Infinite Jordan Permutation Groups

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To my parents

To my husband
To my son

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(لَئِن شَكَرتُم لَأَزِيدَنَّكُم)

"If you give thanks, I will certainly grant you more." Holy Qura'an, Ibrahim (14:7).

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Abstract

If G is a transitive permutation group on a set X, then G is a *Jordan group* if there is a partition of X into non-empty subsets Y and Z with $|Z| > 1$, such that the pointwise stabilizer in G of Y acts transitively on Z (plus other non-degeneracy conditions).

There is a classification theorem by Adeleke and Macpherson for the infinite primitive Jordan permutation groups: such group preserves linear-like structures, or tree-like structures, or Steiner systems or a 'limit' of Steiner systems, or a 'limit' of betweenness relations or D-relations. In this thesis we build a structure M whose automorphism group is an infinite oligomorphic primitive Jordan permutation group preserving a limit of D-relations.

In Chapter 2 we build a class of finite structures, each of which is essentially a finite lower semilinear order with vertices labelled by finite D-sets, with coherence conditions. These are viewed as structures in a relational language with relations L, L', S, S', Q, R. We describe possible one point extensions, and prove an amalgamation theorem. We obtain by Fraïssé's Theorem a Fraïssé limit M.

In Chapter 3, we describe in detail the structure M and its automorphism group. We show that there is an associated dense lower semilinear order, again with vertices labelled by (dense) D-sets, again with coherence conditions.

By a method of building an iterated wreath product described by Cameron which is based on Hall's wreath power, we build in Chapter 4 a group $K < \mathrm{Aut}(M)$ which is a Jordan group with a pre-direction as its Jordan set. Then we find, by properties of Jordan sets, that a pre-D-set is a Jordan set for $\mathrm{Aut}(M)$. Finally we prove that the Jordan group $G = \mathrm{Aut}(M)$ preserves a limit of D-relations as a main result of this thesis.

Contents

Chapter 1

Introduction

This chapter consists of an overview of the background from permutation group theory, model theory, and combinatorics for the thesis. It does not contain original results.

The studies of infinite permutation groups gained momentum from around 1980, as is mentioned in [14], since before that the bulk of permutation groups literature was on finite permutation groups. There is an extensive work of classifying the infinite Jordan groups. In 1985, Neumann classified the primitive Jordan permutation groups with cofinite Jordan sets in [35].

Before that, Cameron classified all the infinite permutation groups which are highly homogeneous but not highly transitive in [9]. His work does not explicitly mention Jordan groups, but his classification helps in classifying Jordan groups since the linear-like structures and tree-like structures provide rich sources of Jordan groups.

The interest of classifying the infinite primitive Jordan permutation groups emerged in 1996. On the one hand, Adeleke and Neumann classified the infinite primitive permutation groups which have proper primitive Jordan sets in their paper [4]. On the other hand, Adeleke and Macpherson classified the infinite primitive Jordan permutation groups but without the assumption that the Jordan sets are primitive in [5]. In their classification a group G acting on a set X is highly transitive or preserves on X one of the linear-like structures (as classified by Cameron), the tree-like structures (as described by Adeleke and Neumann), Steiner systems, or a limit of betweenness relations, D-relations or Steiner systems.

The examples on the first three families that are just mentioned above can be found in [6]. The hardest families to find examples of are the last three families. However, there is an example of an infinite Jordan group preserving a limit of Steiner systems given by Adeleke in [1]. This is developed further by Keith Johnson in [28]. An example of an infinite Jordan group preserving a limit of betweenness relations is given by Bhattacharjee and Macpherson in their paper [7]. This example is an ω-categorical structure with more properties. Another example of an infinite Jordan permutation group preserving a limit of betweenness relations is given by Adeleke in his work [2] (work which was done much earlier than [7], and inspired [7]). On the limits of D-relations, Adeleke gives an example of an infinite primitive Jordan permutation group preserving a limit of D-relations, but it has not been verified whether that example is oligomorphic, that is, arises from an ω-categorical structure.

Adeleke and Macpherson, in the end of their paper [5], referred to an interest of classifying oligomorphic primitive Jordan permutation groups, and asked whether it is possible for an infinite primitive oligomorphic Jordan permutation group to preserve a limit of betweenness relations or D-relations. A positive answer has now been found.

Bhattacharjee and Macpherson give an example of an infinite primitive oligomorphic Jordan permutation group preserving a limit of betweenness relations. In this work, we give a constructed example of an infinite primitive Jordan permutation group preserving a limit of D-relations by Fraïssé's construction, following the procedure used by Bhattacharjee and Macpherson in [7]. This example again has oligomorphic automorphism group. In adddition, here, as in [7] and unlike [2], we describe combinatorial structures on which the groups act.

In this chapter, we introduce the background definitions leading to basic understanding for Jordan groups. At the end of this chapter, we mention Fraïssé 's Theorem which is used to construct our example. The bulk of these definitions are taken from [6], and [8]. For the basics of permutation groups, we refer to [27] and [36].

1.1 Permutation Groups

Infinite permutation groups are connected to many other areas of mathematics, for example model theory and combinatorics. For the former, this connection appears by the theorem of Ryll-Nardzewski 1959, Engeler 1959, Svenonius 1959. Part of the importance of infinite permutation groups comes from the strong relation with model theory and combinatorics.

Throughout this thesis, we mean by (G, X) a permutation group G acting on a set X where X is meant to be a countably infinite set.

1.1.1 Generalities

Definition 1.1.1. A *permutation* of a set X is a bijective map $g : X \to X$. We write x^g for the image of x under g.

The set of all permutations on a set X is a group under the operation of composition of mappings. This group is called the *symmetric group* on X and denoted by Sym(X). If X is finite with $|X| = n$ we write S_n for Sym(X).

Definition 1.1.2. Let G be a group and X be a set. An *action* of G on X is a map $X \times G \to X$ written as $(x, g) \mapsto x^g$ such that

(i) for every $g, h \in G$ and $x \in X$, we have $(x^g)^h = x^{gh}$;

(ii) for every $x \in X$, we have $x^1 = x$, where 1 denotes the identity element of the group G.

We say that X is a *G-space* if a group G has an action on X.

Example 1.1.3. 1. Let $G = S_n$ and $X = \{1, 2, \dots, n\}$. Condition (i) holds by the rule of multiplication in S_n. For condition (ii) the identity permutation maps each element of X to itself.

2. The action of a group G on itself by conjugation. That is $g^h = h^{-1}gh$, $g, h \in G$ satisfies the group action axioms.

These examples are mentioned in many references, for example [27], Chapter 10, and more examples can be found there.

Definition 1.1.4. Let X be a G-space and $Y \subset X$. We define the *setwise stabiliser* of Y in G to be $G_{\{Y\}} := \{g \in G : Y^g = Y\}$, and the *pointwise stabiliser* of Y in G to be $G_{(Y)} := \{g \in G : \forall y \in Y (y^g = y)\}$.

Lemma 1.1.5. *([17], Section 1.5) If X is a G-space and $Y \subseteq X$, then the pointwise stabiliser of Y is a normal subgroup of the setwise stabiliser of Y.*

Definition 1.1.6. Suppose that X is a G-space. The *orbit* of an element x in X is the set

$$x^G := \{x^g : g \in G\}.$$

When $x^G = X$ then we say that G acts *transitively* on X, or that X is a *transitive* G-space.

Hence, if all elements of the set X lie in one orbit then we say that a group G is transitive. Said in another way, the group G is said to be transitive if and only if for any distinct $x, y \in X$, there exists $g \in G$ satisfying $x^g = y$.

Definition 1.1.7. (i) For a field F, the group of all invertible $n \times n$ matrices with entries from F is called the *n-dimensional general linear group*, denoted by $\mathrm{GL}(n, F)$, or sometimes by $\mathrm{GL}_n(F)$.

(ii) Given a vector space V over a field F, the *general linear group over V*, denoted by $\mathrm{GL}(V)$ is the group of all automorphisms of V.

(iii) Let Z(GL(n, F)) be the centre of the group GL(n, F) (the group of scalar multiples of the identity matrix). The quotient GL(n, F)/Z(GL(n, F)) is called the *n-dimensional projective general linear group*, denoted by PGL(n, F).

(iv) The *affine general linear group* on a vector space V is the group of all invertible affine transformations such that

$$\text{AGL}(V) := \{T_{M,b} : M \in \text{GL}(V), b \in V\}$$

where $T_{M,b}(x) = xM + b$.

If V has dimension n then V is isomorphic to F^n, and then GL(V) is isomorphic to GL(n, F).

Example 1.1.8. (i) Any set X as a Sym(X)-space is transitive.

(ii) The action of GL$(2, \mathbb{R})$ on $\mathbb{R}^2$ is not transitive, since the zero element $(0, 0)$ is in a single orbit. However, it is transitive on $\mathbb{R}^2 \smallsetminus \{(0, 0)\}$.

(iii) The group GL(V) is transitive on the set of ordered bases of V.

(iv) The group AGL(V) acts transitively on V via the affine transformations.

The definition of transitivity can be extended as follows.

Definition 1.1.9. For a natural number k, a G-space X is said to be *k-transitive* if for any two sets of k distinct points in X, say $x_1, ..., x_k$ and $y_1, ..., y_k$ there exists $g \in G$ such that $x_i^g = y_i$, for all $i = 1, ..., k$. The maximal such k is called the *degree of transitivity*. If G is k-transitive on an infinite set X for any $k \in \mathbb{N}$, then G is said to be *highly transitive*.

Example 1.1.10. (i) For $n \geq 1$ the usual action of S_n on the set $\{1, ..., n\}$ is k-transitive for all $k \leq n$.

(ii) The group GL$(2, \mathbb{R})$ is not 2-transitive on $\mathbb{R}^2 \smallsetminus \{(0, 0)\}$; since a linearly dependent pair cannot be mapped to a linearly independent pair.

The following theorem is used inductively in the other chapters. Its importance comes as it reduces the degree of transitivity.

Theorem 1.1.11 ([8], 3.13)**.** *Let G act transitively on X, and $x \in X$. Then for $k \geq 1$, X is a $(k+1)$-transitive G-space if and only if G_x acts k-transitively on $X \setminus \{x\}$.*

Definition 1.1.12. A G-space X is said to be *k-homogeneous* if for every $Y, Z \subseteq X$ with $\mid X \mid = \mid Z \mid = k$ there is some $g \in G$ such that $Y^g = Z$.

Here, $Y^g := \{x \in X : x = y^g \text{ for some } y \in Y\}$. We keep the same notation used in [8], Chapter 3.

Definition 1.1.13. An infinite G-space is said to be *highly homogeneous* if it is k-homogeneous for every $k \in \mathbb{N}$.

Example 1.1.14. Aut$(\mathbb{Q}, <)$ is highly homogeneous but not highly transitive (it is not 2-transitive), see [8], Example 3(j).

Theorem 1.1.15. *([8], 3.19) If $m \leq k$ and $2k \leq |X|$, then every k-homogeneous group on X is also m-homogeneous.*

Remark 1.1.16. (i) Homogeneity is weaker than transitivity.

(ii) k-transitivity is about ordered k-sets, while k-homogeneity is about unordered k-sets, and so k-transitive groups are k-homogeneous.

Let G act transitively on a set X, with $|X| > 1$. A *congruence*, or *G-congruence*, on X is an equivalence relation on X which is preserved by G (that is, if $x \equiv y$, then $x^g \equiv y^g$ for all $g \in G$). An equivalence class of a congruence is called a *block*. Note that, if $\mathfrak{B}$ is a block, then so is $\mathfrak{B}^g$ for any $g \in G$. There are always two congruences:

equality : $x \equiv y$ if and only if $x = y$ - the classes are singletons;

the *universal* relation : $x \equiv y$ for all $x, y \in X$ where the equivalence class is the whole set X.

A G-congruence is said to be *non-trivial* if there is a class with more than one element, and it is said to be *proper* if there is more than one class.

Equivalently, a subset $Y \subseteq X$ is called a *block* if for every $g \in G$ either $Y \cap Y^g = \varnothing$ or $Y = Y^g$. A block is said to be *non-trivial* if $|Y| > 1$, and *proper* if $Y \neq X$.

Definition 1.1.17. Let X be a transitive G-space. Then X is said to be a *primitive* G-space if it has no G-congruence (i.e. G-invariant equivalence relation) other than the trivial and the universal ones. Equivalently, the action is said to be primitive if there are no proper non-trivial blocks, or if there is no partition of X preserved by G except for the trivial and improper partitions. Otherwise we say the action is *imprimitive*.

Following [8] in Section 4.1, if ρ is a G-congruence on a set X, and $x, y \in X$, then they are said to be ρ-$equivalent$ if they lie in the same ρ-class. The ρ-class containing an element z of X is referred to by $\rho(z)$. That is $\rho(z) := \{x|\ x \equiv z \pmod \rho\}$.

The following lemma is essential in building examples of primitive permutation groups.

Lemma 1.1.18. *([36], Proposition 3.52) Any 2-transitive permutation group is primitive.*

By Theorem 5.3 in [10], all the finite groups that have a 2-transitive action are classified.

The notion of primitivity can be extended to k-primitivity for some $k \in \mathbb{N}$ as in the following definition:

Definition 1.1.19. Let $k \in \mathbb{N}$. A group G acting on a set X is said to be *k-primitive* if it is k-transitive on X, and for all distinct points $x_1, x_2, ..., x_{k-1} \in X$ their pointwise stabiliser $G_{x_1,x_2,...,x_{k-1}}$ is primitive on the set $X \setminus \{x_1, x_2, ..., x_{k-1}\}$.

Using the fact that any 2-transitive group is primitive, we also get

Corollary 1.1.20. *([8], Lemma 4.10) Let G be a k-transitive group, then G is at least $(k-1)$-primitive.*

Theorem 1.1.21. *([8], Theorem 4.7) For a transitive G-space X with $|X| > 1$, the following are equivalent:*

(i) X *is primitive.*

(ii) X *has no non-trivial proper blocks.*

(iii) *For every* $x \in X$*, the subgroup* G_x *is a maximal subgroup of* G*.*

The following lemma is used in the proof of Proposition 3.2.22(iv).

Lemma 1.1.22. *Suppose* σ *and* ρ *are both congruences for a transitive group* H *acting on a set* X*, and for some* $x \in X$*, we have* $x/\sigma \subset x/\rho$*, that is the* σ*-class containing* x *is a proper subset of the* ρ*-class containing* x*. Then* $\sigma \subset \rho$*, and if* σ *is a maximal congruence, then* ρ *is universal.*

Proof. We want to show $y/\sigma \subset y/\rho$. Let $z \in y/\sigma$. By transitivity we pick some $h \in H$ with $y^h = x$. Then $y^h \sigma z^h$ holds as h preserves σ, so $x \sigma z^h$. But $x/\sigma \subset x/\rho$ so $x \rho z^h$. By applying the inverse, $x^{h^{-1}} \rho \, (z^h)^{h^{-1}}$ so $y \; \rho \; z$. ∎

1.1.2 Wreath Product

The concept of the wreath product is needed in Chapter 4. For detailed information we refer to [27] and [36].

There is a generalization of the idea of the direct product such that given any two groups H, N and a homomorphism $\psi : H \to \mathrm{Aut}(N)$, the constructed new group $N \rtimes_\psi H$ is called the *semidirect product* of N by H with respect to ψ defined as follows:

- The underlying set is the Cartesian product $N \times H$;
- The multiplication operation is defined as follows:

$$(n_1, h_1)(n_2, h_2) = (n_1 \psi(h_1)(n_2), h_1 h_2) \text{ for } n_1, n_2 \in N \;, h_1, h_2 \in H.$$

Example 1.1.23. The symmetric group S_3 acting on the set $\{1, 2, 3\}$ is the semidirect product of $N = \langle (123) \rangle$ by $H = \langle (12) \rangle$.

As a non example, (see [36], Section 2.9), consider a cyclic group of order 4, $H = \{1, -1, i, -i\}$. Then $N = \{1, -1\}$ is a normal subgroup of H, but a subgroup K such that $H = N \rtimes K$ does not exist. Indeed, N, H and the trivial group are the only subgroups of H and they do not satisfy the hypothesis of semidirect product.

Example 1.1.24. *([27],19.3)* The dihedral group D_4 is a semidirect product of the cyclic group $\langle x \rangle$ of order 4 by the cyclic group $\langle y \rangle$ of order 2. This group is not a direct product of these two subgroups, that is because $\langle y \rangle$ is not a normal subgroup of G.

Proposition 1.1.25. *([27], 19.4) Let G be a semidirect product of N by H. For each element h of H, the map $\theta_h : N \to N$ defined by $\theta_h(n) = hnh^{-1}$ is an automorphism of N. The map $\theta : H \to \mathrm{Aut}(N)$ defined by $\theta(h) = \theta_h$ is a homomorphism.*

The reader can see this in details in [27], Proposition 19.5.

Definition 1.1.26. *([27], 19.12)* Let G and H be finite groups with H a subgroup of the symmetric group S_n. The *permutation wreath product*, G wr H, is the semi direct product of a normal subgroup N by H, where N is the direct product of n copies of G. Thus the elements of N are n-tuples $(g_1, \dots, g_n)$ with each $g_i \in G$. The automorphism Θ_h of G^n associated with a permutation h in H is then defined by

$$\Theta_h(g_1, \dots, g_n) = (g_{h(1)}, \dots, g_{h(n)}).$$

Example 1.1.27. The wreath product of $\mathbb{Z}_2$ by S_n is the semidirect product of n copies of $\mathbb{Z}_2$ by S_n. That is:

$$\mathbb{Z}_2 \mathrm{wr} S_n = (\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2) \rtimes S_n = \{(a_1, \dots, a_n, \sigma) : a_i \in \mathbb{Z}_2, \sigma \in S_n\}$$

We multiply two elements as follows:

$$(a_1, \dots, a_n, \sigma)(b_1, \dots, b_n, \pi) = (a_1 b_{\sigma(1)}, a_2 b_{\sigma(2)}, \dots, \sigma\pi)$$

where the action of S_n on $\mathbb{Z}_2^n$ is a place permutation by σ.

$$\sigma(a_1, \dots, a_n) = (a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(n)})$$

If, for example, $n = 3$ then take a vector $(a_1, a_2, a_3) = (1, 0, 1)$ and $\sigma \in S_3$ such that $\sigma = (123)$ then $\sigma(a_1, a_2, a_3) = (a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}) = (a_2, a_3, a_1) = (0, 1, 1)$.

Example 1.1.28. Let Γ be the disjoint union of 4 copies of the complete graph K_3 (this is the undirected graph such that each pair of the 3 vertices is connected by a unique edge). We want to see the wreath product of S_3 by the permutation group S_4 as a group acting on Γ. Then Aut(Γ) is the semi-direct product of four copies of S_3 by S_4 where S_4 acts on S_3^4 by permuting coordinates. For instance, if $\pi \in S_4$, say (1234), and $g = (g_1, g_2, g_3, g_4) \in S_3^4$ then $g^\pi = (g_{\pi(1)}, g_{\pi(2)}, g_{\pi(3)}, g_{\pi(4)}) = (g_2, g_3, g_4, g_1)$.

Suppose that C is any abstract group, and D is a group acting on a set Δ. Put $K := \{f | \ f : \Delta \to C\}$ and put $W := CWrD = K \rtimes D$ with multiplication defined by $(f_1, d_1)(f_2, d_2) = (f_1 f_2^{d_1^{-1}}, d_1 d_2)$. The wreath product W has *base* group K and *top* group D.

To see the wreath product as a permutation group, assume that C and D are permutation groups acting on the sets Γ and Δ respectively. The wreath product of C by D acts on the cartesian product of Γ by Δ. The action of the wreath product on $\Gamma \times \Delta$ is given by

$$(\gamma, \delta)^{(f,d)} = (\gamma^{f(\delta)}, \delta^d)$$

where $f \in K$ and $f(\delta) \in C$.

More details can be found in [8], Chapter 8.

The construction of an iterated wreath product indexed by a totally ordered set has been covered by Hall in his paper [24]. A more general case of that has been studied, which is the general wreath product indexed by a partially ordered set, by Holland in his paper [26].

1.2 Linear relational structures

1.2.1 Linear order

The key notion of this thesis is that of *Jordan groups*, which will be introduced in Section 1.5. In Sections 1.2-1.4 we describe key combinatorial and model-theoretic structures where automorphism groups provide examples of Jordan groups. For more background for this section consult [8], Section 11.3.

Definition 1.2.1. (i) a *partially ordered set* is a set X with a relation $\leq$ satisfying the following conditions:

(1) Reflexivity: $(\forall x \in X)(x \leq x)$.

(2) Anti-symmetry: $(\forall x, y \in X)(x \leq y \wedge y \leq x \Rightarrow x = y)$.

(3) Transitivity: $(\forall x, y, z \in X)(x \leq y \wedge y \leq z \Rightarrow x \leq z)$.

(ii) A *linearly ordered* set is a partially ordered set with the following condition:

$(\forall x, y \in X)(x \leq y \vee y \leq x)$.

If $x \leq y$ or $y \leq x$ then we say that x, y are *comparable* elements of X, while otherwise we say they are *incomparable*, and we will write $\parallel$ for the incomparablity relation.

Linearly ordered sets are also called *totally ordered* sets or *chains*. The rationals $(\mathbb{Q}, \leq)$ is an example of a totally ordered set.

1.2.2 Linear betweenness relation

We can derive a ternary relation from the linear order on the set $\mathbb{Q}$ as the following relation:

$$\mathcal{B}(x; y, z) \Leftrightarrow (y \leq x \leq z) \vee (z \leq x \leq y)$$

Geometrically, that means x lies in the path between y and z. This relation is called *a linear betweenness relation*, and denoted by $\mathcal{B}$.

Definition 1.2.2. *([8], Definition 11.6)* A ternary relation $\mathcal{B}$ defined on a set X is said to be a *linear betweennesss relation* if the following hold:

1. $(\forall x \forall y \forall z) B(x; y, z) \Rightarrow B(x; z, y)$;

2. $(\forall x \forall y \forall z) B(x; y, z) \wedge B(y; x, z) \Leftrightarrow x = y$;

3. $(\forall x \forall y \forall z \forall w) B(x; y, z) \Rightarrow B(x; y, w) \vee B(x; z, w)$;

4. $(\forall x \forall y \forall z) B(x; y, z) \vee (B(y; z, x) \vee B(z; x, y))$.

The automorphism group of this relation Aut$(\mathbb{Q}, \mathcal{B})$ is the group preserving or reversing the linear order. It is 2-transitive, but not 2-primitive as the pointwise stabiliser for 0 has two blocks, the positive rationals and the negative rationals. More details are in [8], Section 11.3.2.

1.2.3 Circular order

Another way of arranging elements is to order them on a circle. However, in this order we cannot talk about a binary relation as in the linear order. The order on a circle is a ternary relation $\mathcal{K}(a, b, c)$ which intuitively says that after a, the element b is reached to before c; going anticlockwise.

Definition 1.2.3. A *circular order* (or *cyclic)* is a ternary relation $\mathcal{K}$ defined on a set X satisfying the following conditions:

1. $(\forall a, b, c \in X) \mathcal{K}(a, b, c) \Rightarrow \mathcal{K}(b, c, a)$;

2. $(\forall a, b, c \in X) \mathcal{K}(a, b, c) \wedge \mathcal{K}(b, a, c) \Leftrightarrow a = b \vee c = b \vee c = a$;

3. $(\forall a, b, c, d \in X)\ (\mathcal{K}(a, b, c) \Rightarrow (\mathcal{K}(a, b, d) \vee \mathcal{K}(d, b, c))$;

4. $(\forall a, b, c \in X)\mathcal{K}(a, b, c) \vee \mathcal{K}(b, a, c)$.

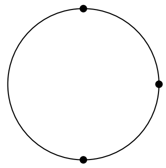

Figure 1.1: $\mathcal{K}(a, b, c)$

Moreover, the circular order is called *dense* if for all distinct $a, b \in X$ there is $c \in X$ such that $\mathcal{K}(a, b, c)$.

There is a strong connection between the linear order and the circular order. For example, if we start with a linearly ordered set, say $\mathbb{Q}$, we can twist the line around the two ends to get a circular order. Hence the circular order on $\mathbb{Q}$ can be defined in terms of the linear order as follows:

$$\mathcal{K}(a, b, c) \Leftrightarrow (a \leq b \leq c) \vee (b \leq c \leq a) \vee (c \leq a \leq b)$$

From this, we see that $\mathcal{K}(a, b, c) \Leftrightarrow \mathcal{K}(b, c, a) \Leftrightarrow \mathcal{K}(c, a, b)$, and for distinct a, b, c, $\mathcal{K}(a, b, c) \Rightarrow \neg\mathcal{K}(b, a, c)$.

On the other hand, the linear order can be recovered from the circular order if we cut a single point of the circular order. That is, if $(X, \mathcal{K})$ is a circular ordering and $a \in X$, then the relation $\leq_a$, defined on $X \setminus \{a\}$ by $x \leq_a y$ if and only if $\mathcal{K}(a, x, y)$, is linear. See Theorem 11.9 of [8].

1.2.4 Separation relation

Another way to arrange elements is to put them on an unoriented circle. A quaternary relation Sep$(a, b; c, d)$ such that a and b separate c from d is called a *separation relation*.

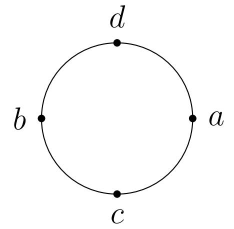

Figure 1.2: Sep$(a, b; c, d)$

Definition 1.2.4 ([8], 11.10)**.** A quaternary relation 'Sep' defined on a set X is a *separation relation* if it satisfies the following for all $a, b, c, d, w \in X$:

(i) Sep$(a, b; c, d) \Rightarrow$ Sep$(b, a; c, d) \wedge$ Sep$(c, d; a, b)$;

(ii) Sep$(a, b; c, d) \wedge$ Sep$(a, c; b, d) \Leftrightarrow b = c \wedge a = d$;

(iii) Sep$(a, b; c, d) \Rightarrow$ Sep$(a, b; c, w) \vee$ Sep$(a, b; d, w)$;

(iv) Sep$(a, b; c, d) \vee$ Sep$(a, c; d, b) \vee$ Sep$(a, d; b, c)$.

On one hand, there is a relationship between the circular order and the separation relation such that if a circular order is given then the quaternary relation defined by

$$(\forall a, b, c, d \in X)\text{Sep}(a, b; c, d) :\Leftrightarrow (\mathcal{K}(a, b, c) \wedge \mathcal{K}(a, d, b)) \vee (\mathcal{K}(a, c, b) \wedge \mathcal{K}(a, b, d))$$

is a separation relation. Conversely, given a separation relation on a set X, there are exactly two corresponding circular orderings and each one is the reverse of the other.

On the other hand, if $\mathcal{B}$ is a linear betweenness relation on a set X, then the quaternary relation defined for all $a, b, c, d \in X$ such as

$$\text{Sep}(a, b; c, d) :\Leftrightarrow (\mathcal{B}(a; b, c) \wedge \mathcal{B}(d; a, b)) \vee (\mathcal{B}(b; c, a) \wedge \mathcal{B}(d; a, b)) \vee (\mathcal{B}(c; a, b) \wedge \neg \mathcal{B}(d; a, b))$$

is a separation relation. On the reverse, given a separation relation on X and fix $a \in X$. Then the $\mathcal{B}$-relation defined such as

$$(\forall b, c, d \in X \smallsetminus \{a\})\mathcal{B}(b; c, d) :\Leftrightarrow \text{Sep}(a, b; c, d)$$

on $X \smallsetminus \{a\}$ is a linear betweenness relation.

The above two paragraphs are what is mentioned in Theorem 11.11 of [8].

We end this section with Cameron's classification theorem mentioned, for example in [8], Theorem 11.12.

Theorem 1.2.5. *Let G be a permutation group acting on an infinite set X, and suppose that G is highly homogeneous but not highly transitive. Then G preserves either a linear order or a circular order or a linear betweenness relation or a separation relation.*

1.3 Steiner Systems

Definition 1.3.1. Let $k \in \mathbb{N}$ with $k \geq 2$. A Steiner k-system consists of a set X of *points* and a set of *blocks* $\mathfrak{B}$ (or *Steiner lines*), where the blocks are subsets of X of the same size (possibly infinite) greater than k satisfying that the number of blocks should be greater than 1, and for k distinct points of X there is a unique block containing them.

If X is finite then the Steiner system is *finite*. Otherwise, the Steiner system is *infinite*.

The number of elements of X is called the *order* of the Steiner system. An intensively studied finite case of Steiner systems is Steiner 2-systems which is known as *Steiner triple systems*. (Note that every Steiner triple system is a Steiner 2-system, but the converse is false.) We also will use such systems to explain the notion of Steiner systems.

The following theorem gives the restrictions on n, l and k for a finite Steiner triple system to exist.

Theorem 1.3.2. *([13], 8.1.2) A Steiner triple system of order n exists if and only if $n \equiv 0, n \equiv 1$ or $n \equiv 3$ (mod 6).*

The condition in the theorem is *necessary* and *sufficient* for this system to exist. The proof of the theorem gives us a recipe for these numbers. Namely, with X the set of points of a Steiner triple system of order n,

(1) Let w be any point in X. Then it lies in $\frac{n-1}{2}$ blocks, which are *triples* in the Steiner triple system.

(2) The set $\mathfrak{B}$ has $\frac{n(n-1)}{6}$ triples.

There are two main methods to construct such Steiner systems. The first method is the *direct construction*. This method is based on having the condition that $n \equiv 3 \pmod 6$ (see Proposition 8.2.1, [13]). Then the set of the points will be $X = \{a_i, b_i, c_i : i \in \mathbb{Z}/(m)\}$ where m is an odd integer such that $n = 3m$, and the blocks take the following forms as stated in section 8.2, [13].

1. The blocks are of the form $a_i a_j b_k$, $b_i b_j b_k$ or $c_i c_j a_k$, where $i, j, k \in \mathbb{Z}/(m)$, $i \neq j$ and $i + j = 2k$ (in $\mathbb{Z}/(m)$.
2. The blocks are of the form $a_i b_i c_i$, $i \in \mathbb{Z}/(m)$).

The following example is taken from [21], and is also mentioned in [13].

Example 1.3.3. Consider a base set X of nine elements, name them $1, 2, 3, 4, 5, 6, 7, 8, 9$. Consider the following subsets $\{1,2,3\}, \{4,5,6\}, \{7,8,9\}, \{1,4,7\}, \{2,5,8\}, \{3,6,9\}, \{1,5,9\}, \{2,6,7\}, \{3,4,8\}, \{1,6,8\}, \{2,4,9\}, \{3,5,7\}$. This is the set of 3-elements subsets of X, call it $\mathfrak{B}$. Take any two elements of X. They will lie in one element of $\mathfrak{B}$. The elements $1, \ldots, 9$ are what we call *points*, and the 3-elements subsets are the *blocks*. This is an example of a Steiner 2-system. Sometimes it is referred to by $S(2, 3, 9)$ where k in the above definition is 2 here, and the 3 refers to the cardinality of each block, and 9 refers to the cardinality of the set X.

The second method is the *recursive construction*. In this method take a set of points X, and start with two blocks and build the rest of the blocks using them. See Section 8.3 of [13] for more details.

From an algebraic point of view we take the projective triple systems as an example. Consider the field $\mathbb{Z}/(2)$, and take a vector space V with dimension d over the field

$\mathbb{Z}/(2)$. Then V is the set of tuples with arity d and $|V| = 2^d$. Take the set of points of Steiner system to be the set of all vectors distinct than the zero-vector, and the set of blocks $\mathfrak{B}$ to be the set of triples satisfying that their sum in the field is zero. Then we get a Steiner triple system of order $2^d - 1$. A familiar example of this is the Fano plane. The blocks are the lines $l_1, \ldots l_7$.

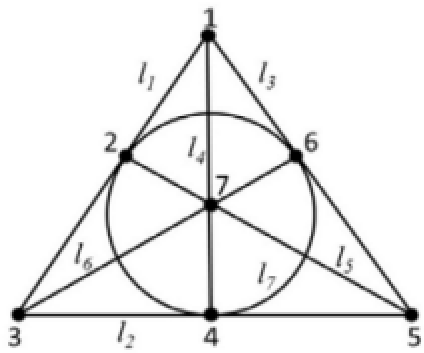

Figure 1.3: Fano Plane

Example 1.3.4. Take the field $\mathbb{Z}/(3)$ and a vector space V of dimension d. Take the set of points to be V itself and the block to be the set of triples which are subsets of X such that their sum is zero in $\mathbb{Z}/(3)$. Then this is a Steiner triple system of order 3^d. It is called an *affine triple system* of dimension d over $\mathbb{Z}/(3)$.

Recall that the set of all 1-dimensional subspaces of a vector space V over a field F is denoted by PG(V), and called the projective space of V.

Example 1.3.5. Let PG(V) be the set of points and the 2-dimensional sub-spaces of V be the set of blocks. This is a Steiner 2-system.

The examples of infinite Steiner systems are more complicated (apart from analogues of the above algebraic examples) and not obvious. For more about infinite Steiner triple systems consult [22], and for a general case, i.e. the blocks are t-elements subset and the base set is countable consult [23].

There is also a notion of 'limit of Steiner systems'. It occurs in the main theorem of Adeleke and Macpherson (see Theorem 1.5.22(j) below), with constructions given by Adeleke in [1] and by Johnson in [28]. I omit details.

1.4 Examples of Tree-like structures

The goal of this section is to describe briefly the notions of treelike relational structures. For more information about these relations, we refer to [6] since we use this as a main reference to this section. Here we rehearse as much as is needed for our work.

1.4.1 Semilinear order

Definition 1.4.1. Let $(X, \leq)$ be a partially ordered set. Then X is said to be a *(lower) semilinearly ordered* set if it satisfies:

(i) For any a in X, the set of points less than a is a totally ordered set, i.e. $(b \leq a \wedge c \leq a) \Rightarrow (b \leq c \vee c \leq b)$.

(ii) $(\forall a, b)(\exists c)(c \leq a \wedge c \leq b)$;

(iii) the set X itself is not totally ordered .

The lower semilinearly ordered sets are sometimes referred to as *trees* as in [18].
The set $(X, \leq)$ is said to be *without endpoints* if it has neither minimal nor maximal element, and it is said to be *dense* if it satisfies that

$$(\forall a, b) a < b \Rightarrow (\exists c)(a < c < b).$$

If we fix an element $p \in X$, then the semilinearly ordered set X will partition into four sets:

(i) $\{x \in X : x > p\}$ and we will call this set Y.

(ii) $\{x \in X : x < p\}$.

(iii) $\{x \in X : x \nleq p \wedge x \ngeq p \wedge x \neq p\}$ (the set of all incomparable elements).

(iv) $\{p\}$.

These four sets are each fixed setwise by $(\mathrm{Aut}(X, \leq))_p$, the stabiliser of p in $\mathrm{Aut}(X, \leq)$.

On the set Y, define an equivalence relation E_p such that

$$xE_py \Leftrightarrow \exists z(p < z \leq x \wedge p < z \leq y).$$

Then E_p is preserved by $(\mathrm{Aut}(X, \leq))_p$.
From now on, when we say a semilinear order we mean a lower semilinear order.

Definition 1.4.2. In the notation above, the *cones* at p are the equivalence classes of the equivalence relation E_p at the node p.

There are two constructed examples of semilinear order in [8], Section 12.1. Also described in [18].

Definition 1.4.3. A 2-homogeneous semilinear order in which all pairs of incomparable elements have no greatest lower bound is said to be of *negative type*. If the greatest lower bound exists for every pair of elements we say it is of a *positive type*.

In our study we work on the positive type.

Let X be a lower semilinearly ordered set. A *lower section* is a subset Y of X such that Y is bounded above, i.e. there is $x \in X$ such that for all $y \in Y$ we have $y \leq x$.

Remark 1.4.4. The (upper) semilinearly ordered sets are defined in similar way by reversing the ordering.

1.4.2 C-relations

Let $(X,<)$ be a semilinearly ordered set. A *maximal chain* in $(X,<)$ is a subset I of X such that:

(i) $(I,<)$ is totally ordered.

(ii) for any J with $I \subsetneq J \subseteq X$, $(J,<)$ is not totally ordered.

Let Y be a set of maximal totally ordered subsets of $(X,<)$ (possibly the set of all maximal totally ordered subsets), and define C on Y by the rule:

$$C(x;y,z) :\Leftrightarrow x \cap y = x \cap z \subset y \cap z$$

The behaviour of maximal chains in a semilinearly ordered set is described by a C-relation, so the name of C-relation comes from " chain".

Definition 1.4.5. Let X be a non-empty set and C a ternary relation on X. Then C is said to be a *C-relation* on X if:

(C1) $(\forall x,y,z)C(x;y,z) \Rightarrow C(x;z,y)$;

(C2) $(\forall x,y,z)C(x;y,z) \Rightarrow \neg C(y;x,z)$;

(C3) $(\forall x,y,z,w)C(x;y,z) \Rightarrow (C(x;w,z) \vee C(w;y,z))$;

(C4) $(\forall x,y)(x \neq y) \Rightarrow C(x;y,y)$;

(C5) $(\forall y,z)(\exists x)C(x;y,z)$;

(C6) $(\forall x,y)(x \neq y) \Rightarrow (\exists z)(y \neq z \wedge C(x;y,z))$.

The relation C is said to be *dense* if

(C7) $(\forall x,y,z)C(x;y,z) \Rightarrow \exists w(C(w;y,z) \wedge C(x;y,w))$.

The ternary relation C defined above (before Definition 1.4.5) on the collection of maximal totally ordered subsets of a semilinear order is a C-relation. This is given by [8], Theorem 12.5.

Example 1.4.6. For any three maximal chains x, y, z in a semilinearly ordered set, two of the sets $x \cap y, x \cap z, y \cap z$ are equal and contained in the third. (See [6], Lemma 11.1).

On the other hand, a semilinear order can be recovered from a C-relation such that the universe of the C-relation is a dense set of maximal chains in the semilinear order, with the natural C-relation. See [6], Theorem 12.4.

1.4.3 B-relations

Definition 1.4.7. A ternary relation B defined on a set X is said to be a *general betweenness relation* if the following hold:

(B1) $(\forall x \forall y \forall z) B(x; y, z) \Rightarrow B(x; z, y)$;

(B2) $(\forall x \forall y \forall z) B(x; y, z) \wedge B(y; x, z) \Leftrightarrow x = y$;

(B3) $(\forall x \forall y \forall z \forall w) B(x; y, z) \Rightarrow B(x; y, w) \vee B(x; z, w)$;

(B4) $(\forall x \forall y \forall z) \neg B(x; y, z) \Rightarrow (\exists w \neq x)(B(w; x, y) \wedge B(w; x, z))$.

