Multiplication Rings and Multiplication Modules

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This work is dedicated to my mother's soul, to my father, to my wife (Ohoud) and my children (Aseel and Almohra) for their endless love, support, and encouragement.

Abstract

This research is devoted to studying a particular class of modules and a specific class of rings that are called *multiplication modules* and *multiplication rings*. Let R be a ring. A left R-module M is called a multiplication module if for every submodule N of M, $N = IM$ for some ideal I of R. If M is a left ideal of R then M is called a *left multiplication ideal*. A ring R is called a *left (resp. right) multiplication ring* if for each pair of ideals I and J of R where $J \subseteq I$, there exists an ideal J' of R such that $J = J'I$ (resp, $J = IJ'$). R is a multiplication ring if it is a left and right multiplication ring.

The thesis is a collection of three papers. Each paper is a chapter in the thesis which investigates an aspect of such classes of modules and rings. The first paper is published in the Journal 'Communications in Algebra' [3], and we have already submitted the second Paper. The preliminaries of all these papers are put together in a joint section.

In the first paper [3], we assume that all rings are commutative. A characterization of multiplication rings with finitely many minimal prime ideals is given: *Each such ring is a finite direct product of rings* $\prod_{i=1}^{n} D_i$ *where* D_i *is either a Dedekind domain or an Artinian, local, principal ideal ring.* In particular, each such ring is a Noetherian ring. As corollaries, subclasses of such rings are described (semiprime, Artinian, semiprime and Artinian, local, domain, etc.).

In the second paper [4], we study multiplication modules over (not necessarily commutative) rings. Several criteria are given for a direct sum of modules to be a multiplication module. For a multiplication noncommutative ring, the following facts are proved: the commutativity of the product of prime ideals and the commutativity of the product of a prime ideal and an ideal that is not contained in it. Moreover, the endomorphisms ring of a multiplication module is studied, and new classes of modules are introduced and studied: *epimorphic module*, *monomorphic*

module, and *automorphic module*.

In the third paper [5], we study multiplication modules over commutative rings in terms of the ideal $\theta(M)$. Several characteristics and properties of a (faithful) multiplication module are given. We prove that $\theta(IM) = \theta(I)\theta(M)$ if M is a faithful multiplication R-module and I is a multiplication ideal of R with zero annihilator. In addition, two cancellation laws are presented. Furthermore, we study the product of two submodules of a (faithful) multiplication module. Several properties and applications of such operation are presented. We present a version of *Chinese Remainder Theorem*, and a version of *Krull's Intersection Theorem*, seen through the paradigm of multiplication modules. Moreover, some cases of embedding a multiplication R-module into R are given.

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Contents

Chapter 1

Introduction

In this chapter, we present an overview of the main results of this work. Any missing terminology is given in the next four chapters.

1.1 Notation

Throughout this thesis, unless otherwise stated, all rings contain 1, and a module means a left module. '$\subset$' means proper inclusion, and $\mathrm{card}(I)$ denotes the cardinality of the set I. $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ stand for the set of natural numbers, the set of integers, and the set of real numbers, respectively. $\mathrm{M_n}(R)$ is the ring of $(n \times n)$ matrices over the ring R, and $\mathrm{E}_{ij} \in \mathrm{M_n}(R)$ are the matrix units.

An R-module is called a *cyclic* if it is 1-generated. For an R-module M, $\mathrm{Cyc}_R(M)$ indicates the set of all its cyclic submodules, and we denote by $\mathrm{ann}_R(M)$ its annihilator. M is called *faithful* if $\mathrm{ann}_R(M) = 0$. For a submodule N of M, the set $[N : M] := \mathrm{ann}_R(M/N) = \{r \in R \,|\, rM \subseteq N\}$ is an ideal of the ring R that contains the *annihilator* of the module M, $\mathrm{ann}_R(M) = [0 : M]$. The ideal $\theta(M) := \sum_{C \in \mathrm{Cyc}_R(M)} [C : M]$ is the companion ideal of an R-module M. Clearly, $\theta(M)$ is an ideal of R, and $\mathrm{ann}_R(M) \subseteq \theta(M)$. If M is an ideal of R then $M \subseteq \theta(M)$. A submodule N of an R-module M is called a *direct summand* of M if $M = N \oplus N'$ for some submodule N' of M. We use $\mathrm{Sub_R}(M)$ to denote the set of all submodules of M, and $\mathrm{Sub_R^{\oplus}}(M)$ to denote the set of all direct summand submodules of M. The set $(\mathrm{Sub}_R(M), \subseteq)$ is a *partially ordered set* (a *poset*, for short). $\mathrm{Tor}(M)$ is the set of all elements of M that are annihilated by regular elements (non-zero-divisors) of R. Clearly, $\mathrm{Tor}(M)$ is a submodule of M. M is called a *torsion-free* module if $\mathrm{Tor}(M) = \{0\}$. $\mathrm{Hom}_R(M, N)$ is the set of all R-module homomorphisms from the R-module M to the R-module N. $\mathrm{End}_R(M)$ is the ring of endomorphisms of an R-module M. $\mathrm{Epi}_R(M)$ and $\mathrm{Mon}_R(M)$ denote the set of all epimorphisms and monomorphisms from M to M, respectively. Obviously, $\mathrm{Epi}_R(M) \subseteq \mathrm{End}_R(M)$ and $\mathrm{Mon}_R(M) \subseteq \mathrm{End}_R(M)$. The group of automorphisms of an R-module M is $\mathrm{Aut}_R(M)$. Clearly, $\mathrm{Aut}_R(M) \subseteq \mathrm{Epi}_R(M)$ and $\mathrm{Aut}_R(M) \subseteq \mathrm{Mon}_R(M)$. For an R-module homomorphism $f : M \to N$, $\ker(f)$ denotes the kernel of f, $\mathrm{im}(f)$ denotes the image of f, and the cokernel of is denoted by $\mathrm{coker}(f)$.

A proper ideal of a ring R is an ideal that is distinct from R. $I \triangleright R$ means that I is an ideal of the ring R. Let $\mathcal{I}(R)$ be the set of ideals of the ring R. Then the set $(\mathcal{I}(R), \subseteq)$ is a poset. A ring R satisfies the a.c.c. if it satisfies the ascending chain condition of the ideals of R, and R satisfies d.c.c. if it satisfies the descending chain condition of the ideals of R. $\mathcal{C}_R$ is the set of all regular elements in R. A nonzero ring R is a domain if it is a commutative ring in which the zero ideal is a prime ideal, i.e., $\mathcal{C}_R = R \setminus \{0\}$. PID indicates the *principal ideals domain*, a domain where all its ideals are cyclic. An ideal I of R is called an idempotent ideal if $I^2 = I$. $\mathrm{Max}(R)$ means the set of all maximal ideals of the ring R, and $\mathrm{Spec}(R)$ means the set of all prime ideals of R. Notice that $\mathrm{Max}(R) \subseteq \mathrm{Spec}(R)$. R is a local ring if R has only one maximal ideal, and we write $(R, \mathfrak{m})$, i.e., R is a local ring with the unique maximal ideal $\mathfrak{m}$. An R-module M is *locally cyclic* if $M_\mathfrak{m}$ is cyclic for every $\mathfrak{m} \in \mathrm{Max}(R)$. A chain in $\mathrm{Spec}(R)$ is a sequence $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$ of prime ideals of R. The *height* of P, $\mathrm{ht}(P)$, is defined as $\mathrm{ht}(P) = \max\{n \mid P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n = P \textit{ is a chain in } \mathrm{Spec}(R)\ \}$. The *Krull dimension*, or, in short, the *dimension* of R, $\dim(R)$, is given by $\dim(R) = \max\{\mathrm{ht}(\mathfrak{m}) \mid \mathfrak{m} \in \mathrm{Max}(R)\}$. Furthermore, the dimension of an R-module M, $\dim_R M$, or $\dim M$, is equal to the dimension of the ring $R/\mathrm{ann}_R(M)$.

Let R be a commutative ring. $\mathrm{T}(R)$ is the *total ring of fractions* of R. An ideal I of a commutative ring R is invertible if $[R :_{T(R)} I]I = R$.

1.2 The thesis outline

An R-module M is called a **multiplication module** if every submodule of M is equal to IM for some ideal I of the ring R. In addition, if M is a left ideal of R then M is called a **left multiplication ideal**. *An R-module M is a multiplication R-module iff $N = [N : M]M$ for every submodule N of M* (Lemma 4.1). The set of all multiplication R-modules is denoted by $\mathrm{Mod}_m(R)$. In case R is a commutative ring, $\mathrm{Mod}_m(R)$ contains R, all cyclic R-modules, and all invertible ideals of R. In the general case, a cyclic module is not necessarily a multiplication module, and therefore the set of all cyclic R-modules is not a subset of $\mathrm{Mod}_m(R)$.

The map $\mu_M : \mathcal{I}(R) \to \mathrm{Sub}_R(M)$, $I \mapsto IM$, respects inclusion, i.e., $I \subseteq J$ implies $IM \subseteq JM$, it is a homomorphism of posets. *An R-module M is a multiplication module iff the map μ_M is a surjection.*

A ring R is said to be a **left multiplication ring** (resp, a **right multiplication ring**) if for every two ideals I and J of R where $J \subseteq I$, there exists an ideal J' of R such that $J = J'I$ (resp, $J = IJ'$). A ring R is called a **multiplication ring** if it is a right and left multiplication ring. In case R is a commutative ring, *R is a multiplication ring iff $\mathcal{I}(R) \subseteq \mathrm{Mod}_m(R)$.* Examples of multiplication rings are Dedekind domains, principal ideal domains and rings all ideals of which are idempotent.

In 1948, Krull [30] orginally introduced the concept of a multiplication commutative ring as a generalization of the concept of Dedekind domain. Larsen and McCarty [34] proved that *if every prime ideal of a commutative ring R is a multiplication ideal then R is a multiplication ring.* In [36], Mott proved that *a multiplication ring with finitely many minimal prime ideals is Noetherian.* Chapter 3 is a classification of multiplication commutative rings with finitely minimal primes.

In 1981, Barnard [14] presented the notion of multiplication modules over commutative rings. The first systematic study of multiplication modules over commutative rings was done by Elbast and Smith [21]. They provided many characterizations and properties of such modules and their submodules. The *fundamental theorem of abelian group* could be described as *every finitely generated $\mathbb{Z}$-module is a finite direct sum of multiplication modules.* Beside getting an approach to transfer the concepts from Ring Theory into Module Theory, this result motivated some authors to study the characteristics and properties of multiplication modules over a commutative ring. For example, Barnard [14], P. F. Smith [42], D. D. Anderson [8], and Y. Alshaniafi and S. Singh [2].

Let M be a finitely generated faithful multiplication module over a commutative ring R. El-bast and Smith in [21], had shown that *$IM = JM$ iff $I = J$ for every two ideals I and J of the ring R.* Alshaniafi and Singh in [2], generalized this result for a faithful multiplication module as follows: *$IM = JM$ iff $I = J$ for every ideals I and J of R that contained in $\theta(M)$.* Such result is called the *cancellation law of multiplication modules.* In Corollary 2.85, we show that there is a bijection between $\mathrm{Sub}_R(M)$ and $\mathcal{I}(\theta(M))$ where M is a faithful multiplication module over a commutative ring R. Naoum in [37, Theorem 3.2], proved that if

M is a finitely generated multiplication module then $\mathrm{End}_R(M) \cong R/\mathrm{ann}_R(M)$. In Lemma 4.23(6), we prove that the endomorphisms ring of a multiplication module over a commutative ring is commutative.

There are only few examples show that a faithful multiplication module over a commutative ring R cannot be embedded into R. Singh and Alshaniafi gave an example for a faithful multiplication module over a von Neuman regular ring that cannot be embedded into R (section 4 in [41]). The question of embedding a faithful multiplication R-module into R has been tackled by El-bast and Smith in [21], and by Singh and Alshaniafi in [41]. Section 5.3 provides some cases of embedding a multiplication module over a commutative ring R into R.

In 2003, Ameri in [6], defined the product of two submodules N and K of a multiplication module M over a commutative ring R, $N.K$, or, NK, as follows:

$$N.K = IJM.$$

Clearly, NK is a submodule of M. In ([6, Theorem 3.4]), he proved that if M is a multiplication module then the presentation of the submodule NK is independent, i.e., if $N = IM = I'M$ and $N = JM = J'M$ where I' and J' are ideals of R then $NK = IJM = I'J'M$. Aziz and Jayaram in [13], provided some applications of the product of submodules of a multiplication module. In most cases, this definition of the product of two submodules of a multiplication R-module M is not efficient to move from $\mathrm{Sub}_R(M)$ into $\mathcal{I}(R)$ and vice verse because the presentation of a submodule of M in the form IM where I is an ideal of R is not unique. In case M is a faithful multiplication R-module, Lemma 2.80 shows that every submodule N of M has a unique representation $I_N M$ where I_N is an ideal of $\theta(M)$. In Corollary 2.85, we establish a bijection between the set of all submodules of M, $\mathrm{Sub}(M)$, and the set of all ideal of R that are contained in $\theta(M)$, $\mathcal{I}(\theta(M))$, i.e., there is (1-1) correspondence between $\mathrm{Sub}(M)$ and $\mathcal{I}(\theta(M))$. This bijection respects inclusion (Corollary 2.82), i.e., if N_1, $N_2 \in \mathrm{Sub}(M)$ such that $N_1 \subseteq N_2$ then $I_{N_1} \subseteq I_{N_2}$ where I_{N_1} and I_{N_2} are the correspondents ideals of N_1 and N_2, respectively in $\mathcal{I}(\theta(M))$. In this situation, the ideal $I_N I_K$ in $\mathcal{I}(\theta(M))$ corresponds the submodule NK, i.e., $N.K := I_N I_K M$. Section 5.2 provides the properties of the product of two submodules of a faithful multiplication module with several applications.

The *algebras of polynomial integro-differential operators* over a field K of characteriastic zero (introduced in [15]),

$$\mathbb{I}_n = K\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n, \int_1, \ldots, \int_n \rangle,$$

have many interesting properties that almost opposite to the ones of the algebra of polynomial differential operators $A_n = K\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$, the *Weyl algebra* (where ∂_i and $\int_i$ are the partial derivations and integrations with respect to the variable x_i). In particular, the algebras $\mathbb{I}_n$ are neither left nor right Noetherian and non-simple. Futhermore, the classical Krull dimension of the algebra $\mathbb{I}_n$ is n and *all ideals of* $\mathbb{I}_n$ *are idempotent ideals*, [15]. Therefore, the algebras $\mathbb{I}_n$ are multiplication rings (Corollary 4.45). This result motivated us to study the class of multiplication modules over noncommutative rings.

There are only very few results in the literature about multiplication modules over noncommutative rings. In [44], Tuganbaev gave some properties of multiplication modules over noncommutative rings, but most of his results are over a *full right invariant ring*, which is a ring that satisfies the condition that every right ideal is an ideal. The class of full right invariant rings is larger than the class of commutative rings, but it is much smaller than the class of noncommutative rings. Chapter 5 is a study of multiplication modules over noncommutative rings.

The main findings of the thesis are as follows:

• **Characterization of multiplication commutative ring with finitely many minimal primes, and its corollaries**.

The next theorem is a description of the class of multiplication rings with finitely many minimal prime ideals.

Theorem 1.1 *Let R be a commutative ring with finitely many minimal prime ideals. Then the ring R is a multiplication ring iff $R \cong \prod_{i=1}^{n} D_i$ is a finite direct product of rings where D_i is either a Dedekind domain or an Artinian, local principal ideal ring.*

The next corollary is a description of semiprime multiplication commutative rings with finitely many minimal prime ideals.

Corollary 1.2 *Let R be a semiprime commutative ring with finitely many minimal prime ideals. Then R is a multiplication ring iff $R \cong \prod_{i=1}^{n} D_i$ is a finite direct product of rings where D_i is either a Dedekind domain or a field.*

Proof. The corollary follows from Theorem 1.1. □

The next corollary is a description of Artinian multiplication commutative rings.

Corollary 1.3 *Let R be an Artinian commutative ring. Then R is a multiplication ring iff it is a finite direct product of Artinian, local, principal ideal rings.*

Proof. The corollary follows from Theorem 1.1. □

The next corollary is a description of semiprime Artinian multiplication commutative rings.

Corollary 1.4 *Let R be a semiprime Artinian commutative ring. Then R is a multiplication ring iff it is a finite direct product of fields.*

Proof. The corollary follows from Corollary 1.2 and Corollary 1.3. □

The next theorem is a description of multiplication domains.

Theorem 1.5 *Let R be a domain. Then R is a multiplication ring iff R is either a field or a Dedekind domain.*

Proof. The theorem follows from Theorem 1.1. □

Corollary 1.6 *Let R be a commutative ring with finitely many minimal prime ideals. Then*

1. *R is a local multiplication ring iff R is either a local Dedekind ring or an Artinian, local, principal ideal ring.*

2. *R is a local multiplication domain iff R is a local Dedekind ring.*

3. *R is a local, Artinian, multiplication ring iff R is an Artinian, local, principal ideal ring.*

In Theorem 3.10, we give a characterization of multiplication commutative rings containing a unique minimal prime ideal and which is not maximal. In this situation, R is a multiplication ring iff it is a Dedekind domain.

• Incomparability of the annihilators.

Proposition 1.7 gives an interesting property of a multiplication module over a noncommutative ring. The proof and corollaries of Proposition 1.7 in Section 4.1.

Proposition 1.7 *Let M be a multiplication R-module and M_1, M_2 be R-modules such that $\mathrm{ann}_R(M_1) \subseteq \mathrm{ann}_R(M_2)$ and the direct sum of R-modules $M_1 \bigoplus M_2$ is an epimorphic image of the R-module M. Then $M_2 = 0$.*

• Five criteria for a direct sum of modules to be a multiplication module.

Let R be a ring (not necessarily commutative). We give several criteria for a direct sum of R-modules to be a multiplication R-module.

To formulate the first criterion (Theorem 1.11) we need to introduce the following concepts.

The intersection, orthogonality and strong orthogonality conditions.

Definition 1.8 *We say that the* **intersection condition** *holds for a direct sum $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ of nonzero R-modules M_λ if for all submodules N of M,*

$$N = \bigoplus_{\lambda\in\Lambda} N \cap M_\lambda.$$

Definition 1.9 *Let $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ be a direct sum of nonzero R-modules with* $\mathrm{card}(\Lambda) \geqslant 2$, $\mathfrak{a}_\lambda = \mathrm{ann}_R(M_\lambda)$ *and $\mathfrak{a}'_\lambda = \cap_{\mu\neq\lambda}\mathfrak{a}_\mu$. We say that the* **orthogonality condition** *holds for the direct sum M if*

$$\mathfrak{a}'_\lambda M_\mu = \delta_{\lambda\mu} M_\mu \text{ for all } \lambda, \mu \in \Lambda$$

where $\delta_{\lambda\mu}$ is the Kronecker delta. Clearly, $\mathfrak{a}'_\lambda \neq 0$ for all $\lambda \in \Lambda$ (since all $M_\lambda \neq 0$). In particular, $\mathfrak{a}_\lambda \neq 0$ for all $\lambda \in \Lambda$.

Definition 1.10 *Let $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$ be a direct sum of nonzero R-modules with* $\mathrm{card}(\Lambda) \geq 2$. *We say that the* **strong orthogonality condition** *holds for the direct sum M if for each set of R-modules $\{N_\lambda\}_{\lambda\in\Lambda}$ such that $N_\lambda \subseteq M_\lambda$, there is a set of of ideals $\{I_\lambda\}_{\lambda\in\Lambda}$ of R such that*

$$I_\lambda M_\mu = \delta_{\lambda\mu} N_\lambda \text{ for all } \lambda, \ \ \mu \in \Lambda.$$

The set of ideals $\{I_\lambda\}_{\lambda\in\Lambda}$ is called an **orthogonalizer** *of $\{N_\lambda\}_{\lambda\in\Lambda}$.*

In particular, the orthogonality condition holds for $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ iff the set of ideals $\{\mathfrak{a}'_\lambda\}_{\lambda\in\Lambda}$ is an orthogonalizer of $\{M_\lambda\}_{\lambda\in\Lambda}$. If the orthogonality condition holds for $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ and $\{I_\lambda\}_{\lambda\in\Lambda}$ is an orthogonalizer of $\{M_\lambda\}_{\lambda\in\Lambda}$ then $I_\lambda \subseteq \mathfrak{a}'_\lambda$ for all $\lambda \in \Lambda$.

Theorem 1.11 is the first criterion for a direct sum of modules to be a multiplication module which is given via the intersection and strong orthogonality conditions.

Theorem 1.11 *Let $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ be a direct sum of nonzero R-modules with* $\mathrm{card}(\Lambda) \geq 2$. *Then M is a multiplication R-module iff the intersection and strong orthogonality conditions hold for the direct sum $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$.*

Furthermore, if $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ is a multiplication R-module then

1. *the R-modules M_λ are multiplication modules, and*
2. *for each submodule N of M and each ideal I of R such that $N = IM$, $N \bigcap M_\lambda = IM_\lambda$ for all $\lambda \in \Lambda$.*

A compressor of a submodule.

Definition 1.12 *Let N be an R-submodule of M. An ideal J of the ring R is called a* **compressor** *of N (in M) if $N = JM$.*

Any sums of compressors of N is a compressor of N. The set of all compressors of N (in M) is denoted by $\mathcal{I}(N, M)$. The set $\mathcal{I}(N, M)$ is a non-empty set iff $[N : M]$ is a compressor of N ($[N : M]M = N$), and in that case $[N : M]$ is the largest compressor of N. Notice that $[N : M]M \subsetneq N$, in general.

Orthogonalizers and their properties.

Let N be a submodule of a multiplication R-module M. Then the set

$$\mathcal{I}(N,M) := \{I \lhd R \,|\, IM = N\} \tag{CINM}$$

is a non-empty set which is closed under addition of ideals (if $I, J \in \mathcal{I}(N,M)$ then $I+J \in \mathcal{I}(N,M)$). The sum

$$I(N,M) = \sum_{I\in\mathcal{I}(N,M)} I \tag{CINM1}$$

is the *largest element* of the set $\mathcal{I}(N,M)$ (w.r.t. inclusion). Clearly, $I(N,M) = [N:M]$.

Let $\mathcal{M} = \{M_\lambda\}_{\lambda\in\Lambda}$ and $\mathcal{N} = \{N_\lambda\}_{\lambda\in\Lambda}$ be sets of R-modules such that $N_\lambda \subseteq M_\lambda$ for all $\lambda \in \Lambda$. Let $\mathcal{I}(\mathcal{N},\mathcal{M})$ be the set of all sets of ideals $\{I_\lambda\}_{\lambda\in\Lambda}$ such that $I_\lambda M_\mu = \delta_{\lambda\mu} N_\lambda$ for all $\lambda, \mu \in \Lambda$, i.e., the set contains all the orthogonalizers for $\mathcal{N}$. In general, the set $\mathcal{I}(\mathcal{N},\mathcal{M})$ could be an empty set. In particular, if $\mathcal{M} = \{M\}$ and $\mathcal{N} = \{N\}$ then $\mathcal{I}(\mathcal{N},\mathcal{M}) = \mathcal{I}(N,M)$.

Lemma 1.13 *Let $\mathcal{M} = \{M_\lambda\}_{\lambda\in\Lambda}$ be a set of R-modules such that their direct sum $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ is a multiplication module. Then for every set of R-modules $\mathcal{N} = \{N_\lambda\}_{\lambda\in\Lambda}$ such that $N_\lambda \subseteq M_\lambda$ for all $\lambda \in \Lambda$, the set $\mathcal{I}(\mathcal{N},\mathcal{M})$ is a non-empty set.*

Proof. The result follows from Theorem 1.11. $\square$

Suppose that $\mathcal{I}(\mathcal{N},\mathcal{M}) \neq \emptyset$. Then the set $\mathcal{I}(\mathcal{N},\mathcal{M})$ is closed under addition (componentwise): if sets $\mathcal{I} = \{I_\lambda\}_{\lambda\in\Lambda}$ and $\mathcal{J} = \{J_\lambda\}_{\lambda\in\Lambda}$ belong to $\mathcal{I}(\mathcal{N},\mathcal{M})$ then $\mathcal{I}+\mathcal{J} = \{I_\lambda + J_\lambda\}_{\lambda\in\Lambda} \in \mathcal{I}(\mathcal{N},\mathcal{M})$. So, the sum in $\mathcal{I}(\mathcal{N},\mathcal{M})$,

$$I(\mathcal{N},\mathcal{M}) := \sum_{\mathcal{I}\in\mathcal{I}(\mathcal{N},\mathcal{M})} \mathcal{I}$$

is the *largest element* of the set $\mathcal{I}(\mathcal{N},\mathcal{M})$ w.r.t. componentwise inclusion, i.e., $\mathcal{I} \subseteq \mathcal{J}$ iff $I_\lambda \subseteq J_\lambda$ for all $\lambda \in \Lambda$. The set $I(\mathcal{N},\mathcal{M})$ is called the **largest orthogonalizer** in $\mathcal{I}(\mathcal{N},\mathcal{M})$.

The next theorem is the second criterion for a direct sum of modules to be a multiplication module.

Theorem 1.14 *Let $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ be a direct sum of nonzero R-modules with* $\mathrm{card}(\Lambda) \geq 2$. *Then M is a multiplication module iff*

1. *the R-modules M_λ, where $\lambda \in \Lambda$, are multiplication modules, and*

2. *for each submodule N of M, $\mathcal{I}(\mathcal{N},\mathcal{M}) \neq \emptyset$ where $\mathcal{N} = \{N \cap M_\lambda\}_{\lambda\in\Lambda}$ and $\mathcal{M} = \{M_\lambda\}_{\lambda\in\Lambda}$, and $N = (\sum_{\lambda\in\Lambda} I_\lambda)M$ for all/some $\{I_\lambda\}_{\lambda\in\Lambda} \in \mathcal{I}(\mathcal{N},\mathcal{M})$.*

Orthogonal set of ideals.

Definition 1.15 *A set of ideals $\{\mathfrak{a}_\lambda\}_{\lambda\in\Lambda}$ of a ring R is called an* orthogonal set of ideals *of R if $\mathfrak{a}_\lambda\mathfrak{a}_\mu = 0$ for all $\lambda \neq \mu$.*

The next theorem is the third criterion for a direct sum of modules to be a multiplication module which is given via orthogonal ideals.

Theorem 1.16 *Let $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ be a direct sum of nonzero R-modules with $\mathrm{card}(\Lambda) \geq 2$, $\mathfrak{a} := \mathrm{ann}_R(M)$ and $\overline{R} := R/\mathfrak{a}$. Then M is a multiplication module iff*

1. *the ring $\overline{R}$ contains a direct sum of nonzero orthogonal ideals $\mathfrak{a}' = \bigoplus_{\lambda\in\Lambda} \mathfrak{a}'_\lambda$ such that $M_\lambda = \mathfrak{a}'_\lambda M$ for all $\lambda \in \Lambda$, and*

2. *for each submodule N of M, $N = \mathfrak{b}'M$ for an ideal $\mathfrak{b}'$ of $\overline{R}$ such that $\mathfrak{b}' = \bigoplus_{\lambda\in\Lambda} \mathfrak{b}'_\lambda$ is a direct sum of ideals $\mathfrak{b}'_\lambda = \mathfrak{b}' \cap \mathfrak{a}'_\lambda$ of $\overline{R}$ for all $\lambda \in \Lambda$.*

Theorem 1.17 is the fourth criterion for a direct sum of modules to be a multiplication module which is given via the orthogonality and intersection conditions.

Theorem 1.17 *Let $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ be a direct sum of nonzero R-modules with $\mathrm{card}(\Lambda) \geq 2$. Then M is a multiplication module iff*

1. *the R-modules M_λ are multiplication modules where $\lambda \in \Lambda$,*

2. *the intersection condition holds for the direct sum $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$, i.e., for any sumbodule N of M, $N = \bigoplus_{\lambda\in\Lambda}(N \cap M_\lambda)$, and*

3. *the orthogonality condition holds for the direct sum $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$, i.e., for all $\lambda, \mu \in \Lambda$, $\mathfrak{a}'_\lambda M_\mu = \delta_{\lambda\mu} M_\mu$.*

$\mathrm{End}_R(M)$-stable submodule of M.

Definition 1.18 *A submodule N of an R-module M is called an* $\mathrm{End}_R(M)$-stable submodule *(resp.,* $\mathrm{End}_R(M)$-invariant submodule*) if $f(N) \subseteq N$ (resp., $f(N) = N$) for every $0 \neq f \in \mathrm{End}_R(M)$.*

Definition 1.19 *We say that the* $\mathrm{End}_R(M)$**-stability condition** *holds for an* R*-module* M *if every submodule* N *of* M *is an* $\mathrm{End}_R(M)$*-stable submodule.*

Theorem 1.20 is the fifth criterion for a direct sum of modules to be a multiplication module given via the orthogonality and $\mathrm{End}_R(M)$-stability conditions.

Theorem 1.20 *Let* $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ *be a direct sum of nonzero* R*-modules with* $\mathrm{card}(\Lambda) \geq 2$*. Then* M *is a multiplication module iff*

1. *the* R*-modules* M_λ *are multiplication modules where* $\lambda \in \Lambda$*,*
2. *every submodule* N *of* M *is an* $\mathrm{End}_R(M)$*-stable submodule, and*
3. *the orthogonality condition holds for the direct sum* $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$*.*

The proofs of the theorems/criteria above are given in Section 4.2. In addition, Section 4.2 contains many applications and results based on these criteria.

Two ideals I and J of a ring R are called *incomparable* if neither $I \subseteq J$ nor $J \subseteq I$. We say that I and J commute if $IJ = JI$.

• Commutativity of prime ideals of a multiplication noncommutative ring.

Theorem 1.21

1. *Let* R *be a left multiplication ring. Then* $PQ = QP$ *for all incomparable prime ideals* P *and* Q *of* R*.*
2. *Let* R *be a left and right multiplication ring. Then*
   (a) $IP = PI$ *for all ideals* I *and* P *such that* $I \nsubseteq P$ *and* P *is a prime ideal of* R*.*
   (b) $PQ = QP$ *for all prime ideals* P *and* Q *of* R*.*

The proof of such theorem with some applications is concluded in Section 4.3.

• Embedding of a projective multiplication R-module into the ring R.

Theorem 1.22 *Let* R *be a commutative ring and* M *be a projective multiplication* R*-module. If there is a regular element* $a \in \mathcal{C}_R$ *such that* aM *is a submodule of a cyclic submodule say* C *of* M *then the* R*-module* M *is isomorphic to an ideal of* R*.*

Section 5.3 contains the proof of Theorem 1.22 and some cases of embedding a multiplication R-module into R.

Let R be a commutative ring and N, K be submodules of an R-module M. Suppose that compressors of N and K in M are existed, i.e., $N = IM$ and $K = JM$ for some ideals I and J of R.

