. To state it, we assume that $u \in \{y, z, (1_I + xy)^{-1}\}$. We also set $u_{(j)} = \Phi^{(j)}u$ for any $j \in \{0, 1, \ldots, |\mathbf{d}|\}$. Note that $y_{(0)} = (1_I + xy)y(1_I + xy)^{-1} \in A$ and $z_{(0)} = xzx^{-1} \in A'$.

Proposition 3.2.14 *For any $K, L \in \mathbb{N}$ and $0 \leq j, j' \leq |\mathbf{d}|$, the elements $u_{(j)}^K, u_{(j')}^L$ are in involution.*

Computations for the Loday bracket

We finish by describing the Loday bracket between generators of A and the functions from Proposition 3.2.12 that commute in $A/[A, A]$. See the discussion before Lemma 3.1.12 for a motivation.

Lemma 3.2.15 *Write $U_\eta = z(1_I + \eta\phi)$, where $\phi = xzx^{-1}z^{-1}$. The left Loday bracket $\{-,-\} : A' \times A' \to A'$ satisfies for any $K \in m\mathbb{N}^\times$*

$$\frac{1}{K}\{U_\eta^K, x\} = -\eta\phi U_\eta^{K-1}zx - xU_\eta^{K-1}z\,, \quad \frac{1}{K}\{U_\eta^K, z\} = -zU_\eta^{K-1}z + U_\eta^{K-1}z^2\,,$$
$$\frac{1}{K}\{U_\eta^K, v_{s,\alpha}\} = 0\,, \qquad \frac{1}{K}\{U_\eta^K, w_{s,\alpha}\} = 0\,.$$

Denote by A_y the algebra A localised at y.

Lemma 3.2.16 *Write $\bar{U}_\eta = y(1_I + \eta\phi)$ with $\phi = (1_I + xy)(1_I + yx)^{-1}$. The left Loday bracket $\{-,-\} : A_y \times A_y \to A_y$ satisfies for any $K \in m\mathbb{N}^\times$*

$$\frac{1}{K}\{\bar{U}_\eta^K, x\} = -\bar{U}_\eta^{K-1} - x\bar{U}_\eta^{K-1}y - \eta\phi\bar{U}_\eta^{K-1}(1_I + yx)\,,$$
$$\frac{1}{K}\{\bar{U}_\eta^K, y\} = -y\bar{U}_\eta^{K-1}y + \bar{U}_\eta^{K-1}y^2\,, \qquad \frac{1}{K}\{\bar{U}_\eta^K, v_{s,\alpha}\} = 0\,, \qquad \frac{1}{K}\{\bar{U}_\eta^K, w_{s,\alpha}\} = 0\,.$$

Lemma 3.2.17 *Write $\hat{U}_\eta = x(1_I + \eta\phi^{-1})$ with $\phi = xzx^{-1}z^{-1}$. The left Loday bracket $\{-,-\} : A' \times A' \to A'$ satisfies for any $K \in m\mathbb{N}^\times$*

$$\frac{1}{K}\{\hat{U}_\eta^K, x\} = -\hat{U}_\eta^{K-1}x^2 + x\hat{U}_\eta^{K-1}x\,, \quad \frac{1}{K}\{\hat{U}_\eta^K, z\} = z\hat{U}_\eta^{K-1} + \eta\phi^{-1}\hat{U}_\eta^{K-1}xz\,,$$
$$\frac{1}{K}\{\hat{U}_\eta^K, v_{s,\alpha}\} = 0\,, \qquad \frac{1}{K}\{\hat{U}_\eta^K, w_{s,\alpha}\} = 0\,.$$

Lemma 3.2.18 *Write $\tilde{U}_\eta = u(1_I + \eta\phi^{-1})$ with $\phi = (1_I + xy)(1_I + yx)^{-1}$ and $u = 1_I + xy$. The left Loday bracket $\{-,-\} : A \times A \to A$ satisfies for any $K \in \mathbb{N}^\times$*

$$\begin{aligned} \frac{1}{K}\{\tilde{U}_\eta^K, x\} =& - \tilde{U}_\eta^{K-1}ux - \eta\, x\phi^{-1}\tilde{U}_\eta^{K-1}u\,, \quad \frac{1}{K}\{\tilde{U}_\eta^K, u\} = -\tilde{U}_\eta^{K-1}u^2 + u\tilde{U}_\eta^{K-1}u\,, \\ \frac{1}{K}\{\tilde{U}_\eta^K, v_{s,\alpha}\} =& 0\,, \qquad \frac{1}{K}\{\tilde{U}_\eta^K, w_{s,\alpha}\} = 0\,. \end{aligned}$$

We can also do the same for the elements given in Proposition 3.2.14.

Lemma 3.2.19 *Write $z_{(j)} = \Phi^{(j)}z$ with $\Phi^{(j)}$ given in (3.42) for $j \in \{0, 1, \ldots, |\mathbf{d}|\}$. The left Loday bracket $\{-,-\} : A' \times A' \to A'$ satisfies for any $K \in m\mathbb{N}^\times$*

$$\begin{aligned} \frac{1}{K}\{z_{(j)}^K, x\} =& - z_{(j)}^K x\,, \quad \frac{1}{K}\{z_{(j)}^K, z\} = 0\,, \\ \frac{1}{K}\{z_{(j)}^K, v_{s,\alpha}\} =& v_{s,\alpha}zz_{(j)}^{K-1}\Phi^{(j)}\,, \quad \frac{1}{K}\{z_{(j)}^K, w_{s,\alpha}\} = -zz_{(j)}^{K-1}\Phi^{(j)}w_{s,\alpha}\,, \quad (s,\alpha) \leq \rho(j)\,, \\ \frac{1}{K}\{z_{(j)}^K, v_{s,\alpha}\} =& 0\,, \quad \frac{1}{K}\{z_{(j)}^K, w_{s,\alpha}\} = 0\,, \quad (s,\alpha) > \rho(j)\,. \end{aligned}$$

Moreover, $\{z_{(j)}^K, z_{(j)}\} = \{z_{(j)}^K, \Phi^{(j)}\} = 0$.

Lemma 3.2.20 *Write $y_{(j)} = \Phi^{(j)}y$ with $\Phi^{(j)}$ given in (3.42) for $j \in \{0, 1, \ldots, |\mathbf{d}|\}$. The left Loday bracket $\{-,-\} : A \times A \to A$ satisfies for any $K \in m\mathbb{N}^\times$*

$$\begin{aligned} \frac{1}{K}\{y_{(j)}^K, x\} =& - y_{(j)}^{K-1}\Phi^{(j)} - y_{(j)}^K x\,, \quad \frac{1}{K}\{y_{(j)}^K, y\} = 0\,, \\ \frac{1}{K}\{y_{(j)}^K, v_{s,\alpha}\} =& v_{s,\alpha}yy_{(j)}^{K-1}\Phi^{(j)}\,, \quad \frac{1}{K}\{y_{(j)}^K, w_{s,\alpha}\} = -yy_{(j)}^{K-1}\Phi^{(j)}w_{s,\alpha}\,, \quad (s,\alpha) \leq \rho(j)\,, \\ \frac{1}{K}\{y_{(j)}^K, v_{s,\alpha}\} =& 0\,, \quad \frac{1}{K}\{y_{(j)}^K, w_{s,\alpha}\} = 0\,, \quad (s,\alpha) > \rho(j)\,. \end{aligned}$$

Moreover, $\{y_{(j)}^K, y_{(j)}\} = \{y_{(j)}^K, \Phi^{(j)}\} = 0$.

Lemma 3.2.21 *Write $u_{(j)} = \Phi^{(j)}u$ with $\Phi^{(j)}$ given in (3.42) for $j \in \{0, 1, \ldots, |\mathbf{d}|\}$ and $u = (1_I + xy)^{-1}$. The left Loday bracket $\{-,-\} : A \times A \to A$ satisfies for any $K \in \mathbb{N}^\times$*

$$\begin{aligned} \frac{1}{K}\{u_{(j)}^K, x\} =& xuu_{(j)}^{K-1}\Phi^{(j)}\,, \quad \frac{1}{K}\{u_{(j)}^K, u\} = 0\,, \\ \frac{1}{K}\{u_{(j)}^K, v_{s,\alpha}\} =& v_{s,\alpha}uu_{(j)}^{K-1}\Phi^{(j)}\,, \quad \frac{1}{K}\{u_{(j)}^K, w_{s,\alpha}\} = -uu_{(j)}^{K-1}\Phi^{(j)}w_{s,\alpha}\,, \quad (s,\alpha) \leq \rho(j)\,, \\ \frac{1}{K}\{u_{(j)}^K, v_{s,\alpha}\} =& 0\,, \quad \frac{1}{K}\{u_{(j)}^K, w_{s,\alpha}\} = 0\,, \quad (s,\alpha) > \rho(j)\,. \end{aligned}$$

Moreover, $\{u_{(j)}^K, u_{(j)}\} = \{u_{(j)}^K, \Phi^{(j)}\} = 0$.

3.3 Remaining proofs

3.3.1 Double brackets for framed Jordan quivers

We successively prove Proposition 3.1.2, Lemma 3.1.3 and Proposition 3.1.4. Most computations rely on a proof by induction based on the identity (3.8). Knowing the double brackets in § 3.1.1 or § 3.1.2, if we want to compute the bracket $\left\{\!\!\left\{\Gamma, c'_\beta\right\}\!\!\right\}$ for some $\Gamma \in A$, we first find $\{\!\{\Gamma, c'_1\}\!\}$ and then show our statement by induction using

$$\{\!\{\Gamma, c'_\alpha\}\!\} = \sum_{\lambda=1}^{\alpha-1} \left(v_\alpha w_\lambda \{\!\{\Gamma, c'_\lambda\}\!\} + \{\!\{\Gamma, v_\alpha w_\lambda\}\!\}\, c'_\lambda\right) + \{\!\{\Gamma, v_\alpha z\}\!\} \,. \tag{3.45}$$

Proof. (Proposition 3.1.2.) We begin with the first equality of (3.9a). We can write

$$\begin{aligned}\{\!\{x, v_\alpha z\}\!\} =& v_\alpha \{\!\{x, z\}\!\} + \{\!\{x, v_\alpha\}\!\}\, z \\ =&\frac{1}{2}(v_\alpha z x \otimes e_0 + v_\alpha \otimes xz + v_\alpha z \otimes x - v_\alpha x \otimes z) + \frac{1}{2}(v_\alpha x \otimes z - v_\alpha \otimes xz) \\ =&\frac{1}{2}(v_\alpha z x \otimes e_0 + v_\alpha z \otimes x)\,,\end{aligned}$$

which proves the first equality in (3.9a) for $\alpha = 1$. Now, we compute

$$\begin{aligned}\{\!\{x, v_\alpha w_\lambda\}\!\} =& v_\alpha \{\!\{x, w_\lambda\}\!\} + \{\!\{x, v_\alpha\}\!\}\, w_\lambda \\ =&\frac{1}{2}(v_\alpha \otimes x w_\lambda - v_\alpha x \otimes w_\lambda) + \frac{1}{2}(v_\alpha x \otimes w_\lambda - v_\alpha \otimes x w_\lambda) = 0\,,\end{aligned}$$

so that if we assume that the first part of (3.9a) is true for any $\lambda < \alpha$, we get from (3.45)

$$\begin{aligned}\{\!\{x, c'_\alpha\}\!\} =& \sum_{\lambda=1}^{\alpha-1} \left(v_\alpha w_\lambda \{\!\{x, c'_\lambda\}\!\} + \{\!\{x, v_\alpha w_\lambda\}\!\}\, c'_\lambda\right) + \{\!\{x, v_\alpha z\}\!\} \\ =& \sum_{\lambda=1}^{\alpha-1} v_\alpha w_\lambda \left(\frac{1}{2} c'_\lambda x \otimes e_0 + \frac{1}{2} c'_\lambda \otimes x\right) + \frac{1}{2}(v_\alpha z x \otimes e_0 + v_\alpha z \otimes x) \\ =&\frac{1}{2}\left(\sum_{\lambda=1}^{\alpha-1} v_\alpha w_\lambda c'_\lambda + v_\alpha z\right) x \otimes e_0 + \frac{1}{2}\left(\sum_{\lambda=1}^{\alpha-1} v_\alpha w_\lambda c'_\lambda + v_\alpha z\right) \otimes x\,,\end{aligned}$$

which is exactly the first equality in (3.9a) by using (3.8). For the second equality, we compute

$$\begin{aligned}\{\!\{z, v_\alpha z\}\!\} =& v_\alpha \{\!\{z, z\}\!\} + \{\!\{z, v_\alpha\}\!\}\, z = \frac{1}{2}(-v_\alpha z^2 \otimes e_0 + v_\alpha \otimes z^2) + \frac{1}{2}(v_\alpha z \otimes z - v_\alpha \otimes z^2) \\ =&\frac{1}{2}(v_\alpha z \otimes z - v_\alpha z^2 \otimes e_0)\,,\end{aligned}$$

which gives the case $\alpha = 1$. Then, we find $\{\!\{z, v_\alpha w_\lambda\}\!\} = 0$ by noticing that the brackets for x or z with v_α and w_λ are exactly the same. We then get by induction that the second equation from (3.9a) holds, too.

To get (3.9b), recall that $a'_\alpha := w_\alpha$. We first compute

$$\begin{aligned}\left\{\!\!\left\{v_\beta z, a'_\alpha\right\}\!\!\right\} =& v_\beta * \left\{\!\!\left\{z, a'_\alpha\right\}\!\!\right\} + \left\{\!\!\left\{v_\beta, a'_\alpha\right\}\!\!\right\} * z \\ =& \frac{1}{2}(e_0 \otimes v_\beta z a'_\alpha - z \otimes v_\beta a'_\alpha) \\ &+ \left[\delta_{\alpha\beta}\, z \otimes e_\infty + \frac{1}{2}\left[o(\beta,\alpha) + \delta_{\alpha\beta}\right]\left(z \otimes v_\beta a'_\alpha + a'_\alpha v_\beta z \otimes e_\infty\right)\right] .\end{aligned}$$

Using $\{\!\{a'_\alpha, v_\beta z\}\!\} = -\{\!\{v_\beta z, a'_\alpha\}\!\}^\circ$, we can write

$$\begin{aligned}\left\{\!\!\left\{a'_\alpha, v_\beta z\right\}\!\!\right\} =& \frac{1}{2}(v_\beta a'_\alpha \otimes z - v_\beta z a'_\alpha \otimes e_0) - \delta_{\alpha\beta}\, e_\infty \otimes z \\ &+ \frac{1}{2}\left[o(\alpha,\beta) - \delta_{\alpha\beta}\right]\left(v_\beta a'_\alpha \otimes z + e_\infty \otimes a'_\alpha v_\beta z\right) \\ =& -\delta_{(\alpha \geqslant \beta)}\left(\frac{1}{2} e_\infty \otimes a'_\alpha v_\beta z + \frac{1}{2} v_\beta z a'_\alpha \otimes e_0 + \delta_{\alpha\beta}\, e_\infty \otimes z\right) \\ &+ \delta_{(\alpha<\beta)}\left(v_\beta a'_\alpha \otimes z - \frac{1}{2} v_\beta z a'_\alpha \otimes e_0 + \frac{1}{2} e_\infty \otimes a'_\alpha v_\beta z\right) ,\end{aligned}$$

recalling that $o(\alpha,\beta) = \delta_{(\alpha<\beta)} - \delta_{(\alpha>\beta)}$. In particular, this yields

$$\left\{\!\!\left\{a'_\alpha, c'_1\right\}\!\!\right\} = -\frac{1}{2} c'_1 a'_\alpha \otimes e_0 - \frac{1}{2} e_\infty \otimes a'_\alpha c'_1 - \delta_{\alpha\beta}\, e_\infty \otimes z$$

which is exactly the case $\beta = 1$ in (3.9b). Next, we need to compute

$$\begin{aligned}\left\{\!\!\left\{a'_\alpha, v_\beta w_\lambda\right\}\!\!\right\} =& v_\beta \left\{\!\!\left\{a'_\alpha, w_\lambda\right\}\!\!\right\} + \left\{\!\!\left\{a'_\alpha, v_\beta\right\}\!\!\right\} w_\lambda \\ =& -\frac{1}{2}\, o(\alpha,\lambda)\left(v_\beta w_\lambda \otimes w_\alpha + v_\beta w_\alpha \otimes w_\lambda\right) \\ &- \delta_{\alpha\beta}\, e_\infty \otimes w_\lambda + \frac{1}{2}\left[o(\alpha,\beta) - \delta_{\alpha\beta}\right]\left(v_\beta a'_\alpha \otimes w_\lambda + e_\infty \otimes a'_\alpha v_\beta w_\lambda\right) ,\end{aligned}$$

and this implies that

$$\begin{aligned}\sum_{\lambda=1}^{\beta-1} \left\{\!\!\left\{a'_\alpha, v_\beta w_\lambda\right\}\!\!\right\} c'_\lambda =& -\sum_{\lambda=1}^{\beta-1}\left[\frac{1}{2} o(\alpha,\lambda)\left(v_\beta w_\lambda \otimes w_\alpha c'_\lambda + v_\beta w_\alpha \otimes w_\lambda c'_\lambda\right) + \delta_{\alpha\beta} e_\infty \otimes w_\lambda c'_\lambda\right] \\ &+ \frac{1}{2}\left[o(\alpha,\beta) - \delta_{\alpha\beta}\right]\sum_{\lambda=1}^{\beta-1}\left(v_\beta a'_\alpha \otimes w_\lambda c'_\lambda + e_\infty \otimes a'_\alpha v_\beta w_\lambda c'_\lambda\right) .\end{aligned}$$

In the case $\alpha \geqslant \beta$ this gives since $w_\alpha = a'_\alpha$

$$\sum_{\lambda=1}^{\beta-1} \left\{\!\!\left\{a'_\alpha, v_\beta w_\lambda\right\}\!\!\right\} c'_\lambda \overset{\alpha \geqslant \beta}{=} -\delta_{\alpha\beta} \sum_{\lambda=1}^{\beta-1} e_\infty \otimes w_\lambda c'_\lambda + \frac{1}{2}\sum_{\lambda=1}^{\beta-1}\left(v_\beta w_\lambda \otimes w_\alpha c'_\lambda - e_\infty \otimes a'_\alpha v_\beta w_\lambda c'_\lambda\right).$$

Otherwise, we just write

$$\sum_{\lambda=1}^{\beta-1} \{\!\{a'_\alpha, v_\beta w_\lambda\}\!\}\, c'_\lambda \overset{\alpha<\beta}{=} -\frac{1}{2}\left(\sum_{\lambda=\alpha+1}^{\beta-1} - \sum_{\lambda=1}^{\alpha-1}\right)\left(v_\beta w_\lambda \otimes w_\alpha c'_\lambda + v_\beta w_\alpha \otimes w_\lambda c'_\lambda\right)$$
$$+\frac{1}{2}\sum_{\lambda=1}^{\beta-1}\left(v_\beta a'_\alpha \otimes w_\lambda c'_\lambda + e_\infty \otimes a'_\alpha v_\beta w_\lambda c'_\lambda\right)$$

Now, assume by induction that (3.9b) holds for any $\lambda < \beta$, that is

$$\{\!\{a'_\alpha, c'_\lambda\}\!\} \overset{\alpha\geqslant\lambda}{=} -\frac{1}{2}c'_\lambda a'_\alpha \otimes e_0 - \frac{1}{2}e_\infty \otimes a'_\alpha c'_\lambda - \delta_{\alpha\lambda}\left(e_\infty \otimes z + \sum_{\gamma=1}^{\lambda-1} e_\infty \otimes a'_\gamma c'_\gamma\right),$$
$$\{\!\{a'_\alpha, c'_\lambda\}\!\} \overset{\alpha<\lambda}{=} -\frac{1}{2}c'_\lambda a'_\alpha \otimes e_0 + \frac{1}{2}e_\infty \otimes a'_\alpha c'_\lambda$$

In the first case, $\alpha \geq \beta$, we find from (3.45) and (3.8)

$$\begin{aligned}\{\!\{a'_\alpha, c'_\beta\}\!\} \overset{\alpha\geqslant\beta}{=} & -\frac{1}{2}\sum_{\lambda=1}^{\beta-1} v_\beta w_\lambda c'_\lambda a'_\alpha \otimes e_0 - \frac{1}{2}\sum_{\lambda=1}^{\beta-1} v_\beta w_\lambda e_\infty \otimes a'_\alpha c'_\lambda \\ & - \delta_{\alpha\beta}\sum_{\lambda=1}^{\beta-1} e_\infty \otimes w_\lambda c'_\lambda + \frac{1}{2}\sum_{\lambda=1}^{\beta-1}\left(v_\beta w_\lambda \otimes w_\alpha c'_\lambda - e_\infty \otimes a'_\alpha v_\beta w_\lambda c'_\lambda\right) \\ & - \left(\frac{1}{2}e_\infty \otimes a'_\alpha v_\beta z + \frac{1}{2}v_\beta z a'_\alpha \otimes e_0 + \delta_{\alpha\beta}\, e_\infty \otimes z\right) \\ = & -\frac{1}{2}c'_\beta a'_\alpha \otimes e_0 - \frac{1}{2}e_\infty \otimes a'_\alpha c'_\beta - \delta_{\alpha\beta} e_\infty \otimes \left(\sum_{\lambda=1}^{\beta-1} a'_\lambda c'_\lambda + z\right)\end{aligned}$$

which coincide with (3.9b). In the second case, we get

$$\begin{aligned}\{\!\{a'_\alpha, c'_\beta\}\!\} \overset{\alpha<\beta}{=} & \sum_{\lambda=1}^{\alpha}\left[-\frac{1}{2}v_\beta w_\lambda c'_\lambda a'_\alpha \otimes e_0 - \frac{1}{2}v_\beta w_\lambda \otimes a'_\alpha c'_\lambda\right] - v_\beta w_\alpha \otimes z \\ & - \sum_{\gamma=1}^{\alpha-1} v_\beta w_\alpha \otimes a'_\gamma c'_\gamma + \sum_{\lambda=\alpha+1}^{\beta-1}\left(-\frac{1}{2}v_\beta w_\lambda c'_\lambda a'_\alpha \otimes e_0 + \frac{1}{2}v_\beta w_\lambda \otimes a'_\alpha c'_\lambda\right) \\ & - \frac{1}{2}\left(\sum_{\lambda=\alpha+1}^{\beta-1} - \sum_{\lambda=1}^{\alpha-1}\right)\left(v_\beta w_\lambda \otimes w_\alpha c'_\lambda + v_\beta w_\alpha \otimes w_\lambda c'_\lambda\right) \\ & + \frac{1}{2}\sum_{\lambda=1}^{\beta-1}\left(v_\beta a'_\alpha \otimes w_\lambda c'_\lambda + e_\infty \otimes a'_\alpha v_\beta w_\lambda c'_\lambda\right) \\ & + \left(v_\beta a'_\alpha \otimes z - \frac{1}{2}v_\beta z a'_\alpha \otimes e_0 + \frac{1}{2}e_\infty \otimes a'_\alpha v_\beta z\right)\end{aligned}$$

which, after some easy manipulations on the sums, yields

$$\begin{aligned}\{\!\{a'_\alpha, c'_\beta\}\!\} \overset{\alpha<\beta}{=} & \frac{1}{2}\sum_{\lambda=1}^{\beta-1}\left(-v_\beta w_\lambda c'_\lambda a'_\alpha \otimes e_0 + e_\infty \otimes a'_\alpha v_\beta w_\lambda c'_\lambda\right) - \frac{1}{2}v_\beta z a'_\alpha \otimes e_0 + \frac{1}{2}e_\infty \otimes a'_\alpha v_\beta z \\ = & -\frac{1}{2}c'_\beta a'_\alpha \otimes e_0 + \frac{1}{2}e_\infty \otimes a'_\alpha c'_\beta\,,\end{aligned}$$

as expected from (3.9b). □

Proof. (Lemma 3.1.3.) First, we compute $\{\!\{v_\alpha, v_\beta z\}\!\}$:

$$\begin{aligned}\{\!\{v_\alpha, v_\beta z\}\!\} &= \{\!\{v_\alpha, v_\beta\}\!\}\, z + v_\beta\, \{\!\{v_\alpha, z\}\!\} \\ &= -\frac{1}{2}\, o(\alpha,\beta)\,(v_\beta \otimes v_\alpha z + v_\alpha \otimes v_\beta z) - \frac{1}{2} v_\beta \otimes v_\alpha z + \frac{1}{2} v_\beta z \otimes v_\alpha\end{aligned}$$

In particular, we get the base of induction with $\beta = 1$,

$$\{\!\{v_\alpha, c_1'\}\!\} \overset{\alpha\geq 1}{=} \frac{1}{2} v_\alpha \otimes c_1' + \frac{1}{2} c_1' \otimes v_\alpha\,, \quad \{\!\{v_1, c_1'\}\!\} = -\frac{1}{2} v_1 \otimes c_1' + \frac{1}{2} c_1' \otimes v_1\,.$$

This is a particular case of our statement for $\beta = 1$. Now, we compute

$$\begin{aligned}\{\!\{v_\alpha, v_\beta w_\lambda\}\!\} &= \{\!\{v_\alpha, v_\beta\}\!\}\, w_\lambda + v_\beta\, \{\!\{v_\alpha, w_\lambda\}\!\} \\ &= -\frac{1}{2}\, o(\alpha,\beta)\,(v_\beta \otimes v_\alpha w_\lambda + v_\alpha \otimes v_\beta w_\lambda) + \delta_{\alpha\lambda} v_\beta \otimes e_\infty \\ &\quad + \frac{1}{2}[o(\alpha,\lambda) + \delta_{\alpha\lambda}]\,(v_\beta \otimes v_\alpha w_\lambda + v_\beta w_\lambda v_\alpha \otimes e_\infty)\,.\end{aligned}$$

Assume by induction that for all $\lambda < \beta$,

$$\{\!\{v_\alpha, c_\lambda'\}\!\} = \frac{1}{2} c_\lambda' \otimes v_\alpha - \frac{1}{2}\,(o(\alpha,\lambda) + \delta_{\alpha\lambda})\, v_\alpha \otimes c_\lambda'\,,$$

then we get by (3.45)

$$\begin{aligned}\{\!\{v_\alpha, c_\beta'\}\!\} =& \frac{1}{2}\sum_{\lambda=1}^{\beta-1} v_\beta w_\lambda c_\lambda' \otimes v_\alpha - \frac{1}{2}\, o(\alpha,\beta) \sum_{\lambda=1}^{\beta-1} \left(v_\beta \otimes v_\alpha w_\lambda c_\lambda' + v_\alpha \otimes v_\beta w_\lambda c_\lambda'\right) \\ &+ \delta_{(\alpha<\beta)} v_\beta \otimes c_\alpha' + \frac{1}{2}\sum_{\lambda=1}^{\beta-1}[o(\alpha,\lambda) + \delta_{\alpha\lambda}] v_\beta \otimes v_\alpha w_\lambda c_\lambda' \\ &- \frac{1}{2}\, o(\alpha,\beta)\,(v_\beta \otimes v_\alpha z + v_\alpha \otimes v_\beta z) - \frac{1}{2} v_\beta \otimes v_\alpha z + \frac{1}{2} v_\beta z \otimes v_\alpha\,.\end{aligned}$$

In the case $\alpha > \beta$ we find

$$\begin{aligned}\{\!\{v_\alpha, c_\beta'\}\!\} &\overset{\alpha>\beta}{=} \frac{1}{2}\sum_{\lambda=1}^{\beta-1} v_\beta w_\lambda c_\lambda' \otimes v_\alpha + \frac{1}{2} v_\beta z \otimes v_\alpha + \frac{1}{2}\sum_{\lambda=1}^{\beta-1} v_\alpha \otimes v_\beta w_\lambda c_\lambda' + \frac{1}{2} + v_\alpha \otimes v_\beta z \\ &= \frac{1}{2} c_\beta' \otimes v_\alpha + \frac{1}{2} v_\alpha \otimes c_\beta'\,.\end{aligned}$$

In the case $\alpha = \beta$ we have

$$\begin{aligned}\{\!\{v_\alpha, c_\beta'\}\!\} &\overset{\alpha=\beta}{=} \frac{1}{2}\sum_{\lambda=1}^{\beta-1} v_\beta w_\lambda c_\lambda' \otimes v_\alpha + \frac{1}{2} v_\beta z \otimes v_\alpha - \frac{1}{2}\sum_{\lambda=1}^{\beta-1} v_\beta \otimes v_\alpha w_\lambda c_\lambda' - \frac{1}{2} v_\beta \otimes v_\alpha z \\ &= \frac{1}{2} c_\beta' \otimes v_\alpha - \frac{1}{2} v_\alpha \otimes c_\beta'\,.\end{aligned}$$

Finally, for $\alpha < \beta$ we get

$$\begin{aligned}\{\!\{v_\alpha, c'_\beta\}\!\} \overset{\alpha<\beta}{=} &\frac{1}{2}\sum_{\lambda=1}^{\beta-1} v_\beta w_\lambda c'_\lambda \otimes v_\alpha - \frac{1}{2}\sum_{\lambda=1}^{\beta-1}\left(v_\beta \otimes v_\alpha w_\lambda c'_\lambda + v_\alpha \otimes v_\beta w_\lambda c'_\lambda\right) + v_\beta \otimes c'_\alpha \\ &+ \frac{1}{2}\left[\sum_{\lambda=\alpha}^{\beta-1} - \sum_{\lambda=1}^{\alpha-1}\right] v_\beta \otimes v_\alpha w_\lambda c'_\lambda - \frac{1}{2}\left(v_\beta \otimes v_\alpha z + v_\alpha \otimes v_\beta z\right) \\ &- \frac{1}{2} v_\beta \otimes v_\alpha z + \frac{1}{2} v_\beta z \otimes v_\alpha \\ = &\frac{1}{2}\sum_{\lambda=1}^{\beta-1} v_\beta w_\lambda c'_\lambda \otimes v_\alpha + \frac{1}{2} v_\beta z \otimes v_\alpha - \frac{1}{2}\sum_{\lambda=1}^{\beta-1} v_\alpha \otimes v_\beta w_\lambda c'_\lambda - \frac{1}{2} v_\alpha \otimes v_\beta z \\ &+ v_\beta \otimes c'_\alpha - \sum_{\lambda=1}^{\alpha-1} v_\beta \otimes v_\alpha w_\lambda c'_\lambda - v_\beta \otimes v_\alpha z\,,\end{aligned}$$

which is exactly $\frac{1}{2}c'_\beta \otimes v_\alpha - \frac{1}{2}v_\alpha \otimes c'_\beta$ since the last three terms cancel out. □

Proof. (Proposition 3.1.4.) It is easier to use the induction in the first variable, that is

$$\{\!\{c'_\alpha, \Gamma\}\!\} = \sum_{\lambda=1}^{\alpha-1}\left(v_\alpha w_\lambda * \{\!\{c'_\lambda, \Gamma\}\!\} + \{\!\{v_\alpha w_\lambda, \Gamma\}\!\} * c'_\lambda\right) + \{\!\{v_\alpha z, \Gamma\}\!\} \tag{3.46}$$

with $\Gamma = c'_\beta$ in our case. By doing so, we can repeatedly use (3.9a), (3.9b) and Lemma (3.1.3). We first compute

$$\begin{aligned}\{\!\{v_\alpha z, c'_\beta\}\!\} &= \{\!\{v_\alpha, c'_\beta\}\!\} * z + v_\alpha * \{\!\{z, c'_\beta\}\!\} \\ &= \frac{1}{2}c'_\beta z \otimes v_\alpha - \frac{1}{2}\left(o(\alpha,\beta) + \delta_{\alpha\beta}\right) v_\alpha z \otimes c'_\beta - \frac{1}{2}c'_\beta z \otimes v_\alpha + \frac{1}{2}c'_\beta \otimes v_\alpha z \\ &= -\frac{1}{2}\left(o(\alpha,\beta) + \delta_{\alpha\beta}\right) v_\alpha z \otimes c'_\beta + \frac{1}{2}c'_\beta \otimes v_\alpha z\,,\end{aligned}$$

which gives in particular $\left\{\!\!\left\{c'_1, c'_\beta\right\}\!\!\right\} = -\frac{1}{2}c'_1 \otimes c'_\beta + \frac{1}{2}c'_\beta \otimes c'_1$. Now we find

$$\begin{aligned}\{\!\{v_\alpha w_\lambda, c'_\beta\}\!\} &= \{\!\{v_\alpha, c'_\beta\}\!\} * w_\lambda + v_\alpha * \{\!\{w_\lambda, c'_\beta\}\!\} \\ &= \frac{1}{2}c'_\beta w_\lambda \otimes v_\alpha - \frac{1}{2}\left(o(\alpha,\beta) + \delta_{\alpha\beta}\right) v_\alpha w_\lambda \otimes c'_\beta - \frac{1}{2}c'_\beta w_\lambda \otimes v_\alpha \\ &\quad + \frac{1}{2}\left(o(\lambda,\beta) - \delta_{\lambda\beta}\right) e_\infty \otimes v_\alpha w_\lambda c'_\beta - \delta_{\lambda\beta}\left(e_\infty \otimes v_\alpha z + \sum_{\gamma=1}^{\beta-1} e_\infty \otimes v_\alpha w_\gamma c'_\gamma\right),\end{aligned}$$

and since the first and third terms cancel out we can write

$$\begin{aligned}\sum_{\lambda=1}^{\alpha-1} \{\!\{v_\alpha w_\lambda, c'_\beta\}\!\} * c'_\lambda = &-\frac{1}{2}\left(o(\alpha,\beta)+\delta_{\alpha\beta}\right)\sum_{\lambda=1}^{\alpha-1} v_\alpha w_\lambda c'_\lambda \otimes c'_\beta \\ &+\frac{1}{2}\sum_{\lambda=1}^{\alpha-1}\left(o(\lambda,\beta)-\delta_{\lambda\beta}\right) c'_\lambda \otimes v_\alpha w_\lambda c'_\beta \\ &-\delta_{(\beta<\alpha)} c'_\beta \otimes v_\alpha z - \delta_{(\beta<\alpha)}\sum_{\gamma=1}^{\beta-1} c'_\beta \otimes v_\alpha w_\gamma c'_\gamma\,.\end{aligned}$$

Now, assume by induction that for all $\lambda < \alpha$,

$$\{\!\{c'_\lambda, c'_\beta\}\!\} \;=\; \frac{1}{2}[o(\lambda,\beta)+\delta_{\lambda\beta}]\left(c'_\beta \otimes c'_\lambda - c'_\lambda \otimes c'_\beta\right)\,,$$

and let us show that this holds for $\lambda = \alpha$. Note that it is exactly (3.11) since in the case $\lambda = \beta$ the two terms cancel out. We find by (3.46)

$$\begin{aligned}\{\!\{c'_\alpha, c'_\beta\}\!\} =&\frac{1}{2}\sum_{\lambda=1}^{\alpha-1}[o(\lambda,\beta)+\delta_{\lambda\beta}]\left(c'_\beta \otimes v_\alpha w_\lambda c'_\lambda - c'_\lambda \otimes v_\alpha w_\lambda c'_\beta\right) \\ &-\frac{1}{2}\left(o(\alpha,\beta)+\delta_{\alpha\beta}\right)\sum_{\lambda=1}^{\alpha-1} v_\alpha w_\lambda c'_\lambda \otimes c'_\beta + \frac{1}{2}\sum_{\lambda=1}^{\alpha-1}\left(o(\lambda,\beta)-\delta_{\lambda\beta}\right) c'_\lambda \otimes v_\alpha w_\lambda c'_\beta \\ &-\delta_{(\beta<\alpha)} c'_\beta \otimes v_\alpha z - \delta_{(\beta<\alpha)}\sum_{\gamma=1}^{\beta-1} c'_\beta \otimes v_\alpha w_\gamma c'_\gamma \\ &-\frac{1}{2}\left(o(\alpha,\beta)+\delta_{\alpha\beta}\right) v_\alpha z \otimes c'_\beta + \frac{1}{2} c'_\beta \otimes v_\alpha z\,.\end{aligned}$$

If $\alpha > \beta$ we find

$$\begin{aligned}\{\!\{c'_\alpha, c'_\beta\}\!\} \overset{\alpha\geq\beta}{=} &\frac{1}{2}\left(\sum_{\lambda=1}^{\beta} - \sum_{\lambda=\beta+1}^{\alpha-1}\right)\left(c'_\beta \otimes v_\alpha w_\lambda c'_\lambda - c'_\lambda \otimes v_\alpha w_\lambda c'_\beta\right) \\ &+\frac{1}{2}\sum_{\lambda=1}^{\alpha-1} v_\alpha w_\lambda c'_\lambda \otimes c'_\beta + \frac{1}{2}\left(\sum_{\lambda=1}^{\beta-1} - \sum_{\lambda=\beta}^{\alpha-1}\right) c'_\lambda \otimes v_\alpha w_\lambda c'_\beta \\ &-\sum_{\lambda=1}^{\beta-1} c'_\beta \otimes v_\alpha w_\lambda c'_\lambda + \frac{1}{2} v_\alpha z \otimes c'_\beta - \frac{1}{2} c'_\beta \otimes v_\alpha z \\ =&-\frac{1}{2}\sum_{\lambda=1}^{\alpha-1} c'_\beta \otimes v_\alpha w_\lambda c'_\lambda - \frac{1}{2} c'_\beta \otimes v_\alpha z + \frac{1}{2}\sum_{\lambda=1}^{\alpha-1} v_\alpha w_\lambda c'_\lambda \otimes c'_\beta + \frac{1}{2} v_\alpha z \otimes c'_\beta\,,\end{aligned}$$

which gives us $-\frac{1}{2}(c'_\beta \otimes c'_\alpha - c'_\alpha \otimes c'_\beta)$. In the other cases,

$$\{\!\{c'_\alpha, c'_\beta\}\!\} \overset{\alpha\lneq\beta}{=} \frac{1}{2}\sum_{\lambda=1}^{\alpha-1}\left(c'_\beta \otimes v_\alpha w_\lambda c'_\lambda - c'_\lambda \otimes v_\alpha w_\lambda c'_\beta\right) - \frac{1}{2}\sum_{\lambda=1}^{\alpha-1} v_\alpha w_\lambda c'_\lambda \otimes c'_\beta$$
$$+ \frac{1}{2}\sum_{\lambda=1}^{\alpha-1} c'_\lambda \otimes v_\alpha w_\lambda c'_\beta - \frac{1}{2} v_\alpha z \otimes c'_\beta + \frac{1}{2} c'_\beta \otimes v_\alpha z\,,$$

and this is precisely $+\frac{1}{2}(c'_\beta \otimes c'_\alpha - c'_\alpha \otimes c'_\beta)$. □

3.3.2 Associated brackets for framed Jordan quivers

We successively prove Lemmae 3.1.5, 3.1.6 and 3.1.7.

Proof. (Lemma 3.1.5.) The first identity in (3.12a) follows from Lemma 3.1.7 with $u = x$. For the second one, we compute with (3.2c) and (3.9a)

$$\{\!\{x, a'_\alpha c'_\beta\}\!\} = \frac{1}{2} e_0 \otimes x a'_\alpha c'_\beta - \frac{1}{2} x \otimes a'_\alpha c'_\beta + \frac{1}{2} a'_\alpha c'_\beta x \otimes e_0 + \frac{1}{2} a'_\alpha c'_\beta \otimes x\,, \tag{3.47}$$

so that we have

$$\left\{\!\!\left\{x^k, a'_\alpha c'_\beta x^l\right\}\!\!\right\} = \sum_{\sigma=1}^{k} x^{\sigma-1} * \{\!\{x, a'_\alpha c'_\beta\}\!\}\, x^l * x^{k-\sigma}$$
$$+ \sum_{\sigma=1}^{k}\sum_{\tau=1}^{l} x^{\sigma-1} * a'_\alpha c'_\beta x^{\tau-1}\{\!\{x, x\}\!\}\, x^{l-\tau} * x^{k-\sigma}\,.$$

Thus, using the double brackets (3.2a) and (3.47)

$$\begin{aligned}\left\{\!\!\left\{x^k, a'_\alpha c'_\beta x^l\right\}\!\!\right\} =& \frac{1}{2}\sum_{\sigma=1}^{k}\sum_{\tau=1}^{l}\left(a'_\alpha c'_\beta x^{k-\sigma+\tau+1} \otimes x^{l-\tau+\sigma-1} - a'_\alpha c'_\beta x^{k-\sigma+\tau-1} \otimes x^{l-\tau+\sigma+1}\right) \\ &+ \frac{1}{2}\sum_{\sigma=1}^{k}\left(x^{k-\sigma} \otimes x^\sigma a'_\alpha c'_\beta x^l - x^{k-\sigma+1} \otimes x^{\sigma-1} a'_\alpha c'_\beta x^l\right. \\ &\left. + a'_\alpha c'_\beta x^{k-\sigma+1} \otimes x^{l+\sigma-1} + a'_\alpha c'_\beta x^{k-\sigma} \otimes x^{l+\sigma}\right).\end{aligned} \tag{3.48}$$

If we apply the multiplication m, only the last two terms do not cancel out and we find that $\{x^k, a'_\alpha c'_\beta x^l\} = k\, a'_\alpha c'_\beta x^{k+l}$.

Finally, to get (3.12b) we split $\left\{\!\!\left\{a'_\gamma c'_\epsilon x^k, a'_\alpha c'_\beta x^l\right\}\!\!\right\}$ as

$$a'_\gamma c'_\epsilon * \left\{\!\!\left\{x^k, a'_\alpha c'_\beta x^l\right\}\!\!\right\} + a'_\alpha c'_\beta \left\{\!\!\left\{a'_\gamma c'_\epsilon, x^l\right\}\!\!\right\} * x^k + \{\!\{a'_\gamma c'_\epsilon, a'_\alpha c'_\beta\}\!\}\, x^l * x^k\,.$$

Let us first reduce the two first terms. From (3.47) and (3.48) we can get

$$\begin{aligned}
S :=& a'_\gamma c'_\epsilon * \left\{\!\!\left\{ x^k, a'_\alpha c'_\beta x^l \right\}\!\!\right\} - \sum_{\tau=1}^{l} a'_\alpha c'_\beta x^{\tau-1} \{\!\{ x, a'_\gamma c'_\epsilon \}\!\}^\circ x^{l-\tau} * x^k \\
=& \frac{1}{2}\sum_{\sigma=1}^{k} \left(x^{k-\sigma} \otimes a'_\gamma c'_\epsilon x^\sigma a'_\alpha c'_\beta x^l - x^{k-\sigma+1} \otimes a'_\gamma c'_\epsilon x^{\sigma-1} a'_\alpha c'_\beta x^l \right) \\
&+ \frac{1}{2}\sum_{\sigma=1}^{k} \left(a'_\alpha c'_\beta x^{k-\sigma+1} \otimes a'_\gamma c'_\epsilon x^{l+\sigma-1} + a'_\alpha c'_\beta x^{k-\sigma} \otimes a'_\gamma c'_\epsilon x^{l+\sigma} \right) \\
&+ \frac{1}{2}\sum_{\sigma=1}^{k}\sum_{\tau=1}^{l} \left(a'_\alpha c'_\beta x^{k-\sigma+\tau+1} \otimes a'_\gamma c'_\epsilon x^{l-\tau+\sigma-1} - a'_\alpha c'_\beta x^{k-\sigma+\tau-1} \otimes a'_\gamma c'_\epsilon x^{l-\tau+\sigma+1} \right) \\
&+ \frac{1}{2}\sum_{\tau=1}^{l} \left(-a'_\alpha c'_\beta x^{\tau} a'_\gamma c'_\epsilon x^k \otimes x^{l-\tau} + a'_\alpha c'_\beta x^{\tau-1} a'_\gamma c'_\epsilon x^k \otimes x^{l-\tau+1} \right) \\
&+ \frac{1}{2}\sum_{\tau=1}^{l} \left(-a'_\alpha c'_\beta x^{k+\tau-1} \otimes a'_\gamma c'_\epsilon x^{l-\tau+1} - a'_\alpha c'_\beta x^{k+\tau} \otimes a'_\gamma c'_\epsilon x^{l-\tau} \right)
\end{aligned}$$

If we apply the multiplication map, then relabel the indices we can write

$$\begin{aligned}
S =& \frac{1}{2}\left[\sum_{\sigma=1}^{k} - \sum_{\sigma=0}^{k-1}\right] x^{k-\sigma} a'_\gamma c'_\epsilon x^\sigma a'_\alpha c'_\beta x^l + \frac{1}{2}\left[\sum_{\sigma=0}^{k-1} + \sum_{\sigma=1}^{k}\right] a'_\alpha c'_\beta x^{k-\sigma} a'_\gamma c'_\epsilon x^{l+\sigma} \\
&+ \frac{1}{2}\left[\sum_{\sigma=0}^{k-1}\sum_{\tau=l} + \sum_{\sigma=0}\sum_{\tau=1}^{l-1} - \sum_{\sigma=1}^{k-1}\sum_{\tau=0} - \sum_{\sigma=k}\sum_{\tau=0}^{l-1}\right] a'_\alpha c'_\beta x^{k-\sigma+\tau} a'_\gamma c'_\epsilon x^{l-\tau+\sigma} \\
&+ \frac{1}{2}\left[\sum_{\tau=0}^{l-1} - \sum_{\tau=1}^{l}\right] a'_\alpha c'_\beta x^{\tau} a'_\gamma c'_\epsilon x^{k+l-\tau} - \frac{1}{2}\left[\sum_{\tau=0}^{l-1} + \sum_{\tau=1}^{l}\right] a'_\alpha c'_\beta x^{k+\tau} a'_\gamma c'_\epsilon x^{l-\tau}\,.
\end{aligned}$$

After simplification we get

$$\begin{aligned}
S =& \frac{1}{2} a'_\gamma c'_\epsilon x^k a'_\alpha c'_\beta x^l - \frac{1}{2} x^k a'_\gamma c'_\epsilon a'_\alpha c'_\beta x^l \\
&+ \frac{1}{2} a'_\alpha c'_\beta x^k a'_\gamma c'_\epsilon x^l + a'_\alpha c'_\beta a'_\gamma c'_\epsilon x^{k+l} + \sum_{\sigma=1}^{k-1} a'_\alpha c'_\beta x^{k-\sigma} a'_\gamma c'_\epsilon x^{l+\sigma} \\
&+ \frac{1}{2}\sum_{\sigma=0}^{k-1} a'_\alpha c'_\beta x^{k+l-\sigma} a'_\gamma c'_\epsilon x^{\sigma} + \frac{1}{2}\sum_{\tau=1}^{l-1} a'_\alpha c'_\beta x^{k+\tau} a'_\gamma c'_\epsilon x^{l-\tau} \\
&- \frac{1}{2}\sum_{\sigma=1}^{k-1} a'_\alpha c'_\beta x^{k-\sigma} a'_\gamma c'_\epsilon x^{l+\sigma} - \frac{1}{2}\sum_{\tau=0}^{l-1} a'_\alpha c'_\beta x^{\tau} a'_\gamma c'_\epsilon x^{k+l-\tau} \\
&+ \frac{1}{2} a'_\alpha c'_\beta a'_\gamma c'_\epsilon x^{k+l} - \frac{1}{2} a'_\alpha c'_\beta x^l a'_\gamma c'_\epsilon x^k \\
&- \frac{1}{2} a'_\alpha c'_\beta x^k a'_\gamma c'_\epsilon x^l - \frac{1}{2} a'_\alpha c'_\beta x^{k+l} a'_\gamma c'_\epsilon - \sum_{\tau=1}^{l-1} a'_\alpha c'_\beta x^{k+\tau} a'_\gamma c'_\epsilon x^{l-\tau}\,.
\end{aligned}$$

We can continue to cancel terms modulo commutators and get

$$\begin{aligned}S=&+\frac{1}{2}a'_\alpha c'_\beta a'_\gamma c'_\epsilon x^{k+l}-\frac{1}{2}a'_\alpha c'_\beta x^{k+l}a'_\gamma c'_\epsilon\\&+\frac{1}{2}\sum_{\sigma=1}^{k-1}\left(a'_\alpha c'_\beta x^{k-\sigma}a'_\gamma c'_\epsilon x^{l+\sigma}+a'_\alpha c'_\beta x^{k+l-\sigma}a'_\gamma c'_\epsilon x^{\sigma}\right)\\&-\frac{1}{2}\sum_{\tau=1}^{l-1}\left(a'_\alpha c'_\beta x^{\tau}a'_\gamma c'_\epsilon x^{k+l-\tau}+a'_\alpha c'_\beta x^{k+\tau}a'_\gamma c'_\epsilon x^{l-\tau}\right)\\=&\frac{1}{2}a'_\alpha c'_\beta a'_\gamma c'_\epsilon x^{k+l}-\frac{1}{2}a'_\gamma c'_\epsilon a'_\alpha c'_\beta x^{k+l}\\&+\frac{1}{2}\left[\sum_{r=1}^{k}-\sum_{r=1}^{l}\right]\left(a'_\alpha c'_\beta x^{r}a'_\gamma c'_\epsilon x^{k+l-r}+a'_\alpha c'_\beta x^{k+l-r}a'_\gamma c'_\epsilon x^{r}\right),\end{aligned}$$

where we added the terms $r=k,l$ in the sums because they cancel out together. Meanwhile, we compute using (3.3b) (which is $\{\!\{a'_\alpha, a'_\beta\}\!\}$), (3.9b) and (3.11)

$$\begin{aligned}&\{\!\{a'_\gamma c'_\epsilon, a'_\alpha c'_\beta\}\!\}\\=&\{\!\{a'_\gamma, a'_\alpha\}\!\}\, c'_\beta * c'_\epsilon + a'_\alpha\, \{\!\{a'_\gamma, c'_\beta\}\!\} * c'_\epsilon + a'_\gamma * \{\!\{c'_\epsilon, a'_\alpha\}\!\}\, c'_\beta + a'_\gamma * a'_\alpha\, \{\!\{c'_\epsilon, c'_\beta\}\!\}\\=&-\frac{1}{2}o(\gamma,\alpha)\left(a'_\gamma c'_\epsilon\otimes a'_\alpha c'_\beta + a'_\alpha c'_\epsilon \otimes a'_\gamma c'_\beta\right)\\&+\frac{1}{2}o(\epsilon,\beta)\left(a'_\alpha c'_\beta\otimes a'_\gamma c'_\epsilon - a'_\alpha c'_\epsilon\otimes a'_\gamma c'_\beta\right)\\&-\frac{1}{2}a'_\alpha c'_\beta a'_\gamma c'_\epsilon\otimes e_0+\frac{1}{2}\left(o(\gamma,\beta)-\delta_{\gamma\beta}\right)a'_\alpha c'_\epsilon\otimes a'_\gamma c'_\beta\\&-\delta_{\gamma\beta}\left(a'_\alpha c'_\epsilon\otimes z+\sum_{\mu=1}^{\beta-1}a'_\alpha c'_\epsilon\otimes a'_\mu c'_\mu\right)\\&+\frac{1}{2}e_0\otimes a'_\gamma c'_\epsilon a'_\alpha c'_\beta-\frac{1}{2}\left(o(\alpha,\epsilon)-\delta_{\alpha\epsilon}\right)a'_\alpha c'_\epsilon\otimes a'_\gamma c'_\beta\\&+\delta_{\alpha\epsilon}\left(z\otimes a'_\gamma c'_\beta+\sum_{\lambda=1}^{\epsilon-1}a'_\lambda c'_\lambda\otimes a'_\gamma c'_\beta\right),\end{aligned}$$

which we have to multiply on the right by x^l (for the outer bimodule structure) and x^k (for the inner bimodule structure). If we do so and apply the multiplication map, we finally get modulo commutators that $\{a'_\gamma c'_\epsilon x^k, a'_\alpha c'_\beta x^l\}$ equals

$$\begin{aligned}
&+\frac{1}{2}\left[\sum_{r=1}^{k}-\sum_{r=1}^{l}\right]\left(a'_\alpha c'_\beta x^r a'_\gamma c'_\epsilon x^{k+l-r}+a'_\alpha c'_\beta x^{k+l-r} a'_\gamma c'_\epsilon x^r\right)\\
&+\frac{1}{2}\,o(\alpha,\gamma)\left(a'_\gamma c'_\epsilon x^k a'_\alpha c'_\beta x^l+a'_\alpha c'_\epsilon x^k a'_\gamma c'_\beta x^l\right)\\
&+\frac{1}{2}o(\epsilon,\beta)\left(a'_\alpha c'_\beta x^k a'_\gamma c'_\epsilon x^l-a'_\alpha c'_\epsilon x^k a'_\gamma c'_\beta x^l\right)\\
&-\frac{1}{2}\left(o(\beta,\gamma)+\delta_{\gamma\beta}\right)a'_\alpha c'_\epsilon x^k a'_\gamma c'_\beta x^l-\delta_{\gamma\beta}\left(a'_\alpha c'_\epsilon x^k z x^l+\sum_{\mu=1}^{\beta-1}a'_\alpha c'_\epsilon x^k a'_\mu c'_\mu x^l\right)\\
&+\frac{1}{2}\left(o(\epsilon,\alpha)+\delta_{\alpha\epsilon}\right)a'_\alpha c'_\epsilon x^k a'_\gamma c'_\beta x^l+\delta_{\alpha\epsilon}\left(z x^k a'_\gamma c'_\beta x^l+\sum_{\lambda=1}^{\epsilon-1}a'_\lambda c'_\lambda x^k a'_\gamma c'_\beta x^l\right)\,,
\end{aligned}$$

which is our claim. □

Proof. (Lemma 3.1.6.) The first identity in (3.13a) follows from Lemma 3.1.7 with $u=z$. Next, remark that if we replace x by z in (3.47), it takes the form

$$\{\!\{z, a'_\alpha c'_\beta\}\!\}=\frac{1}{2}e_0\otimes z a'_\alpha c'_\beta-\frac{1}{2}z\otimes a'_\alpha c'_\beta-\frac{1}{2}a'_\alpha c'_\beta z\otimes e_0+\frac{1}{2}a'_\alpha c'_\beta\otimes z\,,$$

while (3.48) is now written as

$$\begin{aligned}
\left\{\!\!\left\{z^k, a'_\alpha c'_\beta z^l\right\}\!\!\right\}=&\frac{1}{2}\sum_{\sigma=1}^{k}\Big(z^{k-\sigma}\otimes z^\sigma a'_\alpha c'_\beta z^l-z^{k-\sigma+1}\otimes z^{\sigma-1}a'_\alpha c'_\beta z^l\\
&-a'_\alpha c'_\beta z^{k-\sigma+1}\otimes z^{l+\sigma-1}+a'_\alpha c'_\beta z^{k-\sigma}\otimes z^{l+\sigma}\Big)\\
&-\frac{1}{2}\sum_{\sigma=1}^{k}\sum_{\tau=1}^{l}\left(a'_\alpha c'_\beta z^{k-\sigma+\tau+1}\otimes z^{l-\tau+\sigma-1}-a'_\alpha c'_\beta z^{k-\sigma+\tau-1}\otimes z^{l-\tau+\sigma+1}\right)\,.
\end{aligned}$$

Applying the multiplication map on the latter expression clearly yields $\{z^k, a'_\alpha c'_\beta z^l\}=0$. Using the derivation rules, we arrive at the same conclusion for any $k,l\in\mathbb{Z}$. To obtain (3.13b), the proof is similar to the derivation of (3.12b) using the above expressions. □

Proof. (Lemma 3.1.7.) For the first identity, we have that

$$\begin{aligned}
\left\{\!\!\left\{u^k, u^l\right\}\!\!\right\}=&\sum_{\sigma=1}^{k}\sum_{\tau=1}^{l}u^{\sigma-1}*u^{\tau-1}(\{\!\{u,u\}\!\})u^{l-\tau}*u^{k-\sigma}\\
=&\frac{1}{2}\epsilon(u)\sum_{\sigma=1}^{k}\sum_{\tau=1}^{l}(u^{k-\sigma+\tau+1}\otimes u^{l-\tau+\sigma-1}-u^{k-\sigma+\tau-1}\otimes u^{l-\tau+\sigma+1})\,,
\end{aligned}$$

which is easily seen to vanish after applying the multiplication map. Then, using (3.14), we remark that

$$\begin{aligned}\{\!\{u, w_\alpha v_\beta\}\!\} =& w_\alpha \{\!\{u, v_\beta\}\!\} + \{\!\{u, w_\alpha\}\!\} v_\beta \\ =& \frac{1}{2}\left(w_\alpha v_\beta u \otimes e_0 - w_\alpha v_\beta \otimes u + e_0 \otimes u w_\alpha v_\beta - u \otimes w_\alpha v_\beta\right) .\end{aligned} \tag{3.49}$$

Therefore[7]

$$\begin{aligned}\left\{\!\!\left\{u^k, w_\alpha v_\beta u^l\right\}\!\!\right\} =& \frac{1}{2}\epsilon(u) \sum_{\tau=1}^{k}\sum_{\sigma=1}^{l} \Big(w_\alpha v_\beta u^{k-\tau+\sigma+1} \otimes u^{l-\sigma+\tau-1} \\ & \qquad - w_\alpha v_\beta u^{k-\tau+\sigma-1} \otimes u^{l-\sigma+\tau+1}\Big) \\ & + \frac{1}{2}\sum_{\tau=1}^{k}\Big(w_\alpha v_\beta u^{k-\tau+1} \otimes u^{l+\tau-1} - w_\alpha v_\beta u^{k-\tau} \otimes u^{l+\tau} \\ & \qquad + u^{k-\tau} \otimes u^{\tau} w_\alpha v_\beta u^l - u^{k-\tau+1} \otimes u^{\tau-1} w_\alpha v_\beta u^l\Big) .\end{aligned} \tag{3.50}$$

After application of the multiplication map, we get 0 and the second equality follows. To prove that (3.15b) holds, write $\{\!\{w_\gamma v_\epsilon u^k, w_\alpha v_\beta u^l\}\!\}$ as

$$w_\gamma v_\epsilon * \left\{\!\!\left\{u^k, w_\alpha v_\beta u^l\right\}\!\!\right\} + w_\alpha v_\beta \left\{\!\!\left\{w_\gamma v_\epsilon, u^l\right\}\!\!\right\} * u^k + \{\!\{w_\gamma v_\epsilon, w_\alpha v_\beta\}\!\} u^l * u^k .$$

First, we simplify the two first terms. Using (3.49) and (3.50) yields

$$\begin{aligned}T :=& w_\gamma v_\epsilon * \left\{\!\!\left\{u^k, w_\alpha v_\beta u^l\right\}\!\!\right\} - \sum_{\sigma=1}^{l} w_\alpha v_\beta u^{\sigma-1} \{\!\{u, w_\gamma v_\epsilon\}\!\}^\circ u^{l-\sigma} * u^k \\ =& \frac{1}{2}\epsilon(u) \sum_{\tau=1}^{k}\sum_{\sigma=1}^{l}\Big(w_\alpha v_\beta u^{k-\tau+\sigma+1} \otimes w_\gamma v_\epsilon u^{l+\tau-\sigma-1} - w_\alpha v_\beta u^{k-\tau+\sigma-1} \otimes w_\gamma v_\epsilon u^{l+\tau-\sigma+1}\Big) \\ & + \frac{1}{2}\sum_{\tau=1}^{k}\Big(w_\alpha v_\beta u^{k-\tau+1} \otimes w_\gamma v_\epsilon u^{l+\tau-1} - w_\alpha v_\beta u^{k-\tau} \otimes w_\gamma v_\epsilon u^{l+\tau}\Big) \\ & + \frac{1}{2}\sum_{\tau=1}^{k}\Big(u^{k-\tau} \otimes w_\gamma v_\epsilon u^{\tau} w_\alpha v_\beta u^l - u^{k-\tau+1} \otimes w_\gamma v_\epsilon u^{\tau-1} w_\alpha v_\beta u^l\Big) \\ & - \frac{1}{2}\sum_{\sigma=1}^{l}\Big(w_\alpha v_\beta u^{\sigma} w_\gamma v_\epsilon u^k \otimes u^{l-\sigma} - w_\alpha v_\beta u^{\sigma-1} w_\gamma v_\epsilon u^k \otimes u^{l-\sigma+1}\Big) \\ & - \frac{1}{2}\sum_{\sigma=1}^{l}\Big(w_\alpha v_\beta u^{k+\sigma-1} \otimes w_\gamma v_\epsilon u^{l-\sigma+1} - w_\alpha v_\beta u^{k+\sigma} \otimes w_\gamma v_\epsilon u^{l-\sigma}\Big) .\end{aligned}$$

This gives, after multiplication and modulo commutators

$$m \circ T = \frac{1}{2}\epsilon(u) \sum_{\tau=1}^{k}\sum_{\sigma=1}^{l}\Big(w_\alpha v_\beta u^{k-\tau+\sigma+1} w_\gamma v_\epsilon u^{l+\tau-\sigma-1} - w_\alpha v_\beta u^{k-\tau+\sigma-1} w_\gamma v_\epsilon u^{l+\tau-\sigma+1}\Big) ,$$

[7] Note that the summation indices are the opposite of the previous proofs.

because the last four sums cancel out. Relabelling indices, we write

$$\begin{aligned} m\circ T =&\frac{1}{2}\epsilon(u)\left[\sum_{\tau=1}^{k-1}\sum_{\sigma=l}+\sum_{\tau=0}\sum_{\sigma=1}^{l}-\sum_{\tau=k}\sum_{\sigma=1}^{l-1}-\sum_{\tau=1}^{k}\sum_{\sigma=0}\right] w_\alpha v_\beta u^{k-\tau+\sigma} w_\gamma v_\epsilon u^{l+\tau-\sigma} \\ =&\frac{1}{2}\epsilon(u)\left[\sum_{\tau=1}^{k-1} w_\alpha v_\beta u^{k+l-\tau} w_\gamma v_\epsilon u^{\tau}+\sum_{\sigma=1}^{l} w_\alpha v_\beta u^{k+\sigma} w_\gamma v_\epsilon u^{l-\sigma}\right] \\ &-\frac{1}{2}\epsilon(u)\left[\sum_{\sigma=1}^{l-1} w_\alpha v_\beta u^{\sigma} w_\gamma v_\epsilon u^{k+l-\sigma}+\sum_{\tau=1}^{k} w_\alpha v_\beta u^{k-\tau} w_\gamma v_\epsilon u^{l+\tau}\right] \end{aligned}$$

It remains to compute $\{\!\{w_\gamma v_\epsilon, w_\alpha v_\beta\}\!\}$. We can find from (3.3a)–(3.3c)

$$\begin{aligned} \{\!\{w_\gamma v_\epsilon, w_\alpha v_\beta\}\!\} =&-\frac{1}{2}o(\gamma,\alpha)\left(w_\alpha v_\epsilon\otimes w_\gamma v_\beta + w_\gamma v_\epsilon\otimes w_\alpha v_\beta\right) \\ &-\frac{1}{2}o(\beta,\gamma)\left(w_\alpha v_\beta w_\gamma v_\epsilon\otimes e_0 + w_\alpha v_\epsilon\otimes w_\gamma v_\beta\right) \\ &-\delta_{\beta\gamma}\left(w_\alpha v_\epsilon\otimes e_0+\frac{1}{2}w_\alpha v_\epsilon\otimes w_\gamma v_\beta+\frac{1}{2}w_\alpha v_\beta w_\gamma v_\epsilon\otimes e_0\right) \\ &+\delta_{\alpha\epsilon}\left(e_0\otimes w_\gamma v_\beta+\frac{1}{2}w_\alpha v_\epsilon\otimes w_\gamma v_\beta+\frac{1}{2}e_0\otimes w_\gamma v_\epsilon w_\alpha v_\beta\right) \\ &+\frac{1}{2}o(\epsilon,\alpha)\left(e_0\otimes w_\gamma v_\epsilon w_\alpha v_\beta + w_\alpha v_\epsilon\otimes w_\gamma v_\beta\right) \\ &-\frac{1}{2}o(\epsilon,\beta)\left(w_\alpha v_\beta\otimes w_\gamma v_\epsilon + w_\alpha v_\epsilon\otimes w_\gamma v_\beta\right)\ . \end{aligned}$$

Applying the multiplication map, we get

$$\begin{aligned} m\circ\left(\{\!\{w_\gamma v_\epsilon, w_\alpha v_\beta\}\!\}\, u^l * u^k\right) =&-\frac{1}{2}o(\gamma,\alpha)\left(w_\alpha v_\epsilon u^k w_\gamma v_\beta u^l + w_\gamma v_\epsilon u^k w_\alpha v_\beta u^l\right) \\ &-\frac{1}{2}o(\beta,\gamma)\left(w_\alpha v_\beta w_\gamma v_\epsilon u^k u^l + w_\alpha v_\epsilon u^k w_\gamma v_\beta u^l\right) \\ &-\delta_{\beta\gamma}\left(w_\alpha v_\epsilon u^k u^l+\frac{1}{2}w_\alpha v_\epsilon u^k w_\gamma v_\beta u^l+\frac{1}{2}w_\alpha v_\beta w_\gamma v_\epsilon u^k u^l\right) \\ &+\delta_{\alpha\epsilon}\left(u^k w_\gamma v_\beta u^l+\frac{1}{2}w_\alpha v_\epsilon u^k w_\gamma v_\beta u^l+\frac{1}{2}u^k w_\gamma v_\epsilon w_\alpha v_\beta u^l\right) \\ &+\frac{1}{2}o(\epsilon,\alpha)\left(u^k w_\gamma v_\epsilon w_\alpha v_\beta u^l + w_\alpha v_\epsilon u^k w_\gamma v_\beta u^l\right) \\ &-\frac{1}{2}o(\epsilon,\beta)\left(w_\alpha v_\beta u^k w_\gamma v_\epsilon u^l + w_\alpha v_\epsilon u^k w_\gamma v_\beta u^l\right)\ . \end{aligned}$$

Adding $m\circ T$ to this last expression gives $\{w_\gamma v_\epsilon u^k, w_\alpha v_\beta u^l\}$, which finishes the proof. □

3.3.3 Computations using subquivers of the framed Jordan quivers

We prove Propositions 3.1.9 and 3.1.11.

Proof. **(Proposition 3.1.9.)** We choose $\alpha, \eta \in \mathbb{C}$, and prove the statement with $\alpha = \eta'$ to avoid confusion. Hence, for this proof, α is not seen as a parameter running over $1, \dots, d$. Recall that $U_{+,\eta} = u(1+\eta\phi)$. If we write $\{\!\{U_{+,\alpha}, U_{+,\eta}\}\!\} = a' \otimes a''$, we get

$$\frac{1}{KL}\{U_{+,\alpha}^K, U_{+,\eta}^L\} = U_{+,\eta}^{L-1} a' U_{+,\alpha}^{K-1} a'' \mod [A,A]\,, \tag{3.51}$$

after using (2.15) and the derivation property in each argument. Thus, we need to compute

$$\{\!\{u+\alpha u\phi, u+\eta u\phi\}\!\} = \{\!\{u,u\}\!\} + \alpha\,\{\!\{u\phi,u\}\!\} + \eta\,\{\!\{u,u\phi\}\!\} + \alpha\eta\,\{\!\{u\phi,u\phi\}\!\}\;. \tag{3.52}$$

We need the brackets $\{\!\{u,u\}\!\} = \frac{1}{2}\epsilon(u)[u^2\otimes e_0 - e_0\otimes u^2]$ and

$$\{\!\{\phi,a\}\!\} = \frac{1}{2}(a\otimes\phi - e_0\otimes\phi a + a\phi\otimes e_0 - \phi\otimes a)\,, \tag{3.53}$$

for any a which is a word in $\{e_0,\phi,u\}$. The second equality can be obtained by combining (2.19) and Lemma 2.3.14 applied to the subquiver based at $I' = \{0\}$. We find

$$\begin{aligned}
\{\!\{u\phi,u\}\!\} =& u * \{\!\{\phi,u\}\!\} + \{\!\{u,u\}\!\} * \phi\\
=&\frac{1}{2}(u\otimes u\phi - e_0\otimes u\phi u + u\phi\otimes u - \phi\otimes u^2) + \frac{1}{2}\epsilon(u)(u^2\phi\otimes e_0 - \phi\otimes u^2)\,,\\
\{\!\{u,u\phi\}\!\} =& -\frac{1}{2}(u\phi\otimes u - u\phi u\otimes e_0 + u\otimes u\phi - u^2\otimes\phi) - \frac{1}{2}\epsilon(u)(e_0\otimes u^2\phi - u^2\otimes\phi)\,.
\end{aligned}$$

Here, the second equality is obtained by cyclic antisymmetry : $\{\!\{u,u\phi\}\!\} = -\{\!\{u\phi,u\}\!\}^\circ$. We also compute

$$\begin{aligned}
\{\!\{u\phi,u\phi\}\!\} =& \{\!\{u\phi,u\}\!\}\,\phi + u(u * \{\!\{\phi,\phi\}\!\}) + u(-\{\!\{\phi,u\}\!\}^\circ) * \phi\\
=&\frac{1}{2}(u\otimes u\phi^2 - e_0\otimes u\phi u\phi + u\phi\otimes u\phi - \phi\otimes u^2\phi) + \frac{1}{2}\epsilon(u)(u^2\phi\otimes\phi - \phi\otimes u^2\phi)\\
&+\frac{1}{2}(-u\otimes u\phi^2 + u\phi^2\otimes u) - \frac{1}{2}(u\phi^2\otimes u - u\phi u\phi\otimes e_0 + u\phi\otimes u\phi - u^2\phi\otimes\phi)\\
=&\frac{1}{2}(u\phi u\phi\otimes e_0 + u^2\phi\otimes\phi - e_0\otimes u\phi u\phi - \phi\otimes u^2\phi) + \frac{\epsilon(u)}{2}(u^2\phi\otimes\phi - \phi\otimes u^2\phi)\,.
\end{aligned}$$

Write U_η instead of $U_{+,\eta}$, for the rest of the first part of the proof. We can use that $\eta u\phi = U_\eta - u$, and the same holds with α. Hence we can write

$$\begin{aligned}
\alpha\eta\,\{\!\{u\phi,u\phi\}\!\} =& \frac{1}{2}\eta\,u\phi(U_\alpha - u)\otimes e_0 + \frac{1}{2}\eta\,u(U_\alpha - u)\otimes\phi)\\
&-\frac{1}{2}\alpha\,e_0\otimes u\phi(U_\eta - u) - \frac{1}{2}\alpha\,\phi\otimes u(U_\eta - u))\\
&+\frac{1}{2}\epsilon(u)\eta\,u(U_\alpha - u)\otimes\phi - \frac{1}{2}\epsilon(u)\alpha\,\phi\otimes u(U_\eta - u)\,.
\end{aligned}$$

Summing all terms appearing in (3.52), we get after cancellation

$$\begin{aligned}\{\!\{U_\alpha, U_\eta\}\!\} =&\frac{1}{2}\epsilon(u)(u^2\otimes e_0 - e_0\otimes u^2) + \frac{1}{2}\alpha(u\otimes u\phi + u\phi\otimes u) + \frac{1}{2}\alpha\epsilon(u)(u^2\phi\otimes e_0)\\ &-\frac{1}{2}\eta(u\phi\otimes u + u\otimes u\phi) - \frac{1}{2}\eta\epsilon(u)(e_0\otimes u^2\phi) + \frac{1}{2}\epsilon(u)(\eta u U_\alpha\otimes\phi - \alpha\phi\otimes uU_\eta)\\ &+\frac{1}{2}(-\alpha e_0\otimes u\phi U_\eta - \alpha\phi\otimes uU_\eta + \eta u\phi U_\alpha\otimes e_0 + \eta u U_\alpha\otimes\phi)\,.\end{aligned}$$

We use the same trick again to get rid of constants α, η. We also need the modified version $\eta\phi = u^{-1}U_\eta - 1$ which is obtained by multiplying by u^{-1} on the left. We find

$$\begin{aligned}\{\!\{U_\alpha, U_\eta\}\!\} =&+\frac{1}{2}(u\otimes U_\alpha + U_\alpha\otimes u) + \frac{1}{2}\epsilon(u)(uU_\alpha\otimes e_0)\\ &-\frac{1}{2}(U_\eta\otimes u + u\otimes U_\eta) - \frac{1}{2}\epsilon(u)(e_0\otimes uU_\eta)\\ &+\frac{1}{2}\epsilon(u)(uU_\alpha\otimes u^{-1}U_\eta - uU_\alpha\otimes e_0 - u^{-1}U_\alpha\otimes uU_\eta + e_0\otimes uU_\eta)\\ &+\frac{1}{2}(-e_0\otimes U_\alpha U_\eta - u^{-1}U_\alpha\otimes uU_\eta + 2e_0\otimes uU_\eta)\\ &+\frac{1}{2}(U_\eta U_\alpha\otimes e_0 + uU_\alpha\otimes u^{-1}U_\eta - 2uU_\alpha\otimes e_0)\,.\end{aligned}$$

Putting the double bracket in (3.51), we find modulo commutators in $A/[A,A]$ that

$$\begin{aligned}\frac{1}{KL}\{U_\alpha^K, U_\eta^L\} =&+\frac{1}{2}(U_\eta^{L-1}uU_\alpha^K + U_\eta^{L-1}U_\alpha^K u) + \frac{1}{2}\epsilon(u)(U_\eta^{L-1}uU_\alpha^K)\\ &-\frac{1}{2}(U_\eta^L U_\alpha^{K-1}u + U_\eta^L u U_\alpha^{K-1}) - \frac{1}{2}\epsilon(u)(U_\eta^L U_\alpha^{K-1}u)\\ &+\frac{1}{2}\epsilon(u)(U_\eta^L uU_\alpha^K u^{-1} - U_\eta^{L-1}uU_\alpha^K - U_\eta^L u^{-1}U_\alpha^K u + U_\eta^L U_\alpha^{K-1}u)\\ &+\frac{1}{2}(-U_\eta^L u^{-1}U_\alpha^K u + 2U_\eta^L U_\alpha^{K-1}u + U_\eta^L uU_\alpha^K u^{-1} - 2U_\eta^{L-1}uU_\alpha^K)\\ =&\frac{1}{2}(U_\eta^{L-1}U_\alpha^K u - U_\eta^{L-1}uU_\alpha^K) + \frac{1}{2}(U_\eta^L U_\alpha^{K-1}u - U_\eta^L uU_\alpha^{K-1})\\ &+\frac{1}{2}(1+\epsilon(u))(U_\eta^L uU_\alpha^K u^{-1} - U_\eta^L u^{-1}U_\alpha^K u)\,.\end{aligned}$$

Now, notice that $\alpha U_\eta - \eta U_\alpha = (\alpha-\eta)u$. Hence, assuming $\alpha\neq\eta$, the first term in the expression just obtained can be written as

$$U_\eta^{L-1}U_\alpha^K u - U_\eta^{L-1}uU_\alpha^K = \frac{1}{\alpha-\eta}(U_\eta^{L-1}U_\alpha^K(\alpha U_\eta - \eta U_\alpha) - U_\eta^{L-1}(\alpha U_\eta - \eta U_\alpha)U_\alpha^K)$$

which vanishes modulo commutators. It is clear that it also vanishes for $\alpha = \eta$, hence the first term always disappears. Similarly, the second term is zero. Therefore

$$\frac{1}{KL}\{U_\alpha^K, U_\eta^L\} = \frac{1}{2}(1+\epsilon(u))(U_\eta^L uU_\alpha^K u^{-1} - U_\eta^L u^{-1}U_\alpha^K u)\,, \tag{3.54}$$

which vanishes when $\epsilon(u) = -1$.

Finally, we have to show that, if $\epsilon(z) = +1$, $\{U_{-,\alpha}^K, U_{-,\eta}^L\} = 0$ modulo commutators for $U_{-,\alpha} = u(1+\alpha\phi^{-1})$. First, note that we can write

$$\{\!\{u,u\}\!\} = \frac{1}{2}\epsilon(u)[u^2 \otimes e_0 - e_0 \otimes u^2], \;\; \{\!\{\phi^{-1},a\}\!\} = -\frac{1}{2}(a\phi^{-1}\otimes e_0 - \phi^{-1}\otimes a + a\otimes\phi^{-1} - e_0\otimes\phi^{-1}a).$$

Indeed, we have

$$0 = \{\!\{\phi\phi^{-1},a\}\!\} = \phi * \{\!\{\phi^{-1},a\}\!\} + \{\!\{\phi,a\}\!\} * \phi^{-1}$$

which yields $\{\!\{\phi^{-1},a\}\!\} = -\phi^{-1} * \{\!\{\phi,a\}\!\} * \phi^{-1}$. This implies that $\{\!\{\phi^{-1},a\}\!\}$ can be obtained from $\{\!\{\phi,a\}\!\}$ by replacing ϕ by ϕ^{-1} and multiplying by -1 since

$$\{\!\{\phi^{-1},a\}\!\} = -\frac{1}{2}\phi^{-1} * [\phi * E_0(a) + E_0(a) * \phi] * \phi^{-1} = -\frac{1}{2}[\phi^{-1} * E_0(a) + E_0(a) * \phi^{-1}]\,,$$

where we used $\{\!\{\phi,a\}\!\}$ as given by Lemma 2.3.14. Thus, reproducing the proof in the first case with suitable sign changes, we get

$$\frac{1}{KL}\{U_{-,\alpha}^K, U_{-,\eta}^L\} = \frac{1}{2}(-1+\epsilon(z))\left(U_{-,\eta}^L z U_{-,\alpha}^K z^{-1} - U_{-,\eta}^L z^{-1} U_{-,\alpha}^K z\right)\,,$$

and we can conclude. □

Proof. (Proposition 3.1.11.) Without loss of generality, we assume that $0 \leq \beta \leq \alpha \leq d$. To ease notations for the proof, we also set $u_1 = u_{(\alpha)}$, $u_0 = u_{(\beta)}$, $\phi_1 = \Phi_0^{(\alpha)}$ and $\phi_0 = \Phi_0^{(\beta)}$. As in (3.51), we have that

$$\frac{1}{KL}\{u_1^K, u_0^L\} = u_0^{L-1}\,\{\!\{u_1,u_0\}\!\}'\,u_1^{K-1}\,\{\!\{u_1,u_0\}\!\}'' \quad \mathrm{mod}[A,A]\,.$$

Hence, we need to compute $\{\!\{u_1,u_0\}\!\}$ as a first step. Noting that $u_i = \phi_i u$ for $i = 1,2$, we write

$$\{\!\{u_1,u_0\}\!\} = \{\!\{\phi_1,u_0\}\!\} * u + \phi_1 * \{\!\{u,\phi_0\}\!\}\,u + \phi_1 * \phi_0\,\{\!\{u,u\}\!\}\;.$$

By assumption, ϕ_0 and ϕ_1 are moment maps corresponding to the last two quivers in the chain $\bar{Q}_0 \subseteq \bar{Q}_\beta \subseteq \bar{Q}_\alpha$. Hence, we get from Lemma 2.3.15 that

$$\{\!\{\phi_1,u_0\}\!\} = \frac{1}{2}(u_0\otimes\phi_1 - e_0\otimes\phi_1 u_0 + u_0\phi_1\otimes e_0 - \phi_1\otimes u_0)\,,$$
$$\{\!\{\phi_0,u\}\!\} = \frac{1}{2}(u\otimes\phi_0 - e_0\otimes\phi_0 u + u\phi_0\otimes e_0 - \phi_0\otimes u)\,.$$

Using the cyclic antisymmetry, the second equality gives $\{\!\{u,\phi_0\}\!\}$. Now, since we consider $u = y,z$, we have $\{\!\{u,u\}\!\} = -\frac{1}{2}[u^2\otimes e_0 - e_0\otimes u^2]$. Hence, we can group all these terms and get that

$$\{\!\{u_1,u_0\}\!\} = \frac{1}{2}(u_0u_1\otimes e_0 - u_1\otimes u_0 + u_0\otimes u_1 - e_0\otimes u_1u_0)\,,$$

after simplification. This immediately implies that $\{u_1^K, u_0^L\} = 0$. □

3.3.4 Loday brackets for framed Jordan quivers

We successively prove Lemmae 3.1.12, 3.1.13, 3.1.14, 3.1.15, 3.1.16 and 3.1.17.

For the first four results, it is important to remark that we can not use (3.53) to get $\{\!\{\phi, v_\beta\}\!\}$ or $\{\!\{\phi, w_\beta\}\!\}$, since both elements are *not* in the initial algebra defined from $\bar{Q}_0$, for which ϕ is a moment map. However, a short calculation enables us to get these double brackets, and we can write

$$\{\!\{\phi, w_\beta\}\!\} = \frac{1}{2}(e_0 \otimes \phi w_\beta - \phi \otimes w_\beta)\,, \quad \{\!\{\phi, v_\beta\}\!\} = \frac{1}{2}(v_\beta \phi \otimes e_0 - v_\beta \otimes \phi)\,. \tag{3.55}$$

We derive the double bracket involving v_β, the second case being similar. We assume that we work in the localised algebra where x is invertible for the proof, but the equality holds without this assumption. Write $\phi = \phi_+ \phi_-^{-1}$ for $\phi_+ = e_0 + xy$, and $\phi_- = e_0 + yx$, and remark that (3.14) is satisfied for both $u = \phi_+, \phi_-$. Therefore

$$\{\!\{\phi, v_\beta\}\!\} = \{\!\{\phi_+, v_\beta\}\!\} * \phi_-^{-1} - \phi_+ \phi_-^{-1} * \{\!\{\phi, v_\beta\}\!\} * \phi_-^{-1} = \frac{1}{2}(v_\beta \phi_+ \phi_-^{-1} \otimes e_0 - v_\beta \otimes \phi_+ \phi_-^{-1} e_0)\,,$$

as desired.

In each case, we also use (2.38) to get that for any $U, c \in A$ (or A') and $K \in \mathbb{N}^\times$, we have

$$\frac{1}{K}\{U^K, c\} = \{\!\{U, c\}\!\}'\, U^{K-1}\, \{\!\{U, c\}\!\}''\ . \tag{3.56}$$

Hence, the main step is to compute $\{\!\{U, c\}\!\}$ if we want to determine $\{U^K, c\}$.

Proof. (Lemma 3.1.12.) Recall that $U_\eta = z(1 + \eta\phi)$. We can compute

$$\begin{aligned}\{\!\{U_\eta, x\}\!\} =& \{\!\{z, x\}\!\} * (1 + \eta\phi) + \eta z * \{\!\{\phi, x\}\!\} \\ =& \frac{1}{2}(-(e_0 + \eta\phi) \otimes zx - xz(1 + \eta\phi) \otimes e_0 - x(1 + \eta\phi) \otimes z + z(1 + \eta\phi) \otimes x) \\ &+ \frac{1}{2}\eta(x \otimes z\phi - e_0 \otimes z\phi x + x\phi \otimes z - \phi \otimes zx)\,.\end{aligned}$$

We can write $U_\eta - z = \eta z\phi$, and since z is invertible in A', we can use $z^{-1}U_\eta = e_0 + \eta\phi$ or $z^{-1}U_\eta - e_0 = \eta\phi$. Hence

$$\begin{aligned}\{\!\{U_\eta, x\}\!\} =& \frac{1}{2}(-z^{-1}U_\eta \otimes zx - xU_\eta \otimes e_0 - xz^{-1}U_\eta \otimes z + U_\eta \otimes x) \\ &+ \frac{1}{2}(x \otimes (U_\eta - z) - e_0 \otimes (U_\eta - z)x + x(z^{-1}U_\eta - e_0) \otimes z - (z^{-1}U_\eta - e_0) \otimes zx) \\ =& -z^{-1}U_\eta \otimes zx - x \otimes z + e_0 \otimes zx + \frac{1}{2}\left(x \otimes U_\eta - xU_\eta \otimes e_0 + U_\eta \otimes x - e_0 \otimes U_\eta x\right)\ .\end{aligned}$$

In particular, using (3.56), we find

$$\begin{aligned}\frac{1}{K}\{U_\eta^K, x\} =& - z^{-1}U_\eta U_\eta^{K-1}zx - xU_\eta^{K-1}z + U_\eta^{K-1}zx \\ =& - \eta\phi U_\eta^{K-1}zx - xU_\eta^{K-1}z\,.\end{aligned}$$

Next, we compute

$$\begin{aligned}\{\!\{U_\eta, z\}\!\} =& \{\!\{z,z\}\!\} * (1+\eta\phi) + \eta z * \{\!\{\phi, z\}\!\} \\ =& \frac{1}{2}\left((e_0+\eta\phi)\otimes z^2 - z^2(1+\eta\phi)\otimes e_0\right) \\ &+ \frac{1}{2}\eta(z\otimes z\phi - e_0\otimes z\phi z + z\phi\otimes z - \phi\otimes z^2) \\ =& \frac{1}{2}\Big(z^{-1}U_\eta\otimes z^2 - zU_\eta\otimes e_0 + z\otimes(U_\eta - z) - e_0\otimes(U_\eta - z)z \\ &\quad + (U_\eta - z)\otimes z - z^{-1}(U_\eta - z)\otimes z^2\Big) \\ =& - z\otimes z + e_0\otimes z^2 + \frac{1}{2}\left(-zU_\eta\otimes e_0 + z\otimes U_\eta - e_0\otimes U_\eta z + U_\eta\otimes z\right)\,.\end{aligned}$$

This yields $\frac{1}{K}\{U_\eta^K, z\} = -zU_\eta^{K-1}z + U_\eta^{K-1}z^2$.

For the element w_β, we get

$$\begin{aligned}\{\!\{U_\eta, w_\beta\}\!\} =& \{\!\{z, w_\beta\}\!\} * (1+\eta\phi) + \eta z * \{\!\{\phi, w_\beta\}\!\} \\ =& \frac{1}{2}\left((e_0+\eta\phi)\otimes zw_\beta - z(1+\eta\phi)\otimes w_\beta\right) + \frac{1}{2}\eta(e_0\otimes z\phi w_\beta - \phi\otimes zw_\beta) \\ =& \frac{1}{2}\left(z^{-1}U_\eta\otimes zw_\beta - U_\eta\otimes w_\beta + e_0\otimes(U_\eta - z)w_\beta - (z^{-1}U_\eta - e_0)\otimes zw_\beta\right) \\ =& \frac{1}{2}\left(-U_\eta\otimes w_\beta + e_0\otimes U_\eta w_\beta\right)\,,\end{aligned}$$

so that $\{U_\eta^K, w_\beta\} = 0$. Similarly,

$$\begin{aligned}\{\!\{U_\eta, v_\beta\}\!\} =& \{\!\{z, v_\beta\}\!\} * (1+\eta\phi) + \eta z * \{\!\{\phi, v_\beta\}\!\} \\ =& \frac{1}{2}\left(v_\beta z(1+\eta\phi)\otimes e_0 - v_\beta(1+\eta\phi)\otimes z\right) + \frac{1}{2}\eta(v_\beta\phi\otimes z - v_\beta\otimes z\phi) \\ =& \frac{1}{2}\left(v_\beta U_\eta\otimes e_0 - v_\beta z^{-1}U_\eta\otimes z + v_\beta(z^{-1}U_\eta - e_0)\otimes z - v_\beta\otimes(U_\eta - z)\right) \\ =& \frac{1}{2}\left(v_\beta U_\eta\otimes e_0 - v_\beta\otimes U_\eta\right)\,,\end{aligned} \tag{3.57}$$

which gives $\{U_\eta^K, v_\beta\} = 0$. □

Proof. (Lemma 3.1.13.) Recall that $\bar{U}_\eta = y(1+\eta\phi)$. By comparing the double brackets (3.2b), (3.5b), we remark that $\{\!\{y, x\}\!\}$ only differ from $\{\!\{z, x\}\!\}$ by replacing z by y and adding an extra term $-e_0 \otimes e_0$. Hence, by adapting the proof of Lemma 3.1.12 we get

$$\begin{aligned}\frac{1}{K}\{\bar{U}_\eta^K, x\} =& -(e_0+\eta\phi)\bar{U}_\eta^{K-1} - y^{-1}\bar{U}_\eta\bar{U}_\eta^{K-1}yx - x\bar{U}_\eta^{K-1}y + \bar{U}_\eta^{K-1}yx \\ =& -\bar{U}_\eta^{K-1} - x\bar{U}_\eta^{K-1}y - \eta\phi\bar{U}_\eta^{K-1}(e_0+yx)\,.\end{aligned} \tag{3.58}$$

Similarly, the double brackets with y replacing z match so that

$$\frac{1}{K}\{\bar{U}_\eta^K, y\} = -y\bar{U}_\eta^{K-1}y + \bar{U}_\eta^{K-1}y^2\,. \tag{3.59}$$

The same holds for the couple (y, w_β) instead of (z, w_β), or doing it with v_β. This yields $\{\bar{U}_\eta^K, w_\beta\} = 0$ and $\{\bar{U}_\eta^K, v_\beta\} = 0$. □

Proof. (Lemma 3.1.14.) Recall that $\hat{U}_\eta = x(1+\eta\phi^{-1})$. We can compute

$$\begin{aligned}\left\{\!\!\left\{\hat{U}_\eta, x\right\}\!\!\right\} =& \{\!\{x,x\}\!\} * (e_0+\eta\phi^{-1}) - \eta x\phi^{-1} * \{\!\{\phi, x\}\!\} * \phi^{-1} \\ =&\frac{1}{2}(x^2(e_0+\eta\phi^{-1})\otimes e_0 - (e_0+\eta\phi^{-1})\otimes x^2) \\ &-\frac{1}{2}\eta(x\phi^{-1}\otimes x - \phi^{-1}\otimes x^2 + x\otimes x\phi^{-1} - e_0\otimes x\phi^{-1}x)\,.\end{aligned}$$

We use $\hat{U}_\eta - x = \eta x\phi^{-1}$, and that equality after multiplication on the left by x^{-1}. Hence

$$\begin{aligned}\left\{\!\!\left\{\hat{U}_\eta, x\right\}\!\!\right\} =&\frac{1}{2}(x\hat{U}_\eta\otimes e_0 - x^{-1}\hat{U}_\eta\otimes x^2) \\ &+\frac{1}{2}((\hat{U}_\eta - x)\otimes x - x^{-1}(\hat{U}_\eta - x)\otimes x^2 + x\otimes(\hat{U}_\eta - x) - e_0\otimes(\hat{U}_\eta - x)x) \\ =&\, x\otimes x - e_0\otimes x^2 + \frac{1}{2}\left(x\hat{U}_\eta\otimes e_0 - x\otimes\hat{U}_\eta + e_0\otimes\hat{U}_\eta x - \hat{U}_\eta\otimes x\right)\,.\end{aligned}$$

This directly yields

$$\frac{1}{K}\{\hat{U}_\eta^K, x\} = x\hat{U}_\eta^{K-1}x - \hat{U}_\eta^{K-1}x^2\,.$$

Next we find

$$\begin{aligned}\left\{\!\!\left\{\hat{U}_\eta, z\right\}\!\!\right\} =& \{\!\{x,z\}\!\} * (e_0+\eta\phi^{-1}) - \eta x\phi^{-1} * \{\!\{\phi, z\}\!\} * \phi^{-1} \\ =&\frac{1}{2}(zx(e_0+\eta\phi^{-1})\otimes e_0 + (e_0+\eta\phi^{-1})\otimes xz) \\ &+\frac{1}{2}(z(e_0+\eta\phi^{-1})\otimes x - x(e_0+\eta\phi^{-1})\otimes z) \\ &-\frac{1}{2}\eta(z\phi^{-1}\otimes x - \phi^{-1}\otimes xz + z\otimes x\phi^{-1} - e_0\otimes x\phi^{-1}z)\,.\end{aligned}$$

With the previous trick, this gives

$$\left\{\!\!\left\{\hat{U}_\eta, z\right\}\!\!\right\} = \frac{1}{2}(z\hat{U}_\eta \otimes e_0 + x^{-1}\hat{U}_\eta \otimes xz + zx^{-1}\hat{U}_\eta \otimes x - \hat{U}_\eta \otimes z) \\ - \frac{1}{2}(zx^{-1}(\hat{U}_\eta - x) \otimes x - x^{-1}(\hat{U}_\eta - x) \otimes xz + z \otimes (\hat{U}_\eta - x) - e_0 \otimes (\hat{U}_\eta - x)z) \\ = x^{-1}\hat{U}_\eta \otimes xz + z \otimes x - e_0 \otimes xz \\ + \frac{1}{2}(z\hat{U}_\eta \otimes e_0 - \hat{U}_\eta \otimes z - z \otimes \hat{U}_\eta + e_0 \otimes \hat{U}_\eta z)\,,$$

so that

$$\frac{1}{K}\{\hat{U}_\eta^K, z\} = x^{-1}\hat{U}_\eta^K xz + z\hat{U}_\eta^{K-1}x - \hat{U}_\eta^{K-1}xz = z\hat{U}_\eta^{K-1}x + \eta\phi^{-1}\hat{U}_\eta^{K-1}xz\,.$$

After simple computations, we can obtain $\left\{\!\!\left\{\hat{U}_\eta, v_\beta\right\}\!\!\right\} = \frac{1}{2}v_\beta\hat{U}_\eta \otimes e_0 - \frac{1}{2}v_\beta \otimes \hat{U}_\eta$, so $\{\hat{U}_\eta^K, v_\beta\} = 0$ as before. Similarly, we compute $\left\{\!\!\left\{\hat{U}_\eta, w_\beta\right\}\!\!\right\} = \frac{1}{2}(e_0 \otimes \hat{U}_\eta w_\beta - \hat{U}_\eta \otimes w_\beta)$, which also gives $\{\hat{U}_\eta^K, w_\beta\} = 0$. □

Proof. (Lemma 3.1.15.) For simplicity, we work in A' so that $\tilde{U}_\eta = xz(1+\eta\phi)$. First, an easy computation yields

$$\{\!\{xz, x\}\!\} = \frac{1}{2}x^2z \otimes e_0 - \frac{1}{2}e_0 \otimes xzx - \frac{1}{2}xz \otimes x - \frac{1}{2}x \otimes xz\,.$$

We obtain in that way, after using $xz\eta\phi = \tilde{U}_\eta - xz$ and $\eta\phi = (xz)^{-1}\tilde{U}_\eta - e_0$

$$\left\{\!\!\left\{\tilde{U}_\eta, x\right\}\!\!\right\} = \{\!\{xz, x\}\!\} * (1+\eta\phi^{-1}) - \eta xz\phi^{-1} * \{\!\{\phi, x\}\!\} * \phi^{-1} \\ = \frac{1}{2}(x\tilde{U}_\eta \otimes e_0 - (xz)^{-1}\tilde{U}_\eta \otimes xzx - \tilde{U}_\eta \otimes x - x(xz)^{-1}\tilde{U}_\eta \otimes xz) \\ - \frac{1}{2}x[(xz)^{-1}\tilde{U}_\eta - e_0] \otimes xz + \frac{1}{2}[(xz)^{-1}\tilde{U}_\eta - e_0] \otimes xzx \\ - \frac{1}{2}x \otimes [\tilde{U}_\eta - xz] + \frac{1}{2}e_0 \otimes [\tilde{U}_\eta - xz]x\,.$$

Therefore we get by cancelling out terms

$$\frac{1}{K}\{\tilde{U}_\eta^K, x\} = -x(xz)^{-1}\tilde{U}_\eta^K xz - \tilde{U}_\eta^{K-1}xzx + x\tilde{U}_\eta^{K-1}xz \\ = -\tilde{U}_\eta^{K-1}xzx - \eta\, x\phi^{-1}\tilde{U}_\eta^{K-1}xz\,.$$

Second, we get

$$\left\{\!\!\left\{\tilde{U}_\eta, xz\right\}\!\!\right\} = \{\!\{xz, xz\}\!\} * (1+\eta\phi^{-1}) - \eta xz\phi^{-1} * \{\!\{\phi, xz\}\!\} * \phi^{-1} \\ = \frac{1}{2}\left(xz\tilde{U}_\eta \otimes e_0 - (xz)^{-1}\tilde{U}_\eta \otimes (xz)^2\right) \\ - \frac{1}{2}[\tilde{U}_\eta - xz] \otimes xz + \frac{1}{2}[(xz)^{-1}\tilde{U}_\eta - e_0] \otimes (xz)^2 \\ - \frac{1}{2}xz \otimes [\tilde{U}_\eta - xz] + \frac{1}{2}e_0 \otimes [\tilde{U}_\eta - xz]xz\,.$$

We find after simplifying terms that $\frac{1}{K}\{\tilde{U}_\eta^K, xz\} = xz\tilde{U}_\eta^{K-1}xz - \tilde{U}_\eta^{K-1}(xz)^2$. Finally, we can get $\left\{\!\!\left\{\tilde{U}_\eta, v_\beta\right\}\!\!\right\} = \frac{1}{2}v_\beta\tilde{U}_\eta \otimes e_0 - \frac{1}{2}v_\beta \otimes \tilde{U}_\eta$, and $\{\tilde{U}_\eta^K, v_\beta\} = 0$ as before. The same holds for w_β too. $\square$

Proof. (Lemma 3.1.16.) First, note that if $a = x, z, v_\beta, w_\beta$ with $\beta \leq \alpha$, we get from Lemma 2.3.15 that

$$\left\{\!\!\left\{\Phi_0^{(\alpha)}, a\right\}\!\!\right\} = \frac{1}{2}(ae_0 \otimes \Phi_0^{(\alpha)} - e_0 \otimes \Phi_0^{(\alpha)}a + a\Phi_0^{(\alpha)} \otimes e_0 - \Phi_0^{(\alpha)} \otimes e_0 a)\,.$$

Using that $z_{(\alpha)} = \Phi_0^{(\alpha)}z$ together with (3.5a)–(3.5c), we get

$$\begin{aligned}
\{\!\!\{z_{(\alpha)}, x\}\!\!\} &= \frac{1}{2}(xz_{(\alpha)} \otimes e_0 - z_{(\alpha)} \otimes x - e_0 \otimes z_{(\alpha)}x - x \otimes z_{(\alpha)}) \\
\{\!\!\{z_{(\alpha)}, z\}\!\!\} &= \frac{1}{2}(-z \otimes z_{(\alpha)} + zz_{(\alpha)} \otimes e_0 - z_{(\alpha)} \otimes z + e_0 \otimes z_{(\alpha)}z) \\
\{\!\!\{z_{(\alpha)}, w_\beta\}\!\!\} &= \frac{1}{2}(-z_{(\alpha)} \otimes w_\beta + e_0 \otimes z_{(\alpha)}w_\beta - 2z \otimes \Phi_0^{(\alpha)}w_\beta) \\
\{\!\!\{z_{(\alpha)}, v_\beta\}\!\!\} &= \frac{1}{2}(v_\beta z_{(\alpha)} \otimes e_0 - v_\beta \otimes z_{(\alpha)} + 2v_\beta z \otimes \Phi_0^{(\alpha)})\,,
\end{aligned}$$

with $\beta \leq \alpha$. Using that

$$\frac{1}{K}\{z_{(\alpha)}^K, a\} = \{\!\!\{z_{(\alpha)}, a\}\!\!\}'\, z_{(\alpha)}^{K-1}\, \{\!\!\{z_{(\alpha)}, a\}\!\!\}''\,, \tag{3.60}$$

we have the four first cases. For the next two cases, note that if we can show

$$\begin{aligned}
\left\{\!\!\left\{\Phi_0^{(\alpha)}, w_\beta\right\}\!\!\right\} &= \frac{1}{2}(e_0 \otimes \Phi_0^{(\alpha)}w_\beta - \Phi_0^{(\alpha)} \otimes w_\beta)\,, \quad \alpha < \beta\,, \\
\left\{\!\!\left\{\Phi_0^{(\alpha)}, v_\beta\right\}\!\!\right\} &= \frac{1}{2}(v_\beta\Phi_0^{(\alpha)} \otimes e_0 - v_\beta \otimes \Phi_0^{(\alpha)})\,, \quad \alpha < \beta\,,
\end{aligned} \tag{3.61}$$

we can easily obtain that

$$\begin{aligned}
\{\!\!\{z_{(\alpha)}, w_\beta\}\!\!\} &= \frac{1}{2}(-z_{(\alpha)} \otimes w_\beta + e_0 \otimes z_{(\alpha)}w_\beta)\,, \quad \alpha < \beta\,, \\
\{\!\!\{z_{(\alpha)}, v_\beta\}\!\!\} &= \frac{1}{2}(v_\beta z_{(\alpha)} \otimes e_0 - v_\beta \otimes z_{(\alpha)})\,, \quad \alpha < \beta\,,
\end{aligned}$$

and then find last two equalities using (3.60). So it remains to prove (3.61), which we do by induction since $\Phi^{(\alpha)} = \Phi^{(\alpha-1)}(e_0 + w_\alpha v_\alpha)^{-1}$. The base case $\alpha = 0$ is just (3.55), as $\Phi_0^{(0)} = \phi$. Next, remark that for $\alpha < \beta$ we can get from (3.3a)–(3.3c)

$$\begin{aligned}
\{\!\!\{(e_0 + w_\alpha v_\alpha), w_\beta\}\!\!\} &= \frac{1}{2}\left(e_0 \otimes (e_0 + w_\alpha v_\alpha)w_\beta - (e_0 + w_\alpha v_\alpha) \otimes w_\beta\right) \\
\{\!\!\{(e_0 + w_\alpha v_\alpha), v_\beta\}\!\!\} &= \frac{1}{2}\left(v_\beta(e_0 + w_\alpha v_\alpha) \otimes e_0 - v_\beta \otimes (e_0 + w_\alpha v_\alpha)\right)\,.
\end{aligned}$$

Assuming by induction that (3.61) holds for $\alpha - 1$, we can get that

$$\begin{aligned}\left\{\!\!\left\{\Phi_0^{(\alpha)}, w_\beta\right\}\!\!\right\} =& \left\{\!\!\left\{\Phi_0^{(\alpha-1)}, w_\beta\right\}\!\!\right\} * (e_0 + w_\alpha v_\alpha)^{-1} \\ &- \Phi_0^{(\alpha-1)}(e_0 + w_\alpha v_\alpha)^{-1} * \{\!\!\{(e_0 + w_\alpha v_\alpha), w_\beta\}\!\!\} * (e_0 + w_\alpha v_\alpha)^{-1} \\ =& \frac{1}{2}(e_0 \otimes \Phi_0^{(\alpha)} w_\beta - \Phi_0^{(\alpha)} \otimes w_\beta)\,,\end{aligned}$$

as expected. The same holds for v_β with $\alpha < \beta$, which proves our claim.

