Geometrical Dynamics by the Schrödinger Equation and Coherent States Transform

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Abstract

This thesis is concerned with a concept of geometrising time evolution of quantum systems. This concept is inspired by the fact that the Legendre transform expresses dynamics of a classical system through first-order Hamiltonian equations. We consider, in this thesis, coherent state transforms with a similar effect in quantum mechanics: they reduce certain quantum Hamiltonians to first-order partial differential operators. Therefore, the respective dynamics can be explicitly solved through a flow of points in extensions of the phase space. This, in particular, generalises the geometric dynamics of a harmonic oscillator in the Fock-Segal-Bargmann (FSB) space. We describe all Hamiltonians which are geometrised (in the above sense) by Gaussian and Airy beams and exhibit explicit solutions for such systems

This work is dedicated to my mother, wife and my son and to the memory of my father

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Contents

Abstract i

Dedication ii

Acknowledgements iii

Contents iv

List of figures vii

1 Preliminaries **8**

1.1 The Heisenberg group and the shear group $\mathbb{G}$ 8

1.2 Induced representations of the shear group $\mathbb{G}$ 11

1.2.1 Derived representations 13

1.3 The group $\mathbb{G}$ and the Schrödinger group 14

1.3.1 The group $\mathbb{G}$ and the universal enveloping algebra of $\mathfrak{h}$ 19

1.4 Some physical background 20

1.4.1 A mathematical model of QM 20

1.4.2 The Uncertainty Relation 22

1.4.3 Harmonic oscillator and ladder operators 24

1.4.4 Canonical coherent states 26

1.4.5 Dynamics of the harmonic oscillator 27

1.4.6 The FSB space 28

2 Coherent state transform **31**

2.1 The induced coherent state transform and its image 31

2.1.1 The induced coherent state transform of the shear group $\mathbb{G}$ 34

2.1.2 Right shifts and coherent state transform 41

2.2 Characterisation of the image space $L_\phi(\mathbb{G}/Z)$ 44

3 Harmonic oscillator through reduction of order of a PDE **50**

3.1 Harmonic oscillator from the Heisenberg group 50

3.2 Harmonic oscillator from the group $\mathbb{G}$ 56

3.2.1 Geometrical, analytic and physical meanings of new solution . . . 60

3.2.2 Harmonic oscillator Hamiltonian and ladder operators in the space $L_{\phi_E}(\mathbb{G}/Z)$ 63

4 Classification of Hamiltonian operators for geometric dynamics **69**

4.1 Geometrisation of Hamiltonians by Airy coherent states 70

4.2 Example of a geometrisable Hamiltonian and the fiducial vector $\phi_{E,D}$ . . 73

4.2.1 Solving the geometrised Schrödinger equation 74

4.3 Further extensions 77

Appendix **79**

A Algebraic properties of ladder operators 79

B Induced representations of nilpotent Lie groups 83

B.1 The action of a Lie group on a homogeneous space 83

B.2 Induced representations 84

Bibliography **87**

List of figures

1.1 Shear transforms 18

1.2 Squeeze and shear transformations 19

3.1 Shear parameter and analytic continuation 62

4.1 Ground state of harmonic oscillator in a field 70

4.2 Classical orbits in the phase space of the Hamiltonian (4.2.12) 77

Introduction

Hamilton equations describe classical dynamics through a flow on the phase space. This geometrical picture inspires numerous works searching for a similar description of quantum evolution starting from the symplectic structure [42], a curved space-time [13, 35, 41, 75, 86], the differential geometry [14, 16] and the quantizer–dequantizer formalism [15, 86]. A common objective of these works is a conceptual similarity between fundamental geometric objects and their analytical counterparts, for instance, the symplectic structure on the phase space and the derivations of operator algebras. A promising direction that may lead to broader developments into classical-like descriptions of quantum evolution suggests the use of coherent states.

The coherent states were introduced by Schrödinger in 1926 but were not in use until much later [8, 29, 71, 74]. Further developments of the concept of coherent states have manifested a remarkable depth and width [3, 28, 54, 65, 80].

The *canonical* coherent states of the harmonic oscillator have a variety of important properties, for example, semi-classical dynamics, minimal uncertainty, parametrisation by points of the phase space, resolution of the identity, covariance under a group action, etc.

In this thesis, we discuss geometrisation of quantum evolution in the coherent states representation by looking for a simple and effective method to express quantum evolution through a flow of points of some set. More precisely, let the dynamics of a quantum system be defined by a Hamiltonian H and the respective Schrödinger equation

$$\mathrm{i}\hbar\dot{\phi}(t) = H\phi(t). \tag{0.0.1}$$

Geometrisation of (0.0.1) suggested in [15] uses a collection $\{\phi_x\}_{x\in X}$ of coherent states parametrised by points of a set X. Then the solution $\phi_x(t)$ of (0.0.1) for an initial value $\phi_x(0) = \phi_x$ shall have the form

$$\phi_x(t) = \phi_{x(t)}, \tag{0.0.2}$$

where $t : x \mapsto x(t)$ is a one-parameter group of transformations $X \to X$. Recall that the coherent state transform $\tilde{f}(x)$ of a state f in a Hilbert space is defined by

$$f \mapsto \tilde{f}(x) = \langle f, \phi_x \rangle\,. \tag{0.0.3}$$

It is common that a coherent state transform is a unitary map onto a subspace F_2 of $L_2(X, \mathrm{d}\mu)$ for a suitable measure $\mathrm{d}\mu$ on X. If $\{\phi_x\}_{x\in X}$ geometrises a Hamiltonian H in the above sense, then for an arbitrary solution $f(t) = \mathrm{e}^{-\mathrm{i}tH/\hbar} f(0)$ of (0.0.1) we have:

$$\begin{aligned}\tilde{f}(t,x) &= \left\langle \mathrm{e}^{-\mathrm{i}tH/\hbar} f(0), \phi_x \right\rangle \\ &= \left\langle f(0), \mathrm{e}^{\mathrm{i}tH/\hbar}\phi_x \right\rangle = \tilde{f}(0, x(t)).\end{aligned} \tag{0.0.4}$$

Thus, if a family of coherent states geometrises a Hamiltonian H, then the dynamics of any image $\tilde{f}$ of the respective coherent state transform is given by a transformation of variables. Motivation for such a concept is the following example of the canonical coherent states of the harmonic oscillator [8, 28, 29, 71, 74, 80]

Example 0.0.1 *Consider the quantum harmonic oscillator of constant mass m and constant frequency ω:*

$$H = \frac{1}{2m}P^2 + \frac{m\omega^2}{2}Q^2, \tag{0.0.5}$$

where

$$Q\phi(q) = q\phi(q), \qquad P\phi(q) = -\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}q}\phi(q).$$

For the pair of ladder operators

$$a^- = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega Q + \mathrm{i}P), \qquad a^+ = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega Q - \mathrm{i}P), \tag{0.0.6}$$

the above Hamiltonian becomes $H = \omega\hbar(a^+a^- + \frac{1}{2}I)$*. The canonical coherent states,* ϕ_z *(where,* $z = q + \mathrm{i}p$*) of the harmonic oscillator are produced by the action of the "displacement" operator on the vacuum* ϕ_0:

$$\phi_z := \mathrm{e}^{za^+ - \bar{z}a^-}\phi_0 = \mathrm{e}^{-\frac{1}{2}|z|^2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}\phi_n, \tag{0.0.7}$$

where ϕ_0 *is such that* $a^-\phi_0 = 0$ *and* $\phi_n = \frac{1}{\sqrt{n!}}(a^+)^n\phi_0$*. One can then use the spectral decomposition of* H *(i.e. the relation* $H\phi_n = \hbar\omega(n+1/2)\phi_n$*) to obtain evolution in the canonical coherent states representation which takes the form*

$$\mathrm{e}^{-\mathrm{i}tH/\hbar}\phi_z = \mathrm{e}^{-\mathrm{i}\omega t/2}\phi_{z(t)} \tag{0.0.8}$$

where $z(t) = \mathrm{e}^{-\mathrm{i}\omega t}z$ *is a one-parameter group of transformations. These rigid rotations* $z \mapsto \mathrm{e}^{-\mathrm{i}\omega t}z$ *of the phase space are the key ingredients of the dynamics of classical harmonic oscillators. Therefore, for an arbitrary solution* $f(t) = \mathrm{e}^{-\mathrm{i}tH/\hbar}f(0)$ *of* (0.0.1) *for the above harmonic oscillator Hamiltonian we have:*

$$\begin{aligned}\tilde{f}(t,z) &= \left\langle \mathrm{e}^{-\mathrm{i}tH/\hbar}f(0), \phi_z\right\rangle = \left\langle f(0), \mathrm{e}^{\mathrm{i}tH/\hbar}\phi_z\right\rangle \\ &= \mathrm{e}^{-\mathrm{i}\omega t/2}\left\langle f(0), \phi_{z(t)}\right\rangle = \mathrm{e}^{-\mathrm{i}\omega t/2}\tilde{f}(0, \mathrm{e}^{-\mathrm{i}\omega t}z). \end{aligned} \tag{0.0.9}$$

Thus, the classical behaviour of the dynamics in ϕ_z *is completely reflected in the dynamics of its coherent state transform. Nevertheless, the image of such a transform gives rise to the following Hilbert space (a model of the phase space):*

Definition 0.0.2 ([3, 8, 25]) *Let* $z = q + \mathrm{i}p \in \mathbb{C}$*, the Fock-Segal-Bargemann (FSB) space consists of all functions that are analytic on the whole complex plane* $\mathbb{C}$ *and square-integrable with respect to the measure* $\mathrm{e}^{-\pi\hbar|z|^2}\,\mathrm{d}z$*. It is equipped with the inner product*

$$\langle f, g\rangle_F = \int_{\mathbb{C}} f(z)\overline{g(z)}\,\mathrm{e}^{-\pi\hbar|z|^2}\,\mathrm{d}z. \tag{0.0.10}$$

Notably, the ladder operators have the simpler expressions:

$$a^- = \partial_z, \qquad a^+ = zI.$$

Then the harmonic oscillator Hamiltonian on FSB space has the form

$$\tilde{H} = \hbar\omega(z\partial_z + \frac{1}{2}I). \tag{0.0.11}$$

The respective Schrödinger equation is, therefore, a first-order PDE. Hence, one can use, for example, the method of characteristics and obtains the dynamics

$$F(t, z) = \mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t} F(0, \mathrm{e}^{-\mathrm{i}\omega t} z). \tag{0.0.12}$$

This dynamics is exactly the same as (0.0.9). *In other words, the geometric dynamics* (0.0.9) *inherited from that of the corresponding coherent states obeys the Schrödinger equation for the first order Hamiltonian* $\tilde{H}$ (0.0.11).

It was already noted in [15] that even the archetypal canonical coherent states do not geometrise the harmonic oscillator dynamics in the above strict sense (0.0.4) due to the presence of the overall phase factor in the solution (0.0.8). However, the factor is not a minor nuisance but rather a fundamental element: it is responsible for a positive energy of the ground state.

To accommodate such observation with geometrising, we propose the adjusted meaning of geometrisation

Definition 0.0.3 *A collection* $\{\phi_x\}_{x\in X}$ *of coherent states parametrised by points of a manifold* X, *geometrises quantum dynamics, if the time evolution of the coherent state transform* $\tilde{f}$ *is defined by a Schrödinger equation*

$$\mathrm{i}\hbar \frac{\mathrm{d}\tilde{f}}{\mathrm{d}t} = \tilde{H}\tilde{f}, \tag{0.0.13}$$

where $\tilde{H}$ *is a first-order differential operator on* X.

In light of this definition we say that the above canonical coherent states geometrise the Hamiltonian of the harmonic oscillator.

Group representations are a rich source of coherent states [3, 28, 65]. More precisely, let X be the homogeneous space G/H for a group G and its closed subgroup H. Then for a representation ρ of G in a space V and a fiducial vector $\phi \in V$ the collection of coherent states is defined by (see Section 2.1)

$$\phi_x = \rho(\mathsf{s}(x))\phi, \qquad (0.0.14)$$

where $x \in G/H$ and $\mathsf{s} : G/H \to G$ is a section. In this setting, the canonical coherent states of the harmonic oscillator are produced by G being the Heisenberg group [25, 44, 46, 53], H—the centre of G, ρ—the Schrödinger representation and $\phi_{m\omega}(x) = \mathrm{e}^{-\pi\hbar m\omega x^2}$—the Gaussian. So, the above example can be easily adapted to this language as will be seen explicitly in Chapter 3.

The main points of the thesis are outlined as follows.

- We offer a group-theoretic approach to the construction of the first-order differential operator $\tilde{H}$ from Definition 0.0.3. A technical advantage of our method is that it does not require the explicit spectral decomposition of H that is typically used to solve the respective time-dependent Schrödinger equation. Instead, the standard method of characteristics for first-order PDEs becomes an important tool in our investigation. The analytic structure of FSB space is the key source of the simplification of the harmonic oscillator Hamiltonian. Yet, the role of analyticity property in obtaining such a Hamiltonian as a first order differential operator was hidden. Example 0.0.1 will be reconsidered in Chapter 3 within a group-theoretic set-up. As a result of our method, it will be clearer the role of Cauchy–Riemann operator in reducing the order of the harmonic oscillator Hamiltonian in the FSB space, see the last paragraph after (3.1.20). This will resolve the sort of ambiguity in having geometric dynamics (0.0.9) for a second–order differential operator (0.0.5)!

- We apply the method to the harmonic oscillator extending the above example from the Heisenberg group to the minimal three-step nilpotent Lie group, denoted $\mathbb{G}$. The group $\mathbb{G}$ is being viewed as the minimal nilpotent extension of the Heisenberg group $\mathbb{H}$, see Section 1.1. The main advantage is that we are allowed to use Gaussian $\mathrm{e}^{-\pi\hbar Ex^2}$ with arbitrary squeeze parameter $E\,(E > 0)$ [28, 80, 70] as a fiducial vector ϕ_E for a simultaneous geometrisation of all harmonic oscillators with different values of $m\omega$. This is specifically discussed in details in Chapter 3 of this thesis, see the end of Section 1.4 for a further explanation.

- The group $\mathbb{G}$ and its representations provide a wider opportunity of considering various fiducial vectors. For example, we study the fiducial vector (4.1.2) which is the Fourier transform of an Airy wave packet [11] that is useful in paraxial optics [76, 77]. We provide a full classification of all Hamiltonians that can be geometrised by Gaussian and Airy beams according to Definition 0.0.3. For such Hamiltonians we write explicit generic solutions through well-known integral transforms.

The thesis is divided into four chapters. The first chapter presents the group $\mathbb{G}$ together with its main unitary representations that are needed for our approach. Important physical and geometrical aspects related to the group $\mathbb{G}$ structure are also highlighted. Being the simplest three-step nilpotent Lie group, $\mathbb{G}$ is a natural test ground for various constructions in representation theory [17, 44] and harmonic analysis [9, 37]. The group $\mathbb{G}$ was called quartic group in [5, 38, 55] due to its relation to quartic anharmonic oscillator.

The content of the second and the third chapters is based on our work [6]; in the second chapter, we introduce the important notion of coherent states and the respective coherent state transforms from group representation viewpoint. The result which is presented in Corollary 2.1.15 provides a general and largely accessible way of describing properties of the respective image space of such transforms. The fundamental example is the FSB space consisting of analytic functions. We revise this property from the perspective of Corollary 2.1.15 in Example 2.1.16 and deduce the corresponding description of the image space of coherent state transform of $\mathbb{G}$ in Section 2.2. Besides the analyticity-type condition, which relays on a suitable choice of the fiducial vector, we find an additional condition, referred to as *structural condition*, which is completely determined by a Casimir operator of $\mathbb{G}$ and holds for any coherent state transform. Notably, the structural condition coincides with the Schrödinger equation of a free particle. Thereafter, the image space of the coherent state transform of $\mathbb{G}$ is obtained from FSB space through a solution of an initial value problem for a time evolution of a free particle.

The third chapter presents our main technique of reducing the order of a quantum Hamiltonian applied to the harmonic oscillator from the Heisenberg group and the group $\mathbb{G}$. In the case of the Heisenberg group, Section 3.1 we confirm that the geometric dynamics (3.1.20) is the only possibility for the fiducial vector ϕ_E with the matching value of $E = m\omega$. In contrast, Section 3.2 reveals the gain from the larger group $\mathbb{G}$: any minimal uncertainty state can be used as a fiducial vector for a geometrisation of dynamics. We end this chapter with creation and annihilation operators in Section 3.2.2. Their action in terms of the group $\mathbb{G}$ is still connected to Hermite polynomials (but with respect to a complex variable). This can be compared with ladder operators related to squeezed states in [4].

In the final chapter, we provide a complete classification of arbitrary Hamiltonians whose dynamics can be geometrised in the sense of Definition 0.0.3. We give one further example beyond the harmonic oscillator and explicitly solve the respective Schrödinger equation.

Chapter 1

Preliminaries

This chapter is intended to review some known results. We stress the relationship between the group $\mathbb{G}$ and the Heisenberg group $\mathbb{H}$. This relationship suggests a further important relationship between the group $\mathbb{G}$ and the Schrödinger group $\mathbb{S}$ which we explicitly illustrate in Proposition 1.3.2. The connection between the group $\mathbb{G}$ and $\mathbb{S}$ reveals significant geometric and physical phenomena that will also be confirmed from another standpoint in Chapter 3.

Due to its link to the Schrödinger group via shear transformation, what will be seen soon, we may call the group $\mathbb{G}$ the *shear group*.

1.1 The Heisenberg group and the shear group $\mathbb{G}$

The Heisenberg-Weyl algebra, denoted $\mathfrak{h}$, is the two-step nilpotent Lie algebra spanned by elements $\{X, Y, S\}$ with commutation relations:

$$[X, Y] = S, \quad [X, S] = [Y, S] = 0. \tag{1.1.1}$$

Here and in the rest of the thesis the commutator is given by $[A, B] = AB - BA$.

It can be realised by the matrices:

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}; \quad S = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In particular, the element S generates the centre of $\mathfrak{h}$.

The corresponding group is the Heisenberg group , denoted $\mathbb{H}$ [25, 44, 49, 69]. In the *polarised coordinates* (x, y, s) on $\mathbb{H} \sim \mathbb{R}^3$ the group law is [25, § 1.2]:

$$(x, y, s)(x', y', s') = (x + x', y + y', s + s' + xy'). \tag{1.1.2}$$

Let $\mathfrak{g}$ be the three-step nilpotent Lie algebra whose basic elements are $\{X_1, X_2, X_3, X_4\}$ with the following non-vanishing commutators [17, Ex. 1.3.10] [44, § 3.3]:

$$[X_1, X_2] = X_3, \qquad [X_1, X_3] = X_4\,. \tag{1.1.3}$$

In matrix realisation the Lie algebra $\mathfrak{g}$ has the following non-zero basic elements,

$$X_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \qquad X_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix};$$
$$X_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \qquad X_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Clearly, the basic element corresponding to the centre of such a Lie algebra is X_4. The elements X_1, X_3 and X_4 span the above mentioned Heisenberg–Weyl algebra.

Exponentiating the above basic elements in the manner $(x_1, x_2, x_3, x_4) := \exp(x_4X_4)\exp(x_3X_3)\exp(x_2X_2)\exp(x_1X_1)$ (where $x_j \in \mathbb{R}$ and known as *canonical* *coordinates* [44, § 3.3]) leads to a matrix description of the corresponding Lie group, denoted $\mathbb{G}$, being three-step nilpotent Lie group whose elements are of the form

$$(x_1, x_2, x_3, x_4) := \begin{pmatrix} 1 & x_1 & \frac{x_1^2}{2} & x_4 \\ 0 & 1 & x_1 & x_3 \\ 0 & 0 & 1 & x_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The respective group law is

$$\begin{aligned}(x_1, x_2, x_3, x_4)(y_1, y_2, y_3, y_4) = (&x_1 + y_1, x_2 + y_2, x_3 + y_3 + x_1 y_2, \\ &x_4 + y_4 + x_1 y_3 + \tfrac{1}{2} x_1^2 y_2).\end{aligned} \tag{1.1.4}$$

The identity element is $(0, 0, 0, 0)$ and the inverse of an element (x_1, x_2, x_3, x_4) is $(-x_1, -x_2, x_1 x_2 - x_3, x_1 x_3 - \frac{1}{2} x_1^2 x_2 - x_4)$. It is clear that the group $\mathbb{G}$ is not commutative and its centre (the set whose elements commute with all elements of the group) is

$$Z = \{(0, 0, 0, x_4) \in \mathbb{G} : \ x_4 \in \mathbb{R}\}. \tag{1.1.5}$$

This is a one-dimensional subgroup. Another *abelian* subgroup which has the maximal dimensionality is

$$H_1 = \{(0, x_2, x_3, x_4) \in \mathbb{G} : x_j \in \mathbb{R}\}. \tag{1.1.6}$$

This subgroup is of a particular importance because its irreducible representation (which is a character, see Remark 1.1.1 below) induces an irreducible representation of the group $\mathbb{G}$ as will be seen in the next section.

Remark 1.1.1 *Throughout this thesis, a character χ of an abelian subgroup H is a map $\chi : H \to \mathbb{C}$, such that $\chi(h_1 h_2) = \chi(h_1)\chi(h_2)$ and for a unitary character we also have $|\chi(h)| = 1$ and $\bar{\chi}(h) = \chi(h^{-1})$, where $h, h_1, h_2 \in H$.*

A comparison of group laws (1.1.2) and (1.1.4) shows that the Heisenberg group $\mathbb{H}$ is isomorphic to the subgroup

$$\tilde{\mathbb{H}} = \{(x_1, 0, x_3, x_4) \in \mathbb{G} : \ x_j \in \mathbb{R}\} \ \text{by} \ (x, y, s) \mapsto (x, 0, y, s), \ (x, y, s) \in \mathbb{H}\,. \quad (1.1.7)$$

In most cases we will identify $\mathbb{H}$ and $\tilde{\mathbb{H}}$ through the above map. All formulae for the Heisenberg group needed in this thesis will be obtained from this identification. That is, in any formula of $\mathbb{G}$ by setting $x_2 = 0$ we get the group $\mathbb{H}$ counterpart.

1.2 Induced representations of the shear group $\mathbb{G}$

The unitary representations of the group $\mathbb{G}$ can be constructed using inducing procedure (in the sense of Mackey) and Kirillov orbit method, a detailed consideration of this topic is worked out in [44, § 3.3.2]. Here, we construct the needed representations of $\mathbb{G}$ along this induction method which is briefly outlined in Appendix B.

1. For H being the centre $Z = \{(0, 0, 0, x_4) \in \mathbb{G} : \ x_4 \in \mathbb{R}\}$, we have

$$\begin{aligned} \mathsf{p} : \ & (x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3), \\ \mathsf{s} : \ & (x_1, x_2, x_3) \mapsto (x_1, x_2, x_3, 0), \\ \mathsf{r} : \ & (x_1, x_2, x_3, x_4) \mapsto (0, 0, 0, x_4)\,. \end{aligned}$$

   Then,

$$\begin{aligned} &\mathsf{r}((-x_1, -x_2, x_1x_2 - x_3, x_1x_3 - \tfrac{1}{2}x_1^2x_2 - x_4)\mathsf{s}(x_1', x_2', x_3')) \\ &\quad = (0, 0, 0, -x_4 + x_1x_3 - x_1x_3' - \tfrac{1}{2}x_1^2x_2 + \tfrac{1}{2}x_1^2x_2)\,. \end{aligned}$$

   Thus, for a character $\chi(0, 0, 0, x_4) = \mathrm{e}^{2\pi\mathrm{i}\hbar_4 x_4}$ of the centre, formula (B.47) gives the following unitary representation of $\mathbb{G}$ on $L_2(\mathbb{R}^3)$:

$$\begin{aligned} [\tilde{\rho}(x_1, x_2, x_3, x_4)f](x_1', x_2', x_3') &= \mathrm{e}^{2\pi\mathrm{i}\hbar_4(x_4 - x_1x_3 + \frac{1}{2}x_1^2x_2 + x_1x_3' - \frac{1}{2}x_1^2x_2')} \quad (1.2.8) \\ &\quad \times f(x_1' - x_1, x_2' - x_2, x_3' - x_3 - x_1x_2' + x_1x_2). \end{aligned}$$

This representation is reducible and we will discuss its irreducible components below. A restriction of (1.2.8) to the Heisenberg group $\tilde{\mathbb{H}}$ is a variation of the Fock–Segal–Bargmann representation.

2. For the maximal abelian subgroup $H_1 = \{(0, x_2, x_3, x_4) \in \mathbb{G} : x_2, x_3, x_4 \in \mathbb{R}\}$, we have

$$\begin{aligned}
&\mathsf{p} : (x_1, x_2, x_3, x_4) \mapsto x_1 \,, \\
&\mathsf{s} : x \mapsto (x, 0, 0, 0) \,, \\
&\mathsf{r} : (x_1, x_2, x_3, x_4) \mapsto (0, x_2, x_3 - x_1x_2, x_4 - x_1x_3 + \tfrac{1}{2}x_1^2x_2) \,.
\end{aligned}$$

Then,

$$\begin{aligned}
&\mathsf{r}\left((-x_1, -x_2, x_1x_2 - x_3, x_1x_3 - \tfrac{1}{2}x_1^2x_2 - x_4)(x_1', 0, 0, 0)\right) \\
&\quad = (0, -x_2, x_1'x_2 - x_3, -x_4 + x_3x_1' - \frac{1}{2}x_2x_1'^2).
\end{aligned}$$

A generic character of the subgroup H_1 are parametrised by a triple of real constants $(h_2, h_3, \hbar_4)$ where $\hbar_4$ can be identified with the Planck's constant:

$$\chi(0, x_2, x_3, x_4) = \mathrm{e}^{2\pi\mathrm{i}(h_2x_2 + h_3x_3 + \hbar_4x_4)}. \tag{1.2.9}$$

The Kirillov orbit method shows [44, § 3.3.2] that all non-equivalent unitary irreducible representations are induced by characters indexed by $(h_2, 0, \hbar_4)$. For such a character the unitary representation of $\mathbb{G}$ on $L_2(\mathbb{R})$ is, cf. [44, § 3.3, (19)]:

$$[\rho_{h_2\hbar_4}(x_1, x_2, x_3, x_4)f](x_1') = \mathrm{e}^{2\pi\mathrm{i}(h_2x_2 + \hbar_4(x_4 - x_3x_1' + \frac{1}{2}x_2x_1'^2))} f(x_1' - x_1). \tag{1.2.10}$$

This representation is indeed irreducible since its restriction to the Heisenberg group $\tilde{\mathbb{H}}$ coincides with the irreducible Schrödinger representation [25, § 1.3][53] (for irreducibility of the Schrödinger representation see [25, Proposition 1.43]).