• Product of submodules of a (faithful) multiplication module over a commutative ring.

In Section 5.2, we give some properties and applications of the product of two submodules of a faithful multiplication modules over a commutative ring.

The next theorem is a version of *Chinese Remainder Theorem* that fits multiplication module over a commutative ring.

Theorem 1.23 *Let M be a faithful multiplication module over a commutative ring R. If N and K are submodules of M such that $I_N + I_K = \theta(M)$ then $M/NK \cong (M/N) \times (M/K)$.*

The next proposition can be considered as a version of *Krull's Intersection Theorem*, seen through the paradigm of multiplication modules.

Proposition 1.24 *Let R be a commutative ring and M be a faithful Artinian multiplication module. If $N \subseteq \mathrm{rad}(M)$ is a submodule of M. Then there exists $n \in \mathbb{N}$ such that $N^n = 0$, i.e., $\bigcap_{i=1}^{n} N^i = 0$ for some $n \in N$.*

The proof of the above two results are concluded in Section 5.2.

1.3 The thesis structure

The goals of the thesis are:

• to characterize the class of multiplication commutative rings with finitely many minimal prime ideals;

• to give several criteria for a direct sum of modules to be a multiplication module over a noncommutative ring;

• to present some applications of the cancellation law of multiplication modules;

• to study the product of two submodules of a faithful multiplication module over a commutative ring; and,

• to identify the conditions under which a projective multiplication module over a commutative ring R can be embedded into the ring R.

The thesis consists of five chapters and is organized as follows:

In Chapter 2, we collect the necessary definitions, concepts and results that are used in the thesis. Section 2.2 provides a survey on multiplication modules over commutative rings.

In Chapter 3, the class of multiplication commutative rings containing only finitely many minimal prime ideals is studied, Theorem 1.1 is proved, and some corollaries are obtained. In addition, we prove some properties of finitely generated prime ideals with zero annihilators of a multiplication commutative ring (Section 3.1).

The Chapter 4 is organized as follows: Several characterizations and properties of multiplication modules over rings (not necessarily commutative) are given in Section 4.1. Section 4.2 is divided into two parts. The first part includes the proofs of the criteria mentioned above for a direct sum of modules to be a multiplication module. In the second part, applications are given. In Section 4.3, the class of multiplication noncommutative rings is studied. Additionally, some characteristics of a subclass of multiplication modules: *fully-multiplication modules* are given. In Section 4.4, some properties of the endomorphism ring of a multiplication module are obtained. In addition, some classes of modules are presented: *epimorphic modules*, *monomorphic modules* and *automorphic modules* (Definition 4.56). Some characterizations and properties of such modules are given.

Chapter 5 is devoted to studying some aspects of multiplication modules over a commutative ring. Section 5.1 presents some new properties of the companion ideal $\theta(M)$ of a multiplication module M and provides some applications of the cancellation law of multiplication modules. Furthermore, two cancellation laws that depend on the original cancellation law are given (Theorems 5.14 and 5.16), and some well-known results of finitely generated faithful multiplication modules are generalized into faithful multiplication modules (Proposition 2.84 and Corollary 5.23). Also, we give an explicit description of the ideal $\theta(M)$ of a faithful multiplication module M where M is a finite direct sum of submodules (Proposition 5.22). In Section 5.2, the product of two submodules of a faithful multiplication module is studied. Some properties and application of such operation are given. Moreover, we provide a notion of *divisor submodule* and the the idea of *greatest common divisor* of two submodules of a module M. In Section 5.3, some cases of embedding a multiplication module into its ring are presented. Finally, in Section 5.4, we give some properties of multiplication modules over rings: Artinian rings, Noetherian ring, PID, and von Neuman regular ring.

Chapter 2

Background

This chapter consists of two parts. In the first part, we provide the essential background from both commutative and noncommutative algebra that we need for this work. In the second part, we collect some results on multiplication modules over a commutative ring that are used in the proofs of the thesis.

2.1 Review of related topics in Ring and Module Theory

In this section, we recall some relevant concepts and known results in Ring and Module Theory.

2.1.1 Definitions and some properties

Let I and J be ideals of a ring R. The next lemma gives some properties of the ideal $[I : J]$.

Lemma 2.1 *Let R be a commutative ring and I, J and K be ideals of R. Then*

1. $\mathrm{ann}_R(I) \subseteq [J : I]$.
2. *If $K \subseteq I \subseteq J$ then $[K : I] \supseteq [K : J]$.*
3. $[[I : J] : K] = [[I : K] : J] = [I : JK]$.
4. *For any collection $\{J_\lambda \mid \lambda \in \Lambda\}$ of ideals of R such that $I \subseteq J_\lambda$ for all $\lambda \in \Lambda$,*

(a) $[I : \sum_{\lambda\in\Lambda} J_\lambda] = \bigcap_{\lambda\in\Lambda}[I : J_\lambda]$.

(b) $[\bigcap_{\lambda\in\Lambda} J_\lambda : I] = \bigcap_{\lambda\in\Lambda}[J_\lambda : I]$.

Proof. The statements follow from the definition of the ideal $[I : J]$. □

Let R be a commutative ring and M be an R-module. D. Anderson in [8] defined the ideal $\theta(M)$ as follows: $\theta(M) = \sum_{m\in M}[Rm : M]$.

Lemma 2.2 *Let R be a commutative ring and M be an R-module.*

1. $\mathrm{ann}_R(M) \subseteq \theta(M)$.
2. *If I is an ideal of R then $I \subseteq \theta(I)$.*
3. *If N is a submodule of M then $\theta(N) \supseteq \sum_{m\in N}[Rm : M]$.*

Proof. Statements 1 and 2 are trivial. Since $\theta(N) = \sum_{m\in N}[Rm : N] \supseteq \sum_{m\in N}[Rm : M]$, the statement 3 follows. □

The *Jacobson radical* of a commutative ring R, $\mathrm{J}(R)$, is used in studying the *radical* of a multiplication modules over a commutative ring.

Definition 2.3 *Let R be a commutative ring. The Jacobson radical of a ring R, $\mathrm{J}(R) := \bigcap_{\mathfrak{m}\in\mathrm{Max}(R)} \mathfrak{m}$ where $\mathrm{Max}(R)$ is the set of maximal ideals of R.*

The next lemma is a description of the ideal $\mathrm{J}(R)$ where R is a commutative ring.

Lemma 2.4 *([31, Proposition 3.2.3]) Let R be a commutative ring and $r \in R$. Then $r \in \mathrm{J}(R)$ iff $1 - ra$ is a unit in R for every $a \in R$.*

Definition 2.5 *A commutative ring R is called a local ring if it has only one maximal ideal.*

The next lemma is a criterion for a commutative ring to be a local ring.

Lemma 2.6 *([11, Proposition 1.6]) Let R be a commutative ring. R is a local ring iff the set of non units of R is an ideal of R.*

Definition 2.7 *Let R be a commutative ring. The* radical *of I, $\mathrm{rad}(I)$, equals to the set $\{a \in R \mid a^n \in I$ for some $n \in \mathbb{N}\}$.*

Let I be an ideal of a commutative ring R. The next lemma is a description of $\mathrm{rad}(I)$.

Lemma 2.8 *([11, Proposition 1.14]) Let R be a commutative ring and I be an ideal of R. Then $\mathrm{rad}(I) = \bigcap_{I \subseteq P \in \mathrm{Spec}(R)} P$ where $\mathrm{Spec}(R)$ is the set of all prime ideals of R.*

Definition 2.9 *Let R be a commutative ring. The* nil radical*, or, the* prime radical *of a commutative ring R, $\mathrm{nil}(R)$, equals to $\mathrm{rad}(0) = \bigcap_{P \in \mathrm{Spec}(R)} P$.*

Clearly, $\mathrm{nil}(R)$ is the set of all nilpotent elements of R, and $\mathrm{nil}(R) \subseteq J(R)$.

Lemma 2.10 *([11, Proposition 1.11]) Let R be a commutative ring. Then*

1. *If I is an ideal of R such that $I \subseteq \bigcup_{i=1}^{n} P_i$ where $P_i \in \mathrm{Spec}(R)$ and $n \geq 1$ then $I \subseteq P_i$ for some i.*
2. *If $\{I_i \mid 1 = 1, \ldots, n\}$ is a set of ideals of R and $P \in \mathrm{Spec}(R)$ such that $\bigcap_{i=1}^{n} I_i \subseteq P$ then $I_i \subseteq P$ for some i.*

Definition 2.11 *Let R be a commutative ring and I be an ideal of R. An ideal $P \in \mathrm{Spec}(R)$ is called a* minimal prime ideal of I *if $I \subseteq P$ and there is no $p' \in \mathrm{Spec}(R)$ such that $I \subseteq P' \subsetneq P$. A prime ideal P is called a* minimal prime of the ring R *if it is a minimal prime of the zero ideal.*

Definition 2.12 *A proper ideal Q of a ring R is called* primary *if whenever $ab \in Q$ for $a, b \in R$ then either $a \in Q$ or $b \in \sqrt{Q} := \{r \in R \mid r^n \in Q$ for some $n \in \mathbb{N}\}$.*

Clearly, every prime ideal of a commutative ring R is primary.

If Q is a primary ideal then $P := \sqrt{Q}$ is necessarily a prime ideal. It is called *the associated prime ideal of* Q. In this case, Q is called a P-primary ideal.

The next lemma provides some properties of the primary ideals of a commutative ring.

Lemma 2.13 *([11, Proposition 4.1]) Let R be a commutative ring. Then*

1. *An ideal Q of R is primary iff $R/Q \neq 0$ and every zero divisor in R/Q is nilpotent, and*
2. *If Q is a primary ideal then $\mathrm{rad}(Q) := \sqrt{Q}$ is the smallest prime ideal contains Q.*

Definition 2.14 *A nonzero ideal I of R is called an* essential ideal *provided that if $I \cap J = 0$ for some ideal J of R then J must be zero.*

Lemma 2.15 *([29, Theorem 76]) Let R be a commutative ring and M be a finitely generated R-module. If I is an ideal of R such that $IM = M$ then $I + \mathrm{ann}_R(M) = R$.*

Lemma 2.16 *([11, Proposition 2.6])(Nakayama's Lemma) Let R be a commutative ring and M be a finitely generated R-module. If $IM = M$ where I is an ideal of R that is contained in $J(R)$ then $M = 0$.*

Two ideals I and J of a commutative ring R are called *coprime ideals*, or, *comaximal ideals* if $I + J = R$.

Lemma 2.17 *([33, (316, P.69)])(Chinese reminder theorem) Let R be a commutative ring and $\{I_1, \dots, I_n\}$ be a finite set of pairwise coprime ideals of R, i.e., $I_i + I_j = R$ for all $i \neq j$. Then*

$$R/\bigcap_{i=1}^{n} I_i \cong \prod_{i=1}^{n} R/I_i.$$

Definition 2.18 *Let R be a ring. An ideal I of R is called a* semiprime ideal *of R if I satisfies the following condition: if $a^n \in I$ for some $n \in \mathbb{N}$ then $a \in I$. If the zero ideal is a semiprime ideal then R is called a* semiprime ring.

Definition 2.19 *A ring R is called a* simple ring *if it doesn't have a proper ideal besides the zero ideal.*

Definition 2.20 *Let R be a commutative ring. R is said to be von Neuman regular ring if for every element $a \in R$ there exists $b \in R$ such that $a = ba^2$.*

Clearly, If R is a von Neuman regular ring then every ideal I of R is idempotent, i.e., $I^2 = I$.

Definition 2.21 *Let R be a commutative ring and M be an R-module. $N_1 \subsetneq N_2 \subsetneq \cdots \subsetneq N_k$ is called a* chain *of submodules of M of* length *k. The length of the R-module M, $\ell(M)$, is the largest length of any of its chains. If no such largest length exists, we say that M has* infinite length.

Definition 2.22 *A chain of submodules of R-module M: $0 = N_1 \subsetneq N_2 \subsetneq \cdots \subsetneq N_{n-1} \subsetneq N_n = M$ such that N_{i+1}/N_i is simple for $i = 0, \ldots, n-1$ is called a* composition series *of the module M.*

An R-module M has a finite length iff it has a finite composition series: $0 = N_1 \subsetneq N_2 \subsetneq \cdots \subsetneq N_{n-1} \subsetneq N_n = M$, and then $\ell(M) = \sum_{i=0}^{n-1} \ell(N_{i+1}/N_i)$.

2.1.2 Noetherian and Artinian rings

Definition 2.23 *A ring R is said to be a* Noetherian ring *if for every ascending chain of ideals $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$, there exists $k \in \mathbb{N}$ such that $I_k = I_{k+1} = \cdots$. Equivalently,* R is Noetherian iff every ideal of R is finitely generated.

Definition 2.24 *A ring R is said to be an* Artinian ring *if for every descending chain of ideals $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$, there exists $k \in \mathbb{N}$ such that $I_k = I_{k+1} = \cdots$.*

The next lemmas are used in the proof of Theorem 1.1.

Lemma 2.25 *([11, Lemma 7.11])(*Primary decomposition theorem*) Let R be a commutative Noetherian ring. Then every proper ideal of R has a primary decomposition, i.e., every ideal of R is a finite intersectin of primary ideals of R.*

Lemma 2.26 *([11, Lemma 7.11]) Let R be a commutative Noetherian ring. Then*

1. *([19, Chapter 2, Proposition 13]) The prime radical of R,* nil*(R), is the intersection of all minimal prime ideals of R.*

2. *([11, Corollary 7.15]) The prime radical of R, $\mathrm{nil}(R)$, is a nilpotent ideal.*

Lemma 2.27 *([11, Proposition 8.1]) Let R be an Artinian commutative ring. Then $\mathrm{Spec}(R) = \mathrm{Max}(R)$.*

A commutative ring R is called a *zero-dimensional* ring if $\mathrm{Max}(R) = \mathrm{Spec}(R)$.

Lemma 2.28 *([11, Proposition 8.3]) Let R be an Artinian commutative ring. Then R has finitely many maximal ideals.*

Lemma 2.29 *([11, Proposition 8.1]) Let R be a local Artinian commutative ring with a maximal ideal $\mathfrak{m}$. Then the following statement are equivalent.*

1. *The maximal ideal $\mathfrak{m}$ is principal.*
2. *Every ideal of R is principal.*

Lemma 2.30 *([11, Theorem 8.5]) A commutative ring R is Artinian iff R is Noetherian and zero-dimensional.*

Lemma 2.31 *([11, Proposition 8.6]) Every Artinian commutative ring is a finite product of local Artinian rings. This decomposition is unique up to isomorphism.*

Clearly, if $(R, \mathfrak{m})$ be a local Artinian commutative ring then there exists $n \in \mathbb{N}$ such that $0 = \mathfrak{m}^n = \mathfrak{m}^{n+1} = \cdots$. In addition, if R is semiprime then $\mathfrak{m} = 0$, i.e., R is a field.

An element a of a commutative ring R is called a *zero divisor* if there exists $0 \neq b \in R$ such that $ab = 0$. An element of a ring that is not a zero divisor is called *regular*, or a *non-zero-divisor*.

For an R-module M, let $\mathcal{Z}(M) = \{r \in R \mid rm = 0 \text{ for some } 0 \neq m \in M\}$. $\mathbb{Z}(M)$ is the set of all annihilators of non-zero elements in M.

Lemma 2.32 *([29, Theorem 82]) Let R be a Noetherian commutative ring and M be a finitely generated R-module. If I is an ideal of R where $I \subseteq \mathcal{Z}(M)$ then there exists a nonzero element $x \in M$ such that $Ix = 0$.*

Proposition 2.33 *Let R be a Noetherian commutative ring. Then every ideal I of R with zero annihilator contains a non-zero-divisor.*

Proof. Suppose that $I \subseteq \mathcal{Z}(I)$. By Lemma 2.32, there exists $0 \neq a \in I$ such that $aI = 0$ (contradiction to $\mathrm{ann}_R(I) = 0$). So, $I \nsubseteq \mathcal{Z}(I)$, i.e., there is $0 \neq b \in I$ such that $bx \neq 0$ for all $0 \neq x \in I$). Now, suppose that $rb = 0$ for some $r \in R$. Then $(rb)x = b(rx) = 0$ where $x \in I$. Therefore, $rx = 0$ for all $0 \neq x \in I$. So, $r \in \mathrm{ann}_R(I) = 0$, i.e., $r = 0$. Hence, b is a non-zero-divisor. $\square$

Let R be ring and M be an R-module. We say that the module M is a *Noetherian module* if for every ascending chain of submodules of M, $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n \subseteq \cdots$, there exists $k \in \mathbb{N}$ such that $N_k = N_{k+1} = \cdots$, i.e., M is Noetherian module iff M satisfies a.c.c. on submodules of M. M is an *Arinian module* iff M satisfies d.c.c. on submodules of M.

Every Artinian ring is Noetherian, but it is not true that every Artinian module is Noetherian. For example, let $M = \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ is Artinian, but it is not Noetherian as $\mathbb{Z}$-module where p is a prime number.

The next lemma gives a sufficient condition for a module over Noetherian (resp; Artinian) to be Noetherian (resp; Artinian) module.

Lemma 2.34 *([43, Proposition 3.5]) If R is a Noetherian (resp; Artinian) commutative ring and M is a finitely generated R-module then M is Noetherian (resp; Artinian) R-module.*

Lemma 2.35 *([24, Proposition 4.8]) An R-module M has a finite length iff M is both Artinian and Noetherian.*

2.1.3 Multiplicatively closed set and localization

A non-empty subset S of a commutative ring R is called a *multiplicatively closed subset* iff $SS \subseteq S$, $1 \in S$ and $0 \notin S$. Among examples of multiplicatively closed sets are the set of units of the ring R, the set of non-zero-divisors of R and $1 + I$ where I is a proper ideal of R.

Define a relation, $\equiv$, on $R \times S$ which is defined as follows: $(a, s) \equiv (b, t)$ iff $(at - bs)u = 0$ for some $u \in S$. Clearly, this relation is an equivalence relation.

Let a/s denotes the equivalence class of the element (a,s) of $R \times S$. Let $S^{-1}R$ denotes the set of all equivalence classes. Now, we can put a ring structure on $S^{-1}R$ by defining addition and multiplication as follows:

$$(a/s) + (b/t) = (at + bs/st)$$

and

$$(a/s)(b/t) = (ab/st).$$

If R is a domain and $S = R \setminus \{0\}$ then $S^{-1}R$ is called by *the field of fractions* of R. If R is a commutative ring and S is a set of nonzero divisors of R then $S^{-1}R$ is called the *total quotient ring* or the ring of fractions, and it is denoted by $T(R)$.

Let R be a commutative ring and $P \in \mathrm{Spec}(R)$. Then $S = R \setminus P$ is a multiplicatively closed set. We write R_P for $S^{-1}R$ in this case. The process of passing from R to R_P is called *localization* at P. The elements a/s where $a \in P$ form an ideal P_P of R_P. If $b/t \notin P_P$ then $b \notin P$, i.e., $b \in S$, and therefore b/t is a unit in R_P. So, if I is an ideal of R_P such that $I \nsubseteq P_P$ then $I = R_P$, Hence, P_P is a unique maximal ideal of R_P, i.e., (R_P, P_P) is a local ring.

The construction of $S^{-1}R$ can be carried through with an R-module M in place of the ring R. Define a relation $\equiv$ on $M \times S$ as follows: $(m,s) \equiv (m',t)$ iff $u(sm' - tm) = 0$ for some $u \in S$. This relation is an equivalence. Let m/s denotes the equivalence class of the element (m,s). $S^{-1}M$ means the set of all equivalence classes of $\equiv$. $S^{-1}M$ is a $S^{-1}R$-module via the action $S^{-1}R \times S^{-1}M \to S^{-1}M$, $(r/s, m/s') \mapsto rm/ss'$.

The next three lemmas are some of the technical lemmas of the thesis.

Lemma 2.36 *([11, Corollary 3.15]) Let R be a commutative ring and let K and N be submodules of an R-module M. If K is finitely generated then $S^{-1}[N : K] = [S^{-1}N : S^{-1}K]$.*

Lemma 2.37 *([11, Corollary 3.4]) Over a commutative ring R, the operation S^{-1} of localization at S commutes with formation of sums, finite intersection, product, and quotient. Precisely, if N and K are submodules of an R-module M and I is an ideal of R then*

1. $S^{-1}(N + K) = S^{-1}N + S^{-1}K$.
2. $S^{-1}(N \cap K) = S^{-1}N \cap S^{-1}K$.
3. $S^{-1}(IM) = (S^{-1}I)(S^{-1}M)$.

4. $S^{-1}(M/N) \cong S^{-1}M/S^{-1}N$.

Lemma 2.38 *([11, Proposition 3.8]) Let R be a commutative ring and M be an R-module then the following statements are equivalent.*

1. $M = 0$.
2. $M_P = 0$ *for every* $P \in \mathrm{Spec}(R)$.
3. $M_{\mathfrak{m}} = 0$ *for every* $\mathfrak{m} \in \mathrm{Max}(R)$.

2.1.4 Invertible ideals and some specific rings

For a commutative ring R, let S be the set of non-zero-divisors. Then $\mathrm{T}(R) = S^{-1}R$ is called the *total quotient ring* of R, or, *the ring of fractions* of R.

An ideal I of a commutative ring R is called an *invertible ideal* if $[R :_{\mathrm{T}(R)} I]I = R$ where $[R :_{\mathrm{T}(R)} I] = \{x \in \mathrm{T}(R) \mid xI \subseteq R\}$.

Lemma 2.39 *([11, Proposition 9.6]) Let R be a commutative ring. If I is an invertible ideal of R then I is finitely generated.*

Definition 2.40 *A domain R is called a principal ideal domain , PID for short, if every ideal of R is cyclic.*

Definition 2.41 *A domain R is called a discrete valuation domain, DVD for short, if R is a principal ideal domain with a unique maximal ideal $\mathfrak{m}$ such that $\mathfrak{m} \neq 0$.*

Definition 2.42 *A domain R is called a* Dedekind domain *if every nonzero ideal of R is a finite product of prime ideals of R.*

The next lemmas gives criteria for a domain to be a Dedekind domain.

Lemma 2.43 *([11, Corollary 9.4] and [11, Corollary 9.3]) Let R be a domain. Then the following statements are equivalent.*

1. *R is a Dedekind domain.*

2. *Every proper ideal is a unique finite product of prime ideals. (Counted with multiplicity)*

3. *R is Noetherian with $\dim(R) = 1$ such that every local ring $R_{\mathfrak{p}}$ is DVD for every $\mathfrak{p} \in \mathrm{Spec}(R)$.*

Lemma 2.44 *([12, Basic properties 13.1.1] Let R be a domain. Then R is a Dedekind domain iff every ideal of R is projective.*

Definition 2.45 *A domain R is called an* almost Dedekind domain *if $R_{\mathfrak{m}}$ is discrete valuation ring for every $\mathfrak{m} \in \mathrm{Max}(R)$.*

Lemma 2.46 *([40, Remark 1.3]) A Noetherian, almost Dedekind domain is a Dedekind domain.*

2.1.5 The algebras of polynomial integro-differential operators over a field K of characteristic zero

The algebras of polynomial integro-differential operators over a field K of characteristic zero are

$$\mathbb{I}_n = K\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n, \int_1, \dots, \int_n \rangle,$$

where ∂_i and $\int_i$ are the partial derivations and integrations with respect to the variable x_i.

The algebras $\mathbb{I}_n$ have many interesting properties (see [15]). These algebras, $\mathbb{I}_n$, are neither left nor right Noetherian and non-simple. Furthermore, the classical Krull dimension of the algebra $\mathbb{I}_n$ is n.

The next lemma gives some properties of the ideals of $\mathbb{I}_n$ that are used in the thesis.

Lemma 2.47 *([15, Corollary 3.3]) Let $\mathfrak{a}$ be an ideal of $\mathbb{I}_n$. Then*

1. *$\mathfrak{a}$ is an idempotent ideal, i.e. $\mathfrak{a}^2 = \mathfrak{a}$.*

2. *$\mathfrak{a}\mathfrak{b} = \mathfrak{b}\mathfrak{a}$ for every pair of ideals $\mathfrak{b}$ and $\mathfrak{a}$ of $\mathbb{I}_n$.*

2.1.6 Matrix ring

Let $\mathrm{M_n}(R)$ be the ring of all matrices over a commutative ring R. If I is an ideal of the ring R then $\mathrm{M_n}(I) = \{[a_{ij}] \in \mathrm{M_n}(R) \,|\, a_{ij} \in I \text{ for all } 1 \leqslant i, j \leqslant n\}$ is an ideal of the ring $\mathrm{M_n}(R)$.

Lemma 2.48 *([17, Proposition 1.5.4]) Let R be a commutative ring and J be an ideal of $\mathrm{M_n}(R)$. Then there exists an ideal I of R such that $J = \mathrm{M_n}(I)$.*

It follows from Lemma 2.48 that there is (1-1) correspondence between the ideals of the commutative ring R, $\mathcal{I}(R)$, and the ideals of the ring $\mathrm{M_n}(R)$, $\mathcal{I}(\mathrm{M_n}(R))$.

The next lemma is used in the proof of Corollary 4.43.

Lemma 2.49 *Let R be a commutative ring, and let I and J be ideals of R. Then*

1. $\mathrm{M_n}(IJ) = \mathrm{M_n}(I)\mathrm{M_n}(J)$.
2. $I \subseteq J$ *iff* $\mathrm{M_n}(I) \subseteq \mathrm{M_n}(J)$.

Proof. 1. It is clear that $\mathrm{M_n}(IJ) = IJ\mathrm{E}_{11} \oplus IJ\mathrm{E}_{12} \oplus \cdots \oplus IJ\mathrm{E}_{nn}$ where $IJ\mathrm{E}_{ij} = \sum_{k=1}^{n}(I\mathrm{E}_{ik})(J\mathrm{E}_{kj}) \in \mathrm{M_n}(I)\mathrm{M_n}(J)$. So, $\mathrm{M_n}(IJ) \subseteq \mathrm{M_n}(I)\mathrm{M_n}(J) \subseteq \mathrm{M_n}(IJ)$, i.e., $\mathrm{M_n}(IJ) = \mathrm{M_n}(I)\mathrm{M_n}(J)$.

2. It follows from the fact the ideal $\{a_{ij} \,|\, [a_{ij}] \in \mathrm{M_n}(I)\} = I$. □

2.1.7 Direct sum and direct product

Let $\{M_i\}_{i\in I}$ be a set of R-modules. $\bigoplus_{i\in I} M_i$ denotes the *external direct sum* of modules, and defined as $\bigoplus_{i\in I} M_i := \{(m_i)_{i\in I} \,|\, m_i \in M_i \text{ and } m_i = 0 \text{ for almost all } i \in I\}$. If there is no assumption on the number of nonzero component then it is denoted by $\prod_{i\in I} M_i$, and called by the *direct product* of the R-modules M_i. It is clear that if I is finite then $\bigoplus_{i\in I} M_i = \prod_{i\in I} M_i$. On the other hand, the sum of R-submodules of M, $S = \sum_{i\in I} N_i = \bigoplus_{i\in I} N_i$ iff every element of S can be expressed uniquely a finite sum of elements in N_i, $i \in I$ iff $N_i \bigcap (\sum_{j\neq i} N_j) = 0$ for all $i \in I$, and then it is called an *internal direct sum*.

An R-module M is said to be *decomposable* if M is isomorphic to $M_1 \bigoplus M_2$ where M_1 and M_2 are nonzero R-modules. Otherwise, M is called an *indecomposable* module.

Lemma 2.50 *([18, Proposition 17]) Let R be a ring, M be an R-module and $\{N_i\}_{i\in I}$ be a set of R-modules. Then*

1. $\mathrm{Hom}_R(\bigoplus_{i\in I} N_i, M) \cong \prod_{i\in I} \mathrm{Hom}_R(N_i, M)$.
2. $\mathrm{Hom}_R(M, \prod_{i\in I} N_i) \cong \prod_{i\in I} \mathrm{Hom}_R(M, N_i)$.

2.1.8 Idempotents

Let R be a commutative ring. An element e of R is said to be an *idempotent element* if $e^2 = e$.

The elements 1 and 0 are called *trivial idempotents*.

Definition 2.51 *Two idempotents e_1 and e_2 in a commutative ring R are called* orthogonal *if $e_1e_2 = 0$.*

Notice that the sum of two orthogonal idempotents is idempotent, and if e is an idempotent then $1 - e$ is an idempotent, too.

Lemma 2.52 *([10, Corollary 6.20]) Let R be a commutative ring. Then there is $(1-1)$ correspondence between a direct sum decomposition of the ring R, $R = \bigoplus_{i=1}^{n} I_i$ where I_i are ideals of R, and the set of orthogonal idempotents $\{e_1, \ldots, e_n\}$ such that $e_1 + \cdots + e_n = 1$ where $e_i \in I_i$ for all i. Moreover, $I_i = Re_i$, i.e.,* $R = \bigoplus_{i=1}^{n} Re_i$.

Definition 2.53 *An idempotent e in a commutative ring R is called* primitive *if it could not be expressed as a sum of nonzero orthogonal idempotents.*

The next lemma is a criterion in which idempotent is a primitive idempotent.

Lemma 2.54 *Let e be an idempotent element in a commutative ring R. Then e is primitive iff the ideal Re is indecomposable.*

Proof. It follows from Lemma 2.52. □

Lemma 2.55 *([45, Corollary 7.3.6]) Let R be a commutative ring and I be a nilpotent ideal of R. If $1 = e_1 + \cdots + e_n$ is a sum of orthogonal idempotents in R/I then we can write $1 = f_1 + \cdots + f_n$ where $\{f_i\}_{i=1}^n$ is the set of orthogonal idempotents of R such that $f_i + I = e_i$. If e_i are primitive then so are f_i. In such case, we say that the primitive orthogonal idempotents in R/I* can be lifted *to primitive orthogonal idempotents in R.*

Definition 2.56 *A commutative ring R is called a* semiperfect *ring if $R/\mathrm{nil}(R)$ is semisimple and the primitive orthogonal idempotent in $R/\mathrm{nil}(R)$ can be lifted to primitive orthogonal idempotents in R.*

Lemma 2.57 *([39, Lemma 4.1]) Let R be a semiperfect commutative ring. Then R can be written as $Re_1 \bigoplus \cdots \bigoplus Re_n$ where $\{e_i\}_{i=1}^n$ are primitive orthogonal idempotents and Re_i is a local ring for all $1 \leqslant i \leqslant n$.*

The next corollary is used in the proof of Theorem 1.1.