Finally, we want to show that $\{z_{(\alpha)}^K, z_{(\alpha)}\} = \{z_{(\alpha)}^K, \Phi^{(\alpha)}\} = 0$. This follows from the following two double brackets,

$$\begin{aligned}\{\!\!\{z_{(\alpha)}, z_{(\alpha)}\}\!\!\} =& \frac{1}{2}(z_{(\alpha)}^2 \otimes e_0 - e_0 \otimes z_{(\alpha)}^2) \\ \left\{\!\!\left\{z_{(\alpha)}, \Phi^{(\alpha)}\right\}\!\!\right\} =& -\frac{1}{2}(\Phi^{(\alpha)} \otimes z_{(\alpha)} - \Phi^{(\alpha)} z_{(\alpha)} \otimes e_0 + e_0 \otimes z_{(\alpha)}\Phi^{(\alpha)} - z_{(\alpha)} \otimes \Phi^{(\alpha)})\,,\end{aligned}$$

which were obtained in the proof of Proposition 3.1.11. □

Proof. (Lemma 3.1.17.) This is basically the same proof as Lemma 3.1.16 with y replacing z everywhere. The only difference is the double bracket

$$\{\!\!\{y_{(\alpha)}, x\}\!\!\} = -e_0 \otimes \Phi_0^{(\alpha)} + \frac{1}{2}(xy_{(\alpha)} \otimes e_0 - y_{(\alpha)} \otimes x - e_0 \otimes y_{(\alpha)}x - x \otimes y_{(\alpha)})\,,$$

which gives $\frac{1}{K}\{y_{(\alpha)}^K, x\} = -y_{(\alpha)}^{K-1}\Phi_0^{(\alpha)} - y_{(\alpha)}^K x$. □

3.3.5 Double brackets for framed cyclic quivers

We successively prove Propositions 3.2.3 and 3.2.4, Lemma 3.2.5, then Proposition 3.2.6. As in § 3.3.1, we prove the statements by induction using that (3.30) implies

$$\{\!\!\{\Gamma, c'_{s,\alpha}\}\!\!\} = \sum_{\lambda=1}^{\alpha-1}\left(v_{s,\alpha}w_{s,\lambda}\,\{\!\!\{\Gamma, c'_{s,\lambda}\}\!\!\} + \{\!\!\{\Gamma, v_{s,\alpha}w_{s,\lambda}\}\!\!\}\,c'_{s,\lambda}\right) + \{\!\!\{\Gamma, v_{s,\alpha}z\}\!\!\}\,, \qquad (3.62)$$

for any $\Gamma \in A$. Moreover, we can avoid most of the cases by a suitable interpretation of the results from § 3.3.1, as we will see during the proofs[8].

Proof. (Proposition 3.2.3.) We use the double brackets derived in § 3.2.1 and § 3.2.2. First, note that (3.31a) is already computed since $a'_{s,\alpha} = w_{s,\alpha}$. Then, to get (3.31b), we only need to

[8]What we do is more general than the treatment found in [62], where $d_s = 0$ for all $s \in I \setminus \{0\}$.

reproduce what we did in the first part of the proof of Proposition 3.1.2. Let us compute $\{\!\{x, c'_{s,\alpha}\}\!\}$ to show how to deal with the idempotents. We have

$$\begin{aligned}\{\!\{x, v_{s,\alpha}z\}\!\} =& \{\!\{x, v_{s,\alpha}\}\!\}\, z + v_{s,\alpha}\, \{\!\{x, z\}\!\}\\ =&\frac{1}{2}\left(v_{s,\alpha}x \otimes e_s z - v_{s,\alpha} \otimes x e_s z + v_{s,\alpha} z x F_1 + v_{s,\alpha} F_1 x z - v_{s,\alpha} x F_1 z + v_{s,\alpha} z F_1 x\right)\end{aligned}$$

Recalling that $v_{s,\alpha} \in e_\infty A' e_s$, we get $v_{s,\alpha} z x F_1 = v_{s,\alpha} e_s z x F_1$, and since $z \in \oplus_s e_s A' e_{s-1}$ and $x \in \oplus_s e_s A' e_{s+1}$ this is equal to $v_{s,\alpha} e_s z x e_s \otimes e_{s-1}$. We can apply the same method to all four terms given by $\{\!\{x, z\}\!\}$ and get

$$\begin{aligned}\{\!\{x, v_{s,\alpha}z\}\!\} =&\frac{1}{2}\big(v_{s,\alpha} x e_{s+1} \otimes e_s z - v_{s,\alpha} e_s \otimes e_{s-1} x z + v_{s,\alpha} z x e_s \otimes e_{s-1}\\ &+ v_{s,\alpha} e_s \otimes e_{s-1} x z - v_{s,\alpha} x e_{s+1} \otimes e_s z + v_{s,\alpha} z e_{s-1} \otimes e_{s-2} x\big)\\ =&\frac{1}{2}\left(v_{s,\alpha} z x \otimes e_{s-1} + v_{s,\alpha} z \otimes x e_{s-1}\right).\end{aligned}$$

Here, we dropped the idempotents in the first copy of the tensor product since they are determined by the factor $v_{s,\alpha}$. In particular, we get $\{\!\{x, c'_{s,1}\}\!\}$ by setting $c'_{s,1} = v_{s,1}z$. It is not hard to show that $\{\!\{x, v_{s,\alpha}w_{s,\lambda}\}\!\} = 0$ for any $\alpha, \lambda \in \{1, \dots, d_s\}$ so that the claim follows. The double bracket $\{\!\{z, c'_{s,\alpha}\}\!\}$ is left as an exercise as the computations are similar. Finally, (3.31c) is just (3.23b) since $a'_{s,\alpha} = w_{s,\alpha}$. □

Proof. (Proposition 3.2.4.) We begin by treating the case $r \neq s$. We first compute $\{\!\{a'_{s,\alpha}, v_{r,\beta}z\}\!\} = \{\!\{a'_{s,\alpha}, v_{r,\beta}\}\!\}\, z + v_{r,\beta}\, \{\!\{a'_{s,\alpha}, z\}\!\}$ using (3.24) and (3.31a). We have

$$\begin{aligned}\{\!\{a'_{s,\alpha}, v_{r,\beta}z\}\!\} =&\frac{1}{2}o(s,r)\, e_\infty \otimes a'_{s,\alpha} v_{r,\beta} z - \frac{1}{2} v_{r,\beta}(z a'_{s,\alpha} \otimes e_s - a'_{s,\alpha} \otimes e_s z)\\ =&\frac{1}{2}o(s,r)\, e_\infty \otimes a'_{s,\alpha} v_{r,\beta} z - \frac{1}{2}\delta_{(r-1,s)} v_{r,\beta} z a'_{s,\alpha} \otimes e_s\,.\end{aligned}$$

We obtain last equation by noticing that $v_{r,\beta} z a'_{s,\alpha} = v_{r,\beta} z e_{r-1} e_s a'_{s,\alpha}$ vanishes for $s \neq r-1$, while $v_{r,\beta} a'_{s,\alpha} = v_{r,\beta} e_r e_s a'_{s,\alpha}$ is nonzero if and only if $r = s$, which we discard by assumption. In particular,

$$\{\!\{a'_{s,\alpha}, c'_{r,1}\}\!\} = \frac{1}{2}o(s,r)\, e_\infty \otimes a'_{s,\alpha} c'_{r,1} - \frac{1}{2}\delta_{(r,s+1)} c'_{r,1} a'_{s,\alpha} \otimes e_s\,. \qquad (3.63)$$

Next, we compute from (3.24) and (3.23b) (recall $a'_{s,\alpha} = w_{s,\alpha}$)

$$\{\!\{a'_{s,\alpha}, v_{r,\beta} w_{r,\lambda}\}\!\} = \frac{1}{2}o(s,r)\, e_\infty \otimes a'_{s,\alpha} v_{r,\beta} w_{r,\lambda} - \frac{1}{2}o(s,r)\, v_{r,\beta} w_{r,\lambda} \otimes w_{s,\alpha}\,,$$

so that

$$\sum_{\lambda=1}^{\beta-1} \{\!\!\{a'_{s,\alpha}, v_{r,\beta}w_{r,\lambda}\}\!\!\}\, c'_{r,\lambda} = \frac{1}{2}o(s,r)\, e_\infty \otimes \left(\sum_{\lambda=1}^{\beta-1} a'_{s,\alpha}v_{r,\beta}w_{r,\lambda}c'_{r,\lambda}\right) - \frac{1}{2}o(s,r) \sum_{\lambda=1}^{\beta-1} v_{r,\beta}w_{r,\lambda} \otimes w_{s,\alpha}c'_{r,\lambda}\,.$$

Now, assume by induction that for all $\lambda < \beta$

$$\{\!\!\{a'_{s,\alpha}, c'_{r,\lambda}\}\!\!\} = \frac{1}{2}o(s,r)\, e_\infty \otimes a'_{s,\alpha}c'_{r,\lambda} - \frac{1}{2}\delta_{(r,s+1)}c'_{r,\lambda}a'_{s,\alpha} \otimes e_s\,. \tag{3.64}$$

We compute from (3.62)

$$\begin{aligned}\{\!\!\{a'_{s,\alpha}, c'_{r,\lambda}\}\!\!\} =& \frac{1}{2}o(s,r) \sum_{\lambda=1}^{\beta-1} v_{r,\beta}w_{r,\lambda} \otimes a'_{s,\alpha}c'_{r,\lambda} - \frac{1}{2}\delta_{(r,s+1)} \sum_{\lambda=1}^{\beta-1} v_{r,\beta}w_{r,\lambda}c'_{r,\lambda}a'_{s,\alpha} \otimes e_s \\ &+ \frac{1}{2}o(s,r) \left[e_\infty \otimes \left(\sum_{\lambda=1}^{\beta-1} a'_{s,\alpha}v_{r,\beta}w_{r,\lambda}c'_{r,\lambda}\right) - \sum_{\lambda=1}^{\beta-1} v_{r,\beta}w_{r,\lambda} \otimes w_{s,\alpha}c'_{r,\lambda}\right] \\ &+ \frac{1}{2}o(s,r)\, e_\infty \otimes a'_{s,\alpha}v_{r,\beta}z - \frac{1}{2}\delta_{(r,s+1)}v_{r,\beta}za'_{s,\alpha} \otimes e_s \\ =& \frac{1}{2}o(s,r)\, e_\infty \otimes a'_{s,\alpha}v_{r,\beta}\left(\sum_{\lambda=1}^{\beta-1} w_{r,\lambda}c'_{r,\lambda} + z\right) \\ &- \frac{1}{2}\delta_{(r,s+1)}v_{r,\beta}\left(\sum_{\lambda=1}^{\beta-1} w_{r,\lambda}c'_{r,\lambda} + z\right) a'_{s,\alpha} \otimes e_s \\ =& \frac{1}{2}o(s,r)\, e_\infty \otimes a'_{s,\alpha}c'_{r,\beta} - \frac{1}{2}\delta_{(r,s+1)}c'_{r,\beta}a'_{s,\alpha} \otimes e_s\,,\end{aligned}$$

where we used (3.30) to get last equality. This is precisely (3.32) with $r \neq s$.

For $r = s$, it is not hard to see that we can follow the proof from the Jordan case given in § 3.3.1, and establish a result similar to (3.9b) by induction[9] :

$$\begin{aligned}\{\!\!\{a'_{s,\alpha}, c'_{s,\beta}\}\!\!\} =& -\frac{1}{2}c'_{s,\beta}a'_{s,\alpha} \otimes e_s + \frac{1}{2}\left(o(\alpha,\beta) - \delta_{\alpha\beta}\right) e_\infty \otimes a'_{s,\alpha}c'_{s,\beta} \\ &- \delta_{\alpha\beta}e_\infty \otimes e_s\left(z + \sum_{\lambda=1}^{\beta-1} a'_{s,\lambda}c'_{s,\lambda}\right)\,.\end{aligned} \tag{3.65}$$

Noticing that $c'_{s,\beta} = e_\infty c'_{s,\beta}e_{s-1}$ and $a'_{s,\alpha} = e_s a'_{s,\alpha}e_\infty$, we see that the first term vanishes. This is precisely (3.32) for $r = s$. □

[9]For the sceptical reader, the full proof can be found in [62]. For the proof of Lemma 3.2.5 and Lemma 3.2.6, we will also not bother with the full proof of the case $r = s$, which can also be found in [62].

Proof. (Lemma 3.2.5.) Assume that $r \neq s$. We have from (3.23a) that

$$\begin{aligned}\{\!\{v_{s,\alpha}, v_{r,\beta}z\}\!\} &= -\frac{1}{2}o(s,r)\, v_{s,\alpha} \otimes v_{r,\beta}z - \frac{1}{2}(v_{r,\beta}e_s \otimes v_{s,\alpha}z - v_{r,\beta}ze_s \otimes v_{s,\alpha}) \\ &= -\frac{1}{2}o(s,r)\, v_{s,\alpha} \otimes v_{r,\beta}z + \frac{1}{2}\delta_{(r,s+1)}v_{r,\beta}ze_s \otimes v_{s,\alpha}\,,\end{aligned}$$

since $v_{r,\beta} = v_{r,\beta}e_r$ and $r \neq s$. In particular,

$$\{\!\{v_{s,\alpha}, c'_{r,1}\}\!\} = -\frac{1}{2}o(s,r)\, v_{s,\alpha} \otimes c'_{r,1} + \frac{1}{2}\delta_{(r,s+1)}c'_{r,1} \otimes v_{s,\alpha}\,. \tag{3.66}$$

Next, we compute

$$\{\!\{v_{s,\alpha}, v_{r,\beta}w_{r,\lambda}\}\!\} = -\frac{1}{2}o(s,r)\, v_{s,\alpha} \otimes v_{r,\beta}w_{r,\lambda} - \frac{1}{2}o(r,s)\, v_{r,\beta}w_{r,\lambda}v_{s,\alpha} \otimes e_\infty\,.$$

Assuming by induction that for all $\lambda < \beta$

$$\{\!\{v_{s,\alpha}, c'_{r,\lambda}\}\!\} = -\frac{1}{2}o(s,r)\, v_{s,\alpha} \otimes c'_{r,\lambda} + \frac{1}{2}\delta_{(r,s+1)}c'_{r,\lambda} \otimes v_{s,\alpha}\,, \tag{3.67}$$

we obtain by (3.62)

$$\begin{aligned}\{\!\{v_{s,\alpha}, c'_{r,\beta}\}\!\} = &-\frac{1}{2}o(s,r)\sum_{\lambda=1}^{\beta-1} v_{s,\alpha} \otimes v_{r,\beta}w_{r,\lambda}c'_{r,\lambda} + \frac{1}{2}o(s,r)\sum_{\lambda=1}^{\beta-1} v_{r,\beta}w_{r,\lambda}v_{s,\alpha} \otimes c'_{r,\lambda} \\ &-\frac{1}{2}o(s,r)\sum_{\lambda=1}^{\beta-1} v_{r,\beta}w_{r,\lambda}v_{s,\alpha} \otimes c'_{r,\lambda} + \frac{1}{2}\delta_{(r,s+1)}\sum_{\lambda=1}^{\beta-1} v_{r,\beta}w_{r,\lambda}c'_{r,\lambda} \otimes v_{s,\alpha} \\ &-\frac{1}{2}o(s,r)\, v_{s,\alpha} \otimes v_{r,\beta}z + \frac{1}{2}\delta_{(r,s+1)}v_{r,\beta}ze_s \otimes v_{s,\alpha} \\ = &-\frac{1}{2}o(s,r)\, v_{s,\alpha} \otimes c'_{r,\beta} + \frac{1}{2}\delta_{(r,s+1)}c'_{r,\beta} \otimes v_{s,\alpha}\,,\end{aligned}$$

which coincides with our statement. In the case $r = s$, slightly adapting the proof of Lemma 3.1.3 yields

$$\{\!\{v_{s,\alpha}, c'_{s,\beta}\}\!\} = \frac{1}{2}c'_{s,\beta}e_s \otimes v_{s,\alpha} - \frac{1}{2}\left(o(\alpha,\beta) + \delta_{\alpha\beta}\right) v_{s,\alpha} \otimes c'_{s,\beta}\,. \tag{3.68}$$

As $c'_{s,\beta} = c'_{s,\beta}e_{s-1}$, the first term disappears, as desired. □

Proof. (Proposition 3.2.6.) Assume that $r \neq s$. We compute using Lemma 3.2.5 and (3.31b)

$$\begin{aligned}\{\!\{v_{s,\alpha}z, c'_{r,\beta}\}\!\} &= \{\!\{v_{s,\alpha}, c'_{r,\beta}\}\!\} * z + v_{s,\alpha} * \{\!\{z, c'_{r,\beta}\}\!\} \\ &= -\frac{1}{2}o(s,r)\, v_{s,\alpha}z \otimes c'_{r,\beta} + \frac{1}{2}\delta_{(r,s+1)}c'_{r,\beta}z \otimes v_{s,\alpha} \\ &\quad -\frac{1}{2}c'_{r,\beta}z \otimes v_{s,\alpha}e_{r-1} + \frac{1}{2}c'_{r,\beta} \otimes v_{s,\alpha}ze_{r-1} \\ &= -\frac{1}{2}o(s,r)\, v_{s,\alpha}z \otimes c'_{r,\beta}\,,\end{aligned}$$

since $r \neq s$ implies $v_{s,\alpha} z e_{r-1} = 0$. In particular,

$$\{\!\{ c'_{s,1}, c'_{r,\beta} \}\!\} = -\frac{1}{2} o(s,r)\, c'_{s,1} \otimes c'_{r,\beta}\,. \tag{3.69}$$

Next, we get

$$\begin{aligned}
\{\!\{ v_{s,\alpha} w_{s,\lambda}, c'_{r,\beta} \}\!\} &= \{\!\{ v_{s,\alpha}, c'_{r,\beta} \}\!\} * w_{s,\lambda} + v_{s,\alpha} * \{\!\{ w_{s,\lambda}, c'_{r,\beta} \}\!\} \\
&= -\frac{1}{2} o(s,r)\, v_{s,\alpha} w_{s,\lambda} \otimes c'_{r,\beta} + \frac{1}{2}\delta_{(r,s+1)} c'_{r,\beta} w_{s,\lambda} \otimes v_{s,\alpha} \\
&\quad + \frac{1}{2} o(s,r)\, e_\infty \otimes v_{s,\alpha} w_{s,\lambda} c'_{r,\beta} - \frac{1}{2}\delta_{(r,s+1)}\, c'_{r,\beta} w_{s,\lambda} \otimes v_{s,\alpha} e_s \\
&= \frac{1}{2} o(s,r)\, (e_\infty \otimes v_{s,\alpha} w_{s,\lambda} c'_{r,\beta} - v_{s,\alpha} w_{s,\lambda} \otimes c'_{r,\beta})\,.
\end{aligned}$$

Assume by induction that for all $\lambda < \alpha$,

$$\{\!\{ c'_{s,\lambda}, c'_{r,\beta} \}\!\} = -\frac{1}{2} o(s,r)\, c'_{s,\lambda} \otimes c'_{r,\beta}\,. \tag{3.70}$$

We get from the equality corresponding to (3.62) with a development taken in the first component of the double bracket that

$$\begin{aligned}
\{\!\{ c'_{s,\alpha}, c'_{r,\beta} \}\!\} =& \frac{1}{2} o(s,r) \sum_{\lambda=1}^{\alpha-1} (c'_{s,\lambda} \otimes v_{s,\alpha} w_{s,\lambda} c'_{r,\beta} - v_{s,\alpha} w_{s,\lambda} c'_{s,\lambda} \otimes c'_{r,\beta}) \\
& - \frac{1}{2} o(s,r) \sum_{\lambda=1}^{\alpha-1} c'_{s,\lambda} \otimes v_{s,\alpha} w_{s,\lambda} c'_{r,\beta} - \frac{1}{2} o(s,r)\, v_{s,\alpha} z \otimes c'_{r,\beta} \\
=& -\frac{1}{2} o(s,r)\, c'_{s,\alpha} \otimes c'_{r,\beta}\,,
\end{aligned}$$

after using (3.30). In the case $r = s$, use Equations (3.65) and (3.68) instead of $\{\!\{ a'_{s,\alpha}, c'_{s,\beta} \}\!\}$ and $\{\!\{ v_{s,\alpha}, c'_{s,\beta} \}\!\}$ in order to have the same forms for the brackets as in the Jordan case, then reproduce the proof of Proposition 3.1.4 to get

$$\{\!\{ c'_{s,\alpha}, c'_{s,\beta} \}\!\} = \frac{1}{2}(c'_{s,\beta} \otimes c'_{s,\alpha} - c'_{s,\alpha} \otimes c'_{s,\beta})\,,$$

as desired. This means that the vanishing terms that we introduced have cancelled out. □

3.3.6 Other brackets for framed cyclic quivers

We successively prove Lemmae 3.2.7 and 3.2.8.

Proof. (Lemma 3.2.7.) The first identity in (3.35) follows from Lemma 3.2.8. Next, we compute with (3.31a) and (3.31b)

$$\begin{aligned}\{\!\{x, a'_{s,\alpha}c'_{r,\beta}\}\!\} =& a'_{s,\alpha}\{\!\{x, c'_{r,\beta}\}\!\} + \{\!\{x, a'_{s,\alpha}\}\!\} c'_{r,\beta} \\ =& \frac{1}{2}a'_{s,\alpha}c'_{r,\beta}x \otimes e_{r-1} + \frac{1}{2}a'_{s,\alpha}c'_{r,\beta} \otimes xe_{r-1} \\ &+ \frac{1}{2}e_s \otimes xa'_{s,\alpha}c'_{r,\beta} - \frac{1}{2}e_s x \otimes a'_{s,\alpha}c'_{r,\beta}\,.\end{aligned} \tag{3.71}$$

Let $S_1 = \{\!\{x^k, a'_{s,\alpha}c'_{r,\beta}x^l\}\!\}$. We find by developing

$$\begin{aligned}S_1 =& \sum_{\sigma=1}^{k} x^{\sigma-1} * \{\!\{x, a'_{s,\alpha}c'_{r,\beta}\}\!\} x^l * x^{k-\sigma} \\ &+ \sum_{\sigma=1}^{k}\sum_{\tau=1}^{l} x^{\sigma-1} * a'_{s,\alpha}c'_{r,\beta}x^{\tau-1}\{\!\{x, x\}\!\} x^{l-\tau} * x^{k-\sigma}\,.\end{aligned}$$

Using the double brackets (3.71) and (3.37a) for $u = x$ yields

$$\begin{aligned}S_1 =& \frac{1}{2}\sum_{\sigma=1}^{k}\sum_{\tau=1}^{l}\sum_{t\in I} a'_{s,\alpha}c'_{r,\beta}x^{\tau+1}e_{t+1}x^{k-\sigma} \otimes x^{\sigma-1}e_t x^{l-\tau} \\ &- \frac{1}{2}\sum_{\sigma=1}^{k}\sum_{\tau=1}^{l}\sum_{t\in I} a'_{s,\alpha}c'_{r,\beta}x^{\tau-1}e_{t+1}x^{k-\sigma} \otimes x^{\sigma-1}e_t x^{l-\tau+2} \\ &+ \frac{1}{2}\sum_{\sigma=1}^{k}\Big(a'_{s,\alpha}c'_{r,\beta}x^{k-\sigma+1} \otimes x^{\sigma-1}e_{r-1}x^l + a'_{s,\alpha}c'_{r,\beta}x^{k-\sigma} \otimes x^{\sigma}e_{r-1}x^l \\ &\quad + e_s x^{k-\sigma} \otimes x^{\sigma}a'_{s,\alpha}c'_{r,\beta}x^l - e_s x^{k-\sigma+1} \otimes x^{\sigma-1}a'_{s,\alpha}c'_{r,\beta}x^l\Big)\,.\end{aligned} \tag{3.72}$$

If we apply the multiplication m, the summands of the two first sums contain a factor $e_{t+1}x^{k-1}e_t$ which vanishes except when $k \underset{m}{\equiv} 0$ since $e_{t+1}x^{k-1} = x^{k-1}e_{t+k}$. In the latter case, we can omit to write the idempotents as we get a factor $\sum_{s\in I} e_s = 1_I$ and $x1_I = x$. We then see that both sums cancel out. After multiplication, we also get that the last two terms in the third sum always cancel out. However, the first two terms in that sum both equal $a'_{s,\alpha}c'_{r,\beta}x^k e_{r-1}x^l$, and $S_1 = ka'_{s,\alpha}c'_{r,\beta}x^k x^l e_{r+l-1}$. Modulo commutators, we can omit to write the idempotent since $e_{r+l-1}a'_{s,\alpha}$ vanishes unless $r + l - 1 \underset{m}{\equiv} s$, which is the condition for $a'_{s,\alpha}c'_{r,\beta}x^l$ to be nonzero modulo commutators. This finishes to prove (3.35).

We have remarked the important discussion on the role of the idempotents after multiplication. In particular, we now prove (3.36) *assuming that* $l \underset{m}{\equiv} s - (r-1)$ and $k \underset{m}{\equiv} p - (q-1)$, since otherwise the terms will vanish. We will also use without further mention that $a'_{s,\alpha} \in e_s A' e_\infty$ while $c'_{s,\alpha} \in e_\infty A' e_{s-1}$.

We consider the decomposition

$$\left\{\!\!\left\{a'_{p,\gamma}c'_{q,\epsilon}x^k, a'_{s,\alpha}c'_{r,\beta}x^l\right\}\!\!\right\} = \{\!\{a'_{p,\gamma}c'_{q,\epsilon}, a'_{s,\alpha}c'_{r,\beta}\}\!\}\, x^l \,*\, x^k + a'_{s,\alpha}c'_{r,\beta}\left\{\!\!\left\{a'_{p,\gamma}c'_{q,\epsilon}, x^l\right\}\!\!\right\} * x^k$$
$$+ a'_{p,\gamma}c'_{q,\epsilon} * \left\{\!\!\left\{x^k, a'_{s,\alpha}c'_{r,\beta}\right\}\!\!\right\} x^l + a'_{p,\gamma}c'_{q,\epsilon} * a'_{s,\alpha}c'_{r,\beta}\left\{\!\!\left\{x^k, x^l\right\}\!\!\right\}. \tag{3.73}$$

To find the first term of (3.73), we compute $S_{ac} := \left\{\!\!\left\{a'_{p,\gamma}c'_{q,\epsilon}, a'_{s,\alpha}c'_{r,\beta}\right\}\!\!\right\}$ using (3.31c), (3.32) and (3.34)

$$\begin{aligned}
S_{ac} =& -\frac{1}{2}o(p,s)\, a'_{s,\alpha}c'_{q,\epsilon}\otimes a'_{p,\gamma}c'_{r,\beta} - \frac{1}{2}\delta_{ps}o(\gamma,\alpha)(a'_{s,\alpha}c'_{q,\epsilon}\otimes a'_{p,\gamma}c'_{r,\beta} + a'_{p,\gamma}c'_{q,\epsilon}\otimes a'_{s,\alpha}c'_{r,\beta})\\
&+\frac{1}{2}o(p,r)\, a'_{s,\alpha}c'_{q,\epsilon}\otimes a'_{p,\gamma}c'_{r,\beta} - \frac{1}{2}\delta_{(r,p+1)}\, a'_{s,\alpha}c'_{r,\beta}a'_{p,\gamma}c'_{q,\epsilon}\otimes e_p\\
&+\frac{1}{2}\delta_{pr}\left[o(\gamma,\beta)-\delta_{\gamma\beta}\right]\, a'_{s,\alpha}c'_{q,\epsilon}\otimes a'_{p,\gamma}c'_{r,\beta} - \delta_{pr}\delta_{\gamma\beta}\, a'_{s,\alpha}c'_{q,\epsilon}\otimes\left(e_p z + \sum_{\mu=1}^{\beta-1} a'_{p,\mu}c'_{p,\mu}\right)\\
&-\frac{1}{2}o(s,q)\, a'_{s,\alpha}c'_{q,\epsilon}\otimes a'_{p,\gamma}c'_{r,\beta} + \frac{1}{2}\delta_{(q,s+1)}\, e_s\otimes a'_{p,\gamma}c'_{q,\epsilon}a'_{s,\alpha}c'_{r,\beta}\\
&-\frac{1}{2}\delta_{sq}\left(o(\alpha,\epsilon)-\delta_{\alpha\epsilon}\right)\, a'_{s,\alpha}c'_{q,\epsilon}\otimes a'_{p,\gamma}c'_{r,\beta} + \delta_{sq}\delta_{\alpha\epsilon}\left(e_s z + \sum_{\lambda=1}^{\epsilon-1} a'_{s,\lambda}c'_{s,\lambda}\right)\otimes a'_{p,\gamma}c'_{r,\beta}\\
&-\frac{1}{2}o(q,r)\, a'_{s,\alpha}c'_{q,\epsilon}\otimes a'_{p,\gamma}c'_{r,\beta} + \frac{1}{2}\delta_{qr}o(\epsilon,\beta)\left(a'_{s,\alpha}c'_{r,\beta}\otimes a'_{p,\gamma}c'_{q,\epsilon} - a'_{s,\alpha}c'_{q,\epsilon}\otimes a'_{p,\gamma}c'_{r,\beta}\right).
\end{aligned}$$

(Here, we applied the cyclic antisymmetry on $\{\!\{a'_{s,\alpha}, c'_{q,\epsilon}\}\!\}$ to get $\{\!\{c'_{q,\epsilon}, a'_{s,\alpha}\}\!\}$.) We will use $m\circ(S_{ac}x^l \,*\, x^k)$ at the end. Then we compute

$$\begin{aligned}
\{\!\{a'_{p,\gamma}c'_{q,\epsilon}, x\}\!\} =& a'_{p,\gamma} * \{\!\{c'_{q,\epsilon}, x\}\!\} + \{\!\{a'_{p,\gamma}, x\}\!\} * c'_{q,\epsilon}\\
=& -\frac{1}{2}e_{q-1}\otimes a'_{p,\gamma}c'_{q,\epsilon}x - \frac{1}{2}xe_{q-1}\otimes a'_{p,\gamma}c'_{q,\epsilon}\\
&-\frac{1}{2}xa'_{p,\gamma}c'_{q,\epsilon}\otimes e_p + \frac{1}{2}a'_{p,\gamma}c'_{q,\epsilon}\otimes e_p x\,,
\end{aligned}$$

so that the second term from (3.73) becomes

$$\begin{aligned}
&\sum_{\tau=1}^{l} a'_{s,\alpha}c'_{r,\beta}x^{\tau-1}\,\{\!\{a'_{p,\gamma}c'_{q,\epsilon}, x\}\!\}\, x^{l-\tau} \,*\, x^k\\
=& -\frac{1}{2}\left[\sum_{\tau=0}^{l-1} + \sum_{\tau=1}^{l}\right] a'_{s,\alpha}c'_{r,\beta}x^{\tau}e_{q-1}x^k\otimes a'_{p,\gamma}c'_{q,\epsilon}x^{l-\tau}\\
&-\frac{1}{2}\left[\sum_{\tau=1}^{l} - \sum_{\tau=0}^{l-1}\right] a'_{s,\alpha}c'_{r,\beta}x^{\tau}a'_{p,\gamma}c'_{q,\epsilon}x^k\otimes e_p x^{l-\tau}\,.
\end{aligned}$$

Now, remark that by assumptions $k \underset{m}{\equiv} p-(q-1)$, so that $e_{q-1}x^k a'_{p,\gamma} = x^k a'_{p,\gamma}$ and $c'_{q,\epsilon}x^k e_p =$

$c'_{q,\epsilon}x^k$. Hence, applying the multiplication map on the latter expression, we find that

$$\begin{aligned}m \circ (a'_{s,\alpha}c'_{r,\beta} \left\{\!\!\left\{ a'_{p,\gamma}c'_{q,\epsilon}, x^l \right\}\!\!\right\} * x^k) = &-\sum_{\tau=1}^{l-1} a'_{s,\alpha}c'_{r,\beta}x^{k+\tau}a'_{p,\gamma}c'_{q,\epsilon}x^{l-\tau} \\ &- \frac{1}{2}a'_{s,\alpha}c'_{r,\beta}x^k a'_{p,\gamma}c'_{q,\epsilon}x^l - \frac{1}{2}a'_{s,\alpha}c'_{r,\beta}x^{k+l}a'_{p,\gamma}c'_{q,\epsilon} \\ &- \frac{1}{2}a'_{s,\alpha}c'_{r,\beta}x^l a'_{p,\gamma}c'_{q,\epsilon}x^k + \frac{1}{2}a'_{s,\alpha}c'_{r,\beta}a'_{p,\gamma}c'_{q,\epsilon}x^{k+l}\,.\end{aligned}$$

Net, we use (3.71) to compute the third term of (3.73), and it becomes

$$\begin{aligned}&\sum_{\sigma=1}^{k} a'_{p,\gamma}c'_{q,\epsilon}x^{\sigma-1} * \left\{\!\!\left\{ x, a'_{s,\alpha}c'_{r,\beta} \right\}\!\!\right\} x^l * x^{k-\sigma} \\ =&+\frac{1}{2}\left[\sum_{\sigma=0}^{k-1} + \sum_{\sigma=1}^{k}\right] a'_{s,\alpha}c'_{r,\beta}x^{k-\sigma} \otimes a'_{p,\gamma}c'_{q,\epsilon}x^{\sigma}e_{r-1}x^l \\ &+\frac{1}{2}\left[\sum_{\sigma=1}^{k} - \sum_{\sigma=0}^{k-1}\right] e_s x^{k-\sigma} \otimes a'_{p,\gamma}c'_{q,\epsilon}x^{\sigma}a'_{s,\alpha}c'_{r,\beta}x^l\,.\end{aligned}$$

By assumptions, $l \underset{m}{\equiv} s-(r-1)$ so that $e_{r-1}x^l a'_{s,\alpha} = x^l a'_{s,\alpha}$ and $c'_{r,\beta}x^l e_s = c'_{r,\beta}x^l$. Hence, if we take the multiplication map, we can get rid of the idempotents modulo commutators and we write

$$\begin{aligned}m \circ (a'_{p,\gamma}c'_{q,\epsilon} \left\{\!\!\left\{ x^k, a'_{s,\alpha}c'_{r,\beta} \right\}\!\!\right\} x^l) = &\sum_{\sigma=1}^{k-1} a'_{s,\alpha}c'_{r,\beta}x^{k-\sigma}a'_{p,\gamma}c'_{q,\epsilon}x^{l+\sigma} \\ &+ \frac{1}{2}a'_{s,\alpha}c'_{r,\beta}x^k a'_{p,\gamma}c'_{q,\epsilon}x^l + \frac{1}{2}a'_{s,\alpha}c'_{r,\beta}a'_{p,\gamma}c'_{q,\epsilon}x^{k+l} \\ &+ \frac{1}{2}a'_{s,\alpha}c'_{r,\beta}x^l a'_{p,\gamma}c'_{q,\epsilon}x^k - \frac{1}{2}a'_{s,\alpha}c'_{r,\beta}x^{k+l}a'_{p,\gamma}c'_{q,\epsilon}\end{aligned}$$

mod $A'/[A',A']$. Finally, we compute the fourth term of (3.73) :

$$\begin{aligned}&\sum_{\sigma=1}^{k}\sum_{\tau=1}^{l} a'_{p,\gamma}c'_{q,\epsilon}x^{\sigma-1} * a'_{s,\alpha}c'_{r,\beta}x^{\tau-1} \left\{\!\!\left\{ x, x \right\}\!\!\right\} x^{l-\tau} * x^{k-\sigma} \\ =&\frac{1}{2}\sum_{\sigma=1}^{k}\sum_{\tau=1}^{l}\sum_{t\in I} a'_{s,\alpha}c'_{r,\beta}x^{\tau-1}x^2 e_{t+1}x^{k-\sigma} \otimes a'_{p,\gamma}c'_{q,\epsilon}x^{\sigma-1}e_t x^{l-\tau} \\ &-\frac{1}{2}\sum_{\sigma=1}^{k}\sum_{\tau=1}^{l}\sum_{t\in I} a'_{s,\alpha}c'_{r,\beta}x^{\tau-1}e_{t+1}x^{k-\sigma} \otimes a'_{p,\gamma}c'_{q,\epsilon}x^{\sigma-1}e_t x^2 x^{l-\tau}\,,\end{aligned}$$

which becomes, after applying the multiplication map,

$$\begin{aligned}&m \circ (a'_{p,\gamma}c'_{q,\epsilon} * a'_{s,\alpha}c'_{r,\beta} \left\{\!\!\left\{ x^k, x^l \right\}\!\!\right\}) \\ =&\frac{1}{2}\sum_{\sigma=1}^{k}\sum_{\tau=1}^{l}\sum_{t\in I} a'_{s,\alpha}c'_{r,\beta}x^{\tau-1}x^2 e_{t+1}x^{k-\sigma}a'_{p,\gamma}c'_{q,\epsilon}x^{\sigma-1}e_t x^{l-\tau} \\ &-\frac{1}{2}\sum_{\sigma=1}^{k}\sum_{\tau=1}^{l}\sum_{t\in I} a'_{s,\alpha}c'_{r,\beta}x^{\tau-1}e_{t+1}x^{k-\sigma}a'_{p,\gamma}c'_{q,\epsilon}x^{\sigma-1}e_t x^2 x^{l-\tau}\,.\end{aligned}$$

We consider this expression modulo commutators. Note that in the first sum we have a factor $x^{l-\tau}a'_{s,\alpha}c'_{r,\beta}x^{\tau+1} \in e_{s-l+\tau}A'e_{r+\tau}$, so we need t to be such that $t \underset{m}{\equiv} s-l+\tau$ and $t+1 \underset{m}{\equiv} r+\tau \underset{m}{\equiv} (s-l+1)+\tau$ since $l \underset{m}{\equiv} s-(r-1)$. In other words, the only contribution of the sum $t \in I$ which is not trivially vanishing is the one such that $t \underset{m}{\equiv} s-l+\tau$. Then, we can omit to write the idempotents[10]. The same is true in the second sum with $t \underset{m}{\equiv} s-l+\tau-2$. We write

$$\begin{aligned}
&m \circ (a'_{p,\gamma}c'_{q,\epsilon} * a'_{s,\alpha}c'_{r,\beta} \left\{\!\!\left\{x^k, x^l\right\}\!\!\right\}) \\
=&\frac{1}{2}\left[\sum_{\sigma=0}^{k-1}\sum_{\tau=1}^{l} - \sum_{\sigma=1}^{k}\sum_{\tau=0}^{l-1}\right] a'_{s,\alpha}c'_{r,\beta}x^{k-\sigma+\tau}a'_{p,\gamma}c'_{q,\epsilon}x^{l+\sigma-\tau}\,.
\end{aligned}$$

The difference of the sums can be written as

$$\left[\sum_{\sigma=0}^{k-1}\sum_{\tau=1}^{l} - \sum_{\sigma=1}^{k}\sum_{\tau=0}^{l-1}\right] \dots = \left[\sum_{\sigma=0}^{}\sum_{\tau=1}^{l-1} + \sum_{\sigma=0}^{k-1}\sum_{\tau=l}^{} - \sum_{\sigma=1}^{k-1}\sum_{\tau=0}^{} - \sum_{\sigma=k}^{}\sum_{\tau=0}^{l-1}\right] \dots,$$

hence we find

$$\begin{aligned}
&m \circ (a'_{p,\gamma}c'_{q,\epsilon} * a'_{s,\alpha}c'_{r,\beta} \left\{\!\!\left\{x^k, x^l\right\}\!\!\right\}) \\
=&\frac{1}{2}\sum_{\tau=1}^{l-1} a'_{s,\alpha}c'_{r,\beta}x^{k+\tau}a'_{p,\gamma}c'_{q,\epsilon}x^{l-\tau} + \frac{1}{2}\sum_{\sigma=0}^{k-1} a'_{s,\alpha}c'_{r,\beta}x^{k+l-\sigma}a'_{p,\gamma}c'_{q,\epsilon}x^{\sigma} \\
&-\frac{1}{2}\sum_{\sigma=1}^{k-1} a'_{s,\alpha}c'_{r,\beta}x^{k-\sigma}a'_{p,\gamma}c'_{q,\epsilon}x^{l+\sigma} - \frac{1}{2}\sum_{\tau=0}^{l-1} a'_{s,\alpha}c'_{r,\beta}x^{\tau}a'_{p,\gamma}c'_{q,\epsilon}x^{k+l-\tau}\,.
\end{aligned}$$

Summing the last three terms of (3.73) that we derived, we get in $A'/[A', A']$ after simplifications

$$\begin{aligned}
&\{a'_{p,\gamma}c'_{q,\epsilon}x^k, a'_{s,\alpha}c'_{r,\beta}x^l\} - m \circ (S_{ac}x^l * x^k) \\
=&-\frac{1}{2}\sum_{v=1}^{l-1} a'_{s,\alpha}c'_{r,\beta}x^{k+v}a'_{p,\gamma}c'_{q,\epsilon}x^{l-v} + \frac{1}{2}\sum_{v=1}^{k-1} a'_{s,\alpha}c'_{r,\beta}x^{k+l-v}a'_{p,\gamma}c'_{q,\epsilon}x^{v} \\
&+\frac{1}{2}\sum_{v=1}^{k-1} a'_{s,\alpha}c'_{r,\beta}x^{k-v}a'_{p,\gamma}c'_{q,\epsilon}x^{l+v} - \frac{1}{2}\sum_{v=1}^{l-1} a'_{s,\alpha}c'_{r,\beta}x^{v}a'_{p,\gamma}c'_{q,\epsilon}x^{k+l-v} \\
&-\frac{1}{2}a'_{s,\alpha}c'_{r,\beta}x^{k+l}a'_{p,\gamma}c'_{q,\epsilon} + \frac{1}{2}a'_{s,\alpha}c'_{r,\beta}a'_{p,\gamma}c'_{q,\epsilon}x^{k+l}\,,
\end{aligned}$$

[10] It is important to remark that several terms in the sum will vanish after a more careful analysis. For example, in the first sum, we need each couple (σ, τ) to be such that both $c'_{r,\beta}e_{t+1}x^{k+\tau-\sigma+1}a'_{p,\gamma}$ and $c'_{q,\epsilon}x^{l-\tau+\sigma-1}a'_{s,\alpha}$ are not trivially zero, which means $(\tau-\sigma)+k+1 \underset{m}{\equiv} p-(r-1)$ and $-(\tau-\sigma)+l-1 \underset{m}{\equiv} s-(q-1)$. Our aim was only to avoid writing idempotents in the expressions, so that we do not look at these conditions at the moment and we postpone their investigation when we will be looking at their representations on the moduli space. The same holds for the other sums that we have obtained so far.

which we rewrite

$$\begin{aligned}
&\{a'_{p,\gamma}c'_{q,\epsilon}x^k, a'_{s,\alpha}c'_{r,\beta}x^l\} - m\circ(S_{ac}x^l * x^k)\\
=&+\frac{1}{2}\sum_{v=1}^{k-1} a'_{s,\alpha}c'_{r,\beta}x^v a'_{p,\gamma}c'_{q,\epsilon}x^{k+l-v} + \frac{1}{2}\sum_{v=1}^{k-1} a'_{s,\alpha}c'_{r,\beta}x^{k+l-v}a'_{p,\gamma}c'_{q,\epsilon}x^v\\
&-\frac{1}{2}\sum_{v=1}^{l-1} a'_{s,\alpha}c'_{r,\beta}x^v a'_{p,\gamma}c'_{q,\epsilon}x^{k+l-v} - \frac{1}{2}\sum_{v=1}^{l-1} a'_{s,\alpha}c'_{r,\beta}x^{k+l-v}a'_{p,\gamma}c'_{q,\epsilon}x^v\\
&-\frac{1}{2}a'_{s,\alpha}c'_{r,\beta}x^{k+l}a'_{p,\gamma}c'_{q,\epsilon} + \frac{1}{2}a'_{s,\alpha}c'_{r,\beta}a'_{p,\gamma}c'_{q,\epsilon}x^{k+l}\,.
\end{aligned}$$

We can add in the sums the terms corresponding to $v = k$ and $v = l$ as they cancel out. Hence, the latter expression contains precisely the first two terms of (3.36), with two additional terms. Modulo commutators, these two terms vanish with the elements

$$-\frac{1}{2}\delta_{(r,p+1)}\, a'_{s,\alpha}c'_{r,\beta}a'_{p,\gamma}c'_{q,\epsilon}\otimes e_p + \frac{1}{2}\delta_{(q,s+1)}\, e_s\otimes a'_{p,\gamma}c'_{q,\epsilon}a'_{s,\alpha}c'_{r,\beta}$$

of S_{ac} when we add $m\circ(S_{ac}x^l * x^k)$. It now suffices to verify that all the other terms give exactly (3.36). □

Proof. (Lemma 3.2.8.) For the first identity, we get by (3.37a)

$$\begin{aligned}
\left\{\!\!\left\{u^k, u^l\right\}\!\!\right\} =&\sum_{\sigma=1}^{k}\sum_{\tau=1}^{l} u^{\sigma-1} * u^{\tau-1}(\{\!\{u,u\}\!\})u^{l-\tau} * u^{k-\sigma}\\
=&\frac{1}{2}\epsilon(u)\sum_{\sigma=1}^{k}\sum_{\tau=1}^{l}\sum_{s\in I} u^{\tau+1}e_{s+\theta(u)}u^{k-\sigma}\otimes u^{\sigma-1}e_s u^{l-\tau}\\
&-\frac{1}{2}\epsilon(u)\sum_{\sigma=1}^{k}\sum_{\tau=1}^{l}\sum_{s\in I} u^{\tau-1}e_{s+\theta(u)}u^{k-\sigma}\otimes u^{\sigma-1}e_s u^{l-\tau+2}\,.
\end{aligned}$$

If we apply the multiplication map, we get in both sums a factor $e_{s+\theta(u)}u^{k-1}e_s$ which can only be nonzero provided $k\theta(u) \underset{m}{\equiv} 0$ since $e_{s+\theta(u)}u^{k-1} = u^{k-1}e_{s+k\theta(u)}$. In the latter case, we can omit to write the idempotents as usual, and both sums cancel out since each of their summands is just u^{k+l}.

Next we compute using (3.37b)

$$\{\!\{u, w_{s,\alpha}v_{r,\beta}\}\!\} = \frac{1}{2}(e_s\otimes uw_{s,\alpha}v_{r,\beta} - e_s u\otimes w_{s,\alpha}v_{r,\beta} + w_{s,\alpha}v_{r,\beta}u\otimes e_r - w_{s,\alpha}v_{r,\beta}\otimes ue_r)\,.$$

We find in a way similar to (3.72) in the proof of Lemma 3.2.7,

$$\begin{aligned}\left\{\!\!\left\{u^k, w_{s,\alpha}v_{r,\beta}u^l\right\}\!\!\right\} =&\frac{1}{2}\sum_{\sigma=1}^{k}\Big(e_s u^{k-\sigma}\otimes u^\sigma w_{s,\alpha}v_{r,\beta}u^l - e_s u^{k-\sigma+1}\otimes u^{\sigma-1}w_{s,\alpha}v_{r,\beta}u^l \\ &\qquad + w_{s,\alpha}v_{r,\beta}u^{k-\sigma+1}\otimes u^{\sigma-1}e_r u^l - w_{s,\alpha}v_{r,\beta}u^{k-\sigma}\otimes u^\sigma e_r u^l\Big) \\ &+\frac{1}{2}\sum_{\sigma=1}^{k}\sum_{\tau=1}^{l}\sum_{t\in I}\Big(w_{s,\alpha}v_{r,\beta}u^{\tau+1}e_{t+\theta(u)}u^{k-\sigma}\otimes u^{\sigma-1}e_t u^{l-\tau} \\ &\qquad - w_{s,\alpha}v_{r,\beta}u^{\tau-1}e_{t+\theta(u)}u^{k-\sigma}\otimes u^{\sigma-1}e_t u^{l-\tau+2}\Big)\,.\end{aligned}$$

To have nonzero terms modulo commutators, recall that we consider $k = 0 \bmod m$ while $l\theta(u) = s-r \bmod m$ if $u = x, y, z$, while $k, l \in \mathbb{N}$ otherwise. In particular we do not need to write down the idempotents after multiplication, and all terms cancel out.

To prove that (3.39) holds, we assume that $k\theta(u) \underset{m}{\equiv} p-q$ and $l\theta(u) \underset{m}{\equiv} s-r$, as the proof is trivial otherwise since both sides vanish. We decompose

$$\begin{aligned}\left\{\!\!\left\{w_{p,\gamma}v_{q,\epsilon}u^k, w_{s,\alpha}v_{r,\beta}u^l\right\}\!\!\right\} =&w_{p,\gamma}v_{q,\epsilon} * \left\{\!\!\left\{u^k, w_{s,\alpha}v_{r,\beta}u^l\right\}\!\!\right\} + w_{s,\alpha}v_{r,\beta}\left\{\!\!\left\{w_{p,\gamma}v_{q,\epsilon}, u^l\right\}\!\!\right\} * u^k \\ &+ \left\{\!\!\left\{w_{p,\gamma}v_{q,\epsilon}, w_{s,\alpha}v_{r,\beta}\right\}\!\!\right\} u^l * u^l\end{aligned}$$

and see that the double bracket in the first term has just been computed. Writing this first term as T_1, we can write

$$\begin{aligned}m\circ T_1 =&\frac{1}{2}(e_s w_{p,\gamma}v_{q,\epsilon}u^k w_{s,\alpha}v_{r,\beta}u^l - e_s u^k w_{p,\gamma}v_{q,\epsilon}w_{s,\alpha}v_{r,\beta}u^l) \\ &+\frac{1}{2}(w_{s,\alpha}v_{r,\beta}u^k w_{p,\gamma}v_{q,\epsilon}e_r u^l - w_{s,\alpha}v_{r,\beta}w_{p,\gamma}v_{q,\epsilon}u^k e_r u^l) \\ &+\frac{\epsilon(u)}{2}\sum_{\sigma=1}^{k}\sum_{\tau=1}^{l}\sum_{t\in I} w_{s,\alpha}v_{r,\beta}u^{\tau+1}e_{t+\theta(u)}u^{k-\sigma}w_{p\gamma}v_{q\epsilon}u^{\sigma-1}e_t u^{l-\tau} \\ &-\frac{\epsilon(u)}{2}\sum_{\sigma=1}^{k}\sum_{\tau=1}^{l}\sum_{t\in I} w_{s,\alpha}v_{r,\beta}u^{\tau-1}e_{t+\theta(u)}u^{k-\sigma}w_{p\gamma}v_{q\epsilon}u^{\sigma-1}e_t u^{l-\tau+2}\end{aligned}$$

For the first term in the first line, the idempotent tells us that $s = p$, but we know the same from $v_{q,\epsilon}u^k w_{s,\alpha} = v_{q,\epsilon}u^k e_{q+k\theta(u)}e_s w_{s,\alpha}$ by choice of k. Hence, we do not need to write the idempotent, and this is true for the first four terms. In fact, as in the proof of Lemma 3.2.7, the idempotents are also determined by the other terms on the last two lines, so we can skip to write them. Then, the last two sums take the same form as $m\circ T$ in the proof of Lemma 3.1.7, hence we can write after simplification that the following holds modulo commutators

$$\begin{aligned} m\circ T_1 =&\frac{1}{2}(w_{p,\gamma}v_{q,\epsilon}u^k w_{s,\alpha}v_{r,\beta}u^l - w_{p,\gamma}v_{q,\epsilon}w_{s,\alpha}v_{r,\beta}u^{k+l}) \\ &+\frac{1}{2}(w_{p,\gamma}v_{q,\epsilon}u^l w_{s,\alpha}v_{r,\beta}u^k - w_{p,\gamma}v_{q,\epsilon}u^{k+l}w_{s,\alpha}v_{r,\beta}) \\ &+\frac{1}{2}\epsilon(u)\left[\sum_{\tau=1}^{k-1} w_{s,\alpha}v_{r,\beta}u^{k+l-\tau}w_{p,\gamma}v_{q,\epsilon}u^{\tau} + \sum_{\tau=1}^{l} w_{s,\alpha}v_{r,\beta}u^{k+\tau}w_{p,\gamma}v_{q,\epsilon}u^{l-\tau}\right] \\ &-\frac{1}{2}\epsilon(u)\left[\sum_{\sigma=1}^{l-1} w_{s,\alpha}v_{r,\beta}u^{\tau}w_{p,\gamma}v_{q,\epsilon}u^{k+l-\tau} + \sum_{\tau=1}^{k} w_{s,\alpha}v_{r,\beta}u^{k-\tau}w_{p,\gamma}v_{q,\epsilon}u^{l+\tau}\right]. \end{aligned}$$

Using the computation for $\{\!\{u, w_{s,\alpha}v_{r,\beta}\}\!\}$ at the beginning, we easily obtain that

$$\begin{aligned} \left\{\!\!\left\{u^l, w_{p,\gamma}v_{q,\epsilon}\right\}\!\!\right\} =&\frac{1}{2}(e_p\otimes u^l w_{p,\gamma}v_{q,\epsilon} - e_p u^l\otimes w_{p,\gamma}v_{q,\epsilon}) \\ &+\frac{1}{2}(w_{p,\gamma}v_{q,\epsilon}u^l\otimes e_q - w_{p,\gamma}v_{q,\epsilon}\otimes u^l e_q)\,. \end{aligned}$$

Applying the cyclic antisymmetry, we easily compute $m\circ(w_{s,\alpha}v_{r,\beta}\left\{\!\!\left\{w_{p,\gamma}v_{q,\epsilon}, u^l\right\}\!\!\right\}*u^k)$. Modulo commutators, this cancels out with the first four terms in $m\circ T_1$.

Finally, we compute that $\{\!\{w_{p,\gamma}v_{q,\epsilon}, w_{s,\alpha}v_{r,\beta}\}\!\}$ is equal to

$$\begin{aligned} &\frac{1}{2}\left[o(p,r)+o(q,s)-o(p,s)-o(q,r)\right]w_{s,\alpha}v_{q,\epsilon}\otimes w_{p,\gamma}v_{r,\beta} \\ &+\frac{1}{2}\delta_{ps}o(\alpha,\gamma)(w_{s,\alpha}v_{q,\epsilon}\otimes w_{p,\gamma}v_{r,\beta} + w_{p,\gamma}v_{q,\epsilon}\otimes w_{s,\alpha}v_{r,\beta}) \\ &+\frac{1}{2}\delta_{qr}o(\beta,\epsilon)(w_{s,\alpha}v_{q,\epsilon}\otimes w_{p,\gamma}v_{r,\beta} + w_{s,\alpha}v_{r,\beta}\otimes w_{p,\gamma}v_{q,\epsilon}) \\ &+\frac{1}{2}\delta_{qs}[o(\epsilon,\alpha)+\delta_{\epsilon\alpha}](w_{s,\alpha}v_{q,\epsilon}\otimes w_{p,\gamma}v_{r,\beta} + e_s\otimes w_{p,\gamma}v_{q,\epsilon}w_{s,\alpha}v_{r,\beta}) \\ &-\frac{1}{2}\delta_{pr}[o(\beta,\gamma)+\delta_{\beta\gamma}](w_{s,\alpha}v_{q,\epsilon}\otimes w_{p,\gamma}v_{r,\beta} + w_{s,\alpha}v_{r,\beta}w_{p,\gamma}v_{q,\epsilon}\otimes e_r) \\ &+\delta_{qs}\delta_{\epsilon\alpha}\,e_s\otimes w_{p,\gamma}v_{r,\beta} - \delta_{pr}\delta_{\gamma\beta}\,w_{s,\alpha}v_{q,\epsilon}\otimes e_r\,. \end{aligned}$$

Once we compute $m\circ(\{\!\{w_{p,\gamma}v_{q,\epsilon}, w_{s,\alpha}v_{r,\beta}\}\!\}\,u^l * u^k)$ the idempotents can be omitted. The latter expression, modulo commutators, yields the first six lines in (3.39), and the remaining two come from the two last lines of $m\circ T_1$. □

Proof. (Lemma 3.2.10.) Since $(e_s u e_s)^k = e_s u^k e_s$, we have

$$\begin{aligned} \left\{\!\!\left\{(e_s u e_s)^k, (e_r u e_r)^l\right\}\!\!\right\} &= e_s * e_r\left\{\!\!\left\{u^k, u^l\right\}\!\!\right\}e_r * e_s\,, \\ \left\{\!\!\left\{(e_t u e_t)^k, w_{s,\alpha}v_{r,\beta}u^l\right\}\!\!\right\} &= e_t * \left\{\!\!\left\{u^k, w_{s,\alpha}v_{r,\beta}u^l\right\}\!\!\right\} * e_t\,. \end{aligned}$$

Using the double brackets on the right-hand side obtained in the proof of Lemma 3.2.8 for $\epsilon(u)=+1$, $\theta(u)=0$ we can easily conclude that these expressions vanish after applying the multiplication map. □

3.3.7 Computations using subquivers of the framed cyclic quivers

We prove Propositions 3.2.12 and 3.2.14.

Proof. (Proposition 3.2.12.) The idea for the proof is similar to Proposition 3.1.9. Let $\alpha, \eta \in \mathbb{C}$ and recall that $U_{+,\eta} = u(1 + \eta\phi)$. (Clearly, α is not seen as a parameter running over $1, \ldots, d_s$ for some $s \in I$.) If we write $\{\!\{U_{+,\alpha}, U_{+,\eta}\}\!\} = a' \otimes a''$, we can write

$$\frac{1}{KL}\{U_{+,\alpha}^K, U_{+,\eta}^L\} = U_{+,\eta}^{L-1} a' U_{+,\alpha}^{K-1} a'' \mod [A, A]\,, \tag{3.74}$$

So we have to compute

$$\{\!\{u + \alpha u\phi, u + \eta u\phi\}\!\} = \{\!\{u, u\}\!\} + \alpha\,\{\!\{u\phi, u\}\!\} + \eta\,\{\!\{u, u\phi\}\!\} + \alpha\eta\,\{\!\{u\phi, u\phi\}\!\}\,. \tag{3.75}$$

We need the following double brackets for the proof

$$\{\!\{u, u\}\!\} = \frac{1}{2}\epsilon(u)[u^2 F_{\theta(u)} - F_{\theta(u)} u^2]\,, \quad \{\!\{\phi, \gamma\}\!\} = \frac{1}{2}\phi * (\gamma F_0 - F_0\gamma) + \frac{1}{2}(\gamma F_0 - F_0\gamma) * \phi\,,$$

where γ is any word in the letters $\{e_s, x_s, y_s\}$ (with possible inverses). The first equation is (3.37a), while the second equation follows from Lemma 2.3.14 applied to the subquiver based at I (the set of all vertices in the cycle) with (2.19). In fact, we can write

$$\begin{aligned} \{\!\{\phi, u\}\!\} =& \frac{1}{2}(uF_0\phi - F_0\phi u) + \frac{1}{2}(u\phi F_0 - \phi F_0 u)\,, \\ \{\!\{\phi, \phi\}\!\} =& \frac{1}{2}(\phi^2 F_0 - F_0\phi^2)\,, \end{aligned} \tag{3.76}$$

because $\phi \in \oplus_s e_s A e_s$, so $\phi e_s = e_s \phi$. Similarly $u e_s = e_{s-\theta(u)} u$ as $u \in \oplus_s e_s A e_{s+\theta(u)}$, so that

$$\{\!\{u, \phi\}\!\} = \frac{1}{2}\phi(uF_{\theta(u)} - F_{\theta(u)}u) + \frac{1}{2}(uF_{\theta(u)} - F_{\theta(u)}u)\phi\,.$$

We have already the first term in (3.75). For the second term, we compute

$$\begin{aligned} \{\!\{u\phi, u\}\!\} =& u * \{\!\{\phi, u\}\!\} + \{\!\{u, u\}\!\} * \phi \\ =& \frac{1}{2}u * (uF_0\phi - F_0\phi u) + \frac{1}{2}u * (u\phi F_0 - \phi F_0 u) + \frac{1}{2}\epsilon(u)(u^2\phi F_{\theta(u)} - \phi F_{\theta(u)} u^2) \\ =& \frac{1}{2}(uF_{\theta(u)}u\phi - F_{\theta(u)}u\phi u + u\phi F_{\theta(u)}u - \phi F_{\theta(u)}u^2) + \frac{1}{2}\epsilon(u)(u^2\phi F_{\theta(u)} - \phi F_{\theta(u)}u^2) \end{aligned}$$

using that $u * F_0 = \sum_s e_s \otimes u e_s = \sum_s e_s \otimes e_{s-\theta(u)} u = F_{\theta(u)} u$. To get $\{\!\{u, u\phi\}\!\} = -\{\!\{u\phi, u\}\!\}^\circ$, we need the following lemma.

Lemma 3.3.1 *Fix some $r \in \mathbb{N}$ and let $a \in \oplus_s e_s A e_s$, $b_0, b_1 \in \oplus_s e_s A e_{s+r}$ and $c \in \oplus_s e_s A e_{s+2r}$. Then $(b_0 F_r b_1)^\circ = b_1 F_r b_0$ and $(cF_r a)^\circ = aF_r c$.*

Proof. We compute $(b_0 F_r b_1)^\circ = \sum_s e_s b_1 \otimes b_0 e_{s+r} = \sum_s b_1 e_{s+r} \otimes e_s b_0 = b_1 F_r b_0$. The second equality follows similarly. □

Taking $r = \theta(u)$, $b_0 = u$ and $b_1 = u\theta$ gives $(uF_{\theta(u)} u\phi)^\circ = u\phi F_{\theta(u)} u$. In a similar way for the other terms, we find

$$\{\!\{u, u\phi\}\!\} = -\frac{1}{2}(u\phi F_{\theta(u)} u - u\phi u F_{\theta(u)} + uF_{\theta(u)} u\phi - u^2 F_{\theta(u)} \phi) + \frac{1}{2}\epsilon(u)(u^2 F_{\theta(u)}\phi - F_{\theta(u)} u^2 \phi)\,.$$

To get the last double bracket, it remains to compute

$$\begin{aligned}
\{\!\{u\phi, \phi\}\!\} &= \{\!\{u, \phi\}\!\} * \phi + u * \{\!\{\phi, \phi\}\!\} \\
&= \frac{1}{2}\phi(uF_{\theta(u)} - F_{\theta(u)} u) * \phi + \frac{1}{2}(uF_{\theta(u)} - F_{\theta(u)} u)\phi * \phi + \frac{1}{2} u * (\phi^2 F_0 - F_0 \phi^2) \\
&= \frac{1}{2}(\phi u \phi F_{\theta(u)} + u\phi F_{\theta(u)}\phi - \phi F_{\theta(u)} u\phi - F_{\theta(u)} u\phi^2)\,,
\end{aligned}$$

so that we find for $\{\!\{u\phi, u\phi\}\!\} = u\,\{\!\{u\phi, \phi\}\!\} + \{\!\{u\phi, u\}\!\}\,\phi$

$$\begin{aligned}
\{\!\{u\phi, u\phi\}\!\} =& \frac{1}{2}(u\phi u\phi F_{\theta(u)} + u^2\phi F_{\theta(u)}\phi - F_{\theta(u)} u\phi u\phi - \phi F_{\theta(u)} u^2 \phi) \\
&+ \frac{1}{2}\epsilon(u)(u^2\phi F_{\theta(u)}\phi - \phi F_{\theta(u)} u^2\phi)\,.
\end{aligned}$$

We let U_α denote $U_{+,\alpha}$ to ease notations. Using $U_\alpha - u = \alpha u\phi$ and $U_\eta - u = \eta u\phi$, we can write $\alpha\eta\,\{\!\{u\phi, u\phi\}\!\}$ as

$$\begin{aligned}
&\frac{1}{2}\eta u\phi(U_\alpha - u)F_{\theta(u)} + \frac{1}{2}\eta u(U_\alpha - u)F_{\theta(u)}\phi - \frac{1}{2}\alpha F_{\theta(u)} u\phi(U_\eta - u) \\
&- \frac{1}{2}\alpha\phi F_{\theta(u)} u(U_\eta - u) + \frac{1}{2}\epsilon(u)\eta u(U_\alpha - u)F_{\theta(u)}\phi - \frac{1}{2}\epsilon(u)\alpha\phi F_{\theta(u)} u(U_\eta - u)\,.
\end{aligned}$$

Summing all terms in (3.75), we get

$$\begin{aligned}
\{\!\{U_\alpha, U_\eta\}\!\} =& \frac{1}{2}\epsilon(u)(u^2 F_{\theta(u)} - F_{\theta(u)} u^2) + \frac{1}{2}\alpha\epsilon(u)u^2\phi F_{\theta(u)} - \frac{1}{2}\eta\epsilon(u)F_{\theta(u)} u^2\phi \\
&+ \frac{1}{2}\alpha(uF_{\theta(u)} u\phi + u\phi F_{\theta(u)} u) - \frac{1}{2}\eta(u\phi F_{\theta(u)} u + uF_{\theta(u)} u\phi) \\
&+ \frac{1}{2}\eta u\phi U_\alpha F_{\theta(u)} + \frac{1}{2}\eta u U_\alpha F_{\theta(u)}\phi - \frac{1}{2}\alpha F_{\theta(u)} u\phi U_\eta \\
&- \frac{1}{2}\alpha\phi F_{\theta(u)} u U_\eta + \frac{1}{2}\epsilon(u)\eta u U_\alpha F_{\theta(u)}\phi - \frac{1}{2}\epsilon(u)\alpha\phi F_{\theta(u)} u U_\eta\,.
\end{aligned}$$

Repeating the substitution under the form $u^{-1}(U_\alpha - u) = \alpha\phi$, we find

$$\begin{aligned}
\{\!\{U_\alpha, U_\eta\}\!\} =& \frac{1}{2}\epsilon(u)(u^2 F_{\theta(u)} - F_{\theta(u)}u^2) + \frac{1}{2}\epsilon(u)u(U_\alpha - u)F_{\theta(u)} - \frac{1}{2}\epsilon(u)F_{\theta(u)}u(U_\eta - u) \\
&+ \frac{1}{2}(uF_{\theta(u)}(U_\alpha - u) + (U_\alpha - u)F_{\theta(u)}u) \\
&- \frac{1}{2}((U_\eta - u)F_{\theta(u)}u + uF_{\theta(u)}(U_\eta - u)) \\
&+ \frac{1}{2}(U_\eta - u)U_\alpha F_{\theta(u)} + \frac{1}{2}uU_\alpha F_{\theta(u)}u^{-1}(U_\eta - u) \\
&- \frac{1}{2}F_{\theta(u)}(U_\alpha - u)U_\eta - \frac{1}{2}u^{-1}(U_\alpha - u)F_{\theta(u)}uU_\eta \\
&+ \frac{1}{2}\epsilon(u)uU_\alpha F_{\theta(u)}u^{-1}(U_\eta - u) - \frac{1}{2}\epsilon(u)u^{-1}(U_\alpha - u)F_{\theta(u)}uU_\eta \,.
\end{aligned}$$

We get after simplifications

$$\begin{aligned}
\{\!\{U_\alpha, U_\eta\}\!\} =& + \frac{1}{2}(1+\epsilon(u))\left[uU_\alpha F_{\theta(u)}u^{-1}U_\eta - u^{-1}U_\alpha F_{\theta(u)}uU_\eta\right] \\
&+ \frac{1}{2}(uF_{\theta(u)}U_\alpha + U_\alpha F_{\theta(u)}u) - \frac{1}{2}(U_\eta F_{\theta(u)}u + uF_{\theta(u)}U_\eta) \\
&- uU_\alpha F_{\theta(u)} + F_{\theta(u)}uU_\eta + \frac{1}{2}(U_\eta U_\alpha F_{\theta(u)} - F_{\theta(u)}U_\alpha U_\eta) \,.
\end{aligned}$$

If we insert this in (3.74), we get

$$\begin{aligned}
\{U_\alpha^K, U_\eta^L\} =& \frac{1}{2}(1+\epsilon(u))\left[U_\eta^L u U_\alpha^K u^{-1} - U_\eta^L u^{-1} U_\alpha^K u\right] \\
&+ \frac{1}{2}(-U_\eta^{L-1}uU_\alpha^K + U_\eta^{L-1}U_\alpha^K u + U_\eta^L U_\alpha^{K-1}u - U_\eta^L u U_\alpha^{K-1})
\end{aligned} \tag{3.77}$$

all mod $[A, A]$. Indeed, since $U_\alpha \in \oplus_s e_s A e_{s+\theta(u)}$ we find for the first term

$$\begin{aligned}
\sum_s U_\eta^{L-1}uU_\alpha e_{s+\theta(u)}U_\alpha^{K-1}e_s u^{-1}U_\eta =& U_\eta^{L-1}uU_\alpha \left(\sum_s e_{s+\theta(u)}e_{s-(K-1)\theta(u)}\right) U_\alpha^{K-1}u^{-1}U_\eta \\
=& U_\eta^{L-1}uU_\alpha U_\alpha^{K-1}u^{-1}U_\eta
\end{aligned}$$

for $(K-1)\theta(u) \equiv -\theta(u)$ mod m, that is K is divisible by m if $\theta(u) \neq 0$, i.e. when $u = x, y, z$, while $K \geq 1$ for $u = 1_I + xy$. The same argument works for every element, either by inspecting K or L. Now, assuming $\alpha \neq \eta$, we remark that $u = \frac{1}{\alpha-\eta}(\alpha U_\eta - \eta U_\alpha)$. Using this expression in the second line of (3.77), all the terms cancel out. The second line trivially vanishes for $\alpha = \eta$ so that the claim follows as $\epsilon(u) = -1$.

The same proof works when $\epsilon(u) = +1$ to show that $\{U_{-,\alpha}^K, U_{-,\eta}^L\} = 0$ modulo commutators for $U_{-,\alpha} = u(1+\alpha\phi^{-1})$. We only need to notice that $\{\!\{\phi^{-1}, a\}\!\} = -\phi^{-1} * \{\!\{\phi, a\}\!\} * \phi^{-1}$, so we just need to replace in the expression $\{\!\{\phi, a\}\!\}$ the factors ϕ by ϕ^{-1} and multiply by an overall factor -1. Thus, reproducing the proof in the first case with some sign changes, we get

$$\frac{1}{KL}\{U^K_{-,\alpha}, U^L_{-,\eta}\} = \frac{1}{2}(-1+\epsilon(u))\left(U^L_{-,\eta} u U^K_{-,\alpha} u^{-1} - U^L_{-,\eta} u^{-1} U^K_{-,\alpha} u\right) ,$$

modulo commutators. This yields the desired result for $\epsilon(u) = +1$. □

Proof. (Proposition 3.2.14.) First, we assume that $u = y, z$. We consider $K, L \underset{m}{\equiv} 0$. Otherwise the proof is trivial as $u^K_{(j)} \in \oplus_s e_s A e_{s-K}$ (here we need that $u = y, z$) implies that such a term vanishes modulo commutator for K not divisible by m. We also assume without loss of generality that $0 \leq j \leq j' \leq d$.

To ease notations for the proof, we also set $u_1 = u_{(j')}$, $u_0 = u_{(j)}$, $\phi_1 = \sum_s \Phi_s^{(j')}$ and $\phi_0 = \sum_s \Phi_s^{(j)}$. As in (3.74), we have that

$$\frac{1}{KL}\{u_1^K, u_0^L\} = u_0^{L-1}\,\{\!\{u_1, u_0\}\!\}'\, u_1^{K-1}\,\{\!\{u_1, u_0\}\!\}'' \quad \mathrm{mod}[A, A] .$$

Hence, we need to compute $\{\!\{u_1, u_0\}\!\}$ as a first step. We can write

$$\{\!\{u_1, u_0\}\!\} = \{\!\{\phi_1, u_0\}\!\} * u + \phi_1 * \{\!\{u, \phi_0\}\!\}\, u + \phi_1 * \phi_0\, \{\!\{u, u\}\!\} .$$

By assumption, ϕ_0 and ϕ_1 are moment maps corresponding to the last two quivers in the chain $\bar{Q}_0 \subseteq \bar{Q}_j \subseteq \bar{Q}_{j'}$. Hence, we get from Lemma 2.3.15 that

$$\begin{aligned}\{\!\{\phi_1, u_0\}\!\} =& \sum_{s\in I} \frac{1}{2}(u_0 e_s \otimes \phi_{1,s} - e_s \otimes \phi_{1,s} u_0 + u_0 \phi_{1,s} \otimes e_s - \phi_{1,s} \otimes e_s u_0) \\ =& \frac{1}{2}(u_0 F_0 \phi_1 - F_0 \phi_1 u_0 + u_0 \phi_1 F_0 - \phi_1 F_0 u_0) ,\end{aligned}$$

since $\phi_1 = \sum_s e_s \phi_{1,s} e_s$. Using cyclic antisymmetry and the fact that $u \in \oplus_s e_s A e_{s-1}$ (again because $u = y, z$), we get also from Lemma 2.3.15 that

$$\begin{aligned}&\{\!\{u, \phi_0\}\!\} = -\{\!\{\phi_0, u\}\!\}^\circ \\ =& -\frac{1}{2}\sum_s(\phi_{0,s} e_s \otimes e_{s+1} u e_s - \phi_{0,s} u e_{s-1} \otimes e_s + e_s \otimes e_{s+1} u \phi_{0,s} - e_s u e_{s-1} \otimes e_s \phi_{0,s}) \\ =& -\frac{1}{2}(\phi_0 F_{-1} u - \phi_0 u F_{-1} + F_{-1} u \phi_0 - u F_{-1} \phi_0) .\end{aligned}$$

Finally, recall that we have $\{\!\{u, u\}\!\} = -\frac{1}{2}[u^2 F_{-1} - F_{-1} u^2]$. We can group all these terms, and since $F_0 * u = u F_{-1}$, $\phi_1 * F_{-1} = F_{-1} \phi_1$, we get after simplification

$$\{\!\{u_1, u_0\}\!\} = \frac{1}{2}(u_0 u_1 F_{-1} - u_1 F_{-1} u_0 + u_0 F_{-1} u_1 - F_{-1} u_1 u_0) .$$

Therefore,

$$\frac{1}{KL}\{u_1^K,u_0^L\} =\frac{1}{2}\sum_{s\in I}\Big(u_0^L u_1 e_s u_1^{K-1} e_{s+1} - u_0^{L-1} u_1 e_s u_1^{K-1} e_{s+1} u_0 \\ + u_0^L e_s u_1^{K-1} e_{s+1} u_1 - u_0^{L-1} e_s u_1^{K-1} e_{s+1} u_1 u_0\Big)\,,$$

modulo commutators. Since $e_s u_1^{K-1} = u_1^{K-1} e_{s+1}$, all terms cancel out mod $[A,A]$.

In the case $u = (1_I + xy)^{-1}$, note that we have $\{\!\{u,u\}\!\} = -\frac{1}{2}[u^2 F_0 - F_0 u^2]$ after a direct computation using (3.37a) with $1_I + xy$. Thus, the above proof applies also in this case for any $K, L \in \mathbb{N}^\times$ after minor changes, such as replacing F_{-1} by F_0 everywhere since $u \in \oplus_s e_s A e_s$ in this case. □

3.3.8 Loday brackets for framed cyclic quivers

In this subsection, we prove Lemmae 3.2.15, 3.2.16, 3.2.17, 3.2.18, 3.2.19 and 3.2.20. Before proving these results, recall that for $u = x, y, z, 1_I + xy$ we can write

$$\{\!\{\phi, u\}\!\} = \frac{1}{2}(uF_0\phi - F_0\phi u) + \frac{1}{2}(u\phi F_0 - \phi F_0 u)\,,$$

which was obtained as (3.76). We can note as in § 3.3.4 that this equality can not be directly applied to $w_{s,\beta}$ or $w_{s,\beta}$. Rather, we need an explicit computation using that (3.37b) holds for $u = 1_I + xy, 1_I + yx$ to get

$$\{\!\{\phi, v_{s,\beta}\}\!\} = \frac{1}{2}(v_{s,\beta}\phi \otimes e_s - v_{s,\beta} \otimes \phi e_s)\,, \quad \{\!\{\phi, w_{s,\beta}\}\!\} = \frac{1}{2}(e_s \otimes \phi w_{s,\beta} - e_s\phi \otimes w_{s,\beta})\,. \tag{3.78}$$

These brackets will be needed, together with the ones written in § 3.2.1 and § 3.2.2. We also use without further mention that for any $U, c \in A$ (or a suitable localisation) and $K \in \mathbb{N}^\times$, we have

$$\frac{1}{K}\{U^K, c\} = \{\!\{U,c\}\!\}'\, U^{K-1}\, \{\!\{U,c\}\!\}''\,, \tag{3.79}$$

as a direct application of (2.15).

Proof. (Lemma 3.2.15.) We can compute

$$\begin{aligned}\{\!\{U_\eta, x\}\!\} =& \{\!\{z, x\}\!\} * (1_I + \eta\phi) + \eta z * \{\!\{\phi, x\}\!\} \\ =& -\frac{1}{2}(xzF_{-1} + F_{-1}zx - zF_{-1}x + xF_{-1}z) * (1_I + \eta\phi) \\ &+ \frac{1}{2}\eta(z\phi * (xF_0 - F_0x) + z * (xF_0 - F_0x) * \phi) \\ =& -\frac{1}{2}(xz(1_I + \eta\phi)F_{-1} + (1_I + \eta\phi)F_{-1}zx - z(1_I + \eta\phi)F_{-1}x + x(1_I + \eta\phi)F_{-1}z) \\ &+ \frac{1}{2}\eta(xF_{-1}z\phi - F_{-1}z\phi x) + \frac{1}{2}\eta(x\phi F_{-1}z - \phi F_{-1}zx)\,,\end{aligned}$$

where we used that $\phi \in \oplus_s e_s A e_s$ and $z \in \oplus_s e_s A e_{s-1}$ to get

$$xzF_{-1} * (1_I + \eta\phi) = xz(1_I + \eta\phi)F_{-1}\,, \quad z\phi * xF_0 = \sum_s xe_s \otimes z\phi e_s = xF_{-1}z\phi\,,$$

and similar expressions. By definition, $U_\eta = z(1 + \eta\phi)$, so we can write $U_\eta - z = \eta z\phi$. We can also multiply both expressions from the left by z^{-1} to get $\eta\phi = z^{-1}U_\eta - 1_I$ since we work in A'. Thus

$$\begin{aligned}\{\!\{U_\eta, x\}\!\} =& -\frac{1}{2}(xU_\eta F_{-1} + z^{-1}U_\eta F_{-1}zx - U_\eta F_{-1}x + xz^{-1}U_\eta F_{-1}z) \\ &+ \frac{1}{2}(xF_{-1}(U_\eta - z) - F_{-1}(U_\eta - z)x) \\ &+ \frac{1}{2}(xz^{-1}(U_\eta - z)F_{-1}z - z^{-1}(U_\eta - z)F_{-1}zx) \\ =& -z^{-1}U_\eta F_{-1}zx - xF_{-1}z + F_{-1}zx \\ &+ \frac{1}{2}(xF_{-1}U_\eta - xU_\eta F_{-1} - F_{-1}U_\eta x + U_\eta F_{-1}x)\,.\end{aligned}$$

Now, as $U_\eta^{K-1} \in \oplus_s e_{s-1} A e_s$ for $K \in m\mathbb{N}$, $\sum_{s\in I} e_{s-1}U_\eta^{K-1}e_s = U_\eta^{K-1}$ and we find

$$\frac{1}{K}\{U_\eta^K, x\} = -z^{-1}U_\eta U_\eta^{K-1}zx - xU_\eta^{K-1}z + U_\eta^{K-1}zx = -\eta\phi U_\eta^{K-1}zx - xU_\eta^{K-1}z\,.$$

Note that this expression vanishes for $K \notin m\mathbb{N}$, $K \neq 0$, as we expect, so we restrict to $K \in m\mathbb{N}^\times$ for the rest of our discussion. Next, we do this for z and find

$$\begin{aligned}\{\!\{U_\eta, z\}\!\} =& \{\!\{z, z\}\!\} * (1_I + \eta\phi) + \eta z * \{\!\{\phi, z\}\!\} \\ =& -\frac{1}{2}(z^2F_{-1} - F_{-1}z^2) * (1_I + \eta\phi) + \frac{1}{2}\eta(z\phi * (zF_0 - F_0z) + z * (zF_0 - F_0z) * \phi) \\ =& -\frac{1}{2}(z^2(1_I + \eta\phi)F_{-1} - (1_I + \eta\phi)F_{-1}z^2) \\ &+ \frac{1}{2}\eta(zF_{-1}z\phi - F_{-1}z\phi z) + \frac{1}{2}\eta(z\phi F_{-1}z - \phi F_{-1}z^2)\,,\end{aligned}$$

which with the previous trick becomes

$$\begin{aligned}\{\!\{U_\eta, z\}\!\} =& -\frac{1}{2}(zU_\eta F_{-1} - z^{-1}U_\eta F_{-1}z^2) + \frac{1}{2}(zF_{-1}(U_\eta - z) - F_{-1}(U_\eta - z)z) \\ &+ \frac{1}{2}((U_\eta - z)F_{-1}z - (z^{-1}U_\eta - 1_I)F_{-1}z^2) \\ =& F_{-1}z^2 - zF_{-1}z + \frac{1}{2}(zF_{-1}U_\eta - zU_\eta F_{-1} + U_\eta F_{-1}z - F_{-1}U_\eta z)\,.\end{aligned}$$

Hence $\frac{1}{K}\{U_\eta^K, z\} = U_\eta^{K-1}z^2 - zU_\eta^{K-1}z$. For $v_{s,\beta}$ we get

$$\begin{aligned}\{\!\{U_\eta, v_{s,\beta}\}\!\} =& \{\!\{z, v_{s,\beta}\}\!\} * (1_I + \eta\phi) + \eta z * \{\!\{\phi, v_{s,\beta}\}\!\} \\ =& \frac{1}{2}\left(v_{s,\beta}z(1_I + \eta\phi) \otimes e_s - v_{s,\beta}(1_I + \eta\phi) \otimes ze_s\right) \\ &+ \frac{1}{2}\eta(v_{s,\beta}\phi \otimes ze_s - v_{s,\beta} \otimes z\phi e_s) \\ =& \frac{1}{2}\left(v_{s,\beta}z(1_I + \eta\phi) \otimes e_s - v_{s,\beta} \otimes ze_s - v_{s,\beta} \otimes z\phi e_s\right) \\ =& \frac{1}{2}\left(v_{s,\beta}U_\eta \otimes e_s - v_{s,\beta} \otimes U_\eta e_s\right)\,,\end{aligned}$$

which gives $\{U_\eta^K, v_\beta\} = 0$. We find in the exact same way that $\{U_\eta^K, w_\beta\} = 0$. □

Proof. (Lemma 3.2.16.) Note that, as the double bracket $\{\!\{y, x\}\!\}$ can be obtained from $\{\!\{z, x\}\!\}$ by replacing z by y and adding an extra term $-F_{-1}$, we get by adapting the proof of Lemma 3.2.15

$$\begin{aligned}\frac{1}{K}\{\bar{U}_\eta^K, x\} =& -(1_I + \eta\phi)\bar{U}_\eta^{K-1} - y^{-1}\bar{U}_\eta^K yx - x\bar{U}_\eta^{K-1}y + \bar{U}_\eta^{K-1}yx \\ =& -\bar{U}_\eta^{K-1} - x\bar{U}_\eta^{K-1}y - \eta\phi\bar{U}_\eta^{K-1}(1_I + yx).\end{aligned}$$

The double bracket $\{\!\{z, z\}\!\}$ with y replacing z is exactly $\{\!\{y, y\}\!\}$, hence $\frac{1}{K}\{\bar{U}_\eta^K, y\} = -y\bar{U}_\eta^{K-1}y + \bar{U}_\eta^{K-1}y^2$. The same holds for the couple $(y, w_{s,\beta})$ replacing $(z, w_{s,\beta})$, or doing it with $v_{s,\beta}$, so that $\{\bar{U}_\eta^K, w_{s,\beta}\} = 0$ and $\{\bar{U}_\eta^K, v_{s,\beta}\} = 0$. □

Proof. (Lemma 3.2.17.) Recall that we assume $K \in m\mathbb{N}^\times$ so that $\sum_s e_{s+1}\hat{U}_\eta^{K-1}e_s = \hat{U}_\eta^{K-1}$, which we will use under the form $F_1'\hat{U}_\eta^{K-1}F_1'' = \hat{U}_\eta^{K-1}$. Note also that $x\phi^{-1} * F_0 = F_1 x\phi^{-1}$. We compute

$$\begin{aligned}\left\{\!\!\left\{\hat{U}_\eta, x\right\}\!\!\right\} =& \{\!\{x, x\}\!\} * (1_I + \eta\phi^{-1}) - \eta x\phi^{-1} * \{\!\{\phi, x\}\!\} * \phi^{-1} \\ =& \frac{1}{2}(x^2(1_I + \eta\phi^{-1})F_1 - (1_I + \eta\phi)F_1x^2 \\ &- \frac{1}{2}\eta(x\phi^{-1}F_1x - \phi^{-1}F_1x^2 + xF_1x\phi^{-1} - F_1x\phi^{-1}x)\,.\end{aligned}$$

Using that $\hat{U}_\eta = x(1+\eta\phi^{-1})$ and equivalent expressions, we get

$$\begin{aligned}\left\{\!\!\left\{\hat{U}_\eta, x\right\}\!\!\right\} =& \frac{1}{2}(x\hat{U}_\eta F_1 - x^{-1}\hat{U}_\eta F_1 x^2) \\ &- \frac{1}{2}((\hat{U}_\eta - x)F_1 x - x^{-1}(\hat{U}_\eta - x)F_1 x^2 + xF_1(\hat{U}_\eta - x) - F_1(\hat{U}_\eta - x)x) \\ =& xF_1 x - F_1 x^2 + \frac{1}{2}(x\hat{U}_\eta F_1 - \hat{U}_\eta F_1 x - xF_1\hat{U}_\eta + F_1\hat{U}_\eta x)\,,\end{aligned}$$

and we find

$$\frac{1}{K}\{\hat{U}_\eta^K, x\} = x\hat{U}_\eta^{K-1}x - \hat{U}_\eta^{K-1}x^2\,.$$

Next we have

$$\begin{aligned}\left\{\!\!\left\{\hat{U}_\eta, z\right\}\!\!\right\} =& \{\!\!\{x,z\}\!\!\} * (1_I + \eta\phi^{-1}) - \eta x\phi^{-1} * \{\!\!\{\phi, z\}\!\!\} * \phi^{-1} \\ =& \frac{1}{2}(zx(1_I+\eta\phi^{-1})F_1 + (1_I+\eta\phi^{-1})F_1 xz) \\ &+ \frac{1}{2}(-x(1_I+\eta\phi^{-1})F_1 z + z(1_I+\eta\phi^{-1})F_1 x) \\ &- \frac{1}{2}\eta(z\phi^{-1}F_1 x - \phi^{-1}F_1 xz + zF_1 x\phi^{-1} - F_1 x\phi^{-1}z)\,.\end{aligned}$$

Furthermore, using the same trick as before gives

$$\begin{aligned}\left\{\!\!\left\{\hat{U}_\eta, z\right\}\!\!\right\} =& \{\!\!\{x,z\}\!\!\} * (1_I + \eta\phi^{-1}) - \eta x\phi^{-1} * \{\!\!\{\phi, z\}\!\!\} * \phi^{-1} \\ =& \frac{1}{2}(z\hat{U}_\eta F_1 + x^{-1}\hat{U}_\eta F_1 xz - \hat{U}_\eta F_1 z + zx^{-1}\hat{U}_\eta F_1 x) \\ &- \frac{1}{2}(zx^{-1}(\hat{U}_\eta - x)F_1 x - x^{-1}(\hat{U}_\eta - x)F_1 xz + zF_1(\hat{U}_\eta - x) - F_1(\hat{U}_\eta - x)z) \\ =& x^{-1}\hat{U}_\eta F_1 xz + zF_1 x - F_1 xz + \frac{1}{2}(z\hat{U}_\eta F_1 - \hat{U}_\eta F_1 z - zF_1\hat{U}_\eta + F_1\hat{U}_\eta z)\,.\end{aligned}$$

We easily get

$$\frac{1}{K}\{\hat{U}_\eta^K, z\} = x^{-1}\hat{U}_\eta^K xz + z\hat{U}_\eta^{K-1}x - \hat{U}_\eta^{K-1}xz = z\hat{U}_\eta^{K-1} + \eta\phi^{-1}\hat{U}_\eta^{K-1}xz\,.$$

Finally, we can get $\left\{\!\!\left\{\hat{U}_\eta, v_{s,\beta}\right\}\!\!\right\} = \frac{1}{2}v_{s,\beta}\hat{U}_\eta \otimes e_s - \frac{1}{2}v_{s,\beta} \otimes \hat{U}_\eta e_s$, which gives $\{\tilde{U}_\eta^K, v_{s,\beta}\} = 0$. We can do the same for $w_{s,\beta}$ to get $\{\hat{U}_\eta^K, v_{s,\beta}\} = 0$. □

Proof. (Lemma 3.2.18.) We compute in A' instead of A, so that $\sum_s e_s + xy = xz$ and $\tilde{U}_\eta = xz(1+\eta\phi^{-1})$. This does not change the final result, and only ease notations. First, we note that

$$\{\!\!\{xz, x\}\!\!\} = \frac{1}{2}x^2 zF_0 - \frac{1}{2}F_0 xzx - \frac{1}{2}xzF_0 x - \frac{1}{2}xF_0 xz\,.$$

We obtain in that way

$$\left\{\!\!\left\{\tilde{U}_\eta, x\right\}\!\!\right\} = \{\!\{xz, x\}\!\} * (1+\eta\phi^{-1}) - \eta xz\phi^{-1} * \{\!\{\phi, x\}\!\} * \phi^{-1}$$
$$= \frac{1}{2}(x\tilde{U}_\eta F_0 - (xz)^{-1}\tilde{U}_\eta F_0 xzx - \tilde{U}_\eta F_0 x - x(xz)^{-1}\tilde{U}_\eta F_0 xz)$$
$$+ \frac{1}{2}(-x[(xz)^{-1}\tilde{U}_\eta - 1_I]F_0 xz + [(xz)^{-1}\tilde{U}_\eta - 1_I]F_0 xzx)$$
$$+ \frac{1}{2}(-xF_0[\tilde{U}_\eta - xz] + F_0[\tilde{U}_\eta - xz]x)\,.$$

This reduces after simplification to

$$\left\{\!\!\left\{\tilde{U}_\eta, x\right\}\!\!\right\} = -F_0 xzx + xF_0 xz - x(xz)^{-1}\tilde{U}_\eta F_0 xz$$
$$+ \frac{1}{2}(x\tilde{U}_\eta F_0 - xF_0\tilde{U}_\eta + F_0\tilde{U}_\eta x - \tilde{U}_\eta F_0 x)\,,$$

which yields for any $K \in \mathbb{N}^\times$

$$\frac{1}{K}\{\tilde{U}_\eta^K, x\} = -x(xz)^{-1}\tilde{U}_\eta^K xz - \tilde{U}_\eta^{K-1}xzx + x\tilde{U}_\eta^{K-1}xz$$
$$= -\tilde{U}_\eta^{K-1}xzx - \eta\, x\phi^{-1}\tilde{U}_\eta^{K-1}xz\,.$$

Second, we compute $\{\tilde{U}_\eta^K, xz\}$. We get

$$\left\{\!\!\left\{\tilde{U}_\eta, xz\right\}\!\!\right\} = \{\!\{xz, xz\}\!\} * (1+\eta\phi^{-1}) - \eta xz\phi^{-1} * \{\!\{\phi, xz\}\!\} * \phi^{-1}$$
$$= \frac{1}{2}\left(xz\tilde{U}_\eta F_0 - (xz)^{-1}\tilde{U}_\eta F_0 (xz)^2\right)$$
$$+ \frac{1}{2}\left(-[\tilde{U}_\eta - xz]F_0 xz + [(xz)^{-1}\tilde{U}_\eta - 1]F_0 (xz)^2\right)$$
$$+ \frac{1}{2}\left(-xzF_0[\tilde{U}_\eta - xz] + F_0[\tilde{U}_\eta - xz]xz\right)\,.$$

We get by cancelling terms

$$\left\{\!\!\left\{\tilde{U}_\eta, xz\right\}\!\!\right\} = -F_0(xz)^2 + xzF_0 xz + \frac{1}{2}(xz\tilde{U}_\eta F_0 - xzF_0\tilde{U}_\eta + F_0\tilde{U}_\eta xz - \tilde{U}_\eta F_0 xz)\,.$$

This gives us that

$$\frac{1}{K}\{\tilde{U}_\eta^K, xz\} = xz\tilde{U}_\eta^{K-1}xz - \tilde{U}_\eta^{K-1}(xz)^2\,.$$

Finally, we can get $\left\{\!\!\left\{\tilde{U}_\eta, v_{s,\beta}\right\}\!\!\right\} = \frac{1}{2}v_{s,\beta}\tilde{U}_\eta \otimes e_s - \frac{1}{2}v_{s,\beta} \otimes \tilde{U}_\eta e_s$, and $\{\tilde{U}_\eta^K, v_{s,\beta}\} = 0$ as before. The same holds for $w_{s,\beta}$ too. □

For the last two proofs, we use the notations introduced in Remark 3.2.11.