1.2.1 Derived representations

Let ρ be a representation of a Lie group G, the derived representation denoted $\mathrm{d}\rho^X$ and generated by an element X of the corresponding Lie algebra $\mathfrak{g}$ is a representation of $\mathfrak{g}$ in the space of linear operators of a vector space $\mathcal{H}$ and given by

$$\mathrm{d}\rho^X \phi := \frac{\mathrm{d}}{\mathrm{d}t}\rho(\exp tX)\phi\Big|_{t=0}, \tag{1.2.11}$$

where the domain consists of functions $\phi \in \mathcal{H}$ such that the vector-function $g \mapsto \rho(g)\phi$ is infinitely-differentiable function for any $g \in G$. Such functions are called *smooth vectors* and constitute a vector subspace, denoted $\mathcal{D}^\infty$, of $\mathcal{H}$ which can be shown to be dense in $\mathcal{H}$. It is also easy to show that $\mathcal{D}^\infty$ is invariant under $\rho(g)$, for proofs of these properties, see for example [44, Appendix V] [78, Ch. 5] and [17, Appendex A.1].

In our situation, $\mathcal{H}$ is either of $L_2(\mathbb{R}^n)(n = 1, 2, 3)$, in such a case the space $\mathcal{D}^\infty$ coincides with Schwartz space [26, 67] $\mathcal{S}(\mathbb{R}^n)$ which is a dense subspace of $L_2(\mathbb{R}^n)$.

Now, for the basis elements X_j spanning the Lie algebra $\mathfrak{g}$ of the group $\mathbb{G}$ (1.1.3):

$$X_1 = (1, 0, 0, 0), \quad X_2 = (0, 1, 0, 0), \quad X_3 = (0, 0, 1, 0), \quad X_4 = (0, 0, 0, 1)$$

we apply the formula (1.2.11) to the representation $\tilde{\rho}_{\hbar_4}$ of $\mathbb{G}$ on $L_2(\mathbb{R}^3)$ and we obtain:

$$\begin{aligned} \mathrm{d}\tilde{\rho}^{X_1}_{\hbar_4} &= -\partial_1 - x_2\partial_3 + 2\pi \mathrm{i}\hbar_4 x_3 I; & \mathrm{d}\tilde{\rho}^{X_2}_{\hbar_4} &= -\partial_2; \\ \mathrm{d}\tilde{\rho}^{X_3}_{\hbar_4} &= -\partial_3; & \mathrm{d}\tilde{\rho}^{X_4}_{\hbar_4} &= 2\pi \mathrm{i}\hbar_4 I\,, \end{aligned} \tag{1.2.12}$$

here and throughout the thesis "I" denotes the identity operator. For the unitary irreducible representation $\rho_{h_2\hbar_4}$ of $\mathbb{G}$ on $L_2(\mathbb{R})$ (1.2.10) we have:

$$\begin{aligned} \mathrm{d}\rho^{X_1}_{h_2\hbar_4} &= -\frac{\mathrm{d}}{\mathrm{d}y}; & \mathrm{d}\rho^{X_2}_{h_2\hbar_4} &= 2\pi \mathrm{i} h_2 + \mathrm{i}\pi\hbar_4 y^2; \\ \mathrm{d}\rho^{X_3}_{h_2\hbar_4} &= -2\pi \mathrm{i}\hbar_4 y; & \mathrm{d}\rho^{X_4}_{h_2\hbar_4} &= 2\pi \mathrm{i}\hbar_4\, I. \end{aligned} \tag{1.2.13}$$

It is easy to check that the sets of operators (1.2.12) and (1.2.13) represent the Lie algebra $\mathfrak{g}$ (1.1.3) of the group $\mathbb{G}$. Moreover, the domains of these operators are $\mathcal{S}(\mathbb{R}^3)$ and $\mathcal{S}(\mathbb{R})$, respectively.

Moreover, in this thesis physical dimensions will be considered, thus we adhere to the following convention, see [46].

Convention 1.2.1 *Only physical quantities of the same dimension can be added or subtracted. Therefore, mathematical functions such as* $\exp(u) = 1 + u + u^2/2! + \ldots$ *can be naturally constructed out of a dimensionless number u only. Thus, Fourier dual variables, say* x *and* q*, should posses reciprocal dimensions because they have to form the expression* $\mathrm{e}^{\mathrm{i}xq}$*. We assign physical units to coordinates on the Heisenberg group* $\mathbb{H}$*. Precisely, let* M *be a unit of mass,* L *of length and* T*of time. For an element* (x, y, s) *of the Heisenberg group* $\mathbb{H}$*, where* $x, y, s \in \mathbb{R}$*, we adopt the following. To* x *and* y*, we assign physical units* $T/(LM)$ *and* $1/L$*, respectively. These units are reciprocal to that of momentum* p *(measured in* ML/T*) and position* q *(measured in* L*) components of* $(p, q, \hbar)$*. The latter are points parametrising the elements of the dual space* $\mathfrak{h}^*$ *to the Heisenberg-Weyl algebra* $\mathfrak{h}$ *of* $\mathbb{H}$*.*

Remark 1.2.2 *Based on the Convention 1.2.1, we assign the following units to* x_j*; the components of an element* (x_1, x_2, x_3, x_4) *of the group* $\mathbb{G}$*. Namely,* x_1 *has dimension* T/ML *(reciprocal to momentum) and* x_3 *has dimension* $1/L$ *(reciprocal to position.) So, from* (1.1.3) x_2 *is measured in unit* M/T*. Also,* h_2*, the dual to* x_2 *(cf.* (1.2.9)*) has reciprocal dimension to* x_2*, that is, it has dimension* T/M *as well as* $\hbar_4$ *has dimension* ML^2/T *of action which is reciprocal to the dimension of the product* x_1x_3 *or* x_4*. The dimension of* $\hbar_4$ *coincides with that of Planck's constant. Note also that the dimensions of* $\mathrm{d}\rho^{X_j}$ *are, respectively, reciprocal to that of* X_j*.*

1.3 The group $\mathbb{G}$ and the Schrödinger group

An important realisation of the group $\mathbb{G}$ is as a subgroup of the so-called Schrödinger group $\mathbb{S}$. For this purpose we need to recall the notion of the *semidirect product*.

Definition 1.3.1 *Let G and H be two groups and assume that $\gamma : k \to \gamma(k)$ is a homomorphism of H into Aut(G), that is, $\gamma(k) : G \to G$ is an automorphism of G for each k in H. We denote by $G \rtimes_\gamma H$ the semidirect product of G and H which generates a group with group law given by*

$$(g_1, h_1)(g_2, h_2) = (g_1\gamma(h_1)g_2, h_1h_2), \tag{1.3.14}$$

where $g_1, g_2 \in G$ and $h_1, h_2 \in H$.

There are important groups arise as a semidirect product. The Schrödinger group $\mathbb{S}$ is a relevant example, which is the group of symmetries of the Schrödinger equation [39, 63], the harmonic oscillator [64], other parabolic equations [84] and paraxial beams [76]. It is the semidirect product of the Heisenberg group and $\mathrm{SL}_2(\mathbb{R})$—the group of all 2×2 real matrices with the unit determinant [20]:

$$\mathbb{S} = \mathbb{H} \rtimes_{\gamma(A)} \mathrm{SL}_2(\mathbb{R}), \tag{1.3.15}$$

where $\gamma(A)$ is a symplectic automorphism of $\mathbb{H}$:

$$\gamma(A) : (x, y, s) \mapsto \left(ax + by, cx + dy, s - \frac{1}{2}xy + \frac{1}{2}(ax + by)(cx + dy)\right); \tag{1.3.16}$$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ and $(x, y, s) \in \mathbb{H}$.

Denote by $\big((x, y, s), A\big)$ elements of the group $\mathbb{S}$, then the corresponding group law (1.3.14) reads

$$\big((x, y, s), A\big)\big((x', y', s'), B\big) = \big((x, y, s)\gamma(A)\big(x', y', s'\big), AB\big). \tag{1.3.17}$$

Let

$$N = \left\{ \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \ t \in \mathbb{R} \right\} \subset \mathrm{SL}_2(\mathbb{R})$$

and let $n(t)$ denote an element of N, that is, $n(t) := \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$. Then, for this particular case, (1.3.16) becomes

$$\gamma(n(t)) : (x, y, s) \mapsto (x, y + tx, s + \frac{1}{2}tx^2). \tag{1.3.18}$$

Moreover, we have the following result:

Proposition 1.3.2 *The group $\mathbb{G}$ is isomorphic to the subgroup $\mathbb{H} \rtimes_{\gamma(n)} N$ of $\mathbb{S}$.*

Proof

We show that the map $J : \mathbb{G} \to \mathbb{H} \rtimes_{\gamma(n)} N$ given by

$$J(x_1, x_2, x_3, x_4) = \left((x_1, x_3 - x_2x_1, x_4 - \frac{1}{2}x_2x_1^2), n(-x_2) \right),$$

is an isomorphism, where $(x_1, x_3 - x_2x_1, x_4 - \frac{1}{2}x_2x_1^2) \in \mathbb{H}$ and $n(x_2) = \begin{pmatrix} 1 & 0 \\ x_2 & 1 \end{pmatrix} \in N$.

First, we show that J is a homomorphism. Indeed, using the group law (1.3.17) we see that

$$\begin{aligned}
&J(x_1, x_2, x_3, x_4)J(y_1, y_2, y_3, y_4) \\
&= \left((x_1, x_3 - x_2x_1, x_4 - \frac{1}{2}x_2x_1^2), n(-x_2) \right) \left((y_1, y_3 - y_2y_1, y_4 - \frac{1}{2}y_2y_1^2), n(-y_2) \right) \\
&= \Bigg(\left(x_1, x_3 - x_2x_1, x_4 - \frac{1}{2}x_2x_1^2 \right) \gamma(n(-x_2)) \left(y_1, y_3 - y_2y_1, y_4 - \frac{1}{2}y_2y_1^2 \right), \\
&\quad n(-(x_2 + y_2)) \Bigg) \\
&= \Bigg((x_1 + y_1, x_3 - x_1x_2 + y_3 - y_1y_2 - x_2y_1, x_4 - \frac{1}{2}x_1^2x_2 + y_4 - \frac{y_1^2}{2}(y_2 + x_2) \\
&\quad + x_1(y_3 - y_1y_2 - x_2y_1)), n(-(x_2 + y_2)) \Bigg).
\end{aligned} \tag{1.3.19}$$

On the other hand, (using the group law of $\mathbb{G}$ (1.1.4))

$$\begin{aligned}
&J\left((x_1,x_2,x_3,x_4)(y_1,y_2,y_3,y_4)\right)\\
&\quad = J\left(x_1+y_1, x_2+y_2, x_3+y_3+x_1y_2, x_4+y_4+x_1y_3+\frac{1}{2}x_1^2y_2\right)\\
&\quad = \Big(\big(x_1+y_1, x_3+y_3+x_1y_2-(x_2+y_2)(x_1+y_1), x_4+y_4+x_1y_3\\
&\qquad +\frac{1}{2}x_1^2y_2-\frac{1}{2}(x_2+y_2)(x_1+y_1)^2\big), n(-(x_2+y_2))\Big)\\
&\quad = \Big(\big(x_1+y_1, x_3+y_3-x_2x_1-y_2y_1-x_2y_1, x_4+y_4-\frac{1}{2}x_1^2x_2-\frac{y_1^2}{2}(y_2+x_2)\\
&\qquad + x_1(y_3-y_1y_2-x_2y_1)\big), n(-(x_2+y_2))\Big).
\end{aligned}$$

Since the last equality is the same as (1.3.19), the map J is a homomorphism. It remains to show that J is also a bijection. It is clear that J is injective because each component of J contains a linear term of x_j, respectively. Such a map is surjective because each element $((x,y,s),n(t)) \in \mathbb{H} \rtimes_{\gamma(n)} N$ is the image of an element $(x,-t,y-tx,s-\frac{1}{2}tx^2) \in \mathbb{G}$ under J, thus J is bijective. □

The geometrical meaning of the transformation

$$n(x_2)(x_1,x_3) := \begin{pmatrix} 1 & 0 \\ x_2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2x_1+x_3 \end{pmatrix}. \tag{1.3.20}$$

is *shear transform* with the angle $\tan^{-1}(1/x_2)$, see Fig. 1.1.

Note that this also describes a physical picture; it is time-shift Galilean transformation: for a particle with coordinate (position) x_3 and the constant velocity x_1: after a period of time x_2 the particle will still have the velocity x_1 but its new coordinate will be $x_3+x_2x_1$. We shall refer to both geometric and physical interpretations of the shear transform in Section 3.2 in connection with the dynamic of the harmonic oscillator.

Another important group of symplectomorphisms—squeezing—are produced by matrices

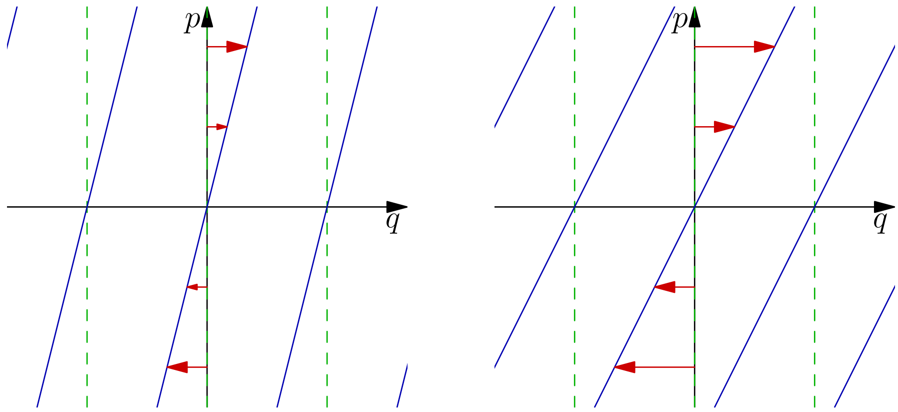

Figure 1.1: Shear transforms: dashed (green) vertical lines are transformed to solid (blue) slanted ones. An interpretation as the dynamic of a free particle: the momentum is constant, the coordinate is changed by an amount proportional to the momentum (cf. different arrows on the same picture) and the elapsed time (cf. the left and right pictures).

$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$. These transformations act transitively on the set of minimal uncertainty states ϕ, such that $\Delta_\phi Q \cdot \Delta_\phi P = \frac{\hbar}{2}$—the minimal value admitted by the Heisenberg–Kennard uncertainity relation, cf. Section 1.4.

Quantum blobs[1] are the smallest phase space units of phase space compatible with the uncertainty principle of quantum mechanics. They are in a bijective correspondence with the squeezed coherent states [28] from standard quantum mechanics of which they are a phase space picture. Quantum blobs have the symplectic group as group of symmetries. In particular, the actions on quantum blobs by the above squeeze and shear (symplectic) transformations brings out a close relationship between these transformations, see Fig. 1.2.

[1] See [19, Definition 8.34] and also [21].

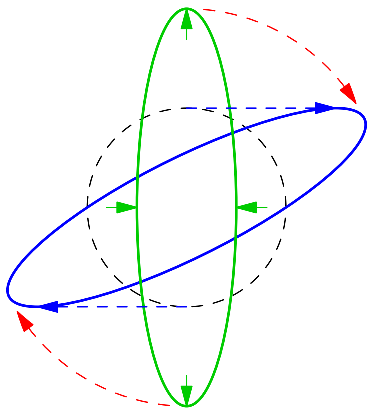

Figure 1.2: Shear transformations (blue) act on blobs as squeeze (green) plus rotation (red), although these transformations are different in general as transformations of $\mathbb{R}^2$ and their effects on quadratic forms coincide.

1.3.1 The group $\mathbb{G}$ and the universal enveloping algebra of $\mathfrak{h}$

A related origin of the group $\mathbb{G}$ is the universal enveloping algebra $\mathcal{H}$ of the Heisenberg–Weyl algebra $\mathfrak{h}$ spanned by elements Q, P and I with $[P,Q]=I$. It is known [84] that the Lie algebra of Schrödinger group can be identified with the subalgebra spanned by the elements $\{Q,P,I,Q^2,P^2,\frac{1}{2}(QP+PQ)\}\subset\mathcal{H}$. This algebra is known as *quadratic algebra* in quantum mechanics [28, § 2.2.4][83, § 17.1]. From the above discussion of the Schrödinger group, the identification

$$X_1\mapsto P,\quad X_2\mapsto \tfrac{1}{2}Q^2,\quad X_3\mapsto Q,\quad X_4\mapsto I \tag{1.3.21}$$

embeds the Lie algebra $\mathfrak{g}$ into $\mathcal{H}$. In particular, the identification $X_2\mapsto\frac{1}{2}Q^2$ was used in physical literature to treat anharmonic oscillator with quartic potential [5, 38, 55]. Furthermore, the group $\mathbb{G}$ is isomorphic to the *Galilei group* via the identification of respective Lie algebras

$$X_1\mapsto -Q,\quad X_2\mapsto \tfrac{1}{2}P^2,\quad X_3\mapsto P,\quad X_4\mapsto I. \tag{1.3.22}$$

We shall note that the consideration of $\mathbb{G}$ as a subgroup of the Schrödinger group or the universal enveloping algebra $\mathcal{H}$ has a limited scope since only representations $\rho_{h_2\hbar_4}$ with $h_2 = 0$ appear as restrictions of representations of Schrödinger group, see [19, Ch. 7][20, Ch. 7] [25, § 4.2].

1.4 Some physical background

Here we briefly review basic elements from quantum mechanics (QM), for more details see for example [12, 33, 60, 72].

1.4.1 A mathematical model of QM

To begin with, we recall that a physical system is described by a state. In quantum mechanics a *state* is meant to be a non-zero vector in a Hilbert space $\mathcal{H}$ whose norm is unity. A quantum *observable* is associated with a self-adjoint operator A on $\mathcal{H}$. In the context of QM, a self-adjoint operator A can be unbounded [33, Ch. 3][68, Ch. 1], thus A requires a certain domain of definition, denoted $\mathrm{Dom}(A)$ such that $\mathrm{Dom}(A)$ is dense in $\mathcal{H}$. In this way, we say that the unbounded operator A is densely defined in $\mathcal{H}$. Density of the domain is a sufficient and necessary condition for the adjoint operator A^* to be well-defined [56, Ch. 10].

For a state ψ, an observable operator A produces a probability distribution with the *expectation value*, denoted $\bar{A}$ and given by

$$\bar{A}\psi = \langle A\psi, \psi\rangle , \qquad \psi \in \mathrm{Dom}(A). \tag{1.4.23}$$

The *dispersion* of A in the state ψ, denoted $\Delta_\psi A$ is defined as the square root of the expectation value of $(A - \bar{A})^2$ and computed as

$$(\Delta_\psi A)^2 = \left\langle (A - \bar{A})^2\psi, \psi\right\rangle = \left\langle (A - \bar{A})\psi, (A - \bar{A})\psi\right\rangle = \left\|(A - \bar{A})\psi\right\|^2 . \tag{1.4.24}$$

Let us consider the Hilbert space $L_2(\mathbb{R})$ of complex-valued function which represents a quantum mechanical model on the real line, also known as the *Schrödinger model*. In such a space the position observable, denoted Q is represented by the self-adjoint operator

$$Q = qI, \tag{1.4.25}$$

on the domain; $\text{Dom}(Q) = \{f \in L_2(\mathbb{R}) : qf(q) \in L_2(\mathbb{R})\}$ which can be shown to be a dense subspace in $L_2(\mathbb{R})$ [33, § 9.8][56, § 10.7][68, § 2.3]. This operator provides a probability distribution of determining the position of a particle on the line. The corresponding expectation value is computed through the integral formula

$$[\bar{Q}\psi](q) = \langle Q\psi, \psi \rangle = \int_{\mathbb{R}} q\psi(q)\overline{\psi(q)}\, \mathrm{d}q = \int_{\mathbb{R}} q|\psi(q)|^2\, \mathrm{d}q.$$

Another important observable operator in the state space $L_2(\mathbb{R})$ is the momentum observable, denoted P and given by the self-adjoint operator

$$P = -\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}q}, \tag{1.4.26}$$

where $\hbar$ is the Planck's constant divided by 2π and has a physical dimension: $energy \times time$. It is self-adjoint on $\text{Dom}(P) = \{f \in L_2(\mathbb{R}) : f'(q) \in L_2(\mathbb{R})\}$ [33, § 9.8][68, § 2.4].

On the Schwartz space $\mathcal{S}(\mathbb{R})$, the operators Q and P are stable and *essentially self adjoint* operators [33, § 9.7][66, § 8.5] and satisfy "the canonical commutation relation" [12, 33, 66]

$$[Q, P] = QP - PQ = \mathrm{i}\hbar I, \tag{1.4.27}$$

since for ψ in the Schwartz space $\mathcal{S}(\mathbb{R})$, we have

$$PQ\psi(q) = -\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}q}(q\psi(q)) = -\mathrm{i}\hbar(\psi(q) + q\psi'(q)) = -\mathrm{i}\hbar\psi(q) + QP\psi(q).$$

That is,

$$[Q, P]\psi(q) = QP\psi(q) - PQ\psi(q) = \mathrm{i}\hbar\psi(q).$$

That is, the operators Q and P do not commute.

1.4.2 The Uncertainty Relation

Theorem 1.4.1 (The Uncertainty Relation [25, 33, 52]) *If A and B are symmetric operators with domains Dom (A), Dom (B) in a Hilbert space $\mathcal{H}$, then*

$$\|(A-a)\psi\| \, \|(B-b)\psi\| \geq \frac{1}{2} \left|\langle (AB-BA)\psi, \psi\rangle\right| \tag{1.4.28}$$

for any ψ in $\mathcal{H}$ such that $\psi \in$ Dom $(AB) \cap$ Dom (BA), where $a, b \in \mathbb{R}$. Equality holds when ψ is a solution of

$$((A-a) + \mathrm{i}k(B-b))\psi = 0, \tag{1.4.29}$$

where k is a real parameter. So, only commuting observables have exact simultaneous measurements.

In particular, for $a = \bar{A}$ and $b = \bar{B}$, we have

$$\Delta_\psi A \; \Delta_\psi B \geq \frac{1}{2} \left|\langle (AB-BA)\psi, \psi\rangle\right|, \tag{1.4.30}$$

and when equality holds, ψ is termed a *minimal uncertainty state* or *coherent state*.

An important consequence of the above theorem is the case when A and B are the position Q and the momentum P observables. In such a situation the relation (1.4.30) is known as the *Heisenberg-Kennard uncertainty relation* [52]:

$$\Delta_\psi Q \; \Delta_\psi P \geq \frac{\hbar}{2}, \tag{1.4.31}$$

for all unit vector $\psi \in L_2(\mathbb{R})$ in $\mathrm{Dom}(QP) \cap \mathrm{Dom}(PQ)$. Equality in (1.4.31) holds when ψ is a solution of the equation

$$((Q-a) + \mathrm{i}k(P-b))\,\psi(q) = 0, \tag{1.4.32}$$

where $a = \bar{Q}$ and $b = \bar{P}$. It can be easily checked that the state ψ needed to satisfy the above equation (1.4.32) is

$$\psi(q) = c \exp\left(\left(\frac{\mathrm{i}b}{\hbar} + \frac{a}{\hbar k}\right) q - \frac{1}{2\hbar k} q^2\right),$$

where c is a constant determined by a normalisation condition.

Let $a = b = 0$, and consider the normalisation of ψ in terms of L_2-norm so that

$$\psi(q) = \left(\frac{1}{\pi\hbar k}\right)^{1/4} \mathrm{e}^{-\frac{1}{2\hbar k}q^2}.$$

Then, we have

$$\begin{aligned}(\Delta_\psi Q)^2 = \left\|(Q - \bar{Q})\psi\right\|^2 &= \int_{\mathbb{R}} |[Q\psi](q)|^2 \, \mathrm{d}q \\ &= \left(\frac{1}{\pi\hbar k}\right)^{1/2} \int_{\mathbb{R}} q^2 \mathrm{e}^{-\frac{1}{\hbar k}q^2} \, \mathrm{d}q \\ &= \frac{\hbar k}{2}.\end{aligned}$$

Thus,

$$\Delta_\psi Q = \sqrt{\frac{\hbar k}{2}}. \tag{1.4.33}$$

For the momentum P we have

$$\begin{aligned}(\Delta_\psi P)^2 = \left\|(P - \bar{P})\psi\right\|^2 &= \int_{\mathbb{R}} \left|\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}q}\psi(q)\right|^2 \, \mathrm{d}q \\ &= \left(\frac{1}{\pi\hbar k}\right)^{1/2} \left(\frac{1}{k}\right)^2 \int_{\mathbb{R}} q^2 \mathrm{e}^{-\frac{1}{\hbar k}q^2} \, \mathrm{d}q \\ &= \frac{\hbar}{2k}.\end{aligned}$$

So,

$$\Delta_\psi P = \sqrt{\frac{\hbar}{2k}}. \tag{1.4.34}$$

Hence,

$$\Delta_\psi Q \, \Delta_\psi P = \sqrt{\frac{\hbar k}{2}} \cdot \sqrt{\frac{\hbar}{2k}} = \frac{\hbar}{2}.$$

It can be easily seen that for $k \neq 1$, the dispersions (1.4.33) and (1.4.34) are *not* equal and one of these is at the expense of the other to maintain the minimum uncertainty relation. This is the case in which one calls ψ a *squeezed state* [70, 79, 80, 81]. It minimizes the uncertainty relation when the dispersions of the respective quantum observables are not equal.