Corollary 2.58 *Let R be a commutative ring where $\mathrm{Spec}(R)$ is finite such that every pair of prime ideals is coprime. If $\mathrm{nil}(R)$ is nilpotent then R is a finite direct sum of local rings. In particular, a Notherian commutative ring with every pair of distinct minimal prime ideals is coprime, is a direct sum of local rings.*

Proof. Let $\mathrm{Spec}(R) = \{P_1, \ldots, P_n\}$. Since every pair of primes is coprime, by (Chinese reminder theorem), Lemma 2.17,

$$R/\mathrm{nil}(R) \cong \prod_{i=1}^{n} R/P_i.$$

Let $\{e'_1, \cdots, e'_n\}$ the correspondent primitive orthogonal idempotents of the direct sum decomposition of $R/\mathrm{nil}(R)$ where $e'_i \in R/P_i$. Since $\mathrm{nil}(R)$ is nilpotent then the primitive orthogonal idempotents in $R/\mathrm{nil}(R)$ can be lifted to primitive orthogonal idempotents in R, $\{e_i\}_{i=1}^n$, by Lemma 2.55, i.e., $R = \prod_{i=1}^{n} Re_i$ where Re_i is a local ring for all $1 \leqslant i \leqslant n$, by Lemma 2.57. □

2.1.9 Endomorphisms ring

Let R be a ring and M be an R-module. *The ring of endomorphisms* of the module M over R is denoted by $\mathrm{End}_R(M)$, and defined as $\mathrm{End}_R(M) := \{f \mid f \in \mathrm{Hom}_R(M, M)\}$. $\mathrm{End}_R(M)$ is a ring with respect to $(f+g)(m) = f(m) + g(m)$ and $(fg)(m) = f(g(m))$ for every $f, g \in \mathrm{End}_R(M)$ and for every $m \in M$.

Let R be a commutative ring and $f : R \to \mathrm{End}_R(M)$, $r \mapsto f_r$ where $f_r : M \to M$, $m \mapsto rm$. Clearly, f is a ring homomorphism. Suppose that M is a faithful R-module. If $r \in \ker(f)$ then $f_r = 0$, i.e. $rm = 0$ for every $m \in M$. Therefore $r = 0$ (since M is faithful). Hence R can be embedded in $\mathrm{End}_R(M)$.

Let $E = \mathrm{End}_R(M)$. Clearly, an R-module M is an E-module with respect to $E \times M \to M$, $(f, m) \mapsto f(m)$. If ${}_RM$ is a faithful module then $R \subseteq E \subseteq \mathcal{E}$ where $\mathcal{E} = \mathrm{End}_E(M)$ (since M is a faithful E-module).

2.1.10 Exact sequence

A sequence of R-modules $N \xrightarrow{f} M \xrightarrow{g} K$ is said to be an *exact sequence* if $\mathrm{im}(f) = \ker(g)$. A *short exact sequence* is an exact sequence of the form $0 \to N \xrightarrow{f} M \xrightarrow{g} K \to 0$. in this case, f is monomorphism and g is epimorphism.

The short exact sequence $0 \to N \xrightarrow{f} M \xrightarrow{g} K \to 0$ is called *isomorphism* to a short exact sequence $0 \to N' \xrightarrow{f} M' \xrightarrow{g} K' \to 0$ if for every R-modules isomorphisms $\theta : N \to N'$ and $\alpha : K \to K'$, there exists an R-module isomorphism $\varphi : M \to M'$.

The short exact sequence $0 \to N \xrightarrow{f} M \xrightarrow{g} K \to 0$ is said to be *split* if either of the following equivalent statements holds:

1. there exists $g' : K \to M$ such that $gg' = I$,
2. there exists $f' : M \to N$ such that $f'f = I$, or
3. there exists an isomorphism of sequence with the sequence

$$0 \to N \xrightarrow{f'} N \oplus K \xrightarrow{g'} K \to 0.$$

.

Lemma 2.59 *([11, Proposition 2.9] Let $0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$ be a short exact sequence of R-modules and N be an R-module. Then*

1. $0 \to \mathrm{Hom}(N, M_1) \xrightarrow{f_\star} \mathrm{Hom}(N, M_2) \xrightarrow{g_\star} \mathrm{Hom}(N, M_3)$ *is exact where* $g_\star(h') = gh'$ *and* $f_\star(h) = fh$ *for every* $h' \in \mathrm{Hom}(N, M_2)$ *and* $h \in \mathrm{Hom}(N, M_1)$.

2. $0 \to \mathrm{Hom}(M_3, N) \xrightarrow{g^\star} \mathrm{Hom}(M_2, N) \xrightarrow{f^\star} \mathrm{Hom}(M_1, N)$ *is exact where* $f^\star(h) = hf$ *and* $g^\star(h') = h'g$ *for every* $h \in \mathrm{Hom}(M_3, N)$ *and* $h' \in \mathrm{Hom}(M_2, N)$.

2.1.11 Free and projective modules.

Let R be a commutative ring and M be an R-module. The set of elements $B = \{m_i\}_{i\in I}$ of M is said to be *linearly independent* over R provided that $\sum_{i\in I} r_i m_i = 0$ with $r_i \in R$ for every $i \in I$ and $r_i = 0$ for almost all i implies that $r_i = 0$ for all i. B is a *basis* of M over R if B is linearly independent over R and generates M. An R-module M is said to be a *free module* if M has a basis.

Lemma 2.60 *([26, Theorem 2.1]) Let R be a commutative a ring and M be an R-module. Then the following statements are equivalent.*

1. *M is a free R-module.*

2. *M is the internal direct sum of a family of cyclic R-modules, each of which is isomorphic to R.*

3. *M is isomorphic to a direct sum of copies of the R-module R.*

Definition 2.61 *An R-module M is said to be* projective *if for every surjective R-module homomorphism $\theta : K \to N$ and every R-module homomorphism $\alpha : M \to N$, there exists an R-module homomorphism $\varphi : M \to K$ such that $\theta\varphi = \alpha$.*

Lemma 2.62 *([46, Proposition 2.2.1]) An R-module M is a projective iff it is a direct summand of a free module.*

The *projective resolution* of M is an exact sequence

$$\cdots \to P_{n+1} \xrightarrow{d_{n+1}} P_n \to \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M$$

where P_j is projective R-module for all j. If there exists n such that $P_j = 0$ for all $j \geq n+1$, then we say that M has finite resolution of length $\leq n$.

Lemma 2.63 *([46, Lemma 2.2.5]) Let R be a ring. Then every R-module has a projective resolution.*

2.1.12 Ext-groups

A sequence of R-modules

$$\cdots \to C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \to \cdots$$

is called a *chain complex* if $d^n d^{n-1} = 0$ for every n. The *cohomology groups* of the chain complex is defined as $H^n(C) := \ker(d^n)/\mathrm{im}(d^{n-1})$.

Let M and N be R-modules. Then the Ext-groups between them are defined as follows:

$$\mathrm{Ext}^n_R(M, N) = H^n(\mathrm{Hom}(Pn, N)), n \geqslant 0$$

where $\cdots \to P_{n+1} \xrightarrow{d_{n+1}} P_n \to \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M$ is any projective resolution of M.

Lemma 2.64 *([32, Theorem 7.1]) Let M_1 and M_2 be two R-modules. Then $\mathrm{Ext}^1_R(M_1, M_2) = 0$ iff every short exact sequence on the form $0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0$ is split.*

2.2 Preliminaries

In this section, we collect some of the results of multiplication modules over commutative rings that are used later.

We start this section with a criterion for a commutative ring to be a multiplication ring.

Lemma 2.65 *([23, Theorem 3.4]) Let R be a commutative ring. Then R is a multiplication ring iff every prime ideal R is a multiplication ideal.*

Lemma 2.66 *Let R be a commutative ring. Then*

1. *Every cyclic R-module is a multiplication module. In particular, the ring R is a multiplication R-module and every principal ideal ring is a multiplication ring.*

2. *Every invertible ideal of R is a multiplication ideal. In particular, a commutative Dedekind domain is a multiplication ring.*

Proof. 1. Any cyclic R-module is isomorphic to a factor ring of R, and statement 1 follows.

2. Suppose that I is an invertible ideal of R. Then $[R :_{T(R)} I]I = I^{\star}I = R$. Let J be an ideal of R such that $J \subseteq I$. Then $J = RJ = (II^{\star})J = (I^{\star}J)I$. As $I^{\star}J$ is an ideal of R, I is a multiplication ideal. □

We need to the following definition to move to Theorem 2.68 which is a criterion for a module over a commutative ring to be a multiplication module.

Definition 2.67 *Let R be a commutative ring, M be an R-module and $\mathfrak{m} \in \mathrm{Max}(R)$,*

1. *$T_{\mathfrak{m}}(M) = \{x \in M : (1-q)x = 0$ for some $q \in \mathfrak{m}\}$. In case $M = T_{\mathfrak{m}}(M)$, M is called **$\mathfrak{m}$-torsion**. Notice that $T_{\mathfrak{m}}(M)$ is submodule of M.*

2. *M is called **$\mathfrak{m}$-cyclic**, if there exists $x \in M$ and $q \in \mathfrak{m}$ such that $(1-q)M \subseteq Rx$.*

Theorem 2.68 *([21, Theorem 1.2]) Let R be a commutative ring. Then an R-module M is a multiplication module iff for every $\mathfrak{m} \in \operatorname{Max}(R)$, either $M = T_{\mathfrak{m}}(M)$ or M is $\mathfrak{m}$-cyclic.*

Lemma 2.69 *([14, Lemma 2]) Let R be a commutative ring and S be a multiplicatively closed subset of a ring R. If M is a multiplication R-module then $S^{-1}M$ is a multiplication $S^{-1}R$-module.*

Here is another criterion for a module over a commutative ring to be a multiplication module.

Theorem 2.70 *([42, Theorem 2]) Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a collection of multiplication R-submodules of M such that $M = \Sigma_{\lambda\in\Lambda} M_\lambda$. Let $A = \Sigma_{\lambda\in\Lambda}[M_\lambda : M]$. Then the following statements are equivalent.*

1. *M is a multiplication module.*
2. *$\operatorname{ann}_R(m) + A = R$ for all $m \in M$.*
3. *For any $\mathfrak{m} \in \operatorname{Max}(R)$, either M is $\mathfrak{m}$-torsion or there exist elements $z \in \cup_{\lambda\in\Lambda} M_\lambda$ and $q \in \mathfrak{m}$ such that $(1-q)M \subseteq Rz$.*

The next theorem is a criterion for a direct sum of modules to be a multiplication module over a commutative ring.

Theorem 2.71 *([21, Theorem 2.2])Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a collection of R-modules such that $\operatorname{card}(\Lambda) \geq 2$ and $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$. Then M is a multiplication module iff*

1. *M_λ is a multiplication module for all $\lambda \in \Lambda$, and*
2. *For every $\lambda \in \Lambda$, there exists an ideal A_λ of R such that $A_\lambda M_\lambda = M_\lambda$ and $A_\lambda M'_\lambda = 0$ where $M'_\lambda = \bigoplus_{\mu\neq\lambda} M_\mu$.*

The next theorem is a characterization of a faithful multiplication module over a commutative ring.

Theorem 2.72 *([21, Theorem 1.6]) Let R be a commutative ring and M be a faithful R-module. Then M is a multiplication module iff M satisfies the following conditions:*

1. *for every set of ideals $\{I_\lambda\}_{\lambda\in\Lambda}$ of R, $\bigcap_{\lambda\in\Lambda}(I_\lambda M) = (\bigcap_{\lambda\in\Lambda} I_\lambda)M$, and*
2. *for every submodule N of M and an ideal A of R such that $N \subset AM$ there exists an ideal B of R such that $B \subset A$ and $N \subseteq BM$.*

The companion ideal, $\theta(M)$, of an R-module M is a useful tool in studying multiplication modules.

The next results are about some characteristics of a multiplication R-module M in terms of the ideal $\theta(M)$.

Lemma 2.73 *Let R be a commutative ring and M be an R-module.*

1. *([9, Lemma 1.1]) If M is a multiplication R-module and N is a submodule of M then $M = \theta(M)M$ and $N = \theta(M)N$.*
2. *([1, Lemma 1.3]) M is a multiplication R-module iff $\theta(M) + \mathrm{ann}_R(m) = R$ for all $m \in M$ iff $Rm = \theta(M)m$ for all $m \in M$.*
3. *([9, Theorem 2.3]) If M is a faithful multiplication R-module then*
   (a) $\theta(M)$ is a multiplication ideal of R.
   (b) $\theta(M)$ is an idempotent ideal of R.
4. *([2, Theorem 1.3]) If M is a faithful multiplication R-module then $\theta(\theta(M)) = \theta(M)$.*

The next three lemmas are technical lemmas of the thesis.

Lemma 2.74 *([21, Corollary 1.4]) Let I be a multiplication ideal of a commutative ring R and M be a multiplication R-module. Then IM is multiplication module.*

Lemma 2.75 *([9, Lemma 2.1]) Let R be a commutative ring and M be multiplication R-module. If I is an ideal of R such that IM is finitely generated then $I \subseteq \theta(M)$.*

Lemma 2.76 *([9, Lemma 2.1]) Let R be a commutative ring and M be a multiplication R-module. If I is a finitely generated ideal of R where $I \subseteq \theta(M)$ then IM is finitely generated.*

The next lemma is a criterion for a multiplication module over a commutative ring R to be finitely generated in terms of the ideal $\theta(M)$.

Lemma 2.77 *([9, Corollary 2.2]) Let M be a multiplication R-module. Then the following statements are equivalent.*

1. *The R-module M is finitely generated.*
2. *$\theta(M) = R$.*
3. *The R-module $\theta(M)$ is finitely generated.*

In particular, every multiplication module over Noetherian commutative ring is finitely generated.

Y. Alshaniafi and S. Singh provide a cancellation law of a faithful multiplication module over a commutative ring R as follows:

Lemma 2.78 *([2, Theorem 1.4]) Let R be a commutative ring and M be a faithful multiplication R-module. If I and J are two ideals of R that are contained in $\theta(M)$ then $IM = JM$ iff $I = J$.*

By Lemmas 2.77 and 2.78, if M is a finitely generated faithful multiplication R-module and I and J are two ideals of R then $IM = JM$ iff $I = J$.

We can generalize Lemma 2.78 without faithfulness condition as follows:

Corollary 2.79 *Let R be a commutative ring and M be a multiplication R-module. If I and J are two ideals of R then $IM = JM$ iff $I\theta(M) + \operatorname{ann}_R(M) = J\theta(M) + \operatorname{ann}_R(M)$.*

Proof. Let $\bar{R} = R/\text{ann}_R(M)$, $\bar{I} = (I + \text{ann}_R(M))/\text{ann}_R(M)$ and $\bar{J} = (J + \text{ann}_R(M))/\text{ann}_R(M)$. Clearly, M is a faithful multiplication $\bar{R}$-module and $\bar{I}M = \bar{J}M$. By Lemma 2.78, $\bar{I}\theta(M)_{\bar{R}} = \bar{J}\theta(M)_{\bar{R}}$. Hence, $I\theta(M) + \text{ann}_R(M) = J\theta(M) + \text{ann}_R(M)$. the converse is obvious. $\square$

As a corollary of Lemma 2.78, we have the following interesting properties which we use widely in the thesis.

Lemma 2.80 *[2, Theorem 1.5] Let R be a commutative ring and M be a faithful multiplication R-module. Then*

1. *If I and J are two ideals of R then $IM = JM$ iff $I \bigcap \theta(M) = J \bigcap \theta(M)$.*
2. *For every ideal I of R, $I \bigcap \theta(M) = I\theta(M)$.*
3. *Every submodule N of M could be written as a unique form IM where I is an ideal of R that is contained in $\theta(M)$.*

We can remark that Lemma 2.78 shows that if M is a faithful multiplication module and $IM = JM$ where I and J are ideals of R then $I \cap \theta(M) = I\theta(M) = J\theta(M) = J \cap \theta(M)$.

Lemma 2.81 *([42, Theorem 9]) Let R be a commutative ring and M be a faithful multiplication R-module and I and J are two ideals of R. Then $IM \subseteq JM$ iff $M = [J : I]M$.*

Corollary 2.82 *Let M be a faithful multiplication R-module and $IM \subseteq JM$ where I and J are ideals of R. Then $I\theta(M) \subseteq J\theta(M)$.*

Proof. By Lemma 2.81, $M = [J : I]M$. Therefore, by Lemma 2.78, $\theta(M) = [J : I]\theta(M)$. Consequently, $I\theta(M) = [J : I]I\theta(M) \subseteq J\theta(M)$. $\square$

In section 4.3, we give an example shows that a submodule of a multiplication module is a multiplication module is not necessary to be multiplication. P. F. Smith [42] addressed the question of under which conditions a submodule of a multiplication module would be a multiplication module in the following lemma.

Lemma 2.83 *([42, Theorem 10]) Let M be a finitely generated faithful multiplication R-module. If N is a submodule of M and I is an ideal of R then*

1. *N is a multiplication module iff $[K : M] = [K : N][N : M]$ for each submodule K of N.*
2. *$I = [IM : M]$.*
3. *IM is a multiplication module iff I is a multiplication ideal of R.*
4. *N is a multiplication module iff $[N : M]$ is a multiplication ideal of R.*

The next proposition is a generalization of Lemma 2.83 to study multiplication submodules of a faithful multiplication module without need the condition of finitely generated.

Proposition 2.84 *Let M be a faithful multiplication R-module. If N is a submodule of M and I is an ideal of R then*

1. *N is a multiplication module iff $[K : M]\theta(M) = [K : N][N : M]\theta(M)$ for each submodule K of N.*
2. *$I\theta(M) = [IM : M]\theta(M)$.*
3. *IM is a multiplication module iff $I\theta(M)$ is a multiplication ideal of R.*
4. *N is a multiplication module iff $[N : M]\theta(M)$ is a multiplication ideal of R.*

Proof. 1. Suppose that N is a multiplication module and K is a submodule of N. Then $K = [K : N]N = [K : N][N : M]M$ and $K = [K : M]M$, as well. Therefore, by Lemma 2.78,

$$[K : N][N : M]\theta(M) = [K : M]\theta(M).$$

Conversely, suppose that $[K : N][N : M]\theta(M) = [K : M]\theta(M)$ for any submodule K of N. Let L be a submodule of N. Then

$$L = [L : M]M = [L : M]\theta(M)M = [L : N][L : M]\theta(M)M = [L : N]N$$

(Since L is a submodule of M and M is a multiplication module), and therefore N is a multiplication module.

2. Since $I\theta(M)M = IM = [IM : M]M = [IM : M]\theta(M)M$, by Lemma 2.78, $I\theta(M) = [IM : M]\theta(M)$.

3. Suppose that $I\theta(M)$ is a multiplication ideal of R. Then $IM = I\theta(M)M$ is a multiplication module, by Lemma 2.74. Conversely, suppose that IM is a multiplication module. Let J be an ideal of R such that $J \subseteq I\theta(M)$. Then $J \subseteq \theta(M)$, and therefore, by Corollary 2.80, $J = J\theta(M)$. By statements 1 and 2,

$$J = J\theta(M) = [JM : M]\theta(M) = [JM : IM][IM : M]\theta(M) = [JM : IM]I\theta(M),$$

and hence, $I\theta(M)$ is a multiplication ideal.

4. It follows from statement 3 (since $N = [N : M]M$). $\square$

Let R be a commutative ring and M be an R-module. $\mathcal{I}(\theta(M))$ denotes by the set of ideals of R that are contained in $\theta(M)$. In case M is a faithful multiplication module over a commutative ring R.

The next corollary shows that there is a (1-1) correspondence between $\mathrm{Sub}_R(M)$ and $\mathcal{I}(\theta(M))$.

Corollary 2.85 *Let R be a commutative ring and M be a faithful multiplication R-module. Then the map $\lambda_\theta : \mathrm{Sub}_R(M) \to \mathcal{I}(\theta(M))$, $N = IM \mapsto I\theta(M)$ is a bijection respects the inclusion, i.e., if $N \subseteq N'$ then $I \subseteq I'$ where I and I' are the correspondent ideals of N and N', respectively in $\mathcal{I}(\theta(M))$.*

Proof : 1. The map λ_θ is well-defined : Suppose $N = IM$. Using the equality $M = \theta(M)M$, I have $N = IM = I\theta(M)M$. Hence,

$$\lambda_\theta(IM) = I\theta(M)\theta(M) = I\theta(M)$$

(since $\theta(M)^2 = \theta(M)$, by Lemma 2.73). So, $\lambda_\theta(IM) \in \mathcal{I}(\theta(M))$. It remains to show that if $N = I^*M$ for another ideal I^* of R then $I\theta(M) = I^*\theta(M)$. It holds, by Lemma 2.80.

2. λ_θ is a surjection : Given $J \in \mathcal{I}(\theta(M))$. Then $J = J\theta(M) = \lambda_\theta(JM)$.

3. λ_θ is an injection : Given two submodules N and N^* of M. Then $N = IM$ and $N^* = I^*M$ for some ideals I and I^* of R. Suppose that $\lambda_\theta(N) = \lambda_\theta(N^*)$, i.e., $I\theta(M) = I^*\theta(M)$, by Lemma 2.80. Then

$$N = IM = I\theta(M)M = I^*\theta(M)M = N^*,$$

i.e., λ_θ is an injection.

Moreover, the existing bijection between $\mathrm{Sub}_R(M)$ and $\mathcal{I}(\theta(M))$ respects the inclusion, by Corollary 2.82. $\square$

The next theorem gives a criterion for a faithful multiplication module to be finitely generated.

Theorem 2.86 *([21, Theorem 3.1]) and ([21, Proposition 3.4]) Let M be a faithful multiplication R-module. Then the following statements are equivalent.*

1. *M is finitely generated.*
2. *For all ideals I and J of R such that $IM \subseteq JM$, $I \subseteq J$.*
3. *for every submodule N of M there exists a unique ideal I of R such that $N = IM$.*
4. *$IM \neq M$ for every proper ideal I of R.*
5. *$QM \neq M$ for every $Q \in \mathrm{Max}(R)$.*
6. *$PM \neq M$ for every minimal prime ideal P of R.*

Corollary 2.87 *Let R be a domain and M be a faithful multiplication R-module. Then M is finitely generated. In particular, a multiplication domain is a Noetherian ring.*

Proof. As R is a domain, R has only one minimal prime which is 0. By Theorem 2.86, M is finitely generated. $\square$

Lemma 2.88 *Let I be an ideal of a ring R. Then I is an invertible ideal iff I is a multiplication ideal which contains a nonzero divisor. In particular, If R is a domain then I is a multiplication ideal iff I is invertible ideal.*

Proof. If I is an invertible ideal then I is a multiplication module, by Lemma 2.66 . Let $K = \mathrm{T}(R)$ be the total ring of fractions of R. As I is an invertible ideal of R, $[R :_K I]I = R$. So, $1 = ab$ where $a \in I$ and $b \in [R :_K I]$. Now, suppose that $ra = 0$ for some $r \in R$. Then $r = r(1) = r(ab) = (ra)b = 0$, i.e., a is a nonzero divisor. Conversely, suppose that I is a multiplication module with a nonzero divisor c. Then $Rc = [Rc : I]I$ (since I is a multiplication ideal).

Now, $R = RcR(1/c) = ([Rc : I]I)RcR(1/c) \subseteq [R :_K I]I \subseteq R$. Therefore, $[R :_K I]I = R$, i.e., I is an invertible ideal. □

It is known that every proper ideal of a commutative ring is contained in a maximal ideal, whereas it is not always true that every proper submodule of an R-module M is contained in a maximal submodule. For example, as a $\mathbb{Z}$-module, $\mathbb{Z}$ is not contained in any maximal $\mathbb{Z}$-submodule of $\mathbb{Q}$.

The next corollary shows that that case holds for a proper submodule of a nonzero multiplication module over a commutative ring.

Corollary 2.89 *([21, Theorem 2.5]) Let R be a commutative ring and M be a nonzero multiplication R-module. Then every proper submodule of M is contained in a maximal submodule of M.*

The next lemma provides an explicit description of maximal submodules of a multiplication module.

Lemma 2.90 *([21, Theorem 2.5]) Let R be a commutative ring and N be a submodule of a nonzero multiplication R-module M. Then N is a maximal submodule of M iff there exists $Q \in \mathrm{Max}(R)$ such that $N = QM$ and $QM \neq M$.*

Let M be an R-module. We recall that the *radical* of M, $\mathrm{rad}(M)$, is the intersection of all maximal submodules of M.

Lemma 2.91 *([21, Corollary 2.6]) Let R be a commutative ring and N be a submodule of a multiplication R-module such that $M = N + \mathrm{rad}(M)$. Then $N = M$.*

Lemma 2.92 *([21, Theorem 2.8]) Let R be a commutative ring with only finitely many maximal ideals. If M is a multiplication R-module then M is cyclic. In particular, if R is an Artinian ring then every multiplication R-module is cyclic.*

Theorem 2.93 *Let M be a faithful multiplication R-module. Then* $\mathrm{rad}(M) = J(R)M$.

Proof. It follows from Lemma 2.90 and Theorem 2.72. □

Definition 2.94 *Let R be a commutative ring. A proper submodule N of a nonzero R-module M is called a* prime submodule *of M if $[N : K] = [N : M]$ for every submodule K of M such that $N \subset K \subseteq M$.*

The next lemma is a criterion for a proper submodule of an R-module M to be prime.

Lemma 2.95 *Let R be a commutative ring. A proper submodule N of an R-module M is prime iff for every $r \in R$ and $m \in M$, if $rm \in N$ and $m \notin N$ then $r \in [N : M]$.*

Proof. ($\Rightarrow$) Suppose that N is a prime submodule and $rm \in N$ such that $m \notin N$. Then $N \subset N + Rm \subseteq M$. It follows that $r \in [N : N + Rm]$, and therefore $r \in [N : M]$.

($\Leftarrow$) Suppose that $N \subset K \subseteq M$, i.e. there exists $x \in K \setminus N$. The goal is to prove that $[N : K] = [N : M]$. Clearly, $[N : M] \subseteq [N : K]$. Now, suppose that $r \in [N : K]$. So, $rx \in N$, and by assumption, $r \in [N : M]$. Hence, $[N : K] \subseteq [N : M]$. □

Theorem 2.96 *([21, Theorem 2.10]) Let R be a commutative ring and M be a multiplication R-module. If $P \in \operatorname{Spec}(R)$ such that $\operatorname{ann}_R(M) \subseteq P$ and $PM \neq M$ then PM is a prime submodule of M.*

Corollary 2.97 *([21, Corollary 2.11]) Let R be a commutative ring and N be a proper submodule of a nonzero multiplication R-module M. Then the following statments are equivalent.*

1. *N is a prime submodule of M.*
2. *$[N : M]$ is a prime ideal of R.*
3. *$N = PM$ for some prime ideal P of R with $\operatorname{ann}(M) \subseteq P$.*

A submodule N of an R-module M is a *minimal prime* submodule if there is a prime submodule N' of M such that $N' \subseteq N$ then $N' = N$.

Theorem 2.92 shows that a multiplication module with only finitely many maximal submodules is cyclic. The next proposition is a companion result.

Proposition 2.98 *([21, Theorem 3.7]) Let R be a commutative ring and M be a faithful multiplication R-module such that M has only finitely many of minimal prime submodules. Then M is finitely generated.*

The next lemma is a description of the endomorphism ring of a finitely generated multiplication module.

Lemma 2.99 *([37, Corollary 3.3]) Let R be a commutative ring and M be a finitely generated multiplication R-module. Then $\mathrm{End}_R(M) \cong R/\mathrm{ann}_R(M)$ is finitely generated.*

Chapter 3

Characterization of multiplication commutative rings with finitely many minimal prime ideals

Throughout this chapter all rings are commutative. A ring R is called a multiplication ring if for every pair of ideals I and J of R where $J \subseteq I$ then $J = I'I$ for some ideal I' of R.

In this chapter, we give a classification of the commutative multiplication rings with finitely many minimal prime ideals.

3.1 Finitely generated prime ideals of a multiplication ring with zero annihilators

In this section, we present some properties of finitely generated prime ideals with zero annihilator of a multiplication ring. The results of this section are used in proofs of the subsequent section.

Proposition 3.1 *Let R be a multiplication ring. Then every finitely generated prime ideal with zero annihilator is a maximal ideal. In particular, if R is a multiplication domain then every nonzero prime ideal of R is a maximal ideal.*

Proof. Let $P \in \mathrm{Spec}(R)$ where P is finitely generated with zero annihilator, and suppose that $P \subsetneq J \subseteq R$ where J is an ideal of R. Since R is a multiplication ring, $P = IJ$ for some ideal I of R. It follows that

$I \subseteq P = IJ \subseteq I$ (since P is a prime ideal and $J \subsetneq P$). Hence, $I = P$. As P is a finitely generated multiplication ideal with zero annihilator and $P = PJ$, by Lemma 2.77 and Lemma 2.78, $J = R$, i.e., P is a maximal ideal of R. The result holds for a multiplication domain because every nonzero ideal is a finitely generated ideal with zero annihilator, by Corollary 2.87. □

Proposition 3.2 *Let R be a multiplication ring. If P is a finitely generated prime ideal of R with zero annihilator then $R \supsetneq P \supsetneq P^2 \supsetneq \cdots \supsetneq P^n \supsetneq \cdots$ is a strictly descending chain of ideals such that all R-modules P^n/P^{n+1} are isomorphic to the simple R-module R/P.*

Proof. Since the ideal P is a finitely generated R-module, so are all its powers P^n, $n \geq 1$.

(i) *All R-modules $\{P^n\}_{n\geq 0}$ have zero annihilator*: Suppose that $rP^n = 0$ for some nonzero element $r \in R$ and $n \geq 0$, we seek a contradiction. Clearly, $n \geq 2$ since $P^0 = R \ni 1$ and $\mathrm{ann}_R(P) = 0$. We assume that n is the least possible. Then $(rP^{n-1})P = 0$, and so, $rP^{n-1} = 0$ (since $\mathrm{ann}_R(P) = 0$), a contradiction.

By Proposition 3.1, the ideal P of R is a maximal ideal. In particular, $R = P^0 \neq P$.

(ii) *The ideals $\{P^n\}_{n\geq 0}$ are distinct*: Suppose this is not the case, we seek a contradiction. we can choose the least natural number $n \geq 0$ such that $P^n = P^{n+1}$. Clearly, $n \geq 1$. The R-modules P^n are finitely generated faithful multiplication modules. By Lemma 2.78, the equality $P^n = P^{n+1}$ implies the equality $P = R$, a contradiction.