Proof. (Lemma 3.2.19.) We will use that for any $a \in A'$

$$\frac{1}{K}\{z_{(j)}^K, a\} = \{\!\{z_{(j)}, a\}\!\}'\, z_{(j)}^{K-1}\, \{\!\{z_{(j)}, a\}\!\}''\,, \tag{3.80}$$

where the double bracket is obtained from the decomposition

$$\{\!\{z_{(j)}, a\}\!\} = \sum_{s\in I} \left\{\!\!\left\{\Phi_s^{(j)}, a\right\}\!\!\right\} * z + \Phi^{(j)} * \{\!\{z, a\}\!\} \,. \tag{3.81}$$

If $a = x, z, v_{r,\beta}, w_{r,\beta}$ with $(r,\beta) \leq \rho(j)$, we get from Lemma 2.3.15 that

$$\left\{\!\!\left\{\Phi_s^{(j)}, a\right\}\!\!\right\} = \frac{1}{2}(ae_s \otimes \Phi_s^{(j)} - e_s \otimes \Phi_s^{(j)} a + a\Phi_s^{(j)} \otimes e_s - \Phi_s^{(j)} \otimes e_s a)\,.$$

Hence, we obtain

$$\sum_{s\in I} \left\{\!\!\left\{\Phi_s^{(j)}, a\right\}\!\!\right\} = \frac{1}{2}(aF_0\Phi^{(j)} - F_0\Phi^{(j)}a + a\Phi^{(j)}F_0 - \Phi^{(j)}F_0 a)\,.$$

Using (3.81) together with (3.27a)–(3.27c), and (3.28) under the form

$$\{\!\{z, w_{r,\beta}\}\!\} = \frac{1}{2}(F_{-1}zw_{r,\beta} - zF_{-1}w_{r,\beta})\,, \quad \{\!\{z, v_{r,\beta}\}\!\} = \frac{1}{2}(v_{r,\beta}zF_{-1} - v_{r,\beta}F_{-1}z)\,, \tag{3.82}$$

we can write that

$$\begin{aligned}
\{\!\{z_{(j)}, x\}\!\} &= \frac{1}{2}(xz_{(j)}F_{-1} - z_{(j)}F_{-1}x - F_{-1}z_{(j)}x - xF_{-1}z_{(j)}) \\
\{\!\{z_{(j)}, z\}\!\} &= \frac{1}{2}(-zF_{-1}z_{(j)} + zz_{(j)}F_{-1} - z_{(j)}F_{-1}z + F_{-1}z_{(j)}z) \\
\{\!\{z_{(j)}, w_{r,\beta}\}\!\} &= \frac{1}{2}(-z_{(j)}F_{-1}w_{r,\beta} + F_{-1}z_{(j)}w_{r,\beta} - 2zF_{-1}\Phi_0^{(j)}w_{r,\beta}) \\
\{\!\{z_{(j)}, v_{r,\beta}\}\!\} &= \frac{1}{2}(v_{r,\beta}z_{(j)}F_{-1} - v_{r,\beta}F_{-1}z_{(j)} + 2v_{r,\beta}zF_{-1}\Phi_0^{(j)})\,,
\end{aligned}$$

with $(r,\beta) \leq \rho(j)$. Here we used that $F_0 * z = zF_{-1}$ and $\Phi^{(j)} * F_{-1} = F_{-1}\Phi^{(j)}$. All these double brackets are sums of terms of the form $bF_{-1}c$ for some $b, c \in A'$. Hence, using (3.80), such terms contribute to the final Loday bracket as

$$\sum_{s\in I} be_{s-1}z_{(j)}^{K-1}e_s c = bz_{(j)}^{K-1}c\,,$$

since $z_{(j)}^{K-1} \in \oplus_s e_s A e_{s+1}$ by assumption on K. Thus, we have the four first cases.

For the next two cases, note that if we can show

$$\begin{aligned}
\sum_{s\in I} \left\{\!\!\left\{\Phi_s^{(j)}, w_{r,\beta}\right\}\!\!\right\} &= \frac{1}{2}(F_0\Phi^{(j)}w_{r,\beta} - \Phi^{(j)}F_0 w_{r,\beta})\,, \quad \rho(j) < (r,\beta)\,, \\
\sum_{s\in I} \left\{\!\!\left\{\Phi_s^{(j)}, v_{r,\beta}\right\}\!\!\right\} &= \frac{1}{2}(v_{r,\beta}\Phi^{(j)}F_0 - v_{r,\beta}F_0\Phi^{(j)})\,, \quad \rho(j) < (r,\beta)\,,
\end{aligned} \tag{3.83}$$

we get by (3.81) and (3.82) that

$$\{\!\{z_{(j)}, w_{r,\beta}\}\!\} = \frac{1}{2}(-z_{(j)}F_{-1}w_{r,\beta} + F_{-1}z_{(j)}w_{r,\beta})\,, \quad \rho(j) < (r,\beta)\,,$$
$$\{\!\{z_{(j)}, v_{r,\beta}\}\!\} = \frac{1}{2}(v_{r,\beta}z_{(j)}F_{-1} - v_{r,\beta}F_{-1}z_{(j)})\,, \quad \rho(j) < (r,\beta)\,.$$

As before, putting these expressions in (3.80) yields $\{z^K_{(j)}, w_{r,\beta}\} = 0$ for $\rho(j) < (r,\beta)$, and the same holds with v instead.

We now prove (3.83) by induction on $j \in \{0,1,\dots,|\mathbf{d}|\}$. The case $j = 0$ holds since it was established as (3.78). For the general case, we remark the relation

$$\sum_{s\in I} \Phi^{(j)}_s = \Phi^{(j-1)}(1_I + w_{\rho(j)}v_{\rho(j)})^{-1} = \left(\Phi^{(j-1)}_s\right)(1_I + w_{\rho(j)}v_{\rho(j)})^{-1}\,,$$

which implies that

$$\sum_{s\in I} \left\{\!\!\left\{\Phi^{(j)}_s, w_{r,\beta}\right\}\!\!\right\} = \sum_{s} \left\{\!\!\left\{\Phi^{(j-1)}_s, w_{r,\beta}\right\}\!\!\right\} * (1_I + w_{\rho(j)}v_{\rho(j)})^{-1}$$
$$- \Phi^{(j-1)}(1_I + w_{\rho(j)}v_{\rho(j)})^{-1} * \{\!\{(1_I + w_{\rho(j)}w_{\rho(j)}), v_{r,\beta}\}\!\} * (1_I + w_{\rho(j)}v_{\rho(j)})^{-1}\,,$$

and a similar expression for v. To compute these, we claim that

$$\{\!\{(1_I + w_{\rho(j)}v_{\rho(j)}), w_{r,\beta}\}\!\} = \frac{1}{2}\left(F_0(1_I + w_{\rho(j)}v_{\rho(j)})w_{r,\beta} - (1_I + w_{\rho(j)}v_{\rho(j)})F_0w_{r,\beta}\right)$$
$$\{\!\{(1_I + w_{\rho(j)}v_{\rho(j)}), v_{r,\beta}\}\!\} = \frac{1}{2}\left(v_{r,\beta}(1_I + w_{\rho(j)}v_{\rho(j)})F_0 - v_{r,\beta}F_0(1_I + w_{\rho(j)}v_{\rho(j)})\right)\,,$$

for all $(r,\beta) > \rho(j)$. Denoting $\rho(j)$ as (t,γ) for some $t \in I$, $1 \le \gamma \le d_t$, we have from the definition of the ordering that either $r = t$ and $\beta > \gamma$, or that $r > t$. In the case $r > t$, these two expressions vanish, in accordance with a direct computation using (3.23b)–(3.23c). In the case $r = t$, $\beta > \gamma$, these expressions can be reduced to

$$\{\!\{(1_I + w_{\rho(j)}v_{\rho(j)}), w_{r,\beta}\}\!\} = \frac{1}{2}\left(e_t \otimes w_{\rho(j)}v_{\rho(j)}w_{r,\beta} - w_{\rho(j)}v_{\rho(j)} \otimes w_{r,\beta}\right)$$
$$\{\!\{(1_I + w_{\rho(j)}v_{\rho(j)}), v_{r,\beta}\}\!\} = \frac{1}{2}\left(v_{r,\beta}w_{\rho(j)}v_{\rho(j)} \otimes e_r - v_{r,\beta} \otimes w_{\rho(j)}v_{\rho(j)}\right)\,,$$

and this is precisely what (3.23a)–(3.24) gives in that case. Assuming by induction that (3.83) holds for $j-1$, we can use the previous expressions to show that (3.83) holds for j, proving the induction step. (One has to use that both $\Phi^{(j-1)}$ and $(1_I + w_{\rho(j)}v_{\rho(j)})^{-1}$ are elements of $\oplus_s e_s A e_s$.)

To finish the proof, note that the double bracket $\{\!\{z_{(j)}, z_{(j)}\}\!\} = \frac{1}{2}(z^2_{(j)}F_{-1} - F_{-1}z^2_{(j)})$ which is obtained at the end of the proof of Proposition 3.2.14 imply that $\{z^K_{(j)}, z_{(j)}\} = 0$. Using the double bracket $\sum_s \left\{\!\!\left\{\Phi_s^{(j)}, a\right\}\!\!\right\}$ obtained at the beginning of the current proof which holds for $a = z_{(j)}$, we get that

$$\sum_{s\in I} \left\{\!\!\left\{z_{(j)}, \Phi_s^{(j)}\right\}\!\!\right\} = -\frac{1}{2}(\Phi^{(j)}F_{-1}z_{(j)} - \Phi^{(j)}z_{(j)}F_{-1} + F_{-1}z_{(j)}\Phi^{(j)} - z_{(j)}F_{-1}\Phi^{(j)})\,.$$

This is in fact the double bracket for $\{\!\!\{z_{(j)}, \Phi^{(j)}\}\!\!\}$, since $\left\{\!\!\left\{\Phi_\infty^{(j)}, z_{(j)}\right\}\!\!\right\} = 0$ as $e_\infty z = 0 = z e_\infty$. Therefore $\{z_{(j)}^K, \Phi^{(j)}\} = 0$. □

Proof. (Lemma 3.2.20.) This is the same proof as Lemma 3.2.19 with y replacing z everywhere, and an additional term $-F_{-1}\Phi^{(j)}$ in the double bracket $\{\!\!\{y_{(\alpha)}, x\}\!\!\}$. □

Proof. (Lemma 3.2.21.) A small computation using (3.37a)–(3.37b) with $1_I + xy$ shows that $u = (1_I + xy)^{-1}$ also satisfies (3.37a)–(3.37b) with $\epsilon(u) = -1$, $\theta(u) = 0$. It is then straightforward that we can adapt the proof of Lemma 3.2.19 by simply replacing z with u and F_{-1} with F_0 in the cases of the brackets with $u, w_{s,\alpha}, v_{s,\alpha}, u_{(j)}, \Phi^{(j)}$.

For the case of x, a direct calculation using $\{\!\!\{1_I + xy, x\}\!\!\}$ obtained at the beginning of the proof of Lemma 3.2.18 shows that

$$\{\!\!\{u, x\}\!\!\} = -u * \{\!\!\{1_I + xy, x\}\!\!\} * u = \frac{1}{2}(uF_0x + F_0ux + xuF_0 - xF_0u)\,.$$

This yields

$$\{\!\!\{u_{(j)}, x\}\!\!\} = \frac{1}{2}(2xuF_0\Phi^{(j)} + xu_{(j)}F_0 - u_{(j)}F_0x + F_0u_{(j)}x - xF_0u_{(j)})\,,$$

leading to $\{u_{(j)}^K, x\} = Kxuu_{(j)}^{K-1}\Phi^{(j)}$. □

Chapter 4

MQVs from the Jordan quiver

In this chapter, we combine the general approach to MQVs and integrability exhibited in § 2.3.3 and § 2.3.4 together with the double quasi-Hamiltonian structure (and related computations) for the double of a framed Jordan quiver derived in Section 3.1. We begin with the general definition of such spaces in Section 4.1, and gather several results that do not depend on the number of framing arrows. This global study is followed by a local investigation, which we begin in Section 4.2 with the simplest type of framing by $d = 1$ additional arrow to the Jordan quiver. Most of the results in that section have appeared in [41, Sect. 3]. We finish by the general case of framing by $d \geq 2$ additional arrows in Section 4.3, which is parallel to [42].

We follow the conventions introduced in Remark 3.1.1.

4.1 General considerations

Consider the quiver $\bar{Q}$ defined in Section 3.1. For $n \in \mathbb{N}^\times$ and $q \in \mathbb{C}^\times$, we form $\widetilde{\alpha} = (1, n)$ and $\widetilde{q} = q^{-n}e_\infty + qe_0$ as in (2.37). A point ρ on the moduli space $\operatorname{Rep}(\mathbb{C}\bar{Q}, \widetilde{\alpha})$ of representations of $\mathbb{C}\bar{Q}$ with dimension $\widetilde{\alpha}$ consists of the vector spaces $\mathcal{V}_0 = \mathbb{C}^n$, $\mathcal{V}_\infty = \mathbb{C}$ together with a linear map $X_a : \mathcal{V}_{h(a)} \to \mathcal{V}_{t(a)}$ for each arrow $a \in \bar{Q}$. To simplify our discussions, we view a point ρ as a $2(d+1)$-uple $(X, Y, V_\alpha, W_\alpha)$ where

$$X, Y \in \operatorname{Mat}_{n\times n}(\mathbb{C}), \quad V_\alpha \in \operatorname{Mat}_{1\times n}(\mathbb{C}), \quad W_\alpha \in \operatorname{Mat}_{n\times 1}(\mathbb{C})\,.$$

Hence we have $\mathcal{X}(x)(\rho) = X$, $\mathcal{X}(y)(\rho) = Y$, $\mathcal{X}(v_\alpha)(\rho) = V_\alpha$ and $\mathcal{X}(w_\alpha)(\rho) = W_\alpha$. It is important to remark that we use the interpretation of Example 2.2.6 : if we follow exactly § 2.2.2, we would view the point ρ as a $2(d+1)$-uple $(\bar{X}, \bar{Y}, \bar{V}_\alpha, \bar{W}_\alpha)$ of elements of $\mathfrak{gl}_{n+1}(\mathbb{C})$ where

$$\bar{X} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & X & \\ 0 & & & \end{pmatrix}, \quad \bar{V}_\alpha = \begin{pmatrix} 0 & & V_\alpha & \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad \bar{W}_\alpha = \begin{pmatrix} 0 & 0 & \dots & 0 \\ & 0 & \dots & 0 \\ W_\alpha & \vdots & & \vdots \\ & 0 & \dots & 0 \end{pmatrix},$$

and $\bar{Y}$ takes the same form as $\bar{X}$ with block Y. We drop the dependence on ρ from now on.

Consider the smooth open subspace $\operatorname{Rep}(\mathbb{C}\bar{Q}, \widetilde{\alpha})^\circ \subset \operatorname{Rep}(\mathbb{C}\bar{Q}, \widetilde{\alpha})$ where

$$\operatorname{Id}_n + XY, \operatorname{Id}_n + YX, \operatorname{Id}_n + W_\alpha V_\alpha \in \operatorname{GL}_n(\mathbb{C}), \quad 1 + V_\alpha W_\alpha \neq 0, \quad \alpha = 1, \dots, d,$$

which we identify to $\operatorname{Rep}(A, \widetilde{\alpha})$. We can then view $\operatorname{Rep}(\Lambda^{\widetilde{q}}, \widetilde{\alpha})$ as the closed subspace where

$$(\operatorname{Id}_n + XY)(\operatorname{Id}_n + YX)^{-1} = q \prod_{\alpha=1,\dots,d}^{\longleftarrow} (\operatorname{Id}_n + W_\alpha V_\alpha), \tag{4.1a}$$

$$\prod_{\alpha=1,\dots,d}^{\longrightarrow} (1 + V_\alpha W_\alpha) = q^{-n}, \tag{4.1b}$$

by applying $\mathcal{X}$ to (3.4a)–(3.4b). Note that (4.1b) follows from (4.1a) by taking determinants, so we omit this condition from now on. Recalling that there exists a $\operatorname{GL}_n(\mathbb{C})$ action by

$$g.(X, Y, V_\alpha, W_\alpha) = (gXg^{-1}, gYg^{-1}, V_\alpha g^{-1}, gW_\alpha), \quad g \in \operatorname{GL}_n(\mathbb{C}), \tag{4.2}$$

we form the MQV

$$\mathcal{C}_{n,q,d} = \operatorname{Rep}\left(\Lambda^{\widetilde{q}}, \widetilde{\alpha}\right) /\!/ \operatorname{GL}_n(\mathbb{C}),$$

as a q-analogue of the Calogero-Moser space $\mathcal{C}_n$ introduced by Wilson [170], which is reviewed in Example 2.1.4. To guarantee that this space is smooth, we apply the regularity criterion of Proposition 2.3.28 for the roots of the Jordan quiver given in Example 2.2.3.

Proposition 4.1.1 *Assume that q is not a root of unity. Then the $\operatorname{GL}_n(\mathbb{C})$ action on $\operatorname{Rep}(\Lambda^{\widetilde{q}}, \widetilde{\alpha})$ is free, and $\mathcal{C}_{n,q,d}$ is a smooth variety of dimension $2nd$ endowed with a non-degenerate Poisson bracket $\{-,-\}_{\mathrm{P}}$.*

The dimension follows from Theorem 2.3.27 because the space is not empty[11]. To compute a Poisson bracket $\{F,G\}_{\mathrm{P}}$ on $\mathcal{C}_{n,q,d}$, recall that we have two equivalent choices. Either we can lift the two functions to $\mathrm{Rep}(A,\widetilde{\alpha})$, and then use Proposition 2.3.26; or we can remark that these two functions are (polynomials in elements) of the form $\mathrm{tr}\,\mathcal{X}(\gamma)$, and we apply (2.36).

4.1.1 Localisation

Consider the subspace $\mathrm{Rep}(\mathbb{C}\bar{Q},\widetilde{\alpha})^\circ \subset \mathrm{Rep}(\mathbb{C}\bar{Q},\widetilde{\alpha})$ where X is invertible. Motivated by § 3.1.2, we construct the matrix $Z = Y + X^{-1}$, together with $\mathbf{A} \in \mathrm{Mat}_{n\times d}(\mathbb{C})$ and $\mathbf{C} \in \mathrm{Mat}_{d\times n}(\mathbb{C})$ which we refer to as the *spin matrices*, and that are defined entry-wise by

$$\mathbf{A}_{i\alpha} = [W_\alpha]_i\,, \quad \mathbf{C}_{\alpha j} = [V_\alpha(\mathrm{Id}_n + W_{\alpha-1}V_{\alpha-1})\dots(\mathrm{Id}_n + W_1V_1)Z]_j\,. \tag{4.3}$$

The α-th column of $\mathbf{A}$ is W_α, so it represents the spin element a'_α by (3.7). Similarly, the α-th row of $\mathbf{C}$ represents c'_α. We get in particular that the moment map equation (4.1a) is equivalent to

$$XZX^{-1} = qZ + q\mathbf{A}\mathbf{C}\,. \tag{4.4}$$

This construction descends to the subspace $\mathcal{C}^\circ_{n,q,d} \subset \mathcal{C}_{n,q,d}$ where X is invertible. In particular, we can understand a point of $\mathcal{C}^\circ_{n,q,d}$ with four matrices instead of $2d+2$. Indeed, given a quadruple $(X,Z,\mathbf{A},\mathbf{C})$ as above, we can recover (X,Y,V_α,W_α) by taking $Y = Z - X^{-1}$, $(W_\alpha)_i = \mathbf{A}_{i\alpha}$, while we form $C_\alpha \in \mathrm{Mat}_{1\times n}(\mathbb{C})$ by $(C_\alpha)_i = \mathbf{C}_{\alpha i}$ and define inductively

$$V_1 = C_1 Z^{-1}\,, \quad V_\alpha = C_\alpha Z^{-1}(\mathrm{Id}_n + W_1V_1)^{-1}\dots(\mathrm{Id}_n + W_{\alpha-1}V_{\alpha-1})^{-1}\,,$$

since all inverses are well-defined by definition of $\mathcal{C}^\circ_{n,q,d}$.

Consider the following functions on $\mathcal{C}^\circ_{n,q,d}$

$$f_k := \mathrm{tr}(X^k)\,, \quad g_k^{\alpha\beta} = \mathrm{tr}(\mathbf{A}E_{\alpha\beta}\mathbf{C}X^k)\,, \quad k \in \mathbb{N},\ \alpha,\beta = 1,\dots,d\,, \tag{4.5}$$

where the matrix $E_{\alpha\beta}$ is the elementary $d\times d$ matrix with entry 1 at (α,β) and zero otherwise. One last benefit of the introduction of the matrices $\mathbf{A}$ and $\mathbf{C}$ is that it will be possible to understand the local Poisson structure by knowing the Poisson bracket between the functions $(f_k, g_k^{\alpha\beta})$, as we will see in Section 4.2 for the case $d=1$, and in Section 4.3 for the case $d \geq 2$.

[11]This is a consequence of the local diffeomorphism that exist in both cases $d=1$ and $d\geq 2$, see below.

Lemma 4.1.2 *For any $\alpha,\beta=1,\dots,d$ and $k,l\geq 1$,*

$$\{f_k,f_l\}_{\mathrm{P}}=0\,,\quad \{f_k,g_l^{\alpha\beta}\}_{\mathrm{P}}=\,k\,g_{k+l}^{\alpha\beta}\,, \tag{4.6a}$$

$$\begin{aligned}
\{g_k^{\gamma\epsilon},g_l^{\alpha\beta}\}_{\mathrm{P}}=&\frac{1}{2}\left(\sum_{r=1}^{k}-\sum_{r=1}^{l}\right)\mathrm{tr}(\mathbf{A}E_{\alpha\beta}\mathbf{C}X^r\mathbf{A}E_{\gamma\epsilon}\mathbf{C}X^{k+l-r})\\
&+\frac{1}{2}\left(\sum_{r=1}^{k}-\sum_{r=1}^{l}\right)\mathrm{tr}(\mathbf{A}E_{\alpha\beta}\mathbf{C}X^{k+l-r}\mathbf{A}E_{\gamma\epsilon}\mathbf{C}X^r)\\
&+\frac{1}{2}o(\alpha,\gamma)\left(\mathrm{tr}(\mathbf{A}E_{\gamma\epsilon}\mathbf{C}X^k\mathbf{A}E_{\alpha\beta}\mathbf{C}X^l)+\mathrm{tr}(\mathbf{A}E_{\alpha\epsilon}\mathbf{C}X^k\mathbf{A}E_{\gamma\beta}\mathbf{C}X^l)\right)\\
&+\frac{1}{2}o(\epsilon,\beta)\left(\mathrm{tr}(\mathbf{A}E_{\alpha\beta}\mathbf{C}X^k\mathbf{A}E_{\gamma\epsilon}\mathbf{C}X^l)-\mathrm{tr}(\mathbf{A}E_{\alpha\epsilon}\mathbf{C}X^k\mathbf{A}E_{\gamma\beta}\mathbf{C}X^l)\right)\\
&+\frac{1}{2}[o(\epsilon,\alpha)+\delta_{\alpha\epsilon}]\ \mathrm{tr}(\mathbf{A}E_{\alpha\epsilon}\mathbf{C}X^k\mathbf{A}E_{\gamma\beta}\mathbf{C}X^l)\\
&-\frac{1}{2}[o(\beta,\gamma)+\delta_{\beta\gamma}]\ \mathrm{tr}(\mathbf{A}E_{\alpha\epsilon}\mathbf{C}X^k\mathbf{A}E_{\gamma\beta}\mathbf{C}X^l)\\
&+\delta_{\alpha\epsilon}\,\mathrm{tr}\left(\left[Z+\sum_{\lambda=1}^{\epsilon-1}\mathbf{A}E_{\lambda\lambda}\mathbf{C}\right]X^k\mathbf{A}E_{\gamma\beta}\mathbf{C}X^l\right)\\
&-\delta_{\beta\gamma}\,\mathrm{tr}\left(\left[Z+\sum_{\mu=1}^{\beta-1}\mathbf{A}E_{\mu\mu}\mathbf{C}\right]X^l\mathbf{A}E_{\alpha\epsilon}\mathbf{C}X^k\right)\,,
\end{aligned} \tag{4.6b}$$

where $o(-,-)$ is the ordering function on d elements defined in Section 1.5.

Proof. A first proof consists in applying $\mathcal{X}$ to Lemma 3.1.5 with $\mathcal{X}(x)=X$, $\mathcal{X}(z)=Z$, $\mathcal{X}(a'_\alpha c'_\beta)=\mathbf{A}E_{\alpha\beta}\mathbf{C}$. Alternatively, we also show[12] how to derive the first equation from Proposition 2.3.26. We lift f_k to

$$\hat{f}_k=\sum X_{i_1i_2}X_{i_2i_3}\dots X_{i_ki_1}\in\mathcal{O}(\mathrm{Rep}(A,\widetilde{\alpha}))\,,$$

where we take the convention that an unlabelled sum means that we sum over repeated indices from 1 to n. We do the same for f_l. Meanwhile, we have by (2.34a) that

$$\{X_{ij},X_{kl}\}_{\mathrm{P}}=\frac{1}{2}X_{kj}^2\delta_{il}-\frac{1}{2}\delta_{kj}X_{il}\,.$$

[12]We advise the reader to compare this with the computations that can be made with the double bracket to derive Lemma 3.1.5. It becomes transparent that computations with the double bracket allow to do computations with $\{-,-\}_{\mathrm{P}}$ in an subscript-free way.

Hence, we compute

$$\begin{aligned}\{\hat{f}_k,\hat{f}_l\}_{\mathrm{P}} =& \sum\sum_{\sigma=1}^{k}\sum_{\tau=1}^{l} X_{i_1 i_\sigma}^{\sigma-1}X_{j_1 j_\tau}^{\tau-1}\{X_{i_\sigma i_{\sigma+1}},X_{j_\tau j_{\tau+1}}\}_{\mathrm{P}} X_{j_{\tau+1}j_1}^{l-\tau}X_{i_{\sigma+1}i_1}^{k-\sigma}\\ =&\frac{1}{2}\sum\sum_{\sigma=1}^{k}\sum_{\tau=1}^{l} X_{j_1 j_\tau}^{\tau-1}X^2_{j_\tau i_{\sigma+1}}X_{i_{\sigma+1}i_1}^{k-\sigma}\ X_{i_1 i_\sigma}^{\sigma-1}\delta_{i_\sigma j_{\tau+1}}X_{j_{\tau+1}j_1}^{l-\tau}\\ &-\frac{1}{2}\sum\sum_{\sigma=1}^{k}\sum_{\tau=1}^{l} X_{j_1 j_\tau}^{\tau-1}\delta_{j_\tau i_{\sigma+1}}X_{i_{\sigma+1}i_1}^{k-\sigma}\ X_{i_1 i_\sigma}^{\sigma-1}X^2_{i_\sigma j_{\tau+1}}X_{j_{\tau+1}j_1}^{l-\tau}\,,\end{aligned}$$

which clearly vanishes. By definition of the Poisson bracket obtained after reduction, this implies that $\{f_k,f_l\}_{\mathrm{P}}=0$. □

If we want to replace X by Z in the previous lemma, it suffices to apply $\mathcal{X}$ to Lemma 3.1.6.

4.1.2 Towards integrability and dynamics

We continue with the notations of § 4.1.1 assuming $\widetilde{\alpha}=(1,n)$. Note that the discussion that follows does not depend on the dimension vector and it would hold for any $\widetilde{\alpha}\in\mathbb{N}^\times\times\mathbb{N}^\times$, except for item 3 in Proposition 4.1.3.

Let $u\in\{x,y,z,e_0+xy\}$, and denote by $U=\mathcal{X}(u)$ the matrix representing u. Introduce the algebra $\mathcal{O}_U$ generated by the functions $\operatorname{tr}U^k$ and $\operatorname{tr}W_\alpha V_\beta U^k$ for any $k\in\mathbb{N}$ and $1\leq\alpha,\beta\leq d$ (which we also see as a sheaf).

Proposition 4.1.3 *The following holds in $\mathcal{C}_{n,q,d}$ (or $\mathcal{C}^\circ_{n,q,d}$ if $U=Z$) :*

1. *The symmetric functions $\{\operatorname{tr}U^k\mid k\in\mathbb{N}\}$ of U are pairwise in involution;*
2. *For any $k,l\in\mathbb{N}$ and $1\leq\alpha,\beta\leq d$, $\{\operatorname{tr}U^k,\operatorname{tr}W_\alpha V_\beta U^l\}_{\mathrm{P}}=0$;*
3. *The algebra $\mathcal{O}_U$ is a Poisson algebra under $\{-,-\}_{\mathrm{P}}$;*
4. *For fixed $\alpha,\beta\in\{1,\dots,d\}$, the subalgebra of $\mathcal{O}_U$ generated by the functions $(\operatorname{tr}U^k)_k$ and $(\operatorname{tr}W_\alpha V_\beta U^k)_k$ is an abelian Poisson subalgebra.*

Proof. First two items follow from Corollary 3.1.8. For the third one, remark that by (2.36) and (3.15b) in Lemma 3.1.7 the Poisson bracket $\{\operatorname{tr}W_\gamma V_\epsilon U^k,\operatorname{tr}W_\alpha V_\beta U^l\}_{\mathrm{P}}$ can be written with functions of the form $\operatorname{tr} W_{\alpha_1} V_{\beta_1} U^{s_1} W_{\alpha_2} V_{\beta_2} U^{s_2}$. But using that $V_{\beta_1} U^{s_1} W_{\alpha_2}$ is a scalar, such a function can be written as $\operatorname{tr}(V_{\beta_1} U^{s_1} W_{\alpha_2}) \operatorname{tr}(V_{\beta_2} U^{s_2} W_{\alpha_1}) \in \mathcal{O}_U$. Hence, the algebra is Poisson. For the fourth item, remark that in (3.15b) we get 0 for $\gamma = \alpha$ and $\epsilon = \beta$. □

Since we assume that Z is invertible when we work with $U = Z$, we can in fact take any $k \in \mathbb{Z}$, and leave this easy verification to the reader. Working with the spin matrices in that case, we get a similar result from Lemma 3.1.6.

Proposition 4.1.4 *The symmetric functions $\{\operatorname{tr} Z^k \mid k \in \mathbb{Z}\}$ of Z are pairwise in involution. Moreover, they Poisson commute with any function $\operatorname{tr} \mathbf{A} E_{\alpha\beta} \mathbf{C} Z^l$, for $l \in \mathbb{Z}$ and $1 \leq \alpha, \beta \leq d$.*

Clearly, both results suggest to look at degenerate integrability. However, we still require some assumptions on the functional independence of the chosen functions to apply Corollary 2.3.39. This will be detailed in Section 4.3.

We write $\Theta^{(0)} = \mathcal{X}(\phi)$ for the matrix that represents the moment map supported at the vertex 0. In other words, $\Theta^{(0)} = (\operatorname{Id}_n + XY)(\operatorname{Id}_n + YX)^{-1}$ on $\mathcal{C}_{n,q,d}$, or $\Theta^{(0)} = XZX^{-1}Z^{-1}$ on $\mathcal{C}^{\circ}_{n,q,d}$. Furthermore, let us restrict our attention to the open subspace $\{\det U \neq 0\}$ of $\mathcal{C}_{n,q,d}$ where U is invertible. Corollary 3.1.10 implies the following.

Proposition 4.1.5 *For any $K \in \mathbb{N}^{\times}$, expand $h^u_K = \frac{1}{K} \operatorname{tr}[U(\operatorname{Id}_n + \eta(\Theta^{(0)})^{-\epsilon(u)})]^K$ in terms of η as $h^u_K = \sum_{k=0}^K h^u_{K,k} \eta^k$. Then all the functions $\{h^u_{K,k} \mid 0 \leq k \leq K\}$ are in involution.*

This time, the result suggests that we can form an integrable system for each possible U. Again, we can not directly apply Corollary 2.3.33 to prove that claim, and we postpone this discussion at the moment. Nevertheless, we can try to obtain one last result before studying the local picture of $\mathcal{C}_{n,q,d}$ to establish integrability : it could be possible that we can explicitly obtain the flows defined by (a lift of) one of the functions $h^u_{K,k}$ in $\operatorname{Rep}(\Lambda^{\widetilde{q}}, \widetilde{\alpha})$, see the comments after Corollary 2.3.39. If we begin with $U = Z$, let $Z_\eta = Z(\operatorname{Id}_n + \eta\Theta^{(0)})$. We get from Lemma 3.1.12 (assuming that we have localised at X)

$$\{h^z_K, X\}_{\mathrm{P}} = -\eta \Phi Z_\eta^{K-1} Z X - X Z_\eta^{K-1} Z\,, \quad \{h^z_K, Z\}_{\mathrm{P}} = -Z Z_\eta^{K-1} Z + Z_\eta^{k-1} Z^2\,,$$

while the brackets with V_β or W_β vanish. However, it does not seem possible to integrate most flows because, after a tedious computations, we can see that $\{h^z_K, Z_\eta\}_{\mathrm{P}} \neq 0$. Thus the matrix Z_η

is not a constant of motion, although all its symmetric functions are by Proposition 4.1.5. If we only look at order 0 in η instead, we get that for the function $h^z_{K,0} = \frac{1}{K} \operatorname{tr} Z^K$ the flows defined by $d/dt_K = \{h^z_{K,0}, -\}_{\mathrm{P}}$ have to satisfy the defining ODEs

$$\frac{dX}{dt_K} = -XZ^K\,, \quad \frac{dZ}{dt_K} = 0\,, \quad \frac{dV_\beta}{dt_K} = 0\,, \quad \frac{dW_\beta}{dt_K} = 0\,.$$

This has the following consequence.

Proposition 4.1.6 *Given the initial condition $(X(0), Z(0), V_\beta(0), W_\beta(0))$, the flow at time t_K defined by the Hamiltonian $\frac{1}{K} \operatorname{tr} Z^K$ for $K \in \mathbb{N}^\times$ is given by*

$$X(t_K) = X(0)\, \exp(-t_K Z(0)^K)\,, \quad Z(t_K) = Z(0)\,, \quad V_\beta(t_K) = V_\beta(0)\,, \; W_\beta(t_K) = W_\beta(0)\,.$$

In particular, the flows descend to complete flows in $\mathcal{C}^\circ_{n,q,d}$.

Recalling the localisation defined in § 4.1.1, we can reintroduce the spin matrices $\mathbf{A}$ and $\mathbf{C}$ since $X(t_K)$ stays invertible, so that we find $\mathbf{A}(t_K) = \mathbf{A}(0)$ and $\mathbf{C}(t_K) = \mathbf{C}(0)$.

For the next case $U = Y$, we write $Y_\eta = Y(\operatorname{Id}_n + \eta\Theta^{(0)})$. We get from Lemma 3.1.13 that (assuming we look at the subspace $\{\det Y \neq 0\}$ in $\operatorname{Rep}(\Lambda^{\widetilde{q}}, \widetilde{\alpha})$)

$$\begin{aligned}
\{h^y_K, X\}_{\mathrm{P}} &= -\, Y_\eta^{K-1} - XY_\eta^{K-1}Y - \eta\Phi Y_\eta^{K-1}(1 + YX)\,,\\
\{h^y_K, Y\}_{\mathrm{P}} &= -\, YY_\eta^{K-1}Y + Y_\eta^{K-1}Y^2\,,
\end{aligned}$$

and the quasi-Poisson brackets with V_β or W_β vanish. Again, we can not hope to integrate all flows explicitly, but considering order 0 in η yields for $h^y_{K,0} = \frac{1}{K} \operatorname{tr} Y^K$ by writing $d/d\tau_K = \{h^y_{K,0}, -\}_{\mathrm{P}}$ that

$$\frac{dX}{d\tau_K} = -Y^{K-1} - XY^K\,, \quad \frac{dY}{d\tau_K} = 0\,, \quad \frac{dV_\beta}{d\tau_K} = 0\,, \quad \frac{dW_\beta}{d\tau_K} = 0\,.$$

It is important to remark that, in the proof of Lemma 3.1.13, the invertibility condition on y is needed to get rid of terms containing a factor η. Hence, these equations are well-defined in $\operatorname{Rep}(\Lambda^{\widetilde{q}}, \widetilde{\alpha})$, also when Y is not invertible.

Proposition 4.1.7 *Given the initial condition $(X(0), Y(0), V_\beta(0), W_\beta(0))$ the flow at time τ_K defined by the Hamiltonian $\frac{1}{K} \operatorname{tr} Y^K$ for $K \in \mathbb{N}^\times$ is given by*

$$\begin{aligned}
X(\tau_K) =& X(0)\, \exp(-\tau_K Y(0)^K) + Y(0)^{-1}[\exp(-\tau_K Y(0)^K) - \operatorname{Id}_n]\,,\\
Y(\tau_K) =& Y(0)\,, \quad W_\beta(\tau_K) = W_\beta(0)\,, \quad V_\beta(\tau_K) = V_\beta(0)\,.
\end{aligned}$$

In particular, the flows descend to complete flows in $\mathcal{C}_{n,q,d}$.

Note that the expression for $X(\tau_k)$ is analytic in $Y(0)$ so does not require its invertibility as we explained above. Both propositions appear in [41] in the case $d=1$, and in [42] for $d \geq 1$.

Now for $U = X$, we write $X_\eta = X(\mathrm{Id}_n + \eta(\Theta^{(0)})^{-1})$ and we work in the subspace $\{\det X \neq 0\}$ in $\mathrm{Rep}(\Lambda^{\widetilde{q}}, \widetilde{\alpha})$. We get from Lemma 3.1.14 that

$$\{h_K^x, X\}_{\mathrm{P}} = X X_\eta^{K-1} X - X_\eta^{K-1} X^2 \,, \quad \{h_K^x, Z\}_{\mathrm{P}} = Z X_\eta^{K-1} X + \eta \phi^{-1} X_\eta^{K-1} X Z \,,$$

and the quasi-Poisson brackets with V_β or W_β vanish. As before, we look at order 0. (We could get rid of the assumption that X is invertible if we look at the dynamics of the matrix Y instead of Z.) For $h_{K,0}^x = \frac{1}{K} \operatorname{tr} X^K$, by writing $d/d\hat{t}_K = \{h_{K,0}^x, -\}_{\mathrm{P}}$, we obtain that

$$\frac{dX}{d\hat{t}_K} = 0\,, \quad \frac{dZ}{d\hat{t}_K} = Z X^K\,, \quad \frac{dV_\beta}{d\hat{t}_K} = 0\,, \quad \frac{dW_\beta}{d\hat{t}_K} = 0\,.$$

This yields the next result.

Proposition 4.1.8 *Given the initial condition* $(X(0), Z(0), V_\beta(0), W_\beta(0))$*, the flow at time* $\hat{t}_K$ *defined by the Hamiltonian* $\frac{1}{K} \operatorname{tr} X^K$ *for* $K \in \mathbb{N}^\times$ *is given by*

$$X(\hat{t}_K) = X(0)\,, \quad Z(\hat{t}_K) = Z(0) \exp(\hat{t}_K X(0)^K)\,, \quad V_\beta(\hat{t}_K) = V_\beta(0)\,, \; W_\beta(\hat{t}_K) = W_\beta(0)\,.$$

In particular, the flows descend to complete flows in $\mathcal{C}_{n,q,d}^\circ$*.*

Finally, for $U = \mathrm{Id}_n + XY$, we denote $\mathrm{Id}_n + XY$ by T, let $T_\eta = T(\mathrm{Id}_n + \eta(\Theta^{(0)})^{-1})$, and write $h_K^T = h_K^{e_0 + xy}$ to ease notations. We can work in $\mathrm{Rep}(\Lambda^{\widetilde{q}}, \widetilde{\alpha})$ since T is already invertible by assumption. We get from Lemma 3.1.15 that

$$\{h_K^T, X\}_{\mathrm{P}} = - T_\eta^{K-1} T X - \eta X (\Theta^{(0)})^{-1} T_\eta^{K-1} T\,, \quad \{h_K^T, T\}_{\mathrm{P}} = T T_\eta^{K-1} T - T_\eta^{K-1} T^2\,,$$

and the quasi-Poisson brackets with V_β or W_β vanish as usual. Reproducing the usual scheme, we get our last statement.

Proposition 4.1.9 *Given the initial condition* $(X(0), T(0), V_\beta(0), W_\beta(0))$*, the flow at time* $\tilde{t}_K$ *defined by the Hamiltonian* $\frac{1}{K} \operatorname{tr} T^K$ *for* $K \in \mathbb{N}^\times$ *satisfies*

$$X(\tilde{t}_K) = \exp(-\tilde{t}_K T^K) X(0)\,, \quad T(\tilde{t}_K) = T(0)\,, \quad V_\beta(\tilde{t}_K) = V_\beta(0)\,, \; W_\beta(\tilde{t}_K) = W_\beta(0)\,.$$

In particular, the flows descend to complete flows in $\mathcal{C}_{n,q,d}$*.*

The completeness of the flows still requires to show that $Y(\tilde{t}_K)$ is well-defined, but we omit those computations. Rather, remark that when we assume that $X(0)$ is invertible, the proposition determines the solution $Y(\tilde{t}_K) = X(\tilde{t}_K)^{-1}[T(\tilde{t}_K) - \mathrm{Id}_n]$ for all time $\tilde{t}_K$. In that case, we have completeness in $\mathcal{C}^{\circ}_{n,q,d}$.

For the previous results, we used the matrix representing the moment map $\phi = \Phi_0^{(0)} \in A_0$ relative to the subquiver $\bar{Q}_0$ consisting only of the arrows $x, y : 0 \to 0$. Recalling the chain of quasi-Hamiltonian algebra (3.16), we consider $\Theta^{(\alpha)} = \mathcal{X}(\Phi_0^{(\alpha)})$ for any $0 \leq \alpha \leq d$. Using (4.1a), we see that in $\mathrm{Rep}(\Lambda^{\widetilde{q}}, \widetilde{\alpha})$

$$\Theta^{(\alpha)} = (\mathrm{Id}_n + XY)(\mathrm{Id}_n + YX)^{-1} \overrightarrow{\prod_{1\leq\beta\leq\alpha}} (\mathrm{Id}_n + W_\beta V_\beta)^{-1} = q \overleftarrow{\prod_{\alpha+1\leq\beta\leq d}} (\mathrm{Id}_n + W_\beta V_\beta)\,, \quad (4.7)$$

where we take empty products to be Id_n. Finally, set $U_{(\alpha)} = \Theta^{(\alpha)} U$ with $U \in \{Y, Z\}$. Proposition 3.1.11 is easily seen to imply the following result.

Proposition 4.1.10 *The functions* $\{\mathrm{tr}\, U_{(\alpha)}^K \mid K \in \mathbb{N},\, 0 \leq \alpha \leq d\}$ *are in involution.*

We would also like to form an integrable system from these functions, which we will discuss later. What we can already do is derive the corresponding flows, in the exact same way as we did before. We leave the geometric translation of Lemmae 3.1.16 and 3.1.17 to the reader, and directly state the results.

Proposition 4.1.11 *Given the initial condition* $(X(0), Z(0), V_\beta(0), W_\beta(0))$*, the flow at time* t_K *defined by the Hamiltonian* $\frac{1}{K}\,\mathrm{tr}\, Z_{(\alpha)}^K$*, for* $K \in \mathbb{N}^\times$ *and* $0 \leq \alpha \leq d$*, is given by*

$$X(t_K) = \exp(-t_K Z_{(\alpha)}(0)^K) X(0)\,, \quad Z(t_K) = Z(0)\,,$$
$$V_\beta(t_K) = V_\beta(0) e^{t_K Z(0) Z_{(\alpha)}(0)^{K-1} \Theta^{(\alpha)}(0)},\; W_\beta(t_K) = e^{-t_K Z(0) Z_{(\alpha)}(0)^{K-1} \Theta^{(\alpha)}(0)} W_\beta(0),\; \beta \leq \alpha,$$
$$V_\beta(t_K) = V_\beta(0)\,,\; W_\beta(t_K) = W_\beta(0)\,,\; \beta > \alpha\,.$$

In particular, the flows descend to complete flows in $\mathcal{C}^{\circ}_{n,q,d}$.

Proposition 4.1.12 *Given the initial condition* $(X(0), Y(0), V_\beta(0), W_\beta(0))$*, the flow at time* τ_K *defined by the Hamiltonian* $\frac{1}{K}\,\mathrm{tr}\, Y_{(\alpha)}^K$*, for* $K \in \mathbb{N}^\times$ *and* $0 \leq \alpha \leq d$*, is given by*

$$X(\tau_K) = \exp(-\tau_K Y_{(\alpha)}(0)^K) X(0) + Y_{(\alpha)}(0)^{-1}[\exp(-\tau_K Y_{(\alpha)}(0)^K) - \mathrm{Id}_n]\Theta^{(\alpha)}\,,$$
$$V_\beta(\tau_K) = V_\beta(0) e^{\tau_K Y(0) Y_{(\alpha)}(0)^{K-1} \Theta^{(\alpha)}(0)},\; W_\beta(\tau_K) = e^{-\tau_K Y(0) Y_{(\alpha)}(0)^{K-1} \Theta^{(\alpha)}(0)} W_\beta(0),\; \beta \leq \alpha,$$
$$Y(\tau_K) = Y(0)\,, \quad V_\beta(\tau_K) = V_\beta(0)\,,\; W_\beta(\tau_K) = W_\beta(0)\,,\; \beta > \alpha\,.$$

In particular, the flows descend to complete flows in $\mathcal{C}_{n,q,d}$.

Let us make some final comments on these expressions for Z, the case for Y being similar. Since $\Theta^{(0)} = XZX^{-1}Z^{-1}$, we have that $\operatorname{tr} Z_{(0)}^K = \operatorname{tr} Z^K$. Because there are no $1 \leq \beta \leq d$ with $\beta \leq 0$, the flows for Z, V_β, W_β given in Propositions 4.1.6 and 4.1.11 are clearly the same. Furthermore, using Proposition 4.1.11 we get that

$$X(t_K) = \exp(-t_K X(0)Z(0)^K X(0)^{-1})X(0) = X(0)\exp(-t_K Z(0)^K)\,,$$

as in Proposition 4.1.6. Similarly we have from the moment map condition $\Theta^{(d)} = q\,\mathrm{Id}_n$ in $\mathcal{C}^\circ_{n,q,d}$ that $\operatorname{tr} Z_{(d)}^K = q^K \operatorname{tr} Z^K$, and we want to compare the flows. Using Proposition 4.1.11 we get for the flow of $\operatorname{tr} Z_{(d)}^K$ that

$$\begin{aligned} X(t_K) =&\ \exp(-q^K t_K Z(0)^K)X(0)\,, \quad Z(t_K) = Z(0)\,, \\ V_\beta(t_K) = V_\beta(0)&e^{q^K t_K Z(0)^K},\ W_\beta(t_K) = e^{-q^K t_K Z(0)^K} W_\beta(0),\ 1 \leq \beta \leq d. \end{aligned}$$

Changing representative after acting by $e^{q^K t_K Z(0)^K} \in \mathrm{GL}_n(\mathbb{C})$ as in (4.2), we precisely get the flow for $q^K \operatorname{tr} Z^K$ obtained from Proposition 4.1.6 with t_K rescaled to $q^K t_K$.

4.2 Simple framing

We first look at the case $d = 1$ where q is not a root of unity, and we write V, W instead of V_1, W_1. We denote $\mathcal{C}_{n,q,d}$ simply by $\mathcal{C}_{n,q}$, and we write $\mathcal{C}^\circ_{n,q}$ for the subspace where X is invertible. In particular, we get the following interpretation for these smooth spaces

$$\begin{aligned} \mathcal{C}_{n,q} =& \{X, Y \in \mathfrak{gl}_n(\mathbb{C}) \,|\, \operatorname{rank}(XY - qYX + (1-q)\,\mathrm{Id}_n) = 1, \det(\mathrm{Id}_n + YX) \neq 0\}/\mathrm{GL}_n(\mathbb{C}), \\ \mathcal{C}^\circ_{n,q} =& \{X, Z \in \mathrm{GL}_n(\mathbb{C}) \,|\, \operatorname{rank}(XZX^{-1}Z^{-1} - q\,\mathrm{Id}_n) = 1\}/\mathrm{GL}_n(\mathbb{C})\,. \end{aligned} \tag{4.8}$$

Indeed, we can discard the case of rank zero since then the condition is empty, e.g. this implies $XZX^{-1}Z^{-1} = q\,\mathrm{Id}_n$ which is impossible by taking determinant. Hence, we can remark that $\mathcal{C}_{n,q}$ is similar to the space of matrices satisfying [91, Proposition 5.2] which are related to the qKP hierarchy. Also, $\mathcal{C}^\circ_{n,q}$ corresponds to the space studied by Fock and Rosly in [80, Appendix], see also [18, 126], which is associated to the Ruijsenaars-Schneider system. We will make these relations precise in the next subsections.

4.2.1 Local Poisson structure

First parametrisation

Let $\mathfrak{h} = \mathbb{C}^n$ with coordinates $x_1, \dots, x_n$ and define $\mathfrak{h}_{\text{reg}}$ to be the open subspace such that

$$\mathfrak{h}_{\text{reg}} = \{x = (x_1, \dots, x_n) \in \mathfrak{h} \mid x_i \neq 0\,,\ x_i \neq x_j\,,\ x_i \neq q x_j \text{ for all } i \neq j\}\,. \tag{4.9}$$

Consider $\mathfrak{h}_I \subset \mathfrak{h}_{\text{reg}} \times \mathbb{C}^n$ such that $1 + p_i x_i \neq 0$ for each i, where we take coordinates $p_1, \dots, p_n$ on $\mathbb{C}^n$. Then, defining the matrices

$$X = \operatorname{diag}(x_1, \dots, x_n)\,, \quad Y = (Y_{ij})\,, \text{ for } Y_{ij} = \delta_{ij} p_j + \delta_{(i \neq j)} \frac{(1-q)(1+p_j x_j)}{x_i - q x_j}\,, \tag{4.10}$$

we see that the matrix $XY - qYX + (1-q)\operatorname{Id}_n$ has rank one and equals $W\tilde{V}$ for $W = (1, \dots, 1)^\top$ and $\tilde{V} = (\tilde{V}_j)$, $\tilde{V}_j = (1 + p_j x_j)$. Moreover, since we can write

$$\operatorname{Id}_n + YX = (1-q)C(X)XP(X)\,, \quad \text{where } C(X)_{ij} = \frac{1}{x_i - q x_j}\,,\ P(X)_{ij} = \delta_{ij}(1 + p_j x_j)\,,$$

we can see that $\operatorname{Id}_n + YX$ is invertible by assumptions on (x_i, p_i). Indeed, X and $P(X)$ are easily seen to be invertible, while to compute $\det C(X)$ we use Cauchy's determinant formula : for $M \in \mathfrak{gl}_n(\mathbb{C})$ with $M_{ij} = (m_i - \mu_j)^{-1}$ we have

$$\det(M) = \frac{\prod_{i<j}(m_i - m_j)(\mu_j - \mu_i)}{\prod_{i,j}(m_i - \mu_j)}\,. \tag{4.11}$$

Hence, for any $(x_i, p_i) \in \mathfrak{h}_I$ we can construct elements $(X, Y) \in \mathcal{C}_{n,q}$ by (4.8), and we can recover that $W = (1, \dots, 1)^\top$ as above while $V = \tilde{V}(\operatorname{Id}_n + YX)^{-1}$. Now, remark that simultaneous permutations in $\mathfrak{h}_I$ as $\tau \cdot (x_i, p_i) = (x_{\tau(i)}, p_{\tau(i)})$ for $\tau \in S_n$ are equivalent to the action on (X, Z) by the corresponding permutation matrix. We obtain in that way a map $\xi : \mathfrak{h}_I / S_n \to \mathcal{C}^\circ_{n,q}$, which is easily seen to be injective. To get surjectivity, we remark that ξ surjects onto the subspace $\mathcal{C}'_{n,q} \subset \mathcal{C}^\circ_{n,q}$ where X is invertible and diagonalisable, with its eigenvalues that define a point in $\mathfrak{h}_{\text{reg}}$.

Lemma 4.2.1 *There is a diffeomorphism $\xi : \mathfrak{h}_I / S_n \to \mathcal{C}'_{n,q}$ such that $\xi(x_i, p_i) = (X, Y, V, W)$ is determined by (X, Y) defined as (4.10).*

In particular, local coordinates are given by diagonal entries of X and Y in the particular form (4.10), as an obvious generalisation of the Calogero-Moser case from Example 2.1.4.

Since X is invertible, we can always define $Z = Y + X^{-1}$ in that case, and $Z = (Z_{ij})$ has the same off-diagonal entries as Y while $Z_{ii} = p_i + x_i^{-1}$. Then, following § 4.1.1, we remark that the functions $f_k = \mathrm{tr}(X^k)$ and $g_k = g_k^{11} = \frac{(1-q)}{q}\,\mathrm{tr}(ZX^k)$ defined in (4.5) are such that

$$\xi^* f_k = \sum_i x_i^k\,, \quad \xi^* g_k = (q^{-1}-1)\sum_i (1+p_i x_i)x_i^{k-1}\,,$$

which are easily seen to define a local coordinate system on $\mathcal{C}'_{n,q}$.

Proposition 4.2.2 *The diffeomorphism $\xi : \mathfrak{h}_I/S_n \to \mathcal{C}'_{n,q}$ is a Poisson morphism for the Poisson bracket $\{-,-\}$ on $\mathfrak{h}_I/S_n$ defined by*

$$\{x_i, x_j\} = 0\,, \tag{4.12a}$$

$$\{x_i, p_j\} = \delta_{ij}(1 + x_i p_j)\,, \tag{4.12b}$$

$$\{p_i, p_j\} = \frac{(1-q)^2(x_i+x_j)(p_i x_i+1)(p_j x_j+1)}{(x_i-x_j)(x_i-qx_j)(x_j-qx_i)}\,, \tag{4.12c}$$

and $\{-,-\}_{\mathrm{P}}$ on $\mathcal{C}'_{n,q}$.

Remark 4.2.3 *Note that* a priori *we do not know that $\{-,-\}$ is a Poisson bracket on $\mathfrak{h}_I/S_n$, but it follows from Section B in the Appendix. We use this argument without further mention throughout the thesis.*

Proof. Since the functions (f_k, g_k) form a local coordinates system, it suffices to prove that

$$\xi^*\{f_k, f_l\}_{\mathrm{P}} = \{\xi^* f_k, \xi^* f_l\}\,, \quad \xi^*\{f_k, g_l\}_{\mathrm{P}} = \{\xi^* f_k, \xi^* g_l\}\,, \quad \xi^*\{g_k, g_l\}_{\mathrm{P}} = \{\xi^* g_k, \xi^* g_l\}\,,$$

see Remark B.3. To compute the brackets on the left-hand sides, we need Lemma 4.1.2. The first identity is then trivial as both sides are zero. For the second,

$$\{\xi^* f_k, \xi^* g_l\} = (q^{-1}-1)\sum_{i,j}\{x_i^k, p_j x_j^l + x_j^{l-1}\} = k\,(q^{-1}-1)\sum_i x_i^{k+l-1}(1+x_i p_i)\,,$$

and we easily get that this coincides with $\xi^*\{f_k, g_l\}_{\mathrm{P}} = k\xi^* g_{k+l}$. For the last identity, we assume for simplicity that $k > l$ and we begin by computing $\{\xi^* g_k, \xi^* g_l\}$ which is up to a factor $(q^{-1}-1)^2$

$$\begin{aligned}
&\sum_{ij}\{p_i x_i^k + x_i^{k-1}, p_j x_j^l + x_j^{l-1}\} \\
=&\sum_{i<j}(1-q)^2\frac{(1+p_i x_i)(1+p_j x_j)}{(x_i-qx_j)(x_j-qx_i)}\frac{x_i+x_j}{x_i-x_j}(x_i^k x_j^l - x_j^k x_i^l) \\
&+ (k-l)\sum_i [x_i^{k+l-1}p_i(1+p_i x_i) + x_i^{k+l-2}(1+p_i x_i)]\,.
\end{aligned}$$

To reduce this expression, we need the following result.

Lemma 4.2.4 *Assume that $a > b$ are positive integers. Then*

$$(x_i + x_j)\,(x_i^a x_j^b - x_i^b x_j^a) = (x_i - x_j)\sum_{t=1}^{a-b}(x_i^{a-t}x_j^{b+t} + x_i^{b+t}x_j^{a-t}).$$

Hence, we can write that $\{\xi^* g_k, \xi^* g_l\}$ equals

$$(q^{-1}-1)^2\sum_{t=1}^{k-l}\left((1-q)^2\sum_{i\neq j}\frac{(1+p_ix_i)x_i^{k-t}(1+p_jx_j)x_j^{l+t}}{(x_i-qx_j)(x_j-qx_i)} + \sum_i x_i^{k+l-2}(1+p_ix_i)^2\right),$$

which is easily seen to be equal to $(q^{-1}-1)^2\sum_{t=1}^{k-l}\xi^*\operatorname{tr}(ZX^{k-t}ZX^{l+t})$, after noticing that $Z_{ij} = (1-q)\frac{1+p_jx_j}{x_i-qx_j}$. Finally, recalling our assumption $k > l$, we rewrite $\{g_k, g_l\}_{\mathrm{P}} = \{g_k^{11}, g_l^{11}\}_{\mathrm{P}}$ in Lemma 4.1.2 as

$$\{g_k^{11}, g_l^{11}\}_{\mathrm{P}} = \sum_{r=l+1}^{k}\operatorname{tr}(\mathbf{AC}X^r\mathbf{AC}X^{k+l-r}) + \operatorname{tr}(ZX^k\mathbf{AC}X^l) - \operatorname{tr}(ZX^l\mathbf{AC}X^k),$$

since the only nonzero terms come from the two first sums and the last two terms. But in our case $d = 1$ we have by (4.4) that $\mathbf{AC} = q^{-1}XZX^{-1} - Z$, so we can write $\{g_k, g_l\}_{\mathrm{P}}$ as

$$\begin{aligned}
&\sum_{t=1}^{k-l}\Big[(1+q^{-2})\operatorname{tr}(ZX^{l+t}ZX^{k-t}) - \frac{1}{q}\operatorname{tr}(ZX^{l+t+1}ZX^{k-t-1}) - \frac{1}{q}\operatorname{tr}(ZX^{l+t-1}ZX^{k-t+1})\Big]\\
&+\frac{1}{q}\operatorname{tr}(ZX^{k+1}ZX^{l-1}) - \operatorname{tr}(ZX^kZX^l) - \frac{1}{q}\operatorname{tr}(ZX^{l+1}ZX^{k-1}) + \operatorname{tr}(ZX^lZX^k)\\
=&\sum_{t=1}^{k-l}(1-q^{-1})^2\operatorname{tr}(ZX^{l+t}ZX^{k-t}),
\end{aligned}$$

and we have indeed $\{\xi^* g_k, \xi^* g_l\} = \xi^*\{g_k, g_l\}_{\mathrm{P}}$. □

It is natural to try to find coordinates such that the bracket defined by (4.12a)–(4.12c) is in canonical form on these new coordinates. To do so, set $x_i' = -(1+p_ix_i)$ and remark that

$$(1-q)^{-1}Y_{ij} = \delta_{ij}\frac{1+x_i'}{(q-1)x_i} - \delta_{(i\neq j)}\frac{x_j'}{x_i-qx_j}$$

coincides with the matrix defined by [91, Eqs (6.4)–(6.5)] after conjugation by $\operatorname{diag}(x_1',\dots,x_n')$. Under the transformation [91, (6.3)] which for us amounts to introduce $y_1,\dots,y_n$ such that

$$1+p_ix_i \;=\; A_i\,\exp(x_iy_i)\,,\quad \text{where } A_i = \prod_{j\neq i}\frac{qx_i-x_j}{x_i-x_j}\,, \tag{4.13}$$

we have a set of canonical coordinates.

Proposition 4.2.5 *The set of local coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ on $\mathfrak{h}_I$ is such that*

$$\{x_i, x_j\} = 0\,, \quad \{x_i, y_j\} = \delta_{ij}\,, \quad \{y_i, y_j\} = 0\,,$$

for the Poisson bracket introduced in Proposition 4.2.2.

Proof. The first identity is (4.12a). That identity together with (4.12b) yields $\{x_i, 1 + p_j x_j\} = \delta_{ij} x_j (1 + p_j x_j)$, and the second identity easily follows. For the third one, a direct computation gives

$$\{A_i(x), y_j\} = \sum_{l \neq i} \frac{A_i(x)\,(1-q)}{q x_i - x_l} \cdot \begin{cases} \delta_{ij} \frac{x_l}{x_i - x_l} & i = j, \\ \\ -\delta_{jl} \frac{x_i}{x_i - x_l} & i \neq j. \end{cases} \tag{4.14}$$

Then, another tedious computation implies for $B_i = A_i(x)\, \exp(x_i y_i)$ that

$$\{B_i, B_j\} = B_i B_j \frac{(1-q)^2 x_j x_i (x_i + x_j)}{(x_i - x_j)(q x_i - x_j)(q x_j - x_i)}\,.$$

Since $B_i = 1 + p_i x_i$, this is equivalent to (4.12c). □

Second parametrisation

As we can see in the proof of Proposition 4.2.2 and the introduction of new coordinates in (4.13), the element $(1 + p_i x_i)$ plays a more important role that p_i on its own. Hence, consider $\mathfrak{h}_{RS} = \mathfrak{h}_{\text{reg}} \times (\mathbb{C}^\times)^n$ with coordinates $\nu_1, \dots, \nu_n$ on $(\mathbb{C}^\times)^n$ and $\mathfrak{h}_{\text{reg}}$ given in (4.9). Then, introduce

$$X = \operatorname{diag}(x_1, \dots, x_n)\,, \quad Z = (Z_{ij})\,, \text{ for } Z_{ij} = (1-q) \frac{\nu_j x_j}{x_i - q x_j}\,. \tag{4.15}$$

Now, the matrix $XZX^{-1} - qZ$ is equal to $W\hat{V}$ for $W = (1, \dots, 1)^\top$ and $\hat{V} = (1-q)(\nu_1, \dots, \nu_n)$, so that $(X, Z) \in \mathcal{C}^\circ_{n,q}$ by (4.8). Indeed, from the definition of $\mathfrak{h}_{\text{reg}}$ we have $\det X \neq 0$ directly, while Z is invertible again using (4.11). We get a variant of Proposition 4.2.2 in that way.

Proposition 4.2.6 *There is a diffeomorphism $\xi : \mathfrak{h}_{RS}/S_n \to \mathcal{C}'_{n,q}$ such that $\xi(x_i, p_i) = (X, Z, V, W)$ is determined by (X, Z) defined as (4.15). Furthermore, it is a Poisson morphism for the Poisson bracket $\{-, -\}$ on $\mathfrak{h}_{RS}/S_n$ defined by*

$$\{x_i, x_j\} = 0\,, \tag{4.16a}$$

$$\{x_i, \nu_j\} = \delta_{ij} x_j \nu_j\,, \tag{4.16b}$$

$$\{\nu_i, \nu_j\} = \frac{(1-q)^2 (x_i + x_j) x_i x_j\, \nu_i \nu_j}{(x_i - x_j)(x_i - q x_j)(x_j - q x_i)}\,, \tag{4.16c}$$

and $\{-,-\}_{\mathrm{P}}$ *on* $\mathcal{C}'_{n,q}$.

Proof. It is similar to the proof of Proposition 4.2.2 since $\nu_j x_j = (1 + p_j x_j)$. □

This time, we can remark that these brackets are similar to [80, (A13)–(A15)][13], and up to changing representative under the action of $g = \mathrm{diag}(\sqrt{\nu_1}, \dots, \sqrt{\nu_n})$ which is locally well-defined, we get that Z takes the form [80, (A.12)]. Hence, we can find log-canonical coordinates following their work, and we introduce

$$s_i = q^{(n-1)/2}\, \nu_i \sqrt{A_i'(x)}\,, \quad A_i'(x) = \prod_{k,k\neq i} \frac{(x_k - x_i)(x_i - x_k)}{(x_k - qx_i)(x_i - qx_k)}\,. \tag{4.17}$$

Since we are taking square roots in (4.17), this parametrisation is only locally defined. We are not willing to discuss the definiteness of $\sqrt{A_i'(x)}$ at all, and we forget about that issue for a moment. (We will see that a third parametrisation is better suited to our study.)

Proposition 4.2.7 *The set of local coordinates* $(x_1, \dots, x_n, s_1, \dots, s_n)$ *on* $\mathfrak{h}_{RS}$ *is such that*

$$\{x_i, x_j\} = 0\,, \quad \{x_i, s_j\} = \delta_{ij} x_i s_j\,, \quad \{s_i, s_j\} = 0\,,$$

for the Poisson bracket introduced in Proposition 4.2.6.

Proof. First two identities are obvious. For the third one, it follows easily from the next result.

Lemma 4.2.8

$$\{A_i'(x), \nu_j\} = \delta_{(i\neq j)}\, A_i'(x)\nu_j \frac{(1-q)^2\, x_i x_j (x_i + x_j)}{(x_j - x_i)(x_i - qx_j)(x_j - qx_i)}\,. \tag{4.18}$$

Last lemma requires an easy but tedious computation left to the reader. □

Using these new variables, we find that

$$\mathrm{Tr}(Z) = \sum_i \nu_i \;=\; \sum_{i=1}^n s_i q^{(1-n)/2} \sqrt{\prod_{k,k\neq i} \frac{(x_k - qx_i)(x_i - qx_k)}{(x_k - x_i)(x_i - x_k)}}\,.$$

Moreover, using (4.11) yields that $\mathrm{tr}(Z + Z^{-1})$ takes the same form, with $(s_i + s_i^{-1})$ instead of s_i. It is claimed in [80] that this is the Ruijsenaars-Schneider Hamiltonian[14], and we can in fact show that taking a suitable representative of the equivalence class, then Z is the complex version of the Lax matrix introduced in [144].

[13] As noted in [126, Remark 2.1], the multiplicative factor corresponding for us to $(q-1)^2$ is missing.

[14] Their identity [80, (A.24)] contains a typo, since all factors except $(s_i + s_i^{-1})$ should be inverted.

Lemma 4.2.9 *For any $(X,Z,V,W) \in \mathcal{C}'_{n,q}$, we can choose a representative such that Z has the form $\mathbf{L} = (\mathbf{L}_{jk})$ with*

$$\mathbf{L}_{jk} \;=\; \exp\left(\beta \frac{\theta_j+\theta_k}{2}\right)\, C_{jk}\,\sqrt{F(r_j)F(r_k)}\,, \tag{4.19}$$

where

$$F(r_j) := \prod_{l\neq j} f(r_j - r_l)\,, \qquad f^2(r) \;:=\; 1 + \frac{\alpha^2}{\sinh^2(r/2)}\,, \tag{4.20a}$$

$$C_{jk} = \left[\cosh\left(\frac{\mu}{2}(r_j-r_k)\right) + ia\sinh\left(\frac{\mu}{2}(r_j-r_k)\right)\right]^{-1}, \qquad a^2 = \alpha^{-2}-1\,. \tag{4.20b}$$

This is the Lax matrix for the trigonometric RS system introduced in [144, Section 4].

Proof. From the representative given in (4.15) such that $Z_{ij} = \frac{(1-q)\nu_j}{x_i x_j^{-1}-q}$, we get by conjugating by $\mathrm{diag}(\sqrt{\nu_1},\dots,\sqrt{\nu_n})$ then using the coordinates (x_i,s_i) that

$$Z_{ij} = \frac{(1-q)}{x_i/x_j - q}\sqrt{s_i s_j}\left[q^{n-1}\prod_{l\neq i}\frac{(x_l-x_i)(x_i-x_l)}{(x_l-qx_i)(x_i-qx_l)}q^{n-1}\prod_{k\neq j}\frac{(x_j-x_k)(x_k-x_j)}{(x_j-qx_k)(x_k-qx_j)}\right]^{-1/4}$$

Introducing coordinates (r_i,θ_i) such that $s_i = e^{\theta_i}$, $x_i = e^{r_i}$, and γ such that $q = e^{2\gamma}$, we can write

$$Z_{ij} = e^{\frac{\theta_i+\theta_j}{2}} e^{\frac{r_j-r_i}{2}} \frac{\sinh(\gamma)}{\sinh\left(\frac{r_j-r_i}{2}+\gamma\right)}\left[\prod_{l\neq i} g(r_l-r_i)\prod_{k\neq j} g(r_j-r_k)\right]^{-1/4} \tag{4.21}$$

with the functions

$$g(r) = \frac{\sinh\left(\frac{r}{2}\right)\sinh\left(\frac{-r}{2}\right)}{\sinh\left(\frac{r}{2}-\gamma\right)\sinh\left(\frac{-r}{2}-\gamma\right)}\,.$$

Conjugating by $\mathrm{diag}(e^{r_1/2},\dots,e^{r_n/2})$ we can get rid of the factor $e^{\frac{r_j-r_i}{2}}$ in Z_{ij}. Now, notice that for $\beta=\mu=1$, $\alpha^2 = -\sinh^2(\gamma)$ we obtain from (4.20a)–(4.20b)

$$C_{jk} = \frac{\sinh(\gamma)}{\sinh\left(\frac{r_k-r_j}{2}+\gamma\right)}\,, \qquad f^2(r) = g^{-1}(r)\,,$$

Hence $\mathbf{L}_{ij}$ given by (4.19) is nothing else than (4.21). □

Again, it is important to remark that this is only true in a neighbourhood of a point due to several manipulations involving square roots.

Third parametrisation

We start again with the Poisson diffeomorphism given by Proposition 4.2.6. This time, we seek a set of log-canonical coordinates that does not require any branch cut, contrary to the coordinates defined in (4.17). This is the approach that was considered in [41]. We choose to set

$$\sigma_i = \nu_i \, A_i''(x) \,, \quad A_i''(x) = \prod_{k,k\neq i} \frac{(x_k - x_i)}{(x_k - q x_i)} \,, \tag{4.22}$$

and we obtain log-canonical coordinates.

Proposition 4.2.10 *The set of local coordinates $(x_1, \dots, x_n, \sigma_1, \dots, \sigma_n)$ on $\mathfrak{h}_{RS}$ is such that*

$$\{x_i, x_j\} = 0 \,, \quad \{x_i, \sigma_j\} = \delta_{ij} x_i \sigma_j \,, \quad \{\sigma_i, \sigma_j\} = 0 \,,$$

for the Poisson bracket introduced in Proposition 4.2.6.

Proof. First two identities are obvious. For the third one, we need the next result.

Lemma 4.2.11

$$\left\{A_i''(x)^{-1}, \nu_j\right\} = \delta_{(i\neq j)} \, A_i''(x)^{-1} \nu_j \frac{(q-1)x_i x_j}{(x_j - x_i)(x_j - q x_i)} \,. \tag{4.23}$$

The proof of this key lemma is left to the reader. □

In these coordinates, the matrices X and $Z = Y + X^{-1}$ of (4.15) take the form

$$X_{ij} = \delta_{ij} x_j \,, \quad Z_{ij} = \sigma_j \frac{(1-q)x_j}{(x_i - q x_j)} \prod_{k,k\neq j} \frac{x_k - q x_j}{x_k - x_j} \,. \tag{4.24}$$

Defining locally canonical coordinates (r_i, φ_i) with $x_i = e^{r_i}$ and $\sigma_i = e^{\varphi_i}$, as well as $q = e^{2\gamma}$, we can rewrite after acting by $\mathrm{diag}(e^{r_1/2}, \dots, e^{r_n/2})$

$$Z_{ij} = -e^{-(n-1)\gamma} \, e^{\varphi_j} \frac{\sinh(\gamma)}{\sinh\left(\frac{r_i - r_j}{2} - \gamma\right)} \prod_{k,k\neq j} \frac{\sinh\left(\frac{r_k - r_j}{2} - \gamma\right)}{\sinh\left(\frac{r_k - r_j}{2}\right)} \,.$$

4.2.2 Integrable systems on the MQV

Finding integrable systems

In order to find integrable systems, we need to look at the functionally independent elements in the families described in § 4.1.2. We begin by a first remark when we look at the elements in involution with any $\operatorname{tr} Z^k$ on $\mathcal{C}^{\circ}_{n,q}$.

Lemma 4.2.12 *For any $k \in \mathbb{N}$, $\operatorname{tr} WVZ^k \in \mathbb{C}[\operatorname{tr} Z^l \mid l \leq k]$. Moreover, for any $k' \leq k$, the element $h^z_{k,k'}$ defined in Proposition 4.1.5 is such that $h^z_{k,k'} \in \mathbb{C}[\operatorname{tr} Z^l \mid l \leq k]$.*

Proof. By choice of q_∞, $\operatorname{tr} WV = VW = q^{-n} - 1 \in \mathbb{C}$. Next, we use the moment map equation (4.4) in the form $XZX^{-1} = qZ + qWVZ$. This gives $\operatorname{tr} WVZ = (q^{-1} - 1) \operatorname{tr} Z$ and raising the moment map relation to the power $k \geq 2$ before taking its trace yields

$$-kq^k \operatorname{tr} WVZ^k = (q^k - 1) \operatorname{tr} Z^k + \ldots + q^k \operatorname{tr}(WVZ)^k .$$

Now, remark that any term on the right-hand side is a trace of some matrix containing either no or at least two products WV. We can rewrite the latter terms as products of $\operatorname{tr} WVZ^l$ with $l < k$ using that a factor VZ^lW is a scalar equal to $\operatorname{tr} WVZ^l$. For example, $\operatorname{tr}(WVZ)^k = (VZW)^k = (\operatorname{tr} WVZ)^k$. This proves the first part of the lemma by induction. For the second part, remark that we can rewrite

$$h^z_k = \frac{1}{k} \operatorname{tr}(Z + \eta XZX^{-1})^k = \frac{1}{k} \operatorname{tr}((1 + \eta q)Z + \eta qWVZ)^k ,$$

so that by a similar induction $h^z_k \in \mathbb{C}[\operatorname{tr} Z^l, \operatorname{tr} WVZ^l \mid l \leq k]$. Taking the element at order k' in η is $h^z_{k,k'}$, which thus satisfies the same property. We conclude by the first part. □

Hence in the particular case $d = 1$, we have nothing else than the involutive family $(\operatorname{tr} Z^k)$ from Proposition 4.1.3, Proposition 4.1.4[15] and Proposition 4.1.5. This is, in fact, true for any $u \in \{x, y, z, e_0 + xy\}$ as we see now.

Lemma 4.2.13 *For any $k \in \mathbb{N}$ and $k' \leq k$, $\operatorname{tr} WVU^k, h^U_{k,k'} \in \mathbb{C}[\operatorname{tr} U^l \mid l \in \mathbb{Z}]$, when we work on the open subset of $\mathcal{C}_{n,q}$ where U is invertible.*

[15] In the case $d = 1$ we just have $\operatorname{tr} \mathbf{AC}Z^l = \operatorname{tr} WVZ^{l+1}$, so it gives nothing more than Proposition 4.1.3.

Proof. We remark that in the case $\epsilon(u) = -1$, we can write from the invertibility condition that the moment map takes the form $AUA^{-1}U^{-1} = q(\mathrm{Id}_n + VW)$ for some $A \in \mathrm{GL}_n$, and the left-hand side in particular is Φ_0. Thus $\Phi_0 U = qU + qVWU$ and we can reproduce the proof of Lemma 4.2.12 to get our claim that the desired elements are in $\mathbb{C}[\operatorname{tr} U^l \mid l \in \mathbb{N}]$.

If $\epsilon(u) = +1$, we write the moment map as $UBU^{-1}B^{-1} = q(\mathrm{Id}_n + VW)$ for some $B \in \mathrm{GL}_n$. In a similar way $\Phi_0^{-1}U = qU + qWVU$ and the proof of Lemma 4.2.12 works again but this time for $\mathbb{C}[\operatorname{tr} U^l \mid l \leq 0]$. □

By the same reasoning, Proposition 4.1.10 gives nothing more than $(\operatorname{tr} Z^k)$ or $(\operatorname{tr} Y^k)$. We now establish how many elements are functionally independent.

Proposition 4.2.14 *At a generic point of $\mathcal{C}_{n,q}$, there are n functionally independent elements inside the algebra $\mathbb{C}[\operatorname{tr} U^l \mid l \in \mathbb{Z}]$.*

Proof. We have seen in § 4.2.1 that the matrix $U = X$ can be generically parametrised by the n-uple $(x_1, \dots, x_n)$ of its eigenvalues up to permutation. These eigenvalues form a point in $\mathfrak{h}_{\mathrm{reg}}$, a n-dimensional space where they are pairwise distinct. This implies that the Jacobian matrix J of $F = (\operatorname{tr} X, \dots, \frac{1}{n}\operatorname{tr} X^n)$ has x_j^{i-1} for (i,j) entry, hence J is a Vandermonde matrix. It is invertible since $x_i \neq x_j$ for $i \neq j$, hence the functions forming F are functionally independent on $\mathcal{C}'_{n,q}$.

In the other cases $U = Y, Z, \mathrm{Id}_n + XY$ the proof is similar. Indeed, since $\mathcal{C}_{n,q}$ is connected by [126], we can parametrise generically a point of by the eigenvalues of U with n other coordinates, then repeat the argument. □

As a corollary of this result, we can generically integrate the flows associated to the vector field $\{\operatorname{tr} U^l, -\}_{\mathrm{P}}$ by quadrature, see § 2.1.2. This local integration is an alternative to the global one derived in § 4.1.2, and was remarked in this form in [41].

Local expressions

We use the third parametrisation and the log-canonical coordinates (x_i, σ_i) of Proposition 4.2.10 to write down on $\mathcal{C}'_{n,q}$ some of the functions in each family, and the system of ODEs they define. This subsection is parallel to the end of [41, Section 3].

We do not discuss the family $(\operatorname{tr} X^k)_k$ since the functions $\operatorname{tr} X^k = \sum_{i=1}^n x_i^k$ do not yield interesting systems of ODEs in local coordinates. So we begin with the elements $(\operatorname{tr} Z^k)_k$, for which we set $G_k^{1,1} = \operatorname{tr} Z^k$. We can write from (4.24)

$$G_k^{1,1} = \sum_{i_1,\dots,i_k=1}^{n} \frac{(1-q)\sigma_{i_2}x_{i_2}}{x_{i_1}-qx_{i_2}} \cdots \frac{(1-q)\sigma_{i_1}x_{i_1}}{x_{i_k}-qx_{i_1}} \prod_{a=1}^{k} \left(\prod_{j_a; j_a\neq i_a} \frac{x_{j_a}-qx_{i_a}}{x_{j_a}-x_{i_a}} \right), \tag{4.25}$$

for any $k \in \mathbb{N}$. In particular, denoting $\frac{d}{dt} = \{-, G_1^{1,1}\}_{\mathrm{P}}$, we get

$$\frac{dx_k}{dt} = x_k\sigma_k \prod_{j;j\neq k} \frac{x_j-qx_k}{x_j-x_k}, \qquad \frac{d\sigma_k}{dt} = \sum_{i\neq k} \sigma_i\sigma_k \frac{(1-q)x_ix_k}{(x_k-x_i)^2} \prod_{j;j\neq i,k} \frac{x_j-qx_i}{x_j-x_i},$$

where we can use Lemma 4.2.11 with σ_j instead of ν_j to compute the second term. To obtain $(x_i(t), \sigma_i(t))$ at generic time t, it then suffices to integrate locally the flow by forming an integrable system from Proposition 4.2.14, or to put the element $(X(t), Z(0), V(0), W(0))$ obtained from Proposition 4.1.6 with $t = t_1$ in the form of the third parametrisation described in § 4.2.1. In other words, we act on $(X(t), Z(0), V(0), W(0))$ by the matrix g_t that puts $X(t)$ in diagonal form (then act by a diagonal matrix that puts $g_tW(0)$ equal to $(1,\dots,1)^\top$). This was observed in [144, Sect. 5] for the Ruijsenaars-Schneider Hamiltonian, which is equivalent to $\operatorname{tr}(Z+Z^{-1})$ as explained with the second parametrisation in § 4.2.1.

For the family $(\operatorname{tr} Y^k)_k$, we can see these elements as deformations of $(\operatorname{tr} Z^k)_k$ on $\mathcal{C}'_{n,q}$ since $Y = Z - X^{-1}$. Hence, we can write for $H_k^{1,1} = \operatorname{tr} Y^k$ that

$$H_1^{1,1} = G_1^{1,1} - \sum_{i=1}^{n} \frac{1}{x_i}, \quad H_2^{1,1} = G_2^{1,1} - 2\sum_{i=1}^{n} \frac{\sigma_i}{x_i} \prod_{j;j\neq i} \frac{x_j-qx_i}{x_j-x_i} + \sum_{i=1}^{n} \frac{1}{x_i^2},$$

and continue for each $k \geq 3$. We then easily get for $\frac{d}{dt} = \{-, H_1^{1,1}\}_{\mathrm{P}}$, that

$$\frac{dx_k}{dt} = x_k\sigma_k \prod_{j;j\neq k} \frac{x_j-qx_k}{x_j-x_k}, \qquad \frac{d\sigma_k}{dt} = -\frac{\sigma_k}{x_k} + \sum_{i\neq k} \sigma_i\sigma_k \frac{(1-q)x_ix_k}{(x_k-x_i)^2} \prod_{j;j\neq i,k} \frac{x_j-qx_i}{x_j-x_i}.$$

The functions $H_k^{1,1}$ first appeared in [91] to define the motion of the zeros for the tau function of the qKP hierarchy, where we need the coordinates (x_i, y_i) given in Proposition 4.2.5. In the above coordinates, they can be seen as the classical version of operators introduced in [21, 167, 168], see [34] for details.

Finally, we want to look at the functions $F_k^{1,1} = \operatorname{tr}(\mathrm{Id}_n + XY)^k$. We can remark that on $\mathcal{C}'_{n,q}$ we have from (4.24) that

$$(ZX)_{ij} = \tilde{\sigma}_j \frac{(1-q)x_j}{(x_i-qx_j)} \prod_{k,k\neq j} \frac{x_k-qx_j}{x_k-x_j},$$

where $\tilde{\sigma}_j = \sigma_j x_j$. But an easy consequence of Proposition 4.2.10 is that the coordinates $(x_i, \tilde{\sigma}_i)_i$ have the same Poisson brackets as $(x_i, \sigma_i)_i$. Hence $F_k^{1,1} = \operatorname{tr}(ZX)^k$ defines the same family as $G_k^{1,1}$ after the reparametrisation $(x_i, \sigma_i)_i \mapsto (x_i, \tilde{\sigma}_i)_i$, so we do not need to discuss this family.

Remark 4.2.15 *In the above formulas, we have always chosen X to be in diagonal form. We could instead put Z in diagonal form and obtain that the couple (Z, X) takes the form*

$$Z = \operatorname{diag}(z_1, \dots, z_n), \quad X_{ij} = \tilde{\sigma}_j \frac{(1-q^{-1})z_j}{z_i - q^{-1}z_j} \prod_{k \neq j} \frac{z_k - q^{-1}z_j}{z_k - z_j}, \tag{4.26}$$

for log-canonical coordinates $(z_i, \tilde{\sigma}_i)$. This can be compared to (X, Z) in (4.24), and we see that now X plays the role of the Lax matrix. The functions $G_k^{1,1} = \operatorname{tr} Z^k$ become trivial, while $\operatorname{tr} X^k$ are the symmetric functions associated to the RS system. This astonishing property can be seen as the duality of the complex trigonometric RS system. Duality properties were first investigated by Ruijsenaars in the (more complicated) real setting [141, 142, 143].

4.3 Multiple framings

In the spin case $d \geq 2$ where q is not a root of unity, we work in $\mathcal{C}^\circ_{n,q,d}$ with the matrices $(X, Z, \mathbf{A}, \mathbf{C})$ introduced in § 4.1.1 in order to understand the local structure in § 4.3.1. We only return to the elements $(V_\alpha, W_\alpha)_\alpha$ when discussing the possible integrable systems on the space in § 4.3.3.

4.3.1 Local Poisson structure

Recall the spaces $\mathfrak{h} = \mathbb{C}^n$ and $\mathfrak{h}_{\text{reg}}$ given by (4.9), that are introduced in § 4.2.1. We consider the open subspace $\mathcal{C}'_{n,q,d} \subset \mathcal{C}^\circ_{n,q,d}$ which is such that for any equivalence class of quadruple $(X, Z, \mathbf{A}, \mathbf{C}) \in \mathcal{C}'_{n,q,d}$, the matrix X is diagonalisable with eigenvalues in $\mathfrak{h}_{\text{reg}}$, and when we choose a representative with X in diagonal form, the matrix $\mathbf{A}$ is such that the entries in each of its rows sum up to a nonzero value. Hence, we can always pick a representative such that $\sum_\alpha \mathbf{A}_{i\alpha} = 1$ in $\mathcal{C}'_{n,q,d}$, and there is still the freedom to act by permutation matrices.

Following [42], we take $(\mathbf{a}^\alpha)^\top, \mathbf{c}^\alpha \in \mathfrak{h}$ for $\alpha = 1, \ldots, d$. We define $\mathfrak{h}_{sp} \subset \mathfrak{h}_{\mathrm{reg}} \times \mathfrak{h}^d \times \mathfrak{h}^d$ to be the subspace such that on global coordinates $(x_i, \mathbf{a}_i^\alpha, \mathbf{c}_i^\alpha)$ we require

$$\sum_\alpha \mathbf{a}_i^\alpha = 1\,, \quad \det(B),\, \det(\mathrm{Id}_n + W'_\beta V'_\beta) \neq 0\,, \tag{4.27}$$

where the matrices B, W'_β, V'_β are defined as follows. We form the matrix $B = (B_{ij})$ by

$$B_{ij} = q \frac{\sum_\alpha \mathbf{a}_i^\alpha \mathbf{c}_j^\alpha}{x_i x_j^{-1} - q} \tag{4.28}$$

and we let $W'_\beta \in \mathrm{Mat}_{n\times 1}(\mathbb{C})$ and $C'_\beta \in \mathrm{Mat}_{1\times n}(\mathbb{C})$ be defined by $(W'_\beta)_i = \mathbf{a}_i^\beta$, $(C'_\beta)_i = \mathbf{c}_i^\beta$. Furthermore, we define inductively

$$V'_\beta = C'_\beta B^{-1} (\mathrm{Id}_n + W'_1 V'_1)^{-1} \ldots (\mathrm{Id}_n + W'_{\beta-1} V'_{\beta-1})^{-1}\,.$$

This space has dimension $2nd$. We define a map $\xi : \mathfrak{h}_{sp} \to \mathcal{C}^\circ_{n,q,d}$ which associates to $(x_i, \mathbf{a}_i^\alpha, \mathbf{c}_i^\alpha)$ the equivalence class of the element $(X, Z, \mathbf{A}, \mathbf{C})$, where

$$\begin{aligned} &X = \mathrm{diag}(x_1, \ldots, x_n)\,, \quad Z = B\,, \\ &\mathbf{A} = (\mathbf{A}_{i\alpha})\,,\ \mathbf{C} = (\mathbf{C}_{\alpha i}) \quad \text{with } \mathbf{A}_{i\alpha} = \mathbf{a}_i^\alpha\,,\ \mathbf{C}_{\alpha i} = \mathbf{c}_i^\alpha\,. \end{aligned} \tag{4.29}$$

It is readily seen that (4.4) is satisfied, and the invertibility of Z and the elements $\mathrm{Id}_n + W_\alpha V_\alpha$ comes from (4.27), as we can remark that each pair (V_α, W_α) is nothing else than (V'_α, W'_α). Note that we have an S_n action on $\mathfrak{h}_{sp}$ given by $\tau \cdot (x_i, \mathbf{a}_i^\alpha, \mathbf{c}_i^\alpha) = (x_{\tau^{-1}(i)}, \mathbf{a}^\alpha_{\tau^{-1}(i)}, \mathbf{c}^\alpha_{\tau^{-1}(i)})$, which corresponds to the action by a permutation matrix of GL_n on $(X, Z, \mathbf{A}, \mathbf{C})$. We obtain in that way the following result.

