Representations of reflection monoids

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Abstract

Reflection groups have been investigated since the nineteenth century and now have fundamental relevance for various strands of mathematics, including Lie groups, Lie algebra, and Weyl groups. Recently, Everitt and Fountain [19] formulated the notion of reflection monoids, a generalisation of the idea of reflection groups to the semigroup theory. In particular, they introduced the family of Boolean reflection monoids of types $\mathcal{A}_{n-1}$ and $\mathcal{B}_n$, where type $\mathcal{A}_{n-1}$ is isomorphic to the symmetric inverse monoid, and type $\mathcal{B}_n$ is isomorphic to the monoid of partial signed permutations.

The last quarter of a century has witnessed a resurgence in the interest in representations of semigroups. The principal approach to identifying these representations is the Clifford-Munn correspondence, the underlying idea of which is that irreducible representations are in one-to-one correspondence with irreducible representations of the maximal subgroups. For type $\mathcal{A}_{n-1}$ Boolean reflection monoids, the maximal subgroups are symmetric groups S_k ($k \leq n$), while for type $\mathcal{B}_n$, the maximal subgroups are signed permutation groups B_k ($k \leq n$). These signed permutation groups, and their properties, can be found scattered in the literature. In this thesis, we collect and reformulate these properties in a form that is analogous to the symmetric group S_n.

We employ the Clifford-Munn correspondence to provide an explicit account of the ordinary irreducible representations of types $\mathcal{A}_{n-1}$ and $\mathcal{B}_n$ Boolean reflection monoids, utilising combinatorial objects called Young tableaux. Preliminary to this, we also prove, for an arbitrary finite inverse monoid, that induced and reduced representations of an irreducible representation, derived from employing the Clifford-Munn correspondence, are themselves irreducible.

Contents

List of Figures

Preface

The purpose of this thesis is to provide an explicit description of all irreducible representations of certain reflection monoids, focusing specifically on types $\mathcal{A}_{n-1}$ and $\mathcal{B}_n$ Boolean reflection monoids. The decision was made to commence each of the thesis' sections with an outline of notable historical issues, and to accompany this with an overview of the contents of the respective sections. A summary of the contents of the chapters is presented below.

Chapter 1 provides contextual information relating to semigroups, and illuminates $[J]^{\ngeq}$, a semigroup ideal that plays a vital role in Chapter 4. A proof is presented for an alternative characterisation of such an ideal. Paired with this, a proof is provided for the statement that for any element s belonging to a finite semigroup S, if a product es is not $\mathcal{J}$-related with an idempotent e, then es lies in any $\mathcal{J}$-class, which is strictly less than the $\mathcal{J}$-class J_e. Alongside these proofs, an illustration of the structural features of inverse and regular semigroups is outlined, and the section ends with an emphasis on the position of a product of a pair of distinct idempotents in a finite inverse semigroup.

Chapter 2 is principally concerned with reflection groups and reflection monoids. An in-depth study of the properties of reflection monoids, as well as the isomorphism between type $\mathcal{A}_{n-1}$ Boolean reflection monoids and the symmetric inverse monoid along with the isomorphism between type $\mathcal{B}_n$ Boolean reflection monoids and the monoid of signed permutations are presented. Such isomorphisms are valuable for the subsequent account of irreducible representations of the monoids. The section concludes by proving the unpublished result of Everitt-Fountain, which relates to the maximal subgroups of reflection monoids.

Chapter 3 first discusses representations of finite groups, followed by an investigation of Specht modules for the symmetric groups S_n. In addition, the reformulation of the properties of type $\mathcal{B}_n$ reflection groups with respect to the signed permutation groups B_n is presented. More specifically, the proof of the sign (signature) of the elements, the generator elements, and the illustration of the conjugacy classes are addressed in this chapter. Thereafter, a detailed account of the Specht modules for the signed permutation groups is provided, and several assertions in the literature are proven.

The Clifford-Munn correspondence is utilised in Chapter 4 to investigate the irreducible representations of inverse semigroups, outlining the apex concept and providing a proof for its alternative formulation. A proof is also provided for Theorem 4.2.8, which states that the representation of a maximal subgroup, derived from the reduction of the irreducible representation of a finite inverse semigroup, is irreducible. Alongside this, Theorem 4.2.11 is proven, which demonstrates that the representation of a finite inverse semigroup, derived from inducing the irreducible representation of a maximal subgroup, is irreducible. Notably, these results are extensions of the arguments that were utilised in [26] to prove a corresponding result, specifically regarding the finite transformation of semigroups.

Chapter 5 provides a survey of Grood's approach for deriving the representations of the symmetric inverse monoid, and one of the thesis' main results, the Clifford-Munn correspondence is utilised to provide an explicit account of these representations.

Following this, Chapter 6 illustrates the monoid of partial signed permutations, alongside the link-cycle decomposition for partial signed permutations. Crucially, this result can be considered an extension of Munn's decomposition of a partial permutation. The chapter ends with the presentation of another new result, generated by applying the Clifford-Munn correspondence to yield explicit descriptions of irreducible representations over the complex field of the monoids of partial signed permutations.

Finally, the thesis ends with a broad overview of potential areas for further research. Some of the avenues for further study are highlighted as worthwhile research pursuits, which are expected to contribute to the development of reflection monoids.

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Declaration

I declare that this thesis is a presentation of original work and I am the sole author. This work has not previously been presented for an award at this, or any other, University. All sources are acknowledged as References.

Chapter 1 presents a review of the fundamental notions utilised during this thesis. I mostly follow [39], [48], and [27]. The author's secondary conclusions are stated in Propositions 1.1.20 and 1.2.23. I am grateful to Professor John Fountain and Professor Victoria Gould for their hints and advice.

Chapter 2 refers to the background materials on reflections and reflection groups sourced from [40] and [44] as well as material from [19] and [20] relating to reflection monoids. I obtained permission from Everitt and Fountain to state their unpublished result and provide a proof for it.

In Chapter 3, all fundamental assertions of representation of finite groups are from [43] and [17]. The contents of Specht modules for the symmetric group S_n are introduced from [75], [41] and [24]. The argument used in Example 3.1.2 was supplied by Dr.Michael Bate for the notes [18]. Although some properties of the signed permutations group B_n featured at the beginning of the final section appear in several mathematical structures, the reformulation of such assertions in the context of the signed permutation group B_n as well as all proofs provided are the author's own. Further, all fundamental results for Specht modules for the signed permutation group B_n are obtained from from [53] and [7]. The illustrations, proofs of assertions, and examples used in this section are all the author's own.

Further, the contents of Chapter 4 are sourced from [26], [27], [18] and [78]. Examples 4.1.1 and 4.2.2 are specifically from [18], and are referenced accordingly. The illustration of the reduction/induction processes in the context of the finite inverse monoids as well as all proofs presented in Section 4.2 are the author's own. The primary conclusions outlined in Theorems 4.2.8 and 4.2.11 expand on work by Ganyushkin and Mazorchuk [26] on finite transformation semigroups. Proposition 4.2.22 expands on Everitt's conclusion [18] for symmetric inverse monoids I_n.

In Chapter 5, Grood's work presented in Section 5.1 is from [33], while Section 5.2 presents original content.

Chapter 6 contains original content relating to Specht modules for the monoid of partial signed permutations.

Chapter 1

Introduction to the semigroup theory

1.1 Basic notations

A relatively new development in the history of mathematics is the theory of semigroups. Although the term “semigroup” first appeared in the literature in 1904, a solid theoretical foundation was not in place until the 1940s [38]. In subsequent years, mathematicians increasingly began working on the semigroup theory. In 1951, Green [31] published a paper explicating a series of equivalence relations that are now referred to as the five Green's relations. These equivalence relations conceptualise a semigroup's elements with respect to the principal ideals they generate, and since then, they have emerged as a crucial technique for investigating semigroups.

Beginning in the 1960s, Clifford and Preston [12, 13] published what are now considered foundational textbooks for the algebraic theory of semigroups; since then, a wealth of general texts have emerged [32, 39, 45, 48], as well as some focussing on particular aspects of semigroup theory [29, 48, 63]. The purpose of this section is to present several of the main semigroup ideas that will be drawn on throughout this dissertation. Specifically, Green's relations and a pair of two-sided ideals are examined, since both are pivotal for the subsequent chapters. In addition, a proof is provided for a minor result (Proposition 1.1.20), since this will aid in the subsequent examination of the irreducibility of representations of maximal semigroups deduced by induction (Section 4.2.2).

Definition 1.1.1. [39] A semigroup S is a pair $(S, \cdot)$ where S is a non-empty set and $\cdot$ is an associative binary operation on S.

On occasion, we write $s \cdot s'$ or omit the binary operation $\cdot$ and write ss' as a shortcut.

Definition 1.1.2. [39] A monoid S is a semigroup with an identity $1 \in S$, where $1s = s = s1$, for all $s \in S$.

Observe that the identity of a monoid is always unique. If a semigroup S does not contain an identity 1, we can always adjoin an element 1 to S to construct a monoid $S \cup \{1\}$ by extending the binary operation in the following manner:

$$\begin{aligned} &s \cdot s' \in S, && \text{for all } s, s' \in S \\ &1 \cdot s = s = s \cdot 1, && \text{for all } s \in S \\ &1 \cdot 1 = 1. \end{aligned}$$

Then, the following convention is adopted

$$S^1 = \begin{cases} S, & \text{if } S \text{ contains an identity element,} \\ S \cup \{1\}, & \text{otherwise.} \end{cases}$$

Definition 1.1.3. [32] A left ideal (dually, right ideal) of a semigroup S is a non-empty subset $I \subseteq S$ such that $S^1 I = I$ (dually, $IS^1 = I$). If I is a left and right ideal simultaneously, then it is called an ideal (or a two-sided ideal).

If a semigroup S does not contain an identity, then for any $a \in S$, the set

$$Sa = \{sa : s \in S\}$$

does not necessarily contain a. The subset $S^1 a \subseteq S$ can be written as

$$S^1 a = Sa \cup \{a\}.$$

Dually,

$$aS^1 = aS \cup \{a\}.$$

Similarly,

$$S^1 a S^1 = SaS \cup Sa \cup aS \cup \{a\}.$$

As stated in [39, Section 2.1], the above three subsets of S are respectively called the principal left (right, two-sided) ideal generated by a. Notice that an element a belongs to $S^1 a$, aS^1, and $S^1 a S^1$; thus, adjoining such an identity guarantees that principal ideals generated by $a \in S^1$ do indeed contain a.

Definition 1.1.4. [39] Let S be a semigroup. A non-empty subset T of S is said to be a subsemigroup if it is closed under the binary operation; that is $T^2 \subseteq T$.

Definition 1.1.5. [39] Let S be a monoid with identity 1. A non-empty subset T of S is said to be a submonoid of S if it is a subsemigroup with identity 1.

Definition 1.1.6. [39] An element $e \in S$ is called an idempotent if $ee = e$ (equivalently, $e^2 = e$).

Denote the set of idempotents in S by $E(S)$. There is an order $\leq$ among elements of $E(S)$ defined by

$$e \leq f \text{ if and only if } ef = fe = e.$$

Definition 1.1.7. [39] Let S and S' be semigroups. Then, a map $\phi : S \longrightarrow S'$ is said to be a semigroup homomorphism if, for all $s, t \in S$,

$$(st)\phi = (s\phi)(t\phi).$$

Definition 1.1.8. [39] Let S and S' be monoids, with identity elements 1_S and $1_{S'}$ respectively. Then, a map $\phi : S \longrightarrow S'$ is called a monoid homomorphism if it is a semigroup homomorphism, and

$$1_S\phi = 1_{S'}.$$

A homomorphism ϕ from S into itself is called an endomorphism.

In view of the principal ideals presented above, the binary relations $\leq_{\mathcal{R}}, \leq_{\mathcal{L}}$ and $\leq_{\mathcal{J}}$ on S are defined in the following manner:

$$\begin{aligned} a \leq_{\mathcal{R}} b &\quad \text{if and only if} \quad aS^1 \subseteq bS^1; \\ a \leq_{\mathcal{L}} b &\quad \text{if and only if} \quad S^1a \subseteq S^1b; \\ a \leq_{\mathcal{J}} b &\quad \text{if and only if} \quad S^1aS^1 \subseteq S^1bS^1. \end{aligned} \tag{1.1}$$

It is evident that the above relations are reflexive and transitive; thus, the relations $\leq_{\mathcal{R}}$, $\leq_{\mathcal{L}}$, and $\leq_{\mathcal{J}}$ are quasi-orders (or Green's pre-orders) [73, Appendix A.1]. The equivalence relations affiliated with the preceding pre-orders, which are referred to as Green's relations, are introduced below:

Definition 1.1.9 (Green's relations). Let S be a semigroup and $a, b \in S$. Then, Green's equivalence relations $\mathcal{R}, \mathcal{L}$, and $\mathcal{J}$ are defined in the following manner:

(i) $a \mathrel{\mathcal{R}} b$ if and only if $aS^1 = bS^1$;

(ii) $a \mathrel{\mathcal{L}} b$ if and only if $S^1a = S^1b$;

(iii) $a \mathrel{\mathcal{J}} b$ if and only if $S^1aS^1 = S^1bS^1$.

In other words, (i) implies that two elements, a and b, are $\mathcal{R}$-related if and only if both generate the same right ideal. A similar statement can be provided when a and b are $\mathcal{L}$-related or $\mathcal{J}$-related. The next proposition is an alternative characterisation of the above equivalence relations.

Proposition 1.1.10. [39] *Let S be a semigroup and $a, b \in S$. Then,*

(i) $a \mathcal{R} b$ if and only if there exists $s, s' \in S^1$ such that $as = b$ and $bs' = a$;

(ii) $a \mathcal{L} b$ if and only if there exists $s, s' \in S^1$ such that $sa = b$ and $s'b = a$;

(iii) $a \mathcal{J} b$ if and only if there exists $s, s', t, t' \in S^1$ such that $sas' = b$ and $tbt' = a$.

Definition 1.1.11. [39] A relation ψ on a semigroup S is said to be left-compatible if for every $s, s', t \in S$,

$$s \ \psi \ s' \text{ implies } ts \ \psi \ ts'.$$

Dually, a right-compatible relation can be defined. Further, ψ is called compatible if for all $s, s', t, t' \in S$,

$$s \ \psi \ t \text{ and } s' \ \psi \ t' \implies ss' \ \psi \ tt'.$$

A left- (right-) compatible equivalence relation is called a left (right) congruence. A compatible equivalence relation is called a congruence. If $a \mathcal{R} b$, then $aS^1 = bS^1$ and then $saS^1 = sbS^1$ for some $s \in S$. It follows that $sa \mathcal{R} sb$ and hence, $\mathcal{R}$ is a left congruence. A symmetric observation holds for $\mathcal{L}$.

Definition 1.1.12. [39] Let ϕ and ψ be binary relations on a set T. Then, the composition of relations $\phi \circ \psi$ is determined by

$$\phi \circ \psi = \{(a, c) \in T \times T : \exists \ b \in T \text{ such that } a \ \phi \ b \text{ and } b \ \psi \ c\}.$$

Consider a partial order on a set of all equivalence relations on a set S. Let $\phi \vee \psi$ be the join of equivalence relations ϕ and ψ.

Proposition 1.1.13. [39] *Let ϕ and ψ be equivalences on a set S and $a, c \in S$. Then $(a, c) \in \phi \vee \psi$ if and only if, for some $n \in \mathbb{N}$, there exist elements $b_1, b_2, \ldots, b_{2n-1} \in S$ such that*

$$a \ \phi \ b_1, \ b_1 \ \psi \ b_2, \ b_2 \ \phi \ b_3, \ldots, b_{2n-1} \ \psi \ c.$$

The proof of the assertion can be found in [39, Proposition 1.5.11].

Proposition 1.1.14. [39] *Let ϕ and ψ be equivalences on a set S such that $\phi \circ \psi = \psi \circ \phi$. Then, $\phi \vee \psi = \phi \circ \psi$.*

The proof of the result appears in [39, Corollary 1.5.12].

Proposition 1.1.15. [39] *The equivalence relations $\mathcal{R}$ and $\mathcal{L}$ commute:*

$$\mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}.$$

The proof of the above result appears in [39, Proposition 2.1.3].

There are also two more Green's relations induced by the equivalence relations $\mathcal{R}$ and $\mathcal{L}$. Define the equivalence relations

$$\mathcal{H} := \mathcal{R} \bigcap \mathcal{L} \;\text{ and }\; \mathcal{D} := \mathcal{R} \vee \mathcal{L}\text{-the join of } \mathcal{L} \text{ and } \mathcal{R} \text{ relations.}$$

In fact, with a consideration of Propositions 1.1.15 and 1.1.14, the Green's relation $\mathcal{D}$ can also be characterised as

$$\mathcal{D} = \mathcal{R} \vee \mathcal{L} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}. \tag{1.2}$$

This enables us to rephrase the definition of the relation $\mathcal{D}$ as follows:

$$\begin{aligned} a \;\mathcal{D}\; b \;\text{ if and only if there exists } s \in S \text{ such that } \;a \;\mathcal{R}\; s \;\mathcal{L}\; b.\\ \text{if and only if there exists } t \in S \text{ such that } \;a \;\mathcal{L}\; t \;\mathcal{R}\; b. \end{aligned} \tag{1.3}$$

Moreover, the relation $\mathcal{H}$ can also be described as

$$a \;\mathcal{H}\; b \;\text{ if and only if }\; a \;\mathcal{R}\; b \text{ and } a \;\mathcal{L}\; b. \tag{1.4}$$

In view of Proposition 1.1.10, it is clear that $\mathcal{R} \subseteq \mathcal{J}$ and $\mathcal{L} \subseteq \mathcal{J}$. Moreover, $\mathcal{D} \subseteq \mathcal{J}$ as $\mathcal{D} = \mathcal{R} \vee \mathcal{L}$, and $\mathcal{H} \subseteq \mathcal{R}$ and $\mathcal{H} \subseteq \mathcal{L}$ by definition.

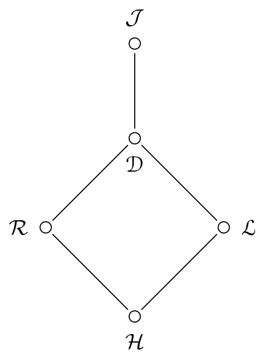


Figure 1.1: Hasse diagram of Green's equivalence relations of a semigroup

The next assertion shows that, in certain cases, the $\mathcal{D}$- and $\mathcal{J}$-relations coincide.

Proposition 1.1.16. [39] *If S is a finite semigroup, then $\mathcal{D} = \mathcal{J}$.*

The proof of the above assertion can be found in [39, Proposition 2.1.4].

As $\mathcal{H}$, $\mathcal{R}$, $\mathcal{L}$, $\mathcal{D}$, and $\mathcal{J}$ are all equivalence relations, each one of them naturally splits a semigroup S into equivalence classes, namely, the $\mathcal{H}$-, $\mathcal{R}$-, $\mathcal{L}$-, $\mathcal{D}$-, $\mathcal{J}$-classes of S. In other words, if $s \in S$, the $\mathcal{J}$-class containing s, for instance, is the set of all elements in S that are $\mathcal{J}$-related to s :

$$J_s = \{t \in S : t \;\mathcal{J}\; s\};$$

and all equivalence classes corresponding to the other Green's relations are obtained in a similar manner. It is also worth to mentioning that the inclusion order presented in (1.1) contributes in introducing a partial order among their corresponding equivalence classes as shown below.

$$\begin{aligned} R_a \leq R_b \quad &\text{if} \quad aS^1 \subseteq bS^1; \\ L_a \leq L_b \quad &\text{if} \quad S^1a \subseteq S^1b; \\ J_a \leq J_b \quad &\text{if} \quad S^1aS^1 \subseteq S^1bS^1. \end{aligned} \tag{1.5}$$

It is evident that for all $a \in S$ and $s, s' \in S^1$,

$$L_{sa} \leq L_a, \ R_{as} \leq R_a, \ J_{ss'} \leq J_s, \ J_{ss'} \leq J_{s'}, \ J_{sas'} \leq J_a. \tag{1.6}$$

If $a \in S$, we can redenote the principal (two-sided) ideal S^1aS^1 generated by the element a as $J(a)$. The following result appears in [39, Section 3.1], and gives an alternative characterisation of $J(a)$. The proof is provided here.

Proposition 1.1.17. [39] *Let S be a semigroup and J be a $\mathcal{J}$-class of S with $a \in J$. Then, $J(a) = \bigcup_{J' \leq J} J'$.*

Proof. If $b \in J(a)$, then there exists $s, s' \in S$ such that $b = sas'$. Consider the $\mathcal{J}$-class $J_{sas'}$ to which sas' belongs. It follows that $J_{sas'} \leq J_a (= J)$ by (1.6). Thus, the $\mathcal{J}$-class $J_{sas'}$ is one of the J' that is less than or equal to J; therefore, it belongs to the union. In particular, $b = sas' \in \bigcup_{J' \leq J} J'$; hence, $J(a) \subset \bigcup_{J' \leq J} J'$. To show the other direction, let $c \in \bigcup_{J' \leq J} J'$. Thus, there is a J' in the union such that $c \in J' \leq J$. In other words, $J_c \leq J_a$; thus, $J(c) \subseteq J(a)$ by (1.5). In particular, $c \in J(c) \subseteq J(a)$. Hence, $\bigcup_{J' \leq J} J' \subseteq J(a)$, and then we obtain the equality. $\square$

Definition 1.1.18. [73] A semigroup S is called stable if both

$$s \ \mathcal{J} \ sx \iff s \ \mathcal{R} \ sx \text{ and also } s \ \mathcal{J} \ xs \iff s \ \mathcal{L} \ xs,$$

for all $s, x \in S$.

Theorem 1.1.19. [73] Every finite semigroup is stable.

The proof of the assertion can be found in [73, Theorem A.2.4]. The next assertions are of considerable use in Section 4.2.1.

Proposition 1.1.20. *Let S be a finite semigroup and $e \in S$ be an idempotent. Then, for all $s \in S$, if $es \notin R_e$, then $es \in J'$ where $J' < J_e$.*

Proof. Consider S^1esS^1; the ideal generated by es. Observe that $S^1esS^1 \subseteq S^1eS^1$; this implies that $es \in J_{es} \leq J_e$ by (1.6). It suffices to show that es does not lie in the $\mathcal{J}$-class J_e. Since S is finite and $es \notin R_e$, then $es \notin J_e$ by Theorem 1.1.19. Hence, $es \in J' < J_e$. $\square$

According to [27], let S be a finite monoid and J be a (fixed) $\mathcal{J}$-class of S. Define the following set:

$$[J]^{\ngeq} = \{s \in S : J \nsubseteq J(s)\}.$$

Note that such a collection forms an ideal of S for the following reason: if $t \in [J]^{\ngeq}$, then $J \nsubseteq J(t)$. Fix $s, s' \in S$ and consider the $\mathcal{J}$-class $J_{sts'}$. By (1.6), we know that $J_{sts'} \leq J_t$, and this implies that $J(sts') \subseteq J(t)$ by (1.5). Since $J \nsubseteq J(t)$, then in particular $J \nsubseteq J(sts')$. Thus, $sts' \in [J]^{\ngeq}$ and hence $[J]^{\ngeq}$ is an ideal. The following proposition provides an alternative characterisation of the ideal $[J]^{\ngeq}$.

Proposition 1.1.21. *Let S be a finite monoid and J be a (fixed) $\mathcal{J}$-class of S. Then,* $[J]^{\ngeq} = \bigcup\limits_{J'' \ngeq J} J''$.

Thus, $[J]^{\ngeq}$ is the ideal of all elements of S that are not $\mathcal{J}$-above some (hence any) elements of J.

Proof. Let $a \in [J]^{\ngeq}$; that is, $J \nsubseteq J(a)$. In view of Proposition 1.1.17, we obtain that J does not belong to $\bigcup_{J' \leq J_a} J'$. In other words, $J \nleq J_a$. Thus, J_a is one of J'' in the $\bigcup_{J'' \ngeq J} J''$. In particular, $a \in \bigcup_{J'' \ngeq J} J''$. Hence, $[J]^{\ngeq} \subseteq \bigcup_{J'' \ngeq J} J''$. Conversely, pick any $b \in \bigcup_{J'' \ngeq J} J''$; thus, there is a J'' such that $b \in J'' \ngeq J$. In other words, $J \nleq J_b (= J'')$. Let $r \in J$, and rewrite the $\mathcal{J}$-class J as J_r; thus, $J_r \nleq J_b$. Hence, $J(r) \nsubseteq J(b)$ by (1.5). This implies that $J \nsubseteq J(b)$ as $J \subseteq J(r)$. Therefore, $b \in [J]^{\ngeq}$ by definition, and then $\bigcup_{J'' \ngeq J} J'' \subseteq [J]^{\ngeq}$. Hence, we obtain the equality. $\square$

Fix a $\mathcal{J}$-class J and $a \in J$. Then, Figure 1.2 illustrates the ideals $[J]^{\ngeq}$ and $J(a)$ among the ordering of the $\mathcal{J}$-classes of a semigroup S.

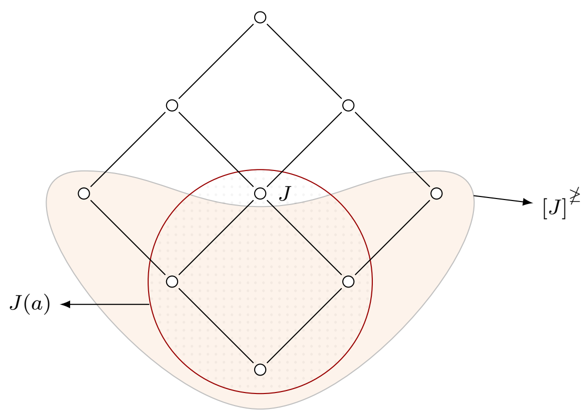


Figure 1.2: Generic picture of the ordering of $\mathcal{J}$-classes of S with $J(a)$ and $[J]^{\ngeq}$

The remainder of this section investigates the structure of $\mathcal{D}$-classes of a semigroup S. In view of the the definition of the equivalence relation $\mathcal{D}$ presented in (1.2), each $\mathcal{D}$-class in a semigroup S is a union of $\mathcal{R}$-classes as well as $\mathcal{L}$-classes. Moreover, the intersection of an $\mathcal{R}$-class and an $\mathcal{L}$-class might be empty or an $\mathcal{H}$-class. Assume that the intersection of an $\mathcal{R}$-class R_a and an $\mathcal{L}$-class L_b is non-trivial; that is, there exists an element $c \in S$ such that $c \in R_a \bigcap L_b$. It obviously implies that $a \mathcal{R} c \mathcal{L} b$, and then both a and b belong to the $\mathcal{D}$-class D. It immediately follows that $R_a \subseteq D$ and $L_b \subseteq D$.

In contrast, with a careful consideration of the definition of $\mathcal{D}$, $a \mathcal{D} b$ if and only if there exists $c \in S$ such that $a \mathcal{R} c \mathcal{L} b$. Thus, $c \in R_a \bigcap L_b$; that is, $R_a \bigcap L_b \neq \emptyset$. As $\mathcal{D}$ is symmetric, $L_a \bigcap R_b \neq \emptyset$. In fact, the above observations suggest the following:

- Whenever an $\mathcal{R}$-class and an $\mathcal{L}$-class intersect, they must be contained in the same $\mathcal{D}$-class.

- Every $\mathcal{R}$-class and $\mathcal{L}$-class that belong to the same $\mathcal{D}$-class intersect non-trivially.

It turns out that any $\mathcal{D}$-class can be visualised as an "eggbox", in which the rows represent $\mathcal{R}$-classes, the columns represent $\mathcal{L}$-classes, and the cells represent $\mathcal{H}$-classes. The order of these rows and columns may be placed arbitrarily.

Definition 1.1.22. [39] Let S be a semigroup and $s \in S^1$. A map $\psi_s : S \longrightarrow S$ defined by $(a)\psi_s = as$ for all $a \in S$, is called a right translation. Dually, a map $\phi_t : S \longrightarrow S$ defined by $(a)\phi_t = ta$ for all $a \in S$, is called a left translation.

The next lemma describes more about the structure of $\mathcal{D}$-classes. It suggests that multiplication by some favourable semigroup elements produces a one-to-one correspondence between $\mathcal{L}$-classes containing the same $\mathcal{D}$-class.

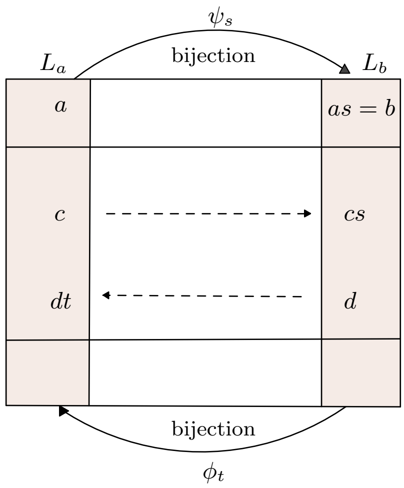
Lemma 1.1.23 (Green's Lemma). [26] *Let S be a semigroup and $a, b \in S$ such that $a \mathcal{R} b$. Let $s, t \in S$ where $as = b$ and $bt = a$. Then*

(i) The mapping $\psi_s : c \mapsto cs$ sends the $\mathcal{L}$-class L_a onto the $\mathcal{L}$-class L_b, and the mapping $\phi_t : d \mapsto dt$ sends the $\mathcal{L}$-class L_b onto the $\mathcal{L}$-class L_a.

(ii) $\psi_s : L_a \longrightarrow L_b$ and $\phi_t : L_b \longrightarrow L_a$ are mutually inverse; hence bijections; that is, $\psi_s\phi_t$ is the identity map on L_a, and dually $\phi_t\psi_s$ is the identity map on L_b.

(iii) Both maps ψ_s and ϕ_t preserve $\mathcal{R}$-classes; that is, for all $c \in L_a$ and $d \in L_b$, we have $c \mathcal{R} (c)\psi_s$ and $d \mathcal{R} (d)\phi_t$.

The proof of the above assertion appears in [39, Lemma 2.1.1]. The figure below demonstrates the statements of Green's Lemma 1.1.23.



Observation 1.1.24. It immediately follows from Green's lemma that any two $\mathcal{L}$-classes contained in a $\mathcal{D}$-class have the same cardinality. Moreover, both maps ψ_s and ϕ_t are mutually inverse bijections between the $\mathcal{H}$-classes belonging to L_a and L_b. If $x \in L_a$, consider the $\mathcal{H}$-class H_x containing x and observe that the restriction of ψ_s into H_x yields a bijection $\psi_s|_{H_x} : H_x \longrightarrow H_{xs}$, with inverse $\phi_t|_{H_{xs}} : H_{xs} \longrightarrow H_x$. Specifically, if $x = a$, then $\psi_s|_{H_a} : H_a \longrightarrow H_b$ and $\phi_t|_{H_b} : H_b \longrightarrow H_a$ are mutually inverse bijections. Therefore, any two $\mathcal{H}$-classes contained in a $\mathcal{D}$-class have the same cardinality.

The assertion below is a dual version of Lemma 1.1.23.

Lemma 1.1.25. [26] *Let S be a semigroup and $a, b \in S$ such that $a \mathcal{L} b$. Let $r, q \in S$ where $ra = b$ and $qb = a$. Then,*

(i) The mapping $\psi_r : c \mapsto rc$ sends the $\mathcal{R}$-class R_a onto the $\mathcal{R}$-class R_b, and the mapping $\phi_q : d \mapsto qd$ sends the $\mathcal{R}$-class R_b onto the $\mathcal{R}$-class R_a.

(ii) $\psi_r : R_a \longrightarrow R_b$ and $\phi_q : R_b \longrightarrow R_a$ are mutually inverse bijections.

(iii) Both maps ψ_r and ϕ_q preserve $\mathcal{L}$-classes; that is, for all $c \in R_a$ and $d \in R_b$, we have

$$c \mathcal{L} (c)\psi_r \quad \text{and} \quad d \mathcal{L} (d)\phi_q.$$

Analogous comments to Observation 1.1.24 apply to Green's lemma 1.1.25.

Proposition 1.1.26. [39] *Every idempotent e in a semigroup S is a left identity for the $\mathcal{R}$-class R_e and a right identity for the $\mathcal{L}$-class L_e.*

Proof. For any $c \in L_e$, there exists $s \in S^1$ such that $c = se$; then, consider

$$ce = (se)e = se^2 = se = c.$$

Thus, e is a right identity for every element belonging to the $\mathcal{L}$-class L_e. A dual argument can apply to show $ed = d$ for any $d \in R_e$. $\square$

The assertion below characterises the maximal subgroups within a semigroup S.

Theorem 1.1.27 (**Maximal Subgroup Theorem**). Let e be an idempotent in a semigroup S. Then, the $\mathcal{H}$-class H_e is the maximal subgroup of S with identity e.

The proof of the theorem appears in [39, Corollary 2.2.9 and Theorem 2.2.5].

Remark 1.1.28. If $e \mathcal{R} a$, then $e \cdot a = a$ and $a = ex$ for some $x \in S$. In view of Observation 1.1.24, there is a bijection between a maximal subgroup H_e and the $\mathcal{H}$-class H_a in which every element belongs to the $\mathcal{H}$-class H_a can be uniquely written as a product ga where $g \in H_e$. Figure 1.3 illustrates such a bijection, and this remark will play a vital role in Section 4.2

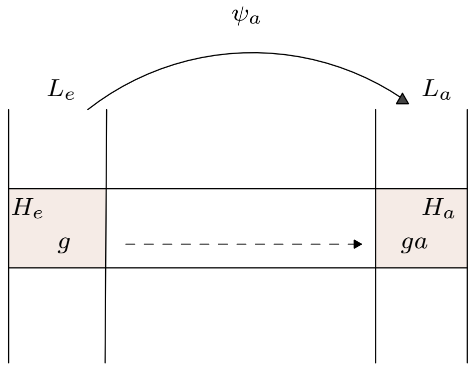


Figure 1.3: $(-)a : H_e \longrightarrow H_a$.

Definition 1.1.29. [39] Let S be a semigroup with an identity 1. Then, the group of units in S is the set U, which consists of all elements $s \in S$, such that $ss^{-1} = 1 = s^{-1}s$ for some $s^{-1} \in S$.

1.2 Regular and inverse semigroups

This section illuminates a pair of fundamental semigroup examples and follows on directly from the previous section's discussion of semigroup theory. Specifically, regular semigroups, introduced by Green [31], are investigated alongside inverse semigroups invented independently by Vagner and Preston [38]. This section also covers the partial order among semigroup elements, several instances of inverse semigroups, and investigates the condition under which the monoid of partial linear isomorphisms $ML(V)$ and the symmetric inverse monoid I_n can be considered factorizable.

Following this, the section explores a minor result (Proposition 1.2.23) that will play a part in a proof in a subsequent assertion. The result itself will address the position of the product of a pair of distinct $\mathcal{J}$-related idempotents of a finite inverse semigroup. Finally, after having stated and presented proofs for the series of assertions characterising regular and inverse semigroups, the implications of each for the symmetric inverse monoid I_n are considered.

Definition 1.2.1. [39] Let S be a semigroup. An element $s \in S$ is regular if there exists an element $s' \in S$ such that $ss's = s$. In addition, a semigroup S is called regular if each element is regular.

It follows that every idempotent is regular and all units are regular.

Proposition 1.2.2. [39] *Let s be a regular element in S. Then, every element belonging to the $\mathcal{D}$-class D_s is regular.*

Proof. As s is a regular element, we have $ss's = s$. If $t \in D_s$, then there exists $a \in S^1$ such that $s \mathrel{\mathcal{R}} a \mathrel{\mathcal{L}} t$. Since $s \mathrel{\mathcal{R}} a$, there exists $b, c \in S^1$ such that $s = ab$ and $a = sc$. Observe that

$$a = sc = (ss's)c = (ss')sc = ss'a = abs'a = ara,$$

where $r = bs'$. Thus, a is regular. Since $a \mathrel{\mathcal{L}} t$, a similar argument yields that t is regular; hence, the result holds. □

The preceding assertion indicates that either all elements of a $\mathcal{D}$-class are regular or none of them are. Moreover, any $\mathcal{D}$-class containing an idempotent is regular. Contrarily, a regular $\mathcal{D}$-class contains at least one idempotent, since if $s \in \mathcal{D}$ with $ss's = s$ for some $s' \in S^1$, then both ss' and $s's$ are idempotents. It immediately follows that every $\mathcal{L}$-class and every $\mathcal{R}$-class in a regular $\mathcal{D}$-class must contain an idempotent.

Define the full transformation semigroup $(T_X, \circ)$ to be the semigroup consisting of all maps from X into X under composition. If X is finite, we denote T_X by

T_n instead, where n is the size of X. Such a semigroup is regular for the following reason: Suppose $\sigma \in T_n$, and let us construct a map $\tau \in T_n$ such that $\sigma\tau\sigma = \sigma$. The approach of such a construction can be explained in the following manner:

- Let $j_1, j_2, \ldots, j_k$ be the images of σ. For each $j_r \in im(\sigma)$ with $1 \leq r \leq k$, let $\mathrm{Fib}(j_r)$ be the fibre (the preimage) of j_r.
- Partition the codomain of σ into blocks $B_1, B_2, \ldots, B_k$ in which each block contains only one image point j_r of σ. Note that such a partition is not unique.
- Consider the fibres of j_r and fix an element l_r from each fibre $\mathrm{Fib}(j_r)$.
- τ assigns each block B_r with the selected elements l_r. Such an assigning means that every element belonging to a block maps to a selected element l_r.

Hence, the constructed map τ satisfies the condition of regularity.

Definition 1.2.3. [39] Let s be an element of a semigroup S. An element $s^* \in S$ is said to be an inverse of s if and only if

$$ss^*s = s \quad \text{and} \quad s^*ss^* = s^*.$$

Remark 1.2.4. We intend to draw the distinction between inverses in a monoid and inverses in a group by using s^* as an inverse of s in the sense of a monoid, and s^{-1} as an inverse of s in the sense of a group.

An element with an inverse is obviously regular. Moreover, if s is a regular element, then there exists $t \in S$ such that $s = sts$. Let $s^* = tst$ and notice that

$$s^*ss^* = (tst)s(tst) = t(sts)(tst) = ts(tst) = t(sts)t = tst = s^*,$$

and

$$ss^*s = s(tst)s = (sts)ts = sts = s.$$

It follows that every regular element s has an inverse s^*. The example below gives us a clue that a regular element may not generally have a unique inverse. Thus, we denote the set of all inverses of s by $V(s)$.

Example 1.2.5. Consider the full transformation semigroup T_4 and choose the maps σ, τ, and α as shown in Figure 1.4. Then, a direct calculation shows that τ and α are both inverses of σ.

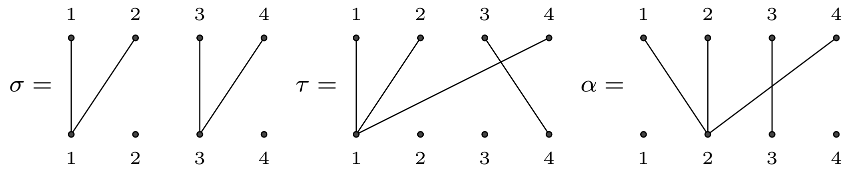
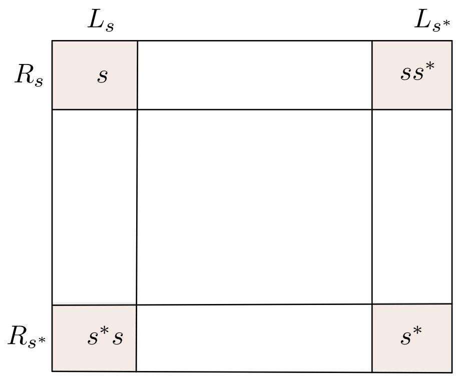


Figure 1.4: A map σ with its inverses.

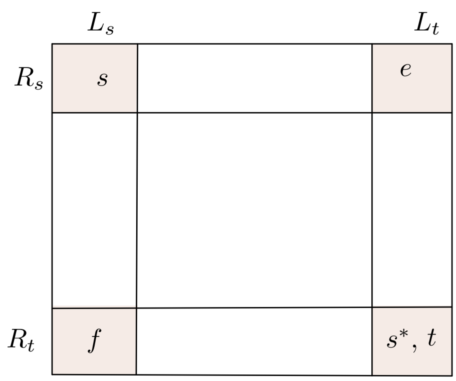
Observe that if $s \in S$ is an element that lies in a regular $\mathcal{D}$-class D, then each inverse s^* of s must belong to the same $\mathcal{D}$-class D as $s \mathrel{\mathcal{R}} ss^* \mathrel{\mathcal{L}} s^*$ and $s \mathrel{\mathcal{L}} s^*s \mathrel{\mathcal{R}} s^*$. The assertion below investigates the locations of the inverses in $\mathcal{D}$-class.

Theorem 1.2.6. [39] Let $s \in S$ be an element that belongs to a regular $\mathcal{D}$-class D.

(i) If $s^* \in V(s)$, then $s^* \in D$ and the $\mathcal{H}$-classes $R_s \cap L_{s^*}$, $L_s \cap R_{s^*}$ contain the idempotents ss^* and s^*s, respectively:



(ii) If $t \in D$ where $R_s \cap L_t$ and $L_s \cap R_t$ contain idempotents e and f respectively, then the $\mathcal{H}$-class H_t contains an inverse s^* of s, where $ss^* = e$ and $s^*s = f$.



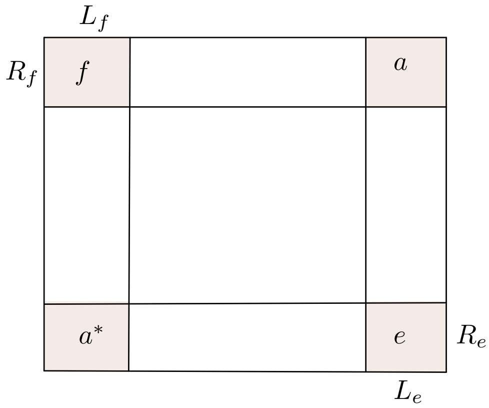
(iii) An $\mathcal{H}$-class H contains at most one inverse of s.

The proof of the assertion appears in [39, Theorem 2.3.4]. The proposition below shows how the $\mathcal{H}$-classes lying in the same (regular) $\mathcal{D}$-class behave.

Proposition 1.2.7. [39] *Let H_e and H_f be two $\mathcal{H}$-classes belonging to the same $\mathcal{D}$-class where e and f are idempotents. Then $H_e \cong H_f$.*

Proof. Since H_e and H_f lie in the same $\mathcal{D}$-class D, then both idempotents e and f particularly lie in D. It follows that $R_f \cap L_e \neq \emptyset$, so let $a \in R_f \cap L_e$. Then $fa = ae = a$ by Proposition 1.1.26. In view of Theorem 1.2.6(i), there exists a unique inverse a^* belongs to $L_f \cap R_e$; thus, $a^*f = ea^* = a^*$. Further, by Theorem 1.2.6(ii), we have

$$aa^* = f, \ a^*a = e. \tag{1.7}$$



As $fa = a$, utilising Lemma 1.1.23 and Remark 1.1.28 yields the following bijection

$$\begin{aligned} \psi_a|_{H_f} &: H_f \longrightarrow H_a \\ h &\mapsto ha \qquad \text{for all } h \in H_f. \end{aligned}$$

Moreover, since $a^*a = e$, then by Lemma 1.1.25, we obtain the following bijection:

$$\begin{aligned} \phi_{a^*}|_{H_a} &: H_a \longrightarrow H_e. \\ r &\mapsto a^*r \qquad \text{for all } r \in H_a. \end{aligned}$$

Therefore, the composition map $\gamma = \psi_a|_{H_f} \circ \phi_{a^*}|_{H_a}$ is a bijection from H_f onto H_e defined as follows: For all $h \in H_f$,

$$(h)\gamma = a^*ha,$$

where its inverse γ^{-1} is given by $(g)\gamma^{-1} = aga^*$ with $g \in H_e$. Figures A and B below illustrate both bijections. If $h, k \in H_f$, then

$$(h)\gamma(k)\gamma = (a^*ha)(a^*ka) = a^*h(aa^*)ka = a^*hfka = a^*hka = (hk)\gamma,$$

as f is the identity of the group H_f. Hence, γ is an isomorphism. $\square$

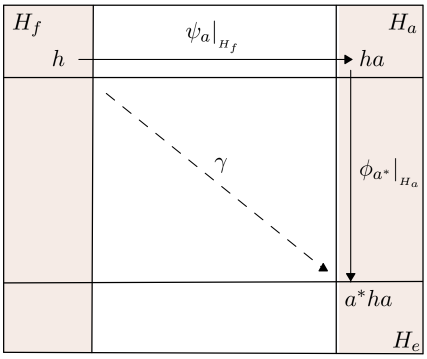


Fig A. γ: $H_f \longrightarrow H_e$

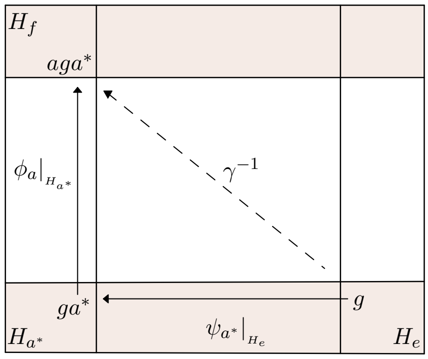


Fig B. γ^{-1}: $H_e \longrightarrow H_f$

Definition 1.2.8. [39] A semigroup S is called inverse if, for all $s \in S$, there is a unique element $s^* \in S$, such that $ss^*s = s$ and $s^*ss^* = s^*$.

The result below provides an alternative characterisation for inverse semigroups.

Theorem 1.2.9. [39] Let S be a semigroup. Then the following statements are equivalent:

(i) S is an inverse semigroup.

(ii) S is regular and its idempotents commute.

(iii) Every $\mathcal{L}$-class and $\mathcal{R}$-class contains exactly one idempotent.

The proof of the assertion can be found in [39, Theorem 5.1.1]. The next proposition presents a few additional properties of inverse semigroups.

Proposition 1.2.10. [48] *Let S be an inverse semigroup. Then,*

(i) $e^* = e$ *for all* $e \in S$;

(ii) $(a^*)^* = a$ *for all* $a \in S$;

(iii) $(a_1 a_2 \ \cdots \ a_n)^* = a_n^* a_{n-1}^* \ \cdots \ a_1^*$ *for all* $a_1, a_2, \ldots, a_n \in S$ *with* $n \geq 2$;

(iv) If e is an idempotent, then for all $s \in S$, the element s^*es is an idempotent;

(v) For any idempotent e and any $s \in S$, there exists an idempotent $f \in S$ such that $es = sf$.

(vi) For any idempotent e and any $s \in S$, there exists an idempotent $f \in S$ such that $se = fs$.

Proof. (i), (ii), and (iv) are straightforward. For (iii), let $n = 2$, and note that

$$a_1a_2(a_2^*a_1^*)a_1a_2 = a_1(a_2a_2^*)(a_1^*a_1)a_2 = a_1(a_1^*a_1)(a_2a_2^*)a_2 = a_1a_2,$$

as idempotents commute. Further,

$$a_2^*a_1^*(a_1a_2)a_2^*a_1^* = a_2^*(a_1^*a_1)(a_2a_2^*)a_1^* = a_2^*(a_2a_2^*)(a_1^*a_1)a_1^* = a_2^*a_1^*.$$

Thus $(a_1a_2)^* = a_2^*a_1^*$. The general case holds by induction. To show (v), consider the idempotent s^*es given by (iv) and observe that

$$s \cdot (s^*es) = ss^*es = ess^*s = es.$$

Putting $f = s^*es$ yields the result. Similarly, we prove (vi). $\square$

Definition 1.2.11. [39] Let S be an inverse semigroup. Define a partial order relation $\leq$ among the elements as follows: if $s, t \in S$, then $s \leq t$ if and only if there exists an idempotent $e \in S$ such that $s = et$.

Proposition 1.2.12. [48] *Let S be an inverse semigroup. Then,*

(i) $s \leq t$ if and only if there exists an idempotent $f \in S$ such that $s = tf$.

(ii) For any $s, t, r, u \in S$, if $s \leq t$ and $r \leq u$, then $sr \leq tu$.

Proof. (i) As $s \leq t$, we have $s = et$. It follows that $et = tf$ for some idempotent $f \in S$, by Proposition 1.2.10(v). Thus the result follows. The other direction follows from Proposition 1.2.10(vi). For (ii), since $s \leq t$ and $r \leq u$, then $s = et$ and $r = fu$, where e and f are any idempotents. Hence, $sr = etfu$, and by Proposition 1.2.10(v), $sr = egtu$ where g is an idempotent; that is, $sr = (eg)tu$. It follows that $sr \leq tu$. $\square$

Proposition 1.2.13. [48] *Let S be an inverse semigroup and $a, b \in S$ then*

(i) $a \mathcal{R} b$ if and only if $aa^* = bb^*$;

(ii) $a \mathcal{L} b$ if and only if $a^*a = b^*b$.

Proof. We only show (i), as the proof of (ii) dually holds. Suppose $a \mathcal{R} b$. Then by Proposition 1.1.10(i), there are $r, u \in S^1$ such that $au = b$ and $a = br$. Consider

$$aa^* = br(br)^* = brr^*b^*. \tag{1.8}$$

Observe that, as $brr^* = b(rr^*)$ and rr^* is idempotent, it follows by Proposition 1.2.12(i) that $brr^* \leq b$. Similarly, as $rr^*b^* = (rr^*)b^*$, we have $rr^*b^* \leq b^*$. Thus, in view of Proposition 1.2.12(2), we obtain

$$(brr^*)(rr^*b^*) \leq bb^* \iff b(rr^*r)r^*b^* \leq bb^* \iff brr^*b^* \leq bb^*.$$

Therefore, (1.8) becomes $aa^* \leq bb^*$. Dually, $bb^* \leq aa^*$, and thus $aa^* = bb^*$. $\square$

Proposition 1.2.14. [48] *Let S be an inverse semigroup. Then any $\mathcal{D}$-class D has an equal number of rows and columns.*

Proof. Since S is inverse, it is known from Theorem 1.2.9(iii) that every $\mathcal{L}$-class and $\mathcal{R}$-class contains only one idempotent. As the order of rows and columns is arbitrary, their positions can be rearranged so that all $\mathcal{H}$-classes containing idempotents appear in the "diagonal" of the eggbox D. Hence, any $\mathcal{D}$-class D has a square shape. $\square$

Example 1.2.15. Given a non-empty set X, a partial permutation is defined as a bijection $\sigma : dom\ \sigma \mapsto im\ \sigma$, where $dom\ \sigma$ and $im\ \sigma$ are subsets of X. Since σ is a bijection, there is a map $\sigma^* : im\ \sigma \mapsto dom\ \sigma$, such that

$$\sigma\sigma^* = id_{dom\sigma}, \quad \sigma^*\sigma = id_{im\sigma}.$$

Clearly, $\sigma\sigma^*\sigma = \sigma$ and $\sigma^*\sigma\sigma^* = \sigma^*$. The set I_X of all partial permutations of X forms a monoid under the composition of partial maps, called the symmetric inverse monoid on X, where if $\sigma, \tau \in I_X$, then

$$dom\ \sigma\tau = (im\ \sigma \bigcap dom\ \tau)\sigma^*,\ im\ \sigma\tau = (im\ \sigma \bigcap dom\ \tau)\tau,$$

and $\sigma\tau$ is the map composition $dom\ \sigma\tau \mapsto im\ \sigma\tau$. If $X = \{1, 2, ..., n\}$, then write I_n instead of I_X. In fact, [39, Theorem 5.1.5] shows that the idempotents of I_X are partial identities id_Y, where $Y \subseteq X$, and they commute. Hence, I_X is an inverse monoid.

Example 1.2.16. [19] Let V be a vector space over $\mathbb{R}$. A partial linear isomorphism of V is a linear isomorphism $\alpha : Y \mapsto Y'$ where $Y,\ Y'$ are subspaces of V. Let $ML(V)$ be the set of the partial linear isomorphisms of V. Since $\alpha : Y \mapsto Y'$ is a linear isomorphism, there is a map $\alpha^* : Y' \mapsto Y$ with

$$\alpha\alpha^* = id_Y, \quad \alpha^*\alpha = id_{Y'}.$$

Obviously, $\alpha\alpha^*\alpha = \alpha,\ \alpha^*\alpha\alpha^* = \alpha^*$. In fact, the set $ML(V)$ forms a monoid under the composition of partial maps; that is, if $\alpha : Y \mapsto Y'$, and $\beta : Z \mapsto Z'$, then the composition map $\alpha\beta$ has the following domain:

$$dom\ \alpha\beta = (im\ \alpha \bigcap dom\ \beta)\alpha^* = (Y' \bigcap Z)\alpha^* = Y'\alpha^* \bigcap Z\alpha^* = Y \bigcap Z\alpha^*, \tag{1.9}$$

where $Z\alpha^*$ is the image of Z under the partial isomorphism α^*, and it has the following image:

$$im\ \alpha\beta = (im\ \alpha \bigcap dom\ \beta)\beta = (Y' \bigcap Z)\beta = Y'\beta \bigcap Z\beta = Y'\beta \bigcap Z'.$$

The idempotents of $ML(V)$ are partial identities on the subspaces of V. Moreover, it is clear that $ML(V) \subset I_V$, since a vector space V can be considered as a set and its subspaces as subsets; thus, it turns out that $ML(V)$ is a submonoid of I_V. Hence, for every $\alpha \in ML(V)$, α has a unique inverse α^*; then $ML(V)$ is an inverse (sub)monoid of I_V, called the general linear monoid on V.

Notice that the interpretation of the partial order relation $\leq$ in I_n is described in the following manner: $\alpha \leq \beta$ if and only if α is a restriction of β; that is, $\alpha \leq \beta$ if and only if $dom\ \alpha \subseteq dom\ \beta$ and $x\alpha = x\beta$ for all $x \in dom\ \alpha$, since $\alpha \leq \beta$ requires the existence of an idempotent e such that $\alpha = e\beta$. Moreover, $e\alpha = ee\beta = e\beta = \alpha$. Thus, observe that

$$\alpha\alpha^*\beta = (e\alpha)\alpha^*\beta = e(\alpha\alpha^*)\beta = (\alpha\alpha^*)e\beta = \alpha\alpha^*\alpha = \alpha.$$

Therefore,

$$\begin{aligned} \alpha \leq \beta &\iff \alpha = \alpha\alpha^*\beta, \\ &\iff \alpha = id_{dom\alpha}\beta, \\ &\iff dom\ \alpha \subseteq dom\ \beta \ \text{ and } \ x\alpha = x\beta \text{ for all } x \in dom\ \alpha, \\ &\iff \alpha = \beta_{dom\alpha}. \end{aligned}$$

Similar result holds for $ML(V)$.

It must be noted that the units of the symmetric inverse monoid I_X are the elements of the symmetric group S_X, whereas the units of $ML(V)$ are the elements of the general linear group $GL(V)$.

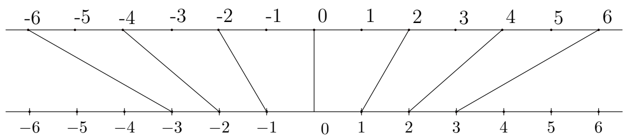
Definition 1.2.17. [39] An inverse monoid S is said to be factorizable if for all elements $s \in S$, there is a unit $g \in U$ such that $s \leq g$. In other words, for all $s \in S$, *thereexists* a unit $g \in U$ and an idempotent $e \in E$ such that $s = eg$; that is, $S = EU$.

Notice that if M is an inverse submonoid of $\mathcal{I}_n$, then M is factorizable if and only if every element $\alpha \in M$ is a restriction of a unit $g \in M$.

Definition 1.2.18. Let V be a vector space over $\mathbb{R}$ and $g \in GL(V)$ be a linear isomorphism. If Y is a subspace of V, then the restriction of g to Y, denoted by g_Y, is a partial linear isomorphism $Y \mapsto Yg$ defined by

$$(v)g_Y = \begin{cases} vg & v \in Y, \\ \text{undefined} & v \notin Y. \end{cases}$$

It must be noted that if V is a finite dimensional vector space, then for any partial isomorphism $\alpha : X \to Y$, where X and Y are subspaces of V, there always exists a (full) linear isomorphism $g : V \to V$ whose restriction to X is α; that is, $g_X = \alpha$. However, if V has infinite dimension, then finding such a linear isomorphism $g : V \to V$ is not necessarily possible. For example, let V be the real vector space with basis $\mathbb{Z}$ and U be a proper subspace spanned by $2\mathbb{Z}$. Define a map $f : U \to V$ on bases by $f : 2m \mapsto m$, with $m \in \mathbb{Z}$.



Then $f : U \to V$ is a linear isomorphism; hence, it is onto. Consequently, f cannot be extended to a linear isomorphism $\hat{f} : V \to V$.

Note that the above discussion provides a clue with regard to whether the general linear monoid $ML(V)$ could be factorizable. Indeed, the general linear monoid $ML(V)$ is factorizable if and only if the vector space V is finite dimensional [19, Section 7], because otherwise we cannot guarantee that every partial linear isomorphism $\alpha \in ML(V)$ is a restriction of a unit $g \in GL(V)$. Similarly, the symmetric inverse monoid I_X is factorizable if and only if the set X is finite [48, Proposition 2.1.1]. This is because if X is a finite set and $\sigma \in I_X$ is a partial bijection, then

$$|X \setminus dom\ \sigma| = |X \setminus im\ \sigma|\,.$$

This implies that a bijective map $\tau : X \setminus dom\ \sigma \to X \setminus im\ \sigma$ can be found. Hence, $\sigma \cup \tau : X \to X$ is a permutation of X that extends σ. However, if X is infinite, then the example can be used (here, consider $2\mathbb{Z}$ and $\mathbb{Z}$ as the sets) to show that it cannot be promised that every partial permutation $\sigma \in I_X$ can be a restriction of a bijection (permutation) on X.

The next assertion is regarded as the analogous inverse semigroup assertion to Cayley's Theorem in group theory:

Theorem 1.2.19 (Wagner-Preston Theorem). Let S be an inverse semigroup. Then, there exists a set X and a monomorphism ϕ from S into I_X.

The proof of the above result can be found in [39, Theorem 5.1.7].

This section concludes with the interpretation of some of the notions presented earlier in this chapter in the special case of I_n. The benefit of such an illustration will become clear in Section 5.2.

Proposition 1.2.20 (Green's relations for I_n). *Let I_n be the symmetric inverse monoid. Then,*

(i) $(\sigma, \tau) \in \mathcal{R}$ *if and only if* $dom(\sigma) = dom(\tau)$;

(ii) $(\sigma, \tau) \in \mathcal{L}$ *if and only if* $im(\sigma) = im(\tau)$;

(iii) $(\sigma, \tau) \in \mathcal{D}$ *if and only if there exists* $\gamma \in I_n$, *where* $dom(\sigma) = dom(\gamma)$ *and* $im(\gamma) = im(\tau)$; *and alternatively, there exists* $\gamma' \in I_n$, *with* $im(\gamma') = im(\sigma)$, *and* $dom(\gamma') = dom(\tau)$;

(iv) $(\sigma, \tau) \in \mathcal{J}$ *if and only if* $|dom(\sigma)| = |dom(\tau)|$ *if and only if* $|im(\sigma)| = |im(\tau)|$.

The proof of the above result was obtained by Reilly [69, Lemma 2] and subsequently by Munn [55, Lemma 1.2]. Such a result contributes to simplifying the description of any $\mathcal{J}$-class of I_n, as all we need is to illuminate the domains of partial permutations and partition the symmetric inverse monoid I_n, based on all possible sizes of these domains. Observe that the largest domain size we could have is n, and the smallest domain size is zero. Therefore, we may label the $\mathcal{J}$-classes of I_n as follows.

$$J_{_0},\ J_{_1},\ \cdots,\ J_{_k},\ \cdots,\ J_{_{n-1}},\ J_{_n}, \tag{1.10}$$

where $J_{_0}$ indicates that the $\mathcal{J}$-class consists of the zero partial permutation $0 : \emptyset \to \emptyset$, and $J_{_k}$ indicates that the $\mathcal{J}$-class consists of all partial permutations $\sigma \in I_n$ whose domains are subsets $Y \subseteq \{1, \ \ldots, \ n\}$ with size k.

Further, since I_n is a finite inverse monoid, $\mathcal{D}$ and $\mathcal{J}$ coincides, and by Proposition 1.2.14, each $\mathcal{J}$-class $J_{_k}$ consists of an equal numbers of rows and columns. The rows of a $\mathcal{J}$-class $J_{_k}$ are $\mathcal{R}$-classes, and its columns are $\mathcal{L}$-classes. However, since any two partial permutations σ and τ are $\mathcal{R}$-related if and only if they have the same domain, then all partial permutations placed in an $\mathcal{R}$-class must have the same domain of size k. Thus, the rows of the $\mathcal{J}$-class can be labelled by all possible domains of σ's with size k.

Similarly, two partial permutations are $\mathcal{L}$-equivalent if they have the same image; hence, the columns of the $\mathcal{J}$-class are labelled by all possible images of γ with size k. It follows that the columns and the rows of the $\mathcal{J}$-class $J_{_k}$ are labelled by all subsets $Y \subset \{1, \ \ldots, \ n\}$ with $|Y| = k$, and $0 \leq k \leq n$. Then, the number of rows and columns inside a $\mathcal{J}$-class $J_{_k}$ are $\binom{n}{k}$.

In view of Theorem 1.1.27, an $\mathcal{H}$-class H with idempotent e forms a maximal subgroup H_e, and any idempotent $e \in I_n$ is a partial identity on a set $Y = \{y_1, \ldots, y_k\}$; e fixes Y point-wise, and it is undefined in $\{1, \ \ldots, \ n\} \setminus Y$. It follows that a maximal subgroup H_e is isomorphic to the symmetric group S_k. By considering Theorem 1.2.9(iii) and Proposition 1.2.14, it is deduced that there are $\binom{n}{k}$ maximal subgroups S_k belonging in $J_{_k}$-class.

In subsequent chapters, the significant contribution of such groups to obtaining the irreducible representations of I_n will be shown. Further, it must also be noted that the $\mathcal{J}$-classes of I_n are linearly (totally) ordered by the sizes of the domains [21, Section 3], and hence they can be drawn as depicted in Figure 1.5.

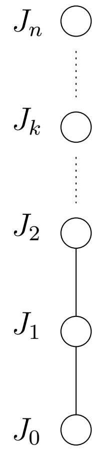


Figure 1.5: The $\mathcal{J}$-classes of I_n.

Remark 1.2.21. Consider an idempotent $e \in J_k$, and recall that e is a partial identity id_X on X where $X \subseteq \{1, \cdots, n\}$ with $|X| = k$. Consider the $\mathcal{R}$-class of e and observe that if $\sigma \in R_e$, then $\mathrm{dom}(\sigma) = X = \mathrm{dom}(e)$. Let $\sigma : X \longrightarrow Y$ where $Y \subseteq \{1, \ldots, n\}$ with $|Y| = k$. For any $\tau \in I_n$, the composition $\sigma\tau$ lies in R_e if and only if $dom(\sigma\tau) = dom(e) = X$ if and only if $Y = im(\sigma) \subseteq \mathrm{dom}(\tau)$. Hence,

$$\sigma\tau \in R_e \text{ if and only if } Y \subseteq \mathrm{dom}(\tau). \tag{1.11}$$

The interpretation of the ideals presented in Proposition 1.1.17 and Proposition 1.1.21 for the symmetric inverse monoid I_n are illustrated in the following example.

Example 1.2.22. Consider the symmetric inverse monoid I_n, and fix a $\mathcal{J}$-class J_k such that $\gamma \in J_k$ with $1 \leq k \leq n$. Consider a partial map $\alpha\gamma\beta$ where $\alpha, \beta \in I_n$. By the composition $\alpha\gamma\beta$ presented in Figure 1.6, we deduce that $|\mathrm{dom}(\alpha\gamma\beta)| \leq |\mathrm{dom}(\gamma)| = k$. Thus, the partial map $\alpha\gamma\beta$ belongs to a $\mathcal{J}$-class J' where $J' \leq J_k$.

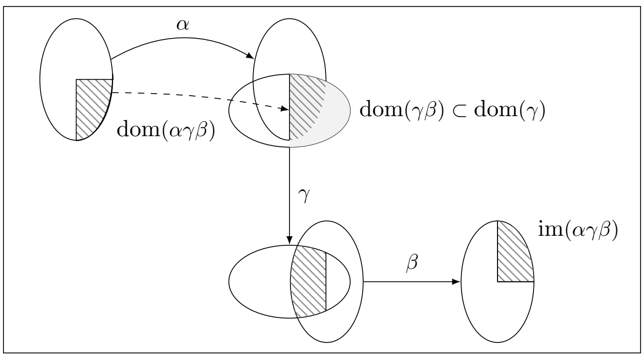


Figure 1.6: The composition of the partial map $\alpha\gamma\beta$

Hence $J(\gamma) = S^1\gamma S^1 = \bigcup_{J' \leq J_k} J' = \bigcup_{i=0}^{k} J_i$. Moreover, in view of Proposition 1.1.21 and the total order among $\mathcal{J}$-classes of I_n, we obtain

$$[J_k]^{\ngeq} = \bigcup_{J'' \ngeq J_k} J'' = \bigcup_{i=1}^{k} J_{i-1}.$$

Both concepts are illustrated in Figure 1.7:

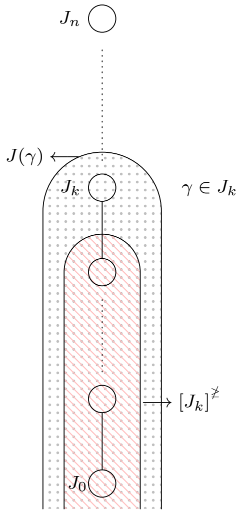


Figure 1.7: The illustration of $J(\gamma)$ and $[J_k]^{\ngeq}$ of I_n.

Henceforth, all semigroups S are **finite** and the relation $\mathcal{J}$ is written for both $\mathcal{J}$ and $\mathcal{D}$ when no confusion should arise. The result below will be of considerable use in Section 5.2.2.

Proposition 1.2.23. *Let S be a finite inverse semigroup, and e, f be distinct $\mathcal{J}$-related idempotents. Then, their product ef does not lie in the same $\mathcal{J}$-class.*

Proof. Suppose e and f are two distinct $\mathcal{J}$-related idempotents such that $e \mathrel{\mathcal{J}} f \mathrel{\mathcal{J}} ef$. By Theorem 1.2.19, we can embed S into I_S using a morphism ϕ. Observe that if g is an idempotent in S, then $g\phi = (g^2)\phi = (g\phi)^2$; thus, $g\phi$ is an idempotent in I_S. Moreover, if $x, y \in S$ such that $x \mathrel{\mathcal{L}} y$, then $x = zy$ and $y = z'x$ for some $z, z' \in S$. By applying ϕ, we have $x\phi = (zy)\phi = z\phi y\phi$, and $y\phi = (z'x)\phi = z'\phi x\phi$; thus, $x\phi \mathrel{\mathcal{L}} y\phi$. The argument for $\mathcal{R}$ is dual. Hence, if $x \mathrel{\mathcal{J}} y$, then there exists $c \in S$ such that $x \mathrel{\mathcal{L}} c \mathrel{\mathcal{R}} y$; it follows that $x\phi \mathrel{\mathcal{L}} c\phi \mathrel{\mathcal{R}} y\phi$. Therefore, $x\phi \mathrel{\mathcal{J}} y\phi$. Now, it follows from our assumption that

$$e\phi \mathrel{\mathcal{J}} f\phi \mathrel{\mathcal{J}} (e\phi f\phi). \tag{1.12}$$

Observe that $e\phi$ and $f\phi$ are distinct idempotents (partial identities maps) in I_S as ϕ is injection and $e \neq f$. Thus, $\mathrm{im}(e\phi) \neq \mathrm{dom}(f\phi)$. By the composition $e\phi f\phi$ presented in Figure 1.8, we deduce $|\mathrm{dom}(e\phi f\phi)| < |\mathrm{dom}(e\phi)|$.