(iii) *For all $n \geq 0$, the R-modules P^n/P^{n+1} are isomorphic to the simple R-module R/P*: Recall that P is a maximal ideal of the ring R. Hence, the R-module R/P is simple. Clearly, the R-modules P^n/P^{n+1} are R/P-modules and R/P is a field. By the statement (ii), the R-modules P^n/P^{n+1} are nonzero. To prove that the statement (iii) holds it suffices to show that the R-module P^n/P^{n+1} is simple. Given an ideal J of R such that $P^{n+1} \subsetneq J \subseteq P^n$, we have to show that $J = P^n$. The ring R is a multiplication ring. So, the inclusions $P^{n+1} \subseteq J$ and $J \subseteq P^n$ yield the equalities $P^{n+1} = IJ$ and $J = J'P^n$ for some ideals I and J' of R. Therefore, $P^{n+1} = IJ'P^n$, and, by Lemma 2.78, $P = IJ'$. Hence, either $P = I$ or $P = J'$ (since P is a prime ideal). Hence, either $P^{n+1} = PJ$ or $J = P^{n+1}$. The second case is not possible, by the choice of J. So, $P^{n+1} = PJ$. Then, by Lemma 2.78, $J = P^n$, as required. □

Proposition 3.3 *Let R be a multiplication domain and P be a nonzero prime ideal. Then $IP = I \cap P$ for every ideal I of R such that $I \nsubseteq P$.*

Proof. Since R is a multiplication domain, every ideal of R is finitely generated, by Corollary 2.87. Since $I \cap P \subseteq I$ and I is a multiplication ideal of R, $I \cap P = I'I$ for some ideal I' of R. Also, there is an ideal $I^\star$ of R such that $IP = I^\star(I \cap P)$ (since $IP \subseteq I \cap P$ and $I \cap P$ is a multiplication ideal). So, $PI = I^\star I' I$. By Lemma 2.77 and Lemma 2.78, $P = I^\star I'$. Since P is a prime ideal then either $I^\star = P$ or $I' = P$. If $I^\star = P$ then $IP = (I \cap P)P$, and, by Lemma 2.77 and Lemma 2.78, $I = I \cap P$, and so $I \subseteq P$ (a contradiction). Therefore $I' = P$, and hence $I \cap P = IP$. □

Proposition 3.4 *Let R be a multiplication domain. Then for each $P \in \mathrm{Spec}(R)$, P^n is a P-primary ideal for any $n \in \mathbb{N}$.*

Proof. As R is a multiplication domain, every ideal is finitely generated, by Corollary 2.87. Suppose that $IJ \subseteq P^n$ and $I \nsubseteq P^n$. We have to show that $J \subseteq P$. Since R is a multiplication ring, $IJ = KP^n$ for some ideal K of R. Now, since $I \nsubseteq P^n$, there exists a natural number n' such that $n' < n$ and $I \subseteq P^{n'}$ and $I \nsubseteq P^{n'+1}$ (notice that $P^0 = R$). So, there exists an ideal $I^\star$ of R such that $I = I^\star P^{n'}$. As $IJ = KP^n = I^\star P^{n'} J$ and $P^{n'}$ is a finitely generated multiplication ideal with zero annihilator, $KP^{n-n'} = I^\star J \subseteq P$, by Lemma 2.78. As $I^\star \nsubseteq P$ and P is a prime ideal, $J \subseteq P$. Hence, P^n is P-primary. □

The following three lemmas are obvious.

Lemma 3.5 *Let $R = \prod_{i \in I} R_i$ be a direct product of rings. Then R is a multiplication ring iff all rings R_i are multiplication rings.*

Lemma 3.6 *Let R be a multiplication ring and I be an ideal of R. Then R/I is a multiplication ring.*

Lemma 3.7 *Let R be a multiplication ring. Then $S^{-1}R$ is a multiplication ring where S is a multiplicatively closed subset of R.*

The next theorem is a description of local multiplication rings with nilpotent maximal ideal which are not fields.

Theorem 3.8 *Let $(R, \mathfrak{m})$ be a local ring and $\mathfrak{m}$ is nilpotent. Then the ring R is a multiplication ring iff it is an Artinian, principal ideal ring. If so, then $\mathfrak{m} = (x)$ for some element x of R and $\{(x^i) \mid i = 0, 1, \dots, \nu+1\}$ are the only distinct ideals of R where $\mathfrak{m}^{\nu+1} = 0$ and $\mathfrak{m}^{\nu} \neq 0$, and $\mathrm{Ass}(R) = \{\mathfrak{m}\}$.*

Proof. ($\Rightarrow$) Let $K = R/\mathfrak{m}$ (the residue field of $\mathfrak{m}$). Then $V = \mathfrak{m}/\mathfrak{m}^2$ is a vector space over K.

(i) $\dim_K(V) = 1$: Given a nonzero subspace U of V. We have to show that $U = V$. Clearly, $U = I/\mathfrak{m}^2$ for some ideal I such that $\mathfrak{m}^2 \subsetneq I \subseteq \mathfrak{m}$. Since the ring R is a multiplication ring, $I = J\mathfrak{m}$ for some ideal J of R which is necessarily equal to R (since $\mathfrak{m}^2 \subsetneq I$ and $(R, \mathfrak{m})$ is a local ring), i.e., $I = \mathfrak{m}$, and so $U = V$, as required.

(ii) $\mathfrak{m} = (x)$ for some $x \in R$: Fix an element $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. By statement (i), $\mathfrak{m} = Rx + \mathfrak{m}^2$. Then $\mathfrak{m} = Rx + (Rx + \mathfrak{m})^2 = Rx + \mathfrak{m}^3 = \cdots = Rx + Rx^{\nu} + \mathfrak{m}^{\nu+1} = Rx$ since $\mathfrak{m}^{\nu+1} = 0$.

(iii) *The ring R is Artinian*: By the statement (ii), the length $\ell(R)$ of the R-module R is equal to $\sum_{i=0}^{\nu+1} \ell(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \nu + 1 < \infty$, and the statement (iii) follows.

(iv) $\{(x^i) \mid i = 0, 1, \dots, \nu + 1\}$ *are the only distinct ideals of R; in particular, R is a principal ideal ring*: Let I be a nonzero ideal of R. We may assume that $I \neq R$, i.e., $I \subseteq \mathfrak{m}$. Then there exists a unique natural number i such that $I \subseteq \mathfrak{m}^i$ but $I \nsubseteq \mathfrak{m}^{i+1}$. We claim that $I = \mathfrak{m}^i$. Fix an element $y \in I$ such that $y \in \mathfrak{m}^i \setminus \mathfrak{m}^{i+1}$. Since $\mathfrak{m}^i = (x^i)$, $y = x^i u$ for some element $u \in R$ such that $u \notin \mathfrak{m}$ (since $y \notin \mathfrak{m}^{i+1}$), i.e., u is a unit of R. Then $\mathfrak{m}^i = (x^i) = (x^i u) = (y) \subseteq I \subseteq \mathfrak{m}^i$, and so $I = \mathfrak{m}^i = (x^i)$. If $(x^s) = (x^t)$ for some natural numbers s and t such that $0 \leqslant s \leqslant t \leqslant \nu + 1$ then $s = t$, by the Nakayama Lemma, and the statements (iv) follows.

($\Leftarrow$) Since the ring R is a principal ideal ring, it is a multiplication ring. $\square$

3.2 Proof of Theorem 1.1

Theorem 1.1 is a classification of multiplication rings with finitely many minimal prime ideals.

We start this section with the following result which is used in the proof of Theorem 3.10.

Lemma 3.9 *([7, Theorem 1.1]) Let R be a local ring. Then all multiplication R-modules are cyclic. In particular, if R is a multiplication ring then all ideals of R are cyclic.*

For a ring R, we denote by $\mathrm{Min}(R)$ and $\mathrm{Max}(R)$ the sets of minimal prime and maximal ideals of R, respectively.

The next theorem is a description of multiplication rings that have a unique minimal prime ideal which is not maximal. The theorem is used in the proof of Theorem 1.1.

Theorem 3.10 *Let R be a ring such that $\mathrm{Min}(R) = \{\mathfrak{p}\}$ and $\mathfrak{p}$ is not a maximal ideal. Then R is a multiplication ring iff R is a Dedekind domain. If so, then $\mathfrak{p} = 0$.*

Proof. ($\Rightarrow$) (i) $\mathfrak{p} = 0$: Since $\mathfrak{p}$ is a unique minimal prime ideal of the ring R and it is not maximal, it is properly contained in every maximal ideal of R. Let $\mathfrak{m} \in \mathrm{Max}(R)$. Then $\mathfrak{p} \subsetneq \mathfrak{m}$ and $\mathfrak{p} = \mathfrak{a}\mathfrak{m}$ for some ideal $\mathfrak{a}$ of R such that $\mathfrak{p} \subseteq \mathfrak{a}$ (since R is a multiplication ring). Hence $\mathfrak{a} = \mathfrak{p}$ (since $\mathfrak{p}$ is a prime ideal and $\mathfrak{p} \subsetneq \mathfrak{m}$), i.e., $\mathfrak{p} = \mathfrak{p}\mathfrak{m}$. Then localizing at $\mathfrak{m}$, we have the equality of $R_\mathfrak{m}$-modules, $\mathfrak{p}_\mathfrak{m} = \mathfrak{p}_\mathfrak{m}\mathfrak{m}_\mathfrak{m}$. The ring R is a multiplication ring hence so is the local ring $(R_\mathfrak{m}, \mathfrak{m}_\mathfrak{m})$. By Lemma 3.9, the $R_\mathfrak{m}$-module $\mathfrak{p}_\mathfrak{m}$ is cyclic. By applying the Nakayama Lemma to the equality $\mathfrak{p}_\mathfrak{m} = \mathfrak{p}_\mathfrak{m}\mathfrak{m}_\mathfrak{m}$, we must have $\mathfrak{p}_\mathfrak{m} = 0$ for all $\mathfrak{m} \in \mathrm{Max}(R)$. Therefore, $\mathfrak{p} = 0$.

(ii) *R is a domain* (by the statement (i)).

(iii) *All maximal ideals of R has height* 1: This statement follows from the statement (ii) and Proposition 3.1.

(iv) *For every maximal ideal $\mathfrak{m}$, $R_\mathfrak{m}$ is a discrete valuation ring*: The ring $(R_\mathfrak{m}, \mathfrak{m}' = \mathfrak{m}_\mathfrak{m})$ is a local multiplication domain. By Lemma 3.9, every ideal is 1-generated. In particular, $\mathfrak{m}' = (x)$ for some element $x \in R$. We have to show that every proper ideal I of $R_{\mathfrak{m}'}$ ($I \neq 0, R_\mathfrak{m}$) is equal to $x^i R_\mathfrak{m}$ for some $i \geqslant 1$. There is a unique natural number $i \geqslant 1$ such that $I \subseteq \mathfrak{m}'^i$ but $I \subsetneq \mathfrak{m}'^{i+1}$. Notice that $I = yR$ for some $y \in \mathfrak{m}'^i \setminus \mathfrak{m}'^{i+1}$. Then $y = x^i u$ for some $u \in R_\mathfrak{m} \setminus \mathfrak{m}'$, a unit of $R_\mathfrak{m}$. Hence, $I = yR_\mathfrak{m} = x^i u R_\mathfrak{m} = x^i R_\mathfrak{m}$.

(v) *R is a Dedekind domain*: This follows from the statement (iv) and [11, Theorem 9.3].

($\Leftarrow$) Recall that every nonzero ideal of a Dedekind domain R is a unique finite product of maximal ideals. Hence, every ideal of R is a multiplication module, i.e., R is a multiplication ring. $\square$

Example. Let K be a field. Then the local ring $R = K[x]_{(x)}[y,z]/(y^2, yz, z^2) = D \bigoplus Dy \bigoplus Dz$ is not a multiplication ring where $D = K[x]_{(x)}$ is a local Dedekind domain, and $\mathfrak{p} = Dy \bigoplus Dz$ is a unique minimal prime which is not a maximal ideal and $\mathfrak{p}^2 = 0$.

Proof. If R were a multiplication ring then, by Theorem 3.10, $\mathfrak{p} = 0$, a contradiction. $\square$

Proof of Theorem 1.1. ($\Rightarrow$) Recall that the set $\mathrm{Min}(R)$ is a finite set then R is a Noetherian ring, by [36, Theorem 11], and so, the prime radical $\mathfrak{n} = \cap_{\mathfrak{p}\in\mathrm{Spec}(R)}\mathfrak{p}$ is a nilpotent ideal. Let $\mathcal{M} = \mathrm{Min}(R) \cap \mathrm{Max}(R) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_t\}$ and $\mathcal{M}' = \mathrm{Min}(R) \setminus \mathrm{Max}(R) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$.

(i) *If $\mathcal{M}' = \emptyset$, i.e., $\mathrm{Min}(R) = \mathrm{Max}(R) = \mathcal{M} = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_t\}$ then $R \cong \prod_{\mathfrak{m}_i\in\mathrm{Min}(R)} R_i$ is a product of Artinian, local principal ideal rings*: Since $\mathrm{Min}(R) = \mathrm{Max}(R)$, the ring R is an Artinian ring. Hence it is a finite direct product of Artinian local rings, say $R = \prod_{i=1}^{n} R_i$. Since R is a multiplication ring, so are the rings R_i. By Theorem 3.8, the rings R_i are Artinian, local, principal ideal ring.

Till the end of the proof we assume that $\mathcal{M}' \neq \emptyset$, i.e., $\overline{\mathcal{M}} := \mathrm{Max}(R) \setminus \mathcal{M} = \mathrm{Max}(R) \setminus \mathrm{Min}(R) \neq \emptyset$.

(ii) *Every maximal ideal $\mathfrak{m} \in \overline{\mathcal{M}}$ contains a unique minimal prime ideal $\mathfrak{p}(\mathfrak{m})$ that necessarily belongs to $\mathcal{M}'$*: The maximal ideal of R contains at least one minimal prime ideal, say, $\mathfrak{p} = \mathfrak{p}(\mathfrak{m})$. Suppose that $\mathfrak{p}'$ is another minimal prime ideal that is contained in $\mathfrak{m}$, we seek a contradiction. The ring $R_\mathfrak{m}$ is a local multiplication ring with the maximal ideal $\mathfrak{m}' = \mathfrak{m}R_\mathfrak{m}$.

Claim: $\mathfrak{p}_\mathfrak{m} = 0$ and $\mathfrak{p}'_\mathfrak{m} = 0$.

By Lemma 3.9, every ideal of the ring $R_\mathfrak{m}$ is 1-generated. In particular, $\mathfrak{m}' = (x)$ and $\mathfrak{p}_\mathfrak{m} = (x')$ for some elements $x, x' \in R$. Since $\mathfrak{p}_\mathfrak{m} \subseteq \mathfrak{m}'$ and $R_\mathfrak{m}$ is a multiplication ring, we must have $\mathfrak{p}_\mathfrak{m} = \mathfrak{a}\mathfrak{m}'$ for some ideal $\mathfrak{a}$ of $R_\mathfrak{m}$ that contains

$\mathfrak{p}_\mathfrak{m}$. Since $\mathfrak{m}' \nsubseteq \mathfrak{p}_\mathfrak{m}$ and $\mathfrak{p}_\mathfrak{m}$ is a prime ideal, we must have $\mathfrak{a} \subseteq \mathfrak{p}_\mathfrak{m}$, i.e., $\mathfrak{a} = \mathfrak{p}_\mathfrak{m}$, and so $\mathfrak{p}_\mathfrak{m} = \mathfrak{p}_\mathfrak{m}\mathfrak{m}'$. Since $\mathfrak{p}_\mathfrak{m}$ is a finitely generated $R_\mathfrak{m}$-module and $(R_\mathfrak{m}, \mathfrak{m}')$ is a local ring, $\mathfrak{p}_\mathfrak{m} = 0$, by the Nakayama Lemma. The proof of the claim is complete.

Since $\mathfrak{p} \neq \mathfrak{p}'$ and $\mathfrak{p}, \mathfrak{p}' \subseteq \mathfrak{m}$, we must have $\mathfrak{p}_\mathfrak{m} \neq \mathfrak{p}'_\mathfrak{m}$ which contradicts to the fact $\mathfrak{p}_\mathfrak{m} = 0 = \mathfrak{p}'_\mathfrak{m}$, by the Claim.

For each $\mathfrak{p}_i \in \mathcal{M}'$, let $\mathcal{V}(\mathfrak{p}_i) = \{\mathfrak{m} \in \mathrm{Max}(R) \,|\, \mathfrak{p}_i \subseteq \mathfrak{m}\} = \{\mathfrak{m} \in \overline{\mathcal{M}} \,|\, \mathfrak{p}_i \subseteq \mathfrak{m}\}$.

(iii) $\mathrm{Max}(R) = \mathcal{M} \coprod \mathcal{V}(\mathfrak{p}_1) \coprod \ldots \coprod \mathcal{V}(\mathfrak{p}_s)$, *a disjoint union*: The statement (iii) follows from the statement (ii).

(iv) *All minimal prime ideals of* R *are co-prime ideals*: Recall that $\mathrm{Min}(R) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s, \mathfrak{m}_1, \ldots, \mathfrak{m}_t\}$ and $\mathfrak{m}_1, \ldots, \mathfrak{m}_t$ are maximal ideals. So, it suffices to show that $\mathfrak{p}_i + \mathfrak{p}_j = R$ for all $i \neq j$, but this follows from the statement (iii). In more detail, if $\mathfrak{p}_i + \mathfrak{p}_j \neq R$ then there is a maximal ideal that contains both $\mathfrak{p}_i$ and $\mathfrak{p}_j$, a contradiction (see the statement (iii)).

(v) $R/\mathfrak{n} \cong \prod_{\mathfrak{p} \in \mathrm{Min}(R)} R/\mathfrak{p}$: This fact follows from the statement (iv).

Let $1 = \sum_{\mathfrak{p} \in \mathrm{Min}(R)} e_\mathfrak{p}$ be the corresponding sum of orthogonal primitive idempotents in $R/\mathfrak{n}$. Since the set of minimal primes is a finite set, the ring R is a Noetherian ring, by [36, Theorem 11]. Hence, $\mathfrak{n}$ is a nilpotent ideal. So, we can lift the decomposition above to $1 = \sum_{\mathfrak{p} \in \mathrm{Min}(R)} e'_\mathfrak{p}$, a sum of primitive orthogonal idempotents in R. So,

$$R \cong \prod_{\mathfrak{p} \in \mathrm{Min}(R)} R(\mathfrak{p})$$

where $R(\mathfrak{p}) := e'_\mathfrak{p} R$ are local rings with unique minimal prime ideals by the statement (ii). Since R is a multiplication ring, the rings $R(\mathfrak{p})$ are also multiplication rings, by Lemma 3.5. Now, the implication $(\Rightarrow)$ follows from Theorem 3.8 and Theorem 3.10.

$(\Leftarrow)$ This implication follows from Lemma 3.5, Theorem 3.8 and Theorem 3.10. $\square$

Chapter 4

Multiplication modules over noncommutative rings

Throughout this chapter R is a ring (not necessarily commutative), unless stated otherwise.

In this chapter, we present some properties and characterizations of multiplication modules over an arbitrary ring, we give several criteria for a direct sum of modules to be a multiplication module, and we provide some properties of multiplication noncommutative rings. Furthermore, we present some properties of the endomorphisms ring of a multiplication module, and we introduce and study some new classes of modules: epimorphic modules, monomorphic modules, and automorphic modules.

4.1 Multiplication modules over (not necessarily commutative) ring

In this section, several characterizations and properties of the class of multiplication modules over an arbitrary ring are given. The results of this section are used in proofs of the subsequent sections.

Lemma 4.1 *An R-module M is a multiplication module iff $N = [N : M]M$ for any submodule N of M.*

Proof. ($\Rightarrow$) Let N be a submodule of the multiplication module M. Then there exists an ideal I of R such that $N = IM$. Hence,

$$N = IM \subseteq [N : M]M \subseteq N.$$

This implies that $N = [N : M]M$.

($\Leftarrow$) This is implication is obvious. $\square$

Let N be a submodule of M. Then $[N : M]M \subseteq N$, and so we have a short exact sequence of modules

$$0 \to [N : M]M \to N \to N/[N : M]M \to 0. \qquad \text{(NMM)}$$

Lemma 4.2 *An R-module M is a multiplication module iff $C = [C : M]M$ for every $C \in \mathrm{Cyc}_R(M)$.*

Proof. ($\Rightarrow$) Lemma 4.1.

($\Leftarrow$) Suppose that $C = [C : M]M$ for every cyclic submodule C of M. Let N be a submodule of M and $I = \sum_{C \in \mathrm{Cyc}_R(N)}[C : M]$. Then

$$IM = \sum_{C \in \mathrm{Cyc}_R(N)} [C : M]M = \sum_{C \in \mathrm{Cyc}_R(N)} C = N.$$

Hence, M is multiplication module. $\square$

Proposition 4.3 *Any homomorphic image of a multiplication module is a multiplication module.*

Proof. Let M be a multiplication R-module and $f : M \to N$ be an R-epimorphism. For each submodule K of N, $f^{-1}(K) = IM$ for some ideal I of R. Now, $K = f(f^{-1}(K)) = f(IM) = If(M) = IN$. Hence, N is multiplication module. $\square$

Proposition 4.4 *Let M be an R-module. Then M is a multiplication R-module if the following two conditions hold:*

1. *$\cap_{\lambda \in \Lambda} I_\lambda M = (\cap_{\lambda \in \Lambda} I_\lambda)M$ for every non-empty set of ideals $\{I_\lambda \,|\, \lambda \in \Lambda\}$ of R, and*

2. *for any submodule N of M and an ideal I of R such that $N \subset IM$, there exists an ideal J of R such that $J \subset I$ and $N \subseteq JM$.*

Proof. Let N be a submodule of M and S be the set of ideals I' of R such that $N \subseteq I'M$. Clearly, $R \in S$. The ideal $A = \bigcap_{I' \in S} I'$ is the smallest element in S since, by condition 1, $N \subseteq \bigcap_{I' \in S} I'M = AM$. Now, suppose that $N \subset AM$, we seek a contradiction. Then, by condition 2, there exists an ideal B of R such that $B \subset A$ and $N \subseteq BM$. Therefore $B \in S$. This contradicts to the minimality of A and hence, $N = AM$, i.e., M is a multiplication module. □

Proposition 4.5 is a criterion for a module to be a multiplication module.

Proposition 4.5 *Let M be an R-module. Then M is a multiplication R-module iff for every nonzero submodule N of M, M/N is a multiplication module such that $[N : M] \nsubseteq \mathrm{ann}_R(M)$.*

Proof. ($\Rightarrow$) Let N be a nonzero submodule of M. Then $N = [N : M]M$ (Lemma 4.1). Since $N \neq 0$, we must have $[N : M] \nsubseteq \mathrm{ann}_R(M)$. By Proposition 4.3, the factor module M/N is a multiplication module.

($\Leftarrow$) Let N be a nonzero submodule of the R-module M and $I = [N : M]$. By the assumption, $IM \neq 0$ (since $I \nsubseteq \mathrm{ann}_R(M)$) and M/IM is a multiplication module. So, by Lemma 4.1,

$$N/IM = [N/IM : M/IM](M/IM) = [N : M](M/IM) = 0,$$

and therefore $N = IM$. Hence, M is a multiplication module. □

Proposition 4.6 *Let M be a multiplication R-module. Then*

1. *If N and K are submodules of M such that $M/N \cong M/K$ then $N = K$.*
2. *If $f : M \to R$ is an R-homomorphism then for every $m \in M$, $f(m)M \subseteq Rm$.*

Proof. 1. Since M is a multiplication module and $M/N \cong M/K$, $N = [N : M]M = \mathrm{ann}_R(M/N)M = \mathrm{ann}_R(M/K)M = [K : M]M = K$.

2. Since M is a multiplication module, $Rm = IM$ for some ideal I of R. Now, $f(m)M \subseteq f(Rm)M = f(IM)M = If(M)M \subseteq IM = Rm$. Hence, $f(m)M \subseteq Rm$. □

Proposition 4.7 *Let M be a semisimple R-module such that $[N : M] \nsubseteq \mathrm{ann}_R(M)$ for every simple submodule N of M. Then M is a multiplication module.*

Proof. Let K be a submodule of M. Since M is a semisimple module, $K = \bigoplus_{i\in I} M_i$ is a direct sum of simple submodules M_i of M. As M_i is simple and $[M_i : M] \nsubseteq \operatorname{ann}(M)$, $[M_i : M]M = M_i$. Now,

$$K = \sum_{i\in I} M_i = \sum_{i\in I}[M_i : M]M = (\sum_{i\in I}[M_i : M])M.$$

Hence, M is a multiplication module, by Lemma 4.1. $\square$

Let M be an R-module. Anderson in [8], defined the ideal $\theta(M) = \sum_{C\in \mathrm{Cyc}_R(M)}[C : M]$ where R is a commutative ring. In case I is an ideal of R, it is clear that $I \subseteq \theta(I)$.

Lemma 4.8 *Let M be a multiplication R-module. Then $M = \theta(M)M$.*

Proof. Since M is a multiplication R-module, $M = \sum_{C\in \mathrm{Cyc}_R(M)} C = \sum_{C\in \mathrm{Cyc}_R(M)}[C : M]M = (\sum_{C\in \mathrm{Cyc}_R(M)}[C : M])M = \theta(M)M$, by Lemma 4.2. $\square$

The next lemma provides a sufficient condition for a multiplication module to be finitely generated.

Lemma 4.9 *Let M be a multiplication R-module. If $\theta(M)$ is a finitely generated R-module then the R-module M is finitely generated.*

Proof. Since $\theta(M) = \sum_{C\in \mathrm{Cyc}_R(M)}[C : M]$ and the R-module $\theta(M)$ is finitely generated, $\theta(M) = \sum_{i=1}^{n} R\theta_i$ for some elements $\theta_i \in [C_i : M]$ where C_i are cyclic submodules of M. Now, by Lemma 4.8,

$$M = \theta(M)M = \sum_{i=1}^{n} R\theta_i M \subseteq \sum_{i=1}^{n} C_i \subseteq M,$$

i.e., $M = \sum_{i=1}^{n} C_i$ is a finitely generated R-module. $\square$

Corollary 4.10 *If R is a left Noetherian ring then every multiplication R-module is finitely generated.*

Proof. The corollary follows from Lemma 4.9. □

Proof of Proposition 1.7. Let $f : M \to M_1 \bigoplus M_2$ be an epimorphism and p_1, p_2 be the projections of the module $M_1 \bigoplus M_2$ onto M_1 and M_2, respectively. For $i = 1, 2$, let $f_i = p_i f$ and $K_i = \ker(f_i)$. Then $M/K_i \cong M_i$. So,

$$[K_1 : M] = \mathrm{ann}_R(M_1) \subseteq \mathrm{ann}_R(M_2) = [K_2 : M].$$

Since the R-module M is a multiplication module, we have $K_1 = [K_1 : M]M \subseteq [K_2 : M]M = K_2$, by Lemma 4.1. Let $k_2 \in M_2$. Then $(0, k_2) = f(m)$ for some element $m \in M$. Clearly, $f_1(m) = p_1(0, k_2) = 0$, i.e., $m \in K_1$. Since $K_1 \subseteq K_2$, $0 = f_2(m) = p_2 f(m) = p_2(0, k_2) = k_2$, i.e., $M_2 = 0$. □

Corollary 4.11 *Let M be a multiplication R-module and M_1, M_2 be R-modules such that* $\mathrm{ann}_R(M_1) = \mathrm{ann}_R(M_2)$ *and the direct sum of R-modules $M_1 \bigoplus M_2$ is an epimorphic image of the R-module M. Then $M_1 = M_2 = 0$.* □

Proof. The corollary follows from Proposition 1.7. □

Proposition 4.12 shows that every multiplication module M does not admit a direct summand which is isomorphic to M.

Corollary 4.12 *Let R be a ring and M be a multiplication R-module. If $M \cong M \oplus N$ where N is an R-module then $N = 0$.*

Proof. Since $M \cong M \bigoplus N$, $\mathrm{ann}_R(M) = \mathrm{ann}_R(N) \cap \mathrm{ann}_R(M)$. So, $\mathrm{ann}_R(M) \subseteq \mathrm{ann}_R(N)$. Hence, by Proposition 1.7, $N = 0$. □

Ideals $\{\mathfrak{a}_i \,|\, i \in I\}$ of a ring R are called *incomparable* if $\mathfrak{a}_i \nsubseteq \mathfrak{a}_j$ for all distinct elements $i, j \in I$.

Corollary 4.13 *Let M be a multiplication R-module and the direct sum of nonzero R-modules $\bigoplus_{i \in I} M_i$ with* $\mathrm{card}(I) \geq 2$ *is an epimorphic image of M. Then the set of ideals* $\{\mathrm{ann}_R(M_i) \;|\, i \in I\}$ *are incomparable. In particular, all the ideals* $\{\mathrm{ann}_R(M_i) \,|\, i \in I\}$ *are distinct and the modules $\{M_i \,|\, i \in I\}$ are not pairwise isomorphic. In particular,* $\mathrm{ann}_R(M_i) \neq 0$ *for all $i \in I$.*

Proof. Let $\mathfrak{a}_i = \mathrm{ann}_R(M_i)$. Suppose that $\mathfrak{a}_i \subseteq \mathfrak{a}_j$ for some $i \neq j$. Then the direct sum $M_i \bigoplus M_j$ is an epimorphic image of M such that $\mathfrak{a}_i \subseteq \mathfrak{a}_j$. By Proposition 1.7, $M_j = 0$, a contradiction. □

Corollary 4.14 *Let a direct sum of R-modules $M = \bigoplus_{i \in I} M_i$ be a multiplication module with* $\mathrm{card}(I) \geq 2$. *Then the set of ideals* $\{\mathrm{ann}_R(M_i) \mid i \in I\}$ *are incomparable. In particular, none of the direct summands M_i is a faithful R-module, i.e.,* $\mathrm{ann}_R(M_i) \neq 0$ *for all $i \in I$.*

Proof. The corollary follows from Corollary 4.13. □

For a module N and a set I, we denote by $N^{(I)}$ a direct sum of I copies of N.

Corollary 4.15 *Let M be a multiplication R-module. Then every nonzero factor module of M cannot be of the type $N^{(I)}$ for some nonzero R-module N and a set I of cardinality ≥ 2.*

Proof. This follows from Proposition 1.7. □

Corollary 4.16 *Let M be a nonzero multiplication R-module. If M is a free R-module then $M \cong R$.*

Proof. If M is free R-module, i.e., $M \cong R^{(I)}$ for some I then, by Corollary 4.15, the set I must be a single element and hence $M \cong R$. □

Let R be a ring and $\widehat{R}$ be the set of isomorphism classes of its simple modules. Let M be a semisimple R-module. Then $M = \bigoplus_{V \in \widehat{R}} M(V)$ where $M(V)$ is the sum of all simple submodules of M isomorphic to V. The module $M(V)$ is called the *isotypic component* of M corresponding to V, or, briefly, the V-*isotypic component* of M.

Corollary 4.17 *Let M be a multiplication R-module. Every semisimple factor module of M is a direct sum of non-isomorphic simple modules, (i.e., each isotypic component is a simple module) with incomparable annihilators.*

Proof. The corollary follows from Corollary 4.13. □

Corollary 4.18 *Let $0 \to M_1 \to M \to M_2 \to 0$ be a short exact sequence of R-modules where M_1 and M_2 are non-zero R-modules and $\mathfrak{a}_i = \mathrm{ann}_R(M_i)$ for $i = 1, 2$. If M is a multiplication module and either $\mathfrak{a}_1 \subseteq \mathfrak{a}_2$ or $\mathfrak{a}_2 \subseteq \mathfrak{a}_1$ then the short exact sequence is not split. In particular,* $\mathrm{Ext}^1_{\mathrm{R}}(\mathrm{M}_2, \mathrm{M}_1) \neq 0$.