Proposition 4.3.1 *The map $\xi : \mathfrak{h}_{sp}/S_n \to \mathcal{C}'_{n,q,d}$ given by (4.29) defines a diffeomorphism.*

Remark 4.3.2 *The $(x_i, \mathbf{a}_i^\alpha, \mathbf{c}_i^\alpha)$ extend to a local coordinate system on the connected component of $\mathcal{C}^\circ_{n,q,d}$ containing $\mathcal{C}'_{n,q,d}$. However, we do not know if they extend to $\mathcal{C}^\circ_{n,q,d}$ as it is not known if that space is connected for $d > 1$. For $d = 1$, it was proved by Oblomkov [126].*

Our next step is to investigate if we can extend the map ξ to a Poisson morphism, as we did in the case $d = 1$ with Propositions 4.2.2 and 4.2.6. To do so, remark that the functions defined in (4.5) can be written in local coordinates as

$$\xi^* f_k := \sum_i x_i^k\,, \quad \xi^* g_k^{\alpha\beta} = \sum_i \mathbf{a}_i^\alpha \mathbf{c}_i^\beta x_i^k\,, \quad \sum_\alpha \xi^* g_k^{\alpha\beta} = \sum_i \mathbf{c}_i^\beta x_i^k\,. \tag{4.30}$$

It follows from these expressions that their differentials span the cotangent space at a generic point. We computed their Poisson structure as Lemma 4.1.2, which allows us to prove the expected Poisson property of the morphism ξ of Proposition 4.3.1.

Proposition 4.3.3 *The map $\xi : \mathfrak{h}_{sp}/S_n \to \mathcal{C}'_{n,q,d}$ from Proposition 4.3.1 extends to a Poisson diffeomorphism for the Poisson bracket $\{-,-\}$ defined on $\mathfrak{h}_{sp}/S_n$ by*

$$\{x_i,x_j\} = 0\,, \quad \{\mathbf{a}_i^\alpha, x_j\} = 0\,, \quad \{\mathbf{c}_i^\alpha, x_j\} = -\delta_{ij}\mathbf{c}_i^\alpha x_j\,, \tag{4.31a}$$

$$\begin{aligned}\{\mathbf{a}_j^\gamma,\mathbf{a}_i^\alpha\} =& \frac{1}{2}\delta_{(i\neq j)}\frac{x_j+x_i}{x_j-x_i}(\mathbf{a}_j^\gamma\mathbf{a}_i^\alpha + \mathbf{a}_i^\gamma\mathbf{a}_j^\alpha - \mathbf{a}_j^\gamma\mathbf{a}_j^\alpha - \mathbf{a}_i^\gamma\mathbf{a}_i^\alpha) + \frac{1}{2}o(\alpha,\gamma)(\mathbf{a}_j^\gamma\mathbf{a}_i^\alpha + \mathbf{a}_i^\gamma\mathbf{a}_j^\alpha) \\ &+ \frac{1}{2}\sum_{\sigma=1}^d o(\gamma,\sigma)\mathbf{a}_i^\alpha(\mathbf{a}_j^\gamma\mathbf{a}_i^\sigma + \mathbf{a}_i^\gamma\mathbf{a}_j^\sigma) - \frac{1}{2}\sum_{\kappa=1}^d o(\alpha,\kappa)\mathbf{a}_j^\gamma(\mathbf{a}_j^\kappa\mathbf{a}_i^\alpha + \mathbf{a}_i^\kappa\mathbf{a}_j^\alpha)\,,\end{aligned} \tag{4.31b}$$

$$\begin{aligned}\{\mathbf{c}_j^\epsilon,\mathbf{a}_i^\alpha\} =& \delta_{\epsilon\alpha}B_{ij} - \mathbf{a}_i^\alpha B_{ij} + \frac{1}{2}\delta_{(i\neq j)}\frac{x_j+x_i}{x_j-x_i}\mathbf{c}_j^\epsilon(\mathbf{a}_j^\alpha - \mathbf{a}_i^\alpha) - \delta_{(\alpha<\epsilon)}\mathbf{a}_i^\alpha\mathbf{c}_j^\epsilon \\ &- \mathbf{a}_i^\alpha\sum_{\lambda=1}^{\epsilon-1}\mathbf{a}_i^\lambda(\mathbf{c}_j^\lambda - \mathbf{c}_j^\epsilon) + \delta_{\epsilon\alpha}\sum_{\lambda=1}^{\epsilon-1}\mathbf{a}_i^\lambda\mathbf{c}_j^\lambda + \frac{1}{2}\sum_{\kappa=1}^d o(\alpha,\kappa)\mathbf{c}_j^\epsilon(\mathbf{a}_j^\kappa\mathbf{a}_i^\alpha + \mathbf{a}_i^\kappa\mathbf{a}_j^\alpha)\,,\end{aligned} \tag{4.31c}$$

$$\begin{aligned}\{\mathbf{c}_j^\epsilon,\mathbf{c}_i^\beta\} =& \frac{1}{2}\delta_{(i\neq j)}\frac{x_j+x_i}{x_j-x_i}(\mathbf{c}_j^\epsilon\mathbf{c}_i^\beta + \mathbf{c}_i^\epsilon\mathbf{c}_j^\beta) + \mathbf{c}_i^\beta B_{ij} - \mathbf{c}_j^\epsilon B_{ji} + \frac{1}{2}o(\epsilon,\beta)(\mathbf{c}_i^\epsilon\mathbf{c}_j^\beta - \mathbf{c}_j^\epsilon\mathbf{c}_i^\beta) \\ &+ \mathbf{c}_i^\beta\sum_{\lambda=1}^{\epsilon-1}\mathbf{a}_i^\lambda(\mathbf{c}_j^\lambda - \mathbf{c}_j^\epsilon) - \mathbf{c}_j^\epsilon\sum_{\mu=1}^{\beta-1}\mathbf{a}_j^\mu(\mathbf{c}_i^\mu - \mathbf{c}_i^\beta)\,,\end{aligned} \tag{4.31d}$$

and the Poisson bracket $\{-,-\}_{\mathrm{P}}$ on $\mathcal{C}'_{n,q,d}$. In (4.31b)–(4.31d), $o(-,-)$ is the ordering function on d elements defined in Section 1.5.

We skip the proof of Proposition 4.3.3 and return to it at the end of this subsection.

Due to the moment map relation in the form (4.4), we are more interested in the entries of the product $S = \mathbf{AC}$ than the entries of the corresponding two matrices. Set $g_{ij} = \sum_\alpha \mathbf{a}_i^\alpha\mathbf{c}_j^\alpha$ for all i,j. Hence, under the isomorphism ξ, Z is given by the matrix B with entries $q\frac{g_{ij}x_j}{x_i - qx_j}$ as defined in (4.28). We find the following result.

Proposition 4.3.4 *The Poisson bracket $\{-,-\}$ satisfies the identities $\{x_i, g_{kl}\} = \delta_{il} x_i g_{kl}$ and*

$$\begin{aligned}\{g_{ij}, g_{kl}\} =& \frac{1}{2} g_{ij} g_{kl} \left[\delta_{(i\neq k)} \frac{x_i + x_k}{x_i - x_k} + \delta_{(j\neq l)} \frac{x_j + x_l}{x_j - x_l} + \delta_{(k\neq j)} \frac{x_k + x_j}{x_k - x_j} + \delta_{(l\neq i)} \frac{x_l + x_i}{x_l - x_i}\right] \\ &+ \frac{1}{2} g_{il} g_{kj} \left[\delta_{(i\neq k)} \frac{x_i + x_k}{x_i - x_k} + \delta_{(j\neq l)} \frac{x_j + x_l}{x_j - x_l} + \frac{x_k + q x_j}{x_k - q x_j} - \frac{x_i + q x_l}{x_i - q x_l}\right] \\ &+ \frac{1}{2} g_{ij} g_{il} \left[\delta_{(i\neq k)} \frac{x_k + x_i}{x_k - x_i} + \frac{x_i + q x_l}{x_i - q x_l}\right] + \frac{1}{2} g_{ij} g_{jl} \left[\delta_{(j\neq k)} \frac{x_j + x_k}{x_j - x_k} - \frac{x_j + q x_l}{x_j - q x_l}\right] \\ &+ \frac{1}{2} g_{kj} g_{kl} \left[\delta_{(i\neq k)} \frac{x_k + x_i}{x_k - x_i} - \frac{x_k + q x_j}{x_k - q x_j}\right] + \frac{1}{2} g_{lj} g_{kl} \left[\delta_{(i\neq l)} \frac{x_i + x_l}{x_i - x_l} + \frac{x_l + q x_j}{x_l - q x_j}\right] .\end{aligned} \tag{4.32}$$

In particular, the commutative subalgebra of $\mathcal{O}(\mathfrak{h}_{sp})$ generated by the elements (x_i, g_{kl}) and localised at $\{x_i - x_j, x_i - q x_j \mid i \neq j\}$ is a Poisson subalgebra.

We will describe the relation between these Poisson brackets and the Arutyunov–Frolov conjecture in § 4.3.2. Now, we proceed to the proof of Propositions 4.3.3 and 4.3.4.

Poisson diffeomorphism : proof of Proposition 4.3.3

We noticed from the local expressions given in (4.30) that the differentials of the functions $(f_k, g_k^{\gamma\epsilon}, \sum_\gamma g_k^{\gamma\epsilon})$ generate the cotangent space at a generic point of $\mathcal{C}'_{n,q,d}$. Hence, our aim is to show that

$$\xi^* \{F_1, F_2\}_{\mathrm{P}} = \{\xi^* F_1, \xi^* F_2\} , \qquad \text{for any } F_1, F_2 = f_k, g_k^{\gamma\epsilon}, \sum_\gamma g_k^{\gamma\epsilon} . \tag{4.33}$$

Indeed, it follows that that this equality will hold when evaluated on a subset of $2nd$ functionally independent functions[16]. We can then conclude by using Remark B.3.

To write down the terms involved in (4.33), we apply ξ^* to the identities from Lemma 4.1.2 for the left-hand sides, and use the expressions (4.30) together with the brackets (4.31a)–(4.31d) in the

[16] We could omit the third type of functions $\sum_\gamma g_k^{\gamma\epsilon}$ from our discussion, since their Poisson brackets are obtained by summing those for $g_k^{\gamma\epsilon}$ over $\gamma = 1, \ldots, d$. However, we prefer to keep the computations involving these functions during the proof, so that the reader can have a better idea of the calculations that were needed to discover the brackets (4.31a)–(4.31d) in the first place. Indeed, as can be seen from the proof, for each new equality of the form (4.33) that we want to obtain, we just require one new bracket on $\mathfrak{h}_{sp}/S_n$. Hence, the original computations consisted in finding for which bracket we get an equality. For example, when I wanted to establish $\xi^*\{f_k, \sum_l g_l^{\alpha\beta}\}_{\mathrm{P}} = \{\xi^* f_k, \xi^* \sum_l g_l^{\alpha\beta}\}$, I already knew that $\{x_i, x_j\} = 0$ and I wanted to determine what $\{x_i, \mathbf{c}_j^\beta\}$ is in order to get the equality.

right-hand sides. Note that in local coordinates, we use $\xi^* X_{ij} = \delta_{ij} x_i$, $\xi^*(\mathbf{A}E_{\alpha\beta}\mathbf{C})_{ij} = \mathbf{a}_i^\alpha \mathbf{c}_j^\beta$ together with $\xi^* Z_{ij} = B_{ij}$.

We reproduce the computations from [42]. First, we note that $\{x_i, x_j\} = 0$ implies $\xi^*\{f_k, f_l\}_{\mathrm{P}} = \{\xi^* f_k, \xi^* f_l\}$ since both expressions vanish. Second, recall that by assumption $\sum_\alpha \mathbf{a}_i^\alpha = 1$ for all i. Thus, from $\{x_i, \mathbf{c}_j^\beta\} = \delta_{ij} x_i \mathbf{c}_j^\beta$,

$$\begin{aligned}
\sum_\alpha \{\xi^* f_k, \xi^* g_l^{\alpha\beta}\} &= \sum_{i,j=1}^n \{x_i^k, \mathbf{c}_j^\beta x_j^l\} = \sum_{i=1}^n k\, x_i^{k+l} \mathbf{c}_i^\beta \,, \\
\sum_\alpha \xi^* \{f_k, g_l^{\alpha\beta}\}_{\mathrm{P}} &= k \sum_\alpha \xi^* \operatorname{Tr}(\mathbf{A}E_{\alpha\beta}\mathbf{C}X^{k+l}) = k \sum_{i=1}^n \mathbf{c}_i^\beta x_i^{k+l} \,,
\end{aligned}$$

and we get $\xi^*\{f_k, \sum_\alpha g_l^{\alpha\beta}\}_{\mathrm{P}} = \{\xi^* f_k, \xi^* \sum_\alpha g_l^{\alpha\beta}\}$. Third, without summing, we get again that $\xi^*\{f_k, g_l^{\alpha\beta}\}_{\mathrm{P}} = \{\xi^* f_k, \xi^* g_l^{\alpha\beta}\}$ using $\{\mathbf{a}_i^\alpha, x_j\} = 0$. This finishes the first case.

Next, we establish (4.33) when the two functions are of the form $g_k^{\gamma\epsilon}$ or $\sum_\gamma g_k^{\gamma\epsilon}$. To obtain the left-hand side of (4.33) in those cases, we use that Lemma 4.1.2 implies

$$\begin{aligned}
\xi^*\{g_k^{\gamma\epsilon}, g_l^{\alpha\beta}\}_{\mathrm{P}} =& \frac{1}{2}\left(\sum_{r=1}^k - \sum_{r=1}^l\right) \sum_{i,j=1}^n \left(\mathbf{a}_j^\alpha \mathbf{c}_i^\beta x_i^r \mathbf{a}_i^\gamma \mathbf{c}_j^\epsilon x_j^{k+l-r} + \mathbf{a}_j^\alpha \mathbf{c}_i^\beta x_i^{k+l-r} \mathbf{a}_i^\gamma \mathbf{c}_j^\epsilon x_j^r\right) \\
&+ \frac{1}{2} o(\alpha,\gamma) \sum_{i,j=1}^n \left(\mathbf{a}_i^\gamma \mathbf{c}_j^\epsilon x_j^k \mathbf{a}_j^\alpha \mathbf{c}_i^\beta x_i^l + \mathbf{a}_i^\alpha \mathbf{c}_j^\epsilon x_j^k \mathbf{a}_j^\gamma \mathbf{c}_i^\beta x_i^l\right) \\
&+ \frac{1}{2} o(\epsilon,\beta) \sum_{i,j=1}^n \left(\mathbf{a}_j^\alpha \mathbf{c}_i^\beta x_i^k \mathbf{a}_i^\gamma \mathbf{c}_j^\epsilon x_j^l - \mathbf{a}_i^\alpha \mathbf{c}_j^\epsilon x_j^k \mathbf{a}_j^\gamma \mathbf{c}_i^\beta x_i^l\right) \\
&+ \frac{1}{2}[o(\epsilon,\alpha) + \delta_{\alpha\epsilon}] \sum_{i,j=1}^n \mathbf{a}_i^\alpha \mathbf{c}_j^\epsilon x_j^k \mathbf{a}_j^\gamma \mathbf{c}_i^\beta x_i^l \\
&- \frac{1}{2}[o(\beta,\gamma) + \delta_{\beta\gamma}] \sum_{i,j=1}^n \mathbf{a}_i^\alpha \mathbf{c}_j^\epsilon x_j^k \mathbf{a}_j^\gamma \mathbf{c}_i^\beta x_i^l \\
&+ \delta_{\alpha\epsilon} \sum_{i,j=1}^n \left(B_{ij} + \sum_{\lambda=1}^{\epsilon-1} \mathbf{a}_i^\lambda \mathbf{c}_j^\lambda\right) x_j^k \mathbf{a}_j^\gamma \mathbf{c}_i^\beta x_i^l \\
&- \delta_{\beta\gamma} \sum_{i,j=1}^n \left(B_{ji} + \sum_{\mu=1}^{\beta-1} \mathbf{a}_j^\mu \mathbf{c}_i^\mu\right) x_i^l \mathbf{a}_i^\alpha \mathbf{c}_j^\epsilon x_j^k \,.
\end{aligned} \tag{4.34}$$

We now want to prove that

$$\sum_{\gamma,\alpha=1}^d \xi^*\{g_k^{\gamma\epsilon}, g_l^{\alpha\beta}\}_{\mathrm{P}} = \sum_{i,j=1}^n \{\mathbf{c}_j^\epsilon x_j^k, \mathbf{c}_i^\beta x_i^l\} \,. \tag{4.35}$$

Using (4.31a) and (4.31d), the right-hand side of (4.35) can be read as

$$\begin{aligned}
(4.35)_{RHS} =& \sum_{i,j=1}^{n} \left(\{\mathbf{c}_j^\epsilon, x_i^l\} x_j^k \mathbf{c}_i^\beta + \{x_j^k, \mathbf{c}_i^\beta\} \mathbf{c}_j^\epsilon x_i^l + \{\mathbf{c}_j^\epsilon, \mathbf{c}_i^\beta\} x_j^k x_i^l \right) \\
=&(k-l) \sum_{i=1}^{n} \mathbf{c}_i^\epsilon, \mathbf{c}_i^\beta x_i^{k+l} + \frac{1}{2} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} x_j^k x_i^l \frac{x_j + x_i}{x_j - x_i} (\mathbf{c}_j^\epsilon \mathbf{c}_i^\beta + \mathbf{c}_i^\epsilon \mathbf{c}_j^\beta) \\
&+ \sum_{i,j=1}^{n} x_j^k x_i^l (\mathbf{c}_i^\beta B_{ij} - \mathbf{c}_j^\epsilon B_{ji}) + \frac{1}{2} o(\epsilon, \beta) \sum_{i,j=1}^{n} x_j^k x_i^l (\mathbf{c}_i^\epsilon \mathbf{c}_j^\beta - \mathbf{c}_j^\epsilon \mathbf{c}_i^\beta) \\
&+ \sum_{i,j=1}^{n} x_j^k x_i^l \mathbf{c}_i^\beta \sum_{\lambda=1}^{\epsilon-1} \mathbf{a}_i^\lambda (\mathbf{c}_j^\lambda - \mathbf{c}_j^\epsilon) - \sum_{i,j=1}^{n} x_j^k x_i^l \mathbf{c}_j^\epsilon \sum_{\mu=1}^{\beta-1} \mathbf{a}_j^\mu (\mathbf{c}_i^\mu - \mathbf{c}_i^\beta).
\end{aligned}$$

The left-hand side of (4.35) can be written from (4.34), after summing over α, γ and using the normalisation $\sum_\gamma \mathbf{a}_i^\gamma = 1$ from (4.27) when possible. We get

$$\begin{aligned}
(4.35)_{LHS} =& \frac{1}{2} \sum_{i,j=1}^{n} \mathbf{c}_i^\beta \mathbf{c}_j^\epsilon \left(\sum_{r=1}^{k} - \sum_{r=1}^{l} \right) \left(x_i^r x_j^{k+l-r} + x_i^{k+l-r} x_j^r \right) && (4.36a) \\
&+ \frac{1}{2} \sum_{i,j=1}^{n} \mathbf{c}_j^\epsilon \mathbf{c}_i^\beta x_j^k x_i^l \sum_{\alpha,\gamma=1}^{d} o(\alpha,\gamma) \left(\mathbf{a}_i^\gamma \mathbf{a}_j^\alpha + \mathbf{a}_i^\alpha \mathbf{a}_j^\gamma \right) && (4.36b) \\
&+ \frac{1}{2} o(\epsilon, \beta) \sum_{i,j=1}^{n} x_j^k x_i^l \left(\mathbf{c}_j^\beta \mathbf{c}_i^\epsilon - \mathbf{c}_j^\epsilon \mathbf{c}_i^\beta \right) && (4.36c) \\
&+ \frac{1}{2} \sum_{\alpha=1}^{d} [o(\epsilon,\alpha) + \delta_{\alpha\epsilon}] \sum_{i,j=1}^{n} \mathbf{a}_i^\alpha \mathbf{c}_j^\epsilon x_j^k \mathbf{c}_i^\beta x_i^l && (4.36d) \\
&- \frac{1}{2} \sum_{\gamma=1}^{d} [o(\beta,\gamma) + \delta_{\beta\gamma}] \sum_{i,j=1}^{n} \mathbf{c}_j^\epsilon x_j^k \mathbf{a}_j^\gamma \mathbf{c}_i^\beta x_i^l && (4.36e) \\
&+ \sum_{i,j=1}^{n} x_j^k x_i^l (B_{ij} \mathbf{c}_i^\beta - B_{ji} \mathbf{c}_j^\epsilon) && (4.36f) \\
&+ \sum_{i,j=1}^{n} \sum_{\lambda=1}^{\epsilon-1} \mathbf{a}_i^\lambda \mathbf{c}_j^\lambda x_j^k \mathbf{c}_i^\beta x_i^l - \sum_{i,j=1}^{n} \sum_{\mu=1}^{\beta-1} \mathbf{a}_j^\mu \mathbf{c}_i^\mu x_i^l \mathbf{c}_j^\epsilon x_j^k \,, && (4.36g)
\end{aligned}$$

To reduce this expression further, remark that by definition of the ordering function $o(-,-)$

$$\sum_{\alpha,\gamma=1}^{d} o(\alpha,\gamma) \left(\mathbf{a}_i^\gamma \mathbf{a}_j^\alpha + \mathbf{a}_i^\alpha \mathbf{a}_j^\gamma \right) = \sum_{\alpha<\gamma} \left(\mathbf{a}_i^\gamma \mathbf{a}_j^\alpha + \mathbf{a}_i^\alpha \mathbf{a}_j^\gamma \right) - \sum_{\alpha>\gamma} \left(\mathbf{a}_i^\gamma \mathbf{a}_j^\alpha + \mathbf{a}_i^\alpha \mathbf{a}_j^\gamma \right) = 0 \,,$$

after relabelling the indices in the second sum, so that (4.36b) disappears. Then, write (4.36a) as

$$(4.36a) = (k-l) \sum_{i=1}^{n} \mathbf{c}_i^\beta \mathbf{c}_i^\epsilon x_i^{k+l} + \frac{1}{2} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \mathbf{c}_i^\beta \mathbf{c}_j^\epsilon \left(\sum_{r=1}^{k} - \sum_{r=1}^{l} \right) \left(x_i^r x_j^{k+l-r} + x_i^{k+l-r} x_j^r \right) \,,$$

so that the sum for $i \neq j$ can be written as (here we assume $k > l$, the case $k < l$ is exactly the same)

$$
\begin{aligned}
&\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \mathbf{c}_i^{\beta}\mathbf{c}_j^{\epsilon} \sum_{r=l+1}^{k} \frac{x_i - x_j}{x_i - x_j}\left(x_i^r x_j^{k+l-r} + x_i^{k+l-r}x_j^r\right)\\
=&\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \mathbf{c}_i^{\beta}\mathbf{c}_j^{\epsilon} \frac{1}{x_i - x_j}\left(x_i^{k+1}x_j^{l} - x_i^{l+1}x_j^{k} + x_i^{k}x_j^{l+1} - x_i^{l}x_j^{k+1}\right)\\
=&\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \mathbf{c}_i^{\beta}\mathbf{c}_j^{\epsilon} \frac{x_i + x_j}{x_i - x_j}\left(x_i^{k}x_j^{l} - x_i^{l}x_j^{k}\right) = -\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n} x_i^{l}x_j^{k}\frac{x_i + x_j}{x_i - x_j}\left(\mathbf{c}_j^{\beta}\mathbf{c}_i^{\epsilon} + \mathbf{c}_i^{\beta}\mathbf{c}_j^{\epsilon}\right) ,
\end{aligned}
\tag{4.37}
$$

after relabelling indices to obtain last equality. Finally, let us look at the terms in (4.36d), (4.36e) and (4.36g) together. They can be written as

$$
\frac{1}{2}\sum_{i,j=1}^{n} x_j^k x_i^l \left(\left[\sum_{\alpha\geq\epsilon} - \sum_{\alpha=1}^{\epsilon-1}\right]\mathbf{a}_i^{\alpha}\mathbf{c}_j^{\epsilon}\mathbf{c}_i^{\beta} - \left[\sum_{\gamma\geq\beta} - \sum_{\gamma=1}^{\beta-1}\right]\mathbf{c}_j^{\epsilon}\mathbf{a}_j^{\gamma}\mathbf{c}_i^{\beta} + 2\sum_{\lambda=1}^{\epsilon-1}\mathbf{a}_i^{\lambda}\mathbf{c}_j^{\lambda}\mathbf{c}_i^{\beta} - 2\sum_{\mu=1}^{\beta-1}\mathbf{a}_j^{\mu}\mathbf{c}_i^{\mu}\mathbf{c}_j^{\epsilon}\right)
$$

and if we split the sum $\sum_{\alpha\geq\epsilon}$ as $\sum_{\alpha\geq\epsilon} = \sum_{\alpha=1}^{d} - \sum_{\alpha=1}^{\epsilon-1}$ and do the same with the sum $\sum_{\gamma\geq\beta}$, we get after using the conditions $\sum_{\alpha}\mathbf{a}_i^{\alpha} = 1$ (and the same for γ)

$$
\begin{aligned}
&\sum_{i,j=1}^{n} x_j^k x_i^l \left(-\sum_{\alpha=1}^{\epsilon-1}\mathbf{a}_i^{\alpha}\mathbf{c}_j^{\epsilon}\mathbf{c}_i^{\beta} + \sum_{\gamma=1}^{\beta-1}\mathbf{c}_j^{\epsilon}\mathbf{a}_j^{\gamma}\mathbf{c}_i^{\beta} + \sum_{\lambda=1}^{\epsilon-1}\mathbf{a}_i^{\lambda}\mathbf{c}_j^{\lambda}\mathbf{c}_i^{\beta} - \sum_{\mu=1}^{\beta-1}\mathbf{a}_j^{\mu}\mathbf{c}_i^{\mu}\mathbf{c}_j^{\epsilon}\right)\\
=&\sum_{i,j=1}^{n} x_j^k x_i^l \left(\sum_{\lambda=1}^{\epsilon-1}\mathbf{a}_i^{\lambda}(\mathbf{c}_j^{\lambda} - \mathbf{c}_j^{\epsilon})\mathbf{c}_i^{\beta} - \sum_{\mu=1}^{\beta-1}\mathbf{a}_j^{\mu}(\mathbf{c}_i^{\mu} - \mathbf{c}_i^{\beta})\mathbf{c}_j^{\epsilon}\right) .
\end{aligned}
$$

Summing together all the terms, we have reduced the left-hand side of (4.35) to the form

$$
\begin{aligned}
(4.35)_{LHS} =&(k-l)\sum_{i=1}^{n}\mathbf{c}_i^{\beta}\mathbf{c}_i^{\epsilon}x_i^{k+l} - \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n} x_i^{l}x_j^{k}\frac{x_i + x_j}{x_i - x_j}\left(\mathbf{c}_j^{\beta}\mathbf{c}_i^{\epsilon} + \mathbf{c}_i^{\beta}\mathbf{c}_j^{\epsilon}\right)\\
&+\frac{1}{2}o(\epsilon,\beta)\sum_{i,j=1}^{n} x_j^k x_i^l\left(\mathbf{c}_j^{\beta}\mathbf{c}_i^{\epsilon} - \mathbf{c}_j^{\epsilon}\mathbf{c}_i^{\beta}\right) + \sum_{i,j=1}^{n} x_j^k x_i^l (B_{ij}\mathbf{c}_i^{\beta} - B_{ji}\mathbf{c}_j^{\epsilon})\\
&+\sum_{i,j=1}^{n} x_j^k x_i^l \left(\sum_{\lambda=1}^{\epsilon-1}\mathbf{a}_i^{\lambda}(\mathbf{c}_j^{\lambda} - \mathbf{c}_j^{\epsilon})\mathbf{c}_i^{\beta} - \sum_{\mu=1}^{\beta-1}\mathbf{a}_j^{\mu}(\mathbf{c}_i^{\mu} - \mathbf{c}_i^{\beta})\mathbf{c}_j^{\epsilon}\right) .
\end{aligned}
$$

This is precisely the right-hand side of (4.35).

Before showing the next case, let us remark using (4.37) that we can obtain the following identity for any functions $\chi, \chi' : \{1, \dots, n\} \to \mathbb{C}[\mathfrak{h}_{sp}]$

$$\begin{aligned} &\frac{1}{2} \sum_{i,j=1}^{n} \chi(i)\chi'(j) \left(\sum_{r=1}^{k} - \sum_{r=1}^{l} \right) \left(x_i^r x_j^{k+l-r} + x_i^{k+l-r} x_j^r \right) \\ =&(k-l) \sum_{i=1}^{n} \chi(i)\chi'(i) x_i^{k+l} - \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} x_i^l x_j^k \frac{x_i + x_j}{x_i - x_j} \left(\chi(j)\chi'(i) + \chi(i)\chi'(j) \right) . \end{aligned} \tag{4.38}$$

In the second case, we have to show

$$\sum_{\gamma=1}^{d} \xi^* \{ g_k^{\gamma\epsilon}, g_l^{\alpha\beta} \}_{\mathrm{P}} = \sum_{i,j=1}^{n} \{ \mathbf{c}_j^{\epsilon} x_j^k, \mathbf{a}_i^{\alpha} \mathbf{c}_i^{\beta} x_i^l \} . \tag{4.39}$$

Using (4.31a), (4.31c)–(4.31d), the right-hand side of (4.39) can be read as

$$\begin{aligned} (4.39)_{RHS} =& \sum_{i,j=1}^{n} \Big(\{ \mathbf{c}_j^{\epsilon}, x_i^l \} x_j^k \mathbf{a}_i^{\alpha} \mathbf{c}_i^{\beta} + \{ x_j^k, \mathbf{c}_i^{\beta} \} \mathbf{c}_j^{\epsilon} \mathbf{a}_i^{\alpha} x_i^l + \{ \mathbf{c}_j^{\epsilon}, \mathbf{c}_i^{\beta} \} x_j^k x_i^l \mathbf{a}_i^{\alpha} + \{ \mathbf{c}_j^{\epsilon}, \mathbf{a}_i^{\alpha} \} x_j^k x_i^l \mathbf{c}_i^{\beta} \Big) \\ =&(k-l) \sum_{i=1}^{n} \mathbf{c}_i^{\epsilon} \mathbf{a}_i^{\alpha} \mathbf{c}_i^{\beta} x_i^{k+l} + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} x_j^k x_i^l \frac{x_j + x_i}{x_j - x_i} (\mathbf{c}_i^{\epsilon} \mathbf{c}_j^{\beta} \mathbf{a}_i^{\alpha} + \mathbf{c}_j^{\epsilon} \mathbf{c}_i^{\beta} \mathbf{a}_j^{\alpha}) \\ &+ \frac{1}{2} \sum_{\kappa=1}^{d} o(\alpha, \kappa) \sum_{i,j=1}^{n} x_j^k x_i^l \mathbf{c}_i^{\beta} \mathbf{c}_j^{\epsilon} (\mathbf{a}_j^{\kappa} \mathbf{a}_i^{\alpha} + \mathbf{a}_i^{\kappa} \mathbf{a}_j^{\alpha}) \\ &+ \frac{1}{2} o(\epsilon, \beta) \sum_{i,j=1}^{n} x_j^k x_i^l (\mathbf{c}_i^{\epsilon} \mathbf{c}_j^{\beta} - \mathbf{c}_j^{\epsilon} \mathbf{c}_i^{\beta}) \mathbf{a}_i^{\alpha} \\ &- \sum_{i,j=1}^{n} x_j^k x_i^l \mathbf{a}_i^{\alpha} \mathbf{c}_j^{\epsilon} \sum_{\mu=1}^{\beta-1} \mathbf{a}_j^{\mu} (\mathbf{c}_i^{\mu} - \mathbf{c}_i^{\beta}) - \delta_{(\alpha<\epsilon)} \sum_{i,j=1}^{n} x_j^k x_i^l \mathbf{c}_i^{\beta} \mathbf{a}_i^{\alpha} \mathbf{c}_j^{\epsilon} \\ &+ \delta_{\epsilon\alpha} \sum_{i,j=1}^{n} x_j^k x_i^l \mathbf{c}_i^{\beta} \left(B_{ij} + \sum_{\lambda=1}^{\epsilon-1} \mathbf{a}_i^{\lambda} \mathbf{c}_j^{\lambda} \right) - \sum_{i,j=1}^{n} x_j^k x_i^l \mathbf{c}_j^{\epsilon} B_{ji} \mathbf{a}_i^{\alpha} , \end{aligned}$$

after some easy simplifications. Meanwhile, we sum (4.34) over γ to get the left-hand side of (4.39). Hence, we can split $(4.39)_{LHS}$ as (4.40a)–(4.40c) and (4.41a)–(4.41b) where

$$(k-l) \sum_{i=1}^{n} \mathbf{a}_i^{\alpha} \mathbf{c}_i^{\beta} \mathbf{c}_j^{\epsilon} x_i^{k+l} - \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} x_j^k x_i^l \frac{x_i + x_j}{x_i - x_j} \left(\mathbf{a}_i^{\alpha} \mathbf{c}_j^{\beta} \mathbf{c}_i^{\epsilon} + \mathbf{a}_j^{\alpha} \mathbf{c}_i^{\beta} \mathbf{c}_j^{\epsilon} \right) \tag{4.40a}$$

$$+ \frac{1}{2} \sum_{\gamma=1}^{d} o(\alpha, \gamma) \sum_{i,j=1}^{n} x_j^k x_i^l \mathbf{c}_j^{\epsilon} \mathbf{c}_i^{\beta} \left(\mathbf{a}_i^{\gamma} \mathbf{a}_j^{\alpha} + \mathbf{a}_i^{\alpha} \mathbf{a}_j^{\gamma} \right) \tag{4.40b}$$

$$+ \frac{1}{2} o(\epsilon, \beta) \sum_{i,j=1}^{n} x_j^k x_i^l \mathbf{a}_i^{\alpha} \left(\mathbf{c}_j^{\beta} \mathbf{c}_i^{\epsilon} - \mathbf{c}_j^{\epsilon} \mathbf{c}_i^{\beta} \right) , \tag{4.40c}$$

$$+\frac{1}{2}[o(\epsilon,\alpha)+\delta_{\alpha\epsilon}]\sum_{i,j=1}^{n}x_j^k x_i^l \mathbf{c}_j^\epsilon \mathbf{a}_i^\alpha \mathbf{c}_i^\beta - \frac{1}{2}\sum_{\gamma=1}^{d}[o(\beta,\gamma)+\delta_{\beta\gamma}]\sum_{i,j=1}^{n}x_j^k x_i^l \mathbf{a}_i^\alpha \mathbf{c}_j^\epsilon \mathbf{a}_j^\gamma \mathbf{c}_i^\beta \tag{4.41a}$$

$$+\delta_{\alpha\epsilon}\sum_{i,j=1}^{n}x_j^k x_i^l \mathbf{c}_i^\beta \left(B_{ij}+\sum_{\lambda=1}^{\epsilon-1}\mathbf{a}_i^\lambda \mathbf{c}_j^\lambda\right) - \sum_{i,j=1}^{n}x_j^k x_i^l \left(B_{ji}+\sum_{\mu=1}^{\beta-1}\mathbf{a}_j^\mu \mathbf{c}_i^\mu\right)\mathbf{a}_i^\alpha \mathbf{c}_j^\epsilon\,, \tag{4.41b}$$

where we used (4.38) to rewrite the first line of (4.34) in order to obtain (4.40a). By rearranging terms in (4.41a) and (4.41b), we can write

$$\begin{aligned}
&(4.41a)+(4.41b)\\
&= +\frac{1}{2}[1-2\delta_{(\alpha<\epsilon)}]\sum_{i,j=1}^{n}x_j^k x_i^l \mathbf{c}_j^\epsilon \mathbf{a}_i^\alpha \mathbf{c}_i^\beta - \frac{1}{2}\sum_{\gamma=1}^{d}[1-2\delta_{(\beta>\gamma)}]\sum_{i,j=1}^{n}x_j^k x_i^l \mathbf{a}_i^\alpha \mathbf{c}_j^\epsilon \mathbf{a}_j^\gamma \mathbf{c}_i^\beta\\
&\quad +\delta_{\alpha\epsilon}\sum_{i,j=1}^{n}x_j^k x_i^l \mathbf{c}_i^\beta \left(B_{ij}+\sum_{\lambda=1}^{\epsilon-1}\mathbf{a}_i^\lambda \mathbf{c}_j^\lambda\right) - \sum_{i,j=1}^{n}x_j^k x_i^l B_{ji}\mathbf{a}_i^\alpha \mathbf{c}_j^\epsilon - \sum_{i,j=1}^{n}x_j^k x_i^l \mathbf{a}_i^\alpha \mathbf{c}_j^\epsilon \sum_{\mu=1}^{\beta-1}\mathbf{a}_j^\mu \mathbf{c}_i^\mu\\
&= -\delta_{(\alpha<\epsilon)}\sum_{i,j=1}^{n}x_j^k x_i^l \mathbf{c}_j^\epsilon \mathbf{a}_i^\alpha \mathbf{c}_i^\beta + \sum_{i,j=1}^{n}x_j^k x_i^l \mathbf{a}_i^\alpha \mathbf{c}_j^\epsilon \sum_{\mu=1}^{\beta-1}\mathbf{a}_j^\mu(\mathbf{c}_i^\beta - \mathbf{c}_i^\mu)\\
&\quad +\delta_{\alpha\epsilon}\sum_{i,j=1}^{n}x_j^k x_i^l \mathbf{c}_i^\beta \left(B_{ij}+\sum_{\lambda=1}^{\epsilon-1}\mathbf{a}_i^\lambda \mathbf{c}_j^\lambda\right) - \sum_{i,j=1}^{n}x_j^k x_i^l B_{ji}\mathbf{a}_i^\alpha \mathbf{c}_j^\epsilon\,,
\end{aligned}$$

where we used again the condition $\sum_{\gamma=1}^{d}\mathbf{a}_j^\gamma = 1$. It is not hard to see that adding (4.40a)–(4.40c) to this last expression gives that $(4.39)_{LHS}$ and $(4.39)_{RHS}$ coincide.

In the third case, we need to prove that

$$\xi^*\{g_k^{\gamma\epsilon}, g_l^{\alpha\beta}\}_{\mathrm{P}} = \sum_{i,j=1}^{n}\{\mathbf{a}_j^\gamma \mathbf{c}_j^\epsilon x_j^k, \mathbf{a}_i^\alpha \mathbf{c}_i^\beta x_i^l\}\,. \tag{4.42}$$

We need the bracket $\{\mathbf{a}_j^\gamma, \mathbf{c}_i^\beta\}$, obtained by antisymmetry in (4.31c). Namely

$$\begin{aligned}
\{\mathbf{a}_j^\gamma, \mathbf{c}_i^\beta\} =& -\delta_{\beta\gamma}B_{ji} + \mathbf{a}_j^\gamma B_{ji} - \frac{1}{2}\delta_{(j\neq i)}\frac{x_i+x_j}{x_i-x_j}\mathbf{c}_i^\beta(\mathbf{a}_i^\gamma - \mathbf{a}_j^\gamma) + \delta_{(\gamma<\beta)}\mathbf{a}_j^\gamma \mathbf{c}_i^\beta\\
&+\mathbf{a}_j^\gamma\sum_{\mu=1}^{\beta-1}\mathbf{a}_j^\mu(\mathbf{c}_i^\mu - \mathbf{c}_i^\beta) - \delta_{\beta\gamma}\sum_{\mu=1}^{\beta-1}\mathbf{a}_j^\mu \mathbf{c}_i^\mu - \frac{1}{2}\sum_{\sigma=1}^{d}o(\gamma,\sigma)\mathbf{c}_i^\beta(\mathbf{a}_i^\sigma \mathbf{a}_j^\gamma + \mathbf{a}_j^\sigma \mathbf{a}_i^\gamma)\,.
\end{aligned}$$

The right-hand side of (4.42) is given by

$$\begin{aligned}
(4.42)_{RHS} =& \sum_{i,j=1}^{n}x_j^k x_i^l\left(\{\mathbf{a}_j^\gamma, \mathbf{a}_i^\alpha\}\mathbf{c}_j^\epsilon \mathbf{c}_i^\beta + \{\mathbf{a}_j^\gamma, \mathbf{c}_i^\beta\}\mathbf{c}_j^\epsilon \mathbf{a}_i^\alpha\right)\\
&+\sum_{i,j=1}^{n}x_j^k x_i^l\left(\{\mathbf{c}_j^\epsilon, \mathbf{a}_i^\alpha\}\mathbf{a}_j^\gamma \mathbf{c}_i^\beta + \{\mathbf{c}_j^\epsilon, \mathbf{c}_i^\beta\}\mathbf{a}_j^\gamma \mathbf{a}_i^\alpha\right) + (k-l)\sum_{i=1}^{n}\mathbf{a}_i^\gamma \mathbf{c}_i^\epsilon \mathbf{a}_i^\alpha \mathbf{c}_i^\beta x_i^{k+l}\,.
\end{aligned}$$

Hence, using (4.31a)–(4.31d) we can write $(4.42)_{RHS}$ as

$$
\begin{aligned}
&(k-l)\sum_{i=1}^{n}\mathbf{a}_i^\gamma\mathbf{c}_i^\epsilon\mathbf{a}_i^\alpha\mathbf{c}_i^\beta x_i^{k+l}+\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}x_j^kx_i^l\frac{x_j+x_i}{x_j-x_i}(\mathbf{c}_i^\epsilon\mathbf{c}_j^\beta\mathbf{a}_j^\gamma\mathbf{a}_i^\alpha+\mathbf{c}_j^\epsilon\mathbf{c}_i^\beta\mathbf{a}_i^\gamma\mathbf{a}_j^\alpha)\\
&+\frac{1}{2}o(\alpha,\gamma)\sum_{i,j=1}^{n}x_j^kx_i^l\mathbf{c}_j^\epsilon\mathbf{c}_i^\beta(\mathbf{a}_j^\gamma\mathbf{a}_i^\alpha+\mathbf{a}_i^\gamma\mathbf{a}_j^\alpha)+\frac{1}{2}o(\epsilon,\beta)\sum_{i,j=1}^{n}x_j^kx_i^l\mathbf{a}_i^\alpha\mathbf{a}_j^\gamma(\mathbf{c}_i^\epsilon\mathbf{c}_j^\beta-\mathbf{c}_j^\epsilon\mathbf{c}_i^\beta)\\
&+[\delta_{(\gamma<\beta)}-\delta_{(\alpha<\epsilon)}]\sum_{i,j=1}^{n}x_j^kx_i^l\mathbf{a}_j^\gamma\mathbf{c}_j^\epsilon\mathbf{c}_i^\beta\mathbf{a}_i^\alpha\\
&+\delta_{\epsilon\alpha}\sum_{i,j=1}^{n}x_j^kx_i^l\mathbf{c}_i^\beta\mathbf{a}_j^\gamma\left(B_{ij}+\sum_{\lambda=1}^{\epsilon-1}\mathbf{a}_i^\lambda\mathbf{c}_j^\lambda\right)-\delta_{\beta\gamma}\sum_{i,j=1}^{n}x_j^kx_i^l\mathbf{c}_j^\epsilon\mathbf{a}_i^\alpha\left(B_{ji}+\sum_{\mu=1}^{\beta-1}\mathbf{a}_j^\mu\mathbf{c}_i^\mu\right)\,.
\end{aligned}
$$

This is obtained by simplifying terms without any non obvious manipulation. Now, remark that we can write $o(\epsilon,\alpha)=\delta_{(\epsilon<\alpha)}-\delta_{(\epsilon>\alpha)}=1-\delta_{\epsilon\alpha}-2\delta_{(\epsilon>\alpha)}$, so that

$$
\frac{1}{2}[o(\epsilon,\alpha)+\delta_{\epsilon\alpha}-o(\beta,\gamma)-\delta_{\beta\gamma}]=[\delta_{(\gamma<\beta)}-\delta_{(\alpha<\epsilon)}]\,.
$$

We can also repeat the argument in (4.38) backwards, this time with $\chi(i)=\mathbf{c}_i^\beta\mathbf{a}_i^\gamma$ and $\chi'(j)=\mathbf{a}_j^\alpha\mathbf{c}_j^\epsilon$. Incorporating these two facts in the above expression for$(4.42)_{RHS}$, we get

$$
\begin{aligned}
(4.42)_{RHS}=&\frac{1}{2}\left(\sum_{r=1}^{k}-\sum_{r=1}^{l}\right)\sum_{i,j=1}^{n}\mathbf{a}_j^\alpha\mathbf{c}_i^\beta\mathbf{a}_i^\gamma\mathbf{c}_j^\epsilon\left(x_i^rx_j^{k+l-r}+x_i^{k+l-r}x_j^r\right)\\
&+\frac{1}{2}o(\alpha,\gamma)\sum_{i,j=1}^{n}x_j^kx_i^l\mathbf{c}_j^\epsilon\mathbf{c}_i^\beta(\mathbf{a}_j^\gamma\mathbf{a}_i^\alpha+\mathbf{a}_i^\gamma\mathbf{a}_j^\alpha)\\
&+\frac{1}{2}o(\epsilon,\beta)\sum_{i,j=1}^{n}x_j^kx_i^l\mathbf{a}_i^\alpha\mathbf{a}_j^\gamma(\mathbf{c}_i^\epsilon\mathbf{c}_j^\beta-\mathbf{c}_j^\epsilon\mathbf{c}_i^\beta)\\
&+\frac{1}{2}[o(\epsilon,\alpha)+\delta_{\epsilon\alpha}-o(\beta,\gamma)-\delta_{\beta\gamma}]\sum_{i,j=1}^{n}x_j^kx_i^l\mathbf{a}_j^\gamma\mathbf{c}_j^\epsilon\mathbf{c}_i^\beta\mathbf{a}_i^\alpha\\
&+\delta_{\epsilon\alpha}\sum_{i,j=1}^{n}x_j^kx_i^l\mathbf{c}_i^\beta\mathbf{a}_j^\gamma\left(B_{ij}+\sum_{\lambda=1}^{\epsilon-1}\mathbf{a}_i^\lambda\mathbf{c}_j^\lambda\right)\\
&-\delta_{\beta\gamma}\sum_{i,j=1}^{n}x_j^kx_i^l\mathbf{c}_j^\epsilon\mathbf{a}_i^\alpha\left(B_{ji}+\sum_{\mu=1}^{\beta-1}\mathbf{a}_j^\mu\mathbf{c}_i^\mu\right)\,.
\end{aligned}
$$

This is precisely (4.34) which, by definition, is $(4.42)_{LHS}$.

Computations with the Poisson brackets : proof of Proposition 4.3.4

Below, we reproduce the analogous result that appears in [42]. Recall that, by definition, $g_{ij} = \sum_{\alpha=1}^{d} \mathbf{a}_i^\alpha \mathbf{c}_j^\alpha$. Hence, to obtain the Poisson bracket $\{g_{ij}, g_{kl}\}$, we first need to compute $\{\mathbf{a}_i^\alpha, g_{kl}\}$ and $\{\mathbf{c}_j^\alpha, g_{kl}\}$, which is given by the following result.

Lemma 4.3.5 *For any* $\epsilon, \gamma = 1, \dots, d$ *and* $j, k, l = 1, \dots, n$, *writing* $(\mathbf{ac})_{kl} = \sum_\alpha \mathbf{a}_k^\alpha \mathbf{c}_l^\alpha$,

$$\begin{aligned} \{\mathbf{c}_j^\epsilon, (\mathbf{ac})_{kl}\} =&(B_{kj}\mathbf{c}_l^\epsilon - B_{jl}\mathbf{c}_j^\epsilon) + (B_{lj} - B_{kj})(\mathbf{ac})_{kl} + \frac{1}{2}\delta_{(j\neq k)}\frac{x_j + x_k}{x_j - x_k}\mathbf{c}_j^\epsilon((\mathbf{ac})_{jl} - (\mathbf{ac})_{kl}) \\ &+ \frac{1}{2}\delta_{(j\neq l)}\frac{x_j + x_l}{x_j - x_l}(\mathbf{c}_j^\epsilon(\mathbf{ac})_{kl} + \mathbf{c}_l^\epsilon(\mathbf{ac})_{kj}) + \frac{1}{2}\mathbf{c}_l^\epsilon(\mathbf{ac})_{kj} - \frac{1}{2}\mathbf{c}_j^\epsilon(\mathbf{ac})_{jl} \\ &+ (\mathbf{ac})_{kl}\sum_{\lambda=1}^{\epsilon-1}(\mathbf{c}_j^\lambda - \mathbf{c}_j^\epsilon)(\mathbf{a}_l^\lambda - \mathbf{a}_k^\lambda)\,, \end{aligned} \tag{4.43a}$$

$$\begin{aligned} \{\mathbf{a}_i^\gamma, (\mathbf{ac})_{kl}\} =&\mathbf{a}_i^\gamma B_{il} - \mathbf{a}_k^\gamma B_{il} + \frac{1}{2}\delta_{(i\neq k)}\frac{x_i + x_k}{x_i - x_k}(\mathbf{a}_k^\gamma - \mathbf{a}_i^\gamma)((\mathbf{ac})_{il} - (\mathbf{ac})_{kl}) \\ &+ \frac{1}{2}\delta_{(i\neq l)}\frac{x_i + x_l}{x_i - x_l}(\mathbf{ac})_{kl}(\mathbf{a}_l^\gamma - \mathbf{a}_i^\gamma) + \frac{1}{2}\mathbf{a}_i^\gamma(\mathbf{ac})_{il} - \frac{1}{2}\mathbf{a}_k^\gamma(\mathbf{ac})_{il} \\ &+ \frac{1}{2}\sum_{\sigma=1}^{d} o(\gamma,\sigma)(\mathbf{ac})_{kl}[\mathbf{a}_i^\gamma(\mathbf{a}_k^\sigma - \mathbf{a}_l^\sigma) + \mathbf{a}_i^\sigma(\mathbf{a}_k^\gamma - \mathbf{a}_l^\gamma)]\,. \end{aligned} \tag{4.43b}$$

Proof. We compute from (4.31c)–(4.31d) with the normalisation $\sum_\alpha \mathbf{a}_k^\alpha = 1$ that

$$\begin{aligned} \{\mathbf{c}_j^\epsilon, (\mathbf{ac})_{kl}\} =& \sum_{\alpha=1}^{d}\left(\{\mathbf{c}_j^\epsilon, \mathbf{a}_k^\alpha\}\mathbf{c}_l^\alpha + \mathbf{a}_k^\alpha\{\mathbf{c}_j^\epsilon, \mathbf{c}_l^\alpha\}\right) \\ =&\mathbf{c}_l^\epsilon B_{kj} - (\mathbf{ac})_{kl}B_{kj} + \frac{1}{2}\delta_{(j\neq k)}\frac{x_j + x_k}{x_j - x_k}\mathbf{c}_j^\epsilon((\mathbf{ac})_{jl} - (\mathbf{ac})_{kl}) - \sum_{\alpha=1}^{\epsilon-1}\mathbf{c}_l^\alpha\mathbf{a}_k^\alpha\mathbf{c}_j^\epsilon \\ &- (\mathbf{ac})_{kl}\sum_{\lambda=1}^{\epsilon-1}\mathbf{a}_k^\lambda(\mathbf{c}_j^\lambda - \mathbf{c}_j^\epsilon) + \mathbf{c}_l^\epsilon\sum_{\lambda=1}^{\epsilon-1}\mathbf{a}_k^\lambda\mathbf{c}_j^\lambda + \frac{1}{2}\sum_{\alpha=1}^{d}\sum_{\kappa=1}^{d}o(\alpha,\kappa)\mathbf{c}_l^\alpha\mathbf{c}_j^\epsilon(\mathbf{a}_j^\kappa\mathbf{a}_k^\alpha + \mathbf{a}_k^\kappa\mathbf{a}_j^\alpha) \\ &+ \frac{1}{2}\delta_{(j\neq l)}\frac{x_j + x_l}{x_j - x_l}(\mathbf{c}_j^\epsilon(\mathbf{ac})_{kl} + \mathbf{c}_l^\epsilon(\mathbf{ac})_{kj}) + (\mathbf{ac})_{kl}B_{lj} - \mathbf{c}_j^\epsilon B_{jl} \\ &+ \frac{1}{2}\sum_{\alpha=1}^{d}o(\epsilon,\alpha)\mathbf{a}_k^\alpha(\mathbf{c}_l^\epsilon\mathbf{c}_j^\alpha - \mathbf{c}_j^\epsilon\mathbf{c}_l^\alpha) + (\mathbf{ac})_{kl}\sum_{\lambda=1}^{\epsilon-1}\mathbf{a}_l^\lambda(\mathbf{c}_j^\lambda - \mathbf{c}_j^\epsilon) - \sum_{\alpha=1}^{d}\sum_{\mu=1}^{\alpha-1}\mathbf{a}_k^\alpha\mathbf{c}_j^\epsilon\mathbf{a}_j^\mu(\mathbf{c}_l^\mu - \mathbf{c}_l^\alpha)\,. \end{aligned} \tag{4.44}$$

Our aim is to reduce some of these thirteen terms, mostly using properties of the ordering function

$o(-,-)$. Summing the fourth, sixth and eleventh terms of (4.44) together yields

$$\begin{aligned}
&-\sum_{\lambda=1}^{\epsilon-1}\mathbf{a}_k^\lambda\mathbf{c}_l^\lambda\mathbf{c}_j^\epsilon+\sum_{\lambda=1}^{\epsilon-1}\mathbf{a}_k^\lambda\mathbf{c}_j^\lambda\mathbf{c}_l^\epsilon+\frac{1}{2}\left[\sum_{\lambda=\epsilon+1}^{d}-\sum_{\lambda=1}^{\epsilon-1}\right](\mathbf{a}_k^\lambda\mathbf{c}_j^\lambda\mathbf{c}_l^\epsilon-\mathbf{a}_k^\lambda\mathbf{c}_l^\lambda\mathbf{c}_j^\epsilon)\\
=&\frac{1}{2}\sum_{\substack{\lambda=1\\ \lambda\neq\epsilon}}^{d}(\mathbf{a}_k^\lambda\mathbf{c}_j^\lambda\mathbf{c}_l^\epsilon-\mathbf{a}_k^\lambda\mathbf{c}_l^\lambda\mathbf{c}_j^\epsilon)=\frac{1}{2}((\mathbf{ac})_{kj}\mathbf{c}_l^\epsilon-(\mathbf{ac})_{kl}\mathbf{c}_j^\epsilon)\,.
\end{aligned}$$

The fifth and twelfth terms of (4.44) give

$$-(\mathbf{ac})_{kl}\sum_{\lambda=1}^{\epsilon-1}\mathbf{a}_k^\lambda(\mathbf{c}_j^\lambda-\mathbf{c}_j^\epsilon)+(\mathbf{ac})_{kl}\sum_{\lambda=1}^{\epsilon-1}\mathbf{a}_l^\lambda(\mathbf{c}_j^\lambda-\mathbf{c}_j^\epsilon)=(\mathbf{ac})_{kl}\sum_{\lambda=1}^{\epsilon-1}(\mathbf{a}_l^\lambda-\mathbf{a}_k^\lambda)(\mathbf{c}_j^\lambda-\mathbf{c}_j^\epsilon)\,.$$

Relabelling indices, we transform the seventh term from (4.44) as

$$\begin{aligned}
&\frac{1}{2}\left[\sum_{\alpha=1}^{d}\sum_{\kappa=\alpha+1}^{d}-\sum_{\alpha=1}^{d}\sum_{\kappa=1}^{\alpha-1}\right]\mathbf{c}_l^\alpha\mathbf{c}_j^\epsilon(\mathbf{a}_j^\kappa\mathbf{a}_k^\alpha+\mathbf{a}_k^\kappa\mathbf{a}_j^\alpha)\\
=&\frac{1}{2}\sum_{\alpha=1}^{d}\sum_{\mu=1}^{\alpha-1}\mathbf{c}_j^\epsilon(\mathbf{c}_l^\mu\mathbf{a}_j^\alpha\mathbf{a}_k^\mu+\mathbf{c}_l^\mu\mathbf{a}_k^\alpha\mathbf{a}_j^\mu-\mathbf{c}_l^\alpha\mathbf{a}_j^\mu\mathbf{a}_k^\alpha-\mathbf{c}_l^\alpha\mathbf{a}_k^\mu\mathbf{a}_j^\alpha)\\
=&\frac{1}{2}\sum_{\alpha=1}^{d}\sum_{\mu=1}^{\alpha-1}\mathbf{c}_j^\epsilon(\mathbf{c}_l^\mu-\mathbf{c}_l^\alpha)(\mathbf{a}_j^\alpha\mathbf{a}_k^\mu+\mathbf{a}_k^\alpha\mathbf{a}_j^\mu)\,,
\end{aligned}$$

which can be summed with the thirteenth term in (4.44) to yield

$$\begin{aligned}
&\frac{1}{2}\sum_{\alpha=1}^{d}\sum_{\mu=1}^{\alpha-1}\mathbf{c}_j^\epsilon(\mathbf{c}_l^\alpha-\mathbf{c}_l^\mu)\left(2\mathbf{a}_k^\alpha\mathbf{a}_j^\mu-\mathbf{a}_j^\alpha\mathbf{a}_k^\mu-\mathbf{a}_k^\alpha\mathbf{a}_j^\mu\right)=\frac{1}{2}\sum_{\alpha=1}^{d}\sum_{\mu=1}^{\alpha-1}\mathbf{c}_j^\epsilon(\mathbf{c}_l^\alpha-\mathbf{c}_l^\mu)\left(\mathbf{a}_k^\alpha\mathbf{a}_j^\mu-\mathbf{a}_j^\alpha\mathbf{a}_k^\mu\right)\\
=&\frac{1}{2}\sum_{\alpha=1}^{d}\sum_{\mu=1}^{\alpha-1}\mathbf{c}_j^\epsilon(\mathbf{c}_l^\alpha-\mathbf{c}_l^\mu)\mathbf{a}_k^\alpha\mathbf{a}_j^\mu-\frac{1}{2}\sum_{\alpha=1}^{d}\sum_{\mu=\alpha+1}^{d}\mathbf{c}_j^\epsilon(\mathbf{c}_l^\mu-\mathbf{c}_l^\alpha)\mathbf{a}_j^\mu\mathbf{a}_k^\alpha\\
=&\frac{1}{2}\sum_{\alpha=1}^{d}\sum_{\substack{\mu=1\\ \mu\neq\alpha}}^{d}\mathbf{c}_j^\epsilon(\mathbf{c}_l^\alpha-\mathbf{c}_l^\mu)\mathbf{a}_k^\alpha\mathbf{a}_j^\mu=\frac{1}{2}\sum_{\alpha=1}^{d}\sum_{\mu=1}^{d}\mathbf{c}_j^\epsilon(\mathbf{a}_k^\alpha\mathbf{c}_l^\alpha\mathbf{a}_j^\mu-\mathbf{a}_j^\mu\mathbf{c}_l^\mu\mathbf{a}_k^\alpha)=\frac{1}{2}\mathbf{c}_j^\epsilon\left((\mathbf{ac})_{kl}-(\mathbf{ac})_{jl}\right)\,.
\end{aligned}$$

Introducing the different terms back in (4.44), we find

$$\begin{aligned}
\{\mathbf{c}_j^\epsilon,(\mathbf{ac})_{kl}\}=&(\mathbf{c}_l^\epsilon B_{kj}-\mathbf{c}_j^\epsilon B_{jl})+(B_{lj}-B_{kj})(\mathbf{ac})_{kl}+\frac{1}{2}\delta_{(j\neq k)}\frac{x_j+x_k}{x_j-x_k}\mathbf{c}_j^\epsilon((\mathbf{ac})_{jl}-(\mathbf{ac})_{kl})\\
&+\frac{1}{2}\delta_{(j\neq l)}\frac{x_j+x_l}{x_j-x_l}(\mathbf{c}_j^\epsilon(\mathbf{ac})_{kl}+\mathbf{c}_l^\epsilon(\mathbf{ac})_{kj})+\frac{1}{2}(\mathbf{c}_l^\epsilon(\mathbf{ac})_{kj}-\mathbf{c}_j^\epsilon(\mathbf{ac})_{jl})\\
&+(\mathbf{ac})_{kl}\sum_{\lambda=1}^{\epsilon-1}(\mathbf{c}_j^\lambda-\mathbf{c}_j^\epsilon)(\mathbf{a}_l^\lambda-\mathbf{a}_k^\lambda)\,,
\end{aligned}$$

as desired. For the second identity, we get from (4.31b)-(4.43)

$$
\begin{aligned}
&\{\mathbf{a}_i^\gamma, (\mathbf{ac})_{kl}\} = \sum_{\alpha=1}^{d} \left(\{\mathbf{a}_i^\gamma, \mathbf{a}_k^\alpha\}\mathbf{c}_l^\alpha + \mathbf{a}_k^\alpha\{\mathbf{a}_i^\gamma, \mathbf{c}_l^\alpha\}\right) \\
=&\frac{1}{2}\delta_{(i\neq k)}\frac{x_i+x_k}{x_i-x_k}(\mathbf{a}_i^\gamma - \mathbf{a}_k^\gamma)((\mathbf{ac})_{kl} - (\mathbf{ac})_{il}) + \frac{1}{2}\sum_{\alpha=1}^{d} o(\alpha,\gamma)(\mathbf{a}_i^\gamma\mathbf{a}_k^\alpha + \mathbf{a}_k^\gamma\mathbf{a}_i^\alpha)\mathbf{c}_l^\alpha \\
&+\frac{1}{2}(\mathbf{ac})_{kl}\sum_{\sigma=1}^{d} o(\gamma,\sigma)(\mathbf{a}_i^\gamma\mathbf{a}_k^\sigma + \mathbf{a}_k^\gamma\mathbf{a}_i^\sigma) - \frac{1}{2}\sum_{\alpha=1}^{d}\sum_{\kappa=1}^{d} o(\alpha,\kappa)\mathbf{a}_i^\gamma\mathbf{c}_l^\alpha(\mathbf{a}_i^\kappa\mathbf{a}_k^\alpha + \mathbf{a}_k^\kappa\mathbf{a}_i^\alpha) \\
&-\mathbf{a}_k^\gamma B_{il} + \mathbf{a}_i^\gamma B_{il} + \frac{1}{2}\delta_{(i\neq l)}\frac{x_i+x_l}{x_i-x_l}(\mathbf{ac})_{kl}(\mathbf{a}_l^\gamma - \mathbf{a}_i^\gamma) + \sum_{\alpha=\gamma+1}^{d}\mathbf{a}_k^\alpha\mathbf{a}_i^\gamma\mathbf{c}_l^\alpha \\
&+\sum_{\alpha=1}^{d}\sum_{\mu=1}^{\alpha-1}\mathbf{a}_k^\alpha\mathbf{a}_i^\gamma\mathbf{a}_i^\mu(\mathbf{c}_l^\mu - \mathbf{c}_l^\alpha) - \mathbf{a}_k^\gamma\sum_{\mu=1}^{\gamma-1}\mathbf{a}_i^\mu\mathbf{c}_l^\mu - \frac{1}{2}\sum_{\sigma=1}^{d} o(\gamma,\sigma)(\mathbf{ac})_{kl}(\mathbf{a}_l^\sigma\mathbf{a}_i^\gamma + \mathbf{a}_i^\sigma\mathbf{a}_l^\gamma)\,.
\end{aligned}
\tag{4.45}
$$

The second, eighth and tenth terms of (4.45) become

$$
\begin{aligned}
&\frac{1}{2}\left[\sum_{\alpha=1}^{\gamma-1} - \sum_{\alpha=\gamma+1}^{d}\right](\mathbf{a}_i^\gamma\mathbf{a}_k^\alpha\mathbf{c}_l^\alpha + \mathbf{a}_k^\gamma\mathbf{a}_i^\alpha\mathbf{c}_l^\alpha) + \sum_{\alpha=\gamma+1}^{d}\mathbf{a}_i^\gamma\mathbf{a}_k^\alpha\mathbf{c}_l^\alpha - \sum_{\alpha=1}^{\gamma-1}\mathbf{a}_k^\gamma\mathbf{a}_i^\alpha\mathbf{c}_l^\alpha \\
=&\frac{1}{2}\sum_{\substack{\alpha=1\\ \alpha\neq\gamma}}^{\gamma-1}(\mathbf{a}_i^\gamma\mathbf{a}_k^\alpha\mathbf{c}_l^\alpha - \mathbf{a}_k^\gamma\mathbf{a}_i^\alpha\mathbf{c}_l^\alpha) \;=\; \frac{1}{2}\mathbf{a}_i^\gamma(\mathbf{ac})_{kl} - \frac{1}{2}\mathbf{a}_k^\gamma(\mathbf{ac})_{il}\,.
\end{aligned}
$$

We write the third and eleventh terms of (4.45) as

$$
\frac{1}{2}\sum_{\sigma=1}^{d} o(\gamma,\sigma)(\mathbf{ac})_{kl}\big(\mathbf{a}_i^\gamma(\mathbf{a}_k^\sigma - \mathbf{a}_l^\sigma) + \mathbf{a}_i^\sigma(\mathbf{a}_k^\gamma - \mathbf{a}_l^\gamma)\big)\,.
$$

Now, we transform the fourth term of (4.45) :

$$
\begin{aligned}
&-\frac{1}{2}\left[\sum_{\alpha=1}^{d}\sum_{\kappa=\alpha+1}^{d} - \sum_{\alpha=1}^{d}\sum_{\kappa=1}^{\alpha-1}\right]\mathbf{a}_i^\gamma(\mathbf{a}_i^\kappa\mathbf{a}_k^\alpha\mathbf{c}_l^\alpha + \mathbf{a}_k^\kappa\mathbf{a}_i^\alpha\mathbf{c}_l^\alpha) \\
=&\frac{1}{2}\sum_{\alpha=1}^{d}\sum_{\kappa=1}^{\alpha-1}\mathbf{a}_i^\gamma(\mathbf{a}_i^\kappa\mathbf{a}_k^\alpha\mathbf{c}_l^\alpha + \mathbf{a}_k^\kappa\mathbf{a}_i^\alpha\mathbf{c}_l^\alpha - \mathbf{a}_i^\alpha\mathbf{a}_k^\kappa\mathbf{c}_l^\kappa - \mathbf{a}_k^\alpha\mathbf{a}_i^\kappa\mathbf{c}_l^\kappa) \\
=&\frac{1}{2}\sum_{\alpha=1}^{d}\sum_{\kappa=1}^{\alpha-1}\mathbf{a}_i^\gamma(\mathbf{c}_l^\alpha - \mathbf{c}_l^\kappa)(\mathbf{a}_i^\kappa\mathbf{a}_k^\alpha + \mathbf{a}_k^\kappa\mathbf{a}_i^\alpha)\,.
\end{aligned}
$$

This can be summed with the ninth term of (4.45) to give

$$\begin{aligned}
&\frac{1}{2}\sum_{\alpha=1}^{d}\sum_{\kappa=1}^{\alpha-1}\mathbf{a}_i^\gamma(\mathbf{c}_l^\kappa-\mathbf{c}_l^\alpha)(2\mathbf{a}_i^\kappa\mathbf{a}_k^\alpha-\mathbf{a}_i^\kappa\mathbf{a}_k^\alpha-\mathbf{a}_k^\kappa\mathbf{a}_i^\alpha)=\frac{1}{2}\sum_{\alpha=1}^{d}\sum_{\kappa=1}^{\alpha-1}\mathbf{a}_i^\gamma(\mathbf{c}_l^\kappa-\mathbf{c}_l^\alpha)(\mathbf{a}_i^\kappa\mathbf{a}_k^\alpha-\mathbf{a}_k^\kappa\mathbf{a}_i^\alpha)\\
=&\frac{1}{2}\sum_{\alpha=1}^{d}\sum_{\kappa=1}^{\alpha-1}\mathbf{a}_i^\gamma(\mathbf{c}_l^\kappa-\mathbf{c}_l^\alpha)\mathbf{a}_i^\kappa\mathbf{a}_k^\alpha-\sum_{\alpha=1}^{d}\sum_{\kappa=\alpha+1}^{d}\mathbf{a}_i^\gamma(\mathbf{c}_l^\alpha-\mathbf{c}_l^\kappa)\mathbf{a}_k^\alpha\mathbf{a}_i^\kappa\\
=&\frac{1}{2}\sum_{\alpha=1}^{d}\sum_{\substack{\kappa=1\\ \kappa\neq\alpha}}^{\alpha-1}\mathbf{a}_i^\gamma(\mathbf{c}_l^\kappa-\mathbf{c}_l^\alpha)\mathbf{a}_i^\kappa\mathbf{a}_k^\alpha=\frac{1}{2}\sum_{\alpha=1}^{d}\sum_{\substack{\kappa=1\\ \kappa\neq\alpha}}^{\alpha-1}\mathbf{a}_i^\gamma(\mathbf{a}_i^\kappa\mathbf{c}_l^\kappa\mathbf{a}_k^\alpha-\mathbf{a}_k^\alpha\mathbf{c}_l^\alpha\mathbf{a}_i^\kappa)=\frac{1}{2}\mathbf{a}_i^\gamma((\mathbf{ac})_{il}-(\mathbf{ac})_{kl})\,.
\end{aligned}$$

We have transformed (4.45) such that

$$\begin{aligned}
\{\mathbf{a}_i^\gamma,(\mathbf{ac})_{kl}\}=&(\mathbf{a}_i^\gamma-\mathbf{a}_k^\gamma)B_{il}+\frac{1}{2}\delta_{(i\neq k)}\frac{x_i+x_k}{x_i-x_k}(\mathbf{a}_k^\gamma-\mathbf{a}_i^\gamma)((\mathbf{ac})_{il}-(\mathbf{ac})_{kl})\\
&+\frac{1}{2}\delta_{(i\neq l)}\frac{x_i+x_l}{x_i-x_l}(\mathbf{a}_l^\gamma-\mathbf{a}_i^\gamma)(\mathbf{ac})_{kl}+\frac{1}{2}\mathbf{a}_i^\gamma(\mathbf{ac})_{il}-\frac{1}{2}\mathbf{a}_k^\gamma(\mathbf{ac})_{il}\\
&+\frac{1}{2}\sum_{\sigma=1}^{d}o(\gamma,\sigma)(\mathbf{ac})_{kl}\big(\mathbf{a}_i^\gamma(\mathbf{a}_k^\sigma-\mathbf{a}_l^\sigma)+\mathbf{a}_i^\sigma(\mathbf{a}_k^\gamma-\mathbf{a}_l^\gamma)\big)\,,
\end{aligned}$$

which finishes the proof. □

Now, to establish Proposition 4.3.4, we have from Lemma 4.3.5

$$\begin{aligned}
\{g_{ij},g_{kl}\}=&\sum_{\gamma=1}^{d}\Big(\{\mathbf{a}_i^\gamma,(\mathbf{ac})_{kl}\}\mathbf{c}_j^\gamma+\mathbf{a}_i^\gamma\{\mathbf{c}_j^\gamma,(\mathbf{ac})_{kl}\}\Big)\\
=&((\mathbf{ac})_{ij}-(\mathbf{ac})_{kj})B_{il}+\frac{1}{2}\delta_{(i\neq k)}\frac{x_i+x_k}{x_i-x_k}((\mathbf{ac})_{kj}-(\mathbf{ac})_{ij})((\mathbf{ac})_{il}-(\mathbf{ac})_{kl})\\
&+\frac{1}{2}\delta_{(i\neq l)}\frac{x_i+x_l}{x_i-x_l}((\mathbf{ac})_{lj}-(\mathbf{ac})_{ij})(\mathbf{ac})_{kl}+\frac{1}{2}(\mathbf{ac})_{ij}(\mathbf{ac})_{il}-\frac{1}{2}(\mathbf{ac})_{kj}(\mathbf{ac})_{il}\\
&+\frac{1}{2}\sum_{\gamma=1}^{d}\sum_{\sigma=1}^{d}o(\gamma,\sigma)(\mathbf{ac})_{kl}\big(\mathbf{a}_i^\gamma\mathbf{c}_j^\gamma(\mathbf{a}_k^\sigma-\mathbf{a}_l^\sigma)+\mathbf{a}_i^\sigma(\mathbf{a}_k^\gamma-\mathbf{a}_l^\gamma)\mathbf{c}_j^\gamma\big)\\
&+B_{kj}(\mathbf{ac})_{il}-B_{jl}(\mathbf{ac})_{ij}+(B_{lj}-B_{kj})(\mathbf{ac})_{kl}\\
&+\frac{1}{2}\delta_{(j\neq k)}\frac{x_j+x_k}{x_j-x_k}(\mathbf{ac})_{ij}((\mathbf{ac})_{jl}-(\mathbf{ac})_{kl})\\
&+\frac{1}{2}\delta_{(j\neq l)}\frac{x_j+x_l}{x_j-x_l}((\mathbf{ac})_{ij}(\mathbf{ac})_{kl}+(\mathbf{ac})_{il}(\mathbf{ac})_{kj})\\
&+\frac{1}{2}(\mathbf{ac})_{il}(\mathbf{ac})_{kj}-\frac{1}{2}(\mathbf{ac})_{ij}(\mathbf{ac})_{jl}+(\mathbf{ac})_{kl}\sum_{\gamma=1}^{d}\sum_{\lambda=1}^{\gamma-1}\mathbf{a}_i^\gamma(\mathbf{c}_j^\lambda-\mathbf{c}_j^\gamma)(\mathbf{a}_l^\lambda-\mathbf{a}_k^\lambda)\,.
\end{aligned}\tag{4.46}$$

Again, we used the fact that $\sum_{\gamma=1}^{d}\mathbf{a}_i^\gamma=1$. The sums in the third line of (4.46) can be reexpressed as follows :

$$\begin{aligned}
&\frac{1}{2}(\mathbf{ac})_{kl}\sum_{\gamma=1}^{d}\left[\sum_{\sigma=\gamma+1}^{d}-\sum_{\sigma=1}^{\gamma-1}\right]\left(\mathbf{a}_i^\gamma\mathbf{c}_j^\gamma(\mathbf{a}_k^\sigma-\mathbf{a}_l^\sigma)+\mathbf{a}_i^\sigma(\mathbf{a}_k^\gamma-\mathbf{a}_l^\gamma)\mathbf{c}_j^\gamma\right)\\
=&\frac{1}{2}(\mathbf{ac})_{kl}\sum_{\gamma=1}^{d}\left[\sum_{\sigma=\gamma+1}^{d}-\sum_{\sigma=1}^{\gamma-1}\right]\left(\mathbf{a}_i^\gamma\mathbf{c}_j^\gamma(\mathbf{a}_k^\sigma-\mathbf{a}_l^\sigma)-\mathbf{a}_i^\gamma(\mathbf{a}_k^\sigma-\mathbf{a}_l^\sigma)\mathbf{c}_j^\sigma\right),
\end{aligned}$$

after swapping the labels $\sigma \leftrightarrow \gamma$ in the second term of the sum. This is nothing else than

$$\frac{1}{2}(\mathbf{ac})_{kl}\sum_{\gamma=1}^{d}\left[\sum_{\sigma=\gamma+1}^{d}-\sum_{\sigma=1}^{\gamma-1}\right]\mathbf{a}_i^\gamma(\mathbf{c}_j^\gamma-\mathbf{c}_j^\sigma)(\mathbf{a}_k^\sigma-\mathbf{a}_l^\sigma)\,.$$

Summing with the last term of (4.46), we get

$$\begin{aligned}
&\frac{1}{2}(\mathbf{ac})_{kl}\sum_{\gamma=1}^{d}\left[\sum_{\lambda=\gamma+1}^{d}-\sum_{\lambda=1}^{\gamma-1}\right]\mathbf{a}_i^\gamma(\mathbf{c}_j^\gamma-\mathbf{c}_j^\lambda)(\mathbf{a}_k^\lambda-\mathbf{a}_l^\lambda)+(\mathbf{ac})_{kl}\sum_{\gamma=1}^{d}\sum_{\lambda=1}^{\gamma-1}\mathbf{a}_i^\gamma(\mathbf{c}_j^\lambda-\mathbf{c}_j^\gamma)(\mathbf{a}_l^\lambda-\mathbf{a}_k^\lambda)\\
=&\frac{1}{2}(\mathbf{ac})_{kl}\sum_{\gamma=1}^{d}\sum_{\substack{\lambda=1\\ \lambda\neq\gamma}}^{d}\mathbf{a}_i^\gamma(\mathbf{c}_j^\gamma-\mathbf{c}_j^\lambda)(\mathbf{a}_k^\lambda-\mathbf{a}_l^\lambda)\\
=&\frac{1}{2}(\mathbf{ac})_{kl}\sum_{\gamma=1}^{d}\sum_{\lambda=1}^{d}\mathbf{a}_i^\gamma\left(\mathbf{c}_j^\gamma\mathbf{a}_k^\lambda-\mathbf{c}_j^\gamma\mathbf{a}_l^\lambda-\mathbf{a}_k^\lambda\mathbf{c}_j^\lambda+\mathbf{a}_l^\lambda\mathbf{c}_j^\lambda\right)\\
=&\;\frac{1}{2}(\mathbf{ac})_{kl}\big((\mathbf{ac})_{ij}-(\mathbf{ac})_{ij}-(\mathbf{ac})_{kj}+(\mathbf{ac})_{lj}\big)=\frac{1}{2}(\mathbf{ac})_{kl}(\mathbf{ac})_{lj}-\frac{1}{2}(\mathbf{ac})_{kl}(\mathbf{ac})_{kj}\,.
\end{aligned}$$

If we reintroduce the elements $g_{ij}=(\mathbf{ac})_{ij}$, we can then rewrite (4.46) as

$$\begin{aligned}
\{g_{ij},g_{kl}\}_{\mathrm{P}}=&g_{ij}B_{il}-g_{kj}B_{il}+\frac{1}{2}\delta_{(i\neq k)}\frac{x_i+x_k}{x_i-x_k}(g_{kj}g_{il}-g_{kj}g_{kl}-g_{ij}g_{il}+g_{ij}g_{kl})\\
&+\frac{1}{2}\delta_{(i\neq l)}\frac{x_i+x_l}{x_i-x_l}(g_{lj}g_{kl}-g_{ij}g_{kl})+\frac{1}{2}g_{ij}g_{il}-\frac{1}{2}g_{kj}g_{il}+\frac{1}{2}g_{kl}g_{lj}-\frac{1}{2}g_{kl}g_{kj}\\
&+B_{kj}g_{il}-B_{jl}g_{ij}+B_{lj}g_{kl}-B_{kj}g_{kl}+\frac{1}{2}\delta_{(j\neq k)}\frac{x_j+x_k}{x_j-x_k}(g_{ij}g_{jl}-g_{ij}g_{kl})\\
&+\frac{1}{2}\delta_{(j\neq l)}\frac{x_j+x_l}{x_j-x_l}(g_{ij}g_{kl}+g_{il}g_{kj})+\frac{1}{2}g_{il}g_{kj}-\frac{1}{2}g_{ij}g_{jl}\\
=&g_{ij}\left(B_{il}+\frac{1}{2}g_{il}\right)-g_{kj}\left(B_{il}+\frac{1}{2}g_{il}\right)+g_{il}\left(B_{kj}+\frac{1}{2}g_{kj}\right)-g_{ij}\left(B_{jl}+\frac{1}{2}g_{jl}\right)\\
&+g_{kl}\left(B_{lj}+\frac{1}{2}g_{lj}\right)-g_{kl}\left(B_{kj}+\frac{1}{2}g_{kj}\right)+\frac{1}{2}\delta_{(i\neq l)}\frac{x_i+x_l}{x_i-x_l}(g_{lj}g_{kl}-g_{ij}g_{kl})\\
&+\frac{1}{2}\delta_{(i\neq k)}\frac{x_i+x_k}{x_i-x_k}(g_{kj}g_{il}-g_{kj}g_{kl}-g_{ij}g_{il}+g_{ij}g_{kl})\\
&+\frac{1}{2}\delta_{(j\neq k)}\frac{x_j+x_k}{x_j-x_k}(g_{ij}g_{jl}-g_{ij}g_{kl})+\frac{1}{2}\delta_{(j\neq l)}\frac{x_j+x_l}{x_j-x_l}(g_{ij}g_{kl}+g_{il}g_{kj})\,,
\end{aligned}$$

after rearranging terms. Next, notice that for any i, j

$$B_{ij} + \frac{1}{2}g_{ij} = \frac{qg_{ij}x_j}{x_i - qx_j} + \frac{1}{2}g_{ij} = \frac{1}{2}\frac{x_i + qx_j}{x_i - qx_j}g_{ij}\,,$$

so that

$$\begin{aligned}\{g_{ij}, g_{kl}\}_{\mathrm{P}} =& \frac{1}{2}\frac{x_i + qx_l}{x_i - qx_l}g_{ij}g_{il} - \frac{1}{2}\frac{x_i + qx_l}{x_i - qx_l}g_{kj}g_{il} + \frac{1}{2}\frac{x_k + qx_j}{x_k - qx_j}g_{il}g_{kj} - \frac{1}{2}\frac{x_j + qx_l}{x_j - qx_l}g_{ij}g_{jl} \\ &+ \frac{1}{2}\frac{x_l + qx_j}{x_l - qx_j}g_{kl}g_{lj} - \frac{1}{2}\frac{x_k + qx_j}{x_k - qx_j}g_{kl}g_{kj} + \frac{1}{2}\delta_{(i\neq l)}\frac{x_i + x_l}{x_i - x_l}(g_{lj}g_{kl} - g_{ij}g_{kl}) \\ &+ \frac{1}{2}\delta_{(i\neq k)}\frac{x_i + x_k}{x_i - x_k}(g_{kj}g_{il} - g_{kj}g_{kl} - g_{ij}g_{il} + g_{ij}g_{kl}) \\ &+ \frac{1}{2}\delta_{(j\neq k)}\frac{x_j + x_k}{x_j - x_k}(g_{ij}g_{jl} - g_{ij}g_{kl}) + \frac{1}{2}\delta_{(j\neq l)}\frac{x_j + x_l}{x_j - x_l}(g_{ij}g_{kl} + g_{il}g_{kj})\,.\end{aligned}$$

It is not hard to see that, by grouping terms together, we get (4.32).

4.3.2 Relation to the Arutyunov–Frolov conjecture

We can consider locally the coordinates $(q_i)_i$ instead of $(x_i)_i$, which we define by $x_i = e^{2q_i}$. Similarly, we put $q = e^{-2\gamma}$ for some $\gamma \in \mathbb{C}^\times \setminus i\pi\mathbb{Q}$. Then, we can see that Proposition 4.3.4 can be written as

$$\begin{aligned}\{q_i, q_k\} =& 0\,, \quad \{g_{ij}, q_k\} = -\frac{1}{2}\delta_{jk}g_{ij}\,, \\ \{g_{ij}, g_{kl}\} =& \frac{1}{2}[\coth(q_{ik}) + \coth(q_{jl}) + \coth(q_{kj}) + \coth(q_{li})]g_{ij}g_{kl} \\ &+ \frac{1}{2}[\coth(q_{ik}) + \coth(q_{jl}) + \coth(q_{kj} + \gamma) - \coth(q_{il} + \gamma)]g_{il}g_{kj} \\ &+ \frac{1}{2}[\coth(q_{ki}) + \coth(q_{il} + \gamma)]g_{ij}g_{il} + \frac{1}{2}[\coth(q_{jk}) - \coth(q_{jl} + \gamma)]g_{ij}g_{jl} \\ &+ \frac{1}{2}[\coth(q_{ki}) - \coth(q_{kj} + \gamma)]g_{kj}g_{kl} + [\coth(q_{il}) + \coth(q_{lj} + \gamma)]g_{lj}g_{kl}\end{aligned}$$

where we write $q_{ij} = q_i - q_j$, and we take the convention that a term with vanishing denominator is omitted. We can readily see that if we set $\mathbf{f}_{ij} = g_{ij}$ and consider the Poisson bracket $\{-,-\}_A = 2\{-,-\}$, these expressions are nothing else than (1.13a)–(1.13b), i.e. the Poisson brackets conjectured by Arutyunov and Frolov. The analogy can be pushed further. With those coordinates, the matrix B defined by (4.28) has entries $\frac{e^{-\gamma}}{2}\frac{e^{-q_{ij}}\mathbf{f}_{ij}}{\sinh(q_{ij}+\gamma)}$, so that $2e^{\gamma}B$ is gauge equivalent to the Lax matrix $\mathbf{L}$ defined by (1.12). Hence, $\operatorname{tr} B$ should define the equation of motion for the trigonometric spin RS system up to a multiplicative constant, which we prove in § 4.3.3. If we continue to look only at the Poisson structure for the moment, it remains to see that (4.31a)–(4.31d) yield (1.11a)–(1.11d) when we degenerate to the rational case. To do so, we put $x_i = e^{2\beta q_i}$ and $q = e^{-2\beta\gamma}$. Then, in the limit $\beta \to 0$, we get that $2\beta B_{ij} \to \frac{\mathbf{f}_{ij}}{q_{ij}+\gamma}$, which are precisely the entries of the matrix $\mathbf{L}$ of Arutyunov and Frolov in the rational case. In a similar way, note that $2\beta\coth(\beta q_{ij}) \to 1/q_{ij}$ in this limit. Now, if we rescale the bracket by a factor 2β before taking the limit in (4.31a)–(4.31d), we get (1.11a)–(1.11d) as expected. Thus the terms containing the ordering of the spins disappear in the rational case, and only the terms with a factor $\coth(q_{ji})$ are visible in this limit. This was expected since the extra terms between the spin variables are a pure consequence of the fusion process underlying the quasi-Hamiltonian formalism we used, while this formalism is not needed in the rational case which can be obtained by Hamiltonian reduction.

Finally, we could ask if it is possible to obtain the Poisson brackets between the non-normalised coordinates (a_i^α, c_i^α), see Section 1.3. To do so, recall that we work on the space $\mathfrak{h}_{sp}$ introduced in § 4.3.1 with coordinates $(x_i, \mathbf{a}_i^\alpha, \mathbf{c}_i^\alpha)_{i\alpha}$ subject to $\sum_\alpha \mathbf{a}_i^\alpha = 1$ and other conditions, see (4.27). We can get these elements from non-normalised coordinates $(x_i, a_i^\alpha, c_i^\alpha)_{i\alpha}$ of $\widetilde{\mathfrak{h}_{sp}} = \mathfrak{h}_{\text{reg}} \times \mathfrak{h}^d \times \mathfrak{h}^d$ through the generically well-defined map $\tilde{N} : \widetilde{\mathfrak{h}_{sp}} \to \mathfrak{h}_{sp}$ given by

$$\tilde{N}(x_i, a_i^\alpha, c_i^\alpha) = (x_i, \mathbf{a}_i^\alpha, \mathbf{c}_i^\alpha)\,, \quad \text{where } \mathbf{a}_i^\alpha = \frac{a_i^\alpha}{\sum_\beta a_i^\beta}\,, \quad \mathbf{c}_i^\alpha = c_i^\alpha \sum_\beta a_i^\beta\,. \tag{4.47}$$

We would like to define a Poisson bracket on $\widetilde{\mathfrak{h}_{sp}}$ that induces (4.31a)–(4.31d).

Lemma 4.3.6 *The map $\tilde{N} : \widetilde{\mathfrak{h}_{sp}} \to \mathfrak{h}_{sp}$ intertwines the Poisson bracket $\{-,-\}$ on $\mathfrak{h}_{sp}$ defined in Proposition 4.3.3 and the antisymmetric biderivation $\{-,-\}'$ on $\widetilde{\mathfrak{h}_{sp}}$ given by*

$$\{x_i, x_j\}' = 0\,, \quad \{a_i^\alpha, x_j\}' = 0\,, \quad \{c_i^\alpha, x_j\}' = -\delta_{ij} c_i^\alpha x_j\,, \tag{4.48a}$$

$$\{a_j^\gamma, a_i^\alpha\}' = \frac{1}{2}\delta_{(i\neq j)}\frac{x_j + x_i}{x_j - x_i} a_i^\gamma a_j^\alpha + \frac{1}{2} o(\alpha,\gamma)(a_j^\gamma a_i^\alpha + a_i^\gamma a_j^\alpha)\,, \tag{4.48b}$$

$$\{c_j^\epsilon, a_i^\alpha\}' = \delta_{\epsilon\alpha}\left[B_{ij}^\vee + \sum_{\mu=1}^{\epsilon-1} a_i^\mu c_j^\mu\right] - \delta_{(\alpha<\epsilon)} a_i^\alpha c_j^\epsilon\,, \tag{4.48c}$$

$$\{c_j^\epsilon, c_i^\beta\}' = \frac{1}{2}\delta_{(i\neq j)}\frac{x_j + x_i}{x_j - x_i} c_i^\epsilon c_j^\beta + \frac{1}{2} o(\epsilon,\beta)(c_i^\epsilon c_j^\beta - c_j^\epsilon c_i^\beta)\,, \tag{4.48d}$$

where $o(-,-)$ is the ordering function on d elements defined in Section 1.5, and $B_{ij}^\vee = q\frac{\sum_\alpha a_i^\alpha c_j^\alpha}{x_i x_j^{-1} - q}$.

Proof. Write $D_i = \sum_\beta a_i^\beta$. It is not difficult to show that $(x_i, a_i^\alpha D_i^{-1}, c_i^\alpha D_i)$ satisfy relations (4.31a)–(4.31d). One has to remark that $D_i^{-1} B_{ij}^\vee D_j = B_{ij}$ for the matrix B defined in (4.28).

Hence $\tilde{N}(B^\vee) = B$. Then, the only non obvious identity to use is that

$$\sum_{\sigma,\kappa=1}^{d} o(\kappa,\sigma)\,(a_j^\kappa a_i^\sigma + a_i^\kappa a_j^\sigma) = 0\,,$$

which we showed using the skewsymmetry of $o(-,-)$ in the proof of Proposition 4.3.3. □

Note the easier form of the expressions (4.48a)–(4.48d) compared to their normalised version (4.31a)–(4.31d). Introducing the functions $f_{ij} = \sum_\alpha a_i^\alpha c_j^\alpha$ on $\widetilde{\mathfrak{h}_{sp}}$ such that $\tilde{N}(f_{ij}) = \mathbf{f}_{ij} = g_{ij}$, we can also compute their brackets as an analogue of Proposition 4.3.4.

Lemma 4.3.7 *For any $i,j,k,l = 1,\ldots,n$, $\{f_{ij}, x_k\}' = -\delta_{jk} f_{ij} x_k$ and*

$$\{f_{ij}, f_{kl}\}' = \frac{1}{2} f_{il} f_{kj} \left[\frac{2qx_j}{x_k - qx_j} - \frac{2qx_l}{x_i - qx_l} + \delta_{(i\neq k)} \frac{x_i + x_k}{x_i - x_k} + \delta_{(j\neq l)} \frac{x_j + x_k}{x_j - x_l} \right]. \tag{4.49}$$

In particular, the commutative subalgebra of $\mathcal{O}(\widetilde{\mathfrak{h}_{sp}})$ generated by (x_k, f_{ij}) and localised at $\{x_i - x_j, x_i - qx_j \mid i \neq j\}$ is stable under $\{-,-\}'$.

Proof. In a way similar to Lemma 4.3.5, we find that

$$\begin{aligned}
\{a_i^\gamma, f_{kl}\}' &= -\, a_k^\gamma B_{il}^\vee + \frac{1}{2}\delta_{(i\neq k)} \frac{x_i + x_k}{x_i - x_k} a_k^\gamma f_{il} + \frac{1}{2} a_i^\gamma f_{kl} - \frac{1}{2} a_k^\gamma f_{il}\,, \\
\{c_i^\epsilon, f_{kl}\}' &= c_l^\epsilon B_{ki}^\vee + \frac{1}{2}\delta_{(i\neq l)} \frac{x_i + x_l}{x_i - x_l} c_l^\epsilon f_{ki} + \frac{1}{2} c_l^\epsilon f_{ki} - \frac{1}{2} c_i^\epsilon f_{kl}\,.
\end{aligned}$$

From $\{g_{ij}, f_{kl}\}' = \sum_\gamma (\{a_i^\gamma, f_{kl}\}' c_j^\gamma + a_i^\gamma \{c_j^\gamma, f_{kl}\}')$, we can then compute (4.49) easily. □

Though these brackets are easier to deal with, we have been unable to prove that Jacobi identity is satisfied for any triple of generators. We conjecture that this is the case, so that $\{-,-\}'$ would be a Poisson bracket. It is then an interesting problem to try to extract the momentum p_i from the elements $(c_i^\alpha)_\alpha$ for each i. In the rational case, this is already done in [16].

4.3.3 Integrable systems

For any $U \in \{X, Y, Z, \mathrm{Id}_n + XY\}$, we have candidates for integrability from Propositions 4.1.3 and 4.1.5. We will study these two cases separately. As, in each case, the elements $\operatorname{tr} U^k$ are part of the family to consider, we first look at the local expression of such functions.

Local expressions

We use the isomorphism of Proposition 4.3.1 to write the matrices in local coordinates, and we omit to write the map ξ from now on. For example, we write that on $\mathcal{C}'_{n,q,d}$ we have $Z_{ij} = q\frac{g_{ij}x_j}{x_i - qx_j}$ though this is in fact $\xi^* Z_{ij}$.