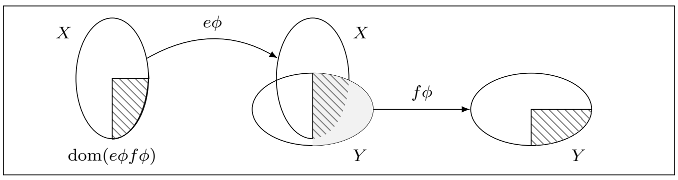
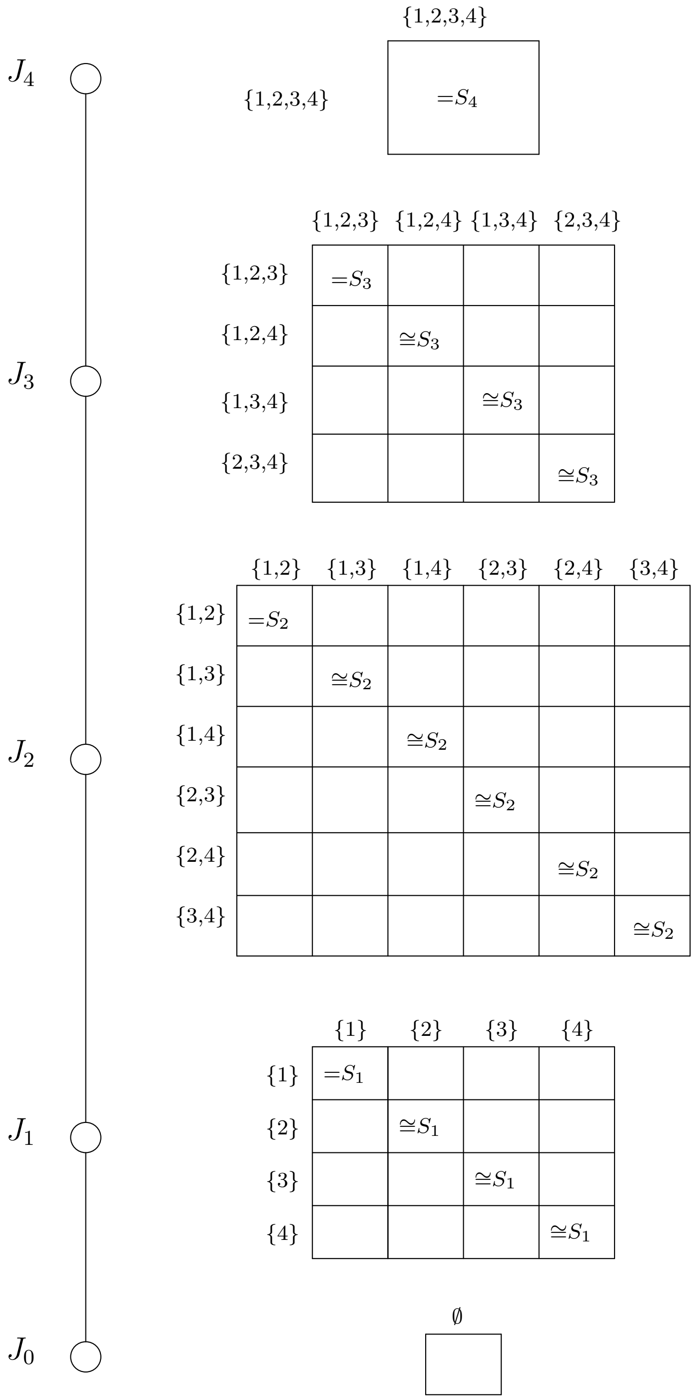


Figure 1.8: The composition of the partial identities $e\phi f\phi$

In other words, $e\phi f\phi$ lies in the $\mathcal{J}$-class that is lower than the $\mathcal{J}$-class containing $e\phi$. This contradicts the result (1.12) that is deduced from the assumption. Hence, the product $ef \notin J_e$. $\square$

Example 1.2.24. Let $n = 3$. The following diagram illustrates the $\mathcal{J}$-classes of the symmetric inverse monoid I_4.



Chapter 2

Introduction to reflection groups and reflection monoids

2.1 Reflections and reflection groups

Harold Coxeter [14] initially classified finite Euclidean reflection groups in 1934. Many years later, in 1954, the theory of finite complex reflection groups was established by Shepherd and Todd. Such groups can also arise from a root system, a collection of non-zero vectors in a Euclidean space satisfying certain properties. The latter concept is at the heart of the theory of Lie groups and Lie algebras, and particularly for the representation of semisimple Lie algebras.

This section provides a brief illustration of certain reflection groups which are relevant to the forthcoming chapters, and it is organized as follows. The section begins with a review of the reflections over a Euclidean vector space. Subsequently, we will explore reflection groups and offer proofs of various properties that emerge from the literature. We then also explore the concept of root systems and demonstrate how they can be utilised to transfer back and forth from finite reflection groups. We will finish the section by outlining three root systems and their relevant reflection groups.

Definition 2.1.1. [74] A Euclidean vector space V is a vector space over the real numbers $\mathbb{R}$ with an inner product defined on it.

Definition 2.1.2. [74] Let V be a Euclidean vector space. Any vectors v and u are said to be orthogonal if and only if $\langle v, u\rangle = 0$.

Definition 2.1.3. [74] Let V be a vector space over $\mathbb{R}$. A linear hyperplane in V is an $(n-1)$-dimensional subspace of V. Alternatively, for any non-zero vector $v \in V$, a subspace consists of all vectors that are orthogonal to a vector v, is also called the linear hyperplane determined by v; that is,

$$H_v = \{u \in V : \langle v, u\rangle = 0\}.$$

Throughout this thesis, we write mapping symbols on the right.

Definition 2.1.4. [40] Let V be a Euclidean vector space and H_v be a linear hyperplane orthogonal to the non-zero vector v. A reflection s_v in H_v is a linear map $V \to V$, such that

- $(u)s_v = u$, for all $u \in H_v$,
- $(v)s_v = -v$.

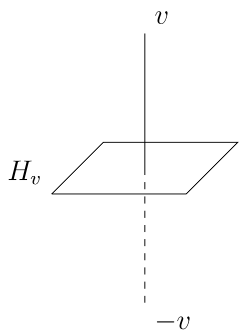


Figure 2.1: Effect of a reflection on a hyperplane H_v and a vector v.

Note that, since a reflection s_v is a linear map, then $(\lambda w + \mu w')s_v = \lambda(w)s_v + \mu(w')s_v$ for every $w, w' \in V$, $\lambda, \mu \in \mathbb{R}$, and $(\mathbf{0})s_v = \mathbf{0}$. Indeed, a reflection s_v is a bijective linear map, and for every reflection s_v, $(w)s_vs_v = w$; that is, $s_v^2 = id$ (the identity map). For all $0 \neq \lambda \in \mathbb{R}$, we have $s_v = s_{\lambda v}$.

Remark 2.1.5. [44] From the definition of a reflection, every reflection s_v is determined by a vector v. However, we can also define a reflection with respect to a reflecting hyperplane H by $s_{_H}$ instead of s_v as the following: A reflection $s_{_H}$ is a linear map $V \to V$ such that

- $(u)s_{_H} = u$, for all $u \in H$,
- $(v)s_{_H} = -v$, if v is perpendicular to H.

The lemma below suggests a formula for a reflection.

Lemma 2.1.6. [40] *Let s_v be a reflection determined by the non-zero vector $v \in V$ and H_v be its reflecting hyperplane. Then, for all $w \in V$, we have:*

$$(w)s_v = w - \frac{2\langle w, v\rangle}{\langle v, v\rangle}\, v.$$

Proof. By choosing a sensible basis, we can verify the formula by considering the effect of the right hand side on it. Alternatively, since the vector $w - \frac{\langle w,v\rangle}{\langle v,v\rangle}\, v \in V$, then

$$\langle w - \frac{\langle w, v\rangle}{\langle v, v\rangle} v, v\rangle = \langle w, v\rangle - \frac{\langle w, v\rangle}{\langle v, v\rangle}\langle v, v\rangle = \langle w, v\rangle - \langle w, v\rangle = 0.$$

This implies that $w - \frac{\langle w,v\rangle}{\langle v,v\rangle}\ v \in H_v$ and $(w - \frac{\langle w,v\rangle}{\langle v,v\rangle}\ v)s_v = w - \frac{\langle w,v\rangle}{\langle v,v\rangle}\ v$. Hence,

$$\begin{aligned}
(w)s_v &= (w - \frac{\langle w,v\rangle}{\langle v,v\rangle}\ v + \frac{\langle w,v\rangle}{\langle v,v\rangle}\ v)s_v,\\
&= (w - \frac{\langle w,v\rangle}{\langle v,v\rangle}\ v)s_v + \frac{\langle w,v\rangle}{\langle v,v\rangle}\ (v)s_v,\\
&= w - \frac{\langle w,v\rangle}{\langle v,v\rangle}\ v - \frac{\langle w,v\rangle}{\langle v,v\rangle}\ v,\\
&= w - \frac{2\langle w,v\rangle}{\langle v,v\rangle}\ v.
\end{aligned}$$

□

Definition 2.1.7. [74] A linear map $g : V \to V$ is said to be an orthogonal map if and only if $\langle wg, w'g\rangle = \langle w, w'\rangle$ for all $w, w' \in V$.

Using the formula in Lemma 2.1.6 and the inner product of V, it is easy to convince ourselves that every reflection is an orthogonal map; that is, $\langle ws_v, w's_v\rangle = \langle w, w'\rangle$. It follows that a reflection s_v preserves the length; that is,

$$ws_v = u \ \text{ implies } \ \|u\| = \|w\|\,. \tag{2.1}$$

Definition 2.1.8. [40] A reflection group W is a group generated by finitely many reflections.

Example 2.1.9. Let D_4 be the group of symmetries of a square. It contains four reflections $\{s_1, s_2, s_3, s_4\}$ and four rotations. Notice that each element of D_4 can be written as a product of reflections. The reason is clear for the reflections as each is self-generated.

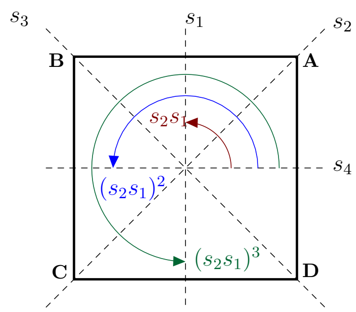


Figure 2.2: Rotations in D_4 are the powers of the reflections s_1 and s_2.

However, for the rotations, we need to think about the square as being divided into eight chambers where each chamber is bounded by two reflections. If we choose any chamber and compose its reflections together, we will obtain the quarter turn rotation counter-clockwise; that is, if we choose the chamber which is bounded by the reflections s_1 and s_2, then the first quarter turn rotation counter-clockwise is

s_2s_1. Observe that the other rotations are the powers of s_2s_1; hence, D_4 is a reflection group.

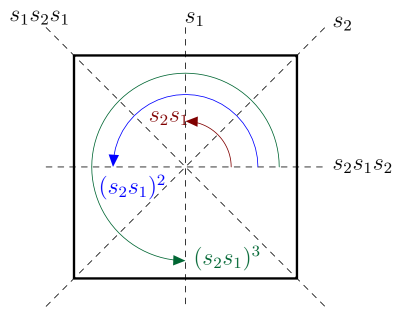


Figure 2.3: All elements of D_4 are composition of the reflections s_1 and s_2

In fact, the reflections s_1 and s_2 can generate the whole group. Thus,

$$D_4 = \langle s_1, s_2 \rangle = \{1,\ s_1,\ s_2,\ s_1s_2s_1,\ s_2s_1s_2,\ s_2s_1,\ (s_2s_1)^2,\ (s_2s_1)^3\}.$$

Notice that if v is a vector that corresponds to a reflection s_v and $g : V \to V$ is an orthogonal map, then the image of the vector v under g is a vector that corresponds to a reflection and is called s_{vg}. The following lemma illustrates that the reflection s_{vg} of the vector vg is the conjugate of a reflection s_v by g.

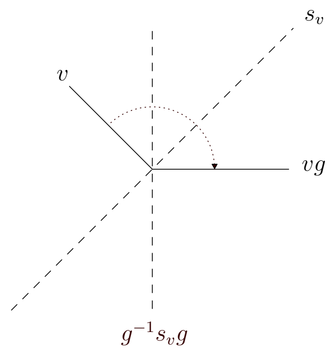


Figure 2.4: vg corresponds to the reflection $g^{-1}s_vg$.

Lemma 2.1.10. [40] *Let $g : V \to V$ be an orthogonal map. If $v \in V$ and s_v is the corresponding reflection, then $s_{vg} = g^{-1}s_vg$.*

Proof. Since s_{vg} is a reflection determined by a vector vg, then it must fix everything in the hyperplane H_{vg} and send the vector vg to the negative of itself. To show that $g^{-1}s_vg$ is precisely s_{vg}, we need to check that $g^{-1}s_vg$ is a reflection that sends the vector vg to $-vg$ and fixes everything in H_{vg}. Firstly,

$$(vg)g^{-1}s_vg = v(gg^{-1})s_vg = v(s_vg) = (vs_v)g = -vg.$$

Therefore, the reflection $g^{-1}s_vg$ sends the vector vg to $-vg$. Since the vector v is perpendicular to the reflecting hyperplane H_v, we have $\langle u, v\rangle = 0$ for all $u \in H_v$. Also, since g is an orthogonal map, then it preserves the inner product; that is $\langle ug, vg\rangle = \langle u, v\rangle = 0$. Therefore, $ug \in H_{vg}$. Notice that

$$(ug)g^{-1}s_vg = u(gg^{-1})s_vg = us_vg = ug.$$

Hence, the reflection $g^{-1}s_vg$ fixes $ug \in H_{vg}$. □

According to Remark 2.1.5, we can rewrite the reflection $g^{-1}s_vg$ with respect to the reflecting hyperplane H as follows:

$$s_{Hg} = g^{-1}s_H\ g. \tag{2.2}$$

Corollary 2.1.11. *If W is a finite reflection group and $g \in W$, then $s_v \in W$ if and only if $s_{vg} \in W$.*

Proof. $s_{vg} \in W \iff g^{-1}s_vg \in W \iff s_v \in W$ since a group W is closed under conjugation. □

Henceforth, all reflection groups are finite and all vector spaces V are finite dimensional.

Consider the symmetries of a square presented in Example 2.1.9, and all the reflections s_i with $1 \leq i \leq 4$ (not only the generating ones). Let $v_i, -v_i$ be the vectors that are orthogonal to each reflecting hyperplane H_i of a reflection s_i as illustrated in Figure 2.5.

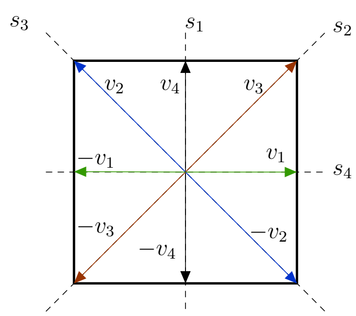


Figure 2.5: vectors v_i, orthogonal to hyperplanes H_i, with their negatives $-v_i$

Then, if we collect all such vectors, for each reflection s_i in a set Φ, we observe the following properties:

- All the vectors in Φ are nonzero vectors.
- Fix $v \in \Phi$, and notice that the only vectors in Φ that belong to the line span of v are v and $-v$; that is,

$$\lambda v \in \Phi \text{ if and only if } \lambda = \pm 1.$$

- Fix $v \in \Phi$, and observe that the reflection s_v associated with v permutes Φ; that is $(w)s_v \in \Phi$ for all $w \in \Phi$.

In fact, such an observation motivates the following concept.

Definition 2.1.12. [40] Let V be a finite dimensional Euclidean vector space. A finite set Φ of nonzero vectors in V is called a root system if and only if the following conditions are satisfied:

(i) $\Phi \bigcap \mathbb{R}v = \{\pm v\}$ for any $v \in \Phi$.

(ii) $(\Phi)s_v = \Phi$ for all $v \in \Phi$.

The reason for considering such a concept goes back to the relationship between a root system and a finite reflection group. In other words, given a root system Φ, we can obtain a finite Euclidean reflection group $W(\Phi)$ and vice versa, as stated below.

Proposition 2.1.13. [44] *Let W be a finite reflection group acting on the finite dimensional Euclidean vector space V. Then a root system Φ given by W is obtained by*

$$\Phi = \Big\{ \tfrac{\pm v}{\|v\|} : v \text{ are vectors associated with all reflections } s_v \in W \Big\}.$$

Moreover, if $\Phi \subset V$ is a root system, then a finite reflection group $W(\Phi)$ arising from Φ is determined by

$$W(\Phi) = \langle s_v \mid \ v \in \Phi \rangle.$$

Proof. Let W be a finite reflection group, and T be the set of all reflections belonging to W (not solely the generating ones). For each reflection $s_v \in T$, consider the vectors $\frac{\pm v}{\|v\|}$, and observe that both vectors have length 1. Then the set

$$\Phi = \{ \tfrac{\pm v}{\|v\|} : s_v \in W \}$$

is a root system for W as Φ is finite and clearly satisfies condition (i) of Definition 2.1.12. Let $w, u \in \Phi$. Since $s_w, s_u \in W$, then $s_{_{(w)s_u}} \in W$ by Corollary 2.1.11. It follows that $(w)s_u$ has a unit length, as shown in (2.1); thus, $(w)s_u \in \Phi$. Hence, condition (ii) holds. Conversely, let $\Phi \subset V$ be a root system with $|\Phi| = r$. For each $\pm v \in \Phi$, we obtain a reflection $s_v = s_{-v}$. Let $S = \{s_v : v \in \Phi\}$; then

$$W(\Phi) = \langle s_v \mid \ s_v \in S \rangle \subset GL(V)$$

is the reflection group generated by S. It remains to show that $W(\Phi)$ is finite. Let U be the subspace of V spanned by Φ. It follows that $V = U \oplus U^{\perp}$ where $U^{\perp}$ is the orthogonal complement of U. Since $v \in \Phi \subset U$, then a reflection s_v associated with v fixes $U^{\perp}$ pointwise. Thus, each $g \in W$ acts trivially on $U^{\perp}$ because g is a composition of some reflections $s_v \in W$. In view of property (ii) in Definition 2.1.12, any reflection s_v permutes the root system Φ. This allows us to define a group homomorphism $\psi : W \longrightarrow S_{\Phi}$ where each $g \in W$ gives a permutation $(g)\psi$ of elements of Φ. Notice that $\mathrm{Ker}(\psi) = \{id_W\}$, where id_W is the identity reflection since, for any $g \in \mathrm{Ker}(\psi)$, we have $(g)\psi = id \in S_{\Phi}$ and this implies that $g \in W$ preserves every element $v \in \Phi$. It follows that g fixes U pointwise; thus, g fixes V pointwise. Hence, $g = id_W$. By the first isomorphism theorem, $W \cong im(\psi) \subseteq S_{\Phi}$. This implies that $|W| \leq r!\,$. $\square$

We will discuss below some examples for root systems Φ and their corresponding finite reflection groups $W(\Phi)$.

Example 2.1.14. Let V be a Euclidean vector space with an orthonormal basis $\{v_1, \ldots, v_n\}$.

(i) **Root system $\mathcal{A}_{n-1}$**

Let $\mathcal{A}_{n-1} \subset V$ where

$$\mathcal{A}_{n-1} = \{\pm(v_i - v_j) : \text{and } \; i, j = 1, 2, \ldots, n\}.$$

Let us now verify that $\mathcal{A}_{n-1}$ satisfies the axioms of a root system. It is clear that the set $\mathcal{A}_{n-1}$ is finite with size $2\binom{n}{2}$. For the second axiom, consider the vector $v_i - v_j$ and its line span $\lambda(v_i - v_j)$. Any vector of $\mathcal{A}_{n-1}$ is a scalar multiple of $(v_i - v_j)$ if and only if the positions of the nonzero entries match the i-th and j-th positions in $v_i - v_j$. But the only roots in $\mathcal{A}_{n-1}$ that satisfy this condition are $\pm(v_i - v_j)$. To show $(\mathcal{A}_{n-1})s_v = \mathcal{A}_{n-1}$ for all $v = v_i - v_j \in \mathcal{A}_{n-1}$, let us first examine the action of s_v on the basis vectors $\{v_1, \ldots, v_n\}$. Using the reflection formula shown in Lemma 2.1.6, we have

$$(v_k)s_v = \begin{cases} v_i & \text{if } \; k = j \\ v_j, & \text{if } \; k = i \\ v_k & \text{if } \; \text{otherwise.} \end{cases} \tag{2.3}$$

Therefore, depending on the value of k, a reflection s_v permutes the basis vectors $\{v_1, \ldots, v_n\}$ by interchanging (v_i, v_j) and fixing all other basis vectors. In other words, a reflection s_v permutes $\{v_1, \ldots, v_n\}$ precisely as a transposition (i, j) permutes $\{1, \ldots, n\}$. As a reflection s_v is a linear map, it follows that

$$(v_k - v_\ell)s_v = (v_k)s_v - (v_\ell)s_v \in \mathcal{A}_{n-1},$$

for all $v_k - v_\ell \in \mathcal{A}_{n-1}$, and this implies $(\mathcal{A}_{n-1})s_v \subseteq \mathcal{A}_{n-1}$. Furthermore,

$$\mathcal{A}_{n-1} = ((\mathcal{A}_{n-1})s_v)s_v \subseteq (\mathcal{A}_{n-1})s_v.$$

Thus, for each $v \in \mathcal{A}_{n-1}$, a reflection s_v permutes $\mathcal{A}_{n-1}$. Hence, $\mathcal{A}_{n-1}$ is a root system. By Proposition 2.1.13, for each vector $v_i - v_j \in \mathcal{A}_{n-1}$, consider a reflection $s_{v_i - v_j}$ and then construct the group generated by these reflections as follows.

$$W(\mathcal{A}_{n-1}) = \langle s_{v_i - v_j} : i \neq j \rangle.$$

Such a group is called the reflection group of type $\mathcal{A}_{n-1}$. Recall that each generator $s_{v_i - v_j}$ of $W(\mathcal{A}_{n-1})$ only interchanges the basis vectors v_i and v_j and fixes the others as clarified above. It follows that each generator $s_{v_i - v_j}$ is a permutation "transposition" of $\{v_1, \ldots, v_n\}$ and every element $w \in W(\mathcal{A}_{n-1})$ is a composition of these generators. Hence, the finite reflection group $W(\mathcal{A}_{n-1})$ arising from $\mathcal{A}_{n-1}$ is a symmetric group S_n acting on V by permuting $\{v_1, \ldots, v_n\}$.

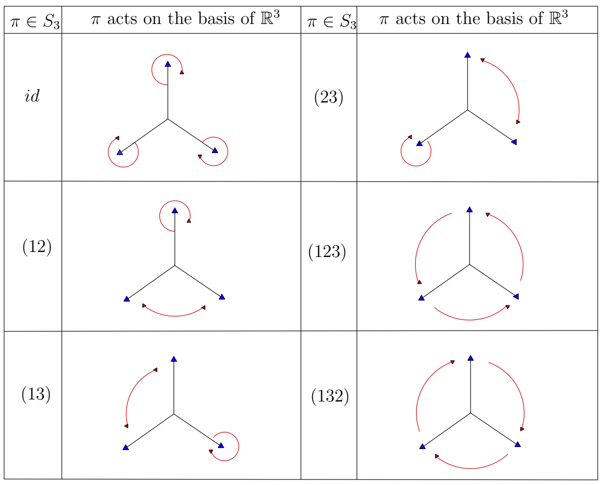


Figure 2.6: The action of S_3 on $\mathbb{R}^3$

(ii) **Root system $\mathcal{B}_n$**

Let $\mathcal{B}_n \subset V$ where

$$\mathcal{B}_n = \{\pm v_i \pm v_j : i \neq j\} \cup \{\pm v_i\} \text{ with } i, j = 1, 2, \ldots, n.$$

Clearly, $\mathcal{B}_n$ is finite with size $2n^2$, and for all $v \in \mathcal{B}_n$ and $\lambda \in \mathbb{R}$, $\lambda v \in \mathcal{B}_n$ if and only if $\lambda = \pm 1$. Moreover, we know the action of a reflection $s_{v_i - v_j}$, on the basis vectors from (2.3). Similarly, by formula 2.1.6, the action of

$s_{v_i+v_j} = s_{-v_i-v_j}$ on the basis vectors can be deduced as

$$(v_k)s_{v_i+v_j} = \begin{cases} -v_i & \text{if } \ k = j \\ -v_j, & \text{if } \ k = i \\ v_k & \quad \text{otherwise,} \end{cases} \tag{2.4}$$

while the action of $s_{v_i} = s_{-v_i}$ on the basis vectors is obtained by

$$(v_k)s_{v_i} = \begin{cases} -v_i & \text{if } \ k = i \\ v_k, & \text{if } \ k \neq i. \end{cases} \tag{2.5}$$

Utilising all the above cases, $(\mathcal{B}_n)s_v = \mathcal{B}_n$, where $v \in \mathcal{B}_n$. Hence, the set $\mathcal{B}_n$ is a root system, and then using Proposition 2.1.13, the group generated by these reflections is obtained by

$$\begin{aligned} W(\mathcal{B}_n) &= \langle s_v : v \in \mathcal{B}_n \rangle. \\ &= \langle s_{v_i-v_j}, s_{v_i+v_j}, s_{v_i} : i \neq j \rangle, \end{aligned} \tag{2.6}$$

and called the reflection group of type $\mathcal{B}_n$. In fact, the generating set of $W(\mathcal{B}_n)$ is smaller than the set of generators shown in (2.6) for the reason below. Since

$$v_i + v_j = (v_i - v_j) + 2v_j = (v_i - v_j) - \frac{2\ \langle v_i - v_j, v_j \rangle}{\langle v_j, v_j \rangle} v_j = (v_i - v_j)s_{v_j}, \tag{2.7}$$

then, using Lemma 2.1.10, we have

$$s_{v_i+v_j} = s_{((v_i-v_j)s_{v_j})} = s_{v_j}^{-1} s_{(v_i-v_j)} s_{v_j} = s_{v_j} s_{(v_i-v_j)} s_{v_j}. \tag{2.8}$$

In other words, each reflection $s_{v_i+v_j}$ can be written as a product of the other generators of $W(\mathcal{B}_n)$. Therefore, $W(\mathcal{B}_n)$ can be redrafted as follows.

$$W(\mathcal{B}_n) = \langle s_{v_i-v_j}, 1 \leq i \neq j \leq n \ ; \ s_{v_i}, 1 \leq i \leq n \rangle. \tag{2.9}$$

It should be noted that the action of certain reflections presented in (2.5) does not only permute basis vectors $\{v_1, \ldots, v_n\}$ among themselves. However, it sends some of the basis vectors v_i to $-v_i$. This indeed suggests considering the set $\{\pm v_1, \ldots, \pm v_n\}$ and examining the action of $W(\mathcal{B}_n)$ on it. Observe that the generators of $W(\mathcal{B}_n)$ can be viewed as permutations of $\{\pm v_1, \ldots, \pm v_n\}$ with a property that $(-v_i)s_v = -(v_i)s_v$ for all $v \in \mathcal{B}_n$ as shown in Figure 2.7.

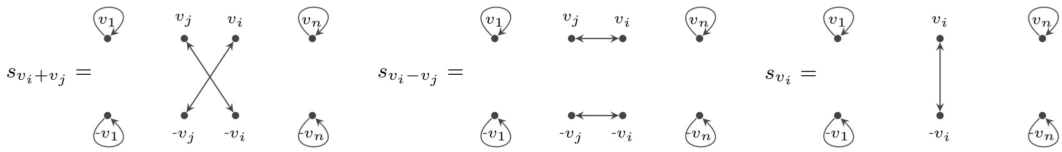


Figure 2.7: Generators of a reflection group $W(\mathcal{B}_n)$.

This implies that the composition σ of the generators also permutes $\{\pm v_1, \ldots, \pm v_n\}$ and satisfies $(-v_i)\sigma = -(v_i)\sigma$. It turns out that

$$W(\mathcal{B}_n) \cong \{\ \tau \in S_{\{\pm 1,\ldots,\pm n\}} : (-x)\tau = -(x)\tau \text{ with } x \in \{\pm 1, \ldots, \pm n\}\}, \quad (2.10)$$

where the set in the right side of (2.10) is called the signed permutation group B_n, and it is intensively investigated in Section 3.3 of this thesis.

(iii) **Root system $\mathcal{D}_n$**

Let $\mathcal{D}_n$ be a subset of the root system $\mathcal{B}_n$ such that

$$\mathcal{D}_n = \{\pm v_i \pm v_j : i \neq j \text{ and } i, j = 1, \ldots, n\}.$$

Clearly, it is a root system with size $4\binom{n}{2}$, and a reflection group $W(\mathcal{D}_n)$ is a subgroup of $W(\mathcal{B}_n)$ that is determined by

$$\begin{aligned} W(\mathcal{D}_n) &= \langle s_{v_i-v_j}, s_{v_i+v_j}, s_{-v_i-v_j}, s_{-v_i+v_j} : i \neq j\rangle; \\ &= \langle s_{v_i-v_j}, s_{v_i+v_j} : 1 \leq i \neq j \leq n\rangle. \end{aligned}$$

According to [40, Section 1.1], the reflection group $W(\mathcal{D}_n)$ arising from the root system of type $\mathcal{D}_n$ is isomorphic to the group of even signed permutations B_n^e where

$$\begin{aligned} B_n^e = \Big\{\ \tau \in & S_{\{\pm 1,\cdots,\pm n\}} : (-x)\tau = -(x)\tau;\ x \in \{\pm 1, \cdots, \pm n\} \text{ and} \\ & |\{x \in \{1, \cdots, n\} : (x)\tau \in \{-1, \cdots, -n\}\}| \text{ is even}\Big\}. \end{aligned}$$

The latter condition appearing in the definition of B_n^e means that the number of positive integers whose images are negative is even. The reader can consult [19, Section 5] for more details about B_n^e and [44, Section 2.2] for justifying the isomorphism.

(iv) There are also other well-known root systems

$$E_6,\ E_7,\ E_8,\ F_4,\ H_3,\ H_4,\ I_2(m).$$

The reader can consult [40, Chapter 2] for more details about these root systems, and their reflection groups can be constructed in a similar manner as just demonstrated.

2.2 Monoids of partial linear isomorphisms and reflection monoids

The symmetric group S_n has been the focus of extensive study as the permutation group of the set $[n] = \{1, 2, \ldots, n\}$. The study also takes the form of, among others, a reflection group which is generated by reflections in the hyperplanes of a finite dimensional Euclidean space. Similarly, the symmetric inverse monoid I_X, whose elements are the partial bijections, takes other versions. If the symmetric group S_n represents the group of symmetries of $[n]$, then the symmetric inverse monoid I_n should be considered for the partial symmetries. One interesting shortcoming is that the symmetric inverse monoid I_X is not considered a monoid generated by certain partial reflections; that is, no generalisation of reflection groups to the partial case was known.

Recently, Everitt and Fountain [19, Section 2] put forward a proposal to identify the symmetric inverse monoid I_X as a "partial" reflection monoid. The authors described reflection monoids as a certain factorizable inverse monoid generated by partial reflections. A reflection group G and a collection of domain subspaces with favourable behaviour (a so-called system S) can also define a reflection monoid $M(G, S)$. The development of such monoids forms the focus of this section. It will also discuss in detail a number of notable outcomes obtained by the aforementioned authors. This section is structured to have four subsections. In the first subsection, certain subspaces that form the domains of the partial isomorphisms, with a particular emphasis on two examples associated with these subspaces, are discussed. In the second subsection, the monoids of partial linear isomorphisms $M(G, S)$ and reflection monoids, alongside several of their key attributes, are characterised. Finally, in the last subsection, Green's relations for $M(G, S)$ and maximal subgroups are considered.

2.2.1 Systems of subspaces

This subsection is devoted to introducing essential subspaces that form a domain of the partial isomorphisms. The reader can consult [19, Section 2 and 3] for more details.

Recall from Example 1.2.16 that

$$ML(V) = \{\sigma : Y \longrightarrow Y' \text{ an isomorphism where } Y, Y' \subseteq V\}$$

is the monoid of all partial linear isomorphisms of V. Let $\alpha : Y \mapsto Y'$, and $\beta : Z \mapsto Z'$. Then, a careful consideration of the domain $Y \bigcap Z\alpha^*$ in the composition

$\alpha\beta \in ML(V)$, described in (1.9), suggests the following.

Definition 2.2.1. [19] Let V be a vector space over $\mathbb{R}$ and G be a subgroup of $GL(V)$. A set S of subspaces of V is called a system in V for G if and only if

(i) $V \in S$,

(ii) for each $Y \in S$ and $g \in G$, we have $Yg \in S$, and

(iii) if $Y, Z \in S$, then $Y \cap Z \in S$.

Utilising the alternative descriptions of the preceding reflection groups, we will discuss some examples of systems of subspaces.

Example 2.2.2 (Boolean systems).

(i) **Boolean system for S_n**

Let V be a Euclidean vector space with an orthonormal basis $\{v_1, \ldots, v_n\}$. Stemming from Figure 2.6, the symmetric group $S_n \subset GL(V)$ acts on V by permuting the coordinates; that is, for all $\sigma \in S_n$, define $g_\sigma \in GL(V)$ by:

$$v_i \, . \, g_\sigma = v_{i\sigma}.$$

Let

$$X(L) = \bigoplus_{l \in L} \mathbb{R} v_l \tag{2.11}$$

be a subspace of V where $L \subseteq [n] = \{1, \ldots, n\}$, and define $\mathcal{B} = \{X(L) :$ for all $L \subseteq [n]\}$. Notice that $X(\emptyset) = \mathbf{0}$, and if $L = [n]$, then $X(L) = V$. Moreover, for any two subspaces $X(L)$, $X(L') \in \mathcal{B}$, we have

$$X(L) \cap X(L') = X(L \cap L'), \tag{2.12}$$

with $L, L' \subseteq [n]$. Also, for all $g_\sigma \in S_n$,

$$X(L) g_\sigma = X(L\sigma). \tag{2.13}$$

Therefore, $\mathcal{B}$ is indeed a system. We call such the system of subspaces, generated by all coordinate hyperplanes and their intersections, the Boolean system in V for S_n. For instance, let $\mathbb{R}^2$ be a vector space with an orthonormal basis $\{v_1, v_2\}$. Since all the subsets of [2] are $\{L_1 = \phi, L_2 = \{1\}, L_3 = \{2\}, L_4 = \{1, 2\}\}$, then all possible subsets of the basis are $\{\phi, \{v_1\}, \{v_2\}, \{v_1, v_2\}\}$. Notice that dim $X(L_i) = |L_i|$, and the subspaces of V are

$$X(L_1) = \mathbf{0}$$

$$X(L_2) = \mathbb{R}v_1$$
$$X(L_3) = \mathbb{R}v_2$$
$$X(L_4) = \mathbb{R}v_1 \oplus \mathbb{R}v_2.$$

Hence,

$$\mathcal{B} = \{X(L_1), X(L_2), X(L_3), X(L_4)\},$$

is the Boolean system in $\mathbb{R}^2$ for S_2. Below is an illustration of the Boolean system in $\mathbb{R}^3$ for S_3.

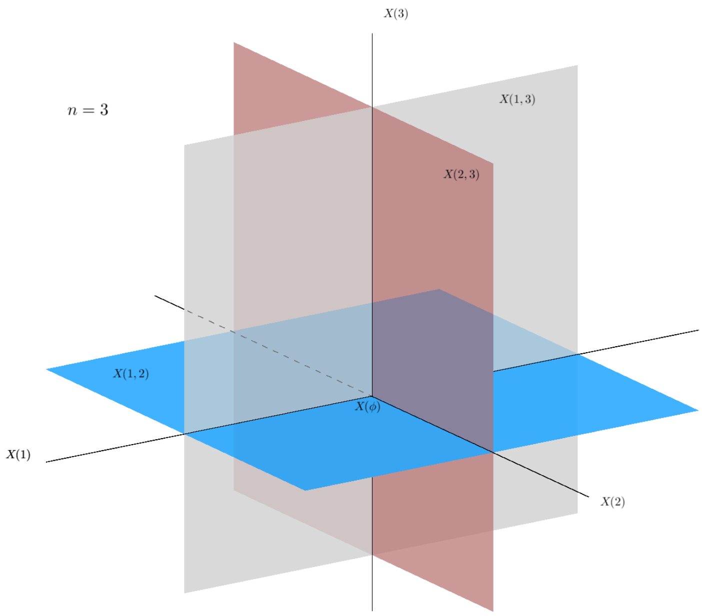


Figure 2.8: All subspaces of $\mathbb{R}^3$, spanned by a subset of the given basis, forms the Boolean system for S_3

(ii) **Boolean system for B_n**

Let V be a Euclidean vector space with an orthonormal basis $\{v_1, \ldots, v_n\}$. Let the signed permutation group $B_n \subset GL(V)$ act on V by

$$v_i \,.\, f_\pi = \begin{cases} v_{i\pi}, & \text{if } \; i\pi > 0 \\ v_{-i\pi} := -v_{i\pi}, & \text{if } \; i\pi < 0, \end{cases}$$

where $\pi \in B_n$. Let $X(L)$ as defined in (2.11). In view of the above example, $X([n]) = V$ and equation (2.12) holds. Since $\mathbb{R}(-v_i) = \mathbb{R}(v_i)$, the last condition

$$X(L)f_\pi = X(L\pi)$$

holds. Therefore, $\mathcal{B} = \{X(L) : \text{for all } L \subseteq [n]\}$ is the Boolean system for B_n.

(iii) **Boolean system for B_n^e**

As B_n^e is a subgroup of B_n, any system $\mathcal{B}$ for B_n is automatically a system for B_n^e. Thus, $\mathcal{B}$ is the Boolean system for B_n^e.

According to [19, Section 5], there will never be a Boolean system $\mathcal{B}$ for any reflection group $W(\Phi)$ arising from root systems presented in Example 2.1.14(iv).

Example 2.2.3 (**Arrangement system**). Let V be a Euclidean vector space with an orthonormal basis $\{v_1, \ldots, v_n\}$ and $W \subset GL(V)$ be a finite reflection group. We know that each reflection $s \in W$ has a reflecting hyperplane H associated with it. Let $\mathcal{A}$ be the set of all the reflecting hyperplanes H of W; that is,

$$\mathcal{A} = \{H : \text{ for each reflection } s \in W\}.$$

Let $\mathcal{H}$ be the collection of all the possible intersections of the hyperplanes H in $\mathcal{A}$, and the vector space V; that is,

$$\mathcal{H} = V \ \cup \ \{ \textstyle\bigcap_i H_i : \ H_i \in \mathcal{A}\}.$$

We claim that $\mathcal{H}$ is a system in V for the reflection group W. To see this, observe that $V \in \mathcal{H}$. Suppose $\bigcap_i H_i$ and $\bigcap_j H_j \in \mathcal{H}$ for some reflecting hyperplanes H_i and H_j in $\mathcal{A}$. Then,

$$(\textstyle\bigcap_i H_i) \ \cap \ (\bigcap_j H_j) \in \mathcal{H}.$$

If $\bigcap_i H_i \in \mathcal{H}$ and $g \in W$, we want to show that $(\bigcap_i H_i)g \in \mathcal{H}$. Let us first show that, if $H \in \mathcal{A}$ and $g \in W$, then $Hg \in \mathcal{A}$. Since $H \in \mathcal{A}$, then its associated reflection $s_{_H}$ belongs to W. Using Lemma 2.1.10 and equation (2.2), we have a reflection

$$s_{_{Hg}} = g^{-1} s_{_H} g.$$

Hence, $s_{_{Hg}} \in W$ for W is a group, and then its reflecting hyperplane Hg is in $\mathcal{A}$. Let $H_1, H_2,\ \ldots\ , H_r$ be reflecting hyperplanes in $\mathcal{A}$. Since $g \in W$ (bijective map), then

$$(\textstyle\bigcap_{i=1}^r H_i)g = \bigcap_{i=1}^r H_i g.$$

However, for all $1 \leq i \leq r$, each $H_i g \in \mathcal{A}$, so $\bigcap_{i=1}^r H_i g \in \mathcal{H}$. Thus, $(\bigcap_{i=1}^r H_i)g \in \mathcal{H}$.

2.2.2 Reflection monoids

This subsection is devoted to the study of the monoid of partial linear isomorphisms $M(G, S)$ and reflection monoids. We provide a preliminary overview of some significant properties of the monoids of partial linear isomorphisms. The reader can consult ([19], [20]) for more details.

Definition 2.2.4. [19] Let G be a subgroup of $GL(V)$ and S be a system in V for G. A submonoid of $ML(V)$ defined by

$$M(G,S) := \{g_{_Y} : g \in G, Y \in S\}$$

is called a monoid of partial linear isomorphisms given by a group G and a system S.

Definition 2.2.5. [19] A submonoid $M \subset ML(V)$ is said to be a reflection monoid if M is of the form $M(W,S)$ for W is a reflection group and S is a system for W.

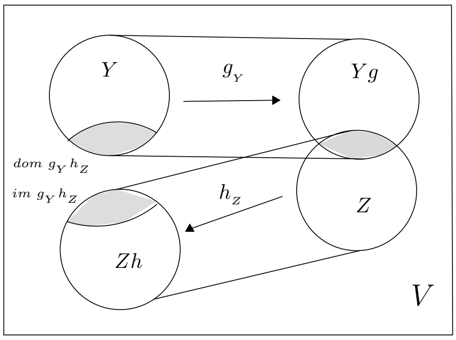
Let us shed more light on the structure of $M(G,S)$ and pave the way to reprove the striking results of Proposition 2.2.13 and Corollary 2.2.15, which give us an alternative characterisation of reflection monoids. The next proposition tells us how to compose two partial linear isomorphisms in $M(G,S)$.

Proposition 2.2.6. [19] *If $g_{_Y},\ h_{_Z} \in M(G,S)$, where $g,h \in G \leq GL(V)$ and $Y,Z \in S$, then*

$$dom\ (g_{_Y} h_{_Z}) = Y \bigcap Zg^{-1}\ ,\ \ and \quad im\ (g_{_Y} h_{_Z}) = (Y \bigcap Zg^{-1})gh,$$

and

$$g_{_Y} h_{_Z} = (gh)_{_{Y \cap Zg^{-1}}}. \tag{2.14}$$



Proof. If $g_{_Y}, h_{_Z} \in M(G,S)$ where

$$g_{_Y} : Y \mapsto Yg\ ,\ h_{_Z} : Z \mapsto Zh,$$

then

$$\begin{aligned} \text{dom}\ (g_{_Y} h_{_Z}) &= \{v \in V : v \in \text{dom}\ g_{_Y} \text{ and } (v)g \in \text{dom}\ h_{_Z}\}, \\ &= \{v \in V : v \in Y \text{ and } (v)g \in Z\}, \\ &= \{v \in V : v \in Y \text{ and } v \in Zg^{-1}\}, \end{aligned}$$

$$= Y \cap Zg^{-1}.$$

Furthermore,

$$\begin{aligned}\mathrm{im}(g_Y h_Z) &= (Y \cap Zg^{-1})g_Y h_Z, \\ &= ((Y \cap Zg^{-1})g_Y)h_Z, \\ &= ((Y \cap Zg^{-1})g)h_Z, \\ &= (Yg \cap Z)h_Z, \\ &= (Yg \cap Z)h.\end{aligned}$$

However,

$$(Y \cap Zg^{-1})gh = (Y)gh \cap (Zg^{-1})gh = (Yg)h \cap (Zg^{-1}g)h = (Yg \cap Z)h.$$

Therefore, im $(g_Y h_Z) = (Y \cap Zg^{-1})gh$. Thus, $g_Y h_Z$ is the map composition

$$Y \cap Zg^{-1} \mapsto (Y \cap Zg^{-1})gh.$$

If $x \in Y \cap Zg^{-1}$, then it is clear that $x(g_Y h_Z) = (xg_Y)h_Z = (xg)h_Z = x(gh)$. Hence, $g_Y h_Z = (gh)_{Y \cap Zg^{-1}}$. □

Observation 2.2.7. Since g_Y is a partial linear isomorphism $Y \to Yg$, then there is a partial linear isomorphism $(g_Y)^* : Yg \to Y$ such that

$$g_Y \ (g_Y)^* = id_Y \quad , \quad (g_Y)^* \ g_Y = id_{Yg}$$

Figure 2.9 shows that we are allowed to write $(g_Y)^*$ as follows:

$$(g_Y)^* = (g^{-1})_{Yg}. \tag{2.15}$$

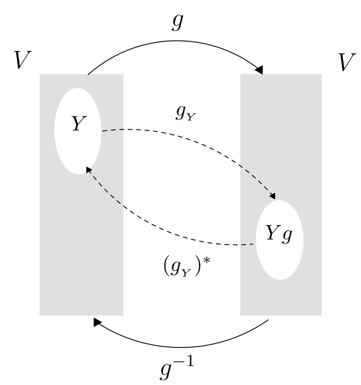


Figure 2.9: $(g^{-1})_{Yg}$ is the partial linear isomorphism $Yg \to Y$.

Notice that the right side of equality (2.15) is a linear isomorphism $g^{-1} \in G$ that is restricted to a subspace $Yg \in S$. Hence, $(g^{-1})_{Yg} \in M(G,S)$. Moreover, $(g_Y)^*$ is an inverse of g_Y, because

$$\begin{aligned}
g_Y\ (g_Y)^*\ g_Y &= g_Y\ (g^{-1})_{Y_g}\ g_Y \\
&= (g_Y\ (g^{-1})_{Y_g})\ g_Y \\
&= ((gg^{-1})_{Y \cap Ygg^{-1}})\ g_Y \\
&= (id_{Y \cap Y})\ g_Y \\
&= id_Y\ g_Y \\
&= g_Y.
\end{aligned}$$

In addition,

$$\begin{aligned}
(g_Y)^*\ g_Y\ (g_Y)^* &= (g^{-1})_{Y_g}\ g_Y\ (g^{-1})_{Y_g} \\
&= ((g^{-1})_{Y_g}\ g_Y)\ (g^{-1})_{Y_g} \\
&= ((g^{-1}g)_{Yg \cap Y(g^{-1})^{-1}})\ (g^{-1})_{Y_g} \\
&= (id_{Y_g \cap Y_g})\ (g^{-1})_{Y_g} \\
&= id_{Y_g}\ (g^{-1})_{Y_g} \\
&= (id\ g^{-1})_{Y_g \cap Ygid^{-1}} \\
&= (g^{-1})_{Y_g \cap Y_g} \\
&= (g^{-1})_{Y_g} \\
&= (g_Y)^*.
\end{aligned}$$

Remark 2.2.8. If we take any two different linear isomorphisms $g, h \in G$ and two subspaces $Y, Z \in S$, it is crucial to know when they have the same partial linear isomorphisms. Obviously, partial linear isomorphisms are equal if they have the same domain and images and the same effect on the domain:

$$\text{dom } (g_Y) = \text{dom } (h_Z) \Leftrightarrow Y = Z.$$

Furthermore, for all $v \in Y$, $vg_Y = vh_Z \Leftrightarrow vg = vh \Leftrightarrow vgh^{-1} = v$; that is, when gh^{-1} fixes Y pointwise. In fact, this leads us to define the isotropy group.

Definition 2.2.9. [19] Given a subspace Y of V, the set G_Y consists of all elements $g \in GL(V)$ that fixes Y pointwise is called the isotropy group of Y :

$$G_Y = \{g \in GL(V) : vg = v\ ,\ \forall\ v \in Y\}.$$

Utilising the above definition, we have

$$g_{_Y} = h_{_Z} \iff Y = Z \text{ and } gh^{-1} \in G_Y. \tag{2.16}$$

Observation 2.2.10. If $g_Y,\ h_Z \in M(G,S)$, then by (2.14), their composition is $g_{_Y} h_{_Z} = (gh)_{_{Y \cap Zg^{-1}}}$. Since $g \in G$ and $Y, Z \in S$, then $Zg^{-1} \in S$ and $Y \cap Z_{g^{-1}} \in S$ by the axioms (ii) and (iii) in Definition 2.2.1. Therefore, the map gh is restricted to a subspace $Y \cap Z_{g^{-1}} \in S$; hence, $g_Y h_Z \in M(G,S)$. Consider the identity linear isomorphism $id \in G$, and restrict it to any subspace $Y \in S$; we will obtain the partial identity $id_{_Y} \in M(G,S)$. All elements $id_{_Y}$ with $Y \in S$ satisfy $(id_{_Y})^2 = id_{_Y}$, and are the idempotents. Let us show why they are the only idempotents in $M(G,S)$. In other words, for any $g_{_Y} \in M(G,S)$ where $g \in G$ and $Y \in S$, if $g_{_Y}^2 = g_{_Y}$, then $g_{_Y} = id_{_Y}$. To see this, notice that

$$g_{_Y} = g_{_Y}^2 = g_{_Y} \cdot g_{_Y} = (g^2)_{_{Y \cap Yg^{-1}}}.$$

In view of (2.16), we have $Y \cap Yg^{-1} = Y$ and then by applying g to both sides, we obtain $Yg \subseteq Y$. Moreover, since g is bijection and Y is finite, we have $Yg = Y$. In view of (2.16), we also acquire that for all $y \in Y$, $(yg)g = yg$. However, as g is one-to-one, we have $yg = y$ for all $y \in Y$. Hence all idempotents in $M(G,S)$ are partial identities on Y. In particular, the identity map $id \in G$ is also an element of $M(G,S)$ for $V \in S$, and it is the identity of $M(G,S)$; that is, for all $g_{_Y} \in M(G,S)$

$$g_{_Y}\ id_{_V} = (g\ id)_{_{Y \cap Vg^{-1}}} = g_{_{(Y \cap V)}} = g_{_Y}.$$

Similarly, $id_{_V}\ g_{_Y} = g_{_Y}$. Finally, since $M(G,S) \subset ML(V)$, the associative law holds in the composition of partial linear isomorphisms. Hence, $M(G,S)$ is a submonoid of $ML(V)$.

Observe that for all $g_{_Y} \in M(G,S)$, there is a map $(g_{_Y})^*$ with $(g_{_Y})^* = (g^{^{-1}})_{_{Yg}}$ that satisfies

$$g_{_Y}\ (g_{_Y})^*\ g_{_Y} = g_{_Y} \quad \text{and} \quad (g_{_Y})^*\ g_{_Y}\ (g_{_Y})^* = (g_{_Y})^*.$$

Further, idempotents in $M(G,S)$ commute as they are restrictions of the identity linear map to any subspace $Y \in S$. Hence, $M(G,S)$ is an inverse monoid.

It is highly beneficial to know the interpretation of the partial order relation in an inverse submonoid $M(G,S)$. The next observation tells us precisely when $g_{_Y} \leq h_{_Z}$ for any $g_{_Y},\ h_{_Z} \in M(G,S)$.

Observation 2.2.11. If $g_{_Y},\ h_{_Z} \in M(G,S)$, then $g_{_Y} \leq h_{_Z}$ if and only if there is an idempotent $id_X \in M(G,S)$, such that

$$g_Y = id_X h_Z \iff g_Y = (id\ h)_{X \cap Z id^{-1}} \iff g_Y = h_{X \cap Z}.$$

However, according to Remark 2.2.8, $g_Y = h_{X \cap Z}$ implies that $Y = X \cap Z$ and gh^{-1} fixes Y pointwise; that is,

$$Y \subseteq Z \text{ and } \forall\ y \in Y,\ v(gh^{-1}) = v \iff vg = vh \iff g_Y = h_Y.$$

Thus,

$$g_Y \leq h_Z \iff Y \subseteq Z \text{ and } g_Y = h_Y.$$

Proposition 2.2.12. [19] *Let V be a vector space over $\mathbb{R}$, $G \leq GL(V)$ and S is a system of subspaces. Then an inverse monoid $M(G,S)$ is factorizable.*

Proof. In order to show that $M(G,S)$ is factorizable, we need to check that $M(G,S) = EU$ where E is the set of idempotents and U is the set of units. Since the whole space $V \in S$, then all (full) linear isomorphisms $g \in G$ are in $M(G,S)$; thus they form the group of units inside $M(G,S)$. It should be noted that each partial linear isomorphism g_Y is the product of a partial identity id_Y and a linear isomorphism g, since

$$id_Y\ g = (id\ g)_{Y \cap V id^{-1}} = g_{Y \cap V} = g_Y.$$

Hence, $M(G,S)$ is factorizable. ☐

The following proposition classifies a factorizable inverse monoid $M \subset ML(V)$ and asserts that it is the monoid of the partial linear isomorphisms $M(G,S)$ under some restrictions of G and S.

Proposition 2.2.13. [19] *If V is a vector space over $\mathbb{R}$, then $M \subset ML(V)$ is a factorizable inverse submonoid if and only if $M = M(G,S)$ where G is the group of units of M and $S = \{dom\ \sigma \mid \sigma \in M\}$ is the system in V for G.*

Proof. If $M = M(G,S)$ where G is the group of units of M and $S = \{dom\ \sigma \mid \sigma \in M\}$ is the system (see below), then in view of the preceding proposition, M is a factorizable inverse submonoid of $ML(V)$. Conversely, suppose M is a factorizable inverse submonoid of $ML(V)$, G is the group of units of M and $S = \{dom\ \sigma \mid \sigma \in M\}$ is a collection of subspaces of V. We first need to form a monoid of the partial linear isomorphisms $M(G,S)$ and deduce that $M = M(G,S)$. To construct $M(G,S)$, it suffices to show that $S = \{dom\ \sigma \mid \sigma \in M\}$ is indeed the system. Since the identity map $id : V \mapsto V$ belongs to $ML(V)$, and M is a submonoid of $ML(V)$, then $id : V \mapsto V$ is also in M; thus, $V \in S$ since $V = \text{dom } id$. Suppose that $Y,\ Z \in S$, where $Y = \text{dom } \sigma$ and $Z = \text{dom } \tau$ for some $\sigma,\ \tau \in M$, and consider the partial linear isomorphism $\sigma\sigma^*\tau\tau^* \in M$, where σ^* and τ^* are the inverses of σ, τ respectively. Since

$$\sigma\sigma^*\tau\tau^* = id_Y id_Z = id_{Y\cap Z},$$

then $\mathrm{dom}(\sigma\sigma^*\tau\tau^*) = Y \cap Z$. However, $Y \cap Z = \mathrm{dom}\sigma \cap \mathrm{dom}\tau$; thus, $\mathrm{dom}\sigma \cap \mathrm{dom}\tau \in S$. Let $g \in G$ be a unit. If $Y \in S$, then $Y =$ dom σ for some σ. Consider a partial linear isomorphism $g^{-1}\sigma \in M$; since

$$dom(g^{-1}\sigma) = V \cap Y(g^{-1})^{-1} = V \cap Yg = Yg,$$

then, $Yg \in S$. However, $Yg = \mathrm{dom}(\sigma)g$; hence, $\mathrm{dom}(\sigma)g \in S$. Therefore, $S = \{dom\ \sigma \mid \sigma \in M\}$ is the system in V for G, and this allows us to form the monoid of partial linear isomorphisms, as follows:

$$M(G, S) = \{g_Y \mid g \in G \text{ (group of units) and } Y = \mathrm{dom}\ \sigma \in S\}.$$

To show $M = M(G, S)$, pick any partial linear isomorphism $g_Y \in M(G, S)$ in which $g \in G$ and $Y =$ dom σ for some $\sigma \in M$. Consider $\sigma\sigma^* g \in M$ and notice that

$$\sigma\sigma^* g = id_Y g = (id\ g)_{Y \cap V id^{-1}} = g_{Y\cap V} = g_Y.$$

Therefore, $g_Y \in M$, and then $M(G, S) \subseteq M$. Hence, $M(G, S)$ is a factorizable inverse submonoid of M. Conversely, let $\sigma \in M$. Since M is a factorizable inverse submonoid of $ML(V)$, then $M = EG$, where E is the set of idempotents and G is the group of units of M. In particular, $\sigma = id_Y g$, with $id_Y \in E$ and $g \in G$. Since both id_Y and g belong to M, their domains Y and V are in S. Thus, $id_Y,\ g \in M(G, S)$, and then their product $id_Y g \in M(G, S)$ as well. Hence, $M \subseteq M(G, S)$, and $M = M(G, S)$. $\square$

Definition 2.2.14. [19] A partial reflection of a real vector space V is a partial linear isomorphism of the form s_Y, where s is a (full) reflection and Y is a subspace of V.

In view of Definition 2.1.8, we know that a reflection group is only a group generated by reflections. However, the corresponding assertion for a reflection monoid needs to be more than a monoid generated by partial reflections. The following corollary identifies when a monoid generated by partial reflections can be called a reflection monoid.

Corollary 2.2.15. [19] *A submonoid $M \subset ML(V)$ is a reflection monoid if and only if M is a factorizable inverse (sub)monoid generated by partial reflections.*

Proof. Suppose $M \subset ML(V)$ is a reflection monoid, then by Definition 2.2.5, M is of the form $M(W, S)$, where $W = \langle \mathcal{S} \rangle$ is a reflection group and $\mathcal{S}$ is a set of reflections. In view of Proposition 2.2.12, M is a factorizable inverse monoid. It remains to show that $M = M(W, S)$ is indeed generated by partial reflections. Let $\sigma \in M$; then σ has the form of g_Y, where $g \in W$ and $Y \in S$. However, M is a factorizable inverse monoid. Therefore, $\sigma = g_Y = id_Y g$, where id_Y is an idempotent and $g \in W$ is a unit. Since W is a reflection group and $g \in W$, then g is a product of reflections; that is $g = s_1 s_2 \dots s_k$, $s_i \in \mathcal{S}$ with $i \in \{1, 2, \cdots, k\}$. Therefore,

$$id_Y g = id_Y s_1 s_2 \dots s_k = (id_Y s_1) s_2 \dots s_k = (s_1)_Y s_2 \dots s_k,$$

where $(s_1)_Y$ is a partial reflection. Hence, any σ is a composition of partial reflections.

For the other direction, suppose that M is a factorizable inverse (sub)monoid of $ML(V)$ generated by partial reflections. Then, according to Proposition 2.2.13, $M = M(G, S)$, where G is the group of units of M and $S = \{dom\ \sigma \mid \sigma \in M\}$ is a system. Notice that each element in M is a product of partial reflections. In particular, the elements in the group of units G are of the form

$$g = (s_1)_{Y_1} (s_2)_{Y_2} \cdots\ (s_i)_{Y_i} \cdots\ (s_k)_{Y_k}, \tag{2.17}$$

where $(s_1)_{Y_1}, (s_2)_{Y_2}, \dots, (s_i)_{Y_i}, \dots, (s_k)_{Y_k}$ are generating partial reflections. We want to show that G is a reflection group, which means that each $g \in G$ is a product of (full) reflections; that is, the domain of $(s_i)_{Y_i}$ for all $1 \le i \le k$, in (2.17) is the whole space V. The way we prove it is by contradiction: suppose some of the generating partial reflections $(s_i)_{Y_i}$ in (2.17) have domains proper subspaces $Y_i \subsetneq V$. If we compute the domain of the product on the right side of (2.17), we obtain

$$Y_1 \bigcap Y_2 (s_1)^{-1} \bigcap\ \cdots\ \bigcap Y_i (s_1 s_2\ \dots\ s_{i-1})^{-1} \bigcap\ \cdots\ \bigcap Y_k (s_1 s_2\ \dots\ s_{k-1})^{-1},$$

which is equivalent to

$$Y_1 \bigcap Y_2 s_1 \bigcap\ \cdots\ \bigcap Y_i (s_{i-1}\ \dots\ s_2 s_1) \bigcap\ \cdots\ \bigcap Y_k (s_{k-1}\ \dots\ s_2 s_1). \tag{2.18}$$

However, $Y_i(s_{i-1}\ \dots\ s_2 s_1)$ is a proper subspace of V because $Y_i \subsetneq V$ is a proper subspace, and the product $s_{i-1}\ \dots\ s_2 s_1$ is a linear isomorphism. Therefore, the intersection of domains in (2.18) is a proper subspace of V as well since it is contained in $Y_i(s_{i-1} \dots s_2 s_1)$. It turns out that the domain of the product on the right side of (2.17) is a proper subspace of V, whereas the domain of g is V because it is a unit by assumption; hence, a contradiction exists. Therefore, g is just a product of full reflections, and then G is a reflection group. Let $\mathcal{F}$ be the collection of generating partial reflections for M. Then $G = \langle \bar{F} \rangle$ where $\bar{F} \subseteq \mathcal{F}$. □

Example 2.2.16. Let V be a Euclidean vector space with an orthonormal basis $\{v_1, \dots, v_n\}$. Let the symmetric group $S_n \subset GL(V)$ act on V as discussed in Example 2.2.2(i):

$$v_i \cdot g_\sigma = v_{(i\sigma)}.$$

Recall that $\mathcal{B} = \{X(L) : L \subseteq [n]\}$ is the Boolean system in V for S_n. Hence, by Definition 2.2.5, we construct a reflection monoid given by a reflection group S_n and a system $\mathcal{B}$ as follows:

$$M(S_n, \mathcal{B}) = \{(g_\sigma)_{X(L)} \mid g_\sigma \in S_n,\ X(L) \in \mathcal{B}\}.$$

We call $M(S_n, \mathcal{B})$ the Boolean (reflection) monoid of type $\mathcal{A}_{n-1}$. Similarly, the Boolean system $\mathcal{B} = \{X(L) : L \subseteq [n]\}$ is also a system in V for B_n and B_n^e; hence, $M(B_n, \mathcal{B})$ and $M(B_n^e, \mathcal{B})$ are the Boolean (reflection) monoids of type $\mathcal{B}_n$ and $\mathcal{D}_n$ respectively. The reader can consult [19, Section 5] for more details about the latter Boolean monoids.

The following proposition gives us nice combinatorial descriptions of the above Boolean monoids.

Proposition 2.2.17. [19] *The Boolean monoid $M(S_n, \mathcal{B})$ of type $\mathcal{A}_{n-1}$ is isomorphic to the symmetric inverse monoid I_n.*

The detailed proof of the above proposition can be found in [19, Proposition 3.1].

Proposition 2.2.18. [19] *The Boolean monoid $M(B_n, \mathcal{B})$ of type $\mathcal{B}_n$ is isomorphic to the monoid of partial signed permutations MB_n where*

$$MB_n = \Big\{\sigma : X \xrightarrow{bijection} Y : X, Y \subseteq [\pm n],\ \textit{where } x \in X \iff -x \in X \textit{ and } (-x)\sigma = -(x)\sigma\Big\}.$$

The full proof of the proposition appears in [19, Proposition 5.1]. We provide an in-depth study of MB_n in Section 6.1 of this thesis. The reader will notice that Proposition 2.2.17 and Proposition 2.2.18 will be utilised to explicitly describe the irreducible representations of the Boolean monoids of types $\mathcal{A}_{n-1}$ and $\mathcal{B}_n$.

Proposition 2.2.19. [19] *The Boolean monoid $M(B_n^e, \mathcal{B})$ of type $\mathcal{D}_n$ is isomorphic to the monoid of partial even signed permutations MB_n^e where*

$$\begin{aligned} MB_n^e = \Big\{ &\sigma : X \xrightarrow{bijection} Y : X, Y \subseteq [\pm n],\ \textit{where } x \in X \iff -x \in X \\ &\textit{and } (-x)\sigma = -(x)\sigma \textit{ and the number of } x, \textit{with } (x)\sigma \textit{ is negative, is even}\Big\}. \end{aligned}$$

The detailed proof of the assertion can also be found in [19, Proposition 5.1].

Example 2.2.20. Let V be a Euclidean vector space with an orthonormal basis $\{v_1, \ldots, v_n\}$ and $W \subset GL(V)$ be a finite reflection group. Suppose that $\mathcal{A}$ is a set of all the reflecting hyperplanes H corresponding to each reflection s and that $\mathcal{H}$ is the collection of all possible intersections of the hyperplanes H in $\mathcal{A}$ along with the vector space V. We have seen in Example 2.2.3 that

$$\mathcal{H} = V \cup \{ \bigcap H_i : \ H_i \in \mathcal{A}\},$$

forms a system in V for the associated reflection group W. Consequently, $M(W, \mathcal{H})$ is a reflection monoid given by a reflection group W and a system $\mathcal{H}$, and is called the Coxeter arrangement monoid.

2.2.3 Green's relations for $M(G, S)$ and maximal subgroups

This subsection illustrates the Green's relations for the monoid of partial linear isomorphisms $M(G, S)$ and investigates its maximal subgroups.

Proposition 2.2.21. [19] *Let $\sigma, \tau \in M(G, S)$ with $\sigma = g_Y$ and $\tau = h_Z$ where $g, h \in G$ and $Y, Z \in S$. Then,*

(i) *$\sigma \,\mathcal{R}\, \tau$ if and only if $Y = Z$;*

(ii) *$\sigma \,\mathcal{L}\, \tau$ if and only if $Yg = Zg$;*

(iii) *$\sigma \,\mathcal{D}\, \tau$ if and only if $Y \in ZG$; in other words, σ and τ are $\mathcal{D}$-related if and only if subspaces Y and Z lie in the same orbit of the action of the group G.*

Observe that, in view of the above assertion, the rows and columns of any $\mathcal{D}$-class of $M(G, S)$ are indexed by the subspaces of the system S of a fixed dimension. In addition, according to (i) and (ii), an $\mathcal{H}$-class $H_{Y,Z}$ in $M(G, S)$ is the set of all maps g_Y that have the same given domain Y and the same given image $Z = Yg$;

$$H_{Y,Z} = \{g_Y : Z = Yg, \text{ where } Y, Z \in S\}. \tag{2.19}$$

Since any $\mathcal{H}$-class that contains an idempotent forms a maximal subgroup and an idempotent in $M(G, S)$ is of the form id_Y, a partial identity on Y, it follows that a maximal subgroup in $M(G, S)$ can be characterised as

$$M^Y := H_{Y,Y} = \{g_Y : Y = Yg, \text{ with } Y \in S\} \tag{2.20}$$

with identity id_Y. In other words, M^Y contains all partial maps that have the same given domain and image. Another interesting maximal subgroup of $M(G, S)$ is the group of units

$$M^V = \{g_V : V = Vg, \text{ with } V \text{ is the whole vector space}\}; \tag{2.21}$$

that is the group G. Specifically, if $M(W, S)$ is a reflection monoid, then M^Y is a maximal subgroup and M^V is the reflection group W. Interestingly, unlike a general finite inverse semigroup, all maximal subgroups M^Y of a reflection monoid $M(W, S)$ can be obtained from the group of units; the reflection group W. Such an observation is an unpublished result due to Everitt and Fountain, and we provide the proof below.

Proposition 2.2.22. *Consider the following two subgroups of W :*

$$W^Y = \{g \in W : Yg = Y\};$$

the invariant subgroup of Y and the isotropy subgroup of Y :

$$W_Y = \{g \in W : vg = v \text{ for all } v \in Y\}.$$

Then each maximal subgroup M^Y is determined by taking the subquotient of the subgroup W^Y by the isotropy group W_Y :

$$M^Y \cong \frac{W^Y}{W_Y}.$$

Proof. It is clear that $W_Y \subset W^Y$. Let us define a map

$$W^Y \xrightarrow{\psi} M^Y \quad \text{by}$$
$$g \mapsto g_Y.$$

Observe that ψ is a group homomorphism for the following reason: Let $g, h \in W^Y$; that is $Yg = Y = Yh$. Then,

$$g\psi\ h\psi = g_Y\ h_Y = (gh)_{Y \cap Yg^{-1}} = (gh)_{Y \cap Y} = (gh)_Y = (gh)\psi,$$

and ψ is surjective because, for all $g_Y \in M^Y$, we have $Yg = Y$; thus, there exists $g \in W^Y$ with $g\psi = g_Y$. Moreover,

$$\begin{aligned}
\operatorname{Ker}(\psi) &= \{g : g\psi = id_Y\} \\
&= \{g : g_Y = id_Y\} \\
&= \{g : g\ id^{-1} \in G_Y\} \quad \text{[By (2.16)]} \\
&= \{g : vg = v \ \ \forall\ v \in Y\} \\
&= W_Y.
\end{aligned}$$

The result then follows by the first isomorphism theorem. □

Chapter 3

Representation theory of finite groups

3.1 General theory of linear representations

This section provides a preliminary overview of key contextual information that will be built on in the thesis. The decision was made to refrain from providing an in-depth account of all relevant background materials, and instead, a brief illumination of some fundamental concepts, such as the simplicity and semisimplicity of a group representation, is given. The symmetric group S_n was chosen as an example to illustrate these concepts. At this stage, suppositions are not mainly made on the field $\mathbb{K}$.