Proof. If the short exact sequence were split then $M \cong M_1 \oplus M_2$. By Corollary 4.13, the ideals $\mathfrak{a}_1$ and $\mathfrak{a}_2$ would be incomparable as M is a multiplication module, a contradiction. □

4.2 Criteria for a direct sum of modules to be a multiplication module

In the first part of this section, the proofs of the five criteria stated in the Introduction are given for a direct sum of modules to be a multiplication module. In the second part, applications are given.

Proof of Theorem 1.11. ($\Rightarrow$) Since M is a multiplication module, for each submodule N of M there is an ideal I of R such that

$$N = IM = I(\bigoplus_{\lambda\in\Lambda} M_\lambda) = \bigoplus_{\lambda\in\Lambda} IM_\lambda \subseteq \bigoplus_{\lambda\in\Lambda} N \bigcap M_\lambda \subseteq N,$$

i.e., $N = \bigoplus_{\lambda\in\Lambda} N \bigcap M_\lambda$ and $N \bigcap M_\lambda = IM_\lambda$, and so, the intersection condition holds for M.

Let $\{N_\lambda\}_{\lambda\in\Lambda}$ be a set of R-modules such that $N_\lambda \subseteq M_\lambda$ for all $\lambda \in \Lambda$. Clearly, N_λ is a submodule of M. So, there exists an ideal I_λ of R such that $N_\lambda = I_\lambda M = \bigoplus_{\mu\in\Lambda} I_\lambda M_\mu$, and so $I_\lambda M_\mu = \delta_{\lambda\mu} N_\lambda$ for all λ, $\mu \in \Lambda$, i.e., the strong orthogonality condition holds.

($\Leftarrow$) Let N be a submodule of M. We have to show that $N = IM$ for some ideal I of R. By the intersection condition, $N = \bigoplus_{\lambda\in\Lambda} N_\lambda$ where $N_\lambda = N \bigcap M_\lambda \subseteq M_\lambda$. For the set $\{N_\lambda\}_{\lambda\in\Lambda}$, let $\{I_\lambda\}_{\lambda\in\Lambda}$ be a set of ideals that satisfies the strong orthogonality condition ($I_\lambda M_\mu = \delta_{\lambda\mu} N_\mu$ for all $\lambda, \mu \in \Lambda$). Then $I = \sum_{\lambda\in\Lambda} I_\lambda$ is an ideal of R such that

$$IM = \sum_{\lambda,\mu\in\Lambda} I_\lambda M_\mu = \sum_{\lambda,\mu\in\Lambda} \delta_{\lambda\mu} N_\mu = \sum_{\lambda\in\Lambda} N_\lambda = \bigoplus_{\lambda\in\Lambda} N_\lambda = N,$$

as required. $\square$

Lemma 4.19 *Suppose that a direct sum of nonzero R-modules $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ is a multiplication module. Let N be a submodule of M, $\mathcal{N} = \{N_\lambda := N \bigcap M_\lambda\}_{\lambda\in\Lambda}$ and $\mathcal{M} = \{M_\lambda\}_{\lambda\in\Lambda}$. Then for all $\{I_\lambda\}_{\lambda\in\Lambda} \in \mathcal{I}(\mathcal{N}, \mathcal{M})$, $N = (\sum_{\lambda\in\Lambda} I_\lambda)M$.*

Proof. $(\sum_{\lambda\in\Lambda} I_\lambda)M = \sum_{\lambda,\mu\in\Lambda} I_\lambda M_\mu = \sum_{\lambda,\mu\in\Lambda} \delta_{\lambda\mu} N_\lambda = \sum_{\lambda\in\Lambda} N_\lambda = N$, by Theorem 1.11. $\square$

The next theorem is an explicit description of the largest orthogonalizer $I(\mathcal{N}, \mathcal{M})$ in $\mathcal{I}(\mathcal{N}, \mathcal{M})$.

Theorem 4.20 *Suppose that a direct sum $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ of nonzero R-modules with* $\mathrm{card}(\Lambda) \geq 2$ *is a multiplication module. Let $\mathfrak{a}_\lambda = \mathrm{ann}_R(M_\lambda)$, $M'_\lambda := \bigoplus_{\mu\neq\lambda} M_\mu$ and $\mathfrak{a}'_\lambda = \mathrm{ann}_R(M'_\lambda) = \bigcap_{\mu\neq\lambda}\mathfrak{a}_\mu$. Let N be a submodule of M, $\mathcal{N} = \{N_\lambda := N\bigcap M_\lambda\}_{\lambda\in\Lambda}$ and $\mathcal{M} = \{M_\lambda\}_{\lambda\in\Lambda}$. Then $I(\mathcal{N},\mathcal{M}) = \{I_\lambda\}_{\lambda\in\Lambda}$ where $I_\lambda = I(N_\lambda, M_\lambda)\bigcap\mathfrak{a}'_\lambda$ for all $\lambda\in\Lambda$ (M_λ is a multiplication R-module as an epimorphic image of the multiplication R-module M, so $I(N_\lambda, M_\lambda)$ makes sense).*

Proof. Let $\mathcal{J} = \{J_\lambda\}_{\lambda\in\Lambda} \in \mathcal{I}(\mathcal{N},\mathcal{M})$. Then $J_\lambda \subseteq I_\lambda$ for all $\lambda\in\Lambda$, by the maximality of $I(\mathcal{N},\mathcal{M})$. By the very definition, $I_\lambda \subseteq I'_\lambda := I(N_\lambda, M_\lambda)\bigcap\mathfrak{a}'_\lambda$ for all $\lambda\in\Lambda$. To finish the proof of the theorem it suffices to show that $\{I'_\lambda\}_{\lambda\in\Lambda} \in \mathcal{I}(\mathcal{N},\mathcal{M})$. For all $\lambda\neq\mu$, $I'_\lambda M_\mu = 0$ (since $I'_\lambda \subseteq \mathfrak{a}_\mu$). Finally,

$$N_\lambda = I_\lambda M_\lambda \subseteq I'_\lambda M_\lambda \subseteq I(N_\lambda, M_\lambda)M_\lambda = N_\lambda.$$

Hence, $N_\lambda = I'_\lambda M_\lambda$, as required. □

Proof of Theorem 1.14. ($\Rightarrow$) If M is a multiplication module then so is every module M_λ (since M_λ is an epimorphic image of M, Proposition 4.3), and so the condition 1 holds. The condition 2 holds, by Lemma 4.19.

($\Leftarrow$) Let N be a submodule of M. Since M_λ is a multiplication module for all $\lambda\in\Lambda$, $I(N_\lambda, M_\lambda)$ makes since. Let $I'_\lambda = I(N_\lambda, M_\lambda)\cap\mathfrak{a}'_\lambda$ where $\mathfrak{a}'_\lambda = \mathrm{ann}_R(\bigoplus_{\mu\neq\lambda} M_\mu)$. Then $\{I'_\lambda\} \in \mathcal{I}(\mathcal{N},\mathcal{M})$, i.e., $\mathcal{I}(\mathcal{N},\mathcal{M}) \neq \emptyset$, and therefore, by condition 2, $N = (\sum_{\lambda\in\Lambda} I_\lambda)M$. Hence, M is a multiplication module. □

The next corollary is an explicit description of the element $I(\mathcal{M},\mathcal{M})$.

Corollary 4.21 *Suppose that a direct sum of nonzero R-modules $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ is a multiplication module where* $\mathrm{card}(\Lambda) \geq 2$, *$\mathfrak{a}_\lambda = \mathrm{ann}_R(M_\lambda)$ and $\mathfrak{a} = \mathrm{ann}_R(M)$. Let $\mathcal{M} = \{M_\lambda\}_{\lambda\in\Lambda}$. Then*

1. *$I(\mathcal{M},\mathcal{M}) = \{\mathfrak{a}'_\lambda\}_{\lambda\in\Lambda}$ and $\mathfrak{a}'_\lambda := \cap_{\mu\neq\lambda}\mathfrak{a}_\mu \neq \cap_{\mu\in\Lambda}\mathfrak{a}_\mu = \mathrm{ann}_R(M)$ for all $\lambda\in\Lambda$.*

2. *Let $\pi : R \to \overline{R} = R/\mathfrak{a}$, $r \rightarrowtail \overline{r} := r + \mathfrak{a}$. Then $\sum_{\lambda\in\Lambda}\overline{\mathfrak{a}'_\lambda} = \bigoplus_{\lambda\in\Lambda}\overline{\mathfrak{a}'_\lambda}$ in $\overline{R}$ and $\overline{\mathfrak{a}'_\lambda} \neq 0$ for all $\lambda\in\Lambda$ where $\{\overline{\mathfrak{a}'_\lambda}\}$ is a set of orthogonal ideals.*

3. *$M = \overline{\mathfrak{a}'}M$ where $\overline{\mathfrak{a}'} = \sum_{\lambda\in\Lambda}\overline{\mathfrak{a}'_\lambda}$. In particular, $\overline{\mathfrak{a}'_\lambda}M_\mu = \delta_{\lambda\mu}M_\mu$ for all $\lambda,\mu\in\Lambda$.*

4. *For each submodule N of M, $N = \mathfrak{b}'M$ for some ideal $\mathfrak{b}'$ of $\overline{R}$ such that $\mathfrak{b}' = \bigoplus_{\lambda\in\Lambda} \mathfrak{b}'_\lambda$ is a direct sum of ideals $\mathfrak{b}'_\lambda$ of $\overline{R}$ such that $\mathfrak{b}'_\lambda \subseteq \overline{\mathfrak{a}'_\lambda}$ for all $\lambda \in \Lambda$.*

Proof. 1. By Theorem 4.20, $I(\mathcal{M}, \mathcal{M}) = \{I_\lambda\}_{\lambda\in\Lambda}$ where

$$I_\lambda = I(M_\lambda, M_\lambda) \cap \mathfrak{a}'_\lambda = R \cap \mathfrak{a}'_\lambda = \mathfrak{a}'_\lambda.$$

Since $\mathrm{card}(\Lambda) \geqslant 2$ and $M_\lambda \neq 0$ for all $\lambda \in \Lambda$, $\mathfrak{a}'_\lambda \neq 0$ for all $\lambda \in \Lambda$, by Theorem 1.11.

2. Suppose that the sum $\sum_{\lambda\in\Lambda} \overline{\mathfrak{a}'_\lambda}$ is not a direct sum. Then there is a nonzero element $a_\lambda \in \overline{\mathfrak{a}'_\lambda}$ that can be written as a sum $\sum_{\mu\neq\lambda} a_\mu$ for some elements $a_\mu \in \overline{\mathfrak{a}'_\mu}$. As the $\overline{R}$-module M is faithful, $0 \neq a_\lambda M = a_\lambda M_\lambda = (\sum_{\mu\neq\lambda} a_\mu M_\lambda) = 0$, a contradiction. Hence, the sum $\sum_{\lambda\in\Lambda} \overline{\mathfrak{a}'_\lambda}$ is a direct sum, and, by statement 1, $\overline{\mathfrak{a}'_\lambda} \neq 0$ for all $\lambda \in \Lambda$.

3. $\overline{\mathfrak{a}'}M = \sum_{\lambda,\mu\in\Lambda} \mathfrak{a}'_\lambda M_\mu = \sum_{\lambda,\mu\in\Lambda} \delta_{\lambda\mu} M_\lambda = \sum_{\lambda\in\Lambda} M_\lambda = M$, by statement 1.

4. This statement follows from Theorem 4.20. □.

Proof of Theorem 1.16. (⇒) This implication follows from Corollary 4.21.

(⇐) It suffices to show that the conditions of Theorem 1.11 are satisfied for the module $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ where $M_\lambda = \mathfrak{a}'_\lambda M$. Let N be a submodule of M. By condition 2, there is an ideal $\mathfrak{b}'$ such that $N = \mathfrak{b}'M = \bigoplus_{\lambda\in\Lambda} \mathfrak{b}'_\lambda M$ where $\mathfrak{b}'_\lambda = \mathfrak{b}' \cap \mathfrak{a}'_\lambda$. Let $N_\lambda = \mathfrak{b}'_\lambda M$. Then

$$N = \bigoplus_{\lambda\in\Lambda} N_\lambda \subseteq \bigoplus_{\lambda\in\Lambda} (N \cap M_\lambda) \subseteq N,$$

i.e., $N_\lambda = N \cap M_\lambda$, and so, the condition 1 of Theorem 1.11 holds. By the very definition of N_λ, $N_\lambda = \mathfrak{b}'_\lambda M = \mathfrak{b}'_\lambda M_\lambda$, and for all $\lambda \neq \mu$,

$$\mathfrak{b}'_\lambda M_\mu \subseteq \mathfrak{a}'_\lambda M_\mu = \mathfrak{a}'_\lambda \mathfrak{a}'_\mu M = 0 \cdot M = 0,$$

i.e., $\mathfrak{b}'_\lambda M_\mu = 0$. So, the condition 2 of Theorem 1.11 holds, as required. □

Proof of Theorem 1.17. (⇒) This implication follows from Theorem 1.11 and Corollary 4.21.

(⇐) Let N be a submodule of M. Then, by the intersection condition, $N = \bigoplus_{\lambda\in\Lambda} N_\lambda$ where $N_\lambda = N \cap M_\lambda$. Since M_λ is a multiplication module, $N_\lambda = J_\lambda M_\lambda$

for an ideal J_λ of R. Then we have $N_\lambda = J_\lambda \mathfrak{a}'_\lambda M_\lambda$, by the orthogonality condition. Let $I_\lambda = J_\lambda \mathfrak{a}'_\lambda$. Then the conditions 1 and 2 of Theorem 1.11 are satisfied since $\mathfrak{a}'_\lambda \mathfrak{a}'_\mu \subseteq \mathfrak{a} = \mathrm{ann}_R(M)$, and so,

$$I_\lambda M_\mu = I_\lambda I_\mu M_\mu \subseteq \mathfrak{a}'_\lambda \mathfrak{a}'_\mu M_\mu \subseteq \mathfrak{a} M = 0.$$

Therefore the implication ($\Leftarrow$) follows from Theorem 1.11. $\square$

The next corollary is a criterion for a direct sum of simple modules to be a multiplication modules.

Corollary 4.22 *Let $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$ be a direct sum of simple R-modules. We keep the notation of Corollary 4.21. Then M is a multiplication R-module iff for all $\lambda, \mu \in \Lambda$, $\mathfrak{a}'_\lambda M_\mu = \delta_{\lambda\mu} M_\mu$.*

Proof. ($\Rightarrow$) By Corollary 4.21, $I(\mathcal{M}, \mathcal{M}) = \{\mathfrak{a}'_\lambda\}_{\lambda \in \Lambda}$ where $\mathcal{M} = \{M_\lambda\}_{\lambda \in \Lambda}$, and the implication follows.

($\Leftarrow$) Suppose that $\mathfrak{a}'_\lambda M_\mu = \delta_{\lambda\mu} M_\mu$ for all $\lambda, \mu \in \Lambda$. Clearly, the simple R-modules $\{M_\lambda\}_{\lambda \in \Lambda}$ are pairwise non-isomorphic. So, if N is a submodule of M then $N = \bigoplus_{\lambda \in \Lambda} N \cap M_\lambda$. Let

$$\mathrm{Supp}(N) = \{\lambda \in \Lambda \mid N \cap M_\lambda \neq 0, \text{ i.e., } N \cap M_\lambda = M_\lambda\}.$$

Then

$$N = \bigoplus_{\lambda \in \mathrm{Supp}(N)} M_\lambda = \bigoplus_{\lambda \in \mathrm{Supp}(N)} \mathfrak{a}'_\lambda M_\lambda = \Big(\sum_{\lambda \in \mathrm{Supp}(N)} \mathfrak{a}'_\lambda\Big) M. \quad \square$$

The next lemma introduces some properties of the endomorphisms ring of a multiplication module.

Lemma 4.23 *Let M be a multiplication R-module. Then*

1. $\mathrm{Epi}_R(M) = \mathrm{Aut}_R(M)$.
2. *The $\mathrm{End}_R(M)$-stability condition holds for the R-module M. In particular, if $f \in \mathrm{End}_R(M)$ then for all $g \in \mathrm{End}_R(M)$, $g(\mathrm{im}(f)) \subseteq \mathrm{im}(f)$.*
3. *If N is a submodule of M then N is an $\mathrm{Epi}_R(M)$-invariant submodule, i.e., for every $f \in \mathrm{Epi}_R(M)$, $f(N) = N$.*

4. *If* $M = \bigoplus_{i\in I} M_i$ *and* $f \in \mathrm{Hom}_R(M, N)$ *where* N *is an* R*-module then* $f(M) = \bigoplus_{i\in I} f(M_i)$.

5. *If* $M = \bigoplus_{i\in I} M_i$ *then* $\mathrm{End}_R(M) = \prod_{i\in I} \mathrm{End}_R(M_i)$*, i.e., the inclusion* $\prod_{i\in I} \mathrm{End}_R(M_i) \subseteq \mathrm{End}_R(M)$ *is an equality. In particular,* $\mathrm{Hom}_R(M_i, M_j) = 0$ *for all* $i \neq j$*; and* $\mathrm{Aut}_R(M) \cong \prod_{i\in I} \mathrm{Aut}_R(M_i)$.

6. *If* R *is a commutative ring and* C *is a cyclic submodule of the* R*-module* M *then for every* $f \in \mathrm{End}_R(M)$*,* $f|_C = r_C : C \to C$*,* $m \mapsto rm$ *for some element* $r = r(f) \in R$*. Futhermore, the ring* $\mathrm{End}_R(M)$ *is a commutative ring.*

Proof. 1. Let $f \in \mathrm{Epi}_R(M)$. Then $M/\ker(f) \cong M$. Since M is a multiplication module,

$$\ker(f) = [\ker(f) : M]M = \mathrm{ann}_R(M/\ker(f))M = \mathrm{ann}_R(M)M = 0,$$

and therefore $f \in \mathrm{Aut}_R(M)$.

2. Let N be a submodule of M and $f \in \mathrm{End}_R(M)$. Then $N = IM$ for some ideal I of R. So,

$$f(N) = f(IM) = If(M) \subseteq IM = N,$$

i.e., N is an $\mathrm{End}_R(M)$-stable submodule.

3. Let N be a submodule of M and $f \in \mathrm{Epi}_R(M)$. By statement 2, $f(N) \subseteq N$. By statement 1, there exists $g \in \mathrm{Aut}_R(M)$ such that $gf = fg = 1$, and therefore

$$N = 1N = g(f(N)) \subseteq f(N) \subseteq N,$$

i.e., $N = f(N)$. Hence, N is an $\mathrm{Epi}_R(M)$-invariant submodule.

4. By Theorem 1.11, $\ker(f) = \bigoplus_{i\in I}(\ker(f) \cap M_i)$. Hence $f(M) \cong M/\ker(f) = \bigoplus_{i\in I} M_i/\ker(f) \cap M_i \cong \bigoplus_{i\in I} f(M_i)$, and so, $f(M) = \bigoplus_{i\in I} f(M_i)$.

5. Statement 5 follows from statement 2 since $f(M_i) \subseteq M_i$ for all $i \in I$ and $f \in \mathrm{End}_R(M)$.

6. Statement 6 follows from statement 2. □

Proof of Theorem 1.20. (⇒) It follows from Theorem 1.17 and Lemma 4.23.

(⇐) In view of Theorem 1.17, it suffices to prove that the intersection condition holds for the direct sum $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$. For each $\lambda \in \Lambda$, let $j_\lambda : M \to M$

be a composition of the projection homomorphism $M \to M_\lambda$ and the inclusion homomorphism $M_\lambda \to M$. Clearly, $j_\lambda \in \mathrm{End}_R(M)$, and

$$N \subseteq \sum_{\lambda \in \Lambda} (N \cap M_\lambda) = \bigoplus_{\lambda \in \Lambda} (N \cap M_\lambda) \subseteq N$$

since N is $\mathrm{End}_R(M)$-stable, i.e., $N = \bigoplus_{\lambda \in \Lambda} (N \cap M_\lambda)$, and so, the intersection condition holds. Hence, by Theorem 1.17, the R-module M is a multiplication module. $\square$

The refinement condition.

Definition 4.24 *Let M be an R-module. We say that two of its decompositions into direct sum of submodules, $M = \bigoplus_{i \in I} M_i$ and $M = \bigoplus_{j \in J} N_j$, satisfy the* **refinement condition** *if $M = \bigoplus_{i \in I, j \in J} M_i \cap N_j$. We say that the module M satisfies the **refinement condition** if every two of its direct sum decomposition satisfy the refinement condition.*

Let M be an R-module and $E = \mathrm{End}_R(M)$. Then the R-module M is a direct sum of its submodules iff the identity map $1 : M \to M$, $m \mapsto m$ is a *sum of orthogonal idempotents*, that is $1 = \sum_{\lambda \in \Lambda} e_\lambda$ where $e_\lambda e_\mu = \delta_{\lambda\mu} e_\lambda$ and for every element $m \in M$, $e_\lambda m = 0$ for all but finitely many λ.

In more detail, if $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$ then $1 = \sum_{\lambda \in \Lambda} e_\lambda$ where e_λ is the projection onto M_λ. Conversely, if $1 = \sum_{\lambda \in \Lambda} e_\lambda$ is a sum of orthogonal idempotents then $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$ where $M_\lambda = e_\lambda M$.

Proposition 4.25 is a criterion for a module to satisfy the refinement condition.

Proposition 4.25 *Let M be an R-module and $E = \mathrm{End}_R(M)$. Then the R-module M satisfies the refinement condition iff for any two sums of orthogonal idempotents in E, $1 = \sum_{i \in I} e_i$ and $1 = \sum_{j \in J} f_j$, $e_i f_j = f_j e_i$ for all $i \in I$ and $j \in J$.*

Proof. ($\Rightarrow$) Suppose that an R-module M satisfies the refinement condition. Let $1 = \sum_{i \in I} e_i$ and $1 = \sum_{j \in J} f_j$ be sums of orthogonal idempotents in E. Then $M = \bigoplus_{i \in I} M_i = \bigoplus_{j \in J} M_j$ where $M_i = e_i M$ and $M_j = f_j M$. Since the R-module satisfies the refinement condition,

$$M = \bigoplus_{i \in I, j \in J} M_i \cap M_j,$$

and $1 = \sum_{i\in I, j\in J} e_i f_j$ is the correspondent sum of orthogonal idempotents such that $e_i f_j = f_j e_i$ for all $i \in I$ and $j \in J$.

($\Leftarrow$) Suppose that $M = \bigoplus_{i\in I} M_i$ and $M = \bigoplus_{j\in J} M_j$ and $1 = \sum_{i\in I} e_i$, $1 = \sum_{j\in J} f_j$ are the correspondent sum of orthogonal idempotents. Since $e_i f_j = f_j e_i$ for all $i \in I$ and $j \in J$,

$$1 = \sum_{i\in I, j\in J} e_i f_j$$

is a sum of orthogonal idempotents. Hence, $e_i f_j M = M_i \cap M_j$ and $M = \bigoplus_{i\in I, j\in J}(M_i \cap M_j)$. □

Definition 4.26 *For an R-module M, let* $\mathrm{Dec}(M) = \mathrm{Dec}_R(M)$ *be the set of all its direct sum decompositions. We say that a direct sum decomposition $\bigoplus_{i\in I} M_i$ is* **finer** *than a direct sum decomposition $\bigoplus_{j\in J} N_j$ and write $\bigoplus_{i\in I} M_i \geq \bigoplus_{j\in J} N_j$ if $I = \coprod_{j\in J} I_j$ is disjoint union of non-empty subsets I_j such that for each $j \in J$, $N_j = \bigoplus_{i\in I_j} M_i$.*

The set $(\mathrm{Dec}(M), \geq)$ is a partially ordered set (a poset, for short). Let $\mathrm{maxDec}(M)$ be a set of maximal elements of $\mathrm{Dec}(M)$.

Definition 4.27 $\mathrm{ds.dim}(M) = \sup\{\mathrm{card}(I) \mid M = \bigoplus_{i\in I} M_i \in \mathrm{Dec}(M)\}$ *is called the* ***direct sum decomposition dimension****.*

If an R-module M satisfies the refinement condition and $\mathrm{maxDec}(M) \neq \emptyset$ then $\mathrm{maxDec}(M)$ contains a unique decomposition, say $\bigoplus_{i\in I} M_i$, and so, $\mathrm{ds.dim}(M) = \mathrm{card}(I)$.

Corollary 4.28 *Let M be a multiplication module such that $M = \bigoplus_{i\in I} M_i = \bigoplus_{j\in J} N_j$ and L be a submodule of M. Then*

1. *$M = \bigoplus_{i\in I, j\in J} M_i \cap N_j$, i.e., every multiplication module satisfies the refinement condition, and*

2. *$L = \bigoplus_{i\in I, j\in J} L \cap M_i \cap N_j$.*

Proof. By Theorem 1.11, $M_i = \bigoplus_{j\in I} M_i \cap N_j$ for all $i \in I$. Then $M = \bigoplus_{i\in I} M_i = \bigoplus_{i\in I, j\in J} M_i \cap N_j$, and statement 1 holds. Statement 2 follows from Theorem 1.11 and statement 1. □

Definition 4.29 *Let R be a ring. The* ***ideal uniform dimension*** *of R,* $\mathrm{iu.dim}(R)$, *is the supremum of cardinalities of sets I such that for some set of ideals $\{\mathfrak{a}_i\}_{i\in I}$ of R, $\sum_{i\in I} \mathfrak{a}_i = \bigoplus_{i\in I} \mathfrak{a}_i$.*

Proposition 4.30 *Let M be a multiplication R-module. Then* $\mathrm{ds.dim}(M) \leq \mathrm{iu.dim}(R/\mathrm{ann}_R(M))$.

Proof. The proposition follows from Corollary 4.21. $\square$

Definition 4.31 *Let R be a ring. Then* $\mathrm{m.dim}(R) = \sup\{\mathrm{ds.dim}(M) \mid M$ is a multiplication R-module$\}$ *is called the* ***multiplication dimension****.*

Corollary 4.32 *Let R be a ring. Then* $\mathrm{m.dim}(R) \leq \sup\{\mathrm{iu.dim}(R/\mathrm{ann}_R(M)) \mid M$ is a multiplication R-module$\} \leq \sup\{\mathrm{iu.dim}(R/\mathrm{ann}_R(M)) \mid M$ is an R-module$\}$.

Proof. The corollary follws from Proposition 4.30. $\square$

Multiplication modules are a unique direct sum of indecomposable modules.

The next proposition shows that there is a unique decomposition of a multiplication module as a direct sum of indecomposable modules (if such decomposition exists).

Proposition 4.33 *Let M be a multiplication R-module. Suppose that $M = \bigoplus_{i\in I} M_i = \bigoplus_{j\in J} N_j$ are direct sums of indecomposable R-modules. Then there is a bijection $\sigma : I \to J$ such that $M_i = N_{\sigma(i)}$ for all $i \in I$.*

Proof. By Theorem 1.11, $M_i = \bigoplus_{j\in J} M_i \cap N_j$ for all $i \in I$. The module M_i is indecomposable. So, $M_i = N_{\sigma(i)}$ for a unique $\sigma(i) \in J$. If $i \neq i'$ then $\sigma(i) \neq \sigma(i')$, i.e., the map

$$\sigma : I \to J,\ i \mapsto \sigma(i),$$

is an injection. By symmetry, there is an injection $\tau : J \to I$, $j \mapsto \tau(j)$, such that $N_j = M_{\tau(j)}$. Clearly, $\sigma\tau(j) = j$ and $\tau\sigma(i) = i$ for all $j \in J$ and $i \in I$. So, $\sigma = \tau^{-1}$, and the result follows. $\square$

Definition 4.34 *The set of nonzero R-modules $\{M_\lambda\}_{\lambda\in\Lambda}$ where* $\mathrm{card}(\Lambda) \geqslant 2$ *is called* **homomorphically independent** *if $\mathrm{Hom}_R(M_\lambda, M_\mu) = 0$ for all $\lambda \neq \mu$ where $\lambda, \mu \in \Lambda$.*

We say that a direct sum $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ **has enough complements** *if for every direct summand K of M, there is a subset $\Lambda' \subseteq \Lambda$ such that $M = \bigoplus_{\lambda\in\Lambda'} M_\lambda \bigoplus K$.*

Proposition 4.35 *Let a direct sum of nonzero R-modules* $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ *where* $\mathrm{card}(\lambda) \geqslant 2$ *be a multiplication R-module. Then*

1. *The set* $\{M_\lambda\}_{\lambda\in\Lambda}$ *is homomorphically independent.*

2. *If* M_λ *is indecomposable and* K *is a nonzero direct summand submodule of* M *then either* $M_\lambda \subseteq K$ *or* $M_\lambda \cap K = 0$*. Moreover, if* $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ *is direct sum of indecomposable modules then* $\mathrm{Sub}^{\oplus}_{\mathrm{R}}(M) = \{\bigoplus_{\lambda\in\Lambda'} M_\lambda \mid \Lambda' \subseteq \Lambda\}$*, i.e.,* M *has enough complements.*

3. *If all R-modules* M_λ*,* $\lambda \in \Lambda$*, have direct sum with enough complements then the R-module* M *has enough complements.*

Proof. 1. It follows from Lemma 4.23.

2. If K is a nonzero direct summand submodule of M then there exists a submodule K' of M such that $M = K \bigoplus K'$. By Theorem 1.11,

$$M_\lambda = M_\lambda \cap K \bigoplus M_\lambda \cap K'.$$

So, either $M_\lambda \subseteq K$ or $M_\lambda \cap K = 0$ (since M_λ is indecomposable). Therefore, by Theorem 1.11, $K = \bigoplus_{\lambda\in\Lambda}(M_\lambda \cap K) = \bigoplus_{\lambda\in\Lambda'} M_\lambda$ where $\Lambda' \subseteq \Lambda$.