The betweenness relation is called *unending in all its directions* (where direction here is the literal English word) if for any two points x, y in a set X one can find an element w such that x is between y and w. That is

$$(\forall x, y)(\exists w \neq x) B(x; y, w)$$

And it is called *dense* if for any x, y there is an element which lies between them, i.e.

$$(\forall x, y)(x \neq y)(\exists z \neq x, y) B(z; x, y).$$

It is said of *positive type* if

$$(\forall x, y, z)(\exists w)(B(w; x, y) \wedge B(w; y, z) \wedge B(w; x, z))$$

The closed interval $[x, y]$ can be defined in terms of the betweenness relation such that

$$[x, y] := \{z \in X : B(z; x, y)\}$$

If $(\forall x, y, z \in X)B(x; y, z) \vee B(y; x, z) \vee B(z; x, y)$, we write $B\{x, y, z\}$, and then x, y, z are called *collinear*. A *linear* set of (X, B) is a subset Y of X with the property that any $x, y, z \in Y$ are collinear.

Definition 1.4.8. The subset Y of a B-set X is said to be a *convex* if $(\forall y, z \in Y)(\forall x \in X)(B(x; y, z) \to x \in Y)$.

Definition 1.4.9. Let (X, B) be a B-set. Let $x \in X$ and define the following equivalence relation on $X \setminus \{x\}$:

$$R_x(y, z) :\Leftrightarrow [x, y] \cap [x, z] \cap (X - \{x\}) \neq \varnothing$$

We call the equivalence classes of the relation R_x *branches*, and we denote the branch containing a by $\bar{a}$.

Those equivalence classes are called *components* of X determined by x in [6].

If, at x, there are three or more R_x-classes then x is called a *ramification point*. If a, b, c are in three distinct R_x-classes at x then x is denoted by $\mathrm{ram}(a, b, c)$.

Definition 1.4.10. Let X be a B-set. A *line* is a maximal linear subset. By a *half-line* is meant a set which is nonempty proper lower section in one of the two linear orderings of some line in X.

Let Y be a subset of the B-set (X, B), x be a point of X. Then Y is said to *lie in one direction* from x if

$$(\forall w_1, w_2 \in Y)(B(w_1; w_2, x) \vee B(w_2; w_1, x))$$

Lemma 1.4.11 ([6], Lemma 16.5)**.** *Let X be a B-set unending in all its directions. A subset Y of X is a half-line if and only if it is convex and unbounded and lies in one direction from some point of X.*

Define a binary relation E on the set of half lines of X by the rule that Y_1 , Y_2 are to be E-related if $Y_1 \cap Y_2$ is unbounded. This is an equivalence relation (see [6], Theorem 16.7). We define a *direction* of X to be an E-class of half-lines. In the finite case directions are called *leaves* or *end points*.

Theorem 1.4.12 ([6],Theorem 16.8)**.** *Let (X, B) be a B-set, $x \in X$ and d be any direction of X. Then x is contained in a unique half-line which lies in one direction from x and is contained in the direction d.*

Theorem 1.4.13 ([6] ,Theorem 19.4)**.** *For a B-set X, let r_1, r_2 be distinct ramification points. There are unique branches Y of r_1 and Y' of r_2 such that $X = Y \cup Y'$.*

A B-relation can be defined on a semilinear order $(P, \leq)$ such that for any $x, y, z \in P$, $B(x; y, z)$ holds if one of the following holds:

(i) $y \geq x \wedge \neg(z \geq x)$.

(ii) $\neg(y \geq x) \wedge z \geq x$.

(iii) $x = \mathrm{glb}\{y, z\}$, where glb means the *greatest lower bound*. This means that if the glb of y and z exists then it lies between them.

On the other side, a semilinear order can be recovered from a B-relation. For example, fix a B-set (X, B). Fix a point $a \in X$, and define a relation $\leq_a$ such that $y \leq_a x$ if and only if $B(y; x, a)$. This relation is a partial order and a lower semilinear order on each branch determined by a.

As a permutation group, Aut(X, B) can be 2-transitive, but not 2-primitive and not 3-transitive. See the end of Section 12.3 of [8], or Section 20 of [6].

1.4.4 D-relations

Following Adeleke and Neumann in [6], the D-relations capture the behaviour of directions in B-sets. A D-relation is a quaternary relation which can be derived either from a B-relation or from a C-relation.

Definition 1.4.14. Let $(T,<)$ be a graph-theoretic tree (connected graph without cycles), and without leaves. A *line* is an infinite two-way path of $(T,<)$. A *half line* is an infinite one-way path of $(T,<)$.

Note that the above definition does not contradict Definition 1.4.10, but we explain it here graph theoretically.

Now let X be the set of all half-lines of T, and define an equivalence relation $\sim$ on X by

$$x \sim y \Leftrightarrow x \cap y \text{ contains infinitely many vertices of } T.$$

Also note that this is analogous to the definition of the binary relation E in the previous section but again here is in a graph theory context.

Define $Y := X/\sim$ to be the set of equivalence classes (or set of directions or end points of T), then there is a related 'D-relation' on Y, as defined below.

Note. The set Y is usually referred to as the set of *ends* of $(T,<)$.

Let $x,y,z,w \in Y$ be distinct directions. $D(x,y;z,w)$ holds if and only if there are half lines $\hat{x} \in x$, $\hat{y} \in y$, $\hat{z} \in z$, $\hat{w} \in w$ such that $\hat{x} \cup \hat{y}$ is a line, $\hat{z} \cup \hat{w}$ is a line, $(\hat{x} \cup \hat{y}) \cap (\hat{z} \cup \hat{w}) = \varnothing$ and there is a path between the meeting points of $\hat{x}$, $\hat{y}$ and $\hat{z}$, $\hat{w}$. See Figure 1.4

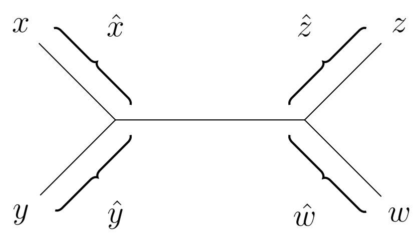

Figure 1.4: $D(x,y;z,w)$

The internal nodes that are the meeting point for at least 3 half-lines (in the infinite case) or at least 3 leaves (in the finite case) are called *ramification points*.

Hence, a finite set with D-relation is a set of leaves connected by edges with no internal vertices of degree 2, and no cycles satisfying the quaternary relation D. This finite graph theoretic tree captures the behaviour of the D-relation, and that is what Cameron described in his paper [11], Sections 3 and 4.

Lemma 1.4.15. *Let T be a graph theoretic tree with n leaves, where $n \geq 3$, and no internal vertices of degree 2. Then there are at most $n-2$ ramification points.*

Proof. By induction on the number leaves. If T has 3 leaves, then the tree T is obtained by allowing the leaves to meet at only one ramification point.

For the induction hypothesis, assume the statement in the lemma holds for n. Let T have $n+1$ leaves with v is a leaf. Let $T' = T \setminus \{v\}$. Then, by the induction hypothesis, T' has $\leq n-2$ ramification points. If v is added on an edge of T' to create a ramification point, then T has $\leq n-1$ ramification points. If v is added to an existing ramification point of T' then T has $\leq n-2$ ramification points. Hence the result follows. ∎

Definition 1.4.16. A quaternary relation $D(x,y;z,w)$ on X is a *D-relation* if for all $x,y,z,w, \in X$:

(D1) $D(x,y;z,w) \Rightarrow D(y,x;z,w) \wedge D(x,y;w,z) \wedge D(z,w;x,y)$;

(D2) $D(x,y;z,w) \Rightarrow \neg D(x,z;y,w)$;

(D3) $D(x,y;z,w) \Rightarrow (\forall a \in X) D(a,y;z,w) \vee D(x,y;z,a)$;

(D4) $(x \neq z \wedge y \neq z) \Rightarrow D(x,y;z,z)$;

(D5) $(x,y,z \text{ distinct}) \Rightarrow (\exists t)(z \neq t \wedge D(x,y;z,t))$. If the D-relation on X satisfies $D1$-$D5$, we say that X is a *proper D-set*.

The D-set is said to be *dense* if

(D6) $D(x,y;z,w) \Rightarrow (\exists a \in X) D(a,y;z,w) \wedge D(x,a;z,w) \wedge D(x,y;a,w) \wedge D(x,y;z,a)$.

The example derived above from a tree, namely (Y, D), is an example of a D-relation which is not dense.

Remark 1.4.17. (i) The notation D comes from the meaning of D-set which is a set of abstract directions.

(ii) Directions suggest ends or points at infinity.

(iii) The elements of any D-relation are directions of an underlying B-relation. See [6], Section 23.

We can see from $D2$ that $\neg D(x, y; y, w)$ holds, hence by $D1$, if one of x, y coincides with one of z, w then $\neg D(x, y; z, w)$.

Lemma 1.4.18. *Let (X, D) be a D-relation. Let $x, y, z, w \in X$. Then*

$$(\forall a \in X)(D(x, y; z, w) \wedge D(x, y; w, a) \rightarrow D(x, y; z, a)).$$

This lemma is ($D9$) in [6], Section 22.

Lemma 1.4.19. *([6], 23.1) If B is a general betweenness relation on X, x, y, z, w are directions in X and $\overleftrightarrow{xz}$, $\overleftrightarrow{xw}$, $\overleftrightarrow{yz}$, $\overleftrightarrow{yw}$ are lines, (the notation $\overleftrightarrow{xz}$ refers to the unique line whose two directions are x, y, similarly for the other lines), then $|\overleftrightarrow{xz} \cap \overleftrightarrow{xw} \cap \overleftrightarrow{yz} \cap \overleftrightarrow{yw}| > 1$ if and only if either there is a branch Y such that $x, y \in Y$ and $z, w \notin Y$ or there is a branch Y such that $x, y \notin Y$ and $z, w \in Y$.*

Given a B-set, there is a natural D-relation on the set of directions, defined as follows:

Theorem 1.4.20 ([6], Theorem 23.2)**.** *Fix a B-set X and a set of directions Y of X. Then there is a D-relation defined on Y as follows: for every $x, y, z, w \in Y$, $D(x, y; z, w) :\Leftrightarrow \{$ there is a branch Y such that $x, y \in Y$ and $z, w \notin Y$ or there is a branch Y such that $x, y \notin Y$ and $z, w \in Y\}$. Then (X, D) is a D-set.*

On the other hand, a B-relation can be recovered from a D-relation. We do not include this here as it depends on a number of notions that we do not explain. To see this consult [6], Section 26.

Note that given a finite D-set, a unique (up to isomorphism) finite rooted tree (in the graph theoretic sense) without vertices of degree 2 can be recovered. Indeed, by Section 9 of [11], the leaves are the points that cannot be in between any two points, in the sense of betweenness relation, and the given D-relation determines the tree with no divalent vertices by Proposition 3.1 in [11]. We bring the reader's attention that the author in [11] uses the notation $ab|cd$ and we use $D(a,b;c,d)$ to refer to a D-relation, and he does not mention the axioms of D-relations that defined in [6], Section 22.

Adeleke and Neumann also commented that a finite combinatorial tree can be recovered from a D-set in [6], Remark 26.5.

Also, a D-set can be obtained from a C-set in two methods; stated in [6], Theorem 23.4, and Theorem 23.5. The latter says for a C-set X

$$(\forall x,y,z,w \in X) D(x,y;z,w) \Leftrightarrow: (C(x;z,w) \wedge C(y;z,w)) \vee (C(z;x,y) \wedge C(w;x,y))$$

Conversely, given a D-relation on a set X. A C-relation can be recovered by fixing a point x_0 of the D-set X, and on $X_0 = X \setminus \{x_0\}$ let C_0 be a ternary relation such that $C_0(x;y,z)$ if and only if $D(x_0,x;y,z)$. Then (X_0,C_0) is a C-set. See Theorem 22.1 of [6]. This process can be reversed: given a C-set on $X \setminus \{x_0\}$, one constructs a D-set on X.

There are concepts of *structural partition*, *components*, *convex halves* and *irreducible* components. We explain here the notion of structural partition. For the other notions we advise the reader to consult [6], Section 28. These concepts are needed to describe the Jordan sets of the automorphism group of a D-set.

Definition 1.4.21. Let (X,D) be a D-set. A partition of X as a disjoint union $\bigcup\{Y | Y \in S\}$ of nonempty subsets is associated with an equivalence relation E. The partition, or the equivalence relation, will be called a *structural partition* with *sectors* Y if

(1) $|S| \geq 3$;

(2) $(\forall Y \in S)(w_1, w_2 \in Y \wedge w_3, w_4 \notin Y \rightarrow D(w_1, w_2; w_3, w_4))$;

(3) w_1, w_2, w_3, w_4 distinct mod E then $\neg D\{w_1, w_2, w_3, w_4\}$, i.e. there is no D-relation on the elements w_1, w_2, w_3, w_4.

Consider (X, D) as a graph-theoretic tree. Define an equivalence relation F_a on the set of leaves of X such that the shortest paths from a to s_1 and from a to s_2 both pass through, at least, another point.

Definition 1.4.22. The *branches* at a in a D-set are the equivalence classes for the equivalence relation F_a .

Note that this is analogous to Definition 1.4.9.

Definition 1.4.23. Define an equivalence relation on the set of branches of a B-set at the ramification point r, by $aRb \Leftrightarrow a, b$ lie in the same branch at r. We denote the equivalence class of a by $\bar{a}$.

This is heavily used in Chapter 2.

1.5 Jordan groups and previous results

The French mathematician, Camille Jordan, introduced the underlying concept of this thesis in the 1870s, see for example, [33], Theorem 1.1. It later became known as a *Jordan group*. These groups have been studied extensively in the last three decades of the 20th century. The finite primitive Jordan groups were classified in 1980s (see [16], [29] and [35]) based on the classification of finite simple groups, while the infinite primitive Jordan groups were classified in 1990s after a much earlier classification by Cameron in 1976 (see [8], Theorem 11.12) for the highly homogeneous but not highly transitive groups.

There is a body of work in classifying all the infinite primitive Jordan groups by Adeleke and Neumann in [6], [3] and [4], and Adeleke and Macpherson in [5] and Adeleke in [1]. Behind the study of Jordan groups lies important applications to model theory and to group theory. To see examples on that consult [33].

Now we introduce the notion of Jordan sets. We refer to [8], and for extensive study about the finite Jordan groups we refer to [35].

Definition 1.5.1. Let $Y \cup Z$ form a partition of a G-space X with $\mid Z \mid > 1$. If there is a subgroup H of G that fixes every point of Y and is transitive on Z, then Z is called a *Jordan set* for G in X and Y is called a *Jordan complement*.

Example 1.5.2. Let Z be any set in X such that $\mid Z \mid \geq 2$ and $G = \mathrm{Sym}(X)$ then Z is a Jordan set with $H = G_{(X \setminus Z)}$.

Example 1.5.3. Let X be the set of rationals and $G = \mathrm{Aut}(\mathbb{Q}, \leq)$ and define $Z := \{x \in \mathbb{Q} : x > 0\}$. Then Z is a Jordan set. More generally, any open convex subset of $\mathbb{Q}$ of size greater than 1 is a Jordan set for G.

In the previous definition, Z is a *primitive Jordan set* if H can be chosen to be primitive on Z. Also, it is said to be an *imprimitive* Jordan set if any such H is imprimitive.

Definition 1.5.4. If the group G is $(k+1)$-transitive and Z is any cofinite subset with $\mid X \setminus Z \mid = k$ then Z is automatically a Jordan set. Such Jordan sets will be said to be *improper*, all others are *proper*.

Definition 1.5.5. If X is a transitive G-space and there is a proper Jordan set for G in X then G is said to be a *Jordan group*.

There are some familiar examples of finite primitive Jordan groups such as the projective and affine linear groups over finite fields and the Mathieu groups M_{22}, M_{23}, M_{24}, Aut(M_{22}) which we do not explain here, but can be seen in [17], Chapter 6.

Further information about the classification of finite Jordan groups can be found in [35]. However, we include some examples.

Example 1.5.6. *([8], 11.2)* Take a vector space V of dimension $d > 2$ over a field F. Consider the group GL(V) of non-singular linear transformations from V to itself. The group GL(V) is not transitive on V. Let W be a proper subspace of V. The complement of the subspace W is a Jordan set for GL(V). We want to see that the complement $V \setminus W$ of a subspace is a Jordan set. Take a basis $e_1, \ldots, e_k$ of W and let v_1, $v_2 \in V \setminus W$. Complete $e_1, \ldots, e_k, v_1$ to a basis $B = \{e_1, \ldots, e_k, v_1, u_1, \ldots, u_r\}$ of V and $B' = \{e_1, \ldots, e_k, v_2, u'_1, \ldots, u'_r\}$ of V. As GL(V) is transitive on ordered bases of V (see exercise 7(i) of [8]), there is an element $g \in$ GL(V) fixing $e_1, \ldots, e_k$, mapping v_1 to v_2 and u_i to u'_i. This g fixes W pointwise. Similarly, the group AGL(V) is a Jordan group with the complement of an affine subspace as a Jordan set. A proof, when the field is $\mathbb{Q}$, can be seen in [30] after Definition 1.10.

In the proof of the following example we follow Macpherson in [32].

Example 1.5.7. Let G = Aut($\mathbb{Q}, \leq$). Then G is primitive since it is 2-homogeneous. Let $x, y \in \mathbb{Q}$ with $x < y$ and $I := (x, y)$. Choose $i, j \in I$. Since the theory of dense linear order without endpoints is ω-categorical (see Definition 1.6.7 below), there are isomorphisms $\phi : (x, i) \cap \mathbb{Q} \to (x, j) \cap \mathbb{Q}$ and $\psi : (i, y) \cap \mathbb{Q} \to (j, y) \cap \mathbb{Q}$. Let σ be the permutation of $\mathbb{Q}$ extending ϕ and ψ, taking i to j, and fixing the rest of $\mathbb{Q}$ pointwise. Then $\sigma \in$ Aut$(\mathbb{Q}, \leq)_{(\mathbb{Q} \setminus I)}$ and $i^\sigma = j$. It is easily checked that the group induced on I by $G_{(\mathbb{Q} \setminus I)}$ is 2-homogeneous, so I is a primitive Jordan set.

Example 1.5.8. Let $|Y| > 1$, $|Z| > 1$. Assuming (H, Y) and (K, Z) are transitive permutation groups. Then the wreath product $HWrK$ acts transitively but imprimitively on the set $Y \times Z$. For any $z \in Z$ the set $\{(y, z) : y \in Y\}$ is a Jordan set. Actually, the wreath product is an example of an imprimitive Jordan group.

Example 1.5.9. Let R be the random graph, that is, the unique countable graph with the property that for any two finite disjoint sets U, W of vertices, there is a vertex adjacent to every vertex of U and to no vertex of W. Then Aut(R) is not a Jordan group. Indeed, suppose for a contradiction that it is a Jordan group. Then we can partition the vertex set of R into non-empty sets V_1, V_2 such that V_2 is a Jordan set. Let $x \in V_1$. Then as V_2 is a

Jordan set, either $R(x) \supseteq V_2$, or $R(x) \cap V_2 = \varnothing$, and without loss of generality, we assume the former. Let $w_1, w_2 \in V_2$ be distinct and put

$$Z := \{y : y \text{ adjacent to } w_1 \text{ and } \ y \text{ nonadjacent to } \ w_2\}.$$

Then if $y \in Z$ then there is no $g \in G_y$ with $w_1^g = w_2$, so as V_2 is a Jordan set we must have $y \in V_2$. Thus $Z \subseteq V_2$. Let u be a vertex adjacent to w_1 but not to x or w_2. Then $u \in Z$, so $u \in V_2$. However, $u \notin R(x)$ and $R(x) \supseteq V_2$, so $u \in V_1$. This is a contradiction.

The following lemma is heavily used, since many arguments apply properties of the family of all Jordan sets.

Lemma 1.5.10. *([5], Lemma 2.2.1) If Y_1, Y_2 are Jordan sets, and $Y_1 \cap Y_2 \neq \varnothing$, then $Y_1 \cup Y_2$ is a Jordan set.*

The following definition is taken from [4].

Definition 1.5.11. .

(a) A *typical pair* is a pair of subsets $Y_1,\ Y_2$ of X such that $Y_1 \nsubseteq Y_2,\ Y_2 \nsubseteq Y_1,\ Y_1 \cap Y_2 \neq \varnothing$.

(b) A family of sets $\{Y_i :\ i \in I\}$ will be said to be *connected* if for any $i, i' \in I$ there exists $j_0, \dots, j_l \in I$ such that $j_0 = i, j_l = i'$ and $Y_{j_{r-1}} \cap Y_{j_r} \neq \varnothing$ for all $1 \leq r \leq l$.

A special case from Lemma 3.2 in [4] was considered in [7], which is the following

Lemma 1.5.12. *Consider a connected system of Jordan sets $\{Z_i :\ i \in I\}$ for a permutation group G on a set X. Then their union over I is a Jordan set.*

In particular, the union of a typical pair of Jordan sets is a Jordan set, as noticed in Lemma 1.5.10.

One of the results about Jordan groups preserving Steiner systems is the following lemma that we need in Chapter 4, Lemma 4.1.25.

Lemma 1.5.13. *[[5], Theorem 2.2.5] Let G be a Jordan group acting on a set X, and let $2 \leq n \in \mathbb{N}$. Then for any distinct $n+1$ elements of X, if there is a Jordan set in X which contains the $n+1$-th element, and excludes the first n elements, then for any $2 \leq k \leq n$ there is no G-invariant Steiner k-system on X.*

Definition 1.5.14. Given two equivalence relations E_1 and E_2 on a set X, we say that E_1 *refines* E_2 if each E_2-class can be written as a union of E_1-classes.

Example 1.5.15 ([6], Theorem 6.9)**.** Let G be the automorphism group of a semilinear order; that is $G = \text{Aut}(P, \leq)$. The possible Jordan sets are of the following forms:

- the cones of the semilinear order at a point. (See Definition 1.4.2).
- any union of cones at a ramification point (i.e. branching point).
- any union of sequence of cones $(c_i : i \in I)$ where I is a totally ordered set and $i < j \Leftrightarrow c_i \subset c_j$.

Example 1.5.16 ([6], Theorem 14.9)**.** Assume that (X, C) is a C-set. Consider $\text{Aut}(X, C)$ such that it is 2-transitive. Then there are several forms of the Jordan sets of the group $\text{Aut}(X, C)$. We mention, for example, the following:

Let $x \in X$, and define a binary relation S_x on $X \setminus \{x\}$ by putting $yS_xz \Leftrightarrow C(x; y, z)$. Then S_x is an equivalence relation, and each S_x-class is a Jordan set. A union of S_x-classes can also be a Jordan set. For more details see [6], page 53.

Example 1.5.17 ([6], Theorem 20.3)**.** Let (X, B) be a B-set, and assume that $\text{Aut}(X, B)$ is 2-transitive. Then every branch (as defined in 1.4.9) is a Jordan set. Unions of branches at a ramification point as well as unions of chains of branches are Jordan sets of $\text{Aut}(X, B)$.

Example 1.5.18. *([6], Theorem 28.6.)* D-sets (X, D) exist such that the group $\text{Aut}(X, D)$ is a Jordan group with a Jordan set $Y \subset X$, for example, Y is a union of two or more branches of a structural partition λ.

The achievement of this work is to test that the built Jordan group in Chapter 4 preserves a *limit of D-relations* as is defined in the following sense:

Definition 1.5.19. If (G, X) is an infinite Jordan group we say that G preserves a *limit of betweenness relations* if there are: a linearly ordered set $(J, \leq)$ with no upper bound in J, a strictly increasing chain $(Y_i : i \in J)$ of subsets of X and an increasing chain $(H_i : i \in J)$ of subgroups of G such that the following hold:

(i) for each $i, H_i = G_{(X \setminus Y_i)}$, and H_i is transitive on Y_i and has a unique non-trivial maximal congruence σ_i on Y_i;

(ii) for each i, $(H_i, Y_i/\sigma_i)$ is a 2-transitive but not 3-transitive Jordan group preserving a betweenness relation;

(iii) $\bigcup(Y_i : i \in J) = X$;

(iv) $(\bigcup(H_i : i \in J), X)$ is a 2-primitive but not 3-transitive Jordan group;

(v) $\sigma_i \supseteq \sigma_j|_{Y_i}$ if $i < j$;

(vi) $\bigcap(\sigma_i : i \in J)$ is equality in Y;

(vii) $(\forall g \in G)(\exists i_0 \in J)(\forall i > i_0)(\exists j \in J)(Y_i^g = Y_j \wedge g^{-1}H_i g = H_j)$;

(viii) for any $x \in X, G_x$ preserves a C-relation on $X \setminus \{x\}$.

Definition 1.5.20. A *limit of D-relations* is defined in the same way, but replacing a betweenness relation by a D-relation in the condition (ii).

In [4], Adeleke and Neumann classified the primitive permutation groups that have primitive Jordan sets, and Adeleke and Macpherson in [5] classified the infinite primitive Jordan groups without the assumption that the Jordan sets are primitive. We include these classifications below.

Theorem 1.5.21 (Adeleke and Neumann Classification, [4])**.** *Suppose that G is a primitive permutation group that has primitive proper Jordan sets. If G is not highly transitive then there is a G-invariant relation R on X which is one of*

(a) a linear order (R then is binary);

(b) a linear betweenness relation (R is ternary);

(c) a cyclic order (R is ternary);

(d) a cyclic separation relation (R is quaternary);

(e) a semilinear order (R is binary);

(f) a general betweenness relation (R is ternary);

(g) a C-relation (R is ternary);

(h) a D-relation (R is quaternary).

Theorem 1.5.22. *[Classification of Infinite Jordan Groups, [5]] Let (G, X) be an infinite primitive Jordan group. Then either G is highly transitive on X or G preserves on X one of the following structures:*

(a) a dense linear order;

(b) a dense circular order;

(c) a dense linear betweenness relation;

(d) a dense separation relation;

(e) a dense semilinear order;

(f) a dense general betweenness relation (induced from semilinear order);

(g) a C-relation;

(h) a D-relation;

(i) a Steiner system;

(j) a limit of dense general betweenness relations, D-relations or Steiner systems.

We do not define here the notion of limit of Steiner systems, see, for example, [1], Definition 2.2.5.

Examples of Jordan groups preserving the relations in the parts (a) to (h) can be found in [4], [3], [6], and [5]. On the family in part (i), Adeleke in his paper [1] gives an example of an infinite Jordan group which is 3-transitive, not 3-primitive and preserves a limit of Steiner 2-systems, but does not preserve the relations in parts $(a), (b), (d), (g), (h)$ in the above theorem, or non-trivial Steiner system (i.e. the blocks have greater than k elements, and there is more than one block). Johnson in [28] gives a k-transitive not $(k+1)$-transitive example, for any $k > 1$. On the last class there are two non-isomorphic examples of infinite Jordan groups. The first is an $\aleph_0$-categorical (Definition 1.6.7) structure whose automorphism group preserves a limit of betweenness relations, given by Bhattacharjee and Macpherson in their paper [7], and the second is a group with infinitely many orbits on triples preserving a limit of betweenness relations in [2]. Moreover, Adeleke gives an example of a group preserving a limit of D-relations, but does not verify whether or not it is oligomorphic (Definition 1.6.10).

In this thesis we construct a 2-primitive oligomorphic Jordan permutation group preserving a limit of D-relations but not preserving a structure of types (a) – (i) .

1.6 Some model theory

For this section we use the references [25] (Chapter 1), [32] and [34] (Chapter 1).

Definition 1.6.1. By a *language* we mean a collection of relation symbols R_i with i ranging through some set I, with arities n_i, a collection of function symbols f_j indexed by a set J such that each f_j has m_j variables, and a collection of constant symbols c_k indexed by $k \in K$.

As examples of languages we mention the language of rings, $\mathscr{L}_{\text{Rings}} = \{+, -, ., 0, 1\}$ where $+$ and $.$ are binary function symbols, where $-$ is a unary function symbol, and 0 and 1 are constants, and there are no relation symbols. The language of graphs consists of one binary relation symbol, $\mathscr{L}_{\text{graphs}} = \{R\}$ which is the adjacency between two vertices, i.e. if there is an edge between the vertex u and the vertex v then they are related by the relation R. Such a language, i.e. a language consisting of only relation symbols is called a *relational* language.

Definition 1.6.2. Let $\mathscr{L}$ be a language. By an *$\mathscr{L}$-structure* $\mathcal{M}$, we mean a nonempty set M, called the *universe* or *underlying set* of $\mathcal{M}$, and for each $i \in I$ a set $R_i(M) \subseteq M^{n(i)}$, where $n(i)$ is the arity of the relation R_i, for each $j \in J$ a function $f_j(M) : M^{m(j)} \to M$, where $m(j)$ is the arity of f_j, and for each $k \in K$ an element $c_k(M) \in M$.

In the previous definition, $R_i(M)$ is called the *interpretation* of R_i in M, $f_j(M)$ the *interpretation* of f_j in M, and $c_k(M)$ the *interpretation* of c_k in M.

For example, consider the language of groups $\mathscr{L}_{\text{groups}} = \{., 1, ^{-1}\}$, then a group G can be considered as an $\mathscr{L}$-structure $\mathcal{M} = (G, ., 1, ^{-1})$ where $c(M)$ is 1, $f_1(M)$ is $^{-1}$ and the . is $f_2(M)$.

Note. In the next chapters, we do not distinguish the notation of the structure from its domain, and we will use M for both of them.

An *$\mathscr{L}$-embedding* of a structure $\mathcal{M}$ into a structure $\mathcal{N}$ is a function $\phi : M \to N$ which is injective and preserves the interpretation of all the symbols of the language. An *isomorphism* is a surjective embedding. Isomorphisms $f : M \to M$ are called *automorphisms of* M .

Definition 1.6.3. Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathscr{L}$-structures. We say that $\mathcal{M}$ is a *substructure* of $\mathcal{N}$ if $M \subseteq N$, for each relation $R \in \mathscr{L}$, $R(M) = R(N) \cap M^n$, where n is the arity of R, for each function symbol f of $\mathscr{L}$, $f(M)$ is the restriction of $f(N)$ to M^m, where m is the arity of f, and for each constant symbol $c \in \mathscr{L}$, $c(M) = c(N)$. We write $\mathcal{M} < \mathcal{N}$ to mean $\mathcal{M}$ is a substructure of $\mathcal{N}$ and $\mathcal{M} \neq \mathcal{N}$. We also say $\mathcal{N}$ is an *extension* of $\mathcal{M}$.

For example $(\mathbb{Z}, +, 0)$ is a substructure of $(\mathbb{R}, +, 0)$. Also, consider $\mathscr{L}_{\text{graphs}} = \{R\}$, $\mathcal{M} = (M, R(M))$ and $\mathcal{N} = (N, R(N))$ are $\mathscr{L}$-structures, then $\mathcal{M} \leq \mathcal{N}$ if it is an "induced subgraph".

Definition 1.6.4. (i) Any variable or constant symbol in a language $\mathscr{L}$ is called an *$\mathscr{L}$-term*. If $t_1, \ldots, t_n$ are terms of $\mathscr{L}$ and f is an n-ary function symbol of $\mathscr{L}$, then $f(t_1, \ldots, t_n)$ is a term of $\mathscr{L}$.

(ii) If $t_1, \ldots, t_n$ be $\mathscr{L}$-terms and R is an n-ary relation symbol of $\mathscr{L}$ then $R(t_1, \ldots, t_n)$ is called an *atomic $\mathscr{L}$-formula*. Likewise, $t_1 = t_2$ is an atomic formula. If ϕ, ψ are $\mathscr{L}$-formulas and x is a variable, then the conjunction $(\phi \wedge \psi)$, the disjunction $(\phi \vee \psi)$, the negation $(\neg\phi)$, the implication $(\phi \to \psi)$, the existential $(\exists x \phi)$, and the universal $(\forall x \phi)$ are *$\mathscr{L}$-formulas*.

The terms and the formulas of a language $\mathscr{L}$ are defined inductively.

Definition 1.6.5. (i) A variable x of a language is called a *free variable* if it does not appear in the scope of $\forall$ or $\exists$. Otherwise x is called a *bound* variable.

(ii) A formula with no free variables is called a *sentence*.

In an $\mathscr{L}$-structure $\mathcal{M}$, a sentence is either true or false, according to an inductive definition of truth which we omit. The set of all $\mathscr{L}$-sentences ϕ such that ϕ is true in $\mathcal{M}$, written as $\mathcal{M} \models \phi$, is called the *theory of* $\mathcal{M}$, denoted by $T(M)$. If T is a set of sentences, we say $\mathcal{M}$ is a *model* of T, and write $\mathcal{M} \models T$, if $\mathcal{M} \models \phi$ for all $\phi \in T$.

Example 1.6.6. (i) $y = x + 1$ is an atomic formula in which $y, x, 1$ and $x + 1$ are terms.

(ii) $\forall x(x > 0 \to \exists y(x = y^2))$ is a sentence, as all instances of x, y are quantified.

Definition 1.6.7. A structure $\mathcal{M}$ is $\aleph_0$-*categorical* or (ω-*categorical*) if

(i) M is countably infinite; and

(ii) if $\mid M \mid = \mid N \mid = \aleph_0$ and $T(\mathcal{M}) = T(\mathcal{N})$, i.e. they satisfy the same set of sentences, then they are isomorphic.

Example 1.6.8. The structure $(\mathbb{Q}, <)$ is $\aleph_0$-categorical, essentially by Cantor's theorem that any two countable dense linear order without endpoints are isomorphic.

Example 1.6.9. Let $\mathcal{M}$ be a structure with $|M| = \aleph_0$ and E be an equivalence relation with finitely many equivalence classes and no finite classes. Then (M, E) is ω-categorical.

Following Cameron in [15], if G acts on X, an element of G acts component-wise on the set X^n of all n-tuples of points of X.

Definition 1.6.10. A group G acting on a set X is said to be *oligomorphic* in its action on X if G has finitely many orbits on X^k, the set of all k-tuples of X, for every natural number k.

For more about oligomorphic groups and permutation groups, the reader can see [12] and [17].

Lemma 1.6.11 ([8], 9.7)**.** *Let $X^{\{k\}}$ denote the set of all k-element subsets of X. Then G is oligomorphic in its action on X if and only if it has finitely many orbits on $X^{\{k\}}$ for every natural number k.*

Example 1.6.12. Aut$(\mathbb{Q}, <)$ is oligomorphic on $\mathbb{Q}$.

Examples 1.6.8 and 1.6.12 are analogous. This is what the following theorem will tell us about. It provides the connection between model theory and permutation groups.

Theorem 1.6.13. *[Ryll- Nardzewski 1959, Engeler 1959, Svenonius 1959] Let $\mathcal{M}$ be a countably infinite structure. Then the following are equivalent*

1. *$\mathcal{M}$ is $\aleph_0$-categorical;*

2. *Aut($\mathcal{M}$) is oligomorphic on $\mathcal{M}$.*

In addition to the natural examples of ω-categorical structures such as the pure set X with automorphism group Sym(X), and $(\mathbb{Q}, \leq)$, there are two methods to construct oligomorphic groups; by building a new structure from an old one, and by the Fraïssé construction. Consult [19] for details.

Definition 1.6.14. A relational structure M is *homogeneous* if

1. M is countable, and

2. whenever U and V are finite substructures of M and $f : U \to V$ is an isomorphism, there is $\hat{f} \in$ Aut(M) extending f. Equivalently, $\hat{f}|_U = f$.

Example 1.6.15. Consider a finite undirected graph R of five vertices, say the Pentagon, such that the vertices are numbered $\{1, 2, 3, 4, 5\}$ and the edges $\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}$ and $\{5, 1\}$. Take two induced subgraphs R_1 of vertices $\{1, 4, 5\}$ and the edges $\{1, 5\}, \{4, 5\}$, and R_2 with the vertices $\{2, 3, 4\}$ and the edges $\{2, 3\}, \{3, 4\}$. Consider the isomorphism $\phi : R_1 \to R_2$ such that $\phi(1) = 4,\ \phi(4) = 2$, and $\phi(5) = 3$. This can be extended if we add the rest of the domain $\{2, 3\}$ and define an isomorphism from R to itself, say ψ such that it keeps the images of $\{1, 4, 5\}$ as in ϕ but we add $\{2, 3\}$ to get the isomorphism $\psi : R \to R$ such that $\psi(1) = 4, \psi(4) = 2, \psi(5) = 3, \psi(2) = 5, \psi(3) = 1$. It is clear that $\psi|_{R_1} = \phi$.

Remark 1.6.16. It can be noticed that there are two definitions through the thesis of the notion of homogeneous; one in the sense of permutation groups and the second is in the sense of model theory. The context will help to decide which one of them is meant.

Example 1.6.17. The structure $(\mathbb{Q}, <)$ is homogeneous in the sense of last definition of homogeneity and Aut$(\mathbb{Q}, <)$ is highly homogeneous in the sense of permutation groups.

Example 1.6.18. *([14], Theorem 2.3)* Let R be the random graph. Then R is homogeneous.

Due to Fraïssé, see [20], there is a flexible technique to build homogeneous structures as follows:

Definition 1.6.19. The *age* Age(M) of a relational structure M is the class of all finite relational structures embeddable (as induced substructures) in M.

Let $\mathscr{C}$ be a non empty class of finite $\mathscr{L}$-structures, where $\mathscr{L}$ is a relational language, satisfying the following properties:
(1) $\mathscr{C}$ is closed under isomorphism, i.e. if $A \in \mathscr{C}$ and B is isomorphic to A then $B \in \mathscr{C}$.
(2) $\mathscr{C}$ is closed under substructures, i.e. if $B \in \mathscr{C}$ and $A < B$ then $A \in \mathscr{C}$. (Hereditary property)
(3) whenever $A, B \in \mathscr{C}$ there is $D \in \mathscr{C}$ such that $A \leq D$ and $B \leq D$. (Joint Embedding Property).
(4) whenever $A, B_1, B_2 \in \mathscr{C}$ and $f_i : A \to B_i, i = 1, 2$ are embeddings, there exist $D \in \mathscr{C}$ and embeddings $g_i : B_i \to D, i = 1, 2$ such that for all $a \in A$ we have $g_1 \circ f_1(a) = g_2 \circ f_2(a)$. (Amalgamation Property).
Then we say that $\mathscr{C}$ is a (Fraïssé) *amalgamation class*.