Squeezed states turn out to be of a prominent role in quantum optics [28, 70]. They first appeared in connection with applications to quantum optics in the work of Yuen [85] under the name *two-photons*. A systematic way of obtaining a squeezed state involves the action of a unitary operator, the so-called *squeeze operator*, introduced in [73] while the name "squeeze operator" was given by Hollenhorst [36]. This has provided a way of generalising squeezed states which falls within a group theoretical framework based on the representations of the special unitary group $\mathrm{SU}(1,1)$ and its Lie algebra [4, 32, 82].

In this thesis, a certain connection to Gaussian with arbitrary squeeze will be seen in the next chapter.

1.4.3 Harmonic oscillator and ladder operators

A one-dimensional harmonic oscillator of mass m and frequency ω has the classical Hamiltonian (energy)

$$h = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}q^2, \tag{1.4.35}$$

where q and p are its position and momentum, respectively. The classical Hamiltonian h is understood as a function in the phase space $\mathbb{R}^2$ of points (q,p). The quantised version (Weyl quantisation [2]) of h is presented by a self-ajoint operator and called the observable of energy and given by

$$H = \frac{1}{2m}P^2 + \frac{m\omega^2}{2}Q^2, \tag{1.4.36}$$

where as before

$$Q\phi(q) = q\phi(q), \qquad P\phi(q) = -\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}q}\phi(q).$$

The operator H is self-adjoint on Dom$\,(H)$ = Dom$\,(P^2)$ $\cap$ Dom$\,(Q^2)$=Dom$\,(P^2)$ [33, § 9.9] [68, § 2.5].

[2] Quantisation is a rule of passing from classical mechanics to quantum mechanics [34, Ch. 13][25, 53].

In quantum mechanics, the eigenvalues of a quantum observable are interpreted as the measured values of such an observable. For example, the eigenvalues of a Hamiltonian are the measured energies of the corresponding quantum system. The problem of finding these eigenvalues in the case of harmonic oscillator is completely solved via algebraic approach. Precisely, for the specific H (1.4.36) of constant mass m and frequency ω, one takes the advantage of the ladder operators, as defined in Appendix A where the parameter λ has the specific value: $\lambda = \sqrt{m\omega}$ and so,

$$a^- = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega Q + \mathrm{i}P), \qquad a^+ = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega Q - \mathrm{i}P). \tag{1.4.37}$$

Here, we restrict the operators P and Q to the Schwartz space $\mathcal{S}(\mathbb{R})$. Then, the relation $[Q, P] = \mathrm{i}\hbar I$ implies that

$$\begin{aligned} a^-a^+ &= \frac{1}{2m\omega\hbar}(P^2 + m^2\omega^2Q^2 + m\omega\hbar I) = \frac{1}{\omega\hbar}(H + \frac{1}{2}\omega\hbar I); \\ a^+a^- &= \frac{1}{2m\omega\hbar}(P^2 + m^2\omega^2Q^2 - m\omega\hbar I) = \frac{1}{\omega\hbar}(H - \frac{1}{2}\omega\hbar I). \end{aligned}$$

Thus,

$$H = \frac{\hbar\omega}{2}(a^-a^+ + a^+a^-),$$

which can be shown to be essentially self-adjoint on $\mathcal{S}(\mathbb{R})$ [33, § 9.9]. Moreover, from $[a^-, a^+] = I$ we arrive at

$$H = \hbar\omega a^+a^- + \frac{\hbar\omega}{2}I. \tag{1.4.38}$$

Finally, using (A.37) the spectrum of H (1.4.36) is determined from

$$H\phi_n = \hbar\omega(n + \frac{1}{2})\phi_n. \tag{1.4.39}$$

The vacuum vector ϕ_0 is the solution of $a^-\phi_0 = 0$. Precisely, $\frac{1}{\sqrt{2m\omega\hbar}}(m\omega q + \hbar\frac{\mathrm{d}}{\mathrm{d}q})\phi_0(q) = 0$ which has the solution

$$\phi_0(q) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \mathrm{e}^{-\frac{1}{2}\frac{m\omega}{\hbar}q^2}. \tag{1.4.40}$$

Then, $\phi_n(q) = \frac{1}{\sqrt{n!}}(a^+)^n\phi_0(q) = \frac{1}{\sqrt{n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{1}{\sqrt{2m\omega\hbar}}\right)^n(m\omega q - \hbar\frac{\mathrm{d}}{\mathrm{d}q})^n \mathrm{e}^{-\frac{1}{2}\frac{m\omega}{\hbar}q^2} = \frac{1}{\sqrt{n!}}\left(\frac{1}{\sqrt{2m\omega\hbar}}\right)^n H_n(\sqrt{\frac{m\omega}{\hbar}}q)\phi_0$, where H_n are the *Hermite polynomials* of order n:

$$H_n(y) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k n!}{k!(n-2k)!}(2y)^{n-2k} \tag{1.4.41}$$

note that $\lfloor\cdot\rfloor$ is the floor function (i.e. for any real x, $\lfloor x\rfloor$ is the greatest integer n such that $n \leq x$.) The functions $\phi_n(q)$ constitute the standard basis of $L_2(\mathbb{R})$ [33, Ch. 11].

The non-negative number n is called the *quantum number*. The value $n = 0$ corresponds to the lowest state energy

$$\frac{1}{2}\hbar\omega \tag{1.4.42}$$

which corresponds to the ground state or the vacuum ϕ_0.

1.4.4 Canonical coherent states

The canonical coherent states, ϕ_z $(z = q + \mathrm{i}p)$ of the harmonic oscillator are produced by the the action of the "displacement" operator on the vacuum ϕ_0 (or $\phi_{m\omega}$ in case we want to emphasize the dependence on the particular value $m\omega$):

$$\phi_z(x) = \mathrm{e}^{za^+ - \bar{z}a^-}\phi_0(x) = \mathrm{e}^{-\frac{1}{2}|z|^2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}\phi_n(x), \tag{1.4.43}$$

where a^+, a^- are given by (1.4.37) and $\bar{z}$ is the complex-conjugate of z. Usually, the canonical coherent states are denoted $|z\rangle$. A list of fundamental properties of these states is given in [28], see also [54, 70]. Among these is that $a^-|z\rangle = z|z\rangle$, so one may even regard this property as a definition of coherent states, i.e. the coherent states are the eigenstates of the annihilation operator. From this, one can easily deduce the expectation value of Q and P in such coherent states from the real and imaginary parts of the eigenvalue z, respectively [83, Ch. 23]. Indeed, the expectation values of Q and P in the coherent state $|z\rangle$ are $\sqrt{\frac{2\hbar}{m\omega}}\Re(z)$ and $\sqrt{2\hbar m\omega}\Im(z)$, respectively. Moreover, by expressing Q and P in terms of a^- and a^+, one can then simply evaluate the expectation values of Q^2 and P^2 which lead to obtain the respective dispersions being $\Delta Q = \sqrt{\frac{\hbar}{2m\omega}}$ and $\Delta P = \sqrt{\frac{m\omega\hbar}{2}}$. Thus, $\Delta Q\,\Delta P = \frac{\hbar}{2}$, that is, the canonical coherent states minimise the uncertainty relation. Therefore, $|z\rangle$ must be related to a Gaussian; one can show [62, Ch.3] that (1.4.43) reduces to

$$|z\rangle := \phi_z(x) = \pi^{-1/4} \exp\left(\frac{1}{2}z^2 - \frac{1}{2}|z|^2 - (\sqrt{m\omega/(2\hbar)}x - z)^2\right),$$

known as *Gaussian wave packets*.

1.4.5 Dynamics of the harmonic oscillator

The time evolution in quantum mechanics (in Schrödinger picture) is determined via the time-dependent Schrödinger equation which takes the form

$$\mathrm{i}\hbar\dot{\psi}(q,t) = H\psi(q,t), \tag{1.4.44}$$

where H is the Hamiltonian observable and ψ is termed *wavefunction*.

With regard to the above harmonic oscillator Hamiltonian H, we can use the spectrum of H:

$$H\phi_n(q) = \hbar\omega(n + \frac{1}{2})\phi_n(q) \tag{1.4.45}$$

to obtain the dynamic of the system as follows.

Since $\{\phi_n\}$ is an orthonormal basis of $L_2(\mathbb{R})$ one can write

$$\phi(q,t) = \sum_{n=0}^{\infty} a_n(t)\phi_n(q). \tag{1.4.46}$$

Then, after substituting into (1.4.44) and using (1.4.45) one gets $a_n(t) = a_n(0)\mathrm{e}^{-\mathrm{i}\omega(n+\frac{1}{2})t}$. Hence, the dynamic is given by

$$\phi(q,t) = \sum_{n=0}^{\infty} a_n(0)\mathrm{e}^{-\mathrm{i}\omega(n+\frac{1}{2})t}\phi_n(q). \tag{1.4.47}$$

The process of obtaining such a solution depends on knowing the spectrum of H. So, in other complicated systems it may be difficult to proceed this way.

In a similar way, one can also obtain evolution in the canonical coherent states representation which takes the form

$$\mathrm{e}^{-\mathrm{i}tH/\hbar}|z\rangle = \mathrm{e}^{-\mathrm{i}\omega t/2}|\mathrm{e}^{-\mathrm{i}\omega t}z\rangle. \tag{1.4.48}$$

Or,

$$\mathrm{e}^{-\mathrm{i}tH/\hbar}\phi_z = \mathrm{e}^{-\mathrm{i}\omega t/2}\phi_{z(t)} \tag{1.4.49}$$

where $z(t) = \mathrm{e}^{-\mathrm{i}\omega t}z$ is a one-parameter group of transformations. This shows that the dynamic in canonical coherent states, or the expectation values of the displacement in canonical coherent states behave in a manner similar to the displacement of classical oscillator.

1.4.6 The FSB space

A transition from the configuration space $\mathbb{R}$ to the phase space $\mathbb{R}^2$ in quantum mechanics is performed by the coherent states transform [19, 20, 25]:

$$\mathcal{W}_{\phi_0} : f \mapsto \langle f, \phi_z\rangle := \tilde{f}(z)$$

where $f \in L_2(\mathbb{R})$ and ϕ_z is the canonical coherent states (1.4.43). The image of this map gives rise to the following Hilbert space:

Definition 1.4.2 ([3, 8, 25]) *Let $z = q + \mathrm{i}p \in \mathbb{C}$, the Fock-Segal-Bargemann (FSB) space consists of all functions that are analytic on the whole complex plane $\mathbb{C}$ and square-integrable with respect to the measure $\mathrm{e}^{-\pi\hbar|z|^2}\,\mathrm{d}z$. It is equipped with the inner product*

$$\langle f, g\rangle_F = \int_{\mathbb{C}} f(z)\overline{g(z)}\,\mathrm{e}^{-\pi\hbar|z|^2}\,\mathrm{d}z. \tag{1.4.50}$$

This space has several advantages over the state space $L_2(\mathbb{R})$. In particular, the dynamics of the harmonic oscillator H has a geometrical description that comes in agreement with the classical counterpart.

Starting from the fact that the ladder operators have the simpler expressions:

$$a^- = \partial_z, \qquad a^+ = zI,$$

where the domain of these operators consist of the space of analytic polynomials. It can be easily verified that $[a^-, a^+] = I$ and $(a^-)^* = a^+$, where "*" is the adjoint of an operator with respect to the inner product $\langle \cdot, \cdot \rangle_F$. The vacuum Φ_0 (i.e. the solution to $a^-\Phi_0 = 0$) in this space is just a constant $\Phi_0(z) = c$, where c is chosen so that $\|\Phi_0\|_F = 1$ and the "exited states" Φ_n are easily seen to be monomials: $\Phi_n(z) = \frac{1}{\sqrt{n!}}(a^+)^n\Phi_0 = \frac{c}{\sqrt{n!}}z^n$. The harmonic oscillator Hamiltonian on FSB space has the form

$$H = \hbar\omega(z\partial_z + \frac{1}{2}I). \tag{1.4.51}$$

So, the dynamic calculated through the Schrödinger equation (which is just a first-order PDE) is

$$F(t, z) = \mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t}F(0, \mathrm{e}^{-\mathrm{i}\omega t}z). \tag{1.4.52}$$

The coordinate transformation here represents a rotation in the phase space $\mathbb{R}^2 \sim \mathbb{C}$ which reflects the classical picture of the dynamic of a classical harmonic oscillator calculated via Hamilton's equations [30]. This nature of such dynamic in this space is clearly inherited from the dynamic of the corresponding coherent states (1.4.48). From another standpoint we also observe that such a geometric nature is a result of the Schrödinger equation being a first-order PDE. These together explain, once again, the formulation of Definition 0.0.3.

Note also that although ladder operators technique completely solves the spectral problem for the harmonic oscillator, it should be observed that:

1. The ladder operators (1.4.37) (and subsequently the eigenvectors ϕ_n) depend on the parameter $m\omega$. They *are not useful* for a harmonic oscillator with a different value of $m'\omega'$.

2. The explicit dynamic (1.4.47) of an arbitrary state ϕ *is not transparent* disregarding the prior difficulty in finding the decomposition $\phi = \sum_n c_n \phi_n$ over the orthonormal basis of eigenvectors ϕ_n. Despite the fact that this dynamic in FSB space is presented in a geometric fashion (1.4.52), this presentation still relays on the vacuum $\phi_{m\omega}$ (and, thus, all other coherent states ϕ_z (1.4.43)) having the given value of $m\omega$ as before.

Metaphorically, the traditional usage of the ladder operators and vacuum $\phi_{m\omega}$ is like a key, which can unlock only the matching harmonic oscillator with the same value of $m\omega$. However, the method we use in Chapter 3 makes possible *an extension of the traditional framework, which allows to use any minimal uncertainty state ϕ_E ($E > 0$) as a vacuum (or fiducial) for a harmonic oscillator with a different value of $m\omega$* to obtain geometric dynamics similar to (1.4.52), cf. Section 3.2.

Chapter 2

Coherent state transform

We consider here the coherent state transform which plays an important role in mathematics and physics. If this transformation is reduced from the group $\mathbb{G}$ to the Heisenberg group it coincides with the Fock–Segal–Bargmann type transform. In this connection, we highlight a certain technical aspect arises in the case of the group $\mathbb{G}$ regarding square-integrability notion, see Remark 2.1.7. The principal result in this chapter is the physical characterisation of the image space of an induced coherent state transform of the group $\mathbb{G}$, Section 2.2.

2.1 The induced coherent state transform and its image

Let G be a Lie group with a left Haar measure $\mathrm{d}g$ and ρ a unitary irreducible representation of the group G in a Hilbert space $\mathcal{H}$. Then, we define the coherent state transform as follows.

Definition 2.1.1 ([3, 49]) *For a fixed vector $\phi \in \mathcal{H}$ called a fiducial vector*[1] *(aka vacuum*

[1]Fiducial vector is a general term [54, Ch. 1][7] and is meant to be an arbitrary unit vector; it can be called vacuum vector or ground state in the context of ladder operators that mentioned earlier.

vector, ground state, mother wavelet), the coherent state transform, denoted $\mathcal{W}_\phi$, of a vector $f \in \mathcal{H}$ is given by:

$$[\mathcal{W}_\phi f](g) = \langle f, \rho(g)\phi \rangle \,, \qquad g \in G.$$

We denote the image space of such a transform by $L_\phi(G)$.

Definition 2.1.2 *The irreducible representation ρ is called square-integrable if for every $\psi, \phi \in \mathcal{H}$, the function $[\mathcal{W}_\phi \psi](g)$ is in $L_2(G, \mathrm{d}g)$. That is,*

$$\|\mathcal{W}_\phi \psi\|_2^2 = \int_G |\, \langle \psi, \rho(g)\phi \rangle \,|^2 \, \mathrm{d}g < \infty. \tag{2.1.1}$$

The coherent state transform may not produce a square-integrable function on the entire group, that is, (2.1.1) may not hold. Take for example the case where G is a nilpotent Lie group, then it is known that $\mathcal{W}_\phi$ is not square-integrable [17, § 4.5]. However, in such a situation it is still possible to define a coherent state transform on a suitable homogeneous space that results in square-integrable function in the following manner.

According to [3, § 8.4][65, § 2] let a fiducial vector $\phi \in \mathcal{H}$ be a joint eigenvector of $\rho(h)$ for all h in a subgroup H of G. That is,

$$\rho(h)\phi = \chi(h)\phi \quad \text{for all} \quad h \in H, \tag{2.1.2}$$

where χ is a character of H, see Remark 1.1.1. Then, we see that

$$[\mathcal{W}_\phi f](gh) = \langle f, \rho(gh)\phi \rangle = \langle f, \rho(g)\rho(h)\phi \rangle = \overline{\chi}(h)[\mathcal{W}_\phi f](g). \tag{2.1.3}$$

This indicates that the coherent state transform is entirely defined via its values indexed by points of $X = G/H$. This motivates the following definition of the coherent state transform on the homogeneous space $X = G/H$:

Definition 2.1.3 *([3, § 8.4][53, § 5.1]) For a group G, a closed subgroup H of G, a section $\mathsf{s} : G/H \to G$, a unitary irreducible representation ρ of G in a Hilbert space $\mathcal{H}$*

and a fiducial vector ϕ satisfying (2.1.2)*, we define the induced coherent state transform $\mathcal{W}_\phi$ from $\mathcal{H}$ to a space of functions (the image space of $\mathcal{W}_\phi$) $L_\phi(G/H)$ by the formula*

$$[\mathcal{W}_\phi f](x) = \langle f, \rho(\mathsf{s}(x))\phi\rangle, \qquad x \in G/H\,. \tag{2.1.4}$$

The family of vectors indexed by x:

$$\phi_x = \rho(\mathsf{s}(x))\phi \tag{2.1.5}$$

is called coherent states [3, 65].

Proposition 2.1.4 *([3, § 8.4][49, § 5.1] [53, § 5.1]) Let G, H, ρ, ϕ and $\mathcal{W}_\phi$ be as in Definition 2.1.3 and χ be a character from* (2.1.2)*. Then, the induced coherent state transform intertwines ρ and $\tilde{\rho}$:*

$$\mathcal{W}_\phi \rho(g) = \tilde{\rho}(g)\mathcal{W}_\phi, \tag{2.1.6}$$

where $\tilde{\rho}$ is a representation induced from the character χ of the subgroup H.

In particular, (2.1.6) *means that the image space $L_\phi(G/H)$ of the induced coherent state transform is invariant under $\tilde{\rho}$.*

The case of the Heisenberg group $\mathbb{H}$ is the leading example of the application of the induced coherent state transform see also [18]:

Example 2.1.5 *Let us consider the the following form of Schrödinger representation as it will be used in the rest of the thesis:*

$$\sigma_\hbar(x, y, s)f(x') = \mathrm{e}^{2\pi \mathrm{i}\hbar(s - x'y)} f(x' - x). \tag{2.1.7}$$

For the centre of $\mathbb{H}$, $Z = \{(0, 0, s) \in \mathbb{H} : s \in \mathbb{R}\}$, we see that

$$\sigma_\hbar(0, 0, s)f(x') = \mathrm{e}^{2\pi \mathrm{i}\hbar s} f(x'), \tag{2.1.8}$$

that is, the property (2.1.2) *is satisfied for the character of the centre* $\chi(0, 0, s) = e^{2\pi i\hbar s}$. *Thus, for the corresponding homogeneous space* $\mathbb{H}/Z \sim \mathbb{R}^2$, *we consider a section* s : $\mathbb{H}/Z \to \mathbb{H}$; $\mathsf{s} : (x, y) \mapsto (x, y, 0)$. *Then, for* $f, \phi \in L_2(\mathbb{R})$, *the respective induced coherent state transform is*

$$\begin{aligned} [\mathcal{W}_\phi f](x, y) &= \langle f, \sigma_\hbar\left(\mathsf{s}(x, y)\right)\rangle \\ &= \langle f, \sigma_\hbar(x, y, 0)\rangle \\ &= \int_{\mathbb{R}} f(x')\overline{\sigma_\hbar(x, y, 0)\phi(x')}\, \mathrm{d}x' \\ &= \int_{\mathbb{R}} f(x')e^{2\pi i\hbar x' y}\overline{\phi}(x' - x)\, \mathrm{d}x'. \end{aligned} \tag{2.1.9}$$

From the last integral, one may notice that this is just a composition of an version formula of Fourier transform and measure preserving a change of variables and hence $\mathcal{W}_\phi$ *defines an* L_2*-function on* $\mathbb{H}/Z \sim \mathbb{R}^2$*, more details are found in [25, § 1.4].*

In the time-frequency analysis, the above transform is known as the short-time Fourier transform and ϕ*, in such context, is called the window function [31, Ch. 3].*

2.1.1 The induced coherent state transform of the shear group $\mathbb{G}$

On the same footing as above we explicitly calculate an induced coherent state transform $\mathcal{W}_\phi$ of $\mathbb{G}$.

For the subgroup H being the centre Z of $\mathbb{G}$; $Z = \{(0, 0, 0, z) \in \mathbb{G} : z \in \mathbb{R}\}$, the representation $\rho_{h_2\hbar_4}$ (1.2.10) and the character $\chi(0, 0, 0, z) = e^{2\pi i\hbar_4 z}$ of Z, any function $\phi \in L_2(\mathbb{R})$ satisfies the eigenvector property (2.1.2). Thus, for the respective homogeneous space $\mathbb{G}/Z \sim \mathbb{R}^3$ and the section $\mathsf{s} : \mathbb{G}/Z \to \mathbb{G}$; $\mathsf{s}(x_1, x_2, x_3) =$

$(x_1, x_2, x_3, 0)$, the induced coherent state transform is:

$$\begin{aligned}[\mathcal{W}_\phi f](x_1, x_2, x_3) &= \langle f, \rho_{h_2\hbar_4}(\mathsf{s}(x_1, x_2, x_3))\phi\rangle \\ &= \langle f, \rho_{h_2\hbar_4}(x_1, x_2, x_3, 0)\phi\rangle \\ &= \int_{\mathbb{R}} f(y)\overline{\rho_{h_2\hbar_4}(x_1, x_2, x_3, 0)\phi(y)}\,\mathrm{d}y \\ &= \int_{\mathbb{R}} f(y)\mathrm{e}^{-2\pi\mathrm{i}(h_2x_2+\hbar_4(-x_3y+\frac{1}{2}x_2y^2))}\overline{\phi}(y-x_1)\,\mathrm{d}y \\ &= \mathrm{e}^{-2\pi\mathrm{i}h_2x_2}\int_{\mathbb{R}} f(y)\mathrm{e}^{-2\pi\mathrm{i}\hbar_4(-x_3y+\frac{1}{2}x_2y^2)}\overline{\phi}(y-x_1)\,\mathrm{d}y. \end{aligned} \tag{2.1.10}$$

The last integral is a composition of the following three unitary operators of $L_2(\mathbb{R}^2)$:

1. The change of variables

$$T : F(x_1, y) \mapsto F(y, y - x_1)\,, \tag{2.1.11}$$

where $F(x_1, y) := (f \otimes \overline{\phi})(x_1, y) = f(x_1)\overline{\phi}(y)$, that is, F is defined on the tensor product $L_2(\mathbb{R}) \otimes L_2(\mathbb{R})$ which is isomorphic to $L_2(\mathbb{R}^2)$ [66];

2. the operator of multiplication by a unimodular function $\psi_{x_2}(x_1, y) = \mathrm{e}^{-\pi\mathrm{i}\hbar_4x_2y^2}$

$$M_{x_2} : F(x_1, y) \mapsto \mathrm{e}^{-\pi\mathrm{i}\hbar_4x_2y^2}F(x_1, y), \quad x_2 \in \mathbb{R}; \tag{2.1.12}$$

3. and the partial inverse Fourier transform in the second variable

$$[\mathcal{F}_2F](x_1, x_3) = \int_{\mathbb{R}} F(x_1, y)\mathrm{e}^{2\pi\mathrm{i}\hbar_4yx_3}\,\mathrm{d}y. \tag{2.1.13}$$

Thus, $[\mathcal{W}_\phi f](x_1, x_2, x_3) = \mathrm{e}^{-2\pi\mathrm{i}h_2x_2}[\mathcal{F}_2 \circ M_{x_2} \circ T]F(x_1, x_3)$ and we obtain

Proposition 2.1.6 *For a fixed $x_2 \in \mathbb{R}$, the map $f \otimes \overline{\phi} \mapsto [\mathcal{W}_\phi f](\cdot, x_2, \cdot)$ is a unitary operator from $L_2(\mathbb{R}) \otimes L_2(\mathbb{R})$ onto $L_2(\mathbb{R}^2)$.*

Such an induced coherent state transform also respects the Schwartz space, that is, if $f, \phi \in \mathcal{S}(\mathbb{R})$ then $[\mathcal{W}_\phi f](\cdot, x_2, \cdot) \in \mathcal{S}(\mathbb{R}^2)$. This is because $\mathcal{S}(\mathbb{R}^2)$ is invariant under each operator (2.1.11)–(2.1.13).