3. For every $\lambda \in \Lambda$, let $M_\lambda = \bigoplus_{i\in I_\lambda} N_i$. Clearly, $\bigoplus_{\lambda\in\Lambda}(\bigoplus_{i\in I_\lambda} N_i)$ is a direct sum decompositionof M. Let K be a direct summand of M, i.e., there exists a submodule K' of M such that $M = K \bigoplus K'$. By Theorem 1.11, $M_\lambda = M_\lambda \cap K \bigoplus M_\lambda \cap K'$, i.e., $M_\lambda \cap K$ is a direct summand of M_λ. So, by statement 2, $M_\lambda \cap K = \bigoplus_{i\in I'_\lambda \subseteq I_\lambda} N_i$. Therefore, by Theorem 1.11, $K = \bigoplus_{\lambda\in\Lambda}(K \cap M_\lambda) = \bigoplus_{\lambda\in\Lambda}(\bigoplus_{i\in I'_\lambda \subseteq I_\lambda} N_i)$, i.e., M has enough complements. □

Corollary 4.36 *Let a direct sum of nonzero R-modules* $M = \bigoplus_{i\in I} M_i$ *where* $\mathrm{card}(I) \geqslant 2$ *be a multiplication R-module. If* N *is an indecomposable submodule of* M *then* $N \subseteq M_i$ *for some* $i \in I$*.*

Proof. By Theorem 1.11, $N = \bigoplus_{i\in I}(N \cap M_i)$. Since N is an indecomposable submodule, $N = N \bigcap M_i$ for some $i \in I$, i.e., $N \subseteq M_i$ for some $i \in I$. □

Corollary 4.37 *Let* $M = \bigoplus_{i\in I} M_i$ *be a direct sum of indecomposable R-submodules of* M *where* $\mathrm{card}(I) \geqslant 2$*. If* M *is a multiplication module and* N *is an indecomposable direct summand of* M *then* $N = M_i$ *for some* $i \in I$*.*

Proof. Since N is a direct summand of M, there exists a submodule K of M such that $M = N \bigoplus K$. By Theorem 1.11, $N = \bigoplus_{i\in I}(N \bigcap M_i)$. It follows that there exists $i \in I$ such that $N = N \bigcap M_i$ (since N is indecomposable). Again, by Theorem 1.11, since $M = N \bigoplus K$ and M_i is a submodule of M,

$$M_i = (M_i \bigcap N) \bigoplus (M_i \bigcap K) = N \bigoplus (M_i \bigcap K).$$

It follows that $M_i = N$ (since M_i is indecomposable and $N \neq 0$). $\square$

Definition 4.38 *An R-module M satisfies the* **direct sum cancellation property** *if $M = K \bigoplus L = K \bigoplus L'$ where K, L and L' are R-modules then $L = L'$.*

Lemma 4.39 *Every multiplication module satisfies the direct sum cancellation property.*

Proof. Let M be a multiplication R-module such that $M = K \bigoplus L = K \bigoplus L'$ where K, L and L' are R-modules. Then, by Theorem 1.11, $L = (K \cap L) \oplus (L \cap L') = L \cap L'$ which follows that $L \subseteq L'$. Similarly, $L' \subseteq L$. $\square$

Definition 4.40 *Let R be a ring. An R-module M satisfies the* **summand property** *if $K + L$ and $K \cap L$ are also direct summands of M for all direct summands K and L of M.*

Proposition 4.41 *Every multiplication module satisfies the summand property.*

Proof. Let K and L be direct summand submodules of M. Then $M = L \oplus L^* = K \oplus K^*$ for some submodule L^* and K^* of M. Since M is a multiplication module,

$$K = (K \cap L) \oplus (K \cap L^*),$$

by Theorem 1.11. Therefore $K \cap L$ is a direct summand submodule of K. Hence, $K \cap L$ is a direct summand submodule of M. Now, since $K + L$ is a submodule of $M = L \oplus L^*$, $K + L = L \oplus (K \cap L^*)$, by Theorem 1.11. Since K is a direct summand submodule of M and L^* is a submodule of M, $K \cap L^*$ is a direct summand submodule of L^*, by Theorem 1.11, i.e., $L^* = (K \cap L^*) \oplus K'$ where $K' = K^* \cap L^*$. So,

$$M = L \oplus L^* = L \oplus ((K \cap L^*) \oplus K') = (L \oplus (K \cap L^*)) \oplus K' = (K + L) \oplus K'.$$

Hence, $K + L$ is a direct summand submodule of M. Hence, M satisfies the summand property. $\square$

Let M be a multiplication module with a direct sum decomposition. The next corollary gives an intersection decomposition for every submodule of M.

Corollary 4.42 *Let $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ be a direct sum of nonzero R-modules where* card$(I) \geqslant 2$. *If M is a multiplication module then for each submodule N of M, $N = \bigcap_{\lambda\in\Lambda}(N + M'_\lambda)$ where $M'_\lambda = \bigoplus_{\mu\neq\lambda} M_\mu$.*

Proof. By Theorem 1.11, $N = \bigoplus_{\lambda\in\Lambda} N \bigcap M_\lambda$. So, for every $\lambda \in \Lambda$, $N + M'_\lambda = N \cap M_\lambda \oplus M'_\lambda$, and the corollary follows. □

4.3 Multiplication rings

Recall that a ring R is a multiplication ring iff for every two ideals $I \subseteq J$ of R, $I = KJ = JK'$ for some ideals K and K' of R.

Corollary 4.43 *Let R be a multiplication commutative ring. Then* $\mathrm{M_n}(R)$ *is a multiplication ring.*

Proof. Let I and J be ideals of $\mathrm{M_n}(R)$ such that $I \subseteq J$. Then, by Lemma 2.48, there exists ideals I' and J' of R such that $I = \mathrm{M_n}(I')$ and $J = \mathrm{M_n}(J')$. By Lemma 2.49, $I' \subseteq J'$ (since $I \subseteq J$), and therefore there exists an ideal L of R such that $I' = LJ'$ (Since R is a multiplication ring). Therefore, by Lemma 2.49,

$$\mathrm{M_n}(I') = \mathrm{M_n}(LJ') = \mathrm{M_n}(L)\mathrm{M_n}(J'),$$

and hence, $\mathrm{M_n}(R)$ is a multiplication ring. □

Lemma 4.44 *Let R be a ring such that all its ideals are idempotent ideals. Then R is a multiplication ring.*

Proof. Let I and J be ideals of R such that $J \subseteq I$. Then $J = J^2 \subseteq JI \subseteq J$. i.e., $J = JI$. Hence, I is a left multiplication module. Similarly, I is a right multiplication module, and hence, R is a multiplication ring. □

Corollary 4.45 *The algebras $\mathbb{I}_n$, $n \geq 1$, of polynomial integro-differential operators over a field of characteristic zero are (left and right) multiplication rings.*

Proof. By [15, Corollary 3.3(3)], every ideal of all $\mathbb{I}_n$ is an idempotent ideal. So, the result follows from Lemma 4.44. □

Lemma 4.46 *Let R be a left multiplication ring and I be an ideal of R. Then $IP \subseteq PI$ for all prime ideals P of R such that $I \nsubseteq P$.*

Proof. Let P be a prime ideal of R such that $I \nsubseteq P$. The ring R is a left multiplication ring. So, the inclusion of ideals $IP \subseteq I$ implies that $JI = IP \subseteq P$ for some ideal J of R. Since $I \nsubseteq P$ and the ideal P is prime, we must have $J \subseteq P$, and so $IP = JI \subseteq PI$. □

Proof of Theorem 1.21. 1. The prime ideals P and Q are incomparable. So, by Lemma 4.46, $QP \subseteq PQ \subseteq QP$, and so $PQ = QP$.

2(a). The ring R is a left multiplication ring. So, by Lemma 4.46, $IP \subseteq PI$ for all prime ideals P of R such that $I \nsubseteq P$. The ring R is a right multiplication ring. So, its opposite ring is a left multiplication ring. Hence, $PI \subseteq IP$ for all prime ideals P of R such that $I \nsubseteq P$. Hence, $IP = PI$.

2(b). We can assume that $P \neq Q$ otherwise the equality $PQ = QP$ is obvious. Then either $P \nsubseteq Q$ or $Q \nsubseteq P$, hence $PQ = QP$, by the statement (a). □

Example. Let R be the ring of upper triangular 3×3 matrices over a field K. The ring R is not a multiplication ring.

Proof. Let E_{ij} be the matrix units. Then the ideals $P_1 = (E_{11}, E_{33})$ and $P_2 = (E_{11}, E_{22})$ are prime ideals of the ring R such that $P_1 \neq P_2$ (since $R = KE_{22} \oplus P_1 = KE_{33} \oplus P_2$. Since $P_1 P_2 \neq P_2 P_1$, the ring R is not a multiplication ring, by Theorem 1.21. □

Corollary 4.47 *Let R be a multiplication ring. Then $I \cap P = IP = PI$ for all ideals I and P such that $I \nsubseteq P$ and P is a prime ideal of R.*

Proof. Since I is a multiplication ideal and $I \cap P$ is a submodule of I, $I \cap P = JI$ for some ideal J of R. As $JI \subseteq P$ and $I \nsubseteq P$, J must be contained in P. So, by Theorem 1.21,

$$I \cap P = JI \subseteq PI = IP \subseteq I \cap P,$$

i.e., $I \cap P = IP = PI$. □

Let R be a ring and I be an ideal of R. We denote by $\mathrm{V}(I)$ the set of all prime ideals of R that contain the ideal I.

Corollary 4.48 *Let R be a multiplication ring and I be an ideal of R. If $J = \prod_{i=1}^{n} P_i$ where P_i is a prime ideal such that $P_i \notin \mathrm{V}(I)$ for all $1 \leqslant i \leqslant n$ then $IJ = JI$.*

Proof. It follows from Theorem 1.21. □

Remark. We proved in Corollary 4.45 that $\mathbb{I}_n$ are multiplication rings. V. Bavula in [15], proved that the product of ideals of $\mathbb{I}_n$ is commutative. So, there is a subclass of the class of multiplication noncommutative rings such that the product of ideals is commutative. Let ς denotes such subclass.

Let $R \in \varsigma$ and I and J be two ideals of R such that $I \subseteq J$ then $I = I\theta(J)$. For,

$$I = LJ = L\theta(J)J = \theta(J)(LJ) = \theta(J)I$$

for some ideal L of R (since $\theta(J)$ is an ideal of R).

Fully-multiplication modules.

Definition 4.49 *An R-module M is called a* fully-multiplication *module if every submodule of M is a multiplication module.*

If M is a fully-multiplication R-module then every submodule of M is so. But, it is not true that every multiplication module is a fully-multiplication module.

Example. Let $R = K[x, y]/(x^2, xy, y^2)$. Since R is a commutative ring, R is a multiplication R-module. Let $I = (\overline{x}, \overline{y})$ and $J = (\overline{x})$. Since the ring (R, I) is a local ring, $I^2 = 0$ and $0 \neq J \subseteq I$, the R-module I is not a multiplication module (since otherwise there is an ideal J' of R such that $0 \neq J = J'I \subseteq I^2 = 0$, a contradiction). □

The next proposition gives a sufficient condition for a multiplication module to be a fully-multiplication module.

Proposition 4.50 *Let M be a multiplication R-module. If $IN = N \cap IM$ for every submodule N of M and every ideal I of R then M is a fully-multiplication module.*

Proof. Let N be a submodule of M and K a submodule of N. Then K is a submodule of M, and therefore $K = IM$ for some ideal I. By the assumption, $K = K \cap N = IM \cap N = IN$. Hence, N is a multiplication module. □

Corollary 4.51 *Let R be a commutative multiplication ring. Then every multiplication R-module is fully-multiplication.*

Proof. It follows from Lemma 2.74. □

The next corollary gives a criterion for a faithful multiplication module over a commutative ring to be a fully-multiplication module in terms of the ideal $\theta(M)$.

Corollary 4.52 *Let R be a commutative ring and M be a faithful multiplication R-module. Then*

1. *If $N = IM$ is a multiplication submodule of M where I is an ideal of $\theta(M)$ then I is a multiplication ideal.*
2. *M is a fully-multiplication module iff $\theta(M)$ is a fully-multiplication R-module iff every prime ideal that contained in $\theta(M)$ is a multiplication ideal. In addition, if M is finitely generated then M is a fully-multiplication module iff R is a multiplication ring.*

Proof. 1. Let J be an ideal of R such that $J \subseteq I$. Then $JM \subseteq IM = N$. So,

$$JM = J^{\star}(IM) = (J^{\star}I)M$$

(since $N = IM$ is a multiplication module). By Lemma 2.78, $J = J^{\star}I$, i.e., I is a multiplication ideal.

2. Suppose that M is a fully-multiplication module. Let I be an R-submodule of $\theta(M)$. Then IM is a submodule of M, and therefore it is a multiplication module (since M is a fully-multiplication module). So, by 1, I is a multiplication submodule of $\theta(M)$. Conversely, suppose that $\theta(M)$ is a fully-multiplication R-module and N be a submodule of M. Then, by Corollary 2.80, $N = IM$ for some ideal of R such that $I \subseteq \theta(M)$. So, I is a multiplication ideal. Hence, by Lemma 2.74, $IM = N$ is a multiplication module, i.e., M is a fully multiplication module, as required. The second equivalence in the statement follows from ([23, Theorem 3.4]). In case, M is finitely generated then $\theta(M) = R$, by Lemma 2.77, and hence, M is a fully-multiplication module iff R is a multiplication ring. □

4.4 The ring of endomorphisms of a multiplication module

In the first part, we give some properties and applications of the endomorphisms ring of a multiplication module. In the second part, we introduce new classes of modules: epimorphic modules, monomorphic modules and automorphic modules.

Let M be a multiplication module, $E = \mathrm{End}_R(M)$ and $\mathcal{E} = \mathrm{End}_E(M)$. If R is a commutative ring then there is a natural ring homomorphism

$$R \to E, \quad r \mapsto (x \mapsto rx) \tag{4.1}$$

with kernel $\mathrm{ann}_R(M)$. In particular, every E-module is also an R-module.

The following proposition is a description of E-submodules and $\mathcal{E}$-submodules of M.

Proposition 4.53 *Let R be a commutative ring, M be a multiplication R-module, $E = \mathrm{End}_R(M)$ and $\mathcal{E} = \mathrm{End}_E(M)$. Then*

1. $\mathrm{Sub}_R(M) = \mathrm{Sub}_E(M)$.
2. *For each submodule N of the R-module M, $[N : M]_R M = [N : M]_E M = N$ where $[N : M]_R = \mathrm{ann}_R(M/N)$ and $[N : M]_E = \mathrm{ann}_E(M/N)$.*
3. *For each submodule N' of the E-module M, $[N' : M]_R M = [N' : M]_E M = N'$.*
4. *M is a faithful multiplication module over the commutative ring E.*
5. *If M is finitely generated E-module then $\mathcal{E} = E$.*

Proof. 1. By Lemma 4.23.(2), every R-submodule of M is an E-submodule. By (4.1), the converse is also true.

2. The R-module M is a multiplication module, hence $[N : M]_R M = N$. Now,

$$N = EN = E[N : M]_R M \subseteq [N : M]_E M \subseteq N,$$

by the statement 1. Therefore, $[N : M]_E M = N$ (since $E[N : M]_R$ is an ideal of the ring E and $E[N : M]_R M = N$).

3. Statement 3 follows from statements 1 and 2.

4. Recall that the ring E is a commutative ring, Lemma 4.23.(6). Statement 4 follows from statement 3.

5. By statement 4, M is a faithful finitely generated multiplication module over the commutative ring E. So, by Lemma 2.99, the inclusion $E \subseteq \mathcal{E}$ is equality. □

Theorem 4.54 is a comparability theorem for R-endomorphisms of a multiplication R-module M.

Theorem 4.54 *Let R be a ring, M be an Artinian multiplication R-module and $f \in \mathrm{End}_R(M)$. Then for all $n \gg 0$ (all but finitely many), the ideals of the ring R, $\mathrm{ann}_R(\ker(f^n))$ and $\mathrm{ann}_R(\mathrm{im}(f^n))$, are comparable iff either $f \in \mathrm{Aut}_R(M)$ or f is nilpotent.*

Proof. ($\Rightarrow$) The R-module M is an Artinian module, therefore,

$$M = \ker(f^n) + \mathrm{im}(f^n)$$

for all $n \gg 0$. Let $x \in \ker(f^n) \cap \mathrm{im}(f^n)$. Since $\mathrm{im}(f^n)$ is a multiplication module, $Rx = If^n(M)$ for some ideal I of R. Now, since $x \in \ker(f^n)$,

$$0 = f^n(Rx) = f^n(If^n(M)) = If^{2n}(M) = If^n(M) = Rx.$$

Therefore,

$$M = \ker(f^n) \oplus \mathrm{im}(f^n).$$

By Proposition 1.7, if the ideals of the ring R, $\mathrm{ann}_R(\ker(f^n))$ and $\mathrm{ann}_R(\mathrm{im}(f^n))$, are comparable then either $\ker(f^n) = 0$ or $\mathrm{im}(f^n) = 0$, i.e., either either $f \in \mathrm{Aut}_R(M)$ or f is nilpotent.

($\Leftarrow$) This implication is obvious. $\square$

Let R be a commutative ring and M be an R-module. Then for all $r \in R$, $r_M : M \to M$, $m \mapsto rm$ is an endomorphism of M.

Proposition 4.55 *Let R be a commutative ring and M be a multiplication R-module such that $\mathrm{ann}_R(M)$ is a prime ideal of R. Then r_M is either zero or monomorphism for every $r \in R$.*

Proof. As $\ker(r_M)$ is a submodule of M and M is a multiplication module, $\ker(r_M) = IM$ for some ideal I of R. So,

$$0 = r_M(IM) = r(IM) = rIM,$$

i.e., $rI \subseteq \mathrm{ann}_R(M)$, and so either $r \in \mathrm{ann}_R(M)$ or $I \subseteq \mathrm{ann}_R(M)$ (since $\mathrm{ann}_R(M)$ is a prime ideal), and so $\ker(r_M) = 0$. Hence, r_M is either zero or monomorphism for every $r \in R$. $\square$

Epimorphic, monomorphic and automorphic modules.

Definition 4.56 *An R-module M is called* **epimorphic** *if every nonzero endomorphism of M is epimorphism, i.e., $\mathrm{End}_R(M) = \mathrm{Epi}_R(M) \cup \{0\}$.*

An R-module M is called **monomorphic** *if every nonzero endomorphism of M is monomorphism, i.e.,* $\mathrm{End}_R(M) = \mathrm{Mon}_R(M) \cup \{0\}$.

An R-module M is called **automorphic** *if every nonzero endomorphism of M is automorphism, i.e.,* $\mathrm{End}_R(M) = \mathrm{Aut}_R(M) \cup \{0\}$.

Lemma 4.57 *An R-module M is automorphic iff $\mathrm{End}_R(M)$ is a division ring.*

Proof. The lemma is obvious. □

Theorem 4.58 *Let R be a ring and M be an R-module. If M is an epimorphic multiplication module then M is automorphic.*

Proof. Since M is a multiplication R-module, $\mathrm{Epi}_R(M) = \mathrm{Aut}_R(M)$, by Lemma 4.23 (1). Since M is epimorhic, $\mathrm{End}_R(M) = \mathrm{Epi}_R(M) \cup \{0\} = \mathrm{Aut}_R(M) \cup \{0\}$. Hence, M is automorphic. □

Theorem 4.59 *Let M be an epimorphic R-module, $E = \mathrm{End}_R(M)$ and $\mathfrak{m} = \{f \in E \,|\, \ker(f) \neq 0\}$. Then*

1. *$(E, \mathfrak{m})$ is a local ring such that $E/\mathfrak{m}$ is a division ring.*
2. *$(1+\mathfrak{m})^\star = \mathrm{Aut}_R(M) \cap (1+\mathfrak{m})$ where $(1+\mathfrak{m})^\star$ is a group of units of the multiplicative monoid $(1+\mathfrak{m})$.*
3. *The ring epimorphism $\pi : E \longrightarrow E/\mathfrak{m}$, $f \longmapsto \bar{f} := f + \mathfrak{m}$ induces the group homomorphism $\pi' : \mathrm{Aut}_R(M) \longrightarrow (E/\mathfrak{m})^\star$, $f \longmapsto \bar{f} := f + \mathfrak{m}$ where $(E/\mathfrak{m})^\star = (E/\mathfrak{m}) \setminus \{0\}$ is the group of units of the division ring $E/\mathfrak{m}$ (see, statement 1). Then π' is a group epimorphism with $\ker(\pi') = (1+\mathfrak{m})^\star$. In particular, $(1+\mathfrak{m})^\star$ is a normal subgroup of $\mathrm{Aut}_R(M)$ and $(E/\mathfrak{m})^\star \cong \mathrm{Aut}_R(M)/(1+\mathfrak{m})^\star$.*

Proof. 1. Clearly, $E := \mathrm{Epi}_R(M) \cup \{0\} = \mathrm{Aut}_R(M) \sqcup \mathfrak{m}$ and $E\mathfrak{m} \subseteq \mathfrak{m}$. Now,

$$\mathfrak{m}E \subseteq \mathfrak{m}\mathrm{Aut}_R(M) \cup \mathfrak{m}\mathfrak{m} \subseteq \mathfrak{m} \cup \mathfrak{m} = \mathfrak{m}.$$

Hence, $\mathfrak{m}$ is an ideal of the ring E, i.e., E is a local ring. Since $E = \mathrm{Aut}_R(M) \sqcup \mathfrak{m}$ is a local ring, the factor ring $E/\mathfrak{m}$ is a division ring (since any nonzero element of $E/\mathfrak{m}$ is invertible).

2. The inclusion $(1+\mathfrak{m})^\star \subseteq \mathrm{Aut}_R(M) \cap (1+\mathfrak{m})$ is obvious. Let $f \in \mathrm{Aut}_R(M) \cap (1+\mathfrak{m})$.We have to show that $f \in (1+\mathfrak{m})^\star$, i.e., $f^{-1} \in 1+\mathfrak{m}$. Clearly, $f = 1+m$ for some $m \in \mathfrak{m}$. Then

$$1 = f^{-1}f = f^{-1}(1+m) = f^{-1} - m'$$

where $m' = -f^{-1}m \in \mathfrak{m}$. Hence, $f^{-1} = 1+m' \in 1+\mathfrak{m}$, as required.

3. Since ${\pi'}^{-1}((E/\mathfrak{m})^\star) \overset{\text{st.1}}{=} {\pi'}^{-1}((E/\mathfrak{m}) \setminus \{0\}) \subseteq \mathrm{Aut}_R(M)$ (as $E = \mathrm{Aut}_R(M) \sqcup \mathfrak{m}$), the group homomorphism ${\pi'}^{-1}$ is an epimorphism with

$$\ker(\pi') = ({\pi'}^{-1})(\bar{1}) = \pi^{-1}(1) \cap \mathrm{Aut}_R(M) = (1+\mathfrak{m}) \cap \mathrm{Aut}_R(M) \overset{\text{st.2}}{=} (1+\mathfrak{m})^\star,$$

and therefore $(1+\mathfrak{m})^\star$ is a normal subgroup, and so $(E/\mathfrak{m})^\star \cong \mathrm{Aut}_R(M)/(1+\mathfrak{m})^\star$. $\square$

For an R-module M, let $\mathcal{I}(M) = \bigcap_{0\neq f\in \mathrm{End}_R(M)} \mathrm{im}(f)$ and $\mathfrak{n}(M) = \{f \in \mathrm{End}_R(M) \mid f(M) = \mathcal{I}(M)\} \cup \{0\}$. It is clear that $\mathfrak{n}$ is a right ideal of $\mathrm{End}_R(M)$. If M is a multiplication R-module then $\mathfrak{n}$ is an ideal of $\mathrm{End}_R(M)$ (Since M satisfies the $\mathrm{End}_R(M)$-stability condition, by Lemma 4.23(2)).

Lemma 4.60 *Let M be an R-module. Then*

1. *M is epimorphic iff $\mathcal{I}(M) = M$ iff $\mathfrak{n}(M) = \mathrm{End}_R(M)$.*
2. *If M is a multiplication R-module then $\mathcal{I}(M)$ is an $\mathrm{End}_R(M)$-stable submodule.*

Proof. 1. It is trivial, by Definition 4.56.

2. It follows from Lemma 4.23 (2). $\square$

Proposition 4.61 *Let M be an R-module such that $\mathcal{I}(M) \neq 0$ and $\mathfrak{n}(M) \neq 0$. Then the R-module $\mathcal{I}(M)$ is epimorphic.*

Proof. Take an element $0 \neq f \in \mathfrak{n}(M)$. Let $0 \neq \alpha \in \mathrm{End}_R(\mathcal{I}(M))$. Then

$$\mathcal{I}(M) \supseteq \alpha(\mathcal{I}(M)) = \alpha f(M) \supseteq \mathcal{I}(M)$$

since $0 \neq \alpha f \in \mathrm{End}_R(M)$, and so $\alpha(\mathcal{I}(M)) = \mathcal{I}(M)$. Hence, $\mathcal{I}(M)$ is an epimorphic R-module. $\square$

The next corollary is a description of the ideal $\mathfrak{n}(\mathcal{I}(M))$.

Corollary 4.62 *Let M be an R-module such that $\mathcal{I}(M) \neq 0$ and $\mathfrak{n}(M) \neq 0$. Then $\mathfrak{n}(\mathcal{I}(M)) = \mathrm{End}_R(\mathcal{I}(M))$.*

Proof. It follows from Proposition 4.61 and Lemma 4.60 (1). □

Proposition 4.63 *Let M be a multiplication R-module such that N be an essential submodule. If $f_{|N} \in \mathrm{Mon}(N)$ then $f \in \mathrm{Mon}(M)$.*

Proof. Since M is a multiplication module and N is a submodule of M, by Lemma 4.23 (2), N is an $\mathrm{End}_R(M)$-stable submodule, i.e., $f_{|N} \in \mathrm{End}_R(N)$. It is clear that $\ker(f_{|N}) = \ker(f) \cap N$. So, if $f_{|N} \in \mathrm{Mon}(N)$ then

$$0 = \ker(f_{|N}) = \ker(f) \cap N.$$

Therefore $\ker(f) = 0$ (since N is an essential submodule). Hence, $f \in \mathrm{Mon}(M)$. □

Corollary 4.64 *Let M be a multiplication R-module and N be an essential submodule of M. If for every $f \in \mathrm{End}_R(M)$, $f_{|N} \in \mathrm{Mon}(N)$ then M is monomorphic module.*

Proof. It follows from Proposition 4.63. □

Chapter 5

Multiplication modules over commutative rings

Throughout the chapter R is a commutative ring.

In this chapter, we present some applications of the cancellation law of multiplication modules, we generalize some known results, and we give some cases of embedding of a multiplication module into its ring. Also, we study the product of two submodules of a (faithful) multiplication module. Several properties and applications of such operation are presented. Furthermore, we study multiplication modules over some rings.

5.1 Multiplication modules and the ideal $\theta(M)$

In this section, we give some characterizations of a multiplication module in terms of the ideal $\theta(M)$, we present some applications of the cancellation law of multiplication modules over commutative rings, we give an explicit description of the minimal prime submodules of a faithful multiplication module, and we present two cancellation laws which depends on the original cancellation law in their proofs. Furthermore, we generalize some known results.

Let $\mathfrak{m}$ be a maximal ideal of a ring R. Definition 2.67 shows that the maximal ideal $\mathfrak{m}$ identifies two subclasses of R-modules which are called $\mathfrak{m}$-cyclic modules and $\mathfrak{m}$-torsion modules.

We recall that an R-module M is *locally cyclic* if $M_{\mathfrak{m}}$ is cyclic $R_{\mathfrak{m}}$-module for every $\mathfrak{m} \in \mathrm{Max}(R)$.

The next corollary presents some characterizations of multiplication modules.

Corollary 5.1 *Let M be an R-module and $\mathfrak{m} \in \mathrm{Max}(R)$. Then*

1. *M is a multiplication module iff for every $\mathfrak{m} \in \mathrm{Max}(R)$ such that $\theta(M) \subseteq \mathfrak{m}$, $M_{\mathfrak{m}} = 0$.*
2. *If M is multiplication module then M is $\mathfrak{m}$-torsion iff $M = \mathfrak{m}M$.*
3. *If M is finitely generated locally cyclic module then M is a multiplication module.*
4. *If M is finitely generated multiplication module then for every $\mathfrak{m} \in \mathrm{Max}(R)$, M is $\mathfrak{m}$-cyclic.*

Proof.

1. ($\Rightarrow$) Suppose that M is a multiplication module and $\mathfrak{m} \in \mathrm{Max}(R)$ such that $\theta(M) \subseteq \mathfrak{m}$. Then, by Lemma 2.73(1),

$$M = \theta(M)M \subseteq \mathfrak{m}M \subseteq M,$$

i.e., $M = \mathfrak{m}M$. So, for every $x \in M$,

$$Rx = IM = I(\mathfrak{m}M) = \mathfrak{m}(IM) = \mathfrak{m}x$$

where I is an ideal of R (since M is a multiplication module). It follows that there exists $q \in \mathfrak{m}$ such that $(1-q)x = 0$. Therefore $(Rx)_{\mathfrak{m}} = 0$ for every $x \in M$, and hence, $M_{\mathfrak{m}} = 0$.

($\Leftarrow$) Suppose that $\mathfrak{m} \in \mathrm{Max}(R)$. Then we have two cases. If $\theta(M) \subseteq \mathfrak{m}$ then, by assumption, $M_{\mathfrak{m}} = 0$. So, $(Rx)_{\mathfrak{m}} = 0$ for every $x \in M$, i.e., there exists $q \in \mathfrak{m}$ such that $(1-q)x = 0$, and hence, M is $\mathfrak{m}$-torsion. If $\theta(M) \nsubseteq \mathfrak{m}$ then there exists $y \in M$ such that $[Ry : M] \nsubseteq \mathfrak{m}$. By maximality of $\mathfrak{m}$, $[Ry : M] + \mathfrak{m} = R$. It follows that there exists $q' \in \mathfrak{m}$ such that $(1-q')M \subseteq Ry$, and hence, M is $\mathfrak{m}$-cyclic. Hence, by Theorem 2.68, M is a multiplication module.

2. ($\Leftarrow$) Suppose that $M = \mathfrak{m}M$. Since M is a multiplication module,

$$Rm = IM = I\mathfrak{m}M = \mathfrak{m}(IM) = \mathfrak{m}m$$

for every $m \in M$ where I is an ideal of R. Therefore, there exists $a \in \mathfrak{m}$ such that $(1-a)m = 0$, i.e., M is $\mathfrak{m}$-torsion.

($\Rightarrow$) Suppose that M is $\mathfrak{m}$-torsion. Then for every $m \in \mathfrak{m}$, there exists $q \in \mathfrak{m}$ such that $(1-q)m = 0$, i.e., $m = qm$. So, $M \subseteq \mathfrak{m}M \subseteq M$, i.e., $M = \mathfrak{m}M$.

