Beyond Regular Semigroups

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Abstract

The topic of this thesis is the class of *weakly U-abundant semigroups*. This class is very wide, containing inverse, orthodox, regular, ample, adequate, quasi-adequate, concordant, abundant, restriction, Ehresmann and weakly abundant semigroups. A semigroup S with subset of idempotents U is weakly U-abundant if every $\widetilde{\mathcal{R}}_U$-class and every $\widetilde{\mathcal{L}}_U$-class contains an idempotent of U, where $\widetilde{\mathcal{R}}_U$ and $\widetilde{\mathcal{L}}_U$ are relations extending the well known Green's relations $\mathcal{R}$ and $\mathcal{L}$. We assume throughout that our semigroups satisfy a condition known as the Congruence Condition (C).

We take several approaches to weakly U-abundant semigroups. Our first results describe those that are analogous to completely simple semigroups. Together with an existing result of Ren this determines the structure of those weakly U-abundant semigroups that are analogues of completely regular semigroups, that is, they are *superabundant*. Our description is in terms of a semilattice of rectangular bands of monoids.

The second strand is to aim for an extension of the Hall-Yamada theorem for orthodox semigroups as spined products of inverse semigroups and fundamental orthodox semigroups. To this end we consider weakly B-orthodox semigroups, where B is a band. We note that if B is a semilattice then a weakly B-orthodox semigroup is exactly an Ehresmann semigroup. We provide a description of a weakly B-orthodox semigroup S as a spined product of a fundamental weakly $\overline{B}$-orthodox semigroup S_B (depending only on B) and S/γ_B, where $\overline{B}$ is isomorphic to B and γ_B is the analogue of the least inverse congruence on an orthodox semigroup. This result is an analogue of the Hall-Yamada theorem for orthodox semigroups. In the case that B is a normal band, or S is weakly B-superabundant, we find a closed form δ_B for γ_B, which simplifies our result to a straightforward form.

For the above to work smoothly in the case S is weakly B-superabundant, we need to find a canonical fundamental weakly B-superabundant subsemigroup of S_B. This we do, and give the corresponding answers in the case of the Hall semigroup W_B and a number of intervening semigroups.

We then change our direction. A celebrated result of Nambooripad shows that regular semigroups are determined by ordered groupoids built over a regular biordered set. Our aim, achieved at the end of the thesis, is to extend Nambooripad's work to *weakly U-regular* semigroups, that is, weakly U-abundant semigroups with (C) and U generating a regular subsemigroup whose set of idempotents is U.

As an intervening step we consider weakly B-orthodox semigroups in this light. We take two approaches. The first is via a new construction of an *inductive generalised category over a band*. In doing so we produce a new approach to characterising orthodox semigroups, by using *inductive generalised groupoids*. We show that the category of weakly B-orthodox semigroups is isomorphic to the category of inductive generalised categories over bands. Our approach is influenced by that of Nambooripad, however, there are significant differences in strategy, the first being the introduction of generalised categories and the second being that it is more convenient to consider (generalised) categories equipped with pre-orders, rather than with partial orders. Our work may be regarded as extending a result of Lawson for Ehresmann semigroups. We also examine the trace of a weakly B-orthodox semigroup, which is a primitive weakly B-orthodox semigroup.

We then take a more 'traditional' approach to weakly B-orthodox semigroups via *band categories* and *weakly orthodox categories over a band*, equipped with two pre-orders. We show that the category of weakly B-orthodox semigroups is equivalent to the category of weakly orthodox categories over bands. To do so we must substantially adjust Armstrong's method for concordant semigroups.

Finally, we consider the most general case of weakly U-regular semigroups. Following Nambooripad's theorem, which establishes a correspondence between algebraic structures (inverse semigroups) and ordered structures (inductive groupoids), we build a correspondence between the category of weakly U-regular semigroups and the category of weakly regular categories over regular biordered sets, equipped with two pre-orders.

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Preface

The aim of this thesis is to investigate weakly U-abundant semigroups, using both techniques developed for regular and abundant semigroups, and new ones. The relevant definitions concerning the classes of semigroups in question are given in Chapter 2.