Remark 2.1.7 *As already mentioned that the coherent state transform on a nilpotent Lie group, cf. Definition 2.1.1, does not produce an L_2-function on the entire group but may rather do on a certain homogeneous space. For the Heisenberg group $\mathbb{H}$ and the homogeneous space $\mathbb{H}/Z$, the respective induced coherent state transform defines an L_2-function on $\mathbb{H}/Z$, see Example 2.1.5. In the context of an induced coherent state transform of $\mathbb{G}$, two types of modified square-integrability are considered [17, § 4.5]: modulo the group's center and modulo the kernel of the representation. The first notion is not applicable to the group $\mathbb{G}$: the induced coherent state transform* (2.1.10) *does not define a square-integrable function on $\mathbb{G}/Z \sim \mathbb{R}^3$ or a larger space $\mathbb{G}/\ker\rho_{h_2\hbar_4}$. On the other hand, the representation $\rho_{h_2\hbar_4}$ is square-integrable modulo the subgroup $H = \{(0, x_2, 0, x_4) \in \mathbb{G} : x_2, x_4 \in \mathbb{R}\}$. However, the theory of α-admissibility [3, § 8.4], which is supposed to work for such a case, reduces the consideration to the Heisenberg group since $\mathbb{G}/H \sim \mathbb{H}/Z$. It shall be seen later* (3.2.30) *that the action of $(0, x_2, 0, 0) \in H$ will be involved in important physical and geometrical aspects of the harmonic oscillator and shall not be factored out. Our study provides an example of the theory of wavelet transform with non-admissible mother wavelets [32, 45, 47, 48, 87].*

In view of the above mentioned insufficiency of square integrability modulo the subgroup $H = \{(0, x_2, 0, x_4) \in \mathbb{G} : x_2, x_4 \in \mathbb{R}\}$, we make the following:

Definition 2.1.8 *For a fixed unit vector $\phi \in L_2(\mathbb{R})$, let $L_\phi(\mathbb{G}/Z)$ denote the image space of the induced coherent state transform $\mathcal{W}_\phi$* (2.1.10) *equipped with the family of inner products parametrised by $x_2 \in \mathbb{R}$*

$$\langle u, v \rangle_{x_2} := \int_{\mathbb{R}^2} u(x_1, x_2, x_3)\, \overline{v(x_1, x_2, x_3)}\, \hbar_4 \, \mathrm{d}x_1 \mathrm{d}x_3 \,. \tag{2.1.14}$$

The respective norm is denoted by $\|u\|_{x_2}$.

The factor $\hbar_4$ in the measure $\hbar_4 \, \mathrm{d}x_1 \mathrm{d}x_3$ makes it dimensionless, which is a natural physical requirement, see Remark 1.2.2.

It follows from Proposition 2.1.6 that $\|u\|_{x_2} = \|u\|_{x_2'}$ for any x_2, $x_2' \in \mathbb{R}$ and $u \in L_\phi(\mathbb{G}/Z)$. In the usual way [25, (1.42)] the isometry from Proposition 2.1.6 implies the following *orthogonality relation*.

Corollary 2.1.9 *Let* f_1, f_2, ϕ_1, $\phi_2 \in L_2(\mathbb{R})$ *then:*

$$\langle \mathcal{W}_{\phi_1} f_1, \mathcal{W}_{\phi_2} f_2 \rangle_{x_2} = \langle f_1, f_2 \rangle \, \overline{\langle \phi_1, \phi_2 \rangle} \qquad \textit{for any } x_2 \in \mathbb{R}\,. \tag{2.1.15}$$

Corollary 2.1.10 *Let* $\phi \in L_2(\mathbb{R})$ *have unit norm, then the induced coherent state transform* $\mathcal{W}_\phi$ *is an isometry from* $(L_2(\mathbb{R}), \|\cdot\|)$ *to* $(L_\phi(\mathbb{G}/Z), \|\cdot\|_{x_2})$.

Proof

It is an immediate consequence of the previous corollary. Alternatively, for $f \in L_2(\mathbb{R})$:

$$\|f\|_{L_2(\mathbb{R})} = \left\|f \otimes \overline{\phi}\right\|_{L_2(\mathbb{R}^2)} = \|\mathcal{W}_\phi f\|_{x_2}\,,$$

as follows from the isometry $\mathcal{W}_\phi : L_2(\mathbb{R}) \to L_2(\mathbb{R}^2)$ in Proposition 2.1.6. □

Proposition 2.1.11 *The following formula represents the adjoint of* $\mathcal{W}_\phi$ *(in the weak sense) with respect to the inner product* (2.1.14) *parametrised by* x_2*:*

$$[\mathcal{M}_\phi(x_2)u](t) = \int_{\mathbb{R}^2} u(x_1, x_2, x_3)\rho_{h_2\hbar_4}(x_1, x_2, x_3, 0)\phi(t)\,\hbar_4\,\mathrm{d}x_1\,\mathrm{d}x_3. \tag{2.1.16}$$

Proof

Let $f, \phi \in \mathcal{S}(\mathbb{R})$ and $u(\cdot, x_2, \cdot) \in \mathcal{S}(\mathbb{R}^2)$, then

$$\begin{aligned}
\langle \mathcal{W}_\phi f, u \rangle_{x_2} &= \int_{\mathbb{R}^2} [\mathcal{W}_\phi f](x_1, x_2, x_3)\overline{u(x_1, x_2, x_3)}\,\hbar_4\,\mathrm{d}x_1\,\mathrm{d}x_3 \\
&= \int_{\mathbb{R}^2} \left\langle f, \rho_{h_2\hbar_4}(x_1, x_2, x_3, 0)\phi \right\rangle \overline{u(x_1, x_2, x_3)}\,\hbar_4\,\mathrm{d}x_1\,\mathrm{d}x_3 \\
&= \left\langle f, \int_{\mathbb{R}^2} u(x_1, x_2, x_3)\rho_{h_2\hbar_4}(x_1, x_2, x_3, 0)\phi\,\hbar_4\,\mathrm{d}x_1\,\mathrm{d}x_3 \right\rangle \\
&= \langle f, \mathcal{M}_\phi(x_2)u \rangle\,.
\end{aligned}$$

□

Corollary 2.1.12 *An inverse of the unitary operator $\mathcal{W}_\phi$ (in the weak sense) is given by its adjoint $\mathcal{M}_\phi(x_2)$* (2.1.16) *for $\|\phi\| = 1$.*

Proof

Generally, for an analysing vector ϕ and a reconstructing vector ψ both in $\mathcal{S}(\mathbb{R})$ and for any $f, g \in \mathcal{S}(\mathbb{R})$ the orthogonality condition (2.1.15) implies:

$$\begin{aligned} \langle \mathcal{M}_\psi(x_2) \circ \mathcal{W}_\phi f, g \rangle &= \langle \mathcal{W}_\phi f, \mathcal{W}_\psi g \rangle_{x_2} \\ &= \langle f, g \rangle \langle \psi, \phi \rangle \\ &= \langle \langle \psi, \phi \rangle f, g \rangle . \end{aligned}$$

Thus, $\mathcal{M}_\psi(x_2) \circ \mathcal{W}_\phi = \langle \psi, \phi \rangle I$ and if $\langle \psi, \phi \rangle \neq 0$, then $\mathcal{M}_\psi(x_2)$ is a left inverse of $\mathcal{W}_\phi$ up to a factor. It is clear that if $\psi = \phi$, then $\mathcal{M}_\phi(x_2)$ is exactly a left inverse. □
Moreover, we have the following result as a direct consequence of Proposition 2.1.4.

Corollary 2.1.13 *The induced coherent state transform $\mathcal{W}_\phi$* (2.1.10) *intertwines $\rho_{h_2\hbar_4}$ with the restriction of the following representation (see* (1.2.8)*) on the image space of $\mathcal{W}_\phi$:*

$$\begin{aligned} [\tilde{\rho}_{h_4}(y_1, y_2, y_3, y_4) f](x_1, x_2, x_3) = {} & \mathrm{e}^{2\pi \mathrm{i}\hbar_4 (y_4 - y_1 y_3 + \frac{1}{2} y_1^2 y_2 + y_1 x_3 - \frac{1}{2} y_1^2 x_2)} \\ & \times f(x_1 - y_1, x_2 - y_2, x_3 - y_1 x_2 + y_1 y_2 - y_3) . \end{aligned} \tag{2.1.17}$$

Proof

We show this through the following straightforward calculation.

$$
\begin{aligned}
&[\mathcal{W}_\phi \rho_{h_2\hbar_4}(y_1, y_2, y_3, y_4)k](x_1, x_2, x_3)\\
&\quad = \left\langle \rho_{h_2\hbar_4}(y_1, y_2, y_3, y_4)k, \rho_{h_2\hbar_4}(x_1, x_2, x_3, 0)\phi \right\rangle\\
&\quad = \left\langle k, \rho_{h_2\hbar_4}((y_1, y_2, y_3, y_4)^{-1})\rho_{h_2\hbar_4}(x_1, x_2, x_3, 0)\phi \right\rangle\\
&\quad = \left\langle k, \rho_{h_2\hbar_4}\left(x_1 - y_2, x_2 - y_2, x_3 - y_3 + y_1 y_2 - x_2 y_1, \right.\right.\\
&\qquad \left.\left. y_1 y_3 - \frac{1}{2} y_1^2 y_2 - y_4 - x_3 y_1 + \frac{1}{2} y_1^2 x_2 \right)\phi \right\rangle\\
&\quad = \mathrm{e}^{-2\pi \mathrm{i} \hbar_4 (-y_4 + y_1 y_3 - \frac{1}{2} y_1^2 y_2 - y_1 x_3 + \frac{1}{2} y_1^2 x_2)}\\
&\qquad \times \left\langle k, \rho_{h_2\hbar_4}(x_1 - y_1, x_2 - y_2, x_3 - y_1 x_2 + y_1 y_2 - y_3, 0)\phi \right\rangle\\
&\quad = \mathrm{e}^{2\pi \mathrm{i} \hbar_4 (y_4 - y_1 y_3 + \frac{1}{2} y_1^2 y_2 + y_1 x_3 - \frac{1}{2} y_1^2 x_2)}\\
&\qquad \times [\mathcal{W}_\phi k](x_1 - y_1, x_2 - y_2, x_3 - y_1 x_2 + y_1 y_2 - y_3)\\
&\quad = \tilde{\rho}_{\hbar_4}(y_1, y_2, y_3, y_4)[\mathcal{W}_\phi k](x_1, x_2, x_3).
\end{aligned}
$$

□

We consider the representation (2.1.17) restricted to the image space $L_\phi(\mathbb{G}/Z)$, which is easily seen to be unitary, that is, $\left\| \tilde{\rho}_{\hbar_4}(y_1, y_2, y_3, y_4) f \right\|_{x_2} = \|f\|_{x_2}$, where $f(x_1, x_2, x_3) \in L_\phi(\mathbb{G}/Z)$.

We recall, here, the notion of the Lie derivative as this will be essential for our method.

Definition 2.1.14 *Let G be a Lie group and $\mathfrak{g}$ the corresponding Lie algebra. Then, the* Lie derivative *(left invariant vector fields) denoted $\mathfrak{L}^X$, for an element X of the Lie algebra $\mathfrak{g}$ is computed through the derived right regular representation:*

$$
[\mathfrak{L}^X F](g) = \frac{\mathrm{d}}{\mathrm{d}t} F(g \exp tX) \bigg|_{t=0} \tag{2.1.18}
$$

for any differentiable function F on G.

We want to calculate this for a function in the image space of the coherent sate transform of the shear group $\mathbb{G}$. However, functions in such a space possess the property

$$F(x_1, x_2, x_3, x_4 + z) = \mathrm{e}^{-2\pi \mathrm{i}\hbar_4 z} F(x_1, x_2, x_3, x_4) \qquad \text{for all } \ z \in \mathbb{R}. \tag{2.1.19}$$

Indeed, for the group $\mathbb{G}$ and its unitary irreducible representation (1.2.10), let $F = \mathcal{W}_\phi f$ for $f \in L_2(\mathbb{R})$. Then, we can easily see that (follows also from (2.1.3) when H is the centre of $\mathbb{G}$)

$$\begin{aligned}
[\mathcal{W}_\phi f](x_1, x_2, x_3, x_4 + z) &= \left\langle f, \rho_{h_2\hbar_4}(x_1, x_2, x_3, x_4 + z)\phi \right\rangle \\
&= \left\langle f, \rho_{h_2\hbar_4}(0,0,0,z)\rho_{h_2\hbar_4}(x_1, x_2, x_3, x_4)\phi \right\rangle \\
&= \mathrm{e}^{-2\pi \mathrm{i}\hbar_4 z} \left\langle f, \rho_{h_2\hbar_4}(x_1, x_2, x_3, x_4)\phi \right\rangle \\
&= \mathrm{e}^{-2\pi \mathrm{i}\hbar_4 z} [\mathcal{W}_\phi f](x_1, x_2, x_3, x_4).
\end{aligned}$$

Taking this property into account, we calculate $\mathfrak{L}^{X_j}$, where X_j being an element of the basis of the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$

$$X_1 = (1,0,0,0), \quad X_2 = (0,1,0,0), \quad X_3 = (0,0,1,0), \quad X_4 = (0,0,0,1).$$

The role of the above property is best seen when calculating $\mathfrak{L}^{X_2}$:

$$\begin{aligned}
\mathfrak{L}^{X_2} F(x_1, x_2, x_3, x_4) &= \frac{\mathrm{d}}{\mathrm{d}t} F((x_1, x_2, x_3, x_4) \exp tX_2)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t} F((x_1, x_2, x_3, x_4)(0, t, 0, 0))\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t} F\big(x_1, x_2 + t, x_3 + x_1 t, x_4 + \frac{1}{2} x_1^2 t\big)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{-\pi \mathrm{i}\hbar_4 t x_1^2} F(x_1, x_2 + t, x_3 + x_1 t, x_4)\Big|_{t=0} \\
&= (\partial_2 + x_1\partial_3 - \pi \mathrm{i}\hbar_4 x_1^2) F(x_1, x_2, x_3, x_4).
\end{aligned}$$

Similarly, we calculate $\mathfrak{L}^{X_j}$ $(j = 1, 3, 4)$ and we obtain

$$\begin{aligned}
\mathfrak{L}^{X_1} &= \partial_1; & \mathfrak{L}^{X_2} &= \partial_2 + x_1\partial_3 - \mathrm{i}\pi\hbar_4 x_1^2 I; \\
\mathfrak{L}^{X_3} &= \partial_3 - 2\pi \mathrm{i}\hbar_4 x_1 I; & \mathfrak{L}^{X_4} &= -2\pi \mathrm{i}\hbar_4 I\,.
\end{aligned} \tag{2.1.20}$$

One can readily check that

$$[\mathfrak{L}^{X_1}, \mathfrak{L}^{X_2}] = \mathfrak{L}^{X_3}, \quad [\mathfrak{L}^{X_1}, \mathfrak{L}^{X_3}] = \mathfrak{L}^{X_4},$$

in agreement with the Lie algebra non-vanishing commutator relations (1.1.3).

2.1.2 Right shifts and coherent state transform

Recall that the right regular representation of a group G, denoted $R(g)$, acts on functions defined on the group G in the following way:

$$R(g) : f(g') \mapsto f(g'g), \qquad g \in G.$$

In particular, it is an immediate to see that

$$R(g)[\mathcal{W}_\phi f](g') = [\mathcal{W}_\phi f](g'g) = \langle f, \rho(g'g)\phi\rangle = \langle f, \rho(g')\rho(g)\phi\rangle = [\mathcal{W}_{\rho(g)\phi} f](g'). \tag{2.1.21}$$

That is, the coherent state transform $\mathcal{W}_\phi$ intertwines the right shift with the action of ρ on the fiducial vector ϕ. This observation leads to the following result that will play a central role in exploring the nature of the image space of the coherent state transform and will be a recurrent theme of our investigation.

Corollary 2.1.15 *(Analyticity of the coherent state transform, [49, § 5]) Let G be a group and $\mathrm{d}g$ be a measure on G. Let ρ be a unitary representation of G, which can be extended by integration to a vector space V of functions or distributions on G. Let a fiducial vector $\phi \in \mathcal{H}$ satisfy the equation*

$$\rho(d)\phi := \int_G d(g)\, \rho(g)\phi \,\mathrm{d}g = 0, \tag{2.1.22}$$

for a fixed distribution $d(g) \in V$. Then, any coherent state transform $\tilde{v}(g') = \langle v, \rho(g')\phi\rangle$ obeys the condition:

$$R(\bar{d})\tilde{v} = 0, \qquad \textit{where} \quad R(\bar{d}) = \int_G \bar{d}(g)\, R(g)\,\mathrm{d}g\,, \tag{2.1.23}$$

with R being the right regular representation of G and $\bar{d}(g)$ is the complex conjugation of $d(g)$.

Important and well-known functional spaces whose members enjoy property of analyticity such as Fock-Segal-Bargmann space and Hardy space, may arise through applications of this result. Here we shall demonstrate a relevant situation regarding the Heisenberg group, further examples can be found in [49, § 5][52] [53, § 5.3].

Example 2.1.16 (Gaussian and analyticity) *Consider again the following form of the Schrödinger representation:*

$$\sigma_\hbar(x', y', s')f(y) = \mathrm{e}^{2\pi\mathrm{i}\hbar(s'-yy')} f(y - x').$$

For the basic elements of the Heisenberg–Weyl algebra,

$$X = (1, 0, 0), \qquad Y = (0, 1, 0), \qquad S = (0, 0, 1),$$

the infinitesimal generators are

$$\mathrm{d}\sigma_\hbar^X = -\frac{\mathrm{d}}{\mathrm{d}y}, \qquad \mathrm{d}\sigma_\hbar^Y = -2\pi\mathrm{i}\hbar yI, \qquad \mathrm{d}\sigma_\hbar^S = 2\pi\mathrm{i}\hbar I.$$

For the purpose of applying the above result in its integral version note that

$$\begin{aligned}
\mathrm{d}\sigma_\hbar^X\phi(y) &= \frac{\mathrm{d}}{\mathrm{d}t}\sigma_\hbar(\exp tX)\phi(y)\bigg|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\sigma_\hbar(t, 0, 0)\phi(y)\bigg|_{t=0} \\
&= \int_{\mathbb{R}^3} \delta(x', y', s')\frac{\partial}{\partial x'}\sigma_\hbar(x', y', s')\phi(y)\,\mathrm{d}x'\mathrm{d}y'\mathrm{d}s'. \\
&= -\int_{\mathbb{R}^3} \frac{\partial}{\partial x'}\delta(x', y', s')\sigma_\hbar(x', y', s')\phi(y)\,\mathrm{d}x'\mathrm{d}y'\mathrm{d}s'. \\
&= \sigma_\hbar(-\delta_1')\phi(y),
\end{aligned}$$

where δ is the Dirac delta distribution and δ_1' is its partial derivative with respect to first component and likewise $\mathrm{d}\sigma_\hbar^Y = \sigma_\hbar(-\delta_2')$. Similarly, we may view the Lie derivative $\mathfrak{L}^X$ (2.1.18) *but with R in place of $\sigma_\hbar$ and $\mathcal{W}_\phi k$ in place of ϕ.*

Now, the Gaussian

$$\phi_E(y) = \mathrm{e}^{-\pi\hbar E y^2}, \quad E > 0 \tag{2.1.24}$$

is a null solution of the annihilation operator

$$\mathrm{i}\mathrm{d}\sigma_\hbar^X - \mathrm{i}E(\mathrm{i}\mathrm{d}\sigma_\hbar^Y) = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}y} - 2\pi\mathrm{i}\hbar E y. \tag{2.1.25}$$

This matches condition (2.1.22) *for the distribution*

$$d(x', y', s') = -\mathrm{i}\delta_1'(x', y', s') - E\delta_2'(x', y', s').$$

Then, any element f in the respective image space of the induced coherent state transform, $f(x, y) = [\mathcal{W}_{\phi_E} k](x, y)$, for $k \in L_2(\mathbb{R})$, is annihilated by the operator:

$$\begin{aligned} \mathcal{D} := R(\bar{d}) &= -\mathrm{i}\mathfrak{L}^X + E\mathfrak{L}^Y \\ &= -\mathrm{i}\partial_x + E\partial_y - 2\pi\mathrm{i}\hbar E x I. \end{aligned} \tag{2.1.26}$$

Yet, from Example 2.1.5 for ϕ being ϕ_E (2.1.24)*, the induced coherent state transform becomes*

$$[\mathcal{W}_{\phi_E} k](x, y) = \mathrm{e}^{\pi\mathrm{i}\hbar x y - \frac{\pi\hbar}{2E}(E^2x^2 + y^2)} V_E(x, y), \tag{2.1.27}$$

where

$$V_E(x, y) = \int_{\mathbb{R}} k(x')\mathrm{e}^{2\pi\hbar(Ex+\mathrm{i}y)x' - \frac{\pi\hbar}{2E}(Ex+\mathrm{i}y)^2}\mathrm{e}^{-\pi\hbar E x'^2}\,\mathrm{d}x'. \tag{2.1.28}$$

This integral represents a Fock–Segal–Bargmann type transform [25, 44, 49, 61].

Now, since $\mathcal{D}f(x, y) = 0$, for $f = \mathcal{W}_{\phi_E} k$, that is

$$(-\mathrm{i}\partial_x + E\partial_y - 2\pi\mathrm{i}\hbar E x I)\mathrm{e}^{\pi\mathrm{i}\hbar x y - \frac{\pi\hbar}{2E}(E^2x^2+y^2)} V_E(x, y) = 0, \tag{2.1.29}$$

clearly,

$$(-\mathrm{i}\partial_x + E\partial_y) V_E(x, y) = 0.$$

This, in turn, shows that $V_E(z)$ is analytic in $z = Ex + \mathrm{i}y$. Precisely, $\partial_{\bar{z}} V_E(z) = 0$, where $\partial_{\bar{z}} = \frac{1}{2}(\frac{1}{E}\partial_x + \mathrm{i}\partial_y)$ is a Cauchy–Riemann type operator. Moreover, the left-hand side of *equality* (2.1.27) *defines a function in* $L_2(\mathbb{R}^2)$*, according to Example 2.1.5. Furthermore,* $|\mathrm{e}^{\pi \mathrm{i}\hbar x y}|^2 = 1$ *and therefore the function* $V_E(x, y)$ *is square-integrable with respect to the measure* $\mathrm{e}^{-\frac{\pi\hbar}{E}(E^2x^2+y^2)}\mathrm{d}x\mathrm{d}y$*. In short, the induced coherent state transform* $\mathcal{W}_{\phi_E}$ *(when* ϕ_E *is a Gaussian) gives rise to a space consisting of functions* $V_E(z)$ *that are analytic in the entire complex plane* $\mathbb{C}$ *and square-integrable with respect to the measure* $\mathrm{e}^{-\frac{\pi\hbar}{E}|z|^2}\mathrm{d}z$*. This is exactly the structure of a FSB space, see Section 1.4, Definition 1.4.2.*

2.2 Characterisation of the image space $L_\phi(\mathbb{G}/Z)$

To give a description of the image space $L_\phi(\mathbb{G}/Z)$ of the respective induced coherent state transform (2.1.10), we employ Corollary 2.1.15. This requires a particular choice of a fiducial vector ϕ such that ϕ lies in $L_2(\mathbb{R})$ and ϕ is a null solution of an operator of the form (2.1.22). For simplicity, we consider the following linear combination of generators of the Lie algebra $\mathfrak{g}$, cf. (1.2.13):

$$\begin{aligned} \mathrm{d}\rho_{h_2\hbar_4}^{\mathrm{i}X_1+\mathrm{i}DX_2+EX_3} &= \mathrm{i}\mathrm{d}\rho_{h_2\hbar_4}^{X_1} + \mathrm{i}D\,\mathrm{d}\rho_{h_2\hbar_4}^{X_2} + E\,\mathrm{d}\rho_{h_2\hbar_4}^{X_3} \\ &= -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}y} - \pi\hbar_4 D y^2 - 2\pi\mathrm{i}E\hbar_4 y - 2\pi h_2 D\,, \end{aligned} \tag{2.2.30}$$

where D and E are some real constants. It is clear that the function

$$\phi_{E,D}(y) = c\exp\left(\frac{\pi\mathrm{i}D\hbar_4}{3}y^3 - \pi E\hbar_4 y^2 + 2\pi\mathrm{i}Dh_2 y\right), \tag{2.2.31}$$

is a generic solution where c is a constant determined via normalisation. Moreover, square integrability of $\phi_{E,D}$ requires that $E\hbar_4$ is strictly positive. It is sufficient here to use the simpler fiducial vector corresponding to the value[2] $D = 0$, cf. (2.1.25):

$$\phi_E(y) = (2h_2E)^{1/4}\mathrm{e}^{-\pi E\hbar_4 y^2}, \qquad \hbar_4 > 0,\ E > 0. \tag{2.2.32}$$

[2]The case $D \neq 0$ will be considered in the last chapter.

Remark 2.2.1 *The factor* $(2h_2E)^{1/4}$ *makes* ϕ_E *normalised with respect to the* L_2*-norm:* $\|f\|^2 = \int_{\mathbb{R}} |f(y)|^2 \sqrt{\frac{\hbar_4}{h_2}}\,\mathrm{d}y$*. Following the Convention 1.2.1, the exponent in* $\phi_E(y)$ *shall be dimensionless. Therefore,* E *has to be of dimension* M/T *which follows from the fact that* y *has dimension* $T/(ML)$ *and* $\hbar_4$ *has dimension* ML^2/T*. As a result, we attach the factor* $\sqrt{\frac{\hbar_4}{h_2}}$ *to the measure so the measure is dimensionless. Note that from Remark 1.2.2* h_2 *has dimension* T/M*.*

Since the function $\phi_E(y)$ (2.2.32) is a null-solution of the operator (2.2.30) with $D = 0$, the image space $L_{\phi_E}(\mathbb{G}/Z)$ can be described through the respective derived right regular representation (Lie derivatives) (2.1.18). Specifically, Corollary 2.1.15 with the distribution

$$d(x_1, x_2, x_3, x_4) = -\mathrm{i}\,\delta_1'(x_1, x_2, x_3, x_4) - E\,\delta_3'(x_1, x_2, x_3, x_4)\,,$$

matches (2.2.30). Thus, any function f in $L_{\phi_E}(\mathbb{G}/Z)$ for ϕ_E (2.2.32) satisfies

$$\mathcal{C}f(x_1, x_2, x_3) = 0 \tag{2.2.33}$$

for the partial differential operator:

$$\mathcal{C} = \left(-\mathrm{i}\mathfrak{L}^{X_1} + E\mathfrak{L}^{X_3}\right) = -\mathrm{i}\partial_1 + E\partial_3 - 2\pi\mathrm{i}\hbar_4 E x_1\,, \tag{2.2.34}$$

where Lie derivatives (2.1.20) are used.