3. Let N be a submodule of M and $\mathfrak{m} \in \mathrm{Max}(R)$. Then $N_{\mathfrak{m}}$ is a submodule of $M_{\mathfrak{m}}$. By assumption $M_{\mathfrak{m}}$ is cyclic, i.e., it is a multiplication module. Therefore

$$N_{\mathfrak{m}} = [N_{\mathfrak{m}} : M_{\mathfrak{m}}]M_{\mathfrak{m}} = [N : M]_{\mathfrak{m}}M_{\mathfrak{m}} = ([N : M]M)_{\mathfrak{m}},$$

by Lemma 2.36 (since M is finitely generated). Hence, $N = [N : M]M$, as required. □

4. Let $\mathfrak{m}$ be any maximal ideal in R. Since M is finitely generated multiplication R-module, $\theta(M) = R$, by Lemma 2.77. Therefore $\theta(M) \nsubseteq \mathfrak{m}$, i.e., there exists $x \in M$ such that $[Rx : M] \nsubseteq \mathfrak{m}$. So, there exists $q \in \mathfrak{m}$ such that $(1-q)M \subseteq Rx$, i.e., M is $\mathfrak{m}$-cyclic.

Characterizations of a multiplication R-module M in terms of the ideal $\theta(M)$.

Lemma 5.2 *Let M be a multiplication R-module. Then M is a locally cyclic module, i.e., for every $\mathfrak{m} \in \mathrm{Max}(R)$, $M_{\mathfrak{m}}$ is cyclic. In particular, if $\theta(M) \subseteq \mathfrak{m}$ then $M_{\mathfrak{m}} = 0$.*

Proof. Let $\mathfrak{m} \in \mathrm{Max}(R)$. Then we have two cases. If $\theta(M) \subseteq \mathfrak{m}$ then, by Corollary 5.1(1), $M_{\mathfrak{m}} = 0$. If $\theta(M) \nsubseteq \mathfrak{m}$ then there exists $x \in M$ such that $[Rx : M] \nsubseteq \mathfrak{m}$. Therefore, by maximality of $\mathfrak{m}$, $[Rx : M] + \mathfrak{m} = R$. So, $1 = a + r$ for some elements $a \in \mathfrak{m}$ and $r \in [Rx : M]$ which implies that $(1-a)M \subseteq Rx$, i.e., M is $\mathfrak{m}$-cyclic. Therefore $M_{\mathfrak{m}} \subseteq (Rx)_{\mathfrak{m}} \subseteq M_{\mathfrak{m}}$, i.e., $M_{\mathfrak{m}} = (Rx)_{\mathfrak{m}}$. Hence, M is locally cyclic. □

Example: Every ideal of an almost Dedekind domain is locally cyclic (Definition 2.45). There are some examples of almost Dedekind domains which are not Noetherian (see [35]). Let $D^{\star}$ be an almost Dedekind domain which is not Noetherian. Then there exists an ideal I of $D^{\star}$ such that I is not finitely generated. Suppose that I is a multiplication ideal of $D^{\star}$. Then, by Lemma 2.87, I is finitely generated, a contradiction. Hence, I is a locally cyclic ideal which is not multiplication. □

Lemma 5.3 *Let M be a multiplication R-module and $\mathfrak{m} \in \mathrm{Max}(R)$. Then $(1+\mathfrak{m})^{-1}M$ is a cyclic $(1+\mathfrak{m})^{-1}R$-module. In particular, if $\theta(M) \subseteq \mathfrak{m}$ then $(1+\mathfrak{m})^{-1}M = 0$.*

Proof. Let $\mathfrak{m} \in \mathrm{Max}(R)$. Then we have two cases. If $\theta(M) \subseteq \mathfrak{m}$ then $M = \mathfrak{m}M$. So, by Corollary 5.1(2), M is $\mathfrak{m}$-torsion, i.e., for every $x \in M$ there exists $a \in \mathfrak{m}$ such that $(1-a)x = 0$. Therefore $(1+\mathfrak{m})^{-1}M = 0$. If $\theta(M) \nsubseteq \mathfrak{m}$ then there exists $y \in M$ such that $[Ry : M] \nsubseteq \mathfrak{m}$. So, by maximality of $\mathfrak{m}$, $\mathfrak{m}+[Ry : M] = R$. Thus $1 = q + a$ where $q \in \mathfrak{m}$ and $a \in [Ry : M]$. Therefore $(1 - q)M = aM \subseteq Ry$, i.e., M is $\mathfrak{m}$-cyclic. So, $(1+\mathfrak{m})^{-1}M \subseteq (1+\mathfrak{m})^{-1}Ry \subseteq (1+\mathfrak{m})^{-1}M$, i.e., $(1+\mathfrak{m})^{-1}M = (1 + \mathfrak{m})^{-1}Rx$, and hence, $(1 + \mathfrak{m})^{-1}M$ is a cyclic $(1 + \mathfrak{m})^{-1}R$-module. □

Theorem 5.4 *Let M be an R-module. The following statements are equivalent.*

1. *M is multiplication module.*
2. *For every $\mathfrak{m} \in \mathrm{Max}(R)$ such that $\theta(M) \subseteq M$, $M_{\mathfrak{m}} = 0$.*
3. *For every $\mathfrak{m} \in \mathrm{Max}(R)$ such that $M_{\mathfrak{m}} \neq 0$, $M_{\mathfrak{m}}$ is a cyclic $R_{\mathfrak{m}}$-module, and for every submodule N of M, $[N : M]_{\mathfrak{m}} = [N_{\mathfrak{m}} : M_{\mathfrak{m}}]$.*
4. *For every $\mathfrak{m} \in \mathrm{Max}(R)$ such that $\theta(M) \subseteq M$, $(1 + \mathfrak{m})^{-1}M = 0$.*
5. *For every $\mathfrak{m} \in \mathrm{Max}(R)$ such that $(1 + \mathfrak{m})^{-1}M \neq 0$, $(1 + \mathfrak{m})^{-1}M$ is a cyclic $(1+\mathfrak{m})^{-1}R$-module, and for every submodule N of M, $(1+\mathfrak{m})^{-1}[N : M] = [(1 + \mathfrak{m})^{-1}N : (1 + \mathfrak{m})^{-1}M]$.*
6. *For all $x \in M$, $Rx = \theta(M)x$.*

Proof. $(1 \Rightarrow 2)$ It follows from Corollary 5.1(1).

$(2 \Rightarrow 3)$ Let $\mathfrak{m} \in \mathrm{Max}(R)$ such that $M_{\mathfrak{m}} \neq 0$. So, by statement 2, $\theta(M) \nsubseteq \mathfrak{m}$, i.e., there exists $x \in M$ such that $[Rx : M] \nsubseteq \mathfrak{m}$. By maximality of $\mathfrak{m}$, $[Rx : M] + \mathfrak{m} = R$. It follows that there exists $q' \in \mathfrak{m}$ such that $(1 - q')M \subseteq Rx$. It follows that $M_{\mathfrak{m}} \subseteq (Rx)_m \subseteq M_{\mathfrak{m}}$, i.e., $M_{\mathfrak{m}} = (Rx)_{\mathfrak{m}}$, and hence, $M_{\mathfrak{m}}$ is a cyclic $R_{\mathfrak{m}}$-module. Now, let N be a submodule of M and $q \in \mathfrak{m}$. If $r \in [N : (1 - q)M]$ then $r(1 - q) \in [N : M]$. So, $(r/1) \in [N : M]_{\mathfrak{m}}$. Hence, $[N : (1 - q)M]_{\mathfrak{m}} \subseteq [N : M]_{\mathfrak{m}}$. Now, we have

$$[N : M]_{\mathfrak{m}} \supseteq [N : (1-q)M]_{\mathfrak{m}} \supseteq [N : Rx]_{\mathfrak{m}} = [N_{\mathfrak{m}} : (Rx)_{\mathfrak{m}}] = [N_{\mathfrak{m}} : M_{\mathfrak{m}}] \supseteq [N : M]_{\mathfrak{m}},$$

and therefore $[N : M]_{\mathfrak{m}} = [N_{\mathfrak{m}} : M_{\mathfrak{m}}]$.

$(3 \Rightarrow 6)$ Suppose that $Rx \neq \theta(M)x$ for some $x \in M$. Then, by Lemma 2.18, $\mathrm{ann}_R(x)+\theta(M) \neq R$. So, there exists $\mathfrak{m} \in \mathrm{Max}(R)$ such that $\mathrm{ann}_R(x)+\theta(M) \subseteq \mathfrak{m}$ which follows that $M_{\mathfrak{m}} \neq 0$ (since $(Rx)_{\mathfrak{m}} \neq 0$). Therefore, by statement 3, $M_{\mathfrak{m}}$ is cyclic, i.e., there exists $y \in M$ such that $M_\mathfrak{m} = (Ry)_\mathfrak{m}$. Again, by statement 3, we have

$$\theta(M)_\mathfrak{m} \supseteq [Ry : M]_\mathfrak{m} = [(Ry)_\mathfrak{m} : M_\mathfrak{m}] = [(Ry)_\mathfrak{m} : (Ry)_\mathfrak{m}] = R_\mathfrak{m},$$

a contradiction (since $\theta(M) \subseteq \mathfrak{m}$), and hence, $\theta(M)x = Rx$.

($1 \Rightarrow 4$) It follows from Lemma 5.3.

($4 \Rightarrow 5$) Let $\mathfrak{m} \in \mathrm{Max}(R)$ such that $(1+\mathfrak{m})^{-1}M \neq 0$. So, by statement 2, $\theta(M) \nsubseteq \mathfrak{m}$, i.e., there exists $y \in M$ such that $[Rx : M] \nsubseteq \mathfrak{m}$. By maximality of $\mathfrak{m}$, $[Rx : M] + \mathfrak{m} = R$, i.e., there exists $q' \in \mathfrak{m}$ such that $(1-q')M \subseteq Rx$. It follows that $(1+\mathfrak{m})^{-1}M \subseteq (1+\mathfrak{m})^{-1}(Rx) \subseteq (1+\mathfrak{m})^{-1}M$, i.e., $(1+\mathfrak{m})^{-1}M = (1+\mathfrak{m})^{-1}(Rx)$, and hence, $(1+\mathfrak{m})^{-1}M$ is a cyclic $(1+\mathfrak{m})^{-1}R$-module. Now, let N be a submodule of M and $q \in \mathfrak{m}$. If $r \in [N : (1-q)M]$ then $r(1-q) \in [N : M]$. So, $(r/1) \in (1+\mathfrak{m})^{-1}[N : M]$. Hence,

$$(1+\mathfrak{m})^{-1}[N : (1-q)M] \subseteq (1+\mathfrak{m})^{-1}[N : M].$$

Now, we have $(1+\mathfrak{m})^{-1}[N : M] \supseteq (1+\mathfrak{m})^{-1}[N : (1-q)M] \supseteq (1+\mathfrak{m})^{-1}[N : Rx] = [(1+\mathfrak{m})^{-1}N : (1+\mathfrak{m})^{-1}(Rx)] = [(1+\mathfrak{m})^{-1}N : (1+\mathfrak{m})^{-1}M] \supseteq (1+\mathfrak{m})^{-1}[N : M]$, and hence, $(1+\mathfrak{m})^{-1}[N : M] = [(1+\mathfrak{m})^{-1}N : (1+\mathfrak{m})^{-1}M]$.

($5 \Rightarrow 6$) Suppose that $Rx \neq \theta(M)x$ for some $x \in M$. As $\mathrm{ann}_R(x) \subseteq \mathfrak{m}$, $(1+\mathfrak{m})^{-1}M \neq 0$. So, by statement 4, $(1+\mathfrak{m})^{-1}M$ is cyclic, i.e., there exists $y \in M$ such that $(1+\mathfrak{m})^{-1}M = (1+\mathfrak{m})^{-1}(Ry)$. Again, by statement 4, we have

$$(1+\mathfrak{m})^{-1}\theta(M) \supseteq (1+\mathfrak{m})^{-1}[Ry : M] = [(1+\mathfrak{m})^{-1}(Ry) : (1+\mathfrak{m})^{-1}M] =$$

$$[(1+\mathfrak{m})^{-1}(Ry) : (1+\mathfrak{m})^{-1}(Ry)] = (1+\mathfrak{m})^{-1}R,$$

a contradiction (since $\theta(M) \subseteq \mathfrak{m}$), and hence, $\theta(M)x = Rx$.

($6 \Rightarrow 1$) It follows from Lemma 2.73(2). $\square$

We recall that a nonzero R-module M is called a *second module* if $\mathrm{ann}_R(M) = \mathrm{ann}_R(M/N)$ for all proper submodules N of M. Moreover, M is called a *prime module* if $\mathrm{ann}_R(M) = \mathrm{ann}_R(N)$ for all proper submodules N of M.

Proposition 5.5 *Let M be a nonzero multiplication module. Then*

1. *If M is a second module then M is simple.*

2. *If M is a prime module then M is finitely generated.*

Proof. 1. Let N be a proper submodule of M. As M is a second multiplication module,

$$N = [N : M]M = \mathrm{ann}_R(M/N)M = \mathrm{ann}_R(M)M = 0.$$

Hence, M is simple.

2. Let $0 \neq m \in M$. Then, by Lemma 2.73,

$$R = \mathrm{ann}_R(Rm) + \theta(M) = \mathrm{ann}_R(M) + \theta(M) = \theta(M)$$

(since M is prime module). Hence, by Lemma 2.77, M is finitely generated. □

Proposition 5.6 *Let N be a finitely generated submodule of a multiplication R-module M. If $\mathrm{ann}_R(N) \subseteq \theta(M)$ then M is finitely generated. In particular, if $\mathrm{ann}_R(N) = \mathrm{ann}_R(M)$ then M is finitely generated.*

Proof. As M is a multiplication module, $N = \theta(M)N$, by Lemma 2.73(1). Since N is finitely generated and $\mathrm{ann}_R(N) \subseteq \theta(M)$, by Lemma 2.15,

$$R = \theta(M) + \mathrm{ann}_R(N) = \theta(M).$$

Therefore, by Lemma 2.77, M is finitely generated. □

Proposition 5.7 *Let M be a faithful multiplication R-module. Then for every $x \in M$, $[\mathrm{ann}_R(Rx)M : M] = \mathrm{ann}_R(Rx)$.*

Proof. Clearly, $[0 : x] \subseteq [[0 : x]M : M]$. Let $r \in [[0 : x]M : M]$. Then $rM \subseteq [0 : x]M$, and therefore, by Lemma 2.18,

$$r\theta(M) \subseteq [0 : x]\theta(M) \subseteq [0 : x].$$

Since M is a multiplication R-module, by Lemma 2.73(1), $Rx = \theta(M)x$. So, $x = sx$ for some $s \in \theta(M)$. Now, $rx = r(sx) = (rs)x = 0$ (since $rs \in r\theta(M) \subseteq [0 : x]$). Hence, $r \in [0 : x]$, i.e., $[0 : x] = [[0 : x]M : M]$. □

Faithful multiplication submodules of a finitely generated faithful multiplication module.

Lemma 2.80(3), gives a description of a submodule of a faithful multiplication. The next corollary describes a faithful multiplication submodule of a finitely generated faithful multiplication module.

Corollary 5.8 *Let M be a finitely generated faithful multiplication R-module. Then every faithful multiplication submodule N of M can be written as $N = IM$ for some multiplication ideal I of R with zero annihilator such that $I \subseteq \theta(N)$. This representation is unique.*

Proof. $\theta(N) = \sum_{m \in N}[Rm : N] \supseteq \sum_{m \in N}[Rm : M]$. So,

$$\theta(N)M \supseteq \sum_{m \in N}[Rm : M]M = \sum_{m \in N} Rm = N.$$

Since N is a faithful multiplication module, $\theta(N)$ is a multiplication ideal, by Lemma 2.73(3). Therefore, by Lemma 2.74, $\theta(N)M$ is a multiplication module. Since N is a submodule of $\theta(N)M$, $N = J(\theta(N)M) = (J\theta(N))M$ for some ideal J of R, i.e., $N = IM$ for some ideal I of R with $I \subseteq \theta(N)$. To show that I is a multiplication ideal, suppose that I' is an ideal of R such that $I' \subseteq I$. Then $I'M \subseteq IM = N$. As N is a multiplication module, there exists an ideal J' of R such that

$$I'M = J'N = J'(IM) = (J'I)M \quad (\star)$$

Since M is a finitely generated multiplication module, $\theta(M) = R$, by Lemma 2.77. So, I' and $J'I$ are contained in $\theta(M)$. By $(\star)$ and cancellation law (Lemma 2.78), $I' = J'I$, and hence, I is a multiplication ideal. Uniqueness follows from the cancellation law as $\theta(M) = R$. $\square$

For a faithful R-module M, let $\mathrm{M_F.Sup}(M) = \{N \mid N \text{ is a faithful multiplication submodule of } M\}$ and $I_\theta = \{I \mid I \text{ is a faithful multiplication ideal of } R \text{ and } I \subseteq \theta(IM)\}$.

Corollary 5.9 *Let M be a finitely generated faithful multiplication R-module. Then there exists a bijection between $\mathrm{M_F.Sup}(M)$ and I_θ.*

Proof. Since M is a finitely generated faithful multiplication module, every faithful multiplication submodule N can be written by a unique way as IM such that I is a faithful multiplication ideal with $I \subseteq \theta(N)$, by Corollary 5.8. Let

$$f : \mathrm{M_F.Sup}(M) \to I_\theta, \ N \longmapsto I$$

where I is a multiplication ideal with zero annihilator and $N = IM$ where $I \subseteq \theta(N)$. This map is a bijection for:

1. By Corollary 5.8, it is well-defined.

2. f is a surjection: Given $I \in I_\theta$. Then, by Lemma 2.74, IM is a faithful multiplication module (as M is faithful multiplication module), and therefore $IM \in \mathrm{M_F.Sup}(M)$.

3. f is an injection: Given two ideals N and $N' \in \mathrm{M_F.Sup}(M)$ such that $f(N) = f(N')$. Then $N = IM$ and $N' = I'M$ for some faithful multiplication ideals I and I' of R such that $I \subseteq \theta(N)$ and $I' \subseteq \theta(N')$, respectively. Since $f(N) = f(N')$, $I = I'$. So $N = IM = I'M = N'$, and hence, f is an injection. □

The ideal $\theta(M)$ of a (faithful) multiplication R-module M.

Proposition 5.10 *Let M be a faithful multiplication R-module. Then for every prime ideal P of R, either $\theta(M) \subseteq P$ or $\theta(M) + P = R$. Moreover, if $\theta(M) \nsubseteq P$ then $R/P\theta(M) \cong (R/P) \times (R/\theta(M))$.*

Proof. Suppose that $\theta(M) \nsubseteq P$. Then there exists $r \in \theta(M) \setminus P$. Since M is a faithful multiplication R-module, $\theta(M)$ is multiplication ideal of R, by Corollary 2.73(1). Therefore, by Lemma 2.73(2) and Lemma 2.73(4),

$$R = \theta(\theta(M)) + [0 : r] = \theta(M) + [0 : r].$$

As $[0 : r]r = 0 \in P$ and $r \notin P$, $[0 : r] \subseteq P$. Hence, $\theta(M) + P = R$. The statement $R/P\theta(M) \cong (R/P) \times (R/\theta(M))$ follows from Lemma 2.17. □

Proposition 5.11 *Let M be a faithful multiplication R-module. Then $\theta(M)$ is an essential ideal of R.*

Proof. Let I be an ideal of R such that $\theta(M) \cap I = 0$. So, by Theorem 2.72, $0 = (\theta(M) \cap I)M = \theta(M)M \cap IM = M \cap IM = IM$. Therefore $I = 0$ (since M is faithful), and hence, $\theta(M)$ is an essential ideal of R. □

Let I be a multiplication ideal with zero annihilator in a ring R and M be a faithful multiplication R-module. The next corollary provides a description of the ideal $\theta(IM)$.

Corollary 5.12 *Let M be a faithful multiplication R-module. If I is a multiplication ideal of R with zero annihilator then $\theta(IM) = \theta(I)\theta(M) = \theta(I) \cap \theta(M)$. Furthermore, IM is finitely generated iff M and I are both finitely generated.*

Proof. Since M is a faithful multiplication R-module and I is a multiplication ideal of R, IM is a faithful multiplication module, by Lemma 2.74. So, by Lemma 2.73(1),

$$\theta(IM)IM = IM = \theta(I)I\theta(M)M = \theta(I)\theta(M)(IM).$$

Since IM is a faithful multiplication module and $\theta(I)\theta(M) \subseteq \theta(IM)$, $\theta(I) \cap \theta(M) = \theta(I)\theta(M) = \theta(IM)$, by Lemmas 2.80(2) and by Lemma 2.78. The another equivalence follows from Lemma 2.77. □

Proposition 5.13 *Let M be a multiplication R-module. If N is a faithful multiplication R-submodule of M then $\theta(N) \subseteq \theta(M)$. In particular, if M has a finitely generated faithful multiplication submodule then M is finitely generated.*

Proof. Since N is a multiplication module, $N = \theta(N)N$, by Lemma 4.8. By Lemma 2.73, $N = \theta(N)\theta(M)N$ (since N is a submodule of a multiplication module M). Therefore, by Lemma 2.78, $\theta(N) = \theta(N)\theta(M)$ (since N is a faithful multiplication module). By Corollary 2.80 (since M is a faithful multiplication module),

$$\theta(N) = \theta(M)\theta(N) = \theta(M) \cap \theta(N),$$

and hence, $\theta(N) \subseteq \theta(M)$. In case N is finitely generated, $\theta(N) = R$, by Lemma 2.77, and therefore $\theta(M) = R$, i.e. M is finitely generated, by Lemma 2.77. □

Further two cancellation laws.

In Theorem 5.14 and Theorem 5.16, we give two cancellation laws which rely on the original cancellation law (Lemma 2.78) in their proofs.

Theorem 5.14 (The first companion cancellation law) *Let M be a faithful multiplication module. If I is a faithful multiplication ideal of R and $IN = IK$ where N and K are submodules of M then $N\theta(I) = K\theta(I)$. Moreover, if I is finitely generated then $N = K$.*

Proof. As M is a multiplication module and $IN = IK$, $I[N : M]M = I[K : M]M$. Therefore, by Lemma 2.78,

$$I[N : M]\theta(M) = I[K : N]\theta(M)$$

(Since M is a faithful multiplication module). Again, by Lemma 2.78 (since I is a faithful multiplication ideal),

$$[N : M]\theta(M)\theta(I) = [K : M]\theta(M)\theta(I).$$

So, $N\theta(I) = K\theta(I)$. If I is finitely generated then $\theta(I) = R$, by Lemma 2.77, and hence, $N = K$. □

Corollary 5.15 *Let R be a multiplication domain and M be a faithful multiplication module. If N and K are R-submodules of M such that $IN = IK$ for some ideals I of R then $N = K$.*

Proof. It follows from Theorem 5.14 (since every ideal of the ring R is finitely generated faithful multiplication ideal, by Corollary 2.87). □

Theorem 5.16 (The second companion cancellation law) *Let M be a faithful multiplication module and I be a faithful multiplication ideal of R where IM is finitely generated. If $IN = IK$ where N and K are submodules of M then $N = K$.*

Proof. As M is a multiplication module and $IN = IK$, $I[N : M]M = I[K : M]M$. So, by Lemma 2.78,

$$I[N : M]\theta(M) = I[K : M]\theta(M)$$

(Since M is a faithful multiplication module). Again, by Lemma 2.78 (As I is a faithful multiplication ideal),

$$[N : M]\theta(M)\theta(I) = [K : M]\theta(M)\theta(I).$$

By Corollary 5.12, $[N : M]\theta(IM) = [K : M]\theta(IM)$, and therefore $[N : M] = [K : M]$, by Lemmas 2.74 and 2.77. Hence, $N = K$ (since M is a multiplication module). □

Corollary 5.17 *Let M be a faithful multiplication R-module and I be a finitely generated multiplication ideal with zero annihilator such that $I \subseteq \theta(M)$. If $IN = IK$ where N and K are submodules of M then $N = K$.*

Proof. It follows from Theorem 5.16 and Lemma 2.76. □

Minimal prime submodules of a nonzero faithful multiplication module.

We recall that a proper submodule N of an R-module M is a minimal prime submodule if it is a prime submodule and if there is a prime submodule N' such that $N' \subseteq N$ then $N' = N$.

Proposition 5.18 *Let N be a proper submodule of a nonzero faithful multiplication R-module. Then N is a minimal prime submodule iff $N = PM$ for some minimal prime ideal P of R such that $\theta(M) \nsubseteq P$.*

Proof. (⇒) Let N be a minimal prime submodule of M. Then, by Corollary 2.97, $N = PM$ for some prime ideal P of R such that $PM \neq M$. Clearly, $\theta(M) \nsubseteq P$. To prove P is a minimal prime ideal of R, suppose that there is $P' \in \mathrm{Spec}(R)$ such that $P' \subseteq P$. It follows that $P'M \subseteq PM = N$ and $P'M$ is a prime submodule of M, by Corollary 2.97. So, by minimality of N, $P'M = PM$. Therefore $P'\theta(M) = P\theta(M)$, by Lemma 2.78. Since $\theta(M) \nsubseteq P$, by Proposition 5.10, $P + \theta(M) = R$. It follows that

$$P' = P'P + P'\theta(M) = P'P + P\theta(M) = P(P' + \theta(M)).$$

Since P' is a prime ideal of R, either $P \subseteq P'$ or $P' + \theta(M) \subseteq P'$. If $P' + \theta(M) \subseteq P'$ then $\theta(M) \subseteq P'$, a contradiction. Therefore $P \subseteq P'$, i.e., $P' = P$, and hence, P is a minimal prime of R.

(⇐) Suppose that P is a minimal prime ideal of R such that $\theta(M) \nsubseteq P$. Then, by Corollary 2.97, PM is a prime submodule of M. To prove PM is a minimal prime submodule, suppose that $N' = P'M \subseteq PM$ where P' is a prime ideal with $P'M \neq M$. So, by Corollary 2.82, $P'\theta(M) \subseteq P\theta(M) \subseteq P$. Since P is a prime ideal and $\theta(M) \nsubseteq P$, $P' \subseteq P$, i.e., $P' = P$ (since P is a minimal prime ideal), and hence, PM is a minimal prime submodule of M. □

The next proposition is a multiplication module's version of Lemma 2.10.

Proposition 5.19 *Let M be a faithful multiplication R-module. If $N_1, N_2, \ldots, N_n$ are submodules of M and $K = PM$ is a prime submodule of M where P is a prime ideal of R such that $\bigcap_{i=1}^{n} N_i \subseteq K$ then $N_i \subseteq K$ for some $1 \leq i \leq n$.*

Proof. Since M is a multiplication module, $N_i = I_iM$ for some ideal I_i of R. By Theorem 2.72, $K = PM \supseteq \bigcap_{i=1}^{n} N_i = \bigcap_{i=1}^{n}(I_iM) = (\bigcap_{i=1}^{n} I_i)M$. Therefore, by Corollary 2.82,

$$(\bigcap_{i=1}^{n} I_i)\theta(M) \subseteq P\theta(M) \ \ (\star).$$

Notice that, by Theorem 2.72, $\bigcap_{i=1}^{n}(I_i \bigcap \theta(M))M = (\bigcap_{i=1}^{n} I_i)\theta(M)M = \bigcap_{i=1}^{n}(I_iM)$ implies that $\bigcap_{i=1}^{n}(I_i \bigcap \theta(M)) = (\bigcap_{i=1}^{n} I_i)\theta(M)$, by Lemma 2.78. So, by $\star$,

$$\bigcap_{i=1}^{n}(I_i \bigcap \theta(M)) \subseteq P\theta(M) \subseteq P.$$

Therefore, by Lemma 2.10, $I_i \bigcap \theta(M) \subseteq P$ for some $1 \leq i \leq n$. Hence, by Theorem 2.72,

$$(I_i \bigcap \theta(M))M = I_iM = N_i \subseteq PM = K,$$

as required. □

Corollary 5.20 *Let M be a faithful multiplication R-module with finitely many maximal submodules. If $\operatorname{rad}(M) \subseteq K$ for some prime submodule K of M then K is a maximal submodule. In particular, if $\operatorname{rad}(M) = 0$ then every prime submodule is a maximal submodule.*

Proof. It follows from Proposition 5.19. □

Let N and K be submodules of an R-module M. It is not always that $[N : M] = [N : K][K : M]$ is true (Always $[K : N][N : M] \subseteq [K : M]$).

Corollary 5.21 *Let N be a multiplication submodule of a faithful multiplication R-module M. If K is a submodule of N such that $[K : M] \subseteq \theta(M)$ then $[K : M] = [K : N][N : M]$. In particular, if K is finitely generated submodule N then the result holds.*

Proof. Since M and N are multiplication modules and K is a submodule of N,

$$[K : M]M = K = [K : N]N = [K : N][N : M]M,$$

and therefore Lemma 2.78,

$$[K : M]\theta(M) = [K : N][N : M]\theta(M).$$

Since $[K : M] \subseteq \theta(M)$, by Corollary 2.80,

$$[K : M]\theta(M) = [K : M].$$

Since $[K : N][N : M] \subseteq [K : M]$,

$$[K : N][N : M]\theta(M) = [K : N][N : M],$$

and hence, $[K : M] = [K : N][N : M]$.

If $K = [K : M]M$ is a finitely generated then, by Lemma 2.75, $[K : M] \subseteq \theta(M)$, and hence, the result holds. □

The ideal $\theta(M)$ where M is a multiplication module and has a direct sum decomposition.

The next proposition gives an explicit description of the ideal $\theta(M)$ where M is a finite direct sum of its submodules and it is a faithful multiplication module.

Proposition 5.22 *Let M be a faithful multiplication R-module. If $M = N \bigoplus K$ for some R-submodules N and K of R. Then*

1. *$\theta(M) = [N : M]\theta(M) \bigoplus [K : M]\theta(M)$. Moreover, the ideals $[N : M]\theta(M)$ and $[K : M]\theta(M)$ are multiplication ideals.*
2. *If $I \subseteq \theta(M)$ then $I = ([N : M] \cap I) \bigoplus ([K : M] \cap I)$*
3. *$\theta(M) = \sum_{x\in N}[Rx : M] \bigoplus \sum_{y\in K}[Ry : M]$.*
4. *$[N : M]\theta(M)$ is an idempotent ideal of R.*

Proof. 1. Since M is a multiplication module,

$$M = N \bigoplus K = [N : M]M \bigoplus [K : M]M = ([N : M] \bigoplus [K : M])M.$$

By Lemma 2.78,

$$\theta(M) = ([N : M] \bigoplus [K : M])\theta(M) = [N : M]\theta(M) \bigoplus [K : M]\theta(M).$$

As M is a faithful multiplication module, $\theta(M)$ is a multiplication ideal, by Lemma 2.73(3), and therefore, by Theorem 1.14, the statement 1 holds.

2. As M is a faithful multiplication module, $\theta(M)$ is a multiplication ideal, by Lemma 2.73(3). So, by Theorem 1.11 and by Lemma 2.80(2),

$I = (I \cap [N : M]\theta(M)) \bigoplus (I \cap [K : M]\theta(M)) = (I \cap [N : M] \cap \theta(M)) \bigoplus (I \cap [K : M] \cap \theta(M)) = ([N : M] \cap I) \bigoplus ([K : M] \cap I)$.