Definition 3.1.1. [77] Let G be a finite group and V be a finite dimensional vector space over a field $\mathbb{K}$. Let $GL(V)$ be the group of all invertible linear maps on V. A representation of G over $\mathbb{K}$ is a homomorphism $\phi : G \longrightarrow GL(V)$.

We call dim V the degree of ϕ. Notice that, since ϕ is a homomorphism, it follows that, for all $g, h \in G$, $(gh)\phi = (g)\phi(h)\phi$. Moreover, if 1 is the identity of G and id is the identity linear map in $GL(V)$, then $(1)\phi = id$. Hence, $(g^{-1})\phi = (g\phi)^{-1}$ as

$$id = (1)\phi = (gg^{-1})\phi = (g)\phi(g^{-1})\phi.$$

Now, for all $v \in V$ and $g, g' \in G$, the product $v(g\phi) \in V$ and homomorphism defined above yield that

$$v((gg')\phi) = v(g\phi)(g'\phi).$$

Further, as $(1)\phi$ is the identity map, it follows that $v(1\phi) = v$ for all $v \in V$, and the features of the linear maps in $GL(V)$ show that for all $v, v' \in V$, $k \in \mathbb{K}$ and $g \in G$, we have

$$(kv)(g\phi) = k(v(g\phi)) \quad \text{and} \quad (v + v')(g\phi) = v(g\phi) + v'(g\phi)$$

In fact, such an observation allows us to view a representation ϕ of G as a G-module or $\mathbb{K}G$-module, as defined below.

Definition 3.1.2. [43] Let G be a finite group and V be a finite dimensional vector space over $\mathbb{K}$. Then, V is a right G-module if there exists a mapping $V \times G \longrightarrow V$ such that $(v, g) \mapsto vg \in V$, where $v \in V$ and $g \in G$, and it satisfies the following conditions for all $v, u \in V$, $k \in \mathbb{K}$ and $g, g' \in G$:

(i) $v(gg') = (vg)g'$,

(ii) $v1_G = v$, where 1_G is the identity of a group G,

(iii) $(kv)g = k(vg)$,

(iv) $(v + v')g = vg + v'g$.

Example 3.1.3.

(i) Consider the symmetric group S_n, and let V be a $\mathbb{C}$-vector space with basis $\{v_1, \ldots, v_n\}$. We know from Section 2.2.1 that elements of S_n can be thought of as linear maps acting on V by permuting the coordinates; that is, for all $\sigma \in S_n$,

$$(v_i)\sigma = v_{i\sigma}.$$

Observe that, for all i with $1 \leq i \leq n$, $(v_i)\sigma \in V$ and $(v_i)id = v_i$ where id is the identity permutation. Moreover, for all $\sigma, \tau \in S_n$, $(v_i)\sigma\tau = v_{i(\sigma\tau)} = v_{(i\sigma)\tau} = (v_i\sigma)\tau$. In addition, extending this action linearly to V yields

$$(\sum_i c_i v_i)\sigma = \sum_i c_i(v_i\sigma).$$

Hence, V is an S_n-module. In fact, such an action is called a permutation module for S_n.

(ii) Let G also be the symmetric group S_n and V be a vector space over $\mathbb{C}$. Define the action of S_n on V as follows: For all $v \in V$ and $\sigma \in S_n$,

$$v\sigma = \begin{cases} v, & \text{if } \sigma \text{ is an even permutation,} \\ -v, & \text{if } \sigma \text{ is an odd permutation.} \end{cases}$$

Observe that $v\sigma \in V$ as V is a vector space and $v', -v \in V$. Furthermore, the identity permutation id is even; thus $(v)id = v$ and $(kv)\sigma = k(v\sigma)$. In addition, for all $v, v' \in V$, we have

$$v\sigma + v'\sigma = \begin{cases} v, & \text{if } \sigma \in A_n, \\ -v, & \text{otherwise.} \end{cases} + \begin{cases} v', & \text{if } \sigma \in A_n, \\ -v', & \text{otherwise.} \end{cases}$$

$$= \begin{cases} v + v', & \text{if } \sigma \in A_n, \\ -(v + v'), & \text{otherwise.} \end{cases}$$

$$= (v + v')\sigma.$$

Let us now show that $v(\sigma\tau) = (v\sigma)\tau$. Consider $v(\sigma\tau)$ and suppose $\sigma\tau$ is an even permutation. Then, $v = v(\sigma\tau)$. In view of our assumption on $\sigma\tau$, we have two cases, as follows: either σ and τ are both even permutations or they are both odd. In the first case, $(v\sigma)\tau = v\tau = v$, and in the second case, $(v\sigma)\tau = (-v)\tau = -(-v) = v$. Thus, $v(\sigma\tau) = v = (v\sigma)\tau$. Let us now assume that $\sigma\tau$ is an odd permutation. Hence $-v = v(\sigma\tau)$. Moreover, the assumption also requires that either σ is even and τ is odd or vice versa. Without loss of generality, let σ be even and τ be odd. Then, $(v\sigma)\tau = v\tau = -v$. Hence, $v(\sigma\tau) = (v\sigma)\tau$. In fact, the above discussion shows that V is an S_n-module, which is commonly called a sign representation. If dim $V = 1$, then it is called the signed representation.

Definition 3.1.4. [43] A one-dimensional vector space V over $\mathbb{K}$ with $vg = v$ for all $v \in V$ and $g \in G$ is called the trivial G-module (trivial representation of G).

Definition 3.1.5. [43] Let V be a G-module. A subspace $U \subseteq V$ is called a G-submodule of V if and only if it is a subspace of V and is invariant under the action of G; that is, for all $u \in U$ and $g \in G$, $ug \in U$.

Definition 3.1.6. [43] A G-module V is called simple (or irreducible) if and only if V is nonzero and the only submodules of V are V and $\mathbf{0}$. Otherwise, V is reducible.

The example below appears in [18] and illustrates the two previous concepts.

Example 3.1.7. Let G be the symmetric group S_n and V be a permutation module over $\mathbb{C}$ with basis $\{v_1, \cdots, v_n\}$. Consider the following two subspaces of V:

(a) The subspace spanned by the vector $v_1 + \cdots + v_n$; that is,

$$U = Span_{\mathbb{C}}\ \{v_1 + v_2 + \cdots + v_n\}.$$

It is obvious that U is an S_n-submodule of V, and it is simple as $\dim U = 1$. Thus, V is a reducible S_n-representation and U is a trivial S_n-module.

(b) The hyperplane W with equation $x_1 + x_2 + \cdots + x_n = 0$; that is,

$$W = \Big\{ w = \sum_{i=1}^{n} c_i v_i \in V : \sum_{i=1}^{n} c_i = 0 \Big\}.$$

It should be noted that W is an S_n-submodule of V, since for all $w = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$, where $c_1 + c_2 + \cdots + c_n = 0$ and all $\sigma \in S_n$, we have

$$\begin{aligned}(w)\sigma &= (c_1 v_1 + c_2 v_2 + \cdots + c_n v_n)\sigma \\ &= c_1 v_{1\sigma} + c_2 v_{2\sigma} + \cdots + c_n v_{n\sigma},\end{aligned}$$

where $c_1 + c_2 + \cdots + c_n = 0$ as well. It turns out that W is indeed a simple S_n-submodule. The following argument was supplied by Dr.Michael Bate for the notes [18]. Let $w \in W$ with $w \neq \mathbf{0}$. Suppose first that w has the same coefficients; that is, $w = \sum_{i=1}^{n} c v_i$. Thus, $\sum_{i=1}^{n} c = 0$ as $w \in W$. Observe that

$$\sum_{i=1}^{n} c = 0 \iff nc = 0 \iff c = 0, \qquad [\ char(\mathbb{C}) \nmid n \text{ as } char(\mathbb{C}) = 0]$$

and this implies that $w = \mathbf{0}$. Therefore, we acquire a contradiction with the assumption that w is a nonzero vector. Hence, each $w \in W$ has at least two distinct scalar multiples when it is written as a linear combination of basis vectors; that is,

$$w = c_1 v_1 + \cdots + c_r v_r + \cdots + c_s v_s + \cdots + c_n v_n, \tag{3.1}$$

where $c_r \neq c_s$ for some $1 \leq r < s \leq n$. Since the elements of S_n are permutations, then for any two numbers $r, s \in [n]$, there always exists a permutation that sends them to two consecutive numbers. In other words, for all $i \in [n]$, there is $\sigma_i \in S_n$ such that $r\sigma_i = i$ and $s\sigma_i = i + 1$. However, by applying σ_i to w, we have

$$\begin{aligned}(w)\sigma_i &= c_1 v_{1\sigma_i} + \cdots + c_r v_{r\sigma_i} + \cdots + c_s v_{s\sigma_i} + \cdots + c_n v_{n\sigma_i} \\ &= c_1 v_{1\sigma_i} + \cdots + c_r v_i + c_s v_{i+1} + \cdots + c_n v_{n\sigma_i}.\end{aligned}$$

Observe that the scalar multiples of the i-th and $(i + 1)$-th basis vectors in $(w)\sigma_i$ are still distinct. Let us relabel them as $c_r = c_i$ and $c_s = c_{i+1}$:

$$(w)\sigma_i = c_1 v_{1\sigma_i} + \cdots + c_i v_i + c_{i+1} v_{i+1} + \cdots + c_n v_{n\sigma_i}.$$

Now, considering any submodule W' of W that contains w yields that $(w)\sigma_i \in W'$. Thus, an S_n-submodule W' contains both w and $(w)\sigma_i$ for all $i \in [n]$. If we write

$(w)\sigma_i$ in terms of its coordinates $(c_1, \ldots, c_i, c_{i+1}, \ldots, c_n)$ and consider the vector $(w)\sigma_i(i, i+1) - (w)\sigma_i \in W'$, we have

$$\begin{aligned}
(w)\sigma_i(i, i+1) - (w)\sigma_i &= (c_1, \ldots, c_i, c_{i+1}, \ldots, c_n)(i, i+1) - (c_1, \ldots, c_i, c_{i+1}, \ldots, c_n) \\
&= (c_1, \ldots, c_{i+1}, c_i, \ldots, c_n) - (c_1, \ldots, c_i, c_{i+1}, \ldots, c_n) \\
&= (0, \ldots, 0, c_{i+1} - c_i, c_i - c_{i+1}, 0, \ldots, 0) \\
&= \zeta(0, \ldots, 0, 1, -1, 0, \ldots, 0),
\end{aligned}$$

where $\zeta = c_{i+1} - c_i$ and $\zeta \neq 0$, as $c_i \neq c_{i+1}$. Thus, $(w)\sigma_i(i, i+1) - (w)\sigma_i = \zeta(v_i - v_{i+1})$, and then, $v_i - v_{i+1} \in W' \subseteq W$, for all i with $1 \leq i \leq n-1$. Now, we claim that $\{v_i - v_{i+1},\ 1 \leq i \leq n-1\}$ forms a basis of W. To see this, notice that the set $\{v_i - v_{i+1},\ 1 \leq i \leq n-1\}$ is independent since if

$$c_1(v_1 - v_2) + c_2(v_2 - v_3) + \cdots + c_{n-1}(v_{n-1} - v_n) = 0,$$

then $c_1v_1 + (c_2 - c_1)v_2 + (c_3 - c_2)v_3 + \cdots + (c_{n-1} - c_{n-2})v_{n-1} + (-c_{n-1})v_n = 0$. As $\{v_1, \cdots, v_n\}$ is the basis of V, hence it is independent. It follows that

$$c_1 = c_2 - c_1 = c_3 - c_2 = c_{n-1} - c_{n-2} = -c_{n-1} = 0.$$

Thus, $c_1 = c_2 = \cdots = c_{n-1} = 0$. Moreover, these vectors $\{v_i - v_{i+1}\}$ spans $(n-1)$-dimensional subspace of W :

$$\operatorname{Span}_{\mathbb{C}}\{v_i - v_{i+1},\ s1 \leq i \leq n-1\} \leq W$$

However, W is a hyperplane so dim $W = n - 1$. Thus, $\operatorname{Span}_{\mathbb{C}}\{v_i - v_{i+1},\ 1 \leq i \leq n-1\} = W$. This means that W' contains a basis for W; hence, W is a simple S_n-module.

Definition 3.1.8. [43] Let V and V' be two G-modules. A function $\vartheta : V \longrightarrow V'$ is called a G-homomorphism if ϑ is a linear map and

$$(vg)\vartheta = (v\vartheta)g \text{ for all } v \in V,\ g \in G.$$

If ϑ is a G-homomorphism and invertible, then it is a G-isomorphism, and V and V' are isomorphic G-modules.

Proposition 3.1.9. [43] *Let V and V' be two G-modules and $\vartheta : V \longrightarrow V'$ be a G-homomorphism. Then,*

(i) *Ker ϑ is a G-submodule of V.*

(ii) *Im ϑ is a G-submodule of V'.*

Recall that if V is a vector space and $V_1, \cdots, V_m$ are subspaces of V, then

$$V_1 + \cdots + V_m = \{v_1 + \cdots + v_m : v_i \in V_i \text{ where } 1 \leq i \leq m\}.$$

Observe that $V_1 + \cdots + V_m$ is a subspace of V.

Definition 3.1.10. [74] A vector space V is the direct sum of a family $\{V_i : 1 \leq i \leq m\}$ of subspaces of V if and only if

(i) $V = \sum_i V_i$, and

(ii) For each i, where $1 \leq i \leq m$,

$$V_i \bigcap \left(\sum_{j \neq i} V_j\right) = \mathbf{0}$$

Then, V is denoted $\bigoplus_i V_i$, with $1 \leq i \leq m$.

Definition 3.1.11. [74] Let V be a G-module and suppose that as a vector space

$$V = \bigoplus_i^m V_i,$$

where the V_i are G-submodules of V. Then V is called a direct sum of submodules.

Definition 3.1.12. [43] A G-module V is said to be semisimple (or a completely reducible representation) if it is a direct sum of simple G-submodules (or irreducible subrepresentations) V_i with $1 \leq i \leq m$.

We will now state a fundamental result in the representation theory of finite groups.

Theorem 3.1.13 (Maschke's Theorem)**.** Let G be a finite group and $\mathbb{K}$ be a field whose characteristic does not divide the order of G. If V is any G-module and V' is a G-submodule of V, then V has a submodule V'' such that $V = V' \oplus V''$.

A full proof of the above statement appears in [17, Theorem 18.1.1]. Notice that if the field is the complex numbers, then the hypothesis of Maschke's Theorem holds.

Corollary 3.1.14. *Let G be a finite group and $\mathbb{K}$ be a field such that char$(\mathbb{K}) \nmid |G|$. Then, every finite dimensional G-module V is semisimple (completely reducible).*

Proof. Let V be a non-zero G-module V. We will prove the result utilising induction on the dimension of V. If dim $V = 1$, then the result holds as every 1-dimensional G-module is simple. Let dim $V = n$, and suppose every G-module of dimension less than n is isomorphic to a direct sum of simple G-submodules. Then, we have two cases:

- Case 1. If V is a simple G-module, then the result holds.
- Case 2. If V is not a simple G-module, then V has a non-zero proper submodule V'. Thus, by Maschke's Theorem, V has a G-submodule complement V'' such that $V = V' \oplus V''$. Notice that dim $V' < n$ and dim $V'' < n$; hence, by induction, both V' and V'' are direct sums of some simple G-submodules V_i' and V_j'' respectively. In other words,

$$V' = V_1' \oplus \cdots \oplus V_k' \text{ and } V'' = V_1'' \oplus \cdots \oplus V_l''.$$

Therefore, V is a direct sum of simple G-submodules, and the induction is completed.

□

Example 3.1.15. Let G be the symmetric group S_n and V be the permutation module V illustrated in Example 3.1.7. Then $V = U \oplus W$ where U and W are S_n-modules described in the same example.

Lemma 3.1.16 (Schur's Lemma)**.** *Let V and V' be simple G-modules over the complex field $\mathbb{C}$. If $\vartheta : V \longrightarrow V'$ is a G-homomorphism, then either ϑ is a G-isomorphism, or $(v)\vartheta = \mathbf{0}$ for all $v \in V$.*

The proof of the lemma appears in [43, Lemma 9.1].

We will now close this section with stating a striking theorem in the representation theory of finite groups that provides a connection between the number of inequivalent irreducible representations and that of conjugacy classes of a finite group G.

Theorem 3.1.17. Let G be a finite group and $\mathbb{C}$ be the field of complex numbers. Then, the number of inequivalent irreducible representations of G is equal to the number of conjugacy classes of G.

The proof of the above assertion can be found in [17, Theorem 18.2.10]. Throughout the remaining chapters, we work over the field of complex numbers $\mathbb{C}$ unless otherwise specified.

3.2 Specht modules for the symmetric group S_n

The question of how many representations we need to look for was dealt with in the previous section, where we stated that one representation should be found for each conjugacy class. Despite this, an explicit one-to-one correspondence has not generally been identified between the conjugacy classes and irreducible representations. Nevertheless, in the case of the finite symmetric group S_n and a few other groups, it is possible to capitalise on our knowledge of the association between the conjugacy classes and elegant combinatorial descriptions, thus creating corresponding irreducible representations. That is to say, distinct from the general case of finite groups, there exists a natural strategy by which the parameterisation of irreducible representations can take place. To be more precise, this strategy relies on the set which is responsible for parametrising the conjugacy classes (i.e. partitions of n, or as would be equivalent, n-size Young diagrams).

Based on his work in the field of invariant theory, the British mathematician Alfred Young [81] devised Young diagrams and Young tableaux in 1901. In the following years, Young and Frobenius [23] independently demonstrated how Young tableaux provide data pertaining to representations of symmetric groups, and from then on, these combinatorial objects have performed critical functions in numerous mathematical branches.

With a focus on combinatorial issues, this section provides a concise overview of the fundamental representation theory of the symmetric group. While basic information about Specht modules is presented, we dedicate a significant portion of this section to ensuring that the reader is pointed toward more effective sources for proofs of this kind. Alongside this, examples are given to elucidate key concepts, and three primary irreducible representations for the symmetric group are outlined.

Definition 3.2.1. [17] Let $\pi \in S_n$. The cycle structure (shape or type) of a permutation π is an expression $(\lambda_1, \ \dots \ , \lambda_r)$, where $\lambda_1 \geq \lambda_2 \geq \ \dots \ \geq \lambda_r$ and there is a λ_i for each cycle of length λ_i when a permutation π is decomposed as a disjoint product of cycles (including cycles of length one).

Note that a cycle shape for $\pi \in S_n$ can also be called a partition λ of n. A partition of an integer n can be defined as given below.

Definition 3.2.2. [75] A partition of a positive integer $n \in \mathbb{Z}^+$ is a tuple $\lambda = (\lambda_1, \lambda_2, \ \dots \ , \lambda_m)$ such that

- $\lambda_i \in \mathbb{Z}^+$ for all $1 \leq i \leq m$,
- $\lambda_1 \geq \lambda_2 \geq \ \dots \ \geq \lambda_m$,

- $\lambda_1 + \lambda_2 + \ \ldots \ + \lambda_m = n$.

We write $\lambda \vdash n$ to denote that λ is a partition of n.

We denote $\sum_{i=1}^{m} \lambda_i$ by $|\lambda|$ and allow λ to be the empty set. It becomes clear that the partitions of n are canonically associated with the cycle shapes of S_n. For instance, the partition $(4, 2, 1)$ of 7 is associated with the permutations on 7 letters with 1 four-cycle, 1 two-cycle and 1 one-cycle.

Proposition 3.2.3. [43] *Let π be a cycle $(a_1, a_2, \cdots, a_m)$ of length m and $\tau \in S_n$. Then, a permutation*

$$\tau^{-1}\pi\tau = (a_1\tau, a_2\tau, \cdots, a_m\tau)$$

is also of length m.

Proposition 3.2.4. *Two permutations in S_n are conjugate if and only if they have the same cycle structure.*

The proof of the above assertion appears in [17, Proposition 11].

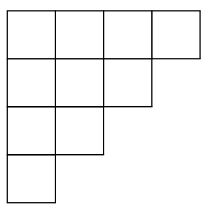
Corollary 3.2.5. [17] *For each positive integer n, the number of different partitions of n is equal to the number of conjugacy classes of S_n.*

Proof. There is a one-to-one correspondence between the conjugacy classes of S_n and the cycle structure, and every cycle shape is a partition. □

Recall that the number of conjugacy classes of S_n is equal to the number of irreducible representations of S_n over $\mathbb{C}$ by Theorem 3.1.17. Hence, the irreducible representations of S_n over the complex field are parametrised by the partitions of n. The following combinatorial objects are fundamental tools in computing the irreducible representations of the symmetric group S_n.

Definition 3.2.6. [24] Given a partition $\lambda = (\lambda_1, \lambda_2, \ \ldots \ , \lambda_m)$ of n, a Young diagram of shape λ is a collection of non-increasing, left-justified, empty boxes arranged in rows such that there are λ_i boxes in the ith row.

For example, the following is the Young diagram of shape $(4, 3, 2, 1)$:



Definition 3.2.7. [75] Given a partition λ of n, a Young tableau t of shape λ is obtained by filling in the boxes of a λ-Young diagram with the numbers $\{1, 2, \ldots, n\}$, such that different boxes contain distinct numbers.

Clearly, there are $n!$ distinct Young tableaux for each λ. Moreover, each Young tableau corresponds to a permutation; that is, if we decompose a permutation into a product of disjoint cycles, then the numbers in each cycle form the entries of rows in a tableau. For instance, a permutation $\pi = (13)(2)(4) \in S_4$ can be presented in a tableau as follows:

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \text{or} \quad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

Further, the symmetric group S_n acts on a λ-Young tableau t by permuting the entries. For instance,

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 5 & 4 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \cdot (12)(354) = \begin{array}{|c|c|}\hline 2 & 5 \\ \hline 4 & 3 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

The definitions below are crucial in determining the irreducible representations of S_n utilising Young tableaux.

Definition 3.2.8. [24] Given a Young tableau t of shape $\lambda = (\lambda_1, \lambda_2, \ \dots \ , \lambda_m)$, define the row group R_t to be the subgroup of S_n consisting of all the permutations that preserve the rows of t.

Observe that $R_t \cong S_{\lambda_1} \times S_{\lambda_2} \times \ldots \times S_{\lambda_m}$.

Definition 3.2.9. [75] Let $c_1, c_2, \ldots, c_l$ be the columns of a λ-Young tableau t. Define the column group C_t to be the subgroup of S_n consisting of all the permutations that preserve the columns c_i of t.

It should be noted that $C_t = S_{c_1} \times S_{c_2} \times \ldots \times S_{c_l}$.

Proposition 3.2.10. [75] *Let t be a λ-tableau and $\pi \in S_n$. Then,*

(i) $C_{t\pi} = \pi^{-1} C_t \pi$,

(ii) $R_{t\pi} = \pi^{-1} R_t \pi$.

The proof of the above assertion appears in [75, Lemma 2.3.3].

Let Y^λ be the set of all λ-Young tableaux t. Define an equivalence relation on Y^λ as follows:

$t_1 \sim t_2$ if and only if the entries of each row in t_1 are the same as the entries in the corresponding row of t_2.

We say that t_1 and t_2 are then row equivalent. The following definition allows us to partition the set Y^λ into distinct classes:

Definition 3.2.11. [75] Given a Young tableau t of shape λ, define a Young tabloid $\{t\}$ to be the set of all Young tableaux of shape λ that are row equivalent to t; that is, a Young tabloid $\{t\}$ is the orbit of Young tableau t under the action of the row group.

$$\{t\} = \{(t)\sigma : \ \sigma \in R_t\}.$$

It is sometimes denoted by omitting the vertical lines between boxes, as stated in [24, Section 7.2].

For instance, if $(3,1) \vdash 4$ and $t = \young(123,4)$ is a $(3,1)$-tableau, then

$$\{t\} = \left\{ \young(123,4), \young(132,4), \young(213,4), \young(231,4), \young(312,4), \young(321,4) \right\}.$$

Observe that the symmetric group S_n acts on the set of tabloids by the following formula

$$\{t\} \ . \ \pi = \{t\pi\} \text{ for all } \pi \in S_n, \tag{3.2}$$

and such an action is clearly well defined [24, Section 7.2]. Notice that if $\pi \in R_t \subseteq S_n$, then π stabilises the Young tabloid $\{t\}$.

Definition 3.2.12. Let $\lambda \vdash n$. Define M^λ to be the complex vector space whose basis is the set of distinct λ-tabloids $\{t\}$.

A typical element of M^λ is a formal $\mathbb{C}$-linear combination of tabloids $\{t\}$ of shape λ.

Lemma 3.2.13. [75] *If $\lambda \vdash n$ and $\lambda = (\lambda_1, \lambda_2, \ \ldots \ , \lambda_m)$, then the dimension of M^λ is determined by the following formula:*

$$dim \ M^\lambda = \frac{n!}{\lambda_1! \ \lambda_2! \ \ \ldots \ \ \lambda_m!} \ .$$

In view of the action presented in (3.2), the symmetric group S_n acts on the set of distinct tabloids $\{t\}$, and by extending such an action linearly, M^λ becomes a right S_n-module.

Example 3.2.14. Let $n = 3$ and fix $\lambda = (2,1) \vdash 3$. Then, the set of all 6 distinct λ-tableaux and the S_3-module $M^{(2,1)}$ can be determined as follows:

$$Y^{(2,1)} = \left\{ t_{12} = \young(23,1), t_{13} = \young(32,1), t_{21} = \young(13,2), t_{23} = \young(31,2), t_{31} = \young(12,3), t_{32} = \young(21,3) \right\}.$$

$$M^{(2,1)} = \mathbb{C}\text{-}\left[\left\{\begin{array}{|c|c|}\hline 2&3\\\hline 1\\\cline{1-1}\end{array}\right\}, \left\{\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\right\}, \left\{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\right\}\right].$$

More generally, if $\lambda = (n-1, 1) \vdash n$, let

$$v_j = \left\{\begin{array}{|c|c|c|c|c|c|}\hline 1&2&\cdots&\hat{j}&\cdots&n\\\hline j\\\cline{1-1}\end{array}\right\}$$

be a basis $(n-1,1)$-tabloid of $M^{(n-1,1)}$, where j refers to the entry in the bottom row with $1 \leq j \leq n$ and $\hat{j}$ in the upper row means that the number j omitted. Observe that the tabloid v_j only varies when the entry in the bottom row varies. In other words, the bottom row entry uniquely determines the basis vectors and

$$M^{(n-1,1)} = \mathbb{C}\text{-}\left[\ v_1 = \left\{\begin{array}{|c|c|c|c|}\hline 2&3&\cdots&n\\\hline 1\\\cline{1-1}\end{array}\right\}, \cdots, v_n = \left\{\begin{array}{|c|c|c|c|}\hline 1&2&\cdots&n\text{-}1\\\hline n\\\cline{1-1}\end{array}\right\}\ \right].$$

Thus, there are n distinct basis vectors of $M^{(n-1,1)}$. In view of the S_n-action presented in the formula (3.2), for all $\pi \in S_n$,

$$v_j \cdot \pi = \left\{\begin{array}{|c|c|c|c|c|c|}\hline 1\pi&2\pi&\cdots&\hat{j\pi}&\cdots&n\pi\\\hline j\pi\\\cline{1-1}\end{array}\right\} = v_{j\pi}.$$

It follows that $M^{(n-1,1)}$ is the S_n-permutation representation, as described in Example 3.1.3(i).

The following definition give us a clue for figuring out all of the irreducible representations of S_n.

Definition 3.2.15. [75] Fix $\lambda \vdash n$. For each Young tableau t of shape λ, there is an element $e_t \in M^\lambda$ defined by the formula

$$e_t = \sum_{\sigma \in C_t} sgn(\sigma)\ \{t\}\sigma.$$

Such an element is called a λ-polytabloid.

The following lemma illustrates how the symmetric group S_n permutes the set $\{e_t : t \text{ is a Young tableau of shape } \lambda\}$.

Lemma 3.2.16. *For any Young tableau t and permutation $\pi \in S_n$, we have*

$$e_t\ .\ \pi = e_{t\pi}$$

Proof. For all $\pi \in S_n$, we have

$$\begin{aligned} e_t\ .\ \pi &= \Big(\sum_{\sigma \in C_t} sgn(\sigma)\ \{t\}\sigma\Big).\ \pi \\ &= \sum_{\sigma \in C_t} sgn(\sigma)\ \Big(\{t\sigma\}.\ \pi\Big) && \text{[By (3.2) and } M^\lambda \text{ is an } S_n\text{-module]} \end{aligned}$$

$$= \sum_{\sigma \in C_t} sgn(\sigma) \Big(\{t\}.\ \pi\pi^{-1}\sigma\pi\Big) \qquad [\text{As } C_t \leq S_n]$$

$$= \sum_{\sigma \in C_t} sgn(\sigma)\ \{t\}.\ \pi\ (\pi^{-1}\sigma\pi)$$

$$= \sum_{\sigma \in C_t} sgn(\pi^{-1}\sigma\pi)\ \{t\}.\ \pi\ (\pi^{-1}\sigma\pi) \qquad [\text{As } sgn(\pi^{-1}\sigma\pi) = sgn\ \pi^{-1}\ sgn\ \sigma\ sgn\ \pi]$$

$$= \sum_{\tau \in \pi^{-1}C_t\pi} sgn(\tau)\ \{t\}.\ \pi\ \tau \qquad [\tau = \pi^{-1}\sigma\pi]$$

$$= \sum_{\tau \in C_{t\pi}} sgn(\tau)\ \{t\pi\}.\ \tau \qquad [\text{By Proposition 3.2.10}(i)]$$

$$= e_{t\pi}.$$

□

Definition 3.2.17. [24] For any partition λ of n, define the Specht module S^λ to be the subspace of M^λ spanned by the elements e_t, where t is taken over all the Young tableaux of shape λ; that is,

$$S^\lambda = \{c_1 e_{t_1} + \ \dots\ + c_{n!} e_{t_{n!}} :\ c_i \in \mathbb{C},\ t_i \text{ is a Young tableau } \forall\ 1 \leq i \leq n!\ \}.$$

By extending the action presented in the preceding lemma, the Specht module S^λ is invariant under S_n; hence, S^λ is an S_n-submodule of M^λ. In the next example, we illustrate various Specht modules S^λ.

Example 3.2.18.

(i) Let $n = 3$ and $\lambda = (3) \vdash 3$. Then, there are 3! distinct (3)-tableaux.

$$Y^{(3)} = \Big\{\boxed{1|2|3}, \boxed{1|3|2}, \boxed{2|1|3}, \boxed{2|3|1}, \boxed{3|1|2}, \boxed{3|2|1}\Big\}$$

Notice that all these tableaux of shape (3) generate the same tabloid $\{\boxed{1|2|3}\}$. Further, the column group C_t for all $t \in Y^{(3)}$ has size 1. Thus, any (3)-tableau in $Y^{(3)}$ gives the same polytabloid; that is,

$$e_{\boxed{1|2|3}} = +\{\boxed{1|2|3}\}\ id = \{\boxed{1|2|3}\}$$

Hence,

$$S^{(3)} = Span_{\mathbb{C}}\Big\{e_{\boxed{1|2|3}}\Big\}.$$

However, a polytabloid $e_{\boxed{1|2|3}}$ is indeed preserved by S_3 as for all $\pi \in S_3$,

$$\{\boxed{1|2|3}\} \cdot \pi = \{\boxed{1|2|3}\}.$$

It follows that the Specht module $S^{(3)}$ is the 1-dimensional trivial S_3-module. More generally, if $\lambda = (n)$, then

$$S^{(n)} = Span_{\mathbb{C}}\left\{e_{\begin{array}{|c|c|c|c|}\hline 1 & 2 & \cdots & n\\\hline\end{array}}\right\}$$

is the trivial S_n-module.

(ii) Let $n = 3$ and $\lambda = (1, 1, 1) \vdash 3$. Then, there are 3! distinct $(1, 1, 1)$-tableaux.

$$Y^{(1,1,1)} = \left\{t_1 = \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array},\ t_2 = \begin{array}{|c|}\hline 1\\\hline 3\\\hline 2\\\hline\end{array},\ t_3 = \begin{array}{|c|}\hline 2\\\hline 1\\\hline 3\\\hline\end{array},\ t_4 = \begin{array}{|c|}\hline 2\\\hline 3\\\hline 1\\\hline\end{array},\ t_5 = \begin{array}{|c|}\hline 3\\\hline 1\\\hline 2\\\hline\end{array},\ t_6 = \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}\right\}$$

It is worth knowing that, for each $(1, 1, 1)$-tableau t_i with $1 \le i \le 6$, the column group $C_{t_i} = S_3$; thus, each polytabloid e_{t_i} constructed by any of the above $(1, 1, 1)$-tableaux t_i is a sum of 3! distinct $(1, 1, 1)$-tabloids. For instance,

$$\begin{aligned} e_{t_1} &= +\left\{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\right\} id - \left\{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\right\}(12) - \left\{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\right\}(13) - \left\{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\right\}(23) + \left\{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\right\}(123) + \left\{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\right\}(132). \\ &= +\left\{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\right\} - \left\{\begin{array}{|c|}\hline 2\\\hline 1\\\hline 3\\\hline\end{array}\right\} - \left\{\begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}\right\} - \left\{\begin{array}{|c|}\hline 1\\\hline 3\\\hline 2\\\hline\end{array}\right\} + \left\{\begin{array}{|c|}\hline 2\\\hline 3\\\hline 1\\\hline\end{array}\right\} + \left\{\begin{array}{|c|}\hline 3\\\hline 1\\\hline 2\\\hline\end{array}\right\} \end{aligned}$$

Thus, the Specht module

$$S^{(1,1,1)} = Span_{\mathbb{C}}\{e_{t_1}, e_{t_2}, e_{t_3}, e_{t_4}, e_{t_5}, e_{t_6}\}.$$

Observe that for all $\pi \in S_3$ and $t_i \in Y^{(1,1,1)}$ we have

$$\begin{aligned} e_{t_i} \cdot \pi &= \Big(\sum_{\sigma \in C_{t_i}} sgn(\sigma)\ \{t_i\}\sigma\Big) \cdot \pi \\ &= \sum_{\sigma \in S_3} sgn(\sigma)\ \{t_i\}\sigma\pi \\ &= \sum_{\sigma \in S_3} sgn(\sigma\pi)\ sgn(\pi)\ \{t_i\}\sigma\pi && \begin{array}{l} sgn(\sigma\pi) = sgn(\sigma)sgn(\pi) \\ sgn(\sigma) = sgn(\sigma\pi)sgn(\pi) \end{array} \\ &= sgn(\pi) \sum_{\sigma \in S_3} sgn(\sigma\pi)\{t_i\}\sigma\pi \\ &= sgn(\pi) \sum_{\sigma\pi \in S_3 \cdot \pi} sgn(\sigma\pi)\{t_i\}\sigma\pi \\ &= sgn(\pi) \sum_{\tau \in S_3} sgn(\tau)\ \{t_i\}\tau && [\text{Set } \ \tau = \sigma\pi] \\ &= sgn(\pi)\ e_{t_i} \end{aligned}$$

Note that for all $t_i \in Y^{(1,1,1)}$, $e_{t_i} = \pm\, e_{t_1}$. Hence, $S^{(1,1,1)} = Span_{\mathbb{C}}\{e_{t_1}\}$ is indeed the 1-dimensional sign S_3-module. More generally, if $\lambda = (\underbrace{1, 1, \cdots, 1}_{n}) \vdash n$, then the Specht module $S^{(1,1,\cdots,1)}$ is the 1-dimensional sign S_n-module.

(iii) Let $n = 3$ and $\lambda = (2,1) \vdash 3$. We illustrated in Example 3.2.14 the 3! distinct (3)-tableaux. Let us relabel a tableau in a set $Y^{(2,1)}$ by $t_{ji} = \begin{array}{|c|c|}\hline i & k \\ \hline j \\ \cline{1-1}\end{array}$ where j, i refer to the first column entries and j in particular is the lower row entry. Since a column group $C_{t_{ji}} = \{id = (i)(j)(k), (i,j)(k)\}$, we have,

$$e_{t_{ji}} = \left\{\begin{array}{|c|c|}\hline i & k \\ \hline j \\ \cline{1-1}\end{array}\right\} - \left\{\begin{array}{|c|c|}\hline j & k \\ \hline i \\ \cline{1-1}\end{array}\right\} = v_j - v_i.$$

Therefore, the Specht module

$$\begin{aligned} S^{(2,1)} &= Span_{\mathbb{C}}\{e_{t_{12}}, e_{t_{13}}, e_{t_{21}}, e_{t_{23}}, e_{t_{31}}, e_{t_{32}}\} \\ &= Span_{\mathbb{C}}\{v_j - v_i : 1 \leq j \neq i \leq 3\}. \end{aligned}$$

Notice that $e_{t_{ji}} = -e_{t_{ij}}$.

Remark 3.2.19. [75] If $\lambda, \mu \vdash n$, where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$, then $\lambda = \mu \iff m = l$ and $\lambda_i = \mu_i$ for all i.

We are now ready to state the main theorem in this section.

Theorem 3.2.20. [75] Given a partition λ of n, the Specht modules

$$S^\lambda = Span_{\mathbb{C}}\{e_t : t \text{ is a tableau of shape } \lambda\}$$

form a complete list of inequivalent irreducible representations of S_n over $\mathbb{C}$ and

$$S^\lambda \cong S^\mu \text{ if and only if } \lambda = \mu.$$

The complete proof of the above theorem requires some assertions and properties that we chose not to address for the purpose of shortening. The reader can consult [75, Section 2.4] for more details.

In view of Examples 3.2.18 (ii) and (iii), the spanning set $\{e_t : t \text{ is a } \lambda\text{-Young tableau}\}$ of the Specht module S^λ is not in general an independent set, and thus, it is not a basis for S^λ. The next definition determines a basis for the Specht module S^λ.

Definition 3.2.21. [75] For any $\lambda \vdash n$, a standard tableau t is a Young tableau of shape λ, where the entries in each row and column are increasing.

Theorem 3.2.22. Given a partition λ of n, the set

$$\{e_t : t \text{ runs through all standard tableaux of shape } \lambda\}$$

forms a basis for S^λ.

The proof of the independence of λ-standard polytobloids e_t appears in [75, Section 2.5], and [75, Section 2.6] shows that such a set spans the Specht module S^λ.

Let f^λ denote the number of λ-standard tableaux. Then, as a consequence of the preceding theorem, we have the result below.

Corollary 3.2.23. *Let $\lambda \vdash n$. Then, dim $S^\lambda = f^\lambda$.*

Example 3.2.24. In view of Example 3.2.18 (ii), the standard tableaux of shape $(2,1)$ are $\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$ and $\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$. As a result, the $(2,1)$-polytabloids associated with them are

$$\begin{aligned} e_{21} &= \left\{\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\right\} - \left\{\begin{array}{|c|c|}\hline 2&3\\\hline 1\\\cline{1-1}\end{array}\right\} \\ &= v_2 - v_1 \end{aligned} \qquad\qquad \begin{aligned} e_{31} &= \left\{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\right\} - \left\{\begin{array}{|c|c|}\hline 3&2\\\hline 1\\\cline{1-1}\end{array}\right\} \\ &= v_3 - v_1 \end{aligned}$$

Hence, by Theorem 3.2.22, we write the the Specht module $S^{(2,1)}$ as

$$\begin{aligned} S^{(2,1)} &= \mathbb{C}\text{-}\{e_{21}, e_{31}\} \\ &= \mathbb{C}\text{-}\{v_2 - v_1, v_3 - v_1\}. \end{aligned}$$

More generally, let $n \geq 2$; if $\lambda = (n-1,1) \vdash n$, then

$$\begin{aligned} S^{(n-1,1)} &= \mathbb{C}\text{-}\{e_{21}, e_{31}, \cdots, e_{n1}\} \\ &= \mathbb{C}\text{-}\{v_j - v_1 : 2 \leq j \leq n\}. \end{aligned}$$

and dim $S^{(n-1,1)} = n-1$. Interestingly, $S^{(n-1,1)}$ is the hyperplane W of the S_n-permutation module $M^{(n-1,1)}$ presented in Example 3.2.14, and the standard tableaux give us the basis $\{v_j - v_1 : 2 \leq j \leq n\}$ that is different from the basis we discussed in Example 3.1.7(b).

We end this section by exhibiting the hook formula, which contributes to determining the number of standard tableaux of a given diagram of shape λ.

Definition 3.2.25. [75] Fix $\lambda \vdash n$. Let b_{ij} be the box in a Young diagram placed in the ith row and jth column. Define the hook length of a box b_{ij} to be the number of boxes occurring below or to the right of box b_{ij}, counting the box itself; the hook length of box b_{ij} is denoted by $h(b_{ij})$.

Example 3.2.26. Fix a partition $\lambda = (3,2,1) \vdash 6$, and notice that the hook length of the box b_{21} is equal to 3.

$$\begin{array}{|c|c|c|}\hline &&\\\hline \bullet&\bullet\\\cline{1-2} \bullet\\\cline{1-1}\end{array}$$

Theorem 3.2.27. [75] Given a partition λ of n and the associated Young diagram, the dimension of the Specht module S^λ is determined by the following formula:

$$\dim\,(S^\lambda) = \frac{n!}{\prod_{ij} h(b_{ij})}.$$

The full proof of the lemma can be found in [75, Section 3.10].

Example 3.2.28. If $\lambda = (3, 2, 1)$, then the hook lengths of the Young diagram below are placed inside the boxes of the array:

$$\begin{array}{|c|c|c|}\hline 5 & 3 & 1 \\ \hline 3 & 1 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array}$$

Hence, the dimension of $S^{(3,2,1)} = \frac{6!}{5\cdot 3^2\cdot 1^3} = 16$.

3.3 Specht modules for the signed permutation group B_n

The signed permutation group B_n [19] has been described as the reflection group of type $\mathcal{B}_n$, hypoctahedral group B_n, Weyl group of type $\mathcal{B}_n$, Coxeter group of type $\mathcal{B}_n$ and symmetries of n dimensional cube. The application of the various descriptions, assertions and properties of this group in diverse contexts complicates efforts to unite them with their proofs under a single umbrella. As a consequence of the diverse descriptions of the group, the irreducible representations have also been clearly characterised in various contexts [3, 4, 28, 30, 36].

The purpose of this section is to represent and prove some assertions in the context of the signed permutation group B_n, as well as in a manner that is useful to us. Then, by utilising the combinatorial objects known as Young tableaux, the explicit descriptions of irreducible representations over the complex field of this group are studied. In fact, the decision to adopt Morris's approach [53] was determined by its similarity to James's diagrammatic approach [41, 75] of describing irreducible representations over the complex field of the symmetric groups, as discussed in the previous section. This approach is especially attractive because it utilises familiar concepts, such as Young tableaux, Young tabloids, polytabloids, as well as row and column groups, which have familiar counterparts in the representation of the symmetric group.

However, we encountered the obstacle of Morris' [53] failure to report the proof of these results, requiring us to reprove some of them in a manner comparable to the corresponding assertions in the symmetric group and its representations. Indeed, the similarity in the construction of these irreducible representations for S_n and B_n would simplify the explicit description of irreducible representations over $\mathbb{C}$ of the Boolean reflection monoids of type $\mathcal{A}_{n-1}$ and $\mathcal{B}_n$, as presented in the next chapters.

Definition 3.3.1. [19] Let S_{2n} be the group of bijections on the set $[\pm n] = \{\pm 1, \ldots, \pm n\}$. The signed permutation group B_n is defined by

$$B_n \coloneqq \{\sigma \in S_{2n} : (-i)\sigma = -(i)\sigma, \text{ for all } i \in [\pm n]\}.$$

Definition 3.3.2. [1] A product of two l-cycles in S_{2n} of the form $(a_1, \ldots, a_l)(-a_1, \ldots, -a_l)$, where $1 \leq l \leq n$ and $a_i \in [\pm n]$, is called a positive cycle.

Definition 3.3.3. [1] A $2l$-cycle in S_{2n} of the form $(a_1, \ldots, a_l, -a_1, \ldots, -a_l)$, where $1 \leq l \leq n$ and $a_i \in [\pm n]$, is called a negative cycle.

Definition 3.3.4. [1] The length of a negative $2l$-cycle or positive two l-cycles is equal to l.

Proposition 3.3.5. *Any signed permutation $\sigma \in B_n$ can be written uniquely as a disjoint product of positive and negative cycles.*

Proof. Let us first prove that any positive or negative cycle belongs to B_n. Let $\alpha \in S_{2n}$ be a positive two l-cycles $\alpha = (a_1, \ldots, a_l)(-a_1, \ldots, -a_l)$, and let $\beta \in S_{2n}$ be a negative $2l$-cycles $\beta = (a_1, \ldots, a_l, -a_1, \ldots, -a_l)$. It is clear that $(-a_i)\alpha = -(a_i)\alpha$ for all i, since we know by definition that

$$(-a_i)\alpha = \begin{cases} -a_{i+1}, & \forall\ i = 1, \ldots, l-1, \\ -a_1, & i = l. \end{cases} \quad \text{and} \quad (a_i)\alpha = \begin{cases} a_{i+1}, & \forall\ i = 1, \ldots, l-1, \\ a_1, & i = l. \end{cases}$$

Similarly, $(-a_i)\beta = -(a_i)\beta$ for all i, since

$$(-a_i)\beta = \begin{cases} -a_{i+1}, & \forall\ i = 1, \ldots, l-1, \\ a_1, & i = l. \end{cases} \quad \text{and} \quad (a_i)\beta = \begin{cases} a_{i+1}, & \forall\ i = 1, \ldots, l-1, \\ -a_1, & i = l. \end{cases}$$

Hence, positive and negative cycles belong in B_n. Let us choose a number $a_1 \in [\pm n]$ and $\sigma \in B_n$, and consider the repeated images of a_1 under the signed permutation σ; then, the orbit of a_1 is determined as follows:

$$a_1,\ (a_1)\sigma,\ (a_1)\sigma^2,\ (a_1)\sigma^3,\ \cdots, \tag{3.3}$$

where $a_2 = (a_1)\sigma, a_3 = (a_1)\sigma^2, a_4 = (a_1)\sigma^3$ and so on. However, as the set $[\pm n]$ is finite, the above orbit must be finite; that is, there must exist some power q such that $(a_1)\sigma^q$ is indeed one of the images appearing in (3.3). In other words, there exists $q > p$ such that $(a_1)\sigma^q = (a_1)\sigma^p$; thus, $(a_1)\sigma^{q-p} = a_1$. Let $q - p = l$; then, we have l distinct elements of $[\pm n]$, forming the cycle $(a_1, a_2, \cdots, a_l)$, and (3.3) can be illustrated as in Figure 3.1.

$$a_1 \longmapsto \overbrace{(a_1)\sigma^1}^{=a_2} \longmapsto \overbrace{(a_1)\sigma^2}^{=a_3} \longmapsto \cdots \longmapsto \overbrace{(a_1)\sigma^{(q-p)-1}}^{=a_l}$$

Figure 3.1: An illustration of the cycle $(a_1, a_2, \cdots, a_l)$.

Indeed, a careful consideration of these images appearing in the cycle yields the following enquiry: Is there a pair of images that are negatives of each other? The answer needs to be discussed in two cases.

Case 1. There is no such pair of images in Figure 3.1 in which each is a negative of the other. In particular, $-a_1$ does not appear in the cycle. Consequently, by taking the images of $-a_1$ under σ repeatedly and utilising the property $(-r)\sigma = -(r)\sigma$ for all $r \in [\pm n]$, we obtain another cycle $(-a_1, -a_2, \cdots, -a_l)$. Hence, we determine the first positive cycle

$$(a_1, a_2, \cdots, a_l)(-a_1, -a_2, \cdots, -a_l).$$

Case 2. Suppose we have two powers s, t of σ where $s < t$ and both $(a_1)\sigma^t$ and $(a_1)\sigma^s$ are the first pair occurring in Figure 3.2 such that $(a_1)\sigma^t = -(a_1)\sigma^s$.

$$a_1 \mapsto (a_1)\sigma^1 \mapsto \cdots \mapsto (a_1)\sigma^s \mapsto \cdots \mapsto (a_1)\sigma^t \mapsto \cdots \mapsto \overbrace{(a_1)\sigma^{(q-p)-1}}^{=a_l}$$

Figure 3.2: Another illustration of the cycle $(a_1, a_2, \cdots, a_l)$.

In other words, none of the images

$$\{a_1, (a_1)\sigma, \ldots, (a_1)\sigma^s, \ldots, (a_1)\sigma^{t-1}\}$$

is a negative of others. Let $(a_1)\sigma^s = b_1$ and rearrange the cycle to start from b_1 instead of a_1 as follows:

$$\overbrace{(a_1)\sigma^s}^{=b_1}, \overbrace{(a_1)\sigma^{s+1}}^{=b_2}, \ \cdots, \ \overbrace{(a_1)\sigma^{t-1}}^{=b_l}, \overbrace{(a_1)\sigma^t}^{=-b_1}, \ \cdots \tag{3.4}$$

Note that the image of the next one is

$$(a_1)\sigma^{t+1} = ((a_1)\sigma^t)\sigma = (-(a)\sigma^s)\sigma = -((a)\sigma^s)\sigma = -(a)\sigma^{s+1} = -b_2;$$

this means that $(a_1)\sigma^{t+1}$ is indeed the negative of the second image in (3.4). If we continue taking the images of the next ones, we will end up listing the negatives of all the others. Thus, (3.4) becomes

$$b_1 \overset{\sigma}{\mapsto} b_2 \overset{\sigma}{\mapsto} \ \cdots \overset{\sigma}{\mapsto} b_l \overset{\sigma}{\mapsto} -b_1 \overset{\sigma}{\mapsto} -b_2 \ \cdots \overset{\sigma}{\mapsto} \ -b_l.$$

Observe that if we apply σ to $-b_l$, then we obtain

$$(-b_l)\sigma = -(b_l)\sigma = -((a_1)\sigma^{t-1})\sigma = -(a_1)\sigma^t = b_1.$$

Therefore, we establish a negative cycle $(b_1, b_2, \cdots, b_l, -b_1, -b_2, \cdots, -b_l)$. We choose another number $c \in [\pm n]$ that does not appear in either of the current positive and negative cycles and consider both cases again to produce the others. We repeat these processes until the set $[\pm n]$ is exhausted. $\square$

Example 3.3.6. Let $\sigma = (1, -2) \in S_{2(2)}$. Note that -1 and 2 both are fixed by σ.

In addition, it is clear that σ is neither a positive nor a negative cycle, and it does not belong to B_2, as $(1)\sigma = -2$ while $(-1)\sigma = -1$, and $(-1)\sigma \neq -(1)\sigma$.

Definition 3.3.7. [1] A positive transposition has the form $(a, b)(-a, -b)$, and a negative transposition has the form $(c, -c)$, where $a, b, c \in [\pm n]$.

Note that, as stated in [1], positive transpositions generate a subgroup of B_n that is isomorphic to the symmetric group S_n, and negative transpositions generate a normal subgroup of B_n isomorphic to $\mathbb{Z}_2^n$.

Proposition 3.3.8. $B_n \cong \mathbb{Z}_2^n \rtimes S_n$.

The proof of this proposition appears in [4]. Using the above assertion, we have

Proposition 3.3.9. [6] *The order of the signed permutation group B_n is equal to $2^n n!$.*

Example 3.3.10. If $n = 2$, then the order of $B_2 = 8$, and the elements of B_2 can be written as positive and negative cycles with their products, as follows:

$$B_2 = \{id, (1,-1), (2,-2), (1,-1)(2,-2), (1,2)(-1,-2), (1,-2)(-1,2), (1,2,-1,-2), (1,-2,-1,2)\},$$

where the identity $id = (1)(-1)(2)(-2)$. It should be noted that elements of B_2 can also be illustrated as shown below.

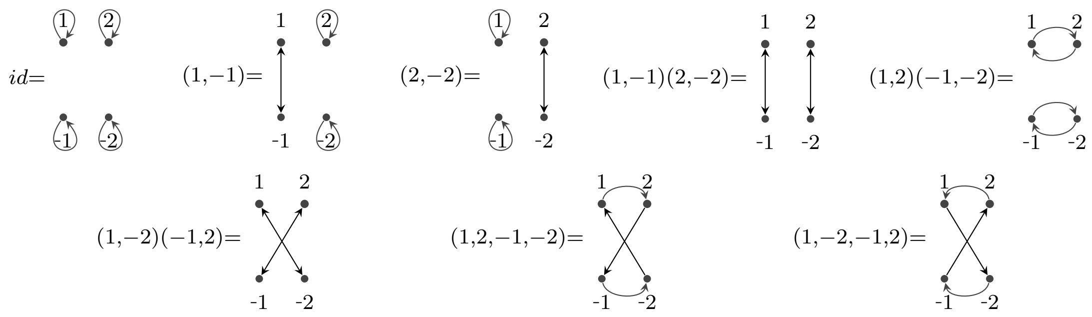


Figure 3.3: Alternative illustration of elements of B_2.

The next result follows from Proposition 3.3.8. However, we intended to prove it just to know how positive and negative cycles decompose into positive and negative transpositions.

Proposition 3.3.11. *The positive and negative transpositions generate the signed permutation group B_n.*

Proof. In view of Proposition 3.3.5, every signed permutation $\sigma \in B_n$ may be written uniquely as a product of positive and negative cycles. Therefore, it suffices to show this for positive and negative cycles. Any positive cycle can be written as a product of positive transpositions, since for all $l \geqslant 2$,

$$(a_1, \ldots, a_l)(-a_1, \ldots, -a_l) = (a_{l-1}, a_l)(-a_{l-1}, -a_l) \cdot (a_{l-2}, a_{l-l})(-a_{l-2}, -a_{l-1}) \cdots (a_1, a_2)(-a_1, -a_2) \ .$$

Furthermore, any negative cycle can be written as a product of a negative transposition and positive transpositions, since for all $l \geqslant 2$,

$$(a_1, \ldots, a_l, -a_1, \ldots, -a_l) = (a_l, -a_l) \cdot (a_{l-1}, a_l)(-a_{l-1}, -a_l) \cdot (a_{l-2}, a_{l-l})(-a_{l-2}, -a_{l-1}) \cdots (a_1, a_2)(-a_1, -a_2).$$

Hence, each signed permutation $\sigma \in B_n$ can be written as a product of positive and negative transpositions. $\square$

Note that these transpositions need not be disjoint, and the decomposition is not unique.

Example 3.3.12. A positive cycle $(1, -2, 3)(-1, 2, -3) \in B_3$ can be written as

$$(1, -2, 3)(-1, 2, -3) = (-2, 3)(2, -3) \cdot (1, -2)(-1, 2),$$

while a negative cycle $(1, -2, -3, -1, 2, 3)$ can be expressed as

$$(1, -2, -3, -1, 2, 3) = (-3, 3)(-2, -3)(2, 3)(1, -2)(-1, 2).$$

In contrast, note that a negative cycle $(1, 2, 3, -1, -2, -3) \in B_5$ can be expressed as a product of transpositions in distinct ways, as follows:

$$\begin{aligned}(1, 2, 3, -1, -2, -3) &= (3, -3)(2, 3)(-2, -3)(1, 2)(-1, -2) \\ &= (3, -3)(2, 3)(-2, -3)(1, 2)(-1, -2)(4, 5)(-4, -5)(4, 5)(-4, -5).\end{aligned}$$

However, the parity of the number of positive and negative transpositions in the decompositions is the same, no matter how $(1, 2, 3, -1, -2, -3)$ is written. Let us investigate why this is true in general for any $\sigma \in B_n$.

Let us introduce the multivariable polynomial $P(x_1, \ldots, x_n) \in K[x_1, \ldots, x_n]$, where

$$P(x_1, \ldots, x_n) = x_1 \cdots x_n \prod_{1 \leq i < j \leq n} (x_i - x_j)(x_i + x_j).$$

Now, for each $\sigma \in B_n$, let σ act on P by permuting the variables exactly as σ permutes indices; that is,

$$P\sigma := P(x_{1\sigma}, \ldots, x_{n\sigma}),$$

where we adopt the convention that $x_{-j} = -x_j$. We claim that for all $\sigma \in B_n$, the signed permutation σ permutes the terms of P and may change some of their signs; that is,

$$P\sigma = \begin{cases} +P \\ -P \end{cases}$$

To see this, take a signed permutation $\sigma \in B_n$ and apply it to the multivariable polynomial P; that is,

$$P\sigma = \left(x_1 \cdots \ x_n \prod_{1 \le i < j \le n} (x_i - x_j)(x_i + x_j) \right) \sigma.$$

Let us now discuss what σ does for each term in P. In the first term, each x_i will be sent under σ to $\pm x_{i\sigma}$, depending on whether $i\sigma > 0$ or $i\sigma < 0$; that is,

$$x_i \overset{\sigma}{\mapsto} \pm x_{i\sigma}.$$

In the other terms, the effect of σ relies on what it does for each pair of parentheses $(x_i - x_j)(x_i + x_j)$. To clarify, the image of a factor $(x_i - x_j)(x_i + x_j)$ under σ; that is,

$$(x_i - x_j)(x_i + x_j) \overset{\sigma}{\mapsto} (x_{i\sigma} - x_{j\sigma})(x_{i\sigma} + x_{j\sigma}),$$

depends on whether $i\sigma$ or $j\sigma$ is positive or negative. Set $i\sigma = k$ and $j\sigma = l$, and recall that $i < j$ by assumption. Let us consider all the possibilities of $i\sigma$ and $j\sigma$, described below.

Case 1. Suppose $i\sigma > 0$ and $j\sigma > 0$. Then,

$$\begin{aligned} [(x_i - x_j)(x_i + x_j)]\sigma &= (x_{i\sigma} - x_{j\sigma})(x_{i\sigma} + x_{j\sigma}) \\ &= (x_k - x_l)(x_k + x_l) \\ &= \begin{cases} (x_k - x_l)(x_k + x_l), & \text{if } \ k < l, \\ -(x_k - x_l)(x_k + x_l), & \text{otherwise.} \end{cases} \end{aligned}$$

Case 2. Suppose, without loss of generality, $i\sigma < 0$ and $j\sigma > 0$. Then,

$$\begin{aligned} [(x_i - x_j)(x_i + x_j)]\sigma &= (x_{i\sigma} - x_{j\sigma})(x_{i\sigma} + x_{j\sigma}) \\ &= (x_{_{-k}} - x_l)(x_{_{-k}} + x_l) \\ &= (-x_k - x_l)(-x_k + x_l) \\ &= -(x_k + x_l)(-x_k + x_l) \\ &= (x_k + x_l)(x_k - x_l) \end{aligned}$$

$$= \begin{cases} (x_k - x_l)(x_k + x_l), & \text{if } \; k < l, \\ -(x_k - x_l)(x_k + x_l), & \text{otherwise.} \end{cases}$$

Thus, we will obtain the same results as above, in which the permutation σ will permute the factors $(x_i - x_j)(x_i + x_j)$ of the polynomial P, and simultaneously, it may change some of their signs.

Case 3. Suppose $i\sigma < 0$ and $j\sigma < 0$. Then,

$$\begin{aligned} [(x_i - x_j)(x_i + x_j)]\sigma &= (x_{i\sigma} - x_{j\sigma})(x_{i\sigma} + x_{j\sigma}) \\ &= (x_{_{-k}} - x_{_{-l}})(x_{_{-k}} + x_{_{-l}}) \\ &= (-x_k + x_l)(-x_k - x_l) \\ &= -(-x_k + x_l)(x_k + x_l) \\ &= (x_k - x_l)(x_k + x_l) \\ &= \begin{cases} (x_k - x_l)(x_k + x_l), & \text{if } \; k < l, \\ -(x_k - x_l)(x_k + x_l), & \text{otherwise.} \end{cases} \end{aligned}$$

Thus, the factors $(x_i - x_j)(x_i + x_j)$ of the polynomial P will be again permuted by σ, and at the same time, this may change the signs of some of the other factors. Note that, in all cases, we will consider the images of all factors $(x_i - x_j)(x_i + x_j)$ for all i and j, where $1 \leq i < j \leq n$ under σ. Furthermore, we will consider the action of σ on the first term, and then we will take the overall effects with pluses and minuses. Hence, we can definitively conclude that the action of B_n on the polynomial P is either a positive or negative value of P.

For any $\sigma \in B_n$, define the sign of σ as follows:

$$sgn(\sigma) := \begin{cases} +1, & P\sigma = P, \\ -1, & P\sigma = -P, \end{cases} \tag{3.5}$$

Example 3.3.13. Let $n = 3$, and $\sigma = (1, -2, 3, -1, 2, -3) \in B_3$. Then,

$$P(x_1, x_2, x_3) = x_1 x_2 x_3 (x_1 - x_2)(x_1 + x_2)(x_1 - x_3)(x_1 + x_3)(x_2 - x_3)(x_2 + x_3).$$

Clearly, $P\sigma = -P$, and then $sgn \ \sigma = -1$.

Proposition 3.3.14. *Let $\sigma \in B_n$ be a positive or negative transposition. Then, $sgn(\sigma) = -1$.*

Proof. Let $\sigma = (c, -c)$ be a negative transposition. Then, for any $i \in \{1, 2, \ \ldots \ , n\}$, if $i = c$, the action of $P\sigma$ is determined by the action σ on the first and the second terms of P. Let us see the effect of σ on the first term

$$\begin{aligned}[x_1 \ \cdots \ x_i \ \cdots \ x_n](c,-c) &= x_1 \ \cdots \ x_{-c} \ \cdots \ x_n \\ &= -x_{1,} \ \cdots \ x_c \ \cdots \ x_n.\end{aligned}$$

This means that all other variables x_r will be fixed by σ, and the only one to be changed is x_i. The action of σ on the other terms is as follows:

$$\begin{aligned}[(x_i - x_j)(x_i + x_j)](c,-c) &= (x_{_{-c}} - x_j)(x_{_{-c}} + x_j) \\ &= (-x_c - x_j)(-x_c + x_j) \\ &= (x_c + x_j)(x_c - x_j),\end{aligned}$$

and all other factors are fixed by σ. Hence, by taking the overall effects with pluses and minuses for both terms in P, we will obtain $P\sigma = -P$, and then $sgn(\sigma) = -1$. Similarly, we obtain the same result if $j = c$. Now, let $\sigma = (a,b)(-a,-b)$ be a positive transposition. For any $i, j \in \{1, 2, \ \ldots \ , n\}$, if $i = a$ and $j = b$ with $i < j$, then the action of σ on the first term of P will be as follows:

$$[x_1 \ \cdots \ x_i \ \cdots x_j \cdots \ x_n](a,b)(-a,-b) = x_1 \ \cdots \ x_b \ \cdots x_a \cdots \ x_n;$$

that is, x_a and x_b are swapped, with no sign change, and all other x_r with $r \in \{1, \cdots, n\} \backslash \{a, b\}$ are fixed. However, the action of $\sigma = (a,b)(-a,-b)$ where $a, b > 0$ with $a < b$ on other terms yields the following cases:

Case 1. The factors $(x_i - x_j)(x_i + x_j)$ with neither i nor j equal to a nor b, are fixed.

Case 2. For the factors with precisely one index in $\{a, b\}$, we will have three sub-cases:

(a) If other index is between $\{a, b\}$, then we notice that

$$\begin{aligned}[(x_a - x_j)(x_a + x_j)](a,b)(-a,-b) &= (x_b - x_j)(x_b + x_j) \\ &= -(x_j - x_b)(x_j + x_b), \qquad (3.6)\end{aligned}$$

as $a \leq i < j \leq b$ by assumption. Moreover,

$$\begin{aligned}[(x_i - x_b)(x_i + x_b)](a,b)(-a,-b) &= (x_i - x_a)(x_i + x_a) \\ &= -(x_a - x_i)(x_a + x_i). \qquad (3.7)\end{aligned}$$

Hence, the two sign changes in (3.6) and (3.7) cancel each other out.

(b) If the other index is larger than b, then we notice that

$$[(x_a - x_j)(x_a + x_j)](a, b)(-a, -b) = (x_b - x_j)(x_b + x_j) \tag{3.8}$$

and

$$[(x_b - x_j)(x_b + x_j)](a, b)(-a, -b) = (x_a - x_j)(x_a + x_j) \tag{3.9}$$

Hence, the factors in (3.8) and (3.9) are are swapped, with no sign change.

(c) If the other index is smaller than a, then we notice that

$$[(x_i - x_a)(x_i + x_a)](a, b)(-a, -b) = (x_i - x_b)(x_i + x_b), \tag{3.10}$$

and

$$[(x_i - x_b)(x_i + x_b)](a, b)(-a, -b) = (x_i - x_a)(x_i + x_a). \tag{3.11}$$

Hence, the factors in (3.10) and (3.11) are are swapped, with no sign change.

Case 3. For the factor with $i = a$ and $j = b$, we have

$$\begin{aligned}[(x_a - x_b)(x_a + x_b)](a, b)(-a, -b) &= (x_b - x_a)(x_b + x_a) \\ &= -(x_a - x_b)(x_a + x_b).\end{aligned}$$

Hence, by taking the overall effects with pluses and minuses for both terms in P, we will obtain $P\tau = -P$, and then $sgn(\tau) = -1$. Similarly, for $(-a, b)(a, -b)$. □

Proposition 3.3.15. *The sign of a permutation σ defines a homomorphism; that is, the map between the signed permutation group B_n and multiplicative group $\{-1, 1\}$*

$$sgn : B_n \longrightarrow \{-1, 1\}$$

is a homomorphism: $sgn(\sigma\tau) = sgn(\sigma)sgn(\tau)$, for all $\sigma, \tau \in B_n$.

Proof. In view of (3.5), we know that if $P\sigma = P$, then $sgn(\sigma) = 1$, and if $P\sigma = -P$, then $sgn(\sigma) = -1$. Therefore, $P\sigma = sgn(\sigma)P$. Consequently, we have

$$P(\sigma\tau) = sgn(\sigma\tau)\ P. \tag{3.12}$$

In contrast, we also know that

$$P(\sigma\tau) = (P\sigma)\ \tau = sgn(\sigma)\ P\tau = sgn(\sigma)sgn(\tau)P. \tag{3.13}$$

Thus, by comparing (3.12) and (3.13), we have $sgn(\sigma\tau)P = sgn(\sigma)sgn(\tau)P$; hence, $sgn(\sigma\tau) = sgn(\sigma)sgn(\tau)$. $\square$

In view of Proposition 3.3.11 and the homomorphism property of the sign map, consider the following: Let $\sigma = \tau_1\tau_2 \cdots \tau_k$, where τ_i are positive and negative transpositions for all $1 \leq i \leq k$. Then,

$$\begin{aligned} sgn(\sigma) &= sgn(\tau_1\tau_2 \cdots \tau_k) \\ &= sgn(\tau_1)sgn(\tau_2) \cdots sgn(\tau_k). \end{aligned}$$

However, we know from Proposition 3.3.14 that $sgn(\tau_i) = -1$ for all $1 \leq i \leq k$. Thus, we can conclude that

$$sgn(\sigma) = (-1)^k, \tag{3.14}$$

where k is the number of positive and negative transpositions in a decomposition of $\sigma \in B_n$. In fact, equation (3.14) can be considered another way of determining the sign of a permutation $\sigma \in B_n$. The only point we still need to examine is that the parity of the number of positive and negative transpositions in a decomposition does not differ, no matter how σ is written as a product of positive and negative transpositions.