Remark 2.2.2 *Due to the explicit similarity to the Heisenberg group case with the Cauchy–Riemann equation, Example 2.1.16, we call* (2.2.33) *the analyticity condition for the coherent state transform. Indeed, it can be easily verified that for the fiducial vector* ϕ_E (2.2.32)*, the induced coherent state transform* (2.1.10) *becomes*

$$[\mathcal{W}_{\phi_E}k](x_1, x_2, x_3) = \exp\left(-2\pi\mathrm{i}h_2x_2 + \pi\mathrm{i}\hbar_4x_1x_3 - \frac{\pi\hbar_4}{2E}(E^2x_1^2 + x_3^2)\right) B_{x_2}(x_1, x_3)$$

where

$$B_{x_2}(x_1,x_3) = \int_{\mathbb{R}} \mathrm{e}^{-\pi \mathrm{i}\hbar_4 x_2 y^2} k(y) \mathrm{e}^{\frac{-\pi\hbar_4}{2E}(Ex_1+\mathrm{i}x_3)^2+2\pi\hbar_4(Ex_1+\mathrm{i}x_3)y-\pi\hbar_4 E y^2}\,\mathrm{d}y$$
$$= [(V_E \circ M_{x_2})k](x_1,x_3)$$

with V_E *being the Fock–Segal–Bargmann type transform* (2.1.28) *and* M_{x_2} *is a multiplication operator by* $\mathrm{e}^{-\pi\hbar_4 x_2 y^2}$. *Thus, by condition* (2.2.34) *we have*

$$[-\mathrm{i}\partial_1 + E\partial_3 - 2\pi\mathrm{i}\hbar_4 E x_1]\left\{\exp\left(-2\pi\mathrm{i}h_2 x_2 + \pi\mathrm{i}\hbar_4 x_1 x_3 - \frac{\pi\hbar_4}{2E}(E^2x_1^2+x_3^2)\right)B_{x_2}(x_1,x_3)\right\} = 0.$$

That is,

$$(-\mathrm{i}\partial_1 + E\partial_3)B_{x_2}(x_1,x_3) = 0, \tag{2.2.35}$$

which can be written as

$$\partial_{\bar{z}} B_{x_2}(z) = 0, \tag{2.2.36}$$

where $z = Ex_1 + \mathrm{i}x_3$ *and* $\partial_{\bar{z}} = \frac{1}{2}(\frac{1}{E}\partial_1 + \mathrm{i}\partial_3)$*—a Cauchy–Riemann type operator. Thus,* $B_{x_2}(z)$ *is entire on the complex plane* $\mathbb{C}$*. As such, the induced coherent state transform of* $\mathbb{G}$ *gives rise to the space consisting of analytic functions* $B_{x_2}(z)$ *which are square-integrable with respect to the measure* $\mathrm{e}^{-\frac{\pi\hbar_4}{E}|z|^2}\mathrm{d}z$.

Remark 2.2.3 *Note that in the previous remark if* $x_2 = 0$*, then we obtain the formula for the induced coherent state transform of the Heisenberg group, Example 2.1.5 and Example 2.1.16.*

A notable difference between the group $\mathbb{G}$ and the Heisenberg group is the presence of an additional condition, which is satisfied by any function $f \in L_{\phi_E}(\mathbb{G}/Z)$ for any fiducial vector ϕ_E. Indeed, another auxiliary condition beside (2.2.34) comes naturally from the group structure. Recall that the Casimir operators are those in the respective universal enveloping algebra of $\mathfrak{g}$ which commute with every element in the Lie algebra $\mathfrak{g}$. Thus, if C is a Casimir operator, then $\mathrm{d}\rho^C_{h_2\hbar_4}$ has the form $c\,I$ where $c \in \mathbb{C}$, by Schur's Lemma [3, 27], since it commutes with the irreducible representation $\mathrm{d}\rho^X_{h_2\hbar_4}$ where $X \in \mathfrak{g}$. This means that $(\mathrm{d}\rho^C_{h_2\hbar_4} - cI)\phi_E(y) = 0$ for any $\phi_E \in L_2(\mathbb{R})$. Therefore, by Corollary 2.1.15, we can conclude that any function f in $L_{\phi_E}(\mathbb{G}/Z)$ vanishes for the operator $\mathfrak{L}^C - \bar{c}I$. Precisely, in our case $C = X_3^2 - 2X_2X_4$ is the only (up to a scalar factor) "non-trivial" Casimir operator, see [17, Ex. 3.3.9] [44, § 3.3.1] [1, 2]. Then, from (1.2.13) it can be easily checked that the corresponding operator acts as a multiplication operator by $8\pi^2 h_2\hbar_4$ on $L_2(\mathbb{R})$:

$$\mathrm{d}\rho^{X_3^2-2X_2X_4}_{h_2\hbar_4}\phi_E(y) = \left((\mathrm{d}\rho^{X_3}_{h_2\hbar_4})^2 - 2\mathrm{d}\rho^{X_2}_{h_2\hbar_4}\mathrm{d}\rho^{X_4}_{h_2\hbar_4}\right)\phi_E(y) = 8\pi^2 h_2\hbar_4\phi_E(y).$$

That is,

$$\left((\mathrm{d}\rho^{X_3}_{h_2\hbar_4})^2 - 2\mathrm{d}\rho^{X_2}_{h_2\hbar_4}\mathrm{d}\rho^{X_4}_{h_2\hbar_4} - 8\pi^2 h_2\hbar_4 I\right)\phi_E(y) = 0. \tag{2.2.37}$$

From here we can proceed in either of ways:

1. Corollary 2.1.15 with the distribution

   $$d(x_1,x_2,x_3,x_4) = \delta^{(2)}_{33}(x_1,x_2,x_3,x_4) - 2\,\delta'_2(x_1,x_2,x_3,x_4)\cdot\delta'_4(x_1,x_2,x_3,x_4)\,,$$

   asserts that the image $f \in L_{\phi_E}(\mathbb{G}/Z)$ of the coherent state transform $\mathcal{W}_{\phi_E}$ is annihilated by the respective Lie derivatives operator

   $$\mathcal{S}f(x_1,x_2,x_3) = 0, \tag{2.2.38}$$

   where, cf. (2.1.20):

   $$\begin{aligned}\mathcal{S} &= (\mathfrak{L}^{X_3})^2 - 2\mathfrak{L}^{X_2}\mathfrak{L}^{X_4} - 8\pi^2 h_2\hbar_4 I \qquad (2.2.39)\\ &= \partial^2_{33} + 4\pi\mathrm{i}\hbar_4\partial_2 - 8\pi^2 h_2\hbar_4 I\,.\end{aligned}$$

2. The representation, cf. (1.2.12):

   $$\mathrm{d}\tilde{\rho}^C_{\hbar_4} = (\mathrm{d}\tilde{\rho}^{X_3}_{\hbar_4})^2 - 2\mathrm{d}\tilde{\rho}^{X_2}_{\hbar_4}\mathrm{d}\tilde{\rho}^{X_4}_{\hbar_4} \tag{2.2.40}$$

of the Casimir element C takes the constant value $8\pi^2 h_2 \hbar_4$ on $L_{\phi_E}(\mathbb{G}/Z)$. Note that this produces exactly (2.2.39) because for the Casimir operator the left and the right actions of the group coincide.

The relation (2.2.38) will be called the *structural condition* because it is determined by the structure of the group $\mathbb{G}$ and its Casimir operator.

Note that (2.2.39) is the Schrödinger equation of a free particle with the time-like parameter x_2. Thus, the structural condition is the generator of the quantised version of the classical dynamics (1.3.20) of a free particle represented by the shear transform, see the discussion of this after (1.3.20).

Summing up, the physical characterisation of the $L_{\phi_E}(\mathbb{G}/Z)$ is as follows.

1. The restriction of a function $f \in L_{\phi_E}(\mathbb{G}/Z)$ to the plane $x_2 = 0$ (a model of the phase space) coincides with the FSB image of the respective state.

2. The function f is a continuation from the plane $x_2 = 0$ to $\mathbb{R}^2 \times \mathbb{R}$ (the product of phase space and timeline) by free time-evolution of the quantum system.

This physical interpretation once more explains the identity $\|f\|_{x_2} = \|f\|_{x_2'}$ for any x_2, x_2' in $\mathbb{R}$: the energy of a free state is constant in time.

Now, we specify the ground states and the respective induced coherent state transforms, which will be used below.

Example 2.2.4 *Consider our normalised fiducial vector* $\phi_E(y) = (2h_2E)^{1/4}\mathrm{e}^{-\pi\hbar_4 E y^2}$ *with a parameter* $E > 0$*. Then, we calculate the induced coherent state transform for* *a minimal uncertainty states $\phi_q(y)$ ($q>0$) as follows* :

$$
\begin{aligned}
[\mathcal{W}_{\phi_E}\phi_q](x_1,x_2,x_3) &= \sqrt{2\hbar_4}(qE)^{1/4}\int_{\mathbb{R}} \mathrm{e}^{-\pi\hbar_4 q y^2}\mathrm{e}^{-2\pi\mathrm{i}(h_2x_2+\hbar_4(-x_3y+\frac{1}{2}x_2y^2))} \\
&\qquad \times \mathrm{e}^{-\pi\hbar_4 E(y-x_1)^2}\,\mathrm{d}y \quad \textit{(see Remark 2.2.1 for the measure)} \\
&= \sqrt{2\hbar_4}(qE)^{1/4}\mathrm{e}^{-\pi\hbar_4 E x_1^2-2\pi\mathrm{i}h_2x_2}\int_{\mathbb{R}} \mathrm{e}^{-\pi\hbar_4(\mathrm{i}x_2+E+q)y^2} \\
&\qquad \times \mathrm{e}^{2\pi\hbar_4(Ex_1+\mathrm{i}x_3)y}\,\mathrm{d}y \\
&= \sqrt{2}(qE)^{1/4}\frac{\exp\left(-\pi h_4 E x_1^2-2\pi\mathrm{i}h_2x_2-\pi\hbar_4\frac{(-\mathrm{i}Ex_1+x_3)^2}{\mathrm{i}x_2+E+q}\right)}{\sqrt{\mathrm{i}x_2+E+q}}.
\end{aligned}
\tag{2.2.41}
$$

Clearly, the function (2.2.41) *is dimensionless and satisfies conditions* (2.2.33) *and* (2.2.38)*. It shall be seen in Section 3.2.2 that such a function is an eigenstate (vacuum state) of the harmonic oscillator Hamiltonian acting on the image space $L_{\phi_E}(\mathbb{G}/Z)$ of the respective induced coherent state transform. Moreover, this function represents a minimal uncertainty state in the space $L_{\phi_E}(\mathbb{G}/Z)$ for any $x_2\in\mathbb{R}$ and any $E>0$.*

Chapter 3

Harmonic oscillator through reduction of order of a PDE

In this chapter we devise an approach applied to the harmonic oscillator through reducing a PDE order for geometrisation of the dynamics in the sense of Definition 0.0.3. First, we use the Heisenberg group and find that geometrisation condition completely determines which fiducial vector needs to be used. The treatment of the group $\mathbb{G}$ provides the wider opportunity: any minimal uncertainty state can be used for the coherent state transform with geometric dynamics in the result. At the end we provide eigenfunctions and respective ladder operators.

3.1 Harmonic oscillator from the Heisenberg group

Here we use a simpler case of the Heisenberg group to illustrate the technique which will be used later. The fundamental importance of the harmonic oscillator stimulates exploration of different approaches. Analysis based on the pair of ladder operators elegantly produces the spectrum and the eigenvectors, see Section 1.4. However, this technique is essentially based on a particular structure of the Hamiltonian of harmonic oscillator expressed in terms of the Heisenberg–Weyl algebra. Thus, it loses its efficiency in other cases. In contrast, our method is applicable for a large family of examples since it has a more general nature as will be seen in Chapter 4.

Before proceeding, we shall briefly adopt our discussion on the harmonic oscillator, Section 1.4, to the language of representations of the Heisenberg group $\mathbb{H}$. Indeed, the Stone-von Neumann theorem [25, § 1.5][33, Ch. 14][44, § 2.2.6] ensures that CCR (1.4.27) provides an irreducible representation of the Heisenberg-Weyl algebra (1.1.1). The corresponding group is the Heisenberg group $\mathbb{H}$ with the group law (1.1.2), it has the unitary irreducible Schrödinger representation (constructed along the induction procedure, cf. (1.2.10) for $x_2 = 0$) on $L_2(\mathbb{R})$:

$$\sigma_\hbar(x, y, s) f(x') = \mathrm{e}^{2\pi \mathrm{i}\hbar(s - x'y)} f(x' - x). \tag{3.1.1}$$

The corresponding infinitesimal generators are (see Example 2.1.16)

$$\mathrm{d}\sigma_\hbar{}^X = -\frac{\mathrm{d}}{\mathrm{d}x'}, \quad \mathrm{d}\sigma_\hbar{}^Y = -2\pi \mathrm{i}\hbar x' I, \quad \mathrm{d}\sigma_\hbar^S = 2\pi \mathrm{i}\hbar I\,, \tag{3.1.2}$$

with $[\mathrm{d}\sigma_\hbar^X, \mathrm{d}\sigma_\hbar^Y] = \mathrm{d}\sigma_\hbar^S = 2\pi \mathrm{i}\hbar I$. These operators are skew-symmetric, thus we need to multiply each by the complex unit "i" to get a self-adjoint operator in a proper domain of $L_2(\mathbb{R})$ where these operators are well-defined. It suffices to consider the Schwartz space $\mathcal{S}(\mathbb{R}) \subset L_2(\mathbb{R})$ on which these operators are essentially self-adjoint operators.

With regard to the above form of Schrödinger representation, we recall from Convention 1.2.1 that x is measured in unit $T/(ML)$ while y is measured in unit $1/L$, so x' shall have physical unit $T/(ML)$. Thus, the self-adjoint operators $\mathrm{i}\mathrm{d}\sigma_\hbar{}^X$ and $\mathrm{i}\mathrm{d}\sigma_\hbar{}^Y$ may be viewed as momentum and position observables, respectively.

Let

$$L^- = \frac{1}{\sqrt{2\hbar m\omega}}(X - \mathrm{i}m\omega Y), \qquad L^+ = \frac{1}{\sqrt{2\hbar m\omega}}(-X - \mathrm{i}m\omega Y).$$

Then, we have the dimensionless pair of ladder operators

$$a^- = \mathrm{d}\sigma_\hbar^{L^-} = \frac{1}{\sqrt{2\hbar m\omega}}(-\mathrm{i}(\mathrm{i}\mathrm{d}\sigma_\hbar^X) - m\omega(\mathrm{i}\mathrm{d}\sigma_\hbar^Y)); \tag{3.1.3}$$

$$a^+ = \mathrm{d}\sigma_\hbar^{L^+} = \frac{1}{\sqrt{2\hbar m\omega}}(\mathrm{i}(\mathrm{i}\mathrm{d}\sigma_\hbar^X) - m\omega(\mathrm{i}\mathrm{d}\sigma_\hbar^Y)). \tag{3.1.4}$$

In this setting, the harmonic oscillator Hamiltonian corresponds to the following element of the respective universal enveloping algebra:

$$L_H = \frac{\hbar\omega}{2}(L^-L^+ + L^+L^-) = -\frac{1}{2m}X^2 - \frac{m\omega^2}{2}Y^2. \tag{3.1.5}$$

This generates the harmonic oscillator Hamiltonian operator acting on $L_2(\mathbb{R})$

$$H = \mathrm{d}\rho_\hbar^{L_H} = \frac{1}{2m}(\mathrm{i}\mathrm{d}\sigma_\hbar^X)^2 + \frac{m\omega^2}{2}(\mathrm{i}\mathrm{d}\sigma_\hbar^Y)^2, \tag{3.1.6}$$

which is the Weyl quantised version of the classical Hamiltonian: $h = \frac{1}{2}(p^2/m + m\omega^2 q^2)$.

In the context of the group $\mathbb{H}$, the (induced) coherent state transform $\mathcal{W}_{\phi_E}$ for a fixed ϕ_E as defined in the preceding Chapter, is used to transfer information from the state space $L_2(\mathbb{R})$ (the Schrödinger model, see (2.1.9)) to a closed subspace of $L_2(\mathbb{R}^2)$ (a FSB model—the induced coherent state transform image):

$$\mathcal{W}_{\phi_E} : f \mapsto \langle f, \sigma_\hbar(x, y, 0)\phi_E\rangle := \tilde{v}(x, y)$$

It intertwines the Schrödinger representation (3.1.1) with the following FSB representation of $\mathbb{H}$ on $L_2(\mathbb{R}^2)$:

$$[\tilde{\sigma}_\hbar(x, y, s)f](x', y') = \mathrm{e}^{2\pi\mathrm{i}\hbar(s - xy + xy')} f(x' - x, y' - y). \tag{3.1.7}$$

That is,

$$\mathcal{W}_\phi \sigma_\hbar(x, y, s) = \tilde{\sigma}_\hbar(x, y, s)\mathcal{W}_\phi. \tag{3.1.8}$$

The representation (3.1.7) is a representation of $\mathbb{H}$ induced from a character of the centre of $\mathbb{H}$, cf. (1.2.8) for $x_2 = 0$. The infinitesimal generators of the respective basic elements of $\mathfrak{h}$ are

$$\mathrm{d}\tilde{\sigma}_\hbar^X = -\partial_x + 2\pi\mathrm{i}\hbar y I, \quad \mathrm{d}\tilde{\sigma}_\hbar^Y = -\partial_y, \quad \mathrm{d}\tilde{\sigma}_\hbar^S = 2\pi\mathrm{i}\hbar I\,, \tag{3.1.9}$$

with $[\mathrm{d}\tilde{\sigma}_\hbar^X, \mathrm{d}\tilde{\sigma}_\hbar^Y] = \mathrm{d}\tilde{\sigma}_\hbar^S = 2\pi\mathrm{i}\hbar I$.

Remark 3.1.1 *For computations convenience in Subsection 3.2.2, we will consider as observables the self-adjoint operators:* $\frac{\mathrm{i}}{\sqrt{2\pi}}\mathrm{d}\tilde{\sigma}_{\hbar}^{X}$ *and* $\frac{\mathrm{i}}{\sqrt{2\pi}}\mathrm{d}\tilde{\sigma}_{\hbar}^{Y}$ *so that* $[\frac{\mathrm{i}}{\sqrt{2\pi}}\mathrm{d}\tilde{\sigma}_{\hbar}^{Y}, \frac{\mathrm{i}}{\sqrt{2\pi}}\mathrm{d}\tilde{\sigma}_{\hbar}^{X}] = \mathrm{i}\hbar I$. *These observables have physical dimensions* ML/T *and* L, *respectively. Again we restrict these operators on the Schwartz space* $\mathcal{S}(\mathbb{R}^2)$ *which is a subspace of the image space* $L_{\phi_E}(\mathbb{H}/Z)$ *of the induced coherent state transform, see [25, § 1.4]. The same is true in case of the group* $\mathbb{G}$, *see the paragraph after Proposition 2.1.6.*

Due to the intertwining property (3.1.8), the quantised harmonic oscillator Hamiltonian H (3.1.6) acts on the FSB–like space $L_{\phi_E}(\mathbb{H}/Z)$ by (we still use the same notation H)

$$\begin{aligned} H &= \frac{1}{2m}\left(\frac{\mathrm{i}\mathrm{d}\tilde{\sigma}_{\hbar}^{X}}{\sqrt{2\pi}}\right)^2 + \frac{m\omega^2}{2}\left(\frac{\mathrm{i}\mathrm{d}\tilde{\sigma}_{\hbar}^{Y}}{\sqrt{2\pi}}\right)^2 \\ &= -\frac{1}{4\pi m}\partial_{xx}^2 - \frac{m\omega^2}{4\pi}\partial_{yy}^2 + \frac{\mathrm{i}\hbar}{m}y\partial_x + \frac{\pi\hbar^2}{m}y^2 I. \end{aligned} \tag{3.1.10}$$

Now, let us consider in the image space $L_{\phi_E}(\mathbb{H}/Z)$ the time evolution of the harmonic oscillator which is defined by the time-dependent Schrödinger equation, Section 1.4,

$$\mathrm{i}\hbar\dot{f}(t,x,y) = Hf(t,x,y), \tag{3.1.11}$$

for $f(t,\cdot,\cdot)$ in $L_{\phi_E}(\mathbb{H}/Z)$ for all t and H (3.1.10). Our aim is to describe the dynamic $f(t,x,y)$ in geometric terms by lowering the order of the differential operator (3.1.10).

Indeed, the structure of $L_{\phi_E}(\mathbb{H}/Z)$ enables to make a reduction to the order of the differential operator (3.1.10) using the analyticity condition. Based on Example 2.1.16, for the Gaussian $\phi_E(y) = \mathrm{e}^{-\pi\hbar E y^2}$ ($E > 0$), the respective image of the induced coherent state transform, $f = \mathcal{W}_{\phi_E}k$ where $k \in L_2(\mathbb{R})$, is annihilated by the operator (the analyticity condition (2.1.26)):

$$\begin{aligned} \mathcal{D} &= -\mathrm{i}\mathfrak{L}^X + E\mathfrak{L}^Y \\ &= -\mathrm{i}\partial_x + E\partial_y - 2\pi\mathrm{i}\hbar E x I\,. \end{aligned} \tag{3.1.12}$$

From the analyticity condition (2.1.26) of the induced coherent state transform, the operator $(A\partial_x + B\partial_y + CI)(-\mathrm{i}\mathfrak{L}^X + E\mathfrak{L}^Y)$ vanishes on any $f \in L_{\phi_E}(\mathbb{H}/Z)$ for any A, B and C. Thus, we can adjust the Hamiltonian (3.1.10) by adding such an operator:

$$H_r = H + (A\partial_x + B\partial_y + CI)(-\mathrm{i}\mathfrak{L}^X + E\mathfrak{L}^Y) \tag{3.1.13}$$

through which the coefficients A, B and C to be determined to eliminate the second-order derivatives in H_r. Thus, it will be a first-order differential operator equal to H on $L_{\phi_E}(\mathbb{H}/Z)$. To achieve this, we need to take

$$A = \frac{\mathrm{i}}{4\pi m}, \quad B = \frac{\omega}{4\pi}, \quad E = m\omega.$$

Note that the value of E is uniquely defined and consequently the corresponding vacuum vector $\phi_E(y) = \mathrm{e}^{-\pi\hbar E y^2}$ is fixed. Furthermore, to obtain a geometric action of H_r in the Schrödinger equation we need purely imaginary coefficients of the first-order derivatives in H_r. This implies that $C = -\frac{\mathrm{i}\hbar\omega}{2}x$, with the final result:

$$\begin{aligned} H_r = & \left(\frac{1}{2m}\left(\frac{\mathrm{i}d\tilde{\sigma}_\hbar^X}{\sqrt{2\pi}}\right)^2 + \frac{m\omega^2}{2}\left(\frac{\mathrm{i}d\tilde{\sigma}_\hbar^Y}{\sqrt{2\pi}}\right)^2 \right) \\ & + (\frac{\mathrm{i}}{4\pi m}\partial_x + \frac{\omega}{4\pi}\partial_y - \frac{\mathrm{i}\hbar\omega}{2}xI)(-\mathrm{i}\mathfrak{L}^X + m\omega\mathfrak{L}^Y) \\ = & \frac{\mathrm{i}\hbar}{m}y\partial_x - \mathrm{i}\hbar m\omega^2 x\partial_y + \left(\frac{1}{2}\hbar\omega + \frac{\pi\hbar^2}{m}(y^2 - m^2\omega^2x^2)\right) I\,. \end{aligned} \tag{3.1.14}$$

Note that operators (3.1.10) *and* (3.1.14) *are not equal in general but have the same restriction to the kernel of the auxiliary analytic condition* (3.1.12). Thus, the Schrödinger equation (3.1.11) becomes equivalent to the first-order PDE

$$\mathrm{i}\hbar\dot{f}(t,x,y) - H_r f(t,x,y) = 0 \tag{3.1.15}$$

on the space of functions that satisfy the analyticity condition (3.1.12), where H_r is given by (3.1.14).

Now, equation (3.1.11) can be solved easily through the following steps:

1. Since we want to determine the dynamic in $L_{\phi_E}(\mathbb{H}/Z)$ for all t, $f(t,x,y)$ must satisfy the analyticity condition. That is,

$$(-\mathrm{i}\partial_x + m\omega\partial_y - 2\pi\mathrm{i}\hbar m\omega x I)\, f(t,x,y) = 0. \tag{3.1.16}$$

   This is a simple linear first-order PDE and we use the method of characteristics to obtain the following general solution

$$f(t,x,y) = \mathrm{e}^{-\pi hm\omega x^2} f_1\,(t, m\omega x + \mathrm{i}y)\,, \tag{3.1.17}$$

   where f_1 is an arbitrary analytic function. It is analytic in $m\omega x + \mathrm{i}y$ because the factor $\mathrm{e}^{-\pi hm\omega x^2}$ "peels" the operator $(-\mathrm{i}\partial_x + m\omega\partial_y - 2\pi\mathrm{i}\hbar m\omega x I)$ to the Cauchy–Riemann operator, $-\mathrm{i}\partial_x + m\omega\partial_y$ on f_1.