Theorem 1.6.20 (Fraïssé 's Theorem)**.** *(i) Let $\mathscr{C}$ be a class of finite structures satisfying the above four conditions. Then*

(a) there is a homogeneous $\mathscr{L}$-structure (called the Fraïssé limit) whose finite substructures are (up to isomorphism) exactly the members of $\mathscr{C}$;

(b) any two homogeneuous $\mathscr{L}$-structures as in (a) are isomorphic.
(ii) Conversely, if M is a homogeneous $\mathscr{L}$-structure, then the class of finite $\mathscr{L}$-structures which are isomorphic to substructures of M satisfies (1)-(4).

Here is an application of Fraïssé's theorem.

Lemma 1.6.21 ([8], Lemma 14.6)**.** *For every $k \in \mathbb{N}$, there is a k-transitive but not $(k+1)$-transitive permutation group on a countably infinite set.*

Note that this is not the case for finite permutation groups; any 6-transitive permutation group on a finite set is S_n or A_n in its natural action.

Example 1.6.22. If M is the Fraïssé limit of the class of all finite 3-hypergraphs, then Aut(M) is 2-transitive, but not 3-transitive.

1.7 Summary of The Results

The first interesting primary result that we have got from this work is that there is a nice amalgamation class of trees of D-sets, denoted by $\mathscr{D}$. We obtained that by amalgamating one-point extensions in $\mathscr{D}$ over a substructure $A \in \mathscr{D}$, and by a version of Fraïssé's Theorem we built a structure M, which is the work of Chapter 2. Second, we analyse the Fraïssé limit M of the class $\mathscr{D}$ as the result of Chapter 3. In Chapter 4 we then show that $G = \text{Aut}(M)$ is a Jordan group for which the 'pre-D-sets' are Jordan sets. As a main result of the thesis, we prove that G preserves a limit of D-relations.

Chapter 2

Trees of D-sets

In this chapter, we construct a relational structure M, and prove that it is ω-categorical by showing that $\mathrm{Aut}(M)$ is oligomorphic. In subsequent chapters we will prove that $\mathrm{Aut}(M)$, in its action on M, preserves a limit of D-relations. We build M in a language $\mathscr{L}$ with six relation symbols.

The structure M is built by a variant of Fraïssé amalgamation, described for example in [19], based on a class $\mathcal{C}$ of finite $\mathscr{L}$-structures (which does not have the hereditary property) and a class $\mathcal{E}$ of embeddings between members of $\mathcal{C}$, with the amalgamation property.

A key step (see Section 2.2) is to describe "one-point extensions" of members of $\mathcal{C}$, and then (Section 2.3) to prove that they can be amalgamated.

Very roughly, each member of $\mathcal{C}$ can be viewed as a finite lower semilinear order $(T, \leq)$, with each vertex $\nu \in T$ labelled by a finite D-set $D(\nu)$ equipped with a D-relation D_ν. There are two important families of maps (f_ν) and $(g_{\mu\nu})$ which describe how the D-sets are interconnected. The universe of a member of $\mathscr{D}$ can be viewed as the D-set labelling the root vertex ρ of $(T, \leq)$, equipped with extra structure determined by all the (f_ν) and $(g_{\mu\nu})$ with $\mu > \rho$.

2.1 Construction of *M*

Notations. Let $(T, \leq)$ be a finite lower semilinear order with a root ρ. Label each vertex ν of T by a finite D-set $D(\nu)$ and a defined D-relation D_ν on $D(\nu)$. We view $D(\nu)$ as the set of leaves of a finite unrooted tree $\overline{D(\nu)}$ (in the graph-theoretic sense) without dyadic vertices (vertices of degree 2), and with D_ν defined in the natural way (this fact can be seen in [6], Remark 26.5). So $D_\nu(a, b; c, d)$ holds if the path from a to b and the path from c to d are disjoint. Vertices of degree at least 3 of $\overline{D(\nu)}$ are called *ramification points*. We shall refer to elements of $\overline{D(\nu)}$ as *nodes*. The ramification point with the three nodes x, y, z will be denoted by $\text{ram}(x, y, z)$. By a *successor* of a vertex $\nu \in T$ we mean a vertex that comes immediately above ν, and we shall denote the set of successors of ν by $\text{succ}(\nu)$. For each element $\nu \in T$ which is not maximal, define a bijection f_ν from the set of immediate successors of ν to the set of ramification points $\overline{D(\nu)}$ which we will denote by $\text{Ram}(D(\nu))$. For each ramification point r of $\overline{D(\nu)}$ there is an equivalence relation E_r on $D(\nu)$ such that two leaves w_1, w_2 of $D(\nu)$ are E_r-equivalent if the unique paths from r to w_1 and from r to w_2 have at least two common nodes. The E_r-classes will be called *branches* at r (this is an analogous to Definition 1.4.9, and also we will denote the branch at r containing a by $\bar{a}$). For each $r \in \overline{D(\nu)}$, one of these branches will be distinguished, and called the *special branch at* r. For $\omega \in T$ with $\omega \in \text{succ}(\nu)$, define the bijection $g_{\omega\nu}$ (Definition 2.1.1) such that it maps the leaves in the D-set $D(\omega)$ to the collection of the non-special branches at the corresponding ramification point $f_\nu(\omega)$ of the immediately lower D-set $D(\nu)$. The induced tree with the above structure is called *a tree of D-sets.*

We shall use the Roman letters $x, y, z, w, u, v, \dots$ for elements of directions and branches of a D-set, while the letter $r, r', r'', \dots, r_1, r_2, \dots$ for the ramification points. The Greek letters $\alpha, \nu, \mu, \dots$ refers to the vertices of the tree while we retain the letter ρ for the root.

It is worth noting that the notations in the following paragraph are the notations that are used in [7]. Our construction is based on a lower semilinear order which is called a *tree* in Droste's book [18].

We start with a lower semilinear order T with root ρ. The labelling unrooted tree $\overline{D(\rho)}$ of the root ρ contains ramification points of a number equal to the number of successors in T of the root ρ. Each ramification point has a special branch identified by the relation L that we define later. For example, consider the D-set $D(\rho)$. Focus on a ramification point, r say. So ρ has a successor $f_\rho^{-1}(r)$ in the tree and it is labelled by a D-set consisting of a number of leaves equal to the number of branches around r minus the special one. In pictures we always refer to the special branch by an arrow entering to the ramification point, where the branches are as defined above.

The tree $\overline{D(f_\rho^{-1}(r))}$ has ramification points whose number depends on the number of the successors for the vertex $f_\rho^{-1}(r)$. For instance, if $f_\rho^{-1}(r)$ has two successors then the leaves in $D(f_\rho^{-1}(r))$ create two ramification points, and this is what the bijection f_ν in the following paragraph does. The labelling D-set of a leaf of T has no ramification points, so the endpoints appear in a path.

We define a bijection $f_\nu : \text{succ}(\nu) \to \text{Ram}(D(\nu))$ between the set of successors of the vertex ν in T and the nodes of ramification of the D-set $D(\nu)$. Each ramification point in $D(\nu)$ picks out a special branch, and for $r \in \text{Ram}(D(\nu))$ if $\omega = f_\nu^{-1}(r)$ then there is a bijection $g_{\omega\nu}$ from $D(\omega)$ to the set of non-special branches at r (in the D-set $D(\nu)$).

Note that we will use τ to refer to a tree of D-sets, where T is used to refer to a semilinear order; that is τ refers to the whole structure consisting of $(T, \leq)$, the labelling D-sets, and the maps f_ν and $g_{\omega\nu}$.

Definition 2.1.1. (i) Let $\nu_0, \dots, \nu_m$ be vertices of the semilinear order$(T, \leq)$ of a tree of D-sets τ such that $\nu_0 < \cdots < \nu_m$. Then $(\nu_0, \dots, \nu_m)$ is a *chain of successors* if each vertex belongs to the set of successors of its predecessor, i.e. $\nu_{i+1} \in \text{succ}(\nu_i)$ where $i \in \{0, \dots, m-1\}$.

(ii) Given the chain $(\nu_0, \dots, \nu_m)$, there is a map $g_{\nu_m\nu_0}$ which we define by induction such that it maps the directions of the D-set $D(\nu_m)$ to a union of branches at a fixed ramification point of $D(\nu_0)$. Let $a \in D(\nu_m)$, define

$$g_{\nu_m\nu_0}(a) := \{x \in D(\nu_0) : \exists y \in g_{\nu_m\nu_{m-1}}(a)(x \in g_{\nu_{m-1}\nu_0}(y))\}.$$

The following diagram is an example of a tree of D-sets. Note that in $D(\rho)$, as indicated by the arrows, x lies in the special branch at r' and u lies in the special branch at r, and in $D(\nu)$, $\bar{x}$ lies in the special branch at r_1, and also in the special branch at r'_1.

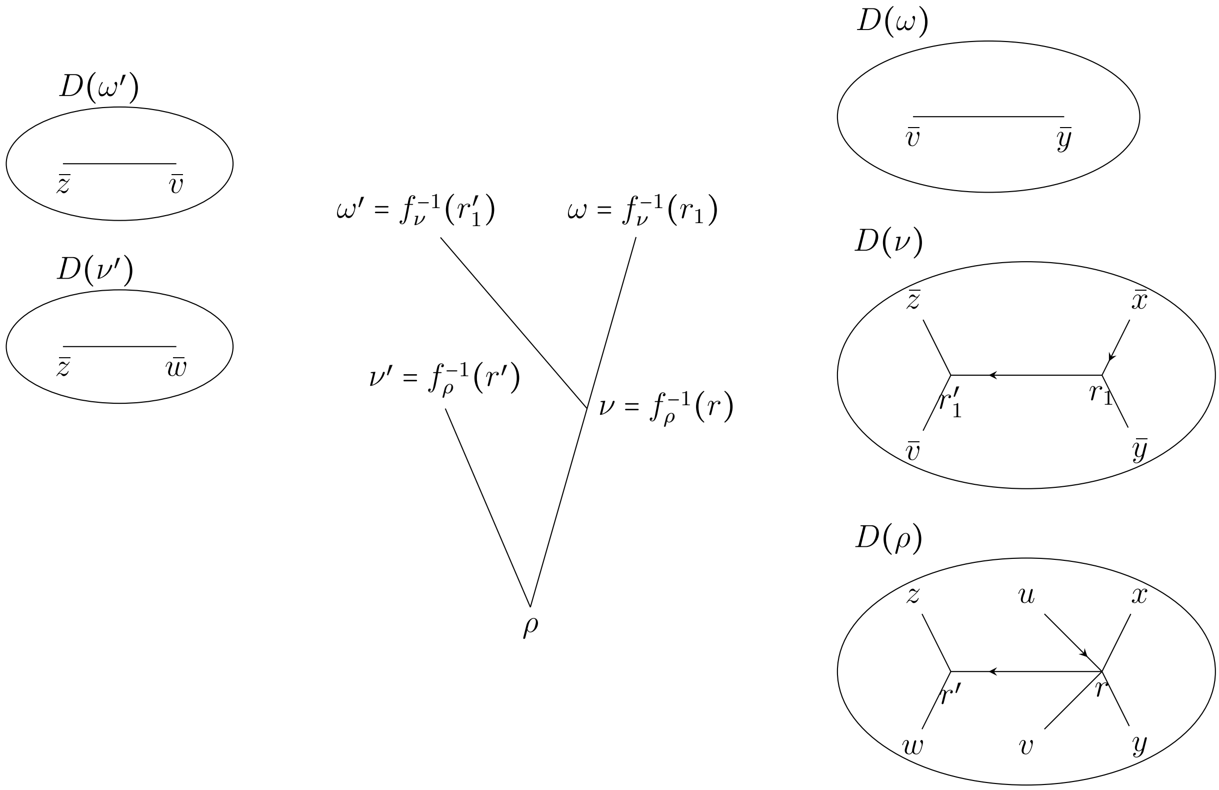

Figure 2.1: Tree of D-sets

Definition 2.1.2. Let τ, τ' be two trees of D-sets. An *isomorphism between trees of D-sets* is an isomorphism between the corresponding two lower semilinear orders $\phi : (T, \leq) \to (T', \leq)$ and, for any vertex $\nu \in T$, an isomorphism ψ_ν from $D(\nu)$ to $D(\phi(\nu))$, which sends the directions and the ramification points in $D(\nu)$ to the directions and the ramification points in $D(\phi(\nu))$ respectively. The maps ψ_ν are required to map special branches to special branches, and to commute with the maps f_ν and $g_{\omega\nu}$.

Suppose τ and τ' are trees of D-sets with semilinear orders $(T, \leq)$, $(T', \leq)$ with roots ρ and ρ', and bijections f_ν, $g_{\nu\mu}$ and $f'_{\nu'}$, $g'_{\nu'\mu'}$ for τ and τ' respectively. Suppose that $\phi : \tau \to \tau'$ is an isomorphism of trees of D-sets. For example, in τ the root is ρ, and say that an immediate successor is ν_1, then the labelling D-set $D(\nu_1)$ has leaves with correspondence to the branches around the ramification point $f_\rho(\nu_1)$ in the root D-set such that if we call the leaves in $D(\nu_1)$ to be $a_1, \ldots, a_{m-1}$ then at $f_\rho(\nu_1)$ we have $g_{\nu_1\rho}(a_1), \ldots, g_{\nu_1\rho}(a_{m-1})$ as branches.

On the other hand, τ' has the root $\phi(\rho)$ and its vertices are the images of the vertices of τ. As above, take an immediate successor of the root ρ' which is $\phi(\nu_1)$, then the labelling D-set $D(\phi(\nu_1))$ has leaves with correspondence with the branches around the ramification point $f'_{\phi(\rho)}(\phi(\nu_1))$ in the root D-set $D(\phi(\rho))$ such that if we call the leaves in $D(\phi(\nu_1))$ as $a'_1, \ldots a'_{m-1}$, then at $f'_{\phi(\rho)}(\phi(\nu_1))$ we have $g'_{\phi(\nu_1)\phi(\rho)}(a'_1), \ldots, g'_{\phi(\nu_1)\phi(\rho)}(a'_{m-1})$, where $a'_i = \psi_\nu(a_i),\ i \in \{1, \ldots, m-1\}$.

Since ψ is a map defined between D-sets, then if we take $\psi_\rho : D(\rho) \to D(\phi(\rho))$ such that it takes nodes to nodes, then under these notations, we can see that $\psi_\rho(f_\rho(\nu_1)) = f'_{\phi(\rho)}(\phi(\nu_1))$ and $\psi_\rho(g_{\nu_1\rho}(a_1)) = g'_{\phi(\nu_1)\phi(\rho)}(a'_1)$.

We say that a D-set $D(\nu)$ *omits* the element $u \in D(\mu)$ if there is no direction $\bar{x}$ of $D(\nu)$ such that $u \in g_{\nu\mu}(\bar{x})$ where ν is a successor of μ. This means that u is in the special branch in $D(\mu)$ at the ramification point corresponding to the vertex ν.

Remember that we mean by $\bar{x}$ an equivalence class of the relation 'being in the same branch' that we defined in Definition 1.4.22, i.e. $\bar{x}$ is a direction higher up where x is a representative of all the elements lying in the corresponding branch at a lower D-set.

We now present a first order language $\mathscr{L}$ so that any tree of D-sets can be viewed as an $\mathscr{L}$-structure. Let τ be a tree of D-sets. The domain of the structure is the set of directions of the root Dset of τ. Our language $\mathscr{L}$ consists of a ternary relation L, two quaternary relations L' and S, a five-ary relation S', a six-ary relation R and a seven-ary relation Q. We define them on the domain of the root D-set of τ such that:

(i) $L(x;y,z)$ holds in τ if either

(a) x,y,z lie in distinct branches at some node r of the root D-set $D(\rho)$, and the branch containing x is special at r, see Figure 2.2, or

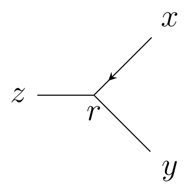

Figure 2.2: $L(x;y,z)$

(b) there is a D-set $D(\nu)$ with a ramification point r, and directions $\bar{x},\bar{y},\bar{z}$ lying in distinct branches at r with $\bar{x}$ lying in the special branch at r, such that $x \in g_{\nu\rho}(\bar{x}), y \in g_{\nu\rho}(\bar{y}), z \in g_{\nu\rho}(\bar{z})$.

We say in (ia) that $D(\rho)$ *witnesses* $L(x;y,z)$, and in (ib) that $D(\nu)$ *witnesses* $L(x;y,z)$. We use the semi-colon to distinguish the special branch in the first place, while there is symmetry on the other two.

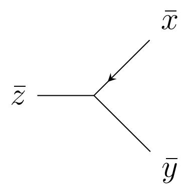

Figure 2.3: $L(x;y,z)$ in $D(\nu)$

(ii) Let $x,y,z,w \in D(\rho)$ be distinct. Then $S(x,y;z,w)$ holds, written $\tau \vDash S(x,y;z,w)$, if one of the following holds

(a) In the root D-set, with universe denoted $D(\rho)$, and a D-relation denoted D_ρ we have $D_\rho(x,y;z,w)$.

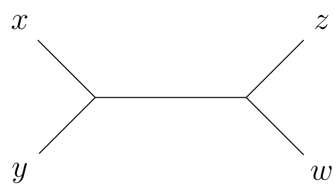

Figure 2.4: $S(x, u; z, w)$

(b) x, y, z, w lie in distinct non-special branches at node r of $D(\rho)$, and there is some vertex $\nu \geq f_\rho^{-1}(r)$ such that $D(\nu)$ contains distinct $\bar{x}, \bar{y}, \bar{w}, \bar{z}$ such that $D_\nu(\bar{x}, \bar{y}; \bar{w}, \bar{z})$ holds in $D(\nu)$, and $x \in g_{\nu\rho}(\bar{x}), y \in g_{\nu\rho}(\bar{y}), w \in g_{\nu\rho}(\bar{w}), z \in g_{\nu\rho}(\bar{z})$.

We say in (iia) that $D(\rho)$ *witnesses* $S(x, y; z, w)$, and in (iib) that $D(\nu)$ *witnesses* $S(x, y; z, w)$.

Note. The relation S captures the behaviour of D-relations except that in S we do not allow equality among its parameters, i.e. axiom $(D4)$ of Definition 1.4.16 does not hold for S. We use the semi-colon to reflect the symmetry between the first two parameters and the last two.

We now describe how an instance of S can be witnessed by a D-set other than the root.

Suppose that x, y, z, w, v lie in four distinct branches at a ramification point of $D(\rho)$, where each branch contains other elements. For example, the branch containing x, we will say in such situation the *branch of* x, contains in addition to x itself, a finite number of leaves say $x_1, \ldots, x_n$, the branch of y has, in addition to y, the leaves $y_1, \ldots, y_m$, the branch of w has more l elements $w_1, \ldots, w_l$, the branch of z (assumed to be special) has p elements $z_1, \ldots, z_p$, and the branch of v has, of course v and $v_1, \ldots, v_k$, where all of m, n, l, k, p are natural numbers which can be equal or not. See, as an example, Figure 2.5 below in which there is one more element in each of the branches of x, y, z, w, v. By considering a suitable equivalence relation 'lying in the same branch' (see Definition 1.4.23), each of the branches of $x,\ y,\ w$

and v will be shown higher as one endpoint by using a representative of that branch, and we write $\bar{x}, \bar{y}, \bar{w}$ and $\bar{v}$ respectively.

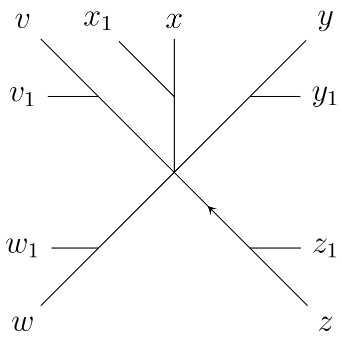

Figure 2.5

Then consider a chain of successors $(\nu_0, \ldots, \nu_m)$ with $\nu_0 = \rho$ and distinct $\bar{x}, \bar{y}, \bar{v}, \bar{w} \in D(\nu_m)$ and $x \in g_{\nu_m \nu_0}(\bar{x}),\ y \in g_{\nu_m \nu_0}(\bar{y}),\ v \in g_{\nu_m \nu_0}(\bar{v}),\ w \in g_{\nu_m \nu_0}(\bar{w})$. Then $S(\bar{x}, \bar{y}; \bar{v}, \bar{w})$ can be witnessed in $D(\nu_m)$ if $\bar{x}, \bar{y}$ are distinct at their meeting point and $\bar{v}, \bar{w}$ are distinct at their meeting point, as in Figure 2.6.

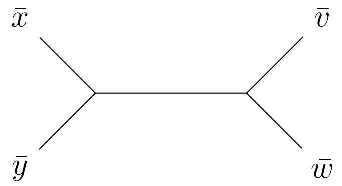

Figure 2.6

(iii) $Q(x, y; z, w : p; q, s)$ holds in τ if there is some D-set in which the relations $S(x, y; z, w)$ and $L(p; q, s)$ are both witnessed. We interpret this as S and L *happen in the same D-set*. This is an example picture.

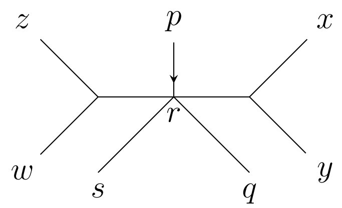

Figure 2.7: $Q(x, y; z, w : p; q, s)$

(iv) $R(x; y, z : p; q, s)$ holds in τ if there is some D-set in which the relations $L(x; y, z)$ and $L(p; q, s)$ are both witnessed. We interpret this as *two L-relations happen in the same D-set*. This is an example picture.

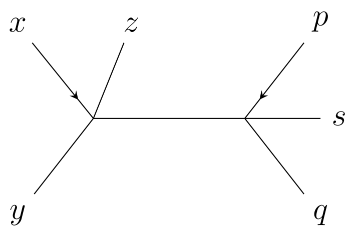

Figure 2.8: $R(x; y, z : p; q, s)$

(v) $L'(x; y, z; u)$ holds in τ if in the D-set $D(\nu)$ witnessing $L(x; y, z)$, the branch containing u is omitted. We then say that the D-set $D(\nu)$ *witnesses* the relation $L'(x; y, z; u)$. We use the first semi-colon as in L above, and the second one to distinguish the omitted element. For example, in Figure 2.1, $L'(x; y, z; u)$ is witnessed in $D(\nu)$.

(vi) $S'(x, y; z, w; u)$ holds if in the D-set $D(\nu)$ witnessing $S(x, y; z, w)$, the branch containing u is omitted. We then say that the D-set $D(\nu)$ witnesses the relation $S'(x, y; z, w; u)$. Also, the semi-colon here is as in L' but replacing L by S. Again, in Figure 2.1, $S'(x, y; z, v; u)$ is witnessed in $D(\nu)$.

Note that, by the definition, the relations L', S' cannot be witnessed in the root D-set, because there are no omitted branches.

Remark 2.1.3. For any distinct x, y, z, u, v, w

(i) $L(x; y, z) \Leftrightarrow L(x; z, y)$.

(ii) $L(x; y, z) \Rightarrow \neg L(y; x, z) \wedge \neg L(z; x, y)$.

(iii) $S(x, y; z, w) \Leftrightarrow S(z, w; x, y)$.

Note: When we say that one of the above relations hold in the structure A we mean it holds in some D-set of A. We may thus view a finite tree of D-sets as an $\mathscr{L}$-structure whose universe is the set of directions of the root D-set. We use symbols like $A, B, C, E, \ldots$ for such finite $\mathscr{L}$-structures. Also we write $A < B$ if A is a substructure of B. We use symbols such as τ for a tree of D-sets when we view it as presented at the start of this section, rather than as an $\mathscr{L}$-structure.

Let $\mathscr{D}$ be the collection of all finite $\mathscr{L}$-structures arising from trees of D-sets as described before.

Lemma 2.1.4. *Let $A \in \mathscr{D}$ and $x, y, z \in A$ be distinct, then $L(x; y, z) \vee L(y; x, z)$ $\vee L(z; x, y)$ holds in A.*

Proof. Choose a vertex ν of T such that $|D(\nu)|$ is minimal subject to there being distinct $\bar{x}, \bar{y}, \bar{z} \in D(\nu)$ with $x \in g_{\nu\rho}(\bar{x})$, $y \in g_{\nu\rho}(\bar{y})$, and $z \in g_{\nu\rho}(\bar{z})$ (we allow the special case when $\nu = \rho$, and $x = \bar{x}, y = \bar{y}, z = \bar{z}$). We may suppose that $\bar{x}, \bar{y}, \bar{z}$ lie in distinct branches at some ramification point r of $D(\nu)$. If none of these branches is special at r, then ν has a successor $\nu' = f_{\nu}^{-1}(r)$ and there are distinct $x^*, y^*, z^* \in D(\nu')$ with $\bar{x} \in g_{\nu'\nu}(x^*)$, $\bar{y} \in g_{\nu'\nu}(y^*)$ and $\bar{z} \in g_{\nu'\nu}(z^*)$ then $x \in g_{\nu'\rho}(x^*)$, $y \in g_{\nu'\rho}(y^*)$, and $z \in g_{\nu'\rho}(z^*)$, and $|D(\nu')| < |D(\nu)|$, contradicting the minimality of $|D(\nu)|$. Thus, one of $\bar{x}, \bar{y}, \bar{z}$, say $\bar{x}$, lies in a special branch at r of $D(\nu)$, and then $L(x; y, z)$ holds, as required. ∎

From now on, we will denote the disjunctions in Lemma 2.1.4 as $L\{x, y, z\}$.

The following lemma says that for distinct x, y, z, w the relations $L(x; y, z)$ and $S(x, y; z, w)$ are witnessed in at most one D-set.

Lemma 2.1.5. *If $A \in \mathscr{D}$ and $x, y, z \in A$, then if $A \vDash L(x; y, z)$ then it is witnessed in one D-set of A. Also, for $p, q, s, t \in A$ if $A \vDash S(p, q; s, t)$ then it is witnessed in one D-set.*

Proof. Suppose $x, y, z \in A$ are distinct, and that $L(x; y, z)$ is witnessed in the D-set $D(\nu)$ with $\nu > \rho$; that is, there are distinct $\bar{x}, \bar{y}, \bar{z} \in D(\nu)$ meeting at ramification point r, with $\bar{x}$ special at r and $x \in g_{\nu\rho}(\bar{x}), y \in g_{\nu\rho}(\bar{y}), z \in g_{\nu\rho}(\bar{z})$. (We ignore the degenerate case where

$\nu = \rho$, which is easier but similar). Now if $\omega > \nu$ then $L(x;y,z)$ is not witnessed in $D(\omega)$, since there is no $x^* \in D(\omega)$ with $\bar{x} \in g_{\omega\nu}(x^*)$ - that is, x is omitted in $D(\omega)$.

If $\omega < \nu$ then there is a ramification point r' of $D(\omega)$ such that $g_{\nu\omega}(\bar{x}), g_{\nu\omega}(\bar{y}), g_{\nu\omega}(\bar{z})$ lie in distinct non-special branches at r', so $D(\omega)$ does not witness $L(x;y,z)$. Finally, suppose ω is incomparable to ν, and ν, ω have greatest lower bound μ in the structure tree. Then there are distinct ramification points r', r'' of $D(\mu)$ such that ν lies above a successor $\nu' = f_\mu^{-1}(r')$ of μ, and ω lies above a successor $\omega' = f_\mu^{-1}(r'')$. Wherever r'' lives, at least two of $g_{\nu\mu}(\bar{x}), g_{\nu\mu}(\bar{y}), g_{\nu\mu}(\bar{z})$ lie in the same branch at r'', so $D(\omega)$ cannot witness $L(x;y,z)$. Similarly for the relation S. ∎

Lemma 2.1.6. *Let $A \in \mathscr{D}$ have a root ρ. Then the relation D_ρ on $D(\rho)$ satisfies the following:*

for all $x,y,z,w \in D(\rho)$, $D_\rho(x,y;z,w) \Leftrightarrow ((x = y) \vee (z = w) \wedge \{x,y\} \cap \{z,w\} = \varnothing) \vee (x,y,$ z,w are all distinct and $S(x,y;z,w) \wedge (\forall t)(\neg S'(x,y;z,w;t)))$.

Proof. Suppose $D_\rho(x,y,z,w)$ with x,y,z,w distinct. Then $S(x,y;z,w)$ holds as it is assumed that there is a D-relation witnessed in the root D-set. As $D(\rho)$ contains all elements of A and is the only D-set witnessing $S(x,y;z,w)$, we have $(\forall t)\neg S'(x,y;z,w;t)$.

Conversely, suppose $S(x,y;z,w) \wedge (\forall t)(\neg S'(x,y;z,w;t))$ holds. For a contradiction, we will assume that the D-set witnessing $S(x,y;z,w)$ is not the root. Then there is a lower D-set in which x,y,z,w lie in distinct branches at a ramification r. As $S(x,y,z,w)$ is witnessed higher up, none of $x,y;z,w$ lies in a special branch at r. Since each ramification point has a special branch, some t lies in the special branch at r. Then $S'(x,y;z,w;t)$ holds, which is a contradiction. ∎

Focus on a ramification point in the root D-set of the substructure A of $\mathscr{D}$, say r. Then the labelling D-set of the successor $f_\rho^{-1}(r)$, which is $D(f_\rho^{-1}(r))$, has endpoints (leaves) of number equal to the number of distinct branches at r in the base minus the special one, as it is dropped. This successor $f_\rho^{-1}(r)$ is considered as a root for a new structure called A_r.

Note that we use the subscript r to refer to the ramification point r in the root D-set of A, and the universe of A_r is the set of the non-special equivalence classes for the equivalence relation in Definition 1.4.23 and $|A_r| < |A|$ (see Lemma 2.1.10 (iv)). Note that A_r is a quotient of a subset of A by the relation F_r (see Definition 1.4.22). We can continue doing this by the same method until we reach a leaf and we stop there because the labelling D-set has no ramification points and the elements just form a path.

Lemma 2.1.7. *Suppose $A \in \mathscr{D}$ and ν, ν' are incomparable vertices of the structure tree on A. Then it can not happen that $x, y, z \in A$ have $\bar{x}, \bar{y}, \bar{z}$ distinct in both $D(\nu)$ and $D(\nu')$.*

Proof. Let ν, ν' be two incomparable vertices in the structure tree of a structure $A \in \mathscr{D}$. For convenience, suppose that ν, ν' are distinct successors of ρ. In the root D-set $D(\rho)$ let $r = f_\rho(\nu)$ and $r' = f_\rho(\nu')$ be two ramification points. Suppose that $\bar{x}, \bar{y}, \bar{z}$ are distinct in A_r, so x, y, z lie in distinct non-special branches at r. As $r \neq r'$, it follows that at least two of x, y, z lie in the same branch at r', so at least two of $\bar{x}, \bar{y}, \bar{z}$ are equal in $A_{r'}$.

■

Definition 2.1.8. *Suppose $\sigma = f_\rho^{-1}(r)$. Let $a \in g_{\sigma\rho}(\bar{a})$, $b \in g_{\sigma\rho}(\bar{b})$, $c \in g_{\sigma\rho}(\bar{c})$, $d \in g_{\sigma\rho}(\bar{d})$, $y \in g_{\sigma\rho}(\bar{y})$, $w \in g_{\sigma\rho}(\bar{w})$, $z \in g_{\sigma\rho}(\bar{z})$, then the relations L, L', S, S', R, Q are interpreted in A_r as follows*

(i) If $\bar{a}, \bar{b}, \bar{c}$ are distinct, then $A_r \vDash L(\bar{a}; \bar{b}, \bar{c}) \Leftrightarrow A \vDash L(a; b, c)$.

(ii) If $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are distinct, then $A_r \vDash L'(\bar{b}; \bar{c}, \bar{d}; \bar{a}) \Leftrightarrow A \vDash L'(b; c, d; a)$.

(iii) If $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are distinct, then $A_r \vDash S(\bar{a}, \bar{b}; \bar{c}, \bar{d}) \Leftrightarrow A \vDash S(a, b; c, d)$.

(iv) If $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are distinct, then $A_r \vDash S'(\bar{a}, \bar{b}; \bar{c}, \bar{d}; \bar{w}) \Leftrightarrow A \vDash S'(a, b; c, d; w)$.

(v) If $\bar{a}, \bar{b}, \bar{c}$ are distinct and $\bar{y}, \bar{z}, \bar{w}$ are distinct, then $A_r \vDash R(\bar{a}; \bar{b}, \bar{c} : \bar{y}; \bar{z}, \bar{w}) \Leftrightarrow A \vDash R(a; b, c : y; z, w)$.

(vi) If $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ *are distinct and* $\bar{y}, \bar{z}, \bar{w}$ *are distinct, then* $A_r \models Q(\bar{a}, \bar{b}; \bar{c}, \bar{d} : \bar{y}; \bar{z}, \bar{w}) \Leftrightarrow A \models Q(a, b; c, d : y; z, w)$.

Lemma 2.1.9. *The relations in Definition 2.1.8 are well defined.*

Proof.

(i) It is suffices to show that if $\bar{a}, \bar{b}, \bar{c} \in A_r$ are distinct, and $a, a', b, b', c, c' \in A$ with $aF_r a', bF_r b', cF_r c'$ (where F_r is the equivalence relation in Definition 1.4.22), then $A \models L(a; b, c) \Leftrightarrow A \models L(a'; b', c')$, but this is immediate.

We do the same for (ii), (iii), (iv), (v) and (vi).

■

Lemma 2.1.10. *If* $A \in \mathscr{D}$ *and* r *is a ramification point in the root* D*-set, then*

(i) A_r *is isomorphic to a substructure of* A.

(ii) $A_r \in \mathscr{D}$.

(iii) Let τ *be the tree of D-sets corresponding to the* $\mathscr{L}$*-structure A, and* τ_r *be the tree of D-sets corresponding to the* $\mathscr{L}$*-structure* A_r*. Define* $h(\tau)$*, the height of the tree* τ*, to be the number of the vertices in the longest path from a leaf to the root* ρ_A*. Then* $h(\tau_r) < h(\tau)$.

(iv) $|A_r| < |A|$.

Proof.

(i) For each element $\bar{x} \in A_r$, pick $x \in \bar{x}$. Then by Definition 2.1.8 and Lemma 2.1.9, the map $\bar{x} \longmapsto x$ gives an isomorphism from A_r to a substructure of A. (Note that '$x \in \bar{x}$ ' is a bit of an abuse of notation).

(ii) First, the structure tree of A_r is τ_r. The universe of A_r is exactly $D(\rho_r)$ ($\rho_r = f_\rho^{-1}(r)$). For each $\nu' > \rho_r$, the D-set of A_r corresponding to ν' is exactly $D(\nu')$. The correspondences between ramification points of D-sets and successors (given by maps such as $(f_r)_\nu$) in A_r are exactly those induced from A, and likewise the maps $(g_r)_{\mu\nu}$ are those induced from A.

(iii) Let ρ_r be the root of the tree of D-sets τ_r, suppose $h(\tau_r) = k$, and let $\rho_r = \sigma_0 < \sigma_1 < \cdots < \sigma_k$ be a path with k edges from ρ_r to a leaf of τ_r. Let $\rho_A < \rho_r < \sigma_1 < \cdots < \sigma_k$ be a path in τ with $k+1$ edges from ρ_A to σ_k. Then $k > 0$, so $h(\tau) \geq k + 1 > k$, so $h(\tau) > h(\tau_r)$.

(iv) By induction on $h(\tau_r)$ it suffices to observe that if ν is a successor of ρ with $f_\rho(\nu) = r$ then $|A_r| < |A|$. This is immediate. Let $\rho_r = f_\rho^{-1}(r)$. Let m be the number of branches of $D(\rho)$ at r. Then as each such branch contains at least one element of A, $m \leq |A|$, and as A_r 'drops' the special branch at r, $|A_r| = m - 1$.

■

Proposition 2.1.11. *Suppose that τ_1, τ_2 are trees of D-sets with corresponding $\mathscr{L}$-structures A_1, A_2 with the isomorphism $\chi : A_1 \to A_2$. Then χ induces a unique isomorphism $\phi : \tau_1 \to \tau_2$.*

Proof. As the base case, suppose that $h(\tau_1) = 1$. Then τ_1 has just the root ρ_1 and $D(\rho_1)$ has no ramification points, so at most two directions. Thus, as an $\mathscr{L}$-structure, no triples, quadruples of τ_1 satisfy L, S respectively. Since A_2 is isomorphic to A_1, the same holds for τ_2, and thus $h(\tau_2) = 1,\ |D(\rho_2)| = |D(\rho_1)| \leq 2$, and χ induces a unique isomorphism ϕ.

For the induction step, suppose $m := h(\tau_1) \geq 2$. From Lemma 2.1.6 we know that $D_{\rho_1}(x, y; z, w) \Leftrightarrow ((x = y) \vee (z = w) \wedge \{x, y\} \cap \{z, w\} = \varnothing) \vee (x, y, z, w$ are all distinct and $S(x, y; z, w) \wedge (\forall t)(\neg S'(x, y; z, w; t)))$, and the same hold for D_{ρ_2}. Thus, χ induces an isomorphism $D(\rho_1) \to D(\rho_2)$. This extends to a unique isomorphism (which we denote by ϕ) $\overline{D(\rho_1)} \to \overline{D(\rho_2)}$ taking ramification points of $D(\rho_1)$ to those of $D(\rho_2)$.

For each $r \in \mathrm{Ram}(D(\rho_1))$, put $\phi(f_{\rho_1}^{-1}(r)) = f_{\rho_2}^{-1}(r)$, to obtain a bijection $\mathrm{succ}(\rho_1) \to \mathrm{succ}(\rho_2)$. By Lemma 2.1.10, A_r and $A_{\phi(r)}$ lie in $\mathscr{D}$ for each $r \in \mathrm{Ram}(D(\rho_1))$. Write ρ_r and $\rho_{\phi(r)}$ for the roots of the structure trees of A_r and $A_{\phi(r)}$ respectively. We claim that χ induces an isomorphism χ_r from the $\mathscr{L}$-structure on A_r to that on $A_{\phi(r)}$. Indeed, $\phi|_{\overline{D(\rho)}}$ gives a bijection $D(\rho_r) \to D(\rho_{\phi(r)})$, and the fact that it is an isomorphism of $\mathscr{L}$-structures is immediate from Definition 2.1.8.