Fundamental semigroups, that is, regular semigroups having no non-trivial idempotent separating congruences, have played an important role in the structure theory of regular semigroups, especially, in the study of inverse semigroups. As an extension of Munn's approach to inverse semigroups, Hall constructed the fundamental semigroup W_B depending only on B, which is a subsemigroup of $\mathcal{OP}(B/\mathcal{L}) \times \mathcal{OP}^*(B/\mathcal{R})$, where for any partially ordered set X, $\mathcal{OP}(X)$ is the monoid of its order preserving selfmaps, with dual $\mathcal{OP}^*$. Also, he showed that if S is an orthodox semigroup with band of idempotents B, then there exists a morphism $\phi : S \to W_B$ whose kernel is μ, the maximum idempotent separating congruence on S. Consequently, an orthodox semigroup S with band of idempotents B is fundamental if and only if it is isomorphic to a full subsemigroup of W_B. Besides, Hall-Yamada showed that a regular semigroup S with band of idempotents B is an orthodox semigroup if and only if it is isomorphic to the spined product of the Hall semigroup W_B and S/γ, where γ is the least inverse congruence on S.

In 1981, El-Qallali and Fountain [7] generalised this result to abundant semigroups with band of idempotents, and satisfying the idempotent connected condition (IC) [8]. They described such a semigroup S having a band of idempotents B as a spined product of W_B and S/δ_B, where δ_B is the analogue of the least inverse congruence (δ_B is in fact the least type A, or ample, congruence on S). In [46] Ren, Shum and the author similarly extended this approach to describe weakly B-orthodox semigroups with a stronger version (PIC) of (IC). We note that an abundant semigroup has (PIC) if and only if it has (IC). Condition (PIC) is designed so that W_B can be used in the spined product construction; but for weakly B-abundant semigroups it is stronger than (IC). The next step was made by El-Qallali, Fountain and Gould in [6]. They built an analogous theory for weakly B-orthodox semigroups with (IC), or the yet weaker (WIC) [14], this time using semigroups U_B and V_B in place of W_B. Here U_B and V_B are the largest fundamental semigroups containing a band of idempotents B in the given classes. To do this they make heavy use of the congruence δ_B. Most recently, Gomes and Gould [17,18] removed the idempotent connected condition (WIC) (or (IC)) from the results of [6] [7], making use of a completely fresh technology to construct a B-fundamental weakly B-orthodox subsemigroup S_B of $\mathcal{OP}(B/\mathcal{L}) \times \mathcal{OP}^*(B/\mathcal{R})$. The missing step is the spined product result in the case no idempotent connected condition holds. The aim of Chapter 5 is to provide such a result.

From Chapter 6, we change our angle to investigate the connection between algebraic structures and ordered structures. Ehresmann-Schein-Nambooripad (ESN) built a correspondence between inverse semigroups and inductive$_1$ groupoids. Here an inductive$_1$ groupoid is a groupoid equipped with a partial order possessing restrictions and co-restrictions, and the set of idempotents forming a semilattice under the partial order. The subscript is used to distinguish this meaning of the word 'inductive' from others that will appear later in this thesis. Inverse semigroups are precisely regular semigroups in which the idempotents form a semilattice. Consequently, we can regard the set of idempotents of a regular semigroup as a generalisation of a semilattice. This idea is precisely described in the definition of a regular biordered set, introduced by Nambooripad [38]. In that article, Nambooripad set up a connection between regular semigroups and inductive$_2$ groupoids. Such a groupoid is a (functorially) ordered groupoid equipped with the structure of a regular biordered set on its identities, which is compatible with the ordered groupoid structure. In 1988, Nambooripad's work for regular semigroups was extended by Armstrong [1] from regular to concordant semigroups, replacing ordered groupoids by ordered cancellative categories. A concordant semigroup is an abundant semigroup with a regular biordered set of idempotents and satisfying the extra condition of being idempotent connected (IC), which is a condition of a standard type that gives some control over the position of idempotents in products of elements of a semigroup. In 1991, Lawson [32] generalised the ESN theorem in a different direction to Ehresmann semigroups. In his work, he used two partial orders on an Ehresmann semigroup to overcome the lack of the idempotent connected condition, and established a correspondence between Ehresmann semigroups and Ehresmann categories. In Chapters 7 and 9 we concentrate on the connection between weakly B-orthodox semigroups and ordered structures, and in Chapter 10 we move away to the general case of weakly U-regular semigroups. These are weakly U-abundant semigroups with (C) and U generating a regular subsemigroup whose set of idempotents is U.

The structure of the thesis is as follows.

Chapter 1 presents some basic definitions and results related to regular semigroups, biordered sets and categories.

Chapter 2 gives the basic definitions and fundamental notions concerning abundant semigroups and weakly U-abundant semigroups, where U is a subset of idempotents of a semigroup.