Proposition 3.3.16. *If σ can be expressed as products of r and s positive and negative transpositions, then r and s have the same parity.*

Proof. Let $\sigma \in B_n$, and suppose it is decomposed into two distinct expressions of positive and negative transpositions; that is,

$$\sigma = \alpha_1\alpha_2 \cdots \alpha_r \quad \text{and} \quad \sigma = \beta_1\beta_2 \cdots \beta_s,$$

where $\alpha_1 \cdots \alpha_r$ and $\beta_1 \cdots \beta_s$ are positive and negative transpositions. Thus,

$$\beta_1\beta_2 \cdots \beta_s = \alpha_1\alpha_2 \cdots \alpha_r$$

and then,

$$\begin{aligned} id &= (\alpha_1\alpha_2 \cdots \alpha_r) \cdot (\beta_1\beta_2 \cdots \beta_s)^{-1} \\ &= (\alpha_1\alpha_2 \cdots \alpha_r) \cdot (\beta_s^{-1}\beta_{s-1}^{-1} \cdots \beta_1^{-1}) \\ &= \alpha_1\alpha_2 \cdots \alpha_r\ \beta_s\beta_{s-1} \cdots \beta_1. \qquad \text{[each transposition is its own inverse]} \end{aligned}$$

Thus, if we apply sgn to both sides, we have

$$sgn\ (id) = sgn\ (\alpha_1\alpha_2 \cdots \alpha_r\ \beta_s\beta_{s-1} \cdots \beta_1) \tag{3.15}$$

However, we know that $sgn\ (id) = 1$, as $(P)\ id = P$, where P is the multivariable polynomial described above. Thus, equation (3.15) becomes

$$\begin{aligned}1 &= sgn\ (\alpha_1\alpha_2\ \cdots\ \alpha_r)\ sgn\ (\beta_s\beta_{s-1}\ \cdots\ \beta_1)\\ 1 &= (-1)^r(-1)^s\\ 1 &= (-1)^{r+s}.\end{aligned}$$

This implies that $r + s \equiv 0 \mod 2$, and then both r and s are either even or odd. Hence, the parity of the number of positive and negative transpositions is always the same. $\square$

Note that, in Example 3.3.12, $sgn((1, -2, 3)(-1, 2, -3)) = (-1)^2 = 1$, while $sgn(1, 2, 3, -1, -2, -3) = (-1)^3 = -1$.

Definition 3.3.17. Let $\sigma \in B_n$. The signed cycle structure (shape) of a permutation σ is an expression $((\lambda_1, \ \dots\ , \lambda_r)\ ,\ (\mu_1, \ \dots\ , \mu_s))$, where $\lambda_1 \geq \lambda_2 \geq\ \dots\ \geq \lambda_r$, $\mu_1 \geq \mu_2\ \dots\ \geq \mu_s$ and there is a λ_i for each negative cycle of length λ_i and a μ_j for each positive cycle of length μ_j, when a permutation σ is decomposed as a disjoint product of positive and negative cycles (including positive cycles of length one).

We write $|\lambda| = \sum_{i=1}^{r} \lambda_i$, $|\mu| = \sum_{j=1}^{s} \mu_j$. Note that a signed cycle shape for $\sigma \in B_n$ can also be called a complementary partition (λ, μ) of n.

Definition 3.3.18. [1] A complementary partition of a positive integer $n \in \mathbb{Z}^+$ is a pair (λ, μ) of partitions where $\lambda = (\lambda_1, \ \dots\ , \lambda_r)$ is a partition of $|\lambda|$, $\mu = (\mu_1, \ \dots\ , \mu_s)$ is a partition of $|\mu|$, and $|\lambda| + |\mu| = n$. Write $(\lambda, \mu) \vdash n$ for a complementary partition.

Again, we recall that $|\lambda|$ or $|\mu|$ can be zero.

Example 3.3.19. Let us characterise the signed cycle shape of the following permutations:

(i) The identity permutation $id = (1)(-1)(2)(-2)(3)(-3) \in B_3$ has the signed cycle shape $(\lambda, \mu) = (\phi, (1, 1, 1))$, since the identity is always expressed as a product of positive cycles of length one, while no negative cycle exists in the decomposition;

(ii) The permutation $(1, -1) \in B_3$ has the signed cycle shape $(\lambda, \mu) = ((1), (1, 1))$, since it is written as product of a negative cycle of length one and two positive cycles of length one; that is, $(1, -1) = (1, -1) \cdot (2)(-2) \cdot (3)(-3)$;

(iii) The permutation $(1, -3, -1, 3) \in B_3$ has the signed cycle shape $(\lambda, \mu) = ((2), (1))$, since it is a product of a negative cycle of length two and a positive cycle of length one; that is, $(1, -3, -1, 3) = (1, -3, -1, 3) \cdot (2)(-2)$; and

(iv) The permutation $(1,2,-1,-1)(3,-3) \in B_3$ has the signed cycle shape $(\lambda,\mu) = ((2,1),\phi)$, since it is a product of a negative cycle of length two and another negative cycle of length one, while no positive cycle exists in the decomposition.

It becomes clear that the signed cycle shapes of B_n are canonically affiliated with the complementary partitions of n; for instance, the complementary partition $(\lambda,\mu) = ((4,1),(2,1,1))$ of 9 is associated with the signed permutation on nine numbers with one negative cycle of length four, one negative cycle of length one, one positive cycle of length two, and two positive cycles of length one. For example,

$$\alpha = (1,3,-5,-7,-1,-3,5,7) \cdot (4,-4) \cdot (2,-6)(-2,6) \cdot (8)(-8) \cdot (9)(-9).$$

Theorem 3.3.20. Two signed permutations in B_n are conjugate if and only if they have the same cycle structure/complementary partition.

Proof. Suppose that we have two signed permutations $\sigma, \tau \in B_n$ that are conjugate. This means that there exists a signed permutation $\gamma \in B_n$ such that $\gamma^{-1}\sigma\gamma = \tau$. We intend to show that both σ and τ have the same signed cycle structure. Consider a positive (l,l)-cycle $(a_1,\ldots,a_l)(-a_1,\ldots,-a_l)$ and a negative $2l$-cycle $(a_1,\ldots,a_l,-a_1,\ldots,-a_l)$. Then, let us show the following: For any $1 \le l \le n$, we have

(i) $\gamma^{-1}(a_1,\ldots,a_l,-a_1,\ldots,-a_l)\gamma = (a_1\gamma,\ldots,a_l\gamma,-a_1\gamma,\ldots,-a_l\gamma)$,

(ii) $\gamma^{-1}(a_1,\ldots,a_l)(-a_1,\ldots,-a_l)\gamma = (a_1\gamma,\ldots,a_l\gamma)(-a_1\gamma,\ldots,-a_l\gamma)$.

We will show (i). Note that the image of each entry $a_i\gamma$ or $-a_i\gamma$, belongs to a negative cycle on the right-hand side, is expressed as

$$(\pm a_i)\gamma = \begin{cases} \pm a_{i+1}\gamma, & \text{for all } \ 1 \le i \le l-1, \\ \mp a_1\gamma, & \text{if } \ i = l. \end{cases}$$

Let us verify whether the effect of the left-hand side of (i) has the same effect as the right-hand side for the entries. Since, for all $1 \le i \le l-1$,

$$\pm a_i\gamma \xmapsto{\gamma^{-1}} \pm a_i \xmapsto{(a_1,\ldots,a_l,-a_1,\ldots,-a_l)} \pm a_{i+1} \xmapsto{\gamma} \pm a_{i+1}\gamma,$$

and if $i = l$, we have

$$\pm a_l\gamma \xmapsto{\gamma^{-1}} \pm a_l \xmapsto{(a_1,\ldots,a_l,-a_1,\ldots,-a_l)} \mp a_1 \xmapsto{\gamma} \mp a_1\gamma.$$

Thus, the effect of the left-hand side of (i) is the same as that of the right-hand side. Hence, (i) holds, and similarly, we can prove (ii). Now, since every signed permutation $\sigma = \sigma_1 \ \ldots \ \sigma_m$ can be written as a disjoint product of positive and negative σ_j cycles, then

$$\gamma^{-1}\sigma\gamma = \gamma^{-1}\ \sigma_1\sigma_2\ \dots\ \sigma_m\ \gamma = \gamma^{-1}\sigma_1\gamma\ \ \gamma^{-1}\sigma_2\gamma\ \dots\ \gamma^{-1}\sigma_m\gamma,$$

is indeed a disjoint product of positive and negative cycles, where each $\gamma^{-1}\sigma_j\gamma$ has the same length as σ_j by (i) and (ii). In other words, the conjugation of σ by γ is the same as applying γ to each transposition. Consequently, the conjugation of σ by γ will not change the signed cycle structure. Hence, let $\tau = \gamma^{-1}\sigma_1\gamma\ \ \gamma^{-1}\sigma_2\gamma\ \dots\ \gamma^{-1}\sigma_m\gamma$, and then we have $\gamma^{-1}\sigma\gamma = \tau$, which means that σ and τ have the same signed cycle structure.

For the other direction, suppose that both σ and τ have the signed cycle structure; this means that the signed cycle structure of σ and τ is $((\lambda_1, \dots, \lambda_r), (\mu_1, \dots, \mu_s))$, where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r$, and $\mu_1 \geq \mu_2 \dots \geq \mu_s$, and there is a λ_i for each negative cycle of length λ_i and a μ_j for each positive cycle of length μ_j when both σ and τ decompose as a disjoint product of positive and negative cycles. Let us rearrange σ and τ by the signed cycle structure and pair of corresponding cycles, as follows:

$$\sigma = (a_{11},\dots,a_{1\lambda_1},-a_{11},\dots,-a_{1\lambda_1})\cdots(a_{p1},\dots,a_{p\lambda_r},-a_{p1},\dots,-a_{p\lambda_r})\ (b_{11},\dots,b_{1\mu_1})(-b_{11},\dots,-b_{1\mu_1})\cdots(b_{q1},\dots,b_{q\mu_s})(-b_{q1},\dots,-b_{q\mu_s})$$

$$\tau = (c_{11},\dots,c_{1\lambda_1},-c_{11},\dots,-c_{1\lambda_1})\cdots(c_{p1},\dots,c_{p\lambda_r},-c_{p1},\dots,-c_{p\lambda_r})\ (d_{11},\dots,d_{1\mu_1})(-d_{11},\dots,-d_{1\mu_1})\cdots(d_{q1},\dots,d_{q\mu_s})(-d_{q1},\dots,-d_{q\mu_s}),$$

Now, for each $x \in [\pm n]$, define $(x)\gamma$ as the image of x under $\gamma \in S_{2n}$ with respect to the following pairing:

$$\begin{array}{cccccccccccccccc}
(a_{11},\dots, & a_{1\lambda_1}, & -a_{11},\dots, & -a_{1\lambda_1})\dots & (a_{p1},\dots, & a_{p\lambda_r}, & -a_{p1},\dots, & -a_{p\lambda_r}) & (b_{11},\dots, & b_{1\mu_1}) & (-b_{11},\dots, & -b_{1\mu_1})\dots & (b_{q1},\dots, & b_{q\mu_s}) & (-b_{q1},\dots, & -b_{q\mu_s}) \\
\downarrow\gamma & \downarrow\gamma & \downarrow\gamma & \downarrow\gamma & \downarrow\gamma & \downarrow\gamma & \downarrow\gamma & \downarrow\gamma & \downarrow\gamma & \downarrow\gamma & \downarrow\gamma & \downarrow\gamma & \downarrow\gamma & \downarrow\gamma & \downarrow\gamma & \downarrow\gamma \\
(c_{11},\dots & c_{1\lambda_1} & -c_{11},\dots & -c_{1\lambda_1})\dots & (c_{p1},\dots, & c_{p\lambda_r}, & -c_{p1},\dots, & -c_{p\lambda_r}) & (d_{11},\dots, & d_{1\mu_1}) & (-d_{11},\dots, & -d_{1\mu_1})\dots & (d_{q1},\dots, & d_{q\mu_s}) & (-d_{q1},\dots, & -d_{q\mu_s})
\end{array}$$

Note that, as this pairing is not unique, we can attain distinct γ's, but this is not an issue, as our goal is to show the existence. Moreover, $(x)\gamma$ is well defined, since the cycles are disjoint in both σ and τ. Therefore, for all $x, y \in [\pm n]$ with $x \neq y$, we have $(x)\gamma \neq (y)\gamma$. Furthermore, $\gamma \in B_n$; that is, $(-x)\gamma = -(x)\gamma$ for all $x \in [\pm n]$. This follows from the fact that both σ and τ are disjoint products of positive and negative cycles and the pairing. Then, for any two consecutive numbers x, y appearing in σ and z, w appearing in τ, the diagram below commutes.

$$\begin{tikzcd}
x \arrow[r, "\sigma"] \arrow[d, "\gamma"] & y \arrow[d, "\gamma"] \\
z \arrow[r, "\tau"] & w
\end{tikzcd}$$

That is, $\sigma\gamma = \gamma\tau$, which implies that $\gamma^{-1}\sigma\gamma = \tau$ for some $\gamma \in B_n$. $\square$

The lemma below asserts a connection between the number of different complementary partitions (λ, μ) of n and the number of conjugacy classes of B_n.

Lemma 3.3.21. [82] *For each positive integer n, the number of distinct complementary partitions (λ, μ) of n is equal to the number of conjugacy classes of B_n.*

The proof of this lemma is not difficult because of the above theorem pointing out that the conjugacy classes of B_n are determined by the signed cycle structure; that is, complementary partitions of n.

Observe that Theorem 3.3.20 and Lemma 3.3.21 allow us to characterise the members of a given conjugacy class as follows: The conjugacy class parametrised by a complementary partition (λ, μ) consists of all $\sigma \in B_n$ with negative and positive cycles of lengths $\lambda_1, \lambda_2 \ldots, \lambda_r$ and $\mu_1, \mu_2 \ldots, \mu_s$ respectively.

The number of conjugacy classes of B_n is equal to the number of ordinary irreducible representations of B_n by Theorem 3.1.17. Hence, our discussion above emphasises that the irreducible representations of B_n over $\mathbb{C}$ are parametrised by the complementary partitions of n. In fact, we will consider the following combinatorial objects that play a role in computing the ordinary irreducible representations of the signed permutation group B_n.

Definition 3.3.22. [1] Given a complementary partition $(\lambda, \mu) \vdash n$ where $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_s)$, and $|\lambda| + |\mu| = n$. A double Young diagram of shape (λ, μ) is a pair $t = (t_\lambda, t_\mu)$, where t_λ is the Young diagram of shape and λ and t_μ is the Young diagram of shape and μ.

Recall that a Young diagram of shape λ is a collection of non-increasing, left-justified, empty boxes arranged in rows, such that there are λ_i boxes in each ith row.

Example 3.3.23. Let $n = 5$. Then, the double Young diagram corresponding to the complementary partition $((2, 1), (1, 1))$ is

$$t = \left(\ydiagram{2,1}, \ydiagram{1,1} \right).$$

Note that since either $|\lambda|$ or $|\mu|$ might be zero, the double Young diagram corresponding to the complementary partition $((4), \phi)$ is

$$t' = \left(\ydiagram{4}, \phi \right).$$

Definition 3.3.24. [1] Given a complementary partition (λ, μ) of n, a (λ, μ)-Young tableau t is a double diagram t of shape (λ, μ) filled with numbers $[\pm n] = \{\pm 1, \ldots, \pm n\}$, such that different boxes contain distinct numbers and for each $i \in [\pm n]$, either i or $-i$ occurs in t.

Note that we can restate the above condition as follows: if we take the absolute value $|i|$ for each entry i appearing in t, we will obtain the set $\{1, 2, \cdots, n\}$. Moreover, it is worthwhile clarifying that not all members of the set $\{\pm 1, \ldots, \pm n\}$ are filling the Young diagram t; however, it will be filled with exactly half of them, as we only have n boxes in t.

Example 3.3.25. Let $n = 6$, and choose the complementary partition $((3,1),(2))$. Then,

$$\left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 5 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|}\hline -4 & 6 \\ \hline\end{array}\right) \text{ and } \left(\begin{array}{|c|c|c|}\hline -2 & 3 & 6 \\ \hline 1 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|}\hline 5 & 4 \\ \hline\end{array}\right)$$

are two Young tableaux of shape $((3,1),(2))$. However,

$$\left(\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 6 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|}\hline -1 & 6 \\ \hline\end{array}\right)$$

is not a Young tableau, as there is no occurrence of 3 or -3 and 4 or -4.

Definition 3.3.26. Fix $(\lambda, \mu) \vdash n$. Define $Y^{(\lambda,\mu)}$ to be the set of all distinct tableaux t of shape (λ, μ); that is,

$$Y^{(\lambda,\mu)} = \{t = (t_\lambda, t_\mu) : t \text{ runs through all Young tableaux of shape } (\lambda, \mu)\}.$$

Observe that if we fix a complementary partition (λ, μ) of n, then the number of distinct Young tableaux in $Y^{(\lambda,\mu)}$ is indeed equal to $2^n n!$, since there are $n!$ distinct ways of filling the boxes of t with distinct numbers from $\{1, 2, \ldots, n\}$. Moreover, we have two choices in each box: We can either change the entry to its negative or leave it as is. Hence, the number of distinct Young tableaux of shape (λ, μ) corresponds to the number of elements of B_n.

Definition 3.3.27. [53] Fix $(\lambda, \mu) \vdash n$, and define the action of the signed permutation groups B_n on a (λ, μ)-Young tableau t by taking the images of all entries in t under a signed permutation $\sigma \in B_n$.

Proposition 3.3.28. *Let $(\lambda, \mu) \vdash n$ and $Y^{(\lambda,\mu)}$ be the set of all distinct tableaux t of shape (λ, μ). Then, $Y^{(\lambda,\mu)}$ is left invariant under the action of B_n; that is,*

$$Y^{(\lambda,\mu)} \times B_n \longrightarrow Y^{(\lambda,\mu)}.$$

Proof. Let $\sigma \in B_n$ and $t \in Y^{(\lambda,\mu)}$. Then, by Definition 3.3.24, we know that for all $a_i \in t$, we have $\{|a_1|, \ \ldots \ , |a_n|\} = \{1, \ \ldots \ , n\}$, and $|a_i| \neq |a_j|$, as each one corresponds to a number in $\{1, \ \ldots \ , n\}$. To show that $t\sigma \in Y^{(\lambda,\mu)}$, we need to examine that for all $(a_i)\sigma \in t\sigma$, we have $\{|(a_1)\sigma|, \ \ldots \ , |(a_n)\sigma|\} = \{1, \ldots, n\}$. Suppose, for a contradiction, that there exists $j \neq k$ such that $|(a_j)\sigma| = |(a_k)\sigma|$. Therefore, we have two cases, as described below.

Case 1. $(a_j)\sigma = (a_k)\sigma$. However, since σ is a bijection, this implies that $j = k$, and this contradicts the assumption.

Case 2. $(a_j)\sigma = -(a_k)\sigma$. Hence, $(a_j)\sigma = (-a_k)\sigma$, as $\sigma \in B_n$ and for all $a_i \in [\pm n]$, we have $-(a_k)\sigma = (-a_k)\sigma$. It follows that $a_j = -a_k$ as σ is again a bijection. Now, by taking their absolute value, we have $|a_j| = |a_k|$, which is again a contradiction with the fact that, for any two entries $a_j, a_k \in t$, we have $|a_j| \neq |a_k|$, as each corresponds to a number in $\{1, \cdots, n\}$. Thus, the assumption cannot hold, and this implies that $\{|(a_j)\sigma|\colon 1 \leq j \leq n\} = \{1, \ \cdots \ n\}$. Hence, either $(a_j)\sigma$ or $-(a_j)\sigma$ must occur in $t\sigma$ and then $t\sigma \in Y^{(\lambda,\mu)}$. $\square$

The following notation has been stated in [7], and it is important to note, as we will utilise it below. Let $(\lambda, \mu) \vdash n$, and $t = (t_\lambda, t_\mu) \in Y^{(\lambda,\mu)}$. Denote by

$$t(i,j,m) = \begin{cases} t_\lambda(i,j), & m = 1, \\ t_\mu(i,j), & m = 2, \end{cases}$$

where $t_x(i,j)$ is the entry in the ith row and jth column of t_x where $x = \lambda$ or $x = \mu$.

Definition 3.3.29. [1] Given a Young tableau t of shape (λ, μ), define a row permutation of t to be an element $\sigma \in B_n$ which permutes the entries in each row of $t = (t_\lambda, t_\mu)$ and may change the sign of entries in t_μ. That is, $\sigma \in B_n$ is a called a row permutation if and only if it is a permutation of the entries of t_λ and t_μ such that

$$\begin{aligned} t(i,j,1)\sigma &= t(i,k,1), && \text{for some } k, \\ t(i,j,2)\sigma &= \pm\, t(i,k,2), && \text{for some } k. \end{aligned}$$

Similarly, we define a column permutation of t to be an element $\tau \in B_n$ that permutes the entries in each column of $t = (t_\lambda, t_\mu)$ and may change the sign of entries in t_λ; that is, $\tau \in B_n$ is called a column permutation if and only if it is a permutation of the entries of t_λ and t_μ such that

$$\begin{aligned} t(i,j,1)\tau &= \pm\, t(q,j,1), && \text{for some } q, \\ t(i,j,2)\tau &= t(q,j,2), && \text{for some } q. \end{aligned}$$

Observation 3.3.30. It is significant to consider the effect of a row permutation on t_μ, since we do not mean that it is changing the signs of all entries in the row at once. What we infer by changing the sign of t_μ is that it may change the signs of the individual entries of the row and leave the other ones unchanged. Similarly, the same consideration applies for a column permutation.

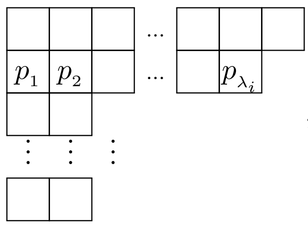
Definition 3.3.31. [1] Let $(\lambda, \mu) \vdash n$ and $t = (t_\lambda, t_\mu) \in Y^{(\lambda,\mu)}$. Define R_t to be the subgroup of B_n consisting of all row permutations of t :

$$R_t = \Big\{\sigma \in B_n : t_\lambda(i,j)\sigma = t_\lambda(i,k),$$
$$t_\mu(i,j)\sigma = \pm\, t_\mu(i,l), \quad \text{for some } k \text{ and } l\Big\}.$$

In fact, the reader should bear in mind that the row group R_t can also be described as follows: If $(\lambda, \mu) \vdash n$, where $\lambda = (\lambda_1, \ \dots \ , \lambda_r)$ and $\mu = (\mu_1, \dots \ , \mu_s)$, then

$$R_t \cong S_{\lambda_1} \times \cdots \times S_{\lambda_r} \times B_{\mu_1} \times \cdots \times B_{\mu_s}, \tag{3.16}$$

where S_{λ_i} represents the symmetric groups on λ_i, with $1 \leq i \leq r$, and B_{μ_j} is the signed permutation group on μ_j, with $1 \leq j \leq s$. In fact, it is not surprising to find such an isomorphism, as we know that for each ith row in the first tableau t_λ, there are λ_i boxes containing numbers, and a row permutation plays a role in permuting these numbers among themselves. Thus, we need to establish a group isomorphic to S_{λ_i}, and a simple way of constructing such a group is as follows: If the entries of the ith row of t_λ are illustrated as

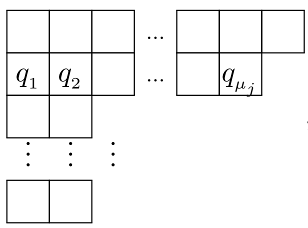


,

then

$$S_{\lambda_i} \cong \langle (p_1, p_2)(-p_1, -p_2), \ \cdots \ , (p_{\lambda_i - 1}, p_{\lambda_i})(-p_{\lambda_i - 1}, -p_{\lambda_i}) \rangle,$$

since S_{λ_i} is a symmetric group inside B_n and it is generated by consecutive transpositions. Indeed, the product $S_{\lambda_1} \times \cdots \times S_{\lambda_r}$ contributes to preserving all the rows of t_λ. However, in the second tableau t_μ, there are μ_j boxes in each jth row. Furthermore, the entries in each jth row of t_μ need not only be permuted among themselves, but they can also have a sign change for each individual entry in that row; thus, we need to identify a group isomorphic to B_{μ_j}. A simple way of constructing such a group is as follows: If the entries of the jth row of t_μ are



,

then

$$B_{\mu_j} \cong \langle \overbrace{(q_1, -q_1), (q_2, -q_2) \ \cdots \ , (q_{\mu_j}, -q_{\mu_j})}^{\text{transpositions generating } \mathbb{Z}_2^{\mu_j}}, \overbrace{(q_1, q_2)(-q_1, -q_2), \ \cdots \ , (q_{\mu_j-1}, q_{\mu_j})(-q_{\mu_j-1}, -q_{\mu_j})}^{\text{transpositions generating } S_{\mu_j}} \rangle,$$

since $B_{\mu_j} \cong \mathbb{Z}_2^{\mu_j} \rtimes S_{\mu_j}$, as clarified in Proposition 3.3.8. Indeed, the part $B_{\mu_1} \times \cdots \times B_{\mu_s}$ of (3.16) will definitely preserve all the rows of t_μ and the sign change for each entry. Similarly, we define the column group C_t as shown below.

Definition 3.3.32. [1] Let $(\lambda, \mu) \vdash n$ and $t = (t_\lambda, t_\mu) \in Y^{(\lambda,\mu)}$. Define C_t to be a subgroup of B_n consisting of all column permutations of t; that is,

$$C_t = \Big\{ \tau \in B_n : t_\lambda(i,j)\tau = \pm\, t_\lambda(k,j), \\ t_\mu(i,j)\tau = t_\mu(l,j), \quad \text{for some } k \text{ and } l \Big\}.$$

Definition 3.3.33. Let $(\lambda, \mu) \vdash n$, where $\lambda = (\lambda_1, \ \ldots \ , \lambda_r)$ and $\mu = (\mu_1, \ldots \ , \mu_s)$, and $t = (t_\lambda, t_\mu) \in Y^{(\lambda,\mu)}$. The partitions $\lambda' = (\lambda'_1, \ \ldots \ , \lambda'_{r'})$ and $\mu' = (\mu'_1, \ldots \ , \mu'_{s'})$ are called the conjugates of λ and μ, respectively, where λ'_i is the number of boxes in the ith column of t_λ and μ'_j is the number of boxes in the jth column of t_μ.

Given the preceding definition, we can describe the column group C_t as follows: Let $(\lambda, \mu) \vdash n$ and λ' and μ' be the conjugate partitions of λ and μ, respectively, where $\lambda' = (\lambda'_1, \ \ldots \ , \lambda'_{r'})$ and $\mu' = (\mu'_1, \ldots \ , \mu'_{s'})$. Then,

$$C_t \cong B_{\lambda'_1} \times \cdots \times B_{\lambda'_{r'}} \times S_{\mu'_1} \times \cdots \times S_{\mu'_{s'}},$$

where $B_{\lambda'_i}$ is the signed permutation group on λ'_i with $1 \le i \le r'$ and $S_{\mu'_j}$ is the symmetric group on μ'_j with $1 \le j \le s'$.

Example 3.3.34. Let $n = 7$ and

$$\left(\begin{array}{|c|c|c|} \hline -2 & 1 & -4 \\ \hline 5 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|} \hline 7 & 3 \\ \hline -6 \\ \cline{1-1} \end{array} \right)$$

be a Young tableau of shape $((3,1),(2,1))$. Then,

$$R_t = \langle (-2,1)(2,-1), (1,-4)(-1,4), (7,3)(-7,-3), (7,-7), (3,-3), (-6,6) \rangle \text{ and}$$
$$C_t = \langle (-2,5)(2,-5), (-2,2), (5,-5), (1,-1), (-4,4), (7,3)(-7,-3) \rangle.$$

Proposition 3.3.35. [53] *Let $\pi \in B_n$. Then, $\pi^{-1} C_t \pi = C_{t\pi}$ and $\pi^{-1} R_t \pi = R_{t\pi}$.*

Proof. We show $\pi^{-1} C_t \pi = C_{t\pi}$. Let $t = (t_\lambda, t_\mu) \in Y^{(\lambda,\mu)}$ and $t_\lambda(i,j)$ be the entry in the ith row and jth column of t_λ, with $t_\mu(i,j)$ being the entry in the ith row and jth column of t_μ. Consider a (λ,μ)-tableau $t\pi$ where $t_\lambda(i,j)\pi$ is the (i,j)-th entry of $t_\lambda \pi$ and $t_\mu(i,j)\pi$ is the (i,j)-th entry of $t_\mu \pi$.

Now for all i, j

$\tau \in C_{t\pi}$ if and only if

$$\begin{aligned} t_\lambda(i,j)\pi\tau &= \pm\, t_\lambda(k,j)\pi, \\ t_\mu(i,j)\pi\tau &= t_\mu(l,j)\pi, \qquad \text{for some } k \text{ and } l. \end{aligned}$$

This also holds if and only if

$$\begin{aligned} t_\lambda(i,j)\pi\tau &= (\pm\, t_\lambda(k,j))\pi, \qquad [\text{ as} - (r)\pi = (-r)\pi \ \text{ for all } r], \\ t_\mu(i,j)\pi\tau &= t_\mu(l,j)\pi, \qquad \text{for some } k \text{ and } l. \end{aligned}$$

This also true if and only if

$$\begin{aligned} t_\lambda(i,j)\pi\tau\pi^{-1} &= \pm\, t_\lambda(k,j), \qquad \text{for some } k, \\ t_\mu(i,j)\pi\tau\pi^{-1} &= t_\mu(l,j), \qquad \text{for some } l. \end{aligned}$$

This also holds if and only if $\pi\tau\pi^{-1} \in C_t$, as $t_x(r,s)$ with $x = \lambda$ or $x = \mu$ are entries in (λ, μ)-tableau t. Hence, we have $\pi C_{t\pi}\pi^{-1} = C_t$ and then $C_{t\pi} = \pi^{-1}C_t\,\pi$. Similarly, we can show $R_{t\pi} = \pi^{-1}R_t\,\pi$, where $\pi \in B_n$. □

Remark 3.3.36. Let $(\lambda, \mu) \vdash n$. Define a relation $\sim$ on the set of (λ, μ)-tableaux $Y^{(\lambda,\mu)}$ as follows:

$$t_1 \sim t_2 \iff \exists\, \sigma \in R_{t_1} \text{ such that } \ t_2 = t_1\sigma,$$

for all $t_1, t_2 \in Y^{(\lambda,\mu)}$. In fact, the above relation $\sim$ is an equivalence relation, as it is reflexive. Moreover, if $t_1 \sim t_2$; that is, $t_2 = t_1\sigma$, then $t_2\sigma^{-1} = t_1$, as σ is a group element. However, $\sigma^{-1} = \sigma^{-1}\sigma^{-1}\sigma \in \sigma^{-1}R_{t_1}\sigma = R_{t_1\sigma} = R_{t_2}$. Hence, the relation $\sim$ is also symmetric. It remains to be shown that it is also transitive. Suppose $t_1 \sim t_2$ and $t_2 \sim t_3$. Then there exists $\sigma \in R_{t_1}$ such that $t_2 = t_1\sigma$, and $\tau \in R_{t_2}$ such that $t_3 = t_2\tau$. Consider $t_3 = t_2\tau = t_1\sigma\tau$. This gives us $t_3 = t_1\sigma\tau$. Now, if we prove that $\tau \in R_{t_1}$, then we have completed the demonstration. Notice that $\tau \in R_{t_2} = R_{t_1\sigma} = \sigma^{-1}R_{t_1}\sigma$; that is, $\tau \in \sigma^{-1}R_{t_1}\sigma$, and then $\tau \in R_{t_1}$. Hence, $t_1 \sim t_3$.

Definition 3.3.37. [53] Fix $(\lambda, \mu) \vdash n$. Define a (λ, μ)-tabloid $\{t\}$ to be the set of all Young tableaux of shape (λ, μ) that are row equivalent to t; that is, a (λ, μ)-tabloid $\{t\}$ is the orbit of Young tableau t under the action of the row group R_t

$$\{t\} := \{s \in Y^{(\lambda,\mu)} : \ s = t\sigma \text{ for some } \ \sigma \in R_t\}.$$

Example 3.3.38. Let $n = 4$, and fix the complementary partition $(\lambda, \mu) = ((1,1), (2))$. Then, the tabloid of

$$t = \left(\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array}, \begin{array}{|c|c|}\hline -2 & -4 \\ \hline \end{array} \right)$$

is equal to

$$\begin{aligned} \{t\} = \Big\{ & \left(\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array}, \begin{array}{|c|c|}\hline -2 & -4 \\ \hline \end{array} \right), \left(\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array}, \begin{array}{|c|c|}\hline -4 & -2 \\ \hline \end{array} \right), \left(\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 2 & -4 \\ \hline \end{array} \right), \left(\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array}, \begin{array}{|c|c|}\hline -2 & 4 \\ \hline \end{array} \right), \\ & \left(\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 2 & 4 \\ \hline \end{array} \right), \left(\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array}, \begin{array}{|c|c|}\hline -4 & 2 \\ \hline \end{array} \right), \left(\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 4 & -2 \\ \hline \end{array} \right), \left(\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 4 & 2 \\ \hline \end{array} \right) \Big\}. \end{aligned}$$

Observe that, as Morris pointed out in [53], the signed permutation group B_n acts on the set of (λ, μ)-tabloids via the following formula:

$$\{t\}\ \sigma = \{t\sigma\} \quad \text{for all } \sigma \in B_n. \tag{3.17}$$

This action is well defined for the following reason: Suppose $\{t\}$ and $\{s\}$ are (λ, μ)-tabloids where $\{t\}$ is a row equivalent to $\{s\}$. Thus, there exists a permutation $\tau \in R_t$ such that $s = t\tau$. Let $\sigma \in B_n$, and consider

$$s\sigma = (t\tau)\sigma = t(\sigma\sigma^{-1})\tau\sigma = t\sigma(\sigma^{-1}\tau\sigma). \tag{3.18}$$

Let $\gamma = \sigma^{-1}\tau\sigma$, and observe that $\gamma \in \sigma^{-1}R_t\sigma$. Using Proposition 3.3.35 results in $\gamma \in R_{t\sigma}$. Hence, we can rewrite equation (3.18) as follows:

$$s\sigma = (t\sigma)\gamma \quad \text{where} \quad \gamma \in R_{t\sigma}.$$

Thus, $\{t\sigma\} = \{(t\sigma)\gamma\} = \{s\sigma\}$. Hence, the action of B_n on (λ, μ)-tabloids is well defined.

Definition 3.3.39. [53] Fix $(\lambda, \mu) \vdash n$. Define $M^{(\lambda,\mu)}$ to be the $\mathbb{C}$-vector space with basis the distinct (λ, μ)-tabloids $\{t\}$.

The elements of $M^{(\lambda,\mu)}$ are the formal $\mathbb{C}$-linear combinations of distinct tabloids $\{t\}$ of shape (λ, μ); that is,

$$M^{(\lambda,\mu)} = \mathbb{C}\text{-}[\{t_1\}\ , \ldots,\ \{t_m\}],$$

where $\{t_i\}$ with $1 \leq i \leq m$ is a complete list of (λ, μ)-tabloids. By extending the preceding action linearly, $M^{(\lambda,\mu)}$ becomes a right $\mathbb{C}B_n$ module or B_n-module.

$$M^{(\lambda,\mu)} \times B_n \subseteq M^{(\lambda,\mu)}.$$

In other words, $M^{(\lambda,\mu)}$ is a representation of the signed permutation group B_n.

Proposition 3.3.40. [53] *Let (λ, μ) be a complementary partition of n where $\lambda = (\lambda_1, \ \ldots \ , \lambda_r)$ and $\mu = (\mu_1, \ldots \ , \mu_s)$. Then,*

$$dim\ M^{(\lambda,\mu)} = 2^{n-|\mu|}\ \frac{n!}{\lambda_1! \ \cdots \ \lambda_r!\, \mu_1! \ \cdots \ \mu_s!}$$

The reason for this is simple, as we know that the size of the set $Y^{(\lambda,\mu)}$, which consists of all distinct (λ,μ)-tableaux, is equal to $2^n n!$. Moreover, each (λ,μ)-tabloid $\{t\}$ is determined by $\{t\} = \{s \in Y^{(\lambda,\mu)} : s = t\sigma \text{ with } \sigma \in R_t\}$. Hence, the number of tableaux in any (λ,μ)-tabloid $\{t\}$ is equal to $|R_t|$. Furthermore, since $R_t \cong S_{\lambda_1} \times \cdots \times S_{\lambda_r} \times B_{\mu_1} \times \cdots \times B_{\mu_s}$, then

$$\begin{aligned} |R_t| &= \lambda_1! \cdots \lambda_r!\ 2^{\mu_1}\mu_1! \cdots 2^{\mu_s}\mu_s! \\ &= \lambda_1! \cdots \lambda_r!\ \mu_1! \cdots \mu_s!\ 2^{|\mu|}. \end{aligned}$$

Since each orbit $\{t\}$ has the same size $|R_t|$, then dividing the number of all (λ,μ)-tableaux in $Y^{(\lambda,\mu)}$ by the number of tableaux in each (λ,μ)-tabloid $\{t\}$ results in the number of distinct (λ,μ)-tabloids; that is,

$$\begin{aligned} \#\ \text{distinct } (\lambda,\mu)\text{-tabloids} &= \frac{2^n n!}{\lambda_1! \cdots \lambda_r!\ \mu_1! \cdots \mu_s!\, 2^{|\mu|}} \\ &= 2^{n-|\mu|}\ \frac{n!}{\lambda_1! \ \cdots \ \lambda_r!\, \mu_1! \ \cdots \ \mu_1!}. \end{aligned}$$

Example 3.3.41. Let $n = 2$, and fix the complementary partition $(\lambda,\mu) = ((1),(1))$. Then, the set $Y^{((1),(1))}$ of all distinct $((1),(1))$-tableaux and the B_2-module $M^{((1),(1))}$ are determined as follows:

$$\begin{aligned} Y^{((1),(1))} &= \Big\{\Big(\boxed{1},\boxed{2}\Big),\Big(\boxed{1},\boxed{-2}\Big),\Big(\boxed{-1},\boxed{-2}\Big),\Big(\boxed{-1},\boxed{2}\Big),\Big(\boxed{2},\boxed{1}\Big),\Big(\boxed{2},\boxed{-1}\Big),\Big(\boxed{-2},\boxed{1}\Big),\Big(\boxed{-2},\boxed{-1}\Big)\Big\}. \\ M^{((1),(1))} &= \mathbb{C}\text{-}\Big[\Big\{\Big(\boxed{1},\boxed{2}\Big)\Big\},\Big\{\Big(\boxed{-1},\boxed{-2}\Big)\Big\},\Big\{\Big(\boxed{2},\boxed{1}\Big)\Big\},\Big\{\Big(\boxed{-2},\boxed{-1}\Big)\Big\}\Big]. \end{aligned}$$

The following assertions indicates how we might determine all the irreducible representations of B_n.

Definition 3.3.42. [53] Fix $(\lambda,\mu) \vdash n$. For each Young tableau t of shape (λ,μ), define the element $e_t \in M^{(\lambda,\mu)}$ as follows:

$$e_t = \sum_{\sigma \in C(t)} sgn(\sigma)\ \{t\}\sigma.$$

Such an element e_t is called a (λ,μ)-polytabloid associated with the tableau $t = (t_\lambda, t_\mu)$.

The result below illustrates how the signed permutation group B_n permutes the set

$$\{e_t :\ t = (t_\lambda, t_\mu) \text{ is a Young tableau of shape } (\lambda,\mu)\}.$$

Proposition 3.3.43. [7] *Fix $(\lambda,\mu) \vdash n$. For any Young tableau t of shape (λ,μ) and signed permutation $\sigma \in B_n$, we have*

$$e_t \, . \, \sigma = e_{t\sigma}.$$

The proof of this proposition is entirely analogous to the proof of Lemma 3.2.16 in which we utilise Proposition 3.3.35 and Definition 3.3.42 to obtain the result.

Definition 3.3.44. [53] For any complementary partition (λ,μ) of n, define the Specht module $S^{(\lambda,\mu)}$ to be the subspace of $M^{(\lambda,\mu)}$ spanned by the elements e_t, where $t = (t_\lambda, t_\mu)$ ranges over all the Young tableaux of shape (λ,μ); that is,

$$S^{(\lambda,\mu)} = \{c_1 e_{t_1} + \ \dots \ + c_q e_{t_q} : \ c_i \in \mathbb{C}, \ t_i \in Y^{(\lambda,\mu)} \text{ for all } \ 1 \le i \le q\}.$$

In view of the preceding proposition, it is clear that the Specht module $S^{(\lambda,\mu)}$ is preserved by the action of B_n, and this justifies the identification of $S^{(\lambda,\mu)}$ as a B_n-submodule of $M^{(\lambda,\mu)}$. Let us now state some essential assertions that can be employed to show the irreducibility of the Specht module $S^{(\lambda,\mu)}$.

Fix $(\lambda,\mu) \vdash n$. Let us define an inner product on $M^{(\lambda,\mu)}$ over the complex field to be a Hermitian linear extension of

$$\langle \{t\}, \{s\} \rangle = \begin{cases} 1, & \{t\} = \{s\}, \\ 0, & \text{otherwise}, \end{cases}$$

that is, if $a = \sum c_t \{t\}$ and $b = \sum c_s \{s\}$ are both vectors in $M^{(\lambda,\mu)}$ with the $c_t, c_s \in \mathbb{C}$, then

$$\langle a, b \rangle = \sum_{s,t} c_t \bar{c}_s \ \langle \{t\}, \{s\} \rangle = \sum_r c_r \bar{c}_r,$$

where the bar represents complex conjugation. The form $\langle \cdot, \cdot \rangle$ is invariant under the action of B_n; that is, for all $\sigma \in B_n$,

$$\langle a\sigma, b\sigma \rangle = \langle a, b \rangle. \tag{3.19}$$

To verify this invariance, it suffices to examine (3.19) for the basis elements of $M^{(\lambda,\mu)}$. If $\sigma \in B_n$, then

$$\langle \{t\}\sigma, \{s\}\sigma \rangle = \begin{cases} 1, & \{t\}\sigma = \{s\}\sigma, \\ 0, & \text{otherwise}, \end{cases}$$

$$
\begin{aligned}
&= \begin{cases} 1, & \{t\} = \{s\}, \qquad \text{[As } \sigma \text{ is a bijection and } \{t\}, \{s\} \text{ are tabloids]} \\ 0, & \text{otherwise,} \end{cases} \\
&= \langle \{t\}, \{s\} \rangle
\end{aligned}
$$

If $W \subseteq M^{(\lambda,\mu)}$, the orthogonal space $W^{\perp}$ can be defined by

$$
W^{\perp} = \{v \in M^{(\lambda,\mu)} : \langle w, v \rangle = 0 \ \text{ for all } w \in W\}.
$$

Observe that, if W is a B_n-submodule of $M^{(\lambda,\mu)}$ and $v \in W^{\perp}$, $\sigma \in B_n$, then

$$
\begin{aligned}
\langle w, v\sigma \rangle &= \langle w\sigma^{-1}, v\sigma\sigma^{-1} \rangle && \text{[As the form is invariant]} \\
&= \langle w\sigma^{-1}, v \rangle \\
&= 0 && [v \in W^{\perp}, w\sigma^{-1} \in W \text{ for } W \text{ is a submodule}].
\end{aligned}
$$

Consequently, $v\sigma \in W^{\perp}$, and then $W^{\perp}$ is also a B_n-submodule of $M^{(\lambda,\mu)}$.

Theorem 3.3.45 (Submodule Theorem [53])**.** Fix $(\lambda, \mu) \vdash n$. Let U be a B_n-submodule of $M^{(\lambda,\mu)}$. Then, either

$$
S^{(\lambda,\mu)} \subseteq U \ \text{ or } \ U \subseteq (S^{(\lambda,\mu)})^{\perp},
$$

where $(S^{(\lambda,\mu)})^{\perp}$ is the orthogonal space of $S^{(\lambda,\mu)}$ in $M^{(\lambda,\mu)}$.

The proof of the assertion appears in [53].

Lemma 3.3.46. [7] *Let (λ, μ) be a complementary partition of n. Then, the Specht module $S^{(\lambda,\mu)}$ is a B_n-irreducible module over the complex field $\mathbb{C}$.*

Proof. Let U be a non-zero B_n-submodule of $S^{(\lambda,\mu)}$. Then, U is specifically a B_n-submodule of $M^{(\lambda,\mu)}$. In view of Theorem 3.3.45, we have either

$$
S^{(\lambda,\mu)} \subseteq U \ \text{ or } \ U \subseteq (S^{(\lambda,\mu)})^{\perp}. \tag{3.20}
$$

However, since $U \subseteq S^{(\lambda,\mu)}$, we can rewrite the expression in (3.20) as follows: We have either

$$
U = S^{(\lambda,\mu)} \ \text{ or } \ U \subseteq S^{(\lambda,\mu)} \cap (S^{(\lambda,\mu)})^{\perp}.
$$

Let us show that $S^{(\lambda,\mu)} \cap (S^{(\lambda,\mu)})^{\perp} = \mathbf{0}$. We know that $\mathbf{0} \subseteq S^{(\lambda,\mu)} \cap (S^{(\lambda,\mu)})^{\perp}$, and if $x \in S^{(\lambda,\mu)} \cap (S^{(\lambda,\mu)})^{\perp}$, then $x \in S^{(\lambda,\mu)}$ and $x \in (S^{(\lambda,\mu)})^{\perp}$. In particular, $\langle x, x \rangle = 0$, and this implies that $x = \mathbf{0}$ as $\langle \cdot, \cdot \rangle$ is a Hermitian inner product. In other words, $S^{(\lambda,\mu)} \cap (S^{(\lambda,\mu)})^{\perp} \subseteq \mathbf{0}$; thus, we obtain the equality, and $S^{(\lambda,\mu)}$ is equal to U. Hence, it is irreducible. $\square$

The result below asserts that the Specht modules for the group of signed permutations B_n are mutually inequivalent; that is,

$$S^{(\lambda,\mu)} = S^{(\lambda',\mu')} \iff (\lambda,\mu) = (\lambda',\mu').$$

where equality of complementary partitions is similar to equality of partitions as stated in Remark 3.2.19.

Theorem 3.3.47. [7] Given a complementary partition (λ,μ) of n. The Specht modules $S^{(\lambda,\mu)}$ form a complete list of inequivlent irreducible representations of B_n over the complex field $\mathbb{C}$.

A detailed proof of this theorem can be found in [7, Theorem 2.6]. Notably, we must point out that Morris [53] and Can [7] considered the field of rational numbers $\mathbb{Q}$ when they established the proof of this vital assertion. However, the reader can easily recognise that the argument they utilise in the proof holds when the field is the complex numbers $\mathbb{C}$. The reader can consult [1, 53], for an explicit description of simple modules of B_n in the modular case.

Example 3.3.48. Let $n = 2$, and fix the complementary partition $(\lambda,\mu) = ((1),(1))$. Then, the set $Y^{((1),(1))}$ of all distinct $((1),(1))$-tableaux is determined as follows:

$$Y^{((1),(1))} = \Big\{\Big(\boxed{1},\boxed{2}\Big),\Big(\boxed{1},\boxed{-2}\Big),\Big(\boxed{-1},\boxed{-2}\Big),\Big(\boxed{-1},\boxed{2}\Big),\Big(\boxed{2},\boxed{1}\Big),\Big(\boxed{2},\boxed{-1}\Big),\Big(\boxed{-2},\boxed{1}\Big),\Big(\boxed{-2},\boxed{-1}\Big)\Big\}.$$

Let us now compute all $((1),(1))$-polytabloids for each $((1),(1))$-tableau in $Y^{((1),(1))}$ as follows:

$t_1 = \Big(\boxed{1},\boxed{2}\Big)$ and $C_{t_1} = \{id,\ (1,-1)\}$. Thus, $e_{t_1} = +\Big\{\Big(\boxed{1},\boxed{2}\Big)\Big\} - \Big\{\Big(\boxed{-1},\boxed{2}\Big)\Big\}$
$t_2 = \Big(\boxed{1},\boxed{-2}\Big)$ and $C_{t_2} = \{id,\ (1,-1)\}$. Thus, $e_{t_2} = +\Big\{\Big(\boxed{1},\boxed{-2}\Big)\Big\} - \Big\{\Big(\boxed{-1},\boxed{-2}\Big)\Big\}$
$t_3 = \Big(\boxed{-1},\boxed{-2}\Big)$ and $C_{t_3} = \{id,\ (1,-1)\}$, Thus, $e_{t_3} = +\Big\{\Big(\boxed{-1},\boxed{-2}\Big)\Big\} - \Big\{\Big(\boxed{1},\boxed{-2}\Big)\Big\}$
$t_4 = \Big(\boxed{-1},\boxed{2}\Big)$ and $C_{t_4} = \{id,\ (1,-1)\}$, Thus, $e_{t_4} = +\Big\{\Big(\boxed{-1},\boxed{2}\Big)\Big\} - \Big\{\Big(\boxed{1},\boxed{2}\Big)\Big\}$
$t_5 = \Big(\boxed{2},\boxed{1}\Big)$ and $C_{t_5} = \{id,\ (2,-2)\}$, Thus, $e_{t_5} = +\Big\{\Big(\boxed{2},\boxed{1}\Big)\Big\} - \Big\{\Big(\boxed{-2},\boxed{1}\Big)\Big\}$
$t_6 = \Big(\boxed{2},\boxed{-1}\Big)$ and $C_{t_6} = \{id,\ (2,-2)\}$, Thus, $e_{t_6} = +\Big\{\Big(\boxed{2},\boxed{-1}\Big)\Big\} - \Big\{\Big(\boxed{-2},\boxed{-1}\Big)\Big\}$
$t_7 = \Big(\boxed{-2},\boxed{1}\Big)$ and $C_{t_7} = \{id,\ (2,-2)\}$, Thus, $e_{t_7} = +\Big\{\Big(\boxed{-2},\boxed{1}\Big)\Big\} - \Big\{\Big(\boxed{2},\boxed{1}\Big)\Big\}$
$t_8 = \Big(\boxed{-2},\boxed{-1}\Big)$ and $C_{t_8} = \{id,\ (2,-2)\}$, Thus, $e_{t_8} = +\Big\{\Big(\boxed{-2},\boxed{-1}\Big)\Big\} - \Big\{\Big(\boxed{2},\boxed{-1}\Big)\Big\}$.

Hence,

$$S^{((1),(1))} = \mathrm{Span}_{\mathbb{C}}\{e_{t_1}, e_{t_2}, e_{t_3}, e_{t_4}, e_{t_5}, e_{t_6}, e_{t_6}, e_{t_7}, e_{t_8}\}.$$

Observe that $e_{t_2} = e_{t_1}$, and $e_{t_3} = -e_{t_1} = e_{t_4}$. Moreover, $e_{t_6} = e_{t_5}$, and $e_{t_7} = -e_{t_5} = e_{t_8}$. This gives us a clue that polytabloids spanning $S^{(\lambda,\mu)}$ are not necessarily independent. The following definition sheds light on a subset of the spanning set $\{e_t : t \text{ is a } (\lambda,\mu)\text{-tableau}\}$ that forms a basis for the Specht module $S^{(\lambda,\mu)}$.

Definition 3.3.49. [7] A (λ,μ)-tableau t is said to be standard if the entries of t are all positive integers, and t_λ, t_μ are both standard tableaux.

Theorem 3.3.50. [7] The set $\{e_t : t$ is a *standard* (λ, μ)-tableau$\}$ is a $\mathbb{C}$-basis for $S^{(\lambda,\mu)}$ and the dimension of $S^{(\lambda,\mu)}$ is the number of standard (λ, μ)-tableaux.

A detailed proof of the assertion appears in [7, Section 3].

Example 3.3.51. In Example 3.3.48, both $t_1 = (\boxed{1}, \boxed{2})$ and $t_5 = (\boxed{2}, \boxed{1})$ are $((1), (1))$-standard tableaux. Using Theorem 3.3.50, $S^{((1),(1))}$ can be written with respect to its basis elements of standard polytabloids $\{e_{t_1},\ e_{t_5}\}$, as follows:

$$S^{((1),(1))} = \mathbb{C}\text{-}[e_{t_1}, e_{t_5}].$$

Chapter 4

Representation theory of semigroups

4.1 Basic notions

The critical role played by the theory of linear representations in investigating finite groups and finite dimensional algebras has been acknowledged for over a century. Contrarily, relatively few applications in the representation theory of semigroups arose and were explored independently in the 1950s by scholars such as Clifford [11], Munn [56, 58, 59] and Ponizovskiĭ [64]. Although it was also developed in the 1960s and 1970s by McAlister [52], Lallement and Petrich [46], as well as Rhodes and Zalcstein (their results were written a long time before they were published [72]), the absence of ready-made applications implied that the representation theory of semigroups subsequently went unexplored [78].

However, beginning with the research of Renner [70, 71], who worked on the theory of linear algebraic monoids, as well as Putcha [67, 68], who computed the irreducible representations of the full transformation monoid, novel explorations of the representation theory of finite semigroups resumed in the 1980s and 1990s. As a result, something of a renaissance started in the representation theory of finite monoids, with numerous articles subsequently being published on the subject. A particularly noteworthy recent text is that of Steinberg [78], which provides a deep overview of each of the aforementioned issues and also presents former and modern techniques for simplifying the theory of characters of such monoids.

The immediate aim of this section is to offer fundamental insights into the theory of semigroup representations, and following this, to provide proofs of several of the related assertions commonly encountered in the literature. We mostly follow [26, Chapter 11].

Definition 4.1.1. [26] Let S be a finite regular monoid and V be a finite dimensional vector space over $\mathbb{C}$. Let $End(V)$ be the monoid of all linear maps on V. An S-representation, or a linear representation of S, is a monoid homomorphism $\psi : S \longrightarrow End(V)$; that is, for any $s \in S$, we have that $(s)\psi \in End(V)$ is a linear map.

We call dim V the degree of ψ. In the following remark, we highlight some assertions which contribute to a better understanding of this representation.

Remark 4.1.2.

(i) In the above definition, S can be any semigroup, which means that it is not strictly necessary to have the terms "monoid", "finiteness", and "regularity" in the definition of a semigroup representation. However, we insist on adding such terms to make the statement of results cleaner.

(ii) The elements of $End(V)$, like those of S, need not have inverses. If S is a group, then the images $(s)\psi$ are invertible as ψ is a homomorphism; hence, $(s)\psi$ lies in $GL(V)-$ a group of invertible linear maps. In particular, if S is a group, the above definition reduces to that of a group representation.

(iii) As indicated in Chapter 1, a monoid homomorphism ψ requires

- For all $s, s' \in S$, $(ss')\psi = s\psi \cdot s'\psi$
- $1_S \xrightarrow{\psi} id_{End(V)}$; that is, ψ must send the identity element of a monoid S to the identity linear map $id : V \longrightarrow V$. Consequently, im $\psi \neq \mathbf{0}$.

(iv) Since S is regular, the regularity is preserved by a homomorphism ψ; that is, for all $s \in S$, there exists s' such that $ss's = s$ and $s'ss' = s'$. Thus,

$$(s)\psi = (ss's)\psi = s\psi \; s'\psi \; s\psi,$$

and then $t = tt't$ where $t = (s)\psi \in S\psi \subseteq End(V)$.

Henceforth, for any $v \in V$, we will abuse the notation $v(s)\psi$ and write $v \cdot s$ or vs instead. In other words, we can think of $s \in S$ as a linear map acting on the vectors of V. This indeed leads us to an alternative way of viewing the S-representation V as an S-module or $\mathbb{C}S$-module, as shown in the following definition.

Definition 4.1.3. [26] Let S be a finite regular monoid and V be a finite dimensional vector space over $\mathbb{C}$. Then, V is an S-module if there exists a mapping $V \times S \longrightarrow V$ such that $(v, s) \mapsto v \cdot s$ where $v \in V$ and $s \in S$, that satisfies the following conditions for all $v, u \in V$, $c \in \mathbb{C}$ and $s, s' \in S$:

(i) $v \cdot (s \cdot s') = (v \cdot s) \cdot s'$,

(ii) $v \cdot 1_S = v$, where 1_S is the identity of S,

(iii) $(cv) \cdot s = c(v \cdot s)$,

(iv) $(v + u) \cdot s = v \cdot s + u \cdot s$.

It should be noted that the above conditions (iii) and (iv) as well as $v \cdot s \in V$ confirm that for all $s \in S$, the function

$$v \longrightarrow v \cdot s$$

is an endomorphism of V.

The concepts of S-modules and S-representations are equivalent; thus, assertions and results associated with representations can be presented in terms of modules or endomorphisms. We frequently present such notions in the module theoretic language, although we occasionally state the equivalent assertions for S-representations. Throughout the remaining chapters, we may also abuse language and utilise the terminologies "representation of S" or "S-representation" when we mean S-module.

Definition 4.1.4. [26] Let V be an S-module. A subspace $U \subset V$ is called an S-submodule of V if and only if it is invariant under the action of S; that is, $u \cdot s \in U$ for all $u \in U$ and $s \in S$.

It is clear that the whole vector space V and zero vector space $\mathbf{0}$ are submodules of any S-module V.

Definition 4.1.5. [26] An S-module V is called simple (or irreducible S-representation of V) if and only if V is nonzero and the only submodules of V are V or $\mathbf{0}$. Otherwise, V is reducible.

The next example, presented in [18], demonstrates the concept of reducibility of certain monoid representations. We redraft it with more details.

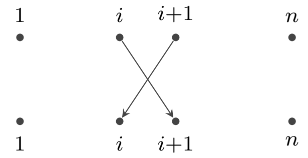
Example 4.1.6.

(i) Let S be the symmetric inverse monoid I_n with $n > 1$, and let V be a finite dimensional vector space over $\mathbb{C}$ with basis $\{v_1, \ldots, v_n\}$. Define a linear map acting on V by partially permuting its basis vectors as follows: for all $\sigma \in I_n$,

$$v_i \cdot \sigma = \begin{cases} v_{i\sigma}, & i \in \text{dom } (\sigma), \\ \mathbf{0}, & \text{otherwise.} \end{cases} \tag{4.1}$$

Consider the following two subspaces of V and let us verify whether both can be I_n-submodules of V below.

(a) The subspace $U = Span_{\mathbb{C}} \{v_1 + v_2 + \cdots + v_n\}$. Consider a partial map $\sigma_i \in I_n$ that permutes two consecutive numbers of $\{1, \dots, n\}$ and is undefined elsewhere; that is,

$\sigma_i =$ 

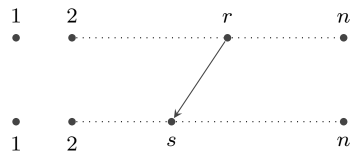
Then $U \times I_n \not\subseteq U$ because

$$(v_1 + \cdots + v_i + v_{i+1} + \cdots + v_n) \cdot \sigma_i = 0 + \cdots 0 + v_{i+1} + v_i + 0 + \cdots + 0 \notin U.$$

Hence, U is not an I_n-submodule.

(b) The hyperplane W consisting of the vectors $\{w = \sum_{i=1}^{n} c_i v_i \in V : \sum_{i=1}^{n} c_i = 0\}$ is also not an I_n-submodule because, if we apply σ_i $(i > 1)$ to the vector $v_i - v_1$, we obtain $(v_i - v_1) \cdot \sigma_i = v_{i+1} \notin W$.

Moreover, the I_n-module V is simple for the following reason: suppose that $W' \subseteq V$ is a nonzero I_n-submodule, we show that $W' = V$. Since $W' \neq \mathbf{0}$, then there exists a vector $w' \in W'$ where $w' \neq \mathbf{0}$; that is, $w' = \sum_{i=1}^{n} c_i v_i$ with $c_r \neq 0$ for some r. Consider the following partial map $\sigma_s \in I_n$:

$\sigma_s =$ 

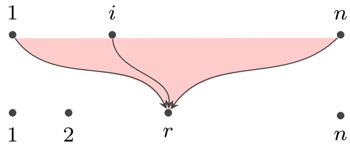
where $1 \leq s \leq n$. As W' is an I_n-submodule, applying σ_s to w' results in $w' \cdot \sigma_s = c_r v_s \in W'$; then, $v_s = \frac{1}{c_r}(w' \cdot \sigma_s) \in W'$. However, since the above observation holds for all s where $1 \leq s \leq n$ and $\{v_s : 1 \leq s \leq n\}$ forms the basis for V, we conclude that $W' = V$. This means that the only I_n-submodules of V are the trivial one and the whole space V. Thus, V is a simple I_n-module called the partial permutation module.

(ii) Let S be the monoid of transformation maps T_n with $n > 1$, and V be a $\mathbb{C}$-vector space with basis $\{v_1, \dots, v_n\}$. Define a linear map acting on the basis vectors of V in the following manner: for all $\sigma \in T_n$,

$$v_i \cdot \sigma = v_{i\sigma}. \tag{4.2}$$

Consider the previous two subspaces of V :

(a) The subspace $U = Span_{\mathbb{C}} \{v_1 + v_2 + \cdots + v_n\}$. Consider the following map $\sigma \in T_n$:

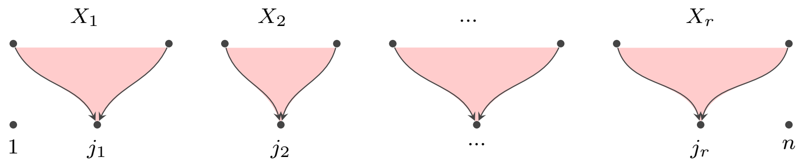
$\sigma =$ 

This means that if we fix $r \in [n]$, then $(i)\sigma = r$, for all $i \in [n]$. Thus,

$$(v_1 + v_2 + \cdots + v_n) \cdot \sigma = v_r + v_r + \cdots + v_r = nv_r \notin U.$$

Hence, the subspace U is not a T_n-submodule of V.

(b) The hyperplane W consisting of $\{w = \sum_{d=1}^{n} c_d v_d \in V : \sum_{d=1}^{n} c_d = 0\}$ is indeed a T_n-submodule of V for the following reason: let $w \in W$, and a map $\sigma \in T_n$ be given:

$\sigma =$ 

where $\mathrm{im}(\sigma) = \{j_1, j_2, \ldots, j_r\} \subset [n]$ and the $\mathrm{dom}(\sigma)$ is a disjoint union of the preimages X_i of j_i with $1 \leq i \leq r$. In other words,

$$\mathrm{dom}(\sigma) = [n] = \bigcup_{i=1}^{r} X_i,$$

where $X_i = \{x \in [n] : (x)\sigma = j_i\}$. The image of w under σ is obtained by

$$w \cdot \sigma = \Big(\sum_{i \in X_1} c_i\Big) v_{j_1} + \cdots + \Big(\sum_{i \in X_r} c_i\Big) v_{j_r},$$

where the sum of its coefficients is

$$\sum_{i \in X_1} c_i + \cdots + \sum_{i \in X_r} c_i = \sum_{d=1}^{n} c_d = 0.$$

Thus, $w \cdot \sigma \in W$, so W is a T_n-submodule of V. Hence, a T_n-module V is not simple and is called a mapping module of T_n.

In fact, W turns out to be a simple T_n-submodule, and to show such an assertion, let us state the result below.

Claim: Let S be a monoid and S' be a submonoid of S. If V is an S-module, then restricting the S-action on V to S' yields that V is an $S'-$module.

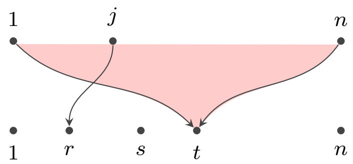
Proof. For every $v \in V$ and $s' \in S'$, we have $v \cdot s' \in V$, as $S' \leq S$ and V is an S-module. Moreover, as S' a submonoid of S, it has the same identity 1 of S, and then $v \cdot 1 = v \in V$. It is also clear that for all $s'_1, s'_2 \in S'$, we have $v \cdot (s'_1 s'_2) = (vs'_1)s'_2 \in V$. Thus, $V \times S' \longrightarrow V$, and hence the desired result holds. $\square$

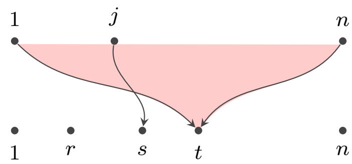
Let us now show that W is a simple T_n-module by exploiting the fact that W has no proper S_n-submodules. In view of the above assertion, since W is a T_n-module, it can also be thought as an S_n-module. Moreover, if W' were a T_n-submodule of T_n-module W, then by applying the above assertion again, we would obtain that W' is an S_n-submodule of the S_n-module W. This means that any proper subspace W' of W, which is invariant under the T_n-action, is certainly invariant under the S_n-action ($S_n \subset T_n$). However, W is a simple S_n-module as shown in Example 3.1.7(b); that is, there is no such proper S_n-submodule of W. Hence, W is a simple T_n-module.

It is also worthwhile noticing that if $n > 2$, the T_n-module V contains no 1-dimensional T_n-submodule: Consider a nonzero T_n-submodule V' of V. Then, there must exist a nonzero element $v' \in V'$ such that

$$v' = c_1v_1 + \cdots + c_jv_j + \cdots + c_nv_n,$$

for some $c_j \neq 0$. Fix $r, s, t \in \{1, \cdots, n\}$ with $r < s < t$ and let σ_r, σ_s be T_n-maps such that

$\sigma_r =$ 

$\sigma_s =$ 

Now, by applying σ_r and σ_s to v', we obtain

$$\begin{aligned} v' \cdot \sigma_r &= c_1v_{(1)\sigma_r} + \cdots + c_jv_{(j)\sigma_r} + \cdots + c_nv_{(n)\sigma_r} \\ &= c_jv_r + lv_t. \end{aligned}$$

$$\begin{aligned} v' \cdot \sigma_s &= c_1v_{1\sigma_s} + \cdots + c_jv_{j\sigma_s} + \cdots + c_nv_{n\sigma_s} \\ &= c_jv_s + lv_t, \end{aligned}$$

where $l = c_1 + \cdots + \hat{c_j} + \cdots + c_n$. Notice that $v' \cdot \sigma_r$ and $v' \cdot \sigma_s$ belong to V' as it is a T_n-submodule. Let us verify that both vectors are linearly independent from each other. Assume that

$$\begin{aligned} c_1(v' \cdot \sigma_r) + c_2(v' \cdot \sigma_s) &= \mathbf{0} \\ c_1(c_jv_r + lv_t) + c_2(c_jv_s + lv_t) &= \mathbf{0} \end{aligned}$$

for some $c_1, c_2 \in \mathbb{C}$ Now, by dividing the r-th entry and s-entry in $v' \cdot \sigma_r$ and $v' \cdot \sigma_s$ respectively by c_j, we have $c_1 v_r + c_2 v_s + ((c_2 + c_2)\frac{l}{c_j})v_t = \mathbf{0}$. It follows that $c_1 = c_2 = 0$. Hence, both vectors are independent and so V' is at least a two-dimensional T_n-submodule.

In view of Definition 3.1.10, we can construct a new S-representations as explained below.

Definition 4.1.7. [74] Let V be a S-module, and suppose that as vector spaces

$$V = \bigoplus_i V_i, \text{ with } 1 \leq i \leq m,$$

where V_i are S-submodules of V. Then, V is called a direct sum of submodules.

Definition 4.1.8. An S-module V is said to be semisimple (or a completely reducible S-representation) if it is a direct sum of simple submodules (or irreducible subrepresentations) V_i with $1 \leq i \leq m$; that is,

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_m.$$

A map that preserves group structures is called a group homomorphism; the corresponding notion for S-representations is called an S-homomorphism, which is defined as given below.

Definition 4.1.9. Let V and V' be two S-modules. A function $\varphi : V \longrightarrow V'$ is said to be an S-homomorphism if φ is a linear map and

$$(vs)\varphi = (v\varphi)s.$$

In other words, φ is a linear map such that for all $s \in S$, the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "(\text{-})s"] \arrow[d, "\varphi"'] & V \arrow[d, "\varphi"] \\ V' \arrow[r, "(\text{-})s"'] & V' \end{tikzcd}$$

We write $\mathrm{Hom}_S(V, V')$ for the set of all S-homomorphisms from V to V', and it is clear that it forms a vector space over $\mathbb{C}$ by defining $\varphi + \varphi'$ and $c\varphi$ in the following manner:

$$v(\varphi + \varphi') = v\varphi + v\varphi',$$
$$(cv)\varphi = c(v\varphi),$$

for all $v \in V$, $c \in \mathbb{C}$ and $\varphi, \varphi' \in \mathrm{Hom}_S(V, V')$. Moreover, φ is an isomorphism if and only if it is invertible.

Proposition 4.1.10. [26] *Let V and V' be two S-representations and $\varphi \in Hom_S(V, V')$. Then,*

(i) *The kernel $Ker(\varphi)$ of φ is an S-submodule of V.*

(ii) *The image $im(\varphi)$ of φ is an S-submodule of V'.*

(iii) *If both V and V' are simple S-modules, then φ is either an S-isomorphism or zero.*

The proof of the above proposition can be found in [26, Proposition 11.1.7]. It is also worth noticing that the assertion mentioned in (iii) is the version of Schur's lemma for a monoid representation.