Since $h(A_r) < h(A)$ (by Lemma 2.1.10, (iv)) it follows by induction that χ induces an isomorphism (denoted ϕ) from the tree of D-sets corresponding to A_r to that corresponding to $A_{\phi(r)}$. Since this holds for all $r \in \mathrm{Ram}(D(\rho_1))$, putting all the data together we have the required isomorphism $\phi : \tau_1 \to \tau_2$, which is clearly uniquely determined. ∎

2.2 One point extensions

We are aiming to build a structure M by amalgamating members of $\mathscr{D}$; it is enough to do the amalgamation of just one point extensions (see 2.3.1) of $\mathscr{L}$-structures in $\mathscr{D}$.

Fix an $\mathscr{L}$-structure $A \in \mathscr{D}$. We want to specify the possible forms of a one point extension $E = A \cup \{e\}$ of A such that $A \cup \{e\} \in \mathscr{D}$ and A is a substructure of $A \cup \{e\}$.

We now describe some one-point extensions:

Type I (Star-like): To obtain τ_E, which is the structure tree on the $\mathscr{L}$-structure E, from τ_A, we add a new root ρ_E under the root ρ_A of the structure tree τ_A, such that $D(\rho_E)$ looks like a star with one ramification point (the centre) and non-special branches corresponding to the end points in the root D-set $D(\rho_A)$ of A, and a special branch e. See for example Figure 2.9.

Since there is only one ramification point in $D(\rho_E)$, it will have form $f_{\rho_E}(\rho_A)$, where ρ_A is the immediate successor of ρ_E. The D-set $D_E := D(\rho_E)$ is a star whose centre is $f_{\rho_E}(\rho_A)$ where the branches are of the form $g_{\rho_A\rho_E}(x)$, x is a direction in

$D(\rho_A)$. The relations on A will also hold in $E := A \cup \{e\}$. Thus, if $a, b, c \in A$ and $A \models L(a; b, c)$ then $E \models L(a; b, c)$; however this is not witnessed in the root D-set of E, and indeed, $E \models L'(a; b, c; e)$. Likewise if $a, b, c, d, \in A$ and $A \models S(a, b; c, d)$, then $E \models S'(a, b; c, d; e)$.

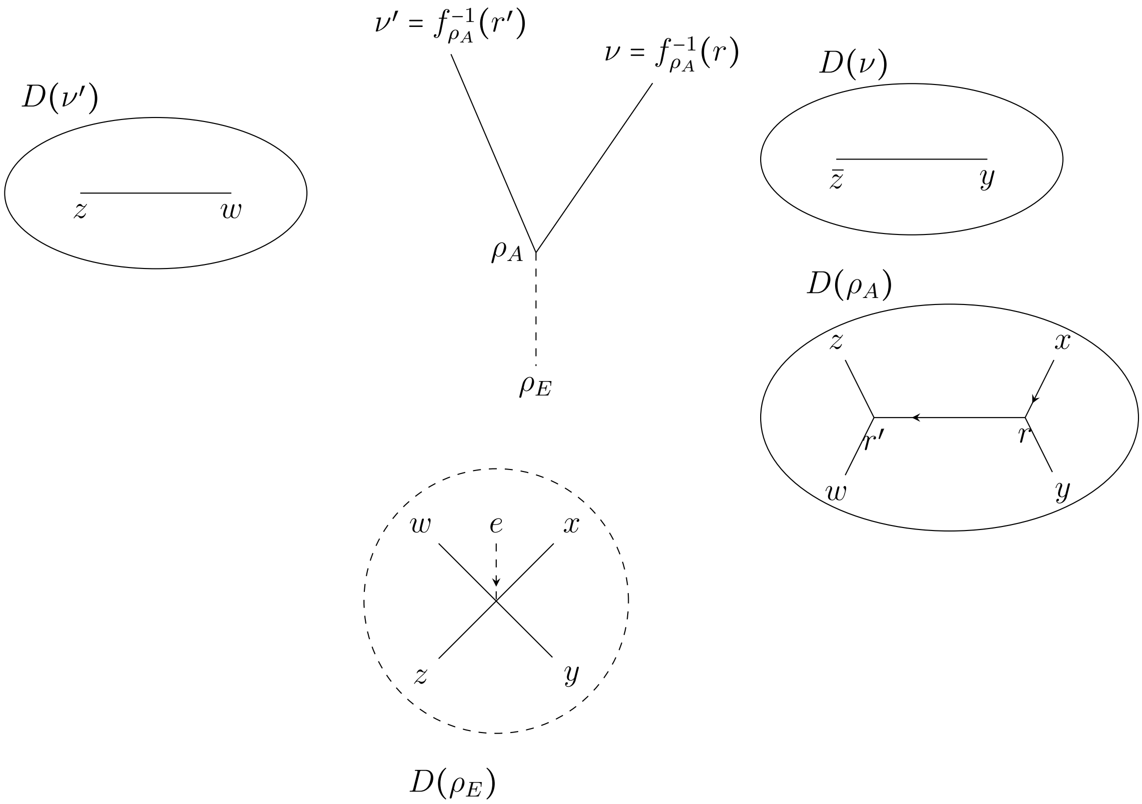

Figure 2.9: One point extension:Type I

Type II : In this type, we assume that the two roots for the two structures A and E are the same, so we call it ρ, and we add the new branch to the existing D-set $D(\rho)$ of the root ρ of τ_A to obtain the root D-set $D(\rho)$ of τ_E. We can do it by two ways:

(a) Add a new direction to an existing ramification point in $D(\rho)$. So the D-set higher up corresponding to that ramification point gets a new end point. If we think of it as a graph, then e is a leaf adjacent to the ramification point. Actually, this process iterates through the structure tree such that if we focus on a ramification point r and call the added direction e, then the successor $f_\rho^{-1}(r)$ will be a root for the substructure isomorphic to E_r of E (before adding e the successor $f_\rho^{-1}(r)$ of ρ is a root of A_r), and the labelling D-set for $f_\rho^{-1}(r)$ gets a new endpoint, and in this case $|E_r| < |E|$ and $A_r < E_r$ as in 2.1.10(iv). Here E_r is the substructure of E that has $f_\rho^{-1}(r)$ as a root.

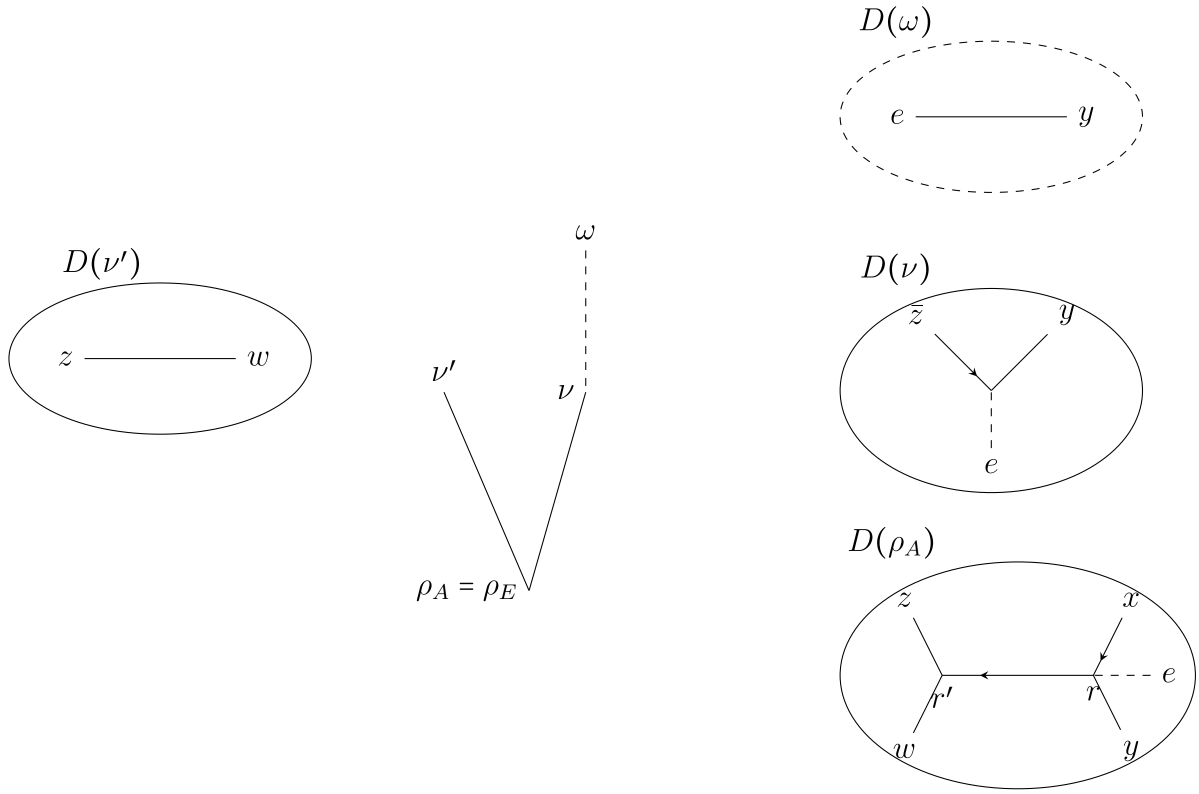

Figure 2.10: One point extension:Type II(a)

(b) Create a new ramification point by adding a vertex on an existing edge in $D(\rho)$, then add a leaf e at this vertex. Here we consider two cases:

(i) e is the special branch at this new ramification point.

(ii) e is not the special branch at this ramification point.

In both cases a new successor is added to the structure tree, but the D-set labelling the new successor has two endpoints, and hence nothing happens further.

Remark 2.2.1. We have to differentiate between these three notations:

- $D_E(A)$: is the root D-relation of E restricted to A.
- D_A or $D(A)$: is the root D-set of A itself, i.e. for the structure tree on A.
- D_E: is the D-set for the root of E.

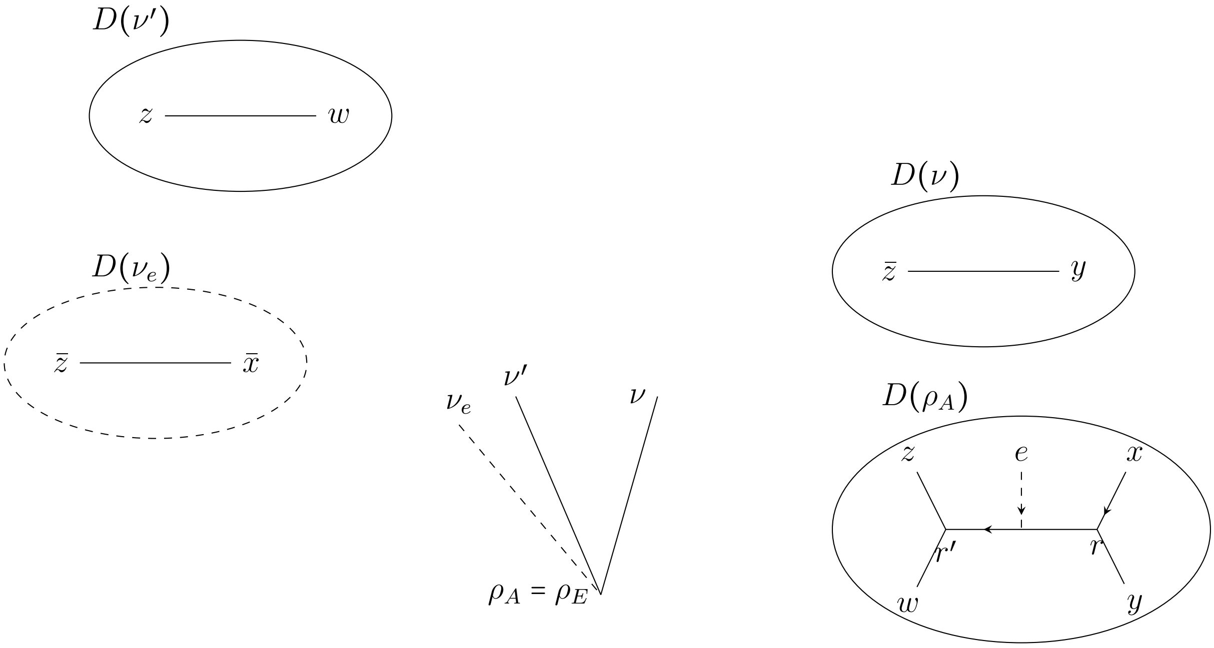

Figure 2.11: One point extension:Type II(b)

Lemma 2.2.2. *If $A, E \in \mathscr{D}$ with $A < E$, and $a, b, c, d \in A$ are all distinct elements, then $D_E(a, b; c, d) \to D_A(a, b; c, d)$.*

Proof. As a, b, c, d are distinct, $D_E(a, b; c, d) \Rightarrow S(a, b; c, d) \wedge (\forall t \in E)\neg S'(a, b; c, d; t) \Rightarrow S(a, b; c, d) \wedge (\forall t \in A)\neg S'(a, b; c, d; t) \Rightarrow D_A(a, b; c, d)$.

∎

Lemma 2.2.3. *If $A \in \mathscr{D}$ and $A \cup \{e\}$ is a Type I extension of A, then $A \cup \{e\} \in \mathscr{D}$ and $A < A \cup \{e\}$.*

Proof. First, it is almost immediate that $E := A \cup \{e\} \in \mathscr{D}$. The structure tree E is obtained from that of A by putting a root ρ_E directly below the root ρ_A of A, and giving the D-set $D(\rho_E)$ the structure of a star with a single ramification point r and leaves corresponding to elements of E, with e special at r. We have $f_{\rho_E}(\rho_A) = r$, and the map $g_{\rho_A \rho_E}$ is just the identity map on A.

Second, we must show that for distinct $a, b, c \in A$ we have $A \vDash L(a; b, c) \Leftrightarrow E \vDash L(a; b, c)$ and similarly for the other relation symbols of $\mathscr{L}$. This is essentially immediate. Indeed, no relation among elements of A is witnessed in the root D-set of E, and such relations hold in E if and only if they are "carried down" from the structure $A = E|_{\rho_A}$ by the map $g_{\rho_A \rho_E}$. ∎

Lemma 2.2.4. *Suppose $A, E \in \mathscr{D}$ with $A < E$, and there is no $e \in E$ such that $A \cup \{e\}$ is a Type I extension of A. Then the root D-set D_A of A, and the structure D_E induced on A by the root D-set D_E of E, denoted $D_E(A)$, are the same.*

Note. We do not here assume that $|E \setminus A| = 1$.

Proof. Let $a, b, c, d \in A$ and $D_E(A)(a, b; c, d)$. Then $D_E(a, b; c, d)$. We may suppose that a, b, c, d are distinct. By Lemma 2.2.2 $D_A(a, b; c, d)$.

Conversely, let $a, b, c, d \in A$ are distinct, and suppose that $D_A(a, b; c, d)$ but $\neg D_E(A)(a, b; c, d)$. We want to show that $D_E(A)(a, b; c, d)$ by a way of contradiction.

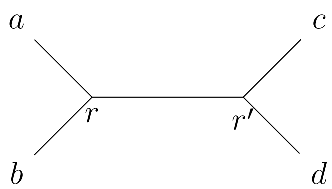

Figure 2.12: $D_A(a, b; c, d)$

So the relation S' holds in E. As $A \vDash S(a,b;c,d)$ and this is not witnessed in the root D-set of E, then there is $e \in E \setminus A$ such that $E \vDash S'(a,b;c,d;e)$. Furthermore, we have the picture in ρ_E, with e special at the shown ramification point in Figure 2.13.

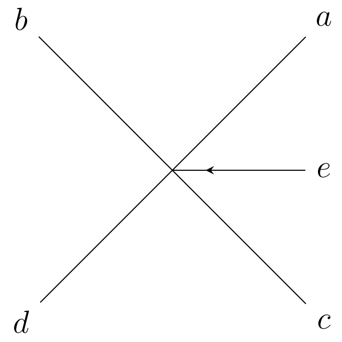

Figure 2.13: D_E

But this picture is a star, and we assume that $A \cup \{e\}$ is not a Type I extension of A. This means that there must be a branch $x \in A$ (hence in E) stopping the star shape, so witnessing that $A \cup \{e\}$ is not a Type I extension of A. We consider the various possible positions of x with respect to a,b,c,d,e in D_E.

Case (1): Suppose x lies in the same branch as c in Figure 2.13. We may suppose (by replacing x by another point if necessary) that one of x,c or d (we may replace d by a or b) is special at ram(x,c,d) in D_E. Since $S(a,d;c,x) \wedge (\forall w \in E)\neg S'(a,d;c,x;w)$ holds in E and hence in A, x must lie in the same branch as c at r' in D_A, with x,c,d meet at ramification point r'' say. Now $E \vDash Q(a,d;c,x:x;c,d) \vee Q(a,d;c,x:c;x,d) \vee Q(a,d;c,x:d;x,c)$, so the same holds in A, that is, one of x,c,d is special at r'' in D_A, say x (the argument is the same in the other cases), see Figure 2.14. Now $A \vDash Q(a,b;c,d:x;c,d)$ but $E \vDash \neg Q(a,b;c,d:x;c,d)$, a contradiction.

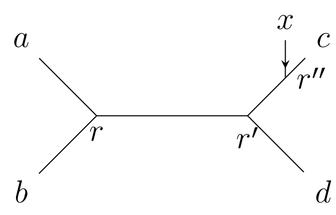

Figure 2.14: Case.1:D_A

Case (2). If x is in the same branch as the special branch e in E, then we will see $S'(a,b;c,d;x)$ holds in E and hence in A, and this is impossible, since we have $D_A(a,b;c,d) \Rightarrow S(a,b;c,d) \wedge (\forall t \in A)\neg S'(a,b;c,d;t)$.

Case (3). Suppose there exists $x' \in A$ in the same branch as x in D_E, see Figure 2.15. We may suppose (by choice of x, x') that one of x, x', d is special at the ramification point r' of x, x', d in D_E. For convenience we suppose $E \vDash L(x; x', d)$, but the other two cases are similar. Thus, $E \vDash Q(x, x'; u, v : x; x', w)$ for any distinct $u, v \in \{a, b, c, d\}$ and any $w \in \{a, b, c, d\}$ so the same holds in A. Furthermore $S(x, x'; u, v)$ (for distinct $u, v \in \{a, b, c, d\}$) is witnessed in the root D-set of A, by Lemma 2.2.2. It follows that in Figure 2.16, x, x' lie in the same branch at r_1, or at r_2 or at a ramification point added between r_1 and r_2, or at ramification point within the direction a (between r_1 and the endpoint a), or within the directions b, c or d.

In particular, there is a ramification point r_3 of D_A which equals $\mathrm{ram}(x, x', w)$ for each $w \in \{a, b, c, d\}$. As x is special at r_3 in A, then $A \vDash Q(a, b; c, d : x; x', a)$, so $E \vDash Q(a, b; c, d : x; x', a)$ contradicting that $S(a, b; c, d)$ is not witnessed in D_E.

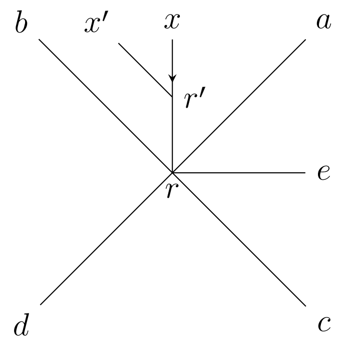

Figure 2.15: Case.3, D_E

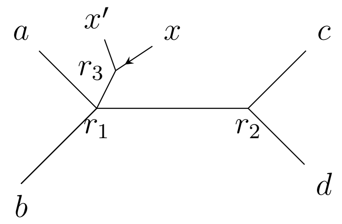

Figure 2.16: Case.3, D_A

■

Lemma 2.2.5. *Suppose $A, E \in \mathscr{D}$ with $A < E$, and there is no $e \in E \setminus A$ such that $A < A \cup \{e\}$ is of Type I. Suppose $a, b, c \in A$ with $L(a; b, c)$. Then $L(a; b, c)$ is witnessed in D_A if and only if it is witnessed in D_E.*

Proof. If $L(a; b, c)$ is not witnessed in D_A then there is $d \in A$ such that $A \models L'(a; b, c; d)$, so $E \models L'(a; b, c; d)$, so $L(a; b, c)$ is not witnessed in D_E.

Conversely, suppose that $L(a; b, c)$ is witnessed in D_A, with a, b, c lying in distinct branches of the ramification point r of D_A. By Lemma 2.2.4 we may identify r with a ramification point of D_E. We suppose for a contradiction that $L(a; b, c)$ is not witnessed in D_E. Then there is $e \in E$ in a special branch at r, distinct from those of a, b, c. Since $E \models L'(a; b, c; e)$, we must have $e \notin A$. Furthermore, as $A < A \cup \{e\}$ is not of Type I and there is no $d \in A$ lying in the same branch as e at r, there must be distinct $p, q, t, s \in A$ with $S(p, q; t, s)$ witnessed in D_A, so in D_E. Then $A \vDash Q(p, q; t, s : a; b, c)$ so $L(a; b, c)$ is witnessed in D_E, a contradiction. ∎

Lemma 2.2.6. *In a Type II extension $A < E = A \cup \{e\}$, if r is a ramification point of the root D-set of A, then the structure A_r is a substructure of E_r.*

Proof. It suffices to check that the relations agree in both A_r and E_r. Let $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{z}, \bar{p}, \bar{q}, \bar{s} \in A_r$, then they are elements of $D(f_\rho^{-1}(r))$. Assume $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are distinct. As $A < E$ is of Type II, by Lemma 2.2.4 r is a ramification point of D_E, and a, b, c lie in distinct branches at r in E. Furthermore, none is special at r in A, as $\bar{a}, \bar{b}, \bar{c}, \bar{d} \in A_r$, and so as it is Type II extension, none is special in E.

Then $A_r \vDash L(\bar{a}; \bar{b}, \bar{c}) \Leftrightarrow A \vDash L(a; b, c) \Leftrightarrow E \vDash L(a; b, c) \Leftrightarrow E_r \vDash L(\bar{a}; \bar{b}, \bar{c})$, and $A_r \vDash S(\bar{a}, \bar{b}; \bar{c}, \bar{d}) \Leftrightarrow A \vDash S(a, b; c, d) \Leftrightarrow E \vDash S(a, b; c, d) \Leftrightarrow E_r \vDash S(\bar{a}, \bar{b}; \bar{c}, \bar{d})$. Also $A_r \vDash Q(\bar{a}, \bar{b}; \bar{c}, \bar{d} : \bar{p}; \bar{q}, \bar{s}) \Leftrightarrow A \vDash Q(a, b; c, d : p; q, s) \Leftrightarrow E \vDash Q(a, b; c, d : p; q, s) \Leftrightarrow$
$E_r \vDash Q(\bar{a}, \bar{b}; \bar{c}, \bar{d} : \bar{p}; \bar{q}, \bar{s})$.
Likewise, $A_r \vDash L'(\bar{b}; \bar{c}, \bar{d}; \bar{a}) \Leftrightarrow A \vDash L'(b; c, d; a) \Leftrightarrow E \vDash L'(b; c, d; a) \Leftrightarrow E_r \vDash L'(\bar{b}; \bar{c}, \bar{d}; \bar{a})$. Let $\bar{w} \in A_r$. Then $A_r \vDash S'(\bar{a}, \bar{b}; \bar{c}, \bar{d}; \bar{w}) \Leftrightarrow A \vDash S'(a, b; c, d; w) \Leftrightarrow E \vDash S'(a, b; c, d; w) \Leftrightarrow E_r \vDash S'(\bar{a}, \bar{b}; \bar{c}, \bar{d}; \bar{w})$. ∎

Lemma 2.2.7. *If $A, E \in \mathscr{D}$ and E is a one point extension of A with $E = A \cup \{e\}$, then (A, E) is of Type I or of Type II.*

Proof. Assume that the extension is not of Type I, so there is no new root under ρ_A with a star D-set. By Lemma 2.2.4, $D_E(A) = D(A)$. By Lemma 2.2.4, we know that the root D-set D_A of A is a substructure of D_E, and hence can identify $\overline{D_A}$ with a subset of $\overline{D_E}$. Furthermore, for $a, b, c \in A$, $L(a; b, c)$ is witnessed in D_A if and only if it is witnessed in D_E by Lemma 2.2.5. Thus, either e is added (in D_E) as a new non-special leaf to an existing ramification point r of D_A, or e is added on a new ramification point r' of an edge of D_A. To prove it is of Type II, we consider the following cases:

Case(i). Suppose that e is added as a new non-special leaf to an existing ramification point r of D_A. Furthermore, using Lemma 2.2.6 we see that A_r is a substructure of E_r, and so by induction, as $|A_r| < |A|$, $A_r < E_r$ is of Type I or Type II. As we assumed that the extension $A < E$ is not of Type I it follows that $A < E$ is a Type II(a) extension.

Case(ii). Suppose that e is added on a new ramification point r' of an edge of D_A. In this case, for E, ρ_A obtains a new successor $\rho_{r'}$ whose D-set has size 2. The structure is otherwise unchanged, and E is an extension of A of Type II(b).

■

Lemma 2.2.8. *Let $A < E$ with $A, E \in \mathscr{D}$. Then there is an element $e \in E \setminus A$ such that $A \cup \{e\} \in \mathscr{D}$.*

Proof. We just showed, by Lemma 2.2.7, that extending a substructure A of E by one element, so that the result lies in $\mathscr{D}$, can be done by only two ways: Type I or Type II.

Firstly, adding e from $E \setminus A$ to A to get $A \cup \{e\}$ in $\mathscr{D}$ by a Type I extension will do the required as we showed in Lemma 2.2.3. Thus, we may suppose there is no such $e \in E$, so Lemma 2.2.4 and Lemma 2.2.5 are applicable.

Suppose there is an edge of A such that E has a ramification point r on the edge and there is $e \in E \setminus A$ and $a, b \in A$ such that a, b, e lie in distinct branches at r. We may suppose (by careful choice of e) that one of a, b, e lies in the special branch at r in E. In this case $A \cup \{e\} \in \mathscr{D}$, a one-point extension of A of Type II(b).

Suppose the configuration of the last paragraph does not occur. Since $D_E(A) = D_A$, there is a ramification point r of A and some $e \in E \setminus A$ lying in a new non-special branch at r of E. Then $A \cup \{e\} \in \mathscr{D}$ and is a one-point extension of A of Type II(a).

■

We will see now that adding one element (n elements) from E to A keeps the extension in the class $\mathscr{D}$, and E can be written as a sequence of one point extensions.

Lemma 2.2.9. *Assume $A < E$ with $A, E \in \mathscr{D}$. Then we may enumerate $E \smallsetminus A$ as $\{e_1, e_2, \ldots, e_n\}$ such that for each $i = 1, \ldots, n$, if E_i is the $\mathscr{L}$-structure of E on $A \cup \{e_1, \ldots, e_i\}$ then $E_i \in \mathscr{D}$.*

Proof. Fix n. We prove by induction on $m < n$ that there are distinct $e_1, \ldots, e_m \in E \smallsetminus A$ such that for each $i = 0, \ldots, m$ the $\mathscr{L}$-structure E_i induced on $A \cup \{e_1, \ldots, e_i\}$ lies in $\mathscr{D}$ (where $E_0 = A$).

The base case $m = 0$ is trivial. Assume the result holds for m. Then by Lemma 2.2.8 applied to $E_m < E$, there is some $e \in E \smallsetminus E_m$ such that $E_m \cup \{e\} \in \mathscr{D}$. Put $e_{m+1} := e$. ■

2.3 Amalgamation Property

Fraïssé's method is based on taking smaller structures, extending them and then amalgamating the extensions. The following lemma is a general lemma that holds for any class of finite structures.

Lemma 2.3.1. *Let $\mathscr{C}$ be a class of finite structures, and suppose that the following hold:*

1. *the class $\mathscr{C}$ has the amalgamation property for one point extensions.*
2. *for any $A, E \in \mathscr{C}$ with $A < E$, we may write $E \smallsetminus A = \{x_1, \ldots, x_n\}$ so that if E_i is the induced substructure of E on $A \cup \{x_1, \ldots, x_i\}$ (for each $i = 1, \ldots, n$), then $E_i \in \mathscr{C}$.*

Then the class $\mathscr{C}$ has the amalgamation property.

Proof. Suppose that $A \cup \{x_1, \ldots, x_m\}$, $A \cup \{y_1, \ldots, y_n\}$ lie in $\mathscr{C}$; we will denote them by X_i and Y_i respectively. Assume E_1 is obtained by adding m elements, say $x_1, \ldots, x_m$, to A, and E_2 is obtained by adding n elements, say $y_1, \ldots, y_n$, to A. Then assume that the one-point extension of A by adding x_1 is X_1, and the one-point extension of A by adding y_1 is Y_1. Then amalgamate X_1 and Y_1 over A. We get E_{11} which is a structure in $\mathscr{C}$. So the one-point extension of X_1 by adding y_1 is E_{11}.

Next step, extend X_1 by a one-point extension via the element x_2 then we get X_2 ($X_2 = X_1 \cup \{x_2\}$). Amalgamate X_2 and E_{11} over X_1 to get E_{21}. (We do not assume that one point extensions amalgamate disjointly, so possibly $E_{11} = X_1$.) Do this process m times to end with E_{m1}. When we look at amalgamating E_{m1} and E_2 we note that the m steps of the amalgamation put y_1 in the common subset that we amalgamate over. Also $|E_2 \setminus Y_1| = n - 1$. Hence it finishes inductively.

∎

Note: The proof of the previous lemma is due to Bhattacharjee and Macpherson in their paper (see the proof of Lemma 3.7 of [7]), but it works under the assumptions of this lemma.

By Lemma 2.2.9 and Lemma 2.3.1 it suffices to prove the amalgamation property for one point extensions.

Lemma 2.3.2. *The class $\mathscr{D}$ has the amalgamation property.*

Proof. We will prove the amalgamation property for one point extensions, and using Lemmas 2.2.9 and 2.3.1 the amalgamation can be done for arbitrary finite extensions. Assume $A < E_1$ and $A < E_2$ with $A, E_1, E_2 \in \mathscr{D}$ such that $E_1 \setminus A = \{e_1\}$ and $E_2 \setminus A = \{e_2\}$. Assume that e_1 and e_2 are distinct. We want to define a structure on $E_1 \cup E_2$ to obtain an element $E \in \mathscr{D}$ such that E_1 and E_2 are induced by E. Let τ_i be the structure tree corresponding to E_i with root ρ_i where $i = 1, 2$. We will consider three cases.

Case i. Suppose that E_1 and E_2 are Type I extensions of A. Then place the root ρ_2 beneath the root ρ_1 such that e_2 is special in $D(\rho_2)$ with e_1 non-special, and in $D(\rho_1)$ the element e_1 is special.

Case ii. Suppose that one of the E_i, say E_1 is of Type I, and E_2 is of Type II. Then place the root ρ_1 under ρ_2 such that $D(\rho_1)$ is a star in which e_1 is special and e_2 is not.

Case iii. Suppose that E_1 and E_2 are of Type II over A. Then we will consider the following four sub-cases.

(1) If e_1, e_2 are added to the same old ramification point r of $D(\rho_A)$ to get E_1, E_2 respectively. Keep them distinct. Then neither of e_1, e_2 is special in the root D-sets $D(\rho_1)$ and $D(\rho_2)$. The root ρ_E of E will have that e_1, e_2 are non-special branches. Then higher up two new end-points are added to the same D-set $D(f_{\rho_E}^{-1}(r))$, and we finish inductively, since $|A_r| < |A|$.

(2) Suppose that e_1 and e_2 are added to distinct ramification points r_1 and r_2 of $D(\rho_A)$. Then again an endpoint will be added to the D-sets corresponding to these ramification points. The structures E_{r_1} and $(E_1)_{r_1}$ will be isomorphic, and E_{r_2} will be isomorphic to $(E_2)_{r_2}$.

(3) Suppose that the branch e_1 is added to an old ramification point r of $D(\rho_A)$, and e_2 creates a new ramification point. Then a new successor (of ρ_A) has trivial D-set in E, and $D(f_{\rho_E}^{-1}(r))$ gets a new endpoint.

(4) If both e_1 and e_2 create new ramification points, then keep them distinct. Hence $D(\rho_E)$ will have two new endpoints and then two new successors each with labelling D-sets of just two elements.

■

The proof of the following lemma is similar to the proof of Lemma 3.6 of [7].

Lemma 2.3.3. *The class $\mathscr{D}$ has the joint embedding property.*

Proof. Take two finite structures $A, B \in \mathscr{D}$ with n, m points respectively. Consider their structure trees of D-sets τ_A, τ_B with roots ρ_A, ρ_B respectively. Build a new tree τ with root ρ such that $D(\rho)$ contains two ramification points r and r' with $n+1$ branches at r, and $m+1$ branches at r', with special branches as shown in the Figure 2.17. The resulting structure E will have E_r isomorphic to A and $E_{r'}$ isomorphic to B.

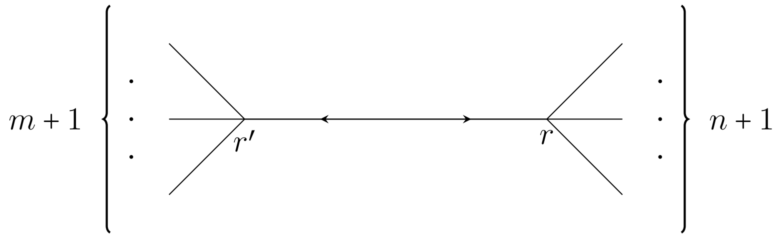

Figure 2.17

∎

Lemma 2.3.4. *The class $\mathscr{D}$ does not satisfy the hereditary property.*

Proof. Take, as an example, a finite structure $C \in \mathscr{D}$ with the elements x, y, z, w, p. Let r =ram(x, y, p, z), r' =ram(z, w, x) with $S(x, y; z, w)$, $L(x; y, z) \wedge L(x; y, w) \wedge L(x; y, p)$ holding at r, and $L(z; w, x) \wedge L(z; w, y) \wedge L(z; w, p)$ holding at r', all in the root D-set $D(\rho_C)$, see Figure 2.18. Assume that $\nu = f^{-1}_{\rho_C}(r)$ and in $D(\nu)$ the relation $L(p; y, z)$ is witnessed at the unique ramification point r'' say. Also let $\nu' = f^{-1}_{\rho_C}(r')$ and $\nu_1 = f^{-1}_{\nu}(r'')$. The two labelling D-sets $D(\nu')$ and $D(\nu_1)$ have just two elements.

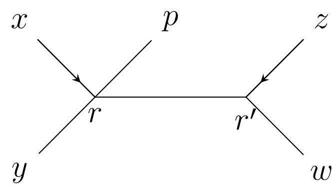

Figure 2.18: ρ_C

Let $A = C \smallsetminus \{x\}$ be a substructure of C. Assume for a contradiction $A \in \mathscr{D}$. Clearly C is not a Type I extension of A, so by Lemma 2.2.4, $D_C(A) = D_A$ (the root D-set of A). Let r_1 = ram(y, p, z) and r_2 = ram(z, w, y) with $L(p; y, z)$ and $L(p; y, w)$ holding at r_1 and $L(z; y, w)$ and $L(z; p, w)$ holding at r_2 as in Figure 2.19. Then $D(p, y; z, w)$ holds in D_C so in D_A. In the labelling D-sets $D(f^{-1}_{\rho_A}(r_1))$ and $D(f^{-1}_{\rho_A}(r_2))$ there are two elements in each one. Then it can be seen that $Q(p, y; z, w : p; y, z)$ is witnessed in a D-set of A (the root) while it is not witnessed in C, which is a contradiction. Hence $A \notin \mathscr{D}$.

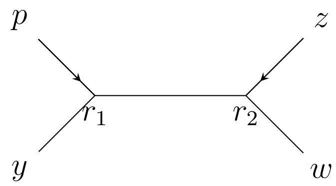

Figure 2.19: ρ_A

■

As the class $\mathscr{D}$ is not closed under the substructure we use a modified version of Fraïssé's Theorem, and here we follow Evans [19].

Definition 2.3.5. Let $\mathscr{D}$ be the class of $\mathscr{L}$-structures that we defined before. Define a collection $\mathcal{E}$ of embeddings $f : A \to D$ where $A, D \in \mathscr{D}$ such that

(i) any isomorphism is in $\mathcal{E}$;

(ii) $\mathcal{E}$ is closed under composition;

(iii) if $f : A \to B$ is in $\mathcal{E}$ and $B \subset D$ is a substructure in $\mathscr{D}$ such that $f(A) \subset B$, then the map obtained by restricting the range of f to B is also in $\mathcal{E}$.

Then we call this collection a *class of $\mathscr{D}$-embeddings*

Note. This definition is also explained in [25], but the author uses the notion of *weakly homogeneous* to explain this.

Consider the following modification for the joint embedding property and the amalgamation property :
(JEP ′) If $A, B \in \mathscr{D}$, there exists $C \in \mathscr{D}$ and embeddings $f : A \to C$ and $g : B \to C$ such that $f, g \in \mathcal{E}$.
(AP ′) Suppose $A, D_1, D_2 \in \mathscr{D}$ and $f_i : A \to D_i$ are embeddings in $\mathcal{E}$. Then there exists $D \in \mathscr{D}$ and embeddings $g_i : D_i \to D$ in $\mathcal{E}$ such that $g_1 f_1 = g_2 f_2$.

Let $\mathcal{E}$ be the class of $\mathscr{D}$-embeddings. For an $\mathscr{L}$-structure M, and a finite substructure $A \in \mathscr{D}$, we say that A is *$\mathcal{E}$-embedded* in M if whenever $B \in \mathscr{D}$ is a finite substructure of M and contains A, the inclusion map from A to B is in $\mathcal{E}$.

Then we use the following version of Fraïssé's Theorem (Theorem 2.10 of [19] in the book [31]):

Theorem 2.3.6. *Suppose that $\mathcal{C}$ is a collection of finite $\mathcal{L}$-structures in which the number of isomorphism types of any finite size is finite. Suppose $\mathcal{E}$ is a class of $\mathcal{C}$-embeddings which satisfies JEP′ and AP′. Then there exists a countable $\mathcal{L}$-structure M with the following properties:*

(a) the class of $\mathcal{E}$-embedded substructures of M is equal to $\mathcal{C}$;

(b) M is a union of a chain of finite $\mathcal{E}$-embedded substructures;

(c) if $A \leq M$ and $\alpha : A \to B$ is in $\mathcal{E}$ then there exists $C \leq M$ containing A and an isomorphism $\beta : B \to C$ such that $\beta\alpha(a) = a$ for all $a \in A$.

Let M be the $\mathscr{L}$-structure built by applying Theorem 2.3.6 to the collection $\mathscr{D}$ and the collection $\mathcal{E}$ of embeddings defined in Definition 2.3.5.