Chapter 3 establishes the structure of completely $\widetilde{\mathcal{J}}_U$-simple semigroups which are weakly U-superabundant semigroups with a single $\widetilde{\mathcal{J}}_U$-class and satisfying the Congruence Condition (C). Here Condition (C) means the relations $\widetilde{\mathcal{R}}_U$ and $\widetilde{\mathcal{L}}_U$ are left and right congruences, respectively. We show that a completely $\widetilde{\mathcal{J}}_U$-simple semigroup is isomorphic to a rectangular band of monoids $M_{i\lambda}$ ($i \in I$, $\lambda \in \lambda$) and satisfying Conditions called (R) and (L). Such conditions give some control over the position of idempotents in the $\widetilde{\mathcal{D}}_U$-class. Finally, we build on an existing result of Ren to show that a weakly U-superabundant semigroup with (C) is a semilattice of rectangular bands of monoids satisfying Conditions (R) and (L).

For the purpose of Chapter 5, we study in **Chapter 4** fundamental weakly B-superabundant semigroups. We find the largest full completely regular subsemigroup of the Hall semigroup W_B, and correspondingly, weakly B-superabundant subsemigroups with (C) of V_B (resp. U_B, S_B).

In **Chapter 5**, we obtain a general structure theorem for weakly B-orthodox semigroups as a spined product, which may be thought of as an analogue of the Hall-Yamada theorem. Our result is rather detailed, but simplifies drastically in the case γ_B, the analogue of the least inverse congruence, has the closed form δ_B, and so in particular, if the band B is normal or S is weakly B-superabundant.

In **Chapter 6** we briefly recall some of the historical achievements such as the

Ehresmann-Schein-Nambooripad (ESN) Theorem, and its many extensions due to Armstrong [1,2], Lawson [32], Meakin [35,36] and Nambooripad [38–40]. These results set up a connection between algebraic structures and ordered structures.

In **Chapter 7** we introduce a new approach. We define inductive generalised categories over bands and pseudo-functors. We show that the category of weakly B-orthodox semigroups is isomorphic to the category of inductive generalised categories. We then turn our attention to some special cases including orthodox semigroups, and in particular recover Lawson's work for Ehresmann semigroups. Our reasoning is briefly as follows. From a regular (concordant) semigroup one can produce a certain ordered category and then endow the category with a so-called pseudo-product. Unfortunately this need not produce the original semigroup: to do so requires factoring by a congruence. Our use of inductive generalised categories circumvents this latter inconvenience. A further point is that we could use partial orders on a semigroup as standard in this area, but to do so would be rather clumsy. It turns out that pre-orders provide an effective method.

In **Chapter 8**, we change our angle a little to discuss the trace of weakly B-orthodox semigroups. We show that the trace of a weakly B-orthodox semigroup is a primitive weakly B-orthodox semigroup and we investigate primitive weakly B-orthodox semigroups via blocked Rees matrix semigroups, which are introduced in [10].

The purpose of **Chapter 9** is to revisit weakly B-orthodox semigroups and provide a correspondence with a class of categories (posscessing two orders) that is more akin to the original approach of Nambooripad and Armstrong. That is, we use triples such as in [1]. A significant point is that we continue to use two pre-orders instead of a partial order in our work and further new tricks to overcome the lack of an idempotent connected condition. It turns out that without the (IC) condition and without the idempotents forming a semilattice, pre-orders provide the most elegant approach. At the end of this chapter, we discuss some special cases including orthodox semigroups, and recover Lawson's work for Ehresmann semigroups.

Chapter 10 focuses on weakly U-regular semigroups. We investigate the correspondence between weakly U-regular semigroups and certain categories, by extending the techniques introduced in Chapter 9.

In **Chapter 11** we are concerned with some special kinds of weakly U-regular semigroups. We recover Armstrong's work for concordant semigroups and Nambooripad's work for regular semigroups.

We tried to keep some homogeneity in notation. Most of the time, we use Greek letters for functions, lower case letters for elements, capital letters for sets and bold face letters for categories. We write functions and functors on the right.

We use the term *morphism* for *homomorphism*. Semigroups are usually denoted by S and monoids by M, but this notation is not frozen: we may also use S for monoids and M for semigroups if needed. In general, we use B and E (or U) to denote a band and a set of idempotents, respectively.

The reader wishing to negotiate a pathway through this thesis can use the following diagram.