Proposition 4.1.11. *Let S be a (finite regular) monoid and V be an S-representation, and write*

$$V = V_1 \oplus \cdots \oplus V_m,$$

a direct sum of irreducible S-subrepresentations V_i, where $1 \leq i \leq m$. If V' is an irreducible subrepresentation of V, then $V' \cong V_i$ for some i.

Proof. Suppose $v' \in V' \subset V$; thus, v' can be expressed uniquely as $v_1 + \cdots + v_m$, where $v_i \in V_i$. Define a map $\beta_i : V' \longrightarrow V_i$ such that $v'\beta_i = v_i$. If we choose i with $v_i \neq \mathbf{0}$ for some $v' \in V'$, we obtain a nonzero map β_i. Notice that since V' is an S-subrepresentation of V, then for all $s \in S$, we have

$$(v's)\beta_i = [(v_1 + \cdots + v_m)s]\beta_i = (v_1 s + \cdots + v_m s)\beta_i = v_i s = (v'\beta_i)s.$$

Moreover, it is clear that β_i is linear; thus, β_i is an S-homomorphism. However, as both V' and V_i are irreducible and $\beta_i \neq 0$, β_i is an S-isomorphism by Proposition 4.1.10(iii). $\square$

Definition 4.1.12. Let S be a finite semigroup and $\mathbb{C}$ be the complex field. Define a vector space $\mathbb{C}S$ where the basis vectors are elements of S; that is, if $S = \{s_1, \ldots, s_n\}$, then let

$$\mathbb{C}S = \Big\{ \sum_{i=1}^{n} c_i s_i : \; c_i \in \mathbb{C} \Big\}$$

where

$$\sum_{i=1}^{n} c_i s_i + \sum_{i=1}^{n} c_i' s_i = \sum_{i=1}^{n} (c_i + c_i') s_i \quad \text{and} \quad c\Big(\sum_{i=1}^{n} c_i s_i\Big) = \sum_{i=1}^{n} (c c_i) s_i,$$

and dim $\mathbb{C}S = n$. Define a multiplication on $\mathbb{C}S$ in the following manner:

$$\Big(\sum_{s\in S} c_s s\Big)\Big(\sum_{s\in S} c_t t\Big) = \sum_{s,t\in S} c_s c_t (st),$$

where $c_s, c_t \in S$. Then $\mathbb{C}S$ is called the semigroup algebra of S over $\mathbb{C}$.

Definition 4.1.13. [22] Let S be a finite regular monoid and $\mathbb{K}$ be a field. Then the semigroup algebra $\mathbb{K}S$ is semisimple when every S-module V over $\mathbb{K}$ is completely reducible.

Theorem 4.1.14 (**Oganesyan** [61])**.** Let S be a finite inverse monoid and $\mathbb{K}$ be a field. Then, the semigroup algebra $\mathbb{K}S$ is semisimple if and only if the characteristic of $\mathbb{K}$ does not divide the order of any subgroup G of S.

The proof of this assertion can be found in [12, Theorem 5.26]; where it is pointed out that such a result was presented in [61]. Further, the above theorem is regarded as the analogous semigroup assertion to Maschke's theorem of finite group representations. It should be noted that it suffices to examine the semisimplicity condition, stated in the theorem, for maximal subgroups. In other words, for all subgroups $G \leq S$,

$$char(\mathbb{K}) \nmid |G| \iff char(\mathbb{K}) \nmid |H_e|\,,$$

for any maximal subgroup H_e of S. To prove this, we will show that if $char(\mathbb{K}) \nmid |H_e|\,$, then $char(\mathbb{K}) \nmid |G|$ by contraposition and the other direction is straightforward. Let $char(\mathbb{K}) = r$ and suppose that $r \mid |G|\,$. Since any subgroup G is contained in a maximal subgroup H_e for some idempotent e, the Lagrange theorem implies that $|G| \mid |H_e|\,$; thus, $r \mid |H_e|\,$. Hence, the proof of the contrapositive infers the original statement.

For instance, in the symmetric inverse monoid I_n, we have seen in Section 1.2 that all maximal subgroups are isomorphic to symmetric groups S_r where $1 \leq r \leq n$. Thus, an I_n-module V is semisimple if and only if $char(\mathbb{K}) \nmid r!$ for all $1 \leq r \leq n$. Consequently, an I_n-module V is semisimple if and only if $char(\mathbb{K}) \nmid n!\,$. Specifically, when the field $\mathbb{K}$ is the complex number $\mathbb{C}$, every I_n-module is semisimple.

In the full transformation monoid T_n, we cannot apply the theorem, as it is not an inverse monoid. However, with the consideration of a T_n-module V illustrated in Example 4.1.6(ii), a hyperplane space W is a simple T_n-submodule of V. Assume

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_m,$$

where V_i are simple T_n-submodules with $1 \leq i \leq m$. Then, by Proposition 4.1.11, $W \cong V_i$ for some i. Thus, V has to be decomposable with only two simple T_n-submodules; W with dimension $n-1$ and a V_j with dimension 1. Observe that if

$n > 2$, then we obtain a contradiction with the fact that there never exists a T_n-subspace of V with dimension 1. Consequently, V can never be semisimple, regardless of the field's characteristic.

4.2 The Clifford-Munn correspondence theorem

Drawing on the syntax of matrix representations and semigroup algebras, Clifford [12, Chapter 5], Munn [56, Section 2] and Ponizovskiĭ [65, Theorem 2.7] provided an account of the simple modules for a finite semigroup. Notably, this description, commonly referred to as the Clifford-Munn correspondence (CMC), is presently regarded as a landmark theorem for the representation of semigroups, and Munn [60] utilised the result to account for all characters of the symmetric inverse semoigroup I_n. As stated in the previous section, Munn's research was then built on by Rhodes and Zalcstein [72] in the following years, both of whom made striking progress in illustrating the notion of a representation of a monoid utilising a linear theoretic terminology.

Ganyushkin and Mazorchuk [26] utilised the CMC in 2008 to derive all simple modules for finite transformation semigroups. In addition, these authors collaborated with Steinberg [27] to publish a brief proof of the CMC, which relied on the use of a free-standing description of the theory of simple modules over the semigroup algebra of a finite semigroup. This research, which drew exclusively on the instruments of associative algebras, was especially noteworthy because it enabled general mathematicians to interact with the results in a way that had previously not been easily possible.

With [18] used as a scaffolding, this section now turns to an elementary demonstration of this theorem. To promote clearness and straightforwardness, we separate our analysis of the CMC into three parts: first, we discuss the reduction process (often called the restriction process), where we examine how to determine the apex of an I_n-partial permutation module; second, we investigate the induction process, where we prove some assertions in the inverse case and give a pair of examples; finally, we present the CMC. Henceforth, S is a finite inverse monoid unless otherwise specified.

4.2.1 The reduction process

The aim of the this subsection is to describe the technical process, which contributes to constructing a representation of a maximal subgroup from an S-representation.

Let V be a simple S-module. Consider the poset of $\mathcal{J}$-classes of S, a generic picture is shown in Figure 4.1.

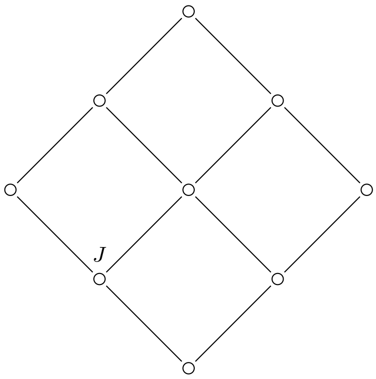


Figure 4.1: Generic picture of $\mathcal{J}$-classes of a finite regular monoid S.

Fix an idempotent e belonging to a $\mathcal{J}$-class J and consider the maximal subgroup G_e with $e \in G_e$. To construct a representation of G_e, we need the following two processes:

(a) Consider the $\mathbb{C}$-subspace $Ve = \{v \cdot e : v \in V\}$ of V.

(b) Define the action of G_e on Ve in the following manner: for all $g \in G_e$ and a vector $v \cdot e \in Ve$,

$$(v \cdot e) \cdot g := v \cdot (eg). \tag{4.3}$$

Such an action leaves the vector space Ve invariant for the following reason:

$$\begin{aligned}(v \cdot e) \cdot g &= v \cdot (eg) && \\ &= v \cdot (ge) && [\text{As } e \text{ is the identity of } G_e] \\ &= (vg) \cdot e && [\text{As } V \text{ is an } S\text{-module and } e, g \in G_e \subset S] \\ &\in Ve. &&\end{aligned}$$

Hence, Ve is a G_e-module. It is also crucial to point out that, up to this point, we have no clue as to whether the produced G_e-module Ve is simple and whether it is a nonzero module or null module. This completes the steps required in the reduction process.

Recall from Chapter 1 that if two idempotents e and f belong to the same $\mathcal{J}$-class J, then in view of Theorem 1.2.6(ii), we know that there exist a and a^* such that

$$a^*a = e, \quad aa^* = f, \quad fa = ae = a, \text{ and } \quad a^* = a^*f = ea^*. \tag{4.4}$$

Moreover, Proposition 1.2.7 and its proof indicate that the maximal subgroups G_e and G_f are isomorphic, and an isomorphism $\gamma : G_f \longrightarrow G_e$, is given by $h \mapsto a^*ha$ for all $h \in G_f$. The following proposition states that for all idempotents e belonging to the same $\mathcal{J}$-class J, the G_e-modules Ve are equivalent up to isomorphism.

Proposition 4.2.1. *For any two idempotents $e, f \in \mathcal{J}$-class J, we have $Ve \cong Vf$.*

Proof. Define a map $\varphi : Vf \longrightarrow Ve$ via $(v \cdot f)\varphi = v \cdot (fa)$, where a is as in (4.4), and

$$v \cdot (fa) = v \cdot (ae) = (v \cdot a) \cdot e \in Ve.$$

Notice that φ is linear because for all $v, v' \in V$, we have

$$\begin{aligned}
(v \cdot f + v' \cdot f)\varphi &= ((v + v') \cdot f)\varphi \qquad \text{[As } V \text{ is an } S\text{-module and } f \in S] \\
&= (v + v') \cdot (fa) \\
&= v \cdot (fa) + v' \cdot (fa) \\
&= (v \cdot f)\varphi + (v' \cdot f)\varphi
\end{aligned}$$

Similarly, $(cv \cdot f)\varphi = c(v \cdot f)\varphi$, for all $c \in \mathbb{C}$. Let us show that the diagram below commutes

$$\begin{tikzcd}
Vf \arrow[r, "(\text{-})h"] \arrow[d, "\varphi"'] & Vf \arrow[d, "\varphi"] \\
Ve \arrow[r, "(\text{-})a^*ha"'] & Ve
\end{tikzcd}$$

If $v \cdot f \in Vf$ and $h \in G_f$, then

$$\begin{aligned}
((v \cdot f) \cdot h)\varphi &= (v \cdot (f \cdot h))\varphi \\
&= (v \cdot (h \cdot f))\varphi \qquad \text{[As } f \text{ is the identity of } G_f] \\
&= ((v \cdot h) \cdot f)\varphi \\
&= (v \cdot h) \cdot fa \\
&= (v \cdot h) \cdot ae \\
&= (v \cdot ha) \cdot e. \qquad (4.5)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
((v \cdot f)\varphi) \cdot a^*ha &= (v \cdot fa) \cdot a^*ha \\
&= v \cdot (faa^*ha) \\
&= v \cdot (ffha) \\
&= v \cdot (fha) \\
&= v \cdot (hfa) \\
&= v \cdot (hae) \\
&= (v \cdot ha) \cdot e. \qquad (4.6)
\end{aligned}$$

Therefore, the equality of (4.5) and (4.6) implies that the above diagram is commutative. Let us construct the inverse map $\psi : Ve \longrightarrow Vf$ via $(v \cdot e)\psi = v \cdot (ea^*)$, where a is as in (4.4), and

$$v \cdot (ea^*) = v \cdot (a^* f) = (v \cdot a^*) \cdot f \in Vf.$$

We will show that $\varphi\psi = id_{_{Vf}}$ and $\psi\varphi = id_{_{Ve}}$ where $id_{_{Vf}}$ and $id_{_{Ve}}$ are the identity maps on Vf and Ve respectively. For all $v \in V$, we have

$$\begin{aligned}
(v \cdot f)\varphi\psi &= (v \cdot (fa))\psi \\
&= (v \cdot (ae))\psi && \text{[By (4.4)]} \\
&= ((v \cdot a) \cdot e)\psi \\
&= (v \cdot a)ea^* \\
&= v \cdot (ae)a^* \\
&= v \cdot (fa)a^* && \text{[By (4.4)]} \\
&= v \cdot faa^* \\
&= v \cdot ff && [\text{As } a \ \mathcal{R} \ f \iff aa^* = f \text{ by Proposition 1.2.13}] \\
&= v \cdot f.
\end{aligned}$$

Similarly, $\psi\varphi = id_{_{Ve}}$. Hence, φ is an isomorphism. □

Definition 4.2.2. [27] Let S be a semigroup and V be an S-module. Then, the set

$$Ann_S(V) = \{s \in S : Vs = \mathbf{0}\},$$

is called the annihilator of V.

Notice that for all $t, t' \in S$ and $s \in Ann_S(V)$, we have

$$V(tst') = Vt\ (st') \subseteq V\ (st') = (Vs)\ t' \subseteq \mathbf{0}\ t' = \mathbf{0}.$$

Thus, $tst' \in Ann_S(V)$ and hence, $Ann_S(V)$ is an ideal.

Proposition 4.2.3. *If s and t are $\mathcal{J}$-related, then $Vs = \mathbf{0}$ if and only if $Vt = \mathbf{0}$.*

Proof. Since $s \ \mathcal{J} \ t$, there are a, a' and b, b' such that $s = ata'$ and $t = bsb'$. Suppose $Vs = \mathbf{0}$; then,

$$Vt = V(bsb') = Vb\ (sb') \subseteq V\ (sb') = (Vs)\ b' = \mathbf{0}\ b' = \mathbf{0}.$$

Similarly, we can show the other direction and then the result holds. □

Remark 4.2.4. In view of the above proposition, either the entire $\mathcal{J}$-class of s is contained in $Ann_S(V)$ or it is completely disjoint from $Ann_S(V)$. Thus, we can alternatively describe $Ann_S(V)$ to be the union of all $\mathcal{J}$-classes of s where $Vs = \mathbf{0}$.

Recall from Section 1.1 that $J(s) = S^1sS^1$ and if we fix a $\mathcal{J}$-class J of S, then $[J]^{\ngeq} = \{s \in S : J \nsubseteq J(s)\}$ is an ideal. It is the union of all $\mathcal{J}$-classes J'' that are not greater than or equal to J. The definition below appears in [56, Section 2] and plays a vital role in the representation theory of semigroups.

Definition 4.2.5 (Apex [27]). Let V be an S-module. A $\mathcal{J}$-class J is said to be the apex of V if $Ann_S(V) = [J]^{\ngeq}$.

The following proposition gives a vital alternative characterisation of the apex of an S-module V that has not been proved in either [27] or [52]. We provide its proof below.

Proposition 4.2.6. *An S-module V has apex J if and only if J is the unique minimal $\mathcal{J}$-class, with respect to the order of $\mathcal{J}$-classes, that does not annihilate V.*

Proof. Suppose J is the apex of an S-module V. We will prove that J is the unique minimal $\mathcal{J}$-class that does not annihilate V. Let X be the set of all $\mathcal{J}$-classes $\bar{J}$ that do not annihilate V; that is, $X = \{\bar{J} : \ Vs \neq \mathbf{0} \ \text{ for some } s \in \bar{J}\}$. Since $J \notin [J]^{\ngeq}$, then $J \notin Ann_S(V)$ as J is the apex. Thus, J does not annihilate V, and hence, $J \in X$ and $X \neq \emptyset$. Let $\tilde{J} \in X$ such that $\tilde{J} \neq J$. It follows that $\tilde{J} \notin Ann_S(V)$. Hence, $\tilde{J} \notin [J]^{\ngeq}$; that is, $\tilde{J}$ does not belong to the union of all $\mathcal{J}$-classes that are not greater than or equal to J. In other words, whenever we choose an arbitrary $\tilde{J} \in X$ where $\tilde{J} \neq J$, we obtain $\tilde{J} > J$. Therefore, J is a minimal $\mathcal{J}$-class with respect to $\mathcal{J}$-classes in X. The uniqueness of J remains to be shown. Suppose that J and J' are both minimal $\mathcal{J}$-classes within X. Since J is minimal, then it is less than or equal to all $\mathcal{J}$-classes in X. Specifically, $J' \geq J$. Similarly, $J \geq J'$ as J' is minimal. Thus, both must be equal to each other. Hence, J is the unique minimal $\mathcal{J}$-class in X.

Conversely, suppose J is the unique minimal $\mathcal{J}$-class that does not annihilate V. We will show that J is the apex of V; that is $Ann_S(V) = [J]^{\ngeq}$. If $s \in Ann_S(V)$, then, $Vs = \mathbf{0}$. Thus, $s \notin J$. In other words, $s \in J'$ where $J' \neq J$. In view of Remark 4.2.4, the $\mathcal{J}$-class J' annihilates V. It follows that $J' \ngeq J$ because of the assumption on J. Hence, $J' \in [J]^{\ngeq}$ by Proposition 1.1.21. Specifically, $s \in [J]^{\ngeq}$, thus, $Ann_S(V) \subseteq [J]^{\ngeq}$. Conversely, if $s \in [J]^{\ngeq}$, then s belongs to $\mathcal{J}$-class J' such that $J' \ngeq J$. However, since J is the minimal $\mathcal{J}$-class that does not annihilate V, then J' must annihilate V. In other words, $Vs = \mathbf{0}$ for $s \in J'$; thus, $s \in Ann_S(V)$. This means that $[J]^{\ngeq} \subseteq Ann_S(V)$, thus, we obtain the equality. Hence, J is the apex of V. □

The theorem below sheds some light on the existence of apexes.

Theorem 4.2.7. [27] Let S be a finite semigroup. Then, every simple S-module V has an apex.

The proof of the above assertion was given independently by Munn [58] and Ponizovskiĭ [66]; the reader is also referred to [78, Theorem 5.5 (ii)] and [27, Theorem 5] for the most recent proofs.

With a consideration of the above assertions, we observe that if we consider a simple S-module V and choose an idempotent representative e from each $\mathcal{J}$-class of S as well as apply the preceding two reduction steps, then we will come up with a list of G_e-modules Ve corresponding to each $\mathcal{J}$-class, as shown in the figure below.

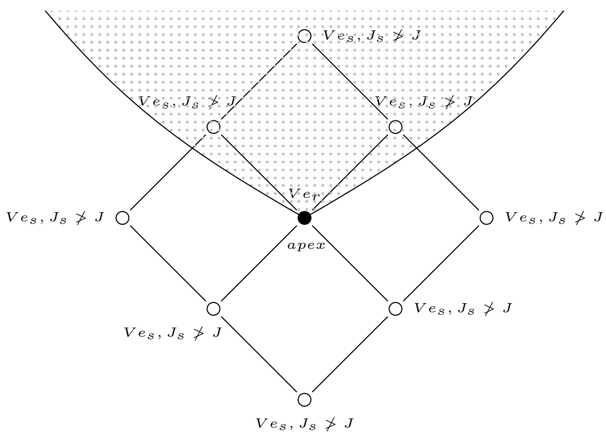


Figure 4.2: The apex $\mathcal{J}$-class J of S-module V.

We will also notice that there exists a unique minimal $\mathcal{J}$-class J $(= J_r)$ called the apex of V with $r \in J$ that satisfies the following conditions:

- Its idempotent representative e_r $(:= e \in J_r)$ determines a nonzero G_{e_r}-module Ve_r;
- For all other $\mathcal{J}$-classes J_s that are greater than the $\mathcal{J}$-class J (with respect to partial ordering $\leq_{\mathcal{J}}$), their idempotent representatives e_s yield nonzero G_{e_s}-modules Ve_s. In other words,

$$\text{For each } e_s \in J_s \text{ where } J_s > J, \text{ we have } Ve_s \neq \mathbf{0};$$

- For all remaining $\mathcal{J}$-classes J_s that are not greater than the $\mathcal{J}$-class J, their idempotent representatives e_s annihilate V. In other words,

$$\text{For each } e_s \in J_s \text{ where } J_s \not> J, \text{ we have } Ve_s = \mathbf{0}.$$

Hence, what has been discussed above suggests that an idempotent representative e_r belonging to the $\mathcal{J}$-class J can also identify the apex of V as stated in [78, Section 5.2], and is unique up to $\mathcal{J}$-equivalence. In fact, the apex J (or $e_r \in J$) of V plays a vital role in semigroup representation theory, as it associates a simple S-module V with a simple G_{e_r}-module Ve_r, as will see below.

Theorem 4.2.8. Let S be a finite inverse monoid. If V is a simple S-module with apex $e_r \in J$, then the G_{e_r}-module Ve_r is simple.

Proof. The approach we adopt to show the result is by using contraposition. Suppose the G_{e_r}-module Ve_r is not simple (i.e. it is reducible); this means that Ve_r contains a nonzero proper G_{e_r}-submodule U :

$$\mathbf{0} \neq U < Ve_r.$$

We will now work to find a nonzero proper S-submodule of V. Consider the $\mathcal{J}$-class J_r. Let I be the set of labels for all $\mathcal{R}$-classes (and $\mathcal{L}$-classes) in J_r and F be a set of representative h_i for each $\mathcal{H}$-class in R_{e_r}, the $\mathcal{R}$-class of e_r :

$$F = \{h_i : i \in I\},$$

where the idempotent $e := e_r$ itself is a representative of the $\mathcal{H}$-class in which G_e is placed. For simplicity, set $\mathcal{H}$-class $H_1 = G_e$.

	1		i		j	
1	$h_1 = e$ G_e	$\cdots$	h_i	$\cdots$	h_j	R_e
	$\vdots$	$\ddots$				
i	h_i^*		e_i			
	$\vdots$			$\ddots$		
j	h_j^*				e_j	

Fix $i \in I$ and consider a subspace of V denoted by $V_i := V \cdot (h_i^* h_i) = Ve_i$, where $e_i = h_i^* h_i$ is an idempotent. Observe that if $i = 1$, then $V_1 = Ve$. As J_r is the apex of V and $e_i \in J_r$, it follows $V_i \neq \mathbf{0}$ by Proposition 4.2.6 and Proposition 4.2.3. Now we claim that for any $i, j \in I$ with $i \neq j$, we have $V_i \bigcap \sum_{i \neq j} V_j = \mathbf{0}$. Since if $v_i \in V_i$, then $v_i = v \cdot e_i$ where $v \in V$. Thus,

$$v_i \cdot e_i = (v \cdot e_i) \cdot e_i = v \cdot e_i^2 = v \cdot e_i = v_i. \tag{4.7}$$

In other words, e_i restricted to a subspace $V_i \subseteq V$ fixes every vector in V_i; therefore, it is the identity linear map on V_i :

$$e_i|_{V_i} = \ id \ \text{ on } V_i.$$

Observe that the intersection of subspaces V_i and $\sum_{i\neq j} V_j$ of V is contained in V_i :

$$(V_i \cap \ \sum_{i\neq j} V_j) := K_i \ \subseteq V_i.$$

Thus, K_i is fixed by e_i as well. Hence,

$$\begin{aligned} K_i = K_i e_i &= (V_i \cap \ \textstyle\sum_{i\neq j} V_j)e_i \\ &\subseteq V_i e_i \cap \ (\textstyle\sum_{i\neq j} V_j)e_i \\ &= V e_i \cap \ \textstyle\sum_{i\neq j} V e_j e_i. \end{aligned} \tag{4.8}$$

Observe that $e_j e_i \in J_{e_j e_i} \leq J_{e_j}$; however, by Proposition 1.2.23, $e_j e_i \notin J_{e_j}$. It follows that the composition $e_j e_i$ belongs to a $\mathcal{J}$-class lower than the apex J_r when $i \neq j$. Therefore, (4.8) becomes

$$K_i \subseteq V e_i \cap \ \textstyle\sum_{i\neq j} V(e_j e_i) = V e_i \cap \mathbf{0} = \mathbf{0}, \tag{4.9}$$

for all i, as J_r is the minimal $\mathcal{J}$-class which does not annihilate V, and this makes the intersection $V_i \cap \ \sum_{i\neq j} V_j$ trivial. This completes the proof of the claim. In view of the result (4.9) and Definition 3.1.10, the sum of the subspaces V_i is direct:

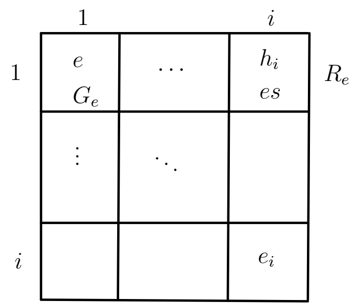
$$\textstyle\sum_{i\in I} V_i = \bigoplus_{i\in I} V_i \ \text{ inside } V.$$

It must be noted that the assumption that U is a G_e-submodule of Ve does not mean that it is also invariant under the S-action; this raises a question regarding the existence of the smallest S-subrepresentation of V which contains U. To answer this, let $\ddot{U}$ be the span of all possible images of all vectors in U under the S-action; that is

$$\ddot{U} := Span_{\mathbb{C}}\{u \cdot s \ : s \in S \text{ and } u \in U\}. \tag{4.10}$$

It is clear that $\ddot{U}$ is an S-invariant subspace of V. Moreover, U is contained in $\ddot{U}$. As $U \subseteq Ve \subseteq V$ and V is an S-module, it follows that $u \cdot 1 = u$ for all $u \in U$. In other words, every vector in U has the form $u \cdot 1$; thus, U consists of linear combinations of $u \cdot 1$ and $U \subseteq \ddot{U}$. Notice that if U' is an S-invariant subspace of V containing U, then $\ddot{U} \subseteq U'$ as if U' contains any $u \in U$, then it contains $u \cdot s$ since it is S-invariant. Moreover, U' is a subspace, so it contains the span of any of its vectors. Therefore, U' contains $\ddot{U}$. All the above discussions confirm that $\ddot{U}$ is the smallest S-submodule of V containing U. However, the existence of an S-submodule of V is not sufficient for asserting that V is reducible. There is another step to examine: $\ddot{U}$ is not all of V. Let us show that $\ddot{U}$ is a proper S-submodule of V.

Observe that as $U \leq Ve(= Ve_r)$, it follows that for all $u \in U$, $s \in S$ and $v \in V$, we have $u \cdot s = (v \cdot e) \cdot s = (v \cdot e^2) \cdot s = (v \cdot e) \cdot es = u \cdot es$. Now, if $es \notin R_e$, then by Proposition 1.1.20, $es \in J'$ where $J' < J_r$. Thus, $u \cdot s = u \cdot es = \mathbf{0}$ as J_r is the apex of V. However, if $es \in R_e$, then by Remark 1.1.28, $es = gh_i$ where $g \in G_e$ and $h_i \in H_{es}$. However, as $h_i \mathcal{L} e_i$ for all i, it follows that $h_i = h_i e_i$, and



$$u \cdot s = u \cdot es = u \cdot gh_i = u \cdot g(h_i e_i) = (u \cdot gh_i) \cdot e_i \in Ve_i = V_i.$$

Hence, we redraft the spanning set of $\ddot{U}$ appearing in (4.10) in the following manner. For all $s \in S$ and $u \in U$,

$$\ddot{U} = Span_{\mathbb{C}}\{u \cdot s \ : \ u \cdot s \in V_i \text{ for some } i\}. \tag{4.11}$$

As i varies, it follows that the spanning set elements $u \cdot s$ are in the direct sum of all possible V_i for all possible i :

$$u \cdot s \in V_i \subset \bigoplus_i V_i = \ V_1 \oplus \bigoplus_{i \neq 1} V_i. \tag{4.12}$$

Further, $es \in R_e$ implies that it can particularly be in any $\mathcal{H}$-class in R_e. If it occurs that $es \in H_1(= G_e)$, then $es = ge$ where $g, e \in G_e$. In other words, $es = g$ as the idempotent e is the identity of G_e. Thus,

$$u \cdot s = u \cdot es = u \cdot g \in U.$$

Since U is a G_e-submodule of $Ve(= V_1)$, we redraft (4.12) as given below:

$$u \cdot s \in U \oplus \bigoplus_{i \neq 1} V_i.$$

As the $u \cdot s$ spans $\ddot{U}$, we obtain

$$\ddot{U} \subseteq U \oplus \bigoplus_{i \neq 1} V_i. \tag{4.13}$$

However, U is a proper G_e-submodule of V_1, which means

$$U \oplus \bigoplus_{i \neq 1} V_i \subsetneq V_1 \oplus \bigoplus_{i \neq 1} V_i.$$

As the direct sum $V_1 \oplus \bigoplus_{i\neq 1} V_i$ is subspace of V, (4.13) becomes

$$\ddot{U} \subseteq U \oplus \bigoplus_{i\neq 1} V_i \subsetneq V_1 \oplus \bigoplus_{i\neq 1} V_i \subseteq V.$$

This proves that $\ddot{U}$ is a proper S-submodule of V. Hence, the proof of the contrapositive infers the original statement. Therefore, $Ve\ (= Ve_r)$ is a simple G_{e_r}-module. $\square$

We can conclude from the above theorems and earlier reduction process that the passage from the simple S-modules V to the simple modules Ve of the maximal subgroup representatives G_e is the reduction via the apex $e \in J$.

Definition 4.2.9 (reduced representation). Let S be a finite inverse monoid and V be an irreducible representation of S with apex J and $e \in J$ be an idempotent. Then, the reduced G_e-representation is obtained by

$$V \downarrow G_e = Ve.$$

We will illustrate the reduction procedure in the following example.

Example 4.2.10. Let S be the symmetric inverse monoid I_n and recall from Section 1.2 that the $\mathcal{J}$-classes J_l of I_n, where $1 \leq l \leq n$, can be ordered totally, as shown in Figure 4.3.

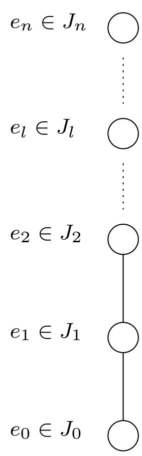


Figure 4.3: All $\mathcal{J}$-classs of I_n.

Let V be the I_n-partial permutation module with basis $\{v_1, \ldots, v_n\}$. In Example 4.1.6 (i), we showed that V is a simple I_n-module. Fix a J_l-class of I_n and choose an idempotent representative $e_l \in J_l$. Consider the maximal subgroup G_{e_l} to which the idempotent e_l belongs. Recall that the maximal subgroup G_{e_l} is isomorphic to the symmetric group S_X where $|X| = l$. Because of the isomorphism and notational simplicity, let $X = \{1, \ldots, l\}$ and notice that the idempotent e_l will fix X pointwise. We will apply the preceding reduction process, seeking to associate the simple I_n-module V with the corresponding module of a maximal subgroup which turns out to be simple, as illustrated below.

- Consider the $\mathbb{C}$-vector space $V_{e_l} = \{v \cdot e_l : v \in V\}$. In view of the I_n-action presented in the formula (4.1), a basis of V_{e_l} is the set $\{v_1, v_2, \ldots, v_l\}$, and hence $\dim(V_{e_l}) = l$.
- Such a vector space V_{e_l} with the action defined in (4.3) yields a G_{e_l}-module V_{e_l}. Since $G_{e_l} \cong S_X$, we have that V_{e_l} is isomorphic to the permutation module of S_X; this is not simple, as stated in Example 3.1.7.

Notice that considering an idempotent representative e_l for all $\mathcal{J}$-classes J_l of I_n where $0 \leq l \leq n$ and applying the above two steps result in a chain of G_{e_l}-modules V_{e_l} corresponding to each J_l-class, as appearing in Figure 4.4.

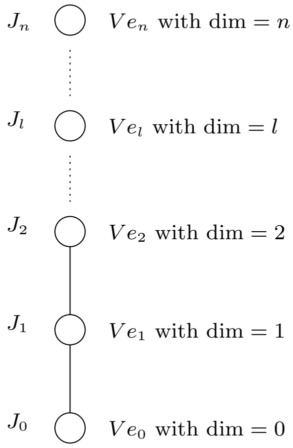


Figure 4.4: All G_{e_i} modules V_{e_i} corresponding to $\mathcal{J}$-class J_l.

Notice that all the following permutation modules $Ve_2, Ve_3, \ldots, Ve_n$ of the maximal subgroups $G_{e_2}, G_{e_3}, \ldots, G_{e_n}$ are neither null nor simple. However, Ve_1 is the nonzero simple module of G_{e_1} as it is one-dimensional, and V_{e_0} is the null module. Thus, it is clear that the J_1-class is the unique minimal $\mathcal{J}$-class that does not annihilate V. Therefore, using Proposition 4.2.6, the J_1-class is the apex of the partial permutation module V, as illustrated in Figure 4.5.

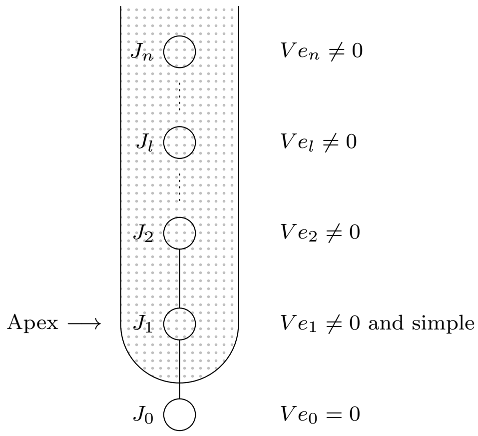


Figure 4.5: The apex of the I_n-module V.

Hence, the simple I_n-partial permutation module V with apex J_1 is affiliated with the simple module $V \downarrow G_{e_1} (= V_{e_1})$ of the maximal subgroup G_{e_1}.

4.2.2 The induction process

The aim of this subsection is to clarify the technical process that contributes to constructing a simple S-module out of a simple module of a maximal subgroup G. This represents an important process, as most of the results of the remaining chapters rely on this subsection.

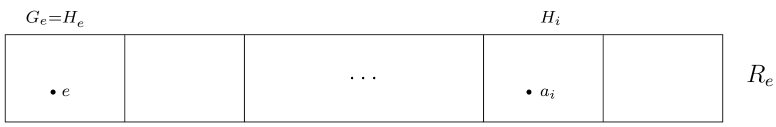
Let S be a finite inverse monoid, and consider the poset of $\mathcal{J}$-classes of S. Fix an idempotent $e \in \mathcal{J}$ and consider R_e, the $\mathcal{R}$-class of e to which the idempotent e and the maximal subgroup G_e belong. To construct an irreducible representation of S, we follow the process illustrated below.

(a) Consider a simple module U of the maximal subgroup G_e.

(b) Let I index the $\mathcal{H}$-classes in R_e and A be a set of representatives a_i of each $\mathcal{H}$-class of the $\mathcal{R}$-class of e :

$$A = \{a_i : i \in I\},$$

where the idempotent e itself is a representative of the $\mathcal{H}$-class in which G_e is placed. Note that each $\mathcal{H}$-class has precisely one representative a_i.



(c) Consider a copy of a simple G_e-module U for each $\mathcal{H}$-class as follows: for each $i \in I$, let U_i be a copy of U corresponding to an $\mathcal{H}$-class where a_i is its representative. Such a copy of U can be constructed in the following manner:

$$U_i = \{u_i := u \otimes a_i : u \in U \text{ and } a_i \in A\}. \tag{4.14}$$

In other words, the elements of such a copy U_i are of the form $u \otimes a_i$ where a_i is fixed and $u \in U$ varies. Note that each copy U_i carries a vector space structure via

$$u_i + u_i' = (u \otimes a_i) + (u' \otimes a_i) = (u + u') \otimes a_i$$
$$cu_i = c(u \otimes a_i) = (cu) \otimes a_i,$$

where $u, u' \in U$ and $c \in \mathbb{C}$. Moreover, dim U_i =dim U. The notation $- \otimes a_i$ is only a tool to indicate in which specific copy of U a vector lies, and serves no purpose apart from that.

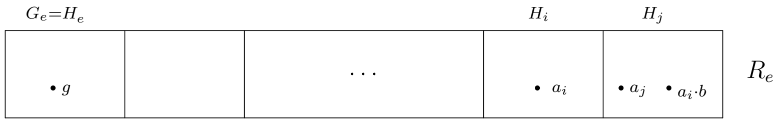
(d) Consider all the copies U_i and define the following vector space:

$$M = \bigoplus_{i \in I} U_i,$$

where a typical element of M is a formal sum $\sum_{i \in I} u_i$ with $u_i \in U_i$. We occasionally denote an element of M by $m = \sum_{i \in I} u_i$. Observe that

$$\dim \bigoplus_{i \in I} U_i = \sum_{i \in I} \dim U_i = \sum_{i \in I} \dim U = |I| \cdot \dim U.$$

(e) For all $b \in S$ and $m \in M$, we let S act on each component $u \otimes a_i$ of a vector m in the following manner. If $a_i \cdot b$ belongs to the $\mathcal{R}$-class of e, then it is worth identifying to which $\mathcal{H}$-class of R_e the element $a_i \cdot b$ belongs. Let $a_i \cdot b \in H_j$ and observe that by Remark 1.1.28, there exists a unique element $g \in G_e$ such that $a_i \cdot b = g \cdot a_j$ with $a_j \in A$.



Hence, we define

$$(u \otimes a_i) \cdot b = \begin{cases} (u \cdot g) \otimes a_j, & \text{if } a_i \cdot b \in R_e \text{ and } a_i \cdot b = g \cdot a_j, \text{with } a_j \in A, \\ 0 & \text{otherwise.} \end{cases} \tag{4.15}$$

Set $U \uparrow S := M$ and observe that $U \uparrow S$ is an S-module induced by a simple G_e-module U. This completes the steps required for the induction process.

The theorem below shows the irreducibility of the S-module $U \uparrow S$.

Theorem 4.2.11. Let S be a finite inverse monoid and U be a simple G_e-module. Then, $U \uparrow S$ is simple.

Proof. We will show the result by choosing any nonzero vector $m \in U \uparrow S$ and proving that any submodule of $U \uparrow S$ containing m must be $U \uparrow S$. Let U be a simple G_e-module and fix an arbitrary nonzero vector $m \in U \uparrow S$ so that $m = \sum_{i \in I} c_i u_i$, with $c_i \in \mathbb{C}$ and $\mathbf{0} \neq u_i \in U_i$ for all $i \in I$. Thus, $m \neq \mathbf{0}$ if a scalar $c_j \neq 0$ exists with $j \in I$. Let Q be the $\mathbb{C}$-span of all the images of the fixed vector m under the elements of S; that is,

$$Q = Span_{\mathbb{C}}\{\ m \cdot s\ \ : s \in S\ \}.$$

Observe that Q is an S-submodule of $U \uparrow S$ as S permutes the spanning set elements among themselves. As $m \cdot 1 = m$, it follows that $m \in Q$. For all $i \in I$, we have

$$a_i \cdot e_i = a_i, \tag{4.16}$$

where e_i is the idempotent belonging to $R_i \cap L_i$, as shown in Figure 4.6. For simplicity, we have re-ordered the $\mathcal{J}$-class so that if $i = 1$, then $e_1 = e$, $U_1 = U$ and $R_1 \cap L_1 = G_e$, the maximal subgroup.

	1		L_i	
1	e G_e	$\cdots$	a_i	R_e
	$\vdots$	$\ddots$		
R_i			e_i	

Figure 4.6: The square eggbox of the $\mathcal{J}$-class of e.

Now, we claim that for all $i \in I$, $u_i \cdot e_i = u_i$. Since $u_i \in U_i$, it has the form $u_i = u \otimes a_i$ with $u \in U$. This means that $u_i \cdot e_i = (u \otimes a_i) \cdot e_i$, but $a_i \cdot e_i = a_i = e \cdot a_i$, as a_i and e are $\mathcal{R}$-related. Therefore, using the S-action that appears in (4.15), we obtain

$$u_i \cdot e_i = (u \otimes a_i) \cdot e_i = (u \cdot e) \otimes a_i = u \otimes a_i = u_i$$

for all $i \in I$. This completes the proof of the claim.

Observe that if $i \neq j$ with $i, j \in I$, then by Proposition 1.2.23, the composition $e_i e_j$ does not belong to the $\mathcal{J}$-class $J_{e_i} (= J_e)$; in particular, $e_i e_j \notin R_e$. Hence,

$$\begin{aligned} m \cdot e_j &= \Big(\sum_{i \in I} c_i u_i\Big) \cdot e_j \\ &= \sum_{i \in I} c_i \; u_i \cdot e_j \\ &= \sum_{i \in I} c_i \; (u_i \cdot e_i) \cdot e_j \qquad \text{[As proved in the claim]} \\ &= \sum_{i \in I} c_i \; u_i \cdot (e_i e_j). \end{aligned} \tag{4.17}$$

Notice that $J_{(a_i e_i e_j)} \leq J_{e_i e_j} \leq J_{e_i} (= J_e)$, but $J_{e_i e_j} \neq J_e$ by Proposition 1.2.23. It follows that $J_{a_i e_i e_j} < J_e$; particularly, $a_i e_i e_j \notin J_e$. Hence, $a_i e_i e_j \notin R_e$. Now, let i run through I and apply the S-action presented in (4.15). Then, the equality (4.17) becomes

$$m \cdot e_j = c_j \; u_j \cdot e_j^2 = c_j \; u_j \cdot e_j = c_j \; u_j \neq \mathbf{0}. \tag{4.18}$$

As $m \cdot e_j \in Q$ and $c_j \neq 0$, it follows that $u_j \in Q$. Moreover, as $e \; \mathcal{R} \; a_j$, we have $ee^* = a_j a_j^*$; that is, $e = a_j a_j^*$. Since $u_j \in Q$ and Q is an S-submodule, we obtain,

$$u_j \cdot a_j^* = (u \otimes a_j) \cdot a_j^*$$

$$
\begin{aligned}
&= (u \cdot e) \otimes e && [\text{As } a_j a_j^* = e = e \cdot e, \text{then apply the action in (4.15)}]\\
&= u \otimes e && [\text{As } u \in U \text{ a } G_e\text{-module}]\\
&= u. && [\text{As } u \otimes e \in U]
\end{aligned}
$$

Thus, $u \in Q$ with $u \neq \mathbf{0}$ as $u_j \neq \mathbf{0}$, and consequently, $u \in U \bigcap Q$. Using the above results, let us now verify that $Q = U \uparrow S$. Recall that Q is an S-submodule of $U \uparrow S$; thus, it is a subspace

$$
Q \leq U \uparrow S = \bigoplus_{i \in I} U_i,
$$

Moreover, U $(= U_1)$ is a subspace of $U \uparrow S$, which implies that $U \bigcap Q$ is also a subspace. Since Q is an S-module and $G_e \subseteq S$, it follows that $Q \times G_e \subseteq Q$; thus, Q can also be viewed as a G_e-module. As U is a G_e-module, it follows that $U \bigcap Q$ is a G_e-submodule of U containing the nonzero vector u. However, U is simple, which means $U \bigcap Q = U$; hence, $U \subseteq Q$. In other words, the first copy of $U (= U_1)$ is contained in Q. We claim that all other copies U_i of U $(i \neq 1)$ are also contained in Q. To see this, observe that for all $i \in I$, we have $U a_i = U_i$, as for $u \in U$,

$$
\begin{aligned}
u \cdot a_i &= (u \otimes e) \cdot a_i && [\text{ As } U \text{ is a copy of itself }]\\
&= (u \cdot e) \otimes a_i && [\text{ As } e \cdot a_i = a_i \in R_e \text{ then use (4.15)}]\\
&= u \otimes a_i.
\end{aligned}
$$

As Q is an S-module, it follows that $U_i = U a_i \subset Q$ for any $i \in I$.

In view of the above result and the fact that Q is a submodule, we obtain $\bigoplus_{i \in I} U_i \subset Q$; that is, $U \uparrow S \subset Q$. Thus, $U \uparrow S = Q$. If W is a nonzero S-submodule of $U \uparrow S$ containing m, then W contains any vector $m \cdot s \in W$ with $s \in S$. It follows that the span of all $m \cdot s$ also belongs to W as W is a subspace.

$$
\mathrm{Span}_{\mathbb{C}}\{\ m \cdot s\ \ : s \in S\ \} \subseteq W.
$$

This implies that $Q = Span_{\mathbb{C}}\{\ m \cdot s\ \ : s \in S\ \} \subseteq W$; however, as $U \uparrow S = Q$, it follows that $U \uparrow S \subseteq W$. We obtained equality, which makes the S-module $U \uparrow S$ simple. $\square$

The following assertion was proved in [26, Proposition 11.2.1] and states that the construction of $U \uparrow S$ does not rely on the choice of the set of representatives A.

Proposition 4.2.12. *Let A and A' be two distinct sets of representatives of $\mathcal{H}$-classes in R_e. Then, the induced S-module $U \uparrow S$ arising from A is isomorphic to the induced S-module $\overline{U \uparrow S}$ arising from A'.*

Proposition 4.2.13. *Let e and f be two idempotents belonging to the same $\mathcal{J}$-class. Then, the S-module $U \uparrow S$ obtained by inducing a G_e-module U is isomorphic to the S-module $U' \uparrow S$ obtained by inducing a G_f-module U'.*

The reader can consult [26, Proposition 11.2.2] for proof of the proposition.

Definition 4.2.14 (induced representation). Let S be a finite inverse monoid, U be an irreducible representation of the maximal subgroup G_e of S, and U_i be the copies of U given by (4.14). Then, the S-representation $U \uparrow S$ obtained by inducing U is given by

$$U \uparrow S = \bigoplus_{i \in I} U_i,$$

and S acts on it, as presented in (4.15).

We illustrate the induction procedure in the example below.

Example 4.2.15. Let $S = I_n$ and J_1 be the $\mathcal{J}$-class consisting of all partial bijections between subsets of $[n] = \{1, \ldots , n\}$ with size 1; that is,

$$J_1 = \{d \in I_n : d : X \longrightarrow Y \text{ with } X, Y \subset [n] \text{ and } |X| = |Y| = 1\}.$$

Fix an idempotent $e : \{j\} \longrightarrow \{j\} \in J_1$ where $j \in [n]$, and consider R_e, the $\mathcal{R}$-class of e to which the idempotent e and G_e maximal subgroup belong. It is obvious that the maximal subgroup G_e is isomorphic to the trivial symmetric group S_1. Indeed, the induction process requires the steps outlined below.

(a) Consider a module U of a trivial group G_e and notice that U is a 1-dimensional module with basis $\{u\}$ where $u \cdot e = u$. Hence, U is simple.

(b) Let $A = \{a_k : \{j\} \longrightarrow \{k\}, k \in [n]\}$ be the set of representatives a_k of each $\mathcal{H}$-class in R_e, where the idempotent e is the representative of H_e. Notice that each $\mathcal{H}$-class in R_e has only one element; thus, there is no choice for the representatives a_k. Moreover, such a representative a_k has the fixed domain $\{j\}$ and the image $\{k\}$ varies with k:

	$\{j\}$			$\{k\}$		
$\{j\}$	G_e $e{:}\{j\}\longrightarrow\{j\}$		$\cdots$	$a_k{:}\{j\}\longrightarrow\{k\}$		R_e

(c) Consider a copy U_k of U for each $\mathcal{H}$-class as given below:

$$U_k = \{u \otimes a_k : u \in U \text{ and } a_k \in A\}.$$

Observe that dim U_k =dim $U = 1$ as $U_k \cong U$.

(d) Consider all the copies U_k and the vector space $\bigoplus\limits_{a_k \in A} U_k$. Observe that

$$\dim \bigoplus_{a_k \in A} U_k = \textstyle\sum_{a_k \in A} \dim U_k = \sum_{a_k \in A} \dim U = \underbrace{1 + \cdots + 1}_{\binom{n}{1}} = n \qquad (4.19)$$

(e) To determine the I_n-action, observe that for all $b \in I_n$, we have $a_k \cdot b \in R_e$ if and only if $\mathrm{dom}(a_k \cdot b) = \mathrm{dom}(e) = \{j\}$. This indeed occurs when k belongs to the domain of b, as discussed in Remark 1.2.21. In other words,

$$a_k \cdot b \in R_e \text{ if and only if } k \in \mathrm{dom}(b).$$

Thus, the partial map $a_k \cdot b$ can be described in the following manner:

$$a_k \cdot b = \begin{cases} a_{kb} : \{j\} \longrightarrow \{kb\}, & \text{if } \; k \in \mathrm{dom}(b), \\ 0 & \text{otherwise.} \end{cases}$$

Further, in view of Remark 1.1.28, we know that there is a unique element $e \in G_e$ such that $a_k \cdot b = a_{kb} = e \cdot a_{kb}$.

	$\{j\}$			$\{k\}$	$\{kb\}$	
$\{j\}$	G_e $e{:}\{j\}\longrightarrow\{j\}$		$\cdots$	$a_k{:}\{j\}\longrightarrow\{k\}$	$a_{kb}{:}\{j\}\to\{kb\}$	R_e

Hence, the I_n-action can be written as

$$(u \otimes a_k) \cdot b = \begin{cases} (u \cdot e) \otimes a_{kb}, & \text{if } \; a_k \cdot b \in R_e \text{ and } a_k \cdot b = e \cdot a_{kb}, \text{with } a_k \in A. \\ 0 & \text{otherwise.} \end{cases}$$

However, as $u \cdot e = u$; thus, the action of I_n becomes

$$(u \otimes a_k) \cdot b = \begin{cases} u \otimes a_{kb}, & \text{if } \; a_k \cdot b \in R_e \text{ and } a_k \cdot b = e \cdot a_{kb}, \text{with } a_k \in A. \\ 0 & \text{otherwise.} \end{cases}$$

Observe that the above formula is precisely the partial permutation action on the n-dimensional I_n-module $\bigoplus\limits_{a_k \in A} U_k$ with basis $\{u \otimes a_k : a_k \in A\}$, and its basis elements $u \otimes a_k$ are partially permuted by elements $b \in I_n$. Notice that we commenced with a simple module U for the trivial group G_e and by inducing it up into I_n, we obtained the simple I_n-module $U \uparrow I_n = \bigoplus\limits_{a_k \in A} U_k$.

4.2.3 The Clifford-Munn correspondence

In this subsection, we will illustrate the Clifford-Munn correspondence (CMC) that states that if S is a finite regular monoid, then the irreducible representations are in one-to-one correspondence with the irreducible representations of the subgroups $G_1, \dots, G_n$, which represent a set of maximal subgroup representatives of the $\mathcal{J}$-classes of the semigroup S. Such a fundamental assertion was developed by Clifford in his earlier results presented in [11] and followed by Munn in his series of papers [58, 59] and [56, Section 2], as well as Ponizovskiĭ [65, Theorem 2.7]. In fact, the semigroup S does not need to be inverse, although all our examples in the upcoming chapters are, and this greatly eases the acquisition of results since the annihilator turns to be trivial (see Proposition 4.2.22 and Theorem 4.2.16).

To state CMC, let $\mathcal{I}rr(S)$ be the set of isomorphism classes of simple S-modules and $\mathcal{I}rr_e(S)$ consist of all simple S-modules that have the apex e;

$$\mathcal{I}rr_e(S) = \{V \in \mathcal{I}rr(S) : \ V \text{ has apex } J \text{ with } e \in J\}.$$

Moreover, let $D = \{e_1, \dots, e_s\}$ be a complete list of idempotent representatives for the $\mathcal{J}$-classes of S. Then, for each representative idempotent $e \in D$, we have the irreducible G_e-modules of the maximal subgroup G_e. Let $\mathcal{I}rr(G_e)$ be the set of all simple G_e-modules. Recall from Section 4.2.2 that if U is a simple G_e-module, then we define a vector space

$$M = \bigoplus_{i \in I} U_i,$$

where U_i is a copy of U constructed as in (4.14). Let $Ann_{L_e}(M)$ be the set of the vectors in M that are annihilated by all elements of the $\mathcal{L}$-class L_e:

$$Ann_{L_e}(M) = \{\ m \in \ M \ : \ m \cdot b = \mathbf{0} \ \text{ for all } \ b \in L_e\}.$$

Theorem 4.2.16 (Clifford-Munn correspondence). Let S be a finite (regular) monoid. Then, for all $V \in \mathcal{I}rr_e(S)$, and $U \in \mathcal{I}rr(G_e)$:

(a) If V is a simple S-module with apex e, then $V \downarrow G_e = Ve$ is a simple G_e-module.

(b) If U is a simple G_e-module, then

$$Ann_{L_e}(M) = \{\ m \in \ M \ : \ m \cdot b = \mathbf{0} \ \text{ for all } \ b \in L_e\}$$

is the unique maximal S-submodule of M and

$$U \uparrow S = {}^{M}\!\big/_{Ann_{L_e}(M)}$$

is the unique simple S-module with apex e such that $(U \uparrow S) \cdot e = U$.

(c) There is a bijection between isomorphism classes $\mathcal{I}rr_e(S)$ of simple S-modules with apex e and isomorphism classes $\mathcal{I}rr(G_e)$ of simple G_e-modules; that is, for all $V \in \mathcal{I}rr_e(S)$, and $U \in \mathcal{I}rr(G_e)$,

$$\begin{array}{ccc} \mathcal{I}rr_e(S) & \xrightarrow[\text{bijection}]{\Phi} & \mathcal{I}rr(G_e), \\ V & \longmapsto & V \downarrow G_e \end{array} \qquad \begin{array}{ccc} \mathcal{I}rr(G_e) & \xrightarrow[\text{bijection}]{\Psi} & \mathcal{I}rr_e(S), \\ U & \longmapsto & U \uparrow S \end{array}$$

where Φ and Ψ are mutual inverses; hence, they are bijections between $\mathcal{I}rr_e(S)$ and $\mathcal{I}rr(G_e)$.

The proof of the theorem can be found in [72, Section 2] as well as [12, Theorem 5.33], and the reader can also consult [78, Theorem 5.5] and [27, Theorem 7] for the most recent proofs.

Below is an illustration of the above theorem with a few schematic pictures. Suppose that S is a finite regular monoid with $\mathcal{J}$-classes visualised as follows:

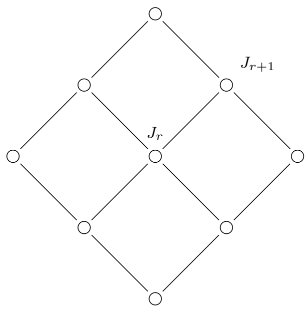


Fig A. Poset of $\mathcal{J}$-classes

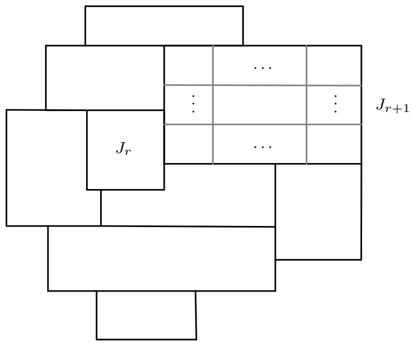
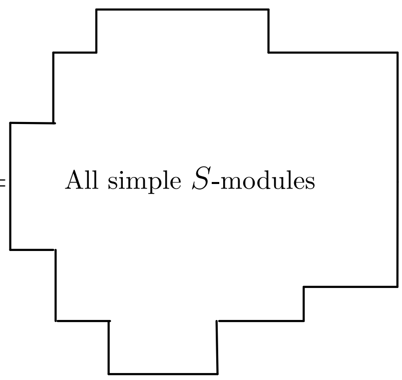


Fig B. Partition of S into $\mathcal{J}$-classes

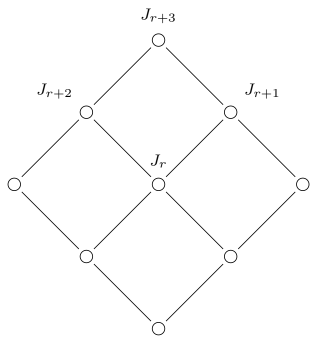
Consider the set $\mathcal{I}rr(S)$ of isomorphism classes of simple S-modules. We will present such a set as in the box below.

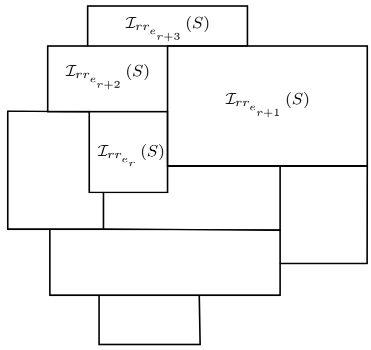
$\mathcal{I}rr(S) =$ 

By Theorem 4.2.7, we know that every simple S-module in $\mathcal{I}rr(S)$ has an apex, and such an apex e_r (or J_r) is unique (up to equivalence) by Proposition 4.2.6. Let us now split up the set $\mathcal{I}rr(S)$ into classes such that each class $\mathcal{I}rr_{e_r}(S)$ consists of all simple S-modules that have apex e_r :

$$\mathcal{I}rr_{e_r}(S) = \{\text{simple } S\text{-modules with apex } e_r\}.$$

We rearrange such classes in a manner that each class corresponds to the placement of their apexes in the poset $\mathcal{J}$-classes as follows:





Observe that the uniqueness of an apex for each simple S-module yields the $\mathcal{I}rr_{e_r}(S)$ partitions $\mathcal{I}rr(S)$. In other words, the set $\mathcal{I}rr(S)$ can be thought of as the disjoint union of all blocks $\mathcal{I}rr_e(S)$ where e is the idempotent representatives for the $\mathcal{J}$-class of S :

$$\mathcal{I}rr(S) = \bigcup_{e \in D} \mathcal{I}rr_e(S), \tag{4.20}$$

where $D = \{e : \ e \text{ is an idempotent representative for each } \mathcal{J}\text{-class}\}$.

Observe that the equation (4.20) gives us a clue that the key to understanding all the simple modules $\mathcal{I}rr(S)$ of a finite monoid S is to understand each individual component $\mathcal{I}rr_e(S)$ of the union. Moreover, Theorem 4.2.16 established the correspondence between the sets $\mathcal{I}rr_e(S)$ of all simple modules of a finite (regular) monoid S with apex e and the sets $\mathcal{I}rr(G_e)$ of all simple modules of the maximal subgroups G_e within S :

$$\mathcal{I}rr_e(S) \longleftrightarrow \mathcal{I}rr(G_e). \tag{4.21}$$

Hence, we conclude the result below.

Corollary 4.2.17. [78] *Let S be a finite (regular) monoid. Let $D = \{e_1, \ldots, e_s\}$ be a complete list of idempotent representatives for the $\mathcal{J}$-classes of S. Then, there is a one-to-one correspondence between $\mathcal{I}rr(S)$ and the disjoint union $\bigcup_{r=1}^{s} \mathcal{I}rr(G_{e_r})$.*

In view of the finiteness of S and disjointness of the union, we deduce the corollary below.

Corollary 4.2.18. *The number of simple S-modules is*

$$|\mathcal{I}rr(S)| = \sum_{r=1}^{s} |\mathcal{I}rr(G_{e_r})| \,.$$

The conclusion that can be drawn from the above discussion is that the irreducible representations of a finite regular monoid S are parametrised by the irreducible representations of the maximal subgroups. Thus, to construct all irreducible representations of S, it suffices to consider all irreducible representations of the maximal subgroups representative G_{e_r} of each $\mathcal{J}$-class J_r and apply the induction process to each individual one.

Stemming from Theorem 4.2.16, we observe that the S-module M presented in the preceding section is factored out by the annihilator $Ann_{L_e}(M)$. This implies that when S is a finite (regular) monoid, we require an extra step in the induction process to obtain the induced simple S-module $U \uparrow S$. We will now shed light on some preliminary results that contribute to proving that such an additional step is unnecessary if S is a finite inverse monoid.

Recall from step (b) of the induction process illustrated in Section 4.2.2 that $A = \{a_i : i \in I\}$ is the set of transversals a_i for all $\mathcal{H}$-classes in the $\mathcal{R}$-class R_e.

Observation 4.2.19. Let S be an inverse monoid and fix a representative $a_k \in R_e$ for some $k \in I$; then, we have $a_k a_k^* = ee^* = e$ by Proposition 1.2.13. Hence,

$$a_k^* = a_k^* a_k a_k^* = a_k^* e. \tag{4.22}$$

This implies that a_k^* and e are $\mathcal{L}$-related; that is, $a_k^* \in L_e$.

Proposition 4.2.20. *Let S be an inverse monoid and $a_i, a_k \in A$ be representatives of two $\mathcal{H}$-classes in R_e, and a_k^* be the inverse of a_k. Then,*

$$a_i a_k^* \in R_e \textit{ if and only if } \; a_i = a_k.$$

Proof. If $a_i = a_k$, then $a_i a_k^* = a_i a_i^*$. Since $a_i \in R_e$, then by Proposition 1.2.13, $a_i a_i^* = e$. Thus, $a_i a_k^* \in R_e$. Conversely, suppose that $a_i a_k^* \in R_e$, and let us show that $a_i = a_k$. In other words, we aim to show that a_i and a_k represent the same $\mathcal{H}$-class. As a_i and a_k are $\mathcal{R}$-related by assumption, it suffices to show that a_i and a_k are $\mathcal{L}$-related. Observe that

$$\begin{aligned}
a_i &= e a_i \\
&= (a_i a_k^* a_k a_i^*) a_i && \text{[By } a_i a_k^* \in R_e \text{ and Proposition 1.2.13]} \\
&= a_i (a_k^* a_k)(a_i^* a_i) \\
&= a_i (a_i^* a_i)(a_k^* a_k) && \text{[As idempotents commutes]} \\
&= (a_i a_k^*) a_k.
\end{aligned}$$

Thus, $a_i = ra_k$ where $r = a_i a_k^*$. Similarly, by interchanging a_i and a_k, we obtain $a_k = sa_i$ where $s = a_k a_i^*$. Consequently, $a_i \mathcal{L} a_k$, and then both a_i and a_k are $\mathcal{H}$-related. As there is only one representative for each $\mathcal{H}$-class in R_e, we have $a_i = a_k$. $\square$

Corollary 4.2.21. *$a_i a_k^* \notin R_e$ if and only if $a_i \neq a_k$.*

For notational simplicity, we redraft the elements of M as $\sum_{i \in I} w_i \otimes a_i$ where each term belongs to a copy U_i. The assertion below suggests that for a finite inverse monoid, the annihilator $Ann_{L_e}(M)$ of M, presented in the above theorem, is trivial.

Proposition 4.2.22. *Let S be a finite inverse monoid. Then, $Ann_{L_e}(M) = \{\mathbf{0}\}$.*

Proof. Let $m \in Ann_{L_e}(M)$; that is, $m = \sum_{i \in I} w_i \otimes a_i \in M$ and for all $b \in L_e$, we have $m \cdot b = \mathbf{0}$. For any representative $a_k \in R_e$, we have $a_k^* \in L_e$. Hence,

$$\begin{aligned}
\mathbf{0} &= m \cdot a_k^* \\
&= \Big(\sum_{i \in I} w_i \otimes a_i \Big) \cdot a_k^* \\
&= \sum_{i \in I} (w_i \otimes a_i) \cdot a_k^* \\
&= (w_k \otimes a_k) \cdot a_k^* && \text{[By Proposition 4.2.20, and formula (4.15)]} \\
&= (w_k \cdot e) \otimes e && \text{[By formula (4.15) and } a_k a_k^* = e = e \cdot e] \\
&= w_k \otimes e. && \text{[As } w_k \in U \text{ (} G_e\text{-module) and } e \text{ is the identity of } G_e]
\end{aligned}$$

This means that the vector $w_k \otimes e$ belonging to copy U, corresponding to the $\mathcal{H}$-class where e is its representative, is a zero vector. However, this occurs precisely when $w_k = \mathbf{0}$ in the G_e-module U. It follows that the corresponding vector $w_k \otimes a_k$ of w_k in the k-th copy U_k of U is also a zero vector. Now, let k vary; we obtain $m = \mathbf{0}$. Hence, the annihilator $Ann_{L_e}(M)$ is trivial. $\square$

Chapter 5

Representation theory of type $\mathcal{A}_{n-1}$ Boolean reflection monoids

5.1 Specht modules for the symmetric inverse monoids Grood's approach

Two pieces of information, namely, the Boolean system $\mathcal{B}$ (a collection of well behaved domain subspaces) and a reflection group $W(\mathcal{A}_{n-1})$, comprise the type $\mathcal{A}_{n-1}$ Boolean monoid. This monoid is also known as the symmetric inverse monoid I_n as stated in Section 2.2. Alternatively, and according to Solomon [76], the symmetric inverse monoid I_n also has a matrix version called the rook monoid R_n. In fact, R_n is the submonoid of the monoid of all the $n \times n$ matrices over $\mathbb{N}$ under the multiplication consisting of all matrices where we have at most one entry of 1 in any row or column and where all the other entries are 0.

Below is an illustration of a partial map α from $n \times n$ matrix $(a_{ij}) \in R_n$. Define a map as follows:

$$(j)\alpha = \begin{cases} i, & \text{if there is a row } i \text{ where its entry } a_{ij} = 1 \\ 0, & \text{otherwise.} \end{cases}$$

The well-definedness of a such map is justified by the definition of R_n.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \cong$$

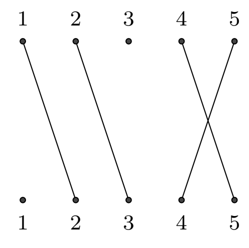


Figure 5.1: A 5×5 matrix corresponds to a map α.

The complex monoid algebra $\mathbb{C}R_n$ was revealed to be semisimple by Munn [58, 60], who determined the irreducible representations of R_n and illustrated that these irreducibles are parameterised by partitions of nonnegative integers r less than or equal to n. Grood, in [33], constructed a Specht module for the rook monoid R_n and, as an extension of her work, Solomon generated a range of character formulas in conjunction with several aesthetically pleasing combinatorics.

This section will offer an insight into Grood's approach of calculating such Specht modules as well as some examples to clarify the assertions. We redrafted the proof of Munn's theorem [60, Theorem 1.1] in relation to the decomposition of any partial permutation as a disjoint product of links and cycles to our benefit. The reader can consult [33] for more details.

Recall that a partition of a positive integer, $r \in \mathbb{Z}^+$, where $0 \leq r \leq n$, is a tuple $\lambda = (\lambda_1, \ \dots \ , \lambda_q)$ such that

(i) $\lambda_i \in \mathbb{Z}^+$ for all $1 \leq i \leq q$.

(ii) $\lambda_1 \geq \lambda_2 \geq \ \dots \ \geq \lambda_q \geq 0$.

(iii) $\lambda_1 + \lambda_2 + \ \dots \ + \lambda_q = r$.

Similar to the statement from Section 3.2 that each partition of n corresponds to a Young diagram, the partitions of r where $0 \leq r \leq n$ can also be associated with a Young diagram.

Definition 5.1.1. Given a partition λ of r where $0 \leq r \leq n$, define a λ_r^n-tableau to be a Young diagram of shape λ filled with r numbers from the set $[n] = \{1, 2, \dots, n\}$ such that different boxes contain distinct numbers.

Note that if $\lambda \vdash r$ where $0 \leq r \leq n$, then there are $n(n-1)(n-2)\cdots(n-r+1)$ distinct λ_r^n-tableaux for each λ. However,

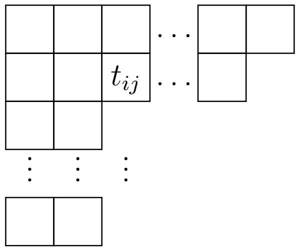
$$\begin{aligned}
n(n-1)(n-2)\cdots(n-r+1) &= \frac{n(n-1)\cdots(n-r+1)(n-r)!}{(n-r)!} \\
&= \frac{n(n-1)\cdots(n-r+1)(n-r)(n-r-1)\cdots \ \ 3\cdot 2\cdot 1}{(n-r)!} \\
&= \frac{n!}{(n-r)!}.
\end{aligned}$$

Hence, the number of distinct λ_r^n-tableaux for each λ is also equal to $\frac{n!}{(n-r)!}$.