Lemma 2.3.7. *Any isomorphism between finite substructures of M which lie in $\mathscr{D}$ extends to an automorphism of M.*

Proof. This follows immediately from Theorem 2.3.6(c). ■

We will call the structure that has the assertion in the above lemma *semi-homogeneous*.

We have built a unique semi-homogeneous countable structure M whose elements embed in $\mathscr{D}$. Note that for now, we do not have a homogeneous structure. All that we have is that any isomorphism between substructures of the class $\mathscr{D}$ can be extended to an automorphism.

Remark 2.3.8. From now on, when we use the phrase 'by semi-homogeneity' we mean that $f : A \to B$ is an isomorphism between substructures of M lying in $\mathscr{D}$, and refer to the existence of an element of Aut(M) extending f.

2.4 Oligomorphicity of *M*

It is known that Fraïssé's construction is a method to build ω-categorical structures. How that works is Fraïssé's construction is used to build homogeneous structures; it is an easy consequence of Ryll-Nardzewski Theorem, and it is known (see [32], Proposition 1.5) that any homogeneous structure over a finite relational language is ω-categorical. The proof of that theorem explains how to get an ω-categorical structure.

As we see from the previous section that the structure M that we build is not homogeneous, we will use the Theorem by Ryll-Nardzewski (see Theorem 1.6.13) and we will show in this section that the automorphism group of M is oligomorphic and hence M is ω-categorical.

We do not yet have a guarantee that Aut(M) is oligomorphic. For example, suppose that M has finite substructures E_i (for $i \in \mathbb{N}$) in the class $\mathscr{D}$, and suppose $|E_1| < |E_2| < |E_3| < \dots$ and that E_i is a substructure of M of smallest size subject to lying in $\mathscr{D}$ and containing a_i, b_i. Then the pairs (a_i, b_i) all lie in distinct orbits of Aut(M) in M^2.
Our next lemma eliminates this problem.

Lemma 2.4.1. *There is a map $f : \mathbb{N} \to \mathbb{N}$ such that for every finite $A \subset M$ there is $F \in \mathscr{D}$ with $A \leq F \leq M$ and $|F| \leq f(|A|)$.*

Proof. By Theorem 2.3.6, A lies in a finite substructure E of M lying in $\mathscr{D}$. We aim to choose F inside E, of minimal size. Let ρ be the root of the structure tree of E, D_E be the corresponding D-set, let D_A be the induced D-set structure on A, and $\overline{D_E}$, $\overline{D_A}$ be the corresponding tree structures. Let $n := |A|$. We shall build F as the union of a finite sequence $A = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq E$. We may suppose that E is chosen minimally, that is, there is no proper substructure of E with $E' \in \mathscr{D}$ and $A \leq E' < E$.

We have $|\mathrm{Ram}(D_A)| \leq n-2$ (by Lemma 1.4.15). We form F_1 by adding, for each ramification point r of D_A such that the special branch of E at r contains no member of A, a member of that special branch. Then $|F_1| \leq |A| + n - 2$, and F_1 contains a special branch at each such ramification point r. Observe that if $|A| = 3$ then $A \in \mathscr{D}$ so by minimality of E, $A = E$.

Next, for each such ramification point r of D_A, let σ be the corresponding successor in the structure tree of E. (We note here that by minimality of E it cannot happen that the elements of A all lie in distinct non-special branches at the same ramification point r of D_E, and thus indeed $|D_\sigma(A)| < |A| = n$). There are at most $n-2$ such σ, and the D-set D_σ of E contains at most $n-1$ elements with representatives in A, giving a D-set $D_\sigma(A)$ of size at most $n-1$, so with at most $n-3$ ramification points. We build F_2 to ensure that there is a special branch at each ramification point of $D_\sigma(A)$, for each σ. This requires adding at most $(n-2)(n-3)$ points to F_1, so $|F_2| \leq |F_1| + (n-2)(n-3)$.

We iterate this process. To build F_2 from F_1, we consider the at most $(n-2)(n-3)$ ramification points of F_2 (of D-sets of successors of ρ), and the corresponding $(n-2)(n-3)$ vertices λ of height 3 in the structure tree of A. The D-set $D_\lambda(E)$ contains at most $(n-2)$ elements with representatives in A, so the corresponding D-set $D_\lambda(A)$ has at most $(n-4)$ ramification points.

Continuing this process, we find that for F_i, each D-set of height i (where ρ has height 1) has at most $n-1-i$ ramification points, and that for $j < i$, each D-set of F_i at height F_j has a special branch at each ramification point. Thus, putting $F := F_{n-3}$, we find that F has a special branch at each ramification point of each D-set, so $F \in \mathscr{D}$. Finally, we see inductively that for each i, $|F_i| = |F_{i-1}| + (n-2)(n-3)\dots(n-(n+1))$.

Thus, we may put $f(n) = (n-2) + (n-2)(n-3) + \cdots + (n-2)(n-3\dots 2) = \sum_{i=1}^{n-1} \frac{(n-2)!}{i!}$.

■

Since A is finite then the induced substructures of A, which are F_i, will be finite, and this is enough to ensure the oligomorphicity of M.

Lemma 2.4.2. *Let M be the Fraïssé limit of a class $\mathcal{C}$ of finite structures in the sense of Theorem 2.3.6. Suppose there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for every finite subset A of M there is $F < M$ with $F \in \mathcal{C}$ and $|F| \leq f(|A|)$. Then M is ω-categorical.*

Proof. Suppose that A is a finite subset of M with k elements. Every such A lies in a member F of $\mathcal{C}$ which is a substructure of M as given in the statement. As the language is finite, and using the bound provided by f, there are finitely many choices of such F (this fact is Exercise 1.2.6 in [25]). Isomorphic structures F lie in the same orbit. As the choices of F are finite then there are finitely many orbits on such sets F. Therefore, as each such F has a finite subset isomorphic to A then the number of orbits on M^k is finite for any k. By 1.6.13, M is ω-categorical.

■

Corollary 2.4.3. Aut*(M) is oligomorphic*

Proof. This follows from the above lemma (and was part of its proof). ■

Chapter 3

Analysing the Fraïssé Limit

Throughout this chapter, let M be the structure built in Chapter 2, and put $G = \mathrm{Aut}(M)$. In the previous chapter, we studied finite trees of D-sets. Here, we want to show that M can also be viewed as a "tree of D-sets". We have to construct the structure tree of M - in the language of model theory, we interpret it in M. The structure tree will be a dense semilinear order without maximal or minimal elements, so in particular there will be no notion of 'root' or of 'successor'. The vertices of the tree are labelled by representatives of some equivalence relation (it will be the relation P (Definition 3.2.1)). Also the elements of the D-sets are equivalence classes of the equivalence relation E_{xyzw} (see 3.2.9).

3.1 Automorphism of *M*

As the language $\mathscr{L}$ consists of six relations, to deal with the automorphism group is hard. To make it easier we write the relations L', S', Q, R in terms of L, S.

Lemma 3.1.1. *Let* $x, y, z, w \in M$. *Then* $M \vDash (\forall x, y, z, w) L'(x; y, z; w) \leftrightarrow [L(x; y, z) \wedge L(w; y, z) \wedge L(w; x, z) \wedge L(w; x, y) \wedge \neg S(x, w; y, z)]$.

Proof. ($\Rightarrow$) Suppose that $L'(x; y, z; w)$ holds in M. Pick a finite substructure $A \in \mathscr{D}$ such that $x, y, z, w \in A < M$. Then there is a D-set containing x, y, z, w with w considered as a special branch at the ramification point $r = \text{ram}(x, y, z)$ (so all of x, y, z, w meet at the same ramification point r). We may assume that this D-set is the root D-set $D(\rho)$ where ρ is the root of the structure tree on A. So $L(w; z, y) \wedge L(w; x, z) \wedge L(w; x, y)$ are witnessed in this root D-set. Then the labelling D-set of the vertex $f_\rho^{-1}(r)$ witnesses $L(x; y, z)$ and omits w, and clearly $A \vDash \neg S(x, w; y, z)$, so $M \vDash \neg S(x, w; y, z)$.

($\Leftarrow$) In a finite structure $A \in \mathscr{D}$ with $x, y, z, w \in A < M$, suppose that $L(x; y, z) \wedge L(w; y, z) \wedge L(w; x, z) \wedge L(w; x, y) \wedge \neg S(x, w; y, z)$. We aim to show $M \vDash L'(x; y, z; w)$. We may suppose (by choosing A as small as possible) that the root D-set $D(\rho)$ of A is the only one containing x, y, z, w as distinct elements, i.e. lying in distinct directions.

Suppose first that $S(x, y; z, w)$ is witnessed in this D-set. Let $r_1 = \text{ram}(x, y, z)$, and $r_2 = \text{ram}(x, z, w)$. See Figure 3.1.

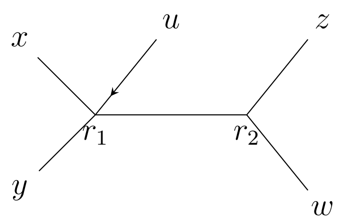

Figure 3.1

Since $L(w; x, y)$, we see that x (and y) cannot be special at r_1. And since $L(x; y, z)$, we see that w cannot be special at r_1. Thus, some other direction u (as depicted) must be special at r_1. Then since z and w are identified in $f_\rho^{-1}(r_1)$ we cannot have $L(x; y, z) \wedge L(w; x, y)$, a contradiction.

Thus, x, y, z, w all lie in different branches at the same ramification point r of $D(\rho)$. We may suppose further (by the minimality of the choice of A) that one of x, y, z, w is special at r. Since $L(w; y, z) \wedge L(w; x, z) \wedge L(w; x, y)$, this must be w, with $L(x; y, z)$ witnessed in a higher D-set of A. Thus $A \vDash L'(x; y, z; w)$, so $M \vDash L'(x; y, z; w)$ ∎

In the language $\mathscr{L}$ there are two relations which describe “happening in the same D-set”, namely Q and R. In the following lemma we re-write them and the relation S' in terms of L and S.

Lemma 3.1.2. *Let $x, y, z, w, p, q, s, t \in M$. Then*

(i) $M \models R(x; y, z : p; q, s) \leftrightarrow [L(x; y, z) \wedge L(p; q, s) \wedge (\forall t)(L'(x; y, z; t) \Leftrightarrow L'(p; q, s; t))]$.

(ii) $M \models S'(x, y; z, w; t) \leftrightarrow \bigwedge\limits_{\substack{u,v \in \{x,y,z,w\} \\ u \neq v}} R(t; x, y : t; u, v) \wedge \bigwedge\limits_{\substack{u,v,s \in \{x,y,z,w\} \\ L(u;v,s)}} \neg R(t; x, y : u; v, s)$ $\wedge S(x, y; z, w)$.

(iii) $M \models Q(x, y; z, w : p; q, s) \leftrightarrow [S(x, y; z, w) \wedge L(p; q, s) \wedge (\forall t)(S'(x, y; z, w; t) \Leftrightarrow L'(p; q, s; t))]$.

Proof.

(i) $\Rightarrow$) Suppose that $M \models R(x; y, z : p; q, s)$, and let $A \in \mathscr{D}$ be any finite substructure of M containing x, y, z, p, q, s. Then $A \models R(x; y, z : p; q, s)$, so from the way R was defined in Chapter 2, Section 1, $L(x; y, z)$ and $L(p; q, s)$ must be witnessed in the same D-set of A. It follows immediately that $A \models (\forall t)(L'(x; y, z; t) \Leftrightarrow L'(p; q, s; t))$. Since this hold for any A, it holds in M.

$\Leftarrow$) Suppose that M satisfies $L(x; y, z) \wedge L(p; q, s) \wedge (\forall t)(L'(x; y, z; t) \Leftrightarrow L'(p; q, s; t))$, and let $A \in \mathscr{D}$ be a finite substructure of M containing x, y, z, p, q, s. Then as $M \models L(p; q, s) \wedge L(x; y, z)$, these L-relations are witnessed in distinct comparable D-sets of A, or incomparable D-sets of A, or in the same D-set of A.

If $L(x; y, z)$ and $L(p; q, s)$ are witnessed in distinct comparable D-sets of A, say $L(p; q, s)$ below $L(x; y, z)$, then there is some $t \in A$ with (namely $t = p$) $A \models L'(x; y, z; t) \wedge \neg L'(p; q, s; t)$, a contradiction.

Suppose that $L(p; q, s)$ and $L(x; y, z)$ are witnessed in incomparable D-sets of A. Then we may suppose (replacing A by a substructure if necessary) that in the root D-set $D(\rho)$ of A, there are distinct ramification points r_1 and r_2 such that x, y, z lie in distinct branches at r_1 and p, q, s lie in distinct branches at r_2. We now see that for all possible choices of special branches at r_1 and r_2, $A \vDash (\exists t)\neg(L'(x; y, z; t) \Leftrightarrow L'(p; q, s; t))$. Thus, $L(p; q, s)$ and $L(x; y, z)$ are witnessed in the same D-set of A, so $A \vDash R(x; y, z : p; q, s)$, as required.

(ii) $\Rightarrow$) Suppose $M \vDash S'(x, y; z, w; t)$, and let $A \in \mathscr{D}$ be a substructure of M containing x, y, z, w in distinct non-special branches of some ramification point r of the root D-set, an $t \in A$ is in the special branch at r. Then as $A \vDash S'(x, y; z, w; t)$, there is a D-set of A witnessing $S(x, y; z, w)$ and omitting t. Then in the root D-set $\bigwedge\limits_{\substack{u,v \in \{x,y,z,w\} \\ u \neq v}} R(t; x, y : t; u, v)$ holds.

It is readily seen that $L(t; x, y)$ and $L(u; v, s)$, where $u, v, s \in \{x, y, z, w\}$ cannot hold in the same D-set of A, hence $\bigwedge\limits_{\substack{u,v,s \in \{x,y,z,w\} \\ L(u;v,s)}} \neg R(t; x, y : u; v, s)$ is witnessed in A, and hence also in M.

$\Leftarrow$) Assume, for a contradiction, that $\neg S'(x, y; z, w; t)$ holds. Then in a finite structure $A \in \mathscr{D}$ with $x, y, z, w, t \in A < M$, $A \vDash \neg S'(x, y; z, w; t)$. Then there is a D-set of A witnessing $S(x, y; z, w)$ and containing t. By considering the possible positions for t in the D-set witnessing $S(x, y; z, w)$, and the various possibilities of the special branch at r_1, r_2, r_3, r_4, r_5, as in Figure 3.2, we will find, at least one relation contradicting $\bigwedge\limits_{\substack{u,v \in \{x,y,z,w\} \\ u \neq v}} R(t; x, y : t; u, v)$ or $\bigwedge\limits_{\substack{u,v,s \in \{x,y,z,w\} \\ L(u;v,s)}} \neg R(t; x, y : u; v, s)$.

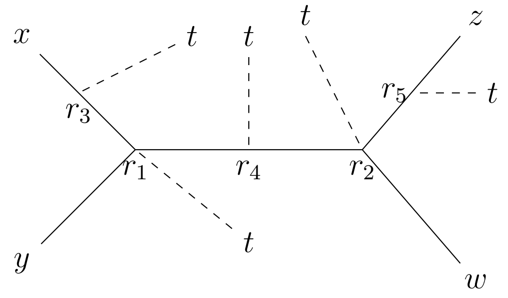

Figure 3.2

(iii) $\Rightarrow$) This is clear, as in (i).

$\Leftarrow$) Assume $M \vDash S(x, y; z, w) \wedge L(p; q, s) \wedge (\forall t)(S'(x, y; z, w; t) \Leftrightarrow L'(p; q, s; t))$. Let $A \in \mathscr{D}$ be any finite substructure of M containing x, y, z, w, p, q, s. Then $S(x, y; z, w)$ and $L(p; q, s)$ are witnessed in the same D-set of A (as we aim to show) or distinct comparable D-sets of A, or incomparable D-sets of A.

Suppose $S(x, y; z, w)$ and $L(p; q, s)$ are witnessed in distinct comparable D-sets of A, say $S(x, y; z, w)$ below $L(p; q, s)$ (the other case being similar). Let $D(\nu)$ be the D-set witnessing $S(x, y; z, w)$, let r be a ramification point of $D(\nu)$, and suppose that $L(p; q, s)$ is witnessed in or above the D-set of the vertex $f_\nu^{-1}(r)$. Let t be in the special branch at r. Then $A \vDash \neg S'(x, y; z, w; t) \wedge L'(p; q, s; t)$, a contradiction.

Likewise, the case when $S(x, y; z, w)$ and $L(p; q, s)$ are witnessed in distinct incomparable D-sets of A is eliminated as in the proof of (i), the right to left implication. Thus, they are witnessed in the same D-set of A, so $A \vDash Q(x, y; z, w : p; q, s)$, $M \vDash Q(x, y; z, w : p; q, s)$, as required.

■

It follows from Lemma 3.1.1 and Lemma 3.1.2 that G consists exactly of the permutations of M which preserve L and S.

Note. Remember that we use the notation $L\{x, y, z\}$ as an abbreviation for the formula: $L(x; y, z) \vee L(y; x, z) \vee L(z; x, y)$.

Lemma 3.1.3. *The group G has the following properties:*

(i) 3*-homogeneous.*

(ii) 2*-transitive.*

(iii) primitive.

(iv) 2*-primitive.*

(v) not 3*-transitive.*

(vi) not 4*-homogeneous.*

Proof.

(i) Let $A = \{x, y, z\}$ and $A' = \{x', y', z'\}$ be 3-element subsets of M. Then by Lemma 2.1.4, $A \vDash L\{x, y, z\}$ and $A' \vDash L\{x', y', z'\}$. Observe that the induced structures on A and A' lie in $\mathscr{D}$, since any 3-element substructure of any member of $\mathscr{D}$ lies in $\mathscr{D}$, and M is a union of a chain of members of $\mathscr{D}$. Then, without loss of generality, let $L(x; y, z)$ and $L(x'; y', z')$ hold in the root D-set of A and A' respectively. It is easily seen that the map $g : A \to A'$ with $(x, y, z)^g = (x', y', z')$ is an isomorphism. Hence, by Lemma 2.3.7, g extends to some $g' \in G$.

(ii) Suppose $x, y, x', y' \in M$ with $x \neq y$ and $x' \neq y'$. Let A be the induced structure on $\{x, y\}$, and A' be that on $\{x', y'\}$. Then $A, A' \in \mathscr{D}$ (the structure trees have just the root, with a 2-element D-set), and the map $g : A \to A'$ given by $(x, y)^g = (x', y')$ is an isomorphism. By Lemma 2.3.7 g extends to an element of G, as required.

(iii) This follows from (*ii*), by Lemma 1.1.18.

(iv) Since G is 2-transitive, it remains to check that for the point a the group G_a is primitive on $M \setminus \{a\}$ (this is by Definition 1.1.19). For that we are going to show that there is no proper non-trivial G_a-congruence on $M \setminus \{a\}$. So for the fixed a it suffices for us to show the following are not equivalence relations:

(a) $E_a(x,y) \Leftrightarrow L(a;x,y) \vee x = y$. It is not an equivalence relation because the transitivity (in the sense of equivalence relation) is violated. Indeed, assume $L(a;x,y) \wedge L(a;y,z)$. We may choose a, y, x to be distinct at a ramification point r with a as a special, and z lies in the same branch as x as in the following picture (in a finite substructure of M).

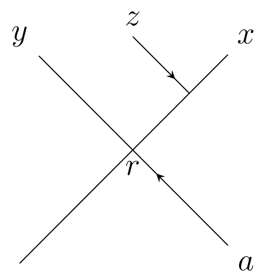

Figure 3.3

Therefore, $\neg L(a;x,z) \wedge x \neq z$. So E_a is not a transitive relation.

(b) $F_a(x,y) \Leftrightarrow L(x;a,y) \vee x = y$. It is not an equivalence relation because $L(x;a,y)$ does not imply $L(y;a,x)$, which means that x, y cannot be exchanged while a is fixed, so E_a is not a symmetric relation.

(c) $F'_a(x,y) \Leftrightarrow L(x;a,y) \vee L(y;a,x) \vee x = y$. This is not an equivalence relation, for in Figure 3.4 below we have $F'_a(x,y) \wedge F'_a(y,z) \wedge \neg F'_a(x,z)$.

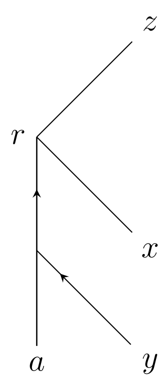

Figure 3.4

(v) This follows by Lemma 2.1.4 because there is a structure in $\mathscr{D}$ with the elements

$\{x, y, z\}$ such that $L(x; y, z)$ holds, and then there is no $g \in G$ such that $(x, y, z)^g = (y, x, z)$.

(vi) Let $A = \{x, y, z, w\}$ with $S(x, y; z, w)$ and $L(x; y, z) \wedge L(x; y, w) \wedge L(z; w, x) \wedge L(z; w, y)$, see Figure 3.5 for the root D-set. Consider another finite structure $A' = \{x', y', z', w'\}$ such that $S(x', y'; z', w')$ holds with $L(z'; x', y') \wedge L(z'; x', w') \wedge L(z'; w', y')$, as in Figure 3.6 below. If there is $g \in G$ with $\{x, y, z, w\}^g = \{x', y', z', w'\}$, then because of the relation S we must have $\{x, y\}^g = \{x', y'\}$ or $\{x, y\}^g = \{z', w'\}$, and it is easily seen that either way, g does not preserve all the relations L, Q, R.

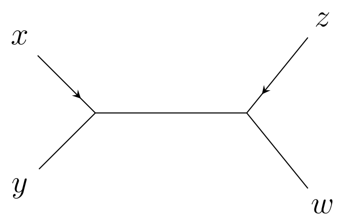

Figure 3.5

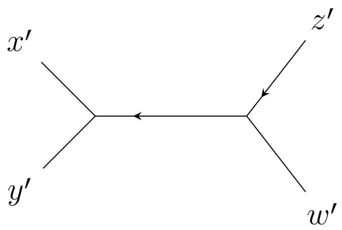

Figure 3.6

■

3.2 Construction

Definition 3.2.1. Define an 8-place relation P intended to describe the fact that the relations $S(x, y; z, w)$ and $S(p, q; u, v)$ *happen in the same* D-set as follows:

$(\forall x, y, z, w, p, q, u, v) P(x, y; z, w : p, q; u, v) \leftrightarrow S(x, y; z, w) \wedge S(p, q; u, v) \wedge (\forall t)(S'(x, y; z, w; t) \leftrightarrow S'(p, q; u, v; t))$.

It is easily seen that if $A \in \mathscr{D}$ then $A \vDash S(x, y; z, w) \wedge S(p, q; u, v) \wedge (\forall t)(S'(x, y; z, w; t) \leftrightarrow S'(p, q; u, v; t))$ if and only if $S(x, y; z, w)$ and $S(p, q; u, v)$ are witnessed in the same D-set of A.

Lemma 3.2.2. *If $A, C \in \mathscr{D}$ with $A < C$ and $x, y, z, w, p, q, s, t \in A$, then*

$$A \vDash P(x, y; z, w : p, q; s, t) \Leftrightarrow C \vDash P(x, y; z, w : p, q; s, t)$$

Proof. ($\Leftarrow$) Since the formula defining P is universal, and it holds in the bigger structure, it necessarily holds in the substructure A, and the result follows.

($\Rightarrow$) Let C be a one point extension of A such that $C = A \cup \{e\}$. We argue that in all possible cases the relation P is preserved, from A to C. This suffices by Lemma 2.2.9.

Case (i) If C is a Type I extension of A, then in the D-set of the new root of C the elements x, y, z, w, p, q, s, t meet at the unique ramification point (the centre) and a new added special branch, say e, and all of them are in correspondence with the elements in the root D-set of A. Since $P(x, y; z, w : p, q; s, t)$ holds in A then $S(x, y; z, w)$ and $S(p, q; , s, t)$ hold in the same D-set of A. As C is a one point extension of A then by the argument in the proof of Lemma 2.2.3 those two S-relations hold in the same D-set of C, as C contains all of A and one added element.

Case (ii) If C is a Type II extension then, by 2.2.4, $D_C(A) = D(A)$. Since $A \vDash P(x, y; z, w : p, q; s, t)$, $S(x, y; z, w)$ and $S(p, q; s, t)$ are witnessed in the same D-set of A. If this is the root D-set of A, then clearly they are witnessed in the same (root) D-set of C, so $C \vDash P(x, y; z, w : p, q; t, s)$. If it is not the root D-set of A, then in $D(\rho_A)$, the elements x, y, z, w, p, q, s, t all lie in distinct branches of a ramification point r, with none of these branches special, so there is a further element $u \in A$ special at r. There are now various cases, according to whether e is added in a new branch at r, in the same branch as u, in the same branch as one of x, y, z, w, p, q, s, t, or in the same branch at r as some other element of A (and whether it is of Type II(a) or Type II(b)). In each case, we find $C \vDash P(x, y; z, w : p, q; s, t)$, as required. ∎

Note. It follows from Lemma 3.2.2 that if $A < M$ with $A \in \mathscr{D}$, and $x, y, z, w, p, q, s, t \in A$, then $A \vDash P(x, y; z, w : p, q; t, s)$ if and only if $M \vDash P(x, y; z, w : p, q; s, t)$.

Lemma 3.2.3. *(i) Let $S^* := \{(x, y, z, w) \in M^4 : M \vDash S(x, y; z, w)\}$. The relation P defines an equivalence relation on S^*.*

(ii) Let $K := \{(x, y, z) \in M^3 : M \vDash L(x; y, z)\}$. The relation R defines an equivalence relation on K.

Proof.

(i) That, P is an equivalence relation on S^* is obtained directly from the Definition 3.2.1.

(ii) Similar to the proof of (i) using Lemma 3.1.2(i).

∎

Next we recover the tree of D-sets of M using the relation P.

Definition 3.2.4. Let $x, y, z, w \in M$. Given $S(x, y; z, w)$ define I_{xyzw} to be the set of all points of M which belong to the D-set of M in which $S(x, y; z, w)$ is witnessed. That is $I_{xyzw} = \{t : P(x, y; z, w : t, y; z, w) \vee P(x, y; z, w : x, t; z, w) \vee P(x, y; z, w : x, y; t, w) \vee P(x, y; z, w : x, y; z, t)\}$.

It describes the following picture where the dashed lines are the possible places of t:

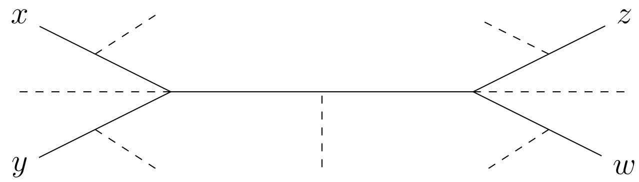

Figure 3.7: I_{xyzw}

Basically, I_{xyzw} consists of all points t such that there is some $A \in \mathscr{D}$ with $x, y, z, w, t \in A < M$ and $S(x, y; z, w)$ and some relation such as $S(t, y; z, w)$ witnessed in the root D-set of A. We call I_{xyzw} a *pre D-set*.

Lemma 3.2.5. *(i) For any pair of quadruples (x, y, z, w) and (p, q, s, t) such that $M \models P(x, y; z, w : p, q; s, t)$, $I_{xyzw} = I_{pqst}$.*

(ii) Let $(x, y, z, w), (p, q, s, t) \in S^*$. If $I_{xyzw} = I_{pqst}$, then $M \models P(x, y; z, w : p, q; s, t)$.

Proof.

(i) Suppose $a \in I_{xyzw}$. We want to show that $a \in I_{pqst}$. Then as $M \models P(x, y; z, w : p, q; s, t)$ there is a finite structure $A \in \mathscr{D}$ containing the points $x, y, z, w, p, q, s, t, a$, and since P is universal $A \models P(x, y; z, w : p, q; s, t)$. This implies that $S(x, y; z, w)$ and $S(p, q; s, t)$ happen in the same D-set of A. But we assume that $a \in I_{xyzw}$, so wherever $S(x, y; z, w)$ holds, one of the disjunctions of I_{xyzw} holds, without loss of generality $P(x, y; z, w : x, a; z, w)$ say. As P is an equivalence relation, then $P(p, q; s, t : x, a; w, z)$ is witnessed. Hence a occurs in the D-set witnessing $S(p, q; s, t)$ so $a \in I_{pqst}$ because wherever it lives it will satisfy one formula of Definition 3.2.4. The result now follows by symmetry.

(ii) Suppose $I_{xyzw} = I_{pqst}$ but $M \models \neg P(x, y; z, w : p, q; s, t)$. Then $M \models S(x, y; z, w) \wedge S(p, q; s, t)$ and without loss of generality there is $u \in M$ such that $M \models S'(x, y; z, w; u) \wedge \neg S'(p, q; s, t; u)$. Choose a finite $A < M$ with $x, y, z, w, p, q, s, t, u \in A$. Then the D-set of A witnessing $S(x, y; z, w)$, say $D(\mu)$, omits u but the D-set $D(\nu)$ witnessing $S(p,q;s,t)$ contains u, so $\nu < \mu$ or ν, μ are incomparable. Either way, we find $u \in I_{pqst} \setminus I_{xyzw}$, a contradiction.

■

We refer to the equivalence classes of P on S^* as *vertices*, and denote the P-class containing (x,y,z,w) as $\langle xyzw \rangle$. We define a partial order $\leq$ on S^*/P by reverse inclusion; that is $\langle xyzw \rangle \leq \langle pqrs \rangle$ if and only if $I_{xyzw} \supseteq I_{pqrs}$. It is immediate from Lemma 3.2.5 that this is well defined. We shall show later (Lemma 3.2.17) that $(S^*/P, \leq)$ is a dense semilinear order without maximal or minimal elements. Before that, we aim to associate a D-set with each vertex of S^*/P. First, it is useful to show that vertices can be identified with triples (via L and R) as well as quadruples (via S and P).

Definition 3.2.6. Let $p,q,s \in M$. Given $L(p;q,s)$, define J_{pqs} to be $\{j : R(p;q,s : j;q,s) \vee R(p;q,s : p;j,s) \vee R(p;q,s : p;j,q) \vee [R(p;q,s : p;j,q) \wedge R(p;q, s : p;j,s)]\}$.

Intuitively, J_{pqs} is the set of all the points of M which belong to the (pre)-D-set of M in which $L(p;q,s)$ is witnessed.

In the following picture, we show all the possible positions that j might be in.

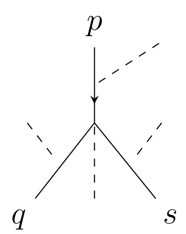

Figure 3.8: J_{pqs}

Lemma 3.2.7. *Let* $x,y,z,w,p,q,s \in M$. *Then*

(i) $M \models Q(x,y;z,w : p;q,s) \Leftrightarrow I_{xyzw} = J_{pqs}$.

(ii) $M \models R(x;y,z : p;q,s) \Leftrightarrow J_{xyz} = J_{pqs}$.

Proof.

(i) $\Rightarrow$) Let $a \in I_{xyzw}$, we want $a \in J_{pqs}$. There is a finite substructure $A < M$, $A \in \mathscr{D}$ containing x, y, z, w, p, q, s, a. Then as $A \vDash Q(x, y; z, w : p; q, s)$, $S(x, y; z, w)$ and $L(p; q, s)$ are witnessed in the same D-set of A. But $a \in I_{xyzw}$, so without loss of generality, suppose $P(x, y; z, w : x, a; z, w)$ holds. Then we have $A \vDash P(x, y; z, w : x, a; z, w) \wedge L(p; q, s)$, so $S(x, a; z, w)$ and $L(p; q, s)$ are witnessed in the same D-set of A, and wherever the position of a, one of the disjunctions of Definition 3.2.6 holds, so $a \in J_{pqs}$. To show that $J_{pqs} \subseteq I_{xyzw}$ let $b \in J_{pqs}$, then without loss of generality $R(p; q, s : b; q, s)$ holds. Let $A \in \mathscr{D}$ contain x, y, z, w, p, q, s, b, so $L(p; q, s)$ and $L(b; q, s)$ in the same D-set of A. But we are given that $S(x, y; z, w) \wedge L(p; q, s)$ in the same D-set of A. So whenever b lies we will have at least one of the disjunctions of the definition of I_{xyzw} hence $b \in I_{xyzw}$.

$\Leftarrow$) Assume, for a contradiction, that $I_{xyzw} = J_{pqs}$ but $M \vDash \neg Q(x, y; z, w : p; q, s)$. Pick a finite $A < M$ with $x, y, z, w, p, q, s \in A \in \mathscr{D}$. Then by 3.1.2(iii), $S(x, y; z, w)$ and $L(p; q, s)$ are witnessed in distinct D-sets of A. If $L(p; q, s)$ is witnessed in a D-set D_μ of A below the D-set D_ν where $S(x, y; z, w)$ is witnessed, then the D-set D_μ contains an element t such that $A \vDash S'(x, y; z, w; t)$. We find $t \in J_{pqs} \setminus I_{xyzw}$. Likewise, if $S(x, y; z, w)$ is witnessed in A below $L(p; q, s)$, then we find $t \in I_{xyzw} \setminus J_{pqs}$.

Now suppose that $S(x, y; z, w)$ and $L(p; q, s)$ hold in M in two incomparable D-sets. Then for the finite structure A let p, q, s, t_1 be distinct at a ramification point r with t_1 the special branch and x, y, z, w, t_2 are distinct at a ramification point r' with t_2 the special branch. At a higher level there will be two incomparable D-sets that witness $S(x, y; z, w)$ and $L(p; q, s)$ and $t_1 \in I_{xyzw} \setminus J_{pqs}$ and $t_2 \in J_{pqs} \setminus I_{xyzw}$ respectively, a contradiction.

(ii) $\Rightarrow$) To show that $J_{xyz} \subset J_{pqs}$, let $b \in J_{xyz}$, so we want $b \in J_{pqs}$. There is a finite substructure $A < M$, $A \in \mathscr{D}$ containing x, y, z, p, q, s, b. Since R is in the language,

$A \vDash R(x; y, z : p; q, s)$, so $L(x; y, z)$ and $L(p; q, s)$ happen in the same D-set of A. But $b \in J_{xyz}$ so, without loss of generality, let $R(x; y, z : b; y, z)$ hold. Then we have $A \vDash R(x; y, z : b; y, z) \wedge R(x; y, z : p; q, s)$, so by the transitivity in the sense of equivalence relations, this implies $R(p; q, s : b; q, s)$, i.e. $b \in J_{pqs}$. Similarly we show that $J_{pqs} \subset J_{xyz}$.

$\Leftarrow$) Assume, for a contradiction $J_{xyz} = J_{pqs}$ and $\neg R(x; y, z : p; q, s)$. Then there is a finite substructure $A \in \mathscr{D}$ such that $x, y, z, p, q, s \in A < M$. Since $A \vDash \neg R(x; y, z : p; q, s)$, $L(x; y, z)$ and $L(p; q, s)$ happen in different D-sets of A. We suppose first they are comparable, so (without loss of generality) there is $t \in A$ such that $A \vDash L'(x; y, z; t)$ and t lies in the D-set of A witnessing $L(p; q, s)$. We see easily that $t \in J_{pqs} \smallsetminus J_{xyz}$. On the other hand, if $L(p; q, s)$ and $L(x; y, z)$ happen in two incomparable D-sets, then a lower D-set contains p, q, s in distinct branches at a ramification point, r say, with t_1 as the special branch and x, y, z are in distinct branches at another ramification point, r' say, with t_2 as the special branch. So higher up in the structure tree we have $L'(p; q, s; t_1)$ and $L'(x; y, z; t_2)$ hence $t_2 \in J_{pqs} \smallsetminus J_{xyz}$.

■

The importance of the previous lemma is to give us the ability to express any vertex in $(S^*/P, \leq)$ by a triple, and then any D-set can be referred to by that triple. Indeed, by 3.2.7(ii), R is an equivalence relation on the set of all triples of M satisfying L, and by 3.2.7(i), each R-class can be identified with a P-class, i.e. a vertex. Thus, if $I_{xyzw} = J_{pqs}$ then we may refer to the vertex $\langle xyzw \rangle$ of S^*/P as $\langle pqs \rangle$. Then $\langle pqs \rangle = \langle p'q's' \rangle$ if and only if $M \vDash R(p; q, s : p'; q', s')$.

Lemma 3.2.8. *Let $x, y, z, u, v, w \in M$. Then $J_{xyz} = J_{uvw} \Leftrightarrow \langle xyz \rangle = \langle uvw \rangle$.*

Proof. $J_{xyz} = J_{uvw}$ if and only if $R(x; y, z : u; v, w)$ (by Lemma 3.2.7(ii)) if and only if $\langle xyz \rangle = \langle uvw \rangle$ (as noted above). ■

Definition 3.2.9. Define a relation E_{xyzw} on I_{xyzw} such that

$$uE_{xyzw}v \Leftrightarrow [(\forall l)(\forall m)(\forall n)P(x,y;z,w:l,m;n,u) \leftrightarrow P(x,y;z,w:l,m;n,v)))].$$

Lemma 3.2.10. *(i) Suppose $I_{xyzw} = I_{x'y'z'w'}$. Then $E_{xyzw} = E_{x'y'z'w'}$.*

(ii) E_{xyzw} is an equivalence relation on I_{xyzw}, and is invariant under $G_{\{I_{xyzw}\}}$.

Proof.

(i) By Lemma 3.2.5, we have $P(x,y;z,w:x',y';z',w')$. Since P is an equivalence relation on S^*, the result follows immediately.

(ii) Both assertions are immediate from the definition of E_{xyzw}, noting part (i) and that P is G-invariant.

■

Definition 3.2.11. (i) Given I_{xyzw} and E_{xyzw}, define R_{xyzw} to be the quotient I_{xyzw}/E_{xyzw}, so elements of R_{xyzw} are E_{xyzw}-classes of elements of M. We use the notation $[m]$ to refer to the element of R_{xyzw} containing the element $m \in M$ (when the underlying equivalence relation E_{xyzw} is clear). We call the elements of R_{xyzw} *directions*.

(ii) Let $[u],[v],[t],[s] \in R_{xyzw}$. Write $D_{xyzw}([u],[v];[t],[s]) \Leftrightarrow ([u]=[v] \wedge [u] \notin \{[s],[t]\}) \vee ([t]=[s] \;\wedge\; [t] \notin \{[u],[v]\}) \vee P(x,y;z,w:u,v;t,s)$.

(iii) We call the elements of R_{xyzw} *directions*, when viewed as elements of R_{xyzw}, and *pre-directions*, when viewed as subsets of M.