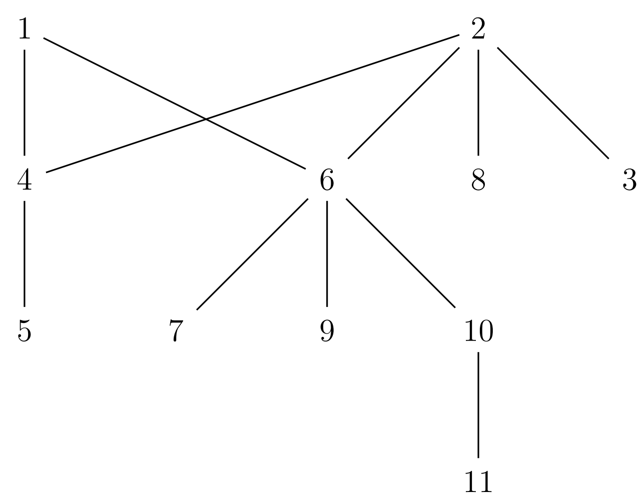

Figure 1: The structure of this thesis

The main result of Chapter 9 is in fact a special case of that of Chapter 10. However, Chapter 9 introduces many of the new techniques required, but in the more concrete content of a band, rather than a regular biordered set. The reader wishes to avoid the full technicalities of Chapter 10 may wish to focus on Chapter 9.

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Author's Declaration

Chapters 1 and 2 mainly present definitions and results by other authors. Chapter 1 is concerned with fundamental information about semigroup theory, biordered sets and category theory, mostly from [26], [38] and [34]. In Chapter 2, results about abundant semigroups and weakly U-abundant semigroups are introduced from [8], [10], [14], [32], [33] and [44]. Examples provided and results about weakly B-orthodox semigroups are from a joint paper [20] with Gould.

In Chapter 3, some results are stated and used from [44]. This chapter forms a part of [51].

In Chapter 4, the fundamental orthogroup in the Hall semigroup and its generalisation in U_B, V_B and S_B are constructed via two different methods. One method contributes to [49]. We are grateful to Professor Mária B. Szendrei for her advice with regard to the orthodox case and for giving us one technique.

Chapter 5 is a part of [51]. Examples provided are from [17], which are referenced accordingly.

Chapter 6 provides background information about the connection between algebraic structures and ordered structures, mostly from [1, 2], [32], [35, 36] and [38–40].

The contents of Chapter 7 is from a joint paper [20] with Gould. The main result of this chapter is an extension of Lawson's work for Ehresmann semigroups [32].

In Chapter 8, definitions and results by Fountain [10] on blocked Rees matrix semigroups are presented. Results in Section 8.4 form a part of the joint paper [20] with Gould.

In Chapters 9 and 10, we use an adjustment of Armstrong's method for concordant semigroups [1] to build our results. Our modifications are to compensate for the lack of an idempotent connected condition and the fact on semigroups are only *weakly U-abundant*. Some of proofs are similar to those of Armstrong. The contents of Chapter 9 will form a paper [50].

In Chapter 11, we use the main result of Chapter 10 to recover Armstrong's work for concordant semigroups and Nambooripad's work for regular semigroups.

Chapter 1

Basic Theory I

In this chapter, we mainly present certain basic definitions and results mostly taken from [26], [38] and [34]. For further details of semigroup theory, we refer the reader to [26], for biordered sets to [38] and for category theory to [34].

1.1 Relations

Let X be a non-empty set. A subset ρ of $X \times X$ is called a (*binary*)*relation* on X. It is worth specifically mentioning three special relations on X: the *empty subset* ϕ of $X \times X$, the *universal* relation $X \times X$ and the *identity* relation

$$1_X = \{(x, x) : x \in X\}.$$

Let $\mathcal{B}(X)$ denote the set of all binary relations on X. We define a binary operation $\circ$ on $\mathcal{B}(X)$ by the rule that for any $\rho, \sigma \in \mathcal{B}(X)$,

$$\rho \circ \sigma = \{(x, y) \in X \times X : (\exists z \in X)(x, z) \in \rho \text{ and } (z, y) \in \sigma\}.$$

Lemma 1.1. *[26] The set $\mathcal{B}(X)$ forms a semigroup under $\circ$.*

Let ρ be a relation on X. We define the *domain* $\mathrm{dom}(\rho)$ of ρ by

$$\mathrm{dom}\ (\rho) = \{x \in X : (\exists y \in X)(x, y) \in \rho\}$$

and the *co-domain* $\operatorname{ran}(\rho)$ of ρ by

$$\operatorname{ran}(\rho) = \{y \in X : (\exists x \in X)(x, y) \in \rho\}.$$

For any $x \in X$, we define

$$x\rho = \{y \in X : (x, y) \in \rho\}.$$

The inverse of a relation ρ on X is the relation ρ^{-1} on X defined by

$$\rho^{-1} = \{(y, x) \in X \times X : (x, y) \in \rho\}.$$

We pause to remark that for all $\rho, \rho_1, \ldots, \rho_n$ in $\mathcal{B}(X)$,

$$(\rho^{-1})^{-1} = \rho,$$

$$(\rho_1 \circ \rho_2 \circ \ldots \circ \rho_n)^{-1} = \rho_n^{-1} \circ \ldots \circ \rho_2^{-1} \circ \rho_1^{-1}$$

and

$$\operatorname{dom}(\rho^{-1}) = \operatorname{ran}(\rho),\ \operatorname{ran}(\rho^{-1}) = \operatorname{dom}(\rho).$$

If ρ is a relation on X, we shall usually write '$x\ \rho\ y$' for '$(x, y) \in \rho$'.