Example 5.1.2. Let $r = 3$ and $n = 4$ and fix $\lambda = (2, 1) \vdash 3$. There are 24 distinct λ_3^4-tableaux illustrated in set T_r^λ, as follows:

$$\begin{aligned}
T_r^\lambda = \Bigg\{ & \begin{array}{|c|c|}\hline 3&4\\\hline 1\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 4&3\\\hline 1\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2&3\\\hline 1\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 3&2\\\hline 1\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2&4\\\hline 1\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 4&2\\\hline 1\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 3&1\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 4&1\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 3&4\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 4&3\\\hline 2\\\cline{1-1}\end{array}, \\
& \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2&1\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&4\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 4&1\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2&4\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 4&2\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline 4\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2&1\\\hline 4\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&3\\\hline 4\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 3&1\\\hline 4\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2&3\\\hline 4\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 3&2\\\hline 4\\\cline{1-1}\end{array} \Bigg\}.
\end{aligned}$$

Let t_{ij} be the entry in box B_{ij} of λ^n_r-tableau t,



Now define the action of any partial permutation $b \in I_n$ on λ^n_r-tableau t, as follows:

$$(t_{ij}) \ . \ \pi = \begin{cases} (t\pi)_{ij}, & \text{if } \ t_{ij} \in \operatorname{dom}(\pi) \text{ for all } i, j \\ 0, & \text{otherwise.} \end{cases} \tag{5.1}$$

Definition 5.1.3. Given $\lambda \vdash r,$ where $0 \leq r \leq n$, define a λ^n_r-tabloid $\{t\}$ to be the set of all λ^n_r-tableaux that are row-equivalent to t. In other words, we do not care about the order of elements in each row in t.

Definition 5.1.4. Given $\lambda \vdash r$ where $0 \leq r \leq n$, define $\mathcal{M}^\lambda$ as the complex vector space whose basis is the set of all distinct λ^n_r-tabloids $\{t\}$.

Note that the action of I_n on the basis element λ^n_r-tabloid $\{t\}$ will be defined by

$$\{t\} \ . \ b = \begin{cases} \{tb\}, & \text{if } \ \text{all entries of } t \text{ belong to } \operatorname{dom}(b) \\ 0, & \text{otherwise.} \end{cases} \tag{5.2}$$

Extending this action to vector space $\mathcal{M}^\lambda$ implies that $\mathcal{M}^\lambda$ is an I_n- module.

Example 5.1.5. Considering Example 5.1.2 yields that the complex vector space $\mathcal{M}^\lambda$ is determined as follows:

$$\begin{aligned} \mathcal{M}^\lambda = \mathbb{C}\text{-}\Big[& \left\{\begin{smallmatrix}3&4\\1&\end{smallmatrix}\right\}, \left\{\begin{smallmatrix}2&3\\1&\end{smallmatrix}\right\}, \left\{\begin{smallmatrix}2&4\\1&\end{smallmatrix}\right\}, \left\{\begin{smallmatrix}1&3\\2&\end{smallmatrix}\right\}, \left\{\begin{smallmatrix}1&4\\2&\end{smallmatrix}\right\}, \left\{\begin{smallmatrix}3&4\\2&\end{smallmatrix}\right\}, \\ & \left\{\begin{smallmatrix}1&2\\3&\end{smallmatrix}\right\}, \left\{\begin{smallmatrix}1&4\\3&\end{smallmatrix}\right\}, \left\{\begin{smallmatrix}2&4\\3&\end{smallmatrix}\right\}, \left\{\begin{smallmatrix}1&2\\4&\end{smallmatrix}\right\}, \left\{\begin{smallmatrix}1&3\\4&\end{smallmatrix}\right\}, \left\{\begin{smallmatrix}2&3\\4&\end{smallmatrix}\right\} \Big]. \end{aligned}$$

Definition 5.1.6. Fix $\lambda \vdash r$ where $0 \leq r \leq n$, and a λ^n_r-tableau t. Let C_j be the numbers in the jth column of t and let $X = \bigcup C_j$. It follows that $|X| \ = r$. Let us now define the group

$$C_t = S_{C_1} \times S_{C_2} \times \ldots \times S_{C_l}.$$

Notice that C_t is a subgroup within the J-class labelled by r. More precisely, it is a subgroup of $S_X \subseteq I_n$. Also note that every element in C_t preserves the columns of t.

Observe that for all $\sigma \in C_t$, we can define $\operatorname{sgn}(\sigma)$ as the sign function of the symmetric group S_X.

Definition 5.1.7. Let $\lambda \vdash r$, where $0 \leq r \leq n$. For each λ^n_r-tableau, define the element of $\mathcal{M}^\lambda$ as follows:

$$e_t = \sum_{\sigma \in C_t} sgn(\sigma)\ \{t\}\sigma.$$

Notice that in this definition, whenever we apply $\sigma \in C_t$ to a λ^n_r-tabloid $\{t\}$, the product will never be zero because σ is a permutation of the numbers in t. This means that every entry in t belongs to the domain of σ.

Definition 5.1.8. For any $\lambda \vdash r$ where $0 \leq r \leq n$, define R^λ to be the subspace of $\mathcal{M}^\lambda$ spanned by elements e_t, where t runs through all the λ^n_r-tableaux; that is,

$$R^\lambda = \{c_1 e_{t_1} + \ \ldots\ + c_i e_{t_i} :\ c_i \in \mathbb{C},\ t_i \text{ is a } \lambda^n_r\text{-tableau for all } i\}.$$

Example 5.1.9.
Consider Example 5.1.2 and fix a λ^4_3-tableau $t = \begin{smallmatrix}3&4\\1&\end{smallmatrix}$. Then, the column group C_t is obtained as follows:

$$C_t = \{id = (1)(3)(4), (13)\}.$$

Hence,

$$\begin{aligned} e_{\begin{smallmatrix}3&4\\1&\end{smallmatrix}} &= \sum_{\sigma \in C(t)} sgn(\sigma)\ \left\{\begin{smallmatrix}3&4\\1&\end{smallmatrix}\right\}\sigma \\ &= +\left\{\begin{smallmatrix}3&4\\1&\end{smallmatrix}\right\}id - \left\{\begin{smallmatrix}3&4\\1&\end{smallmatrix}\right\}(13) \\ &= \left\{\begin{smallmatrix}3&4\\1&\end{smallmatrix}\right\} - \left\{\begin{smallmatrix}1&4\\3&\end{smallmatrix}\right\} \in \mathcal{M}^\lambda. \end{aligned}$$

Observe that the permutation $(13)(2)(4) \in S_4 \subset I_4$ also preserves the columns of t but does not belong to the column group C_t. Now, by computing all e_t where $t \in T^\lambda_r$, as illustrated in Example 5.1.2, we acquire a $\mathbb{C}$-subspace R^λ of $\mathcal{M}^\lambda$ as follows:

$$\begin{aligned} R^\lambda = Span_{\mathbb{C}}\Big\{ & e_{\begin{smallmatrix}3&4\\1&\end{smallmatrix}}, e_{\begin{smallmatrix}4&3\\1&\end{smallmatrix}}, e_{\begin{smallmatrix}2&4\\1&\end{smallmatrix}}, e_{\begin{smallmatrix}4&2\\1&\end{smallmatrix}}, e_{\begin{smallmatrix}2&3\\1&\end{smallmatrix}}, e_{\begin{smallmatrix}3&2\\1&\end{smallmatrix}}, e_{\begin{smallmatrix}1&3\\2&\end{smallmatrix}}, e_{\begin{smallmatrix}3&1\\2&\end{smallmatrix}}, e_{\begin{smallmatrix}1&4\\2&\end{smallmatrix}}, e_{\begin{smallmatrix}4&1\\2&\end{smallmatrix}}, e_{\begin{smallmatrix}3&4\\2&\end{smallmatrix}}, e_{\begin{smallmatrix}4&3\\2&\end{smallmatrix}} \\ & e_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}}, e_{\begin{smallmatrix}2&1\\3&\end{smallmatrix}}, e_{\begin{smallmatrix}1&4\\3&\end{smallmatrix}}, e_{\begin{smallmatrix}4&1\\3&\end{smallmatrix}}, e_{\begin{smallmatrix}2&4\\3&\end{smallmatrix}}, e_{\begin{smallmatrix}4&2\\3&\end{smallmatrix}}, e_{\begin{smallmatrix}1&2\\4&\end{smallmatrix}}, e_{\begin{smallmatrix}2&1\\4&\end{smallmatrix}}, e_{\begin{smallmatrix}1&3\\4&\end{smallmatrix}}, e_{\begin{smallmatrix}3&1\\4&\end{smallmatrix}}, e_{\begin{smallmatrix}2&3\\4&\end{smallmatrix}}, e_{\begin{smallmatrix}3&2\\4&\end{smallmatrix}} \Big\}. \end{aligned}$$

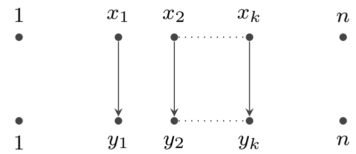
Notice that since

$$e_{\begin{smallmatrix}k&m\\l&\end{smallmatrix}} = -e_{\begin{smallmatrix}l&m\\k&\end{smallmatrix}},$$

where $k, l, m \in \{1, 2, 3, 4\}$, the spanning set $\{e_t : t \in T^\lambda_r\}$ is not a basis for R^λ because it is not linearly independent.

Munn [60, Theorem 1.1] has developed a simple technique for the decomposition of a partial permutation. We will rephrase it here in a manner which is suitable for our purposes. Let $\pi \in I_n$ and $\pi : X \xrightarrow{bij} Y$, where $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_k\}$ are subsets of $[n]$.

For notational simplicity, we define b as $(x_r)\pi = y_r$ for all $1 \leq r \leq k$.

$\pi =$ 

Now consider the following cases:

Case 1. If $X = Y$, then b is a permutation in S_X, and we know by Proposition **??** that it can be expressed uniquely as a disjoint product of cycles.

Case 2. If $X \neq Y$ and $X \cap Y \neq \emptyset$, then choose any $x_r \in X \setminus X \cap Y$ and consider its image under π. Now, $(x_r)\pi$ belongs to either $X \cap Y$ or $Y \setminus X \cap Y$. Notice that $(x_r)\pi$ can never be in $X \setminus X \cap Y$, as Y is the set of images of π. Thus, we have two sub-cases.

(I) If $(x_r)\pi \in X \cap Y$; thus $(x_r)\pi \in X$. Let $(x_r)\pi := x_{r+1}$ and keep taking the images under π as long as the images remain in $X \cap Y$. However, as $X \cap Y$ is a finite subset of $[n]$ containing distinct elements and π is a bijection, we will eventually obtain an element $y_s \in Y \setminus X \cap Y$. Hence, we produce a sequence of numbers from $[n]$ of which the first is $x_r \in X \setminus X \cap Y$ and the last is $y_l \in Y \setminus X \cap Y$:

$$x_r \xrightarrow{\pi} x_{r+1} \xrightarrow{\pi} x_{r+2} \xrightarrow{\pi} \cdots \xrightarrow{\pi} x_s \xrightarrow{\pi} y_l. \tag{5.3}$$

Let us denote such a sequence by $[x_r, x_{r+1}, \ \ldots \ , x_s, y_l]$ and call it a link.

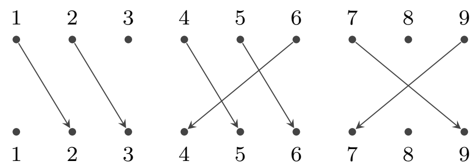
(II) If $(x_r)b \in Y \setminus X \cap Y$, then we obtain a sequence $[x_r, y_r]$ of length two, where $y_r = (x_r)\pi$.

Observe that if set $X \setminus X \cap Y$ contains more than one element, then we can produce another link. We will repeat the entire procedure of producing links until set $X \setminus X \cap Y$ is exhausted. If there exist some $x_r \in X \cap Y$ that are not contained in any produced link, which means their images under π never belonged to $Y \setminus X \cap Y$, then their images must belong to $X \cap Y$. Such numbers, which remain at the intersection $X \cap Y$ after considering their images and are not contained in the produced links, will be permuted among themselves. Thus, they form permutations which can also be expressed as disjoint products of cycles. The nature of the preceding process ensures that the above decomposition is unique.

The following assertion recapitulates the above observations:

Theorem 5.1.10. [60] Every partial permutation of a finite set can be expressed uniquely as a product of links and disjoint cycles.

Example 5.1.11. Let $\pi \in I_9$ such that $\pi : X \xrightarrow{bij} Y$ where $X = \{1, 2, 4, 5, 6, 7, 9\}$ and $Y = \{2, 3, 4, 5, 6, 7, 9\}$ and π is defined as follows:

$\pi =$ 

Notice that $X \cap Y = \{2, 4, 5, 6, 7, 9\}$ and $X \setminus X \cap Y = \{1\}$ as well as $Y \setminus X \cap Y = \{3\}$. Hence, π can be decomposed into a link and two cycles, as follows:

$$\pi = [1, 2, 3](4, 5, 6)(7, 9).$$

Importantly, the number 8 belongs neither to $X \setminus X \cap Y$ nor to $Y \setminus X \cap Y$; hence, we may omit it or write it as a link of length one [8]. Thus, π may also be expressed as

$$\pi = [1, 2, 3](4, 5, 6)(7, 9)[8].$$

Definition 5.1.12. Let $\pi \in I_n$. Then, define $\widehat{\pi}$ as an element of the symmetric group S_n obtained by changing each link appearing in π to the coressponding cycle.

Notice that if $\pi \in S_n$, then $\pi = \widehat{\pi}$. For instance, if $\pi = [1, 2, 3](4, 5, 6)(7, 9)[8]$, then $\widehat{\pi} = (1, 2, 3)(4, 5, 6)(7, 9)(8) \in S_9$.

The following propositions play a role in making the vector space R^λ an I_n-module.

Proposition 5.1.13. *Let $\pi \in I_n$ and t be a λ_r^n-tableau. If $t\pi \neq 0$, then $t\pi = t\widehat{\pi}$.*

Proposition 5.1.14. *Suppose $\pi \in I_n$, and t be a λ_r^n-tableau. If $t\pi = 0$, then $e_t\pi = 0$. Otherwise, $e_t\pi = e_{t\widehat{\pi}}$.*

The above proposition ensures the following result:

Corollary 5.1.15. *Let $\lambda \vdash r$, where $0 \leq r \leq n$. The vector space R^λ is an I_n-module.*

The following assertion indicates that the I_n-module R^λ is irreducible, and it gives us a complete list of I_n-irreducible modules.

Theorem 5.1.16. Let $\lambda \vdash r$ where $0 \leq r \leq n$. Then, R^λ is an irreducible I_n-module and, whenever any two I_n-modules R^λ and R^μ are isomorphic, their partitions λ and μ must be equal.

Definition 5.1.17. A λ_r^n-tableau is standard if and only if the entries in each row and column are increasing.

In the following theorem, Grood determines the basis vectors for an I_n-module R^λ.

Theorem 5.1.18. Let $\lambda \vdash r$ where $0 \leq r \leq n$. Then the following set forms a basis for R^λ :

$$\{e_t : t \text{ runs through all } \lambda_r^n\text{-standard tableaux}\},$$

and dim $R^\lambda = \binom{n}{r} \cdot f_\lambda$ where f_λ is the number of λ_r^n-standard tableaux.

In Example 5.1.9, we computed a Specht module R^λ for I_n and verified that the spanning set is not a basis for R^λ. However, by considering all λ_r^n-standard tableaux

$$\left\{ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} \right\},$$

we rewrite the Specht module R^λ for I_n in terms of its basis elements as follows:

$$R^\lambda = \mathbb{C}\text{-} \left[e_{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}}, e_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}}, e_{\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}}, e_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1}\end{array}}, e_{\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}}, e_{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}}, e_{\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}}, e_{\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}} \right].$$

5.2 Specht modules for the symmetric inverse monoids using CMC

The purpose of this section is to consider explicit descriptions of the ordinary irreducible representations of the Boolean reflection monoid of type $\mathcal{A}_{n-1}$ (the symmetric inverse monoid I_n) utilising combinatorial objects called Young tableaux. The principle approach for accomplishing this is the Clifford-Munn correspondence that states that, if S is an inverse monoid, then the irreducible representations are in one-to-one correspondence with the irreducible representations of the subgroups $G_1, \ldots, G_n$, which represent a set of maximal subgroup representatives of the $\mathcal{J}$-classes of the semigroup S. As illustrated in Section 4.2.2, the passage from the irreducible modules of the maximal subgroups to the irreducible modules of semigroup S is called induction.

The maximal subgroups G_i of the symmetric inverse monoid I_n are the symmetric subgroups S_r $(r \leq n)$ of S_n. The irreducible modules here were derived from the classical work of Munn [60]; however, it is reasonable to anticipate that the Specht modules of the maximal subgroups will be subject to induction in such a way so as to produce generalised Specht modules for I_n.

Although this was achieved recently by Grood, as presented in the preceding section, the Clifford-Munn correspondence was not employed for this purpose. In fact, Grood began her combinatorial construction of these irreducible modules from a unique starting point, using an independent method to obtain the modules in question. In a manner that was less explicit than Grood's approach, Steinberg [77, Section 9.4] later illustrated Specht modules for the symmetric inverse monoid I_n by means of the induction of the Specht modules of the maximal subgroups. Tabloids and polytabloids were not examined in detail in his work.

This section includes a more explicit description of the Specht modules for the Boolean monoid of type $\mathcal{A}_{n-1}$ resulting from the induction of the Specht modules of the maximal subgroups. Such an approach combines the explicit nature of Grood and the Clifford-Munn correspondence. We give further precise information arising from the induction techniques and portray how tabloids and polytabloids appear, as well as how the symmetric inverse monoid I_n acts on them.

The isomorphism between the generalized Specht modules for the Boolean monoid type of $\mathcal{A}_{n-1}$ resulting from induction and the Specht modules for the rook monoid acquired in Grood's work is verified. The Specht modules for the Boolean monoid $M(S_4, \mathcal{B})$ is calculated at the close of this section by means of the Specht modules for the symmetric inverse monoid I_4.

Consider the $\mathcal{J}$-classes J_r of I_n and recall from Section 4.2.2 that for each J_r-class of I_n where $0 \leq r \leq n$, if we choose an idempotent $e \in J_r$ and consider the maximal subgroup G_e, then the induction of a representation of G_e into I_n results in such a representation of I_n. Let us discuss this process in detail:

Fix an idempotent $e \in J_r$, where

$$(x)e = \begin{cases} x & x \in X = \mathrm{dom}(e) \subseteq [n], \text{ and } |X| = r \\ \text{undefined} & \text{otherwise.} \end{cases}$$

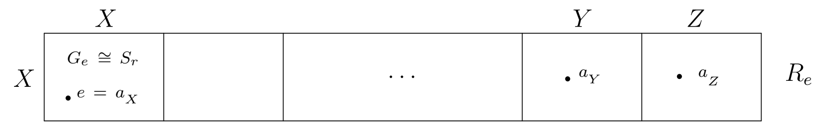
Consider R_e as the $\mathcal{R}$-class of e to which the idempotent e and the G_e maximal subgroup belong. Recall that the maximal subgroup G_e is isomorphic to the symmetric subgroup S_r $(r \leq n)$ of S_n. In view of the isomorphism and for notational simplicity, consider $X = \{1, 2, \ldots\ r\}$. Recall that the irreducible representations of S_r over $\mathbb{C}$ are parametrised by the λ partitions of r.

Fix $\lambda \vdash r$ and let t be a Young tableau of shape λ with entries from X. Consider M^λ to be the $\mathbb{C}$-vector space with basis distinct λ-tabloids $\{t\}$ whose ij-th entries $t_{ij} \in X$; that is,

$$M^\lambda = \mathbb{C}\text{-}\Big[\{t^1\}\ , \ldots , \{t^m\}\Big], \tag{5.4}$$

where $\{t^l\}$ with $1 \leq l \leq m$ is a complete list of distinct λ-tabloids. In Section 3.2, we stated that M^λ is a representation of the symmetric group S_r that is reducible. Note that inducing M^λ into I_n results in a representation of the symmetric inverse monoid I_n, which is also reducible as we will see later. Indeed, the induction process requires the following steps:

(a) Fix a representation M^λ of S_r. Take a representative a_Y of each $\mathcal{H}$-class of the $\mathcal{R}$-class of e, considering the idempotent e itself representative of the $\mathcal{H}$-class in which S_r is placed.



Henceforth, we will fix the following choice for a_Y: a_Y is a partial map whose domain is $X = [r]$ and image is $Y = \{y_1, y_2, \ldots, y_r\}$, where $y_1 < y_2 < \cdots < y_r$, and $Y \subseteq [n]$, with $|Y| = r$, and

$$(x)a_Y = \begin{cases} y_x & x \in X, \\ \text{undefined} & \text{otherwise.} \end{cases} \tag{5.5}$$

A representative a_Y for an $\mathcal{H}$-class labelled by Y can be illustrated as follows:

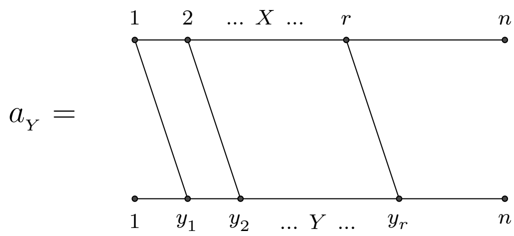
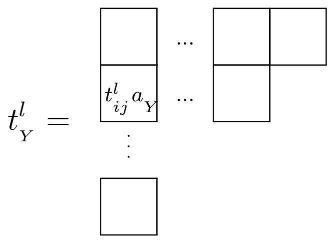


Figure 5.2: A representative a_Y for an $\mathcal{H}$-class labelled by Y.

Similarly to (5.1), define the action of any partial map $b \in I_n$ on a λ-tableau t as follows:

$$(t_{ij})b = \begin{cases} (tb)_{ij}, & \text{if } \ t_{ij} \in \mathrm{dom}(b) \text{ for all } i, j \\ 0, & \text{otherwise} \end{cases} \tag{5.6}$$

In particular, as the entries of t^l belong to X, if we apply a partial map a_Y to t^l, we obtain a λ-Young tableau $t^l_Y := t^l \cdot a_Y$ where the entries are in Y. In other words, the entries of a λ-Young tableau t^l_Y are $(t^l_{ij})a_Y$ where t^l_{ij} is an element of X appearing in row i and column j of t^l.



We will now highlight some similar definitions to ones discussed in the previous section:

Definition 5.2.1. Fix $\lambda \vdash r$, where $0 \leq r \leq n$. Let t_Y be any λ-Young tableau whose entries are $(t_{ij} \cdot a_Y)$ where $Y \subseteq [n]$ with $|Y| = r$. Then the column group C_{t_Y} is a Young subgroup of a maximal subgroup S_Y, preserving the columns of t_Y :

$$C_{t_Y} = \{\sigma \in S_Y : \sigma \text{ leaves the columns of } t_Y \text{ invariant}\}. \tag{5.7}$$

Similarly, define the row group R_{t_Y} as follows:

Definition 5.2.2. Fix $\lambda \vdash r$, where $0 \leq r \leq n$. Let t_Y be any λ-Young tableau whose entries are $(t_{ij} \cdot a_Y)$ where $Y \subseteq [n]$ with $|Y| = r$. Then the row group R_{t_Y} is a Young subgroup of a maximal subgroup S_Y preserving the rows of t_Y :

$$R_{t_Y} = \{\tau \in S_Y : \tau \text{ leaves the rows of } t_Y \text{ invariant}\}. \tag{5.8}$$

Our chief aim of leaving the columns or rows invariant is to allow the entries in the same column or row to be permuted among themselves. Observe that both C_{t_Y} and R_{t_Y} are subgroups within the J-class labelled by r; specifically, they are subgroups of $S_Y \subset I_n$. Moreover, not all the elements of I_n that stabilise the columns or rows of t_Y are contained in C_{t_Y} or R_{t_Y} respectively.

(b) Consider a copy of M^λ for each $\mathcal{H}$-class as follows: let M_Y^λ be a copy of M^λ corresponding to an $\mathcal{H}$-class where a_Y is its representative. The approach in constructing such a copy M_Y^λ is that, for each basis element $\{t^l\} \in M^\lambda$ presented in (5.4), let $\{t_Y^l\} := \{t^l \cdot a_Y\}$ be a tabloid in M_Y^λ. In other words, $\{t_Y^l\}$ can also be described as the orbit of the λ-tableau t_Y^l under the row group action $R_{t_Y^l}$. Thus, M_Y^λ is the $\mathbb{C}$-vector space with basis λ-tabloids $\{t_Y^l\}$ whose ij-th entries are $(t_{ij}^l \cdot a_Y)$. Note that the entries of any basis vector λ-tabloid $\{t_Y^l\}$ of M_Y^λ are elements from Y. Further, whenever we alter a subset Y to another subset Y' of $[n]$ with $|Y'| = r$, we acquire another copy of M^λ corresponding to another $\mathcal{H}$-class. Since M_Y^λ is a copy of M^λ, they all have the same dimension.

(c) Consider all the copies M_Y^λ where $Y \subseteq [n]$ with $|Y| = r$ and define the following vector space:

$$M^\lambda \uparrow I_n = \bigoplus_Y M_Y^\lambda.$$

Observe that the dimension of the direct sum is the sum of the dimensions of each M_Y^λ; that is,

$$\dim \bigoplus_Y M_Y^\lambda = \sum_Y \dim M_Y^\lambda = \binom{n}{r} \cdot \dim M^\lambda,$$

since there are $\binom{n}{r}$ subsets Y of $[n]$ where $|Y| = r$ and $M^\lambda \cong M_Y^\lambda$ for all Y.

(d) Define the action of I_n on $M^\lambda \uparrow I_n$ as follows: for all $b \in I_n$ and any basis vector $\{t_Y^l\} \in M_Y^\lambda$,

$$\{t_Y^l\} \cdot b = \begin{cases} \text{has } ij\text{-th entry } t_{ij}^l(a_Y \cdot b), & \text{if } a_Y \cdot b \in R_e, \\ 0 & \text{otherwise.} \end{cases} \tag{5.9}$$

In view of Remark 1.2.21, we know that $a_Y \cdot b$ belongs to R_e if and only if $Y \subseteq \mathrm{dom}(b)$. Thus, we can redraft the above formula as follows:

$$\{t_Y^l\} \cdot b = \begin{cases} \text{has } ij\text{-th entry } t_{ij}^l(a_Y \cdot b), & \text{if } Y \subseteq \mathrm{dom}(b), \\ 0 & \text{otherwise.} \end{cases} \tag{5.10}$$

Let us show that the above action is well-defined. In view of Definition 5.2.2, two λ-tabloids are row-equivalent if and only if their sets of entries in each row are identical; that is, we do not care about the entries' position in each row. Let Y^i be the set of all entries in the i-th row of a tabloid $\{t^l_{_Y}\}$ and observe that $\{t^l_{_Y}\} = \{s^l_{_Y}\}$ implies that

$$\{t^l_{_Y}\} = \begin{array}{l} \hline \quad Y^1 \quad \\ \hline \quad Y^2 \\ \cline{1-1} \vdots \\ \hline \\ \hline \end{array} = \{s^l_{_Y}\}$$

Thus, if $Y \subseteq \mathrm{dom}(b)$, then applying a partial map b to both sides requires applying b to each set Y^i in each row; that is,

$$\{t^l_{_Y}\} \cdot b = \begin{array}{l} \hline \quad Y^1b \quad \\ \hline \quad Y^2b \\ \cline{1-1} \vdots \\ \hline \\ \hline \end{array} = \{t^l_{_Y}b\} \quad \text{and} \quad \{s^l_{_Y}\} \cdot b = \begin{array}{l} \hline \quad Y^1b \quad \\ \hline \quad Y^2b \\ \cline{1-1} \vdots \\ \hline \\ \hline \end{array} = \{s^l_{_Y}b\} \tag{5.11}$$

Hence, the action of b in formula (5.9) is well-defined.

Since $a_{_Y} \cdot b$ belongs to the $\mathcal{R}$-class of e, let us point out to which $\mathcal{H}$-class of R_e that $a_{_Y} \cdot b$ belongs. In order to answer the enquiry, let us first determine the image of a composition $a_{_Y} \cdot b$ as follows:

$$\mathrm{im}(a_{_Y} \cdot b) = (\mathrm{im}\ a_{_Y} \cap \mathrm{dom}(b))b = (Y \cap \mathrm{dom}(b))b = Yb,$$

as $Y \subseteq \mathrm{dom}(b)$. Hence, $a_{_Y} \cdot b \in R_e$ if and only if $a_{_Y} \cdot b$ has the domain X and image Yb. As $Y \subseteq [n]$ with $|Y| = r$ and b is a partial map a "bijection", $Yb \subseteq [n]$ with $|Yb| = r$. Thus, $a_{_Y} \cdot b$ is a partial map that is precisely placed in the $\mathcal{H}$-class of R_e labelled by Yb.

Note that the representative of the $\mathcal{H}$-class of R_e labelled by Yb is a partial map $a_{_{Yb}}$ whose domain is X and whose image is $Yb = \{y'_{_1}, y'_{_2}, \ldots, y'_{_r}\}$ where $y'_{_1} < y'_{_2} < \cdots < y'_{_r}$. It follows that both partial maps $a_{_Y} \cdot b$ and $a_{_{Yb}}$ are in the $\mathcal{H}$-class labelled by Yb, as illustrated in the following diagram.

$$\begin{array}{c|c|c|c|c|c|c} \multicolumn{1}{c}{} & \multicolumn{1}{c}{X} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{Y} & \multicolumn{1}{c}{Yb} & \multicolumn{1}{c}{} \\ \cline{2-6} X & \begin{matrix} G_e \cong S_r \\ \bullet\ g \end{matrix} & & \cdots & \bullet\ a_{_Y} & \begin{matrix} \bullet\ (a_{_Y} \cdot b) \\ \bullet\ a_{_{Yb}} \end{matrix} & R_e \\ \cline{2-6} \end{array}$$

This draws our attention to the relationship between $a_{_Y} \cdot b$ and $a_{_{Yb}}$.

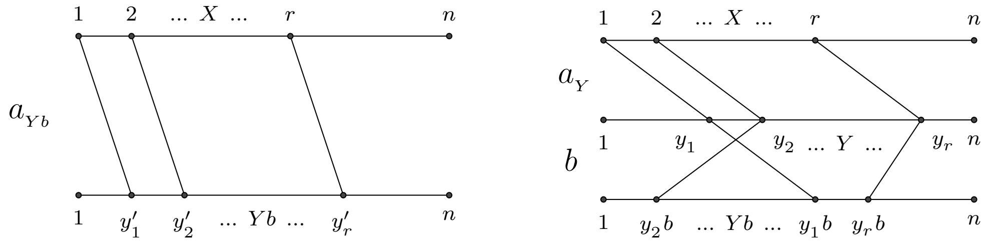


Figure 5.3: A partial map a_{Yb} versus a partial map $a_Y \cdot b$.

In view of Remark 1.1.28, there must exist a unique permutation $g \in S_r$ such that $a_Y \cdot b = g \cdot a_{Yb}$ holds. Thus, if $a_Y \cdot b \in R_e$, then $\{t^l_Y\} \cdot b$ has ij-th entries

$$t^l_{ij} \cdot (a_Y \cdot b) = t^l_{ij} \cdot (g \cdot a_{Yb}) = (t^l_{ij} \cdot g) \cdot a_{Yb}. \tag{5.12}$$

On the other hand, a tabloid $\{(t^l g)_{Yb}\}$ also has ij-th entries $(t^l_{ij} g) \cdot a_{Yb}$. This indeed allows us to produce an improved version of (5.9) as follows:

$$\{t^l_Y\} \cdot b = \begin{cases} \{(t^l g)_{Yb}\}, & Y \subseteq \operatorname{dom} b,\ a_Y \cdot b = g \cdot a_{Yb} \text{ and } \ g \in S_r \\ 0 & \text{otherwise.} \end{cases} \tag{5.13}$$

Therefore, we now know how I_n acts on the basis vector $\{t^l_Y\}$ of M^λ_Y for any $Y \subseteq [n]$ with $|Y| = r$. Extending this action linearly reveals how I_n acts on $\bigoplus_Y M^\lambda_Y$.

Observation 5.2.3. Since t^l is a λ-tableau with entries from $[r]$ and $g \in S_r$, then $t^l g$ is another λ-tableau filled with entries from $[r]$. We also have $Yb \subseteq [n]$ with $|Yb| = r$ as b is a bijection; thus, $\{(t^l g)_{Yb}\} \in M^\lambda_{Yb}$, where M^λ_{Yb} is another summand of $M^\lambda \uparrow I_n$.

The above observation concludes all essential steps required in the induction process. Given all that has been discussed so far, we state the following theorem.

Theorem 5.2.4. Fix $\lambda \vdash r$, where $0 \leq r \leq n$, and consider a representation M^λ of a maximal subgroup S_r of I_n over the complex field $\mathbb{C}$. Then, M^λ induces to an I_n representation $M^\lambda \uparrow I_n$ determined by a vector space

$$M^\lambda \uparrow I_n = \bigoplus_Y M^\lambda_Y, \quad Y \subseteq [n] \ \text{ with } |Y| = r,$$

and the action described as follows: For all basis vectors $\{t^l_Y\}$ of $M^\lambda \uparrow I_n$ where $1 \leq l \leq m$ and a partial map $b \in I_n$,

$$\{t^l_Y\} \cdot b = \begin{cases} \{(t^l g)_{Yb}\}, & Y \subseteq \operatorname{dom}(b),\ a_Y \cdot b = g \cdot a_{Yb} \text{ and } \ g \in S_r \\ 0 & \text{otherwise,} \end{cases}$$

where $t^{'}$ is a λ-tableau that forms a distinct basis element $\{t^{'}\}$ of M^{λ}.

The following theorem shows that there is an isomorphism between the I_n-module $M^{\lambda} \uparrow I_n$ obtained by the induction and the I_n-module $\mathcal{M}^{\lambda}$ introduced by Grood.

Theorem 5.2.5. Let $\lambda \vdash r$, where $0 \leq r \leq n$, and let M^{λ} be a representation of a maximal subgroup S_r. Then, for all λ and for all $r \leq n$, $M^{\lambda} \uparrow I_n$ is isomorphic to $\mathcal{M}^{\lambda}$ where $\mathcal{M}^{\lambda}$ is the I_n-module with basis the set of all distinct λ_r^n-tabloids $\{t\}$;

$$M^{\lambda} \uparrow I_n \cong \mathcal{M}^{\lambda}.$$

Proof. Let us first show that they are isomorphic as vector spaces by showing that they have the same dimension. We already computed the dimension of $M^{\lambda} \uparrow I_n$ in step (c) of the induction process as dim $M^{\lambda} \uparrow I_n = \binom{n}{r} \cdot$ dim M^{λ}. In view of Lemma 3.2.13, we have dim $M^{\lambda} = r!\,/\kappa$, where $\kappa = \lambda_1!\ \lambda_2!\ \ldots\ \lambda_m!$. Hence, dim $M^{\lambda} \uparrow I_n = \frac{n!}{(n-r)!\cdot\kappa}$. Let us determine the dimension of $\mathcal{M}^{\lambda}$. We know that there are $n!\,/(n-r)!$ ways to fill in the boxes of a λ_r^n-Young diagram with r distinct numbers from $\{1, 2, \ldots, n\}$. In other words, the size of set T_r^{λ} is equal to $n!\,/(n-r)!$. Observe that each λ_r^n-tabloid $\{t\} \in \mathcal{M}^{\lambda}$ is the orbit of t under the action of the row group R_t :

$$\{t\} = \{t\sigma : \sigma \in R_t\}.$$

Hence, the number of tableaux in any λ_r^n-tabloid $\{t\}$ is equal to $|R_t| = \kappa$ since $R_t \cong S_{\lambda_1} \times S_{\lambda_2} \ldots S_{\lambda_m}$. Therefore, dividing the number of all λ_r^n-tableaux in T_r^{λ} by the number of tableaux in any λ_r^n-tabloid $\{t\}$ yields the number of distinct λ_r^n-tabloids, which is $\frac{n!}{(n-r)!\cdot\kappa}$. Hence, we can infer that there is an isomorphism φ between $M^{\lambda} \uparrow I_n$ and $\mathcal{M}^{\lambda}$ is defined as follows:

$$\begin{aligned} \varphi\colon M^{\lambda} \uparrow I_n &\longrightarrow \mathcal{M}^{\lambda} \\ \{t^{'}_{_Y}\} &\longmapsto \{t^{''}\}, \end{aligned} \tag{5.14}$$

where $t^{''} = t^{'}_{_Y}$. Thus, φ is a vector space isomorphism. To show that φ gives an I_n-module isomorphism it remains to check the action. Notice that for all $b \in I_n$ and $\{t^{'}_{_Y}\} \in M^{\lambda} \uparrow I_n$, if $Y \subseteq$ dom b, then by formula (5.13), we have $\{t^{'}_{_Y}\}\ .\ b = \{(t^{'}g)_{_{Yb}}\}$. Thus, $\{(t^{'}g)_{_{Yb}}\} \cdot \varphi = \{t'\}$, where $t' = (t^{'}g)_{_{Yb}}$. Moreover, $\{t^{'}_{_Y}\}\ .\ \varphi = \{t^{''}\}$, where $t^{''} = t^{'}_{_Y}$. In view of formula (5.2), we have $\{t^{''}\} \cdot b = \{t^{''}b\}$. Note that by formula (5.12), both tabloids $\{t^{''}b\}$ and $\{t'\}$ have ij-th entry $(t^{'} \cdot a_{_Y}) \cdot b$. Hence the following diagram commutes:

$$\begin{array}{ccc} M^{\lambda} \uparrow I_n & \xrightarrow{(\text{-})b} & M^{\lambda} \uparrow I_n \\ \varphi\big\downarrow & & \big\downarrow\varphi \\ \mathcal{M}^{\lambda} & \xrightarrow{\quad(\text{-})b\quad} & \mathcal{M}^{\lambda} \end{array}$$

Therefore, I_n acts on $M^{\lambda} \uparrow I_n$ in the same way it acts on $\mathcal{M}^{\lambda}$. $\square$

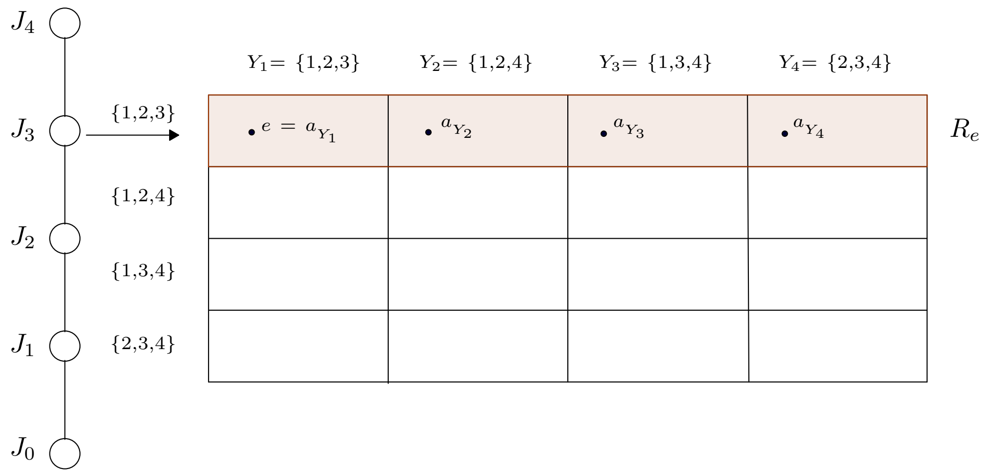
Example 5.2.6. Let $r = 3$ and $n = 4$. Fix $\lambda = (2,1) \vdash 3$, we know from Example 3.2.14 that M^{λ} can be written in terms of basis vectors as follows:

$$M^{(2,1)} = \mathbb{C}\text{-}\left[\left\{\begin{array}{|c|c|}\hline 2&3\\\hline 1\\\cline{1-1}\end{array}\right\}, \left\{\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\right\}, \left\{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\right\}\right]$$

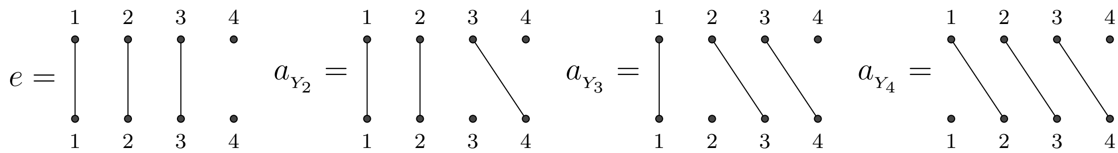
Recall that $M^{(2,1)}$ is a permutational representation of the symmetric group S_3. Fix an idempotent $e \in J_3$, where

$$(x)e = \begin{cases} x & if\ x \in \{1,2,3\}, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Consider the $\mathcal{R}$-class of e " R_e" as follows:



Now, choose the representatives for each $\mathcal{H}$-class in R_e as follows:



Hence, we can determine the copies of M^{λ} using the above representatives for each $\mathcal{H}$-class as follows:

$$M^{(2,1)}_{Y_1} = \mathbb{C}\text{-}\left[\left\{\begin{array}{|c|c|}\hline 2&3\\\hline 1\\\cline{1-1}\end{array}\right\}, \left\{\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\right\}, \left\{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\right\}\right], \quad M^{(2,1)}_{Y_2} = \mathbb{C}\text{-}\left[\left\{\begin{array}{|c|c|}\hline 2&4\\\hline 1\\\cline{1-1}\end{array}\right\}, \left\{\begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1}\end{array}\right\}, \left\{\begin{array}{|c|c|}\hline 1&2\\\hline 4\\\cline{1-1}\end{array}\right\}\right]$$

$$M^{(2,1)}_{Y_3} = \mathbb{C}\text{-}\left[\left\{\begin{array}{|c|c|}\hline 3&4\\\hline 1\\\cline{1-1}\end{array}\right\}, \left\{\begin{array}{|c|c|}\hline 1&4\\\hline 3\\\cline{1-1}\end{array}\right\}, \left\{\begin{array}{|c|c|}\hline 1&3\\\hline 4\\\cline{1-1}\end{array}\right\}\right], \quad M^{(2,1)}_{Y_4} = \mathbb{C}\text{-}\left[\left\{\begin{array}{|c|c|}\hline 3&4\\\hline 2\\\cline{1-1}\end{array}\right\}, \left\{\begin{array}{|c|c|}\hline 2&4\\\hline 3\\\cline{1-1}\end{array}\right\}, \left\{\begin{array}{|c|c|}\hline 2&3\\\hline 4\\\cline{1-1}\end{array}\right\}\right].$$

Thus,

$$M^{(2,1)} \uparrow I_4 = \mathbb{C}\text{-}\Big[\Big[\Big\{\ytableaushort{23,1}\Big\}, \Big\{\ytableaushort{13,2}\Big\}, \Big\{\ytableaushort{12,3}\Big\}\Big] \bigoplus \Big[\Big\{\ytableaushort{24,1}\Big\}, \Big\{\ytableaushort{14,2}\Big\}, \Big\{\ytableaushort{12,4}\Big\}\Big] \bigoplus$$
$$\Big[\Big\{\ytableaushort{34,1}\Big\}, \Big\{\ytableaushort{14,3}\Big\}, \Big\{\ytableaushort{13,4}\Big\}\Big] \bigoplus \Big\{\Big\{\ytableaushort{34,2}\Big\}, \Big\{\ytableaushort{24,3}\Big\}, \Big\{\ytableaushort{23,4}\Big\}\Big] \Big].$$

Notice that the Clifford-Munn correspondence only guarantees correspondence between irreducible representations of maximal subgroups and irreducible representations of semigroups. Although M^λ is reducible, we attempted to induce it to ascertain how the basis vector of such a module could be presented and how I_n acts on it. Our primary aim is inducing the Specht module S^λ for a maximal subgroup S_r into I_n, $0 \le r \le n$ since it is the only irreducible module for such a maximal subgroup.

In view of the action presented in (5.10), the restriction of I_n action to the maximal subgroup $S_{_Y}$ plays a role in the following definition. Such a definition is the crucial prerequisite we require in order to obtain the Specht module for I_n.

Definition 5.2.7. Fix $\lambda \vdash r$, where $0 \le r \le n$, and $Y \subseteq [n]$ with $|Y| = r$. For each Young tableau t of shape λ, determine a λ-tableau $t_{_Y}$ and then define an element $e^t_{_Y} \in M^\lambda_{_Y}$ as follows:

$$e^t_{_Y} = \sum_{\sigma \in C_{t_Y}} sgn(\sigma)\ \{t_{_Y}\}\sigma. \tag{5.15}$$

Observe that for all $\sigma \in C_{t_Y}$, we define $\mathrm{sgn}(\sigma)$ to be the sign function of $S_{_Y}$. The following example illustrates what the vector $e^t_{_{Y_4}}$ of $M^{(2,1)}_{_{Y_4}}$ looks like.

Example 5.2.8. As shown in Example 5.2.6, let $r = 3$ and $n = 4$. Fix $\lambda = (2,1) \vdash 3$ and choose the Young tableau $t = \ytableaushort{12,3}$ and $Y_4 = \{2,3,4\} \subset [4]$. Therefore, we can determine that the λ-Young tableau $t_{_{Y_4}} = \ytableaushort{23,4}$. Note that $C_{t_{Y_4}} = \{id\ ,\ (24)\}$, and hence the vector $e^t_{_{Y_4}}$ can be obtained as follows:

$$\begin{aligned} e^t_{_{Y_4}} &= \sum_{\sigma \in C_{t_{Y_4}}} sgn(\sigma)\ \Big\{\ytableaushort{23,4}\Big\} \cdot \sigma \\ &= +\Big\{\ytableaushort{23,4}\Big\} - \Big\{\ytableaushort{43,2}\Big\} \in M^{(2,1)}_{_{Y_4}}. \end{aligned}$$

Recall from Definition 3.2.17, the Specht module S^λ for the symmetric group S_r can be defined as follows: For any partition λ of r, where $0 \le r \le n$, the Specht module S^λ is the subspace of M^λ spanned by the elements e^t, where t runs through all Young tableaux of shape λ; that is,

$$S^\lambda = Span_{\mathbb{C}}\ \{e^t :\ t \text{ is a tableau filled with numbers from } \{1, \ldots,\ r\}\}.$$

Moreover, S^λ is an S_r irreducible submodule of M^λ by Theorem 3.2.20. In fact, S^λ was considered because whenever an irreducible representation S^λ of S_r is induced into I_n, we acquire a corresponding irreducible representation of I_n as stated in the Clifford-Munn correspondence theorem.

Let us now examine all four steps again to induce S^λ up to I_n.

(a) Fix a Specht module S^λ for the symmetric group S_r. Take a representative $a_{_Y}$ from each $\mathcal{H}$-class of the $\mathcal{R}$-class of e, considering the idempotent e a representative of the $\mathcal{H}$-class in which S_r is placed.

(b) Consider a copy of S^λ for each $\mathcal{H}$-class as follows: let $S^\lambda_{_Y}$ be a copy of S^λ corresponding to an $\mathcal{H}$-class with a representative $a_{_Y}$. In fact, $S^\lambda_{_Y}$ is defined as the subspace of $M^\lambda_{_Y}$ spanned by all the elements $e^t_{_Y}$, where t runs through all Young tableaux of shape λ; that is,

$$S^\lambda_{_Y} = Span_{\mathbb{C}}\ \{e^t_{_Y} :\ t \text{ a } \lambda\text{-tableau filled with numbers from } \{1, \ldots, r\}\}. \tag{5.16}$$

Here, the subset Y is fixed, and we only let t vary. Importantly, whenever we alter a subset Y to another subset Z of $[n]$ with $|Z| = r$, we attain another copy S^λ_Z of S^λ corresponding to the $\mathcal{H}$-class labelled by Z. Moreover, since $S^\lambda_{_Y}$ is a copy of S^λ, then both respect their vector space structures. Therefore, the spanning set

$$\{e^t_{_Y} :\ t \text{ is a } \lambda\text{-Young tableau filled with numbers from } \{1, \ldots, r\}\},$$

is not linearly independent; thus, it cannot be a basis for $S^\lambda_{_Y}$. However, if we choose t to run through all standard tableaux of shape λ and adapt the representative $a_{_Y}$ chosen for each $\mathcal{H}$-class, as stated in (5.5), then the set

$$\{e^t_{_Y} :\ t \text{ is a standard tableau filled with numbers from } \{1, \ldots, r\}\}$$

forms a basis for $S^\lambda_{_Y}$. Hence, we can present $S^\lambda_{_Y}$ by utilising its basis elements. However, we will maintain our consideration of $S^\lambda_{_Y}$ as it appears in (5.16) for the remainder of this section unless stated otherwise.

(c) Consider all copies $S^\lambda_{_Y}$ where $Y \subseteq [n]$ with $|Y| = r$, and define the following vector space:

$$S^\lambda \uparrow I_n = \bigoplus_Y S^\lambda_{_Y}.$$

Now $S^\lambda_{_Y}$ is a subspace of $M^\lambda_{_Y}$ for all $Y \subseteq [n]$ with $|Y| = r$; thus, $\bigoplus\limits_Y S^\lambda_{_Y}$ is also a subspace of $\bigoplus\limits_Y M^\lambda_{_Y}$. Furthermore, with a consideration of standard tableaux, we know that the dimension of each summand $S^\lambda_{_Y}$ is equal to the dimension of S^λ, and there are $\binom{n}{r}$ subsets Y of $[n]$ in which $|Y| = r$. Hence, we can determine the dimension of $\bigoplus\limits_Y S^\lambda_{_Y}$ as follows:

$$\dim \bigoplus_Y S^\lambda_{_Y} = \sum_Y \dim S^\lambda_{_Y} = \binom{n}{r} \cdot \dim S^\lambda. \tag{5.17}$$

The last step in the induction process reveals how the symmetric inverse monoid I_n acts on such a vector. However, let us first lay the groundwork by proving the lemmas that help to demonstrate such an action on the induced module $S^\lambda \uparrow I_n$.

In Section 5.1, we explored how every partial map of a finite set can be written uniquely as a product of links and disjoint cycles. We also know that if $b \in I_n$, then $\hat{b}$ is an element in $S_n \subset I_n$, which can be determined by changing every link in b into a cycle. Moreover, it is crucial to point out that the last number in any link of $b \in I_n$ does not belong to the domain of b, and the first number in a link does not belong to the image of b.

Lemma 5.2.9. *Fix $\lambda \vdash r$, where $0 \leq r \leq n$ and $Y \subseteq [n]$ with $|Y| = r$. Let $b \in I_n$, and $t_{_Y}$ be a λ-tableau whose entries belong to Y. If $Y \subseteq$ dom b, then*

(1) $t_{_Y} \cdot b = t_{_Y} \cdot \hat{b}$

(2) $\{t_{_Y}\} \cdot b = \{t_{_Y}\} \cdot \hat{b}$

(3) $e^t_{_Y} \cdot b = e^t_{_Y} \cdot \hat{b}$

Proof. (1) Let us denote the entries of λ-tableau $t_{_Y}$ by $\overline{t_{ij}}$ and write $b \in I_n$ as a product of links and disjoint cycles. If $Y \subseteq$ dom (b), then none of the $\overline{t_{ij}}$ entries of tableau $t_{_Y}$ appear as the last number in any link of b; that is, the $\overline{t_{ij}}$ entries that do appear in tableau $t_{_Y}$ have the same image under b as they do under $\hat{b}$. Hence, $\overline{t_{ij}}\ \hat{b} = \overline{t_{ij}}\ b$ for all i, j. Thus, (1) holds.

(2) This is straightforward by (1) and (5.11).

(3) Since

$$e^t_{Y} = \sum_{\sigma \in C_{t_Y}} sgn(\sigma)\ \{t_Y\}\sigma,$$

each summand $\{t_Y\sigma\}$ of e^t_Y is clearly a tabloid whose entries are precisely the same as the entries of t_Y but in a different order. In other words, the entries of each summand $\{t_Y\sigma\}$ still belong to dom(b). Hence,

$$\begin{aligned}
e^t_Y \cdot b &= \Big(\sum_{\sigma \in C_{t_Y}} sgn(\sigma)\ \{t_Y\}\sigma \Big) \cdot b, \\
&= \Big(\sum_{\sigma \in C_{t_Y}} sgn(\sigma)\ \{t_Y\sigma\} \Big) \cdot b, \\
&= \sum_{\sigma \in C_{t_Y}} sgn(\sigma)\ \Big(\{t_Y\sigma\} \cdot b \Big), \\
&= \sum_{\sigma \in C_{t_Y}} sgn(\sigma)\ \Big(\{h_Y\} \cdot b \Big), && [\text{ Put } h_Y = t_Y\sigma] \\
&= \sum_{\sigma \in C_{t_Y}} sgn(\sigma)\ \Big(\{h_Y\} \cdot \hat{b} \Big), && [\text{ By } (2)] \\
&= \Big(\sum_{\sigma \in C_{t_Y}} sgn(\sigma)\ \{h_Y\} \Big) \cdot \hat{b}, \\
&= \Big(\sum_{\sigma \in C_{t_Y}} sgn(\sigma)\ \{t_Y\sigma\} \Big) \cdot \hat{b}, \\
&= e^t_Y \cdot \hat{b}.
\end{aligned}$$

Thus, (3) holds. □

Observation 5.2.10. Let $\lambda \vdash r$, where $0 \leq r \leq n$ and $Y \subseteq [n]$ with $|Y| = r$. Let t_Y be a λ-Young tableau with entries from Y and recall that the column group C_{t_Y} is a subgroup of a maximal subgroup $S_Y \subset I_n$. Define an injective map

$$\sigma \in S_Y \hookrightarrow \acute{\sigma} \in S_n \quad \text{such that}$$

$$(y)\acute{\sigma} = \begin{cases} (y)\sigma & y \in Y, \\ y & \text{otherwise.} \end{cases}$$

In other words, the injective map $\acute{\sigma}$ embeds a maximal subgroup S_Y into the group of units S_n of I_n. Hence, this embedding allows both $\sigma \in S_Y$ and $\acute{\sigma} \in S_n$ to have the same sign.

Example 5.2.11. Let $r = 5$ and $n = 7$, and fix $\lambda = (3, 2) \vdash 5$. Choose

$Y = \{2,3,4,5,6\} \subset [7]$ with $|Y| = 5$. Choose the $(3,2)$-Young tableau

$$t_Y = \begin{array}{|c|c|}\hline 2 & 5 \\ \hline 4 & 3 \\ \hline 6 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

Notice that $C_{t_Y} = S_{\{2,4,6\}} \times S_{\{3,5\}}$, and let the permutation

$$\begin{aligned}\sigma &= (4,6,2)(3,5) \\ &= (6,2)(4,6)(3,5) \in C_{t_Y} \subset S_Y.\end{aligned}$$

Thus, the permutation

$$\begin{aligned}\acute{\sigma} &= (4,6,2)(3,5)(1)(7) \\ &= (6,2)(4,6)(3,5)(1)(7) \in S_7.\end{aligned}$$

Moreover, $sgn(\sigma) = sgn(\acute{\sigma}) = (-1)^3 = -1$.

In view of the above observation and considering S_n as a maximal subgroup of I_n, we have the following lemma.

Lemma 5.2.12. *Fix $\lambda = (\lambda_1, \ldots, \lambda_q) \vdash r$, where $0 \le r \le n$, and let t_Y be a λ-Young tableau whose entries belong to Y, where $Y \subseteq [n]$ with $|Y| = r$. Then, for all $\pi \in S_n$, we have*

(1) $\pi^{-1} C_{t_Y} \pi = C_{t_Y \pi}$.

(2) $sgn(\pi^{-1}\sigma\pi) = sgn(\sigma), \ \forall\, \sigma \in C_{t_Y}$.

Proof. (1) Let $c_1, c_2, \ldots, c_d$ be the columns of a λ-tableau t_Y. Then,

$$\begin{aligned}\sigma \in C_{t_Y} &\implies (c_j)\sigma = c_j \quad \forall\, j = 1,2,\ldots,d, \\ &\implies (c_j)\sigma\pi = c_j\pi \quad \forall\, j = 1,2,\ldots,d \text{ and for all } \pi \in S_n, \\ &\implies (c_j)\pi\pi^{-1}\sigma\pi = c_j\pi, \\ &\implies c_j\pi \ \ \pi^{-1}\sigma\pi = c_j\pi, \\ &\implies \pi^{-1}\sigma\pi \in C_{(t_Y\pi)},\end{aligned}$$

and this implies that $\pi^{-1} C_{t_Y} \pi \subseteq C_{(t_Y\pi)}$. Conversely,

$$\begin{aligned}\sigma \in C_{(t_Y\pi)} &\implies (c_j\pi)\sigma = c_j\pi \quad \forall\, j = 1,2,\ldots,d, \\ &\implies c_j \ \pi\sigma\pi^{-1} = c_j, \ \forall\, j = 1,2,\ldots,d, \\ &\implies \pi\sigma\pi^{-1} \in C_{t_Y}, \\ &\implies \sigma \in \pi^{-1} C_{t_Y} \pi \\ &\implies C_{(t_Y\pi)} \subseteq \pi^{-1} C_{t_Y} \pi.\end{aligned}$$

Hence, $\pi^{-1}C_{t_Y}\pi = C_{t_Y\pi}$ for all $\pi \in S_n$.
(2) S_Y can be viewed as a subgroup of S_n, so for all $\sigma \in C_{t_Y} \leq S_Y$, we have

$$\begin{aligned} sgn(\pi^{-1}\sigma\pi) &= sgn(\pi^{-1})\ sgn(\sigma)\ sgn(\pi) \\ &= sgn(\sigma), \end{aligned}$$

since $sgn(\pi^{-1}) = sgn(\pi)$. □

The above discussion makes several noteworthy contributions to the investigation of how I_n acts on $S^\lambda \uparrow I_n$. Thus, let us proceed with the last step of the induction process of I_n by considering the following lemma:

Lemma 5.2.13. *Fix $\lambda \vdash r$, where $0 \leq r \leq n$, and let t_Y be a λ-Young tableau filled with numbers from $Y \subseteq [n]$ with $|Y| = r$. Then, for all $b \in I_n$ and $e^t_Y \in S^\lambda \uparrow I_n$, we have*

$$e^t_Y \cdot b = \begin{cases} e^{tg}_{Yb}, & Y \subseteq dom\ b \text{ and } a_Y \cdot b = g \cdot a_{Yb} \text{ where } g \in S_r \\ 0 & otherwise. \end{cases} \tag{5.18}$$

Proof. Suppose that $Y \subseteq$ dom b, then by Lemma 5.2.9 (3), we have

$$\begin{aligned} e^t_Y \cdot b &= e^t_Y \cdot \hat{b} \\ &= \Big(\sum_{\sigma\in C_{t_Y}} sgn(\sigma)\ \{t_Y\}\sigma\Big) \cdot \hat{b}, \\ &= \sum_{\sigma\in C_{t_Y}} sgn(\sigma)\ \{t_Y\}\ \sigma\ \cdot \hat{b}, \\ &= \sum_{\sigma\in C_{t_Y}} sgn(\sigma)\ \{t_Y\}\ \hat{b}\ \hat{b}^{-1}\ \sigma\ \hat{b}, && [\text{As } \hat{b} \in S_n,\ and\ C_{t_Y} \leq S_Y \subseteq S_n] \\ &= \sum_{\sigma\in C_{t_Y}} sgn(\sigma)\ \{t_Y\}\ \hat{b}\quad (\hat{b}^{-1}\ \sigma\ \hat{b}) \\ &= \sum_{\sigma\in C_{t_Y}} sgn(\hat{b}^{-1}\ \sigma\ \hat{b})\ \{t_Y\}\ \hat{b}\quad (\hat{b}^{-1}\ \sigma\ \hat{b}), && [\text{By Lemma 5.2.12 (2)}] \\ &= \sum_{\rho\in\hat{b}^{-1}C_{t_Y}\hat{b}} sgn(\rho)\ \{t_Y\}\ \hat{b}\ \cdot \rho, && [\rho = \hat{b}^{-1}\ \sigma\ \hat{b}] \\ &= \sum_{\rho\in C_{(t_Y\cdot\hat{b})}} sgn(\rho)\ \{t_Y\}\ \hat{b}\ \cdot \rho, && [\text{By Lemma 5.2.12 (1)}] \\ &= \sum_{\rho\in C_{(t_Y\cdot b)}} sgn(\rho)\ \{t_Y\}\ b\ \cdot \rho, && [\text{By Lemma 5.2.9 (1) and (2)}] \\ &= \sum_{\rho\in C_{(tg)_{Yb}}} sgn(\rho)\ \{(tg)_{Yb}\}\ \rho, && [\text{By formula (5.13) and } t_Y \cdot b = (tg)_{Yb}] \\ &= e^{tg}_{Yb}. \end{aligned}$$

By contrast, if $Y \nsubseteq \operatorname{dom} b$, then by formula (5.13), we have $\{t_{_Y}\} \cdot b = 0$, which means that at least one entry $(t_{ij} \cdot a_{_Y})$ of $t_{_Y}$ does not belong to the domain of b. Moreover, since

$$e^t_{_Y} = \sum_{\sigma \in C_{t_Y}} sgn(\sigma)\ \{t_{_Y}\}\sigma,$$

each summand of $e^t_{_Y}$ has precisely the same entries as those of tableau $t_{_Y}$ but in a different order. In particular, entry $(t_{ij} \cdot a_{_Y})$ still belongs to every summand in $e^t_{_Y}$. Therefore, if we apply a partial map b to a ploytabloid $e^t_{_Y}$, then each summand will be sent to zero. Hence, $e^t_{_Y} \cdot b = 0$. $\square$

Observation 5.2.14. In (5.18), since t is a λ-tableau with entries from $X = \{1, \ldots, r\}$ and $g \in S_{_r}$, tg is another λ-tableau filled with entries from X. Also notice that $Yb \subseteq [n]$ with $|Yb| = r$ since b is a bijection; thus, $e^{tg}_{_{Yb}}$ is a summand of $S^\lambda \uparrow I_n$. As I_n is an inverse monoid, there is no need for a further step; hence, the Clifford-Munn correspondence asserts that $S^\lambda \uparrow I_n$ is an irreducible representation for I_n. By Theorem 3.2.20, if $\lambda \neq \mu$, then $S^\lambda \ncong S^\mu$. Consequently, both $S^\lambda \uparrow I_n$, and $S^\mu \uparrow I_n$ are inequivalant irreducible representations of I_n for all $\lambda \neq \mu$.

In view of all that has been discussed so far, we state the following main result.

Theorem 5.2.15. Fix $\lambda \vdash r$, where $0 \leq r \leq n$, and consider a Specht module S^λ for the maximal subgroup representative S_r of a $\mathcal{J}$-class J_r of I_n. Then, S^λ induces to an I_n-representation $S^\lambda \uparrow I_n$ called the Specht module for I_n and determined as follows:

$$S^\lambda \uparrow I_n = \bigoplus_Y S^\lambda_{_Y}, \quad Y \subseteq [n] \text{ with } |Y| = r,$$

where $S^\lambda_{_Y}$ is given by (5.16). Moreover, for each vector $e^t_{_Y} \in S^\lambda \uparrow I_n$ and partial permutation $b \in I_n$,

$$e^t_{_Y} \cdot b = \begin{cases} e^{tg}_{_{Yb}}, & Y \subseteq \operatorname{dom} b \text{ and } a_{_Y} \cdot b = g \cdot a_{_{Yb}} \text{ where } g \in S_r \\ 0 & \text{otherwise.} \end{cases}$$

The next theorem shows that there is an isomorphism between the Specht modules $S^\lambda \uparrow I_n$ for I_n produced by utilising the Clifford-Munn correspondence and the Specht modules R^λ for I_n produced by Grood.

Theorem 5.2.16. Let $\lambda \vdash r$, where $0 \leq r \leq n$, and let S^λ be a Specht module for S_r. Then, $S^\lambda \uparrow I_n$ is isomorphic as an I_n-module to R^λ, where R^λ is a Specht module for I_n spanned by polytabloids e_t such that t ranges over all λ_r^n-tableaux:

$$S^\lambda \uparrow I_n \cong R^\lambda.$$

Proof. To show that $S^\lambda \uparrow I_n$ and R^λ are isomorphic as I_n modules, recall that Theorem 5.2.5 shows that there is an I_n module isomorphism φ between $M^\lambda \uparrow I_n$ and $\mathcal{M}^\lambda$; that is, for all $v \in M^\lambda \uparrow I_n$ and $b \in I_n$, we have

$$\begin{aligned} &M^\lambda \uparrow I_n \xrightarrow{\varphi} \mathcal{M}^\lambda \\ &(v \cdot b)\ \varphi = (v\varphi) \cdot b. \end{aligned} \tag{5.19}$$

Moreover, Lemma 5.2.13 shows that $S^\lambda \uparrow I_n$ is an I_n submodule of $M^\lambda \uparrow I_n$. Contrariwise, Proposition 5.1.14 confirms that R^λ is a submodule of $\mathcal{M}^\lambda$ as well. The following diagram illustrates the isomorphism

$$\begin{tikzcd} M^\lambda \uparrow I_n \arrow[r, "\overset{\cong}{module}", "\varphi"'] & \mathcal{M}^\lambda \\ S^\lambda \uparrow I_n \arrow[u, hook] \arrow[r, "\bar{\varphi}"] & R^\lambda. \arrow[u, hook] \end{tikzcd}$$

Note that if we restrict φ to $S^\lambda \uparrow I_n$, then this restriction, denoted by $\bar{\varphi} := \varphi|_{S^\lambda \uparrow I_n}$, maps $S^\lambda \uparrow I_n$ to R^λ because for all $e^t_{Y} \in S^\lambda \uparrow I_n$ and $b \in I_n$, we have the following:

$$\begin{aligned}
(e^t_{Y} \cdot b)\ \varphi|_{S^\lambda \uparrow I_n} &= \Big(\Big(\sum_{\sigma \in C_{t_Y}} sgn(\sigma)\ \{t_{Y}\}\sigma\Big) \cdot\ b\Big)\ \varphi \\
&= \Big(\Big(\sum_{\sigma \in C_{t_Y}} sgn(\sigma)\ \{t_{Y}\}\sigma\Big) \cdot\ \varphi\Big)\ b && [\text{ By (5.19) }] \\
&= \Big(\sum_{\sigma \in C_{t_Y}} sgn(\sigma)\ \Big(\{t_{Y}\}\sigma\Big) \cdot\ \varphi\Big)\ b && [\ \varphi \text{ an isomorphism }] \\
&= \Big(\sum_{\sigma \in C_{t_Y}} sgn(\sigma)\ \Big(\{t_{Y}\}\varphi\Big) \cdot \sigma\Big)\ b && [\ \sigma \in C_{t_Y} \leq S_{Y} \subseteq I_n\ ] \\
&= \Big(\sum_{\sigma \in C_t} sgn(\sigma)\ \{t\}\sigma\Big)\ b && [\text{ By (5.14), and } t = t_{Y}\ ] \\
&= e_{t} \cdot b \in R^\lambda.
\end{aligned}$$

Thus, the image of $S^\lambda \uparrow I_n$ under action I_n is contained in R^λ. Moreover, this restriction inherits module map properties from φ. It remains to show that $\bar{\varphi}$ is a bijection. Observe that φ is a one-to-one map as it is an isomorphism; thus, $\bar{\varphi}$ is a one-to-one map as well because it is a restriction. Thus $\bar{\varphi}$ maps $S^\lambda \uparrow I_n$ isomorphically onto a submodule U of R^λ. In other words, dim $S^\lambda \uparrow I_n =$ dim U. In view of Theorem 5.1.18 and equality (5.17), we have dim $S^\lambda \uparrow I_n =$ dim R^λ. This yields that $\bar{\varphi}$ is onto, so it is an isomorphism. $\square$

Example 5.2.17. Let $r = 3$, $n = 4$ and fix $\lambda = (2,1) \vdash 3$. We know from Example 3.2.14 that the set of all 6 distinct λ-tableaux and the S_3-module $M^{(2,1)}$ can be determined as follows:

$$Y^{(2,1)} = \left\{ t_1 = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}, t_2 = \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}, t_3 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}, t_4 = \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}, t_5 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, t_6 = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \right\}.$$

$$M^{(2,1)} = \mathbb{C}\text{-}\left[\left\{ \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} \right\}, \left\{ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \right\}, \left\{ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \right\} \right].$$

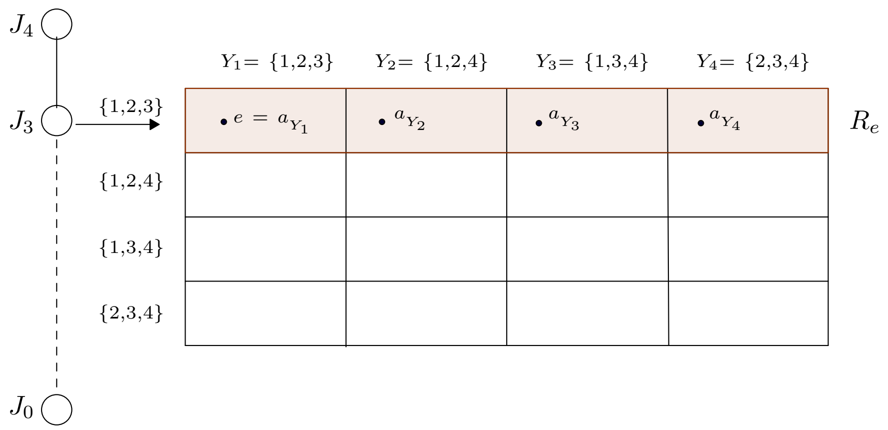
In Example 3.2.18(iii), we also obtained a Specht module $S^{(2,1)}$ for S_3 as follows:

$$S^{(2,1)} = \mathrm{Span}_{\mathbb{C}}\{e^{t_1}, e^{t_2}, e^{t_3}, e^{t_4}, e^{t_5}, e^{t_6}\}.$$

Fix an idempotent $e \in J_3$, where

$$(x)e = \begin{cases} x & x \in [3] = \{1,2,3\}, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Consider the $\mathcal{R}$-class of e " R_e" as appears below:



Now, choose the same representatives for each $\mathcal{H}$-class in R_e as those in Example 5.2.6

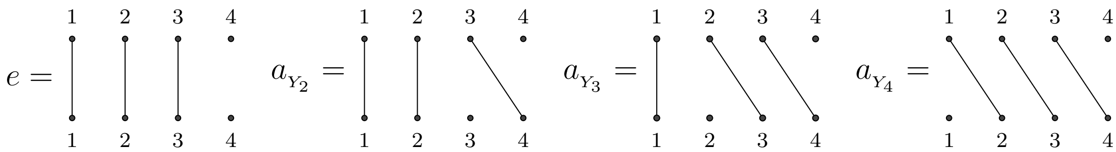


Figure 5.4: Representatives of $\mathcal{H}$-classes of the R_e.