We will see that R_{xyzw} is precisely the D-set associated with the vertex $\langle xyzw \rangle$ of the structure tree of M.

Lemma 3.2.12. *(i) The relation D_{xyzw} is well-defined on R_{xyzw}.*

(ii) The structure (R_{xyzw}, D_{xyzw}) *is a dense proper* D*-set.*

(iii) The relation D_{xyzw} *is* $G_{\{I_{xyzw}\}}$*-invariant.*

Proof.

(i) Suppose $[u],[v],[t],[s] \in R_{xyzw}$ are distinct, and $u' \in [u], v' \in [v], t' \in [t]$ and $s' \in [s]$. We have

$$(\forall m \forall n\ \forall l)(P(x,y;z,w:m,n;l,u) \leftrightarrow P(x,y;z,w:m,n;l,u'))$$

$$(\forall m \forall n\ \forall l)(P(x,y;z,w:m,n;l,v) \leftrightarrow P(x,y;z,w:m,n;l,v'))$$

$$(\forall m \forall n\ \forall l)(P(x,y;z,w:m,n;l,t) \leftrightarrow P(x,y;z,w:m,n;l,t'))$$

$$(\forall m \forall n\ \forall l)(P(x,y;z,w:m,n;l,s) \leftrightarrow P(x,y;z,w:m,n;l,s'))$$

Thus, (using symmetry conditions on the variables in P) $P(x,y;z,w:u,v;t,s) \leftrightarrow P(x,y;z,w:u',v;t,s) \leftrightarrow P(x,y;z,w:u',v';t,s) \leftrightarrow P(x,y;z,w:u',v';t',s) \leftrightarrow P(x,y;z,w:u',v';t',s')$, as required.

(ii) We want to show that conditions $(D1)$ – $(D6)$ of Definition 1.4.16 hold. Axioms $(D1)$, $(D2)$, $(D3)$ and $(D4)$ follow immediately from corresponding conditions on S, inherited via P. For $(D5)$, suppose that $[u],[v],[t] \in R_{xyzw}$ are distinct. Pick finite $A \in \mathscr{D}$ with $x,y,z,w,u,v,t \in A < M$. We may suppose that $S(x,y;z,w)$ is witnessed in the root D-set of A. By semi-homogeneity, A has a Type II(b) extension $A < A' = A \cup \{s\}$ such that $S(u,v;t,s)$ is witnessed in the root D-set of A', with $A' < M$. Then $M \vDash P(x,y;z,w:u,v;t,s)$, and we have $D_{xyzw}(u,v;t,s)$.

The argument is similar for $(D6)$. Suppose $[u],[v],[t],[s] \in R_{xyzw}$ with $D_{xyzw}([u],[v];[t],[s])$, and for convenience we suppose them distinct. Pick $A \in \mathscr{D}$ with $x,y,z,w,u,v,t,s \in A < M$. Then $A \vDash P(x,y;z,w:u,v;t,s)$, and we may suppose $S(x,y;z,w)$ and $S(u,v;t,s)$ are witnessed in the root D-set of A. Now, by semi-homogeneity (Remark 2.3.8), A has a Type II(b) extension $A < A' = A \cup \{a\} < M$, as depicted in Figure 3.9.

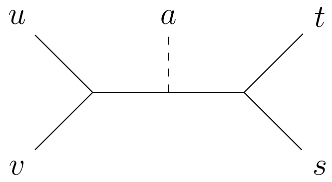

Figure 3.9

Then

$$A \vDash S(a, v; t, s) \wedge S(u, a; t, s) \wedge S(u, v; a, s) \wedge S(u, v; t, a),$$

so in M we have (putting $D = D_{xyzw}$)

$$D([a], [u]; [t], [s]) \wedge D([u], [a]; [t], [s]) \wedge D([u], [v]; [a], [s]) \wedge D([u], [v]; [t], [a])$$

as required.

(iii) Suppose $D_{xyzw}([u], [v]; [t], [s])$. Then $P(x, y; z, w : u, v; t, s)$. Let $g \in G_{\{I_{xyzw}\}}$, and let $x' = x^g, y' = y^g, z' = z^g, w' = w^g, u' = u^g, v' = v^g, t' = t^g, s' = s^g$. Then $I_{xyzw} = I_{x'y'z'w'}$ so $P(x, y; z, w : x', y'; z', w')$. Also as g preserves P, $P(x', y'; z', w' : u', v'; t', s')$. So as P is an equivalence relation, $P(x, y; z, w : u', v'; t', s')$, hence $D_{xyzw}([u'], [v']; [t'], [s'])$ as required.

■

Definition 3.2.13. Define an equivalence relation E_{pqs} on J_{pqs}, putting

$$uE_{pqs}v \Leftrightarrow (\exists x, y, z, w)(J_{pqs} = I_{xyzw} \wedge uE_{xyzw}v).$$

Observe that if $J_{pqs} = J_{p'q's'}$ then $E_{pqs} = E_{p'q's'}$.

Lemma 3.2.14. *The relation E_{pqs} is an equivalence relation on J_{pqs}, and equals E_{xyzw} where $J_{pqs} = I_{xyzw}$.*

Proof. This is immediate. For example, to see transitivity, suppose $tE_{pqs}u$ and $uE_{pqs}v$. Then there are $x, y, z, w, x', y', z', w'$ such that $J_{pqs} = I_{xyzw} = I_{x'y'z'w'}$ and $tE_{xyzw}u$ and

$uE_{x'y'z'w'}v$. Since $I_{xyzw} = I_{x'y'z'w'}$, by Lemma 3.2.10(i) we have $E_{xyzw} = E_{x'y'z'w'}$ so $uE_{xyzw}v$ and hence $tE_{xyzw}v$, where $tE_{pqs}v$. ■

Definition 3.2.15. Given J_{pqs} and the equivalence relation E_{pqs} on it, we define R_{pqs} to be the quotient J_{pqs}/E_{pqs}. Elements of R_{pqs} are the E_{pqs}-classes of points of M. As before, we use the notation $[m]$ to refer the element of R_{pqs} containing the element $m \in M$.

We will see that R_{pqs} is precisely the D-set associated with the vertex $\langle pqs \rangle$ of the structure tree of M. We call the elements of R_{pqs} *directions*.

The above equivalence relation allows us to refer to the D-set that witnesses I_{xyzw} by J_{pqs} where $Q(x, y; z, w : p; q, s)$ holds.

Lemma 3.2.16. *(i) If $L(p; q, s)$ holds in M then there are $x, y, z, w \in M$ such that $Q(x, y; z, w : p; q, s)$.*

(ii) If $S(x, y; z, w)$ holds in M then there are $p, q, s \in M$ such that $Q(x, y; z, w : p; q, s)$.

Proof.

(i) First observe that the induced L-structure on $\{p, q, s\}$ lies in $\mathscr{D}$. Pick $A < M$ with $A \in \mathscr{D}$, containing distinct elements $p', q', s', x', y', z', w'$ such that $L(p'; q', s')$ and $S(x', y'; z', w')$ are witnessed in the root D-set of A. Then $A \vDash Q(x', y'; z', w' : p'; q', s')$ so $M \vDash Q(x', y'; z', w' : p'; q', s')$. By 3-homogeneity (Lemma 3.1.3(i)) there is $g \in G$ with $(p', q', s')^g = (p, q, s)$. Put $x := x'^g, y := y'^g, z = z'^g, w := w'^g$. Then $M \vDash Q(x, y; z, w : p; q, s)$, as required.

(ii) Similar to (i).

■

It follows from Lemmas 3.2.7 and 3.2.16 that if $L(p; q, s)$ then there are $x, y, z, w \in M$ such that $J_{pqs} = I_{xyzw}$. Furthermore, by Lemma 3.2.14 E_{pqs} then equals E_{xyzw}. Thus, we may identify R_{pqs} with R_{xyzw}. As noted in Lemma 3.2.12, R_{xyzw} carries a D-set structure. If $Q(x,y;z,w:p;q,s)$, we say that $L(p;q,s)$ is *witnessed* in the D-set R_{xyzw} (or equivalently, in the D-set R_{pqs}).

Likewise, $S(x,y;z,w)$ is then witnessed in the D-set R_{xyzw}, or likewise in R_{pqs}. If $L(p;q,s)$ is witnessed in R_{xyzw}, and the E_{xyzw}-classes of p,q,s lie in distinct branches at the ramification point r, we say that the branch at r containing p/E_{xyzw} is the *special branch* at r. Let $\text{Ram}(R_{xyz})$ denote the set of ramification points of R_{xyz}. If $a_1,\ldots,a_n \in R_{xyz}$ (for $n \geq 3$) lie in distinct branches at $r \in \text{Ram}(R_{xyz})$, we write $r = \text{ram}(a_1,\ldots,a_n)$.

Lemma 3.2.17. *The partial order $(S^*/P,\leq)$ is a lower semilinear order. Furthermore, and if $\langle xyzw\rangle,\langle pqts\rangle$ are incomparable elements there is a vertex $\langle abcd\rangle$ such that $\langle abcd\rangle = \inf\{\langle xyzw\rangle,\langle pqts\rangle\}$, so S^*/P is a meet-semilattice. In addition, it has no maximal or minimal elements, and is dense.*

Proof. We show the semilinearity via Claims 1 and 2 below:

Claim 1. Given two sets I_{xyzw} and I_{pqts} such that no one contains the other then there is another set I_{abcd} containing both.

Proof. Let $A < M$ with $x,y,z,w,p,q,t,s \in A \in \mathscr{D}$. Then (by putting additional points into A if necessary) $S(x,y;z,w)$ and $S(p,q;t,s)$ are witnessed in incomparable D-sets of A, and we may suppose these lie in cones (at the root ρ of A) corresponding to distinct ramification points r_1,r_2 of $D(\rho)$. There are $a,b,c,d \in A$ with $S(a,b;c,d)$ witnessed in $D(\rho)$, such that x,y,z,w are in distinct branches at the ramification point r_1 and p,q,t,s are in distinct distinct branches at r_2, as depicted.

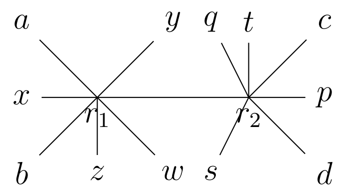

Figure 3.10: D_ρ

We claim $I_{xyzw} \subseteq I_{abcd}$. Let $v \in I_{xyzw}$. We may suppose $v \in A$. Then, without loss of generality, we have $P(x,y;z,w : x,v;z,w)$. Since v appears in a higher level, that means it is in a distinct branch at $r_1 = \text{ram}(x,y,z,w)$, and since a,b,c,d are in the root then x,z,w,v will be in distinct branches at r_1. This implies that wherever v lives in the root and since we have $S(a,b;c,d)$ then $v \in I_{abcd}$ so $I_{xyzw} \subset I_{abcd}$. Similarly we have $I_{pqts} \subset I_{abcd}$. Moreover, this shows that $\langle abcd \rangle$ is the infimum in S^*/P of $\langle xyzw \rangle$ and $\langle pqts \rangle$.

■

Claim 2. Assume $\langle xyzw \rangle$ and $\langle pqts \rangle$ are incomparable. There is no I_{lmno} contained in both I_{pqts} and I_{xyzw}.

Proof. Assume there are $l,m,n,o \in M$ with $I_{lmno} \subseteq I_{xyzw} \cap I_{pqst}$. By assumption, there are $a,b \in M$ with $a \in I_{xyzw} \setminus I_{pqst}$ and $b \in I_{pqst} \setminus I_{xyzw}$. Let $A < M$ be finite with $x,y,z,w,p,q,s,t,l,m,n,o,p \in A \in \mathscr{D}$. Let $S(x,y;z,w)$ and $S(p,q;s,t)$ be witnessed in A by D-sets D_{ν_1} and D_{ν_2} respectively, and $S(l,m;n,o)$ by D_μ. Then ν_1, ν_2 are incomparable (due to the existence of a,b) but $\mu \geq \nu_1$ and $\mu \geq \nu_2$ (as $I_{lmno} \subseteq I_{xyzw} \cap I_{pqst}$). This is impossible, as the structure tree of A is semilinearly ordered. ■

Claim 3. $(S^*/P, \leq)$ has no greatest or least element, and it is dense where the density here is in the sense of semilinear orders.

Proof. Let $\langle xyzw \rangle \in S^*/P$. We may suppose that the structure induced on $\{x,y,z,w\}$ lies in $\mathscr{D}$. Choose a structure $A \in \mathscr{D}$ containing x',y',z',w',p,q,s,t as depicted, (in the root D-set):

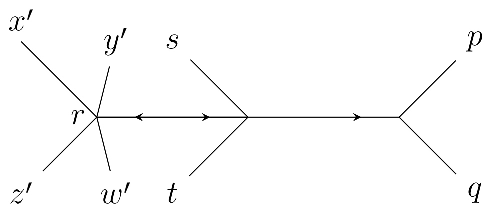

Figure 3.11: D_{ρ_A}

so the map $(x, y, z, w) \longmapsto (x', y', z', w')$ is an (L, S)-isomorphism, $S(x', y'; z', w')$ is witnessed in the successor D-set D_μ of the root D_ρ of A corresponding to the ramification point r, and p, q, s, t are as shown in D_ρ. We may suppose also A_r is isomorphic to the structure induced on $\{x, y, z, w\}$ via an isomorphism ϕ inducing $(x', y', z', w') \longmapsto (x, y, z, w)$.

We may suppose $A \leq M$. Then by semi-homogeneity, ϕ extends to some $g \in G$. Clearly $\langle pqst \rangle < \langle x'y'z'w' \rangle$, and it follows that $\langle p^g q^g s^g t^g \rangle < \langle xyzw \rangle$, as required. A similar argument shows that S^*/P has no greatest element under $\leq$.

The argument for density is a similar application of semi-homogeneity. Assume $\langle xyzw \rangle < \langle pqst \rangle$. We may find a finite substructure A of M containing x, y, z, w, p, q, s, t, such that $S(x, y; z, w)$ is witnessed in the root D-set D_ρ of A, which has a ramification point r at which p, q, s, t lie in distinct non-special branches. We may suppose that there are $l, m, n, o \in A$ such that p, q, s, t, l, m, n, o all lie in distinct non-special branches at r, that $S(l, m; n, o)$ is witnessed in the successor D_μ corresponding to r, and that p, q, s, t lie in distinct non-special branches at a ramification point of D_μ. It follows that $\langle xyzw \rangle < \langle lmno \rangle < \langle pqst \rangle$, as required.

■

At this point we know that M has an interpretable meet semi-lattice which is dense with no maximal or minimal elements, that each vertex is coded by a quadruple $\langle xyzw \rangle$ and a triple $\langle pqs \rangle$, and that corresponding to each vertex $\langle pqs \rangle$ there is a dense D-set D_{pqs} with universe J_{pqs}/E_{pqs}.

Our next task is to identify analogues for M of the maps f_μ and $g_{\mu\nu}$ for members of $\mathscr{D}$.

We define a bijection $f_{\langle xyz \rangle}$ from the set of cones at $\langle xyz \rangle$ to $\mathrm{Ram}(R_{xyz})$. We first need the following lemma.

Lemma 3.2.18. *Suppose $x, y, z, p, q, s \in M$. Then*

(i) Let $\langle pqs \rangle < \langle xyz \rangle$. Then $E_{pqs}\big|_{J_{xyz}}$ refines E_{xyz}.

(ii) *Suppose* $\langle xyz \rangle < \langle pqs \rangle$. *Let* p, q, s *lie in distinct branches at the ramification point* r *of* R_{xyz}, *and let* $u, v \in J_{pqs}$ *be* E_{pqs}*-inequivalent. Then* u, v *lie in distinct branches at* r.

Proof.

(i) Let $u, v \in J_{xyz}$ with $uE_{pqs}v$. Then there are p', q', s', t' such that $J_{pqs} = I_{p'q's't'}$. Pick x', y', z', w' such that $J_{xyz} = I_{x'y'z'w'}$. We must show $uE_{x'y'z'w'}v$, that is, for every $l, m, n \in M$, $P(x', y'; z', w' : l, m; n, u) \Leftrightarrow P(x', y'; z', w' : l, m; n, v)$. This follows by considering finite $A \in \mathscr{D}$ with $A < M$ and A containing all the above elements.

(ii) We prove the contrapositive, so suppose we have in R_{xyz} a diagram such as the following.

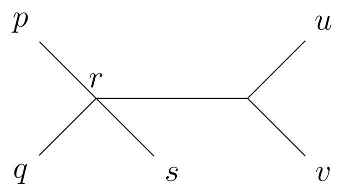

Figure 3.12

As in (i), pick p', q', s', t' such that $J_{pqs} = I_{p'q's't'}$. We must again show $uE_{p'q's't'}v$, that is, if $l, m, n \in M$, $P(p', q'; s', t' : l, m; n, u) \Leftrightarrow P(p', q'; s', t' : l, m; n, v)$. Again, this can be argued in finite substructures of M lying in $\mathscr{D}$.

∎

Now suppose $\langle xyz \rangle < \langle pqs \rangle$. Then by the last lemma p, q, s are inequivalent modulo E_{xyz}, so there is a ramification point r of R_{xyz} such that the E_{xyz}-classes of p, q, s lie in distinct branches at r. Put $f_{\langle xyz \rangle}(\langle pqs \rangle) = r$.

Lemma 3.2.19. (i) *In the above notation, the value of* $f_{\langle xyz \rangle}(\langle pqs \rangle)$ *depends only on the cone at* $\langle xyz \rangle$ *containing* $\langle pqs \rangle$.

(ii) *$f_{\langle xyz\rangle}$ determines a bijection between the set of cones at $\langle xyz\rangle$ and the set of ramification points of R_{xyz}.*

Proof.

(i) *Claim 1.* If $a, b \in J_{pqs}$ are inequivalent modulo E_{pqs}, then they lie in distinct branches at r.

Proof. This is immediate from Lemma 3.2.18(ii). ■

Claim 2. If $\langle xyz\rangle < \langle pqs\rangle < \langle p'q's'\rangle$ then $f_{\langle xyz\rangle}(\langle pqs\rangle) = f_{\langle xyz\rangle}(\langle p'q's'\rangle)$.

Proof. In this situation, p', q', s' are inequivalent modulo $E_{p'q's'}$ and hence modulo E_{pqs} (Lemma 3.2.18(i)) so lie in distinct branches at r. (by Claim 1). ■

Claim 3. If $\langle pqs\rangle$ and $\langle p'q's'\rangle$ are incomparable but in the same cone at $\langle xyz\rangle$, then $f_{\langle xyz\rangle}(\langle pqs\rangle) = f_{\langle xyz\rangle}(\langle p'q's'\rangle)$.

Proof. Pick p'', q'', s'' with $\langle xyz\rangle < \langle p''q''s''\rangle < \langle pqs\rangle$, and $\langle xyz\rangle < \langle p''q''s''\rangle < \langle p'q's'\rangle$. By Claim 2, $f_{\langle xyz\rangle}(\langle pqs\rangle = f_{\langle xyz\rangle}(\langle p''q''s''\rangle) = f_{\langle xyz\rangle}(\langle p'q's'\rangle)$. ■

Part (i) follows.

(ii) To see that $f_{\langle xyz\rangle}$ is surjective, let $r \in \text{Ram}(R_{xyz})$ and choose $p, q, s \in M$ such that modulo E_{xyz} they lie in distinct non-special branches at r. Then $\langle xyz\rangle < \langle pqs\rangle$, by considering finite substructures of M. It follows that $f_{\langle xyz\rangle}(\langle pqs\rangle) = r$.

For injectivity, suppose $\langle pqs\rangle$, $\langle p'q's'\rangle$ lie in distinct cones at $\langle xyz\rangle$. Suppose that there is a finite $A \in \mathscr{D}$ with $x, y, z, p, q, s, p', q', s' \in A < M$ such that p, q, s and p', q', s' meet at the same ramification point in the D-set in which $L(x; y, z)$ holds. Then (e.g. by considering a sequence of one-point extensions between them) this holds in any A' with $A < A' < M$. It follows by semi-homogeneity that there are $u, v, w \in M$ with $\langle xyz\rangle < \langle uvw\rangle$ and $\langle uvw\rangle < \langle pqs\rangle$ and $\langle uvw\rangle < \langle p'q's'\rangle$, so $\langle pqs\rangle$ and $\langle p'q's'\rangle$ lie in the same cone of S^*/P at $\langle xyz\rangle$.

■

■

Lemma 3.2.20. *Let $\langle xyz\rangle < \langle pqs\rangle$, and let $[m]$ be a pre-direction of R_{pqs}. Let $r := f_{\langle xyz\rangle}(\langle pqs\rangle) \in Ram(R_{xyz})$. Then there is a unique set t of branches of R_{xyz} at r such that $[m] = \cup\cup t$.*

Proof. Consider a finite structure $A \in \mathscr{D}$ containing x, y, z, p, q, s, m. Consider the vertex $\langle xyz\rangle < \langle pqs\rangle$, and $r := f_{\langle xyz\rangle}(\langle pqs\rangle)$. Let t be the set of branches $\{t_1, \dots, t_n\}$ at r that corresponds to the direction $[m]$. Each pre-branch of t_i, $i \in \{1, \dots, n\}$ consists of a collection of pre-directions, say

$$t_1 = \{u_1^{(1)}, u_2^{(1)}, \dots, u_{m_1}^{(1)}\}, m_1 \in \mathbb{N}$$
$$t_2 = \{u_1^{(2)}, u_2^{(2)}, \dots, u_{m_2}^{(2)}\}, m_2 \in \mathbb{N}$$
$$\vdots$$
$$t_n = \{u_1^{(n)}, u_2^{(n)}, \dots, u_{m_n}^{(n)}\}, m_n \in \mathbb{N}$$

then

$$\bigcup t = \bigcup_{i=1}^{n} t_i = t_1 \cup \dots \cup t_n = \{u_1^{(1)}, u_2^{(1)}, \dots, u_{m_1}^{(1)}, \dots, u_1^{(n)}, u_2^{(n)}, \dots, u_{m_n}^{(n)}\}$$

hence $[m] \cap A = (\bigcup\bigcup\{t_1, \dots, t_n\}) \cap A$. Since this holds for any $A' \in \mathscr{D}$ with $A < A' < M$, the result follows. ■

Define $g_{\langle pqs\rangle\langle xyz\rangle}([m]) = t$ where $[m] = \cup\cup t$. So $g_{\langle pqs\rangle\langle xyz\rangle}$ is a map from the directions of R_{pqs} to the power set of the set of branches at r. From the definition we see that if $[m] \neq [m']$ then $g_{\langle pqs\rangle\langle xyz\rangle}([m]) \cap g_{\langle pqs\rangle\langle xyz\rangle}([m']) = \varnothing$.

Lemma 3.2.21. *The map $g_{\langle pqs\rangle\langle xyz\rangle}$ is well defined.*

Proof. The point essentially is that in the proof of Lemma 3.2.20, if $\langle pqs\rangle = \langle p'q's'\rangle$, then in any finite structure $A \in \mathscr{D}$ with $x, y, z, p, q, s, p', q', s', m \in A < M$, the set t of branches depends just on the direction of m in the D-set witnessing $L(p;q,s)$, and on the maps $g_{\mu\nu}$ where μ codes in A the D-set witnessing $L(p;q,s)$, and ν the D-set witnessing $L(x;y,z)$. ∎

Proposition 3.2.22. *(i) The group $G_{\langle xyz\rangle}$ is transitive on the internal nodes of R_{xyz}. i.e. on the set of ramification points in R_{xyz}.*

(ii) (a) The stabiliser $G_{\langle xyz\rangle}$ is transitive on J_{xyz}.

(b) The group G is transitive on the semilinear order S^*/P.

(c) The group G is transitive on the set $\mathscr{X}$, where $\mathscr{X} = \bigcup_{xyzw} R_{xyzw}$, the union of all the sets of directions in the structure M.

(iii) The group $G_{\{J_{xyz}\}}$ induces a 2-transitive group on the set of directions of R_{xyz}, i.e. is transitive on the set of pairs of distinct directions.

(iv) The equivalence relation E_{xyzw} is the unique maximal $G_{\{I_{xyzw}\}}$-congruence on I_{xyzw}.

Proof.

(i) Assume r, r' are two ramification points of R_{xyz} with x, y, z and p, q, s as triples lying in distinct branches around them respectively with $L(p;q,s)$ witnessed in R_{xyz}. We want to find some $g \in G_{(\langle xyz\rangle)}$ such that $r^g = r'$. Then $R(x;y,z:p;q,s)$ holds. By 3-homogeneity there is $g \in G$ such that $\{x,y,z\}^g = \{p,q,s\}$. As $L(x;y,z)$ and $L(p;q,s)$ hold, then $x^g = p$ and $\{y,z\}^g = \{q,s\}$. Since, by Lemma 3.2.7(ii), $J_{xyz} = J_{pqs}$ so $\langle xyz\rangle = \langle pqs\rangle$, so $g \in G_{\langle xyz\rangle}$, and g preserves the D-relation on R_{xyz}, so $r^g = r'$.

(ii) (a) Consider a finite substructure A with the elements $x, y, z, u \in A < M$ and let $L(x;y,z)$ be witnessed in the root D-set of A with x, y, z in distinct branches at the ramification point r. To show the transitivity we want $g \in G_{\langle xyz\rangle}$ such that for $u \in J_{xyz}$, $u^g = x$. There are 3 cases to be considered:

Case 1. If u is in the same branch at r as x, then $L(x;y,z)$ and $L(u;y,z)$ are witnessed in R_{xyz}, hence $\langle xyz \rangle = \langle uyz \rangle$. By semi-homogeneity of G there is $g \in G$ such that $(u,y,z)^g = (x,y,z)$. As $J_{xyz} = J_{uyz}$ then the D-set is fixed and hence, by Lemma 3.2.8, $\langle xyz \rangle$ is fixed so $g \in G_{\langle xyz \rangle}$.

Case 2. If x,y,z,u are distinct at a ramification point r, then $L(x;y,z) \wedge L(x;u,y) \wedge L(x;u,z)$ hold and by semi-homogeneity there exists w in the same branch as u at r such that $L(u;w,z)$ is witnessed at r' in the same D-set i.e. $R(x;y,z : u;w,z)$ holds as in picture 3.13. This is because we can find a structure in M with the elements x',y',z',u',w' and $L(u';w',z')$ and $L(x';y',z')$ such that the substructure on x',y',z',u' is isomorphic to the one on x,y,z,u, so there is $g \in G$ such that $(x',y',z',u')^g = (x,y,z,u)$. Put $w = w'^g$. Therefore, $L(x;y,z)$ and $L(u;w,z)$ hold, so by semi-homogeneity, there is $g' \in G$ such that $(u,w,z)^{g'} = (x,y,z)$. As $R_{uwz} = R_{xyz}$, g' fixes the D-set and hence fixes $\langle xyz \rangle$, so $g' \in G_{\langle xyz \rangle}$.

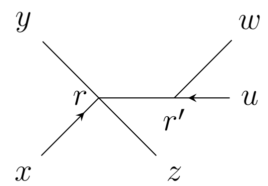

Figure 3.13

Case 3. Consider u in the same branch as z (the same if it is in the same branch as y). If u is a special at ram(u,y,z) such that $L(u;y,z)$ holds, then, by semi-homogeneity, there is some $g \in G$ that fixes y,z and takes u to x, and as $R_{xyz} = R_{uyz}$ then the D-set is fixed, hence $\langle xyz \rangle$ is fixed, so $g \in G_{\langle xyz \rangle}$. Otherwise, by semi-homogeneity as argued in Case 2, there is some w such that $L(u;w,y)$ is witnessed in R_{xyz}. Again, there is $g \in G$ with $(u,w,y)^g = (x,y,z)$, as required.

(b) Let $\langle xyz \rangle$, $\langle pqs \rangle \in (S^*/P, \leq)$. Then $M \vDash L(x;y,z) \wedge L(p;q,s)$ so by semi-homogeneity there is $g \in G$ with $(p,q,s)^g = (x,y,z)$. Then $\langle pqs \rangle = \langle xyz \rangle$.

(c) This follows from (a) and (b).

(iii) Let $[p] \neq [q]$ be distinct directions of R_{xyz} with $[p] = p/E_{xyz}, [q] = q/E_{xyz}$ and put $[x] = x/E_{xyz}, [y] = y/E_{xyz}$. It suffices to show there is $g \in G_{\{J_{xyz}\}}$ with $([x],[y])^g = ([p],[q])$. Choose $s \in M$ such that $R(x;y,z:p;q,s)$ holds - this exists by semi-homogeneity. Using 3-homogeneity (Lemma 3.1.3(i)) there is $g \in G$ with $\{x,y,z\}^g = \{p,q,s\}$, put $x^g = p, y^g = q, z^g = s$ with $L(x;y,z)$ and $L(p;q,s)$. Since $R(x;y,z:p;q,s)$ holds, g fixes J_{xyz} setwise, so g preserves E_{xyz} so fixes R_{xyz} setwise, and clearly $([x],[y])^g = ([p],[q])$.

(iv) First, $I_{xyzw} = J_{pqs}$. Thus, maximality of E_{xyzw} follows immediately from 2-transitivity of $G_{\{J_{pqs}\}}$ on $R_{pqs} = J_{pqs}/E_{pqs}$, and this was proved in (iii).

It remains to prove that E_{pqs} is the unique maximal $G_{\{J_{pqs}\}}$-congruence. To see this, suppose E^* is a $G_{\{J_{pqs}\}}$-congruence on J_{pqs} and $E^* \notsubseteq E_{pqs}$. We may suppose pE^*q. Let $p' \in J_{pqs}$ with $pE_{pqs}p'$. Then $L(p;q,s) \wedge L(p';q,s)$, and furthermore the map $(p,q,s) \longmapsto (p',q,s)$ preserves an L-relation witnessed in R_{pqs}. It follows by semi-homogeneity that there is a $g \in G$ with $(p,q,s)^g = (p',q,s)$. Then $J_{pqs}^g = J_{pqs}$, and as $q^g = q$, g fixes $E^*(q)$ setwise, so as pE^*q we have $p'E^*q$. Thus $p/E_{pqs} \subset E^*(q)$. Hence $E_{pqs} \subset E^*$ and it follows that E^* is universal, as required.

■

Observe that the class $\mathscr{D}$ does not have the hereditary property (Lemma 2.3.4). Corresponding to this, we believe that the structure M is not homogeneous, but have not proved this.

Chapter 4

Jordan Group

As our goal is to show that the automorphism group of the structure M that we built is a Jordan group preserving a limit of D-relations, in this chapter we investigate what our Jordan sets might be. In fact, we find more Jordan sets than we need (using the properties of Jordan sets, see Section 1.5). Then we show that G, where $G = \mathrm{Aut}(M)$, is a Jordan group. In Section 4.2 we prove that G satisfies the requirements of Definition 1.5.19 to get the main result of this work.

4.1 Jordan Sets

Recall that if H is a permutation group on X a Jordan set for H is a set Z such that whenever $X = Y \cup Z$, $|Z| > 1$, $Y \cap Z = \varnothing$ and $H_{(Y)}$ is transitive on Z. A transitive group with a proper Jordan set is called a Jordan group. In this chapter we find a Jordan set to show that G is a Jordan group.

Recall that if (X, B) is a B-set and $x \in X$, *branches* mean the equivalence classes for the equivalence relation defined on $X \setminus \{x\}$ such that $R_x(y,z) :\Leftrightarrow [x,y] \cap [x,z] \cap (X - \{x\}) \neq \varnothing$. There is a similar definition of *branches* for D-sets, see 1.4.22. Hence branches are subsets of B-sets and D-sets.

Also, recall that each pair (R_{xyzw}, D_{xyzw}) is a D-set of M, and an E_{xyzw}-class is a *direction* in R_{xyzw}.

Definition 4.1.1. (i) A subset $\hat{U}$ of M is said to be a *pre-branch* if there are $x, y, z, w \in M$ with $S(x, y; z, w)$ and a branch U in R_{xyzw} such that $\hat{U} = \{w \in M : [w] \in U\} = \bigcup\{[w] : [w] \in U\}$, i.e. the union of all E_{xyzw}-classes in one branch at some ramification point.

(ii) We say that $\hat{U}$ is a *pre-branch* at a ramification point if the corresponding U is a branch at that ramification point in some D-set (R_{xyzw}, D_{xyzw}).

Remark 4.1.2. The elements of the labelling D-sets are the directions in the sense of betweenness relations. So we take a subset $\hat{U}$ of M and we call it a *pre-branch* if in a particular D-set which witnesses $S(x, y; z, w)$ there is a branch U in R_{xyzw} such that $\hat{U}$ is the union of the E_{xyzw}-classes of the elements in that particular branch. Hence the branch U is a set of E_{xyzw}-classes, while the pre-branch is a subset of M. Given a D-set R_{xyzw} we put $\hat{R}_{xyzw} = \bigcup R_{xyzw}$, the corresponding subset of M.

The relation L is not defined on the set of directions of M. So we will define $\tilde{L}([x]; [y], [z])$ as follows:

Suppose there is a D-set $R = J_{xyz}/E_{xyz}$ and $u, v, w \in M$ such that $L(u; v, w)$ is witnessed in R. Let $[u], [v], [w]$ be the corresponding directions of R. Then put $\tilde{L}([u]; [v], [w])$. This relation on R is well-defined, by the following lemma.

Lemma 4.1.3. *If $[u] = [u'], [v] = [v']$ and $[w] = [w']$ in the D-set R, then $L(u; v, w)$ is witnessed in R if and only if $L(u'; v', w')$ is witnessed in R.*

Proof. Assume $L(u; v, w)$ is witnessed in R (in M). Suppose $u' E_{uvw} u$, $v' E_{uvw} v$ and $w' E_{uvw} w$. Then in any finite $A \in \mathscr{D}$ containing u, u', v, v', w, w', the D-set witnessing $L(u; v, w)$ witnesses $L(u'; v', w')$. Hence $R(u; v, w : u'; v', w')$ holds in A, and hence in M. So $L(u'; v', w')$ is witnessed in R. ■

Now, the goal is to show that each pre-branch $\hat{U}$ is a Jordan set for $G = \text{Aut}(M)$ in its action on M. In order to do that we first show that each pre-direction is a Jordan set.

Lemma 4.1.4. *(i) If $[n]$ is a pre-direction, there is a unique vertex j_n of τ (the structure tree) such that $[n]$ is a pre-direction of $D(j_n)$.*

(ii) Let j_n be the vertex for which $[n]$ is a pre-direction. Then the stabiliser of the pre-direction $[n]$ is a subgroup of the stabiliser of the vertex j_n, i.e. $G_{\{[n]\}} \leq G_{j_n}$.

Proof.

(i) It suffices to observe that $[n]$ is a pre-direction of only one D-set (namely R_{j_n}). We know that if $i < j_n$ that $[n]$ is a union of a set of pre-branches of R_i. So $[n]$ cannot be a pre-direction of two comparable D-sets. And an easy argument with finite structures shows directions of two incomparable D-sets cannot give equal pre-directions.

(ii) This follows immediately from (i).

■

Fix a direction $[n]$ of M (so $n \in M$). Let j_n be the unique vertex of the structure tree of M, whose D-set R_{j_n} has $[n]$ as a pre-direction. Define $I := \{i \in J : i < j_n\}$ where J is a chain in the structure tree, and for each $i \in I$ let R_i be the D-set indexed by i. Let D_i denote the corresponding D-relation D_{xyzw}, where $R_i = R_{xyzw}$. Then I carries a total order <, where $i < j \Leftrightarrow \hat{R}_j \subset \hat{R}_i$.

As usual, we may write each R_i in the form R_{xyzw} or R_{pqs} (and likewise for each $\hat{R}_i$). For each $i \in I$, let $r_i = f_i(j_n)$, the ramification point of R_i corresponding to the cone at i containing j_n.

For each $i \in I$, there is, by Lemma 3.2.20, a set S_i of branches at r_i such that $g_{j_n i}([n]) = \bigcup \bigcup S_i$.

We consider the induced structure on the subset $[n]$ of M. First, for each $i \in I$, there is an equivalence relation F_i on $[n]$ defined by

$$d_1 F_i d_2 \Leftrightarrow d_1,\ d_2 \text{ lie in the same pre-branch of } \hat{R}_i \text{ at } r_i.$$

Also for each $i \in I$, let E_i be the equivalence relation E_{xyzw} (restricted to $[n]$, where $R_i = R_{xyzw}$).

Lemma 4.1.5. *If $i, j \in I$ with $i < j$, then $E_i \subset F_i \subset E_j \subset F_j$.*

Proof. Take a particular pre-branch at r_i in $\hat{R}_i$ lying in $[n]$, say $\hat{U}_i$. By the definition of the relation F_i the pre-branch $\hat{U}_i$ is an F_i-class. Since each pre-branch corresponds to a branch and it is a union of pre-directions then all elements in the same direction lie in the same branch so $E_i \subset F_i$. Similarly we have $E_j \subset F_j$. To show that $F_i \subset E_j$, we see that if $[m]$ is a pre-direction for some R_j where $j \in I$ with $j > i$, then $[m]$ is a union of pre-branches of $\hat{R}_i$ at r_i. That gives us all the pre-directions higher contain F_i-classes. ■

The following definition and lemma are rephrasing for Theorem 22.1 in [6] to suit the context here.

Definition 4.1.6. For each $i \in I$, define a C-relation C_i on $\bigcup S_i$ (so on the set of directions of R_i lying in the branches of S_i) as follows: if $[x], [y], [z] \in \bigcup S_i$, then $C_i([z]; [x], [y]) \leftrightarrow D_i([x], [y]; [z], [w])$ for any pre-direction $[w]$ of R_i lying outside $\bigcup S_i$.

See Figure 4.1 below.

Lemma 4.1.7. *The relation C_i induces a C-relation on each member of S_i.*

Proof. See [6], Theorem 22.1. ■

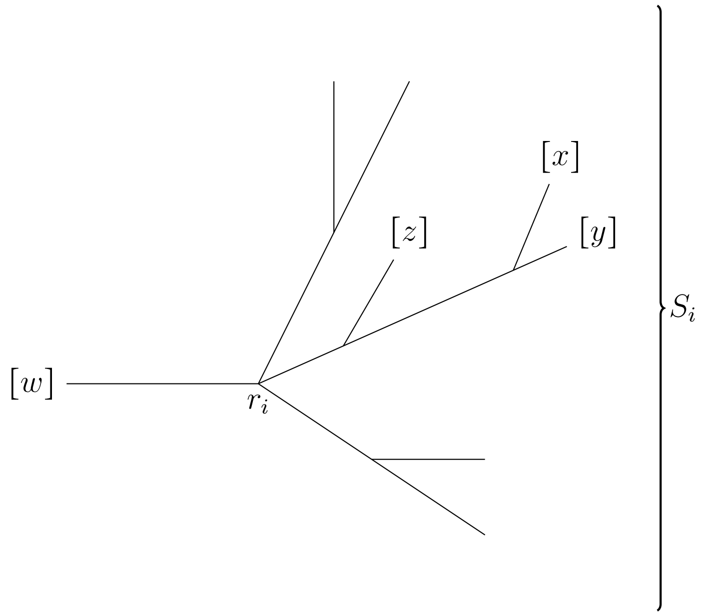

Figure 4.1: R_i

For each $i \in I$ there is such S_i as described before; that is because in each D-set R_i the pre-direction $[n]$ of R_{j_n} corresponds to a collection of branches of R_i at r_i.