We say that a relation ρ on X is *reflexive* if $x\ \rho\ y$ for all x in X, *symmetric* if $x\ \rho\ y$ implies $y\ \rho\ x$ for all x, y in X, *anti-symmetric* if $x\ \rho\ y$ and $y\ \rho\ x$ together imply $x = y$, and *transitive* if $x\ \rho\ y$ and $y\ \rho\ z$ together imply that $x\ \rho\ z$ for all $x, y, z \in X$.

A relation ρ on X is said to be a *pre-order* if it is reflexive and transitive. A pre-order is also sometimes called a *quasi-order*. In this thesis, we prefer to call it a pre-order.

A relation ρ on X is called a *partial order* if it is reflexive, anti-symmetric and transitive.

A relation ρ is an *equivalence* relation on X if it is reflexive, symmetric and transitive. If ρ is an equivalence relation, then the sets $x\rho$ are called *ρ-classes* or *equivalence classes* containing x, where $x \in X$. The set of all ρ-classes of X is said to be the *quotient set of X by ρ* and is denoted by X/ρ. Clearly, the mapping $\rho^\natural : X \to X/\rho$ defined by

$$x\rho^\natural = x\rho \qquad \big(x \in X\big)$$

is well-defined. We shall call it the *natural mapping* associated to ρ.

If $\{\rho_i : i \in I\}$ is a non-empty family of equivalences on a set X, then it is easy to see that the intersection $\bigcap\{\rho_i : i \in I\}$ is again an equivalence. If ρ is a relation on X, then the family of equivalence relations containing ρ is non-empty, since $X \times X$ is one such equivalence, and so the intersection of all the equivalences on X containing ρ is an equivalence, that is, the unique minimum equivalence on X containing ρ. We shall call it the equivalence on X *generated by* ρ and denote it by ρ^e. It is routine to show that

$$\rho^e = [\rho \cup \rho^{-1} \cup 1_X]^t,$$

where $R^t = \bigcup_{n=1}^{\infty} R^n$ is the *transitive closure* of an arbitrary relation R. If ρ and σ are equivalences on X, then the family of equivalence relations containing ρ and σ is non-empty, as $X \times X$ is one such equivalence. By definition, $(\rho \cup \sigma)^e$ is the least equivalence containing ρ and σ. We will denote it by $\rho \vee \sigma$.

An extremely useful result is that:

Lemma 1.2. *[26] If ρ and σ are equivalences on a set X such that $\rho \circ \sigma = \sigma \circ \rho$, then $\rho \vee \sigma = \rho \circ \sigma$.*

Note that if ρ is a pre-order on X, then the relation $\equiv_\rho$ on X given by

$$x \ \equiv_\rho \ y \text{ if and only if } x \ \rho \ y \text{ and } y \ \rho \ x \qquad \big(x, y \in X\big),$$

is an equivalence relation. Since ρ is a pre-order, that is, ρ is reflexive and transitive, it immediately leads to $\equiv_\rho$ being an equivalence relation. For any $x \in X$, we will use $[x]$ to denote the $\equiv_\rho$-class of X containing x.

In addition, the relation $\preceq_\rho$ on $X/\equiv_\rho$ defined by

$$[x] \ \preceq_\rho \ [y] \text{ if and only if } x \ \rho \ y$$

is a well-defined partial order. Since if $x \ \equiv_\rho \ x'$, $y \ \equiv_\rho \ y'$ and $x \ \rho \ y$, then $x' \ \rho \ x \ \rho \ y \ \rho \ y'$, and so $x' \ \rho \ y'$, so that the choice of x and y does not matter. Hence $\preceq_\rho$ is well-defined.

It is easy to see that $\preceq_\rho$ is reflexive and transitive as ρ is reflexive and transitive. To show that $\preceq_\rho$ is anti-symmetric, we suppose that $[x] \preceq_\rho [y]$ and $[y] \preceq_\rho [x]$. Then $x \; \rho \; y$ and $y \; \rho \; x$, so that $x \equiv_\rho y$, that is, $[x] = [y]$.