Notice that inducing $S^{(2,1)}$ into I_4 requires that the copies of $S^{(2,1)}$ be obtained for each $\mathcal{H}$-class of R_e using the above representatives. Considering the first representative a_{Y_1} yields the following:

$$\begin{aligned} S^{(2,1)}_{Y_1} &= \mathrm{Span}_{\mathbb{C}}\{e^{t_1}_{Y_1}, e^{t_2}_{Y_1}, e^{t_3}_{Y_1}, e^{t_4}_{Y_1}, e^{t_5}_{Y_1}, e^{t_6}_{Y_1}\}. \\ &= \mathrm{Span}_{\mathbb{C}}\{e^{t_1}, e^{t_2}, e^{t_3}, e^{t_4}, e^{t_5}, e^{t_6}\}. \end{aligned}$$

Consider the second representative a_{Y_2}, and let us compute $S^{(2,1)}_{Y_2}$ as follows:

$$\begin{aligned}
t_{1_{Y_2}} &= \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 \\ \cline{1-1}\end{array} \text{ and } C_{t_{1_{Y_2}}} = \{id = (1)(2)(4),\ (12)\}. \text{ Thus, } e^{t_1}_{Y_2} = +\left\{\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 \\ \cline{1-1}\end{array}\right\} - \left\{\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}\right\} \\
t_{2_{Y_2}} &= \begin{array}{|c|c|}\hline 4 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} \text{ and } C_{t_{2_{Y_2}}} = \{id = (1)(2)(4),\ (14)\}. \text{ Thus, } e^{t_2}_{Y_2} = +\left\{\begin{array}{|c|c|}\hline 4 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}\right\} - \left\{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1}\end{array}\right\} \\
t_{3_{Y_2}} &= \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} \text{ and } C_{t_{3_{Y_2}}} = \{id = (1)(2)(4),\ (12)\}. \text{ Thus, } e^{t_3}_{Y_2} = +\left\{\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}\right\} - \left\{\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 \\ \cline{1-1}\end{array}\right\} \\
t_{4_{Y_2}} &= \begin{array}{|c|c|}\hline 4 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \text{ and } C_{t_{4_{Y_2}}} = \{id = (1)(2)(4),\ (24)\}. \text{ Thus, } e^{t_4}_{Y_2} = +\left\{\begin{array}{|c|c|}\hline 4 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\right\} - \left\{\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 4 \\ \cline{1-1}\end{array}\right\} \\
t_{5_{Y_2}} &= \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1}\end{array} \text{ and } C_{t_{5_{Y_2}}} = \{id = (1)(2)(4),\ (14)\}. \text{ Thus, } e^{t_5}_{Y_2} = +\left\{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1}\end{array}\right\} - \left\{\begin{array}{|c|c|}\hline 4 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}\right\} \\
t_{6_{Y_2}} &= \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 4 \\ \cline{1-1}\end{array} \text{ and } C_{t_{6_{Y_2}}} = \{id = (1)(2)(4),\ (24)\}. \text{ Thus, } e^{t_6}_{Y_2} = +\left\{\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 4 \\ \cline{1-1}\end{array}\right\} - \left\{\begin{array}{|c|c|}\hline 4 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\right\}
\end{aligned}$$

Hence, the copy of $S^{(2,1)}$, corresponding to the $\mathcal{H}$-class labelled by Y_2 is

$$S^{(2,1)}_{Y_2} = \mathrm{Span}_{\mathbb{C}}\{e^{t_1}_{Y_2}, e^{t_2}_{Y_2}, e^{t_3}_{Y_2}, e^{t_4}_{Y_2}, e^{t_5}_{Y_2}, e^{t_6}_{Y_2}\}.$$

Consider the third representative a_{Y_3}, and compute $S^{(2,1)}_{Y_3}$ as follows:

$$\begin{aligned}
t_{1_{Y_3}} &= \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 \\ \cline{1-1}\end{array} \text{ and } C_{t_{1_{Y_3}}} = \{id = (1)(3)(4),\ (13)\}. \text{ Thus, } e^{t_1}_{Y_3} = +\left\{\begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 \\ \cline{1-1}\end{array}\right\} - \left\{\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}\right\} \\
t_{2_{Y_3}} &= \begin{array}{|c|c|}\hline 4 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} \text{ and } C_{t_{2_{Y_3}}} = \{id = (1)(3)(4),\ (14)\}. \text{ Thus, } e^{t_2}_{Y_3} = +\left\{\begin{array}{|c|c|}\hline 4 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}\right\} - \left\{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}\right\} \\
t_{3_{Y_3}} &= \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \text{ and } C_{t_{3_{Y_3}}} = \{id = (1)(3)(4),\ (13)\}. \text{ Thus, } e^{t_3}_{Y_3} = +\left\{\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}\right\} - \left\{\begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 \\ \cline{1-1}\end{array}\right\} \\
t_{4_{Y_3}} &= \begin{array}{|c|c|}\hline 4 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \text{ and } C_{t_{4_{Y_3}}} = \{id = (1)(3)(4),\ (34)\}. \text{ Thus, } e^{t_4}_{Y_3} = +\left\{\begin{array}{|c|c|}\hline 4 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\right\} - \left\{\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 4 \\ \cline{1-1}\end{array}\right\} \\
t_{5_{Y_3}} &= \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} \text{ and } C_{t_{5_{Y_3}}} = \{id = (1)(3)(4),\ (14)\}. \text{ Thus, } e^{t_5}_{Y_3} = +\left\{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}\right\} - \left\{\begin{array}{|c|c|}\hline 4 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}\right\} \\
t_{6_{Y_3}} &= \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 4 \\ \cline{1-1}\end{array} \text{ and } C_{t_{6_{Y_3}}} = \{id = (1)(3)(4),\ (34)\}. \text{ Thus, } e^{t_6}_{Y_3} = +\left\{\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 4 \\ \cline{1-1}\end{array}\right\} - \left\{\begin{array}{|c|c|}\hline 4 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\right\}
\end{aligned}$$

Hence, the copy of $S^{(2,1)}$ that corresponds to the $\mathcal{H}$-class labelled by Y_3 is

$$S^{(2,1)}_{Y_3} = \mathrm{Span}_{\mathbb{C}}\{e^{t_1}_{Y_3}, e^{t_2}_{Y_3}, e^{t_3}_{Y_3}, e^{t_4}_{Y_3}, e^{t_5}_{Y_3}, e^{t_6}_{Y_3}\}.$$

Consider the last representative a_{Y_4}, and compute $S^{(2,1)}_{Y_4}$ as follows:

$$\begin{aligned}
t_{1_{Y_4}} &= \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} \text{ and } C_{t_{1_{Y_4}}} = \{id = (2)(3)(4),\ (23)\}. \text{ Thus, } e^{t_1}_{Y_4} = +\left\{\begin{array}{|c|c|}\hline 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}\right\} - \left\{\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}\right\} \\
t_{2_{Y_4}} &= \begin{array}{|c|c|}\hline 4 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \text{ and } C_{t_{2_{Y_4}}} = \{id = (2)(3)(4),\ (24)\}. \text{ Thus, } e^{t_2}_{Y_4} = +\left\{\begin{array}{|c|c|}\hline 4 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\right\} - \left\{\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}\right\} \\
t_{3_{Y_4}} &= \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \text{ and } C_{t_{3_{Y_4}}} = \{id = (2)(3)(4),\ (23)\}. \text{ Thus, } e^{t_3}_{Y_4} = +\left\{\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}\right\} - \left\{\begin{array}{|c|c|}\hline 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}\right\}
\end{aligned}$$

$t_{4_{Y_4}} = \ytableaushort{42,3}$ and $C_{t_{4_{Y_4}}} = \{id = (2)(3)(4),\ (34)\}$. Thus, $e^{t_4}_{Y_4} = +\left\{\ytableaushort{42,3}\right\} - \left\{\ytableaushort{32,4}\right\}$
$t_{5_{Y_4}} = \ytableaushort{23,4}$ and $C_{t_{5_{Y_4}}} = \{id = (2)(3)(4),\ (24)\}$. Thus, $e^{t_5}_{Y_4} = +\left\{\ytableaushort{23,4}\right\} - \left\{\ytableaushort{43,2}\right\}$
$t_{6_{Y_4}} = \ytableaushort{32,4}$ and $C_{t_{6_{Y_4}}} = \{id = (2)(3)(4),\ (34)\}$. Thus, $e^{t_6}_{Y_4} = +\left\{\ytableaushort{32,4}\right\} - \left\{\ytableaushort{42,3}\right\}$

Therefore, the copy of $S^{(2,1)}$ that corresponds to the $\mathcal{H}$-class labelled by Y_4 is

$$S^{(2,1)}_{Y_4} = \mathrm{Span}_{\mathbb{C}}\{e^{t_1}_{Y_4}, e^{t_2}_{Y_4}, e^{t_3}_{Y_4}, e^{t_4}_{Y_4}, e^{t_5}_{Y_4}, e^{t_6}_{Y_4}\}.$$

Hence, we acquire a Specht module $S^{(2,1)} \uparrow I_4$ for I_4 as follows:

$$S^{(2,1)} \uparrow I_4 = S^{(2,1)}_{Y_1} \bigoplus S^{(2,1)}_{Y_2} \bigoplus S^{(2,1)}_{Y_3} \bigoplus S^{(2,1)}_{Y_4}.$$

However, by Example 3.2.24, the Specht module $S^{(2,1)}$ for S_3 can also be written as a linear combination of the two basis vectors involving $(2,1)$-standard tableaux:

$$S^{(2,1)} = \mathbb{C}\text{-}\left[e_{\ytableaushort{13,2}},\ e_{\ytableaushort{12,3}}\right]$$

Thus, we can redraft the copies of $S^{(2,1)}$ with respect to $(2,1)$-standard tableaux:

$$S^{(2,1)}_{Y_1} = \mathbb{C}\text{-}\left[e_{\ytableaushort{13,2}},\ e_{\ytableaushort{12,3}}\right] \quad , \quad S^{(2,1)}_{Y_2} = \mathbb{C}\text{-}\left[e_{\ytableaushort{14,2}},\ e_{\ytableaushort{12,4}}\right]$$

$$S^{(2,1)}_{Y_3} = \mathbb{C}\text{-}\left[e_{\ytableaushort{14,3}},\ e_{\ytableaushort{13,4}}\right] \quad , \quad S^{(2,1)}_{Y_4} = \mathbb{C}\text{-}\left[e_{\ytableaushort{24,3}},\ e_{\ytableaushort{23,4}}\right].$$

Hence, we also obtain a Specht module $S^{(2,1)} \uparrow I_4$ for the symmetric inverse monoid I_4 written with respect to basis vectors involving $(2,1)$-standard tableaux as follows:

$$S^{(2,1)} \uparrow I_4 = \mathbb{C}\text{-}\left[\left[e_{\ytableaushort{13,2}},\ e_{\ytableaushort{12,3}}\right] \bigoplus \left[e_{\ytableaushort{14,2}},\ e_{\ytableaushort{12,4}}\right] \bigoplus \left[e_{\ytableaushort{14,3}},\ e_{\ytableaushort{13,4}}\right] \bigoplus \left[e_{\ytableaushort{24,3}},\ e_{\ytableaushort{23,4}}\right]\right].$$

In the following example, we will compute Specht modules for the symmetric inverse monoid I_4 by inducing all Specht modules for the maximal subgroup representatives of $\mathcal{J}$-classes J_r, where $0 \le r \le 4$.

Example 5.2.18. Let $n = 4$, and consider all the partitions λ for all nonnegative integer r, where $0 \le r \le 4$. Let us illustrate all Specht modules $S^{\lambda} \uparrow I_4$ for all $\lambda \vdash r$, as follows:

Let $r = 0$. Then the only partition λ of 0 is the empty set $\emptyset$, and the corresponding

Specht module is

$$S^{\emptyset} = \mathbf{0}, \quad \text{zero vector space.}$$

The corresponding Specht module for I_4 is

$$S^{\emptyset} \uparrow I_4 = \mathbf{0}.$$

Let $r = 1$; then, the only possible partition of 1 is $\lambda = (1)$, and the Specht module for the maximal subgroup representative S_1 of J_1 is

$$S^{(1)} = \mathbb{C}\ [e_{\boxed{1}}].$$

However, the corresponding Specht module for I_4 is

$$S^{(1)} \uparrow I_4 = \mathbb{C}\text{-}\Big[\Big[e_{\boxed{1}}\Big] \bigoplus \Big[e_{\boxed{2}}\Big] \bigoplus \Big[e_{\boxed{3}}\Big] \bigoplus \Big[e_{\boxed{4}}\Big]\Big].$$

Observe that $e_{\boxed{i}} = +\{\boxed{i}\}$ for all i.

Let $r = 2$. Then, all possible partitions of 2 are $\lambda = (2)$ and $\lambda = (1, 1)$.

Case (I) If $\lambda = (2)$, then the corresponding Specht module for the maximal subgroup representative S_2 of J_2, expressed in its basis vector involving a (2)-standard tableau is

$$S^{(2)} = c\ [e_{\boxed{1}\boxed{2}}], \quad c \in \mathbb{C}.$$

Furthermore, the corresponding Specht module for I_4 is

$$S^{(2)} \uparrow I_4 = \mathbb{C}\text{-}\Big[[e_{\boxed{1}\boxed{2}}] \bigoplus [e_{\boxed{1}\boxed{3}}] \bigoplus [e_{\boxed{1}\boxed{4}}] \bigoplus [e_{\boxed{2}\boxed{3}}] \bigoplus [e_{\boxed{2}\boxed{4}}] \bigoplus [e_{\boxed{3}\boxed{4}}]\Big].$$

Observe that $e_{\boxed{i}\boxed{j}} = +\{\boxed{i}\boxed{j}\}$.

Case (II) if $\lambda = (1, 1)$, then the corresponding Specht module for the maximal subgroup representative S_2 of J_2 expressed in its basis vector involving a $(1, 1)$-standard tableau is

$$S^{(1,1)} = c\ e_{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}}, \quad c \in \mathbb{C}.$$

The corresponding Specht module for I_4 is

$$S^{(1,1)} \uparrow I_4 = \mathbb{C}\text{-}\left[\left[e_{\begin{smallmatrix}\boxed{1}\\\boxed{2}\end{smallmatrix}}\right] \bigoplus \left[e_{\begin{smallmatrix}\boxed{1}\\\boxed{3}\end{smallmatrix}}\right] \bigoplus \left[e_{\begin{smallmatrix}\boxed{1}\\\boxed{4}\end{smallmatrix}}\right] \bigoplus \left[e_{\begin{smallmatrix}\boxed{2}\\\boxed{3}\end{smallmatrix}}\right] \bigoplus \left[e_{\begin{smallmatrix}\boxed{2}\\\boxed{4}\end{smallmatrix}}\right] \bigoplus \left[e_{\begin{smallmatrix}\boxed{3}\\\boxed{4}\end{smallmatrix}}\right]\right].$$

Notice that in both Specht modules, $e_{\begin{smallmatrix}\boxed{i}\\\boxed{j}\end{smallmatrix}} = +\left\{\begin{smallmatrix}\boxed{i}\\\boxed{j}\end{smallmatrix}\right\} - \left\{\begin{smallmatrix}\boxed{j}\\\boxed{i}\end{smallmatrix}\right\}$.

....

Let $r = 3$. Then, all possible partitions of 3 are $\lambda = (3)$, $\lambda = (1,1,1)$ and $\lambda = (2,1)$.

Case (I) If $\lambda = (3)$, then the corresponding Specht module for the maximal subgroup representative S_3 of J_3 expressed in its basis vector involving a (3)-standard tableau is

$$S^{(3)} = c\ [e_{\boxed{1}\boxed{2}\boxed{3}}], \quad c \in \mathbb{C}.$$

The corresponding Specht module for I_4 is

$$S^{(3)} \uparrow I_4 = \mathbb{C}\text{-}\Big[[e_{\boxed{1}\boxed{2}\boxed{3}}] \bigoplus [e_{\boxed{1}\boxed{2}\boxed{4}}] \bigoplus [e_{\boxed{1}\boxed{3}\boxed{4}}] \bigoplus [e_{\boxed{2}\boxed{3}\boxed{4}}]\Big].$$

Observe that $e_{\boxed{i}\boxed{j}\boxed{k}} = +\{\boxed{i}\boxed{j}\boxed{k}\}$.

Case (II) If $\lambda = (1,1,1)$, then the corresponding Specht module for the maximal subgroup representative S_3 of J_3 expressed in its basis vector involving a $(1,1,1)$-standard tableau is

$$S^{(1,1,1)} = c\ e_{\begin{smallmatrix}\boxed{1}\\\boxed{2}\\\boxed{3}\end{smallmatrix}} \quad c \in \mathbb{C}.$$

The corresponding Specht module for I_4 is obtained by

$$S^{(1,1,1)} \uparrow I_4 = \mathbb{C}\text{-}\left[\left[e_{\begin{smallmatrix}\boxed{1}\\\boxed{2}\\\boxed{3}\end{smallmatrix}}\right] \bigoplus \left[e_{\begin{smallmatrix}\boxed{1}\\\boxed{2}\\\boxed{4}\end{smallmatrix}}\right] \bigoplus \left[e_{\begin{smallmatrix}\boxed{1}\\\boxed{3}\\\boxed{4}\end{smallmatrix}}\right] \bigoplus \left[e_{\begin{smallmatrix}\boxed{2}\\\boxed{3}\\\boxed{4}\end{smallmatrix}}\right]\right].$$

Observe that $e_{\begin{smallmatrix}\boxed{i}\\\boxed{j}\\\boxed{k}\end{smallmatrix}} = \sum_{\sigma \in S_{\{i,j,k\}}} sgn(\sigma)\left\{\begin{smallmatrix}\boxed{i}\\\boxed{j}\\\boxed{k}\end{smallmatrix}\right\}\sigma$.

Case (III) If $\lambda = (2,1)$, then the corresponding Specht module for the maximal subgroup representative S_3 of J_3 expressed in its basis vectors is

$$S^{(2,1)} = \mathbb{C}\text{-}\left[e_{\begin{smallmatrix}1&3\\2&\end{smallmatrix}}, e_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}}\right].$$

In contrast, the corresponding Specht module for I_4 is determined as follows:

$$S^{(2,1)} \uparrow I_4 = \mathbb{C}\text{-}\left[\left[e_{\begin{smallmatrix}1&3\\2&\end{smallmatrix}}, e_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}}\right] \bigoplus \left[e_{\begin{smallmatrix}1&4\\2&\end{smallmatrix}}, e_{\begin{smallmatrix}1&2\\4&\end{smallmatrix}}\right] \bigoplus \left[e_{\begin{smallmatrix}1&4\\3&\end{smallmatrix}}, e_{\begin{smallmatrix}1&3\\4&\end{smallmatrix}}\right] \bigoplus \left[e_{\begin{smallmatrix}2&4\\3&\end{smallmatrix}}, e_{\begin{smallmatrix}2&3\\4&\end{smallmatrix}}\right]\right].$$

It is clear that $e_{\begin{smallmatrix}i&k\\j&\end{smallmatrix}} = \left\{\begin{smallmatrix}i&k\\j&\end{smallmatrix}\right\} - \left\{\begin{smallmatrix}j&k\\i&\end{smallmatrix}\right\}$.

....

Let $r = 4$. Then, all possible partitions of 4 are $\lambda = (4)$, $\lambda = (1,1,1,1)$, $\lambda = (2,2)$, $\lambda = (2,1,1)$ and $\lambda = (3,1)$.

Case (I) If $\lambda = (4)$, then the corresponding Specht module for the group of units S_4 of J_4 expressed in its basis vector involving a (4)-standard tableau is

$$S^{(4)} = c\ [e_{\begin{smallmatrix}1&2&3&4\end{smallmatrix}}], \quad c \in \mathbb{C}.$$

The corresponding Specht module for I_4 is obtained by

$$S^{(4)} \uparrow I_4 = c\ [e_{\begin{smallmatrix}1&2&3&4\end{smallmatrix}}], \quad c \in \mathbb{C}.$$

Notice that $e_{\begin{smallmatrix}i&j&k&l\end{smallmatrix}} = \left\{\begin{smallmatrix}i&j&k&l\end{smallmatrix}\right\}$.

Case (II) If $\lambda = (1,1,1,1)$, then the corresponding Specht module for the group of units S_4 of J_4 expressed in its basis vector involving a $(1,1,1,1)$-standard tableau is

$$S^{(1,1,1,1)} = c\ e_{\begin{smallmatrix}1\\2\\3\\4\end{smallmatrix}}, \quad c \in \mathbb{C}.$$

Hence, the corresponding irreducible module for I_4 is

$$S^{(1,1,1,1)} \uparrow I_4 = c\ e_{\begin{smallmatrix}1\\2\\3\\4\end{smallmatrix}}, \quad c \in \mathbb{C}.$$

Obviously $e_{\begin{smallmatrix}i\\j\\k\\l\end{smallmatrix}} = \sum_{\sigma \in S_{\{i,j,k,l\}}} sgn(\sigma)\left\{\begin{smallmatrix}i\\j\\k\\l\end{smallmatrix}\right\}\sigma.$

Case (III) If $\lambda = (3,1)$, then the corresponding Specht module for the group of units S_4 of J_4 expressed in its basis vectors involving $(3,1)$-standard tableaux is

$$S^{(3,1)} = \mathbb{C}\text{-}\left[e_{\ytableaushort{134,2}},\ e_{\ytableaushort{124,3}},\ e_{\ytableaushort{123,4}}\right],$$

and the corresponding irreducible module for I_4 is obtained as follows:

$$S^{(3,1)} \uparrow I_4 = \mathbb{C}\text{-}\left[e_{\ytableaushort{134,2}},\ e_{\ytableaushort{124,3}},\ e_{\ytableaushort{123,4}}\right].$$

Observe that $e_{\ytableaushort{ikl,j}} = +\left\{\ytableaushort{ikl,j}\right\} - \left\{\ytableaushort{jkl,i}\right\}$.

Case (III) If $\lambda = (2,2)$, then the corresponding Specht module for the group of units S_4 of J_4 expressed in its basis vectors involving $(2,2)$-standard tableaux is

$$S^{(2,2)} = \mathbb{C}\text{-}\left[e_{\ytableaushort{12,34}},\ e_{\ytableaushort{13,24}}\right].$$

and the corresponding irreducible module for I_4 is determined by

$$S^{(2,2)} \uparrow I_4 = \mathbb{C}\text{-}\left[e_{\ytableaushort{12,34}},\ e_{\ytableaushort{13,24}}\right].$$

Observe that $e_{\ytableaushort{ik,jl}} = \sum_{\sigma \in S_{\{i,j\}} \times S_{\{k,l\}}} sgn(\sigma)\left\{\ytableaushort{ik,jl}\right\}\sigma$.

Case (IV) If $\lambda = (2,1,1)$, then the corresponding Specht module for the group of units S_4 of J_4 expressed in its basis vectors involving $(2,1,1)$-standard tableaux is

$$S^{(2,1,1)} = \mathbb{C}\text{-}\left[e_{\ytableaushort{14,2,3}},\ e_{\ytableaushort{12,3,4}},\ e_{\ytableaushort{13,2,4}}\right].$$

and the corresponding irreducible module for I_4 is

$$S^{(2,1,1)} \uparrow I_4 = \mathbb{C}\text{-}\left[e_{\ytableaushort{14,2,3}},\ e_{\ytableaushort{12,3,4}},\ e_{\ytableaushort{13,2,4}}\right].$$

It is clear that $e_{\begin{array}{|c|c|}\hline i & l \\ \hline j \\ \cline{1-1} k \\ \cline{1-1}\end{array}} = \sum_{\sigma \in S_{\{i,j,k\}}} sgn(\sigma) \left\{ \begin{array}{|c|c|}\hline i & l \\ \hline j \\ \cline{1-1} k \\ \cline{1-1}\end{array} \right\} \sigma.$

It is no surprise that for all $\lambda \vdash 4$, S^{λ} and $S^{\lambda} \uparrow I_4$ are identical as $r = n = 4$.

Chapter 6

Representation theory of type $\mathcal{B}_n$ Boolean reflection monoids

6.1 The monoid of partial signed permutations MB_n

The combination of both the Boolean system $\mathcal{B}$ and the reflection group $W(\mathcal{B}_n)$ of type $\mathcal{B}_n$ yields the reflection monoids known as the Boolean monoids $M(W(\mathcal{B}_n), \mathcal{B})$, as discussed in Section 2.2. Alternatively, and according to Everitt and Fountain in [19, Proposition 5.1], these are also known as the monoids of partial signed permutations MB_n. This section is primarily concerned with the detailed study of the monoid of partial signed permutations MB_n. This section establishes some significant assertions and properties which contribute to a better understanding of such monoids. As part of this study, we aim to analyse and in turn gain a much more in-depth insight into the $\mathcal{J}$-classes of MB_n and determine their maximal subgroups. We will then pinpoint the processes involved in the decomposition of any partial signed permutation as a disjoint product of positive links and positive and negative cycles.

Consider the signed permutations group B_n, and recall that $\alpha \in B_n$ is a signed permutation on $[\pm n]$ satisfying the property that $(-r)\alpha = -(r)\alpha$ for all $r \in [\pm n]$. Choose a subset $X \subseteq [\pm n]$ such that whenever $+x \in X$, $-x \in X$ as well. Notice that the restriction of α to X, denoted by $\alpha|_X$, is a partial signed permutation $X \longrightarrow X\alpha$ defined by

$$(x)\alpha|_X = \begin{cases} (x)\alpha & x \in X, \\ \text{undefined} & x \notin X, \end{cases}$$

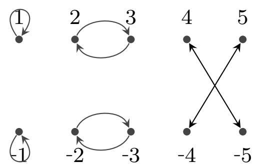
where $(-x)\alpha = -(x)\alpha$ is satisfied for all $x \in X$. It is worth noting that the result of such restriction is not unique; there may be another signed permutation $\beta \in B_n$ such that when we restrict it to some subset $X \subseteq [\pm n]$ such that whenever $+x \in X$, $-x \in X$ we acquire the same partial signed permutation:

$$\alpha|_X = \beta|_X.$$

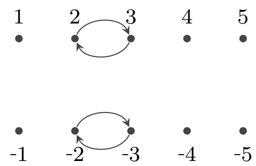
However, this obstacle can be overcome in the following manner: for all signed permutations $\alpha, \beta \in B_n$ and $X, Y \subseteq [\pm n]$, two partial signed permutations are the same if they satisfy the following:

$$\alpha|_X = \beta|_Y \iff X = Y \text{ and } (x)\alpha = (x)\beta, \text{ for all } x \in X.$$

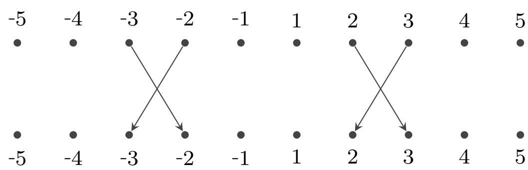
Example 6.1.1. Let $n = 5$ and $\alpha \in B_5$ such that

$\alpha =$ 

Let $X = \{\pm 2, \pm 3\} \subset [\pm 5]$. Then, the restriction $\alpha|_X$, which is defined only on X and is undefined elsewhere, results in a partial signed permutation $\alpha|_X$, as illustrated below:

$\alpha_X =$ 

We can also illustrate a partial signed permutation $\alpha|_X$ as follows:

$\alpha_X =$ 

The latter illustration will be used in computing irreducible representations for the monoid MB_n in the next section.

Definition 6.1.2. [19] The monoid of partial signed permutations MB_n is defined as

$$MB_n = \{\sigma : X \xrightarrow{bij} Y : X, Y \subseteq [\pm n], \text{ where}$$
$$x \in X \iff -x \in X \text{ and } (-x)\sigma = -(x)\sigma\},$$

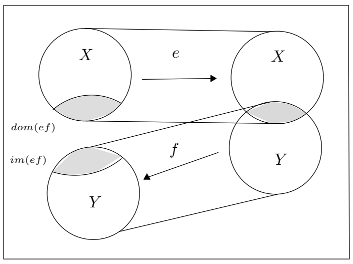
with a similar composition stated in Proposition 2.2.6; that is, if $\sigma : X_1 \longrightarrow Y_1$ and $\tau : X_2 \longrightarrow Y_2$ where $X_1, X_2, Y_1, Y_2 \subseteq [\pm n]$, then the composition $\sigma\tau$ is defined on the following domain:

$$
\begin{aligned}
dom(\sigma\tau) &= \{x \in [\pm n] : x \in dom(\sigma) \text{ and } x\sigma \in dom(\tau)\}; \\
&= \{x \in [\pm n] : x \in dom(\sigma) \text{ and } x \in dom(\tau)\sigma^*\}; \\
&= dom(\sigma) \cap dom(\tau)\sigma^*; \\
&= X_1 \cap X_2\sigma^*,
\end{aligned}
$$

where $\sigma^* : Y_1 \longrightarrow X_1$. Moreover, since $(-r)\sigma = -(r)\sigma$ and $(-r)\tau = -(r)\tau$, its composition satisfies this property as well; that is $(-r)(\sigma\tau) = -(r)(\sigma\tau)$
From the composition described above, it is obvious that

$$dom(\sigma\tau) \subseteq dom(\sigma). \tag{6.1}$$

Observation 6.1.3. Since for every partial signed permutation $\sigma : X_1 \longrightarrow Y_1$, there is a partial signed permutation $\sigma^* : Y_1 \to X_1$ belonging to MB_n such that $\sigma\sigma^*\sigma = \sigma$. Thus, MB_n is a regular monoid whose identity id is a signed permutation fixing every number in $[\pm n]$. Further, the group of units of MB_n is the group of signed permutations B_n, and the idempotents in MB_n are all partial signed identities on $X \subseteq [\pm n]$; that is, all $e \in MB_n$ such that e fixes $X \subseteq [\pm n]$ point-wise and is undefined in $[\pm n] \setminus X$. Let $X, Y \subseteq [\pm n]$ and e, f be idempotents on X and Y, respectively. Let us show that the idempotents commute; that is $ef = fe$. Consider the diagram below



and notice that since $e = e^*$ and $X \cap Y \subseteq X$, we have

$$\mathrm{dom}(ef) = (\mathrm{im}(e) \cap \mathrm{dom}(f))e^* = (X \cap Y)e^* = (X \cap Y)e = X \cap Y.$$

Thus, $\mathrm{im}(ef) = (X \cap Y)ef = X \cap Y$ as $X \cap Y \subseteq X, Y$ and e, f are idempotents. Hence, $\mathrm{dom}(ef) = X \cap Y = \mathrm{im}(ef)$. Similarly, $\mathrm{dom}(fe) = X \cap Y = \mathrm{im}(fe)$. Now, for all $\pm r \in X \cap Y$, we have $(\pm r)(ef) = ((\pm r)e)f = (\pm r)f = \pm r$. Similarly, $(\pm r)(ef) = \pm r$. Thus, the idempotents commute. Alternatively, for each partial signed permutation σ, its σ^* is clearly unique. Hence, MB_n is an inverse monoid.

We deduce Green's equivalence relations for the monoids of partial signed permutations MB_n from the isomorphisms mentioned in Propositions 2.2.18 and 2.2.21; however, it is also worthwhile to discuss why Green's equivalence relations hold for MB_n :

Proposition 6.1.4. *Let MB_n be the monoids of partial signed permutations. Then,*

(i) $(\sigma, \tau) \in \mathcal{R}$ *if and only if* $dom(\sigma) = dom(\tau)$;

(ii) $(\sigma, \tau) \in \mathcal{L}$ *if and only if* $im(\sigma) = im(\tau)$;

(iii) $(\sigma, \tau) \in \mathcal{D}$ *if and only if there exists* $\gamma \in MB_n$, *where* $dom(\sigma) = dom(\gamma)$ *and* $im(\gamma) = im(\tau)$, *and alternatively, there exists* $\gamma' \in MB_n$, *with* $im(\gamma') = im(\sigma)$, *and* $dom(\gamma') = dom(\tau)$; *and*

(iv) $(\sigma, \tau) \in \mathcal{J}$ *if and only if* $|dom(\sigma)| = |dom(\tau)|$ *if and only if* $|im(\sigma)| = |im(\tau)|$.

Proof. (i) Suppose $(\sigma, \tau) \in \mathcal{R}$; then, by Proposition 1.1.10(i), there exists $\gamma, \xi \in MB_n$ such that $\sigma = \tau\gamma$ and $\tau = \sigma\xi$. Thus, in consideration of (6.1), we have

$$dom(\sigma) = dom(\tau\gamma) \subseteq dom\ (\tau);$$

that is, $dom(\sigma) \subseteq dom\ (\tau)$. Similarly, we have $dom(\tau) \subseteq dom\ (\sigma)$, so $dom(\sigma) = dom(\tau)$. Conversely, if $dom(\sigma) = dom(\tau)$, then we can write σ as $\sigma = \tau\tau^*\sigma$ because $\tau\tau^*$ is the partial signed identity on $dom(\tau)$. Put $\gamma = \tau^*\sigma$, then we obtain a partial signed permutation $\gamma \in MB_n$ such that $\sigma = \tau\gamma$. Using a similar argument, we have $\tau = \sigma\xi$ for some $\xi \in MB_n$. Thus, both σ and τ are $\mathcal{R}$-related.

(ii) It can be proved in the same manner as (i).

(iii) Suppose that $(\sigma, \tau) \in \mathcal{D}$; then, by (1.3), there exists $\gamma \in MB_n$ such that $\sigma \mathrel{\mathcal{R}} \gamma$ and $\gamma \mathrel{\mathcal{L}} \tau$. Hence, using (i) and (ii), we obtain $dom(\sigma) = dom(\gamma)$ and $\mathrm{im}(\gamma) = \mathrm{im}(\tau)$. The reverse assertion follows in a similar manner.

(iv) Suppose that $(\sigma, \tau) \in \mathcal{J}$; then, by definition both σ and τ generate the same two-sided ideal; that is, there exist $\gamma_1, \gamma_2, \zeta_1, \zeta_1 \in MB_n$ such that $\sigma = \gamma_1\tau\gamma_2$ and $\tau = \zeta_1\sigma\zeta_2$. Consider $\sigma = \gamma_1 \cdot (\tau\gamma_2)$, and notice that by (6.1), we have $dom(\gamma_1 \cdot (\tau\gamma_2)) \subseteq dom(\gamma_1)$. Moreover, with careful consideration of $dom(\gamma_1 \cdot (\tau\gamma_2))$, we have that for all $x \in dom(\gamma_1 \cdot (\tau\gamma_2))$,

$$x \in dom(\gamma_1) \text{ and } x\gamma_1 \in dom(\tau\gamma_2). \tag{6.2}$$

However, it is also known that $dom(\tau\gamma_2) \subseteq dom(\tau)$. Thus, (6.2) can be rewritten as follows: for all $x \in dom(\gamma_1 \cdot (\tau\gamma_2))$,

$$x \in dom(\gamma_1) \text{ and } x\gamma_1 \in dom(\tau).$$

Now, since γ_1 is a bijective map and $dom(\gamma_1\tau\gamma_2) \subseteq dom(\gamma_1)$, the restriction of γ_1 into $dom(\gamma_1\tau\gamma_2)$ induces a one-to-one map into $dom(\tau)$; this is illustrated in Figure 6.1.

$$\gamma_1|_{dom(\gamma_1\tau\gamma_2)} : dom(\gamma_1\tau\gamma_2) \xrightarrow{1-1} dom(\tau).$$

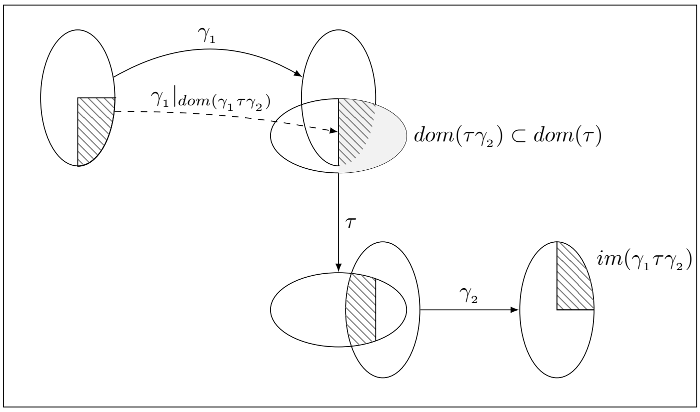


Figure 6.1: $\gamma_1|_{dom(\gamma_1\tau\gamma_2)}$ is a one-to-one map into $dom(\tau)$.

Thus, the injective map between two sets, $dom(\gamma_1\tau\gamma_2)$ and $dom(\tau)$, implies that

$$|dom(\gamma_1\tau\gamma_2)| \leq |dom(\tau)|\,; \tag{6.3}$$

that is, $|dom(\sigma)| \leq |dom(\tau)|$. Using a similar argument for $\tau = \zeta_1\sigma\zeta_2$, we acquire the reverse inequality, and then $|dom(\sigma)| = |dom(\tau)|$.

Conversely, suppose we have two partial signed permutations $\sigma, \tau \in MB_n$, where $\sigma : X_1 \longrightarrow Y_1$, $\tau : X_2 \longrightarrow Y_2$, and $|dom(\sigma)| = |dom(\tau)|$. Let $X_1 = \{\pm x_1, \cdots, \pm x_k\}$ and $X_2 = \{\pm x'_1, \cdots, \pm x'_k\}$. The aim is to show that there exist $\gamma_1, \gamma_2, \zeta_1, \zeta_2 \in MB_n$ such that $\sigma = \gamma_1\tau\gamma_2$ and $\tau = \zeta_1\sigma\zeta_2$. It suffices to show that $\sigma = \gamma_1\tau\gamma_2$ and $\tau = \zeta_1\sigma\zeta_2$ can then be shown in a similar manner. Since $|X_1| = |X_2|$, then there exists a bijection $\gamma_1 : X_1 \longrightarrow X_2$ such that $(x_r)\gamma_1 = x'_r$ and $(-x_r)\gamma_1 = -x'_r$ with $1 \leq r \leq k$. It also follows that $|Y_1| = |Y_2|$ as σ and τ are bijections. Consequently, a bijection must exist between subsets Y_1 and Y_2. Define a map γ_2 as follows:

$$\gamma_2 : Y_2 \longrightarrow Y_1, \qquad \text{where} \quad (\pm y)\gamma_2 := (\pm y)\tau^*\gamma_1^*\sigma.$$

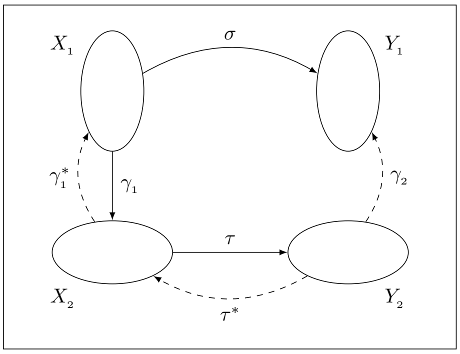


Figure 6.2: γ_2 is a composition of τ^* with γ_1^* and σ.

Clearly, γ_2 is a bijective map, as it is a composition of bijections. The only point that needs to be verified is the property $(-y)\gamma_2 = -(y)\gamma_2$.

$$
\begin{aligned}
(-y)\gamma_2 &= (-y)\tau^*\gamma_1^*\sigma \\
&= (-y)\tau^*(\gamma_1^*\sigma) \\
&= -(y)\tau^*\gamma_1^*\sigma \\
&= (-(y)\tau^*)\gamma_1^*\sigma \\
&= (-(y)\tau^*\gamma_1^*)\sigma \\
&= -(y)\tau^*\gamma_1^*\sigma \\
&= -(y)\gamma_2.
\end{aligned}
$$

Thus, we obtain $\gamma_1\tau\gamma_2 = \sigma$. Similarly, it can be shown $\tau = \zeta_1\sigma\zeta_2$. Hence, $\sigma \ \mathcal{J} \ \tau$. Notice that since $\sigma \ \mathcal{J} \ \tau \iff |\mathrm{dom}(\sigma)| = |\mathrm{dom}(\tau)|$ and σ, τ are bijections, we have $|\mathrm{im}(\sigma)| = |\mathrm{dom}(\sigma)| = |\mathrm{dom}(\tau)| = |\mathrm{im}(\tau)|$. □

Proposition 6.1.5. *On the monoid of partial signed permutations MB_n, the relations $\mathcal{D}$ and $\mathcal{J}$ coincide.*

The last property in Proposition 6.1.4 plays a vital role in describing the $\mathcal{J}$-classes of the monoid MB_n. To determine the $\mathcal{J}$-classes, we need to illuminate the domains of partial signed permutations and then partition the monoid MB_n based on all possible sizes of these domains. However, upon careful consideration of the property, which emphasises that whenever $+x \in \mathrm{dom}(\sigma)$, $-x \in \mathrm{dom}(\sigma)$ as well for all $\sigma \in MB_n$, we deduce that the domain of any partial signed permutation σ always has an even size. Consequently, the largest domain is $2n$, and the smallest domain is zero. Hence, the $\mathcal{J}$-classes of the monoid MB_n can be labelled in the following manner:

$$
J_0, \ J_2, \ \ldots, \ J_{2k}, \ \ldots, \ J_{2n-2}, \ J_{2n}, \tag{6.4}
$$

where J_0 indicates that the $\mathcal{J}$-class consists of the zero partial signed permutation, and J_{2k} indicates that the $\mathcal{J}$-class consists of all partial signed permutations $\sigma \in MB_n$ whose domains are subsets $X \subset [\pm n]$ with size $2k$ which satisfy the following property: for all $x \in X, -x \in X$. Further, by the $\mathcal{J}$-class labelling listed in (6.4), it is clear that the number of all $\mathcal{J}$-classes of MB_n is $n+1$, beginning with J_0 and ending with J_{2n}. Notice that if $\sigma \in J_{2k}$, then σ is a bijection between subsets of $[\pm n]$ with the same size $2k$; that is,

$$
|\mathrm{dom}(\sigma)| = |\mathrm{im}(\sigma)| = 2k.
$$

Therefore, the sizes of the σ images are also even.

Now, in view of the coincidence of the relations $\mathcal{J}$ and $\mathcal{D}$, the $\mathcal{J}$-class J_{2k} can be visualised as an eggbox. In addition, since MB_n is an inverse monoid, each $\mathcal{J}$-class

J_{2k} consists of equal numbers of rows and columns by Proposition 1.2.14. Recall that the rows of an eggbox are $\mathcal{R}$-classes and its columns are $\mathcal{L}$-classes. However, since any two partial signed permutations σ, τ are $\mathcal{R}$-related if and only if they have the same domain, then all partial signed permutations placed in an $\mathcal{R}$-class must have the same domain. This allows us to label the rows of an eggbox J_{2k} by all possible domains of σ with size $2k$. In other words, the rows of J_{2k} can be labelled with all subsets of $X \subseteq [\pm n]$ with size $2k$. Similarly, as two partial signed permutations γ and ζ are $\mathcal{L}$-related if and only if they have the same images, it follows that all partial signed permutations located in an $\mathcal{L}$-class have the same images. Consequently, we can label the columns of J_{2k} by all possible images of γ with size $2k$. In other words, the columns of J_{2k} can be labelled with all subsets $Y \subseteq [\pm n]$ with size $2k$.

However, the following question needs to be raised: how many rows/columns are there in the $\mathcal{J}$-class J_{2k}? In order to answer this enquiry, we need to pay close attention to the domain of any partial signed permutation $\sigma \in MB_n$. As the domain of σ is a subset of $[\pm n]$ with size $2k$ such that whenever $x \in \text{dom}(\sigma)$, $-x \in \text{dom}(\sigma)$ as well, let $\text{dom}(\sigma) = X = \{\pm x_1, \cdots, \pm x_k\}$. Recall the property $(-x_r)\sigma = -(x_r)\sigma$ for all $1 \leq r \leq k$, which says that wherever x_r goes under the action of σ, the negative of x_r will be its mirror image. In other words, the negatives of all x_r numbers will be permuted by σ in the same manner that their positives x_r are permuted by σ. Thus, all we need to be wary of is selecting all possible subsets $\{x_1, \cdots, x_k\}$ of $[+n] = \{1, \cdots, n\}$ with size k and then consider their corresponding negatives. This implies that we consider $\binom{n}{k}$ subsets of $[+n]$ of size k, and by considering their negatives, we have $\binom{n}{k}$ subsets of $[\pm n]$ of size $2k$ labelling the rows and columns of the $\mathcal{J}$-class J_{2k}. Hence, we have $\binom{n}{k}$ rows and $\binom{n}{k}$ columns for the $\mathcal{J}$-class J_{2k}.

As indicated in Chapter 1, each row intersects column, thereby creating cells called $\mathcal{H}$-classes, and there is only one $\mathcal{H}$-class in each row and column containing a unique idempotent, since MB_n is an inverse monoid. Further, we know by Observation 6.1.3 that an idempotent $f \in J_{2k}$ is a partial signed identity on $X = \{\pm x_1, \cdots, \pm x_k\}$; f fixes X point-wise and it is undefined in $[\pm n] \setminus X$.

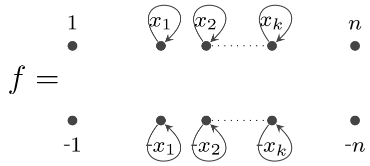


Figure 6.3: Idempotent f belongs to J_{2k}.

As $\text{dom}(f) = X = \text{im}(f)$, this suggests how a maximal subgroup G_f in J_{2k} can be described. It is indeed an $\mathcal{H}$-class in which its domain and image are the same; that is,

$$G_f = \{\sigma : X \xrightarrow{bij} X : X \subseteq [\pm n], \text{ where } |X| = 2k \text{ and } (-i)\sigma = -(i)\sigma\}. \tag{6.5}$$

In view of Definition 3.3.1, it is evident that a maximal subgroup G_f is isomorphic to the signed permutation group B_k. We rearrange the rows and columns of the $\mathcal{J}$-class J_{2k} in such a manner that all maximal subgroups appear in the diagonal of the J_{2k}. After doing so, $\binom{n}{k}$ maximal subgroups appear in J_{2k}, and each one is isomorphic to the others. Hence, it is reasonable to only consider a maximal subgroup representative for each $\mathcal{J}$-class J_{2k} of MB_n, where $0 \leq k \leq n$.

The following proposition suggests how to order the $\mathcal{J}$-classes of MB_n.

Proposition 6.1.6. *$J_{2k} \leq J_{2l}$ if and only if $2k \leq 2l$ for all $0 \leq k, l \leq n$.*

Proof. Suppose $\tau \in J_{2k}$, $\sigma \in J_{2l}$ and $J_{2k} \leq J_{2l}$. We want to show that $2k \leq 2l$. Since $\tau \in J_{2k}$, τ is a partial signed map whose domain is of the size $2k$. Set $S^1 = MB_n$, and consider the set $S^1\tau S^1$. Recall that by (6.3), we have $|\mathrm{dom}(\gamma_1\tau\gamma_2)| \leq |\mathrm{dom}(\tau)|$ for all $\gamma_1, \gamma_2 \in S^1$. Hence, the set $S^1\tau S^1$ may be described in the following manner:

$$S^1\tau S^1 = \{\gamma_1\tau\gamma_2 : \ |\mathrm{dom}(\gamma_1\tau\gamma_2)| \leq 2k, \ \text{ for all } \gamma_1, \gamma_2 \in S^1\}.$$

Similarly, since $\sigma \in J_{2l}$, σ is a partial signed map whose domain is of the size $2l$, then,

$$S^1\sigma S^1 = \{\gamma_1\sigma\gamma_2 : \ |\mathrm{dom}(\gamma_1\sigma\gamma_2)| \leq 2l, \ \text{ for all } \gamma_1, \gamma_2 \in S^1\}.$$

However, since $J_{2k} \leq J_{2l}$, then $S^1\tau S^1 \subseteq S^1\sigma S^1$, and this implies $2k \leq 2l$. The other direction easily follows from the assumption. $\square$

The preceding proposition accounts for the fact that all the $\mathcal{J}$-classes of MB_n are linearly (totally) ordered by the sizes of the domains; thus, we can visualise all $\mathcal{J}$-classes J_{2k} of MB_n where $0 \leq k \leq n$ as a chain beginning with J_0 and ending with J_{2n} as follows:

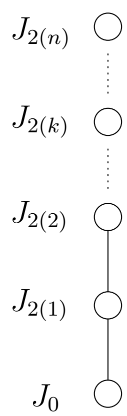
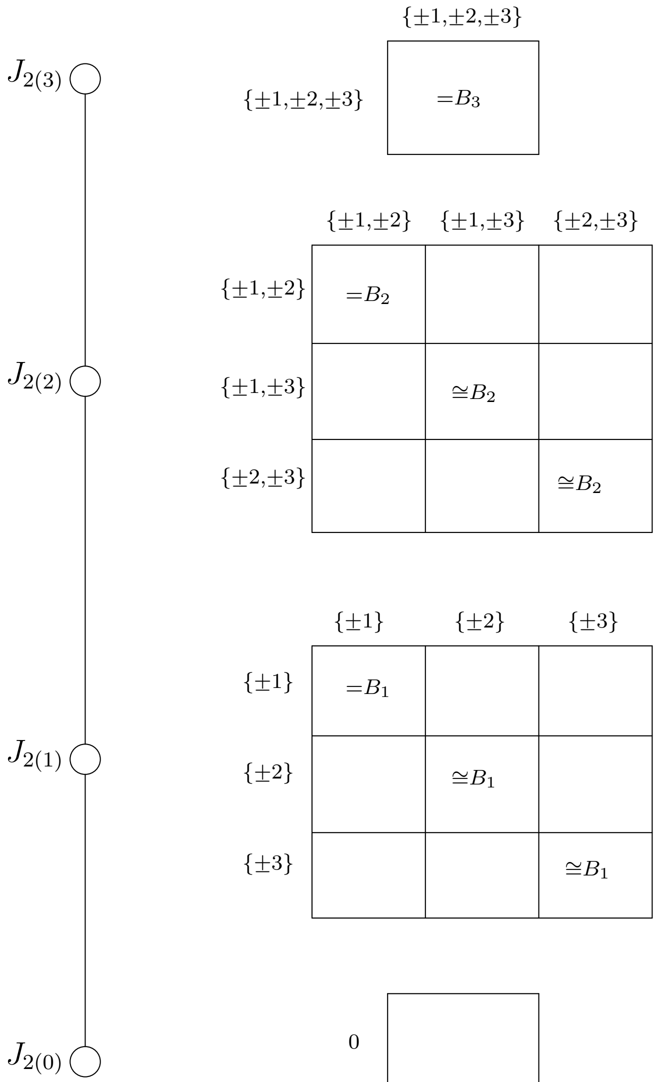


Figure 6.4: All $\mathcal{J}$-classes of MB_n.

Example 6.1.7. Let $n = 3$. The following diagram illustrates the $\mathcal{J}$-classes of the monoid of partial signed permutations MB_3.



Remark 6.1.8. Consider an idempotent $f \in J_{2k}$, as illustrated in Figure 6.3, and the $\mathcal{R}$-class of f. Notice that if $\sigma \in R_f$, then $\text{dom}(\sigma) = X = \text{dom}(f)$. Thus, σ can be written as $\sigma : X \longrightarrow Y$ where $Y \subseteq [\pm n]$ with $|Y| = 2k$. For all $\tau \in MB_n$, $\sigma \cdot \tau$ belongs to R_f requires that

$$\text{dom}(\sigma \cdot \tau) = \text{dom}(f) = X.$$

Now, observe that

$$\begin{aligned} \text{dom}(\sigma \cdot \tau) &= (\text{im } \sigma \cap \text{dom}(\tau))\sigma^* \\ X &= (Y \cap \text{dom}(\tau))\sigma^*. \end{aligned} \tag{6.6}$$

By multiplying both sides of (6.6) from the right by σ, we obtain

$$X\sigma = (Y \cap \mathrm{dom}(\tau))\ \sigma^*\sigma$$
$$Y = (Y \cap \mathrm{dom}(\tau))id_{_Y}.$$
$$Y = Y \cap \mathrm{dom}(\tau).$$

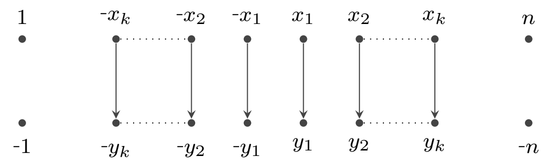
Hence, $Y \subseteq \mathrm{dom}(\tau)$. Conversely, if $Y \subseteq \mathrm{dom}(\tau)$, then it is straightforward that $\sigma \cdot \tau \in R_{_f}$. Hence,

$$\sigma \cdot \tau \in R_{_f} \text{ if and only if } \ Y \subseteq \mathrm{dom}(\tau). \tag{6.7}$$

All the propositions and discussions above contribute to a better understanding of the structure of all $\mathcal{J}$-classes of the monoid of partial signed maps MB_n.

Theorem 5.1.10 shows how a partial map belonging to I_n can be expressed uniquely as a product of links and disjoint cycles. Let us end this section by proving the corresponding assertion in the context of the monoids of partial signed permutations MB_n. As a first step towards investigating the decomposition of a partial signed permutation, let $\sigma \in MB_n$ and $\sigma : X \xrightarrow{bij} Y$, where $X = \{\pm x_1, \cdots, \pm x_k\}$ and $Y = \{\pm y_1, \cdots, \pm y_k\}$ are subsets of $[\pm n]$ and $1 \leq k \leq n$.

For notational simplicity, we define σ as $(\pm x_r)\sigma = \pm y_r$ for all $1 \leq r \leq k$. Thus, it is obvious that for all x_r, $(-x_r)\sigma = -(x_r)\sigma$.

$\sigma =$ 

Now consider the following cases:

Case 1. If $X = Y$, then σ is a signed permutation lying in B_X and by Proposition 3.3.5, it can be expressed uniquely as a disjoint product of positive and negative cycles.

Case 2. If $X \neq Y$ and $X \cap Y \neq \emptyset$, then choose any $x_r \in X \setminus X \cap Y$, and consider its image under σ. Now, either $(x_r)\sigma$ belongs to $X \cap Y$, or it belongs to $Y \setminus X \cap Y$. Observe that $(x_r)\sigma$ can never be in $X \setminus X \cap Y$ as Y is the set of images of σ. Thus, we have two sub-cases.

(I) If $(x_r)\sigma \in X \cap Y$; that is $(x_r)\sigma \in X$. Let $(x_r)\sigma := x_{r+1}$, and keep taking the images under σ as long as the images remain in $X \cap Y$. However, as $X \cap Y$ is a finite subset of $[\pm n]$ containing distinct elements and σ is a bijection, we eventually end up obtaining element $y_s \in Y \setminus X \cap Y$. Hence, a sequence of numbers is produced from $[\pm n]$, in which the first number is $x_r \in X \setminus X \cap Y$ and the last number is $y_l \in Y \setminus X \cap Y$:

$$x_r \xrightarrow{\sigma} x_{r+1} \xrightarrow{\sigma} x_{r+2} \xrightarrow{\sigma} \cdots \xrightarrow{\sigma} x_s \xrightarrow{\sigma} y_l. \tag{6.8}$$

Let us denote such a sequence by $[x_r, x_{r+1}, \ \ldots \ , x_s, y_l]$. Observe that the first number x_r in a sequence is only in $\mathrm{dom}(\sigma)$ but not in $\mathrm{im}(\sigma)$. However, the last number y_l is only in $\mathrm{im}(\sigma)$ but not in $\mathrm{dom}(\sigma)$. Meanwhile, all other numbers in between belong to both $\mathrm{dom}(\sigma)$ and $\mathrm{im}(\sigma)$.

Claim: $-x_r$ also belongs to $X \setminus X \cap Y$.

Proof. We know necessarily that $-x_r \in X$. Suppose, with the aim of obtaining a contradiction, that $-x_r \in Y$; then, x_r must belong to Y because whenever $-x_r \in Y$, x_r must be in Y as well. Thus, we have $x_r \notin X \setminus X \cap Y$, which contradicts our choice that $x_r \in X \setminus X \cap Y$. Hence, $-x_r \notin Y$, so $-x_r$ must be in $X \setminus X \cap Y$. $\square$

Now, by taking an image of $-x_r$ under σ and considering the images in (6.8) as well as the property that for all x_r, $(-x_r)\sigma = -(x_r)\sigma$, we can obtain another sequence:

$$[-x_r, -x_{r+1}, \ \ldots \ , -x_s, -y_l].$$

Observe that the first number $-x_r$ of the sequence is only in $\mathrm{dom}(\sigma)$ and not in $\mathrm{im}(\sigma)$ and all other numbers except the last one belong to both $\mathrm{dom}(\sigma)$ and $\mathrm{im}(\sigma)$. We only need to verify that the last number $-y_l$ of the sequence belongs to $\mathrm{im}(\sigma)$ but not to $\mathrm{dom}(\sigma)$. Since $y_l \in Y$, $-y_l \in Y$ as well. Suppose, with the aim of a contradiction, that $-y_l \in X$, then $y_l \in X$. Thus, we attain a contradiction as $y_l \in Y \setminus X \cap Y$. Hence, $-y_l \notin X$ and then $-y_l \in Y \setminus X \cap Y$; that is, it only belongs to $\mathrm{im}(\sigma)$ and not to $\mathrm{dom}(\sigma)$. Such a pair of sequences,

$$[x_r, x_{r+1}, \ \ldots \ , x_s, y_l][-x_r, -x_{r+1}, \ \ldots \ , -x_s, -y_l],$$

is called a positive link.

(II) If $(x_r)\sigma \in Y \setminus X \cap Y$, then we obtain a sequence $[x_r, y_r]$ of length two, where $y_r = (x_r)\sigma$. Thus, by the same argument used in (I), we also acquire the second sequence $[-x_r, -y_r]$. Hence, we have a two-element-positive link $[x_r, y_r][-x_r, -y_r]$.

In both sub-cases (I and II), if the set $X \setminus X \cap Y$ has more than one element, then we can produce another positive link. We persist going through the entire procedure of producing positive links until the set $X \setminus X \cap Y$ vanishes. Notice that if there exist some $x_r \in X \cap Y$ that are not contained in any produced positive link, it means that their images under σ never belong to $Y \setminus X \cap Y$, and their images must belong to $X \cap Y$. In fact, such numbers, which remain at the intersection $X \cap Y$ after considering their images and are not contained in produced positive links, will produce positive or negative cycles. In other words, the elements of the intersection $X \cap Y$ either appear in positive links or appear as signed permutations that can also be written as disjoint products of positive or negative cycles, as stated in Proposition 3.3.5.

Case 3. If $X \neq Y$ and $X \cap Y = \emptyset$, then the image of any $x_r \in X$ under σ will only be in Y. Hence, we obtain a two-element-positive link $[x_r, y_r][-x_r, -y_r]$ for all $x_r \in X$. Importantly, the virtue of the preceding steps guarantees that the above decomposition is unique.

We recapitulate these observations in a theorem, described below

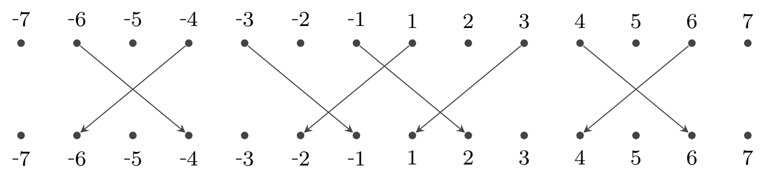
Theorem 6.1.9. Every partial signed permutation in MB_n may be expressed uniquely as a product of positive links and disjoint positive and negative cycles.

Observation 6.1.10. In view of the attribute of the produced sequence in (6.8), we confirm that a negative link does not exist; this is because if there were such a link, then there would be a sequence of the following form:

$$[x_r,\ x_{r+1},\ \ldots\ ,\ y_l,\ -x_r,\ -x_{r+1},\ \ldots\ ,\ -y_l].$$

Observe that all numbers occurring in the above sequence, including y_l, belong to $\mathrm{dom}(\sigma)$, except for the last number $-y_l$, which belongs to $\mathrm{im}(\sigma)$ and not to $\mathrm{dom}(\sigma)$. Thus, we obtain a contradiction with the fact that whenever $y_l \in dom(\sigma)$, $-y_l$ must be in $dom(\sigma)$ as well. Hence, no such link exists.

Example 6.1.11. Let $\alpha \in MB_7$ such that $\alpha : X \xrightarrow{bij} Y$, where $X = \{\pm1, \pm3, \pm4, \pm6\}$ and $Y = \{\pm1, \pm2, \pm4, \pm6\}$, and α is defined in the following manner:

$\alpha =$ 

Observe that $X \cap Y = \{\pm1, \pm4, \pm6\}$ and $X \setminus X \cap Y = \{\pm3\}$; note also that $Y \setminus X \cap Y = \{\pm2\}$. Hence, α can be decomposed into a positive link and a positive cycle, as follows.

$$[3, 1, -2][-3, -1, 2](4, 6)(-4, -6).$$

6.2 The Specht module for the monoids of partial signed permutations MB_n

This section examines explicit accounts of irreducible representations over the complex field of the monoids of partial signed permutations MB_n, which draw on the combinatorial objects referred to as Young tableaux. The Clifford−Munn correspondence is the main way in which to achieve this. The key starting point is to examine the irreducible representations of the maximuml subgroups G_i which constitute signed permutations subgroups B_k of B_n where $(k \leq n)$.

Induction is the main tool utilised for turning irreducible representations of these maximal subgroups into irreducible representations of the monoid MB_n. This approach is completely analogous to the one which was outlined in Section 5.2, although the inputs and outputs in each case are different. At the same time, we give some examples of Specht modules for the monoid of partial signed permutations MB_3.

Consider the $\mathcal{J}$-classes J_{2k} of MB_n where $0 \leq k \leq n$. Recall from Section 4.2.2 that for each $\mathcal{J}$-class J_{2k} of MB_n, if we choose an idempotent $f \in J_{2k}$ and consider the maximal subgroup G_f, then the induction of an irreducible representation of G_f into MB_n results in such an irreducible representation of MB_n. In the following account, we will pave the way to go through this process in detail:

Fix an idempotent $f \in J_{2k}$; this means that f is a partial signed identity on X that fixes X point-wise, and it is undefined in $[\pm n] \setminus X$, where $X \subseteq [\pm n]$ with $|X| = 2k$.

$$(r)f = \begin{cases} r & r \in X,\ X \subseteq [\pm n], \\ \text{undefined} & \text{otherwise.} \end{cases}$$

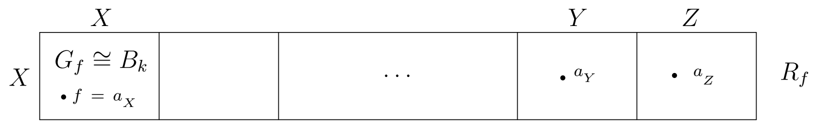
Consider R_f as the $\mathcal{R}$-class of f, to which the idempotent f belongs. Recall that the maximal subgroup G_f is isomorphic to the signed permutation subgroups B_k of B_n. Because of the isomorphism and notational simplicity, consider $X = \{\pm 1, \ldots, \pm k\}$. Recall from Section 3.3 that the irreducible representations of B_k over the complex field are parametrised by the complementary partitions (λ, μ) of k.

Fix $(\lambda, \mu) \vdash k$ and let t be a Young tableau of shape (λ, μ), with entries from $X = \{\pm 1, \ldots, \pm k\}$. Thus, t is a double Young diagram (t_λ, t_μ), where the ij-th entries $x_z(i, j)$ belong to X, and $z = \lambda$ or $z = \mu$ and satisfies the condition that, for all $x \in X$, either x or $-x$, but not both, occurs in t. Consider $M^{(\lambda,\mu)}$ the $\mathbb{C}$-vector space with basis the distinct (λ, μ)-tabloids $\{t\}$; that is,

$$M^{(\lambda,\mu)} = \mathbb{C}\text{-}[\{t^1\}\ , \ldots,\ \{t^m\}] \tag{6.9}$$

where $\{t^r\}$ with $1 \leq r \leq m$ is a complete list of distinct (λ, μ)-tabloids. Recall from Section 3.3 that $M^{(\lambda,\mu)}$ is a reducible representation of B_k. Observe that inducing $M^{(\lambda,\mu)}$ into MB_n results in a representation of the monoid of partial signed permutation MB_n, which is also reducible, as we will see below. Indeed, the induction process requires the following steps:

(a) Fix a representation $M^{(\lambda,\mu)}$ of B_k. Take a representative a_Y of each $\mathcal{H}$-class of the $\mathcal{R}$-class of f, considering the idempotent f as representative of the $\mathcal{H}$-class in which B_k is placed.



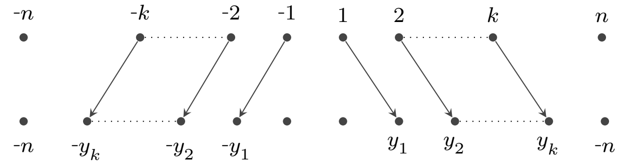
Henceforth, we will fix the following choice for a_Y: a_Y is a partial signed map with the domain $X = [\pm k]$ and image $Y = \{\pm y_1, \ldots, \pm y_k\}$, where

$$-y_k < \cdots < -y_2 < -y_1 < y_1 < y_2 < \cdots < y_k, \tag{6.10}$$

and $Y \subseteq [\pm n]$, with $|Y| = 2k$. In addition,

$$(x)a_Y = \begin{cases} y_x & x \in X \text{ and } x > 0, \\ -y_{-x} & x \in X \text{ and } x < 0, \\ \text{undefined} & \text{otherwise.} \end{cases} \tag{6.11}$$

In fact, a representative a_Y for an $\mathcal{H}$-class labelled by Y can be illustrated in the following manner:

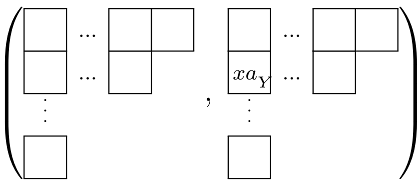
$a_Y =$ 

The above illustration of such a partial signed map a_Y ensures the satisfaction of the property $(-x)a_Y = -(x)a_Y$, where $x \in X$. Let us define the action of a partial signed map $b \in MB_n$ on any (λ, μ)-Young tableau t as follows:

$$t\ .\ b = \begin{cases} x_z(i,j) \cdot b = xb_z(i,j), & \text{if } x_z(i,j) \in \mathrm{dom}(b) \text{ for all } i, j \textit{ and } z = \lambda \textit{ or } z = \mu \\ 0, & \text{otherwise} \end{cases} \tag{6.12}$$

The only property that needs to be verified is that for all entries $x \in t$, only $(x)b$ or $-(x)b$ occurs in (λ, μ)-Young tableau $t \cdot b$. Suppose, with the aim of a contradiction, that there were $x \in t$ such that $(x)b$ and $-(x)b$ occurred simultaneously in tb. Then, we would have $(-x)b \in tb$ as $(-x)b = -(x)b$, and this requires that $-x \in t$. Hence, we obtain a contradiction with the fact that as long as $x \in t$, $-x \notin t$.