Lemma 4.1.8. *The group $G_{\{[n]\},i}$ preserves the equivalence relations E_i and F_i on $[n]$.*

Proof. Let $g \in G_{\{[n]\},i}$. As $G_{\{[n]\},i} \leq G_i$, g fixes $\hat{R}_i$ and R_i setwise and preserves D_i, and as $[n]$ is a union of pre-branches at r_i and g fixes $[n]$ setwise, g preserves the partition of $\hat{R}_i$ into pre-branches at r_i. The result follows. ∎

For each $i \in I$, let $\hat{U}_i$ be a pre-branch of $\hat{R}_i$ at r_i and G^{U_i} be the group induced by $G_{\{\hat{U}_i\}}$ on $\hat{U}_i/E_i$. This group is $G_{\{\hat{U}_i\}}^{\hat{U}_i/E_i}$ and, again, for ease we write it as G^{U_i}.

Lemma 4.1.9. *(i) The group G^{U_i} has six orbits on ordered pairs of E_i-inequivalent elements of $\hat{U}_i$.*

(ii) The group G^{U_i} is transitive on $\hat{U}_i$.

Proof.

(i) We will argue this based on the position of the special branch at ramification points within $\hat{U}_i$.

Case.1 Let $[x],[y] \in U_i$, then $x,y \in \hat{U}_i$, and assume that there is $[z] \in R_i \setminus U_i$, (so $z \in \hat{R}_i \setminus \hat{U}_i$), such that $L(z;x,y)$ holds at $r = \text{ram}(x,y,z)$. Then, by semi-homogeneity, for any distinct $[x'],[y'] \in U_i$, then $x',y' \in \hat{U}_i$, with $L(z;x',y')$ holding at $r' = \text{ram}(x',y',z)$, there is $g \in G$ such that $(x,y,z)^g = (x',y',z)$. This g can be chosen to fix $\hat{U}_i$. Indeed, pick $w \in \hat{R}_i \setminus \hat{U}_i$ such that $L(w;z,x)$ (so w is in the special branch at $r_i = \text{ram}(z,w,x)$). Then by semi-homogeneity choose $g \in G$ with $(x,y,z,w)^g = (x',y',z,w)$. As $S(x,y;z,w)$ and $S(x',y';z,w)$ in R_i, g fixes R_i, and because g fixes z,w and $x^g \in \hat{U}_i$ then it fixes $\hat{U}_i$ as a set. This case gives one orbit on ordered pairs from $\hat{U}_i$. See Figure 4.2.

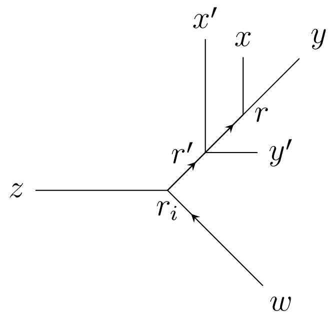

Figure 4.2: Case.1

Case.2 Assume that one of the branches containing x,y is special at $r = \text{ram}(x,y,z)$, say the one containing x, see Figure 4.3. Again, by semi-homogeneity, for any distinct $[x'],[y'] \in U_i$ with, for example, $L(x';y',z)$ holding at r', (the dashed lines for x' mean the possible places for x') then there is $g \in G$ such that $(x,y,z)^g = (x',y',z)$. As above, g can be chosen to fix $\hat{U}_i$, as for $[w] \in R_i \setminus U_i$ with $L(w;z,x)$ there is $g \in G$, by semi-homogeneity, such that $(x,y,z,w)^g = (x',y',z,w)$, and such g fixes U_i. Thus there is $g \in G_{\{U_i\}}$ such that $(x,y,z)^g = (x',y',z)$. Observe that this case gives one orbit on unordered pairs, but two orbits on ordered pairs.

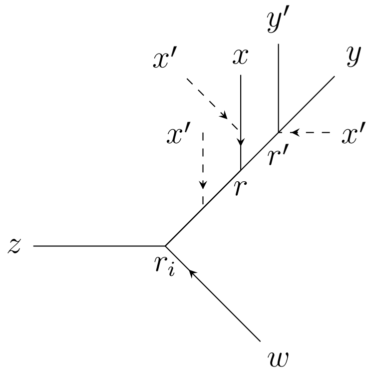

Figure 4.3: Case.2

Case.3 Suppose the special branch at $r = \mathrm{ram}(x, y, z, u)$ within $\hat{U}_i$ is another one, i.e. neither the branch containing x nor y or z, say u as in Figure 4.4. Let $[u'] \in U_i$ such that for some $[x'], [y'] \in U_i$ the relation $L(u'; x', y')$ holds at $r' = \mathrm{ram}(x', y', z, u')$. Pick $[w] \in R_i \setminus U_i$ as in the picture. Assume that $L(x; y, z)$ and $L(x'; y', z)$, or $L(y; x, z)$ and $L(y'; x', z)$, or $L(z; x, y)$ and $L(z; x', y')$. In each of these three cases, by semi-homogeneity there is $g \in G$ with $(x, y, u, z, w)^g = (x', y', u', z, w)$. Such g fixes U_i, so this case gives three further G^{U_i}-orbits on ordered pairs from U_i.

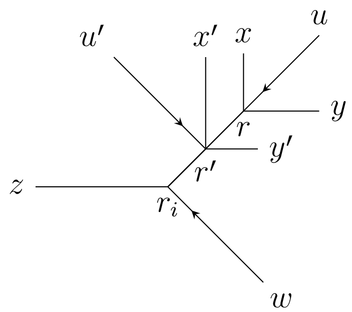

Figure 4.4: Case.3

(ii) Let $x, x' \in \hat{U}_i$ and $z \notin \hat{U}_i$ in a D-set $\hat{R}_i$. We want to find $g \in G^{U_i}$ such that $x^g = x'$. Choose $y, z \in \hat{R}_i \setminus \hat{U}_i$ so that $y \neq z$, $r_i = \mathrm{ram}(x, y, z)$, and $L(y; x, z)$ holds. Then $L(y; x', z)$ holds too at r_i, so by semi-homogeneity there is $g \in G$ such that $(x, y, z)^g = (x', y, z)$, hence $x^g = x'$. So g fixes the D-set hence fixes i. As the two triples determine r_i, g fixes r_i and because $x, x' \in \hat{U}_i$ then $g \in G_{\{\hat{U}_i\}}$ so G^{U_i} is transitive on U_i.

■

Remark 4.1.10. In the proof of part (i) above, colour the ramification points r into two colours, red and green depending on the position of the special branch, such that if z lies in the special branch at r we colour the ramification point by red, otherwise we colour it by green. We see that G^U is transitive on ramification points of each colour, so has two orbits on ramification points of $\hat{U}$.

For the F_i-class U_i, the set $\hat{U}_i/E_i$ carries a C-relation structure (as is mentioned in Corollary 4.1.7) with induced relations L and S, and the binary relations in the following definition.

Definition 4.1.11. Consider a D-set $\hat{R}_i$ and a pre-branch $\hat{U}_i$ at r_i with $x, y, z \in \hat{R}_i$ and $x, y \in \hat{U}_i$, but $z \notin \hat{U}_i$. Let $r = \text{ram}(x, y, z)$. We define the following binary relations:

1. $P_1(x, y)$ if z is special at r.
2. $P_2(x, y)$ if x is special at r (so $P_2(y, x)$ if y is special at r).
3. $P_3(x, y)$ if none of x, y, z is special at r and $L(z; x, y)$ holds in the D-set of the vertex corresponding to r higher up.
4. $P_4(x, y)$ if none of x, y, z is special at r and $L(x; y, z)$ holds in a D-set of a vertex of the cone corresponding to r higher up.

For each such U_i, the group G^{U_i} is transitive on $\hat{U}_i/E_i$ by Lemma 4.1.9.

Lemma 4.1.12. *The binary relations P_1, P_2, P_3, P_4 are preserved by G^{U_i}, and are orbits of G^{U_i} on pairs of E_i-inequivalent elements of $\hat{U}_i$.*

Proof. This follows immediately from the argument in Lemma 4.1.9(i) such that Case 1 implies P_1 is an orbit, Case 2 applies to P_2 and Case 3 applies to P_3 and P_4. ■

Lemma 4.1.13. *The $G_{\{\hat{U}_i\}}$-congruence E_i on $\hat{U}_i$ is maximal.*

Proof. We want to show that $G_{\{\hat{U}_i\}}$ is primitive on $U_i = \hat{U}_i/E_i$. Suppose that $x, y \in \hat{U}_i$ are inequivalent modulo E_i, and that xE^*y for some equivalence relation E^*. Let $z \in \hat{R}_i \setminus \hat{U}_i$ and let $r = \text{ram}(x, y, z)$. By considering various configurations for x, y, z, we show $E^*(x) = \hat{U}_i$, i.e. that E^* is universal.

Case 1. Assume z is special at r. Let $x' \in \hat{U}_i$. We consider the various possible positions of x' in the following sub-cases:

(a) Suppose that x' is in the same branch as x at r. By Lemma 4.1.9(i) there is $g \in G^{U_i}$ such that $(x, y)^g = (x', y)$. As g fixes the $E^*([y])$ (since it fixes y, it fixes the E^*-class of y) and $x \in E^*([y])$ (by the assumption) then $x' \in E^*(y)$. Hence $x' \in E^*(x)$, thus $E^*(x)$ contains all the branch at r containing x'.

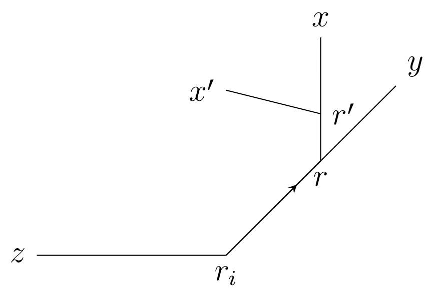

Figure 4.5: Case 1.(a)

(b) If x' is in distinct pre-branch at r there is $g \in G^{U_i}$ taking y to x' and fixing x, hence fixing $E^*([x])$. As $[x]E^*[y]$ and $y^g = x'$ then $E^*([x])$ contains the branch at r containing x', so $x' \in E^*(x)$. And again thus $E^*(x)$ contains all the branch at r containing x'.

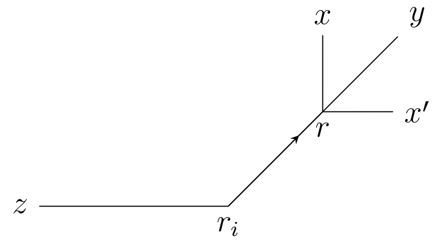

Figure 4.6: Case 1.(b)

(c) Consider a ramification point r' strictly between r_i and r such that $r' = \text{ram}(z, x, x')$ with $L(z; x, x') \wedge L(z; y, x')$. By Lemma 4.1.9($i$) there is $g \in G_{\{U_i\}}$ with $(x, y)^g = (x, x')$. Then g fixes $E^*(x)$ as a set, so as $y \in E^*(x)$, also $x' \in E^*(x)$. Thus $E^*(x)$ contains the branch at r' containing x', and likewise that containing x.

In Case (c), it follows that $E^*(x)$ contains all branches at r' other that the special one containing z. Since such a ramification point r' can be chosen coinitially in U_i, i.e. arbitrarily close to to r_i, it follows that $E^*(x) = \hat{U}_i$.

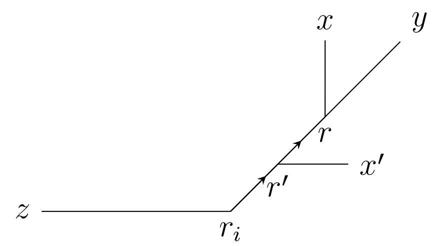

Figure 4.7: Case 1.(c)

Case 2. Suppose that the special branch at r is the one containing x or containing y, say the branch containing x (the argument is similar if the branch containing y is special) such that $L(x; y, z)$ holds at r and $[x]E^*[y]$. Let $x' \in \hat{U}_i$. We consider various possibilities regarding the place of x':

(a) Suppose x' lies in the branch containing x at r. By Lemma 4.1.9(i) there is $g \in G_{\{\hat{U}_i\}}$ with $(x, y)^g = (x', y)$. Since g fixes y it fixes $E^*(y) = E^*(x)$, so $x'E^*y$. Thus $E^*(y)$

contains the whole branch at r containing x'. A similar argument applies if x' lies in the branch at r containing y, or in another branch at r.

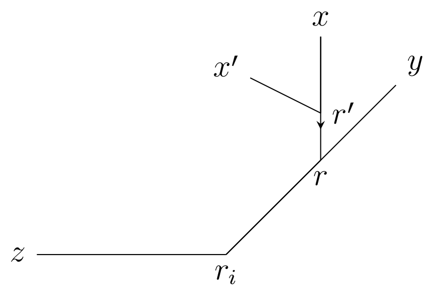

Figure 4.8: Case 2.(a)

(b) For any ramification point r^* between r_i and r, there is r' as depicted in Figure 4.9. At such r', choose a branch x'. Then by Lemma 4.1.9(i) there is $g \in G_{\{\hat{U}_i\}}$ with $(x,y)^g = (x,x')$. This is because we have $L(x;y,z)$ at r and $L(x;x',z)$ at r'. As x is fixed, $E^*(x)$ is fixed. Then the branch containing x at r is mapped to the branch containing x at r'. Hence the branch containing x at r' lies in $E^*([x])$. The whole branch at r_i is a union of such branches so lies in $E^*([x])$.

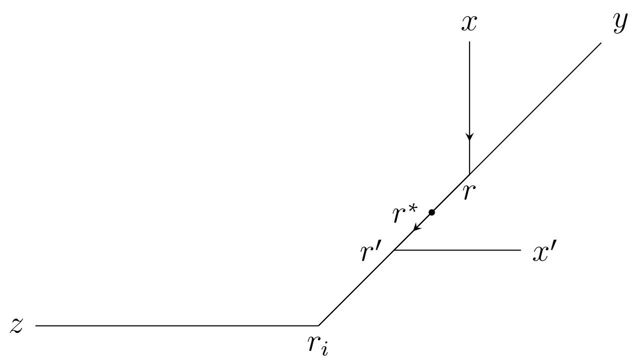

Figure 4.9: Case $2.b$

Therefore, wherever the arbitrary element lies within U_i it will be in $E^*([x])$, so E^* is universal.

Case 3. Suppose that the special branch at r contains none of x, y or z. Arguing as in Cases 1 and 2, and using Lemma 4.1.9(i), we see that $E^*(y)$ contains the branch containing x, and the branch at r containing y. Furthermore, we can find a ramification point r' in $\hat{U}_i$ arbitrarily close to r_i, and $g \in G^{U_i}$ fixing x and taking x' to y where $r' = \text{ram}(z, x, x')$. Then $E^*(x)$ contains the branch at r' containing x, and thus contains the whole of $\hat{U}_i$. Hence E^* is universal.

∎

Consider three vertices $i, j, j' \in I$ such that $i < j < j'$ and their corresponding D-sets $R_i, R_j, R_{j'}$ respectively. Then focus on an F_i-class (it contains E_i-classes). This F_i-class will be contained in a single E_j-class which is contained in an F_j-class, and this F_j-class will be contained in an $E_{j'}$-class, so this $E_{j'}$-class contains the F_i-class and each E_k-class where $k > j'$ will contain the F_i-class. That means the classes are going to be coarser if we go higher in the structure tree. In particular, we have the following.

Lemma 4.1.14. *Given an F_i-class $\hat{U}_i$, the intersection of the E_j-classes containing $\hat{U}_i$ (for $j > i$) is just U_i.*

Proof. We want to show that $F_i = \bigcap_{j>i} E_j$. It is clear that $F_i \subseteq \bigcap_{j>i} E_j$ and E_j $(j > i)$ contains F_i from the above paragraph. Conversely, suppose $\neg u F_i v$. We want to find $j > i$ such that $\neg u E_j v$. Let $a \in M$ lie in the special branch at r_i. Consider a finite structure $A \in \mathscr{D}$ containing elements a', u', v', w', s', t' in distinct branches at a ramification point r at the root D-set (with a' special), such that in a higher D-set we have $L(w'; s', t')$, with u', v', w', s', t' again in distinct branches at a ramification point. We may suppose $A \leq M$. By semi-homogeneity there is $g \in G$ with $(a', u', v')^g = (a, u, v)$. The relation $L(w'^g; s'^g, t'^g)$ will be witnessed in a D-set R_j with $j > i$, and we have $\neg u E_j v$.

∎

Recall that if $i < j_n$ then S_i is the set of branches of $\hat{R}_i$ at the ramification point r_i which corresponds to the direction $[n]$.

Lemma 4.1.15. *Let $u, v_1, \ldots v_m$ be distinct elements of $[n]$. Then there is a greatest i such that u is E_i-inequivalent to each of $v_1, \ldots, v_m$ and for such i the element u will be F_i-equivalent to at least one of v_j where $j \in \{1, \ldots, m\}$.*

Proof. Find $i_0 < j_n$ containing elements $w, s, t, u, v_1, \ldots, v_m$ all lying in distinct branches at the ramification point r_{i_0} of the D-set R_{i_0}, with $w, s, t \notin [n]$, and with $L(w; s, t)$ witnessed in R_{j_n}. Consider finite $A \leq M$ with $A \in \mathscr{D}$ and $w, s, t, u, v_1, \ldots, v_m$ lying in distinct branches at a ramification point of the root D-set.

By considering the structure of A, we see that there is i with $i_0 < i < j_n$ such that at r_i, u is in the same branch as at least one of the v_i, but in a distinct direction to each. (Working in A, consider the D-sets in the structure tree between the root and the D-set witnessing $L(w; s, t)$, and the corresponding ramification points; there will be a least D-set such that u lies in the same branch as some v_j at the relevant ramification point).

■

Lemma 4.1.16. *Let g be a permutation of M which is the identity on $M \setminus [n]$, and for each $i \in I$ preserves the equivalence relation E_i, the relations L and S on $[n]$, and the C-relation induced by C_i on each F_i-class of $[n]$. Then $g \in G$.*

Proof. By Lemma 4.1.14 and the assumption that g preserves the relations E_i, then g preserves each $F_i\big|_{[n]}$ and hence each F_i. It is enough to show that g preserves L and S on M, since we have seen in Lemma 3.1.1 and Lemma 3.1.2 that all the relations in the language can be written in terms of L and S. Hence, we will divide the proof into two parts; part A for the proof of preserving the relation L, and part B for the proof of preserving the relation S.

Part A. To prove that $g \in G$ preserves L, we argue in four cases:

Case I. If $x, y, z \in [n]$, then $L(x; y, z) \leftrightarrow L(x^g; y^g, z^g)$ follows immediately by the hypothesis that g preserves L on $[n]$.

Case II. Let $x \in [n]$, $y, z \in M \setminus [n]$. Let R be the D-set in which $L\{x, y, z\}$ is witnessed with x, y, z lying in distinct branches at the ramification point r of R. We want $\{x, y, z\}$ and $\{x^g, y, z\}$ to satisfy the same L-relation. Since the position of the D-set R is not known we will consider the possible cases based on where the D-set R witnessing $L\{x, y, z\}$ could be.

Sub-case 1. Assume that the D-set R is R_{j_n}, and let $L\{x,y,z\}$ hold, witnessed in R. Now x, x^g lie in the same element of R, and y, z lie in two other distinct elements of R, fixed by g. It is therefore immediate that x^g, y, z satisfy the same L-relation as x, y, z.

Sub-case 2. Assume that the D-set R is lower than R_{j_n}, and has the corresponding subset for the pre-direction $[n]$, denoted S_i. However, $L(x; y, z)$ cannot be witnessed in this D-set at r_i, because $x \in [n]$ so cannot lie in the special branch at r_i. But it is possible to see $L(y; x, z)$ at r_i, see Figure 4.10 (the same for $L(z; x, y)$) and since $x^g \in S_i$, as every thing is fixed outside $[n]$ and only moved within $[n]$, then $x^g \in [n]$, and the relation $L(y; x^g, z)$ holds.

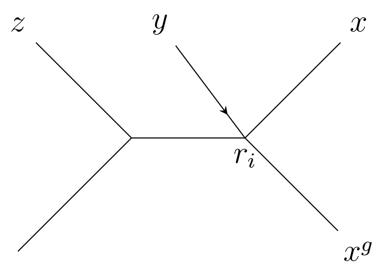

Figure 4.10

If $L(x; y, z)$ holds in R_i such that x is special at another ramification point r_i' not within S_i, again because $x^g \in S_i$, see Figure 4.11, and since x and x^g lie in the same branch at r_i' we get $L(x^g; y, z)$. However if $L(y; x, z)$ holds at r_i' (the same for $L(z; x, y)$), then x and x^g will be in the same branch at r_i' and it is readily seen that $L(y; x^g, z)$ holds.

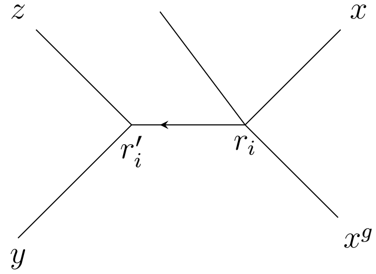

Figure 4.11

Sub-case 3. Assume that the D-set R is higher than R_{j_n}. Now the direction containing x in R contains the whole of $[n]$, so is fixed by g, as are y and z. It follows that x, y, z and x^g, y, z satisfy the same L-relation.

Sub-case 4. Suppose the D-set R corresponds to the vertex k of the structure tree with k incomparable with j_n. Let $i = \inf\{j_n, k\}$, so $i \in I$. We may suppose that the cone of k at i (in the structure tree) corresponds to the ramification point r' of R_i; then $r' \neq r_i$. Since $L\{x, y, z\}$ is witnessed in R, x, y, z lie in distinct non-special branches at r'. Hence, as $y, z \notin S_i$, it follows that r' cannot be in a meeting point for the branches in S_i, and we have, for example, the picture below in R_i.

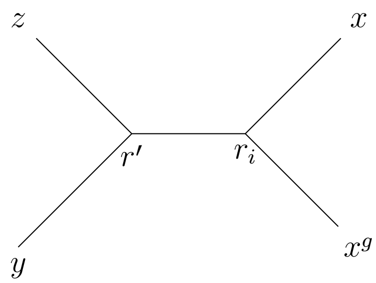

Figure 4.12

Now, x, x^g lie in the same branch at r' so in the same pre-direction of R, so the same L-relation holds among x, y, z and x^g, y, z.

Case III. If $x, y \in [n]$ and $z \in M \setminus [n]$ we will consider the sub-cases as before.

Sub-case 1. Suppose that the D-set R is R_{j_n}. The relations $L\{x, y, z\}$ is not witnessed because x and y are in the same direction in R_{j_n}.

Sub-case 2. Suppose that the D-set R is lower than R_{j_n}, say $R = R_i$. If $\neg x F_i y$ at r_i, then the relation $L(x; y, z)$ or $L(y; x, z)$ cannot be witnessed at r_i because neither x nor y can be special at r_i. If $L(z; x, y)$ holds at r_i then $L(z; x^g, y^g)$ is witnessed in R_i (x^g, y^g are in distinct branches at r_i because F_i is preserved on $[n]$).

If $x F_i y$ and $L(x; y, z)$ is witnessed at r (see Figure 4.13 below) then we want to see $L(x^g; y^g, z)$ holds (the same if $L(y; x, z)$ holds). For, we know that g preserves L on $[n]$, from the hypothesis, so there is $t \in [n]$ such that $tF_i x \wedge tF_i y$ and $L(x;y,t)$ holds. Since L is preserved on elements of $[n]$, and g preserves the C-relation on each F_i-class of $[n]$ we get $L(x^g;y^g,t^g)$, then $L(x;y,z) \leftrightarrow L(x;y,t)$ and $L(x^g;y^g,z^g) \leftrightarrow L(x^g;y^g,t^g)$ as g preserves C. Thus, if $L(x;y,z)$ then $L(x^g;y^g,z^g)$, so $L(x;y,t) \leftrightarrow L(x^g;y^g,t^g)$ as $x, y, t \in [n]$, so $L(x^g;y^g,t^g)$ (as $z^g = z$), as required.

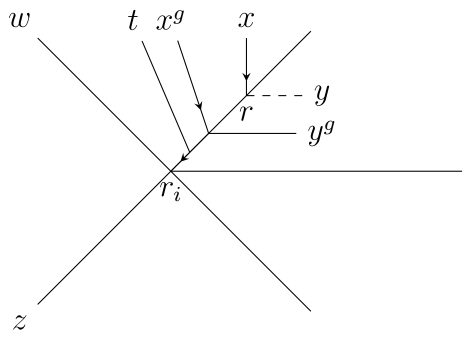

Figure 4.13

Sub-case 3. Suppose that the D-set R higher than R_{j_n}, say R_k, then $L\{x,y,z\}$ cannot be witnessed because x, y are in the same direction in R_k.

Sub-case 4. Assume that R is the D-set of the vertex k incomparable with j_n, and put $i = \inf\{k, j_n\}$. Then the cone of k at i corresponds to a ramification point r' of R_i distinct from r_i, and as x, y, z lie in distinct branches of R_i at r', we must have $r' \in S_i$, as in the diagram,

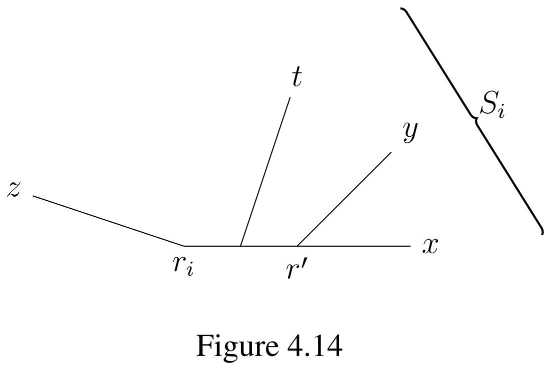

Figure 4.14

Choose t as depicted, in the same branch as z at r' and the same branch as x at r_i. As g

preserves L on $[n]$ and the C-relation on F_i-classes, we have

$$L(x;y,z) \Leftrightarrow L(x;y,t) \Leftrightarrow L(x^g;y^g,t^g) \Leftrightarrow L(x^g;y^g,z)$$

and likewise for other permutations of $\{x,y,z\}$.

Case IV. If $x,y,z \notin [n]$, then as g is the identity on $M \setminus [n]$ we have $L(x;y,z) \leftrightarrow L(x^g;y^g,z^g)$, and likewise for the other orderings of $\{x,y,z\}$.

Part B. To prove that g preserves S, we argue in four cases. Again, let R be the D-set in which S is witnessed, with $S(x,y;z,w)$.

Case I. If $x,y,z,w \in [n]$, then by the hypothesis, g preserves S on $[n]$ so $S(x,y;z,w) \leftrightarrow S(x^g,y^g;z^g,w^g)$.

Case II. Suppose that $x \in [n]$ and $y,z,w \in M \setminus [n]$.

Sub-case 1. Suppose that the D-set R is R_{j_n}. Then let $S(x,y;z,w)$ holds, then as $xE_{j_n}x^g$ it is easily seen that $S(x^g,y;z,w)$ holds.

Sub-case 2. Suppose that the D-set R is lower than R_{j_n}, say $R = R_i$. If $S(x,y;z,w)$ holds, then as $x,x^g \in [n]$, because g permutes elements of $[n]$, and as $[n]$ is a union of pre-branches at r_i the relation $S(x^g,y;z,w)$ holds.

Sub-case 3. Suppose that the D-set R is higher than R_{j_n}. If $S(x,y;z,w)$ holds then $S(x^g,y;z,w)$ holds as x,x^g lie in the same direction of R.

Sub-case 4. Suppose that R is the D-set of vertex k incomparable with j_n. As before, let $i = inf\{j_n,k\}$, and let r' be the ramification point of R_i corresponding to the cone of k, so $r' \neq r_i$. As x,y,z,w are in distinct directions of R, they are in distinct branches at r', so $r' \notin S_i$. Now as $x^g \in S_i$, x and x^g lie in the same branch at r', so $S(x^g,y;z,w)$ holds, witnessed in R.

Case III. Suppose that $x,y \in [n]$ and $z,w \in M \setminus [n]$.

Sub-case 1. Assume that the D-set R is R_{j_n}. The relation $S(x,y;z,w)$ is violated, since x,y are E_{j_n}-equivalent and so lie in the same direction of R_{j_n}.

Sub-case 2. Assume that the D-set R is lower than R_{j_n}. If $\neg xF_iy$ then as $S(x,y;z,w)$ is witnessed in R_i, z and w must lie in the same branch at r_i, and we have $S(x^g,y^g;z,w)$. If xF_iy, then $x^gF_iy^g$, and as $z,w\notin S_i$ we again have $S(x^g,y^g;z,w)$.

Sub-case 3. Assume that the D-set R is higher than R_{j_n}. Then x,y will be in the same direction of R, hence $S(x,y;z,w)$ is not witnessed.

Sub-case 4. Suppose that R is the D-set of vertex k incomparable with j_n, put $i=\inf\{j_n,k\}$, and let r' be the ramification point of R_i corresponding to the cone at i of k. Then because $r'\neq r_i$ and $x,y\in S_i$ and $z,w\notin S_i$, it is not possible that x,y,z,w lie in distinct branches at r', so not possible that $S(x,y;z,w)$ is witnessed in R.

Case IV. Suppose that $x,z\in[n]$ and $y,w\in M\setminus[n]$. Sub-cases 1,3,4 are handled exactly as in Case III above.

For sub-case 2, as S_i is a union of branches at r_i, and $x,z\in S_i$ and $y,w\notin S_i$, we cannot have $S(x,y;z,w)$ witnessed in R_i.

Case V. Suppose that $x,y,z\in[n]$ and $w\in M\setminus[n]$.

Sub-case 1. Suppose that the D-set R is R_{j_n}, then $S(x,y;z,w)$ is not witnessed in R.

Sub-case 2. Suppose that the D-set R is lower than R_{j_n}, say $R=R_i$. If $S(x,y;z,w)$ holds such that xF_iy and $\neg zF_i\{x,y\}$ then $x^gF_iy^g$ and $\neg z^gF_i\{x^g,y^g\}$. So $S(x^g,y^g;z^g,w)$ holds. If $S(x,y;z,w)$ holds such that x,y,z are F_i-equivalent with $C(z;x,y)$ (as this F_i-class) then since g preserves C, then $C(z^g;y^g,x^g)$ holds (on the corresponding F_i-class), so $S(x^g,y^g;z^g,w)$ holds.

Sub-case 3. If the D-set R is higher than R_{j_n}, then $S(x,y;z,w)$ is not witnessed since x,y,z are in the same direction of R.

Sub-case 4. Suppose that R is the D-set of the vertex k incomparable with j_n, let $i=\inf\{j_n,k\}$, and let r' be the ramification point of R_i corresponding to the cone at i of k, so $r'\neq r_i$. Since $x,y,z\in S_i$ and $w\notin S_i$ and x,y,z,w lie in distinct branches at r', we must have $r'\in S_i$. Choose t as depicted (so in the same F_i-class as x,y,z).

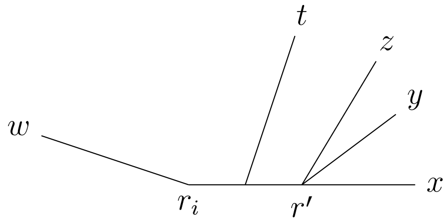

Figure 4.15

Then x^g, y^g, z^g, t^g will be in the same configuration (possibly in a different F_i-class), as g preserves C and F_i. Also $S(x,y;z,t) \Leftrightarrow S(x^g,y^g;z^g,t^g)$, as g preserves S on $[n]$. Thus

$$S(x,y;z,w) \Leftrightarrow S(x,y;z,t) \Leftrightarrow S(x^g,y^g;z^g,t^g) \Leftrightarrow S(x^g,y^g;z^g,w)$$

as required.

Case VI. If $x,y,z,w \notin [n]$, then as g is the identity on $M \setminus [n]$, $S(x,y;z,w) \Leftrightarrow S(x^g,y^g;z^g,w^g)$.

■

Remark 4.1.17. In [7], there is a similar version of the previous lemma, but there is a missing assumption. In the proof of claim 6 of Proposition 5.6 there needs to be a statement saying that g preserves the semilinear order relation on the E_i-classes.

The proof of the following lemma is quite similar to the proof of Lemma 5.6 in [7], and we follow the same procedure.

Lemma 4.1.18. *Each pre-direction $[n]$ is a Jordan set of G.*

Proof. To show this, we want to define a group $K \leq G$ which is transitive on $[n]$ and fixes the complement $M \setminus [n]$. We want to construct K as an iterated wreath product of groups of automorphisms of C-relations.

Write $[n] = \{u_i : i \in \omega\}$. For each $u \in [n]$, $i \in I$, put $[u]_i = \{x \in M : xF_iu\}$. Also for each $i \in I$, let $V_i = [u_0]_i/E_i$ (the branch at r_i containing u_0). Let $e_i := u_0/E_i \in V_i$. Define

$$\Omega := \{f : I \to \bigcup_{i \in I} V_i : f(i) \in V_i \text{ for all } i, \text{ supp}(f) \text{ finite}\}$$

where $\mathrm{supp}(f) = \{i \in I : f(i) \neq e_i\}$.

For each $i \in I$ and $u \in [n]$, define $A^i(u) = [u]_i/E_i$ (the branch at r_i containing u). We aim to find a system of maps $\phi^i_V : V \to V_i$, where $i \in I$ and V ranges through branches $[u]_i/E_i$ for $u \in [n]$.

Given such maps, define $\chi : [n] \to \Omega$ by $\chi(u)(i) = \phi^i_{A^i(u)}(u/E_i)$ for all $u \in [n]$ and $i \in I$. We need to define the maps ϕ^i_V so that χ is a bijection. Define $\chi(u_0)$ so that $\chi(u_0)(i) = e_i$ for all $i \in I$.

Suppose that $\chi(u_0), \dots, \chi(u_{k-1})$ have been defined. We may suppose that each map $\phi^i_{A^i(u_l)}$ has been defined, for all $l < k$, and all $i \in I$.

Let i_k be the largest $i \in I$ such that u_k is E_i-inequivalent to u_l for each $l < k$. Hence, by Lemma 4.1.15, there is some $l < k$ such that $u_l F_{i_k} u_k$, so $u_l F_i u_k$ for all $i \geq i_k$. Now by assumption $\phi^i_{A^i(u_l)}$ has been defined for all $i \in I$, so $\phi^i_{A^i(u_k)}$ has been defined for all $i \geq i_k$, but not for $i < i_k$. For $i < i_k$, choose $g \in G$ such that $(A^i(u_k))^g = V_i$ and $([u_k]_i/E_i)^g = e_i$ (this exists, since G is transitive on the set of branches and induces a transitive group on each branch). Then put $\phi^i_{A^i(u_k)}(u/E_i) = (u/E_i)^g$, for all $i < i_k$ and $u F_i u_k$. Observe that the maps $\phi^i_{A^i(u_l)}$ are now defined for all $l \leq k$ and all $i \in I$.

Claim 1. With the maps $\phi^i_{A^i(u)}$ so defined, we have $\chi(u_k) \in \Omega$ for each $k \in \omega$.

Proof. This is by induction on k. It is immediate that $\chi(u_0) \in \Omega$, so assume it holds for all $l < k$. By construction, as $\phi^i_{A^i(u_k)}$ is a bijection $[u_k]_i/E_i \to V_i$, we have $\chi(u_k)(i) \in V_i$. We must show $\mathrm{supp}(\chi(u_k))$ is finite. There is $l < k$ such that for $i > i_k$, $\chi(u_k)(i) = \chi(u_l)(i)$, so $\mathrm{supp}(\chi(u_k)) \cap \{j \in I : j > i_k\} = \mathrm{supp}(\chi(u_l)) \cap \{j \in I : j > i_k\}$, so by induction is finite. By construction, $\chi(u_k)(i) = e_i$ for all $i < i_k$, and the claim follows. ■

Claim 2. $\chi : [n] \to \Omega$ is a bijection.

Proof. We first show that χ is injective. So suppose $l < k$. We must show $\chi(u_l) \neq \chi(u_k)$. Pick i such that $u_k F_i u_l$ and $\neg u_k E_i\, u_l$. Then $[u_k]_i = [u_l]_i$, but $[u_k]_i/E_i \neq [u_l]_i/E_i$, so as $A^i(u_k) = A^i(u_l)$, $\chi(u_k)(i) = \phi^i_{A^i(u_k)}(u_k/E_i) \neq \phi^i_{A^i(u_l)}(u_l/E_i) = \chi(u_l)(i)$.

To see surjectivity, suppose for a contradiction that χ is not surjective, and let $f \in \Omega \setminus \text{Range}(\chi)$ have minimal support, with $\text{supp}(f) = \{i_1, \dots, i_t\}$ where $i_1 < \dots < i_t$. Define $f' \in \Omega$ where $f'(i_1) = e_{i_1}$, and $f'(j) = f(j)$ for all $j \neq i_1$.

By minimality of $\text{supp}(f)$, there is $u \in [n]$ with $\chi(u) = f'$. Let $v = f(i_1) \in V_{i_1}$, and let k be least such that u_k lies in the E_i-class $(\phi^i_{A^{i_1}(u)})^{-1}(v)$.

To obtain a contradiction and thereby to prove surjectivity, it suffices to prove

Sub-claim 1. $\chi(u_k) = f$.