2. We take the solution (3.1.17) and substitute in the reduced Schrödinger equation (3.1.15) which becomes even simpler in f_1:

$$\left(\mathrm{i}\hbar\partial_t - \hbar\omega z\partial_z - (\frac{\pi\hbar^2}{m}z^2 + \frac{1}{2}\hbar\omega)I\right) f_1(t,z) = 0 \tag{3.1.18}$$

   where $z = m\omega x + \mathrm{i}y$. Again the method of characteristics provides us with the following general solution

$$f_1(t,z) = \mathrm{e}^{-\frac{\mathrm{i}\omega}{2}t - \frac{\pi\hbar}{2m\omega}z^2} f_2(\mathrm{e}^{-\mathrm{i}\omega t}z), \tag{3.1.19}$$

   where f_2 is an arbitrary analytic function in $z = m\omega x + \mathrm{i}y$.

3. Finally, we make a substitution of (3.1.19) into (3.1.17) to get the general solution

$$f(t,x,y) = \exp\left(-\frac{\mathrm{i}\omega}{2}t + \pi\mathrm{i}\hbar xy - \frac{\pi\hbar}{2m\omega}(m^2\omega^2x^2 + y^2)\right) f_2\left(\mathrm{e}^{-\mathrm{i}\omega t}(m\omega x + \mathrm{i}y)\right) \tag{3.1.20}$$

   which satisfies (3.1.15) and certainly satisfies the analyticity condition (3.1.12).

Hence, it is also a general solution of the main equation (3.1.11) with the Hamiltonian (3.1.10). As known (see the end of Section 1.4) this solution uses a coordinate transformation that geometrically corresponds to a uniform rotation of the phase space, cf. (1.4.52). This is a classical dynamics of the harmonic oscillator.

We can easily see that the reduction has been achieved due to the analyticity condition. Specifically, the second-order derivative terms in (3.1.10) represent a Laplacian, thus it vanishes on analytic functions—a standard result from complex analysis. This clearly explains the reason behind the simple form of a harmonic oscillator Hamiltonian on FSB space being a first-order differential operator, cf. Section 1.4.

3.2 Harmonic oscillator from the group $\mathbb{G}$

Here we obtain an exact solution of the Schrödinger equation for the harmonic oscillator in the space $L_{\phi_E}(\mathbb{G}/Z)$. It is achieved by the reduction of the order of the corresponding differential operator in a manner illustrated on the Heisenberg group in the previous section 3.1. A new feature of this case is that we need to use both operators (2.2.34) and (2.2.39) simultaneously.

The quantum harmonic oscillator Hamiltonian acting in $L_{\phi_E}(\mathbb{G}/Z)$ is given by (cf. (3.1.10) and the previous paragraph to it):

$$\begin{aligned} H &= \frac{1}{2m}\left(\frac{\mathrm{i}d\tilde{\rho}^{X_1}_{\hbar_4}}{\sqrt{2\pi}}\right)^2 + \frac{m\omega^2}{2}\left(\frac{\mathrm{i}d\tilde{\rho}^{X_3}_{\hbar_4}}{\sqrt{2\pi}}\right)^2 \\ &= -\frac{1}{4\pi m}\partial^2_{11} - \frac{1}{4\pi m}x_2^2\partial^2_{33} - \frac{m\omega^2}{4\pi}\partial^2_{33} - \frac{1}{2\pi m}x_2\partial^2_{13} \\ &\quad + \frac{\mathrm{i}\hbar_4}{m}x_3\partial_1 + \frac{\mathrm{i}\hbar_4}{m}x_2x_3\partial_3 - \frac{1}{2m}(-\mathrm{i}\hbar_4 x_2 - 2\pi\hbar_4^2 x_3^2)I\,. \end{aligned} \tag{3.2.21}$$

Note that if $x_2 = 0$, then we get exactly (3.1.10) the Heisenberg group case. Although, the Hamiltonian (3.2.21) seems a bit alienated in comparison with the Heisenberg group case, we can still adjust it by using conditions (2.2.33) and (2.2.38) as follows. We write

$$H_r = H + (A\partial_1 + B\partial_2 + C\partial_3 + KI)\mathcal{C} + F\mathcal{S}\,. \tag{3.2.22}$$

To eliminate all second order derivatives one has to take $A = \frac{\mathrm{i}}{4\pi m}$, $B = 0$, $C = \frac{\mathrm{i}}{2\pi m}(\frac{-\mathrm{i}}{2}E + x_2)$ and $F = -\frac{1}{4\pi m}(\mathrm{i}x_2 + E)^2 + \frac{m\omega^2}{4\pi}$. A significant difference from the Heisenberg group case is that there is no particular restrictions on the parameter E. This allows us to use functions $\mathrm{e}^{-\pi\hbar_4 E y^2}$, with any $E > 0$ as fiducial vectors. Such functions are known as squeezed states, cf. Section 1.4.

To make the action of the first order operator geometric via the reduced Schrödinger equation, we make coefficients in front of the first-order derivatives ∂_1 and ∂_3 imaginary. For this we put $K = \frac{\mathrm{i}\hbar_4}{2m}x_1(-E + 2\mathrm{i}x_2)$. The final result is

$$\begin{aligned}
H_r &= H + \left(\frac{\mathrm{i}}{4\pi m}\partial_1 + \frac{\mathrm{i}}{2\pi m}(\frac{-\mathrm{i}}{2}E + x_2)\partial_3 + \frac{\mathrm{i}\hbar_4}{2m}x_1(-E + 2\mathrm{i}x_2)\right)\mathcal{C} \\
&\quad + \left(\frac{1}{4\pi m}(x_2 + \mathrm{i}E)^2 + \frac{m\omega^2}{4\pi}\right)\mathcal{S} \\
&= \frac{\mathrm{i}\hbar_4}{m}\left((x_3 + x_1x_2)\partial_1 - \left((\mathrm{i}x_2 + E)^2 - m^2\omega^2\right)\partial_2 - (E^2x_1 - x_2x_3)\partial_3\right) \\
&\quad - \frac{\hbar_4}{2m}\left(-8\mathrm{i}\pi h_2 E x_2 - \mathrm{i}x_2 + 4\pi h_2 x_2^2 - 2\pi\hbar_4 x_3^2\right. \\
&\quad \left. + 4\pi h_2 m^2\omega^2 - E - 4\pi h_2 E^2 - 4\mathrm{i}\pi E\hbar_4 x_1^2 x_2 + 2\pi\hbar_4 E^2 x_1^2\right) I
\end{aligned} \tag{3.2.23}$$

where $\mathcal{C}$ and $\mathcal{S}$ are given by (2.2.34) and (2.2.39). Thus, the Schrödinger equation for the harmonic oscillator

$$\mathrm{i}\hbar_4\partial_t f(t, x_1, x_2, x_3) - Hf(t, x_1, x_2, x_3) = 0, \tag{3.2.24}$$

is equivalent to the first-order linear PDE

$$\mathrm{i}\hbar_4\partial_t f(t, x_1, x_2, x_3) - H_r f(t, x_1, x_2, x_3) = 0, \tag{3.2.25}$$

for f in the image space $L_{\phi_E}(\mathbb{G}/Z)$. Following the same scheme as in the case of the Heisenberg group, we can achieve a generic solution to (3.2.24) via the following steps:

1. We start from a general formula for $f(t, x_1, x_2, x_3)$ that satisfies the analyticity condition (2.2.33) for all t. That is, we solve the equation:

$$(-\mathrm{i}\partial_1 + E\partial_3 - 2\pi\mathrm{i}\hbar_4 E x_1) f(t, x_1, x_2, x_3) = 0. \tag{3.2.26}$$

Since this is a first-order linear PDE, then by the method of characteristics we obtain the following general solution

$$f(t, x_1, x_2, x_3) = \mathrm{e}^{-\pi\hbar_4 E x_1^2} f_1(t, Ex_1 + \mathrm{i}x_3, \mathrm{i}x_2 + E), \tag{3.2.27}$$

where f_1 is an arbitrary function, analytic with respect to $Ex_1 + \mathrm{i}x_3$. Similar to the case of the Heisenberg group, f_1 is analytic in $Ex_1 + \mathrm{i}x_3$ because the operator of multiplication

$$\mathcal{G} : f_1(t, x_1, x_2, x_3) \mapsto \mathrm{e}^{-\pi\hbar_4 E x_1^2} f_1(t, x_1, x_2, x_3)$$

intertwines the operator (2.2.34) with the Cauchy–Riemann type operator:

$$\mathcal{G} \circ \mathcal{C} = (E\partial_3 - \mathrm{i}\partial_1) \circ \mathcal{G}.$$

2. Upon substitution of (3.2.27) into the reduced Schrödinger equation (3.2.25) with H_r (3.2.23), we obtain a simplified first-order PDE

$$\begin{aligned}\Bigg(\mathrm{i}\hbar_4\partial_t + \frac{\hbar_4}{m}\Bigg(zw\partial_z + (w^2 - m^2\omega^2)\partial_w \\ + (\pi\hbar_4 z^2 + 2\pi h_2(w^2 - m^2\omega^2) + \frac{1}{2}w)I\Bigg)\Bigg) f_1(t, z, w) = 0\end{aligned} \tag{3.2.28}$$

where

$$z = Ex_1 + \mathrm{i}x_3, \quad w = \mathrm{i}x_2 + E.$$

Again, by the method of characteristics, we obtain the following general form of the function f_1

$$\begin{aligned}f_1(t, z, w) = \frac{\sqrt{E + m\omega}}{\sqrt{w + m\omega}} \exp\left(\frac{\mathrm{i}\omega}{2}t - 2\pi h_2 w - \pi\hbar_4 \frac{z^2}{w + m\omega}\right) \\ f_2\left(\mathrm{e}^{\mathrm{i}\omega t}\frac{z}{w + m\omega}, \mathrm{e}^{2\mathrm{i}\omega t}\frac{m\omega - w}{m\omega + w}\right).\end{aligned} \tag{3.2.29}$$

Thus, substituting f_1 (3.2.29) into (3.2.27) we get, in terms of the original coordinates x_j,

$$\begin{aligned} f(t,x_1,x_2,x_3) = {} & \frac{\sqrt{E+m\omega}}{\sqrt{\mathrm{i}x_2+E+m\omega}} \\ & \times \exp\left(\frac{-\mathrm{i}\omega}{2}t - \pi\hbar_4 E x_1^2 - 2\pi\mathrm{i}h_2x_2 + \pi\hbar_4\frac{(Ex_1+\mathrm{i}x_3)^2}{\mathrm{i}x_2+E+m\omega}\right) \\ & \times f_2\left(\mathrm{e}^{-\mathrm{i}\omega t}\frac{Ex_1+\mathrm{i}x_3}{\mathrm{i}x_2+E+m\omega}, \mathrm{e}^{-2\mathrm{i}\omega t}\frac{m\omega-(\mathrm{i}x_2+E)}{m\omega+(\mathrm{i}x_2+E)}\right). \end{aligned} \tag{3.2.30}$$

At this stage, formula (3.2.30) satisfies (3.2.25) with the reduced Hamiltonian (3.2.23), and of course the analyticity condition (3.2.26). Yet, it is not a solution of (3.2.24) with the original Hamiltonian (3.2.21). We need the following step.

3. In the final step we request that (3.2.30) satisfy the structural condition (2.2.38). This results in a heat-like equation in terms of f_2:

$$\partial_v f_2(u,v) = -\frac{1}{8\pi\hbar_4 m\omega}\partial^2_{uu} f_2(u,v) \tag{3.2.31}$$

where

$$u = \frac{Ex_1+\mathrm{i}x_3}{\mathrm{i}x_2+E+m\omega}, \quad v = \frac{m\omega-(\mathrm{i}x_2+E)}{m\omega+(\mathrm{i}x_2+E)}.$$

The solution of (3.2.31) is formally given by:

$$f_2(u,v) = \left(\frac{2m\omega\hbar_4}{v}\right)^{1/2}\int_{\mathbb{R}} g(\xi)\,\mathrm{e}^{2\pi\hbar_4 m\omega\frac{(u-\xi)^2}{v}}\,\mathrm{d}\xi, \tag{3.2.32}$$

where g is the initial condition $g(u) = f_2(u,0)$ and we use analytic extension from the real variable v to some neighbourhood of the origin in the complex plane—see the discussion in the next section.

Thus, with f_2 as in (3.2.32), formula (3.2.30) yields a generic solution to (3.2.24) with the Hamiltonian (3.2.21).

Remark 3.2.1 *Note that the above initial value of f_2, related to $v = 0$, corresponds to $x_2 = 0$ and $E = m\omega$. Thus, it indicates that this general solution is obtained from the unsqueezed states of Fock–Segal–Bargmann space, that is, the case related to the Heisenberg group.*

3.2.1 Geometrical, analytic and physical meanings of new solution

Analysing the new solution (3.2.30) we immediately note that it reduces by the substitution $x_2 = 0$ (no shear) and $E = m\omega$ (predefined non-squeezing for the Heisenberg group) into the solution (3.1.20). Thus, it is interesting to look into the meaning of formula (3.2.30) for other values. This can be deconstructed as follows.

The first factor, shared by any solution, is responsible for

1. adding the value $\frac{1}{2}\hbar_4\omega$ to every integer multiple of $\hbar_4\omega$ eigenvalue;
2. peeling the second factor to analytic function;
3. proper L_2-normalisation.

The first variable in the function f_2 of the second factor produces integer multiples of $\hbar_4\omega$ in eigenvalues. The rotation dynamics of the second variable alternate the shear parameter as follows. Points $\mathrm{i}x_2 + E$ form a vertical line on the complex plane. The Cayley-type transformation

$$\mathrm{i}x_2 + E \;\mapsto\; \frac{m\omega - (\mathrm{i}x_2 + E)}{m\omega + (\mathrm{i}x_2 + E)} \tag{3.2.33}$$

maps the vertical line $\mathrm{i}x_2 + E$ into the circle with the centre $\frac{-E}{m\omega+E}$ and radius $\frac{m\omega}{m\omega+E}$ (therefore passing -1). Rotations of a point of this circle around the origin creates circles centred at the origin and a radius between $c = \left|\frac{m\omega-E}{m\omega+E}\right|$ and 1, see Fig. 3.1.

Let a function f_2 from (3.2.32) have an analytic extension from the real values of v into a (possibly punctured) neighbourhood of the origin of a radius R in the complex plane. An example of functions admitting such an extension are the eigenfunctions of the harmonic oscillator (3.2.42) considered in the next section. In order for the solution (3.2.30) to be well-defined for all values of t one needs the following inequality to hold

$$\left| \frac{m\omega - (\mathrm{i}x_2 + E)}{m\omega + (\mathrm{i}x_2 + E)} \right| < R\,. \tag{3.2.34}$$

It implies the allowed range of E around the special value $m\omega$:

$$\frac{1-R}{1+R} m\omega < E < \frac{1+R}{1-R} m\omega. \tag{3.2.35}$$

For every such E, the respective allowed range of x_2 around 0 can be similarly deduced from the required inequality (3.2.34), see the arc drawn by thick pen on Fig. 3.1.

The existence of bounds (3.2.35) for possible squeezing parameter E shall be expected from the physical consideration. The integral formula (3.2.32) produces for a real v a solution of the irreversible heat-diffusion equation for the time-like parameter u. However, its analytic extension into the complex plane will include also solutions of time-reversible Schrödinger equation for purely imaginary v. Since the rotation of the second variable in (3.2.30) requires all complex values of v with fixed $|v|$, only a sufficiently small neighbourhood (depending on the "niceness" of an initial value $f_2(u, 0)$) is allowed. Also note that a rotation of a squeezed state in the phase space breaks the minimal uncertainty condition at certain times, however the state periodically "re-focus" back to the initial minimal uncertainty shape [81].

If a solution $f_2(u, v)$, $v \in \mathbb{R}$ of (3.2.32) does not permit an analytic expansion into a neighbourhood of the origin, then two analytic extensions $f_2^{\pm}(u, v)$, for $\Im v > 0$ and $\Im v < 0$ respectively, shall exist. Then, the dynamics in (3.2.30) will experience two distinct jumps for all values of t such that

$$\Re \left(\mathrm{e}^{-2\mathrm{i}\omega t} \frac{m\omega - (\mathrm{i}x_2 + E)}{m\omega + (\mathrm{i}x_2 + E)} \right) = 0\,. \tag{3.2.36}$$

Analysis of this case and its physical interpretation may require further investigation.

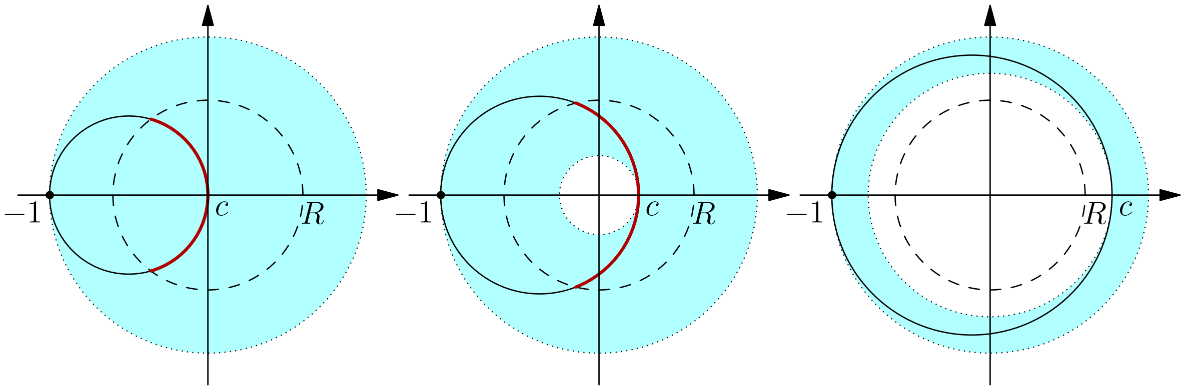

Figure 3.1: Shear parameter and analytic continuation. The solid circle is the image of the line $\mathrm{i}x_2 + E$ under the Cayley-type transformation (3.2.33). The shadowed region with dotted boundary (the annulus with radii c and 1) is obtained from the solid circle under rotation around the origin. The dashed circle of the radius R bounds the domain of the analytic continuation of the solution (3.2.32).

The left picture corresponds to $E = m\omega$ (thus $c = 0$)—there always exists a part of the shaded region inside the circle of a radius R (even for $R = 0$).

The middle picture represents a case of some E within the bound (3.2.35)—there is an arc (drawn by a thick pen) inside of the dashed circle. The arc corresponds to values of x_2 such that the solution (3.2.30) is meaningful.

The right picture illustrates the squeeze parameter E, which is outside of the range (3.2.35). For such a state, which is squeezed too much, no values of x_2 allow to use the region of the analytic continuation within the dashed circle.

3.2.2 Harmonic oscillator Hamiltonian and ladder operators in the space $L_{\phi_E}(\mathbb{G}/Z)$

For determining a complete set of eigenvectors of the harmonic oscillator Hamiltonian (3.2.21), we consider ladder operators technique as explained in Appendix A. To this end, let

$$L^+ = \sqrt{\frac{m\omega}{\pi\hbar_4}}\left(\frac{\mathrm{i}}{2m\omega}X_1 - \frac{1}{2}X_3\right) \quad \text{and} \quad L^- = \sqrt{\frac{m\omega}{\pi\hbar_4}}\left(\frac{\mathrm{i}}{2m\omega}X_1 + \frac{1}{2}X_3\right),$$

which represent two elements of the Lie algebra $\mathfrak{g}$ corresponding to the shear group $\mathbb{G}$. Using the derived representation formulae (1.2.12) we have the following dimensionless operators—linear combination of infinitesimal generators of the Lie algebra $\mathfrak{g}$:

$$\begin{aligned}
a^+ := \mathrm{d}\tilde{\rho}_{\hbar_4}^{L^+} &= \sqrt{\frac{m\omega}{\pi\hbar_4}}\left(\frac{1}{2m\omega}(\mathrm{id}\tilde{\rho}_{\hbar_4}^{X_1}) + \frac{\mathrm{i}}{2}(\mathrm{id}\tilde{\rho}_{\hbar_4}^{X_3})\right)\\
&= \frac{\mathrm{i}}{2\sqrt{\pi m\omega\hbar_4}}(-\partial_1 - (x_2 + \mathrm{i}m\omega)\partial_3 + 2\pi\mathrm{i}\hbar_4 x_3\, I);\\
a^- := \mathrm{d}\tilde{\rho}_{\hbar_4}^{L^-} &= \sqrt{\frac{m\omega}{\pi\hbar_4}}\left(\frac{1}{2m\omega}(\mathrm{id}\tilde{\rho}_{\hbar_4}^{X_1}) - \frac{\mathrm{i}}{2}(\mathrm{id}\tilde{\rho}_{\hbar_4}^{X_3})\right)\\
&= \frac{\mathrm{i}}{2\sqrt{\pi m\omega\hbar_4}}(-\partial_1 - (x_2 - \mathrm{i}m\omega)\partial_3 + 2\pi\mathrm{i}\hbar_4 x_3\, I)\,.
\end{aligned}$$

Since, $[(2\pi)^{-1/2}\,\mathrm{id}\tilde{\rho}_{\hbar_4}^{X_3}, (2\pi)^{-1/2}\,\mathrm{id}\tilde{\rho}_{\hbar_4}^{X_1}] = \mathrm{i}\hbar_4 I$, which gives a realisation of the canonical commutation relation in the Hilbert space $L_{\phi_E}(\mathbb{G}/Z)$, one can immediately verify the commutator relation

$$[a^-, a^+] = I\,. \tag{3.2.37}$$

Thus, all properties of the ladder operators included in Appendix A are valid for a^-, a^+ here. Notably, the Hamiltonian (3.2.21) is expressible in terms of the above ladder operators:

$$H = \hbar_4\omega(a^+a^- + \frac{1}{2}\,I)\,. \tag{3.2.38}$$

Thus the following commutators hold:

$$[H, a^+] = \hbar_4\omega a^+, \quad [H, a^-] = -\hbar_4\omega a^-.$$

Moreover, the creation and the annihilation operators are adjoint of each other:

$$(a^-)^* = a^+ \tag{3.2.39}$$

where "*" indicates the adjoint of an operator in terms of the inner product defined by (2.1.14). Then, from (3.2.38) and (3.2.39) we see that $H^* = H$.

Corollary 3.2.2 *The function (see* (2.2.41) *for $q = m\omega$)*

$$\Phi_0(x_1, x_2, x_3) = \frac{\sqrt{2}(m\omega E)^{1/4}}{\sqrt{\mathrm{i}x_2 + E + m\omega}} \exp\left(-\pi\hbar_4 E x_1^2 - 2\pi\mathrm{i}h_2 x_2 + \pi\hbar_4 \frac{(Ex_1 + \mathrm{i}x_3)^2}{\mathrm{i}x_2 + E + m\omega}\right), \tag{3.2.40}$$

represents a vacuum vector in the space $L_{\phi_E}(\mathbb{G}/Z)$. That is,

$$a^-\Phi_0 = 0.$$

Proof

The vacuum Φ_0 in the image space $L_{\phi_E}(\mathbb{G}/Z)$ must be of the form $\Phi_0 = \mathcal{W}_{\phi_E} f$ for some $f \in L_2(\mathbb{R})$ where $\phi_E(y) = (2h_2 E)^{1/4}\mathrm{e}^{-\pi\hbar_4 E y^2}$ with a parameter $E > 0$. We can simply find f for which $a^-\Phi_0 = 0$ as follows. The vacuum Φ_0 is defined as the null solution of the annihilation operator:

$$a^-\Phi_0 = \sqrt{\frac{m\omega}{\pi\hbar_4}} \left(\frac{1}{2m\omega}(\mathrm{i}\mathrm{d}\tilde{\rho}_{\hbar_4}^{X_1}) - \frac{\mathrm{i}}{2}(\mathrm{i}\mathrm{d}\tilde{\rho}_{\hbar_4}^{X_3})\right)\Phi_0 = 0.$$

That is,

$$\left(\mathrm{i}\mathrm{d}\tilde{\rho}_{\hbar_4}^{X_1} + m\omega \mathrm{d}\tilde{\rho}_{\hbar_4}^{X_3}\right)\mathcal{W}_{\phi_E} f = 0.$$

But, the intertwining property, Corollary 2.1.13, implies that

$$\begin{aligned}
&\mathrm{i}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \tilde{\rho}_{\hbar_4}(\exp tX_1)\mathcal{W}_{\phi_E} f + m\omega \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \tilde{\rho}_{\hbar_4}(\exp tX_3)\mathcal{W}_{\phi_E} f \\
&= \mathrm{i}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{W}_{\phi_E}\rho_{h_2\hbar_4}(\exp tX_1) f + m\omega \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{W}_{\phi_E}\rho_{h_2\hbar_4}(\exp tX_3) f \\
&= \mathcal{W}_{\phi_E}\left(\mathrm{i}\mathrm{d}\rho_{h_2\hbar_4}^{X_1} + m\omega \mathrm{d}\rho_{h_2\hbar_4}^{X_3}\right) f
\end{aligned}$$

The latter vanishes if

$$\left(\mathrm{i}\mathrm{d}\rho^{X_1}_{h_2\hbar_4} + m\omega\mathrm{d}\rho^{X_3}_{h_2\hbar_4}\right) f(y) = 0,$$

where $\mathrm{d}\rho^{X_j}_{h_2\hbar_4}$ are given by (1.2.13). Thus, it has the general solution

$$f(y) = (2h_2 m\omega)^{1/4}\mathrm{e}^{-\pi\hbar_4 m\omega y^2} = \phi_{m\omega}(y).$$

Thus, the vacuum is

$$\Phi_0 = \mathcal{W}_{\phi_E}\phi_{m\omega} \tag{3.2.41}$$

which has been explicitly computed, see Example 2.2.4 for $q = m\omega$. □

The vacuum Φ_0 is normalised ($\|\Phi_0\|_{x_2} = 1$) and for the higher order *normalised* states, we put

$$\Phi_j = \frac{1}{\sqrt{j!}}(a^+)^j\Phi_0, \qquad j = 1, 2, \ldots.$$

Orthogonality of Φ_j follows from the fact that H is self-adjoint.