As in the case $d = 1$ discussed in § 4.2.2, we have that $\operatorname{tr} X^k = \sum_i x_i^k$ trivially. Next, we have for $G_k^{1,d} = \operatorname{tr} Z^k$ that

$$G_1^{1,d} = \frac{q}{1-q}\sum_i g_{ii}\,, \quad G_k^{1,d} = q^k \sum_{i_1,\ldots,i_k=1}^{n} \frac{g_{i_1i_2}x_{i_2}}{x_{i_1} - qx_{i_2}} \cdots \frac{g_{i_ki_1}x_{i_1}}{x_{i_k} - qx_{i_1}}\,, \tag{4.50}$$

with $k \geq 2$. If we let $\frac{d}{dt} = \{-, G_1^{1,d}\}_{\mathrm{P}}$, we can find the corresponding set of ODEs on local coordinates $(q_i, \mathbf{a}_i^\alpha, \mathbf{c}_i^\alpha)$, where each q_i satisfies $e^{2q_i} = x_i$. To do so, introduce the function

$$V_{ik} = \frac{2(1-q)x_i x_k}{(x_i - x_k)(x_i - qx_k)} = \coth(q_i - q_k) - \coth((q_i - q_k) + \gamma)\,. \tag{4.51}$$

Hence, $V_{ik} = V^{trig}(q_i - q_k)$ for the trigonometric potential introduced in Section 1.3.

Lemma 4.3.8 *For any $i = 1, \ldots, n$ and $\alpha = 1, \ldots, d$*

$$\frac{dq_i}{dt} = \frac{q}{2(1-q)} g_{ii}\,, \tag{4.52a}$$

$$\frac{d\mathbf{a}_i^\gamma}{dt} = -\frac{q}{2(1-q)} \sum_{k \neq i} V_{ik}(\mathbf{a}_i^\gamma - \mathbf{a}_k^\gamma) g_{ik}\,, \tag{4.52b}$$

$$\frac{d\mathbf{c}_j^\epsilon}{dt} = \frac{q}{2(1-q)} \sum_{k \neq j} (V_{jk}\mathbf{c}_j^\epsilon g_{jk} - V_{kj}\mathbf{c}_k^\epsilon g_{kj})\,. \tag{4.52c}$$

Proof. Note that $\{x_i, \mathrm{Tr}(Z)\} = \frac{q}{1-q}\sum_k \{x_i, f_{kk}\}$, so that the first assertion follows from Proposition 4.3.4. Then, remark that from Lemma 4.3.5

$$\{\mathbf{a}_i^\gamma, g_{kk}\} = (\mathbf{a}_i^\gamma - \mathbf{a}_k^\gamma)B_{ik} + \frac{1}{2}\delta_{(i\neq k)}\frac{x_i + x_k}{x_i - x_k}(\mathbf{a}_k^\gamma - \mathbf{a}_i^\gamma)g_{ik} + \frac{1}{2}(\mathbf{a}_i^\gamma - \mathbf{a}_k^\gamma)g_{ik}\,, \tag{4.53}$$

which is clearly zero for $k = i$. Now, using (4.28) and (4.51),

$$\{\mathbf{a}_i^\gamma, g_{kk}\} = \frac{1}{2}\delta_{(i\neq k)}(\mathbf{a}_k^\gamma - \mathbf{a}_i^\gamma)g_{ik}\left[\frac{-2qx_k}{x_i - qx_k} + \frac{x_i + x_k}{x_i - x_k} - 1\right] = \frac{1}{2}\delta_{(i\neq k)}(\mathbf{a}_k^\gamma - \mathbf{a}_i^\gamma)g_{ik}V_{ik}\,. \tag{4.54}$$

It is easy to obtain the third identity from it. The last one is proved similarly. □

Hence, if we set $\{-,-\}_A = 2\{-,-\}$ and $\mathbf{f}_{ij} = g_{ij}$ as in § 4.3.2, this is precisely (1.9a)–(1.9c) with trigonometric potential up to a factor $(q^{-1}-1)^{-1}$. Thus, $(q^{-1}-1)G_1^{1,d}$ is the Hamiltonian for the trigonometric spin Ruijsenaars-Schneider system introduced by Krichever and Zabrodin [105]. In particular, we can explicitly integrate any of the commuting flows defined by $G_k^{1,d}$ in $\mathcal{C}^\circ_{n,q,d}$ using Proposition 4.1.6. The motion for the matrices (X,Z) under such flows was already determined in [138]. It is also important to remark that, due to our method, we are constrained by the conditions on q given by $q \neq 0$ and $q^N \neq 1$ for any $N \in \mathbb{Z}$. Hence, we can not consider the limiting case $\gamma \to \infty$ in (4.51), which gives the potential $V(q) = \coth(q)$. This case arises in the study of affine Toda solitons, see [33]. A unifying approach in the real setting could be the methods introduced by Luen-Chau Li (see [111] and references therein) or Laszlo Fehér [64, 65], where such a potential can be obtained [111, Sect. 5][64, §6.1].

Then, we look at the elements $H_k^{1,d} = \operatorname{tr} Y^k$. Again, using that $Y = Z - X^{-1}$ on $\mathcal{C}'_{n,q,d}$, we can see each function $H_k^{1,d}$ as a deformation of $G_k^{1,d}$. For $k = 1,2$ we have

$$H_1^{1,d} = G_1^{1,d} - \sum_{i=1}^n \frac{1}{x_i}\,, \qquad H_2^{1,d} = G_2^{1,d} - \frac{2q}{1-q}\sum_{i=1}^n \frac{g_{ii}}{x_i} + \sum_{i=1}^n \frac{1}{x_i^2}\,.$$

We can compute the defining ODEs associated to $\frac{d}{dt} = \{-, H_1^{1,d}\}_{\mathrm{P}}$ from Lemma 4.3.8, and easily get that

$$\begin{aligned}
\frac{dq_i}{dt} &= \frac{q}{2(1-q)} g_{ii}\,,\\
\frac{d\mathbf{a}_i^\gamma}{dt} &= -\frac{q}{2(1-q)} \sum_{k\neq i} V_{ik}(\mathbf{a}_i^\gamma - \mathbf{a}_k^\gamma) g_{ik}\,,\\
\frac{d\mathbf{c}_j^\epsilon}{dt} &= \frac{q}{2(1-q)} \sum_{k\neq j} (V_{jk}\mathbf{c}_j^\epsilon g_{jk} - V_{kj}\mathbf{c}_k^\epsilon g_{kj}) - \frac{\mathbf{c}_j^\epsilon}{x_j}\,.
\end{aligned}$$

This clearly differs from the RS system of ODEs due to the extra term $-\frac{\mathbf{c}_j^\epsilon}{x_j}$ in $d\mathbf{c}_j^\epsilon/dt$. It seems to define a system that is unknown as was pointed out in [42]. We have that the flows associated to any $H_k^{1,d}$ can be explicitly integrated in $\mathcal{C}_{n,q,d}$ using Proposition 4.1.7.

Finally, for $F_k^{1,d} = \operatorname{tr}(\mathrm{Id}_n + XY)^k$, we remark that we can write $F_k^{1,d} = \operatorname{tr}(ZX)^k$. We can write locally $(ZX)_{ij} = q\frac{\tilde{g}_{ij}x_j}{x_i - qx_j}$ for $\tilde{g}_{ij} = g_{ij}x_j$. Now, note that the coordinates $(x_i, \tilde{g}_{ij})_{ij}$ have the same Poisson brackets as $(x_i, g_{ij})_{ij}$ after an easy computation involving Proposition 4.3.4. Hence $F_k^{1,d}$ defines the same function as $G_k^{1,d}$ after the above reparametrisation, and we do not need to discuss this family.

Liouville integrability with a spectral parameter

In the non-spin case $d = 1$, we obtained as Proposition 4.2.14 the existence of n functionally independent elements in the algebra $\mathbb{C}[\operatorname{tr} U^l \mid l \in \mathbb{Z}]$. While this result does not depend on the number $d \geq 1$, we can not form an integrable system for $d \geq 2$ since in that case $n < nd = \frac{1}{2}\dim \mathcal{C}_{n,q,d}$. To get a family of functions in involution of greater dimension, we have to study Proposition 4.1.5. We follow the method given in [62], which is partially based on the proof of integrability for the spin Calogero-Moser case [20, 104].

Remark 4.3.9 *For the remainder of this subsection, we require that $d \leq n$.*

Lemma 4.3.10 *For each U, the commutative algebra generated by the elements*

$$\left\{ h_K^u = \frac{1}{K} \operatorname{tr}[U(\mathrm{Id}_n + \eta \Phi_0^{-\epsilon(u)})]^K \;\middle|\; K \in \mathbb{N},\, \eta \in \mathbb{C} \right\}$$

is an abelian Poisson algebra of dimension $nd - \frac{d(d-1)}{2}$.

For $U = Z$, this is in fact proved in [105]. We follow the ideas of that work to claim Liouville integrability after an extra reduction. Before doing that, we prove Lemma 4.3.10.

Proof. **(Lemma 4.3.10.)** We begin with the case $U = Z$. Note that we can write $(\mathrm{Id}_n + \eta \Phi_0)Z$ as $Z + \eta XZX^{-1}$. This means that, for fixed K, the coefficients for the development of h_K^u in η in front of η^0 and η^K are the same. Hence we have a first constraint for each $K \in \mathbb{N}$.

Next, note that we can also write $(\mathrm{Id}_n + \eta \Phi_0)Z$ as $Z + q\eta(Z+S)$ by (4.4), recalling that $S = \mathbf{AC}$. Up to a constant, this means that we can study the elements in the family $\{\operatorname{tr}(Z + \eta S)^j \,|\, j \in \mathbb{N}, \eta \in \mathbb{C}\}$ (note that this is a different η). Since these functions are obtained from the eigenvalues of $Z + \eta S$, it is sufficient to study the functions for $j = 1, \ldots, n$. Expanding $\operatorname{tr}(Z + \eta S)^j$ in η as $\sum_{l=0}^{j} G_{j,l}\eta^l$, we get $j + 1$ functions. Thus, we get at most $n(n + 1)/2 + n$ functionally independent elements. We already obtained n relations between these functions above. It is not hard to see that these relations are equivalent to be able to rewrite each $G_{j,j}$ in terms of elements $G_{j',l'}$ with strictly smaller j', or strictly smaller l' if $j' = j$.

We get additional constraints because the rank of S is at most $d \leqslant n$. The exact count is obtained by looking at a particular spectral curve. We introduce

$$\Gamma(\eta, \mu) \equiv \det((Z + \eta S) - \mu \,\mathrm{Id}_n) = 0\,. \tag{4.55}$$

We write $\Gamma(\eta,\mu) \equiv \sum_{i=0}^{n} r_i(\eta)\mu^i = 0$, where $r_i(\eta)$ is a symmetric function of order $n-i$ of $Z+\eta S$, so is a function of $\{G_{j,l} | 0 \leqslant l \leqslant j-1,\ 1 \leqslant j \leqslant n-i\}$. In the latter set, we omit $l=j$ because of the constraint obtained above. We can expand each $r_i(\eta)$ in terms of η as $r_i(\eta) = \sum_{s=0}^{n-i} I_{n-i,s}\eta^s$. Hence the set of functionally independent functions is contained in the $\{I_{n-i,s}\}$, which are functions of the $n(n+1)/2$ functions $\{G_{j,l}\}$. To get the exact number of functionally independent functions, we need to know how many relations exist on the $\{I_{n-i,s}\}$. In a neighbourhood of $\eta=\infty$, we can write

$$\Gamma(\eta,\mu) = \prod_{i=1}^{n}(\mu-\mu_i(\eta))\,, \qquad \text{for } \mu_i(\eta) = \eta\nu_i\,, \tag{4.56}$$

for $(\nu_i)_i$ the eigenvalues of S. At a generic point, S has rank d and its nonzero eigenvalues are different, so we can order the $(\nu_i)_i$ so that $\nu_1 < \nu_2 < \ldots < \nu_d$, and $\nu_{d+1} = \ldots = \nu_n = 0$. Thus near $\eta=\infty$ we write $\Gamma(\eta,\mu) = \mu^{n-d}\prod_{i=1}^{d}(\mu-\eta\nu_i)$. From this behaviour at infinity, we require that if we write $\Gamma(\eta,\mu) \equiv \sum_{i=0}^{n}\Gamma_i(\mu)\eta^i$, then $\Gamma_i(\mu) = 0$ for all $i = d+1,\ldots,n$. Each $\Gamma_i(\mu)$ has order $n-i$ as a polynomial in μ whose coefficients are functions of the $\{I_{n-i,s}\}$. We can write $\Gamma_i(\mu) = \sum_{s=0}^{n-i} J_{n-i,s}(I_{k,t})\mu^s$. Thus, their vanishing for $i>d$ is equivalent to imposing

$$\sum_{i=d+1}^{n}(n-i+1) = \sum_{j=0}^{n-d-1}(n-(j+d)) = (n-d)^2 - \frac{(n-d-1)(n-d)}{2} = \frac{(n-d)(n-d+1)}{2}$$

relations. Imposing these constraints on the number of elements $\{I_{n-i,s}\}$, we get

$$\frac{n(n+1)}{2} - \frac{(n-d)(n-d+1)}{2} = nd - \frac{d(d-1)}{2}$$

independent first integrals. Remark that we get nothing by looking at $\mu=\infty$, since at a generic point Z is invertible and diagonalisable.

All the other cases follow this method and yield $nd - \frac{d(d-1)}{2}$ independent functions. For $U = \mathrm{Id}_n + XY$, we remark that this is equivalent to the case treated above since $\mathrm{Id}_n + XY$ can be generically written as the matrix Z up to the change of coordinates $(x_i, g_{ij}) \mapsto (x_i, \tilde{g}_{ij} = g_{ij}x_j)$.

For $U=Y$, remark that we can write $(\mathrm{Id}_n + \eta\Phi_0)Y = (1+q\eta)Y + q\eta SZ^{-1}Y$. After a change of spectral parameter, we can write the elements of the family in terms of $\{\mathrm{tr}(Y+\eta T)^j\}_j$, with $T = SZ^{-1}Y$. Hence, the proof is the same with the pair (Y,T) instead of (Z,S).

For $U=X$, recall that $\Phi_0 = q\,\mathrm{Id}_n + qSZ^{-1}$ is invertible by definition of the space. Since the matrix $SZ^{-1} = \mathbf{A}\mathbf{C}Z^{-1}$ is a product of the $n\times d$ matrix $\mathbf{A}$ with the $d\times n$ matrix

$\mathbf{C}Z^{-1}$, we can use the Woodbury matrix identity to write $\Phi_0^{-1} = q^{-1}\operatorname{Id}_n - q^{-1}T'$ for $T' = \mathbf{A}(\operatorname{Id}_n + q\mathbf{C}Z^{-1}\mathbf{A})^{-1}\mathbf{C}Z^{-1}$. Then we can write $(\operatorname{Id}_n + \eta\Phi_0^{-1})X = (1+q^{-1}\eta)X + q\eta T'X$. We then look at the family $\{\operatorname{tr}(X+\eta T)^j\}_j$, with $T = T'X$, and we redo the proof with the pair (X,T) instead of (Z,S). □

We get from Lemma 4.3.10 that our best hope to obtain an integrable system is to restrict our attention to a space of dimension $2nd - d(d-1)$ where the families of functions descend. Introduce the Lie subgroup (and algebraic subgroup) $\mathcal{H} \subset \operatorname{GL}_d(\mathbb{C})$ whose elements have the vector $(1,\dots,1)$ as eigenvector with eigenvalue 1, as in [105]. This can also be defined as

$$\mathcal{H} = \left\{ h = (h_{\alpha\beta}) \in \operatorname{GL}_d(\mathbb{C}) \,\middle|\, \sum_{\beta=1}^{d} h_{\alpha\beta} = 1 \text{ for all } \alpha \right\}. \tag{4.57}$$

We note that $\sum_\alpha (\mathbf{A}h)_{i\alpha} = 1$ for all i and $h \in \mathcal{H}$. Hence, the action $h \cdot (X,Z,\mathbf{A},\mathbf{C}) = (X,Z,\mathbf{A}h,h^{-1}\mathbf{C})$ is well-defined in $\mathcal{C}'_{n,q,d}$, and we can also show that the system of ODEs defined by $G_1^{1,d}$ in Lemma 4.3.8 (or its variant for $H_1^{1,d}$) is invariant under the action of $\mathcal{H}$.

We define the affine GIT quotient $\mathcal{C}^{\mathcal{H}}_{n,q,d}$ as

$$\mathcal{C}^{\mathcal{H}}_{n,q,d} = \mathcal{C}'_{n,q,d} /\!/ \mathcal{H}\,,$$

for the action given above on a quadruple $(X,Z,\mathbf{A},\mathbf{C}) \in \mathcal{C}'_{n,q,d}$. The functions in $\mathcal{C}^{\mathcal{H}}_{n,q,d}$ are $\mathcal{H}$-invariants, so the coordinate ring is generated by traces of words in (X,Z,S), for $S = \mathbf{A}\mathbf{C}$. Lifting such functions to $\mathcal{C}'_{n,q,d}$, and writing them in coordinates give elements that are polynomials in $(x_i, g_{ij}, (x_i - qx_j)^{-1}, (x_i - x_j)^{-1})$ which are $\mathcal{H}$-invariant and form a Poisson subalgebra by Proposition 4.3.4. Thus, we can define uniquely a Poisson bracket $\{-,-\}^{\mathcal{H}}$ such that the injection $\iota : \mathcal{C}^{\mathcal{H}}_{n,q,d} \to \mathcal{C}'_{n,q,d}$ dual to $\mathcal{O}(\mathcal{C}'_{n,q,d}) \to \mathcal{O}(\mathcal{C}'_{n,q,d})^{\mathcal{H}}$ is a Poisson morphism. This is an algebraic analogue to Proposition 2.1.1. The different families $(h^u_K)_K$ in Proposition 4.1.5 descend to $\mathcal{C}^{\mathcal{H}}_{n,q,d}$ since they are $\mathcal{H}$-invariant, and by definition of the Poisson bracket they stay in involution. For each family, we can equivalently consider as in the proof of Lemma 4.3.10 the elements $\{\operatorname{tr}(U+\eta T)^j\}_j$, where T is a specific matrix which has rank d at a generic point of $\mathcal{C}'_{n,q,d}$. We look at the expansion $\operatorname{tr}(U+\eta T)^j = \sum_{l=0}^{j} H^u_{j,l}\eta^l$.

Theorem 4.3.11 *The elements $\{H^u_{j,l} \mid (j,l) \in J_d\}$ define a completely integrable system on the smooth locus of $\mathcal{C}^{\mathcal{H}}_{n,q,d}$ for $J_d = \{(j,l) \mid j = 1,\dots,n,\ l = 0,\dots,\min(j-1,d)\}$.*

Proof. We show the existence of a non-empty open subset $\overline{\mathcal{S}}$ of $\mathcal{C}'_{n,q,d}$ where $\mathcal{H}$ acts properly and freely in Lemmae 4.3.12 and 4.3.13 below, on which we can perform Poisson reduction as in Proposition 2.1.1 for the analytic structure. In particular, the space of $\mathcal{H}$-orbits in $\overline{\mathcal{S}}$ defines a smooth complex manifold of dimension $2nd - d(d-1)$ inside $\mathcal{C}^{\mathcal{H}}_{n,q,d}$. By construction, a point $(X, Z, \mathbf{A}, \mathbf{C})$ in $\overline{\mathcal{S}}$ is characterised by the fact that all the d-dimensional minors of $\mathbf{A}$ are nonzero. Thus, it is the complement of the Zariski closed subsets defined by having a vanishing minor of dimension d. The subspace $\overline{\mathcal{S}}$ is dense in $\mathcal{C}'_{n,q,d}$, and so does its reduction in $\mathcal{C}^{\mathcal{H}}_{n,q,d}$. We can intersect $\overline{\mathcal{S}}$ with the complement of the level set $\{\det U = 0\}$, and still get a dense subset in $\mathcal{C}^{\mathcal{H}}_{n,q,d}$ after reduction. The elements $\{H^u_{j,l} \mid (j,l) \in J_d\}$ are functionally independent by the argument developed in Lemma 4.3.10, so the statement holds. □

Lemma 4.3.12 *The action of $\mathcal{H}$ is free on the subset of $\mathcal{C}'_{n,q,d}$ where, for each representative $(X, Z, \mathbf{A}, \mathbf{C})$, either $\mathbf{A}$ or $\mathbf{C}$ has rank d.*

Proof. Assume $\mathbf{A}$ has rank d, the proof being the same if we assume the latter for $\mathbf{C}$. By definition, there exists $K = (k_1, \ldots, k_d) \subset \{1, \ldots, n\}$ such that $\bar{\mathbf{A}} = (\mathbf{A}_{k_\alpha \beta})$ is a $d \times d$ matrix which has rank d, so is invertible. If we take some h in the stabiliser of the point $(X, Z, \mathbf{A}, \mathbf{C})$, then in particular $\mathbf{A}h = \mathbf{A}$ and thus $\bar{\mathbf{A}}h = \bar{\mathbf{A}}$. Indeed,

$$(\bar{\mathbf{A}}h)_{\alpha\beta} = \sum_\gamma \mathbf{A}_{k_\alpha \gamma} h_{\gamma\beta} = (\mathbf{A}h)_{k_\alpha \beta} = \mathbf{A}_{k_\alpha \beta} = \bar{\mathbf{A}}_{\alpha\beta} \,. \tag{4.58}$$

Since $\bar{\mathbf{A}}$ is invertible, $h = \mathrm{Id}_d$. □

Lemma 4.3.13 *The action of $\mathcal{H}$ is proper on the subset $\overline{\mathcal{S}} \subset \mathcal{C}'_{n,q,d}$ where, for each representative $(X, Z, \mathbf{A}, \mathbf{C})$, all the minors of dimension d of $\mathbf{A}$ are invertible.*

Proof. We have to show that given sequences $(h_m) \subset \mathcal{H}$, $(X_m, Z_m, \mathbf{A}_m, \mathbf{C}_m) \subset \overline{\mathcal{S}}$ satisfying $(X_m, Z_m, \mathbf{A}_m, \mathbf{C}_m) \to (X, Z, \mathbf{A}, \mathbf{C}) \in \overline{\mathcal{S}}$ and $h_m \cdot (X_m, Z_m, \mathbf{A}_m, \mathbf{C}_m) \to (X', Z', \mathbf{A}', \mathbf{C}') \in \overline{\mathcal{S}}$, then h_m converges in $\mathcal{H}$. Note that trivially $X' = X$ and $Z' = Z$.

For any choice of $K = (k_1, \ldots, k_d) \subset \{1, \ldots, n\}$, we can form $\bar{\mathbf{A}}$ as in Lemma 4.3.12. In particular, we also write $\bar{G}$ for the $d \times d$ matrix obtained in that way from some $n \times d$ matrix $\bar{G}$. We see that $h_m = \bar{\mathbf{A}}_m^{-1} \overline{h_m \cdot \mathbf{A}_m}$, since $\overline{\mathbf{A}_m h_m} = \bar{\mathbf{A}}_m h_m$.

From this, form $h := \bar{\mathbf{A}}^{-1}\bar{\mathbf{A}}'$. This element does not depend on the choice of K : take any two $K = (k_1, \dots, k_d)$, $L = (l_1, \dots, l_d)$ and construct $\bar{\mathbf{A}}^K$ and $\bar{\mathbf{A}}^L$ as before. They are both invertible, so they are related by $\bar{\mathbf{A}}^K = T\bar{\mathbf{A}}^L$ for some $T \in \mathrm{GL}_d(\mathbb{C})$, and the same holds for any $m \in \mathbb{N}$. Forming h^K and h^L, we get

$$h^K = \lim_{m\to\infty} (\bar{\mathbf{A}}^K_m)^{-1}(h_m \cdot \bar{\mathbf{A}}^K_m) = \lim_{m\to\infty} (\bar{\mathbf{A}}^L_m)^{-1}T_m^{-1}T_m(h_m \cdot \bar{\mathbf{A}}^L_m) = h^L \,.$$

To get that $h \in \mathcal{H}$, we first show that $h \in \mathrm{GL}_d(\mathbb{C})$. Since $h := \bar{\mathbf{A}}^{-1}\bar{\mathbf{A}}'$ and both elements on the right-hand side have nonzero determinant, so too has h. Remark that this is where the condition that all minors are invertible occurs : in general, if $\bar{\mathbf{A}}^K$ is invertible for some $K = (k_1, \dots, k_d) \subset \{1, \dots, n\}$, we do not know that $\bar{\mathbf{A}}'^K$ is also invertible but only that $\bar{\mathbf{A}}'^L$ is invertible for some possibly different string $L \subset \{1, \dots, n\}$ of length d.

Second, $h \in \mathcal{H}$ as

$$\sum_\beta h_{\alpha\beta} = \sum_{\gamma,\beta} \bar{\mathbf{A}}^{-1}_{\alpha\gamma}\bar{\mathbf{A}}'_{\gamma\beta} = \sum_\gamma \bar{\mathbf{A}}^{-1}_{\alpha\gamma} 1 \;=\; 1\,.$$

Indeed, when $\sum_\alpha \mathbf{A}_{i\alpha} = 1$ for all i, we have $\sum_\alpha \bar{\mathbf{A}}_{\gamma\alpha} = 1$ for all γ. Moreover, the same holds for its inverse. To show this last assertion, denote $U_\alpha := \sum_\gamma \bar{\mathbf{A}}^{-1}_{\alpha\gamma}$. Then

$$\sum_\alpha \bar{\mathbf{A}}_{\beta\alpha} U_\alpha = \sum_{\alpha,\gamma} \bar{\mathbf{A}}_{\beta\alpha}\bar{\mathbf{A}}^{-1}_{\alpha\gamma} = \sum_\gamma (\mathrm{Id}_d)_{\beta\gamma} = 1\,.$$

If we form the vector $U = (U_\alpha)$, $\bar{\mathbf{A}}U = (1, \dots, 1)^\top$. Since $\bar{\mathbf{A}}$ is invertible, $U_\alpha = 1$ for all α. □

Liouville integrability for specific cases

Assume that $d = 2$ first, and fix $\alpha, \beta \in \{1, 2\}$. Using item 4 in Proposition 4.1.3, the following holds.

Theorem 4.3.14 *For any $U = X, Y, Z, \mathrm{Id}_n + XY$, the elements*

$$\{\mathrm{tr}\, U^k\,,\;\; \mathrm{tr}(W_\alpha V_\beta U^k) \mid k = 1, \dots, n\}$$

form an integrable system on $\mathcal{C}_{n,q,d}$.

Proof. We can generically fix a gauge where U is in diagonal form with distinct nonzero eigenvalues and such that $\sum_\alpha (W_\alpha)_i = 1$ for all i. (The latter condition is analogous to (4.27).)

We can complete the set of $3n$ functions given by the eigenvalues of U and $((W_\alpha)_i,(V_\beta)_i)_i$ by the diagonal entries of X in the above gauge to get local coordinates, see the more general proof of Proposition 4.3.16 below. □

It seems natural to ask if we can extend this set whenever $d>2$ by additional $n(d-2)$ functions in involution. Unfortunately, this task seems difficult as it requires a careful analysis of (3.15b), which we postpone for a moment. Nevertheless, this is not too hard when $n=1$, so we can also give this specific case. The main step is to note that, when we do not take higher powers of U in (3.15b), we can write that

$$\begin{aligned}\{t_{\gamma\epsilon},t_{\alpha\beta}\}_{\mathrm{P}} =&\delta_{\alpha\epsilon}(t_{\gamma\beta}+\frac{1}{2}t_{\gamma\epsilon}t_{\alpha\beta}+\frac{1}{2}t_{\gamma\beta}t_{\alpha\epsilon})-\delta_{\gamma\beta}(t_{\alpha\epsilon}+\frac{1}{2}t_{\gamma\epsilon}t_{\alpha\beta}+\frac{1}{2}t_{\gamma\beta}t_{\alpha\epsilon})\\ &+\frac{1}{2}[o(\gamma,\beta)+o(\epsilon,\alpha)-o(\epsilon,\beta)-o(\gamma,\alpha)](t_{\gamma\epsilon}t_{\alpha\beta}+t_{\gamma\beta}t_{\alpha\epsilon})\,,\end{aligned}$$

where we have set $t_{\gamma\epsilon}=\operatorname{tr}(W_\gamma V_\epsilon)=V_\epsilon W_\gamma$. In particular, $\{t_{\gamma\gamma},t_{\alpha\alpha}\}_{\mathrm{P}}=0$.

Theorem 4.3.15 *If $n=1$, the elements $(\operatorname{tr}U,t_{\alpha\alpha})_{\alpha\neq d}$ form an integrable system for any $d\geq 2$. Hence $\operatorname{tr}U^k$ is Liouville integrable for any $k\in\mathbb{N}^\times$.*

Proof. We have already obtained that they Poisson commute. Since in this case each element W_α,V_α,X,Y is a scalar, the moment map (4.1a) reads $q\prod_\alpha(1+t_{\alpha\alpha})=1$ and we can rewrite t_{dd} as $-1+q^{-1}\prod_{\alpha\neq d}(1+t_{\alpha\alpha})^{-1}$. The group acting in this case is $\mathbb{C}^\times$, so we can generically fix the gauge by the condition $\sum_\alpha W_\alpha=1$, which amounts to $W_d=1-\sum_{\alpha\neq d}W_\alpha$. Thus, at a generic point the functions $(X,Z,W_\alpha,t_{\alpha\alpha})_{\alpha\neq d}$ are coordinates. In particular, the family from the statement contains d functionally independent elements at a generic point. We conclude since $\operatorname{tr}U^k=h_1^k$ for $h_1=\operatorname{tr}U$ in the case $n=1$. □

Degenerate integrability

Recall the definition of the subalgebra $\mathcal{O}_U$ defined before Proposition 4.1.3, which is generated by the functions $(\operatorname{tr}U^k,\operatorname{tr}W_\alpha V_\beta U^k)$. This algebra is defined on the space $\mathcal{C}_{n,q,d}$ (or $\mathcal{C}^\circ_{n,q,d}$ for Z), and is in fact a Poisson algebra by Proposition 4.1.3. As mentioned in [42], we can get the following result[17].

[17] In this subsection and the next one, we can claim integrability only on the connected component of $\mathcal{C}^\circ_{n,q,d}$ containing $\mathcal{C}'_{n,q,d}$, see Remark 4.3.2. Since it is conjectured that $\mathcal{C}^\circ_{n,q,d}$ is connected, we do not emphasise that we work in a connected component for our results.

Proposition 4.3.16 *We can complete the set of functions* $(\operatorname{tr} U^l)_{l=1}^n$ *by* $2nd - 2n$ *elements of* $\mathcal{O}_U$*, such that at a generic point these* $2nd - n$ *elements are functionally independent. Moreover, among these* $2nd - n$ *functions only the* n *elements* $\operatorname{tr} U^l$ *Poisson commute with all the other ones.*

As a direct application of Proposition 4.3.16, we obtain the following result.

Corollary 4.3.17 *For any* $k \in \mathbb{N}^\times$*, there exists a degenerate integrable system containing* $\operatorname{tr} U^k$*.*

Proof. We use Corollaries 2.3.39 and 3.1.8, or we directly use the definition of degenerate integrability. □

Proof. (Proposition 4.3.16.) We begin with the case $U = Z$, and we first introduce a convenient set of local coordinates. Consider the space $\mathfrak{h}^{2(d-1)+1}$ where $\mathfrak{h} = \mathbb{C}^n$ as before, with local coordinates $(z_i, v_{\alpha,i}, w_{\alpha,i})$ for $i = 1, \dots, n$, $1 \leq \alpha < d$. We consider the subspace $\mathfrak{h}_1$ where $z_i \neq 0$, $z_i \neq z_j$ for all $i \neq j$, and for each α, $1 + \sum_i w_{\alpha,i} v_{\alpha,i} \neq 0$. Now, define the matrices

$$Z = \operatorname{diag}(z_1, \dots, z_n), \quad V_\alpha = (v_{\alpha,i})_i, \quad W_\alpha = (w_{\alpha,i})_i\,,$$

with $1 \leq \alpha < d$, and consider $F_{d-1} = q(\operatorname{Id}_n + W_{d-1}V_{d-1}) \dots (\operatorname{Id}_n + W_1 V_1) Z$. This is clearly invertible by assumption on $\mathfrak{h}_1$. Moreover, we claim that F_{d-1} has distinct eigenvalues generically[18], and that there exists $V_d \in \operatorname{Mat}_{1\times n}$ such that $F_d = F_{d-1} + W_d V_d F_{d-1}$ has the same spectrum as Z for $W_d = (1, \dots, 1)^\top$. Indeed, this follows by induction on d from Lemma A.1. Moreover, this lemma tells us that by fixing the matrix X that diagonalises F_d into Z, V_d is uniquely defined.

Since Z is invertible, so is F_d. But F_{d-1} being invertible implies that $(\operatorname{Id}_n + W_d V_d)$ is invertible in its turn. So we can define a point of $\mathcal{C}^\circ_{n,q,d}$ by fixing X, which amounts to choose an eigenbasis for F_d. In other words, around a generic point of $\mathfrak{h}_1$, we can locally complete the elements $(z_i, v_{\alpha,i}, w_{\alpha,i})$ by n additional functions (corresponding to a choice of eigenbasis) such that we define a local coordinate system in $\mathcal{C}^\circ_{n,q,d}$.

[18] To see that this is a non-empty condition, remark that fixing all $v_{\alpha,i} = 0 = w_{\alpha,i}$, we have that $F_{d-1} = \operatorname{diag}(qz_1, \dots, qz_n)$.

We now form the degenerate integrable system. To do so, introduce $t^l_{\gamma\epsilon} = \mathrm{tr}(W_\gamma V_\epsilon Z^l)$. In local coordinates $t^l_{\gamma\epsilon} = \sum_i z_i^l v_{\epsilon,i} w_{\gamma,i}$ for $\gamma, \epsilon \neq d$, and $t^l_{d\epsilon} = \sum_i z_i^l v_{\epsilon,i}$ for $\epsilon \neq d$. We consider

$$T = \left(\left(\frac{1}{k} \mathrm{tr}\, Z^k \right)_k, \left(t^l_{d1}\right)_l, \dots, \left(t^l_{d,d-1}\right)_l, \left(t^l_{11}\right)_l, \dots, \left(t^l_{d-1,1}\right)_l \right) \tag{4.59}$$

where each k ranges over $1, \dots, n$ and l over $0, \dots, n-1$. We write

$$(z, v, w) = ((z_k)_k, (v_{1,k})_k, \dots, (v_{d-1,k})_k, (w_{1,k})_k, \dots, (w_{d-1,k})_k),$$

for this particular order of the subset of coordinates (we omitted those that determine X). Then we can see that the Jacobian matrix takes the form

$$\frac{\partial T}{\partial(z, v, w)} = \begin{pmatrix} V_z & 0 & 0 \\ * & A & 0 \\ * & * & B \end{pmatrix},$$

where $A = \mathrm{diag}(V_z, \dots, V_z)$ is composed of $(d-1)$ diagonal square blocks V_z of size n with entries $(V_z)_{ij} = z_j^{i-1}$ (so V_z is the transpose of a Vandermonde matrix), while $B = \mathrm{diag}(V_1 V_z, \dots, V_1 V_z)$ is also composed of $(d-1)$ blocks for $V_1 = \mathrm{diag}(v_{1,1}, \dots, v_{1,n})$. Since $\det V_z \neq 0$ by assumption and $\det V_1 \neq 0$ if we restrict to the generic subspace where each $v_{1,k} \neq 0$, we get that the functions forming T are generically independent. Therefore, $\dim \mathcal{O}_Z \geq 2nd - n$ at a generic point. We have equality as no element $t^l_{\epsilon\alpha}$ depends on X, hence on the n remaining coordinates.

Finally, we know from the two first items of Proposition 4.1.3 and the independence of $\mathrm{tr}\, Z \dots, \mathrm{tr}\, Z^n$ that the centre of $\mathcal{O}_Z$ has dimension at least n. Assume that the dimension is strictly greater than n. Then the rank r_Z of the Poisson structure on $\mathcal{O}_Z$ is strictly less than $2nd - 2n$. Hence, we see that the rank r of the Poisson structure at a generic point satisfies $r \leq r_Z + 2n$, since we can at most increase the rank by $2n$ when completing the elements of T by another n independent functions. Thus $r < 2nd$, which contradicts that the Poisson bracket is non-degenerate. Hence, the centre of $\mathcal{O}_Z$ has precisely dimension n.

For the other cases, we rewrite the moment map equation in the form

$$\begin{aligned} (X + Y^{-1})Y(X + Y^{-1})^{-1} &= F(V, W)Y, \\ (XZ)X^{-1}(XZ)^{-1} &= F(V, W)X^{-1}, \\ [(\mathrm{Id}_n + XY)Y](\mathrm{Id}_n + XY)^{-1}[(\mathrm{Id}_n + XY)Y]^{-1} &= F(V, W)(\mathrm{Id}_n + XY)^{-1}, \end{aligned}$$

where $F(V,W) = q(\mathrm{Id}_n + W_d V_d)\dots(\mathrm{Id}_n + W_1 V_1)$. We work either on the subspaces $\mathcal{C}^\circ_{n,q,d}$ or $\{\det Y \neq 0\} \subset \mathcal{C}_{n,q,d}$. Then, we can reproduce the proof in the same way. Note that for $U = X$, we use X^{-1} instead of Z, and we diagonalise $F(V,W)X^{-1}$ into X^{-1} using some Z', from which we define a point of $\mathcal{C}^\circ_{n,q,d}$ by setting $Z = X^{-1}Z'$. A similar argument is needed for $U = \mathrm{Id}_n + XY$. □

Liouville integrability : the general case

Recall the notation $U_{(\alpha)} = \Theta^{(\alpha)}U$ where $U \in \{Y,Z\}$, and $\Theta^{(\alpha)} = \mathcal{X}(\Phi^{(\alpha)})$ is given by (4.7) for $0 \leq \alpha \leq d$. We can form the commutative algebra $\mathcal{H}_U$ generated by the functions $\operatorname{tr} U^K_{(\alpha)}$ from Proposition 3.1.11, and this result implies that $\mathcal{H}_U$ is also a Poisson commutative subalgebra under $\{-,-\}_{\mathrm{P}}$. We also denote by $\mathcal{H}_U$ the corresponding sheaf of functions on $\mathcal{C}_{n,q,d}$ (or $\mathcal{C}^\circ_{n,q,d}$ for $U = Z$). In a way similar to [42], we can obtain the following result.

Proposition 4.3.18 *At a generic point, there are nd functionally independent elements in $\mathcal{H}_U$.*

As a direct application of Proposition 4.3.18, we obtain the following result.

Corollary 4.3.19 *For any $k \in \mathbb{N}^\times$, there exists a Liouville integrable system containing $\operatorname{tr} U^k$.*

Proof. (Proposition 4.3.18.) We do the proof for $U = Z$, the other case being similar. Using (4.7), we can write that

$$U_{(\alpha)} = q(\mathrm{Id}_n + W_d V_d)\dots(\mathrm{Id}_n + W_{\alpha+1}V_{\alpha+1})Z\,, \quad \alpha = 0,\dots,d-1\,,$$

and $U_{(d)} = qZ$. We will show that the functions $h_{\alpha,K} = \frac{1}{q^K}\operatorname{tr} U^K_{(\alpha)}$ are functionally independent for $1 \leq \alpha \leq d$ and $K = 1,\dots,n$. In fact, it is more convenient to study these functions under the form $\frac{1}{q^K}\operatorname{tr}[ZU_{(\alpha)}Z^{-1}]^K$, and to ease notations we also write by $U_{(\alpha)}$ the matrix obtained after conjugation, i.e.

$$U_{(\alpha)} = qZ(\mathrm{Id}_n + W_d V_d)\dots(\mathrm{Id}_n + W_{\alpha+1}V_{\alpha+1})\,, \quad \alpha = 0,\dots,d-1\,,$$

again with $U_{(d)} = qZ$. Recalling that $t^l_{\alpha\beta} = \operatorname{tr} W_\alpha V_\beta Z^l$, we get that

$$\begin{aligned} h_{d,K} = \operatorname{tr} Z^K\,, \quad h_{d-1,K} &= \operatorname{tr} Z^K + K t^K_{dd} + P_K[t^l_{dd}]\,, \\ h_{\alpha-1,K} &= \operatorname{tr} Z^K + K t^K_{\alpha\alpha} + P_K[t^l_{\alpha\alpha}] + Q_{\alpha-1,K}\,, \end{aligned} \tag{4.60}$$

where $P_K[t^l_{\alpha\alpha}] \in \mathbb{C}[t^l_{\alpha\alpha} \mid 0 \leq l < K]$, $P_0 = 0$, and the polynomial $Q_{\alpha-1,K}$ satisfies the following property : it is a sum of terms of the form

$$t^{l_1}_{\gamma_1\gamma_2} t^{l_2}_{\gamma_2\gamma_3} \dots t^{l_s}_{\gamma_s\gamma_1},\; l_1 + \ldots + l_s = K,\; \alpha \leq \gamma_i \leq d \text{ with } \gamma_i > \alpha \text{ for at least one } i\,.$$

This implies that $Q_{\alpha-1,K}$ vanishes whenever $V_\beta = 0_{1\times n}$ or $W_\beta = 0_{n\times 1}$ for all $\alpha < \beta \leq d$, so that $h_{\alpha-1,K}$ takes the same form as $h_{d-1,K}$ in such case (with α replacing d).

To show functional independence, let us introduce some local coordinates, in a way similar to Proposition 4.3.16. Consider again $\mathfrak{h}^{2(d-1)+1}$, $\mathfrak{h} = \mathbb{C}^n$, with this time the local coordinates $(z_i, v_{\beta,i}, w_{\beta,i})$ for $i = 1, \ldots, n$, $1 < \beta \leq d$. We consider the subspace $\mathfrak{h}_2$ where $z_i \neq 0$, $z_i \neq z_j$ for all $i \neq j$, and for each β, $1 + \sum_i w_{\beta,i} v_{\beta,i} \neq 0$. Now, define the matrices

$$Z = \operatorname{diag}(z_1, \ldots, z_n), \quad V_\beta = (v_{\beta,i})_i, \quad W_\beta = (w_{\beta,i})_i\,,$$

with $1 < \beta \leq d$, and consider $F'_{d-1} = qZ(\operatorname{Id}_n + W_d V_d) \ldots (\operatorname{Id}_n + W_2 V_2)$. As in Proposition 4.3.16 but now using Lemma A.4 inductively, we can finally get that there exists $V_1 \in \operatorname{Mat}_{1\times n}$ such that $F'_d = F'_{d-1} + F'_{d-1} W_1 V_1$ has the same spectrum as Z for $W_1 = (1, \ldots, 1)^\top$. Moreover, for some fixed eigenbasis represented by $X' \in \operatorname{GL}_n(\mathbb{C})$, we have a unique V_d such that

$$X' Z (X')^{-1} = F'_d = F'_{d-1}(\operatorname{Id}_n + W_1 V_1)\,.$$

Comparing with (4.1a), we can then define a point of $\mathcal{C}^\circ_{n,q,d}$ by setting $X = Z^{-1} X'$. Hence, we can locally complete the elements $(z_i, v_{\beta,i}, w_{\beta,i})$, $1 < \beta \leq d$, to get a coordinate system on a (generic) neighbourhood in $\mathcal{C}^\circ_{n,q,d}$, where we can write

$$\operatorname{tr} Z^k = \sum_i z_i^k, \quad t^k_{\alpha\beta} = \sum_i w_{\alpha,i} v_{\beta,i} z_i^k\,,\; 2 \leq \alpha, \beta \leq d\,. \tag{4.61}$$

Note that one could argue that we end up in a connected component of $\mathcal{C}^\circ_{n,q,d}$ which is different from the one obtained in Proposition 4.3.16 since we have different local coordinates. Given diagonal Z and generic V_γ, W_γ, $1 \leq \gamma < d$, one can get nonzero V_d, W_d by the construction in the proof of Proposition 4.3.16 such that

$$XZX^{-1} = F_{d-1} + W_d V_d F_{d-1} = q(Id_n + W_d V_d) \ldots (\operatorname{Id}_n + W_1 V_1) Z = Z^{-1}(F'_{d-1} + F'_{d-1} W_1 V_1) Z.$$

And these $(X, Z, W_\gamma, V_\gamma)$ define a point of $\mathcal{C}^\circ_{n,q,d}$. Thus, taking V_β, W_β, $1 < \beta \leq d$, and a suitable eigenbasis for F'_{d-1} in the above construction, we recover the exact same V_1, W_1 by uniqueness once we fix $X' = ZX$, hence we define the same point $\mathcal{C}^\circ_{n,q,d}$.

We can now show by (descending) induction on $\alpha = d, \dots, 2$ that $h_{d,K}, \dots, h_{\alpha-1,K}$, $K = 1, \dots, n$, are $nd - n(\alpha - 2)$ independent functions. For $\alpha = d$, we want to prove that $\operatorname{tr} Z^K$ and $h_{d-1,K}$, $K = 1, \dots, n$, are functionally independent. Recall the definition of the latter functions in (4.60). We easily see that we have an isomorphism of the commutative algebras generated by the $h_{d-1,K}$ or the t^K_{dd} with $K = 1, \dots, n$. So it is sufficient to show the functional independence of $\operatorname{tr} Z^K, t^K_{dd}$ with $K = 1, \dots, n$. Using the expressions from (4.61), we just look at the Jacobian matrix of these functions with respect to $(z_i, v_{d,i})_i$ which is the $2n \times 2n$ matrix of the form

$$J := \begin{pmatrix} V & 0_{n\times n} \\ * & A^d \end{pmatrix} \text{ for } V_{ki} = \frac{\partial \operatorname{tr} Z^k}{\partial z_i},\; A^d_{ki} = \frac{\partial t^k_{dd}}{\partial v_{d,i}}. \quad k, i = 1, \dots, n\,.$$

In particular, we have that $V_{ki} = k\, z_i^{k-1}$, $A^d_{ki} = w_{d,i} z_i^k$. For nonzero distinct (z_i) and e.g. $w_{d,i} \neq 0$, $i = 1, \dots, n$, the matrix J has full rank $2n$. By induction on α, the Jacobian matrix J_α for $h_{d,K}, \dots, h_{\alpha,K}$, $K = 1, \dots, n$, with respect to $(z_i, v_{d,i}, \dots, v_{\alpha+1,i})_i$ has full rank $n(d-\alpha+1)$ (under the identification of the functions $(h_{d-1,K})$ and (t^K_{dd}), the matrix J above is J_{d-1}). Hence the Jacobian matrix for $h_{d,K}, \dots, h_{\alpha-1,K}$, $K = 1, \dots, n$, with respect to $(z_i, v_{d,i}, \dots, v_{\alpha,i})_i$ has the form

$$J_{\alpha-1} := \begin{pmatrix} J_\alpha & 0_{n_\alpha\times n} \\ * & C^\alpha \end{pmatrix} \text{ for } n_\alpha = n(d-\alpha+1) \text{ and } C^\alpha_{ki} = \frac{\partial h_{\alpha-1,k}}{\partial v_{\alpha,i}}, \quad k, i = 1, \dots, n\,.$$

We could therefore conclude if C^α has full rank n at some point. If we restrict our attention to a point where $w_{\beta,i} = v_{\beta,i} = 0$ for all $\alpha < \beta \leq d$, it is not hard to see that the differential of $Q_{\alpha-1,K}$ vanishes, hence

$$C^\alpha_{ki} = \frac{\partial \left(k t^k_{\alpha\alpha} + P_k(t^l_{\alpha\alpha})\right)}{\partial v_{\alpha,i}}\,.$$

As in the case $\alpha = d$, we can replace the functions $k t^k_{\alpha\alpha} + P_k(t^l_{\alpha\alpha})$ by $t^k_{\alpha\alpha}$ for $k = 1, \dots, n$ since they define the same algebra. Thus, C^α has full rank n at such a point, if this is true for A^α where $A^\alpha_{ki} = w_{\alpha,i} z_i^k$. Taking any point of the subspace where the z_i are nonzero distinct and $w_{\alpha,i} \neq 0$ for $i = 1, \dots, n$ proves the claim, and we get that C^α has generic rank n. □

Chapter 5

MQVs from cyclic quivers

In this chapter, we rely on the formalism introduced in Section 3.2 to develop a study similar to Chapter 4 in the case of a framed cyclic quiver. We introduce the MQVs of interest in Section 5.1, and obtain some preliminary results about integrability. Next, we restrict our attention in Section 5.2 to these MQVs for specific choices of dimension vectors. In such cases, we can find local coordinates and write the Poisson structure locally. Moreover, we can formulate precise statements regarding both degenerate integrability and Liouville integrability in those cases. We end this chapter with Section 5.3 which details the integrable systems for four families of framings. In § 5.3.2, we look at the special case of a cyclic quiver framed by one arrow, which first appeared in [41]. In § 5.3.3, we replace this unique arrow by several ones, still pointing to the same vertex of the cyclic quiver as in [62]. Finally, we proceed to the general case of an arbitrary framing for a cyclic quiver with $m = 2$ vertices in § 5.3.4.

We follow the conventions introduced in Remark 3.2.1 throughout this chapter.

5.1 General results

We consider the quiver $\bar{Q}_{\mathbf{d}}$ corresponding to $m \geq 2$, $\mathbf{d} \in \mathbb{N}^I$ for $I = \mathbb{Z}/m\mathbb{Z}$, and the ordering defined at the beginning of Section 3.2. As already noted, we choose that $d_0 \neq 0$ up to relabelling vertices.

5.1.1 Definition of the MQVs

We fix $\widetilde{\alpha} = (1, \mathbf{n})$ with $\mathbf{n} = (n_s) \in \mathbb{N}^I$ such that $|\mathbf{n}| = \sum_s n_s > 0$, and $\widetilde{q} = q_\infty e_\infty + \sum_s q_s e_s$ for some $\mathbf{q} = (q_s) \in (\mathbb{C}^\times)^I$ and $q_\infty = \prod_s q_s^{-n_s}$. A point $\rho \in \operatorname{Rep}(\mathbb{C}\bar{Q}_{\mathbf{d}}, \widetilde{\alpha})$ consists of the vector space $\mathcal{V} = \oplus_s \mathcal{V}_s \oplus \mathcal{V}_\infty$ with $\mathcal{V}_s = \mathbb{C}^{n_s}$ for each $s \in I$ and $\mathcal{V}_\infty = \mathbb{C}$, together with $2m + 2|\mathbf{d}|$ matrices given by

$$X_s \in \operatorname{Hom}(\mathcal{V}_{s+1}, \mathcal{V}_s)\,, \quad Y_s \in \operatorname{Hom}(\mathcal{V}_s, \mathcal{V}_{s+1})\,,$$
$$V_{s,\alpha} \in \operatorname{Hom}(\mathcal{V}_s, \mathcal{V}_\infty)\,, \quad W_{s,\alpha} \in \operatorname{Hom}(\mathcal{V}_\infty, \mathcal{V}_s)\,,$$

which respectively represent the arrows $x_s, y_s, v_{s,\alpha}, w_{s,\alpha}$ (under the interpretation given in Example 2.2.6). We identify the point ρ with the $2(m+|\mathbf{d}|)$-uple $(X_s, Y_s, V_{s,\alpha}, W_{s,\alpha})$ to ease our discussion.

The subspace $\operatorname{Rep}(\mathbb{C}\bar{Q}^\circ_{\mathbf{d}}, \widetilde{\alpha})$ is the open affine subset such that the endomorphisms

$$\operatorname{Id}_{\mathcal{V}_s} + X_s Y_s,\ \ \operatorname{Id}_{\mathcal{V}_{s+1}} + Y_s X_s,\ \ \operatorname{Id}_{\mathcal{V}_s} + W_{s,\alpha} V_{s,\alpha},\ \ \operatorname{Id}_{\mathcal{V}_\infty} + V_{s,\alpha} W_{s,\alpha}$$

are invertible for all possible indices. We then view $\operatorname{Rep}(\Lambda^{\mathbf{q}}_{\mathbf{d}}, \widetilde{\alpha})$ as the closed subscheme defined by the conditions

$$(\operatorname{Id}_{\mathcal{V}_s} + X_s Y_s)(\operatorname{Id}_{\mathcal{V}_s} + Y_{s-1} X_{s-1})^{-1} = q_s \overleftarrow{\prod_{1 \leqslant \alpha \leqslant d_s}} (\operatorname{Id}_{\mathcal{V}_s} + W_{s,\alpha} V_{s,\alpha})\,, \quad s \in I\,, \tag{5.1a}$$

representing (3.26a). From our choice of q_∞, the condition that represents (3.26b) follows by taking determinants.

We have a $\operatorname{GL}(\mathbf{n}) = \prod_s \operatorname{GL}_{n_s}(\mathbb{C})$ action on $\operatorname{Rep}(\Lambda^{\mathbf{q}}_{\mathbf{d}}, \widetilde{\alpha})$ given by

$$g \cdot (X_s, Y_s, W_{s,\alpha}, V_{s,\alpha}) = (g_s X_s g_{s+1}^{-1}, g_{s+1} Y_s g_s^{-1}, g_s W_{s,\alpha}, V_{s,\alpha} g_s^{-1})\,, \quad g = (g_s) \in \operatorname{GL}(\mathbf{n})\,. \tag{5.2}$$

We form the MQV $\operatorname{Rep}(\Lambda^{\mathbf{q}}_{\mathbf{d}}, \widetilde{\alpha}) /\!/ \operatorname{GL}(\mathbf{n})$ as the affine GIT quotient defined by $\operatorname{Spec} \mathbb{C}[\operatorname{Rep}(\Lambda^{\mathbf{q}}_{\mathbf{d}}, \widetilde{\alpha})]^{\operatorname{GL}(\mathbf{n})}$. We want to end up with a smooth manifold, which we can prescribe by the regularity criterion of Proposition 2.3.28. To do so, it is useful to introduce the following constants

$$t_s := \prod_{0 \leq \tilde{s} \leq s} q_{\tilde{s}}\,,\ \ s = 0, \ldots, m-1\,, \quad t := t_{m-1}\,, \quad t_{-1} := 1\,. \tag{5.3}$$

We use the identification of I with $\{0, \ldots, m-1\}$, since $t_{m-1} \neq t_{-1}$. In order to state the next result which generalises [41, Proposition 4.5], we say that a m-uple $\mathbf{t} = (t_s) \in (\mathbb{C}^\times)^I$ is *regular* whenever $t_{s_1}^{-1}t_{s_2} \neq t^k$ for any $k \in \mathbb{Z}$, $-1 \leq s_1 < s_2 \leq m-1$ with $(s_1, s_2) \neq (-1, m-1)$, or whenever $t^k \neq 1$ for any $k \in \mathbb{Z}^\times$.

Proposition 5.1.1 *Assume that* $\mathbf{q}$ *is such that* $\mathbf{t}$ *defined through* (5.3) *is regular. Then, provided that it is not empty, the space* $\operatorname{Rep}(\Lambda^{\mathbf{q}}_{\mathbf{d}}, \widetilde{\alpha})/\!/ \operatorname{GL}(\mathbf{n})$ *is a smooth variety of dimension* $2p(\mathbf{n}) = 2\sum_{s\in I} n_s(n_{s+1} + d_s - n_s)$*, endowed with a non-degenerate Poisson bracket* $\{-,-\}_{\mathrm{P}}$.

Proof. Recall from Example 2.2.4 that real roots for a cyclic quiver have the form

$$\rho = \pm(\epsilon_i + \ldots + \epsilon_j) + k\delta\,, \quad 1 \leq i < j \leq m-1,\ k \in \mathbb{Z}\,,$$

while imaginary roots are given by $\rho = k\delta$ with $k \in \mathbb{Z}^\times$, where $\delta = (1, \ldots, 1)$. Hence the regularity condition given in Proposition 2.3.28, which states that $\mathbf{q}^\rho \neq 1$ for any root ρ, can be written either as $(t_{i-1}^{-1}t_j)^{\pm 1}t^k \neq 1$, or as $t^k \neq 1$. The dimension is given by Theorem 2.3.27. □

For the rest of this chapter, we assume that the regularity condition in Proposition 5.1.1 is satisfied.

5.1.2 Towards integrability and dynamics

We continue with the notations of Section 5.1, and we assume that the regularity assumption in Proposition 5.1.1 holds. Below, we use 1_I to denote $\sum_s \operatorname{Id}_{\mathcal{V}_s}$, hence $\operatorname{Id}_{\mathcal{V}} = \operatorname{Id}_{\mathcal{V}_\infty} + 1_I$.

Let $u \in \{x, y, z, 1_I + xy\}$, and denote by $U = \mathcal{X}(u)$ the matrix representing U. The statements that we consider take place in $\operatorname{Rep}(\Lambda^{\mathbf{q}}_{\mathbf{d}}, \widetilde{\alpha})/\!/ \operatorname{GL}(\mathbf{n})$. Note that for $u = z$, they are further restricted on the complement of $\{\det X = 0\}$ which may be empty, see Section 5.2. We introduce the commutative algebra $\mathcal{O}_U$ generated by the functions $\operatorname{tr} U^k$ and $\operatorname{tr} W_{s,\alpha}V_{r,\beta}U^k$ for any indices.

Proposition 5.1.2 *The following results hold :*

1. *The symmetric functions* $\{\operatorname{tr} U^k \mid k \in \mathbb{N}\}$ *of* U *are pairwise in involution;*
2. *For any* $k, l \in \mathbb{N}$ *and for any possible spin indices,* $\{\operatorname{tr} U^k, \operatorname{tr} W_{s,\alpha}V_{r,\beta}U^l\}_{\mathrm{P}} = 0$*;*
3. *The algebra* $\mathcal{O}_U$ *is a Poisson algebra under* $\{-,-\}_{\mathrm{P}}$*;*

4. *For two fixed admissible spin indices* $(s,\alpha),(r,\beta)$*, the subalgebra of* $\mathcal{O}_U$ *generated by the functions* $(\operatorname{tr} U^k)_k$ *and* $(\operatorname{tr} W_{s,\alpha}V_{r,\beta}U^k)_k$ *is an abelian Poisson subalgebra.*

Proof. This is similar to Proposition 4.1.3, and follows from Lemma 3.2.8 and Corollary 3.2.9 in this case. □

Note that some items can be trivial, e.g. when $U \neq 1_I + XY$ and k is not divisible by m we get that $\operatorname{tr} U^k = 0$ so such elements trivially commute with any function. In fact, we can get a stronger statement in the case $U = 1_I + XY$ if we also use Lemma 3.2.10.

Proposition 5.1.3 *The following results hold :*

1. *The functions* $\{\operatorname{tr}(\mathrm{Id}_{\mathcal{V}_s} + X_sY_s)^k \mid k \in \mathbb{N}, s \in I\}$ *are pairwise in involution;*
2. *For any* $k,l \in \mathbb{N}$ *and for any possible spin indices,*

$$\{\operatorname{tr}(\mathrm{Id}_{\mathcal{V}_s} + X_sY_s)^k, \operatorname{tr} W_{p,\alpha}V_{q,\beta}(\mathrm{Id}_{\mathcal{V}_r} + X_rY_r)^l\}_{\mathrm{P}} = 0\,;$$

3. *The commutative algebra* $\mathcal{O}'_U$ *generated by the functions* $\operatorname{tr}(\mathrm{Id}_{\mathcal{V}_s} + X_sY_s)^k$ *and* $\operatorname{tr} W_{p,\alpha}V_{q,\beta}(\mathrm{Id}_{\mathcal{V}_r} + X_rY_r)^l$ *for any indices is a Poisson algebra under* $\{-,-\}_{\mathrm{P}}$*, which contains* $\mathcal{O}_U$*.*

Proof. We just show the inclusion $\mathcal{O}_U \subseteq \mathcal{O}'_U$ which is completely new. Since we can write

$$1_I + XY = \sum_s (\mathrm{Id}_{\mathcal{V}_s} + X_sY_s)\,, \quad W_{p,\alpha}V_{q,\beta}(1_I + XY)^l = W_{p,\alpha}V_{q,\beta}(\mathrm{Id}_{\mathcal{V}_q} + X_qY_q)^l\,,$$

it easily follows. □

Note that for any indices,

$$\operatorname{tr} W_{p,\alpha}V_{q,\beta}(\mathrm{Id}_{\mathcal{V}_r} + X_rY_r)^l = \delta_{qr}\delta_{pr} \operatorname{tr} W_{p,\alpha}V_{q,\beta}(\mathrm{Id}_{\mathcal{V}_r} + X_rY_r)^l = \operatorname{tr} W_{p,\alpha}V_{q,\beta}(1_I + XY)^l\,,$$

so such an element of $\mathcal{O}'_U$ is in $\mathcal{O}_U$. We will show as part of the proof of Corollary 5.2.38 that, when considered as sheaves, $\mathcal{O}'_U$ and $\mathcal{O}_U$ coincide at a generic point.

Write $\Theta = \mathcal{X}(\phi)$ for the matrix that represents the moment map ϕ for the quiver supported at the vertices of I. Hence $\Theta = (1_I + XY)(1_I + YX)^{-1}$, or $\Theta = XZX^{-1}Z^{-1}$ where X is invertible. We also consider the open subspace $\{\det U \neq 0\}$ where U is invertible. Remark that it may be empty. We get the next result from Corollary 3.2.13.

Proposition 5.1.4 *For any $K \in \mathbb{N}^\times$, expand $h^u_K = \frac{1}{K}\operatorname{tr}[U(1_I + \eta\Theta^{-\epsilon(u)})]^K$ in terms of η as $h^u_K = \sum_{k=0}^K h^u_{K,k}\eta^k$. Then all the functions $\{h^u_{K,k} \mid 0 \leq k \leq K\}$ are in involution.*

Again, it can happen that some of these functions are trivially zero.

Propositions 5.1.2, 5.1.3 and 5.1.4 give us candidates to form (degenerate) integrable systems, but we need specific dimension vectors to be able to state results about integrability. For example, if $n_s \neq n_{s+1}$ for some $s \in I$, X_s will never be invertible so that we can not define Z_s, and all the above results corresponding to $u = z$ are not defined on any space. We return to this problem in § 5.2.5 with dimension vector $\widetilde{\alpha} = (1, n, \dots, n)$. Nevertheless, it is interesting to notice that we have explicit expressions for some of the Hamiltonian flows, and they do not depend on the dimension vector.

We begin with the case that may be ill-defined : we pick $U = Z$ and assume for this specific case that $\{\det X \neq 0\}$ is not empty. Let $Z_\eta = Z(1_I + \eta\Theta)$. Then, we obtain from Lemma 3.2.15 that the equations governing the Hamiltonian flows for h^z_K with $K \in m\mathbb{N}^\times$ are given by

$$\{h^z_K, X\}_{\mathrm{P}} = -\eta\Theta Z_\eta^{K-1} Z X - X Z_\eta^{K-1} Z\,, \quad \{h^z_K, Z\}_{\mathrm{P}} = [Z_\eta^{k-1} Z, Z]\,,$$

and $\{h^z_K, V_{s,\alpha}\}_{\mathrm{P}} = \{h^z_K, W_{s,\alpha}\}_{\mathrm{P}} = 0$. As in the Jordan quiver case, it is not clear how to get an exact solution apart from when $\eta = 0$. Then, the flows associated to the Hamiltonian $h^z_{K,0} = \frac{1}{K}\operatorname{tr} Z^K$, $K \in m\mathbb{N}^\times$, satisfy the ODEs (where $d/dt_K = \{h^z_{K,0}, -\}_{\mathrm{P}}$)

$$\frac{dX}{dt_K} = -XZ^K\,, \quad \frac{dZ}{dt_K} = 0\,, \quad \frac{dV_{s,\alpha}}{dt_K} = 0\,, \quad \frac{dW_{s,\alpha}}{dt_K} = 0\,.$$

Proposition 5.1.5 *Given the initial condition $(X(0), Z(0), V_{s,\alpha}(0), W_{s,\alpha}(0))$, the flow at time t_K defined by the Hamiltonian $\frac{1}{K}\operatorname{tr} Z^K$ for $K \in m\mathbb{N}^\times$ is given by*

$$X(t_K) = X(0)\,\exp(-t_K Z(0)^K)\,, \quad Z(t_K) = Z(0)\,,$$
$$V_{s,\alpha}(t_K) = V_{s,\alpha}(0)\,, \quad W_{s,\alpha}(t_K) = W_{s,\alpha}(0)\,.$$

In particular, the flows descend to complete flows inside the subspace $\{\det X \neq 0\}$ of $\operatorname{Rep}(\Lambda^{\mathbf{q}}_{\mathbf{d}}, \widetilde{\alpha}) /\!/ \operatorname{GL}(\mathbf{n})$.

We pick $U = Y$ and set $Y_\eta = Y(1_I + \eta\Theta)$. Then, we obtain from Lemma 3.2.16 that the equations governing the Hamiltonian flows for h^y_K with $K \in m\mathbb{N}^\times$ are given by

$$\{h^y_K, X\}_{\mathrm{P}} = -Y_\eta^{K-1} - \eta\Theta Y_\eta^{K-1}(1_I + YX) - XY_\eta^{K-1}Y\,, \quad \{h^y_K, Y\}_{\mathrm{P}} = [Y_\eta^{k-1}Y, Y]\,,$$

and $\{h^y_K, V_{s,\alpha}\}_{\mathrm{P}} = \{h^y_K, W_{s,\alpha}\}_{\mathrm{P}} = 0$. Again, we consider $\eta = 0$ and the flows associated to the Hamiltonian $h^y_{K,0} = \frac{1}{K}\operatorname{tr} Y^K$, $K \in m\mathbb{N}^\times$, satisfy the ODEs (where $d/d\tau_K = \{h^y_{K,0}, -\}_{\mathrm{P}}$)

$$\frac{dX}{d\tau_K} = -Y^{K-1} - XY^K\,, \quad \frac{dY}{d\tau_K} = 0\,, \quad \frac{dV_{s,\alpha}}{d\tau_K} = 0\,, \quad \frac{dW_{s,\alpha}}{d\tau_K} = 0\,.$$

In fact, we have that these ODEs can be obtained without requiring that Y is invertible, so they are well-defined on *any* MQV satisfying the regularity condition of Proposition 5.1.1.

Proposition 5.1.6 *Given the initial condition* $(X(0), Y(0), V_{s,\alpha}(0), W_{s,\alpha}(0))$ *the flow at time* τ_K *defined by the Hamiltonian* $\frac{1}{K}\operatorname{tr} Y^K$ *for* $K \in m\mathbb{N}^\times$ *is given by*

$$\begin{aligned} X(\tau_K) =& X(0)\, \exp(-\tau_K Y(0)^K) + Y(0)^{-1}[\exp(-\tau_K Y(0)^K) - \mathrm{Id}_n]\,, \\ Y(\tau_K) =& Y(0)\,, \quad W_{s,\alpha}(\tau_K) = W_{s,\alpha}(0)\,, \quad V_{s,\alpha}(\tau_K) = V_{s,\alpha}(0)\,. \end{aligned}$$

In particular, the flows descend to complete flows in $\operatorname{Rep}(\Lambda^{\mathbf{q}}_{\mathbf{d}}, \widetilde{\alpha})/\!/ \operatorname{GL}(\mathbf{n})$.

For $U = X$, we let $X_\eta = X(1_I + \eta\Theta^{-1})$ and we work in the subspace $\{\det X \neq 0\}$. We get from Lemma 3.2.17 that

$$\{h^x_K, X\}_{\mathrm{P}} = -\,[X_\eta^{K-1}X, X]\,, \quad \{h^x_K, Z\}_{\mathrm{P}} = ZX_\eta^{K-1}X + \eta\Theta^{-1}X_\eta^{K-1}XZ\,,$$

while the brackets with $V_{s,\alpha}$ or $W_{s,\alpha}$ vanish. As before, we look at order 0, where we could omit the assumption that X is invertible and get dynamics on any regular MQV. Nevertheless, it is easier to work with Z which is defined under that condition. For $h^x_{K,0} = \frac{1}{K}\operatorname{tr} X^K$, we obtain for any $K \in m\mathbb{N}^\times$ by writing $d/d\hat{t}_K = \{h^x_{K,0}, -\}_{\mathrm{P}}$ that

$$\frac{dX}{d\hat{t}_K} = 0\,, \quad \frac{dZ}{d\hat{t}_K} = ZX^K\,, \quad \frac{dV_{s,\alpha}}{d\hat{t}_K} = 0\,, \quad \frac{dW_{s,\alpha}}{d\hat{t}_K} = 0\,.$$

(If we want to remove the assumption on X, we have to work with $dY/d\hat{t}_K = X^{K-1} + YX^K$ instead of the ODE defined on Z.)

Proposition 5.1.7 *Given the initial condition* $(X(0), Z(0), V_{s,\alpha}(0), W_{s,\alpha}(0))$*, the flow at time* $\hat{t}_K$ *defined by the Hamiltonian* $\frac{1}{K}\operatorname{tr} X^K$ *for* $K \in m\mathbb{N}^\times$ *is given by*

$$X(\hat{t}_K) = X(0)\,, \quad Z(\hat{t}_K) = Z(0)\exp(\hat{t}_K X(0)^K)\,,$$
$$V_{s,\alpha}(\hat{t}_K) = V_{s,\alpha}(0)\,,\ W_{s,\alpha}(\hat{t}_K) = W_{s,\alpha}(0)\,.$$

In particular, the flows descend to complete flows inside the subspace $\{\det X \neq 0\}$ *of* $\operatorname{Rep}(\Lambda^{\mathbf{q}}_{\mathbf{d}}, \widetilde{\alpha})/\!/\operatorname{GL}(\mathbf{n})$.

In the final case $U = 1_I + XY$, we introduce the notations $T = \operatorname{Id}_n + XY$, $T_\eta = T(1_I + \eta\Phi^{-1})$, and $h^T_K = h^{1_I+xy}_K$. The invertibility of T is already assumed to define the MQV, so that the discussion holds without further assumption. We get from Lemma 3.2.18 that

$$\{h^T_K, X\}_{\mathrm{P}} = -\,T_\eta^{K-1}TX - \eta X\Theta^{-1}T_\eta^{K-1}T\,, \quad \{h^T_K, T\}_{\mathrm{P}} = [-T_\eta^{K-1}T, T]\,,$$

and the other brackets vanish. (Note that this does not completely define the flow since we need to know $\{h^T_K, Y\}_{\mathrm{P}}$, but in the cases of interest where X is invertible this is sufficient.) Looking at order 0 in η, we have for $h^T_{K,0} = \frac{1}{K}\operatorname{tr}(1_I + XY)^K$, $K \in \mathbb{N}^\times$, by writing $d/d\tilde{t}_K = \{h^T_{K,0}, -\}_{\mathrm{P}}$ that

$$\frac{dX}{d\tilde{t}_K} = -T^K X\,, \quad \frac{dT}{d\tilde{t}_K} = 0\,, \quad \frac{dV_{s,\alpha}}{d\tilde{t}_K} = 0\,, \quad \frac{dW_{s,\alpha}}{d\tilde{t}_K} = 0\,.$$

Proposition 5.1.8 *Given the initial condition* $(X(0), T(0), V_{s,\alpha}(0), W_{s,\alpha}(0))$*, the flow at time* $\tilde{t}_K$ *defined by the Hamiltonian* $\frac{1}{K}\operatorname{tr} T^K$ *for* $K \in \mathbb{N}^\times$ *satisfies*

$$X(\tilde{t}_K) = \exp(-\tilde{t}_K T^K)X(0)\,,\ T(\tilde{t}_K) = T(0)\,,\ V_{s,\alpha}(\tilde{t}_K) = V_{s,\alpha}(0)\,,\ W_{s,\alpha}(\tilde{t}_K) = W_{s,\alpha}(0).$$

If $X(0)$ *is invertible, this implies that the flows descend to complete flows inside the subspace* $\{\det X \neq 0\}$ *of* $\operatorname{Rep}(\Lambda^{\mathbf{q}}_{\mathbf{d}}, \widetilde{\alpha})/\!/\operatorname{GL}(\mathbf{n})$.

So far, we have only used the embedding $A_0 \subset A$ from (3.41) to get Proposition 5.1.4, and the following integrations. We can in fact look at the full chain in (3.41). Denote by $\Theta^{(j)}_s = \mathcal{X}(\Phi^{(j)}_s) \in \operatorname{Hom}(\mathcal{V}_s, \mathcal{V}_s)$ the matrix representing the component $s \in I$ of the j-th moment map given in (3.42), $j \in \{0,1,\dots,|\mathbf{d}|\}$. We can write in $\operatorname{Rep}(\Lambda^{\mathbf{q}}_{\mathbf{d}},\widetilde{\alpha})/\!/\operatorname{GL}(\mathbf{n})$ that

$$\begin{aligned}\Theta_s^{(j)} =&(\operatorname{Id}_{\mathcal{V}_s}+X_sY_s)(\operatorname{Id}_{\mathcal{V}_s}+Y_{s-1}X_{s-1})^{-1}\prod^{\longrightarrow}_{\substack{1\leqslant\alpha\leqslant d_s\\(s,\alpha)\leq\rho(j)}}(\operatorname{Id}_{\mathcal{V}_s}+W_{s,\alpha}V_{s,\alpha})\\ =&q_s\prod^{\longleftarrow}_{\substack{1\leqslant\alpha\leqslant d_s\\(s,\alpha)>\rho(j)}}(\operatorname{Id}_{\mathcal{V}_s}+W_{s,\alpha}V_{s,\alpha})\end{aligned} \tag{5.4}$$

where we have used the map ρ introduced in Remark 3.2.11. Set $\Theta^{(j)} := \sum_s \Theta_s^{(j)}$, and $U_{(j)} = \Theta^{(j)}U$ with $U \in \{Y, Z, (1_I + XY)^{-1}\}$. The following result directly follows from Proposition 3.2.14.

Proposition 5.1.9 *The functions* $\left\{\operatorname{tr} U_{(j)}^K \mid K\in\mathbb{N},\, j\in\{0,1,\dots,|\mathbf{d}|\}\right\}$ *are in involution.*

Note that $\operatorname{tr} Y_{(j)}^K = \operatorname{tr} Z_{(j)}^K = 0$ trivially when K is not divisible by m, so in those cases we only consider $K \in m\mathbb{N}$. A nice feature of these three families is that we can get explicit expressions for the flows. We state these results which easily follow from Lemmae 3.2.19, 3.2.20 and 3.2.21.

Proposition 5.1.10 *Given the initial condition* $(X(0), Z(0), V_{s,\alpha}(0), W_{s,\alpha}(0))$*, the flow at time* t *defined by the Hamiltonian* $\frac{1}{K}\operatorname{tr} Z_{(j)}^K$*, for* $K\in m\mathbb{N}^\times$ *and* $j\in\{0,1,\dots,|\mathbf{d}|\}$*, is given by*

$$\begin{aligned}&X(t) = \exp(-tZ_{(j)}(0)^K)X(0)\,, \quad Z(t) = Z(0)\,,\\ &V_{s,\alpha}(t) = V_{s,\alpha}(0)e^{tZ(0)Z_{(j)}(0)^{K-1}\Theta^{(j)}(0)}, \quad (s,\alpha)\leq\rho(j),\\ &W_{s,\alpha}(t) = e^{-tZ(0)Z_{(j)}(0)^{K-1}\Theta^{(j)}(0)}W_{s,\alpha}(0), \quad (s,\alpha)\leq\rho(j),\\ &V_{s,\alpha}(t) = V_{s,\alpha}(0)\,,\ W_{s,\alpha}(t) = W_{s,\alpha}(0)\,,\ (s,\alpha)>\rho(j)\,.\end{aligned}$$

In particular, the flows descend to complete flows inside the subspace $\{\det X \neq 0\}$ *of* $\operatorname{Rep}(\Lambda^{\mathbf{q}}_{\mathbf{d}},\widetilde{\alpha})/\!/\operatorname{GL}(\mathbf{n})$.

Proposition 5.1.11 *Given the initial condition* $(X(0), Y(0), V_{s,\alpha}(0), W_{s,\alpha}(0))$*, the flow at time* τ *defined by the Hamiltonian* $\frac{1}{K}\operatorname{tr} Y_{(j)}^K$*, for* $K\in m\mathbb{N}^\times$ *and* $j\in\{0,1,\dots,|\mathbf{d}|\}$*, is given by*

$$\begin{aligned}&X(\tau) = \exp(-\tau Y_{(j)}(0)^K)X(0) + Y_{(j)}(0)^{-1}[\exp(-\tau Y_{(j)}(0)^K) - 1_I]\Theta^{(j)}\,, \quad Y(\tau) = Y(0)\,,\\ &V_{s,\alpha}(\tau) = V_{s,\alpha}(0)e^{\tau Y(0)Y_{(j)}(0)^{K-1}\Theta^{(j)}(0)}, \quad (s,\alpha)\leq\rho(j),\\ &W_{s,\alpha}(\tau) = e^{-\tau Y(0)Y_{(j)}(0)^{K-1}\Theta^{(j)}(0)}W_{s,\alpha}(0), \quad (s,\alpha)\leq\rho(j),\\ &V_{s,\alpha}(\tau) = V_{s,\alpha}(0)\,,\ W_{s,\alpha}(\tau) = W_{s,\alpha}(0)\,,\ (s,\alpha)>\rho(j)\,.\end{aligned}$$

In particular, the flows descend to complete flows in $\operatorname{Rep}(\Lambda^{\mathbf{q}}_{\mathbf{d}}, \widetilde{\alpha})//\operatorname{GL}(\mathbf{n})$.

Proposition 5.1.12 *Given the initial condition* $(X(0), T(0), V_{s,\alpha}(0), W_{s,\alpha}(0))$*, the flow at time* $\tilde{t}$ *defined by the Hamiltonian* $\frac{1}{K}\operatorname{tr} T^K_{(j)}$*, for* $T = (1_I + XY)^{-1}$, $K \in \mathbb{N}^\times$ *and* $j \in \{0, 1, \ldots, |\mathbf{d}|\}$*, is given by*

$$\begin{aligned}
&X(\tilde{t}) = X(0)\exp(\tilde{t}T(0)T_{(j)}(0)^{K-1}\Theta^{(j)})\,, \quad T(\tilde{t}) = T(0)\,,\\
&V_{s,\alpha}(\tilde{t}) = V_{s,\alpha}(0)e^{\tilde{t}T(0)T_{(j)}(0)^{K-1}\Theta^{(j)}(0)}, \quad (s,\alpha) \leq \rho(j),\\
&W_{s,\alpha}(\tilde{t}) = e^{-\tilde{t}T(0)T_{(j)}(0)^{K-1}\Theta^{(j)}(0)}W_{s,\alpha}(0), \quad (s,\alpha) \leq \rho(j),\\
&V_{s,\alpha}(\tilde{t}) = V_{s,\alpha}(0)\,,\ W_{s,\alpha}(\tilde{t}) = W_{s,\alpha}(0)\,,\ (s,\alpha) > \rho(j)\,.
\end{aligned}$$

In particular, the flows descend to complete flows inside the subspace $\{\det X \neq 0\}$ *of* $\operatorname{Rep}(\Lambda^{\mathbf{q}}_{\mathbf{d}}, \widetilde{\alpha})//\operatorname{GL}(\mathbf{n})$.

5.2 MQVs with dimension vector $(1, n\delta)$

Motivated both by § 3.2.2 and the relevance of some generic subspaces to easily get complete flows in § 5.1.2, we are particularly interested in the subspace of $\operatorname{Rep}(\Lambda^{\mathbf{q}}_{\mathbf{d}}, \widetilde{\alpha})//\operatorname{GL}(\mathbf{n})$ where the matrices X_s are invertible. Hence, since this subspace is clearly empty if $n_s \neq n_{s+1}$ for some $s \in I$, we restrict our attention to the case $\widetilde{\alpha} = (1, n\delta)$, where $n \in \mathbb{N}^\times$ and $\delta = (1, \ldots, 1)$ is the basic imaginary root for the cyclic quiver. We then write

$$\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m) := \operatorname{Rep}(\Lambda^{\mathbf{q}}_{\mathbf{d}}, \widetilde{\alpha})//\operatorname{GL}(n\delta)\,,$$

and denote by $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)^\circ$ its open subset where the product $X_0 \ldots X_{m-1}$ is invertible. We can write Proposition 5.1.1 in this case as follows.

Proposition 5.2.1 *Assume that* $t_{s_1}^{-1}t_{s_2} \neq t^k$ *for any* $k \in \mathbb{Z}$, $-1 \leq s_1 < s_2 \leq m-1$ *with* $(s_1, s_2) \neq (-1, m-1)$*, and* $t^k \neq 1$ *for any* $k \in \mathbb{Z}^\times$*. Then* $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)$ *is a smooth symplectic variety of dimension* $2n|\mathbf{d}|$.

Proof. Remark that the space $\mathcal{C}_{n,\mathbf{q},d'}(m)$ corresponding to $d' = (1, 0, \ldots, 0)$ embeds in any $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)$ by setting $W_{s,\alpha} = 0_{n\times 1}$, $V_{s,\alpha} = 0_{1\times n}$ for all $(s,\alpha) \neq (0,1)$. Thus, if we show that $\mathcal{C}_{n,\mathbf{q},d'}(m)^\circ$ is not empty, $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)$ is also not empty and by Proposition 5.1.1 it is a smooth symplectic variety of dimension $2p(n\delta) = 2n|\mathbf{d}|$.

We now use a construction similar to [41, Section 4.1]. By assumption, each X_s is invertible in $\mathcal{C}_{n,\mathbf{q},d'}(m)^\circ$ so up to a change of basis through the $\mathrm{GL}(n\delta)$ action we can choose $X_1 = \ldots = X_{m-1} = \mathrm{Id}_n$ and set $A := X_0$. We also set $B = q_0^{-1}Y_0A$. There remains an overall GL_n action by diagonal embedding inside $\mathrm{GL}(n\delta)$. Using (5.1a) for $s \neq 0$, we obtain that $Y_s = q_sX_s^{-1}Y_{s-1}X_{s-1} + (q_s-1)X_s^{-1}$, which gives by induction that $Y_s = (t_st_0^{-1}-1)\,\mathrm{Id}_n + t_sBA^{-1}$ for $s \neq 0, m-1$ while $Y_0 = t_0BA^{-1}$, $Y_{m-1} = (tt_0^{-1}-1)A^{-1} + tA^{-1}BA^{-1}$. The remaining equation from the moment map at $s=0$ reads

$$(\mathrm{Id}_n + t_0BA^{-1})(tt_0^{-1}\,\mathrm{Id}_n + tA^{-1}B)^{-1} = t_0(\mathrm{Id}_n + W_{0,1}V_{0,1})\,,$$

which we can rewrite as

$$(\mathrm{Id}_n + (t_0B)A^{-1})(\mathrm{Id}_n + A^{-1}(t_0B))^{-1} = t(\mathrm{Id}_n + W_{0,1}V_{0,1})\,.$$

Hence, a point in $\mathcal{C}_{n,\mathbf{q},d'}(m)^\circ$ is equivalent to the quadruple $(t_0B, A^{-1}, V_{0,1}, W_{0,1})$ which defines a point in $\mathcal{C}_{n,q}^\circ$, see Section 4.2, for $q = t$ not a root of unity. (To be precise, we have to check that invertibility conditions in each space imply those in the other, but this is not difficult to see.) As $\mathcal{C}_{n,q}^\circ$ is not empty, the result follows. □

We assume that the condition stated in Proposition 5.2.1 is satisfied for the rest of the chapter.

5.2.1 Matrices after localisation

To reduce the number of matrices defining the space $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)^\circ$, consider $\mathcal{R}^\circ \subset \mathrm{Rep}(\mathbb{C}\bar{Q}_{\mathbf{d}}, \widetilde{\alpha})$ as the subspace with invertibility conditions as above, i.e. with $X_0 \ldots X_{m-1} \in \mathrm{GL}_n(\mathbb{C})$. We define $Z_s = Y_s + X_s^{-1}$ for each $s \in I$ and let $X = \sum_s X_s$, $Y = \sum_s Y_s$, $Z = \sum_s Z_s$.

For each s, we also introduce $\mathbf{A}^s \in \mathrm{Mat}_{n\times d_s}(\mathbb{C})$ and $\mathbf{C}^s \in \mathrm{Mat}_{d_s\times n}(\mathbb{C})$ which we refer to as the s-th *spin matrices*, and that are defined entry-wise by[19]

$$\mathbf{A}^s_{i\alpha} = [W_{s,\alpha}]_i\,, \quad \mathbf{C}^s_{\alpha j} = [V_{s,\alpha}(\mathrm{Id}_n + W_{s,\alpha-1}V_{s,\alpha-1})\ldots(\mathrm{Id}_n + W_{s,1}V_{s,1})Z_{s-1}]_j\,. \tag{5.5}$$

The α-th column of $\mathbf{A}^s$ represents the spin element $a'_{s,\alpha}$, while the α-th row of $\mathbf{C}^s$ represents $c'_{s,\alpha}$. We get in particular that $\mathbf{A}^sE^{sr}_{\alpha\beta}\mathbf{C}^r$ represent the element $a'_{s,\alpha}c'_{r,\beta}$, where $E^{sr}_{\alpha\beta} \in \mathrm{Mat}_{d_s\times d_r}(\mathbb{C})$ is the matrix with entry $+1$ at (α,β) and zero everywhere else. We can clearly reconstruct the

[19] This can be readily compared to the matrices $\mathbf{A}, \mathbf{C}$ in the Jordan case defined with (4.3).

elements $(X_s, Y_s, W_{s,\alpha}, V_{s,\alpha})$ once we are given such $(X, Z, \mathbf{A}^s\mathbf{C}^s)$. Note that the moment map equation (5.1a) is now equivalent to

$$X_s Z_s X_{s-1}^{-1} = q_s Z_{s-1} + q_s \mathbf{A}^s\mathbf{C}^s\,, \quad \text{for all } s \in I\,. \tag{5.6}$$

As a corollary, the space $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)^\circ$ is characterised by the linear data $(X, Z, \mathbf{A}^s, \mathbf{C}^s)$ satisfying (5.6), modulo the action

$$g \cdot (X, Z, \mathbf{A}^s, \mathbf{C}^s) = (gXg^{-1}, gZg^{-1}, g_s\mathbf{A}^s, \mathbf{C}^s g_{s-1}^{-1})\,, \quad g = (g_s) \in \mathrm{GL}(n\delta)\,. \tag{5.7}$$

5.2.2 The slice

To ease notations, we refer to the MQV $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)$ simply as $\mathcal{C}_{n,m}$ and $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)^\circ$ as $\mathcal{C}^\circ_{n,m}$, while they clearly depend on both $\mathbf{d} \in \mathbb{N}^I$ and $\mathbf{q} \in (\mathbb{C}^\times)^I$. We also continue to identify $I = \mathbb{Z}/m\mathbb{Z}$ with $\{0, \dots, m-1\}$.

For each equivalence class $[(X, Z, \mathbf{A}^s, \mathbf{C}^s)]$ representing a point of $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)^\circ$ (i.e. a $\mathrm{GL}(n\delta)$-orbit in the subspace of $\mathcal{R}^\circ$ cut out by (5.6)), we consider an arbitrary representative $(X, Z, \mathbf{A}^s, \mathbf{C}^s)$, and we want to find a particular element $g \in \mathrm{GL}(n\delta)$ such that $g\cdot(X, Z, \mathbf{A}^s, \mathbf{C}^s)$ adopts a particular chosen form.

Introduce for each $s \in I$ the matrix $\mathbf{X}_s = X_0 \dots X_s$, so that $\mathcal{C}^\circ_{n,m} \subset \mathcal{C}_{n,m}$ is the subspace where the product $\mathbf{X} = \mathbf{X}_{m-1}$ is invertible in each equivalence class. Given such a representative $(X, Z, \mathbf{A}^s, \mathbf{C}^s)$ of a point in $\mathcal{C}^\circ_{n,m}$, we can act by $g_1 = (\mathrm{Id}_n, \mathbf{X}_0, \dots, \mathbf{X}_{m-2})$ so that $g_1 \cdot X_s = \mathrm{Id}_n$ for all $s \neq m-1$ while $g_1 \cdot X_{m-1} = \mathbf{X}$. Assume furthermore that $\mathbf{X}$ is diagonalisable with eigenvalues $(x_1, \dots, x_n)$ taking value inside

$$\mathfrak{h}_{\mathrm{reg}} = \{x = (x_1, \dots, x_n) \in \mathfrak{h} \mid x_i \neq 0\,,\ x_i \neq x_j\,,\ x_i \neq t x_j \text{ for all } i \neq j\}\,,$$

that we have already defined in (4.9). Thus, there exists $U \in \mathrm{GL}_n(\mathbb{C})$ such that $U\mathbf{X}U^{-1} = \mathrm{diag}(x_1, \dots, x_n)$. Then acting by $g_\mathfrak{h} = (U, \dots, U)$, we get that $g_\mathfrak{h} g_1 \cdot X_s = \mathrm{Id}_n$ for all $s \neq m-1$ while $g_\mathfrak{h} g_1 \cdot X_{m-1} = \mathrm{diag}(x_1, \dots, x_n)$. Now, consider the open subspace where $a_i \neq 0$, with $a_i := \sum_{\alpha=1}^{d_0} (g_\mathfrak{h} g_1 \cdot \mathbf{A}^0)_{i\alpha}$ for all $i = 1, \dots, n$. This is a non-empty condition as we assume $d_0 > 0$. We can form the matrix $A = \mathrm{diag}(a_1^{-1}, \dots, a_n^{-1})$, then define $g_a = (A, \dots, A)$. We find that $\sum_\alpha (g_a g_\mathfrak{h} g_1 \cdot \mathbf{A}^0)_{i\alpha} = 1$ for each i.

Remark 5.2.2 *We define the subspace $\mathcal{C}'_{n,m} \subset \mathcal{C}^\circ_{n,m}$ as the subset of points such that we can always perform the last two choices. I.e. given an arbitrary representative of a point, we set the matrices $X_0, \dots, X_{m-2}$ to the identity, then put X_{m-1} in diagonal form with entries in $\mathfrak{h}_{\text{reg}}$ defined as (4.9), and after such transformations we have that $\sum_{\alpha=1}^{d_0} \mathbf{A}^0_{i\alpha} \neq 0$. To see that $\mathcal{C}'_{n,m}$ is not empty, remark that it contains a subset isomorphic to $\mathcal{C}'_{n,q}$ for $q = t$ using the proof of Proposition 5.2.1.*

We do a final transformation to have all matrices constituting X in the same form. Consider $\lambda_i \in \mathbb{C}^\times$ such that $\lambda_i^m = x_i$. In particular, $\lambda_i^m \neq \lambda_j^m$ and $\lambda_i^m \neq t\lambda_j^m$ for each $i \neq j$. Then acting by $g_\lambda = (\text{Id}_n, \Lambda, \dots, \Lambda^{m-2})$, where $\Lambda = \text{diag}(\lambda_1^{-1}, \dots, \lambda_n^{-1})$, we have that

$$(\hat{X}, \hat{Z}, \hat{\mathbf{A}}^s, \hat{\mathbf{C}}^s) = g_\lambda g_a g_{\mathfrak{h}} g_1 \cdot (X, Z, \mathbf{A}^s, \mathbf{C}^s) \tag{5.8}$$

satisfies $\hat{X}_s = \text{diag}(\lambda_1, \dots, \lambda_n)$ for each s, and $\sum_{\alpha=1}^{d_0} \hat{\mathbf{A}}^0_{i\alpha} = +1$.

Lemma 5.2.3 *The choice of gauge given by (5.8) completely determines the representative up to an action by the finite group $W = S_n \ltimes \mathbb{Z}_m^n$.*

Proof. It is clear that the choice of diagonal form of $(\hat{X}_s)_s$ depends on both the ordering of $(x_1, \dots, x_n)$, and the choice of m-th root of unity. To be precise, the action of an element $(\sigma, \mathbf{k}) \in S_n \ltimes \mathbb{Z}_m^n$, is represented by the matrix $g = \prod_s g_\sigma g_{\mathbf{k}}^{-s}$, where g_σ is the permutation matrix corresponding to σ while $g_{\mathbf{k}} = \text{diag}(\zeta^{k_1}, \dots, \zeta^{k_n})$ for $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}_m^n$ and ζ is a primitive m-th root of unity. This clearly maps $\hat{X}_s$ to $\text{diag}(\zeta^{k_{\sigma(1)}}\lambda_{\sigma(1)}, \dots, \zeta^{k_{\sigma(n)}}\lambda_{\sigma(n)})$ for any s, so each $\hat{X}_s$ remains in the wanted form. To see that $\sum_{\alpha=1}^{d_0} (g \cdot \hat{\mathbf{A}}^0)_{i\alpha} = 1$, remark that the component of g acting on $\mathcal{V}_0$ is just $g_\sigma g_{\mathbf{k}}^{-0} = g_\sigma$, hence $\sum_{\alpha=1}^{d_0} (g \cdot \hat{\mathbf{A}}^0)_{i\alpha} = \sum_{\alpha=1}^{d_0} (\hat{\mathbf{A}}^0)_{\sigma^{-1}(i),\alpha} = 1$. We conclude as no other element of $\text{GL}(n\delta)$ is in the stabiliser of such a point. □

Note that if we set $\mathbf{a}_i^{s,\alpha} = \hat{\mathbf{A}}^s_{i\alpha}$ and $\mathbf{c}_j^{s,\alpha} = \hat{\mathbf{C}}^s_{\alpha j}$, then the elements $(\lambda_i, \mathbf{a}_i^{s,\alpha}, \mathbf{c}_i^{s,\alpha})$ form a set of local coordinates under the constraint $\sum_{\alpha=1}^{d_0} \mathbf{a}_i^{0,\alpha} = 1$ (and some invertibility conditions that can easily be recovered). Indeed, such data defined from $(\hat{X}, \hat{\mathbf{A}}^s, \hat{\mathbf{C}}^s)$ completely determine the matrices $(\hat{Z}_s)$. To see this, consider the following functions

$$g_{ij}^s := \sum_{\alpha=1}^{d_s} \mathbf{a}_i^{s,\alpha} \mathbf{c}_j^{s,\alpha}\,, \quad i,j = 1, \dots, n, \;\; s \in I\,, \tag{5.9}$$

which are the entries of the matrix $\hat{\mathbf{A}}^s\hat{\mathbf{C}}^s$ (assuming $d_s > 0$, and otherwise we set $g_{ij}^s = 0$). Then, for any $r = 0, \dots, m-1$,

$$(\hat{Z}_r)_{ij} = \sum_{s=0}^{r} \frac{t_r}{t_{s-1}} \frac{\lambda_i^{m+(s-r-1)}\lambda_j^{-(s-r-1)}}{\lambda_i^m - t\lambda_j^m} g_{ij}^s + \sum_{s=r+1}^{m-1} \frac{tt_r}{t_{s-1}} \frac{\lambda_i^{s-r-1}\lambda_j^{m-(s-r-1)}}{\lambda_i^m - t\lambda_j^m} g_{ij}^s \,. \tag{5.10}$$

To show that this relation holds, remark that in our choice of gauge, if we multiply (5.6) at entry (i,j) by $\frac{t}{t_s}(\lambda_j/\lambda_i)^{m-s-1}$ and take the sum over s of all such equations, we get

$$\left(\frac{\lambda_i}{\lambda_j} - t\frac{\lambda_j^{m-1}}{\lambda_i^{m-1}}\right) \hat{Z}_{m-1,ij} = \sum_{s=0}^{m-1} \frac{t}{t_{s-1}} \left(\frac{\lambda_j}{\lambda_i}\right)^{m-s-1} g_{ij}^s \,,$$

which yields in particular

$$(\hat{Z}_{m-1})_{ij} = \sum_{s=0}^{m-1} \frac{t}{t_{s-1}} \frac{\lambda_i^s \lambda_j^{m-s}}{\lambda_i^m - t\lambda_j^m} g_{ij}^s \,. \tag{5.11}$$

This is exactly (5.10) for $r = m-1$. We can then use relation (5.6) at $s = 0$ to get $\hat{Z}_0$, and finally get the other matrices by induction.