Proof. Certainly $\chi(u_k)(i_1) = \phi^{i_1}_{A^{i_1}(u_k)}(u_k/E_{i_1}) = v = f(i_1)$. For $j > i_1$, $\chi(u_k)(j) = \phi^j_{A^j(u_k)}(u_k/E_j) = \phi^j_{A^j(u)}(u/E_j) = f(j)$. Also, $i_k \geq i_1$, for otherwise there is $j < i_1$ and $l < k$ such that $u_l F_j u_k$, and hence $u_l E_{i_1} u_k$ contradicting minimality of k. Hence $\chi(u_k)(j) = e_j = f(j)$ for all $j < i_1$, so indeed $\chi(u_k)(j) = f(j)$ for all j. ■

■

For each $i \in I$, let H_i be the group induced by $G_{\{V_i\}}$ on V_i. For each triple (i, g, h), where $i \in I$, $g : (i, \infty) \to \bigcup_{j>i} V_j$ with $g(j) \in V_j$ for all j, and $h \in H_i$, define the function $x(i, g, h) : \Omega \to \Omega$ as follows:

$$f^{x(i,g,h)}(j) = \begin{cases} f(i)^h & \text{if } j = i \text{ and } f\big|_{(i,\infty)} = g, \\ f(j) & \text{otherwise} \end{cases}$$

Claim 3. Each map $x(i, g, h)$ is a permutation of Ω.

Proof. See [26], Lemma 3.3. ■

Now define K, the *generalized wreath product*, to be the subgroup of $\text{Sym}(\Omega)$ generated by permutations $x(i, g, h)$ where i, g, h are as above. By [24], Lemma 1, the group K is transitive on Ω. Thus, K has an induced transitive action on $[n]$, given by $u^x = \chi^{-1}((\chi(u))^x)$ for all $x \in K$ and $u \in U$. (Note that we keep using Cameron's notation in [11] for the permutation groups K, which also was used in [7]). We extend this action to the whole of M by putting $v^x = v$ for all $v \notin [n]$.

Claim 4. In this action, K is a subgroup of Aut(M).

Proof. It suffices to show that elements $x(i,g,h)$ as above are automorphisms of M, and for this we use Lemma 4.1.16. First, observe

Sub-claim 2. For $u,v \in [n]$, and $i \in I$, $uE_iv \Leftrightarrow \chi(u)(j) = \chi(v)(j)$ for all $j \geq i$.

Proof. If uE_iv then $A^j(u) = A^j(v)$ for all $j \geq i$, so $\chi(u)(j) = \phi^j_{A^j(u)}(u/E_j) = \phi^j_{A^j(v)}(v/E_j) = \chi(v)(j)$ for all $j \geq i$. Conversely, if $\neg uE_iv$, then there is $j \geq i$ such that uF_jv and $\neg uE_jv$. Then $A^j(u) = A^j(v)$, so $\chi(u)(j) = \phi^j_{A^j(u)}(u/E_j) \neq \phi^j_{A^j(v)}(v/E_j) = \chi(v)(j)$, as required. ∎

Since $x(i',g,h)$ acts as a permutation in the single coordinate i', in its action on Ω, it is clear that for $u,v \in [n]$ and $i \in I$, we have $\chi(u)(j) = \chi(v)(j)$ for all $j \geq i$ if and only if $\chi(u)^{x(i',g,h)}(j) = \chi(v)^{x(i',g,h)}(j)$ for all $j \geq i$. Thus, uE_iv if and only if $u^{x(i',g,h)}E_iv^{x(i',g,h)}$, so the maps $x(i',g,h)$ preserve all the equivalence relations E_i. Also, the relations F_i are preserved by these maps as a result of preserving E_i, using Lemma 4.1.14.

For $u,v,w \in [n]$, put

$$\sigma(u,v,w) = \text{Max}\{i : u/E_i, v/E_i, w/E_i \text{ are all distinct}\}.$$

$$\mu(u,v,w) = \text{Max}\{i : u/E_i, v/E_i, w/E_i \text{ are not all equal}\}.$$

Then $\mu(u,v,w) \geq \sigma(u,v,w)$, and $\mu(u,v,w) = \sigma(u,v,w)$ if and only if there is i (namely $\sigma(u,v,w)$) such that u,v,w are F_i-equivalent but not E_i-equivalent.

Suppose $\mu(u,v,w) = \sigma(u,v,w) = i$. Let C_i be as in Definition 4.1.6. Then since the map $\phi^i_{A^i(u)}$ is induced by an element of G, we have

$$C_i(u;v,w) \leftrightarrow C_i(\phi^i_{A^i(u)}(u/E_i); \phi^i_{A^i(v)}(v/E_i), \phi^i_{A^i(w)}(w/E_i)).$$

It follows that under the assumption $\mu(u,v,w) = \sigma(u,v,w) = i$, the fact that $C(u;v,w)$ holds depends just on $\chi(u)(i), \chi(v)(i), \chi(w)(i)$. Similarly, the fact that $L(u;v,w)$ holds depends just on $\chi(u)(i), \chi(v)(i), \chi(w)(i)$. And if u,v,w,z are all

F_i-equivalent but E_i-inequivalent, the fact that $S(u,v;w,z)$ holds depends just on $\chi(u)(i), \chi(v)(i), \chi(w)(i)$ and $\chi(z)(i)$. We call this phenomenon *tail-independence*.

Sub-claim 3. The group K preserves the C-relation on the branches at r_i.

Proof. Suppose u,v,w lie in the same F_i-class but distinct E_i-classes, so $\mu(u,v,w) = \sigma(u,v,w) = i$, and assume $C_i(u;v,w)$ holds in this branch. Let $x = x(i',g,h) \in K$. If $i' > i$, then $\chi(u)(i) = \chi(u^x)(i)$, $\chi(v)(i) = \chi(v^x)(i)$ and $\chi(w)(i) = \chi(w^x)(i)$, so $C_i(u^x;v^x,w^x)$ by tail-independence. If $i = i'$, then $C_i(u^x;v^x,w^x)$ since the action of x in the i^{th} coordinate is induced by an element of G^{V_i} which preserves the C-relation on V_i. If $i' < i$ then $C_i(u^x;v^x,w^x)$ holds by tail-independence. ■

Sub-claim 4. The group K preserves the L-relation and S-relation on the branches at r_i. That is, if $\mu(u,v,w) = \sigma(u,v,w) = i$, then for $x \in K$ we have $L(u;v,w) \Leftrightarrow L(u^x;v^x,w^x)$, and similarly for S.

Proof. This is similar to Sub-claim 3. ■

Sub-claim 5. The group K preserves L on $[n]$.

Proof. Let $u,v,w \in [n]$ be distinct with $L(u;v,w)$. By Sub-claim 4, we may suppose $i = \sigma(u,v,w) < \mu(u,v,w)$. Thus, two of u,v,w are F_i-equivalent and the other F_i-inequivalent to these. We suppose uF_iv and $\neg uF_iw$ (the other case are similar). Pick $z \in A^i(u)$ with $C_i(z;u,v)$, as shown in Figure 4.16.

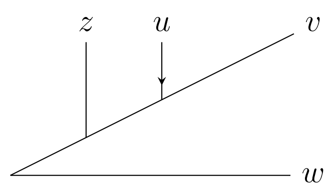

Figure 4.16

Then for $x \in K$, $L(u;v,w) \Leftrightarrow L(u;v,z) \overset{\text{by Sub-claim 4}}{\Longleftrightarrow} L(u^x;v^x,z^x) \Leftrightarrow L(u^x;v^x,w^x)$ (since x preserves the relations E_j, F_j and C_j). ■

Sub-claim 6. The group K preserves S on $[n]$.

Proof. Let $u, v, w, z \in [n]$ be distinct. Let i be greatest such that $u/E_i, v/E_i, w/E_i, z/E_i$ are distinct. Then at least two of u, v, w, z are F_i-equivalent. If all are F_i-equivalent, then K preserves any S-relation among these by Sub-claim 4. If just three of u, v, w, z are F_i-equivalent, then K preserves any S-relation among them by the proof of Sub-claim 5. If say uF_iv and $\neg uF_iw \wedge \neg uF_iz$, then as K preserves F_i, if $x \in K$ we have $u^xF_iv^x \wedge \neg u^xF_iw^x \wedge w^xF_iz^x$. We now see $S(u, v; w, z) \wedge S(u^x, v^x; w^x, z^x)$ as required. ■

By the sub-claims, the conditions of Lemma 4.1.16 are satisfied, completing the proof of Claim 4.

■

It follows that $[n]$ is a Jordan set for G.

■

Proposition 4.1.19. *Each pre-branch is a Jordan set for G in its action on M.*

Proof. Let R be a D-set of M, and let U be a branch of R at a ramification point r. Pick z lying in a branch at r other than U. We may choose a sequence $(r_i : i \in \mathbb{N})$ of ramification points which is coinitial in U, that is, for each ramification point r' in U there is $i \in \mathbb{N}$ such that for all $j \geq i$, r_j lies between r and r'.

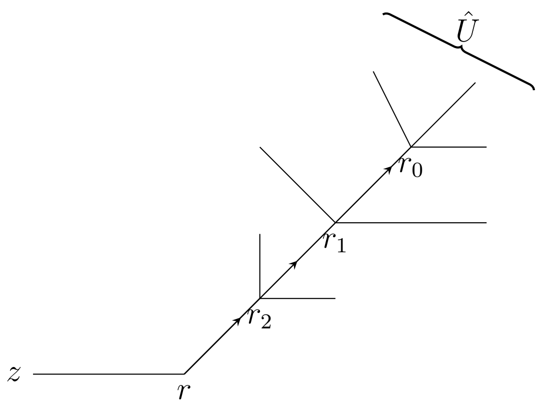

Figure 4.17

We may suppose in addition that r_{i+1} lies between r_i and r for each i, and that z lies in the special branch at r_i for each i. For each i, there is a union T_i of pre-branches at r_i which is a pre-direction of a higher D-set. We may suppose that for each i, r_i is a ramification point of one of the branches of T_{i+1}.

It follows that $T_i \subseteq T_{i+1}$ for each i and that $\bigcup_{i \in \mathbb{N}} T_i = \hat{U}$. Since pre-directions are Jordan sets by Lemma 4.1.18, each T_i is a Jordan set, so $\hat{U}$ is a Jordan set by Lemma 1.5.12. ∎

Corollary 4.1.20. *G_s is primitive on $M \setminus \{s\}$.*

Proof. This is (iv), Lemma 3.1.3. ∎

For the following lemma we will use Lemma 2.2.2 of [5],which we quote here:

Lemma 4.1.21. *Let H be a group acting on a set $\mathfrak{X}$. Consider a collection $\mathscr{K}$ of subsets of $\mathfrak{X}$, such that*

(i) each member $\mathfrak{R} \in \mathscr{K}$ has more than one element;

(ii) for $g \in H$ and $\mathfrak{R} \in \mathscr{K}$, $\mathfrak{R}^g \in \mathscr{K}$;

(iii) if $\mathfrak{R}_1, \mathfrak{R}_2 \in \mathscr{K}$, then $\mathfrak{R}_1 \subseteq \mathfrak{R}_2$ or $\mathfrak{R}_2 \subseteq \mathfrak{R}_1$ or $\mathfrak{R}_1 \cap \mathfrak{R}_2 = \varnothing$;

(iv) if $\mathfrak{u}, \mathfrak{v} \in \mathfrak{X}$ are distinct, then there is $\mathfrak{R} \in \mathfrak{X}$ with $\mathfrak{u}, \mathfrak{v} \in \mathfrak{R}$ and $\mathfrak{R} \neq \mathfrak{X}$;

(v) if $\mathfrak{u}, \mathfrak{v} \in \mathfrak{X}$ are distinct in $\mathfrak{X}$ then there is $\mathfrak{R} \in \mathfrak{X}$ containing $\mathfrak{u}$ but not $\mathfrak{v}$.

Define a ternary relation K such that

$$\forall \mathfrak{u}, \mathfrak{v}, \mathfrak{w} \in \mathfrak{X},\ K(\mathfrak{u}; \mathfrak{v}, \mathfrak{w}) \Leftrightarrow (\exists \mathfrak{R} \in \mathfrak{X})(\mathfrak{v}, \mathfrak{w} \in \mathfrak{R} \wedge \mathfrak{u} \notin \mathfrak{R}).$$

Then K is a C-relation on $\mathfrak{X}$.

Lemma 4.1.22. *There is a G_s-invariant C-relation on $M \setminus \{s\}$.*

Proof. Consider all the pre D-sets that contain s and the pre-branches $\hat{U}$ in these pre D-sets that do not contain s, with the property that s lies in the special branch at the ramification point at which U is a branch. Call this collection $\mathscr{K}$. The elements of this collection are all Jordan sets (by Proposition 4.1.19). Now we check (i)- (v) of the above Lemma, applied to G_s acting on $M \setminus \{s\}$.

(i) For all $\hat{U} \in \mathscr{K}$, $|\hat{U}| > 1$. Indeed, as each branch carries a C-structure such that the pre-directions are the parameters of a C-relation, then each pre-branch has more than one element.

(ii) If $\hat{U} \in \mathscr{K}$ and $g \in G_s$, then $\hat{U}^g \in \mathscr{K}$ by the description of $\mathscr{K}$.

(iii) $\mathscr{K}$ has no typical pair (Definition 1.5.11(a)). First, suppose that $\hat{U}, \hat{V} \in \mathscr{K}$ are pre-branches of the same D-set. Since $\hat{U}, \hat{V}$ both omit the element s of this D-set, it is immediate that $\hat{U}, \hat{V}$ do not form a typical pair.

Next, suppose $\hat{U}, \hat{V} \in \mathscr{K}$ are pre-branches of distinct but comparable D-sets R_i and R_j respectively with R_j below R_i. We may suppose that R_i corresponds to the ramification point r of R_j, and that V is a branch at the ramification point r' of R_j. If $r = r'$, then $\hat{U}$ is a union of pre-branches at r' omitting s, so contains $\hat{V}$ or is disjoint from $\hat{V}$. If r lies in the same branch at r' containing s, then again, $\hat{U}$ either contains $\hat{V}$ or $\hat{U} \cap \hat{V} = \varnothing$. If r is a ramification point lying in $\hat{V}$, then $\hat{U} \subset \hat{V}$. And if r lies in a branch at r' other than V or that containing s, then $\hat{U} \cap \hat{V} = \varnothing$.

Finally, suppose that $\hat{U}$ and $\hat{V}$ are pre-branches of D-sets R_1, R_2 labelling incomparable vertices ν_1, ν_2 of the structure tree. Let $\mu := \inf\{\nu_1, \nu_2\}$, and R be the D-set of μ, and suppose R_i corresponds to the ramification points r_i of R, for $i = 1, 2$. Thus, $\hat{U}$ and $\hat{V}$ correspond to union of pre-branches at r_1 and r_2 respectively of R, omitting s. If, say, r_2 is a ramification point of $\hat{U}$, then r_1 is not a ramification point of V, (otherwise $s \in \hat{U} \cup \hat{V}$), and $V \subset U$. Alternatively, r_2 is not a ramification point of $\hat{U}$, and r_1 is not a ramification point of $\hat{V}$, and in this case $\hat{U} \cap \hat{V} = \varnothing$.

(iv) Choose a D-set R such that the pre-D-set $\hat{R}$ contains u, v, s in distinct pre-directions. There is a ramification point r at R such that s lies in the special pre-branch at r, and u, v lie in the same other pre-branch $\hat{U}$ at r. Then $\hat{U} \in \mathscr{K}$ and contains u, v.

(v) Choose a D-set R such that $\hat{R}$ contains u, v, s in distinct pre-directions, meeting at ramification point r. There is a ramification point r' in the branch at r containing u, such that the branch at r' containing s is special. Let $\hat{U}$ be the pre-branch at r' containing u. Then $\hat{U} \in \mathscr{K}$ and contains u but not v.

Then define a ternary relation C_s such that for every $x, y, z \in M \setminus \{s\}$, the relation $C_s(x; y, z)$ holds if and only if $(\exists U \in \mathscr{K})(y, z \in U \wedge x \notin \mathscr{K})$. Then C_s is a C-relation by Lemma 4.1.21.

■

Lemma 4.1.23. *Each pre-D-set $\hat{R}_i$ is a Jordan set for G.*

Proof. Consider two distinct ramification points r_1, r_2 of R. Let U_{r_1} be the branch at r_1 which includes r_2, and U_{r_2} be the branch at r_2 containing r_1. We know by Proposition 4.1.19 that the corresponding pre-branches are Jordan sets and they form a typical pair, hence by Lemma 1.5.10 their union is a Jordan set and is the whole pre-D-set, so it is a Jordan set. ■

Lemma 4.1.24. *There is no G-invariant separation relation on M*

Proof. Choose a configuration in M as depicted, in some D-set.

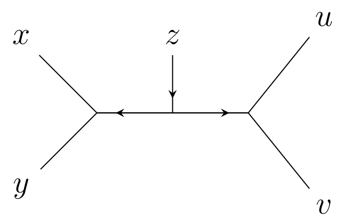

Figure 4.18

By semi-homogeneity there is $g \in G$ inducing $(x)(y)(z)(uv)$. An element with such cycle structure cannot preserve a separation relation.

■

Lemma 4.1.25. *There is no G-invariant Steiner system on M.*

Note: We use the idea of the proof of Lemma 1.5.13, which is also used in the proof of Lemma 6.5 in [7].

Proof. On the contrary, suppose there is a Steiner n-system. Let $s_1, \ldots, s_n$ be distinct in a block $\mathfrak{B}$ and s_{n+1} be in $\mathfrak{B}$. Since we may choose a D-set in which all s_i lie in different branches at a ramification point, there is a pre-branch V containing s_{n+1} and omitting $s_1, \ldots, s_n$. Let $t \in V$. Since V is a Jordan set, there is $g \in G_{(M \setminus V)}$ with $s_{n+1}^g = t$. As g fixes $s_1, \ldots, s_n$, it fixes setwise the unique block $\mathfrak{B}$ containing $s_1, \ldots, s_n$, so as $s_{n+1} \in \mathfrak{B}$, also $t \in \mathfrak{B}$; that is, $V \subseteq \mathfrak{B}$.

Let s^* be an element of $M \setminus \mathfrak{B}$ (hence not in V) and $\mathfrak{B}'$ be a block containing $s_1, \ldots, s_{n-2}, s_{n+1}, s^*$. As $|\mathfrak{B}'| \geq n+1$, there is $s^{**} \in \mathfrak{B}'$ distinct from $s_1, \ldots, s_{n-2}, s_{n+1}, s^*$ and $s^{**} \notin \mathfrak{B}$, so as $V \subseteq \mathfrak{B}$ then $s^{**} \notin V$. But $s_1, \ldots, s_{n-2}, s^*, s^{**}$ are all in $\mathfrak{B}'$ determine $\mathfrak{B}'$. So as $s_{n+1} \in V \cap \mathfrak{B}'$, by the argument of being V a Jordan set above, we get $V \subseteq \mathfrak{B}'$. So $V \subseteq \mathfrak{B} \cap \mathfrak{B}'$. But V is infinite and $|\mathfrak{B} \cap \mathfrak{B}'| = n-1$. This is a contradiction. ■

Lemma 4.1.26. *There is no G-invariant D-relation on M.*

Proof. Suppose, for a contradiction, that there is a G-invariant D-relation D defined on M. Fix x, y, z_0. Find $u_1 \in M \setminus \{x, y, z_0\}$ with $D(u_1, z_0; x, y)$. Note that in the argument below, we should not confuse D with the various D-sets in M coded by the structure tree.

Find a D-set R_1 of M containing u_1, z_0, x, y in distinct branches at the same ramification point r_1, and pick $v_1 \in M$ lying in the pre-branch at r_1 containing z_0, with $L(z_0; v_1, x)$ witnessed in this D-set. See Figure 4.19.

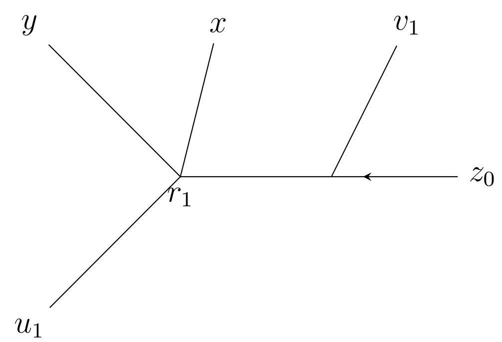

Figure 4.19: The D-set R_1

Let $z_1 \in M \setminus \hat{R}_1$. Choose $h_1, k_1 \in G_{z_0,z_1}$ with $(x, v_1)^{h_1} = (v_1, x)$ and $(u_1, v_1)^{k_1} = (v_1, u_1)$- these exist by semi-homogeneity.

In the D-relation on M, consider the regions P, Q, R, S as depicted.

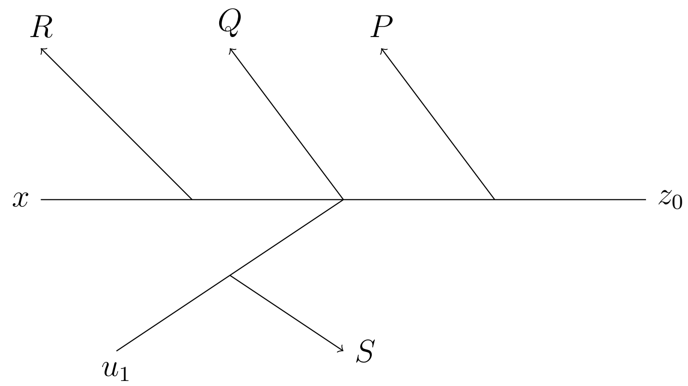

Figure 4.20

Let supp$\langle h_1, k_1\rangle$ denote the set of elements of M moved by some element of the subgroup $\langle h_1, k_1\rangle$ of G generated by h_1 and k_1. If say $v_1 \in R$, then we see that $R \cup S \subseteq$ supp$(k_1) \subseteq$ supp$\langle h_1, k_1\rangle$. If $v_1 \in S$ then $R \cup S \subseteq$ supp$(h_1) \subseteq$ supp$\langle h_1, k_1\rangle$. If $v_1 \in Q$ then $R \subseteq$ supp$(h_1) \subseteq$ supp$\langle h_1, k_1\rangle$, and $S \subseteq$ supp$(k_1) \subseteq$ supp$\langle h_1, k_1\rangle$. Finally, if $v_1 \in P$ then $R, S \subseteq$ supp$(h_1) \subseteq$ supp$\langle h_1, k_1\rangle$. Thus, wherever v_1 lies, $R \cup S \subseteq$ supp$\langle h_1, k_1\rangle$, so as h_1, k_1 fix z_1, so $z_1 \notin R \cup S$. Thus, $z_1 \in P \cup Q$. Since $D(u_1, z_0; x, y)$, $y \in R$, so we have the following picture.

Figure 4.21

Now we iterate this argument with (z_0, x, z_1) in place of (z_0, x, y). Pick $u_2 \in M \setminus \{x, z_0, z_1\}$ with $D(u_2, z_0; x, z_1)$. Find a D-set R_2 of M containing u_2, z_0, x, z_1 in distinct branches at the same ramification point r_2, and pick $v_2 \in M$ lying in the pre-branch at r_2 containing z_0, with $L(z_0; v_2, x)$ witnessed in this D-set. Let $z_2 \in M \setminus \hat{R}_1$. By semi-homogeneity there are $h_2, k_2 \in G_{z_0,z_2}$ with $(x, v_2)^{h_2} = (v_2, x)$ and $(u_2, v_2)^{k_2} = (v_2, u_2)$. Let $x, z_0, u_2, P', Q', R', S'$ replace x, z_0, u_1, P, Q, R, S above. We see that $z_2 \in P' \cup Q'$, and thus the D-relation on M satisfies the following picture.

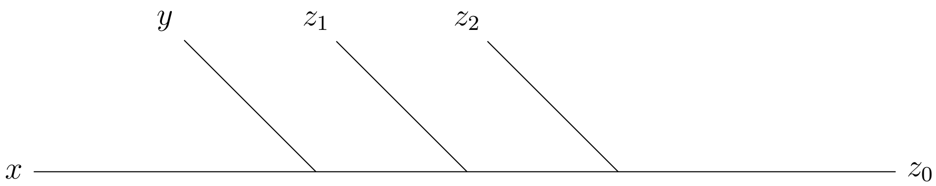

Figure 4.22

Observe that we have $L(z_1; x, z_0) \wedge L(z_2; x, z_0) \wedge L(z_2; x, z_1) \wedge L(z_2; z_0, z_1)$. Thus, by semi-homogeneity, there is $g \in G_{z_1,z_2}$ inducing (x, z_0). Such g does not preserve the D-relation, a contradiction. ∎

4.2 Proof of Main Theorem

In this section, we investigate the requirements to show that $G = \mathrm{Aut}(M, L, S)$ is an infinite primitive Jordan group preserving a limit of D-relations (Definition 1.5.19).

We may view M as an $\mathscr{L}$-structure, or as a structure in just the language with symbols L and S, since the other $\mathscr{L}$-symbols are $\varnothing$-definable in terms of L and S.

Let $\hat{R}$ be a pre-D-set with D-set R, let $H := G_{(M \smallsetminus \hat{R})}$ and let E be the equivalence relation on $\hat{R}$ corresponding to being in the same direction (the equivalence relation identified in Definition 3.2.9). Let D be the induced D-relation on $R = \hat{R}/E$.

Lemma 4.2.1. *In the above notation,*

(i) H preserves E and the relation D;

(ii) H is transitive on $\hat{R}$;

(iii) H is 2-transitive but not 3-transitive on R; and

(iv) E is the unique maximal H-congruence on $\hat{R}$.

Proof.

(i) H preserves E by Lemma 3.2.10(ii) as $H < G_{\{M \smallsetminus \hat{R}\}}$. Also, H preserves D follows from Lemma 3.2.12(iii).

(ii) This follows from Lemma 4.1.23.

(iii) Fix $x_0 \in \hat{R}$. We show that H_{x_0} is transitive on $\hat{R} \smallsetminus \{x_0\}$. Let u, v be distinct elements of $\hat{R} \smallsetminus \{x_0\}$. Choose a ramification point r such that there is a branch U at r containing u, v and omitting x_0. It is known that U is a Jordan set (pre-branches are Jordan sets) so there is $g \in G_{(M \smallsetminus \hat{U})} < H$ with $u^g = v$. However, H is not 3-transitive; for if $u, v, w \in R$ and meet at a ramification point r with $L(u; v, w)$ then there is no element of H inducing $(uv)(w)$.

(iv) We need to show that E is preserved by elements of H which is done in (i). To show the maximality, we show that H is 2-transitive on $\hat{R}/E$ and that is done in part (iii). For the uniqueness, suppose E^* is an H-congruence on $\hat{R}$ and there are $u, v \in \hat{R}$ with $\neg uEv$ and uE^*v. Since pre-directions are Jordan sets, for $v' \in \hat{R}$ if $v'Ev$ there is $g \in H$ fixing $M \smallsetminus (v/E)$ pointwise with $v^g = v'$. As $u^g = u$, g fixes $E^*(u)$, so vE^*v', so $v/E \subset v/E^*$, so E^* contains E properly, hence is universal.

■

Theorem 4.2.2. *G preserves a limit of D-relations on M.*

Proof. Let $G = \mathrm{Aut}(M)$. Then G is an infinite Jordan group acting on M. Indeed, M is a Fraïssé limit of the class $\mathscr{D}$, so it is associated with an infinite structure tree τ. Let J be a maximal chain from τ. Then J is a linear ordered. Let R_j be the D-set indexed by j, for $j \in J$. Then by the paragraph above Definition 3.2.6, for $i, j \in J$ we have $i < j \Leftrightarrow \hat{R}_j \subset \hat{R}_i$. So we get a strictly increasing chain of subsets of M such that they get bigger by going down in the structure tree. Let $\hat{R}_j$ be the pre-D-set corresponding to R_j, let $H_j := G_{(M \setminus \hat{R}_j)}$, and let E_j be the unique maximal H_j congruence on $\hat{R}_j$ as in Lemma 4.2.1(iv). Similarly, $\{H_j : j \in J\}$ is an increasing chain of subgroups of G. Then we want to check the list (i)-(viii) in Definition 1.5.19.

(i) This is (ii) and (iv) in the Lemma 4.2.1 above.

(ii) This is (i) and (iii) in the Lemma 4.2.1 above. Note that since pre-branches are Jordan sets of G, branches are Jordan sets of H, so (H, R) is a Jordan group.

(iii) It is clear that $\bigcup(R_i : i \in J) = M$.

(iv) Let $H := \bigcup_{j \in J} H_j$. Then H is a Jordan group on M, since any pre-branch of R_j is a Jordan set for H_j. The group G is not 3-transitive since it preserves the relation L (and $L(u; v, w) \to \neg L(v; u, w)$), hence H is not 3-transitive.

We now show that H is 2-primitive on M . We first observe a point from Lemma 4.1.18. In the proof of that lemma, if $[n]$ is a pre-direction corresponding to vertex j_n, then for each $j < j_n$ there is a D-set R_j and ramification point r_j such that $[n]$ is a union S_j of branches at r_j. It follows from that proof that for each branch $U \in S_j$ at r_j, the pointwise stabiliser of the complement of $[n]$ induces G^U on U.

Now let $x_0 \in M$, and let ρ be a nontrivial H_{x_0}-congruence on $M \setminus \{x_0\}$. We must show that ρ is universal. Pick distinct $u, v \in M \setminus \{x_0\}$ with $u \neq v$. Choose $j \in J$ such that x_0, u, v lie in distinct pre-directions of R_j. For a contradiction, we may suppose that if $\mathfrak{B}$ is the ρ-class containing u, then $\mathfrak{B}$ is a proper subset of $R_j \setminus \{x_0\}$.

Let r be a ramification point of R_j such that u, v lie in the same pre-branch $\hat{U}$ at r, and x_0 in a different pre-branch. Let C be the C-relation induced on the corresponding branch U at r. Suppose there are distinct $u', v', w' \in \hat{U}$ such that $C(u'/E_j; v'/E_j, w'/E_j)$ and $u' \rho w'$. Let V be the largest branch in U containing v', w' and omitting u'. Then V is a Jordan set, so there is $g \in G_{(M \setminus \hat{V})} < H_{x_0}$ with $(u', w')^g = (u', v')$. Since g fixes u', it follows that $v' \rho w'$. Thus $\mathfrak{B} \cap \hat{R}_j$, is a pre-branch of R_j, the union of a nested sequence of pre-branches of R_j, or a union of more than one pre-branches at some fixed vertex. By choosing j sufficiently low in the structure tree, we may assume that the last one holds, i.e. $\mathfrak{B} \cap \hat{R}_j$ is the union of more than one pre-branch at a ramification point r_j of R_j.

Pick a ramification point r^* of R_j such that elements of $\mathfrak{B}$ and x_0 lie in distinct branches at r^* with the one containing elements of $\mathfrak{B}$ non-special. There is a direction $[n]$ which is a union of pre-branches at r^* including the pre-branch $\hat{V}$ at r^* containing $\mathfrak{B}$, and excluding that containing x_0. Now by the observation above, since $G_{(M \setminus [n])} \leq H$, H induces the full group G^V on V. In particular, there is a ramification point r between r^* and r_j such that H_{x_0} contains an element h with $u^h = u$ and $r_j^h = r$. It follows that $\mathfrak{B}^h \supset \mathfrak{B}$, contradicting that $\mathfrak{B}$ is a block of H_{x_0}. See the following picture.

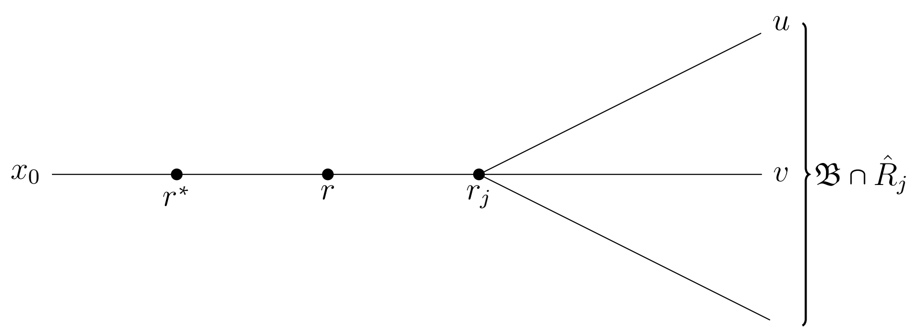

Figure 4.23: R_j

(v) $E_j\big|_{\hat{R}_i} \subseteq E_i$ if $i > j$, by Lemma 4.1.5.

(vi) $\bigcap(E_i : i \in J)$ is equality. Let $u, v \in M$ be distinct. By 2-transitivity of G, there is a D-set R such that u, v lie in distinct directions of R. Choose $j \in J$ such that the corresponding D-set R_j labels a vertex of the structure tree below that of R. Then u, v lie in distinct directions of R_j, so $\neg uE_jv$.

(vii) Given $g \in G$, choose a segment I of J which lies in the common part of J and J^g. Let $i_0 \in I^{g^{-1}} \subseteq J^{-1}$. Then for any $i < i_0$ we have $i^g < i_0^g$ and so $i^g \in I$. Thus $i^g = j$ for some $j \in J$. Hence $g^{-1}H_ig = H_j$ and $R_i^g = R_j$.

(viii) This is by 4.1.22.

Each pre-D-set $\hat{R}_i$ is a Jordan set of G. There is a natural congruence E_i on R_i given by pre-directions. Each D-set $\hat{R}_i/E_i$ has a D-relation defined on it by Definition 3.2.11(ii). It is a Jordan group with branches are Jordan sets. ∎

Theorem 4.2.3. *There is a ternary relation L and a quaternary relation S on a countably infinite set M, such that if $G := \mathrm{Aut}(M)$, then G is oligomorphic, 3-homogeneous, 2-primitive but not 3-transitive or 4-homogeneous on M, and a Jordan group preserving a limit of D-relations on M, and not preserving any of the structures of types $(a)-(i)$ in Theorem 1.5.22.*

Proof. This is by Corollary 2.4.3, Lemma 3.1.3, Lemma 4.1.24, Lemma 4.1.25, Lemma 4.1.26 and Theorem 4.2.2. Note that G cannot preserve a linear or circular order or a linear betweenness relation since it does not preserve a separation relation, G cannot preserve a C-relation since it does not preserve a D-relation, and cannot preserve a semilinear order or general betweenness relation since it is 2-primitive. ∎

Chapter 5

Extensions and Open Problems

We do not have a full theory of the structure M, because we do not have suitable axioms for the relations L and S. If some one can do that then a number of questions arise.

Question 1. Can we recover the structure tree from the two relations L and S?

As an attempt to answer that we try to write axioms for the relations L and S.

Definition 5.0.1. *Let X be a set. Then define a ternary relation L on X such that:*

(L1) $(\forall x, y, z) L(x; y, z) \rightarrow (x \neq y \wedge x \neq z \wedge y \neq z)$;

(L2) $(\forall x, y, z) L(x; y, z) \leftrightarrow L(x; z, y)$*;*

(L3) $(\forall x, y, z) L(x; y, z) \rightarrow \neg L(y; x, z) \wedge \neg L(z; x, y)$*;*

(L4) $(\forall x, y, z$ *distinct)* $L(x; y, z) \vee L(y; x, z) \vee L(z; x, y)$*;*

(L5) $(\forall x, y, z, w) L(x; y, z) \rightarrow (L\{w, y, x\} \wedge L\{w, y, z\}) \vee (L\{w, z, x\} \wedge \{L(w, z, y\}) \vee (L\{w, x, y\} \wedge L\{w, x, z\}) \vee (L(x; y, z) \wedge L(x; y, w) \wedge L(x; z, w))$.

Here $L\{a, b, c\}$ *means* $L(a; b, c) \vee L(b; a, c) \vee L(c; a, b)$ *for distinct* $a, b, c \in \{x, y, z, w\}$.

Then we say (X, L) is an *L-set*.

Definition 5.0.2. *Let X be a set. Then define a quaternary relation S on X such that for all $x, y, z, w \in X$:*

(S1) $S(x, y; z, w) \to (x \neq y \wedge z \neq w \wedge x \neq z \wedge y \neq w)$.

(S2) $S(x, y; z, w) \to S(x, y; w, z) \wedge S(y, x; z, w) \wedge S(z, w; x, y)$.

(S3) $S(x, y; z, w) \to \neg S(x, z; y, w) \wedge \neg S(x, w; z, y) \wedge \neg S(y, z; x, w) \wedge \neg S(y, w; x, z)$.

(S4) $(\forall x, y, z, w, t)(S(x, y; z, w) \wedge \neg L(t; x, y) \wedge \neg L(t; x, y) \wedge \neg L(t; y, z) \wedge \neg L(t; x, w) \wedge \neg L(t; y, w) \wedge \neg L(t; z, w)) \to S(t, y; z, w) \vee S(x, y; z, t)$.

Then we say that (X, S) is an S-*set*.

Are there more axioms we can add to the L-axioms and the S-axioms? However, if Question.1 has a positive answer then we aim to improve Adeleke-Macpherson Theorem (Theorem 1.5.22) to replace "preserves a limit of D-relations" by "preserves an (L, S)-structure."

Question 2. In Definition 1.5.19 of limits of D-relations, can we consider $(J, \leq)$ a semilinearly ordered set rather than linear, and require it to be G-invariant?

As an attempt to tackle this, assume that $\mathcal{F} = \{\Gamma_i : i \in I\}$ a chain of Jordan sets with I is a totally ordered set as in the Definition of a limit of D-relations. Let $\mathcal{F}^*$ be the translate of $\mathcal{F}$, i.e. $\mathcal{F}^* = \{\Gamma^g : \Gamma \in \mathcal{F}, g \in G\}$, where G is an infinite permutation Jordan group. Our goal is to show that $\mathcal{F}^*$ is a semilinearly ordered set by inclusion.

In order to show this we try to prove whenever $\Gamma \in \mathcal{F}^*$, then the set $\{\Delta \in \mathcal{F}^* : \Gamma \subseteq \Delta\}$ is totally ordered by inclusion.

Question 3. (i) Show that the structure M is not homogeneous, i.e. there is some isomorphism between finite substructures of M cannot be extended to an automorphism.

(ii) Can we homogenise the structure M, that is, find finitely many invariant relations such that M becomes homogeneous when symbols are added for these relations.

Question 4. Give an algebraic construction of a limit of D-relations, e.g. via valued fields.

Question 5. Study the model theory of our construction, e.g. investigate whether M satisfies NIP.

Question 6. If M is the structure constructed in Chapters 2 and 3, and $G = \mathrm{Aut}(M)$, let $f(k)$ be the number of orbits of G on the set of k-subsets of M. How fast does the sequence $(f(k))$ grow? Is it bounded above by some exponential function?

Question 7. Is there a relationship between limits of B-relations and limits of Drelations?

Question 8. Could the group G preserve a limit of B-relations, or a limit of Steiner systems?

Question 9. By the procedure that used to build the structure M, can one construct further examples?

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