Corollary 3.2.3 *For*

$$u = \frac{Ex_1 + \mathrm{i}x_3}{\mathrm{i}x_2 + E + m\omega} \quad \textit{and} \quad v = \frac{m\omega - (\mathrm{i}x_2 + E)}{m\omega + (\mathrm{i}x_2 + E)}$$

we have

$$\Phi_j(x_1, x_2, x_3) = \frac{1}{\sqrt{2^j j!}}(v)^{j/2} H_j\left(\sqrt{\frac{-2\pi\hbar_4 m\omega}{v}}\, u\right)\Phi_0 \tag{3.2.42}$$

where

$$H_j(y) = \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} \frac{(-1)^k j!}{k!(j-2k)!}(2y)^{j-2k} \tag{3.2.43}$$

are the Hermite polynomials of order j, see (1.4.41).

Proof

We have

$$\Phi_j = \frac{1}{\sqrt{j!}}(a^+)^j\Phi_0 \overset{(3.2.41)}{=} \frac{1}{\sqrt{j!}}(a^+)^j[\mathcal{W}_{\phi_E}\phi_{m\omega}].$$

Again, Corollary 2.1.13 implies that

$$\begin{aligned}
\frac{1}{\sqrt{j!}}(a^+)^j[\mathcal{W}_{\phi_E}\phi_{m\omega}] &= \frac{1}{\sqrt{j!}}\left(\sqrt{\frac{m\omega}{\pi\hbar_4}}\right)^j \left[\mathcal{W}_{\phi_E}\left(\frac{\mathrm{i}}{2m\omega}\mathrm{d}\rho^{X_1}_{h_2\hbar_4} + \frac{1}{2}\mathrm{d}\rho^{X_3}_{h_2\hbar_4}\right)^j \phi_{m\omega}\right] \\
&= \frac{1}{\sqrt{j!}}\left(\sqrt{\frac{m\omega}{\pi\hbar_4}}\right)^j \\
&\quad \times \left\langle \left(-\frac{\mathrm{i}}{2m\omega}\frac{\mathrm{d}}{\mathrm{d}y} + \pi\hbar_4 y\right)^j \phi_{m\omega}(y), \rho_{h_2\hbar_4}(x_1,x_2,x_3,0)\phi_E(y)\right\rangle \\
&= \frac{(-1)^{j/2}}{\sqrt{2^j j!}}\left\langle H_j\left(\sqrt{2\pi m\omega\hbar_4}\, y\right)\phi_{m\omega}(y), \rho_{h_2\hbar_4}(x_1,x_2,x_3,0)\phi_E(y)\right\rangle \\
&= \frac{(\mathrm{i})^j}{\sqrt{2^j j!}} I(x_1,x_2,x_3)\Phi_0(x_1,x_2,x_3), \qquad (3.2.44)
\end{aligned}$$

where

$$I(x_1,x_2,x_3) = \int_{\mathbb{R}} \mathrm{e}^{-\pi q^2} H_j\left(\alpha q + \beta u\right)\,\mathrm{d}q, \qquad (3.2.45)$$

for $\alpha = \frac{\sqrt{2\pi m\omega}}{\sqrt{\mathrm{i}x_2 + E + m\omega}}$, $\beta = \sqrt{2\pi m\omega\hbar_4}$ and $u = \frac{Ex_1 + \mathrm{i}x_3}{\mathrm{i}x_2 + E + m\omega}$. Now, we explicitly evaluate the above integral as follows:

$$\begin{aligned}
I(x_1,x_2,x_3) &= \int_{\mathbb{R}} \mathrm{e}^{-\pi q^2} H_j\left(\alpha q + \beta u\right)\,\mathrm{d}q \\
&= \int_{\mathbb{R}} \mathrm{e}^{-\pi q^2} \sum_{k=0}^{\lfloor j/2\rfloor} \frac{(-1)^k j!}{k!(j-2k)!}\left(2\left(\alpha q + \beta u\right)\right)^{j-2k}\,\mathrm{d}q \\
&= \sum_{k=0}^{\lfloor j/2\rfloor}\sum_{n=0}^{j-2k} \frac{(-1)^k 2^{j-2k} j!}{k!(j-2k)!}\frac{(j-2k)!}{n!(j-2k-n)!}(\beta u)^{j-2k-n}\alpha^n \int_{\mathbb{R}} \mathrm{e}^{-\pi q^2} q^n\,\mathrm{d}q.
\end{aligned}$$

In the equality previous to the last, we used the closed form of the expansion of the binomial $(\alpha q + \beta u)^{j-2k}$. Moreover, the last integral vanishes unless n is even, so for $n = 2r$ with ($r = 0, 1, \cdots$), we use the identity

$$\int_{\mathbb{R}} \mathrm{e}^{-\pi q^2} q^{2r}\,\mathrm{d}q = \frac{(2r)!}{2^{2r}\pi^r r!}. \qquad (3.2.46)$$

Thus,

$$
\begin{aligned}
I(x_1,x_2,x_3) &= \sum_{k=0}^{\lfloor j/2\rfloor}\sum_{r=0}^{\lfloor j/2\rfloor-k}\frac{(-1)^k 2^{j-2k} j!}{k!(j-2(k+r))!}(\beta u)^{j-2(k+r)}\alpha^{2r}\left(\frac{1}{2^{2r}\pi^r r!}\right)\\
&= \sum_{k=0}^{\lfloor j/2\rfloor}\sum_{r=0}^{\lfloor j/2\rfloor-k}\frac{(-1)^k 2^{r} j!}{r!k!(j-2(k+r))!}\frac{(m\omega)^r}{(\mathrm{i}x_2+E+m\omega)^r}(2\sqrt{2\pi m\omega\hbar_4}u)^{j-2(k+r)}.
\end{aligned}
$$

Let $k+r=s$, the interior sum becomes

$$
\begin{aligned}
&\sum_{k=0}^{\lfloor j/2\rfloor}\sum_{s=k}^{\lfloor j/2\rfloor}\frac{(-1)^k 2^{s-k} j!}{(s-k)!k!(j-2s)!}\frac{(m\omega)^{s-k}}{(\mathrm{i}x_2+E+m\omega)^{s-k}}(2\sqrt{2\pi m\omega\hbar_4}u)^{j-2s}\\
&= \sum_{s=0}^{\lfloor j/2\rfloor}\sum_{k=0}^{s}\left[\frac{j!2^s}{s!(j-2s)!}\frac{s!}{k!(s-k)!}\left(\frac{-1}{2}\right)^k\left(\frac{m\omega}{\mathrm{i}x_2+E+m\omega}\right)^{s-k}\right.\\
&\qquad \left.\times\,(2\sqrt{2\pi m\omega\hbar_4}u)^{j-2s}\right]\\
&= \sum_{s=0}^{\lfloor j/2\rfloor}\left[\frac{j!2^s}{s!(j-2s)!}(2\sqrt{2\pi m\omega\hbar_4}\,u)^{j-2s}\left(-\frac{1}{2}+\frac{(m\omega)}{\mathrm{i}x_2+E+m\omega}\right)^s\right]\\
&= \sum_{s=0}^{\lfloor j/2\rfloor}\left[\frac{j!}{s!(j-2s)!}(2\sqrt{2\pi m\omega\hbar_4}\,u)^{j-2s}v^s\right]\\
&= (\mathrm{i})^{-j}(v)^{j/2}\sum_{s=0}^{\lfloor j/2\rfloor}\frac{j!(-1)^s}{s!(j-2s)!}\left(2\sqrt{\frac{-2\pi m\omega\hbar_4}{v}}\,u\right)^{j-2s}\\
&= (\mathrm{i})^{-j}(v)^{j/2}H_j\left(\sqrt{\frac{-2\pi m\omega\hbar_4}{v}}\,u\right).
\end{aligned}
$$

Thus,

$$I(x_1,x_2,x_3) = (\mathrm{i})^{-j}(v)^{j/2}H_j\left(\sqrt{\frac{-2\pi m\omega\hbar_4}{v}}\,u\right).$$

Finally, upon substitution into (3.2.44), we obtain

$$\Phi_j(x_1,x_2,x_3) = \frac{1}{\sqrt{2^j j!}}(v)^{j/2}H_j\left(\sqrt{\frac{-2\pi m\omega\hbar_4}{v}}\,u\right)\Phi_0.$$

□

Due to the general algebraic properties of ladder operators, Appendix A, we still have

$$a^- \Phi_j = \sqrt{j}\Phi_{j-1}. \tag{3.2.47}$$

Therefore,

$$a^+ a^- \Phi_j = j\Phi_j.$$

Hence, for the Hamiltonian H of the harmonic oscillator (3.2.21) we have

$$H\Phi_j = \hbar_4\omega(j + \tfrac{1}{2})\,\Phi_j.$$

Recall that $\mathcal{C}\Phi_0 = \mathcal{S}\Phi_0 = 0$, see the last paragraph after (2.2.41). Furthermore, it can be verified that both operators $\mathcal{C}$ (2.2.34) and $\mathcal{S}$ (2.2.39) commute with the creation operator a^+ and thus

$$\mathcal{C}\Phi_j = \mathcal{S}\Phi_j = 0, \quad j = 0, 1, 2, \ldots.$$

In other words, the higher-order states Φ_j satisfy the analyticity condition (2.2.33) and the structural condition (2.2.38).

Note that the singularity of eigenfunction (3.2.42) at $v = 0$ is removable due to a cancellation between the first power factor and the Hermite polynomial given by (3.2.43). Moreover, the eigenfunction (3.2.42) has an analytic extension in v to the whole complex plane, thus does not have any restriction on the squeezing parameter E from the inequality (3.2.34).

The eigenfunction $\Phi_j(u, v)$ (3.2.42) at $v = 0$ reduces to the power u^j of the variable u, as can be expected from the connection to the FSB space and the Heisenberg group.

Chapter 4

Classification of Hamiltonian operators for geometric dynamics

So far we have considered a specific case of Hamiltonian operator, the harmonic oscillator's. The corresponding geometric dynamics have been obtained in the space $L_{\phi_E}(\mathbb{G}/Z)$ for the fiducial vector ϕ_E being a squeezed Gaussian, that is, the case of ϕ_E (2.2.31) where $D = 0$. However, certain Hamiltonian operators of the form,

$$H = \sum_{j,k=1}^{3} a_{jk}(\mathrm{i}\mathrm{d}\tilde{\rho}_{\hbar_4}^{X_j})(\mathrm{i}\mathrm{d}\tilde{\rho}_{\hbar_4}^{X_k}), \tag{4.0.1}$$

where $\mathrm{d}\tilde{\rho}^{X_j}$ are as in (1.2.12), can not be reduced to first-order differential operators when such a Gaussian (i.e. the case $D = 0$) is taken as a fiducial vector.

In this chapter, we provide the full classification of the Hamiltonians which can be geometrised by the minimal three-step nilpotent Lie group $\mathbb{G}$ and the *cubic* exponent in the fiducial vector (2.2.31). The latter is the Fourier transform of an Airy wave packet [11] which is useful in paraxial optics [76, 77].

4.1 Geometrisation of Hamiltonians by Airy coherent states

Without lose of generality, let us consider the above form of a Hamiltonian H acting in the image space $L_{\phi_{E,D}}(\mathbb{G}/Z)$ where the fiducial vector $\phi_{E,D}$ is given by (cf. (2.2.31) and Fig.4.1)

$$\phi_{E,D}(y) = c\exp\left(\frac{\pi \mathrm{i} D\hbar_4}{3}y^3 - \pi E\hbar_4 y^2 + 2\pi \mathrm{i} Dh_2 y\right). \tag{4.1.2}$$

Note that in order for $\phi_{E,D}$ to be in $L_2(\mathbb{R})$, the parameter D must be real and as before $E > 0, \hbar_4 > 0$.

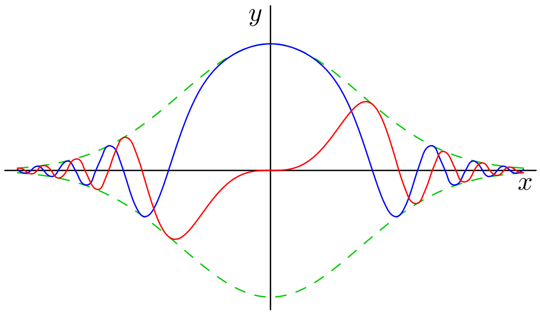

Figure 4.1: Fiducial vector $\phi_{E,D}$ (2.2.31): solid blue and red graphs are its real and imaginary parts respectively. The green dashed envelope is the absolute value $|\phi_{E,D}|$, which coincides with a Gaussian.

Recall that any function f in the image space $L_{\phi_{E,D}}(\mathbb{G}/Z)$ satisfies

$$\mathcal{S}f = 0, \tag{4.1.3}$$

where $\mathcal{S}$ is given by (2.2.39) which we repeat here:

$$\begin{aligned}\mathcal{S} &= (\mathfrak{L}^{X_3})^2 - 2\mathfrak{L}^{X_2}\mathfrak{L}^{X_4} - 8\pi^2 h_2\hbar_4 I \\ &= \partial_{33}^2 + 4\pi \mathrm{i}\hbar_4\partial_2 - 8\pi^2 h_2\hbar_4 I\end{aligned} \tag{4.1.4}$$

and also f satisfies the analyticity condition

$$\mathcal{C}f = 0, \tag{4.1.5}$$

where (see (2.2.30) and the discussion afterwards adapted to the case $D \neq 0$)

$$\begin{aligned} \mathcal{C} &= -\mathrm{i}\mathfrak{L}^{X_1} - \mathrm{i}D\mathfrak{L}^{X_2} + E\mathfrak{L}^{X_3} \\ &= -\mathrm{i}\partial_1 - \mathrm{i}D\partial_2 + (E - \mathrm{i}Dx_1)\partial_3 - \pi\hbar_4(2\mathrm{i}Ex_1 + Dx_1^2)I\,. \end{aligned} \tag{4.1.6}$$

Generally, we may say that the image space $L_{\phi_{E,D}}(\mathbb{G}/Z)$ is annihilated by elements of the left operator ideal $\mathcal{K}$ generated by $\mathcal{C}$ (4.1.6) and $\mathcal{S}$ (4.1.4). Precisely, an element of $\mathcal{K}$ has the form $\mathcal{AC} + \mathcal{BS}$, where $\mathcal{A}, \mathcal{B}$ are any differential operators. Therefore, any two operators of $\mathcal{K}$ have equal restrictions to the space $L_{\phi_{E,D}}(\mathbb{G}/Z)$. Among many such operators we can look for a representative with desired properties which geometrises the dynamic. If the Hamiltonian H (4.0.1) admits, in the sense of Definition 0.0.3, geometric dynamic in Airy-type coherent state $\rho_{h_2\hbar_4}(x_1, x_2, x_3, 0)\phi_{E,D}$ then there is a first-order differential operator, H_r on $L_{\phi_{E,D}}(\mathbb{G}/Z)$ such that $H_r - H$ is in the ideal $\mathcal{K}$ generated by $\mathcal{C}$ (4.1.6) and $\mathcal{S}$ (4.1.4):

$$H_r f(x_1, x_2, x_3) = (H + (A\partial_1 + B\partial_2 + C\partial_3 + K)\mathcal{C} + F\mathcal{S})\, f(x_1, x_2, x_3). \tag{4.1.7}$$

The coefficients A, B, C, K and F are chosen to eliminate all possible second-order derivatives appear in (4.1.7). Note that the possibility of obtaining H_r as a first-order differential operator depends on the values E and D in the fiducial vector (4.1.2) and the respective form of the operator $\mathcal{C}$.

From the relation (4.1.7), a reduction of the order of H is possible when

$$\begin{aligned} A &= \mathrm{i}a_{11}, \\ B &= -\mathrm{i}Da_{11} + \mathrm{i}(a_{21} + a_{12}), \\ C &= a_{11}(2\mathrm{i}x_2 - \mathrm{i}Dx_1 + E) + \mathrm{i}(a_{13} + a_{31}), \\ F &= a_{11}(Dx_1 - x_2 + \mathrm{i}E)^2 - (a_{31} + a_{13})(Dx_1 - x_2 + \mathrm{i}E) + a_{33} \\ K &= -2\pi\hbar_4(a_{13} + a_{31})x_1 + a_{11}\big(- 2\pi\mathrm{i}\hbar_4 Ex_1 - \frac{D}{E} + 5\pi\hbar_4 Dx_1^2 \\ &\quad - 4\pi\hbar_4 x_1 x_2\big) + \frac{a_{21}}{E}. \end{aligned} \tag{4.1.8}$$

The last parameter K is used to get imaginary coefficients in front of ∂_1 and ∂_3 in the Schrödinger equation for H_r. In other words, the choice of K as such will allow to obtain a geometric action in the plane parametrised by points (x_1, x_3). Furthermore, the polynomials (4.1.8) amount to the following restriction on the coefficients a_{jk} of the Hamiltonian (4.0.1):

$$\begin{aligned} a_{12} &= 2Da_{11} - a_{21}, \\ a_{22} &= D^2 a_{11}, \\ a_{23} &= a_{13}D + Da_{31} - a_{32}. \end{aligned} \tag{4.1.9}$$

while, $a_{11}, a_{21}, a_{13}, a_{31}, a_{32}$ and a_{33} are free parameters. Thus, we have obtained the desired classification:

Proposition 4.1.1 *The Hamiltonian* (4.0.1) *can be geometrised over* $\mathbb{G}/Z$ *by Airy-type coherent state from the fiducial vector* (4.1.2) *with* $D \neq 0$ *or by squeezed Gaussian (i.e.* $D = 0$*) if and only if coefficients* a_{jk} *satisfy* (4.1.9).

A direct consequence of this result is the harmonic oscillator Hamiltonian that we discussed in the previous chapter, for $D = 0$ (so, $\phi_{E,D}$ (2.2.31) is Gaussian) and $a_{12} = a_{23} = 0$.

4.2 Example of a geometrisable Hamiltonian and the fiducial vector $\phi_{E,D}$

In light of Proposition 4.1.1, let us consider the matrix (a_{jk}) satisfying condition (4.1.9):

$$(a_{jk}) = \frac{1}{m}\begin{pmatrix} 1 & D & 0 \\ D & D^2 & 0 \\ 0 & 0 & a^2 \end{pmatrix}, \tag{4.2.10}$$

where m is the mass, see the remark below, and $a = m\omega > 0$. Then, the entries of (4.2.10) together with (4.1.9) bring the Hamiltonian (4.0.1) into the following form:

$$H = \frac{1}{m}\left(\left(\mathrm{id}\tilde{\rho}^{X_1}_{\hbar_4}\right) + D\left(\mathrm{id}\tilde{\rho}^{X_2}_{\hbar_4}\right)\right)^2 + \frac{a^2}{m}(\mathrm{id}\tilde{\rho}^{X_3}_{\hbar_4})^2. \tag{4.2.11}$$

It is the Weyl quantisation of the classical Hamiltonian

$$h = \tfrac{1}{m}(p + Dq^2)^2 + \tfrac{a^2}{m}q^2\,. \tag{4.2.12}$$

Explicitly, we have

$$H = -\frac{1}{m}\Big(\partial^2_{11} + (x_2^2 + a^2)\partial^2_{33} + 2x_2\partial^2_{13} + D^2\partial^2_{22} + 2D\partial^2_{12} + 2Dx_2\partial^2_{32} \\ - 4\pi\mathrm{i}\hbar_4 x_3\partial_1 - 4\pi\mathrm{i}\hbar_4 Dx_3\partial_2 - (4\pi\mathrm{i}\hbar_4 x_2x_3 - D)\partial_3 - (4\pi^2\hbar_4^2 x_3^2 + 2\pi\mathrm{i}\hbar_4 x_2)I\Big).$$

Remark 4.2.1 *According to Remark 1.2.2 and noting that D is measured in unit LM^2/T^2, it can be checked that the Hamiltonian H* (4.2.11) *has the physical dimension $M\frac{L^2}{T^2}$. This also justifies our choice of the parameter a_{11} being $\frac{1}{m}$.*

A reduction of the order of H is achieved by a direct substitution of (4.2.10) into (4.1.8) which results in:

$$A = \frac{\mathrm{i}}{m}, \qquad B = \frac{\mathrm{i}D}{m}, \qquad C = \frac{1}{m}\left(2\mathrm{i}x_2 + E - \mathrm{i}Dx_1\right),$$
$$F = \frac{1}{m}\left(Dx_1 - x_2 + \mathrm{i}E\right)^2 + \frac{a^2}{m}, \qquad K = \frac{1}{m}\left(-2\mathrm{i}\pi\hbar_4 Ex_1 + 5\pi\hbar_4 Dx_1^2 - 4\pi\hbar_4 x_1x_2\right).$$

Subsequently, the reduced Hamiltonian H_r (4.1.7) is

$$\begin{aligned}
H_r = &\frac{1}{m}\left(4\pi\mathrm{i}\hbar_4 x_3 - 6\pi\mathrm{i}\hbar_4 D x_1^2 + 4\pi\mathrm{i}\hbar_4 x_1 x_2\right)\partial_1 \\
&+ \frac{1}{m}\Bigg(4\pi\mathrm{i}\hbar_4 D x_3 - 2\pi\mathrm{i}\hbar_4 D^2 x_1^2 + 8\pi\hbar_4 E x_2 - 4\pi\mathrm{i}\hbar_4 D x_1 x_2 + 4\pi\mathrm{i}\hbar_4 x_2^2 \\
&\quad - 4\pi\mathrm{i}\hbar_4 E^2 - 8\pi\hbar_4 D E x_1 + 4\pi\mathrm{i}\hbar_4 a^2\Bigg)\partial_2 \\
&+ \frac{1}{m}\left(2\pi\mathrm{i}\hbar_4 D x_1^2 x_2 - 4\pi\mathrm{i}\hbar_4 E^2 x_1 - 4\pi\mathrm{i}\hbar_4 D^2 x_1^3 + 4\pi\mathrm{i}\hbar_4 x_2 x_3\right)\partial_3 \qquad (4.2.13)\\
&+ \frac{\pi\hbar_4}{m}\Bigg(-8\pi h_2 (D x_1 - x_2 + iE)^2 - 5\pi\hbar_4 D^2 x_1^4 + 4\pi\hbar_4 x_3^2 \\
&\quad + 8\mathrm{i}\pi\hbar_4 E x_1^2 x_2 - 4\pi\hbar_4 E^2 x_1^2 + 2\mathrm{i}x_2 - 2\mathrm{i}D x_1 \\
&\quad - 8\mathrm{i}\pi\hbar_4 E D x_1^3 + 4\pi\hbar_4 D x_1^3 x_2 + 2E - 8\pi h_2 a^2\Bigg) I.
\end{aligned}$$

4.2.1 Solving the geometrised Schrödinger equation

For a function f in the image space $L_{\phi_{E,D}}(\mathbb{G}/Z)$, the Schrödinger equation

$$\mathrm{i}\hbar_4 \partial_t f(t, x_1, x_2, x_3) - H f(t, x_1, x_2, x_3) = 0 \qquad (4.2.14)$$

becomes equivalent to the first-order PDE

$$\mathrm{i}\hbar_4 \partial_t f(t, x_1, x_2, x_3) - H_r f(t, x_1, x_2, x_3) = 0. \qquad (4.2.15)$$

We proceed to solve (4.2.14) in the same manner as we treated the harmonic oscillator in the previous chapter . Namely, the first step is to solve the analyticity condition (4.1.5) for $f(t, x_1, x_2, x_3)$ (using the method of characteristics). Indeed, the following formula represents a general solution of such an equation:

$$\begin{aligned}
f(t, x_1, x_2, x_3) = &\exp\left(\pi\mathrm{i}\hbar_4\left(\mathrm{i}E x_1^2 + \frac{D}{3}x_1^3\right)\right) \qquad (4.2.16)\\
&\times \phi\left(t, D x_1^2 + 2\mathrm{i}E x_1 - 2x_3, D x_1 - x_2 + \mathrm{i}E\right),
\end{aligned}$$

for all t.