Remark 5.2.4 *Recall the residual $W = S_n \ltimes \mathbb{Z}_m^n$ action defined in Lemma 5.2.3. The element $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}_m^n \subset W$ acts as $\mathbf{k} \cdot (\lambda_i, \mathbf{a}_i^{s,\alpha}) = (\zeta^{k_i}\lambda_i, \zeta^{-sk_i}\mathbf{a}_i^{s,\alpha})$, while $\mathbf{k} \cdot \mathbf{c}_i^{s,\alpha} = \zeta^{(s-1)k_i}\mathbf{c}_i^{s,\alpha}$ for $s \neq 0$ and $\mathbf{k} \cdot \mathbf{c}_i^{0,\alpha} = \zeta^{(m-1)k_i}\mathbf{c}_i^{0,\alpha}$. In particular, $\mathbf{k} \cdot g_{ij}^s = \zeta^{s(k_j-k_i)-k_j} g_{ij}^s$ for $s \neq 0$, while $\mathbf{k} \cdot g_{ij}^0 = \zeta^{(m-1)k_j} g_{ij}^0$. In all cases, $g_{ii}^s\lambda_i$ is $\mathbb{Z}_m^n$-invariant.*

5.2.3 Local Poisson structure

Remark 5.2.5 *We continue to denote $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)^\circ$ as $\mathcal{C}_{n,m}^\circ$, and to identify I with $\{0, \dots, m-1\}$.*

It is convenient to consider the following functions on $\mathcal{C}_{n,m}^\circ$

$$f^k := \operatorname{tr}(X^k)\,, \quad g_{s\alpha,r\beta}^l = \operatorname{tr}(\mathbf{A}^s E_{\alpha\beta}^{sr} \mathbf{C}^r X^l)\,, \tag{5.12}$$

for any $k, l \in \mathbb{N}$, $s, r \in I$, $\alpha = 1, \dots, d_s$, and $\beta = 1, \dots, d_r$. Clearly, $f^k = 0$ if k does not satisfy $k \underset{m}{\equiv} 0$, while $g_{s\alpha,r\beta}^l = 0$ whenever the condition $l \underset{m}{\equiv} s - (r-1)$ is not satisfied. Here, the symbol $\underset{m}{\equiv}$ means that we take the equality modulo m, see § 3.2.3. Since we have that $\mathcal{X}(x) = X$, $\mathcal{X}(z) = Z$ and $\mathcal{X}(a'_{s,\alpha}c'_{r,\beta}) = \mathbf{A}^s E_{\alpha\beta}^{sr} \mathbf{C}^r$, where $E_{\alpha\beta}^{sr} \in \operatorname{Mat}_{d_s \times d_r}(\mathbb{C})$ is given by $(E_{\alpha\beta}^{sr})_{\alpha'\beta'} = \delta_{\alpha\alpha'}\delta_{\beta\beta'}$, Lemma 3.2.7 and (2.36) yield the identities for the Poisson bracket $\{-,-\}_{\mathrm{P}}$ on $\mathcal{C}_{n,m}^\circ$ between the functions $(f^k, g_{s\alpha,r\beta}^l)$.

Lemma 5.2.6 *For any possible indices,*

$$\{f^k,f^l\}_{\mathrm{P}}=0\,,\quad \{f^k,g^l_{s\alpha,r\beta}\}_{\mathrm{P}}=\,k\,g^{k+l}_{s\alpha,r\beta}\,, \tag{5.13a}$$

$$\begin{aligned}
\{g^k_{p\gamma,q\epsilon},g^l_{s\alpha,r\beta}\}_{\mathrm{P}}=\;&\frac{1}{2}\left(\sum_{v=1}^{k}-\sum_{v=1}^{l}\right)\mathrm{tr}(\mathbf{A}^sE^{sr}_{\alpha\beta}\mathbf{C}^rX^v\mathbf{A}^pE^{pq}_{\gamma\epsilon}\mathbf{C}^qX^{k+l-v})\\
&+\frac{1}{2}\left(\sum_{v=1}^{k}-\sum_{v=1}^{l}\right)\mathrm{tr}(\mathbf{A}^sE^{sr}_{\alpha\beta}\mathbf{C}^rX^{k+l-v}\mathbf{A}^pE^{pq}_{\gamma\epsilon}\mathbf{C}^qX^{v})\\
&+\frac{1}{2}[o(p,r)-o(p,s)+o(q,s)-o(q,r)]\,\mathrm{tr}(\mathbf{A}^sE^{sq}_{\alpha\epsilon}\mathbf{C}^qX^k\mathbf{A}^pE^{pr}_{\gamma\beta}\mathbf{C}^rX^l)\\
&+\frac{1}{2}\delta_{ps}o(\alpha,\gamma)[\mathrm{tr}(\mathbf{A}^pE^{pq}_{\gamma\epsilon}\mathbf{C}^qX^k\mathbf{A}^sE^{sr}_{\alpha\beta}\mathbf{C}^rX^l)+\mathrm{tr}(\mathbf{A}^sE^{sq}_{\alpha\epsilon}\mathbf{C}^qX^k\mathbf{A}^pE^{pr}_{\gamma\beta}\mathbf{C}^rX^l)]\\
&+\frac{1}{2}\delta_{qr}o(\epsilon,\beta)[\mathrm{tr}(\mathbf{A}^sE^{sr}_{\alpha\beta}\mathbf{C}^rX^k\mathbf{A}^pE^{pq}_{\gamma\epsilon}\mathbf{C}^qX^l)-\mathrm{tr}(\mathbf{A}^sE^{sq}_{\alpha\epsilon}\mathbf{C}^qX^k\mathbf{A}^pE^{pr}_{\gamma\beta}\mathbf{C}^rX^l)]\\
&+\frac{1}{2}\delta_{qs}[o(\epsilon,\alpha)+\delta_{\alpha\epsilon}]\,\mathrm{tr}(\mathbf{A}^sE^{sq}_{\alpha\epsilon}\mathbf{C}^qX^k\mathbf{A}^pE^{pr}_{\gamma\beta}\mathbf{C}^rX^l)\\
&-\frac{1}{2}\delta_{pr}[o(\beta,\gamma)+\delta_{\beta\gamma}]\,\mathrm{tr}(\mathbf{A}^sE^{sq}_{\alpha\epsilon}\mathbf{C}^qX^k\mathbf{A}^pE^{pr}_{\gamma\beta}\mathbf{C}^rX^l)\\
&+\delta_{qs}\delta_{\alpha\epsilon}\,\mathrm{tr}(ZX^k\mathbf{A}^pE^{pr}_{\gamma\beta}\mathbf{C}^rX^l)+\delta_{qs}\delta_{\alpha\epsilon}\sum_{\lambda=1}^{\epsilon-1}\mathrm{tr}(\mathbf{A}^sE^{ss}_{\lambda\lambda}\mathbf{C}^sX^k\mathbf{A}^pE^{pr}_{\gamma\beta}\mathbf{C}^rX^l)\\
&-\delta_{pr}\delta_{\beta\gamma}\,\mathrm{tr}(ZX^l\mathbf{A}^sE^{sq}_{\alpha\epsilon}\mathbf{C}^qX^k)-\delta_{pr}\delta_{\beta\gamma}\sum_{\mu=1}^{\beta-1}\mathrm{tr}(\mathbf{A}^sE^{sq}_{\alpha\epsilon}\mathbf{C}^qX^k\mathbf{A}^pE^{pp}_{\mu\mu}\mathbf{C}^pX^l)\,.
\end{aligned} \tag{5.13b}$$

(See Remark 3.2.1 for the conventions on symbols.) In particular, in order for the elements on which we evaluate the Poisson bracket to be nonzero, we need $k\underset{m}{\equiv}0$ *for* f^k, $l\underset{m}{\equiv}0$ *for* f^l, *while* $l\underset{m}{\equiv}s-(r-1)$ *for* $g^l_{s\alpha,r\beta}$, *and* $k\underset{m}{\equiv}p-(q-1)$ *for* $g^k_{p\gamma,q\epsilon}$.

Let $\mathfrak{h}_{sp,m}\subset\mathbb{C}^{2n|\mathbf{d}|+n}$ denote the subspace of elements $(\lambda_i,\mathbf{a}_i^{s,\alpha},\mathbf{c}_i^{s,\alpha})$ which can be obtained when we pick a representative in the form (5.8) of a point $(X,Z,\mathbf{A}^s,\mathbf{C}^s)\in\mathcal{C}'_{n,m}$. The map $[(X,Z,\mathbf{A}^s,\mathbf{C}^s)]\mapsto(\lambda_i,\mathbf{a}_i^{s,\alpha},\mathbf{c}_i^{s,\alpha})$ has an inverse modulo W by Lemma 5.2.3, and we denote by $\xi:\mathfrak{h}_{sp,m}/W\to\mathcal{C}'_{n,m}$ this inverse. By construction, ξ is a diffeomorphism, and we want to find the Poisson bracket $\{-,-\}$ on $\mathfrak{h}_{sp,m}/W$ such that ξ is a Poisson morphism.

For the functions given in (5.12), we remark that we can write locally

$$\xi^*f^k=m\sum_{i=1}^{n}\lambda_i^k\,,\quad \xi^*g^l_{s\alpha,r\beta}=\sum_{i=1}^{n}\mathbf{a}_i^{s\alpha}\mathbf{c}_i^{r\beta}\lambda_i^l\,,\quad \sum_{\alpha=1}^{d_0}\xi^*g^{l'}_{0\alpha,r\beta}=\sum_{i=1}^{n}\mathbf{c}_i^{r\beta}\lambda_i^{l'}\,, \tag{5.14}$$

assuming that $k\underset{m}{\equiv}0$, while $l\underset{m}{\equiv}s-(r-1)$ and $l'\underset{m}{\equiv}1-r$. From these expressions, it is not difficult to see that the differentials of the functions (taken with any possible indices)

$$f^k\,,\quad \sum_{\alpha}g^{l'}_{0\alpha,r\beta}\,,\quad g^l_{s\alpha,01}\,, \tag{5.15}$$

generate the cotangent space at a generic point of $\mathcal{C}'_{n,m}$. Hence, assuming that we have defined $\{-,-\}$ on $\mathfrak{h}_{sp,m}/W$, it suffices to verify the identity

$$\{\xi^* F_1, \xi^* F_2\} = \xi^* \{F_1, F_2\}_{\mathrm{P}}\,, \tag{5.16}$$

for F_1, F_2 running through the three types of functions in (5.15) in order to get that ξ is a Poisson diffeomorphism, see Remark B.3. This is the strategy for the proof of the next result.

Proposition 5.2.7 *The map $\xi : \mathfrak{h}_{sp,m}/W \to \mathcal{C}'_{n,m}$ is a Poisson diffeomorphism for the Poisson bracket $\{-,-\}$ defined on $\mathfrak{h}_{sp,m}/W$ as follows, with $1 \leq i,j \leq n$.*

For any admissible spins (s,α) and (r,β), we have

$$\{\lambda_i, \lambda_j\} = 0, \quad \{\lambda_i, \mathbf{a}_j^{s\alpha}\} = 0, \quad \{\lambda_i, \mathbf{c}_j^{r\beta}\} = \frac{1}{m}\delta_{ij}\lambda_i \mathbf{c}_j^{r\beta}; \tag{5.17}$$

for any $1 \leq \beta, \epsilon \leq d_0$, we have

$$\begin{aligned}\{\mathbf{c}_j^{0\epsilon}, \mathbf{c}_i^{0\beta}\} =& \frac{1}{2}\delta_{(i\neq j)}\frac{\lambda_j^m + \lambda_i^m}{\lambda_j^m - \lambda_i^m}(\mathbf{c}_j^{0\epsilon}\mathbf{c}_i^{0\beta} + \mathbf{c}_i^{0\epsilon}\mathbf{c}_j^{0\beta}) \\ &+ (Z_{m-1})_{ij}\mathbf{c}_i^{0\beta} - (Z_{m-1})_{ji}\mathbf{c}_j^{0\epsilon} + \frac{1}{2}o(\epsilon,\beta)(\mathbf{c}_i^{0\epsilon}\mathbf{c}_j^{0\beta} - \mathbf{c}_j^{0\epsilon}\mathbf{c}_i^{0\beta}) \\ &+ \mathbf{c}_i^{0\beta}\sum_{\lambda=1}^{\epsilon-1}\mathbf{a}_i^{0\lambda}(\mathbf{c}_j^{0\lambda} - \mathbf{c}_j^{0\epsilon}) - \mathbf{c}_j^{0\epsilon}\sum_{\mu=1}^{\beta-1}\mathbf{a}_j^{0\mu}(\mathbf{c}_i^{0\mu} - \mathbf{c}_i^{0\beta})\,;\end{aligned} \tag{5.18}$$

for any $q \in I \setminus \{0\}$, $1 \leq \epsilon \leq d_q$ and $1 \leq \beta \leq d_0$, we have

$$\begin{aligned}\{\mathbf{c}_j^{q\epsilon}, \mathbf{c}_i^{0\beta}\} =& \frac{q-m}{m}\delta_{ij}\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{0\beta} + \delta_{(i\neq j)}\frac{\lambda_i^m}{\lambda_j^m - \lambda_i^m}\left(\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{0\beta} + \frac{\lambda_j^q}{\lambda_i^q}\mathbf{c}_i^{q\epsilon}\mathbf{c}_j^{0\beta}\right) \\ &- (Z_{m-1})_{ji}\mathbf{c}_j^{q\epsilon} - \mathbf{c}_j^{q\epsilon}\sum_{\mu=1}^{\beta-1}\mathbf{a}_j^{0\mu}(\mathbf{c}_i^{0\mu} - \mathbf{c}_i^{0\beta})\,;\end{aligned} \tag{5.19}$$

for any $r \in I \setminus \{0\}$, $1 \leq \beta \leq d_r$ and $1 \leq \epsilon \leq d_0$, we have

$$\begin{aligned}\{\mathbf{c}_j^{0\epsilon}, \mathbf{c}_i^{r\beta}\} =& \frac{m-r}{m}\delta_{ij}\mathbf{c}_j^{0\epsilon}\mathbf{c}_i^{r\beta} + \delta_{(i\neq j)}\frac{\lambda_j^m}{\lambda_j^m - \lambda_i^m}\left(\mathbf{c}_j^{0\epsilon}\mathbf{c}_i^{r\beta} + \frac{\lambda_j^{-r}}{\lambda_i^{-r}}\mathbf{c}_i^{0\epsilon}\mathbf{c}_j^{r\beta}\right) \\ &+ (Z_{m-1})_{ij}\mathbf{c}_i^{r\beta} + \mathbf{c}_i^{r\beta}\sum_{\lambda=1}^{\epsilon-1}\mathbf{a}_i^{0\lambda}(\mathbf{c}_j^{0\lambda} - \mathbf{c}_j^{0\epsilon})\,;\end{aligned} \tag{5.20}$$

for any $q \in I \setminus \{0\}$ and $1 \leq \beta, \epsilon \leq d_q$, we have

$$\{\mathbf{c}_j^{q\epsilon}, \mathbf{c}_i^{q\beta}\} = \frac{1}{2}\delta_{(i\neq j)}\frac{\lambda_j^m + \lambda_i^m}{\lambda_j^m - \lambda_i^m}\left(\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{q\beta} + \mathbf{c}_i^{q\epsilon}\mathbf{c}_j^{q\beta}\right) + \frac{1}{2}o(\epsilon,\beta)\left(\mathbf{c}_i^{q\epsilon}\mathbf{c}_j^{q\beta} - \mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{q\beta}\right)\,; \tag{5.21}$$

for any $0 < r < q \leq m-1$, *and for any* $1 \leq \beta \leq d_r$, $1 \leq \epsilon \leq d_q$, *we have*

$$\{\mathbf{c}_j^{q\epsilon}, \mathbf{c}_i^{r\beta}\} \overset{q\geq r}{=} \delta_{ij}\frac{q-r}{m}\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{r\beta} + \delta_{(i\neq j)}\frac{\lambda_j^m}{\lambda_j^m - \lambda_i^m}\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{r\beta} + \delta_{(i\neq j)}\frac{\lambda_i^m}{\lambda_j^m - \lambda_i^m}\frac{\lambda_j^{q-r}}{\lambda_i^{q-r}}\mathbf{c}_i^{q\epsilon}\mathbf{c}_j^{r\beta}\,; \tag{5.22}$$

for any $0 < q < r \leq m-1$, *and for any* $1 \leq \beta \leq d_r$, $1 \leq \epsilon \leq d_q$, *we have*

$$\{\mathbf{c}_j^{q\epsilon}, \mathbf{c}_i^{r\beta}\} \overset{q\leq r}{=} \delta_{ij}\frac{q-r}{m}\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{r\beta} + \delta_{(i\neq j)}\frac{\lambda_i^m}{\lambda_j^m - \lambda_i^m}\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{r\beta} + \delta_{(i\neq j)}\frac{\lambda_j^m}{\lambda_j^m - \lambda_i^m}\frac{\lambda_j^{q-r}}{\lambda_i^{q-r}}\mathbf{c}_i^{q\epsilon}\mathbf{c}_j^{r\beta}\,; \tag{5.23}$$

for any $1 \leq \alpha, \epsilon \leq d_0$, *we have*

$$\begin{aligned}\{\mathbf{c}_j^{0\epsilon}, \mathbf{a}_i^{0\alpha}\} =& \frac{1}{2}\delta_{(i\neq j)}\frac{\lambda_j^m + \lambda_i^m}{\lambda_j^m - \lambda_i^m}\mathbf{c}_j^{0\epsilon}(\mathbf{a}_j^{0\alpha} - \mathbf{a}_i^{0\alpha}) - \delta_{(\alpha<\epsilon)}\mathbf{a}_i^{0\alpha}\mathbf{c}_j^{0\epsilon} + \delta_{\epsilon\alpha}\sum_{\lambda=1}^{\epsilon-1}\mathbf{a}_i^{0\lambda}\mathbf{c}_j^{0\lambda} \\ &+ \delta_{\epsilon\alpha}(Z_{m-1})_{ij} - \mathbf{a}_i^{0\alpha}(Z_{m-1})_{ij} - \mathbf{a}_i^{0\alpha}\sum_{\lambda=1}^{\epsilon-1}\mathbf{a}_i^{0\lambda}(\mathbf{c}_j^{0\lambda} - \mathbf{c}_j^{0\epsilon}) \\ &+ \frac{1}{2}\mathbf{c}_j^{0\epsilon}\sum_{\kappa=1}^{d_0}o(\alpha,\kappa)(\mathbf{a}_i^{0\alpha}\mathbf{a}_j^{0\kappa} + \mathbf{a}_j^{0\alpha}\mathbf{a}_i^{0\kappa})\,;\end{aligned} \tag{5.24}$$

for any $s \in I \setminus \{0\}$, $1 \leq \epsilon \leq d_0$ *and* $1 \leq \alpha \leq d_s$, *we have*

$$\begin{aligned}\{\mathbf{c}_j^{0\epsilon}, \mathbf{a}_i^{s\alpha}\} =& \frac{s-m}{m}\delta_{ij}\mathbf{c}_j^{0\epsilon}\mathbf{a}_i^{s\alpha} + \delta_{(i\neq j)}\frac{\lambda_i^m}{\lambda_j^m - \lambda_i^m}\frac{\lambda_j^s}{\lambda_i^s}\mathbf{c}_j^{0\epsilon}\mathbf{a}_j^{s\alpha} - \delta_{(i\neq j)}\frac{\lambda_j^m}{\lambda_j^m - \lambda_i^m}\mathbf{c}_j^{0\epsilon}\mathbf{a}_i^{s\alpha} \\ &- \mathbf{a}_i^{s\alpha}(Z_{m-1})_{ij} - \mathbf{a}_i^{s\alpha}\sum_{\lambda=1}^{\epsilon-1}\mathbf{a}_i^{0\lambda}(\mathbf{c}_j^{0\lambda} - \mathbf{c}_j^{0\epsilon})\,;\end{aligned} \tag{5.25}$$

for any $q \in I \setminus \{0\}$ *and* $1 \leq \epsilon, \alpha \leq d_q$, *we have*

$$\begin{aligned}\{\mathbf{c}_j^{q\epsilon}, \mathbf{a}_i^{q\alpha}\} =& \frac{q}{m}\delta_{ij}\mathbf{c}_j^{q\epsilon}\mathbf{a}_i^{q\alpha} + \delta_{(i\neq j)}\frac{\lambda_i^m}{\lambda_j^m - \lambda_i^m}\mathbf{c}_j^{q\epsilon}\left(\frac{\lambda_j^q}{\lambda_i^q}\mathbf{a}_j^{q\alpha} - \mathbf{a}_i^{q\alpha}\right) \\ &- \delta_{(\alpha<\epsilon)}\mathbf{c}_j^{q\epsilon}\mathbf{a}_i^{q\alpha} + \delta_{\alpha\epsilon}\sum_{\lambda=1}^{\epsilon-1}\mathbf{a}_i^{q\lambda}\mathbf{c}_j^{q\lambda} + \delta_{\alpha\epsilon}(Z_{q-1})_{ij}\,;\end{aligned} \tag{5.26}$$

for any $q \in I \setminus \{0\}$, $1 \leq \epsilon \leq d_q$ *and* $1 \leq \alpha \leq d_0$, *we have*

$$\begin{aligned}\{\mathbf{c}_j^{q\epsilon}, \mathbf{a}_i^{0\alpha}\} =& \frac{1}{2}\delta_{(i\neq j)}\frac{\lambda_j^m + \lambda_i^m}{\lambda_j^m - \lambda_i^m}\mathbf{c}_j^{q\epsilon}\left(\mathbf{a}_j^{0\alpha} - \mathbf{a}_i^{0\alpha}\right) \\ &+ \frac{1}{2}\mathbf{c}_j^{q\epsilon}\sum_{\kappa=1}^{d_0}o(\alpha,\kappa)(\mathbf{a}_i^{0\alpha}\mathbf{a}_j^{0\kappa} + \mathbf{a}_j^{0\alpha}\mathbf{a}_i^{0\kappa})\,;\end{aligned} \tag{5.27}$$

for any $q, s \in I \setminus \{0\}$ *with* $q \neq s$, *and for any* $1 \leq \epsilon \leq d_q$, $1 \leq \alpha \leq d_s$, *we have*

$$\{\mathbf{c}_j^{q\epsilon}, \mathbf{a}_i^{s\alpha}\} \overset{q\neq s}{=} \frac{s}{m}\delta_{ij}\mathbf{c}_j^{q\epsilon}\mathbf{a}_i^{s\alpha} + \delta_{(i\neq j)}\frac{\lambda_i^m}{\lambda_j^m - \lambda_i^m}\mathbf{c}_j^{q\epsilon}\left(\frac{\lambda_j^s}{\lambda_i^s}\mathbf{a}_j^{s\alpha} - \mathbf{a}_i^{s\alpha}\right) - \delta_{(s<q)}\mathbf{c}_j^{q\epsilon}\mathbf{a}_i^{q\alpha}\,; \tag{5.28}$$

for any $1 \leq \alpha, \gamma \leq d_0$*, we have*

$$\begin{aligned}\{\mathbf{a}_j^{0\gamma}, \mathbf{a}_i^{0\alpha}\} =& \frac{1}{2}\delta_{(i\neq j)}\frac{\lambda_j^m + \lambda_i^m}{\lambda_j^m - \lambda_i^m}\left(\mathbf{a}_j^{0\gamma}\mathbf{a}_i^{0\alpha} + \mathbf{a}_i^{0\gamma}\mathbf{a}_j^{0\alpha} - \mathbf{a}_j^{0\gamma}\mathbf{a}_j^{0\alpha} - \mathbf{a}_i^{0\gamma}\mathbf{a}_i^{0\alpha}\right)\\ &+ \frac{1}{2}o(\alpha,\gamma)(\mathbf{a}_j^{0\gamma}\mathbf{a}_i^{0\alpha} + \mathbf{a}_i^{0\gamma}\mathbf{a}_j^{0\alpha})\\ &+ \frac{1}{2}\mathbf{a}_i^{0\alpha}\sum_{\sigma=1}^{d_0} o(\gamma,\sigma)(\mathbf{a}_j^{0\gamma}\mathbf{a}_i^{0\sigma} + \mathbf{a}_i^{0\gamma}\mathbf{a}_j^{0\sigma})\\ &- \frac{1}{2}\mathbf{a}_j^{0\gamma}\sum_{\kappa=1}^{d_0} o(\alpha,\kappa)(\mathbf{a}_i^{0\alpha}\mathbf{a}_j^{0\kappa} + \mathbf{a}_j^{0\alpha}\mathbf{a}_i^{0\kappa})\,;\end{aligned} \tag{5.29}$$

for any $p \in I \setminus \{0\}$ *and* $1 \leq \alpha, \gamma \leq d_p$*, we have*

$$\begin{aligned}\{\mathbf{a}_j^{p\gamma}, \mathbf{a}_i^{p\alpha}\} =& \frac{1}{2}\delta_{(i\neq j)}\frac{\lambda_j^m + \lambda_i^m}{\lambda_j^m - \lambda_i^m}\left(\mathbf{a}_j^{p\gamma}\mathbf{a}_i^{p\alpha} + \mathbf{a}_i^{p\gamma}\mathbf{a}_j^{p\alpha}\right) + \frac{1}{2}o(\alpha,\gamma)(\mathbf{a}_j^{p\gamma}\mathbf{a}_i^{p\alpha} + \mathbf{a}_i^{p\gamma}\mathbf{a}_j^{p\alpha})\\ &- \delta_{(i\neq j)}\frac{\lambda_i^m}{\lambda_j^m - \lambda_i^m}\frac{\lambda_j^p}{\lambda_i^p}\mathbf{a}_j^{p\gamma}\mathbf{a}_j^{p\alpha} - \delta_{(i\neq j)}\frac{\lambda_j^m}{\lambda_j^m - \lambda_i^m}\frac{\lambda_i^p}{\lambda_j^p}\mathbf{a}_i^{p\gamma}\mathbf{a}_i^{p\alpha}\,;\end{aligned} \tag{5.30}$$

for any $s \in I \setminus \{0\}$*,* $1 \leq \gamma \leq d_0$ *and* $1 \leq \alpha \leq d_s$*, we have*

$$\begin{aligned}\{\mathbf{a}_j^{0\gamma}, \mathbf{a}_i^{s\alpha}\} =& \frac{1}{2}\delta_{(i\neq j)}\frac{\lambda_j^m + \lambda_i^m}{\lambda_j^m - \lambda_i^m}\left(\mathbf{a}_j^{0\gamma}\mathbf{a}_i^{s\alpha} - \mathbf{a}_i^{0\gamma}\mathbf{a}_i^{s\alpha}\right)\\ &+ \delta_{(i\neq j)}\frac{\lambda_i^m}{\lambda_j^m - \lambda_i^m}\frac{\lambda_j^s}{\lambda_i^s}\left(\mathbf{a}_i^{0\gamma}\mathbf{a}_j^{s\alpha} - \mathbf{a}_j^{0\gamma}\mathbf{a}_j^{s\alpha}\right)\\ &+ \frac{1}{2}\mathbf{a}_i^{s\alpha}\sum_{\sigma=1}^{d_0} o(\gamma,\sigma)(\mathbf{a}_j^{0\gamma}\mathbf{a}_i^{0\sigma} + \mathbf{a}_i^{0\gamma}\mathbf{a}_j^{0\sigma})\,;\end{aligned} \tag{5.31}$$

for any $p \in I \setminus \{0\}$*,* $1 \leq \gamma \leq d_p$ *and* $1 \leq \alpha \leq d_0$*, we have*

$$\begin{aligned}\{\mathbf{a}_j^{p\gamma}, \mathbf{a}_i^{0\alpha}\} =& \frac{1}{2}\delta_{(i\neq j)}\frac{\lambda_j^m + \lambda_i^m}{\lambda_j^m - \lambda_i^m}\left(\mathbf{a}_j^{p\gamma}\mathbf{a}_i^{0\alpha} - \mathbf{a}_j^{p\gamma}\mathbf{a}_j^{0\alpha}\right)\\ &+ \delta_{(i\neq j)}\frac{\lambda_j^m}{\lambda_j^m - \lambda_i^m}\frac{\lambda_j^{-p}}{\lambda_i^{-p}}\left(\mathbf{a}_i^{p\gamma}\mathbf{a}_j^{0\alpha} - \mathbf{a}_i^{p\gamma}\mathbf{a}_i^{0\alpha}\right)\\ &- \frac{1}{2}\mathbf{a}_j^{p\gamma}\sum_{\kappa=1}^{d_0} o(\alpha,\kappa)(\mathbf{a}_i^{0\alpha}\mathbf{a}_j^{0\kappa} + \mathbf{a}_j^{0\alpha}\mathbf{a}_i^{0\kappa})\,;\end{aligned} \tag{5.32}$$

for any $0 < p < s \leq m-1$*, and for any* $1 \leq \gamma \leq d_p$*,* $1 \leq \alpha \leq d_s$*, we have*

$$\begin{aligned}\{\mathbf{a}_j^{p\gamma}, \mathbf{a}_i^{s\alpha}\} \overset{p\leq s}{=}& -\delta_{ij}\mathbf{a}_j^{p\gamma}\mathbf{a}_i^{s\alpha} + \delta_{(i\neq j)}\frac{\lambda_i^m}{\lambda_j^m - \lambda_i^m}\left(\mathbf{a}_j^{p\gamma}\mathbf{a}_i^{s\alpha} + \frac{\lambda_j^{s-p}}{\lambda_i^{s-p}}\mathbf{a}_i^{p\gamma}\mathbf{a}_j^{s\alpha}\right)\\ &- \delta_{(i\neq j)}\frac{\lambda_i^m}{\lambda_j^m - \lambda_i^m}\frac{\lambda_j^s}{\lambda_i^s}\mathbf{a}_j^{p\gamma}\mathbf{a}_j^{s\alpha} - \delta_{(i\neq j)}\frac{\lambda_j^m}{\lambda_j^m - \lambda_i^m}\frac{\lambda_j^{-p}}{\lambda_i^{-p}}\mathbf{a}_i^{p\gamma}\mathbf{a}_i^{s\alpha}\,;\end{aligned} \tag{5.33}$$

for any $0 < s < p \leq m-1$, *and for any* $1 \leq \gamma \leq d_p$, $1 \leq \alpha \leq d_s$, *we have*

$$\begin{aligned}\{\mathbf{a}_j^{p\gamma}, \mathbf{a}_i^{s\alpha}\} \overset{p\geq s}{=} \; & \delta_{ij}\mathbf{a}_j^{p\gamma}\mathbf{a}_i^{s\alpha} + \delta_{(i\neq j)}\frac{\lambda_j^m}{\lambda_j^m - \lambda_i^m}\left(\mathbf{a}_j^{p\gamma}\mathbf{a}_i^{s\alpha} + \frac{\lambda_j^{s-p}}{\lambda_i^{s-p}}\mathbf{a}_i^{p\gamma}\mathbf{a}_j^{s\alpha}\right) \\ & - \delta_{(i\neq j)}\frac{\lambda_i^m}{\lambda_j^m - \lambda_i^m}\frac{\lambda_j^s}{\lambda_i^s}\mathbf{a}_j^{p\gamma}\mathbf{a}_j^{s\alpha} - \delta_{(i\neq j)}\frac{\lambda_j^m}{\lambda_j^m - \lambda_i^m}\frac{\lambda_j^{-p}}{\lambda_i^{-p}}\mathbf{a}_i^{p\gamma}\mathbf{a}_i^{s\alpha}\,.\end{aligned} \tag{5.34}$$

Remark 5.2.8 *The operation* $\{-,-\}$ *that we consider on* $\mathfrak{h}_{sp,m}/W$ *is only defined as an antisymmetric biderivation in the first place. This is a Poisson bracket because of Lemma B.1.*

Before proceeding to the proof, remark that we can write from Lemma 5.2.6 that

$$\xi^*\{f^k, f^l\}_{\mathrm{P}} = 0\,, \quad \xi^*\{f^k, g^l_{s\alpha,r\beta}\}_{\mathrm{P}} = k\sum_{i=1}^n \mathbf{a}_i^{s\alpha}\mathbf{c}_i^{r\beta}\lambda_i^{k+l}\,. \tag{5.35}$$

Moreover, we also obtain

$$\begin{aligned}\xi^*\{g^k_{p\gamma,q\epsilon}, g^l_{s\alpha,r\beta}\}_{\mathrm{P}} = \; & \frac{1}{2}\left(\overset{\bullet}{\sum_{v=1,\dots,k}} - \overset{\bullet}{\sum_{v=1,\dots,l}}\right)\sum_{i,j}(\mathbf{a}_j^{s\alpha}\mathbf{c}_i^{r\beta}\lambda_i^v\mathbf{a}_i^{p\gamma}\mathbf{c}_j^{q\epsilon}\lambda_j^{k+l-v}) \\ & + \frac{1}{2}\left(\overset{\triangle}{\sum_{v=1,\dots,k}} - \overset{\triangle}{\sum_{v=1,\dots,l}}\right)\sum_{i,j}(\mathbf{a}_j^{s\alpha}\mathbf{c}_i^{r\beta}\lambda_i^{k+l-v}\mathbf{a}_i^{p\gamma}\mathbf{c}_j^{q\epsilon}\lambda_j^v) \\ & + \frac{1}{2}[o(p,r) - o(p,s) + o(q,s) - o(q,r)]\sum_{i,j}(\mathbf{a}_i^{s\alpha}\mathbf{c}_j^{q\epsilon}\lambda_j^k\mathbf{a}_j^{p\gamma}\mathbf{c}_i^{r\beta}\lambda_i^l) \\ & + \frac{1}{2}\delta_{ps}o(\alpha,\gamma)\sum_{i,j}[(\mathbf{a}_i^{p\gamma}\mathbf{c}_j^{q\epsilon}\lambda_j^k\mathbf{a}_j^{s\alpha}\mathbf{c}_i^{r\beta}\lambda_i^l) + (\mathbf{a}_i^{s\alpha}\mathbf{c}_j^{q\epsilon}\lambda_j^k\mathbf{a}_j^{p\gamma}\mathbf{c}_i^{r\beta}\lambda_i^l)] \\ & + \frac{1}{2}\delta_{qr}o(\epsilon,\beta)\sum_{i,j}[(\mathbf{a}_i^{s\alpha}\mathbf{c}_j^{r\beta}\lambda_j^k\mathbf{a}_j^{p\gamma}\mathbf{c}_i^{q\epsilon}\lambda_i^l) - (\mathbf{a}_i^{s\alpha}\mathbf{c}_j^{q\epsilon}\lambda_j^k\mathbf{a}_j^{p\gamma}\mathbf{c}_i^{r\beta}\lambda_i^l)] \\ & + \frac{1}{2}\Big(\delta_{qs}[o(\epsilon,\alpha) + \delta_{\alpha\epsilon}] - \delta_{pr}[o(\beta,\gamma) + \delta_{\beta\gamma}]\Big)\sum_{i,j}(\mathbf{a}_i^{s\alpha}\mathbf{c}_j^{q\epsilon}\lambda_j^k\mathbf{a}_j^{p\gamma}\mathbf{c}_i^{r\beta}\lambda_i^l) \\ & + \delta_{qs}\delta_{\alpha\epsilon}\sum_{i,j}\left((Z_{s-1})_{ij} + \sum_{\lambda=1}^{\epsilon-1}\mathbf{a}_i^{s\lambda}\mathbf{c}_j^{s\lambda}\right)\lambda_j^k\mathbf{a}_j^{p\gamma}\mathbf{c}_i^{r\beta}\lambda_i^l \\ & - \delta_{pr}\delta_{\beta\gamma}\sum_{i,j}\left((Z_{p-1})_{ji} + \sum_{\mu=1}^{\beta-1}\mathbf{a}_j^{p\mu}\mathbf{c}_i^{p\mu}\right)\lambda_i^l\mathbf{a}_i^{s\alpha}\mathbf{c}_j^{q\epsilon}\lambda_j^k\,,\end{aligned} \tag{5.36}$$

where in the sum $\overset{\bullet}{\sum}$ we require $v \underset{m}{\equiv} p-(r-1)$, while for $\overset{\triangle}{\sum}$ we require $v \underset{m}{\equiv} s-(q-1)$. To understand how we get the factor $(Z_{s-1})_{ij}$ when we write $\xi^*\operatorname{tr}(ZX^k\mathbf{A}^pE^{pr}_{\gamma\beta}\mathbf{C}^rX^l)$, remark that $\mathbf{C}^r : \mathcal{V}_{r-1} \to \mathcal{V}_\infty$ so that the element X^lZ in $\mathbf{C}^rX^lZ$ acts as $X_{r-1}\dots X_{l+r-2}Z_{l+r-2}$. By assumption $(r-1)+l=s$ modulo m, so that the element Z in this expression can be replaced by Z_{s-1}. The same observation explains how we get $(Z_{p-1})_{ji}$ in $\xi^* \operatorname{tr}(ZX^l \mathbf{A}^s E^{sq}_{\alpha\epsilon} \mathbf{C}^q X^k)$.

The proof will follow from the different lemmae presented in the next subsections. Indeed, we will show that (5.16) holds for the different functions given in (5.15).

First type of brackets

Lemma 5.2.9 *For any possible indices,*

$$\{\xi^* f^k, \xi^* f^l\} = \xi^*\{f^k, f^l\}_{\mathrm{P}},\ \{\xi^* f^k, \sum_\alpha \xi^* g^l_{0\alpha,r\beta}\} = \xi^*\{f^k, \sum_\alpha g^l_{0\alpha,r\beta}\}_{\mathrm{P}},$$
$$\{\xi^* f^k, \xi^* g^l_{s\alpha,01}\} = \xi^*\{f^k, g^l_{s\alpha,01}\}_{\mathrm{P}}\,.$$

Proof. The first identity in (5.17) implies $\{\xi^* f^k, \xi^* f^l\} = 0$, as desired from (5.35). Similarly, we get

$$\{\xi^* f^k, \sum_\alpha \xi^* g^l_{0\alpha,r\beta}\} = mk \sum_{ij} \lambda_i^{k-1}\lambda_j^l \{\lambda_i, \mathbf{c}_j^{r\beta}\} = k\sum_i \lambda_i^{k+l}\mathbf{c}_i^{r\beta}\,,$$

which is $k\sum_\alpha g^{k+l}_{0\alpha,r\beta}$, hence this is $\xi^*\{f^k, \sum_\alpha g^l_{0\alpha,r\beta}\}_{\mathrm{P}}$ by Lemma 5.2.6. The last identity is checked in the same way. □

Second type of brackets

As a second step, we need to check that

$$\{\sum_\gamma \xi^* g^k_{0\gamma,q\epsilon}, \sum_\alpha \xi^* g^l_{0\alpha,r\beta}\} = \sum_{\alpha,\gamma} \xi^*\{g^k_{0\gamma,q\epsilon}, g^l_{0\alpha,r\beta}\}_{\mathrm{P}} \tag{5.37}$$

for all possible indices. We can first rewrite the left-hand side using (5.17) as

$$(5.37)_{LHS} = \frac{k-l}{m}\sum_i \lambda_i^{k+l}\mathbf{c}_i^{q\epsilon}\mathbf{c}_i^{r\beta} + \sum_{i,j}\lambda_j^k\lambda_i^l\{\mathbf{c}_j^{q\epsilon}, \mathbf{c}_i^{r\beta}\}\,, \tag{5.38}$$

while we get for the right-hand side using (5.36) that

$$
\begin{aligned}
(5.37)_{RHS} =& \frac{1}{2}\left(\sum^{\bullet}_{v=1,\ldots,k} - \sum^{\bullet}_{v=1,\ldots,l}\right)\sum_{i,j}(\mathbf{c}_i^{r\beta}\lambda_i^v\mathbf{c}_j^{q\epsilon}\lambda_j^{k+l-v}) \\
&+ \frac{1}{2}\left(\sum^{\triangle}_{v=1,\ldots,k} - \sum^{\triangle}_{v=1,\ldots,l}\right)\sum_{i,j}(\mathbf{c}_i^{r\beta}\lambda_i^{k+l-v}\mathbf{c}_j^{q\epsilon}\lambda_j^v) \\
&+ \frac{1}{2}[o(0,r)+o(q,0)-o(q,r)]\sum_{i,j}(\mathbf{c}_j^{q\epsilon}\lambda_j^k\mathbf{c}_i^{r\beta}\lambda_i^l) \\
&+ \frac{1}{2}\delta_{qr}o(\epsilon,\beta)\sum_{i,j}[(\mathbf{c}_j^{r\beta}\lambda_j^k\mathbf{c}_i^{q\epsilon}\lambda_i^l)-(\mathbf{c}_j^{q\epsilon}\lambda_j^k\mathbf{c}_i^{r\beta}\lambda_i^l)] \\
&+ \frac{1}{2}\delta_{q0}\sum_{\alpha}[o(\epsilon,\alpha)+\delta_{\alpha\epsilon}]\sum_{i,j}(\mathbf{a}_i^{0\alpha}\mathbf{c}_j^{q\epsilon}\lambda_j^k\mathbf{c}_i^{r\beta}\lambda_i^l) \\
&- \frac{1}{2}\delta_{0r}\sum_{\gamma}[o(\beta,\gamma)+\delta_{\beta\gamma}]\sum_{i,j}(\mathbf{c}_j^{q\epsilon}\lambda_j^k\mathbf{a}_j^{0\gamma}\mathbf{c}_i^{r\beta}\lambda_i^l) \\
&+ \delta_{q0}\sum_{i,j}\left((Z_{m-1})_{ij}+\sum_{\lambda=1}^{\epsilon-1}\mathbf{a}_i^{0\lambda}\mathbf{c}_j^{0\lambda}\right)\lambda_j^k\mathbf{c}_i^{r\beta}\lambda_i^l \\
&- \delta_{0r}\sum_{i,j}\left((Z_{m-1})_{ji}+\sum_{\mu=1}^{\beta-1}\mathbf{a}_j^{0\mu}\mathbf{c}_i^{0\mu}\right)\lambda_i^l\mathbf{c}_j^{q\epsilon}\lambda_j^k\,,
\end{aligned}
$$

where in the sum $\sum^{\bullet}$ we require $v \underset{m}{\equiv} 1-r$, while for $\sum^{\triangle}$ we require $v \underset{m}{\equiv} 1-q$. Here, we used that the fourth line of (5.36) vanishes when we sum over all $\alpha,\gamma = 1,\ldots,d_0$. We clearly see that we have to discuss the possible choices of $r,q = 0,\ldots,m-1$ separately.

Lemma 5.2.10 *For $q = r = 0$ and for any $1 \leq \beta,\epsilon \leq d_0$, we have that (5.37) holds.*

Proof. We use (5.18), and the proof is similar to the corresponding case for a Jordan quiver given in Proposition 4.3.3, so we can omit it. □

Lemma 5.2.11 *For $r = 0$ and for any $q \in I \setminus \{0\}$, $1 \leq \epsilon \leq d_q$, $1 \leq \beta \leq d_0$, we have that (5.37) holds. For $q = 0$ and for any $r \in I \setminus \{0\}$, $1 \leq \beta \leq d_r$, $1 \leq \epsilon \leq d_0$, we have that (5.37) holds.*

Proof. We only need to show that (5.37) holds in the first case, as the second case can be obtained by antisymmetry. Note that for $g^k_{0\gamma,q\epsilon}$ and $g^l_{0\alpha,0\beta}$ to be nonzero, we need $k = k_0m+1-q$ and

$l = l_0 m + 1$ for some $k_0, l_0 \in \mathbb{N}^\times$. In particular, remark that $k < k_0 m + 1$. We can write

$$\begin{aligned}
(5.37)_{RHS} =& \frac{1}{2}\left(\sum_{v_0=0}^{k_0-1} - \sum_{v_0=0}^{l_0}\right)\sum_{i,j} \mathbf{c}_i^{0\beta}\lambda_i^{v_0 m+1}\mathbf{c}_j^{q\epsilon}\lambda_j^{(k_0+l_0-v_0)m+1-q} \\
&+ \frac{1}{2}\left(\sum_{v_0=1}^{k_0} - \sum_{v_0=1}^{l_0}\right)\sum_{i,j} \mathbf{c}_i^{0\beta}\lambda_i^{(k_0+l_0-v_0)m+1}\mathbf{c}_j^{q\epsilon}\lambda_j^{v_0 m+1-q} \\
&- \frac{1}{2}\sum_{\gamma=1}^{d_0}[o(\beta,\gamma)+\delta_{\beta\gamma}]\sum_{i,j}(\mathbf{c}_j^{q\epsilon}\lambda_j^k \mathbf{a}_j^{0\gamma}\mathbf{c}_i^{0\beta}\lambda_i^l) \\
&- \sum_{i,j}\left((Z_{m-1})_{ji} + \sum_{\mu=1}^{\beta-1}\mathbf{a}_j^{0\mu}\mathbf{c}_i^{0\mu}\right)\lambda_i^l \mathbf{c}_j^{q\epsilon}\lambda_j^k\,,
\end{aligned}$$

Indeed, in the first sum, we need $v \underset{m}{\equiv} 1$ so we sum over $v = v_0 m + 1$ but we can not consider $v_0 = k_0$, while for the second $v \underset{m}{\equiv} 1 - q$ and $v = v_0 m + 1 - q < v_0 m + 1$. Using that $[o(\beta,\gamma) + \delta_{\beta\gamma}] = 1 - 2\delta_{(\beta>\gamma)}$, we can write

$$\begin{aligned}
(5.37)_{RHS} =& \frac{1}{2}\sum_{i,j}\mathbf{c}_i^{0\beta}\mathbf{c}_j^{q\epsilon}\lambda_i\lambda_j^{1-q}\,\Sigma_{(k,l)}^{(i,j)} - \frac{1}{2}\sum_{i,j}\mathbf{c}_i^{0\beta}\mathbf{c}_j^{q\epsilon}\lambda_i^{k_0 m+1}\lambda_j^{l_0 m+1-q} \\
&- \frac{1}{2}\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{0\beta} + \sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{q\epsilon}\sum_{\gamma=1}^{\beta-1}\mathbf{a}_j^{0\gamma}\mathbf{c}_i^{0\beta} \\
&- \sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{q\epsilon}\left((Z_{m-1})_{ji} + \sum_{\mu=1}^{\beta-1}\mathbf{a}_j^{0\mu}\mathbf{c}_i^{0\mu}\right)\,,
\end{aligned}$$

where we have set

$$\Sigma_{(k,l)}^{(i,j)} := \left(\sum_{v_0=1}^{k_0} - \sum_{v_0=1}^{l_0}\right)\left(\lambda_i^{v_0 m}\lambda_j^{(k_0+l_0-v_0)m} + \lambda_i^{(k_0+l_0-v_0)m}\lambda_j^{v_0 m}\right)\,. \tag{5.39}$$

To reduce $\Sigma_{(k,l)}^{(i,j)}$, we have the following result similar to (4.38).

Lemma 5.2.12 *If $i = j$, $\Sigma_{(k,l)}^{(i,j)} = (k_0 - l_0)\lambda_i^{(k_0+l_0)m}$, while if $i \neq j$,*

$$\Sigma_{(k,l)}^{(i,j)} = \frac{\lambda_i^m + \lambda_j^m}{\lambda_i^m - \lambda_j^m}\left(\lambda_i^{k_0 m}\lambda_j^{l_0 m} - \lambda_i^{l_0 m}\lambda_j^{k_0 m}\right)\,. \tag{5.40}$$

Using this lemma, we can write

$$\begin{aligned}
(5.37)_{RHS} =& \frac{1}{2}\sum_{i\neq j}\mathbf{c}_i^{0\beta}\mathbf{c}_j^{q\epsilon}\lambda_i\lambda_j^{1-q}\frac{\lambda_i^m + \lambda_j^m}{\lambda_i^m - \lambda_j^m}\left(\lambda_i^{k_0 m}\lambda_j^{l_0 m} - \lambda_i^{l_0 m}\lambda_j^{k_0 m}\right) \\
&+ (k_0 - l_0 - 1)\sum_i \lambda_i^{k+l}\mathbf{c}_i^{0\beta}\mathbf{c}_i^{q\epsilon} - \frac{1}{2}\sum_{i\neq j}\mathbf{c}_i^{0\beta}\mathbf{c}_j^{q\epsilon}(\lambda_i^{k_0 m+1}\lambda_j^{l_0 m+1-q} + \lambda_j^k\lambda_i^l) \\
&- \sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{q\epsilon}(Z_{m-1})_{ji} - \sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{q\epsilon}\sum_{\mu=1}^{\beta-1}\mathbf{a}_j^{0\mu}(\mathbf{c}_i^{0\mu} - \mathbf{c}_i^{0\beta})\,.
\end{aligned}$$

Recalling that $k=k_0m+1-q$ and $l=l_0m+1$, we find

$$
\begin{aligned}
(5.37)_{RHS} =& \frac{1}{2}\sum_{i\neq j}\lambda_j^k\lambda_i^l\frac{\lambda_j^m+\lambda_i^m}{\lambda_j^m-\lambda_i^m}\left(\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{0\beta}+\frac{\lambda_j^q}{\lambda_i^q}\mathbf{c}_i^{q\epsilon}\mathbf{c}_j^{0\beta}\right) \\
&+\left(\frac{k-l}{m}+\frac{q-m}{m}\right)\sum_i \lambda_i^{k+l}\mathbf{c}_i^{0\beta}\mathbf{c}_i^{q\epsilon}-\frac{1}{2}\sum_{i\neq j}\lambda_j^k\lambda_i^l\left(\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{0\beta}+\frac{\lambda_j^q}{\lambda_i^q}\mathbf{c}_i^{q\epsilon}\mathbf{c}_j^{0\beta}\right) \\
&-\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{q\epsilon}(Z_{m-1})_{ji}-\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{q\epsilon}\sum_{\mu=1}^{\beta-1}\mathbf{a}_j^{0\mu}(\mathbf{c}_i^{0\mu}-\mathbf{c}_i^{0\beta})\,.
\end{aligned}
$$

If we sum together the first and third terms of $(5.37)_{RHS}$ just obtained, it is not hard to see that it matches $(5.37)_{LHS}$ after introducing (5.19) in it. □

Lemma 5.2.13 *For any $q\in I\setminus\{0\}$ with $r=q$ and $1\leq\beta,\epsilon\leq d_q$, we have that (5.37) holds.*

Proof. For $r=q$, we consider $k=k_0m+1-q$ and $l=l_0m+1-q$ for arbitrary $k_0,l_0\in\mathbb{N}^\times$. We can directly write

$$
(5.37)_{RHS}=\frac{1}{2}\sum_{i,j}\mathbf{c}_i^{q\beta}\mathbf{c}_j^{q\epsilon}\lambda_i^{1-q}\lambda_j^{1-q}\,\Sigma_{(k,l)}^{(i,j)}+\frac{1}{2}o(\epsilon,\beta)\sum_{i,j}\lambda_j^k\lambda_i^l(\mathbf{c}_j^{r\beta}\mathbf{c}_i^{q\epsilon}-\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{r\beta})\,,
$$

using the symbol $\Sigma_{(k,l)}^{(i,j)}$ introduced as (5.39). From Lemma 5.2.12, this is

$$
\begin{aligned}
(5.37)_{RHS}=&(k_0-l_0)\sum_i\lambda_i^{k+l}\mathbf{c}_i^{q\beta}\mathbf{c}_i^{q\epsilon}+\frac{1}{2}\sum_{i\neq j}\lambda_j^k\lambda_i^l\frac{\lambda_j^m+\lambda_i^m}{\lambda_j^m-\lambda_i^m}(\mathbf{c}_j^{r\beta}\mathbf{c}_i^{q\epsilon}+\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{r\beta}) \\
&+\frac{1}{2}o(\epsilon,\beta)\sum_{i,j}\lambda_j^k\lambda_i^l(\mathbf{c}_j^{r\beta}\mathbf{c}_i^{q\epsilon}-\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{r\beta})\,,
\end{aligned}
$$

and since $k_0-l_0=\frac{k-l}{m}$ we conclude that this equals $(5.37)_{LHS}$ when we use (5.21). □

Lemma 5.2.14 *For any $0<r<q\leq m-1$, $1\leq\beta\leq d_r$ and $1\leq\epsilon\leq d_q$, we have that (5.37) holds. This is also true when $0<q<r\leq m-1$.*

Proof. By antisymmetry, it suffices to prove that (5.37) holds in the first case. We take $k=k_0m+1-q$ and $l=l_0m+1-r$ for $k_0,l_0\in\mathbb{N}^\times$. By assumption, $q>r$ so that $k<k_0m+1-r$

while $l > l_0 m + 1 - q$. Hence, we write

$$(5.37)_{RHS} = \frac{1}{2}\left(\sum_{v_0=1}^{k_0-1} - \sum_{v_0=1}^{l_0}\right)\sum_{i,j}(\mathbf{c}_i^{r\beta}\lambda_i^{v_0 m+1-r}\mathbf{c}_j^{q\epsilon}\lambda_j^{(k_0+l_0-v_0)m+1-q})$$
$$+\frac{1}{2}\left(\sum_{v_0=1}^{k_0} - \sum_{v_0=1}^{l_0}\right)\sum_{i,j}(\mathbf{c}_i^{r\beta}\lambda_i^{(k_0+l_0-v_0)m+1-r}\mathbf{c}_j^{q\epsilon}\lambda_j^{v_0 m+1-q})$$
$$-\frac{1}{2}o(q,r)\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{r\beta},$$

and if we introduce $\Sigma_{(k,l)}^{(i,j)}$ this can be written as

$$\frac{1}{2}\sum_{i,j}\mathbf{c}_i^{r\beta}\lambda_i^{1-r}\mathbf{c}_j^{q\epsilon}\lambda_j^{1-q}\,\Sigma_{(k,l)}^{(i,j)} - \frac{1}{2}\sum_{i,j}\mathbf{c}_i^{r\beta}\lambda_i^{k_0 m+1-r}\mathbf{c}_j^{q\epsilon}\lambda_j^{l_0 m+1-q} + \frac{1}{2}\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{r\beta}.$$

We use Lemma 5.2.12 in order to obtain

$$(5.37)_{RHS} = \frac{1}{2}\sum_{i\neq j}\mathbf{c}_i^{r\beta}\lambda_i^{1-r}\mathbf{c}_j^{q\epsilon}\lambda_j^{1-q}\frac{\lambda_i^m+\lambda_j^m}{\lambda_i^m-\lambda_j^m}\left(\lambda_i^{k_0 m}\lambda_j^{l_0 m} - \lambda_i^{l_0 m}\lambda_j^{k_0 m}\right)$$
$$(k_0-l_0)\sum_i\lambda_i^{k+l}\mathbf{c}_i^{q\epsilon}\mathbf{c}_i^{r\beta} - \frac{1}{2}\sum_{i,j}\mathbf{c}_i^{r\beta}\lambda_i^{k+q-r}\mathbf{c}_j^{q\epsilon}\lambda_j^{l+r-q} + \frac{1}{2}\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{r\beta}$$
$$=\left(\frac{k-l}{m}+\frac{q-r}{m}\right)\sum_i\lambda_i^{k+l}\mathbf{c}_i^{q\epsilon}\mathbf{c}_i^{r\beta}$$
$$+\frac{1}{2}\sum_{i\neq j}\lambda_j^k\lambda_i^l\frac{\lambda_j^m+\lambda_i^m}{\lambda_j^m-\lambda_i^m}\left(\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{r\beta}+\frac{\lambda_j^{q-r}}{\lambda_i^{q-r}}\mathbf{c}_j^{r\beta}\mathbf{c}_i^{q\epsilon}\right)$$
$$+\frac{1}{2}\sum_{i\neq j}\lambda_j^k\lambda_i^l\left(\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{r\beta}-\frac{\lambda_j^{q-r}}{\lambda_i^{q-r}}\mathbf{c}_j^{r\beta}\mathbf{c}_i^{q\epsilon}\right).$$

As in the previous cases, introducing (5.22) in $(5.37)_{LHS}$ yields the same result. □

Third type of brackets

As a third step, we need to check that

$$\{\sum_\gamma \xi^* g^k_{0\gamma,q\epsilon}, \xi^* g^l_{s\alpha,01}\} = \sum_\gamma \xi^*\{g^k_{0\gamma,q\epsilon}, g^l_{s\alpha,01}\}_{\mathrm{P}} \tag{5.41}$$

for all possible couples of indices (q,ϵ) and (s,α). First, we have by (5.17) that the left-hand side is given by

$$(5.41)_{LHS} = \frac{k-l}{m}\sum_i\lambda_i^{k+l}\mathbf{c}_i^{q\epsilon}\mathbf{c}_i^{01}\mathbf{a}_i^{s\alpha} + \sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{a}_i^{s\alpha}\{\mathbf{c}_j^{q\epsilon},\mathbf{c}_i^{01}\} + \sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{c}_i^{01}\{\mathbf{c}_j^{q\epsilon},\mathbf{a}_i^{s\alpha}\}.$$

For the middle term, we can use the expression for $\{\mathbf{c}_j^{q\epsilon},\mathbf{c}_i^{01}\}$ given by (5.18) if $q=0$, or (5.19) otherwise. Hence, we will need to consider these two cases separately, and we will discuss subcases depending on the form of the right-hand side. For the latter, note also that (5.36) gives

$$\begin{aligned}
(5.41)_{RHS} =& \frac{1}{2}\left(\sum_{v=1,\ldots,k}^{\bullet} - \sum_{v=1,\ldots,l}^{\bullet}\right)\sum_{i,j} \mathbf{a}_j^{s\alpha}\mathbf{c}_i^{01}\lambda_i^{v}\mathbf{c}_j^{q\epsilon}\lambda_j^{k+l-v} \\
&+\frac{1}{2}\left(\sum_{v=1,\ldots,k}^{\triangle} - \sum_{v=1,\ldots,l}^{\triangle}\right)\sum_{i,j} \mathbf{a}_j^{s\alpha}\mathbf{c}_i^{01}\lambda_i^{k+l-v}\mathbf{c}_j^{q\epsilon}\lambda_j^{v} \\
&+\frac{1}{2}[-o(0,s)+o(q,s)-o(q,0)]\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{a}_i^{s\alpha}\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{01} \\
&+\frac{1}{2}\delta_{0s}\sum_{\gamma=1}^{d_0}o(\alpha,\gamma)\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{01}[\mathbf{a}_i^{0\gamma}\mathbf{a}_j^{s\alpha}+\mathbf{a}_i^{s\alpha}\mathbf{a}_j^{0\gamma}] \\
&-\frac{1}{2}\delta_{q0}\delta_{(\epsilon\neq 1)}\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{a}_i^{s\alpha}[\mathbf{c}_j^{01}\mathbf{c}_i^{q\epsilon}-\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{01}] \\
&+\frac{1}{2}\delta_{qs}[o(\epsilon,\alpha)+\delta_{\alpha\epsilon}]\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{q\epsilon}\mathbf{a}_i^{s\alpha}\mathbf{c}_i^{01}-\frac{1}{2}\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{q\epsilon}\mathbf{a}_i^{s\alpha}\mathbf{c}_i^{01} \\
&+\delta_{qs}\delta_{\alpha\epsilon}\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{c}_i^{01}\left((Z_{s-1})_{ij}+\sum_{\lambda=1}^{\epsilon-1}\mathbf{a}_i^{s\lambda}\mathbf{c}_j^{s\lambda}\right) \\
&-\sum_{i,j}\lambda_j^k\lambda_i^l(Z_{m-1})_{ji}\mathbf{a}_i^{s\alpha}\mathbf{c}_j^{q\epsilon}\,,
\end{aligned}$$

where in the sum $\overset{\bullet}{\sum}$ we require $v \underset{m}{\equiv} +1$, while for $\overset{\triangle}{\sum}$ we require $v\underset{m}{\equiv} s-q+1$.

Third type of brackets : case $q=0$. We use (5.18) to write $(5.41)_{LHS}$ in this case as

$$\begin{aligned}
(5.41)_{LHS} =& \frac{k-l}{m}\sum_i \lambda_i^{k+l}\mathbf{c}_i^{0\epsilon}\mathbf{c}_i^{01}\mathbf{a}_i^{s\alpha}+\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{c}_i^{01}\{\mathbf{c}_j^{0\epsilon},\mathbf{a}_i^{s\alpha}\} \\
&+\frac{1}{2}\sum_{i\neq j}\lambda_j^k\lambda_i^l\frac{\lambda_j^m+\lambda_i^m}{\lambda_j^m-\lambda_i^m}\mathbf{a}_i^{s\alpha}(\mathbf{c}_j^{0\epsilon}\mathbf{c}_i^{01}+\mathbf{c}_i^{0\epsilon}\mathbf{c}_j^{01}) \\
&-\frac{1}{2}\delta_{(\epsilon\neq1)}\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{a}_i^{s\alpha}(\mathbf{c}_i^{0\epsilon}\mathbf{c}_j^{01}-\mathbf{c}_j^{0\epsilon}\mathbf{c}_i^{01}) \\
&+\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{a}_i^{s\alpha}\left((Z_{m-1})_{ij}\mathbf{c}_i^{01}-(Z_{m-1})_{ji}\mathbf{c}_j^{0\epsilon}\right) \\
&+\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{a}_i^{s\alpha}\mathbf{c}_i^{01}\sum_{\lambda=1}^{\epsilon-1}\mathbf{a}_i^{0\lambda}(\mathbf{c}_j^{0\lambda}-\mathbf{c}_j^{0\epsilon})\,.
\end{aligned} \tag{5.42}$$

Lemma 5.2.15 *For $q=s=0$ and for any $1\leq\alpha,\epsilon\leq d_0$, we have that (5.41) holds.*

Proof. We use (5.24), and the proof is similar to the corresponding case for a Jordan quiver given in Proposition 4.3.3, so we can omit it. □

Lemma 5.2.16 *For $q = 0$, $s \in I \setminus \{0\}$, $1 \leq \alpha \leq d_s$ and $1 \leq \epsilon \leq d_0$, we have that (5.41) holds.*

Proof. We take $k = k_0 m + 1$, $l = l_0 m + s + 1$ for $k_0, l_0 \in \mathbb{N}^\times$, and we can note that $k < k_0 m + s + 1$. We obtain

$$\begin{aligned}
(5.41)_{RHS} =& \frac{1}{2}\left(\sum_{v_0=0}^{k_0} - \sum_{v_0=0}^{l_0}\right)\sum_{i,j} \mathbf{a}_j^{s\alpha}\mathbf{c}_i^{01}\lambda_i^{v_0 m+1}\mathbf{c}_j^{0\epsilon}\lambda_j^{(k_0+l_0-v_0)m+s+1} \\
&+ \frac{1}{2}\left(\sum_{v_0=0}^{k_0-1} - \sum_{v_0=0}^{l_0}\right)\sum_{i,j} \mathbf{a}_j^{s\alpha}\mathbf{c}_i^{01}\lambda_i^{(k_0+l_0-v_0)m+1}\mathbf{c}_j^{0\epsilon}\lambda_j^{v_0 m+s+1} \\
&- \frac{1}{2}\delta_{(\epsilon\neq 1)}\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{a}_i^{s\alpha}[\mathbf{c}_j^{01}\mathbf{c}_i^{0\epsilon} - \mathbf{c}_j^{0\epsilon}\mathbf{c}_i^{01}] \\
&- \frac{1}{2}\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{0\epsilon}\mathbf{a}_i^{s\alpha}\mathbf{c}_i^{01} - \sum_{i,j}\lambda_j^k\lambda_i^l(Z_{m-1})_{ji}\mathbf{a}_i^{s\alpha}\mathbf{c}_j^{0\epsilon}\,.
\end{aligned}$$

Introducing the element $\Sigma_{(k,l)}^{(i,j)}$ from (5.39), we write

$$\begin{aligned}
(5.41)_{RHS} =& \frac{1}{2}\sum_{i,j} \mathbf{a}_j^{s\alpha}\mathbf{c}_i^{01}\lambda_i^1\lambda_j^{s+1}\mathbf{c}_j^{0\epsilon}\Sigma_{(k,l)}^{(i,j)} \\
&- \frac{1}{2}\sum_{i,j} \mathbf{a}_j^{s\alpha}\mathbf{c}_i^{01}\lambda_i^{l_0 m+1}\mathbf{c}_j^{0\epsilon}\lambda_j^{k_0 m+s+1} - \frac{1}{2}\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{0\epsilon}\mathbf{a}_i^{s\alpha}\mathbf{c}_i^{01} \\
&- \frac{1}{2}\delta_{(\epsilon\neq 1)}\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{a}_i^{s\alpha}[\mathbf{c}_j^{01}\mathbf{c}_i^{0\epsilon} - \mathbf{c}_j^{0\epsilon}\mathbf{c}_i^{01}] - \sum_{i,j}\lambda_j^k\lambda_i^l(Z_{m-1})_{ji}\mathbf{a}_i^{s\alpha}\mathbf{c}_j^{0\epsilon}\,.
\end{aligned}$$

We use Lemma 5.2.12 to obtain

$$\begin{aligned}
(5.41)_{RHS} =& (k_0 - l_0 - 1)\sum_{i,j}\lambda_i^{k+l}\mathbf{a}_i^{s\alpha}\mathbf{c}_i^{01}\mathbf{c}_i^{0\epsilon} - \frac{1}{2}\sum_{i\neq j}\lambda_j^k\lambda_i^l\left(\mathbf{c}_j^{0\epsilon}\mathbf{a}_i^{s\alpha}\mathbf{c}_i^{01} + \frac{\lambda_j^s}{\lambda_i^s}\mathbf{a}_j^{s\alpha}\mathbf{c}_i^{01}\mathbf{c}_j^{0\epsilon}\right) \\
&+ \frac{1}{2}\sum_{i\neq j}\lambda_i\lambda_j^{s+1}\frac{\lambda_i^m + \lambda_j^m}{\lambda_i^m - \lambda_j^m}\mathbf{a}_j^{s\alpha}\mathbf{c}_i^{01}\mathbf{c}_j^{0\epsilon}\left(\lambda_i^{k_0 m}\lambda_j^{l_0 m} - \lambda_i^{l_0 m}\lambda_j^{k_0 m}\right) \\
&- \frac{1}{2}\delta_{(\epsilon\neq 1)}\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{a}_i^{s\alpha}[\mathbf{c}_j^{01}\mathbf{c}_i^{0\epsilon} - \mathbf{c}_j^{0\epsilon}\mathbf{c}_i^{01}] - \sum_{i,j}\lambda_j^k\lambda_i^l(Z_{m-1})_{ji}\mathbf{a}_i^{s\alpha}\mathbf{c}_j^{0\epsilon} \\
=& \frac{k - l + s - m}{m}\sum_{i,j}\lambda_i^{k+l}\mathbf{a}_i^{s\alpha}\mathbf{c}_i^{01}\mathbf{c}_i^{0\epsilon} - \frac{1}{2}\sum_{i\neq j}\lambda_j^k\lambda_i^l\left(\mathbf{c}_j^{0\epsilon}\mathbf{a}_i^{s\alpha}\mathbf{c}_i^{01} + \frac{\lambda_j^s}{\lambda_i^s}\mathbf{a}_j^{s\alpha}\mathbf{c}_i^{01}\mathbf{c}_j^{0\epsilon}\right) \\
&+ \frac{1}{2}\sum_{i\neq j}\lambda_j^k\lambda_i^l\frac{\lambda_j^m + \lambda_i^m}{\lambda_j^m - \lambda_i^m}\left(\mathbf{a}_i^{s\alpha}\mathbf{c}_j^{01}\mathbf{c}_i^{0\epsilon} + \frac{\lambda_j^s}{\lambda_i^s}\mathbf{a}_j^{s\alpha}\mathbf{c}_i^{01}\mathbf{c}_j^{0\epsilon}\right) \\
&- \frac{1}{2}\delta_{(\epsilon\neq 1)}\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{a}_i^{s\alpha}[\mathbf{c}_j^{01}\mathbf{c}_i^{0\epsilon} - \mathbf{c}_j^{0\epsilon}\mathbf{c}_i^{01}] - \sum_{i,j}\lambda_j^k\lambda_i^l(Z_{m-1})_{ji}\mathbf{a}_i^{s\alpha}\mathbf{c}_j^{0\epsilon}\,.
\end{aligned}$$

Meanwhile, if we put (5.25) in (5.42), it gives

$$
\begin{aligned}
(5.41)_{LHS} = & \frac{k-l+s-m}{m} \sum_{i,j} \lambda_i^{k+l} \mathbf{a}_i^{s\alpha} \mathbf{c}_i^{01} \mathbf{c}_i^{0\epsilon} - \frac{1}{2}\delta_{(\epsilon \neq 1)} \sum_{i,j} \lambda_j^k \lambda_i^l \mathbf{a}_i^{s\alpha} [\mathbf{c}_j^{01}\mathbf{c}_i^{0\epsilon} - \mathbf{c}_j^{0\epsilon}\mathbf{c}_i^{01}] \\
& - \sum_{i,j} \lambda_j^k \lambda_i^l (Z_{m-1})_{ji} \mathbf{a}_i^{s\alpha} \mathbf{c}_j^{0\epsilon} + \frac{1}{2} \sum_{i \neq j} \lambda_j^k \lambda_i^l \frac{\lambda_j^m + \lambda_i^m}{\lambda_j^m - \lambda_i^m} \mathbf{a}_i^{s\alpha} \mathbf{c}_j^{01} \mathbf{c}_i^{0\epsilon} \\
& + \sum_{i \neq j} \lambda_j^k \lambda_i^l \frac{\lambda_i^m}{\lambda_j^m - \lambda_i^m} \frac{\lambda_j^s}{\lambda_i^s} \mathbf{a}_j^{s\alpha} \mathbf{c}_i^{01} \mathbf{c}_j^{0\epsilon} - \frac{1}{2} \sum_{i \neq j} \lambda_j^k \lambda_i^l \mathbf{a}_i^{s\alpha} \mathbf{c}_i^{01} \mathbf{c}_j^{0\epsilon} ,
\end{aligned}
$$

so the two sides coincide. □

Third type of brackets : case $q > 0$. Let $q \in I \setminus \{0\}$, and use (5.19) to rewrite $(5.41)_{LHS}$ as

$$
\begin{aligned}
(5.41)_{LHS} = & \frac{k-l+q-m}{m} \sum_{i} \lambda_i^{k+l} \mathbf{c}_i^{q\epsilon} \mathbf{c}_i^{01} \mathbf{a}_i^{s\alpha} + \sum_{i,j} \lambda_j^k \lambda_i^l \mathbf{c}_i^{01} \{\mathbf{c}_j^{q\epsilon}, \mathbf{a}_i^{s\alpha}\} \\
& + \sum_{i \neq j} \lambda_j^k \lambda_i^l \frac{\lambda_i^m}{\lambda_j^m - \lambda_i^m} \mathbf{a}_i^{s\alpha} \left(\mathbf{c}_j^{q\epsilon} \mathbf{c}_i^{01} + \frac{\lambda_j^q}{\lambda_i^q} \mathbf{c}_i^{q\epsilon} \mathbf{c}_j^{01} \right) \\
& - \sum_{i,j} \lambda_j^k \lambda_i^l \mathbf{a}_i^{s\alpha} (Z_{m-1})_{ji} \mathbf{c}_j^{q\epsilon} .
\end{aligned} \tag{5.43}
$$

Lemma 5.2.17 *For any $q \in I \setminus \{0\}$ with $s = q$ and $1 \leq \epsilon, \alpha \leq d_q$, we have that (5.41) holds.*

Proof. We take $k = k_0 m + 1 - q$, $l = l_0 m + q + 1$ for $k_0, l_0 \in \mathbb{N}^\times$, so that $k < k_0 m + 1$. Hence

$$
\begin{aligned}
(5.41)_{RHS} = & \frac{1}{2} \left(\sum_{v_0=0}^{k_0-1} - \sum_{v_0=0}^{l_0} \right) \sum_{i,j} \mathbf{a}_j^{q\alpha} \mathbf{c}_i^{01} \mathbf{c}_j^{q\epsilon} \lambda_i^{v_0 m+1} \lambda_j^{(k_0+l_0-v_0)m+1} \\
& + \frac{1}{2} \left(\sum_{v_0=0}^{k_0-1} - \sum_{v_0=0}^{l_0} \right) \sum_{i,j} \mathbf{a}_j^{q\alpha} \mathbf{c}_i^{01} \mathbf{c}_j^{q\epsilon} \lambda_i^{(k_0+l_0-v_0)m+1} \lambda_j^{v_0 m+1} \\
& + \frac{1}{2} [o(\epsilon, \alpha) + \delta_{\alpha\epsilon}] \sum_{i,j} \lambda_j^k \lambda_i^l \mathbf{c}_j^{q\epsilon} \mathbf{a}_i^{q\alpha} \mathbf{c}_i^{01} - \frac{1}{2} \sum_{i,j} \lambda_j^k \lambda_i^l \mathbf{c}_j^{q\epsilon} \mathbf{a}_i^{q\alpha} \mathbf{c}_i^{01} \\
& + \delta_{\alpha\epsilon} \sum_{i,j} \lambda_j^k \lambda_i^l \mathbf{c}_i^{01} \left((Z_{q-1})_{ij} + \sum_{\lambda=1}^{\epsilon-1} \mathbf{a}_i^{q\lambda} \mathbf{c}_j^{q\lambda} \right) - \sum_{i,j} \lambda_j^k \lambda_i^l (Z_{m-1})_{ji} \mathbf{a}_i^{q\alpha} \mathbf{c}_j^{q\epsilon} .
\end{aligned}
$$

Using the element $\Sigma_{(k,l)}^{(i,j)}$ from (5.39), this is

$$
\begin{aligned}
(5.41)_{RHS} = & \frac{1}{2} \sum_{i,j} \mathbf{a}_j^{q\alpha} \mathbf{c}_j^{q\epsilon} \mathbf{c}_i^{01} \lambda_i \lambda_j \Sigma_{(k,l)}^{(i,j)} - \frac{1}{2} \sum_{i,j} \mathbf{a}_j^{q\alpha} \mathbf{c}_j^{q\epsilon} \mathbf{c}_i^{01} \lambda_i \lambda_j \left(\lambda_i^{k_0 m} \lambda_j^{l_0 m} + \lambda_i^{l_0 m} \lambda_j^{k_0 m} \right) \\
& - \delta_{(\alpha < \epsilon)} \sum_{i,j} \lambda_j^k \lambda_i^l \mathbf{c}_j^{q\epsilon} \mathbf{a}_i^{q\alpha} \mathbf{c}_i^{01} + \delta_{\alpha\epsilon} \sum_{i,j} \lambda_j^k \lambda_i^l \mathbf{c}_i^{01} \left((Z_{q-1})_{ij} + \sum_{\lambda=1}^{\epsilon-1} \mathbf{a}_i^{q\lambda} \mathbf{c}_j^{q\lambda} \right) \\
& - \sum_{i,j} \lambda_j^k \lambda_i^l (Z_{m-1})_{ji} \mathbf{a}_i^{q\alpha} \mathbf{c}_j^{q\epsilon} .
\end{aligned}
$$

We use Lemma 5.2.12 to write the first line as

$$(k_0-l_0-1)\sum_i \lambda_i^{k+l}\mathbf{a}_i^{q\alpha}\mathbf{c}_i^{q\epsilon}\mathbf{c}_i^{01} - \frac{1}{2}\sum_{i\neq j}\mathbf{a}_j^{q\alpha}\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{01}\lambda_i\lambda_j\left(\lambda_i^{k_0m}\lambda_j^{l_0m}+\lambda_i^{l_0m}\lambda_j^{k_0m}\right)$$
$$+\frac{1}{2}\sum_{i\neq j}\mathbf{a}_j^{q\alpha}\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{01}\lambda_i\lambda_j\frac{\lambda_i^m+\lambda_j^m}{\lambda_i^m-\lambda_j^m}\left(\lambda_i^{k_0m}\lambda_j^{l_0m}-\lambda_i^{l_0m}\lambda_j^{k_0m}\right)$$
$$=(k_0-l_0-1)\sum_i \lambda_i^{k+l}\mathbf{a}_i^{q\alpha}\mathbf{c}_i^{q\epsilon}\mathbf{c}_i^{01} + \sum_{i\neq j}\lambda_j^k\lambda_i^l\frac{\lambda_i^m}{\lambda_j^m-\lambda_i^m}\frac{\lambda_j^q}{\lambda_i^q}\left(\mathbf{a}_j^{q\alpha}\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{01}+\mathbf{a}_i^{q\alpha}\mathbf{c}_i^{q\epsilon}\mathbf{c}_j^{01}\right).$$

Since $(k_0-l_0-1)m = k+q-(l-q)-m$, we can write

$$(5.41)_{RHS} = \frac{k-l+2q-m}{m}\sum_i \lambda_i^{k+l}\mathbf{a}_i^{q\alpha}\mathbf{c}_i^{q\epsilon}\mathbf{c}_i^{01} - \sum_{i,j}\lambda_j^k\lambda_i^l(Z_{m-1})_{ji}\mathbf{a}_i^{q\alpha}\mathbf{c}_j^{q\epsilon}$$
$$+\sum_{i\neq j}\lambda_j^k\lambda_i^l\frac{\lambda_i^m}{\lambda_j^m-\lambda_i^m}\frac{\lambda_j^q}{\lambda_i^q}\left(\mathbf{a}_j^{q\alpha}\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{01}+\mathbf{a}_i^{q\alpha}\mathbf{c}_i^{q\epsilon}\mathbf{c}_j^{01}\right)$$
$$-\delta_{(\alpha<\epsilon)}\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{q\epsilon}\mathbf{a}_i^{q\alpha}\mathbf{c}_i^{01} + \delta_{\alpha\epsilon}\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{c}_i^{01}\left((Z_{q-1})_{ij}+\sum_{\lambda=1}^{\epsilon-1}\mathbf{a}_i^{q\lambda}\mathbf{c}_j^{q\lambda}\right).$$

For the left-hand side, we use (5.26) and (5.43) to find

$$(5.41)_{LHS} = \frac{k-l+2q-m}{m}\sum_i \lambda_i^{k+l}\mathbf{c}_i^{q\epsilon}\mathbf{c}_i^{01}\mathbf{a}_i^{s\alpha} + \sum_{i\neq j}\lambda_j^k\lambda_i^l\mathbf{c}_i^{01}\frac{\lambda_i^m}{\lambda_j^m-\lambda_i^m}\left(\frac{\lambda_j^q}{\lambda_i^q}\mathbf{a}_j^{q\alpha}-\mathbf{a}_i^{q\alpha}\right)\mathbf{c}_j^{q\epsilon}$$
$$-\delta_{(\alpha<\epsilon)}\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{c}_i^{01}\mathbf{c}_j^{q\epsilon}\mathbf{a}_i^{q\alpha} + \delta_{\alpha\epsilon}\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{c}_i^{01}\left(\sum_{\lambda=1}^{\epsilon-1}\mathbf{a}_i^{q\lambda}\mathbf{c}_j^{q\lambda}+(Z_{q-1})_{ij}\right)$$
$$+\sum_{i\neq j}\lambda_j^k\lambda_i^l\frac{\lambda_i^m}{\lambda_j^m-\lambda_i^m}\mathbf{a}_i^{s\alpha}\left(\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{01}+\frac{\lambda_j^q}{\lambda_i^q}\mathbf{c}_i^{q\epsilon}\mathbf{c}_j^{01}\right) - \sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{a}_i^{s\alpha}(Z_{m-1})_{ji}\mathbf{c}_j^{q\epsilon},$$

which coincides with the right-hand side after simplifications. □

Lemma 5.2.18 *For $s=0$, $q\in I\setminus\{0\}$, $1\leq\epsilon\leq d_q$ and $1\leq\alpha\leq d_0$, we have that (5.41) holds.*

Proof. We take $k=k_0m+1-q$, $l=l_0m+1$ for $k_0,l_0\in\mathbb{N}^\times$, and note that $k<k_0m+1$. Thus

$$(5.41)_{RHS} = \frac{1}{2}\left(\sum_{v_0=0}^{k_0-1}-\sum_{v_0=0}^{l_0}\right)\sum_{i,j}\mathbf{a}_j^{0\alpha}\mathbf{c}_i^{01}\mathbf{c}_j^{q\epsilon}\lambda_i^{v_0m+1}\lambda_j^{(k_0+l_0-v_0)m+1-q}$$
$$+\frac{1}{2}\left(\sum_{v_0=1}^{k_0}-\sum_{v_0=1}^{l_0}\right)\sum_{i,j}\mathbf{a}_j^{0\alpha}\mathbf{c}_i^{01}\mathbf{c}_j^{q\epsilon}\lambda_i^{(k_0+l_0-v_0)m+1}\lambda_j^{v_0m+1-q}$$
$$+\frac{1}{2}\sum_\gamma o(\alpha,\gamma)\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{01}[\mathbf{a}_i^{0\gamma}\mathbf{a}_j^{0\alpha}+\mathbf{a}_i^{0\alpha}\mathbf{a}_j^{0\gamma}]$$
$$-\frac{1}{2}\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{q\epsilon}\mathbf{a}_i^{0\alpha}\mathbf{c}_i^{01} - \sum_{i,j}\lambda_j^k\lambda_i^l(Z_{m-1})_{ji}\mathbf{a}_i^{0\alpha}\mathbf{c}_j^{q\epsilon}.$$

We can write the first two lines together with the first term in the last line as

$$\frac{1}{2}\sum_{i,j} \mathbf{a}_j^{0\alpha}\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{01}\lambda_i\lambda_j^{1-q}\Sigma_{(k,l)}^{(i,j)} - \frac{1}{2}\sum_{i,j} \mathbf{a}_j^{0\alpha}\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{01}\lambda_i^{k_0m+1}\lambda_j^{l_0m+1-q} - \frac{1}{2}\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{q\epsilon}\mathbf{a}_i^{0\alpha}\mathbf{c}_i^{01}\,,$$

so that Lemma 5.2.12 allows us to write these terms in the following form

$$\begin{aligned}
&(k_0-l_0-1)\sum_i \lambda_i^{k+l}\mathbf{c}_i^{q\epsilon}\mathbf{a}_i^{0\alpha}\mathbf{c}_i^{01} - \frac{1}{2}\sum_{i\neq j}\lambda_i^{k+q}\lambda_j^{l-q}\mathbf{c}_j^{q\epsilon}\mathbf{a}_j^{0\alpha}\mathbf{c}_i^{01} - \frac{1}{2}\sum_{i\neq j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{q\epsilon}\mathbf{a}_i^{0\alpha}\mathbf{c}_i^{01}\\
&+\frac{1}{2}\sum_{i\neq j}\mathbf{a}_j^{0\alpha}\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{01}\lambda_i\lambda_j^{1-q}\frac{\lambda_i^m+\lambda_j^m}{\lambda_i^m-\lambda_j^m}\left(\lambda_i^{k_0m}\lambda_j^{l_0m}-\lambda_i^{l_0m}\lambda_j^{k_0m}\right)\\
=&(k_0-l_0-1)\sum_i \lambda_i^{k+l}\mathbf{c}_i^{q\epsilon}\mathbf{a}_i^{0\alpha}\mathbf{c}_i^{01} + \frac{1}{2}\sum_{i\neq j}\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{01}\lambda_j^k\lambda_i^l\left(\frac{\lambda_j^m+\lambda_i^m}{\lambda_j^m-\lambda_i^m}\mathbf{a}_j^{0\alpha}-\mathbf{a}_i^{0\alpha}\right)\\
&+\frac{1}{2}\sum_{i\neq j}\lambda_j^k\lambda_i^l\frac{\lambda_i^m}{\lambda_j^m-\lambda_i^m}\frac{\lambda_j^q}{\lambda_i^q}\mathbf{c}_i^{q\epsilon}\mathbf{c}_j^{01}\mathbf{a}_i^{0\alpha}\,.
\end{aligned}$$

We get in that way

$$\begin{aligned}
(5.41)_{RHS} =& \frac{k-l+q-m}{m}\sum_i \lambda_i^{k+l}\mathbf{c}_i^{q\epsilon}\mathbf{a}_i^{0\alpha}\mathbf{c}_i^{01} + \frac{1}{2}\sum_{i\neq j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{01}\frac{\lambda_j^m+\lambda_i^m}{\lambda_j^m-\lambda_i^m}\left(\mathbf{a}_j^{0\alpha}-\mathbf{a}_i^{0\alpha}\right)\\
&+\sum_{i\neq j}\lambda_j^k\lambda_i^l\frac{\lambda_i^m}{\lambda_j^m-\lambda_i^m}\mathbf{a}_i^{0\alpha}\left(\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{01}+\frac{\lambda_j^q}{\lambda_i^q}\mathbf{c}_i^{q\epsilon}\mathbf{c}_j^{01}\right)\\
&+\frac{1}{2}\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{01}\sum_\gamma o(\alpha,\gamma)[\mathbf{a}_i^{0\gamma}\mathbf{a}_j^{0\alpha}+\mathbf{a}_i^{0\alpha}\mathbf{a}_j^{0\gamma}] - \sum_{i,j}\lambda_j^k\lambda_i^l(Z_{m-1})_{ji}\mathbf{a}_i^{0\alpha}\mathbf{c}_j^{q\epsilon}.
\end{aligned}$$

Introducing (5.27) inside (5.43) easily gives the same expression. □

Lemma 5.2.19 *For any $q,s\in I\setminus\{0\}$ with $q\neq s$, and for any $1\leq\epsilon\leq d_q$, $1\leq\alpha\leq d_s$, we have that (5.41) holds.*

Proof. We take $k=k_0m+1-q$, $l=l_0m+1+s$ for $k_0,l_0\in\mathbb{N}^\times$, and note that k is less than both $k_0m+1+s-q$ and k_0m+1. Thus

$$\begin{aligned}
(5.41)_{RHS} =& \frac{1}{2}\left(\sum_{v_0=0}^{k_0-1}-\sum_{v_0=0}^{l_0}\right)\sum_{i,j}\mathbf{a}_j^{s\alpha}\mathbf{c}_i^{01}\mathbf{c}_j^{q\epsilon}\lambda_i^{v_0m+1}\lambda_j^{(k_0+l_0-v_0)m+1+s-q}\\
&+\frac{1}{2}\left(\sum_{v_0=0}^{k_0-1}-\sum_{v_0=0}^{l_0}\right)\sum_{i,j}\mathbf{a}_j^{s\alpha}\mathbf{c}_i^{01}\mathbf{c}_j^{q\epsilon}\lambda_i^{(k_0+l_0-v_0)m+1}\lambda_j^{v_0m+1+s-q}\\
&+\frac{1}{2}o(q,s)\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{a}_i^{s\alpha}\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{01} - \frac{1}{2}\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{q\epsilon}\mathbf{a}_i^{s\alpha}\mathbf{c}_i^{01} - \sum_{i,j}\lambda_j^k\lambda_i^l(Z_{m-1})_{ji}\mathbf{a}_i^{s\alpha}\mathbf{c}_j^{q\epsilon}.
\end{aligned}$$

Note that the two first terms in the last line sums up together if $q > s$, while they cancel out if $q < s$. For the first line, we can write from (5.39) and Lemma 5.2.12 that it equals

$$\begin{aligned}
&\frac{1}{2}\sum_{i,j} \mathbf{a}_j^{s\alpha}\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{01}\lambda_i\lambda_j^{1-q+s}\Sigma_{(k,l)}^{(i,j)} \\
&-\frac{1}{2}\sum_{i,j} \mathbf{a}_j^{s\alpha}\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{01}\left(\lambda_i^{k_0 m+1}\lambda_j^{l_0 m+1-q+s}+\lambda_i^{l_0 m+1}\lambda_j^{k_0 m+1-q+s}\right) \\
=&(k_0-l_0-1)\sum_i \lambda_i^{k+l}\mathbf{a}_i^{s\alpha}\mathbf{c}_i^{q\epsilon}\mathbf{c}_i^{01}+\frac{1}{2}\sum_{i,j}\lambda_j^k\lambda_i^l\frac{\lambda_j^m+\lambda_i^m}{\lambda_j^m-\lambda_i^m}\left(\frac{\lambda_j^s}{\lambda_i^s}\mathbf{a}_j^{s\alpha}\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{01}+\frac{\lambda_j^q}{\lambda_i^q}\mathbf{a}_i^{s\alpha}\mathbf{c}_i^{q\epsilon}\mathbf{c}_j^{01}\right) \\
&-\frac{1}{2}\sum_{i,j}\lambda_j^k\lambda_i^l\left(\frac{\lambda_j^s}{\lambda_i^s}\mathbf{a}_j^{s\alpha}\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{01}+\frac{\lambda_j^q}{\lambda_i^q}\mathbf{a}_i^{s\alpha}\mathbf{c}_i^{q\epsilon}\mathbf{c}_j^{01}\right).
\end{aligned}$$

Hence, we can write

$$\begin{aligned}
(5.41)_{RHS}=&\frac{k-l+q+s-m}{m}\sum_i \lambda_i^{k+l}\mathbf{a}_i^{s\alpha}\mathbf{c}_i^{q\epsilon}\mathbf{c}_i^{01} \\
&+\sum_{i,j}\lambda_j^k\lambda_i^l\frac{\lambda_i^m}{\lambda_j^m-\lambda_i^m}\left(\frac{\lambda_j^s}{\lambda_i^s}\mathbf{a}_j^{s\alpha}\mathbf{c}_j^{q\epsilon}\mathbf{c}_i^{01}+\frac{\lambda_j^q}{\lambda_i^q}\mathbf{a}_i^{s\alpha}\mathbf{c}_i^{q\epsilon}\mathbf{c}_j^{01}\right) \\
&-\delta_{(q>s)}\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{q\epsilon}\mathbf{a}_i^{s\alpha}\mathbf{c}_i^{01}-\sum_{i,j}\lambda_j^k\lambda_i^l(Z_{m-1})_{ji}\mathbf{a}_i^{s\alpha}\mathbf{c}_j^{q\epsilon},
\end{aligned}$$

which coincides with $(5.41)_{LHS}$ after substituting (5.28) inside (5.43). □

Fourth type of brackets

Finally, we have to check that

$$\{\xi^* g^k_{p\gamma,01}, \xi^* g^l_{s\alpha,01}\} = \xi^*\{g^k_{p\gamma,01}, g^l_{s\alpha,01}\}_{\mathrm{P}} \tag{5.44}$$

for all possible couples of indices (p,γ) and (s,α). Using (5.17) and (5.18), the left-hand side is given by

$$\begin{aligned}
(5.44)_{LHS}=&\sum_{i,j}\lambda_j^k\lambda_i^l\left(\mathbf{a}_j^{p\gamma}\mathbf{c}_i^{01}\{\mathbf{c}_j^{01},\mathbf{a}_i^{s\alpha}\}+\mathbf{c}_j^{01}\mathbf{a}_i^{s\alpha}\{\mathbf{a}_j^{p\gamma},\mathbf{c}_i^{01}\}+\mathbf{c}_j^{01}\mathbf{c}_i^{01}\{\mathbf{a}_j^{p\gamma},\mathbf{a}_i^{s\alpha}\}\right) \\
&+\frac{k-l}{m}\sum_i\lambda_i^{k+l}(\mathbf{c}_i^{01})^2\mathbf{a}_i^{p\gamma}\mathbf{a}_i^{s\alpha}+\sum_{i\neq j}\lambda_j^k\lambda_i^l\frac{\lambda_j^m+\lambda_i^m}{\lambda_j^m-\lambda_i^m}\mathbf{a}_j^{p\gamma}\mathbf{a}_i^{s\alpha}\mathbf{c}_j^{01}\mathbf{c}_i^{01} \\
&+\sum_{i,j}\lambda_j^k\lambda_i^l(Z_{m-1})_{ij}\mathbf{a}_j^{p\gamma}\mathbf{a}_i^{s\alpha}\mathbf{c}_i^{01}-\sum_{i,j}\lambda_j^k\lambda_i^l(Z_{m-1})_{ji}\mathbf{a}_j^{p\gamma}\mathbf{a}_i^{s\alpha}\mathbf{c}_j^{01},
\end{aligned} \tag{5.45}$$

with the brackets in the first line depending on the combination of indices. For the right-hand side, (5.36) gives

$$
\begin{aligned}
(5.44)_{RHS} =& \frac{1}{2}\left(\overset{\bullet}{\sum_{v=1,\ldots,k}} - \overset{\bullet}{\sum_{v=1,\ldots,l}}\right)\sum_{i,j} \mathbf{a}_j^{s\alpha}\mathbf{c}_i^{01}\mathbf{a}_i^{p\gamma}\mathbf{c}_j^{01}\,\lambda_i^v\lambda_j^{k+l-v} \\
&+\frac{1}{2}\left(\overset{\triangle}{\sum_{v=1,\ldots,k}} - \overset{\triangle}{\sum_{v=1,\ldots,l}}\right)\sum_{i,j} \mathbf{a}_j^{s\alpha}\mathbf{c}_i^{01}\mathbf{a}_i^{p\gamma}\mathbf{c}_j^{01}\,\lambda_i^{k+l-v}\lambda_j^{v} \\
&+\frac{1}{2}[o(p,0)-o(p,s)+o(0,s)]\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{a}_j^{p\gamma}\mathbf{c}_j^{01}\mathbf{a}_i^{s\alpha}\mathbf{c}_i^{01} \\
&+\frac{1}{2}\delta_{ps}o(\alpha,\gamma)\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{01}\mathbf{c}_i^{01}(\mathbf{a}_i^{p\gamma}\mathbf{a}_j^{s\alpha}+\mathbf{a}_i^{s\alpha}\mathbf{a}_j^{p\gamma}) \\
&+\frac{1}{2}\left(\delta_{0s}-\delta_{p0}\right)\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{01}\mathbf{a}_j^{p\gamma}\mathbf{c}_i^{01}\mathbf{a}_i^{s\alpha} \\
&+\delta_{0s}\delta_{\alpha 1}\sum_{i,j}\lambda_j^k\lambda_i^l(Z_{m-1})_{ij}\mathbf{a}_j^{p\gamma}\mathbf{c}_i^{01} - \delta_{p0}\delta_{1\gamma}\sum_{i,j}\lambda_j^k\lambda_i^l(Z_{m-1})_{ji}\mathbf{a}_i^{s\alpha}\mathbf{c}_j^{01}\,,
\end{aligned}
$$

where in the sum $\overset{\bullet}{\sum}$ we require $v \underset{m}{\equiv} p+1$, while for $\overset{\triangle}{\sum}$ we require $v \underset{m}{\equiv} s+1$.