1.2 Ordered sets

A *partially ordered set* $(X, \leq)$ is a non-empty set X together with a partial order $\leq$. Let X be a partially ordered set with respect to $\leq$ and let Y be a non-empty subset of X. An element a of Y is called *maximal* if there is no element of Y strictly greater than a, that is, if for any $y \in Y$ we have $a \leq y$, then $y = a$. An element b of Y is called *maximum* if $y \leq b$ for all $y \in Y$. Obviously, a maximum element is maximal, but the converse is not necessarily true. We say that an element x in X is a *lower bound* for Y if $c \leq y$ for every $y \in Y$. If the set of lower bounds of Y is non-empty and has a maximum element d, we refer to d as the *greatest lower bound*, or *meet* of Y. The element d is unique if it exists, and we write

$$d = \bigwedge\{y : y \in Y\}.$$

In particular, if $Y = \{a, b\}$, we denote $d = a \wedge b$. If X is a partially ordered set with respect to $\leq$ and such that $a \wedge b$ exists for every $a, b \in X$, we say that $(X, \leq)$ is a *lower semilattice*. In a lower semilattice $(X, \leq)$ we have that, for all $a, b \in X$,

$$a \leq b \text{ if and only if } a \wedge b = a.$$

Analogously, we define the *least upper bound* or *join*

$$\bigvee\{y : y \in Y\}$$

of a non-empty subset Y of X and an *upper semilattice*.

1.3 Semigroups and Green's relations

1.3.1 Basic definitions

Let S be a non-empty set. A *binary operation* on S is a mapping μ from $S \times S$ into S. The pair (S, μ) is said to be a *semigroup* if μ is associative, that is, for all $x, y, z \in S$, $((x, y)\mu, z)\mu = (x, (y, z)\mu)\mu$. To avoid the notation being rather cumbersome, we shall follow the usual practice in algebra to write $(x, y)\mu$ as xy and usually call xy the *product* of x and y. In this case, the semigroup operation is called *multiplication* and the associative formula may be simply expressed as $(xy)z = x(yz)$.

A semigroup (S, μ) is a pair, but we shall usually say 'S is a semigroup' and assume the binary operation is known.

An element e of S is called an *idempotent* if $e^2 = e$. The set of idempotents of S is denoted by $E(S)$.

An element e of S is said to be a *left* (resp. *right*) *identity* if, for all $x \in S$, $ex = x$ ($xe = x$). An element is an *identity element* or *identity* if it is a left and a right identity. It is easy to see that there exists at most one identity, which will be called *the* identity and denoted by 1.

An element e of S is called a *left* (resp. *right*) *zero* if, for all $x \in S$, $ex = e$ ($xe = e$). An element of S is called a *zero element* or *zero* if it is a left and a right zero. There can be at most one zero, which is then called *the* zero and denoted by 0.

Observe that a left (resp. right) identity is necessary idempotent, and so is a left (resp. right) zero.

A *monoid* is a semigroup with an identity. If S is a semigroup, S^1 denotes the monoid equal to S if S is a monoid and to $S \cup \{1\}$ if S is not a monoid. In the latter case, the operation of S^1 is completed by the rules

$$1x = x1 = x,$$

for all $x \in S^1$. We say that S^1 is S *with an identity adjoined if necessary.*

If S is a semigroup with or without a zero element, we usually use S^0 to denote the semigroup with underlying set $S \cup \{0\}$ and multiplication extending that of S, with

$$0x = x0 = 0 \qquad \big(x \in S\big)$$

and $00 = 0$. We say that S^0 is S *with a zero adjoined.*

A monoid M is a *group* if each of its elements has a group inverse, that is, for all $x \in M$, there exists $x' \in M$ such that $xx' = x'x = 1$. Here, x' is a group inverse of x. Note that if x'' is another group inverse of x, then $x'' = x''1 = x''xx' = 1x' = x'$, and so the group inverse of x is unique, so that we shall say x' is *the group inverse* of x.

A subsemigroup G of a semigroup S is said to be a *subgroup* if G is a group.

A *0-group* is a group G with a zero adjoined.

A semigroup (resp. monoid, group) S is said to be *commutative* if, for all $x, y \in S$, $xy = yx$.

A *band* is a semigroup B in which every element is an idempotent, that is, $x^2 = x$ for all $x \in B$.

In the following, we mention some special bands.

A band B is a *left zero band* if, for all $x, y \in B$, $xy = x$. Symmetrically, a *right zero band* is defined.

A *normal band* is a band satisfying $xyzx = xzyx$ for all $x, y, z \in B$.