In particular, since the entries of t^r belong to X, if we apply a partial signed map $a_{_Y}$ to t^r, we obtain a (λ, μ)-Young tableau $t^r_{_Y} := t^r \cdot a_{_Y}$ where the entries are in Y. In other words, the entries of a (λ, μ)-Young tableau $t^r_{_Y}$ are precisely $(x)a_{_Y}$ where x is an element of X appearing in t^r.

$t^r_{_Y} =$ 

It is also worthwhile pointing out an alternative description of (λ, μ)-tableau $t^r_{_Y}$. If $t^{^r} = (t^{^r}_\lambda, t^{^r}_\mu)$ is a (λ, μ)-tableau where the ij-th entries $x_z(i,j)$ belong to X, and $z = \lambda$ or $z = \mu$, then, a tableau $t^r_{_Y}$ of shape (λ, μ) is a double Young diagram $((t^{^r}_\lambda)_{_Y}, (t^{^r}_\mu)_{_Y})$ where

$$\begin{aligned} (t^{^r}_\lambda)_{_Y} &:= \text{ a tableau of shape } \lambda \text{ whose entries are } \ x_\lambda(i,j) \cdot a_{_Y}, \\ (t^{^r}_\mu)_{_Y} &:= \text{ a tableau of shape } \mu \text{ whose entries are } \ x_\mu(i,j) \cdot a_{_Y} \end{aligned}$$

Let us now draw special attention to some definitions that contribute to understanding the upcoming steps as well as the analogues of the Specht module for the monoid of partial signed permutations MB_n.

Definition 6.2.1. Fix $(\lambda, \mu) \vdash k$, where $0 \leq k \leq n$. Let $t^r_{_Y}$ be a (λ, μ)-Young tableau, where the entries are $x_z(i,j) \cdot a_{_Y}$ and $z = \lambda$ or $z = \mu$ as well as $Y \subseteq [\pm n]$ with $|Y| = 2k$. Then, the row group $R_{t^r_{_Y}}$ is a Young subgroup of a maximal subgroup $B_{_Y}$ that preserves the rows of $t^r_{_Y}$ and may change the signs of the entries of $(t^{^r}_\mu)_{_Y}$; that is,

$$\begin{aligned} R_{t^r_{_Y}} = \Big\{ \sigma \in B_{_Y} : (x_\lambda(i,j)a_{_Y})\sigma = x_\lambda(i,p)a_{_Y}, \\ (x_\mu(i,j)a_{_Y})\sigma = \pm x_\mu(i,q)a_{_Y}, \text{ for some } p \text{ and } q \Big\}. \end{aligned}$$

Notice that what we meant by the possibility of changing the signs of the entries of $(t^{^r}_\mu)_{_Y}$ is that a row group $R_{t^r_{_Y}}$ contains some signed permutations σ that may change the sign of each individual entry of $(t^{^r}_\mu)_{_Y}$ and leave the other ones unchanged. Hence, a signed permutation σ does not necessarily have to change the signs of all entries in the $(t^r_\mu)_Y$ at once to be an element in $R_{t^r_Y}$.

Similarly, the column group $C_{t^r_Y}$ is defined in the following manner;

Definition 6.2.2. Fix $(\lambda, \mu) \vdash k$, where $0 \leq k \leq n$. Let t^r_Y be a (λ, μ)-Young tableau where the entries are $x_z(i,j) \cdot a_Y$ and $z = \lambda$ or $z = \mu$ as well as $Y \subseteq [\pm n]$ with $|Y| = 2k$. Then, the column group $C_{t^r_Y}$ is a Young subgroup of B_Y that preserves the columns of t^r_Y and may change the sign of the entries of $(t^r_\lambda)_Y$; that is,

$$C_{t^r_Y} = \Big\{\tau \in B_Y : (x_\lambda(i,j)a_Y)\tau = \pm\, x_\lambda(p,j)a_Y,$$
$$(x_\mu(i,j)a_Y)\tau = x_\mu(q,j)a_Y, \text{ for some } p \text{ and } q\Big\}.$$

Similar comments regarding sign change apply to the column group $C_{t^r_Y}$ as do to the row group $R_{t^r_Y}$. Observe that both $R_{t^r_Y}$ and $C_{t^r_Y}$ are subgroups of a maximal subgroup B_Y in the $\mathcal{J}$-class labelled by $2k$. Hence, both $R_{t^r_Y}$ and $C_{t^r_Y}$ are placed on the diagonal $\mathcal{H}$-class labelled by Y.

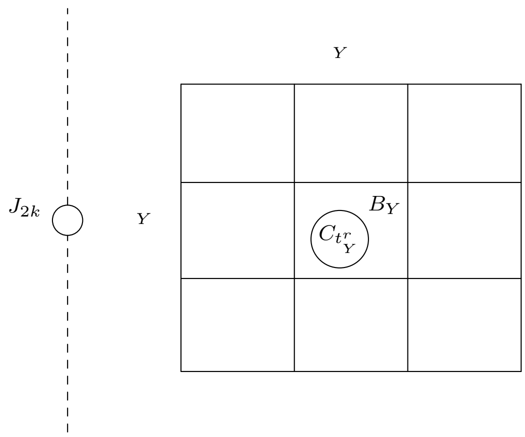


Figure 6.5: Column group $C_{t^r_Y}$ placed in the $\mathcal{H}$-class labelled by Y.

It is also significant to observe that not all the elements of MB_n that preserve the columns of t^r_Y and may change the sign of the entries of $(t^r_\lambda)_Y$ contained in $C_{t^r_Y}$. For instance, if $k \neq n$, we may have a signed permutation $\tau \in B_n \subset MB_n$ that preserves the columns of t^r_Y and may change the sign of the entries of $(t^r_\lambda)_Y$; however clearly, $\tau \notin C_{t^r_Y}$ as $C_{t^r_Y}$ is a subgroup of B_Y with $|Y| = 2k$.

Let us now proceed to the induction process:

(b) Consider a copy of $M^{(\lambda,\mu)}$ for each $\mathcal{H}$-class, as follows: let $M^{(\lambda,\mu)}_Y$ be a copy of $M^{(\lambda,\mu)}$ corresponding to an $\mathcal{H}$-class where a_Y is its representative. The approach to constructing such a copy $M^{(\lambda,\mu)}_Y$ is that for each basis element $\{t^r\} \in M^{(\lambda,\mu)}$ presented in (6.9), where $1 \leq r \leq m$, let $\{t^r_Y\} := \{t^r \cdot a_Y\}$ be a tabloid in $M_Y^{(\lambda,\mu)}$. Note that a tabloid $\{t_Y^r\}$ can also be described as the orbit of t_Y^r under the row group $R_{t_Y^r}$. Further, $M_Y^{(\lambda,\mu)}$ is defined as the $\mathbb{C}$-vector space with basis the (λ,μ)-tabloids $\{t_Y^r\}$. It should be noted that the entries of any basis vector (λ,μ)-tabloid $\{t_Y^r\}$ of $M_Y^{(\lambda,\mu)}$ are elements from Y, and whenever a subset Y is altered to another subset $Z = \{\pm z_1, \dots, \pm z_k\}$ of $[\pm n]$, we acquire another copy of $M^{(\lambda,\mu)}$ corresponding to another $\mathcal{H}$-class. Since $M_Y^{(\lambda,\mu)}$ is a copy of $M^{(\lambda,\mu)}$, they all have the same dimension.

(c) Consider all the copies $M_Y^{(\lambda,\mu)}$, where $Y \subseteq [\pm n]$ with $|Y| = 2k$, and define the following vector space:

$$M^{(\lambda,\mu)} \uparrow MB_n = \bigoplus_Y M_Y^{(\lambda,\mu)}. \tag{6.13}$$

Note that the dimension of the direct sum is the sum of the dimensions of each $M_Y^{(\lambda,\mu)}$; that is,

$$\dim \bigoplus_Y M_Y^{(\lambda,\mu)} = \sum_Y \dim M_Y^{(\lambda,\mu)} = \binom{n}{k} \cdot \dim M_Y^{(\lambda,\mu)} = \binom{n}{k} \cdot \dim M^{(\lambda,\mu)},$$

since there are $\binom{n}{k}$ subsets Y of $[\pm n]$ where $|Y| = 2k$, with the property that, whenever $y \in Y$, $-y \in Y$ and $M^{(\lambda,\mu)} \cong M_Y^{(\lambda,\mu)}$.

(d) Define the action of MB_n on $M^{(\lambda,\mu)} \uparrow MB_n$ in the following manner: for all $b \in MB_n$ and any basis vector $\{t_Y^r\} \in M_Y^{(\lambda,\mu)}$,

$$\{t_Y^r\} \cdot b = \begin{cases} \text{has } ij\text{-th entries } x_z(i,j)(a_Y \cdot b) & \text{if } \ a_Y \cdot b \in R_f, \\ 0 & \text{otherwise,} \end{cases} \tag{6.14}$$

where $z = \lambda$ or $z = \mu$. In view of the result presented in (6.7), $a_Y \cdot b$ belongs to R_f if and only if $Y \subseteq \mathrm{dom}(b)$. Hence, the above formula can be elaborated in the following manner:

$$\{t_Y^r\} \cdot b = \begin{cases} \text{has } ij\text{-th entries } x_z(i,j)(a_Y \cdot b) & \text{if } \ Y \subseteq \mathrm{dom}(b), \\ 0 & \text{otherwise,} \end{cases} \tag{6.15}$$

In order to show that the above action is well-defined, suppose that $\{t_Y^r\}$ and $\{s_Y\}$ are row-related. With a careful consideration of the row group $R_{t_Y^r}$, we note that $\{t_Y^r\} = \{s_Y\}$ requires that the rows' contents Z_i of both $(t_\lambda^r)_Y$ and $(s_\lambda)_Y$ are identical. However, the rows' contents of $(t_\mu^r)_Y$ and $(s_\mu)_Y$ can be identified as follows:

The entries of each i-th row of $(t_\mu^r)_Y$ can be split up into disjoint union sets

$Y_i \bigcup W_i$; where set Y_i contains numbers that are identical to certain numbers in the i-th row of $(s_\mu)_Y$ and set W_i contains all the remaining numbers such that their signs are changed in the corresponding i-th row in $(s_\mu)_Y$.

$$\left(-\,,\ \begin{array}{l}\hline Y_1 \cup W_1 \\ \hline \vdots \\ \hline Y_i \cup W_i \\ \hline \\ \underline{\quad} \end{array}\right) = \{-,(t^r_\mu)_Y\} = \{-,(s_\mu)_Y\} = \left(-\,,\ \begin{array}{l}\hline Y_1 \cup (-W_1) \\ \hline \vdots \\ \hline Y_i \cup (-W_i) \\ \hline \\ \underline{\quad} \end{array}\right)$$

Now, if $Y \subseteq \mathrm{dom}(b)$, then applying $b \in MB_n$ to both tabloids requires applying b to the rows' contents of $(t^r_\lambda)_Y$ and $(s_\lambda)_Y$ as well as the rows' contents of $(t^r_\mu)_Y$ and $(s_\mu)_Y$. In the first case, as the rows' entries of $(t^r_\lambda)_Y$ and $(s_\lambda)_Y$ are identical, the above action is well-defined as for the symmetric inverse monoid I_n [see (5.11)]. Hence, it remains to verify that the MB_n action is also well-defined for the rows' contents of $(t^r_\mu)_Y$ and $(s_\mu)_Y$. Notice that applying b to each i-th row of $(t^r_\mu)_Y$ implies that

$$(Y_i \bigcup W_i)b = Y_ib \bigcup W_ib = Y' \bigcup W \qquad [\text{Put } Y' = Y_ib \text{ and } W = W_ib]. \quad (6.16)$$

However, applying b to each i-th row of $(s_\mu)_Y$ yields that

$$(Y_i \bigcup (-W_i))b = Y_ib \bigcup (-W_i)b = Y_ib \bigcup (-(W_ib)) = Y' \bigcup (-W). \qquad (6.17)$$

Thus, by comparing equations (6.16) and (6.17), it is clear that each row of $(t^r_\mu)_Y \cdot b$ splits up into disjoint union sets $Y' \bigcup W$, where one set is identical to its corresponding in $(s_\mu)_Y \cdot b$ and the other set is the negative of its corresponding in $(s_\mu)_Y \cdot b$.

$$\{-,(t^r_\mu)_Y\} \cdot b = \left(-\,,\ \begin{array}{l}\hline Y_1b \cup W_1b \\ \hline \vdots \\ \hline Y_ib \cup W_ib \\ \hline \\ \underline{\quad} \end{array}\right) = \{-,(t^r_\mu)_Y b\}$$

$$\{-,(s_\mu)_Y\} \cdot b = \left(-\,,\ \begin{array}{l}\hline Y_1b \cup -(W_1)b \\ \hline \vdots \\ \hline Y_ib \cup -(W_i)b \\ \hline \\ \underline{\quad} \end{array}\right) = \{-,(s_\mu)_Y b\}$$

In other words, both $\{t^r_Y\} \cdot b$ and $\{s_Y\} \cdot b$ represent the same tabloid; hence, the action is well-defined.

$$\{t^r_Y\} \cdot b = \{s_Y\} \cdot b \qquad (6.18)$$

Observe that as $a_Y \cdot b \in R_f$, it is reasonable to indicate to which $\mathcal{H}$-class of

R_f that $a_Y \cdot b$ precisely belongs. As a first step towards answering the above enquiry, let us first find the image of composition $a_Y \cdot b$ as follows:

$$\begin{aligned} \mathrm{im}(a_Y \cdot b) &= (\mathrm{im}\ a_Y \cap \mathrm{dom}(b))b \\ &= (Y \cap \mathrm{dom}(b))b \\ &= Yb. \qquad\qquad\qquad [\text{As } \ Y \subseteq \mathrm{dom}(b)] \end{aligned}$$

Therefore, if $a_Y \cdot b \in R_f$, then $a_Y \cdot b$ has the domain X and image Yb. However, since $Y \subseteq [\pm n]$ has $|Y| = 2k$ and b is a partial signed map, "a bijection", then $Yb \subseteq [\pm n]$ has $|Yb| = 2k$ as well. Moreover, $a_Y \cdot b$ satisfies the property that $(-x)a_Y \cdot b = -(x)a_Y \cdot b$, as $a_Y \cdot b$ is a composition of bijections. Hence, $a_Y \cdot b$ is a partial signed map that is precisely placed in the $\mathcal{H}$-class of R_f labelled by Yb. Recall that the representative of the $\mathcal{H}$-class of R_f labelled by Yb is a_{Yb}, which has the domain X and image $Yb = \{\pm y'_1, \ldots, \pm y'_k\}$, where

$$-y'_k < \cdots < -y'_2 < -y'_1 < y'_1 < y'_2 < \cdots < y'_k.$$

Hence, both partial signed maps, $a_Y \cdot b$ and a_{Yb}, are in the $\mathcal{H}$-class labelled by Yb, as illustrated in the diagram below.

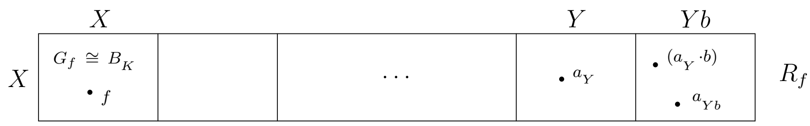


Figure 6.6: Place of partial map a_{Yb} and partial map $a_Y \cdot b$ in the R_f.

This raises a concern regarding the relation between $a_Y \cdot b$ and a_{Yb}.

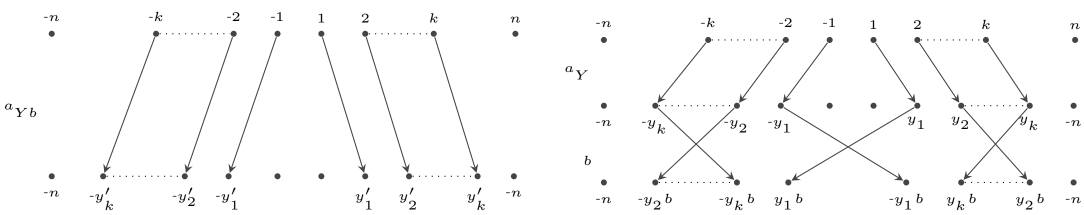


Figure 6.7: Partial map a_{Yb} versus partial map $a_Y \cdot b$.

However, in view of Remark 1.1.28, we deduce the existence of a unique signed permutation $g \in B_k$ such that $a_Y \cdot b = g \cdot a_{Yb}$ holds. Thus, if $a_Y \cdot b \in R_f$, then the product $\{t^r_Y\} \cdot b$ has ij-th entries:

$$x_z(i,j)(a_Y \cdot b) = x_z(i,j) \cdot (g \cdot a_{Yb}) = (x_z(i,j) \cdot g) \cdot a_{Yb}, \tag{6.19}$$

where $z = \lambda$ or $z = \mu$. On the other hand, $\{(t^r g)_{Yb}\}$ also has ij-th entries, $(x_z(i,j)g) \cdot a_{Yb}$. Hence, the discussion above allows us to produce an "improved" version of (6.14), as follows:

$$\{t^r_{_Y}\} \cdot b = \begin{cases} \{(t^r g)_{_{Yb}}\}, & Y \subseteq \text{dom}(b),\ a_{_Y} \cdot b = g \cdot a_{_{Yb}} \text{ and } \ g \in B_k \\ 0 & \text{otherwise,} \end{cases} \tag{6.20}$$

where t^r is a (λ, μ)-tableau that forms a distinct basis element $\{t^r\}$ of $M^{(\lambda,\mu)}$. Therefore, we now know how MB_n acts on the basis vector $\{t^r_{_Y}\}$ of $M^{(\lambda,\mu)}_{_Y}$ for any $Y \subseteq [\pm n]$ such that whenever $r \in Y$, $-r \in Y$ as well and $|Y| = 2k$. In addition, extending this action linearly illustrates how MB_n acts on $\bigoplus\limits_{Y} M^{(\lambda,\mu)}_{_Y}$.

Observation 6.2.3. Since t^r is a (λ, μ)-tableau with entries from $[\pm k]$ and $g \in B_{_k}$, then $t^r g$ is another (λ, μ)-tableau filled with entries from $[\pm k]$. We also have $Yb \subseteq [\pm n]$ with $|Yb| = 2k$ as b is a bijection; thus, $\{(t^r g)_{_{Yb}}\} \in M^{(\lambda,\mu)}_{_{Yb}}$, where $M^{(\lambda,\mu)}_{_{Yb}}$ is another summand of $M^{(\lambda,\mu)} \uparrow MB_n$.

This completes the steps required in the induction process, and all that has been investigated above can be summarised in the theorem below:

Theorem 6.2.4. Fix $(\lambda, \mu) \vdash k$, where $0 \leq k \leq n$, and consider a representation $M^{(\lambda,\mu)}$ of a maximal subgroup B_k. Then, $M^{(\lambda,\mu)}$ induces to an MB_n representation $M^{(\lambda,\mu)} \uparrow MB_n$ determined by a vector space

$$M^{(\lambda,\mu)} \uparrow MB_n = \bigoplus_{Y} M^{(\lambda,\mu)}_{_Y}, \quad Y \subseteq [\pm n] \ \text{ with} \, |Y| = 2k,$$

and the action described as follows: For all basis vectors $\{t^r_{_Y}\}$ of $M^{(\lambda,\mu)} \uparrow MB_n$ and partial signed permutation $b \in MBn$,

$$\{t^r_{_Y}\} \cdot b = \begin{cases} \{(t^r g)_{_{Yb}}\}, & Y \subseteq \text{dom}(b),\ a_{_Y} \cdot b = g \cdot a_{_{Yb}} \text{ and } \ g \in B_k \\ 0 & \text{otherwise,} \end{cases}$$

where t^r is a (λ, μ)-tableau that forms a distinct basis element $\{t^r\}$ of $M^{(\lambda,\mu)}$.

Example 6.2.5. Let $k = 2$ and $n = 3$. Fix the complementary partition $(\lambda, \mu) = ((1), (1)) \vdash 2$. We know from Example 3.3.41 that

$$M^{((1),(1))} = \mathbb{C}\text{-}\Big[\Big\{\Big(\boxed{1}\,,\boxed{2}\Big)\Big\}, \Big\{\Big(\boxed{-1}\,,\boxed{-2}\Big)\Big\}, \Big\{\Big(\boxed{2}\,,\boxed{1}\Big)\Big\}, \Big\{\Big(\boxed{-2}\,,\boxed{-1}\Big)\Big\}\Big].$$

Recall that the signed permutation group

$$B_2 = \{id, (1,-1), (2,-2), (1,-1)(2,-2), (1,2)(-1,-2), (1,-2)(-1,2), (1,2,-1,-2), (1,-2,-1,2)\},$$

and notice that $M^{((1),(1))}$ is a representation of the signed permutation group B_2. Fix an idempotent $f \in J_{2(2)}$, where

$$(x)f = \begin{cases} x & x \in \{\pm 1, \pm 2\}, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Consider the $J_{2(2)}$-class and the $\mathcal{R}$-class of f " R_f" as shown in Figure 6.8. Now,

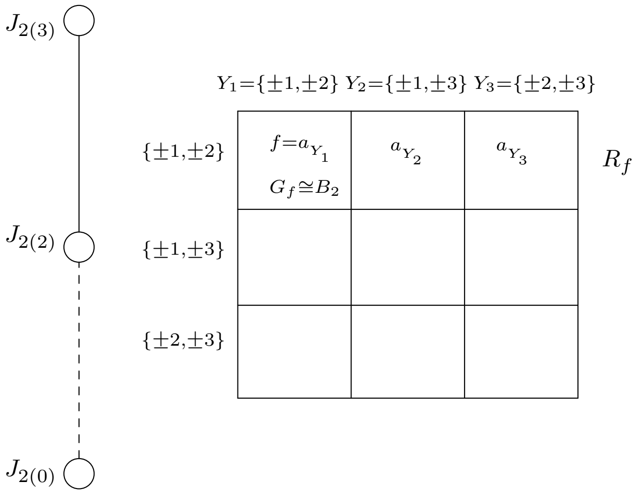


Figure 6.8: $J_{2(2)}$-class and $\mathcal{R}$-class of idempotent f with representatives a_{Y_i}.

select the representatives for each $\mathcal{H}$-class in R_f to be as illustrated in Figure 6.9.

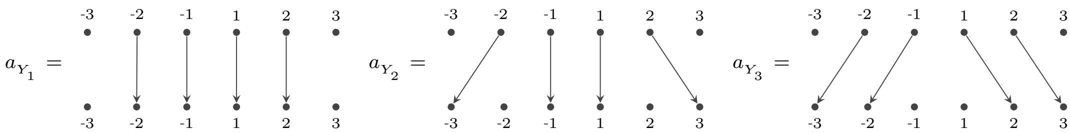


Figure 6.9: Representatives of $\mathcal{H}$-classes of R_f.

Thus, by utilising the above representatives for each $\mathcal{H}$-class, we obtain the following copies of $M^{((1),(1))}$:

$$M^{((1),(1))}_{Y_1} = \mathbb{C}\text{-}\Big[\Big\{\Big(\boxed{1},\boxed{2}\Big)\Big\},\Big\{\Big(\boxed{-1},\boxed{-2}\Big)\Big\},\Big\{\Big(\boxed{2},\boxed{1}\Big)\Big\},\Big\{\Big(\boxed{-2},\boxed{-1}\Big)\Big\}\Big].$$

$$M^{((1),(1))}_{Y_2} = \mathbb{C}\text{-}\Big[\Big\{\Big(\boxed{1},\boxed{3}\Big)\Big\},\Big\{\Big(\boxed{-1},\boxed{-3}\Big)\Big\},\Big\{\Big(\boxed{3},\boxed{1}\Big)\Big\},\Big\{\Big(\boxed{-3},\boxed{-1}\Big)\Big\}\Big].$$

$$M^{((1),(1))}_{Y_3} = \mathbb{C}\text{-}\Big[\Big\{\Big(\boxed{2},\boxed{3}\Big)\Big\},\Big\{\Big(\boxed{-2},\boxed{-3}\Big)\Big\},\Big\{\Big(\boxed{3},\boxed{2}\Big)\Big\},\Big\{\Big(\boxed{-3},\boxed{-2}\Big)\Big\}\Big].$$

Thus,

$$\begin{aligned} M^{((1),(1))} \uparrow MB_3 = \mathbb{C}\text{-}\Big[\; & \Big[\Big\{\Big(\boxed{1},\boxed{2}\Big)\Big\},\Big\{\Big(\boxed{-1},\boxed{-2}\Big)\Big\},\Big\{\Big(\boxed{2},\boxed{1}\Big)\Big\}\Big\{\Big(\boxed{-2},\boxed{-1}\Big)\Big\}\Big] \bigoplus \\ & \Big[\Big\{\Big(\boxed{1},\boxed{3}\Big)\Big\},\Big\{\Big(\boxed{-1},\boxed{-3}\Big)\Big\},\Big\{\Big(\boxed{3},\boxed{1}\Big)\Big\}\Big\{\Big(\boxed{-3},\boxed{-1}\Big)\Big\}\Big] \bigoplus \\ & \Big[\Big\{\Big(\boxed{2},\boxed{3}\Big)\Big\},\Big\{\Big(\boxed{-2},\boxed{-3}\Big)\Big\},\Big\{\Big(\boxed{3},\boxed{2}\Big)\Big\}\Big\{\Big(\boxed{-3},\boxed{-2}\Big)\Big\}\Big] \;\Big]. \end{aligned}$$

Observe that the Clifford-Munn Theorem only guarantees correspondence between irreducible representations of maximal subgroups and irreducible representations of semigroups. Although $M^{(\lambda,\mu)}$ is reducible, we induce it to obtain the sense of how to present the basis vector of this module and how MB_n acts on it. However, we aim to induce up $S^{(\lambda,\mu)}$ the Specht module for B_k where $0 \leq k \leq n$ to MB_n, as it is the irreducible module for such a maximal subgroup B_k.

In view of the action presented in (6.15), the restriction of the MB_n action to the maximal subgroup $B_{_Y}$ plays a vital role in considering the following definition. This definition is a crucial assertion that we require before we resume the induction process to acquire the Specht module for MB_n.

Definition 6.2.6. Let $(\lambda, \mu) \vdash k$, where $0 \leq k \leq n$, and $Y \subseteq [\pm n]$ with $|Y| = 2k$. Fix any (λ, μ)-Young tableau t filled with numbers from $[\pm k]$, and then determine (λ, μ)-tableau $t_{_Y}$ and define an element $e^t_{_Y} \in M^{(\lambda,\mu)}_{_Y}$ in the following manner:

$$e^t_{_Y} = \sum_{\tau \in C_{t_Y}} sgn(\tau)\ \{t_{_Y}\}\tau. \tag{6.21}$$

Call such an element $e^t_{_Y}$ a polytabloid which is determined by t and Y.

Observe that for all $\tau \in C_{t_Y}$, the $\mathrm{sgn}(\tau)$ is defined as the sign function of the maximal subgroup $B_{_Y}$. If $t_{_Y}$ is a standard (λ, μ)-tableau, then we call $e^t_{_Y}$ a standard (λ, μ)-polytabloid associated with the (λ, μ)-tableau $t_{_Y}$. Let us now illustrate an example of how the vector $e^t_{_{Y_3}}$ of $M^{((1),(1))}_{_{Y_3}}$ is identified.

Example 6.2.7. As shown in Example 6.2.5, let $k = 2$ and $n = 3$. Fix the complementary partition $((1), (1)) \vdash 2$ and select the following $((1), (1))$-Young tableau:

$$t = \left(\boxed{1}\,,\boxed{2}\right) \in Y^{((1),(1))}.$$

Choose $Y_3 = \{\pm 2, \pm 3\} \subset [\pm 3]$ and consider the representative $a_{_{Y_3}}$ as illustrated in Example 6.2.5. Let us now determine the $((1), (1))$-Young tableau $t_{_{Y_3}}$ as follows:

$$t_{_{Y_3}} = \left(\boxed{1}\,,\boxed{2}\right) \cdot a_{_{Y_3}} = \left(\boxed{2}\,,\boxed{3}\right).$$

Note that the column group $C_{t_{Y_3}} = \{id = (2)(-2)(3)(-3), (2, -2)\}$. Hence, the polytabloid $e^t_{_{Y_3}}$ associated with the $((1), (1))$-Young tableau $t_{_{Y_3}}$ can be identified by

$$\begin{aligned} e^t_{_{Y_3}} &= \textstyle\sum_{\tau \in C_{t_{Y_3}}} sgn(\tau)\ \left\{\left(\boxed{2}\,,\boxed{3}\right)\right\}\tau \\ &= + \left\{\left(\boxed{2}\,,\boxed{3}\right)\right\} - \left\{\left(\boxed{-2}\,,\boxed{3}\right)\right\} \in M^{((1),(1))}_{_{Y_3}}. \end{aligned}$$

In view of Definition 3.3.44, for any complementary partition (λ, μ) of k, define $S^{(\lambda,\mu)}$ to be the subspace of $M^{(\lambda,\mu)}$ spanned by the elements e^t, where t runs through all the Young tableaux of shape (λ, μ); that is,

$$S^{(\lambda,\mu)} = Span_{\mathbb{C}}\ \{e^t :\ t \text{ is a } (\lambda, \mu)\text{-Young tableau filled with numbers from } [\pm k]\}.$$

Recall that $S^{(\lambda,\mu)}$ is the Specht module for B_k; that is, it is an irreducible representation of the maximal subgroup B_k. Moreover, we know that whenever an irreducible representation $S^{(\lambda,\mu)}$ of B_k is induced into the monoid of partial signed permutation MB_n, we obtain a corresponding irreducible representation of MB_n, as emphasised in the Clifford-Munn correspondence theorem.

Let us now go through all four steps again to induce $S^{(\lambda,\mu)}$ up to MB_n as follows:

(a) Fix an irreducible representation $S^{(\lambda,\mu)}$ of B_k. Take a representative $a_{_Y}$ of each $\mathcal{H}$-class of the $\mathcal{R}$-class of f considering the idempotent f representative of the $\mathcal{H}$-class in which B_k is placed.

(b) Consider a copy of $S^{(\lambda,\mu)}$ for each $\mathcal{H}$-class in the following manner: let $S_{_Y}^{(\lambda,\mu)}$ be a copy of $S^{(\lambda,\mu)}$ corresponding to an $\mathcal{H}$-class with a representative $a_{_Y}$. In fact, $S_{_Y}^{(\lambda,\mu)}$ is defined as a subspace of $M_{_Y}^{(\lambda,\mu)}$ spanned by all the elements $e_{_Y}^{t}$, where t runs through all Young tableaux of shape (λ,μ) :

$$S_{_Y}^{(\lambda,\mu)} = \mathrm{Span}_{\mathbb{C}}\ \{e_{_Y}^{t}:\ t \text{ is a } (\lambda,\mu)\text{-tableau filled with numbers from } [\pm k]\}. \quad (6.22)$$

Observe that in the spanning set of $S_{_Y}^{(\lambda,\mu)}$, we fixed the subset Y, and we only let t vary. Observe that whenever we alter a subset Y to another subset $Z = \{\pm z_{_1},\cdots,\ \pm z_{_k}\}$ of $[\pm n]$, we generate another copy $S_{_Z}^{(\lambda,\mu)}$ of $S^{(\lambda,\mu)}$ corresponding to the $\mathcal{H}$-class labelled by Z. As $S_{_Y}^{(\lambda,\mu)}$ is a copy of $S^{(\lambda,\mu)}$, both respect their vector space structures. It follows that the spanning set $\{e_{_Y}^{t}:\ t$ is a Young tableau of shape $(\lambda,\mu)\}$ of $S_{_Y}^{(\lambda,\mu)}$ is not linearly independent; thus, it does not form a basis for $S_{_Y}^{(\lambda,\mu)}$.

Recall that a (λ,μ)-Young tableau $t=(t_\lambda,\ t_\mu)$ filled with numbers $[\pm k]$ is said to be standard if all the entries of t are positive and both t_λ and t_μ are standard. In view of Theorem 3.3.50, the set $\{e^t:\ t$ is a (λ,μ)-standard tableau$\}$ forms a basis for $S^{(\lambda,\mu)}$. Hence, if we let the (λ,μ)-Young tableau t in (6.22) run through all the standard tableaux of shape (λ,μ), and adapt the choise of a representative $a_{_Y}$ for an $\mathcal{H}$-class labelled by Y as mentioned in (6.10) and (6.11), then the set

$$\{e_{_Y}^{t}:\ t \text{ is a standard tableau filled with numbers from } [\pm k]\} \quad (6.23)$$

forms a basis for $S_{_Y}^{(\lambda,\mu)}$.

Notice that failing to consider the choice of representative $a_{_Y}$ may not contribute to expressing $S_{_Y}^{(\lambda,\mu)}$ in terms of its basis. Thus, our discussion above provides us an alternative approach to present the copy $S_{_Y}^{(\lambda,\mu)}$ of $S^{(\lambda,\mu)}$ in terms of its basis elements. However, we maintain our consideration of the definition of $S_{_Y}^{(\lambda,\mu)}$ as it appears in (6.22) for the remainder of this section unless stated otherwise.

(c) Consider all the copies $S_{_Y}^{(\lambda,\mu)}$ where $Y \subseteq [\pm n]$ with $|Y| = 2k$, and define the following vector space:

$$S^{(\lambda,\mu)} \uparrow MB_n = \bigoplus_{Y} S_{_Y}^{(\lambda,\mu)}. \tag{6.24}$$

Since $S_{_Y}^{(\lambda,\mu)}$ is a subspace of $M_{_Y}^{(\lambda,\mu)}$ for all $Y \subseteq [\pm n]$ with $|Y| = 2k$, thus, $\bigoplus\limits_{Y} S_{_Y}^{(\lambda,\mu)}$ is also a subspace of $\bigoplus\limits_{Y} M_{_Y}^{(\lambda,\mu)}$. Observe that as $S_{_Y}^{(\lambda,\mu)}$ is a copy of $S^{(\lambda,\mu)}$, the dimension of each summand $S_{_Y}^{(\lambda,\mu)}$ is equal to the dimension of $S^{(\lambda,\mu)}$. Thus, the dimension of $S^{(\lambda,\mu)} \uparrow MB_n$ can be deduced as follows:

$$\begin{aligned} \dim S^{(\lambda,\mu)} \uparrow MB_n &= \dim \bigoplus_{Y} S_{_Y}^{(\lambda,\mu)} \\ &= \sum_{Y} \dim S_{_Y}^{(\lambda,\mu)} \\ &= \binom{n}{r} \cdot \dim S^{(\lambda,\mu)}, \end{aligned} \tag{6.25}$$

since there are $\binom{n}{k}$ distinct subsets Y of $[\pm n]$ in which $|Y| = 2k$.

The next step in the investigation is to elucidate how the monoid of partial signed permutations MB_n acts on the vector space $S^{(\lambda,\mu)} \uparrow MB_n$ defined above. However, to describe such an action, we need to consider some assertions and lemmas. Recall from Section 6.1 that every partial signed map $b \in MB_n$ may be written uniquely as a product of positive links and disjoint positive and negative cycles. Furthermore, the last number appearing in a positive link of $b \in MB_n$ does not belong to the domain of b and the first number in a positive link does not belong to the images of b; however, all other numbers in between are in both the domain and image of b.

Definition 6.2.8. Fix $b \in MB_n$ and let $\tilde{b}$ be the element in the group of units $B_n \subset MB_n$ determined by altering every positive link in b into a positive cycle.

For instance, let $n = 7$ and $b = [3, 1, -2][-3, -1, 2](4, 6)(-4, -6) \in MB_7$. Then

$$\begin{aligned} \tilde{b} &= (3, 1, -2)(-3, -1, 2)(4, 6)(-4, -6). \\ &= (3, 1, -2)(-3, -1, 2)(4, 6)(-4, -6)(5)(-5)(7)(-7) \in B_7. \end{aligned}$$

Lemma 6.2.9. *Fix $(\lambda, \mu) \vdash k$, where $0 \le k \le n$, and $Y \subseteq [\pm n]$ with $|Y| = 2k$. Let $b \in MB_n$ and $t_{_Y}$ be a (λ, μ)-Young tableau whose entries from Y. If*

$Y \subseteq dom(b)$, *then*

(1) $t_{_Y} \cdot b = t_{_Y} \cdot \tilde{b}$

(2) $\{t_{_Y}\} \cdot b = \{t_{_Y}\} \cdot \tilde{b}$

(3) $e^t_{_Y} \cdot b = e^t_{_Y} \cdot \tilde{b}$

Proof. (1) Since $t_{_Y}$ is a (λ, μ)-Young tableau, then its entries are of the form $x_z(i,j)a_{_Y}$ with $z = \lambda$ or $z = \mu$. Let $b \in MB_n$, then by Theorem 6.1.9, we can express b uniquely as a product of positive links and disjoint positive and negative cycles. If $Y \subseteq \mathrm{dom}(b)$, then none of the entries $x_z(i,j)a_{_Y}$ with $z = \lambda$ or $z = \mu$ of $t_{_Y}$ appear as the last number in any positive link of b. In other words, the entries $x_z(i,j)a_{_Y}$ with $z = \lambda$ or $z = \mu$ that occur in tableau $t_{_Y}$ have the same image under b as they do under $\tilde{b}$. Hence, $(x_z(i,j)a_{_Y})\,\tilde{b} = (x_z(i,j)a_{_Y})\,b$ with $z = \lambda$ or $z = \mu$ for all i, j. Thus, (1) holds.

(2) Since $Y \subseteq \mathrm{dom}\ b$, the entries of $\{t_{_Y}\} \cdot b$ are $(x_z(i,j)a_{_Y})\,b$ with $z = \lambda$ or $z = \mu$. In view of (6.19), there must exist $g \in B_k$ such that $a_{_Y} \cdot b = g \cdot a_{_{Yb}}$ and for all i, j,

$$x_z(i,j)\ a_{_Y}\ b = x_z(i,j)\ g \cdot a_{_{Yb}}, \tag{6.26}$$

where $z = \lambda$ or $z = \mu$. However, $Yb = Y\tilde{b}$ as $Y \subseteq \mathrm{dom}(b)$, and none of the elements of Y occur as the last numbers in any positive link of b. In other words, both Yb and $Y\tilde{b}$ label the same $\mathcal{H}$-class, and then the representative of the $\mathcal{H}$-class determined by these labels must be the same as well; that is, $a_{_{Yb}} = a_{_{Y\tilde{b}}}$. Thus, equation (6.26) can be rewritten in the following manner: for all i, j in $\{t_{_Y}\} \cdot b$,

$$\begin{aligned} x_z(i,j)\ a_{_Y}\ b &= x_z(i,j)\ g \cdot a_{_{Yb}} \\ &= x_z(i,j)\ g \cdot a_{_{Y\tilde{b}}} \\ &= x_z(i,j)\ a_{_Y} \cdot \tilde{b}, \end{aligned}$$

where $z = \lambda$ or $z = \mu$. Hence, $\{t_{_Y}\} \cdot b = \{t_{_Y}\} \cdot \tilde{b}$.

(3) Since

$$e^t_{_Y} = \sum_{\tau \in C_{t_Y}} sgn(\tau)\ \{t_{_Y}\}\tau,$$

each summand $\{t_{_Y}\tau\}$ of $e^t_{_Y}$ is clearly a (λ, μ)-tabloid with entries from Y. Since $Y \subseteq \mathrm{dom}(b)$, then applying b to $e^t_{_Y}$ requires applying b to each summand $\{t_{_Y}\tau\}$

of $e^t_{_Y}$. However, using the preceding result, we have $\{t_{_Y}\tau\} \cdot b = \{t_{_Y}\tau\} \cdot \tilde{b}$ for each summand $\{t_{_Y}\tau\}$ occurring in $e^t_{_Y}$. Hence, $e^t_{_Y} \cdot b = e^t_{_Y} \cdot \tilde{b}$. $\square$

Remark 6.2.10. Fix $(\lambda, \mu) \vdash k$, where $0 \le k \le n$, and $Y \subseteq [\pm n]$ with $|Y| = 2k$. Let $t_{_Y}$ be a (λ, μ)-Young tableau with entries from Y, and recall that the column group $C_{t_{_Y}}$ is a subgroup of a maximal subgroup $B_{_Y}$. In fact, we can embed a maximal subgroup $B_{_Y}$ within the group of units B_n of MB_n in the following manner: Define an injective map

$$\tau \in B_{_Y} \hookrightarrow \hat{\tau} \in B_n \quad \text{such that}$$

$$(r)\hat{\tau} = \begin{cases} (r)\tau & r \in Y, \\ r & \text{otherwise,} \end{cases}$$

and for all $-r \in Y$, we have $(-r)\hat{\tau} = (-r)\tau = -(r)\tau = -(r)\hat{\tau}$. It should be noted that because of the nature of the embedding, both $\tau \in B_{_Y}$ and $\hat{\tau} \in B_n$ have the same sign.

Example 6.2.11. Let $k = 7$ and $n = 8$, and fix $(\lambda, \mu) = ((3,1),(2,1)) \vdash 7$. Choose $Y = \{\pm 1, \cdots, \pm 7\} \subset [\pm 8]$. In addition, select the $((3,1),(2,1))$-Young tableau

$$t_{_Y} = \left(\begin{array}{|c|c|c|} \hline -2 & 1 & -4 \\ \hline 5 \\ \cline{1-1} \end{array} , \begin{array}{|c|c|} \hline 7 & 3 \\ \hline -6 \\ \cline{1-1} \end{array} \right).$$

It is clear that $C_{t_{_Y}} = \langle (-2,5)(2,-5), (-2,2), (5,-5), (1,-1), (-4,4), (7,-6)(-7,6) \rangle$. Notice that the permutation

$$\tau = (-2,5)(2,-5)\ (1,-1)\ (-4,4)\ (7,-6)(-7,6) \in C_{t_{_Y}} \subset B_{_Y},$$

and the permutation

$$\hat{\tau} = (-2,5)(2,-5)\ (1,-1)\ (-4,4)\ (7,-6)(-7,6)\ (3)(-3)\ (8)(-8) \in B_8.$$

have the same sign as $sgn(\tau) = (-1)^4 = sgn(\hat{\tau})$.

In view of the preceding remark, we obtain the following lemma.

Lemma 6.2.12. *Fix $(\lambda, \mu) \vdash k$, where $0 \le k \le n$, and let $t_{_Y}$ be a (λ, μ)-Young tableau with entries from $Y \subseteq [\pm n]$ where $|Y| = 2k$. Then, for all $\pi \in B_n$, we have*

(1) $\pi^{-1} C_{t_{_Y}} \pi = C_{t_{_Y}\pi}$.

(2) $sgn(\pi^{-1}\tau\pi) = sgn(\tau),\ \forall\ \tau \in C_{t_{_Y}}$.

Proof. (1) The proof is similar to the proof of Proposition 3.3.35. We rephrase it here using our conventions. Let us show that $\pi^{-1}C_{t_Y}\pi = C_{t_Y\pi}$. Recall that $t_Y = ((t_\lambda)_Y, (t_\mu)_Y)$, where

$$
\begin{aligned}
(t_\lambda)_Y &:= \text{ a tableau of shape } \lambda \text{ whose entries are } \ x_\lambda(i,j)\cdot a_Y,\\
(t_\mu)_Y &:= \text{ a tableau of shape } \mu \text{ whose entries are } \ x_\mu(i,j)\cdot a_Y
\end{aligned}
$$

Consider a (λ,μ)-tableau $t_Y\pi$, where $x_\lambda(i,j)a_Y\pi$ is the (i,j)-th entry of $(t_\lambda)_Y\pi$ and $x_\mu(i,j)a_Y\pi$ is the (i,j)-th entry of $(t_\mu)_Y\pi$. Now for all i,j

$\sigma \in C_{t_Y\pi}$ if and only if

$$
\begin{aligned}
x_\lambda(i,j)a_Y\pi\sigma &= \pm\, x_\lambda(p,j)a_Y\pi,\\
x_\mu(i,j)a_Y\pi\sigma &= x_\mu(q,j)a_Y\pi, \qquad\qquad \text{for some } p \text{ and } q.
\end{aligned}
$$

This also holds if and only if

$$
\begin{aligned}
x_\lambda(i,j)a_Y\pi\sigma &= (\pm\, x_\lambda(p,j)a_Y)\pi, \qquad [\text{ as } -(r)\pi = (-r)\pi \text{ for all } r,\ r = x_\lambda(p,j)a_Y]\\
x_\mu(i,j)a_Y\pi\sigma &= x_\mu(q,j)a_Y\pi,
\end{aligned}
$$

This also true if and only if

$$
\begin{aligned}
x_\lambda(i,j)a_Y\pi\sigma\pi^{-1} &= \pm\, x_\lambda(p,j)a_Y,\\
x_\mu(i,j)a_Y\pi\sigma\pi^{-1} &= x_\mu(q,j)a_Y.
\end{aligned}
$$

This also holds if and only if $\pi\sigma\pi^{-1} \in C_{t_Y}$, as $x_z(i,j)a_Y$ with $z=\lambda$ or $z=\mu$ are entries in (λ,μ)-tableau t_Y. Hence, we have $C_{t_Y\pi} = \pi^{-1}C_{t_Y}\pi$.

(2) Since C_{t_Y} is a subgroup of a maximal subgroup B_Y, and B_Y can be viewed as a subgroup of B_n, as discussed in the above remark, then for all $\tau \in C_{t_Y}$, we have

$$
\begin{aligned}
sgn(\pi^{-1}\tau\pi) &= sgn(\pi^{-1})\ sgn(\tau)\ sgn(\pi)\\
&= sgn(\tau),
\end{aligned}
$$

since $sgn(\pi^{-1}) = sgn(\pi)$. □

In fact, all what we have investigated above play a vital role in verifying how MB_n acts on $S^{(\lambda,\mu)} \uparrow MB_n$. Let us now end this section with the the following lemma, which is considered the last step in the MB_n induction process:

Lemma 6.2.13. *Fix $(\lambda,\mu) \vdash k$, where $0 \le k \le n$, and let t_Y be a λ-Young tableau filled with numbers from $Y \subseteq [\pm n]$ with $|Y| = 2k$. Then, for all $b \in$*

MB_n *and* $e^t_{_Y} \in S^{(\lambda,\mu)} \uparrow MB_n$*, we have*

$$e^t_{_Y} \cdot b = \begin{cases} e^{tg}_{_{Yb}}, & \textit{if } Y \subseteq \textit{dom } b \textit{ and } a_{_Y} \cdot b = g \cdot a_{_{Yb}} \textit{ with } g \in B_k \\ 0 & \textit{otherwise.} \end{cases} \tag{6.27}$$

Proof. Suppose that $Y \subseteq \mathrm{dom}(b)$; then, using Lemma 6.2.9 (3), we obtain

$$\begin{aligned}
e^t_{_Y} \cdot b &= e^t_{_Y} \cdot \tilde{b} \\
&= \Big(\sum_{\tau \in C_{t_Y}} sgn(\tau)\ \{t_{_Y}\}\tau \Big) \cdot \tilde{b}, \\
&= \sum_{\tau \in C_{t_Y}} sgn(\tau)\ \{t_{_Y}\}\ \tau\ \cdot \tilde{b}, \\
&= \sum_{\tau \in C_{t_Y}} sgn(\tau)\ \{t_{_Y}\}\ \tilde{b}\ \tilde{b}^{-1}\ \tau\ \tilde{b}, && [\ \tilde{b} \in B_n,\ \textit{and}\ C_{t_Y} \leq B_{_Y} \subseteq B_n\ ] \\
&= \sum_{\tau \in C_{t_Y}} sgn(\tau)\ \{t_{_Y}\}\ \tilde{b}\ \ \ (\tilde{b}^{-1}\ \tau\ \tilde{b}) \\
&= \sum_{\tau \in C_{t_Y}} sgn(\tilde{b}^{-1}\ \tau\ \tilde{b})\ \{t_{_Y}\}\ \tilde{b}\ \ \ (\tilde{b}^{-1}\ \tau\ \tilde{b}), && [\text{By Lemma 6.2.12 (2)}] \\
&= \sum_{\rho \in \tilde{b}^{-1} C_{t_Y} \tilde{b}} sgn(\rho)\ \{t_{_Y}\}\ \tilde{b}\ \cdot \rho, && [\ \rho = \tilde{b}^{-1}\tau\ \tilde{b}\ ] \\
&= \sum_{\rho \in C_{(t_Y \cdot \tilde{b})}} sgn(\rho)\ \{t_{_Y}\}\ \tilde{b}\ \cdot \rho, && [\text{By Lemma 6.2.12 (1)}] \\
&= \sum_{\rho \in C_{(t_Y \cdot b)}} sgn(\rho)\ \{t_{_Y}\}\ b\ \cdot \rho, && [\text{By Lemma 6.2.9 (1) and (2)}] \\
&= \sum_{\rho \in C_{(tg)_{Yb}}} sgn(\rho)\ \{(tg)_{_{Yb}}\}\ \rho, && [\text{By formula (6.20) and as } t_{_Y} \cdot b = (tg)_{_{Yb}}] \\
&= e^{tg}_{_{Yb}}.
\end{aligned}$$

On the other hand, if $Y \nsubseteq \mathrm{dom}(b)$, then using formula (6.20), we obtain $\{t_{_Y}\} \cdot b = 0$; that is, there is at least one entry y of $t_{_Y}$ that does not belong to the domain of b. In consequence, the negative of y never belongs to $\mathrm{dom}(b)$ as well; that is $-y \notin \mathrm{dom}(b)$. Now, since

$$e^t_{_Y} = \sum_{\tau \in C_{t_Y}} sgn(\tau)\ \{t_{_Y}\}\tau,$$

then for each summand $\{t_{_Y}\tau\}$ of $e^t_{_Y}$, the tableau $t_{_Y}\tau$ has the same entries as a tableau $t_{_Y}$ but in a different order, or it may have the entries' negatives of $(t_{_Y})_\lambda$. In other words, either the entry y or $-y$ still belongs to these summands. Thus, by applying b to $e^t_{_Y}$, every summand will be sent to zero under the action of b. Hence, $e^t_{_Y} \cdot b = 0$. $\square$

Observation 6.2.14. In the formula of Lemma 6.2.13, observe that t is a (λ, μ)-tableau with entries from $X = \{\pm 1, \cdots, \pm k\}$ and $g \in B_{_k}$, then tg is another (λ, μ)-tableau filled with entries from X. We also have $Yb \subseteq [\pm n]$ with $|Yb| = 2k$ as b is a bijection; thus, $e^{tg}_{_{Yb}} \in S^{(\lambda,\mu)}_{_{Yb}}$ where $S^{(\lambda,\mu)}_{_{Yb}}$ is a summand of $S^{(\lambda,\mu)} \uparrow MB_n$. In addition, the Clifford-Munn correspondence asserts that $S^{(\lambda,\mu)} \uparrow MB_n$ is an irreducible representation for MB_n. Moreover, Theorem 3.3.47 tells us that whenever $(\lambda, \mu) \neq (\lambda', \mu')$, $S^{(\lambda,\mu)} \ncong S^{(\lambda',\mu')}$. Consequently, $S^{(\lambda',\mu')} \uparrow MB_n$, and $S^{(\lambda',\mu')} \uparrow MB_n$ represent distinct irreducible representations for MB_n for all $(\lambda, \mu) \neq (\lambda', \mu')$.

What we have investigated thus far allows us to make the following assertion:

Theorem 6.2.15. Fix $(\lambda, \mu) \vdash k$, where $0 \leq k \leq n$, and consider the Specht module $S^{(\lambda,\mu)}$ for the signed permutation groups B_k. Then, $S^{(\lambda,\mu)}$ induces to an MB_n representation $S^{(\lambda,\mu)} \uparrow MB_n$, called the Specht module for MB_n and determined as follows:

$$S^{(\lambda,\mu)} \uparrow MB_n = \bigoplus_{Y} S^{(\lambda,\mu)}_{_Y}, \quad Y \subseteq [\pm n] \ \text{ with } |Y| = 2k.$$

Moreover, for each vector $e^t_{_Y} \in S^{(\lambda,\mu)} \uparrow MB_n$ and partial signed permutation $b \in MBn$,

$$e^t_{_Y} \cdot b = \begin{cases} e^{tg}_{_{Yb}}, & \text{if } Y \subseteq \operatorname{dom} b \text{ and } \ a_{_Y} \cdot b = g \cdot a_{_{Yb}} \text{ where } g \in B_k \\ 0 & \text{otherwise.} \end{cases}$$

Example 6.2.16. Let $k = 2$ and $n = 3$. Fix the complementary partition $(\lambda, \mu) = ((1), (1)) \vdash 2$. Recall from Example 3.3.41 that

$$Y^{((1),(1))} = \Big\{t_1 = \big(\boxed{1}, \boxed{2}\big), t_2 = \big(\boxed{1}, \boxed{-2}\big), t_3 = \big(\boxed{-1}, \boxed{-2}\big), t_4 = \big(\boxed{-1}, \boxed{2}\big), t_5 = \big(\boxed{2}, \boxed{1}\big), t_6 = \big(\boxed{2}, \boxed{-1}\big), t_7 = \big(\boxed{-2}, \boxed{1}\big), t_8 = \big(\boxed{-2}, \boxed{-1}\big)\Big\}.$$
$$M^{((1),(1))} = \mathbb{C}\text{-}\Big[\Big\{\big(\boxed{1}, \boxed{2}\big)\Big\}, \Big\{\big(\boxed{-1}, \boxed{-2}\big)\Big\}, \Big\{\big(\boxed{2}, \boxed{1}\big)\Big\}, \Big\{\big(\boxed{-2}, \boxed{-1}\big)\Big\}\Big].$$

From Example 3.3.48, we also obtain the Specht module $S^{((1),(1))}$ for B_2 as given below:

$$S^{((1),(1))} = \operatorname{Span}_{\mathbb{C}}\{e_{t_1}, e_{t_2}, e_{t_3}, e_{t_4}, e_{t_5}, e_{t_6}, e_{t_6}, e_{t_7}, e_{t_8}\}.$$

Fix an idempotent $f \in J_{2(2)}$, where

$$(x)f = \begin{cases} x & x \in \{\pm 1, \pm 2\}, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Consider the $\mathcal{R}$-class R_f of f as shown below.

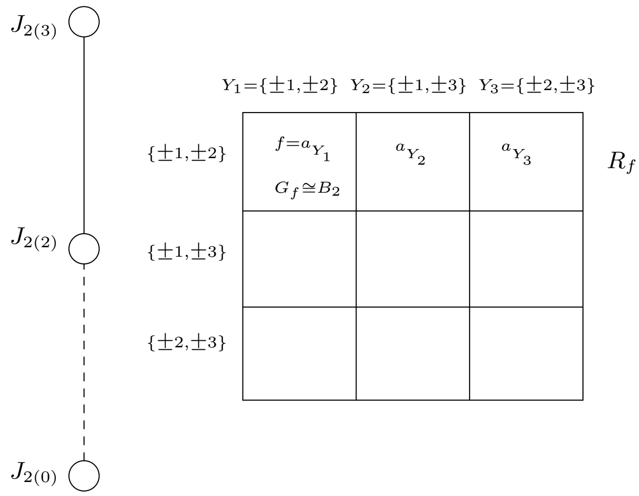


Figure 6.10: $\mathcal{R}$-class of idempotent f with representatives a_{Y_r}.

Now, select the representatives for each $\mathcal{H}$-class in R_f to be as shown below:

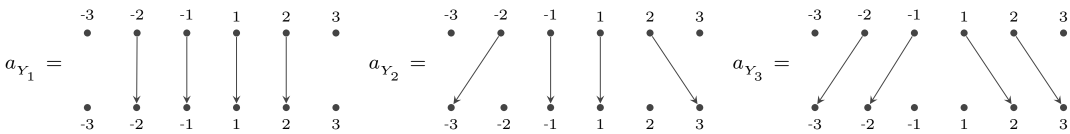


Figure 6.11: Representatives of $\mathcal{H}$-classes of R_f.

Notice that inducing $S^{((1),(1))}$ into MB_3 requires obtaining the copies of $S^{((1),(1))}$ for each $\mathcal{H}$-class of R_f using the above representatives. It is clear that considering the first representative a_{Y_1} yields

$$\begin{aligned} S^{((1),(1))}_{Y_1} &= \mathrm{Span}_{\mathbb{C}}\{e^{t_1}_{Y_1}, e^{t_2}_{Y_1}, e^{t_3}_{Y_1}, e^{t_4}_{Y_1}, e^{t_5}_{Y_1}, e^{t_6}_{Y_1}, e^{t_7}_{Y_1}, e^{t_8}_{Y_1}\}. \\ &= \mathrm{Span}_{\mathbb{C}}\{e_{t_1}, e_{t_2}, e_{t_3}, e_{t_4}, e_{t_5}, e_{t_6}, e_{t_7}, e_{t_8}\}. \end{aligned}$$

Consider the second representative a_{Y_2}, and let us compute $S^{((1),(1))}_{Y_2}$ as

$t_{1_{Y_2}} = \left(\boxed{1},\boxed{3}\right)$ and $C_{t_{1_{Y_2}}} = \{id,\ (1,-1)\}$. Thus, $e^{t_1}_{Y_2} = +\{\left(\boxed{1},\boxed{3}\right)\} - \{\left(\boxed{-1},\boxed{3}\right)\}$
$t_{2_{Y_2}} = \left(\boxed{1},\boxed{-3}\right)$ and $C_{t_{2_{Y_2}}} = \{id,\ (1,-1)\}$. Thus, $e^{t_2}_{Y_2} = +\{\left(\boxed{1},\boxed{-3}\right)\} - \{\left(\boxed{-1},\boxed{-3}\right)\}$
$t_{3_{Y_2}} = \left(\boxed{-1},\boxed{-3}\right)$ and $C_{t_{3_{Y_2}}} = \{id,\ (1,-1)\}$. Thus, $e^{t_3}_{Y_2} = +\{\left(\boxed{-1},\boxed{-3}\right)\} - \{\left(\boxed{1},\boxed{-3}\right)\}$
$t_{4_{Y_2}} = \left(\boxed{-1},\boxed{3}\right)$ and $C_{t_{4_{Y_2}}} = \{id,\ (1,-1)\}$. Thus, $e^{t_4}_{Y_2} = +\{\left(\boxed{-1},\boxed{3}\right)\} - \{\left(\boxed{1},\boxed{3}\right)\}$
$t_{5_{Y_2}} = \left(\boxed{3},\boxed{1}\right)$ and $C_{t_{5_{Y_2}}} = \{id,\ (3,-3)\}$. Thus, $e^{t_5}_{Y_2} = +\{\left(\boxed{3},\boxed{1}\right)\} - \{\left(\boxed{-3},\boxed{1}\right)\}$
$t_{6_{Y_2}} = \left(\boxed{3},\boxed{-1}\right)$ and $C_{t_{6_{Y_2}}} = \{id,\ (3,-3)\}$. Thus, $e^{t_6}_{Y_2} = +\{\left(\boxed{3},\boxed{-1}\right)\} - \{\left(\boxed{-3},\boxed{-1}\right)\}$
$t_{7_{Y_2}} = \left(\boxed{-3},\boxed{1}\right)$ and $C_{t_{7_{Y_2}}} = \{id,\ (3,-3)\}$. Thus, $e^{t_7}_{Y_2} = +\{\left(\boxed{-3},\boxed{1}\right)\} - \{\left(\boxed{3},\boxed{1}\right)\}$
$t_{8_{Y_2}} = \left(\boxed{-3},\boxed{-1}\right)$ and $C_{t_{8_{Y_2}}} = \{id,\ (3,-3)\}$. Thus, $e^{t_8}_{Y_2} = +\{\left(\boxed{-3},\boxed{-1}\right)\} - \{\left(\boxed{3},\boxed{-1}\right)\}$

Hence, the copy of $S^{((1),(1))}$ corresponding to the $\mathcal{H}$-class labelled by Y_2 is

$$S^{((1),(1))}_{Y_2} = \mathrm{Span}_{\mathbb{C}}\{e^{t_1}_{Y_2}, e^{t_2}_{Y_2}, e^{t_3}_{Y_2}, e^{t_4}_{Y_2}, e^{t_5}_{Y_2}, e^{t_6}_{Y_2}, e^{t_7}_{Y_2}, e^{t_8}_{Y_2}\}.$$

Consider the last representative a_{Y_3}, and let us compute $S^{((1),(1))}_{Y_3}$ as

$t_{1_{Y_3}} = \left(\boxed{2}, \boxed{3}\right)$ and $C_{t_{1_{Y_3}}} = \{id,\ (2,-2)\}$. Thus, $e^{t_1}_{Y_3} = +\left\{\left(\boxed{2}, \boxed{3}\right)\right\} - \left\{\left(\boxed{-2}, \boxed{3}\right)\right\}$
$t_{2_{Y_3}} = \left(\boxed{2}, \boxed{-3}\right)$ and $C_{t_{2_{Y_3}}} = \{id,\ (2,-2)\}$. Thus, $e^{t_2}_{Y_3} = +\left\{\left(\boxed{2}, \boxed{-3}\right)\right\} - \left\{\left(\boxed{-2}, \boxed{-3}\right)\right\}$
$t_{3_{Y_3}} = \left(\boxed{-2}, \boxed{-3}\right)$ and $C_{t_{3_{Y_3}}} = \{id,\ (2,-2)\}$. Thus, $e^{t_3}_{Y_3} = +\left\{\left(\boxed{-2}, \boxed{-3}\right)\right\} - \left\{\left(\boxed{2}, \boxed{-3}\right)\right\}$
$t_{4_{Y_3}} = \left(\boxed{-2}, \boxed{3}\right)$ and $C_{t_{4_{Y_3}}} = \{id,\ (2,-2)\}$. Thus, $e^{t_4}_{Y_3} = +\left\{\left(\boxed{-2}, \boxed{3}\right)\right\} - \left\{\left(\boxed{2}, \boxed{3}\right)\right\}$
$t_{5_{Y_3}} = \left(\boxed{3}, \boxed{2}\right)$ and $C_{t_{5_{Y_3}}} = \{id,\ (3,-3)\}$. Thus, $e^{t_5}_{Y_3} = +\left\{\left(\boxed{3}, \boxed{2}\right)\right\} - \left\{\left(\boxed{-3}, \boxed{2}\right)\right\}$
$t_{6_{Y_3}} = \left(\boxed{3}, \boxed{-2}\right)$ and $C_{t_{6_{Y_3}}} = \{id,\ (3,-3)\}$. Thus, $e^{t_6}_{Y_3} = +\left\{\left(\boxed{3}, \boxed{-2}\right)\right\} - \left\{\left(\boxed{-3}, \boxed{-2}\right)\right\}$
$t_{7_{Y_3}} = \left(\boxed{-3}, \boxed{2}\right)$ and $C_{t_{7_{Y_3}}} = \{id,\ (3,-3)\}$. Thus, $e^{t_7}_{Y_3} = +\left\{\left(\boxed{-3}, \boxed{2}\right)\right\} - \left\{\left(\boxed{3}, \boxed{2}\right)\right\}$
$t_{8_{Y_3}} = \left(\boxed{-3}, \boxed{-2}\right)$ and $C_{t_{8_{Y_3}}} = \{id,\ (3,-3)\}$. Thus, $e^{t_8}_{Y_3} = +\left\{\left(\boxed{-3}, \boxed{-2}\right)\right\} - \left\{\left(\boxed{3}, \boxed{-2}\right)\right\}$

Thus, the copy of $S^{((1),(1))}$ corresponding to the $\mathcal{H}$-class labelled by Y_3 is

$$S^{((1),(1))}_{Y_3} = \mathrm{Span}_{\mathbb{C}}\{e^{t_1}_{Y_3}, e^{t_2}_{Y_3}, e^{t_3}_{Y_3}, e^{t_4}_{Y_3}, e^{t_5}_{Y_3}, e^{t_6}_{Y_3}, e^{t_7}_{Y_3}, e^{t_8}_{Y_3}\}.$$

Hence, $S^{((1),(1))} \uparrow MB_3 = S^{((1),(1))}_{Y_1} \bigoplus S^{((1),(1))}_{Y_2} \bigoplus S^{((1),(1))}_{Y_3}$.

It is worthwhile to mention that considering the standard tableaux $t_1 = \left(\boxed{1}, \boxed{2}\right)$ and $t_5 = \left(\boxed{2}, \boxed{1}\right)$ facilitates the acquisition of results. Recall from Example 3.3.51 that the Specht module $S^{((1),(1))}$ for B_2 can be written using its basis elements, the "standard polytabloids" $\{e_{t_1},\ e_{t_5}\}$, as

$$S^{((1),(1))} = \mathbb{C}\text{-}[e_{t_1}, e_{t_5}].$$

Hence, the copies of $S^{((1),(1))}$ for each $\mathcal{H}$-class of R_f are obtained in the following manner:

$$S^{((1),(1))}_{Y_1} = \mathbb{C}\text{-}[e^{t_1}_{Y_1}, e^{t_5}_{Y_1}]$$
$$S^{((1),(1))}_{Y_2} = \mathbb{C}\text{-}[e^{t_1}_{Y_2}, e^{t_5}_{Y_2}]$$
$$S^{((1),(1))}_{Y_3} = \mathbb{C}\text{-}[e^{t_1}_{Y_3}, e^{t_5}_{Y_3}.]$$

Therefore,

$$\begin{aligned} S^{((1),(1))} \uparrow MB_3 &= S^{((1),(1))}_{Y_1} \bigoplus S^{((1),(1))}_{Y_2} \bigoplus S^{((1),(1))}_{Y_3}. \\ &= \mathbb{C}\text{-}\Big[\ [e^{t_1}_{Y_1}, e^{t_5}_{Y_1}] \bigoplus [e^{t_1}_{Y_2}, e^{t_5}_{Y_2}] \bigoplus [e^{t_1}_{Y_3}, e^{t_5}_{Y_3}]\ \Big]. \end{aligned}$$

Chapter 7

Further work on representations of reflection monoids

The even signed permutation group B_n^e [19] (a subgroup of the signed permutation group B_n of index two) is comparable to B_n in terms of being considered the Weyl group of type $\mathcal{D}_n$, Coxeter group of type $\mathcal{D}_n$, and reflection group of type $\mathcal{D}_n$. Since the group has been described in many different ways, its irreducible representations have also been provided in a range of settings [8, 10, 36, 50]. Nevertheless, the representations of the even signed permutation group B_n^e do not appear to have been conveniently characterised in the form of a tableau filled with numbers and, in a manner, resembling James' diagrammatic approach delineated by [41]. Thus, it is suggested that an explicit description of the irreducible representations of B_n^e can be achieved by amending the techniques put forth by Morris and Halicioğlu [35, 37].

It is worthwhile to have investigated the monoid version of the even signed permutation group, known as "the monoid of partial even signed permutations MB_n^e". According to [19], the maximal subgroups of such a monoid are all signed permutation groups B_k ($k < n$) apart from its group of units, which is an even signed permutation group B_n^e. Such an observation is useful for future research because, despite the knowledge possessed regarding the irreducible representations of B_k ($k < n$), the difficulty presented by the lack of a convenient description of the irreducible representations of B_n^e remains. Dealing with this challenge will enable the application of induction to the irreducible representations of the maximal subgroups in order to achieve generalised MB_n^e irreducible representations.

Further, another relevant research topic is the Coxeter arrangement monoids. Comprehension of the maximal subgroups of these monoids is necessary although a difficult problem. Subsequently, the irreducible representations of the Coxeter arrangement monoids can be obtained through the application of the CMC.

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