We have to analyse the different choices of (p,γ) and (s,α). The easiest case is the following one.

Lemma 5.2.20 *For $p = s = 0$ and for any $1 \leq \alpha, \gamma \leq d_0$, we have that (5.44) holds.*

Proof. We use (5.18), (5.24) and (5.29), and the proof is similar to the corresponding case for a Jordan quiver given in Proposition 4.3.3, so we can omit it. □

Fourth type of brackets : case $p = s$ nonzero. We begin with some preliminary results : we remark that we can write (5.25) in the special case of $\epsilon = 0$ as

$$
\begin{aligned}
\{\mathbf{c}_j^{01},\mathbf{a}_i^{s\alpha}\} \overset{s\neq 0}{=} & \frac{s-m}{m}\delta_{ij}\mathbf{c}_j^{01}\mathbf{a}_i^{s\alpha} + \delta_{(i\neq j)}\frac{\lambda_i^m}{\lambda_j^m-\lambda_i^m}\frac{\lambda_j^s}{\lambda_i^s}\mathbf{c}_j^{01}\mathbf{a}_j^{s\alpha} \\
&- \delta_{(i\neq j)}\frac{\lambda_j^m}{\lambda_j^m-\lambda_i^m}\mathbf{c}_j^{01}\mathbf{a}_i^{s\alpha} - \mathbf{a}_i^{s\alpha}(Z_{m-1})_{ij}\,,
\end{aligned}
\tag{5.46}
$$

and by antisymmetry we get

$$
\begin{aligned}
\{\mathbf{a}_j^{p\gamma},\mathbf{c}_i^{01}\} \overset{p\neq 0}{=} & \frac{m-p}{m}\delta_{ij}\mathbf{c}_i^{01}\mathbf{a}_j^{p\gamma} + \delta_{(i\neq j)}\frac{\lambda_j^m}{\lambda_j^m-\lambda_i^m}\frac{\lambda_i^p}{\lambda_j^p}\mathbf{c}_i^{01}\mathbf{a}_i^{p\gamma} \\
&- \delta_{(i\neq j)}\frac{\lambda_i^m}{\lambda_j^m-\lambda_i^m}\mathbf{c}_i^{01}\mathbf{a}_j^{p\gamma} + \mathbf{a}_j^{p\gamma}(Z_{m-1})_{ji}\,.
\end{aligned}
\tag{5.47}
$$

Hence, in the case $p = s$ nonzero, we obtain after simplifications

$$\begin{aligned}(5.44)_{LHS} =& \frac{k-l}{m} \sum_i \lambda_i^{k+l} (\mathbf{c}_i^{01})^2 \mathbf{a}_i^{p\gamma} \mathbf{a}_i^{p\alpha} + \sum_{i,j} \lambda_j^k \lambda_i^l \mathbf{c}_j^{01} \mathbf{c}_i^{01} \{\mathbf{a}_j^{p\gamma}, \mathbf{a}_i^{p\alpha}\} \\ &+ \sum_{i\neq j} \lambda_j^k \lambda_i^l \mathbf{c}_j^{01} \mathbf{c}_i^{01} \mathbf{a}_j^{p\gamma} \frac{\lambda_i^m}{\lambda_j^m - \lambda_i^m} \frac{\lambda_j^p}{\lambda_i^p} \mathbf{a}_j^{p\alpha} + \sum_{i\neq j} \lambda_j^k \lambda_i^l \mathbf{c}_j^{01} \mathbf{c}_i^{01} \mathbf{a}_i^{p\alpha} \frac{\lambda_j^m}{\lambda_j^m - \lambda_i^m} \frac{\lambda_i^p}{\lambda_j^p} \mathbf{a}_i^{p\gamma} .\end{aligned} \tag{5.48}$$

Lemma 5.2.21 *For any $p \in I \setminus \{0\}$ with $s = p$ and $1 \leq \alpha, \gamma \leq d_p$, we have that (5.44) holds.*

Proof. Let $k = k_0 m + p + 1$ and $l = l_0 m + p + 1$ for $k_0, l_0 \in \mathbb{N}^\times$. We can write

$$\begin{aligned}(5.44)_{RHS} =& \frac{1}{2} \left(\sum_{v_0=1}^{k_0} - \sum_{v_0=1}^{l_0} \right) \sum_{i,j} \mathbf{a}_j^{p\alpha} \mathbf{c}_i^{01} \mathbf{a}_i^{p\gamma} \mathbf{c}_j^{01} \, \lambda_i^{p+1} \lambda_j^{p+1} \lambda_i^{v_0 m} \lambda_j^{(k_0+l_0-v_0)m} \\ &+ \frac{1}{2} \left(\sum_{v_0=1}^{k_0} - \sum_{v_0=1}^{l_0} \right) \sum_{i,j} \mathbf{a}_j^{p\alpha} \mathbf{c}_i^{01} \mathbf{a}_i^{p\gamma} \mathbf{c}_j^{01} \, \lambda_i^{p+1} \lambda_j^{p+1} \lambda_i^{(k_0+l_0-v_0)m} \lambda_j^{v_0 m} \\ &+ \frac{1}{2} o(\alpha, \gamma) \sum_{i,j} \lambda_j^k \lambda_i^l \mathbf{c}_j^{01} \mathbf{c}_i^{01} (\mathbf{a}_i^{p\gamma} \mathbf{a}_j^{p\alpha} + \mathbf{a}_i^{p\alpha} \mathbf{a}_j^{p\gamma}) .\end{aligned}$$

We easily recognise $\Sigma_{(k,l)}^{(i,j)}$ in the first two lines, so that with Lemma 5.2.12 we can write

$$\begin{aligned}(5.44)_{RHS} =& (k_0 - l_0) \sum_i \lambda_i^{k+l} (\mathbf{c}_i^{01})^2 \mathbf{a}_i^{p\gamma} \mathbf{a}_i^{p\alpha} + \frac{1}{2} \sum_{i\neq j} \lambda_j^k \lambda_i^l \frac{\lambda_j^m + \lambda_i^m}{\lambda_j^m - \lambda_i^m} \mathbf{c}_j^{01} \mathbf{c}_i^{01} (\mathbf{a}_j^{p\gamma} \mathbf{a}_i^{p\alpha} + \mathbf{a}_i^{p\gamma} \mathbf{a}_j^{p\alpha}) \\ &+ \frac{1}{2} o(\alpha, \gamma) \sum_{i,j} \lambda_j^k \lambda_i^l \mathbf{c}_j^{01} \mathbf{c}_i^{01} (\mathbf{a}_i^{p\gamma} \mathbf{a}_j^{p\alpha} + \mathbf{a}_i^{p\alpha} \mathbf{a}_j^{p\gamma}) .\end{aligned}$$

Since $\frac{k-l}{m} = k_0 - l_0$, we can see that this expression coincides with $(5.44)_{LHS}$ after putting (5.30) in (5.48). □

Fourth type of brackets : case $p = 0$, $s \neq 0$ or $s = 0$, $p \neq 0$. Again, we notice that we can write from (5.24) the simpler expression

$$\begin{aligned}\{\mathbf{c}_j^{01}, \mathbf{a}_i^{0\alpha}\} =& \frac{1}{2} \delta_{(i\neq j)} \frac{\lambda_j^m + \lambda_i^m}{\lambda_j^m - \lambda_i^m} \mathbf{c}_j^{01} (\mathbf{a}_j^{0\alpha} - \mathbf{a}_i^{0\alpha}) + \delta_{1\alpha} (Z_{m-1})_{ij} - \mathbf{a}_i^{0\alpha} (Z_{m-1})_{ij} \\ &+ \frac{1}{2} \mathbf{c}_j^{01} \sum_{\kappa=1}^{d_0} o(\alpha, \kappa) (\mathbf{a}_i^{0\alpha} \mathbf{a}_j^{0\kappa} + \mathbf{a}_j^{0\alpha} \mathbf{a}_i^{0\kappa}) ,\end{aligned} \tag{5.49}$$

and by antisymmetry

$$\begin{aligned}\{\mathbf{a}_j^{0\gamma}, \mathbf{c}_i^{01}\} =& \frac{1}{2} \delta_{(i\neq j)} \frac{\lambda_j^m + \lambda_i^m}{\lambda_j^m - \lambda_i^m} \mathbf{c}_i^{01} (\mathbf{a}_i^{0\gamma} - \mathbf{a}_j^{0\gamma}) - \delta_{1\gamma} (Z_{m-1})_{ji} + \mathbf{a}_j^{0\gamma} (Z_{m-1})_{ji} \\ &- \frac{1}{2} \mathbf{c}_i^{01} \sum_{\sigma=1}^{d_0} o(\gamma, \sigma) (\mathbf{a}_j^{0\gamma} \mathbf{a}_i^{0\sigma} + \mathbf{a}_i^{0\gamma} \mathbf{a}_j^{0\sigma}) .\end{aligned} \tag{5.50}$$

Assuming that $p=0$ and $s\neq 0$ we can write using (5.46) and (5.50) that

$$
\begin{aligned}
(5.44)_{LHS} =& \frac{k-l+s-m}{m}\sum_i \lambda_i^{k+l}(\mathbf{c}_i^{01})^2\mathbf{a}_i^{0\gamma}\mathbf{a}_i^{s\alpha} + \sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{01}\mathbf{c}_i^{01}\{\mathbf{a}_j^{0\gamma},\mathbf{a}_i^{s\alpha}\} \\
&+\sum_{i\neq j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{01}\mathbf{c}_i^{01}\mathbf{a}_j^{s\alpha}\mathbf{a}_j^{o\gamma}\frac{\lambda_i^m}{\lambda_j^m-\lambda_i^m}\frac{\lambda_j^s}{\lambda_i^s} + \frac{1}{2}\sum_{i\neq j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{01}\mathbf{c}_i^{01}\mathbf{a}_i^{s\alpha}\mathbf{a}_i^{0\gamma}\frac{\lambda_j^m+\lambda_i^m}{\lambda_j^m-\lambda_i^m} \\
&-\frac{1}{2}\sum_{i\neq j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{01}\mathbf{a}_j^{o\gamma}\mathbf{c}_i^{01}\mathbf{a}_i^{s\alpha} - \delta_{\gamma 1}\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{01}\mathbf{a}_i^{s\alpha}(Z_{m-1})_{ji} \\
&-\frac{1}{2}\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{01}\mathbf{c}_i^{01}\mathbf{a}_i^{s\alpha}\sum_{\sigma=1}^{d_0} o(\gamma,\sigma)(\mathbf{a}_j^{0\gamma}\mathbf{a}_i^{0\sigma}+\mathbf{a}_i^{0\gamma}\mathbf{a}_j^{0\sigma})\,.
\end{aligned} \tag{5.51}
$$

A similar expression holds for the case $p\neq 0$ and $s=0$, but we will not need it.

Lemma 5.2.22 *For $p=0$ and for any $s\in I\setminus\{0\}$, $1\leq\gamma\leq d_0$ and $1\leq\alpha\leq d_s$, we have that (5.44) holds. This is also true if $s=0$ and $p\in I\setminus\{0\}$.*

Proof. It suffices to prove the case $p=0$, $s\neq 0$ when we use (5.31), and the other case follows by antisymmetry. So let $k=k_0m+1$, $l=l_0m+s+1$ and note that $k<k_0m+s+1$. Hence

$$
\begin{aligned}
(5.44)_{RHS} =& \frac{1}{2}\left(\sum_{v_0=0}^{k_0}-\sum_{v_0=0}^{l_0}\right)\sum_{i,j}\mathbf{a}_j^{s\alpha}\mathbf{c}_i^{01}\mathbf{a}_i^{0\gamma}\mathbf{c}_j^{01}\,\lambda_i^{v_0m+1}\lambda_j^{(k_0+l_0-v_0)m+s+1} \\
&+\frac{1}{2}\left(\sum_{v_0=0}^{k_0-1}-\sum_{v_0=0}^{l_0}\right)\sum_{i,j}\mathbf{a}_j^{s\alpha}\mathbf{c}_i^{01}\mathbf{a}_i^{0\gamma}\mathbf{c}_j^{01}\,\lambda_i^{(k_0+l_0-v_0)m+1}\lambda_j^{v_0m+s+1} \\
&-\frac{1}{2}\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{01}\mathbf{a}_j^{0\gamma}\mathbf{c}_i^{01}\mathbf{a}_i^{s\alpha} - \delta_{1\gamma}\sum_{i,j}\lambda_j^k\lambda_i^l(Z_{m-1})_{ji}\mathbf{a}_i^{s\alpha}\mathbf{c}_j^{01} \\
=& (k_0+l_0-1)\sum_i\lambda_i^{k+l}(\mathbf{c}_i^{01})^2\mathbf{a}_i^{0\gamma}\mathbf{a}_i^{s\alpha} + \frac{1}{2}\sum_{i\neq j}\mathbf{c}_j^{01}\mathbf{a}_j^{s\alpha}\mathbf{c}_i^{01}\mathbf{a}_i^{0\gamma}\lambda_i^1\lambda_j^{s+1}\Sigma_{(k,l)}^{(i,j)} \\
&-\frac{1}{2}\sum_{i\neq j}\lambda_j^k\lambda_i^l\frac{\lambda_j^s}{\lambda_i^s}\mathbf{c}_i^{01}\mathbf{a}_i^{0\gamma}\mathbf{a}_j^{s\alpha}\mathbf{c}_j^{01} - \frac{1}{2}\sum_{i\neq j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{01}\mathbf{a}_j^{0\gamma}\mathbf{c}_i^{01}\mathbf{a}_i^{s\alpha} \\
&-\delta_{1\gamma}\sum_{i,j}\lambda_j^k\lambda_i^l(Z_{m-1})_{ji}\mathbf{a}_i^{s\alpha}\mathbf{c}_j^{01}\,.
\end{aligned}
$$

Using Lemma 5.2.12, we can write the second term as

$$
\frac{1}{2}\sum_{i\neq j}\lambda_j^k\lambda_i^l\frac{\lambda_j^m+\lambda_i^m}{\lambda_j^m-\lambda_i^m}\mathbf{c}_j^{01}\mathbf{c}_i^{01}\left(\mathbf{a}_i^{s\alpha}\mathbf{a}_j^{0\gamma}+\frac{\lambda_j^s}{\lambda_i^s}\mathbf{a}_j^{s\alpha}\mathbf{a}_i^{0\gamma}\right)\,.
$$

Since $(k_0 - l_0)m = k - l + s$, we get

$$(5.44)_{RHS} = \frac{k-l+s-m}{m}\sum_i \lambda_i^{k+l}(\mathbf{c}_i^{01})^2\mathbf{a}_i^{0\gamma}\mathbf{a}_i^{s\alpha} + \sum_{i\neq j}\lambda_j^k\lambda_i^l\frac{\lambda_i^m}{\lambda_j^m-\lambda_i^m}\mathbf{c}_j^{01}\mathbf{a}_j^{0\gamma}\mathbf{c}_i^{01}\mathbf{a}_i^{s\alpha}$$
$$+\sum_{i\neq j}\lambda_j^k\lambda_i^l\frac{\lambda_i^m}{\lambda_j^m-\lambda_i^m}\frac{\lambda_j^s}{\lambda_i^s}\mathbf{c}_j^{01}\mathbf{a}_j^{s\alpha}\mathbf{c}_i^{01}\mathbf{a}_i^{0\gamma} - \delta_{1\gamma}\sum_{i,j}\lambda_j^k\lambda_i^l(Z_{m-1})_{ji}\mathbf{a}_i^{s\alpha}\mathbf{c}_j^{01}.$$

If we put (5.31) inside (5.51), we get the same result. □

Fourth type of brackets : case of distinct $p, s \neq 0$. Let $p, s \in I\setminus\{0\}$ that we assume to be distinct. We can use (5.46) and (5.47) to write

$$\begin{aligned}(5.44)_{LHS} =& \frac{k-p-l+s}{m}\sum_i \lambda_i^{k+l}(\mathbf{c}_i^{01})^2\mathbf{a}_i^{p\gamma}\mathbf{a}_i^{s\alpha} + \sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{01}\mathbf{c}_i^{01}\{\mathbf{a}_j^{p\gamma},\mathbf{a}_i^{s\alpha}\}\\ &+\sum_{i\neq j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{01}\mathbf{c}_i^{01}\mathbf{a}_j^{p\gamma}\mathbf{a}_j^{s\alpha}\frac{\lambda_i^m}{\lambda_j^m-\lambda_i^m}\frac{\lambda_j^s}{\lambda_i^s}\\ &+\sum_{i\neq j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{01}\mathbf{c}_i^{01}\mathbf{a}_i^{p\gamma}\mathbf{a}_i^{s\alpha}\frac{\lambda_j^m}{\lambda_j^m-\lambda_i^m}\frac{\lambda_i^p}{\lambda_j^p}.\end{aligned} \tag{5.52}$$

Lemma 5.2.23 *For any* $p, s \in I\setminus\{0\}$ *with* $p \neq s$*, and for any* $1 \leq \gamma \leq d_p$, $1 \leq \alpha \leq d_s$*, we have that* (5.44) *holds.*

Proof. We prove the case $p < s$ where we need (5.33). For $k = k_0m + p + 1 < k_0m + s + 1$ and $l = l_0m + s + 1$ with $k_0, l_0 \in \mathbb{N}^\times$, we have

$$\begin{aligned}(5.44)_{RHS} =& \frac{1}{2}\left(\sum_{v_0=0}^{k_0} - \sum_{v_0=0}^{l_0}\right)\sum_{i,j}\mathbf{a}_j^{s\alpha}\mathbf{c}_i^{01}\mathbf{a}_i^{p\gamma}\mathbf{c}_j^{01}\,\lambda_i^{v_0m+p+1}\lambda_j^{(k_0+l_0-v_0)m+s+1}\\ &+\frac{1}{2}\left(\sum_{v_0=0}^{k_0-1} - \sum_{v_0=0}^{l_0}\right)\sum_{i,j}\mathbf{a}_j^{s\alpha}\mathbf{c}_i^{01}\mathbf{a}_i^{p\gamma}\mathbf{c}_j^{01}\,\lambda_i^{(k_0+l_0-v_0)m+p+1}\lambda_j^{v_0m+s+1}\\ &-\frac{1}{2}\sum_{i,j}\lambda_j^k\lambda_i^l\mathbf{a}_j^{p\gamma}\mathbf{c}_j^{01}\mathbf{a}_i^{s\alpha}\mathbf{c}_i^{01}\\ =& (k_0+l_0-1)\sum_i\lambda_i^{k+l}(\mathbf{c}_i^{01})^2\mathbf{a}_i^{p\gamma}\mathbf{a}_i^{s\alpha} + \frac{1}{2}\sum_{i\neq j}\mathbf{c}_j^{01}\mathbf{a}_j^{s\alpha}\mathbf{c}_i^{01}\mathbf{a}_i^{p\gamma}\lambda_i^{p+1}\lambda_j^{s+1}\Sigma_{(k,l)}^{(i,j)}\\ &-\frac{1}{2}\sum_{i\neq j}\lambda_j^k\lambda_i^l\frac{\lambda_j^{s-p}}{\lambda_i^{s-p}}\mathbf{c}_i^{01}\mathbf{a}_i^{p\gamma}\mathbf{a}_j^{s\alpha}\mathbf{c}_j^{01} - \frac{1}{2}\sum_{i\neq j}\lambda_j^k\lambda_i^l\mathbf{c}_j^{01}\mathbf{a}_j^{p\gamma}\mathbf{c}_i^{01}\mathbf{a}_i^{s\alpha}.\end{aligned}$$

We see that Lemma 5.2.12 allows us to rewrite the second term as

$$\frac{1}{2}\sum_{i\neq j}\lambda_j^k\lambda_i^l\frac{\lambda_j^m+\lambda_i^m}{\lambda_j^m-\lambda_i^m}\mathbf{c}_j^{01}\mathbf{c}_i^{01}\left(\mathbf{a}_i^{s\alpha}\mathbf{a}_j^{p\gamma} + \frac{\lambda_j^{s-p}}{\lambda_i^{s-p}}\mathbf{a}_j^{s\alpha}\mathbf{a}_i^{p\gamma}\right),$$

So that with $(k_0-l_0)m=(k-p)-(l-s)$, we get

$$(5.44)_{RHS} = \frac{k-p-l+s-m}{m}\sum_i \lambda_i^{k+l}(\mathbf{c}_i^{01})^2\mathbf{a}_i^{0\gamma}\mathbf{a}_i^{s\alpha} + \sum_{i\neq j}\lambda_j^k\lambda_i^l\frac{\lambda_i^m}{\lambda_j^m-\lambda_i^m}\mathbf{c}_j^{01}\mathbf{a}_j^{p\gamma}\mathbf{c}_i^{01}\mathbf{a}_i^{s\alpha} + \sum_{i\neq j}\lambda_j^k\lambda_i^l\frac{\lambda_i^m}{\lambda_j^m-\lambda_i^m}\frac{\lambda_j^{s-p}}{\lambda_i^{s-p}}\mathbf{c}_j^{01}\mathbf{a}_j^{s\alpha}\mathbf{c}_i^{01}\mathbf{a}_i^{p\gamma}\,.$$

By substituting (5.33) in (5.52), we get the same result. □

5.2.4 Generalisation of the Poisson bracket of Arutyunov–Frolov

Before discussing integrability in § 5.2.5 for the dimension vectors $(1,n\delta)$, let us remark that the functions $\operatorname{tr} U^K$, $U\in\{X,Y,Z,1_I+XY\}$, for which we can explicitly write the flows (see Propositions 5.1.5, 5.1.6, 5.1.7 and 5.1.8) can be written on $\mathcal{C}^\circ_{n,m}$ in terms of traces of matrices involving X or Z. At the same time, we have seen in § 5.2.2 that such matrices only depend on the elements $(\lambda_i, g^s_{ij}=\sum_{\alpha=1}^{d_s}\mathbf{a}_i^{s\alpha}\mathbf{c}_j^{s\alpha})$ when restricted to $\mathcal{C}'_{n,m}$. Therefore, it seems natural to try to obtain the Poisson bracket on these elements, which we do now.

Preparation

In this subsection, the method is just an adaptation of the proof of Lemma 4.3.5 with the Poisson brackets from Proposition 5.2.7, so we skip the proofs. We denote the matrix $\hat{Z}_r$ given by (5.10) simply as Z_r.

Lemma 5.2.24 *For any* $1\leq\epsilon,\gamma\leq d_0$ *and* $i,j,k,l=1,\dots,n$,

$$\begin{aligned}\{\mathbf{c}_j^{0\epsilon},g^0_{kl}\} =&(Z_{m-1})_{kj}\mathbf{c}_l^{0\epsilon}-(Z_{m-1})_{jl}\mathbf{c}_j^{0\epsilon}+((Z_{m-1})_{lj}-(Z_{m-1})_{kj})g^0_{kl}\\ &+\frac{1}{2}\delta_{(j\neq k)}\frac{\lambda_j^m+\lambda_k^m}{\lambda_j^m-\lambda_k^m}\mathbf{c}_j^{0\epsilon}(g^0_{jl}-g^0_{kl})+\frac{1}{2}\delta_{(j\neq l)}\frac{\lambda_j^m+\lambda_l^m}{\lambda_j^m-\lambda_l^m}(\mathbf{c}_j^{0\epsilon}g^0_{kl}+\mathbf{c}_l^{0\epsilon}g^0_{kj})\\ &+\frac{1}{2}\mathbf{c}_l^{0\epsilon}g^0_{kj}-\frac{1}{2}\mathbf{c}_j^{0\epsilon}g^0_{jl}+g^0_{kl}\sum_{\lambda=1}^{\epsilon-1}(\mathbf{c}_j^{0\lambda}-\mathbf{c}_j^{0\epsilon})(\mathbf{a}_l^{0\lambda}-\mathbf{a}_k^{0\lambda})\,,\\ \{\mathbf{a}_i^{0\gamma},g^0_{kl}\} =&\mathbf{a}_i^{0\gamma}(Z_{m-1})_{il}-\mathbf{a}_k^{0\gamma}(Z_{m-1})_{il}+\frac{1}{2}\delta_{(i\neq k)}\frac{\lambda_i^m+\lambda_k^m}{\lambda_i^m-\lambda_k^m}(\mathbf{a}_k^{0\gamma}-\mathbf{a}_i^{0\gamma})(g^0_{il}-g^0_{kl})\\ &+\frac{1}{2}\delta_{(i\neq l)}\frac{\lambda_i^m+\lambda_l^m}{\lambda_i^m-\lambda_l^m}g^0_{kl}(\mathbf{a}_l^{0\gamma}-\mathbf{a}_i^{0\gamma})+\frac{1}{2}\mathbf{a}_i^{0\gamma}g^0_{il}-\frac{1}{2}\mathbf{a}_k^{0\gamma}g^0_{il}\\ &+\frac{1}{2}\sum_{\sigma=1}^{d}o(\gamma,\sigma)g^0_{kl}[\mathbf{a}_i^{0\gamma}(\mathbf{a}_k^{0\sigma}-\mathbf{a}_l^{0\sigma})+\mathbf{a}_i^{0\sigma}(\mathbf{a}_k^{0\gamma}-\mathbf{a}_l^{0\gamma})]\,.\end{aligned}$$

Lemma 5.2.25 *For any* $1 \leq \epsilon, \gamma \leq d_0$, $s \in I \setminus \{0\}$, *and* $i, j, k, l = 1, \ldots, n$,

$$\begin{aligned}\{\mathbf{c}_j^{0\epsilon}, g_{kl}^s\} =& \frac{m-s}{m}(\delta_{jl} - \delta_{kj})\mathbf{c}_j^{0\epsilon} g_{kl}^s + ((Z_{m-1})_{lj} - (Z_{m-1})_{kj}) g_{kl}^s \\ &+ \delta_{(j \neq k)} \frac{1}{\lambda_j^m - \lambda_k^m} \mathbf{c}_j^{0\epsilon} \left[\lambda_k^{m-s}\lambda_j^s g_{jl}^s - \lambda_j^m g_{kl}^s\right] \\ &+ \delta_{(j \neq l)} \frac{1}{\lambda_j^m - \lambda_l^m} \left[\lambda_j^m \mathbf{c}_j^{0\epsilon} g_{kl}^s + \lambda_j^{m-s}\lambda_l^s \mathbf{c}_l^{0\epsilon} g_{kj}^s\right] + g_{kl}^s \sum_{\lambda=1}^{\epsilon-1} (\mathbf{c}_j^{0\lambda} - \mathbf{c}_j^{0\epsilon})(\mathbf{a}_l^{0\lambda} - \mathbf{a}_k^{0\lambda}),\end{aligned}$$

$$\begin{aligned}\{\mathbf{a}_i^{0\gamma}, g_{kl}^s\} =& \frac{1}{2}\delta_{(i \neq k)}(\mathbf{a}_k^{0\gamma} - \mathbf{a}_i^{0\gamma}) \left[\frac{2\lambda_i^s \lambda_k^{m-s}}{\lambda_i^m - \lambda_k^m} g_{il}^s - \frac{\lambda_i^m + \lambda_k^m}{\lambda_i^m - \lambda_k^m} g_{kl}^s\right] \\ &+ \frac{1}{2}\delta_{(i \neq l)}(\mathbf{a}_l^{0\gamma} - \mathbf{a}_i^{0\gamma}) \frac{\lambda_i^m + \lambda_l^m}{\lambda_i^m - \lambda_l^m} g_{kl}^s \\ &+ \frac{1}{2}\sum_{\sigma=1}^{d_0} o(\gamma, \sigma) g_{kl}^s [\mathbf{a}_i^{0\gamma}(\mathbf{a}_k^{0\sigma} - \mathbf{a}_l^{0\sigma}) + \mathbf{a}_i^{0\sigma}(\mathbf{a}_k^{0\gamma} - \mathbf{a}_l^{0\gamma})].\end{aligned}$$

Lemma 5.2.26 *For any* $q \in I \setminus \{0\}$, $1 \leq \epsilon, \gamma \leq d_q$, *and* $i, j, k, l = 1, \ldots, n$,

$$\begin{aligned}\{\mathbf{c}_j^{q\epsilon}, g_{kl}^0\} =& \frac{q-m}{m}\delta_{jl}\mathbf{c}_j^{q\epsilon} g_{kl}^0 - (Z_{m-1})_{jl}\mathbf{c}_j^{q\epsilon} + \frac{1}{2}\mathbf{c}_j^{q\epsilon} g_{kl}^0 - \frac{1}{2}\mathbf{c}_j^{q\epsilon} g_{jl}^0 \\ &+ \frac{1}{2}\delta_{(j \neq k)} \frac{\lambda_j^m + \lambda_k^m}{\lambda_j^m - \lambda_k^m} \mathbf{c}_j^{q\epsilon}(g_{jl}^0 - g_{kl}^0) \\ &+ \delta_{(j \neq l)} \frac{1}{\lambda_j^m - \lambda_l^m} \left[\lambda_l^m \mathbf{c}_j^{q\epsilon} g_{kl}^0 + \lambda_l^{m-q}\lambda_j^q \mathbf{c}_l^{q\epsilon} g_{kj}^0\right],\end{aligned}$$

$$\begin{aligned}\{\mathbf{a}_i^{q\gamma}, g_{kl}^0\} =& -\frac{q-m}{m}\delta_{il}\mathbf{a}_i^{q\gamma} g_{kl}^0 + \mathbf{a}_i^{q\gamma}(Z_{m-1})_{il} + \frac{1}{2}\mathbf{a}_i^{q\gamma}(g_{il}^0 - g_{kl}^0) \\ &+ \frac{1}{2}\delta_{(i \neq k)} \frac{\lambda_i^m + \lambda_k^m}{\lambda_i^m - \lambda_k^m}(\mathbf{a}_i^{q\gamma} g_{kl}^0 - \mathbf{a}_i^{q\gamma} g_{il}^0) + \delta_{(i \neq k)} \frac{\lambda_i^{m-q}\lambda_k^q}{\lambda_i^m - \lambda_k^m}(\mathbf{a}_k^{q\gamma} g_{il}^0 - \mathbf{a}_k^{q\gamma} g_{kl}^0) \\ &+ \delta_{(i \neq l)} \frac{1}{\lambda_i^m - \lambda_l^m}(\lambda_i^{m-q}\lambda_l^q \mathbf{a}_l^{q\gamma} g_{kl}^0 - \lambda_l^m \mathbf{a}_i^{q\gamma} g_{kl}^0).\end{aligned}$$

Lemma 5.2.27 *For any* $q \in I \setminus \{0\}$, $1 \leq \epsilon, \gamma \leq d_q$, *and* $i, j, k, l = 1, \ldots, n$,

$$\begin{aligned}\{\mathbf{c}_j^{q\epsilon}, g_{kl}^q\} =& \frac{q}{m}\delta_{jk}\mathbf{c}_j^{q\epsilon} g_{kl}^q + (Z_{q-1})_{kj}\mathbf{c}_l^{q\epsilon} + \frac{1}{2}\mathbf{c}_l^{q\epsilon} g_{kj}^q - \frac{1}{2}\mathbf{c}_j^{q\epsilon} g_{kl}^q \\ &+ \delta_{(j \neq k)} \frac{1}{\lambda_j^m - \lambda_k^m} \mathbf{c}_j^{q\epsilon} \left[\lambda_j^q \lambda_k^{m-q} g_{jl}^q - \lambda_k^m g_{kl}^q\right] \\ &+ \frac{1}{2}\delta_{(j \neq l)} \frac{\lambda_j^m + \lambda_l^m}{\lambda_j^m - \lambda_l^m} \left[\mathbf{c}_j^{q\epsilon} g_{kl}^q + \mathbf{c}_l^{q\epsilon} g_{kj}^q\right],\end{aligned}$$

$$\begin{aligned}\{\mathbf{a}_i^{q\gamma}, g_{kl}^q\} =& -\frac{q}{m}\delta_{il}\mathbf{a}_i^{q\gamma}g_{kl}^q - \mathbf{a}_k^{q\gamma}(Z_{q-1})_{il} + \frac{1}{2}\mathbf{a}_i^{q\gamma}g_{kl}^q - \frac{1}{2}\mathbf{a}_k^{q\gamma}g_{il}^q \\ &+ \frac{1}{2}\delta_{(i\neq k)}\frac{\lambda_i^m+\lambda_k^m}{\lambda_i^m-\lambda_k^m}(\mathbf{a}_i^{q\gamma}g_{kl}^q + \mathbf{a}_k^{q\gamma}g_{il}^q) \\ &- \delta_{(i\neq k)}\frac{1}{\lambda_i^m-\lambda_k^m}\left[\lambda_i^q\lambda_k^{m-q}\mathbf{a}_i^{q\gamma}g_{il}^q + \lambda_i^{m-q}\lambda_k^q\mathbf{a}_k^{q\gamma}g_{kl}^q\right] \\ &+ \delta_{(i\neq l)}\frac{1}{\lambda_i^m-\lambda_l^m}\left[\lambda_i^{m-q}\lambda_l^q\mathbf{a}_l^{q\gamma}g_{kl}^q - \lambda_i^m\mathbf{a}_i^{q\gamma}g_{kl}^q\right]\,.\end{aligned}$$

Lemma 5.2.28 *For any $q,s \in I\setminus\{0\}$, $q<s$, $1\leq \epsilon,\gamma \leq d_q$, and $i,j,k,l=1,\dots,n$,*

$$\begin{aligned}\{\mathbf{c}_j^{q\epsilon}, g_{kl}^s\} \overset{q\leq s}{=}& \left[\frac{q-s}{m}\delta_{jl}+\frac{s}{m}\delta_{jk}\right]\mathbf{c}_j^{q\epsilon}g_{kl}^s + \delta_{(j\neq k)}\frac{1}{\lambda_j^m-\lambda_k^m}\mathbf{c}_j^{q\epsilon}\left[\lambda_j^s\lambda_k^{m-s}g_{jl}^s - \lambda_k^m g_{kl}^s\right] \\ &+ \delta_{(j\neq l)}\frac{1}{\lambda_j^m-\lambda_l^m}\left[\lambda_l^m\mathbf{c}_j^{q\epsilon}g_{kl}^s + \lambda_j^{m+q-s}\lambda_l^{s-q}\mathbf{c}_l^{q\epsilon}g_{kj}^s\right]\,,\end{aligned}$$

$$\begin{aligned}\{\mathbf{a}_i^{q\gamma}, g_{kl}^s\} \overset{q\leq s}{=}& \left[1-\frac{q}{m}\delta_{il}-\delta_{ik}\right]\mathbf{a}_i^{q\gamma}g_{kl}^s + \delta_{(i\neq l)}\frac{1}{\lambda_i^m-\lambda_l^m}\left[\lambda_i^{m-q}\lambda_l^q\mathbf{a}_l^{q\gamma} - \lambda_i^m\mathbf{a}_i^{q\gamma}\right]g_{kl}^s \\ &+ \delta_{(i\neq k)}\frac{1}{\lambda_i^m-\lambda_k^m}\left[\lambda_k^m\mathbf{a}_i^{q\gamma}g_{kl}^s + \lambda_k^{m+q-s}\lambda_i^{s-q}\mathbf{a}_k^{q\gamma}g_{il}^s - \lambda_i^s\lambda_k^{m-s}\mathbf{a}_i^{q\gamma}g_{il}^s - \lambda_i^{m-q}\lambda_k^q\mathbf{a}_k^{q\gamma}g_{kl}^s\right]\,.\end{aligned}$$

Lemma 5.2.29 *For any $q,s \in I\setminus\{0\}$, $q>s$, $1\leq \epsilon,\gamma \leq d_q$, and $i,j,k,l=1,\dots,n$,*

$$\begin{aligned}\{\mathbf{c}_j^{q\epsilon}, g_{kl}^s\} \overset{q> s}{=}& \left[\frac{q-s}{m}\delta_{jl}+\frac{s}{m}\delta_{jk}-1\right]\mathbf{c}_j^{q\epsilon}g_{kl}^s + \delta_{(j\neq k)}\frac{1}{\lambda_j^m-\lambda_k^m}\mathbf{c}_j^{q\epsilon}\left[\lambda_j^s\lambda_k^{m-s}g_{jl}^s - \lambda_k^m g_{kl}^s\right] \\ &+ \delta_{(j\neq l)}\frac{1}{\lambda_j^m-\lambda_l^m}\left[\lambda_j^m\mathbf{c}_j^{q\epsilon}g_{kl}^s + \lambda_j^{q-s}\lambda_l^{m+s-q}\mathbf{c}_l^{q\epsilon}g_{kj}^s\right]\,,\end{aligned}$$

$$\begin{aligned}\{\mathbf{a}_i^{q\gamma}, g_{kl}^s\} \overset{q> s}{=}& \left[\delta_{ik}-\frac{q}{m}\delta_{il}\right]\mathbf{a}_i^{q\gamma}g_{kl}^s + \delta_{(i\neq l)}\frac{1}{\lambda_i^m-\lambda_l^m}\left[\lambda_i^{m-q}\lambda_l^q\mathbf{a}_l^{q\gamma} - \lambda_i^m\mathbf{a}_i^{q\gamma}\right]g_{kl}^s \\ &+ \delta_{(i\neq k)}\frac{1}{\lambda_i^m-\lambda_k^m}\left[\lambda_i^m\mathbf{a}_i^{q\gamma}g_{kl}^s + \lambda_i^{m+s-q}\lambda_k^{q-s}\mathbf{a}_k^{q\gamma}g_{il}^s - \lambda_i^s\lambda_k^{m-s}\mathbf{a}_i^{q\gamma}g_{il}^s - \lambda_i^{m-q}\lambda_k^q\mathbf{a}_k^{q\gamma}g_{kl}^s\right]\,.\end{aligned}$$

The generalised Arutyunov-Frolov brackets

In this subsection, the Poisson brackets (5.54)–(5.57) can be obtained by adapting the proof of Proposition 4.3.4. The main difference is the use of the identities

$$(Z_{m-1})_{ij} + \frac{1}{2}g_{ij}^0 = \sum_{s=1}^{m-1}\frac{t_{r-1}}{t_{s-1}}\frac{\lambda_i^s\lambda_j^{m-s}}{\lambda_i^m-t\lambda_j^m}g_{ij}^s + \frac{1}{2}\frac{\lambda_i^m+t\lambda_j^m}{\lambda_i^m-t\lambda_j^m}g_{ij}^r\,, \tag{5.53a}$$

$$\begin{aligned}(Z_{r-1})_{ij} + \frac{1}{2}g_{ij}^r =& \sum_{s=0}^{r-1}\frac{t_{r-1}}{t_{s-1}}\frac{\lambda_i^{m+s-r}\lambda_j^{r-s}}{\lambda_i^m-t\lambda_j^m}g_{ij}^s + \frac{1}{2}\frac{\lambda_i^m+t\lambda_j^m}{\lambda_i^m-t\lambda_j^m}g_{ij}^r \\ &+ \sum_{s=r+1}^{m-1}\frac{tt_{r-1}}{t_{s-1}}\frac{\lambda_i^{s-r}\lambda_j^{m+r-s}}{\lambda_i^m-t\lambda_j^m}g_{ij}^s\,, \qquad \text{for } r\neq 0\,,\end{aligned} \tag{5.53b}$$

which follow directly from (5.10).

To shorten expressions, we take the convention that all the terms with a vanishing denominator should be omitted. For example $\frac{\lambda_i^m+\lambda_k^m}{\lambda_i^m-\lambda_k^m}$ stands for $\delta_{(i\neq k)}\frac{\lambda_i^m+\lambda_k^m}{\lambda_i^m-\lambda_k^m}$. We begin by writing the Poisson bracket $\{g_{ij}^q, g_{kl}^s\}$ for $q=s=0$. It follows from Lemma 5.2.24 that in this case,

$$\begin{aligned}
\{g_{ij}^0, g_{kl}^0\} = &\frac{1}{2} g_{ij}^0 g_{kl}^0 \left[\frac{\lambda_i^m+\lambda_k^m}{\lambda_i^m-\lambda_k^m} + \frac{\lambda_j^m+\lambda_l^m}{\lambda_j^m-\lambda_l^m} + \frac{\lambda_k^m+\lambda_j^m}{\lambda_k^m-\lambda_j^m} + \frac{\lambda_l^m+\lambda_i^m}{\lambda_l^m-\lambda_i^m}\right] \\
&+ \frac{1}{2} g_{il}^0 g_{kj}^0 \left[\frac{\lambda_i^m+\lambda_k^m}{\lambda_i^m-\lambda_k^m} + \frac{\lambda_j^m+\lambda_l^m}{\lambda_j^m-\lambda_l^m} + \frac{\lambda_k^m+t\lambda_j^m}{\lambda_k^m-t\lambda_j^m} - \frac{\lambda_i^m+t\lambda_l^m}{\lambda_i^m-t\lambda_l^m}\right] \\
&+ \frac{1}{2} g_{ij}^0 g_{il}^0 \left[\frac{\lambda_k^m+\lambda_i^m}{\lambda_k^m-\lambda_i^m} + \frac{\lambda_i^m+t\lambda_l^m}{\lambda_i^m-t\lambda_l^m}\right] + \frac{1}{2} g_{ij}^0 g_{jl}^0 \left[\frac{\lambda_j^m+\lambda_k^m}{\lambda_j^m-\lambda_k^m} - \frac{\lambda_j^m+t\lambda_l^m}{\lambda_j^m-t\lambda_l^m}\right] \\
&+ \frac{1}{2} g_{kj}^0 g_{kl}^0 \left[\frac{\lambda_k^m+\lambda_i^m}{\lambda_k^m-\lambda_i^m} - \frac{\lambda_k^m+t\lambda_j^m}{\lambda_k^m-t\lambda_j^m}\right] + \frac{1}{2} g_{lj}^0 g_{kl}^0 \left[\frac{\lambda_i^m+\lambda_l^m}{\lambda_i^m-\lambda_l^m} + \frac{\lambda_l^m+t\lambda_j^m}{\lambda_l^m-t\lambda_j^m}\right] \\
&+ \sum_{s\neq 0} \frac{t}{t_{s-1}} \frac{\lambda_i^s \lambda_l^{m-s}}{\lambda_i^m - t\lambda_l^m} g_{il}^s (g_{ij}^0 - g_{kj}^0) + \sum_{s\neq 0} \frac{t}{t_{s-1}} \frac{\lambda_k^s \lambda_j^{m-s}}{\lambda_k^m - t\lambda_j^m} g_{kj}^s (g_{il}^0 - g_{kl}^0) \\
&- \sum_{s\neq 0} \frac{t}{t_{s-1}} \frac{\lambda_j^s \lambda_l^{m-s}}{\lambda_j^m - t\lambda_l^m} g_{jl}^s g_{ij}^0 + \sum_{s\neq 0} \frac{t}{t_{s-1}} \frac{\lambda_l^s \lambda_j^{m-s}}{\lambda_l^m - t\lambda_j^m} g_{lj}^s g_{kl}^0 \,.
\end{aligned} \tag{5.54}$$

Note that if $g_{ij}^s = 0$ for all $s \neq 0$, i.e. when $\mathbf{d} = (d, 0, \ldots, 0)$, we recover the Poisson brackets of Arutyunov-Frolov (4.32). We will come back to this relation in § 5.3.3. For $q=0$ and $s\neq 0$, we get from Lemma 5.2.25 that

$$\begin{aligned}
\{g_{ij}^0, g_{kl}^s\} = &\frac{1}{2} g_{ij}^0 g_{kl}^s \left[\frac{\lambda_i^m+\lambda_k^m}{\lambda_i^m-\lambda_k^m} + \frac{2\lambda_j^m}{\lambda_j^m-\lambda_l^m} + \frac{2\lambda_j^m}{\lambda_k^m-\lambda_j^m} + \frac{\lambda_l^m+\lambda_i^m}{\lambda_l^m-\lambda_i^m}\right] \\
&+ \frac{m-s}{m}(\delta_{jl}-\delta_{kj}) g_{kl}^s g_{ij}^0 + g_{il}^s g_{kj}^0 \frac{\lambda_i^s \lambda_k^{m-s}}{\lambda_i^m-\lambda_k^m} + g_{il}^0 g_{kj}^s \frac{\lambda_j^{m-s}\lambda_l^s}{\lambda_j^m-\lambda_l^m} \\
&+ g_{ij}^0 g_{il}^s \frac{\lambda_k^{m-s}\lambda_i^s}{\lambda_k^m-\lambda_i^m} + g_{ij}^0 g_{jl}^s \frac{\lambda_j^s \lambda_k^{m-s}}{\lambda_j^m-\lambda_k^m} \\
&+ \frac{1}{2} g_{kj}^0 g_{kl}^s \left[\frac{\lambda_k^m+\lambda_i^m}{\lambda_k^m-\lambda_i^m} - \frac{\lambda_k^m+t\lambda_j^m}{\lambda_k^m-t\lambda_j^m}\right] \\
&+ \frac{1}{2} g_{lj}^0 g_{kl}^s \left[\frac{\lambda_i^m+\lambda_l^m}{\lambda_i^m-\lambda_l^m} + \frac{\lambda_l^m+t\lambda_j^m}{\lambda_l^m-t\lambda_j^m}\right] \\
&+ \sum_{q\neq 0} \frac{t}{t_{q-1}} \left[\frac{\lambda_l^q \lambda_j^{m-q}}{\lambda_l^m - t\lambda_j^m} g_{lj}^q g_{kl}^s - \frac{\lambda_k^q \lambda_j^{m-q}}{\lambda_k^m - t\lambda_j^m} g_{kj}^q g_{kl}^s\right] .
\end{aligned} \tag{5.55}$$

It is nice to see that some terms are similar in the cases $s=0$ and $s\neq 0$ by comparing (5.54) and (5.55). By antisymmetry, we can find $\{g_{ij}^q, g_{kl}^s\}$ for $q\neq 0$ and $s=0$, which coincide with an explicit computation using Lemma 5.2.26.

Next, we can assume $q, s \neq 0$. If $q = s$ in this case, Lemma 5.2.27 yields

$$\begin{aligned}
\{g_{ij}^s, g_{kl}^s\} =& \frac{1}{2} g_{ij}^s g_{kl}^s \left[\frac{\lambda_i^m + \lambda_k^m}{\lambda_i^m - \lambda_k^m} + \frac{\lambda_j^m + \lambda_l^m}{\lambda_j^m - \lambda_l^m} + \frac{2\lambda_k^m}{\lambda_k^m - \lambda_j^m} + \frac{2\lambda_i^m}{\lambda_l^m - \lambda_i^m} \right] \\
&+ \frac{1}{2} g_{il}^s g_{kj}^s \left[\frac{\lambda_i^m + \lambda_k^m}{\lambda_i^m - \lambda_k^m} + \frac{\lambda_j^m + \lambda_l^m}{\lambda_j^m - \lambda_l^m} + \frac{\lambda_k^m + t\lambda_j^m}{\lambda_k^m - t\lambda_j^m} - \frac{\lambda_i^m + t\lambda_l^m}{\lambda_i^m - t\lambda_l^m} \right] \\
&+ \frac{s}{m} (\delta_{jk} - \delta_{il}) g_{ij}^s g_{kl}^s + g_{ij}^s g_{il}^s \frac{\lambda_k^{m-s} \lambda_i^s}{\lambda_k^m - \lambda_i^m} + g_{ij}^s g_{jl}^s \frac{\lambda_j^s \lambda_k^{m-s}}{\lambda_j^m - \lambda_k^m} \\
&+ g_{kj}^s g_{kl}^s \frac{\lambda_k^s \lambda_i^{m-s}}{\lambda_k^m - \lambda_i^m} + g_{kl}^s g_{lj}^s \frac{\lambda_i^{m-s} \lambda_l^s}{\lambda_i^m - \lambda_l^m} \\
&+ \sum_{q=0}^{s-1} \frac{t_{s-1}}{t_{q-1}} \left[\frac{\lambda_k^{m+q-s} \lambda_j^{s-q}}{\lambda_k^m - t\lambda_j^m} g_{kj}^q g_{il}^s - \frac{\lambda_i^{m+q-s} \lambda_l^{s-q}}{\lambda_i^m - t\lambda_l^m} g_{kj}^s g_{il}^q \right] \\
&+ \sum_{q=s+1}^{m-1} \frac{t t_{s-1}}{t_{q-1}} \left[\frac{\lambda_k^{q-s} \lambda_j^{m+s-q}}{\lambda_k^m - t\lambda_j^m} g_{kj}^q g_{il}^s - \frac{\lambda_i^{q-s} \lambda_l^{m+s-q}}{\lambda_i^m - t\lambda_l^m} g_{kj}^s g_{il}^q \right] .
\end{aligned} \tag{5.56}$$

If $q < s$ are both nonzero, we obtain from Lemma 5.2.28 that

$$\begin{aligned}
\{g_{ij}^q, g_{kl}^s\} \overset{q<s}{=}& \frac{1}{2} g_{ij}^q g_{kl}^s \left[\frac{\lambda_i^m + \lambda_k^m}{\lambda_i^m - \lambda_k^m} + \frac{\lambda_j^m + \lambda_l^m}{\lambda_j^m - \lambda_l^m} + \frac{2\lambda_k^m}{\lambda_k^m - \lambda_j^m} + \frac{2\lambda_i^m}{\lambda_l^m - \lambda_i^m} \right] \\
&+ g_{ij}^q g_{kl}^s \left[\frac{q-s}{m} \delta_{jl} + \frac{s}{m} \delta_{jk} - \frac{q}{m} \delta_{il} - \delta_{ik} \right] \\
&+ g_{il}^q g_{kj}^s \frac{\lambda_j^{m+q-s} \lambda_l^{s-q}}{\lambda_j^m - \lambda_l^m} + g_{kj}^q g_{il}^s \frac{\lambda_i^{s-q} \lambda_k^{m+q-s}}{\lambda_i^m - \lambda_k^m} \\
&+ g_{ij}^q g_{il}^s \frac{\lambda_k^{m-s} \lambda_i^s}{\lambda_k^m - \lambda_i^m} + g_{ij}^q g_{jl}^s \frac{\lambda_j^s \lambda_k^{m-s}}{\lambda_j^m - \lambda_k^m} \\
&+ g_{kj}^q g_{kl}^s \frac{\lambda_k^q \lambda_i^{m-q}}{\lambda_k^m - \lambda_i^m} + g_{lj}^q g_{kl}^s \frac{\lambda_i^{m-q} \lambda_l^q}{\lambda_i^m - \lambda_l^m} .
\end{aligned} \tag{5.57}$$

By antisymmetry, we get the case $q > s$. The latter can also be checked using Lemma 5.2.29. Finally, we have directly that $\{\lambda_i, \lambda_j\} = 0$ and $\{\lambda_i, g_{kl}^s\} = \frac{1}{m} \delta_{ij} \lambda_i g_{kl}^s$ by Lemma 5.2.9. Gathering all these results, we obtain the following.

Proposition 5.2.30 *Up to localisation, the commutative algebra generated by the elements (λ_i, g_{ij}^s) inside $\mathcal{O}[\mathcal{C}'_{n,m}]$ is a Poisson subalgebra.*

5.2.5 Returning to integrability

For any $U \in \{X, Y, Z, 1_I + XY\}$, we have obtained different families of Poisson commuting functions in Propositions 5.1.2, 5.1.3 and 5.1.9, and they all contain the elements $(\operatorname{tr} U^k)_k$. Furthermore, these families are such that we can derive explicit expressions for the Hamiltonian flows of some of their elements in Propositions 5.1.5–5.1.8 and Propositions 5.1.10–5.1.12. In particular, we can always write down the flows of the functions $(\operatorname{tr} U^k)_k$ in these different cases. These results suggest that we should try to build (degenerate) integrable systems containing some of the functions $(\operatorname{tr} U^k)_k$. In fact, we will show that such functions are indeed (degenerately) integrable, and proving this statement will occupy the rest of this subsection. We will omit to write down expressions for the Hamiltonians in terms of local coordinates on $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)'$, as this will be the subject of Section 5.3.

Remark 5.2.31 *In a way similar to the study of integrability on the MQVs associated to a Jordan quiver with general framing that we analysed in § 4.3.3, our statements will only hold in the connected component of $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)^\circ$ containing $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)'$. Here, $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)'$ denotes the subspace defined in Remark 5.2.2.*

Trivial cases

First, assume that $|\mathbf{d}| = 2$. By construction, this means that the space $\mathcal{C}_{n,m}$ is characterised by matrices $X, Y, V_{1,1}, W_{1,1}$ and $V_{\alpha,s}, W_{\alpha,s}$, where (α, s) is either $(1, 2)$ if $d_0 = 2$, or $(\beta, 1)$, $\beta \in I \setminus \{0\}$, if $d_0 = d_\beta = 1$ are the only nonzero components of $\mathbf{d}$. Then, from the fourth item of Proposition 5.1.2, we get in a way similar to Theorem 4.3.14 that the following holds.

Theorem 5.2.32 *If $|\mathbf{d}| = 2$ and $U = X, Y, Z$, the elements*

$$\{\operatorname{tr} U^{km}\,,\ \operatorname{tr}(W_{1,1}V_{1,1}U^{km}) \mid k = 1, \dots, n\}$$

form an integrable system on $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)$ (or $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)^\circ$ for $U = Z$). For $U = 1_I + XY$, this is true with $\operatorname{tr} U^k$ instead of $\operatorname{tr} U^{km}$.

Next, introduce $t_{p\gamma|q\epsilon} = \operatorname{tr}(W_{p,\gamma}V_{q,\epsilon}) = V_{q,\epsilon}W_{p,\gamma}$ (not to be confused with the parameters (t_s)

from (5.3)). Then, when we look at (3.39) for $k = l = 0$, we see that

$$\begin{aligned}\{t_{p\gamma|q\epsilon}, t_{s\alpha|r\beta}\}_{\mathrm{P}} =&\frac{1}{2}\left[o(p,r)+o(q,s)-o(p,s)-o(q,r)\right] t_{p\gamma|q\epsilon}t_{s\alpha|r\beta} \\ &+\frac{1}{2}\delta_{ps}o(\alpha,\gamma)(t_{p\gamma|q\epsilon}t_{s\alpha|r\beta}+t_{s\alpha|q\epsilon}t_{p\gamma|r\beta}) \\ &+\frac{1}{2}\delta_{qr}o(\beta,\epsilon)(t_{p\gamma|q\epsilon}t_{s\alpha|r\beta}+t_{s\alpha|q\epsilon}t_{p\gamma|r\beta}) \\ &+\frac{1}{2}\delta_{qs}[o(\epsilon,\alpha)+\delta_{\epsilon\alpha}](t_{p\gamma|q\epsilon}t_{s\alpha|r\beta}+t_{s\alpha|q\epsilon}t_{p\gamma|r\beta})+\delta_{qs}\delta_{\epsilon\alpha}\, t_{p\gamma|r\beta} \\ &-\frac{1}{2}\delta_{pr}[o(\beta,\gamma)+\delta_{\beta\gamma}](t_{p\gamma|q\epsilon}t_{s\alpha|r\beta}+t_{s\alpha|q\epsilon}t_{p\gamma|r\beta})-\delta_{pr}\delta_{\gamma\beta}\, t_{s\alpha|q\epsilon}\end{aligned}$$

In particular, $\{t_{p\gamma|p\gamma}, t_{s\alpha|s\alpha}\}_{\mathrm{P}} = 0$.

Theorem 5.2.33 *If $n = 1$ and $U = X, Y, Z$, the elements $(\operatorname{tr} U^m, t_{s\alpha|s\alpha})_{(s,\alpha)\neq(1,1)}$ form an integrable system for any $\mathbf{d}$. Hence $\operatorname{tr} U^{km}$ is Liouville integrable for any $k \in \mathbb{N}^\times$. If $U = 1_I + XY$, the same is true with $\operatorname{tr} U^k$ for $k \in \mathbb{N}^\times$.*

Proof. Using the second item of Proposition 5.1.2 and the remark above, these elements Poisson commute. We deal with the case $U = X$ now, and we work on the subspace where X is invertible. In this case each element $W_{s,\alpha}, V_{s,\alpha}, X_s, Z_s$ is a scalar, the moment map (4.1a) reads

$$q_s \prod_{\alpha=1}^{d_s}(1 + t_{s\alpha|s\alpha}) = X_s Z_s X_{s-1}^{-1} Z_{s-1}^{-1}\,, \quad s \in I\,,$$

and by taking the product of these expressions we get that $t\prod_{(s,\alpha)}(1 + t_{s\alpha|s\alpha}) = 1$. Hence, we can rewrite $t_{11|11}$ in terms of the other $t_{s\alpha|s\alpha}$. The group acting in this case is $(\mathbb{C}^\times)^m$, so we can generically fix the gauge by the conditions $X_s = 1$ for $s \neq 0$ and $\sum_\alpha W_{0,\alpha} = 1$, which amounts to $W_{0,1} = 1 - \sum_{\alpha\neq 1} W_{0,\alpha}$.

The moment map at s now allows to write Z_s/Z_{s-1} in terms of the other elements. Thus, we can see that at a generic point the functions $(X_0, Z_0, W_{s,\alpha}, t_{s\alpha|s\alpha})_{(s,\alpha)\neq(0,1)}$ are coordinates. In particular, the family from the statement contains $|\mathbf{d}|$ functionally independent elements at a generic point. The other cases are similar. □

Degenerate integrability

Recall the definition of the commutative algebra $\mathcal{O}_U$ defined in Proposition 5.1.2, which is generated by the functions $\operatorname{tr} U^k$ and $\operatorname{tr}(W_{s,\alpha}V_{r,\beta}U^l)$ and is also a Poisson algebra. This algebra is defined on the space $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)$ (or $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)^\circ$ for $U = Z$), and we use the same notation $\mathcal{O}_U$ to denote the corresponding sheaf of analytic functions.

Proposition 5.2.34 *For $U = X, Y, Z$, we can complete the set of functions $(\operatorname{tr} U^{lm})_{l=1}^n$ by $2n|\mathbf{d}| - 2n$ elements of $\mathcal{O}_U$, such that they are all functionally independent at a generic point. Moreover, among these $2n|\mathbf{d}| - n$ functions only the n elements $\operatorname{tr} U^{lm}$ Poisson commute with all the other ones.*

The case $U = 1_I + XY$ is slightly more involved. To state it, denote by $m_{\mathbf{d}}$ the cardinality of the set $I_{\mathbf{d}} = \{s \in I \mid d_s \neq 0\}$.

Proposition 5.2.35 *We can complete the set of functions $(\operatorname{tr}(1_I + XY)^l)_{l=1}^n$ by $2n|\mathbf{d}| - (m_{\mathbf{d}} + 1)n$ elements of $\mathcal{O}_U$, such that they are all functionally independent at a generic point. Moreover, we can choose these $2n|\mathbf{d}| - m_{\mathbf{d}} n$ functions so that $m_{\mathbf{d}} n$ of them commute with all the other ones.*

As in the one-loop case, we obtain the following result as a direct application of Propositions 5.2.34 and 5.2.35.

Corollary 5.2.36 *For any $k \in \mathbb{N}^\times$, there exists a degenerate integrable system containing $\operatorname{tr} U^{km}$, for $U = X, Y, Z$, or $\operatorname{tr} U^k$ for $U = 1_I + XY$.*

In the proof of Proposition 5.2.34, we will use an inductive argument based on the ordering considered in Remark 3.2.11.

Proof. (Proposition 5.2.34.) We only show the case $U = Z$, the other cases being treated similarly (by using an analogue to the argument at the end of Proposition 4.3.16).

First, we need to introduce a convenient set of local coordinates. Consider the space $\mathfrak{h}^{2(|\mathbf{d}|-1)+1}$ where $\mathfrak{h} = \mathbb{C}^n$, with local coordinates $(z_i, v_{s,\alpha,i}, w_{s,\alpha,i})$ for $i = 1, \dots, n$, $s \in I$, and $1 \leq \alpha \leq d_s$ for $s < \bar{s}$, while[20] $1 \leq \alpha < d_{\bar{s}}$ for $s = \bar{s}$. We consider the subspace $\mathfrak{h}_1$ where $z_i \neq 0$, $z_i \neq z_j$ for all $i \neq j$, and for each (s, α), $1 + \sum_i w_{s,\alpha,i} v_{s,\alpha,i} \neq 0$. Now, define the matrices

$$\begin{aligned} &Z_{m-1} = \operatorname{diag}(z_1, \dots, z_n), \quad Z_s = \operatorname{Id}_n \text{ for } s \neq m-1, \\ &V_{s,\alpha} = (v_{s,\alpha,i})_i, \quad W_{s,\alpha} = (w_{s,\alpha,i})_i\,, \quad \text{for } (s,\alpha) < (\bar{s}, d_{\bar{s}}). \end{aligned} \tag{5.58}$$

[20] See Remark 3.2.11 for the definition of $(\bar{s}, d_{\bar{s}})$.

We can define for any such $(s,\alpha)<(\bar{s},d_{\bar{s}})$ the matrix

$$F_{s,\alpha}=t\left(\overleftarrow{\prod_{r>s}}Z_{r-1}\right)\left(\overleftarrow{\prod_{1\leqslant\gamma\leqslant\alpha}}(\mathrm{Id}_n+W_{s,\gamma}V_{s,\gamma})\right)Z_{s-1}\left(\overleftarrow{\prod_{r<s}}\left[\overleftarrow{\prod_{1\leqslant\beta\leqslant d_r}}(\mathrm{Id}_n+W_{r,\beta}V_{r,\beta})\right]Z_{r-1}\right),$$

where an empty product is set equal to 1. The rightmost factor is always $Z_{-1}:=Z_{m-1}$. (Note that the matrix $F_{\bar{s},d_{\bar{s}}}$ defined in the same way in $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)^\circ$ is nothing else than the product $X_{m-1}Z_{m-1}\dots Z_0X_{m-1}^{-1}$ by (5.1a).) If the elements $(s,\alpha),(r,\beta)$ are such that $(s,\alpha)<(r,\beta)$ and they follow one another in the total ordering, we have from our choice of (Z_s) defined in (5.58) that $F_{r,\beta}=(\mathrm{Id}_n+W_{r,\beta}V_{r,\beta})F_{s,\alpha}$, and they are all clearly invertible.

Let (s_-,α_-) be the element preceding $(\bar{s},d_{\bar{s}})$ in the ordering. We claim that F_{s_-,α_-} has distinct eigenvalues generically. We also claim that for $W=(1,\dots,1)^\top$, this implies that there exists $V\in\mathrm{Mat}_{1\times n}$ such that $F_+=F_{s_-,\alpha_-}+WVF_{s_-,\alpha_-}$ has the same spectrum as $Z_0=Z_{m-1}\dots Z_0$. Indeed, this follows from Lemma A.1 by induction on the order of the (s,α). Moreover, this lemma tells us that by fixing the matrix X_{m-1} that diagonalises F_+ into Z_0, V is uniquely defined.

It remains to show that the elements of $\mathfrak{h}_1$ and a fixed choice of X_{m-1} determine a point in $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)^\circ$. Indeed, we can set $W_{\bar{s},d_{\bar{s}}}=W$, $V_{\bar{s},d_{\bar{s}}}=V$, and define inductively from X_{m-1}

$$\begin{aligned}X_0&=q_0\overleftarrow{\prod_{1\leqslant\alpha\leqslant d_0}}(\mathrm{Id}_n+W_{0,\alpha}V_{0,\alpha})\,Z_{m-1}X_{m-1}\,,\\ X_s&=q_s\overleftarrow{\prod_{1\leqslant\alpha\leqslant d_s}}(\mathrm{Id}_n+W_{s,\alpha}V_{s,\alpha})\,X_{s-1}\,,\quad 0<s<m-1\,.\end{aligned}$$

It is not hard to see that all invertibility conditions of $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)^\circ$ are satisfied as well as (5.1a).

Locally, we can complete the coordinates given by $(z_i,v_{s,\alpha,i},w_{s,\alpha,i})$ generic in $\mathfrak{h}_1$ by n functions that correspond to the choice of eigenbasis determining X_{m-1} (up to permutation). It remains to adapt the counting argument in the proof of Proposition 4.3.16 to the present case, where we want to show that we can find $2n|\mathbf{d}|-n$ functionally independent elements in $\mathcal{O}_Z$. Introduce $t^k_{p\gamma|q\epsilon}=\mathrm{tr}\,W_{p,\gamma}Z^{km+p-q}V_{q,\epsilon}$ for any $k\in\mathbb{N}^\times$ and admissible spins. We have locally that $t^k_{p\gamma|q\epsilon}=\sum_i z_i^k w_{p,\gamma,i}v_{q,\epsilon,i}$ for $(p,\gamma),(q,\epsilon)\neq(\bar{s},d_{\bar{s}})$, and $t^k_{q\epsilon}:=t^k_{\bar{s},d_{\bar{s}}|q\epsilon}=\sum_i z_i^k v_{q,\epsilon,i}$ for $(q,\epsilon)\neq(\bar{s},d_{\bar{s}})$. Now, recall the map ρ defined in Remark 3.2.11 which assigns an admissible spin to all $\{1,\dots,|\mathbf{d}|\}$. As ρ preserves the ordering, $\rho(|\mathbf{d}|)=(\bar{s},d_{\bar{s}})$. We can consider the functions $t^k_{\rho(j)|\rho(j')}$ and $t^k_{\rho(j)}$ for $j,j'\neq|\mathbf{d}|$. We then form

$$T=\left(\left(\frac{1}{k}\,\mathrm{tr}\,Z^{km}\right)_k,\left(t^k_{\rho(1)}\right)_k,\dots,\left(t^k_{\rho(|\mathbf{d}|-1)}\right)_k,\dots\left(t^k_{\rho(1),\rho(1)}\right)_k,\dots,\left(t^k_{\rho(1),\rho(|\mathbf{d}|-1)}\right)_k\right)$$

where each k ranges over $1, \dots, n$. Since $\frac{1}{k}\operatorname{tr} Z^{km} = \sum_i z_i^k$, it is an easy exercise to see that the Jacobian matrix with respect to the coordinates $(z_i, v_{s,\alpha,i}, w_{s,\alpha,i})$ has diagonal form

$$\frac{\partial T}{\partial(z,v,w)} = \begin{pmatrix} V_z & 0 & 0 \\ * & A & 0 \\ * & * & B \end{pmatrix},$$

where A, B are composed of $(|\mathbf{d}|-1)$ diagonal $n \times n$ blocks which are generically invertible. Thus, this matrix has full rank $2n|\mathbf{d}| - n$ as in Proposition 4.3.16.

Finally, we need to show that the functions $\frac{1}{k}\operatorname{tr} Z^{km}$, $1 \leq k \leq n$, are the only ones that commute with the other elements of T. This is similar to the end of the proof of Proposition 4.3.16 : if this statement were false, this would contradict that $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)$ is symplectic. □

Proof. **(Proposition 5.2.35.)** We only explain the differences with the proof of Proposition 5.2.34, as the ideas are similar. Recall that $m_{\mathbf{d}}$ is the cardinality of $I_{\mathbf{d}} = \{s \in I \mid d_s \neq 0\}$.

First, we introduce a convenient set of local coordinates. Consider the space $\mathfrak{h}^{2(|\mathbf{d}|-m_{\mathbf{d}})+m_{\mathbf{d}}}$ where $\mathfrak{h} = \mathbb{C}^n$, with local coordinates $(z_{p,i}, v_{s,\alpha,i}, w_{s,\alpha,i})$ for $i = 1, \dots, n$, $p \in I_{\mathbf{d}}$, $s \in \{p \in I_{\mathbf{d}} \mid d_p > 1\}$ and $1 < \alpha \leq d_s$. We consider the subspace $\mathfrak{h}_1'$ where $z_{p,i} \neq 0$, $z_{p,i} \neq z_{p,j}$ for all $p, q \in I_{\mathbf{d}}$, $1 \leq i, j \leq n$ with $(p,i) \neq (q,j)$, and for each (s,α), $1 + \sum_i w_{s,\alpha,i} v_{s,\alpha,i} \neq 0$. We define the matrices

$$B_p = \operatorname{diag}(z_{p,1}, \dots, z_{p,n}) \text{ for } p \in I_{\mathbf{d}}, \quad V_{s,\alpha} = (v_{s,\alpha,i})_i, \quad W_{s,\alpha} = (w_{s,\alpha,i})_i\,. \tag{5.59}$$

Next, fix $p \in I_{\mathbf{d}}$. For any $1 < \alpha \leq d_p$, we can define the matrix

$$U_{p,\alpha} = (\operatorname{Id}_n + W_{p,d_p} V_{p,d_p}) \dots (\operatorname{Id}_n + W_{p,\alpha} V_{p,\alpha}),$$

and we also set $U_{p,+} = \operatorname{Id}_n$. Put $W_{p,1} = (1, \dots, 1)^\top$. As in the proof of Proposition 5.2.34 with Lemma A.4 in this case, we know that there exists some $V_{p,1} \in \operatorname{Mat}_{1\times n}(\mathbb{C})$ such that $U_{p,1} := B_p U_{p,2}(\operatorname{Id}_n + W_{p,1} V_{p,1})$ has the same spectrum as B_{p-1}. (We take $U_{p,2} = U_{p,+}$ if $d_p = 1$, otherwise the result follows by induction on α.) Moreover, up to fixing the matrix $Y_{p-1} \in \operatorname{GL}_n(\mathbb{C})$ that satisfies

$$Y_{p-1} B_{p-1} Y_{p-1}^{-1} = U_{p,d_p} = B_p(\operatorname{Id}_n + W_{p,d_p} V_{p,d_p}) \dots (\operatorname{Id}_n + W_{p,1} V_{p,1}), \tag{5.60}$$

the covector $V_{p,1}$ is unique. Note that Y_{p-1} depends on n parameters $(c_{p-1,i})$ corresponding to a choice of eigenbasis. This choice is independent for distinct $p, q \in I_{\mathbf{d}}$.

For each $s \in I$ with $d_s = 0$, we set $B_s = B_{s-1}$ inductively. This is well-defined since B_0 was given in (5.59) as $d_0 \neq 0$. We can remark that any matrix $Y_{s-1} \in \mathrm{GL}_n(\mathbb{C})$ satisfying

$$Y_{s-1} B_{s-1} Y_{s-1}^{-1} = B_s \,, \tag{5.61}$$

must be diagonal with nonzero entries. We simply fix $Y_{s-1} = \mathrm{Id}_n$ in such cases.

We claim that the $(2|\mathbf{d}| - m_{\mathbf{d}})n$ functions $(v_{s,\alpha,i}, w_{s,\alpha,i}, z_{p,i})$ of $\mathfrak{h}_1'$ and $nm_{\mathbf{d}}$ parameters $(c_{s-1,i})$ form a local coordinate system around a generic point of $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)$. Indeed, we can put $X_s = (t_s^{-1} B_s - \mathrm{Id}_n) Y_s^{-1}$, so that $(\mathrm{Id}_n + X_s Y_s) = t_s B_s^{-1}$ and $(\mathrm{Id}_n + Y_{s-1} X_{s-1})^{-1} = t_{s-1}^{-1} Y_{s-1} B_{s-1} Y_{s-1}^{-1}$. Then, (5.60)–(5.61) imply that

$$\begin{aligned}
(\mathrm{Id}_n + X_s Y_s)(\mathrm{Id}_n + Y_{s-1} X_{s-1})^{-1} =& q_s B_s^{-1} Y_{s-1} B_{s-1} Y_{s-1}^{-1} \\
=& q_s \overleftarrow{\prod_{1 \leqslant \alpha \leqslant d_s}} (\mathrm{Id}_n + W_{s,\alpha} V_{s,\alpha}) \,,
\end{aligned}$$

which is precisely (5.1a). Hence, we define a point generically inside the subspace $\{\det Y^m \neq 0\}$ of $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)$. The choices of $(B_p, W_{p,1})$ at all $p \in I_{\mathbf{d}}$ and (B_s, Y_{s-1}) at all $s \in I \setminus I_{\mathbf{d}}$ fix the gauge (up to permutations). Therefore, we get local coordinates.

Second, we pick a suitable set of functions. We will work in the algebra $\mathcal{O}_U'$ for convenience (which contains $\mathcal{O}_U$ by Proposition 5.1.3). Hence, we can pick symmetric functions of the elements $\mathrm{Id}_n + X_s Y_s$, $s \in I$. We set $T_s^k = t_s^{-k} \operatorname{tr}(\mathrm{Id}_n + X_s Y_s)^k$ for each $s \in I$, $k \in \mathbb{N}^\times$. We also introduce the following elements $t_{p,\gamma,\epsilon}^k = t_p^{-k} \operatorname{tr} W_{p,\gamma} (\mathrm{Id}_n + X_p Y_p)^k V_{p\epsilon}$, for any $p \in I_{\mathbf{d}}$, $1 \leq \gamma, \epsilon \leq d_p$, and $k \in \mathbb{N}^\times$. In local coordinates, we can write that for any $p \in I_{\mathbf{d}}$, $1 < \gamma, \epsilon \leq d_p$

$$T_s^k = \sum_i z_{p,i}^{-k} \,, \quad t_{p,\gamma,\epsilon}^k = \sum_i w_{p,\gamma,i} v_{p,\epsilon,i} z_{p,i}^{-k} \,, \quad t_{p,1,\epsilon}^k = \sum_i v_{p,\epsilon,i} z_{p,i}^{-k} \,.$$

Hence, for each $p \in I_{\mathbf{d}}$, we consider

$$\tilde{T}_p = \left(\left(\frac{1}{k} T_s^k \right)_k , \left(t_{p,1,d_p}^k \right)_k , \ldots , \left(t_{p,1,2}^k \right)_k , \left(t_{p,d_p,2}^k \right)_k , \ldots , \left(t_{p,2,2}^k \right)_k \right)$$

where each k ranges over $1, \ldots, n$ (we only take the first n functions for $d_p = 1$). With respect to the coordinates $x_q = (z_{q,i}, v_{q,\alpha,i}, w_{q,\alpha,i})$ for $q \in I_{\mathbf{d}}$, it is not hard to see that the Jacobian matrix of $\tilde{T}_p$ vanishes for $p \neq q$, while for $p = q$ it takes the form

$$\frac{\partial \tilde{T}_p}{\partial x_p} = \begin{pmatrix} V_p & 0 & 0 \\ * & A_p & 0 \\ * & * & B_p \end{pmatrix} ,$$

for some $V_p \in \mathrm{Mat}_{n\times n}$, $A \in \mathrm{Mat}_{n_p\times n_p}$ and $B \in \mathrm{Mat}_{n_p\times n_p}$ where $n_p = n(d_p - 1)$. These three matrices are generically invertible, so the matrix $\partial\tilde{T}_p/\partial x_p$ has rank $2nd_p - n$. Therefore, the Jacobian matrix of $T = (\tilde{T}_p)$ with respect to the coordinates (x_q) is constituted of nonzero diagonal blocks of such ranks at a generic point. Hence, we get

$$\sum_{p\in I_{\mathbf{d}}} n(2d_p - 1) = 2n \sum_{p\in I_{\mathbf{d}}} d_p - \sum_{p\in I_{\mathbf{d}}} n = 2n|\mathbf{d}| - nm_{\mathbf{d}}$$

functionally independent elements. Among them, the $nm_{\mathbf{d}}$ functions T_p^k, $p \in I_{\mathbf{d}}$ and $k = 1, \ldots, n$, are Poisson commuting with all the other ones by Proposition 5.1.3. Moreover, we could replace one of the T_p^k by $\mathrm{tr}(1_I + XY)^k$ without changing the functional independence. □

Liouville integrability

We consider for any $j \in \{0, 1, \ldots, |\mathbf{d}|\}$ the elements $\Theta^{(j)}$ and $U_{(j)} = \Theta^{(j)}U$ for $U \in \{Y, Z, (1_I + XY)^{-1}\}$, which we used to state the involutivity of the functions $\mathrm{tr}\, U_{(j)}^K$ in Proposition 5.1.9. Denote by $\mathcal{H}_U$ the commutative algebra generated by the functions $\mathrm{tr}\, U_{(j)}^K$, $j \in \{0, 1, \ldots, |\mathbf{d}|\}$, $K \in \mathbb{N}$, which is therefore an abelian subalgebra of $\mathcal{O}[\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)]$ (or $\mathcal{O}[\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)^\circ]$ for $U = Z$). We also refer to the corresponding sheaf of functions as $\mathcal{H}_U$. The following result is the generalisation of Proposition 4.3.18.

Proposition 5.2.37 *At a generic point, there are $n|\mathbf{d}|$ functionally independent elements in $\mathcal{H}_U$.*

Corollary 5.2.38 *For any $k \in m\mathbb{N}^\times$, there exists a Liouville integrable system containing $\mathrm{tr}\, U^k$. Furthermore, there exists a Liouville integrable system containing $\mathrm{tr}(\mathrm{Id}_n + X_sY_s)^k$ for any $k \in \mathbb{N}^\times$ and $s \in I$.*

Proof. (Corollary 5.2.38.) It suffices to show that such elements are in $\mathcal{H}_U$ at a generic point. For $U = Y, Z$, we get from (5.4) that $U_{(|\mathbf{d}|)}$ has m blocks q_sU_s of size $n \times n$, on the left of the diagonal blocks. Hence, $\mathrm{tr}\, U_{(|\mathbf{d}|)}^k = t^{k/m}\, \mathrm{tr}\, U^k$ for k divisible by m.

For $U = (1_I + XY)^{-1}$, U_0 has diagonal blocks $(\mathrm{Id}_n + XY)(\mathrm{Id}_n + YX)^{-1}(\mathrm{Id}_n + XY)^{-1}$, so clearly any $\mathrm{tr}(1_I + YX)^{-k}$ is in $\mathcal{H}_U$ for $k \in \mathbb{N}$, which then follows for any $k \in \mathbb{Z}$ by using Cayley-Hamilton theorem. But $\mathrm{tr}(1_I + YX)^k = \mathrm{tr}(1_I + XY)^k$ for any $k \in \mathbb{N}$, so using the Cayley-Hamilton theorem again, $\mathrm{tr}(1_I + XY)^k \in \mathcal{H}_U$ for any $k \in \mathbb{Z}$.

To get the last statement, we need that for any $k \in \mathbb{N}$ and $s \in I$

$$q_s^k \operatorname{tr}(\mathrm{Id}_n + X_s Y_s)^{-k} - \operatorname{tr}(\mathrm{Id}_n + Y_{s-1}X_{s-1})^{-k} \in \mathcal{H}_U \,. \tag{5.62}$$

Assuming that (5.62) holds and working on $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)^\circ$, $\operatorname{tr}(\mathrm{Id}_n + Y_{s-1}X_{s-1})^{-k} = \operatorname{tr}(\mathrm{Id}_n + X_{s-1}Y_{s-1})^{-k}$ so if we take linear combinations of (5.62) this yields that $(t^k - 1)\operatorname{tr}(\mathrm{Id}_n + X_{m-1}Y_{m-1})^{-k} \in \mathcal{H}_U$. Since t is not a root of unity by assumption (see Proposition 5.2.1), we get that $\operatorname{tr}(\mathrm{Id}_n + X_s Y_s)^{-k} \in \mathcal{H}_U$ for $s = m-1$, and using (5.62) shows that this holds for all s as $q_s \neq 0$. The statement for all $k \in \mathbb{N}$ then implies it for $k \in \mathbb{Z}$.

We now show (5.62) for each s. By assumption, $d_0 \geq 1$, and $\rho(d_0) = (0, d_0)$. Hence, using the second equality in (5.4) for $j = d_0$ and $s = 0$, we find that

$$\operatorname{tr} U_{(d_0)}^k = \operatorname{tr} U_{(0)}^k + q_0^k \operatorname{tr}(\mathrm{Id}_n + X_0 Y_0)^{-k} - \operatorname{tr}(\mathrm{Id}_n + Y_{m-1}X_{m-1})^{-k} \,,$$

for any $k \in \mathbb{N}$. This is the base step.

Now, consider $0 < s \leq m-1$. If $d_s = 0$ note that $(\mathrm{Id}_n + X_s Y_s) = q_s(\mathrm{Id}_n + Y_{s-1}X_{s-1})$ by (5.1a). Hence there is nothing to prove since (5.62) just vanishes. If $d_s \neq 0$, denote by $j_s \in \{0, 1, \ldots, |\mathbf{d}|\}$ the element such that $\rho(j_s) = (s, d_s)$. Then, using the second equality in (5.4) for $j = j_s$ and s, we find that

$$\operatorname{tr} U_{(j_s)}^k = \operatorname{tr} U_{(j_s - d_s)}^k + q_s^k \operatorname{tr}(\mathrm{Id}_n + X_s Y_s)^{-k} - \operatorname{tr}(\mathrm{Id}_n + Y_{s-1}X_{s-1})^{-k} \,,$$

for any $k \in \mathbb{N}$. Indeed, by assumption $\rho(j_s - d_s) = (r, d_r)$ for some $r < s$, so $U_{(j_s)}$ and $U_{(j_s - d_s)}$ only differ by their s-th diagonal blocks which are

$$q_s(\mathrm{Id}_n + X_s Y_s)^{-1} \text{ for } U_{(j_s)} \,, \quad (\mathrm{Id}_n + X_s Y_s)(\mathrm{Id}_n + Y_{s-1}X_{s-1})^{-1}(\mathrm{Id}_n + X_s Y_s)^{-1} \text{ for } U_{(j_s - d_s)} \,.$$

and the equality easily follows. □

Proof. (Proposition 5.2.37.) We sketch the proof for $U = Z$ and the details can be recovered by adapting Proposition 4.3.18. Moreover, we can assume $|\mathbf{d}| \geq 2$ as in the case $|\mathbf{d}| = 1$ we just need $\operatorname{tr} Z^{km}$ with $k = 1, \ldots, n$.

First, we need to introduce a convenient set of local coordinates. Consider the space $\mathfrak{h}^{2(|\mathbf{d}|-1)+1}$ where $\mathfrak{h} = \mathbb{C}^n$, with local coordinates $(z_i, v_{s,\alpha,i}, w_{s,\alpha,i})$ for $i = 1, \ldots, n$, $s \in I$, and $1 \leq \alpha \leq d_s$, with $(s,\alpha) \neq (0,1)$. We consider the subspace $\mathfrak{h}_2$ where $z_i \neq 0$, $z_i \neq z_j$ for all $i \neq j$, and for each (s,α), $1 + \sum_i w_{s,\alpha,i} v_{s,\alpha,i} \neq 0$. Now, define the matrices

$$Z_{m-1} = \operatorname{diag}(z_1, \dots, z_n), \; Z_s = \operatorname{Id}_n \text{ for } s \neq m-1, \quad V_{s,\alpha} = (v_{s,\alpha,i})_i, \quad W_{s,\alpha} = (w_{s,\alpha,i})_i \,.$$

with indices $(s,\alpha) \neq (1,1)$. Then, as in the proof of Proposition 5.2.34, we can complete the elements of $\mathfrak{h}_2$ to get local coordinates on $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)^\circ$ around a generic point (this requires Lemma A.4 to get $V_{0,1}$).

Recall that the admissible spins (s,α) correspond to some $j \in \{1, \dots, |\mathbf{d}|\}$ under the map ρ of Remark 3.2.11. In particular, $\rho(1) = (0,1)$. Define $h_{j,K} = \operatorname{tr} Z_{(j)}^{Km}$. The proof then consists in showing by (descending) induction on $j = |\mathbf{d}|, \dots, 2$ that the functions $h_{|\mathbf{d}|,K}, \dots, h_{j-1,K}$, $K = 1, \dots, n$, are $n(|\mathbf{d}| - j + 2)$ independent functions. This is done as in Proposition 4.3.18, and we only sketch the induction step. Assume that we have independence for some j. Using the second equality in (5.4), each function $h_{j-1,K}$ depends on all the matrices $(V_{s,\alpha}, W_{s,\alpha})$ with $(s,\alpha) \geq \rho(j)$, while any $h_{l,K}$, $j \leq l \leq |\mathbf{d}|$, depends on those with $(s,\alpha) > \rho(l) \geq \rho(j)$. Thus the elements $T_{j-1} := (h_{j-1,K})_{K=1}^n$ depend on the $2n$ local coordinates $x_{\rho(j)} = (v_{\rho(j),i}, w_{\rho(j),i})$, while any $h_{l,K}$ with $j \leq l \leq |\mathbf{d}|$ does not. It remains to find a point where $\partial T_{j-1}/\partial x_{\rho(j)}$ has rank n to conclude.

The case $U = Y$ is exactly the same. For the introduction of local coordinates in the case $U = 1_I + XY$, we need to follow the argument developed in Proposition 5.2.35 and first introduce $n m_{\mathbf{d}}$ functions $(z_{p,i})$ as entries of the diagonal matrices $\operatorname{Id}_n + X_p Y_p$, $p \in I_{\mathbf{d}}$, instead of the n functions (z_i). □

5.3 Explicit forms for the Hamiltonians

5.3.1 General expressions

Let us work with a fixed dimension vector $(1, n\delta)$ in the setting of Section 5.2. Our aim is to find explicit expressions for some of the functions forming the integrable systems described in § 5.2.5 on the space $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)$. Indeed, we would like to see if such systems can be identified with known ones, in the same way as we obtained the trigonometric Ruijsenaars-Schneider system in Section 4.2 or its spin version in Section 4.3, together with some modifications of them. So, let us consider the local coordinates $(\lambda_i, \mathbf{a}_i^{s,\alpha}, \mathbf{c}_i^{s,\alpha})$ defined in § 5.2.2 on the subspace $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)'$ (see Remark 5.2.2 for its definition).

On the one hand, we will get nothing from the case $U = X$ since the elements $\operatorname{tr} X^{km}$ are just given by $\sum_{i=1}^n \lambda_i^{km}$. On the other hand for $U = Y, Z$, it seems difficult to write the elements

$$G_k^{m,\mathbf{d}} = \operatorname{tr} Z^{km}\,, \quad H_k^{m,\mathbf{d}} = \operatorname{tr} Y^{km}\,,$$

since Z is constituted of the blocks (5.10), and Z^m is obtained by multiplying them. This is similar for $Y = Z - X^{-1}$. We will only come back to these functions in some specific cases given below. Hopefully, the case $U = 1_I + XY$ is easier to deal with, as we do not need to consider U^m or its multiples. Moreover, since $1_I + XY = \sum_{r\in I}(\operatorname{Id}_n + X_r Y_r)$ and we noted that the elements $\operatorname{Id}_n + X_r Y_r$ are integrable as well in § 5.2.5, it suffices to write down such functions.

Set $L_r = \operatorname{Id}_n + X_r Y_r$ for all $r \in I$. Working in $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)'$, we can write $L_r = X_r Z_r$, so that in the gauge given by (5.8) we have from (5.10)

$$(\hat{L}_r)_{ij} = \sum_{s=0}^{r} \frac{t_r}{t_{s-1}} \frac{\lambda_i^{m+(s-r)} \lambda_j^{-(s-r-1)}}{\lambda_i^m - t\lambda_j^m} g_{ij}^s + \sum_{s=r+1}^{m-1} \frac{tt_r}{t_{s-1}} \frac{\lambda_i^{s-r} \lambda_j^{m-(s-r-1)}}{\lambda_i^m - t\lambda_j^m} g_{ij}^s\,, \tag{5.63}$$

where $g_{ij}^s := \sum_{\alpha=1}^{d_s} \mathbf{a}_i^{s,\alpha} \mathbf{c}_j^{s,\alpha}$. In particular, we can write down the Hamiltonians $F_{r,k}^{m,\mathbf{d}} = \operatorname{tr} L_r^k$ from (5.63). For example,

$$F_{r,1}^{m,\mathbf{d}} = \frac{t_r}{1-t} \sum_{s=0}^{r} t_{s-1}^{-1} \sum_{i=1}^{n} f_{ii}^s + \frac{t_r}{1-t} \sum_{s=r+1}^{m-1} tt_{s-1}^{-1} \sum_{i=1}^{n} f_{ii}^s\,, \quad f_{ii}^s = g_{ii}^s \lambda_i\,. \tag{5.64}$$

Here, we introduced the elements f_{ii}^s because they are $\mathbb{Z}_m^n$-invariant by Remark 5.2.4. (As we will shortly see, it is not a coincidence that in the case $\mathbf{d} = (d, 0, \ldots, 0)$, the function given by (5.64) for $r = m-1$ is precisely $G_1^{1,d}$ given in (4.25) with $q = t$.) More generally, we can see that $t_r F_{r+1,1}^{m,\mathbf{d}} - t_{r+1} F_{r,1}^{m,\mathbf{d}} = tt_{r+1} \sum_i f_{ii}^{r+1}$ for $r = 0, \ldots, m-2$. Hence, we can equivalently look at the integrable Hamiltonians

$$F_0 = \sum_{i=1}^n f_{ii}^0\,,\ F_1 = \sum_{i=1}^n f_{ii}^1\,, \ldots,\ F_{m-1} = \sum_{i=1}^n f_{ii}^{m-1}\,.$$

We can evaluate the vector fields that they define on the local coordinates $(\lambda_i, \mathbf{a}_i^{s,\alpha}, \mathbf{c}_i^{s,\alpha})$ using the Poisson brackets obtained in § 5.2.4. To write them, we denote the derivation $\{-, F_r\}$ as $\frac{d}{dt_r}$, and we introduce

$$\tilde{V}_{ij}^r = \frac{\lambda_i^m + \lambda_j^m}{\lambda_i^m - \lambda_j^m} g_{ij}^r - 2(Z_{r-1})_{ij} - g_{ij}^r\,, \tag{5.65}$$

for any $r \in I$.