Let E be a commutative semigroup of idempotents. We define a relation $\leq$ on E by

$$x \leq y \text{ if and only if } xy = x \qquad \big(x, y \in E\big).$$

It is easy to see that $\leq$ is a partial order on E. Indeed if $x \in E$, then $x^2 = x$, and so $x \leq x$ so that $\leq$ is reflexive. Suppose that $x, y \in E$ with $x \leq y$ and $y \leq x$, then $x = xy = yx = y$, so $\leq$ is anti-symmetric. Furthermore, if $x \leq y$ and $y \leq z$ in E, then $x = xy$ and $yz = y$, so that $xz = (xy)z = x(yz) = xy = x$, whence $x \leq z$. Hence, $\leq$ is transitive and so $\leq$ is a partial order on E. We note that for any $x, y \in E$, xy is the greatest lower bound of x and y. Consequently, $(E, \leq)$ becomes a lower semilattice, that is, a partially ordered set in which every pair of elements has a greatest lower bound.

Conversely, if $(E, \leq)$ is a lower semilattice, then E, together with the binary operation $\wedge$, forms a commutative semigroup of idempotents. If $a, b, c \in E$, then

$$(a \wedge b) \wedge c \leq a \wedge b \leq a, \quad (a \wedge b) \wedge c \leq a \wedge b \leq b,$$

and

$$(a \wedge b) \wedge c \ \leq \ c.$$

Thus $(a \wedge b) \wedge c$ is a lower bound of $\{a, b, c\}$. If d is a lower bound of $\{a, b, c\}$, then $d \ \leq \ a$, $d \ \leq \ b$ and $d \ \leq \ c$. Hence, $d \ \leq \ a \wedge b$, and so $d \ \leq \ (a \wedge b) \wedge c$. Thus $(a \wedge b) \wedge c$ is the unique greatest lower bound of $\{a, b, c\}$. Similarly, we may show that $a \wedge (b \wedge c)$ is the unique greatest lower bound of $\{a, b, c\}$. Hence, $a \wedge (b \wedge c) = (a \wedge b) \wedge c$, so that $(E, \wedge)$ is a semigroup. Obviously, $a \wedge a = a$ for all $a \in E$ and $a \wedge b = b \wedge a$ for all $a, b \in E$. Thus $(E, \wedge)$ is a commutative semigroup of idempotents. Moreover, $a \wedge b = a$ if and only if $a \ \leq \ b$.

To sum up, we have:

Proposition 1.3. *[26] Let $(E, \leq)$ be a lower semilattice. Then $(E, \wedge)$ is a commutative semigroup of idempotents and*

$$(\forall a, b \in E) \ a \ \leq \ b \textit{ if and only if } a \wedge b = a.$$

Let $(E, \cdot)$ be a commutative semigroup of idempotents. Then the relation $\leq$ on E defined by

$$a \ \leq \ b \textit{ if and only if } ab = a$$

is a partial order on E with respect to which E is a lower semilattice. In $(E, \leq)$, the meet of a and b is their product ab.

As a consequence of Proposition 1.3, the notions of 'lower semilattice' and 'commutative semigroup of idempotents' are equivalent and interchangeable. We shall use the term *semilattice* to mean either concept, making free and frequent transfers between algebraic structure and ordered structure.

We close this section with the notion of a rectangular band. A band B is said to be *rectangular* if, for all $x, y, z \in B$, $xyz = xz$. For example, given arbitrary non-empty sets I and J, one can define a semigroup operation on $I \times J$ by putting

$$(i, j) \cdot (k, \ell) = (i, \ell).$$

The resulting semigroup $I \times J$ is a rectangular band as for any $(i, j) \in I \times J$,

$$(i, j) \cdot (i, j) = (i, j),$$

and for any (i,j), (λ,μ), $(k,w) \in I \times J$, we have that

$$(i,j)\cdot(\lambda,\mu)\cdot(k,w) = (i,w).$$

Notice that any rectangular band is isomorphic to one so constructed.

We omit the proof of the next lemma as it will be shown in Section 1.3.6.

Lemma 1.4. *[26] Every band is a semilattice of rectangular bands.*

1.3.2 Green's relations

Let S be a semigroup. A relation ρ on S is called *left compatible* if

$$(\forall s,t,a \in S) \qquad (s,t) \in \rho \Rightarrow (as,at) \in \rho;$$

the notion of *right compatible* is defined dually. It is called *compatible* if

$$(\forall s,t,s',t' \in S) \qquad (s,s') \in \rho \text{ and } (t,t') \in \rho \Rightarrow (st,s't') \in \rho.$$

A left (resp. right) compatible equivalence relation is called a *left* (resp. *right*) *congruence*. A compatible equivalence relation is called a *congruence*. It is easy to see that a relation ρ on a semigroup S is a congruence if and only if it is both a left and a right congruence.

It is necessary to mention the theorem below as it will be used in the later chapters.