3. Since $M = N \bigoplus K$ is a multiplication module, N and K are multiplication modules, by Theorem 1.14. Therefore, by Lemma 4.8 and by Corollary 5.21,

$$\begin{aligned} M &= N \bigoplus K = \theta(N)N \bigoplus \theta(K)K \\ &= \sum_{x\in N}[Rx : N]N \bigoplus \sum_{y\in K}[Ry : K]K \\ &= \sum_{x\in N}[Rx : N][N : M]M \bigoplus \sum_{y\in K}[Ry : K][K : M]M \end{aligned}$$

$$= \sum_{x\in N}[Rx : M]M \bigoplus \sum_{y\in K}[Ry : M]M$$

$$= (\sum_{x\in N}[Rx : M] \bigoplus \sum_{y\in K}[Ry : M])M.$$

Therefore, by Lemma 2.78, $\theta(M) = \sum_{x\in N}[Rx : M] \bigoplus \sum_{y\in K}[Ry : M]$.

4. Since $[N : M][K : M]M \subseteq N \cap K = 0$ and M is a faithful module, then $[N : M][K : M] = 0$. By Lemma 2.73(3) and statement 1,

$$\begin{aligned}[N : M]\theta(M) &= [N : M]\theta^2(M)\\ &= [N : M]\theta(M)([N : M]\theta(M) \bigoplus [K : M]\theta(M))\\ &= [N : M]^2\theta^2(M)\\ &= ([N : M]\theta(M))^2,\end{aligned}$$

and therefore $[N : M]\theta(M)$ is an idempotent ideal. □

P. Smith in ([42, Corollary 1 of Theorem 10]) proved that If M is a finitely generated R-module such that $\mathrm{ann}_R(M) = Re$ for some idempotent element e of R then N is a multiplication submodule of M iff $[N : M]$ is a multiplication ideal of R. The next corollary is a generalization of such corollary without need the condition of finitely generated.

Corollary 5.23 *Let M be a multiplication R-module such that $\mathrm{ann}_R(M) = Re$ for some idempotent element e of R. Then N is a multiplication submodule of M iff $[N : M]\theta(M)$ is a multiplication ideal of R.*

Proof. Suppose that N is a multiplication R-submodule of M. Let $S = Re$ and $T = R(1 - e)$. Then $R = S \bigoplus T$, and therefore M is faithful multiplication $T \cong R/S$-module. N is a multiplication T-module for if K is a T-submodule of N then K is an R-submodule of N, and therefore (since N is a multiplication R-module),

$$\begin{aligned}K &= [K : N]N\\ &= [K : N][N : M]M\\ &= ([N : M]S \bigoplus [N : M]T)[K : N]M\\ &= [K : N][N : M]TM = ([K : N]T)N.\end{aligned}$$

So, by Proposition 2.84, $[N :_T M]\theta(M_T)$ is a multiplication ideal of T. Now,

$$\begin{aligned}[N : M]\theta(M) &= ([N : M]S \bigoplus [N : M]T)\theta(M) \\ &= (S \bigoplus [N : M]T)\theta(M) \\ &= S\theta(M) \bigoplus [N : M]T^2\theta(M) \\ &= S\theta(M) \bigoplus [N :_T M]\theta(M_T),\end{aligned}$$

and hence, $[N : M]\theta(M)$ is a multiplication ideal, by Theorem 2.71. The converse is obvious, by Lemma 2.74. □

5.2 Product of two submodules of a faithful multiplication module

Let N and K be submodules of an R-module M. Suppose that $N = IM$ and $K = JM$ for some ideals I and J of R. Ameri in [6], defined the product of N and K, NK, or, N.K, as follows:

$$NK := IJM.$$

If M is a multiplication module M then every submodule N of M is equal to IM for some ideal I of R. This prsentation of N is not always unique. Ameri proved in ([6, Theorem 3.4]), that the presentation of the submodule NK in a multiplication module M is independent, i.e., if $N = IM = I'M$ and $K = JM = J'M$ where I' and J' are ideals of R then $NK = IJM = I'J'M$. Aziz and Jayaram in [13], provided some applications of the product of submodules of a multiplication module.

Lemma 2.80 shows that every submodule N of M has a unique representation $I_N M$ where I_N is an ideal of $\theta(M)$. Corollary 2.85 shows that there is a bijection between the set of all submodules of M, $\mathrm{Sub}(M)$, and the set of all ideal of R that are contained in $\theta(M)$, $\mathcal{I}(\theta(M))$. This bijection respects inclusion (Corollary 2.82), i.e., if $N_1,\ N_2 \in \mathrm{Sub}(M)$ such that $N_1 \subseteq N_2$ then $I_{N_1} \subseteq I_{N_2}$ where I_{N_1} and I_{N_2} are the correspondents ideals of N_1 and N_2, respectively in $\mathcal{I}(\theta(M))$. In that situation, the ideal $I_N I_K$ in $\mathcal{I}(\theta(M))$ corresponds the submodule NK, i.e., $N.K := I_N I_K M$.

In this section, we apply such operation to a faithful multiplication R-module.

The next proposition gives some properties of the product of submodules of a faithful multiplication module.

Proposition 5.24 *Let M be a faithful multiplication R-module. If N, K and L are submodules of M then*

1. *NK is a submodule of M such that $I_{NK} = I_N I_K$.*
2. *$NM = N$.*
3. *$N^n \subseteq N$ where $n \in \mathbb{N}$.*
4. *$NK \subseteq N \cap K$.*
5. *$N(K+L) = NK + NL$.*
6. *If $I_N + I_K = \theta(M)$ then $NK = N \cap K$.*
7. *For every $\mathfrak{p} \in \operatorname{Spec}(R)$, $(NK)_{\mathfrak{p}} = N_{\mathfrak{p}} K_{\mathfrak{p}}$.*
8. *For every $\mathfrak{p} \in \operatorname{Spec}(R)$ and $n \in \mathbb{Z}$, $(N^n)_{\mathfrak{p}} = (N_{\mathfrak{p}})^n$.*

Proof. 1. Since $NK = I_N I_K M$, NK is a submodule of M, and therefore, by Lemma 3.33, $NK = I_{NK} M$. Hence, By cancellation law (Lemma 2.78), $I_{NK} = I_N I_K$ (since $I_{NK} M = I_N I_K M$).

2. Clearly, $I_M = \theta(M)$. So, $NM = I_N I_M M = I_N \theta(M) M = I_N M = N$.

3. $N^n = \overbrace{N \cdots N}^{n \text{ times}} = \overbrace{I_N \cdots I_N}^{n \text{ times}} M = I_N^n M \subseteq I_N M = N$.

4. $NK = I_N I_K M \subseteq (I_N \cap I_K) M \overset{\text{Theorem 2.72}}{=} I_N M \cap I_K M = N \cap K$.

5. $N(K+L) = (I_N M)(I_K M + I_L M) = (I_N M)((I_K + I_L)M = (I_N(I_K + I_L))M = (I_N I_K + I_N I_L)M = NK + NL$.

6. By Lemma 2.73, $N \cap K = \theta(M)(N \cap K) = (I_N + I_K)(I_N M \cap I_K M) \overset{\text{Theorem 2.72}}{=} (I_N + I_K)(I_N \cap I_K)M = ((I_N \cap I_K)I_N + (I_N \cap I_K)I_K)M \subseteq (I_K I_N + I_N I_K)M = I_N I_K M = NK$. So $N \cap K \subseteq NK$. Hence, by statement 4, the result holds.

7. $(NK)_{\mathfrak{p}} = (I_N I_K M)_{\mathfrak{p}} \overset{\text{Lemma 2.37}}{=} I_{N_{\mathfrak{p}}} I_{K_{\mathfrak{p}}} M_{\mathfrak{p}} = N_{\mathfrak{p}} K_{\mathfrak{p}}$ (since $M_{\mathfrak{p}}$ is a faithful multiplication $R_{\mathfrak{p}}$-module, by Lemma 2.69).

8. It follows from statement 7 and statement 3. $\square$

Definition 5.25 *Let M be a faithful multiplication R-module. A submodule N of M is called an* **invertible submodule** *if there exists a submodule $N^\star$ of M such that $NN^\star = M$.*

Corollary 5.26 *Let M be a faithful multiplication R-module. A submodule N of M is invertible iff there exists a submodule $N^\star$ of M such that $I_N I_{N^\star} = \theta(M)$.*

Proof. Suppose that N is an invertible submodule. Then there exists a submodule $N^\star$ of M such that $NN^\star = M$ which implies that $I_N I_{N^\star} M = M$. Therefore, by Lemma 2.78 and Lemma 2.80(2),

$$\theta(M) = I_N I_{N^\star}\theta(M) = I_N I_{N^\star} \cap \theta(M),$$

i.e., $\theta(M) \subseteq I_N I_{N^\star}$. Conversely, suppose that $\theta(M) \subseteq I_N I_{N^\star}$. Then, by Corollary 2.80, $\theta(M) = I_N I_{N^\star} \cap \theta(M) = I_N I_{N^\star}\theta(M)$, and therefore $M = I_N I_{N^\star} M = NN^\star$, i.e., N is invertible.

Proof of Theorem 1.23. As $I_N + I_K = \theta(M)$, by Lemma 5.24 (6), $NK = N \cap K$. Let $f : M/NK \longrightarrow (M/N) \times (M/K)$, $x + NK \mapsto (x + N, x + K)$.

1. f is well-defined: suppose that $x + NK = y + NK$ where x and $y \in M$. Then $x - y \in NK = N \cap K$, i.e., $x - y \in N$ and $x - y \in K$. Therefore $(x + N, x + K) = (y + N, y + K)$.

2. f is homomorphism: for every $x + NK$ and $y + NK \in M/NK$,

$$\begin{aligned} f((x+NK)+(y+NK)) &= f(x+y+NK) \\ &= (x+y+N, x+y+K) \\ &= (x+N, x+K) + (y+N, y+K) \\ &= f(x+NK) + f(y+NK) \end{aligned}$$

. Also for every $r \in R$,

$$\begin{aligned} f(r(x+NK)) &= f(rx+NK) \\ &= (rx+N, rx+K) \\ &= r(x+N, x+K) \\ &= rf(x+NK). \end{aligned}$$

3. f is injection: given $x + NK \in \ker(f)$. Then $f(x + KN) = (x + N, x + K) = (N, K)$. So, $x \in K \cap N = KN$, i.e., $x + KN = KN$, and therefore $\ker(f) = 0$.

4. f is surjection: Since $I_N + I_K = \theta(M)$, $N + K = M$, by Lemma 2.73(1). So, for every $x \in M$, $x = n + k$ where $n \in N$ and $k \in K$. So, every element in $(M/N) \times (M/K)$ is written as $(a_K + N, a_N + K)$ where $a_K \in K$ and $a_N \in N$. Now, $f(a_K + a_N + NK) = (a_K + a_N + N, a_N + a_K + K) = (a_K + N, a_N + K)$. □

Lemma 5.27 *Let M be a finitely generated faithful multiplication R-module and N be a submodule of M where $[N : M]$ is a finitely generated ideal of R. If $NK = N$ for some submodule K of M such that $K \subsetneq \operatorname{rad}(M)$ then $N = 0$.*

Proof. As M is a finitely generated faithful multiplication module, $\theta(M) = R$, by Lemma 2.77, and therefore $I_N = [N : M]$. By assumption, $I_N I_K M = NK = N = I_N M$. Therefore, by Lemma 2.78, $I_N I_K = I_N$ (since M is a finitely generated faithful multiplication R-module). By Theorem 2.93, $\operatorname{rad}(M) = J(R)M$. So, $K = I_K M \subsetneq \operatorname{rad}(M) = J(R)M$. Therefore $I_K \subsetneq J(R)$, by Corollary 2.82. Now, I have $I_N I_K = I_N$ where I_N is finitely generated. So, by Nakayama's Lemma, $I_N = 0$, and hence, $N = 0$, as required. □

Proof of Proposition 1.24. By Lemma 2.92, M is cyclic. So, $M \cong R$ (since M is faithful), and therefore R is Artinian which implies that R is Noetherian. Therefore $[N^n : M]$ is finitely generated. By lemma 5.24, we have

$$N \supseteq N^2 \supseteq \cdots \supseteq \; N^k \supseteq \cdots$$

Since M is artinian, there exists $n \in \mathbb{N}$ such that $NN^n = N^{n+1} = N^n$. Therefore by Lemma 5.27, $N^n = 0$. □

Proposition 5.28 *Let M be a faithful multiplication R-module. Then*

1. *If N and K are multiplication submodules of M then NK is a multiplication submodule of M.*
2. *If N and K are faithful multiplication submodules of M then $\theta(NK) = \theta(I_N)\theta(I_K)\theta(M)$*

Proof. 1. Since N and K are multiplication submodules of M, by Proposition 2.84(3), I_N and I_K are multiplication ideals, and therefore, by Lemma 2.74, $I_N I_K$ is a multiplication ideal of R. Again, by Lemma 2.74, $I_N I_K M = NK$ is a multiplication submodule of M.

2. By Corollary 4.52, I_N and I_k are faithful multiplication ideals. So, by Lemma 2.74, $I_N I_K$ is a faithful multiplication ideal. So, by Corollary 5.12 and Lemma 2.77,

$$\theta(NK) = \theta(I_N I_K M) = \theta(I_N I_K)\theta(M) = \theta(I_N I_K)\theta(M) = \theta(I_N)\theta(I_K)\theta(M).$$

□

Divisors of an R-module.

Definition 5.29 *Let M be an R-module and N be a submodule of M. We say that N* **divides** *M, $N|M$, if $N = IM$ for some ideal I of R. In this case, N is called a* **divisor** *of M.*

If M is a multiplication module then all submodules of M are divisors. If $K|N$ and $K|N'$ then K is called a **common divisor** of N and N'.

Lemma 5.30 *Let M be an R-module. Then*

1. *$0|N$ and $N|N$.*
2. *M is a multiplication module iff every submodule of M is divisor. In addition, if M is fully-multiplication module and N, K are submodules of M such that $N \subseteq K$ then N is a divisor of K.*
3. *If $N|K$ and $K|N$ where K is a finitely generated faithful multiplication module then $K = IN$ where I is an invertible ideal of R.*
4. *If $N|K$ then $JN|K$ for all ideals J of R.*
5. *If $N|K$ and $N|L$ then $N|(IK + JL)$ where I and J are ideals of R.*
6. *If $L|K$ and $N|K$ where K is a faithful multiplication module then $(L \cap N)|K$.*
7. *$N|K$ iff $N = [N : K]K$.*

Proof. The statements 1, 2, 4 and 5 are trivial.

3. Since $N|K$ and $K|N$, $N = IK$ and $K = JN$ where I and J are ideals of R. So, $K = JN = JIK$, and therefore, by Lemma 2.78, $R = JI$, i.e., I is an invertible ideal of R.

6. Since $L|K$ and $N|K$, $L = IK$ and $N = JK$ where I and J are ideals of R. By Theorem 2.72, (since K is a faithful multiplication module), $L \cap N = IK \cap JK = (I \cap J)K$, i.e., $(L \cap N)|K$.

7. Let $N|K$. Then $N = IK$ for some ideal I. Now, $N = IK \subseteq [N : K]K \subseteq N$, i.e., $N = [N : K]K$. The converse is trivial. □

Definition 5.31 *Let M be an R-module, and let N and K be submodules of M. Then the* greatest common divisor submodule*, $\gcd(N, K)$, is a submodule L satisfies the following:*

1. *$L|K$ and $L|N$, and*

2. if $L'|K$ and $L'|N$ where L' is a submodule of L then $L'|L$.

If N and K are multiplication submodules of an R-module M then $\gcd(N, K) = N \cap K$.

Proposition 5.32 *Let N and K be submodules of an R-module M. Then* $\gcd(N, M)$ *is exists. Moreover, it is unique where* $\gcd(N, M) = \sum_{K|N \text{ and } K|M} K$.

Proof. It follows from Definition 5.31. □

5.3 Embedding of a multiplication R-module into the ring R

We recall that there are a very few examples of a faithful multiplication module that cannot be embedded into their rings. In this section, we deduce some cases of embedding of a multiplication module into its ring.

Embedding of a projective multiplication R-module into the ring R.

Proof of Theorem 1.22. We have the following commutative diagram:

$$\begin{tikzcd} & M \arrow[dl, "g"'] \arrow[d, "a_M"] & \\ R \arrow[r, "f"] & C \arrow[r] & 0 \end{tikzcd}$$

where f is an epimorphism (since C is a cyclic R-module), a_M is the R-homomorphism $a_M : M \to M$, $m \mapsto am$, and g is an R-homomorphism such that $a_M = fg$. Since $a \in \mathcal{C}_R$ and M is a projective R-module, the map a_M is a *monomorphism* . Hence, so is the map g. □

The next corollary shows that the cyclic module C in Theorem 1.22 is unique (up to isomorphism).

Corollary 5.33 *The cyclic module C in Theorem 1.22 is isomorphic to the cyclic R-module $R/\mathrm{ann}_R(M)$.*

Proof. In the proof of Theorem 1.22, we have shown that the map $a_M : M \to M$, $m \mapsto am$, is a monomorphism. Then $aM \subseteq C \subseteq M$, and so, $\mathrm{ann}_R(M) = \mathrm{ann}_R(aM) \supseteq \mathrm{ann}_R(C) \supseteq \mathrm{ann}_R(M)$, i.e., $\mathrm{ann}_R(C) = \mathrm{ann}_R(M)$. □

Corollary 5.34 *Let R be a domain. Then every projective multiplication R-module is isomorphic to an ideal of R.*

Proof. Let M be a projective multiplication module. We may assume that $M \neq 0$ (otherwise, the statement is obvious). Let C be a nonzero cyclic submodule of M. Then $C = IM$ for a nonzero ideal I of R (since M is a multiplication R-module). Choose a nonzero element a of I. Then $a \in \mathcal{C}_R$ and $aM \subseteq IM = C$. By Theorem 1.22, M is isomorphic to an ideal of R. □

The next corollary is a description of projective multiplication module over a commutative Dedekind domain.

Corollary 5.35 *For a commutative Dedekind domain R, all projective multiplication modules are precisely ideals of R.*

Proof. Suppose that R is Dedekind domain. Then, by Lemma 2.44, every ideal of a Dedekind domain R is projective, and, by Lemma 2.66(3), every ideal of R is multiplication. Conversely, every projective multiplication module is isomorphic to an ideal of R, by Corollary 5.34. □

Multiplication modules over a principal ideal domain.

For an R-module M, we recall that $\mathrm{T}(M)$ is the set of elements in M which can be annihilated by a regular element of R. If $\mathrm{T}(M) = 0$ then M is called a torsion-free module.

The next lemma is used in the proof of Proposition 5.37.

Lemma 5.36 *([25, Theorem 3.3]) Let R be a principal ideal domain. Then every finitely generated R-module is isomorphic to a direct sum of a free module F and $T(M)$ where $T(M) = \{m \in M \mid rm = 0 \text{ for some regular element } r \in R\}$.*

Proposition 5.37 *Let R be a principal ideal domain and M be a multiplication R-module. Then $M \cong R$.*

Proof. Since $\theta(M)$ is a principal ideal, By Lemma 4.9, M is a finitely generated . So, by Lemma 5.36, $M \cong F \bigoplus T(M)$ where F is a free module. Therefore F is a multiplication module, by Theorem 1.11. So, by Corollary 4.16, F is isomorphic to R. Since $\mathrm{T}(M)$ is a submodule of M and M is a multiplication module, $\mathrm{T}(M) = IM$ for some ideal I of R. So, $M \cong R \bigoplus IM$. Hence, by Proposition 1.7, $IM = 0$ (Since $\mathrm{ann}_R(R) = 0$). □

Embedding a torsion-free multiplication R-module into R.

Proposition 5.38 *Let M be a torsion-free multiplication R-module. If there exists $0 \neq m \in M$ such that $m = cx$ where c is a regular element of R and $x \in M$ then M can be embedded into R. In particular, if R is a domain then every torsion-free multiplication R-module isomorphic to an ideal of R.*

Proof. Since M is a multiplication module, $Rm = IM$ for some ideal I of R. It follows that $cx \in IM$, i.e., $c \in I$. So, for every $m' \in M$, $cm' = rm$ for some $r \in R$. Let $f : M \to R$, $m' \mapsto r'c$ where $r'm = cm'$. If $rc = r'c$ then $r = r'$ (since c is a regular element), and therefore f is well-defined. Clearly, f is homomorphism, and therefore $M/\ker(f) \cong f(M) \subseteq R$. Let $m^\star \in \ker(f)$. Then $0 = f(m^\star) = r^\star c$ for some $r^\star \in R$ where $cm^\star = r^\star m$. Since c is a regular, $r^\star = 0$, i.e., $cm^\star = 0$. So, $m^\star = 0$ (since M is free-torsion), and therefore $\ker(f) = 0$. Hence, M is isomorphic to an ideal of R. □

5.4 Multiplication modules over some rings

In this section, we study multiplication modules over some specific rings: Artinian rings, Noetherian rings, domains, and von Neuman regular rings.

Corollary 5.39 *Let M be an Artinian multiplication R-module. Then*

1. *If M is faithful then R is an Artinian ring.*

2. *M is a Noetherian R-module.*

Proof. 1. Since M is an Artinian multiplication module, M is cyclic, by Theorem 2.92. So, $M \cong R/\text{ann}(M)$. Since M is faithful, R is an Artinian ring.

2. Since M is an Artinian multiplication module, M is cyclic, by Theorem 2.92. So, $M \cong R/\text{ann}_R(M)$. It follows that $R/\text{ann}_R(M)$ is Artinian, and therefore $R/\text{ann}(M)$ is Noetherian. Hence, by Lemma 2.34, M is a Noetherian module. □

Corollary 5.40 *Let R be a ring. If the ring R has a faithful multiplication Noetherian module M then R is Noetherian.*

Proof. Let $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_k \subseteq \cdots$ be an ascending chain of ideals of R. Then $I_1M \subseteq I_2M \subseteq \cdots \subseteq I_kM \subseteq \cdots$ is an ascending chain of submodules of M. Since M is Noetherian, I_iM is finitely generated, and by Lemma 2.75, $I_i \subseteq \theta(M)$ for all $i \in \mathcal{N}$. Also, since M is Noetherian, there exists $n \in \mathbb{N}$ such that $I_nM = I_{n+1} = \cdots$. By cancellation law, Lemma 2.78, $I_n = I_{n+1} = \cdots$, and hence, R is Noetherian. □

For an R-module M, recall that $\mathcal{Z}(M) = \{r \in R \mid rm = 0 \text{ for some } 0 \neq m \in M\}$.

Proposition 5.41 *Let R be a Noetherian domain and M be a nonzero faithful multiplication R-module. Then $\mathcal{Z}(M) = 0$.*

Proof. Suppose that $a \in \mathcal{Z}(M)$, and $I = Ra$. Since M is multiplication module over a Noetherian ring, M is finitely generated, by Lemma 2.77. So, by Lemma 2.32, $Im = 0$ for some $0 \neq m \in M$. Now, As M is a multiplication module, $Rm = JM$ for some ideal J of R. So, $0 = Im = IJM$ which implies that $IJ = 0$ (since M is faithful). Therefore either $J = 0$ or $I = 0$ (since 0 is a prime ideal). If $J = 0$ then $m = 0$ (a contradiction). Hence, $I = Ra = 0$, i.e., $\mathcal{Z}(M) = 0$. □

Proposition 5.42 *Let R be a local Artinian ring with a maximal ideal $\mathfrak{m}$. If $\mathfrak{m}$ is multiplication ideal then R is a multiplication ring.*

Proof. Since R is an Artinian and $\mathfrak{m}$ is a multiplication ideal, by Lemma 2.92, $\mathfrak{m}$ is a principal ideal. Therefore, by Lemma 2.29, every ideal of R is principal, i.e., every ideal of R is multiplication ideal. Hence, R is a multiplication ring. □

Corollary 5.43 *Let R be a Noetherian ring. Then every faithful multiplication ideal is an invertible ideal.*

Proof. It follows from Proposition 2.33 and Lemma 2.88.□

The next theorem is a criterion for a commutative ring to be a domain.

Theorem 5.44 *Let R be a commutative ring. Then R is a domain iff every multiplication ideal is invertible. Moreover, if R is a multiplication domain iff R is a Dedekind domain.*

Proof. ($\Rightarrow$) It follows from Lemma 2.88.

($\Leftarrow$) Let $0 \neq a \in R$ such that $ab = 0$ for some $b \in R$. Since Ra is a multiplication ideal, By assumption, Ra is an invertible ideal. Therefore Ra contains a nonzero divisor, by Lemma 2.88, say $x = ra$ for some $r \in R$. Now,

$$xb = (ra)b = r(ab) = 0,$$

and therefore, b must be zero. Hence a is a non-zero-divisor, i.e., R is a domain. □

Primary submodules of a multiplication module.

Elbast and Smith in [21] investigated prime submodules of a multiplication R-module. By the same manner, we study *primary submodules* of a multiplication module over commutative rings.

Definition 5.45 *A proper submodule N of a nonzero R-module M is called a* **primary submodule** *provided for all $r \in R$ and $x \in M$ if $rx \in N$ such that $x \notin N$ then there exists $n \in \mathbb{N}$ such that $r \in \sqrt{[N : M]}$.*

Clearly, every prime submodule of M is primary, by Lemma 2.95.

Lemma 5.46 *Let N be a proper submodule of a nonzero multiplication R-module M. Then N is a primary submodule iff $[N : M]$ is a primary ideal of R.*

Proof. Suppose that N is a primary submodule of M. Let $ab \in [N : M]$ such that $a \notin [N : M]$ for some $a, b \in R$. Then there exists $m \in M$ such that $am \notin N$, but $b(am) = (ab)m \in N$. As N is primary, there exist $n \in \mathbb{N}$ such that $b^n \in [N : M]$, and therefore $[N : M]$ is a primary ideal of R.

Conversely, suppose that $[N : M]$ is a primary ideal of R. Let $rx \in N$ for some $r \in R$ and $x \in M \setminus N$. As M is a multiplication module, $[Rx : M] \nsubseteq [N : M]$, but, $[Rx : M]Rr \subseteq [N : M]$. Since $[N : M]$ is a primary submodule and $[Rx : M] \nsubseteq [N : M]$, there exists $n \in \mathbb{N}$ such that $r^n \in [N : M]$, and hence, N is a primary submodule. □

Theorem 5.47 *Let N be a proper submodule of a multiplication R-module M. Then N is a primary submodule iff $N = IM$ for some primary ideal I of R with* $\mathrm{ann}_R(M) \subseteq I$ *and $IM \neq M$.*

Proof. The only part is obvious, by lemma 5.46, Take $I = [N : M]$. Conversely, suppose that $N = IM$ for some primary ideal I of R with $\mathrm{ann}_R(M) \subseteq I$ and $IM \neq M$. Let $ax \in N$ where $a \in R$ and $x \in M \setminus N$. The goal is to prove that $a^n \in [N : M]$ for some $n \in \mathbb{N}$. Since M is a multiplication module, $[Rx : M] \nsubseteq I$. We have $Ra[Rx : M]M \subseteq IM = N$. By Corollary 2.82 and Lemma 2.80(2), $Ra[Rx : M] = Ra[Rx : M]\theta(M) \subseteq I\theta(M) \subseteq I$. Since I is a primary ideal and $[Rx : M] \nsubseteq I$, $a^n \in I \subseteq [IM : M] = [N : M]$, as required. □

We recall that a ring R has a *primary decomposition property* if every proper ideal of R is a finite intersection of primary ideals of R.

Let R be a ring. If R has a primary decomposition property then, by Theorem 5.47 and by Theorem 2.72, every submodule of a faithful multiplication R-module is a finite intersection of primary submodules of M, i.e., M has a primary decomposition property, too.

Corollary 5.48 *Let R be a ring and M be a finitely generated faithful multiplication R-module. If M has a primary decomposition property then R has the same property.*

Proof. Let I be an ideal of R. Since M is a multiplication module and IM is a submodule of M, IM has a primay decomposition, i.e.,

$$IM = \bigcap_{i=1}^{n} N_i$$

where N_i is a primary submodule of M for all $1 \leq i \leq n \in \mathbb{N}$. By Theorem 5.47, $N_i = Q_i M$ ifor some primary ideal Q_i of R. So, by Theorem 2.72,

$$IM = \bigcap_{i=1}^{n} Q_i M = (\bigcap_{i=1}^{n} Q_i)M.$$

Therefore, by Lemma 2.78, $I = \bigcap_{i=1}^{n} Q_i$ (Since $\theta(M) = R$, by Lemma 2.77), i.e., I has a primary decomposition. □

Remark: For an R-module M, let $\mathcal{I}^{\star}(R)$ be the set of all ideals I such that $IM \neq M$. If M be a faithful multiplication R-module then, by Corollary 2.97, there a bijection between prime submodules of M and the prime ideals in $\mathcal{I}^{\star}(R)$. Also, by Theorem 5.47, there is a bijection between the primary submodule of M and the primary ideals of $\mathcal{I}^{\star}(R)$. Notice that, by Theorem 2.86, if M is finitely generated then by $\mathcal{I}^{\star}(R) = \mathcal{I}(R) \setminus \{R\}$. In this case, we can refer to an *associated prime submodule* PM of a primary submodule QM such that P is an associated prime ideal of an primary ideal Q where P and $Q \in \mathcal{I}^{\star}(R)$. In some sense, if $K = QM$ is a primary submodule of M then the associated prime submodule of K is $\sqrt{K} = \sqrt{QM} = \sqrt{Q}M$, by Theorem 2.72. □

The next result discusses multiplication modules over a von Neumann regular ring.

Proposition 5.49 *Let R be a von Neumann regular ring and M be a multiplication R-module. Then*

1. *M is a fully-multiplication module.*
2. *If M is faithful then for every $x \in M$, $M = Rx \bigoplus [0 : x]M$. In particular, if M contains an element with zero annihilator then M is cyclic.*

Proof. 1. Let N be a submodule of M. Then $N = IM$ for some ideal I of R. Since R is a von Neumann regular ring, I is an idempotent ideal, and so, by Lemma 4.44, I is a multiplication ideal. Therefore, by Lemma 2.74, $IM = N$ is a multiplication submodule, i.e., M is a fully multiplication module.

2. Since M is a multiplication R-module, $Rx = IM$ for some ideal I of R. As $I^2 = I$, $Rx = IM = I^2M = I(IM) = Ix$. Therefore, by Lemma 2.18, $I + [0 : x] = R$. Let $a \in I \bigcap [0 : x]$. Then $aM \subseteq Rx$ and there exists $b \in R$ such that $a = ba^2$. So,

$$aM = ba^2M \subseteq ba(rx) = 0.$$

Hence, $a = 0$ (since M is faithful), i.e., $R = I \bigoplus [0 : x]$ which implies that $M = IM \bigoplus [0 : x]M = Rx \bigoplus [0 : x]M$. □

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