Lemma 5.3.1 *For any* $1 \leq \gamma, \epsilon \leq d_0$ *and* $i, j = 1, \dots, n$,

$$\begin{aligned} \frac{d\lambda_i}{dt_0} =& \frac{1}{m}\lambda_i f^0_{ii}\,, \\ \frac{d\mathbf{a}^{0\gamma}_i}{dt_0} =& -\frac{1}{2}\sum_{k\neq i}(\mathbf{a}^{0\gamma}_i - \mathbf{a}^{0\gamma}_k)\tilde{V}^0_{ik}\lambda_k\,, \\ \frac{d\mathbf{c}^{0\epsilon}_j}{dt_0} =& -\frac{1}{m}\mathbf{c}^{0\epsilon}_j f^0_{jj} + \frac{1}{2}\sum_{k\neq j}(\mathbf{c}^{0\epsilon}_j\tilde{V}^0_{jk} - \mathbf{c}^{0\epsilon}_k\tilde{V}^0_{kj})\lambda_k\,. \end{aligned}$$

For any $q \in I \setminus \{0\}$, $1 \leq \gamma, \epsilon \leq d_q$ *and* $i, j = 1, \dots, n$,

$$\begin{aligned} \frac{d\mathbf{a}^{q\gamma}_i}{dt_0} =& \frac{m-q}{m}\mathbf{a}^{q\gamma}_i f^0_{ii} + \mathbf{a}^{q\gamma}_i (Z_{m-1})_{ii}\lambda_i - \frac{1}{2}\sum_{k\neq i}\left(\mathbf{a}^{q\gamma}_i\tilde{V}^0_{ik}\lambda_k - \mathbf{a}^{q\gamma}_k\frac{2\lambda_i^{m-q}\lambda_k^q}{\lambda_i^m - \lambda_k^m}f^0_{ik}\right)\,, \\ \frac{d\mathbf{c}^{q\epsilon}_j}{dt_0} =& \frac{q-1-m}{m}\mathbf{c}^{q\epsilon}_j f^0_{jj} - \mathbf{c}^{q\epsilon}_j (Z_{m-1})_{jj}\lambda_j + \frac{1}{2}\sum_{k\neq j}\left(\mathbf{c}^{q\epsilon}_j\tilde{V}^0_{jk}\lambda_k - \mathbf{c}^{q\epsilon}_k\frac{2\lambda_k^{m-q+1}\lambda_j^{q-1}}{\lambda_k^m - \lambda_j^m}f^0_{kj}\right)\,. \end{aligned}$$

Proof. Recall that $f^0_{kk} = g^0_{kk}\lambda_k$ and $\{\lambda_i, \mathbf{c}^{s\alpha}_k\} = \frac{1}{m}\delta_{ik}\lambda_i\mathbf{c}^{s\alpha}_k$ for any (s,α). The first bracket is obvious, while the next two are obtained using that

$$\begin{aligned} \{\mathbf{a}^{0\gamma}_i, g^0_{kk}\} =& \frac{1}{2}\delta_{(i\neq k)}(\mathbf{a}^{0\gamma}_k - \mathbf{a}^{0\gamma}_i)\left[\frac{\lambda_i^m + \lambda_k^m}{\lambda_i^m - \lambda_k^m}g^0_{ik} - g^0_{ik} - 2(Z_{m-1})_{ik}\right]\,, \\ \{\mathbf{c}^{0\epsilon}_j, g^0_{kk}\} =& \frac{1}{2}\delta_{(j\neq k)}\mathbf{c}^{0\epsilon}_j\left[\frac{\lambda_j^m + \lambda_k^m}{\lambda_j^m - \lambda_k^m}g^0_{jk} - g^0_{jk} - 2(Z_{m-1})_{jk}\right] \\ & - \frac{1}{2}\delta_{(j\neq k)}\mathbf{c}^{0\epsilon}_k\left[\frac{\lambda_k^m + \lambda_j^m}{\lambda_k^m - \lambda_j^m}g^0_{kj} - g^0_{kj} - 2(Z_{m-1})_{kj}\right]\,, \end{aligned}$$

as a direct consequence of Lemma 5.2.24. Similarly, the last two brackets are obtained using

$$\begin{aligned} \{\mathbf{a}^{0\gamma}_i, g^0_{kk}\} =& \delta_{ik}\frac{m-q}{m}\mathbf{a}^{q\gamma}_i g^0_{kk} + \delta_{ik}\mathbf{a}^{q\gamma}_i(Z_{m-1})_{ik} + \delta_{(i\neq k)}\mathbf{a}^{q\gamma}_k g^0_{ik}\frac{\lambda_i^{m-q}\lambda_k^q}{\lambda_i^m - \lambda_k^m} \\ & - \frac{1}{2}\delta_{(i\neq k)}\mathbf{a}^{q\gamma}_i\left[\frac{\lambda_i^m + \lambda_k^m}{\lambda_i^m - \lambda_k^m}g^0_{ik} - g^0_{ik} - 2(Z_{m-1})_{ik}\right]\,, \\ \{\mathbf{c}^{0\epsilon}_j, g^0_{kk}\} =& \delta_{jk}\frac{q-m}{m}\mathbf{c}^{q\epsilon}_j g^0_{kk} - \delta_{jk}\mathbf{c}^{q\epsilon}_j(Z_{m-1})_{kk} + \delta_{(j\neq k)}\mathbf{c}^{q\epsilon}_k\frac{\lambda_k^{m-q}\lambda_j^q}{\lambda_k^m - \lambda_j^m}g^0_{kj} \\ & + \frac{1}{2}\delta_{(j\neq k)}\mathbf{c}^{q\epsilon}_j\left[\frac{\lambda_j^m + \lambda_k^m}{\lambda_j^m - \lambda_k^m}g^0_{jk} - g^0_{jk} - 2(Z_{m-1})_{jk}\right]\,, \end{aligned}$$

which follows from Lemma 5.2.26. □

To see the similarity between Lemma 5.3.1 and Lemma 4.3.8, we introduce

$$V^0_{ij} = \frac{\lambda_i^m + \lambda_j^m}{\lambda_i^m - \lambda_j^m} + \frac{\lambda_i^m + t\lambda_j^m}{\lambda_i^m - t\lambda_j^m}\,, \quad U^{0,s}_{ij} = \frac{2t}{t_{s-1}}\frac{\lambda_i^s\lambda_j^{m-s}}{\lambda_i^m - t\lambda_j^m}\,,\ s \neq 0\,, \tag{5.66}$$

so that we can write using (5.10) and (5.65) that $\tilde{V}^0_{ij} = V^0_{ij} g^0_{ij} + \sum_{s\neq 0} U^{0,s}_{ij} g^s_{ij}$. Moreover, we replace $\mathbf{c}^{0\epsilon}_j$ by $\mathbf{c}^{0\epsilon}_j \lambda_j$ for all indices. Then, we obtain $\frac{d\lambda_i^m}{dt_0} = \lambda_i^m f^0_{ii}$ together with

$$\begin{aligned}
\frac{d\mathbf{a}^{0\gamma}_i}{dt_0} =& -\frac{1}{2}\sum_{k\neq i}(\mathbf{a}^{0\gamma}_i - \mathbf{a}^{0\gamma}_k) V^0_{ik} f^0_{ik} - \frac{1}{2}\sum_{k\neq i}\sum_{s\neq 0}(\mathbf{a}^{0\gamma}_i - \mathbf{a}^{0\gamma}_k) U^{0,s}_{ik} f^s_{ik}\,, \\
\frac{d\mathbf{c}^{0\epsilon}_j}{dt_0} =& \frac{1}{2}\sum_{k\neq j}(\mathbf{c}^{0\epsilon}_j V^0_{jk} f^0_{jk} - \mathbf{c}^{0\epsilon}_k V^0_{kj} f^0_{kj}) + \frac{1}{2}\sum_{k\neq j}\sum_{s\neq 0}(\mathbf{c}^{0\epsilon}_j U^{0,s}_{jk} f^s_{jk} - \mathbf{c}^{0\epsilon}_k U^{0,s}_{kj} f^s_{kj})\,.
\end{aligned}$$

Thus, for $f^s_{ij} = 0$ for all $s \neq 0$ with $x_i = \lambda_i^m$, we recover precisely Lemma 4.3.8 up to a multiplicative factor. This is the case when $d_s = 0$ for all $s \neq 0$, and we come back to this relation in § 5.3.3 below. This is also the case if we restrict to the subspace of $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)'$ where all $W_{s,\alpha}, V_{s,\alpha}$ with $s \neq 0$ are set to zero. In the latter case, the flows are complete on this subspace by Proposition 5.1.12.

We can also write the vector fields for $s \neq 0$. The proof is a boring adaptation of Lemma 5.3.1, so we leave the details to the reader.

Lemma 5.3.2 *For any* $1 \leq \gamma, \epsilon \leq d_0$ *and* $i, j = 1, \dots, n$,

$$\begin{aligned}
\frac{d\lambda_i}{dt_s} =& \frac{1}{m}\lambda_i f^s_{ii}\,, \\
\frac{d\mathbf{a}^{0\gamma}_i}{dt_s} =& -\sum_{k\neq i}(\mathbf{a}^{0\gamma}_i - \mathbf{a}^{0\gamma}_k)\frac{\lambda_i^s \lambda_k^{m-s}}{\lambda_i^m - \lambda_k^m} f^s_{ik}\,, \\
\frac{d\mathbf{c}^{0\epsilon}_j}{dt_s} =& -\frac{1}{m}\mathbf{c}^{0\epsilon}_j f^s_{jj} + \sum_{k\neq j}\left(\mathbf{c}^{0\epsilon}_j \frac{\lambda_j^s \lambda_k^{m-s}}{\lambda_j^m - \lambda_k^m} f^s_{jk} - \mathbf{c}^{0\epsilon}_k \frac{\lambda_k^{s+1}\lambda_j^{m-s-1}}{\lambda_k^m - \lambda_j^m} f^s_{kj}\right)\,.
\end{aligned}$$

For any $q \in I \setminus \{0\}$, $q < s$, $1 \leq \gamma, \epsilon \leq d_q$ *and* $i, j = 1, \dots, n$,

$$\begin{aligned}
\frac{d\mathbf{a}^{q\gamma}_i}{dt_s} =& -\frac{q}{m}\mathbf{a}^{q\gamma}_i f^s_{ii} - \sum_{k\neq i}\left(\mathbf{a}^{q\gamma}_i \frac{\lambda_i^s \lambda_k^{m-s}}{\lambda_i^m - \lambda_k^m} - \mathbf{a}^{q\gamma}_k \frac{\lambda_i^{s-q}\lambda_k^{m+q-s}}{\lambda_i^m - \lambda_k^m}\right) f^s_{ik}\,, \\
\frac{d\mathbf{c}^{q\epsilon}_j}{dt_s} =& \frac{q-1}{m}\mathbf{c}^{q\epsilon}_j f^s_{jj} + \sum_{k\neq j}\left(\mathbf{c}^{q\epsilon}_j \frac{\lambda_j^s \lambda_k^{m-s}}{\lambda_j^m - \lambda_k^m} f^s_{jk} - \mathbf{c}^{q\epsilon}_k \frac{\lambda_k^{s-q+1}\lambda_j^{m+q-s-1}}{\lambda_k^m - \lambda_j^m} f^s_{kj}\right)\,.
\end{aligned}$$

For any $1 \leq \gamma, \epsilon \leq d_s$ *and* $i, j = 1, \dots, n$,

$$\begin{aligned}
\frac{d\mathbf{a}^{s\gamma}_i}{dt_s} =& -\frac{s}{m}\mathbf{a}^{s\gamma}_i f^s_{ii} - \mathbf{a}^{s\gamma}_i (Z_{s-1})_{ii}\lambda_i - \frac{1}{2}\sum_{k\neq i}\left(\mathbf{a}^{s\gamma}_i \frac{2\lambda_i^s \lambda_k^{m-s}}{\lambda_i^m - \lambda_k^m} f^s_{ik} - \mathbf{a}^{s\gamma}_k \tilde{V}^s_{ik}\lambda_k\right)\,, \\
\frac{d\mathbf{c}^{s\epsilon}_j}{dt_s} =& \frac{s-1}{m}\mathbf{c}^{s\epsilon}_j f^s_{jj} + \mathbf{c}^{s\epsilon}_j (Z_{s-1})_{jj}\lambda_j + \frac{1}{2}\sum_{k\neq j}\left(\mathbf{c}^{s\epsilon}_j \frac{2\lambda_j^s \lambda_k^{m-s}}{\lambda_j^m - \lambda_k^m} f^s_{jk} - \mathbf{c}^{s\epsilon}_k \tilde{V}^s_{kj}\lambda_k\right)\,.
\end{aligned}$$

For any $q \in I \setminus \{0\}$, $q > s$, $1 \leq \gamma, \epsilon \leq d_q$ *and* $i, j = 1, \ldots, n$,

$$\begin{aligned}
\frac{d\mathbf{a}_i^{q\gamma}}{dt_s} =& \frac{m-q}{m}\mathbf{a}_i^{q\gamma} f_{ii}^s - \sum_{k\neq i}\left(\mathbf{a}_i^{q\gamma}\frac{\lambda_i^s\lambda_k^{m-s}}{\lambda_i^m-\lambda_k^m} - \mathbf{a}_k^{q\gamma}\frac{\lambda_i^{m+s-q}\lambda_k^{q-s}}{\lambda_i^m-\lambda_k^m}\right) f_{ik}^s\,,\\
\frac{d\mathbf{c}_j^{q\epsilon}}{dt_s} =& \frac{q-1-m}{m}\mathbf{c}_j^{q\epsilon} f_{jj}^s + \sum_{k\neq j}\left(\mathbf{c}_j^{q\epsilon}\frac{\lambda_j^s\lambda_k^{m-s}}{\lambda_j^m-\lambda_k^m} f_{jk}^s - \mathbf{c}_k^{q\epsilon}\frac{\lambda_k^{m+s-q+1}\lambda_j^{q-s-1}}{\lambda_k^m-\lambda_j^m} f_{kj}^s\right)\,.
\end{aligned}$$

5.3.2 Simple framing

We consider the MQV $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)$ for $\mathbf{d} = d' = (1, 0, \ldots, 0)$. Then, the local coordinates are given by the $3n$ elements $(\lambda_i, \mathbf{a}_i^{0,1}, \mathbf{c}_i^{0,1})$ under the constraints $\mathbf{a}_i^{0,1} = 1$ for all i. Thus, we only need to look at the elements $(\lambda_i, \mathbf{c}_i)$, where we have set $\mathbf{c}_i = \mathbf{c}_i^{0,1}$. We easily obtain from Proposition 5.2.7 that the Poisson brackets are given by

$$\begin{aligned}
\{\lambda_i, \lambda_j\} =& 0\,, \quad \{\lambda_i, \mathbf{c}_j\} = \frac{1}{m}\delta_{ij}\lambda_i\mathbf{c}_j\,,\\
\{\mathbf{c}_j, \mathbf{c}_i\} =& \delta_{(i\neq j)}\frac{\lambda_j^m+\lambda_i^m}{\lambda_j^m-\lambda_i^m}\mathbf{c}_i\mathbf{c}_j + (Z_{m-1})_{ij}\mathbf{c}_i - (Z_{m-1})_{ji}\mathbf{c}_j\,.
\end{aligned}$$

Using (5.10), we have in this case (we remove the hat from the notation)

$$(Z_r)_{ij} = \frac{t_r}{t_{-1}}\frac{\lambda_i^{m-r-1}\lambda_j^{r+1}}{\lambda_i^m - t\lambda_j^m} g_{ij}^0 = t_r\left(\frac{\lambda_i}{\lambda_j}\right)^{m-r-1}\frac{\lambda_j^m}{\lambda_i^m - t\lambda_j^m}\mathbf{c}_j\,, \quad r \in I\,.$$

Using the case $r = m-1$ with $t = t_{m-1}$, this allows us to get after simplification

$$\{\mathbf{c}_j, \mathbf{c}_i\} = (1-t)^2\delta_{(i\neq j)}\frac{(\lambda_j^m+\lambda_i^m)\lambda_i^m\lambda_j^m}{(\lambda_j^m-\lambda_i^m)(\lambda_i^m-t\lambda_j^m)(\lambda_j^m-t\lambda_i^m)}\mathbf{c}_i\mathbf{c}_j\,.$$

By doing so, we have proved the following result.

Lemma 5.3.3 *Introduce the functions* $x_i = \lambda_i^m$, $\nu_i' = \mathbf{c}_i$. *Then, the Poisson bracket on* (x_i, ν_i') *takes the form* (4.16a)–(4.16c). *The same is true for* (x_i, ν_i) *with* $\nu_i = (1-t)\mathbf{c}_i\lambda_i$.

This lemma implies that there exists a connection to the phase space $\mathcal{C}'_{n,t}$ of the (non-spin) trigonometric RS system introduced in § 4.2.1 (with parameter $t = \prod_s q_s$), and that the two spaces are locally diffeomorphic as Poisson manifolds. In fact, this diffeomorphism is globally defined between $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)^\circ$ and $\mathcal{C}^\circ_{n,t}$, see the proof of Proposition 5.2.1 (and [41, Appendix C] to show that the isomorphism is Poisson). As another consequence of this lemma, we can obtain log-canonical coordinates (x_i, σ_i) as in (4.22).

Let us precise the relation between the integrable systems on $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)$ and the non-spin RS system in a way similar to [41, §4.2]. To do so, it is convenient to break the symmetry between the matrices $X_0,\ldots,X_{m-1}$ by acting with the element $g_X = (\Lambda^{-m},\Lambda^{-m+1},\ldots,\Lambda^{-1})$, $\Lambda = \operatorname{diag}(\lambda_1,\ldots,\lambda_n)$, as in (5.7). We get that $X_s = \operatorname{Id}_n$ for $0\leq s<m-1$, while $X_{m-1}=A$, where $A=\Lambda^m = \operatorname{diag}(x_1,\ldots,x_n)$. Moreover, from the expression of Z_r given above, we get after acting with g_X that

$$Z_r = t_r B\,, r\neq m-1\,,\quad Z_{m-1}=t_{m-1}A^{-1}B\,,\qquad B_{ij}=(1-t)\frac{\nu_j x_j}{x_i - t x_j}\,.$$

The matrices (A,B) are just the elements (X,Z) in (4.15). In particular, this allows us to write down the Hamiltonian of interest in terms of the Lax matrix and the diagonal matrix of positions for the trigonometric RS system.

First, we note that

$$F_{r,k}^{m,d'} = \operatorname{tr}(\operatorname{Id}_n + X_r Y_r)^k = \operatorname{tr}(X_r Z_r)^k = t_r^k \operatorname{tr} B^k\,, \tag{5.67}$$

are just multiples of the functions $G_k^{1,1} = \operatorname{tr} B^k$ given in (4.25). This is precisely the family of the trigonometric RS system which we have already studied.

Next, we have

$$G_k^{m,d'} = \operatorname{tr} Z^{km} = m\, t_{m-1}^k \ldots t_0^k \operatorname{tr}(A^{-1}B^m)^k\,. \tag{5.68}$$

Up to normalisation and using (4.22), we can write $G_1^{m,1}$ as

$$G_1^{m,1} = \sum_{1\leq j_0,\ldots,j_{m-1}\leq n} (\sigma_{j_0}\ldots\sigma_{j_{m-1}}) x_{j_0}^{-1} \prod_{s=0}^{m-1} \frac{t-1}{t - x_{j_s}x_{j_{s+1}}^{-1}} \prod_{s=0}^{m-1}\prod_{a\neq j_s}^{n} \Upsilon_{j_s a}\,, \tag{5.69}$$

where we have set

$$\Upsilon_{ij} = \frac{1-t x_i x_j^{-1}}{1-x_i x_j^{-1}} = \frac{t - x_j x_i^{-1}}{1-x_j x_i^{-1}}\,.$$

It is shown in [41] for $k=1$, and in [34] for any $k\geq 1$, that the (normalised) Hamiltonian $G_k^{m,d'}$ is the quasi-classical limit of the Hamiltonian operator of the twisted Macdonald–Ruijsenaars system in type A_{n-1} defined in [40]. Moreover, the work of Braverman, Etingof and Finkelberg [34] gives a systematic approach to obtain the quantisation of the above integrable systems, as well as the one based on Y which we now describe.

Using that $Y_s = Z_s - X_s^{-1}$, we can find

$$Y_{m-1}\ldots Y_0 = \left(t_{m-1}A^{-1}B - A^{-1}\right)(t_{m-2}B-\operatorname{Id}_n)\ldots(t_0 B - \operatorname{Id}_n)\,.$$

Introducing the matrix polynomial

$$\mathbf{P}(B) = \prod_{s\in I}(B - t_s^{-1}\,\mathrm{Id}_n)\,, \tag{5.70}$$

we can then write

$$H_k^{m,d'} = \operatorname{tr} Y^{km} = m\, t_{m-1}^k \dots t_1^k \, \operatorname{tr}\left(A^{-1}\mathbf{P}(B)\right)^k \,. \tag{5.71}$$

This can be seen as a generalisation of the family $G_k^{m,d'}$. Indeed, since each t_s contains a factor q_0, we get in the limit $q_0 \to \infty$ that $H_k^{m,d'}$ degenerate to $G_k^{m,d'}$ (after normalisation to get $\operatorname{tr}(A^{-1}B^m)^k$). We can also rewrite the functions $H_1^{m,d'}$ in terms of $\{G_1^{l,d'} \mid 0 \le l \le m\}$ after introducing $G_1^{0,d'} = m \operatorname{tr} A^{-1}$. As an example, we have for $m = 2$

$$H_1^{2,d'} = G_1^{2,d'} - \left(1 + \frac{t_1}{t_0}\right) G_1^{1,d'} + G_1^{0,d'} \,. \tag{5.72}$$

In the case $m = 1$, the analogues of these Hamiltonians are related to the qKP hierarchy, see § 4.2.2. In the case $m \ge 2$, a similar relation holds between $H_k^{m,d'}$ and a version of the qKP hierarchy with $\mathbb{Z}_m$-symmetry. See Section 6.1 for further explanations.

So far, we have obtained explicit forms for the Hamiltonians in terms of generic matrices (A, B) satisfying $\operatorname{rank}[ABA^{-1}B^{-1} - t\,\mathrm{Id}_n] = 1$ and A is diagonalised. We could, instead, decide to diagonalise B as described in Remark 4.2.15. In that case, the Hamiltonians $E_k^{m,d'} = \operatorname{tr} X^k$, $F_k^{m,d'} = \operatorname{tr}(1_I + XY)^k = \sum_r F_{r,k}^{m,d'}$ and $H_k^{m,d'}$ are the quasi-classical limit of the generators of the quantised Coulomb branch of a framed quiver gauge theory of Jordan type as explained in [98], see also [35, 36]. Finally, the Hamiltonians $G_k^{m,d'}$ can be seen as the quasi-classical limits of generalised Macdonald operators introduced in [59]. The interested reader can find the details for these claims in [34, 41].

5.3.3 Multiple framings : the spin case

Similarly to § 5.3.2, we will identify locally the MQV $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)^\circ$ in the case $m \ge 2$, $\mathbf{d} = (d, 0, \dots, 0)$, with the MQV $\mathcal{C}_{n,t,d}^\circ$ in the case $m = 1$, $d \in \mathbb{N}^\times$, treated in Section 4.3. To do so, recall that we have local coordinates $(\lambda_i, \mathbf{a}_i^{0,\alpha}, \mathbf{c}_i^{0,\alpha})$ on $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)'$ under the constraint $\sum_{\alpha=1}^d \mathbf{a}_i^{0,\alpha} = 1$. Their Poisson structure is defined in Proposition 5.2.7.

Lemma 5.3.4 *The Poisson bracket given on the elements* $(\lambda_i^m, \mathbf{a}_i^{0,\alpha}, \mathbf{c}_i^{0,\alpha})$ *is the same as the Poisson bracket of* $\mathfrak{h}_{sp}/S_n$ *defined in Proposition 4.3.3. The same is true for the elements* $(\lambda_i^m, \mathbf{a}_i^{0,\alpha}, \mathbf{c}_i^{0,\alpha}\lambda_i)$.

Proof. If we set $x_i = \lambda_i^m$, $\mathbf{a}_i^\alpha = \mathbf{a}_i^{0,\alpha}$ and $\mathbf{c}_j^\alpha = \mathbf{c}_j^{0,\alpha}$, then the entries of the matrix Z_{m-1} given by (5.10) become

$$(Z_{m-1})_{ij} = t\frac{\sum_\alpha \mathbf{a}_i^\alpha \mathbf{c}_j^\alpha x_j}{x_i - tx_j}\,.$$

This is exactly (4.28) for t instead of q. Now, it suffices to see that (5.17) is equivalent to (4.31a), while (5.29), (5.24), (5.18) yield (4.31b)–(4.31d).

Fix $p \in \mathbb{Z}$, $q \in \mathbb{N}^\times$. The transformation $\mathbf{c}_i^\alpha \mapsto \mathbf{c}_i^\alpha x_i^{p/q}$, for all $i = 1, \ldots, n$, preserves (4.31a)–(4.31d). The second part of the statement follows directly. □

As a consequence of the lemma, there is a local Poisson diffeomorphism $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)' \to \mathcal{C}'_{n,t,d}$ given by setting $x_i = \lambda_i^m$, $\mathbf{a}_i^\alpha = \mathbf{a}_i^{0,\alpha}$ and $\mathbf{c}_i^\alpha = \mathbf{c}_i^{0,\alpha}\lambda_i$. Note that $f_{ij} = \sum_\alpha \mathbf{a}_i^\alpha \mathbf{c}_j^\alpha = g_{ij}^0\lambda_j$. Hence, we can use the coordinates $(x_i, \mathbf{a}_i^\alpha, \mathbf{c}_i^\alpha)$ on $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)'$ to write the integrable systems obtained in § 5.2.5. In particular, we would like to investigate the relation with the trigonometric spin RS system.

Local form of the Hamiltonians

As in the case $d = 1$ given in § 5.3.2, it is easier to write the matrices $(X_s, Z_s, \mathbf{A}^0, \mathbf{C}^0)$ in terms of $(A, B, \mathbf{A}, \mathbf{C})$, where

$$A = \operatorname{diag}(x_1, \ldots, x_n)\,, \quad B_{ij} = t\frac{f_{ij}x_j}{x_i - tx_j}\,, \quad \mathbf{A}_{i\alpha} = \mathbf{a}_i^\alpha\,, \quad \mathbf{C}_{\alpha j} = \mathbf{c}_j^\alpha\,, \tag{5.73}$$

parametrise a point of $\mathcal{C}'_{n,t,d}$ by § 4.3.1. Starting from the gauge in (5.8), we act with the element $g_X = (\Lambda^{-m}, \Lambda^{-m+1}, \ldots, \Lambda^{-1})$, $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. We get that $X_s = \operatorname{Id}_n$ for $0 \leq s < m-1$, while $X_{m-1} = A$. From the expression (5.10) of Z_r, we get after acting with g_X that

$$Z_r = t_r B\,, r \neq m-1\,, \quad Z_{m-1} = t_{m-1}A^{-1}B\,.$$

Furthermore, $\mathbf{A}^0 = A^{-1}\mathbf{A}$ and $\mathbf{C}^0 = \mathbf{C}$. Though we did not prove it, this parametrisation is not only local but it comes from a global Poisson diffeomorphism $\mathcal{C}^\circ_{n,t,d} \to \mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)^\circ$ by [62, §5.1].

We now investigate the Hamiltonians of interest and we follow [62, §5.2]. First, we remark that

$$F_{r,k}^{m,\mathbf{d}} = \operatorname{tr}(\mathrm{Id}_n + X_r Y_r)^k = \operatorname{tr}(X_r Z_r)^k = t_r^k \operatorname{tr} B^k \,, \tag{5.74}$$

are just multiples of the functions $G_k^{1,d} = \operatorname{tr} B^k$ given in (4.50). This is precisely the family of the trigonometric spin RS system which we have already studied.

Next, we have

$$G_k^{m,\mathbf{d}} = \operatorname{tr} Z^{km} = m\, t_{m-1}^k \dots t_0^k \operatorname{tr}(A^{-1}B^m)^k \,,$$

which directly generalises the non-spin case given by (5.69). Up to normalisation of the local expressions, we can write

$$G_k^{m,\mathbf{d}} = \sum_{i_1,\dots,i_{km}=1}^{n} \left(\prod_{a=1}^{km} \frac{f_{i_a i_{a+1}} x_{i_{a+1}}}{x_{i_a} - t x_{i_{a+1}}} \right) \left(\prod_{0 \le s \le k-1} x_{i_{sm+1}}^{-1} \right) .$$

Finally, we can get the spin analogue of (5.71) as

$$H_k^{m,\mathbf{d}} = \operatorname{tr} Y^{km} = m\, t_{m-1}^k \dots t_1^k \operatorname{tr}\left(A^{-1}\mathbf{P}(B)\right)^k \,,$$

where we use the matrix polynomial given in (5.70). This allows us to write

$$H_k^{m,\mathbf{d}} = \sum_{i_1,\dots,i_{jm}} \prod_{a=0}^{j-1} x_{i_{a(m+1)+1}} \prod_{s=1}^{m} \left(\frac{x_{i_{am+s+1}} f_{i_{am+s} i_{am+s+1}}}{x_{i_{am+s}} - t x_{i_{am+s+1}}} - t_{s-1}^{-1} \delta_{(i_{am+s}, i_{am+s+1})} \right) .$$

This is also the case for $d \ge 2$ that the family $(H_k^{m,\mathbf{d}})$ generalises the family $(G_k^{m,\mathbf{d}})$, and that we can rewrite the functions $H_1^{m,\mathbf{d}}$ in terms of $\{G_1^{l,\mathbf{d}} \mid 0 \le l \le m\}$. For example, (5.72) holds without any change (except that we use the matrices (5.73)).

To the best of the author's knowledge, all these Hamiltonians appear to be new. At the same time they are straightforward generalisations of the Hamiltonians for the non-spin case $d = 1$ obtained in § 5.3.2 : it suffices to replace the Lax matrix for the trigonometric non-spin RS system B by its spin versions. Hence, it would be interesting to see which connections to other topics in mathematics, such as the ones reviewed in § 5.3.2, could be extended to the cases $d \ge 2$.

More on integrability

So far, we have not used Proposition 5.1.4 in our discussion of integrability for MQVs associated to cyclic quivers. The reason is that it is a more subtle statement to consider. Due to the local isomorphism $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)' \to \mathcal{C}'_{n,t,d}$ in this case $\mathbf{d} = (d, 0, \ldots, 0)$, it is expected that we need an extra reduction as performed in §4.3.3 when $n \geq d$ to obtain an analogue to Theorem 4.3.11. This is indeed true, and we only state the result, leaving to the reader the task to prove the missing steps based on Lemmae 4.3.10, 4.3.12 and 4.3.13. For the complete proof, see [62, §5.4,5.5].

Recall that we introduced in (4.57) the Lie subgroup $\mathcal{H}$ of $\mathrm{GL}_d(\mathbb{C})$ whose elements have the vector $(1, \ldots, 1)$ as eigenvector with eigenvalue 1. Moreover, introduce an $\mathcal{H}$-action on $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)'$ which acts on the $(2m+2)$-uple $(X_s, Z_s, \mathbf{A}^0, \mathbf{C}^0)$ by $h \cdot (X_s, Z_s, \mathbf{A}^0, \mathbf{C}^0) = (X, Z, \mathbf{A}^0 h, h^{-1}\mathbf{C}^0)$. We form the GIT quotient $\mathcal{C}^{\mathcal{H}}_{n,m} = \mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)'//\mathcal{H}$. Fix $U = X, Y, Z, 1_I + XY$.

Theorem 5.3.5 *Consider the functions $\{h^u_{K,k}\}$ defined in Proposition 5.1.4. Among them, we can pick a subset of $nd - \frac{1}{2}d(d-1)$ elements which define a completely integrable system on the smooth locus of $\mathcal{C}^{\mathcal{H}}_{n,m}$.*

Some of these additional Hamiltonians can also be written down locally, see [62, §5.2].

5.3.4 Multiple framings : the $m = 2$ case

Consider the MQV $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)$ in the case $m = 2$, $\mathbf{d} = (d_0, d_1) \in \mathbb{N}^\times \times \mathbb{N}$. If $d_1 = 0$, this corresponds to the case $m = 2$ discussed in §5.3.3. On $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)'$, we have local coordinates $(\lambda_i, \mathbf{a}_i^{s,\alpha}, \mathbf{c}_i^{s,\alpha})$,for (s, α) with $s = 0, 1$ and $1 \leq \alpha \leq d_s$, under the constraints $\sum_{\alpha=1}^{d_0} \mathbf{a}_i^{0,\alpha} = 1$. The Poisson bracket evaluated on these elements can be obtained from Proposition 5.2.7.

The eight matrices $(X_s, Z_s, \mathbf{A}^s, \mathbf{C}^s)$, $s = 0, 1$, which determine a point of $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)'$ can be written down in terms of the local coordinates in the gauge (5.8). We have in particular that $X_0 = X_1 = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ while by (5.10)

$$(Z_0)_{ij} = t_0 \frac{\lambda_i}{\lambda_i^2 - t\lambda_j^2} f^0_{ij} + t \frac{\lambda_j}{\lambda_i^2 - t\lambda_j^2} f^1_{ij}\,, \quad (Z_1)_{ij} = t \frac{\lambda_j}{\lambda_i^2 - t\lambda_j^2} f^0_{ij} + \frac{t}{t_0} \frac{\lambda_i}{\lambda_i^2 - t\lambda_j^2} f^1_{ij}\,,$$

where we have set $f^s_{ij} = g^s_{ij}\lambda_j = \sum_{\alpha=1}^{d_s} \mathbf{a}_i^{s,\alpha}\mathbf{c}_j^{s,\alpha}\lambda_j$. We also set $f^1_{ij} = 0$ when $d_1 = 0$. If we consider a square root $\sqrt{t} \neq 0$, we can write

$$\frac{\lambda_i}{\lambda_i^2 - t\lambda_j^2} = \frac{1}{2}\frac{1}{\lambda_i - \sqrt{t}\lambda_j} + \frac{1}{2}\frac{1}{\lambda_i + \sqrt{t}\lambda_j}\,, \quad \frac{\lambda_j}{\lambda_i^2 - t\lambda_j^2} = \frac{1}{2\sqrt{t}}\frac{1}{\lambda_i - \sqrt{t}\lambda_j} - \frac{1}{2\sqrt{t}}\frac{1}{\lambda_i + \sqrt{t}\lambda_j}\,,$$

and use these expressions for the entries of Z_0, Z_1. We do not write the Hamiltonians for $F^{2,\mathbf{d}}_{s,k}$ as they are not very different from the general discussion in § 5.3.1. For the functions $G^{2,\mathbf{d}}_k = m\,\mathrm{tr}(Z_1 Z_0)^k$ and $H^{2,\mathbf{d}}_k = m\,\mathrm{tr}(Y_1 Y_0)^k$ where $m = 2$, the expressions for $k = 1$ are not too cumbersome. First, we have

$$G^{2,\mathbf{d}}_k = \frac{\sqrt{t}}{2t_0} \sum_{i,j=1}^{n} \left[\frac{t_0 f^0_{ij} + \sqrt{t} f^1_{ij}}{\lambda_i - \sqrt{t}\lambda_j} + \frac{t_0 f^0_{ij} - \sqrt{t} f^1_{ij}}{\lambda_i + \sqrt{t}\lambda_j} \right] \left[\frac{t_0 f^0_{ji} + \sqrt{t} f^1_{ji}}{\lambda_j - \sqrt{t}\lambda_i} - \frac{t_0 f^0_{ji} - \sqrt{t} f^1_{ji}}{\lambda_j + \sqrt{t}\lambda_i} \right] .$$

Indeed, the first factor is $2(Z_0)_{ij}$ and the second is $\frac{2t_0}{\sqrt{t}}(Z_1)_{ji}$. Next, using that $Y_s = Z_s - X_s^{-1}$, we find

$$H^{2,\mathbf{d}}_k = G^{2,\mathbf{d}}_k - \frac{2}{1-t} \sum_{i=1}^{n} \left[(t_0 - t)\frac{f^0_{ii}}{\lambda_i^2} + \left(t + \frac{t}{t_0} \right) \frac{f^1_{ii}}{\lambda_i^2} \right] + \sum_{i=1}^{n} \frac{1}{\lambda_i^2} .$$

We can see the $W = S_n \ltimes \mathbb{Z}_2^n$-invariance of both Hamiltonians from these expressions. The S_n-invariance is obvious. The $\mathbb{Z}_2^n$ action by an element $(k_1, \ldots, k_n)$ is given by Remark 5.2.4 as $\lambda_i \mapsto (-1)^{k_i}\lambda_i$, $f^0_{ij} \mapsto f^0_{ij}$, and $f^1_{ij} \mapsto (-1)^{k_j - k_i} f^1_{ij}$. Thus, each factor in the terms constituting $G^{2,\mathbf{d}}_k$ is multiplied by $(-1)^{k_i}$ for such a transformation, so each term of $G^{2,\mathbf{d}}_k$ is $\mathbb{Z}_2^n$-invariant. The invariance of $H^{2,\mathbf{d}}_k$ directly follows. When we restrict to the case $d_1 = 0$, we recover the Hamiltonians written down in [42, 5.6.4] up to a multiplicative factor.

Chapter 6

Concluding remarks

6.1 New perspectives on integrable systems

In this thesis, we outlined a way of obtaining the completed phase space for the trigonometric spin RS system as a MQV, and we proved integrability of the system. We were also able to formulate the Poisson bracket in terms of local coordinates, which answered a conjecture of Arutyunov and Frolov [16]. Furthermore, we generalised these results to numerous new systems of RS type with trigonometric potential.

Our method relied on Van den Bergh's theory of double quasi-Poisson brackets [162, 163], which we applied to extended cyclic quivers. From this point of view, a natural question consists in determining for which quivers we can obtain integrable systems. This question extends the analogous issue that was formulated in the Hamiltonian case [43, 125], but is still unanswered. In fact, it is not even known which choices of dimension vector for the cyclic quiver (other than the ones in Section 5.2) allow to obtain an integrable system. Noticing that the flows associated to particular functions can be obtained by the projection method (see Propositions 5.1.5–5.1.8) for any dimension vector $\widetilde{\alpha} = (1, \alpha)$ and generic parameter $\widetilde{q} = (\mathbf{q}^{-\alpha}, \mathbf{q})$, we are led to believe that it should be possible to form an integrable system containing any function for which we can explicitly integrate the flows in that way (as far as the flows are defined in a non-empty subset of the corresponding MQV). A possible step in that direction could be to use the reflection functor, as observed in the Hamiltonian case by Silantyev [149]. It would also be important to investigate if the regularity condition on $\mathbf{q}$ that is considered in Proposition 2.3.28 can be relaxed. Indeed, we would like to allow the parameter q considered in Chapter 4 to take the value of some roots of unity, which is forbidden at the moment due to Proposition 4.1.1.

There have been some recent advances in the understanding of (real) RS systems for different kinds of potentials and root systems using a reduction picture, following works of Fehér and co-authors (see e.g. [68, 69, 71, 72, 74]). It would be interesting to know if their complex versions can be obtained from (quasi-)Hamiltonian algebras. Assuming that the latter is possible, and since systems of CM or RS type inherit some duality properties [141, 142, 143] (see also [79, 87], and for some recent developments obtained by reduction [63, 66, 67, 69, 70, 74, 75, 135, 136, 137, 139]), it would be relevant to understand dualities in terms of their underlying algebras. The author made a first step in that direction in [62] for the self-duality of the trigonometric spin RS system. Conversely, we hope that the present thesis can prove to be useful in deriving the real analogues of the systems that we introduced. In fact, the author believes that this work has already contributed to revive some interest for spin RS systems, see [64, 65, 73].

There are multiple perspectives that we will not mention, though they may be relatively important[21]. Rather, we will focus on some specific directions of research, which the author has attempted to follow (at least partially) since the beginning of his PhD in September 2015.

Additional reductions

Recall that in Theorem 4.3.11 (for the trigonometric spin RS system and some modifications) and in Theorem 5.3.5 (for similar systems with additional $\mathbb{Z}_m$-symmetry) we obtained integrable systems not directly at the level of a MQV, but after an extra reduction. We conjecture that this procedure extends to framed cyclic quivers as follows.

Under the notations of Section 5.2, assume that $\mathbf{d} \in \mathbb{N}^I$ is such that $0 \leq d_s \leq n$ for all $s \in I$, with $d_0 \geq 1$ and where n denotes the dimension of the vector spaces $(\mathcal{V}_s)_{s\in I}$. As in § 5.2.1, we consider the subspace $\mathcal{C}_{n,\mathbf{q},\mathbf{d}}(m)^\circ$ of the MQV which we denote simply as $\mathcal{C}^\circ_{n,m}$. It is characterised by the linear data $(X, Z, \mathbf{A}^s, \mathbf{C}^s)$, where $\mathbf{A}^s \in \operatorname{Mat}_{n\times d_s}(\mathbb{C})$ and $\mathbf{C}^s \in \operatorname{Mat}_{d_s\times n}(\mathbb{C})$. We can further restrict to the subspace $\mathcal{C}'_{n,m}$ from Remark 5.2.2, for which we can pick at each point a representative $(\hat{X}, \hat{Z}, \hat{\mathbf{A}}^s, \hat{\mathbf{C}}^s)$ satisfying $\hat{X}_s = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ for each $s \in I$, and where

[21] For example, the possibility of obtaining the quantum version of this work using [93].

$\sum_{\alpha=1}^{d_0} \hat{\mathbf{A}}^0_{i\alpha} = +1$ for each $1 \leq i \leq n$. Introduce the algebraic group

$$\mathcal{H}_0 = \left\{ h = (h_{\alpha\beta}) \in \mathrm{GL}_{d_0}(\mathbb{C}) \,\middle|\, \sum_{\beta=1}^{d_0} h_{\alpha\beta} = 1 \text{ for all } \alpha \right\}, \tag{6.1}$$

and set $\mathcal{H}_s = \mathrm{GL}_{d_s}(\mathbb{C})$ for each $s \in I \setminus \{0\}$. When $d_s = 0$, we set $\mathcal{H}_s = \{1\}$. If we now consider the group $\mathcal{H} = \prod_s \mathcal{H}_{d_s}$, we can define the operation

$$(h_s) \cdot (\hat{X}, \hat{Z}, \hat{\mathbf{A}}^s, \hat{\mathbf{C}}^s) = (\hat{X}, \hat{Z}, \hat{\mathbf{A}}^s h_s, h_s^{-1} \hat{\mathbf{C}}^s) , \quad (h_s) \in \mathcal{H} ,$$

which yields an action of $\mathcal{H}$ onto $\mathcal{C}'_{n,m}$. We can then form the GIT quotient $\mathcal{C}^{\mathcal{H}}_{n,m} = \mathcal{C}'_{n,m}//\mathcal{H}$, on which the functions $(\lambda_i, g^s_{ij} = \sum_{\alpha=1}^{d_s} \mathbf{a}^{s\alpha}_i \mathbf{c}^{s\alpha}_j)$ are well-defined. Assuming that they generate the coordinate ring (up to localisation), we can define uniquely a Poisson bracket $\{-,-\}^{\mathcal{H}}$ on $\mathcal{C}^{\mathcal{H}}_{n,m}$ since the commutative algebra generated by the elements (λ_i, g^s_{ij}) is Poisson using Proposition 5.2.30.

Conjecture 6.1.1 *For each $u \in \{x, y, z, 1_I + xy\}$, we can find in the family of functions $(h^u_{K,k})$ defined in Proposition 5.1.4 exactly $n|\mathbf{d}| - \frac{1}{2}\sum_{s \in I} d_s(d_s - 1)$ functionally independent elements. These elements descend to an integrable system on the smooth part of $\mathcal{C}^{\mathcal{H}}_{n,m}$.*

The assumption that $0 \leq d_s \leq n$ for all $s \in I$ is motivated by Lemmae 4.3.12 and 4.3.13. These results suggest that we have to work on a subspace where each $\mathbf{A}^s$ has all its $d_s \times d_s$ minors which are invertible.

Integrable hierarchies

It was first noted in [2, 48] that the positions of the poles of some $(1 + 1)$-dimensional integrable partial differential equations (such as the Korteweg-de Vries equation) evolve over time as classical particles, which interact as a system of CM type. This was extended to the $(2 + 1)$-dimensional KP equation and CM systems with arbitrary potential [101, 102], then to the multicomponent (or matrix) KP equation with spin CM system [104]. More importantly for us, Wilson understood in the rational case that this correspondence holds for the whole KP hierarchy [170]. In its turn, this was generalised to new versions of the multicomponent KP hierarchy associated to systems of rational CM type [43], which are constructed from extended cyclic quivers using Van den Bergh's Hamiltonian formalism.

In works in progress with O. Chalykh, the author considers the analogue of these constructions in the quasi-Hamiltonian case, i.e. for some of the systems considered in this thesis. We can describe such a relation for the Hamiltonians introduced in § 4.2.2 : the flows of $(G_k^{1,1})_k$ describe the motion of the poles of some solutions to the $2D$ Toda hierarchy, and the same holds for $(H_k^{1,1})_k$ and the qKP hierarchy. The latter generalises the work of Iliev [91]. In the case $m \geq 2$ of a cyclic quiver, a similar relation holds between $(G_k^{m,d'})_k$ (resp. $(H_k^{m,d'})_k$) given in § 5.3.2 and a version of the $2D$ Toda (resp. qKP) hierarchy with $\mathbb{Z}_m$-symmetry. Moreover, we can partially generalise the above cases to multicomponent hierarchies using the Hamiltonians $(G_k^{1,d})_k$ or $(H_k^{1,d})_k$ from § 4.3.3, and get an additional $\mathbb{Z}_m$-symmetry using their analogues $(G_k^{m,\mathbf{d}})_k, (H_k^{m,\mathbf{d}})_k$ (which are defined in full generalities in § 5.3.1) for particular extensions of cyclic quivers. The case $m = 1, d > 1$ then generalises the work of Krichever and Zabrodin [105]. All these results can be seen as an extension of [43] to the q-case.

We give a sketch of the method in the simplest case of the multicomponent hierarchy, without an additional $\mathbb{Z}_m$-symmetry. The idea is similar to [43, Section VI].

We consider $(\tau, \hbar) \in \mathbb{C}^\times \times \mathbb{C}$ and define $A_\tau^\hbar = \mathbb{C}\langle x, y\rangle/(xy - \tau yx - \hbar)$, which is an associative algebra. For $(\tau, \hbar) = (1, 1)$, $A_\tau^\hbar$ is the first Weyl algebra. For $\hbar = 0$ and $\tau \neq 1$, $A_\tau^\hbar$ gives after localisation at x and y the coordinate ring of a quantum torus. This is nothing else than $A'/(e_\infty, \Phi_0 - \tau e_0)$ for A' given in § 3.1.2. If $\hbar = \tau - 1$, $A_\tau^\hbar$ is the first quantised Weyl algebra which, after localisation at $1 + xy$ is just $A/(e_\infty, \Phi_0 - \tau e_0)$ for A given in § 3.1.1. We then introduce the associative algebra

$$\mathcal{P}_\tau^\hbar = \left\{ \sum_{i=-N}^{\infty} p_i(x) y^{-i} \middle| p_i(x) \in \mathbb{C}(x),\, N \in \mathbb{N} \right\}$$

with multiplication induced by that on $A_\tau^\hbar$, i.e. it is a consequence of $xy - \tau yx - \hbar$.

We now fix some $d \in \mathbb{N}^\times$. Let $\mathcal{P} = \mathfrak{gl}_d(\mathbb{C}) \otimes \mathcal{P}_\tau^\hbar$, where we denote $g \otimes 1$ by g for any $g \in \mathfrak{gl}_d(\mathbb{C})$, and $\mathrm{Id}_d \otimes p$ by p for any $p \in \mathcal{P}_\tau^\hbar$. In $\mathcal{P}$, we consider a $(d+1)$-uple $(L, R_\alpha)_\alpha$ of the form

$$L = y + \sum_{i=0}^{\infty} l_i(x) y^{-i} \,, \quad R_\alpha = E_\alpha + \sum_{i=1}^{\infty} r_{\alpha,i}(x) y^{-i} \,, \; 1 \leq \alpha \leq d.$$

Here, $l_i(x), r_{\alpha,i}(x) \in \mathfrak{gl}_d(\mathbb{C}) \otimes \mathbb{C}(x) \subset \mathcal{P}$, while $E_\alpha = E_{\alpha\alpha}$ is an elementary matrix (recall that $(E_{\alpha\beta})_{\gamma\epsilon} = \delta_{\alpha\gamma}\delta_{\beta\epsilon}$). Moreover, we restrict our attention to such $(L, R_\alpha)_\alpha$ satisfying

$$[L, R_\alpha] = 0 \,, \quad R_\alpha R_\beta = \delta_{\alpha\beta} R_\alpha \,, \quad \sum_{\alpha=1}^{d} R_\alpha = \mathrm{Id}_d \,. \tag{6.2}$$

Using the decomposition of an arbitrary $T = \sum_{i=-N}^{\infty} t_i(x)y^{-i} \in \mathcal{P}$ as $T = T_+ + T_-$ for $T_+ = \sum_{i=-N}^{0} t_i(x)y^{-i}$, $T_- = \sum_{i=1}^{\infty} t_i(x)y^{-i}$, we define the *multicomponent* $(\tau, \hbar)$*-hierarchy* as

$$\frac{\partial L}{\partial t_{k\beta}} = [(L^k R_\beta)_+, L] \,, \quad \frac{\partial R_\alpha}{\partial t_{k\beta}} = [(L^k R_\beta)_+, R_\alpha] \,, \tag{6.3}$$

for time variables $t_{k\beta}$, with $k \in \mathbb{N}$ and $1 \leq \beta \leq d$. These derivations pairwise commute, and we seek solutions $(L, R_\alpha)_\alpha$ of (6.3) satisfying (6.2). Note that in the commutative case $(\tau, \hbar) = (1, 0)$ any commutator vanishes and the hierarchy becomes trivial, so we omit this case from now on.

Lemma 6.1.2 *Let* $\Psi = 1 + \sum_{i=1}^{\infty} \psi_i(x)y^{-i}$. *Form* $L = \Psi y \Psi^{-1}$ *and* $R_\alpha = \Psi E_\alpha \Psi^{-1}$. *If* $\frac{\partial \Psi}{\partial t_{k\beta}} = -(L^k R_\beta)_- \Psi$ *for any indices, then* $(L, R_\alpha)_\alpha$ *satisfies* (6.2) *and* (6.3).

Our aim is to construct elements Ψ satisfying Lemma 6.1.2. To do so, our first step is to consider the space $\mathcal{M}$ of matrices $(X, Y, i_\alpha, j_\alpha)_\alpha$, where $X, Y \in \mathfrak{gl}_n(\mathbb{C})$ and $i_\alpha \in \mathrm{Mat}_{n\times 1}(\mathbb{C})$, $j_\alpha \in \mathrm{Mat}_{1\times n}(\mathbb{C})$ for any $1 \leq \alpha \leq d$. We define the subspace $\mathcal{M}_0$ given by

$$\mathcal{M}_0 = \left\{ \tau XY - YX + \hbar \, \mathrm{Id}_n + \sum_{\alpha=1}^{d} i_\alpha j_\alpha = 0_n \right\} \subset \mathcal{M} \,,$$

and we will use the corresponding moduli space $\mathcal{M}_{red} = \mathcal{M}_0 /\!/ \mathrm{GL}_n(\mathbb{C})$, where we use the action $g \cdot (X, Y, i_\alpha, j_\alpha) = (gXg^{-1}, gYg^{-1}, gi_\alpha, j_\alpha g^{-1})$ for any $g \in \mathrm{GL}_n(\mathbb{C})$.

Meanwhile, we consider on $\mathcal{P}_\tau^\hbar \otimes \mathrm{End}(\mathbb{C}^n)$ the elements

$$\hat{X}_\gamma = 1 \otimes X - \gamma x \otimes \mathrm{Id}_n \,, \quad \hat{Y}_\gamma = 1 \otimes \gamma Y - y \otimes \mathrm{Id}_n \,, \quad \gamma \in \mathbb{C}^\times \,,$$

for $X, Y \in \mathfrak{gl}_n(\mathbb{C})$, and we put $\hat{X} = \hat{X}_1$, $\hat{Y} = \hat{Y}_1$. Up to completing the algebra to introduce formal sums, we can also consider

$$\hat{X}_\gamma^{-1} = -\sum_{k \geq 0} (\gamma x)^{-k-1} \otimes X^k \,, \quad \hat{Y}_\gamma^{-1} = -\sum_{k \geq 0} y^{-k-1} \otimes \gamma^k Y^k \,.$$

We now denote $1 \otimes i$ as i for any $i \in \mathrm{Mat}_{n\times 1}(\mathbb{C})$, and $1 \otimes j$ as j for any $j \in \mathrm{Mat}_{1\times n}(\mathbb{C})$. This allows us to define the following elements of $\mathcal{P}$ from any point in $\mathcal{M}$

$$\Psi = 1 - \sum_{\alpha\beta} E_{\alpha\beta} \otimes j_\alpha \hat{X}_\tau^{-1} \hat{Y}^{-1} i_\beta \,, \quad \Psi' = 1 + \sum_{\alpha\beta} E_{\alpha\beta} \otimes j_\alpha \hat{Y}_\tau^{-1} \hat{X}^{-1} i_\beta \,. \tag{6.4}$$

Indeed, for each α, β we have that $j_\alpha \hat{X}_\tau^{-1} \hat{Y} i_\beta, j_\alpha \hat{Y}_\tau^{-1} \hat{X} i_\beta \in \mathcal{P}_\tau^\hbar$. We will be concerned with pairs (Ψ, Ψ') constructed from elements of $\mathcal{M}_0$, since in that case they are inverse to each other in $\mathcal{P}$. Hence, we will denote Ψ' by Ψ^{-1}. More importantly, as the expressions (6.4) are $\mathrm{GL}_n(\mathbb{C})$ invariant, we can associate a pair (Ψ, Ψ^{-1}) to any point of $\mathcal{M}_{red}$.

We now consider an element Ψ obtained from a point of $\mathcal{M}_{red}$ by (6.4), and we define $L = \Psi y \Psi^{-1}$ and $R_\alpha = \Psi E_\alpha \Psi^{-1}$.

Lemma 6.1.3 *The condition $\frac{\partial \Psi}{\partial t_{k\beta}} = -(L^k R_\beta)_- \Psi$ can be identified with the vector field on $\mathcal{M}_{red}$ given (as the image of the vector field of $\mathcal{M}_0$ given) by*

$$\frac{dY}{dt_{k\beta}} = 0\,, \quad \frac{di_\alpha}{dt_{k\beta}} = -\delta_{\alpha\beta} Y^k i_\alpha\,, \quad \frac{dj_\alpha}{dt_{k\beta}} = \delta_{\alpha\beta}\tau^k j_\alpha Y^k\,, \quad \frac{dX}{dt_{k\beta}} = -\sum_{a=0}^{k-1} \tau^{k-a-1} Y^a i_\beta j_\beta Y^{k-a-1}\,.$$

Furthermore, such vector fields commute in $\mathcal{M}_{red}$.

Combined with Lemma 6.1.2, this tells us that we can construct a solution $(L, R_\alpha)_\alpha$ to the hierarchy (6.3) from a point of $\mathcal{M}_{red}$. Moreover, the evolution under time $t_{k\beta}$ of this solution corresponds to the flow in $\mathcal{M}_{red}$ of the corresponding vector field given in Lemma 6.1.3. Due to this relation, we are interested in an explicit solution for the latter flow. We have not been successful to construct it in general, though in some cases it is possible. To state the results that can be obtained, we consider the derivation $\frac{\partial}{\partial t_k} = \sum_\beta \frac{\partial}{\partial t_{k\beta}}$ which is commuting with the whole $(\tau, \hbar)$-hierarchy.

Lemma 6.1.4 *The condition $\frac{\partial \Psi}{\partial t_k} = -(L^k)_- \Psi$ can be identified with the vector field on $\mathcal{M}_{red}$ given (as the image of the vector field of $\mathcal{M}_0$ given) by*

$$\frac{dY}{dt_k} = 0\,, \quad \frac{di_\alpha}{dt_k} = 0\,, \quad \frac{dj_\alpha}{dt_k} = (\tau^k - 1) j_\alpha Y^k\,, \quad \frac{dX}{dt_k} = (\tau^k - 1) X Y^k + \hbar \frac{\tau^k - 1}{\tau - 1} Y^{k-1}\,,$$

for $\tau \neq 1$, while for $\tau = 1$

$$\frac{dY}{dt_k} = 0\,, \quad \frac{di_\alpha}{dt_k} = 0\,, \quad \frac{dj_\alpha}{dt_k} = 0\,, \quad \frac{dX}{dt_k} = k\hbar Y^{k-1}\,.$$

For $(\tau, \hbar) = (1, 1)$, we get the KP hierarchy and $\mathcal{M}_{red}$ is the completed phase space of the rational spin CM system. We then recover from Lemma 6.1.4 a result of [43] which states that these vector fields are Hamiltonian for $H_k = \operatorname{tr} Y^k$.

A first new result is to consider $(\tau, \hbar) = (q^{-1}, q^{-1} - 1)$. We obtain that y is the operator defined by $yx = qxy + (q-1)$, so up to a scaling factor this is the q-difference operator D_q which satisfies $D_q x = qxD_q + 1$. Hence the hierarchy is a multicomponent version of the qKP hierarchy [88, 90, 91]. Meanwhile, we can write for the space that

$$\mathcal{M}_{red} = \left\{ (\mathrm{Id}_n + XY) = q(\mathrm{Id}_n + YX) - q\sum_\alpha i_\alpha j_\alpha \right\} /\!/ \, \mathrm{GL}_n(\mathbb{C}) \,,$$

so that, up to restricting to an open subspace, we can make $\mathcal{M}_{red}$ coincides with the space $\mathcal{C}_{n,q,d}$ considered in Chapter 4 after setting

$$i_\alpha = -W_\alpha \,, \quad j_\alpha = V_\alpha(\mathrm{Id}_n + W_{\alpha-1}V_{\alpha-1}) \dots (\mathrm{Id}_n + W_1V_1)(\mathrm{Id}_n + YX) \,, \tag{6.5}$$

for any $1 \leq \alpha \leq d$. The space $\mathcal{C}_{n,q,d}$ is a Poisson manifold, and we can see that the vector field associated to $(1-q^{-k})H_k^{1,d} = (1-q^{-k})\,\mathrm{tr}\, Y^k$ is given by (see Proposition 4.1.7)

$$\frac{dY}{dt_k} = 0 \,, \quad \frac{dW_\alpha}{dt_k} = 0 \,, \quad \frac{dV_\alpha}{dt_k} = 0 \,, \quad \frac{dX}{dt_k} = (q^{-k}-1)XY^k + (q^{-k}-1)Y^{k-1} \,.$$

It is not hard to see that it coincides with the vector field given in Lemma 6.1.4 with parameters $(\tau, \hbar) = (q^{-1}, q^{-1}-1)$. We can state the above remark as follows.

Proposition 6.1.5 *Consider the map $\mathcal{C}_{n,q,d} \to \mathcal{P} : (X, Y, V_\alpha, W_\alpha) \mapsto \Psi$ obtained from (6.4) and (6.5). Moreover, define $L = \Psi y \Psi^{-1}$, and $R_\alpha = \Psi E_\alpha \Psi^{-1}$ for $1 \leq \alpha \leq d$. Then the Hamiltonian flow given in Proposition 4.1.7 of a point in $\mathcal{C}_{n,q,d}$ under $(1-q^{-k})\,\mathrm{tr}\, Y^k$ is mapped to the flow of the $(d+1)$-uple (L, R_α) under $\frac{\partial}{\partial t_k} = \sum_\beta \frac{\partial}{\partial t_{k\beta}}$ for the multicomponent qKP hierarchy.*

The second new result is to consider $(\tau, \hbar) = (q^{-1}, 0)$. We denote y by z and Y by Z in this case to avoid any confusion with the previous case. Here, the relation $zx = qxz$ gives that z is the difference (or shift) operator[22], so we obtain the positive flows of the $2D$ Toda Lattice hierarchy [160]. (We omit the discussion of the negative flows and the role of the second operator $M = \Psi y^{-1} \Psi^{-1}$.) We can write for the space

$$\mathcal{M}_{red} = \left\{ XZ = qZX - q\sum_\alpha i_\alpha j_\alpha \right\} /\!/ \, \mathrm{GL}_n(\mathbb{C}) \,.$$

Up to restricting to an open subspace, $\mathcal{M}_{red}$ coincides with $\mathcal{C}^\circ_{n,q,d}$ considered in Chapter 4 if we choose to set

$$i_\alpha = -W_\alpha \,, \quad j_\alpha = V_\alpha(\mathrm{Id}_n + W_{\alpha-1}V_{\alpha-1}) \dots (\mathrm{Id}_n + W_1V_1)ZX \,. \tag{6.6}$$

[22] To understand z as a difference operator, we think about x as being of the form q^m for some discrete variable $m \in \mathbb{Z}$, so that $z : m \mapsto m+1$.

As before, since $\mathcal{C}_{n,q,d}$ has a Poisson bracket we can write the vector field associated to the function $(1-q^{-k})G_k^{1,d} = (1-q^{-k})\operatorname{tr} Z^k$ as (see Proposition 4.1.6)

$$\frac{dZ}{dt_k} = 0\,, \quad \frac{dW_\alpha}{dt_k} = 0\,, \quad \frac{dV_\alpha}{dt_k} = 0\,, \quad \frac{dX}{dt_k} = (q^{-k}-1)XZ^k\,.$$

Again, it is precisely the vector field given in Lemma 6.1.4 for $(\tau,\hbar) = (q^{-1},0)$.

Proposition 6.1.6 *Consider the map $\mathcal{C}^\circ_{n,q,d} \to \mathcal{P} : (X,Z,V_\alpha,W_\alpha) \mapsto \Psi$ obtained from (6.4) and (6.6). Moreover, define $L = \Psi z \Psi^{-1}$, and $R_\alpha = \Psi E_\alpha \Psi^{-1}$ for $1 \leq \alpha \leq d$. Then the Hamiltonian flow given in Proposition 4.1.6 of a point in $\mathcal{C}^\circ_{n,q,d}$ under $(1-q^{-k})\operatorname{tr} Z^k$ is mapped to the flow of the $(d+1)$-uple (L,R_α) under $\frac{\partial}{\partial t_k} = \sum_\beta \frac{\partial}{\partial t_{k\beta}}$ for the multicomponent $2D$ Toda hierarchy.*

We emphasise that, for $d>1$, we are unable to identify the other vector fields from Lemma 6.1.3 as being Hamiltonian vector fields on $\mathcal{C}_{n,q,d}$ or $\mathcal{C}^\circ_{n,q,d}$ in the last two cases.

In the case $d=1$, it would be interesting to understand when the space $\mathcal{M}_{red}$ coincides with the moduli space of isomorphism classes of ideals of $A^\hbar_\tau$ (up to localisation). This is known to be true for the Weyl algebra when $(\tau,\hbar) = (1,1)$ [23, 29, 30], and the quantum torus when $(\tau,\hbar) = (q,0)$ with q not a root of unity [27]. An approach based on a general construction for any $(\tau,\hbar)$ has been considered, but turns out to be unsuccessful so far.

Hamiltonian algebra for the elliptic CM system

Recall from Section 1.3 that the spin RS system was originally introduced with an elliptic potential. Since we have shown in this thesis that the trigonometric spin RS system can be obtained from a suitable quasi-Hamiltonian algebra, it is then an interesting question to determine if this can be generalised to the elliptic case. To make a first step in that direction, it seems easier to seek after a Hamiltonian algebra that yields the elliptic CM system, since CM systems can be seen as a degeneration of RS systems. We outline the method developed by O. Chalykh and the author to deal with this case.

We start by considering an elliptic curve $\mathcal{E}$ given by $y^2 = 4x^3 - g_2x - g_3$, to which we add punctures at 0 and some $\mu \neq 0$ (modulo the lattice). Using $\wp(q)$ the Weierstrass $\wp$-function associated to $\mathcal{E}$, we set $\lambda = \wp(\mu)$, $\lambda' = \wp'(\mu)$ and remark that the following functions are well-defined on the punctured curve

$$s = x - \lambda = \wp(q) - \wp(\mu)\,, \quad u = \frac{y+\lambda'}{2s} = \frac{\wp'(q)+\wp'(\mu)}{2(\wp(q)-\wp(\mu))}\,. \tag{6.7}$$

The corresponding algebra $A = A(\mu)$ of functions is given by

$$A = \mathbb{C}[u,s]/\langle u^2 s - \lambda' u = s^2 + a_0 s + a_1\rangle\,, \quad \text{for } a_0 = 3\lambda\,,\ a_1 = 3\lambda^2 - \frac{1}{4}g_2\,. \tag{6.8}$$

We can consider the A-bimodule of double derivations $\mathbb{D}\mathrm{er}(A)$, and construct its tensor algebra $\mathcal{A}$ that we see as an ungraded algebra. It is generated by symbols $u, s, \partial_\mu, \Delta$ under the relations

$$\begin{aligned} &[u,s] = 0\,, \quad u^2 s - \lambda' u = s^2 + a_0 s + a_1\,, \\ &[s,\partial_\mu] = us\Delta + s\Delta u - \lambda'\Delta\,, \quad [u,\partial_\mu] = s\Delta + \Delta s + a_0\Delta - \Delta u^2\,. \end{aligned} \tag{6.9}$$

Reinterpreting Van den Bergh's work [162, 3.2], we obtain the following.

Proposition 6.1.7 *The algebra $\mathcal{A}$ admits a double Poisson bracket and $-\Delta$ is a moment map.*

We can look at the induced Lie bracket $\{-,-\}$ on $\mathcal{A}/[\mathcal{A},\mathcal{A}]$ (see § 2.3.1).

Lemma 6.1.8 *The following holds in $\mathcal{A}/[\mathcal{A},\mathcal{A}]$:*

$$\{\partial_\mu^k, \partial_\mu^l\} = 0\,, \quad \{(u+\alpha s)^k, (u+\beta s)^l\} = 0\,, \quad k,l \in \mathbb{N},\ \alpha,\beta \in \mathbb{C}. \tag{6.10}$$

In particular ∂_μ, u, s (or any linear combination $u + \alpha s$) are involutive elements in $\mathcal{A}$.

We then form the algebra A_μ by identifying the unit in $\mathcal{A}$ with the idempotent e_2 from the path algebra of the quiver $\bar{Q}_0$ depicted in Figure 2. Since the algebra $\mathbb{C}\bar{Q}_0$ is Hamiltonian by Theorem 2.3.5 and so is $\mathcal{A}$ by Proposition 6.1.7, A_μ is also a Hamiltonian algebra. We can present A_μ as the algebra generated by symbols $u, s, \partial_\mu, \Delta, v, w, e_0, e_\infty$ where the first four elements satisfy the relations (6.9) in $\mathcal{A}$, the orthogonal idempotents e_0, e_∞ satisfy $1 = e_0 + e_\infty$, and

$$v = e_\infty v e_0\,, \quad w = e_0 w e_\infty\,, \quad b = e_0 b e_0 \quad \text{for any } b = u, s, \partial_\mu, \Delta\ . \tag{6.11}$$

The algebra A_μ is an algebra over $B = \mathbb{C}e_\infty \oplus \mathbb{C}e_0$, and we can decompose its moment map as $-e_0(\Delta + wv)e_0 + e_\infty v w e_\infty$.

Consider the space of representations of dimension $(\alpha_\infty, \alpha_0) = (1, n)$ which we write $\operatorname{Rep}(\mathrm{A}_\mu, (1, n))$. Assuming that A is smooth in the first place, i.e. when $\mathcal{E}$ is a smooth curve, then $\operatorname{Rep}(\mathrm{A}_\mu, (1, n))$ is a smooth $\mathrm{GL}_n(\mathbb{C})$-variety, with Poisson bracket $\{-,-\}$ defined from $\{\!\{-,-\}\!\}$ by (2.31), and with a moment map given by the matrix Ψ representing $-\Delta + \mu$. We can think of $\operatorname{Rep}(\mathrm{A}_\mu, (1, n))$ as the space of matrices $U, S, B_\mu, D \in \operatorname{Mat}_{n\times n}(\mathbb{C})$, $V \in \operatorname{Mat}_{1\times n}(\mathbb{C})$ and $W \in \operatorname{Mat}_{n\times 1}(\mathbb{C})$ (representing respectively $u, s, \partial_\mu, \Delta, v$ and w) subject to the relations

$$
\begin{aligned}
&US = SU\,, \quad U^2 S - \lambda' U = S^2 + a_0 S + a_1 \operatorname{Id}_n\,, \\
&[S, B_\mu] = USD + SDU - \lambda' D\,, \quad [U, B_\mu] = SD + DS + a_0 D - DU^2\,.
\end{aligned}
\tag{6.12}
$$

The preimage under the moment map Ψ of a point $I_\nu = (-\nu \operatorname{Id}_n, n\nu)$, $\nu \in \mathbb{C}$, corresponds to the level set in $\operatorname{Rep}(\mathrm{A}_\mu, (1, n))$ given by the matrix relations

$$
D + WV = \nu \operatorname{Id}_n\,, \quad VW = n\,\nu\,. \tag{6.13}
$$

By construction, we obtain the following result.

Proposition 6.1.9 *The space $\mathcal{C}_{n,\mu} = \Psi^{-1}(I_\nu)/\!/\operatorname{GL}_n(\mathbb{C})$ is a Poisson variety obtained by Hamiltonian reduction.*

For $\nu \neq 0$, $\mathcal{C}_{n,\mu}$ is a smooth irreducible affine variety of dimension $2n$ by [24, Theorem 1.1], which is called the *elliptic Calogero-Moser space* (with puncture μ and trivial line bundle). Note that, in this original construction, the space is given without the perspective of Hamiltonian reduction.

To understand the relation to the elliptic CM system, we assume that $\nu \neq 0$ from now on and proceed as follows. Let $\mathcal{C}'_{n,\mu}$ be the subspace where the matrix S is semisimple, and each representative of the matrix V has no zero entry when S is in diagonal form. Note that when S is in diagonal form, so is U. We write the diagonal entries of S and U as s_i and u_i respectively. The second relation in (6.12) yields that $u_i^2 s_i - \lambda' u_i = s_i^2 + a_0 s_i + a_1$, that is a pair (u_i, s_i) corresponds to a point on the punctured curve A. Using (6.12) and (6.13), we find that

$$
(s_i - s_j)(B_\mu)_{ij} = (u_i s_i + u_j s_i - \lambda')(\nu \delta_{ij} - w_i)\,, \tag{6.14}
$$

which gives for $i = j$ that $W = \nu(1, \ldots, 1)$. The diagonal entries of B_μ remain free and we let $p_i := \frac{1}{\nu}(B_\mu)_{ii}$. The off-diagonal identities give

$$
(B_\mu)_{ij} = -\nu(u_i s_i + u_j s_i - \lambda')/(s_i - s_j)\,, \quad i \neq j\,.
$$

From our discussion at the beginning of this subsection, we can write the couple (u_i, s_i) as

$$u_i = \frac{\wp'(q_i) + \wp'(\mu)}{2(\wp(q_i) - \wp(\mu))}\,, \quad s_i = \wp(q_i) - \wp(\mu)\,, \tag{6.15}$$

for suitable variables $(q_i)_i$. Therefore $(B_\mu)_{ii} = \nu p_i$, while when $i \neq j$

$$\frac{-2}{\nu}(B_\mu)_{ij} = \frac{\wp'(q_i) - \wp'(\mu)}{[\wp(q_i) - \wp(q_j)]} + \frac{[\wp'(q_j) + \wp'(\mu)][\wp(q_i) - \wp(\mu)]}{[\wp(q_i) - \wp(q_j)][\wp(q_j) - \wp(\mu)]}\,. \tag{6.16}$$

Some easy pole analysis yields the next result.

Lemma 6.1.10 *The matrix B_μ is proportional to L^μ given by $(L^\mu)_{ii} = p_i$ and*

$$L^\mu_{ij} = \frac{\sigma(q_j)\,\sigma(q_i - \mu)\,\sigma(q_j - q_i - \mu)}{\sigma(q_i)\,\sigma(q_j - \mu)\,\sigma(q_j - q_i)\,\sigma(-\mu)} \quad \textit{for } i \neq j\,. \tag{6.17}$$

Up to gauge transformation (see [100, (4.40)–(4.42)]), the matrix L^μ coincides with the Lax matrix for the elliptic Calogero-Moser model discovered by Krichever [102] (which was introduced earlier by Calogero [39] for specific values of the spectral parameter μ). We get that its symmetric functions $\mathrm{tr}(L^\mu)^k$, $k \in \mathbb{N}$, pairwise commute by (6.10) and (2.36). (Indeed, (2.36) also holds in the Hamiltonian case.) If we consider the Poisson structure in local coordinates $(q_i, p_i)_i$, we find the following.

Lemma 6.1.11 *The Poisson bracket on $\mathcal{C}_{n,\mu}$ obtained by Hamiltonian reduction can be written locally as $\{p_i, q_j\} = \delta_{ij}$ while $\{q_i, q_j\} = 0 = \{p_i, p_j\}$.*

Since the local coordinates are canonical, the above discussion implies that we have recovered the usual result stating that the elliptic CM system is Liouville integrable.

Some problems remain unanswered at the moment. First, we would like to be able to identify open subspaces corresponding to different punctures to get the completed phase space for this system. Second, we can obtain the phase space for the spin elliptic CM system by fusing d copies of the algebra $\mathbb{C}\bar{Q}_0$ with the algebra $\mathcal{A}$. This yields a phase space of dimension $2nd$, but we do not know how to form an integrable system containing the functions $\mathrm{tr}\, B_\mu^k$, $k = 1, \ldots, n$. If it is obtained, the next step is to understand the Poisson commutativity of the integrable system at the level of the Hamiltonian algebra directly. Third, (6.10) suggests that for fixed $\alpha \in \mathbb{C}$, $\mathrm{tr}(U + \alpha S)^k$, $k = 1, \ldots, n$, should define an integrable system dual to the elliptic CM system. We have not found a suitable set of local coordinates to understand this second system. Finally, it would be interesting to get the elliptic RS system in a similar way.

6.2 Fifteen years of double brackets : happy birthday!

In our investigation of integrable systems through Chapter 4 and Chapter 5, a crucial role was played by the relations that we derived in Chapter 3, which are all completely determined by the existence of a double (quasi-Poisson) bracket. Hence, the central use of double brackets in the Ruijsenaars-Schneider family has been outlined in this thesis. To conclude this work, it seems interesting to collect the different applications of double brackets to integrable systems, as well as the other topics where such structures have appeared so far. Since the first version of Van den Bergh's pioneering work [162] appeared on the arXiv in 2004, this section gathers the ramifications of double brackets during their first fifteen years in the mathematical community[23].

General study of double brackets

We introduced double brackets over $\mathbb{C}$, but there is a generalisation of double Poisson brackets over an arbitrary ground ring, see [159]. In particular, this allows to define double brackets for Hopf algebras [117]. For the sake of conciseness, we now report results regarding algebras over a field of characteristic zero, as initially introduced by Van den Bergh.

In [161], Van de Weyer gives a first systematic study of double Poisson algebras (see also [9]), and introduces the notion of double Poisson cohomology, as an analogue of Poisson-Lichnerowicz cohomology [112]. This cohomological study has been investigated in more details with Pichereau in [133]. In that paper, the authors show that this cohomology is equivalent to the Hochschild cohomology of free algebras in the case of a linear double Poisson bracket. For a particular linear case, the first two cohomology groups are computed in [3]. Moving to quadratic double Poisson brackets, an explicit classification is obtained in [128] for the free algebra over two elements, and in [150] over three elements. Few examples are known outside free algebras, except for quivers [31, 162]. A natural obstruction is that the definition of double brackets is not suited to commutative algebras : polynomial rings with at least two generators do not have non trivial double Poisson brackets [134].

[23] This is a list that was established to the best of the author's knowledge, and that was completed by 1 March 2019.

Algebraic structures related to double brackets

It is suggested in [58, 129, 145] that we could relax the definition of a double Poisson bracket by removing its derivation property *(D2)* in Definition 2.3.1, which would define a double (Lie) algebra on a vector space, a double analogue of a Lie algebra. In such a case, Jacobi identity takes place in $A^{\otimes 3}$ as the vanishing of the map (2.11). There do not exist simple double algebra structures on a finite-dimensional vector space, except the trivial one [86]. In fact, double algebras are equivalent to solutions of the associative Yang-Baxter equation [128, 145]. This relation is further explored in [129] where the generators also depend on local parameters. Another way to relax the definition of a double Poisson bracket is to drop the cyclic antisymmetry *(D1)* from Definition 2.3.1, as did Arthamonov [13, 15]. Alternatively, we can try to generalise this structure to Lie algebras instead of associative algebras [122].

There has been some interest in trying to obtain double Poisson algebras *up to homotopy*. For example, we can ask the triple bracket (2.11) to be nonzero and to satisfy some relations with the double bracket up to a 4-bracket (and so on...) which leads to the notion of double Poisson-infinity algebra [145]. The definition of a double Poisson bracket can also be extended to DG algebras [26], which turns out to be useful to introduce derived Poisson algebras, a derived version of Crawley-Boevey's (non-commutative) H_0–Poisson structures [52, 53]. The definition can be easily adapted for bimodules over this algebra both in the usual and DG cases [45]. These generalisations of double Poisson brackets appear on Fukaya categories [47], associated to particular Calabi-Yau algebras [45], related to dual Hodge decompositions [28], and extend the usual correspondence between bi-symplectic and double Poisson structures [46].

Other features of double Poisson brackets are that they can be studied in terms of a particular protoperad $\mathcal{DP}ois$ introduced by Leray [108, 110], or that they are related to pre-Calabi-Yau structures. Regarding this second relation, it is possible to construct such structures from (graded) double Poisson brackets on DG categories [175]. In fact, double Poisson brackets (and double Poisson-infinity brackets) classify pre-Calabi-Yau structures (of prescribed types) [76, 92].

More integrable systems

It is quickly mentioned in Remark 2.3.40 that we could look at integrability directly on algebras endowed with a double (quasi-)Poisson bracket. This gives, in fact, a subset of interesting examples of Hamiltonian ODEs on associative algebras. As remarked e.g. in [58] and can easily be seen from (2.38), what plays a central role is the left Loday bracket $\{-,-\} = m \circ \{\!\{-,-\}\!\}$ (seen as a map $A/[A,A] \times A \to A$). In particular, assuming that the system of ODEs is defined by $\{b_1, -\}$, we get from (2.16) that $\{b_2, -\}$ defines a symmetry (i.e. a commuting system of ODEs) whenever $\{b_1, b_2\} \in [A, A]$. Thus, what is important to get symmetries is to understand $\{-,-\}$ as a map on $A/[A,A]$, which is suggested in [128] under the name of trace bracket, a particular example of H_0–Poisson structure. This interpretation allows to reformulate studies such as [127] in terms of double Poisson brackets. Note that a quantisation of trace brackets is proposed in [19], but it does not provide a quantisation of double brackets directly.

Motivated by the previous point, De Sole, Kac and Valeri define a general scheme to obtain non-commutative Hamiltonian PDEs in [58]. They introduce double Poisson vertex algebras, where the underlying associative algebras are differential, and they extend the correspondence between Poisson vertex algebras and Hamiltonian PDEs to the non-commutative setting.

It is possible to recover systems of Calogero-Moser type from this thesis if the double quasi-Poisson brackets are replaced by the double Poisson brackets introduced by Van den Bergh [162], see § 2.3.2. Though this has not been written so far, this is an easy exercise and one can recover existing results that already contain part of the quiver interpretation [32, 43, 118, 155, 156].

Finally, note that it is also possible to introduce non-commutative versions of discrete integrable systems. Indeed, in [81], the pentagram map was expressed in terms of moves defined on a network on a disc or annulus, which are particular examples of quivers with weights assigned to the arrows. In particular, it was found that some quantities are conserved under those moves, and they imply the complete integrability of the pentagram map. In [132], a double bracket is introduced on an algebra A associated to such networks, and it is shown that there are analogous invariants for the moves under the associated Lie bracket on $A/[A,A]$.

Relation to topology

Considering a punctured oriented surface with a marked point on its boundary, Massuyeau and Turaev describe in [115] how to endow its fundamental group with a double quasi-Poisson bracket. Looking at representation spaces, they show that it coincides with the quasi-Poisson structure on the corresponding representation variety for $G = \mathrm{GL}_n$ defined in [6]. They extend their result in [116] to the Pontryagin algebra of an arbitrary smooth oriented manifold with boundary. Note that in the construction of the double quasi-Poisson bracket in [115], the authors first introduce an operation which fails to satisfy the cyclic antisymmetry *(D1)* from Definition 2.3.1, which they correct to obtain a double quasi-Poisson bracket.

It is also possible to define a double bracket on the fundamental groupoid associated to a ribbon graph [13]. An alternative construction is given in [132]. Note that both structures descend to the same induced bracket which coincide with Goldman's Lie bracket. Quite surprisingly, the two operations defined by Massuyeau and Turaev [115] described above also descend to Goldman's bracket. They have been used to relate Goldman-Turaev formality and the Kashiwara-Vergne problem in arbitrary genera [3, 4, 5, 8].

Use in non-commutative geometry

The starting point of [162, 163] is the introduction of a non-commutative version of Poisson and quasi-Poisson geometry. Hence, a natural question is to see which structures depending on a (quasi-)Poisson manifold admit a non-commutative analogue relying on an algebra with a double (quasi-)Poisson bracket. For example, the equivalence of Poisson manifolds and differential graded manifolds of degree $+1$ admits an algebraic analogue [10].

Recall that there exists a graded version of double Poisson brackets $\{\!\{-,-\}\!\}_{\mathrm{SN}}$ satisfying (2.12a)–(2.12c). This structure can be related to a connection on the algebra of double derivations. This can be found in [85], where an extension of the theory of differential operators to noncommutative geometry is given.

A general result in [163] is the introduction of double Lie algebroids, as a double version of Lie algebroids. As noted in [108], it is more convenient to refer to these structures as double Lie-Rinehart algebras, in order to avoid confusion with the different notion of double Lie algebroids associated to double Lie groupoids [113], but also because it naturally extends the algebraic version of Lie algebroids, called Lie-Rinehart algebras. An extensive study of these structures and their shifted version is obtained by Leray [108, 109], while the introduction of non-commutative calculus on double Lie-Rinehart algebras is investigated by Chemla [44]. A next step is to establish which structures depending on a Lie algebroid admit non-commutative analogues. This has been done for Courant algebroids in [11, 77].

Final remarks

Note that we only considered works that *explicitly* use double brackets. For example, though [93, 174] rely on the work of Van den Bergh on double quasi-Poisson brackets associated to quivers, these articles only require the geometric counterpart of the theory (see § 2.3.3) and do not include computations with double brackets. Otherwise, we would also need to collect the extensive literature connected to quiver varieties based on the works [123, 125], since the underlying Poisson structure is given by the Hamiltonian algebra structure of the path algebra of a quiver, see Theorem 2.3.5. We also omitted to mention the applications relying on the symplectic counterpart of double brackets as introduced in [54] (see e.g. [22, 78, 156, 163]), or works depending only on the associated Loday bracket (see e.g. [146, 164]).

Appendices

A Some linear algebra

Lemma A.1 *Let* $Z = \operatorname{diag}(z_1, \dots, z_n)$ *for pairwise distinct* (z_k)*, and put* $W := (1, \dots, 1)^\top$. *Let* F *be an invertible matrix with distinct ordered eigenvalues* $\mu_1 < \dots < \mu_n$*, and fix a corresponding eigenbasis* (e_i)*. Define* $E = (E_{ij})$*, where* $E_{ij} = (e_j)_i$*, and assume that* $E^{-1}W$ *is a vector with nonzero entries.*

(1) There exists $V \in \operatorname{Mat}_{1\times n}$ *such that* $F + WVF$ *and* Z *have the same spectrum.*

(2) For a fixed choice of ordered eigenbasis, the pair (V, X) *satisfying*

$$XZX^{-1} = F + WVF\,,$$

is uniquely determined up to $(\mathbb{C}^\times)^n$ *action* $(c_i) \cdot (V, X) = (V, XC)$*,* $C = \operatorname{diag}(c_1, \dots, c_n)$*.*

Proof. We have that $Fe_i = \mu_i e_i$, and we form E by $E_{ij} = (e_j)_i$ which satisfies $E^{-1}FE = D_\mu = \operatorname{diag}(\mu_1, \dots, \mu_n)$. Note that E is uniquely chosen by assumption.

Remark that we can write for arbitrary $V = (v_1, \dots, v_n)$

$$(E^{-1}WVFE)_{ij} = A_i B_j \mu_j\,, \quad A_i = \sum_{s=1}^{n} E^{-1}_{is}\,,\ B_j = \sum_{t=1}^{n} v_t E_{tj}\,.$$

Here, $A_i \neq 0$ for all i, by assumption on $E^{-1}W$. For any unknown λ, since WVF has rank one,

$$\det(F + WVF - \lambda \operatorname{Id}_n) = \det(E^{-1}FE - \lambda \operatorname{Id}_n)\left[1 + VFE(E^{-1}FE - \lambda \operatorname{Id}_n)^{-1}E^{-1}W\right]$$
$$= \prod_{k=1}^{n}(\mu_k - \lambda)\ \left[1 + \sum_{i=1}^{n} \frac{A_i B_i \mu_i}{\mu_i - \lambda}\right] = \prod_{k=1}^{n}(\mu_k - \lambda) + \sum_{i=1}^{n} A_i B_i \mu_i \prod_{k\neq i}(\mu_k - \lambda)\,.$$

Now, the problem amounts to determine the entries $v_1, \ldots, v_n$ of V such that

$$\prod_{k=1}^{n}(z_k - \lambda) = \prod_{k=1}^{n}(\mu_k - \lambda) + \sum_{t=1}^{n} v_t \sum_{i=1}^{n} E_{ti} A_i \mu_i \prod_{k\neq i}(\mu_k - \lambda)\,. \tag{A.18}$$

We need the different coefficients of this expression as a polynomial in λ to vanish. Introduce for any $l \in \mathbb{N}^\times$ the (signed) symmetric polynomials $(p_{l,\nu}(a_1, \ldots, a_l))_{\nu=0}^{l}$ such that

$$\prod_{k=1}^{l}(a_k - \lambda) = \sum_{\nu=0}^{l} \lambda^\nu\, p_{l,l-\nu}(a_1, \ldots, a_l)\,. \tag{A.19}$$

Hence, expanding (A.18), we can write

$$\begin{aligned}&\sum_{\nu=0}^{n}[p_{n,n-\nu}(z_1, \ldots, z_n) - p_{n,n-\nu}(\mu_1, \ldots, \mu_n)]\lambda^\nu \\ &= \sum_{\nu=0}^{n-1} \lambda^\nu \sum_{i=1}^{n} p_{n-1,n-1-\nu}(\hat{\mu}^i) A_i \mu_i \sum_{t=1}^{n} v_t E_{ti}\,,\end{aligned} \tag{A.20}$$

where we write $(\hat{\mu}^i)$ for the sequence of $n-1$ terms obtained from $(\mu_1, \ldots, \mu_n)$ by omitting the i-th term. In other words, we need to require for $\nu = 0, \ldots, n-1$

$$\begin{aligned}&[p_{n,n-\nu}(z_1, \ldots, z_n) - p_{n,n-\nu}(\mu_1, \ldots, \mu_n)] \\ &= \sum_{t=1}^{n} v_t \sum_{i=1}^{n} E_{ti} A_i \mu_i\, p_{n-1,n-1-\nu}(\hat{\mu}^i)\,.\end{aligned} \tag{A.21}$$

We write this in matrix notation : by introducing $M = (M_{\nu t})_{\nu=0,\ldots,n-1}^{t=1,\ldots,n}$ and $N = (N_\nu)_{\nu=0,\ldots,n-1}$ where the entries are given by

$$M_{\nu t} = \sum_{i=1}^{n} E_{ti} A_i \mu_i\; p_{n-1,n-1-\nu}(\hat{\mu}^i)\,,\;\; N_\nu = p_{n,n-\nu}(z_1, \ldots, z_n) - p_{n,n-\nu}(\mu_1, \ldots, \mu_n), \tag{A.22}$$

the n equations in (A.21) give us nothing else than the matrix equation $MV = N$. Thus, V is unique if M is invertible. Now, remark that if we introduce $M^\circ = (M^\circ_{\nu i})_{\nu=0,\ldots,n-1}^{i=1,\ldots,n}$ with

$$M^\circ_{\nu i} = p_{n-1,n-1-\nu}(\hat{\mu}^i)\,, \tag{A.23}$$

we can write M in terms of a matrix product as

$$M = M^\circ\, \mathrm{diag}(A_1, \ldots, A_n)\, D_\mu\, E^\top\,. \tag{A.24}$$

Thus, part (1) of the lemma follows if we show that M° is invertible. This is Lemma A.3.

For part (2), remark that for a fixed E the above construction gives a unique V. Then, X corresponds to a choice of eigenbasis of $F + WVF$ ordered with respect to the eigenvalues $(z_1, \ldots, z_n)$. □

Remark A.2 *Among all the pairs* (X, V) *such that* Z *and* $qF(\mathrm{Id}_n + WV)$ *have the same spectrum while* X^{-1} *puts* $qF(\mathrm{Id}_n + WV)$ *in diagonal form, knowing* X *fixes* V*. Indeed, assume that we have*

$$XZX^{-1} = qF(\mathrm{Id}_n + WV)\,, \quad XPZP^{-1}X^{-1} = qF(\mathrm{Id}_n + W\tilde{V})\,,$$

for some $V, \tilde{V} \in \mathrm{Mat}_{1\times n}$*, and a permutation matrix* P *distinct from the identity. Then, taking the difference of these equations yields*

$$Z - PZP^{-1} = qX^{-1}FW(V - \tilde{V})X\,.$$

Since the eigenvalues of Z *are pairwise distinct, the matrix on the left has rank at least* 2*. But the matrix on the right has rank at most* 1*, leading to a contradiction.*

Lemma A.3 *The matrix* M° *with entries* $M^\circ_{\nu i} = p_{n-1,n-1-\nu}(\mu_1, \ldots, \hat{\mu}_i, \ldots, \mu_n)$ *is invertible if and only if all* $(\mu_k)_k$ *are distinct.*

Proof. This follows if we can show that

$$\det M^\circ = \prod_{a>b} (\mu_a - \mu_b)\,, \tag{A.25}$$

which is similar to show the analogous statement for a Vandermonde matrix. Indeed, remark that when we substract the b-th column from the a-th column, we get that the new a-th column has entries

$$p_{n-1,n-1-\nu}(\mu_1, \ldots, \hat{\mu}_a, \ldots, \mu_n) - p_{n-1,n-1-\nu}(\mu_1, \ldots, \hat{\mu}_b, \ldots, \mu_n). \tag{A.26}$$

But the latter ν-th entry of the new column is precisely the coefficient of the element λ^ν in

$$\prod_{k\neq a} (\mu_k - \lambda) - \prod_{k\neq b} (\mu_k - \lambda)\,, \tag{A.27}$$

which is clearly divisible by $\mu_a - \mu_b$. Such transformations are compatible with taking determinant, hence (A.25) holds up to a multiplicative constant (because any $\mu_a - \mu_b$ divide the determinant and such a product has maximal degree). Looking at the coefficient for $\mu_2\mu_3^2 \ldots \mu_n^{n-1}$ we can conclude that the constant is $+1$ and (A.25) holds. □

By a straightforward adaptation in the proof of Lemma A.1, we get the next result.

Lemma A.4 *Let* $Z = \mathrm{diag}(z_1, \dots, z_n)$ *for pairwise distinct* (z_k)*, and put* $W := (1, \dots, 1)^\top$*. Let* F *be an invertible matrix with distinct ordered eigenvalues* $\mu_1 < \dots < \mu_n$*, and fix a corresponding eigenbasis* (e_i)*. Define* $E = (E_{ij})$*, where* $E_{ij} = (e_j)_i$*, and assume that* $E^{-1}W$ *is a vector with nonzero entries.*

(1) There exists $V \in \mathrm{Mat}_{1\times n}$ *such that* $F + FWV$ *and* Z *have the same spectrum.*

(2) For a fixed choice of ordered eigenbasis, the pair (V, X) *satisfying*

$$XZX^{-1} = F + FWV\,,$$

is uniquely determined up to $(\mathbb{C}^\times)^n$ *action* $(c_i) \cdot (V, X) = (V, XC)$*,* $C = \mathrm{diag}(c_1, \dots, c_n)$*.*

B An elementary result in Poisson geometry

Consider the following easy result.

Lemma B.1 *Assume that* $\psi : M \to N$ *is an isomorphism of manifolds such that* $\{-,-\}_N$ *is a Poisson bracket on* N*. Furthermore, assume that there exists an antisymmetric biderivation* $\{-,-\}_M$ *on* M *such that for all* $f_1, f_2 \in \mathcal{O}_N$*, we have that* $\psi^*\{f_1, f_2\}_N = \{\psi^* f_1, \psi^* f_2\}_M$*. Then* $\{-,-\}_M$ *is a Poisson bracket.*

Proof. For arbitrary $g_1, g_2, g_3 \in \mathcal{O}_M$, we have that $Jac_M(g_1, g_2, g_3) = \psi^* Jac_N(f_1, f_2, f_3) = 0$ with $f_i = g_i \circ \psi^{-1}$. □

This implies that, if M is endowed with an antisymmetric biderivation $\{-,-\}_M$ such that Lemma B.1 holds locally, then $\{-,-\}_M$ is a Poisson bracket since Jac_M vanishes around every point. Meanwhile, let us look at the following.

Lemma B.2 *Assume that* $\tilde{M}$ *is a manifold on which a finite group* H *acts freely. Assume that there exists an antisymmetric biderivation* $\{-,-\}_{\tilde{M}}$ *preserved by the* H*-action, i.e. for all* $h \in H$*,* $f_1, f_2 \in \mathcal{O}_{\tilde{M}}$*, we have that* $h \cdot \{f_1, f_2\}_{\tilde{M}} = \{h \cdot f_1, h \cdot f_2\}_{\tilde{M}}$*. If the restriction of* $\{-,-\}_{\tilde{M}}$ *on* H*-invariant functions defines a Poisson bracket* $\{-,-\}_M$ *on the manifold* $M = \tilde{M}/H$*, then* $\{-,-\}_{\tilde{M}}$ *is Poisson.*

Proof. Assume that there exist functions f_1, f_2, f_3 and a point $p \in \tilde{M}$ such that $Jac_{\tilde{M}}(f_1, f_2, f_3)(p) \neq 0$. Take a sufficiently small neighbourhood U of p so that $h \cdot U \cap h' \cdot U = \emptyset$ for all $h, h' \in H$, $h \neq h'$, and define for each i the function f_i^H on $V = H \cdot U$ by setting $(f_i^H)_{|h \cdot U}(p') = f_i(h^{-1} \cdot p')$ for each $h \in H, p' \in h \cdot U$. These are clearly H-invariant functions.

We have that the function $F = Jac_{\tilde{M}}(f_1^H, f_2^H, f_3^H)$ does not vanish on V by assumption. Since V is identified with the open set V/H of M, and since the functions (f_i^H) are H-invariant and can be identified with functions f_i' on V/H, we can identify by H-invariance of $\{-,-\}_{\tilde{M}}$ the function F with $Jac_M(f_1', f_2', f_3')$. The latter is always zero as $\{-,-\}_M$ is Poisson, giving a contradiction. □

Combining Lemmae B.1 and B.2, we get the following result : if $\tilde{M}$ is endowed with an antisymmetric biderivation $\{-,-\}_{\tilde{M}}$ which descends to the quotient $\tilde{M}/H$ for some finite group H acting freely on $\tilde{M}$, and if there exists a diffeomorphism ψ between the quotient space $\tilde{M}/H$ and some Poisson manifold which intertwines the brackets, then $\{-,-\}_{\tilde{M}}$ is a Poisson bracket.

Remark B.3 *If we want to apply Lemma B.1, note that we do not need to check that $\psi^*\{f_1, f_2\}_N = \{\psi^* f_1, \psi^* f_2\}_M$ for all $f_1, f_2 \in \mathcal{O}_N$, but we only need to check this on a subset of globally defined functions that are functionally independent (assuming that such a subset of maximal dimension exists). To be precise, if $f_1, \dots, f_n \in \mathcal{O}_N$ are $n = \dim N = \dim M$ functionally independent elements, we only need to check that*

$$\psi^*\{f_k, f_l\}_N = \{\psi^* f_k, \psi^* f_l\}_M\,, \quad \textit{for all } 1 \leq k, l \leq n\,. \tag{B.28}$$

Furthermore, the antisymmetric biderivation $\{-,-\}_M$ on M which satisfies (B.28) is unique. To show this statement, consider another antisymmetric biderivation $\{-,-\}'_M$ such that (B.28) holds, and form the $n \times n$ matrices P, Q with entries

$$P_{kl} = \{\psi^* f_k, \psi^* f_l\}_M - \{\psi^* f_k, \psi^* f_l\}'_M\,, \quad Q_{kl} = \{x_k, x_l\}_M - \{x_k, x_l\}'_M\,,$$

where the (x_i) form a local coordinate system on M. We have that P is identically zero by assumption, and we want to show that this is also the case for Q. Note that the Jacobian matrix V with entries $V_{ki} = \partial \psi^* f_k / \partial x_i$ is generically invertible, since otherwise it would contradict that ψ is an isomorphism and the $f_1, \dots, f_n$ are functionally independent. As we can write

$$\{\psi^* f_k, \psi^* f_l\}_M = \sum_{i,j} \frac{\partial \psi^* f_k}{\partial x_i} \{x_i, x_j\}_M \frac{\partial \psi^* f_l}{\partial x_j}\,,$$

and do the same for $\{-,-\}'_M$*, we have that* $P = VQV^\top$*. Thus* $Q = 0_{n\times n}$ *generically, and hence vanishes since its entries are analytic functions.*

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