Then, the substitution of (4.2.16) into the reduced equation (4.2.15) produces the following significantly simplified equation

$$\begin{aligned}\Big(&\mathrm{i}\hbar_4 m\partial_t + 4\pi\mathrm{i}\hbar_4 u_1u_2\partial_1 + 4\pi\mathrm{i}\hbar_4(u_2^2+a^2)\partial_2\\ &+\left(8\pi^2h_2\hbar_4(u_2^2+a^2)+2\pi\mathrm{i}\hbar_4u_2-\pi^2\hbar_4^2u_1^2\right)\Big)\phi(t,u_1,u_2)=0,\end{aligned} \tag{4.2.17}$$

where

$$u_1 = Dx_1^2 + 2\mathrm{i}Ex_1 - 2x_3, \qquad u_2 = Dx_1 - x_2 + \mathrm{i}E. \tag{4.2.18}$$

Equation (4.2.17) is a first-order PDE whose generic solution is

$$\begin{aligned}\phi(t,u_1,u_2) = {}&\frac{1}{\sqrt{u_2-\mathrm{i}a}}\exp\left(\frac{2a\pi\mathrm{i}}{m}t + 2\pi\mathrm{i}h_2u_2 - \frac{\pi\mathrm{i}\hbar_4}{4}\frac{u_1^2}{u_2-\mathrm{i}a}\right)\\ &\times\psi\left(\mathrm{e}^{\frac{4a\pi\mathrm{i}}{m}t}\frac{u_1}{u_2-\mathrm{i}a},\mathrm{e}^{\frac{8a\pi\mathrm{i}}{m}t}\frac{u_2+\mathrm{i}a}{u_2-\mathrm{i}a}\right).\end{aligned} \tag{4.2.19}$$

At this point, (4.2.16) with ϕ from (4.2.19) solves (4.2.15) for any ψ in (4.2.19) and obviously satisfies the analyticity condition (4.1.5). The function $f(t,u_1,u_2)$ (4.2.16) will also be a generic solution of (4.2.14) if it further satisfies the structural condition (4.1.3). This requirement leads to the following equation:

$$\partial_\eta\psi(\xi,\eta) = \frac{1}{2\pi\hbar_4 a}\partial^2_{\xi\xi}\psi(\xi,\eta), \tag{4.2.20}$$

where

$$\xi = \frac{u_1}{u_2-\mathrm{i}a}, \qquad \eta = \frac{u_2+\mathrm{i}a}{u_2-\mathrm{i}a}. \tag{4.2.21}$$

A generic solution of (4.2.20) is given by the integral:

$$\psi(\xi,\eta) = \left(\frac{a\hbar_4}{2\eta}\right)^{1/2}\int_{\mathbb{R}} k(s)\,\mathrm{e}^{\frac{1}{2}\pi\hbar_4 a\frac{(\xi-s)^2}{\eta}}\,\mathrm{d}s, \tag{4.2.22}$$

where k is determined by initial conditions. The function f (4.2.16) with ϕ (4.2.19) and ψ (4.2.22) represents a generic solution of (4.2.14).

Remark 4.2.2 *Regarding the above integral convergence, the main point is to control the behaviour of the exponential factor at infinity. This requires a certain range for the* *parameter E. Indeed,*

$$\begin{aligned}\left|\mathrm{e}^{-\frac{1}{2}\pi\hbar_4 a\frac{(\xi-s)^2}{\eta}}\right| &= \mathrm{e}^{\Re\left\{-\frac{1}{2}\pi\hbar_4 a\left(\frac{u_1}{u_2-\mathrm{i}a}-s\right)^2\frac{u_2-\mathrm{i}a}{u_2+\mathrm{i}a}\right\}}\\ &= \mathrm{e}^{\Re\left\{-\frac{1}{2}\pi\hbar_4 a\left(\frac{u_1^2}{(u_2-\mathrm{i}a)^2}-\frac{2u_1}{u_2-\mathrm{i}a}s+s^2\right)\frac{u_2-\mathrm{i}a}{u_2+\mathrm{i}a}\right\}}\\ &= \mathrm{e}^{\Re\left\{-\frac{1}{2}\pi\hbar_4 a\left(\frac{u_1^2}{(u_2-\mathrm{i}a)^2}\right)\frac{u_2-\mathrm{i}a}{u_2+\mathrm{i}a}\right\}}\\ &\quad\times \mathrm{e}^{\Re\left\{-\frac{1}{2}\pi\hbar_4 a\left(-\frac{2u_1}{u_2+\mathrm{i}a}s+\frac{u_2-\mathrm{i}a}{u_2+\mathrm{i}a}s^2\right)\right\}}.\end{aligned}$$

The first factor does not contribute to convergence since it is independent of s. While, for the second let us rewrite it in terms of x_j coordinates:

$$\begin{aligned}&\mathrm{e}^{\Re\left\{-\frac{1}{2}\pi\hbar_4 a\left(-\frac{2u_1}{u_2+\mathrm{i}a}s+\frac{u_2-\mathrm{i}a}{u_2+\mathrm{i}a}s^2\right)\right\}}\\ &= \mathrm{e}^{\Re\left\{-\frac{1}{2}\pi\hbar_4 a\left(\frac{-2(Dx_1^2+2\mathrm{i}Ex_1-2x_3)}{Dx_1-x_2+\mathrm{i}(E+a)}s+\frac{Dx_1-x_2+\mathrm{i}(E-a)}{Dx_1-x_2+\mathrm{i}(E+a)}s^2\right)\right\}}\\ &= \mathrm{e}^{-\frac{1}{2}\pi\hbar_4 a\left(\frac{-2(Dx_1-x_2)(Dx_1^2-2x_3)-4(E+a)Ex_1}{(Dx_1-x_2)^2+(E+a)^2}s+\frac{(Dx_1-x_2)^2+E^2-a^2}{(Dx_1-x_2)^2+(E+a)^2}s^2\right)}.\end{aligned}$$

This is dominated by a Gaussian-like decay at infinity, if

$$(Dx_1-x_2)^2+E^2-a^2>0. \tag{4.2.23}$$

This inequality is maintained whenever $E\geq a=m\omega$.

The Hamiltonians (4.2.12) is similar to a charged particle in a magnetic field. In such a case projections of classical dynamics to the configuration space coincide with the dynamics with $D=0$ (no field). However, classical trajectories in the phase space for $D\neq 0$ are significantly different from the rigid rotation of the phase space familiar from the harmonic oscillator case, see Fig. 4.2.

Quantisation of Hamiltonian (4.2.12) may be relevant for paraxial optics [76, 77]. The parameter D in the fiducial vector $\phi_{E,D}$ (4.1.2) is dictated by the respective Hamiltonian, while the squeezing parameter E is not fixed. However, the convergence of the integral (4.2.22) requires that $E\geq a=m\omega$ as already shown in the remark above.

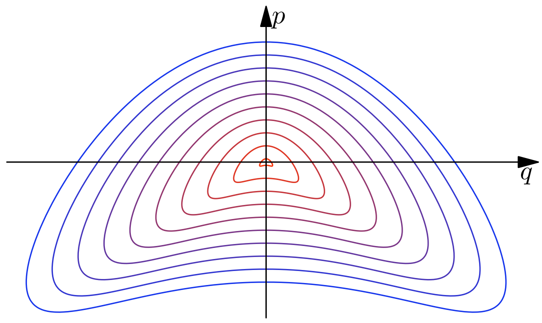

Figure 4.2: Classical orbits in the phase space of the Hamiltonian (4.2.12) .

4.3 Further extensions

The present work provides a further example of numerous cases [32, 45, 47, 48, 87] when the coherent state transform is meaningful and useful beyond the traditional setup of square-integrable representations modulo a subgroup H [3, Ch. 8], see Remark 2.1.7. Specifically, the coherent states parametrised by points of the homogeneous space $\mathbb{G}/H$ are not sufficient to accommodate the dynamics (3.2.30) and (4.2.19).

The method used in this thesis and the construction of coherent states from the group representations is fully determined by a choice of a group G, its subgroup H, a representation ρ and a fiducial vector ϕ. Thus, varying some of these components we obtain different geometrisable Hamiltonians in the sense of Definition 0.0.3. Therefore, our work can be naturally extended as follows.

- In the context of the present group $\mathbb{G}$ and the fiducial vector $\phi_{E,D}$ (4.1.2) one may look for Hamiltonians beyond the quadratic forms (4.0.1). Even in this case, the nature of the ideal algebra of elements $\mathcal{AC} + \mathcal{BS}$ (see the paragraph after (4.1.6)) still suggests a possibility for order-reduction of the respective Hamiltonians.
- Corollary 2.1.15 in such a general version offers a concrete base for a wide applicability. Still in the context of the group $\mathbb{G}$ one may also look for

another fiducial vectors, which will be null-solutions to more complicated analytic conditions than (2.2.30).

- Many different groups can be considered instead of $\mathbb{G}$ with the Schrödinger group [76, 77] to be a very attractive choice. Indeed, a more refined coherent state transform can be achieved by the Schrödinger group $\mathbb{S}$ introduced in Section 1.3 because it is the largest natural group for describing coherent states for the harmonic oscillator, see [1] [25, Ch. 5]. However, as was mentioned at the end of Section 1.3, the smaller group $\mathbb{G}$ has more representations than the larger Schrödinger group. Thus, advantages of each group for geometric description of quantum dynamics needs to be carefully investigated.

Finally, our technique may be extended to differential equations in Banach spaces through the respective adaptation of covariant transforms, see [23, 24, 45].

Appendices

A Algebraic properties of ladder operators

Here we briefly discuss the ladder operators technique which is useful for the study of the eigenvalues and the corresponding eigenvectors of a harmonic oscillator Hamiltonian based just on " the canonical commutation relation"

$$[Q, P] = \mathrm{i}\hbar I. \tag{A.24}$$

Here Q and P are the usual position and momentum operators restricted to the Schwartz space. We only need to use the fact that Q and P are essentially self-adjoint in this case. In fact, all properties we are deriving below neither depend on what a Hilbert space being considered nor the explicit expressions of Q and P. We remark that for general self-adjoint operators, one can use *Weyl relation* [33, § 11.2][66, § 8.5] to avoid the questions about the domain of definition of the relative operators that satisfy (A.24).

We introduce a pair of operators:

$$a^- = \frac{1}{\sqrt{2\hbar}}(\lambda Q + \frac{\mathrm{i}}{\lambda}P), \qquad a^+ = \frac{1}{\sqrt{2\hbar}}(\lambda Q - \frac{\mathrm{i}}{\lambda}P), \tag{A.25}$$

where λ is a positive real parameter. In particular, for $\lambda = \sqrt{m\omega}$ we have the harmonic oscillator Hamiltonian; $H = \frac{1}{2m}P^2 + \frac{m\omega^2}{2}Q^2 = \frac{\hbar\omega}{2}(a^+a^- + a^-a^+)$.

The operators a^- and a^+ are called the *annihilation* operator and the *creation* operator, respectively (together called *ladder operators*.) These operators are conjugate,

$$(a^-)^* = a^+, \tag{A.26}$$

where '*' denotes the adjoint of an operator on $L_2(\mathbb{R})$. The relation $[Q, P] = \mathrm{i}\hbar I$ implies that,

$$[a^-, a^+] = a^-a^+ - a^+a^- = I. \tag{A.27}$$

Let

$$N = a^+a^-.$$

Then, the relation (A.26) results in

$$\langle N\phi, \phi\rangle = \langle a^+a^-\phi, \phi\rangle = \langle a^-\phi, a^-\phi\rangle = \|a^-\phi\|^2 \geq 0. \tag{A.28}$$

Thus, the operator N is positive and hence its eigenvalues (when exist) are non-negative [22, § 4.9]. Furthermore, from (A.27) we can see that

$$[N, a^-] = Na^- - a^-N = (a^+a^-)a^- - a^-(a^+a^-) = (a^+a^- - a^-a^+)a^- = -a^-. \tag{A.29}$$

Similarly,

$$[N, a^+] = a^+. \tag{A.30}$$

A very important consequence of (A.29) is that if ϕ is an eigenvector of N with eigenvalue k, that is, $N\phi = k\phi$, then

$$N(a^-\phi) \overset{\text{(A.29)}}{=} a^-(N - I)\phi = (k-1)a^-\phi,$$

which means that if $a^-\phi \neq 0$, then this is an eigenvector of N with the eigenvalue $k - 1$. Repeating the above calculation for $\phi_1 := a^-\phi$ and so on, we get

$$N((a^-)^m\phi) = (k - m)\phi_m\,,$$

where $\phi_m = (a^-)^m\phi$. This process of applying the operator a^- repeatedly to ϕ has to terminate since otherwise one must pass on to negative eigenvalues of N which contradicts the property of the operator N being positive. Thus, there must exist a certain non-negative integer m_0 such that $(a^-)^{m_0}\phi = 0$. That is, $k - m_0 = 0$, and hence $k = m_0$, which implies that the spectrum of N consists of non-negative integers, in other words, the spectrum of N is discrete.

The vector $\phi_{m_0} := \phi_0$ is called *vacuum vector* which is defined by

$$a^-\phi_0 = 0. \tag{A.31}$$

It is an eigenvector of N with the zero eigenvalue.

On the other hand, the relation (A.30) implies that

$$N(a^+\phi_0) \overset{\text{(A.30)}}{=} a^+(N + I)\phi_0 = a^+\phi_0.$$

That is, $a^+\phi$ is also an eigenvector of N and the corresponding eigenvalue is 1. Similarly, $(a^+)^2\phi_0$ is another eigenvector of N with eigenvalue 2. Hence, the set of eigenvectors of N are of the form

$$\phi_n := \frac{1}{\sqrt{n!}}(a^+)^n\phi_0, \qquad n = 0, 1, 2, \ldots. \tag{A.32}$$

Moreover, it can be shown by induction that

$$[a^-, (a^+)^n] = n(a^+)^{n-1}. \tag{A.33}$$

By virtue of relation (A.33) we can see that

$$\begin{aligned} a^-\phi_n &= \frac{1}{\sqrt{n!}}(a^-(a^+)^n - (a^+)^n a^-)\phi_0 \\ &= \frac{1}{\sqrt{n!}}[a^-, (a^+)^n]\phi_0 \\ &\overset{\text{(A.33)}}{=} \frac{\sqrt{n}}{\sqrt{(n-1)!}}(a^+)^{n-1}\phi_0. \\ &= \sqrt{n}\phi_{n-1}. \end{aligned} \tag{A.34}$$

Now,

$$\begin{aligned}\|\phi_n\|^2 &= \langle \phi_n, \phi_n \rangle \\ &= \left\langle \frac{1}{\sqrt{n}} a^+ \phi_{n-1}, \phi_n \right\rangle \\ &= \frac{1}{\sqrt{n}} \langle \phi_{n-1}, a^- \phi_n \rangle \\ &\overset{(A.34)}{=} \langle \phi_{n-1}, \phi_{n-1} \rangle .\end{aligned}$$

Hence, once the vacuum vector ϕ_0 is normalised then so are all ϕ_n. For orthogonality of $\{\phi_n\}_{n=0}^{\infty}$ note that for any eigenvectors ϕ_m, ϕ_n of the operator N with eigenvalues m, n we have

$$\langle N\phi_m, \phi_n \rangle = m \langle \phi_m, \phi_n \rangle .$$

But, we also have

$$\langle N\phi_m, \phi_n \rangle = \langle \phi_m, N^* \phi_n \rangle = \langle \phi_m, N\phi_n \rangle = n \langle \phi_m, \phi_n \rangle .$$

Thus,

$$(m - n) \langle \phi_m, \phi_n \rangle = 0.$$

From this we see that if $m \neq n$ then $\langle \phi_m, \phi_n \rangle = 0$.

Particularly, for the normalised eigenvectors ϕ_n we have

$$a^- \phi_n = \sqrt{n} \phi_{n-1}; \tag{A.35}$$

$$a^+ \phi_n = \sqrt{n+1} \phi_{n+1} \tag{A.36}$$

which explain the name *ladder* operators. This implies that

$$N\phi_n := a^+ a^- \phi_n = n\phi_n. \tag{A.37}$$

Since $H = \hbar\omega a^+ a^- + \frac{\hbar\omega}{2} I$, the relation (A.37) implies that $H\phi_n = \omega\hbar(n + \frac{1}{2})\phi_n$ from which one determines the spectrum of H. For explicit expression of the vacuum and all ϕ_n see Section 1.4, particularly, Subsection 1.4.3.

B Induced representations of nilpotent Lie groups

B.1 The action of a Lie group on a homogeneous space

Let G be a Lie group and H be a closed subgroup of G, a *homogeneous space* X is defined as the space of left (right) cosets of the subgroup H [27, 44, 49]. That is, $X = G/H = \{gH : \ g \in G\}$, where $gH = \{gh : h \in H\}$. The respective equivalence relation is given as $g' \sim g$ if and only if $g' = gh$, for $h \in H$.

Definition B.1 *Let G and H be as above and let p be the natural projection $\mathsf{p} : G \to G/H$ and $\mathsf{s} : G/H \to G$ be a section, that is, s is a right inverse of p. Then, the left action of the group G on the homogeneous space $X = G/H$ is given by*

$$g \cdot x = \mathsf{p}(g\mathsf{s}(x)), \quad g \in G, \ \ \textit{and} \ \ x \in X, \tag{B.38}$$

where $g\mathsf{s}(x)$ is calculated using the respective group law.

Any element $g \in G$ can be uniquely written as [43, § 13]

$$g = \mathsf{s}(x)h, \quad x = \mathsf{p}(g) \ \text{ and } \ h \in H. \tag{B.39}$$

To see this, let $x = gH$, the natural projection p maps each element $g \in G$ to its equivalence class, $\mathsf{p}(g) = gH$. Since $\mathsf{s}(x) \in G$, there must exist $g' \in G$ such that $\mathsf{s}(x) \sim g'$ which means that

$$\mathsf{s}(x) = g'h_1, \qquad h_1 \in H. \tag{B.40}$$

But, $g' = \mathsf{s}(x)h_1^{-1}$ implies that $g'H = \mathsf{p}(g') = \mathsf{p}(\mathsf{s}(x)h_1^{-1}) = \mathsf{p}(\mathsf{s}(x)) = x$. Thus, $g' \sim g$ and this means $g' = gh_2$, $h_2 \in H$. Now substituting into (B.40) gives

$$g = \mathsf{s}(x)h, \tag{B.41}$$

where $h = (h_2 h_1)^{-1}$ which is an element of H.

In the following example we compute the action $g^{-1} \cdot x$ as this will be used in connection with the induced representations of the group $\mathbb{G}$ below.

Example B.2 *Consider the shear group* $\mathbb{G}$:

- *for the centre* $Z = \{(0,0,0,x_4) \in \mathbb{G} : x_4 \in \mathbb{R}\}$, *observe that* $\mathbb{G} \ni (x_1,x_2,x_3,x_4) = (x_1,x_2,x_3,0)(0,0,0,x_4)$, *this defines maps, according to* (B.39),

$$\mathsf{p} : \mathbb{G} \to \mathbb{G}/Z; \quad \mathsf{p} : (x_1,x_2,x_3,x_4) \mapsto (x_1,x_2,x_3).$$
$$\mathsf{s} : \mathbb{G}/Z \to \mathbb{G}; \quad \mathsf{s} : (x_1,x_2,x_3) \mapsto (x_1,x_2,x_3,0).$$

  Thus, the action of $\mathbb{G}$ *on* $\mathbb{G}/Z \sim \mathbb{R}^3$ *is*

$$(x_1,x_2,x_3,x_4)^{-1} : (x_1',x_2',x_3') \mapsto (x_1'-x_1, x_2'-x_2, x_3'-x_3-x_1x_2'+x_1x_2).$$

- *For the subgroup* $H_1 = \{(0,x_2,x_3,x_4) \in \mathbb{G} : x_j \in \mathbb{R}\}$ *it can be checked that* $(x_1,x_2,x_3,x_4) = (x_1,0,0,0)(0,x_2,x_3-x_1x_2,x_4-x_1x_3+\frac{1}{2}x_2x_1^2)$ *which defines*

$$\mathsf{p} : \mathbb{G} \to \mathbb{G}/H_1; \quad \mathsf{p} : (x_1,x_2,x_3,x_4) \mapsto x_1.$$
$$\mathsf{s} : \mathbb{G}/H_1 \to \mathbb{G}; \quad \mathsf{s} : x \mapsto (x,0,0,0).$$

  So, the action of $\mathbb{G}$ *on* $\mathbb{G}/H_1 \sim \mathbb{R}$ *is*

$$(x_1,x_2,x_3,x_4)^{-1} : x' \mapsto x'-x_1.$$

B.2 Induced representations

The induced representations (in the sense of Mackey) is one of the pillars of the theory of representation [27, 40, 43, 44] with strong connections to physics [10, 57, 58, 59] and further research potential [50, 51, 53]. We briefly outline the method aiming mainly to build representations of the group $\mathbb{G}$.

Let G be a nilpotent Lie group and H be a closed abelian subgroup of G. According to Kirillov [44], unitary representations of G are induced from one-dimensional representations (characters) of its subgroups. Explicitly, for a unitary character χ of the subgroup $H \subset G$, that is, $\chi(h) \in \mathbb{C}$ and for $h, h' \in H$; $\chi(hh') = \chi(h)\chi(h')$ and $|\chi(h)| = 1$, we consider the space $L_2^\chi(G)$ of square-integrable functions F defined on the group G and have the (right) H-covariance property

$$F(gh) = \overline{\chi}(h)F(g), \quad g \in G,\ h \in H, \tag{B.42}$$

where the bar sign indicates the complex conjugation. The space $L_2^\chi(G)$ is invariant under the left G-shifts:

$$\Lambda(g) :\ F(g') \mapsto F(g^{-1}g'), \qquad g, g' \in G. \tag{B.43}$$

The restriction of the left G-shifts (B.43) to the space $L_2^\chi(G)$ is called *induced representation* (from the character χ).

An equivalent realisation of the above induced representation conveniently defined on a smaller function space. Consider the natural projection $\mathsf{p} : G \to G/H$ and a right inverse (section) $\mathsf{s} : G/H \to G$. Different choices of this section lead to equivalent representations below [3, Ch. 4] . As mentioned above, any $g \in G$ has a unique decomposition of the form $g = \mathsf{s}(x)h$, where $x = \mathsf{p}(g) \in G/H$ and $h \in H$. Note that G/H is a left homogeneous space with the G-action defined as in (B.38).

For a character χ of H we can define a *lifting* $\mathcal{L}_\chi : L_2(G/H) \to L_2^\chi(G)$ as follows:

$$[\mathcal{L}_\chi f](g) = \chi(\mathsf{r}(g))f(\mathsf{p}(g)) \qquad \text{where} \quad f(x) \in L_2(G/H) \tag{B.44}$$

where the map $\mathsf{r} : G \to H$ is given through p and s:

$$\mathsf{r}(g) = (\mathsf{s}(x))^{-1}g, \qquad \text{where} \quad x = \mathsf{p}(g) \in G/H. \tag{B.45}$$

Note that the map r gives the solution of the so-called *master equation* [44, Appendix V]:

$$g^{-1}\mathsf{s}(x) = \mathsf{s}(g^{-1}\cdot x)h(x,g) \tag{B.46}$$

in the form $h(x,g) = \mathsf{r}(g^{-1}\mathsf{s}(x))$.

The image space of the lifting $\mathcal{L}_\chi$ satisfies (B.42) and is invariant under left shifts (B.43). We also define the *pulling* $\mathcal{P} : L_2^\chi(G) \to L_2(G/H)$, which is a left inverse of the lifting and explicitly can be given, for example, by $[\mathcal{P}F](x) = F(\mathsf{s}(x))$. Then the induced representation on $L_2(G/H)$ is generated by the formula $\rho_\chi(g) = \mathcal{P} \circ \Lambda(g) \circ \mathcal{L}_\chi$. This representation has the realisation ρ_χ in the space $L_2(G/H)$ by the formula [43, § 13.2.(7)–(9)]:

$$[\rho_\chi(g)f](x) = \bar{\chi}(\mathsf{r}(g^{-1}\mathsf{s}(x)))f(g^{-1}\cdot x)\,, \tag{B.47}$$

where $g \in G$, $x \in G/H$, $h \in H$ and $\mathsf{r} : G \to H$, $\mathsf{s} : G/H \to G$ are maps defined above and "$\cdot$" denotes the action (B.38) of G on G/H.

Since $\bar{\chi}$ is a unimodular multiplier and G/H posses a left invariant Haar measure, because G is nilpotent, formula (B.47) defines a unitary representation of the group G on $L_2(G/H)$, where $L_2(G/H)$ is the Lebesgue space of square-integrable functions on $G/H \sim \mathbb{R}^n$ (n is the dimension of G/H) with the inner product

$$\langle f, g\rangle = \int_{G/H} f(x)\overline{g(x)}\,\mathrm{d}x,$$

where $\mathrm{d}x$ is the Lebesgue measure on $\mathbb{R}^n$.

The map r enjoys the property:

$$\mathsf{r}((g_1g_2)^{-1}\mathsf{s}(x)) = \mathsf{r}(g_2^{-1}\mathsf{s}(g_1^{-1}\cdot x))\mathsf{r}(g_1^{-1}\mathsf{s}(x)). \tag{B.48}$$

This property of the map r is necessary for the following to hold

$$\rho_\chi(g_1)\rho_\chi(g_2) = \rho_\chi(g_1g_2).$$

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Note: the numbers which appear next each reference are to indicate the page numbers where such a reference is mentioned within the text.

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Index

Analyticity of coherent state transform, 44
Annihilation operator, 25, 83

Canonical coherent states, 3, 26
Canonical commutation relation, 22, 83
Casimir operator, 7, 49
Character of a subgroup, 11, 34, 88
Coherent state transform, 34, 43
Coherent states, 23
Creation operator, 25, 83

Derived representation, 14
Displacement operator, 3
Dynamics in canonical coherent states, 28
Dynamics of a harmonic oscillator in configuration space, 28
Dynamics of a harmonic oscillator in FSB space, 30

Fock-Segal-Bargmann (FSB) space, 29, 44
Fock-Segal-Bargmann transform, 45, 48

Harmonic oscillator Hamiltonian, 25, 29
Heisenberg group, 10, 35, 54
Heisenberg-Kennard uncertainty relation, 23
Heisenberg-Weyl algebra, 9, 44
Hermite polynomials, 26, 68
Homogeneous space, 5, 86

Induced coherent state transform, 35
Induced representation, 35, 89

Ladder operators, 66, 86
Lie derivative, 41
Lifting, 89

Minimal uncertainty state, 23

Natural projection, 87

Physical units, 15
Pulling, 89

Quantum observable, 21

Schrödinger equation, 27
Schrödinger group, 16
Schrödinger representation, 35
Section, 5, 87
Semidirect product, 16
Shear transform, 18
Square-integrable representations, 34

Squeezed state, 24

State, 21