Theorem 1.5. *[26] If ρ is a congruence on a semigroup S, then S/ρ is a semigroup with respect to the operation defined by the rule that*

$$(x\rho)(y\rho) = (xy)\rho \qquad \big(\forall x,y \in S\big)$$

and the mapping $\rho^\natural : S \to S/\rho$ defined by

$$x\rho^\natural = x\rho \qquad \big(x \in S\big)$$

is a morphism.

We pause here to make a short comment on Theorem 1.5. It is easy to see that the operation $(x\rho)(y\rho) = (xy)\rho$ is well-defined. Since for all $x,x',y,y' \in S$, if $x \rho x'$ and $y \rho y'$, then $(x, x') \in \rho$ and $(y, y') \in \rho$, so that $(xy, x'y') \in \rho$, that is, $xy \rho x'y'$.

Let S be a semigroup and ρ be a congruence on S. Then we shall call ρ a $\mathcal{K}$-*congruence* if S/ρ is a $\mathcal{K}$-semigroup. For example, if S/ρ is a semilattice, then ρ is called a *semilattice congruence*.

If A and B are subsets of a semigroup S, we write AB for $\{ab : a \in A, b \in B\}$. It is easy to see that

$$(\forall A, B, C \in S)\ (AB)C = A(BC).$$

Hence notation such as ABC, $A_1A_2 \cdots A_n$ are meaningful. To deal with singleton sets, we shall use the notational simplifications that are customary in algebra. For example, we will write Ab for $A\{b\}$.

A non-empty set A of a semigroup S is called a *left ideal* if $SA \subseteq A$, a *right ideal* if $AS \subseteq A$, and a *(two-sided) ideal* if it is both a left and a right ideal. Note that every (left, right) ideal is a subsemigroup; but not every subsemigroup is an ideal. Any semigroup is an ideal of itself.

Let S be a semigroup without identity. For any $a \in S$, Sa will not in general contain a. However,

$$S^1a = Sa \cup \{a\},$$

$$aS^1 = aS \cup \{a\}$$

and

$$S^1aS^1 = SaS \cup Sa \cup aS \cup \{a\}.$$

Notice that S^1a, aS^1 and S^1aS^1 do not contain 1, so they are all subsets of S. Precisely, they are the smallest left, right and two-sided ideals of S containing a, respectively. Commonly, S^1a is called the *principal left ideal generated by a*. Dually, aS^1 is the *principal right ideal generated by a* and S^1aS^1 is the *principal ideal generated by a*.

Built on these ideals mentioned above, we define relations $\leq_{\mathcal{L}}$, $\leq_{\mathcal{R}}$ and $\leq_{\mathcal{J}}$ on a semigroup S as follows: for any $a, b \in S$,

$$a \leq_{\mathcal{L}} b \ \text{ if and only if } S^1a \subseteq S^1b,$$

$$a \leq_{\mathcal{R}} b \ \text{ if and only if } aS^1 \subseteq bS^1,$$

$$a \leq_{\mathcal{J}} b \text{ if and only if } S^1aS^1 \subseteq S^1bS^1.$$

It is easy to see that $\leq_{\mathcal{L}}, \leq_{\mathcal{R}}$ and $\leq_{\mathcal{J}}$ are pre-orders on S, respectively. Also, we note that $\leq_{\mathcal{R}}$ is left compatible. Since if $a, b, c \in S$ are such that $a \leq_{\mathcal{R}} b$, then $aS^1 \subseteq bS^1$, and so $caS^1 \subseteq cbS^1$, that is, $ca \leq_{\mathcal{R}} cb$. Dually, $\leq_{\mathcal{L}}$ is right compatible.

The following lemma is immediate.

Lemma 1.6. *Let S be a semigroup with set of idempotents $E(S)$. For any $e, f \in E(S)$,*

(i) $e \leq_{\mathcal{R}} f$ if and only if $fe = e$;

(ii) $e \leq_{\mathcal{L}} f$ if and only if $ef = e$.

We denote the associated equivalences by $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{J}$, respectively. So for any $a, b \in S$,

$$a \,\mathcal{L}\, b \text{ if and only if } S^1a = S^1b,$$
$$a \,\mathcal{R}\, b \text{ if and only if } aS^1 = bS^1,$$
$$a \,\mathcal{J}\, b \text{ if and only if } S^1aS^1 = S^1bS^1.$$

In addition, we use $\mathcal{H}$ and $\mathcal{D}$ to denote the intersection and join of $\mathcal{L}$ and $\mathcal{R}$, respectively, that is,

$$\mathcal{H} = \mathcal{L} \cap \mathcal{R} \text{ and } \mathcal{D} = \mathcal{L} \vee \mathcal{R}.$$

As usual, these equivalence relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$ are called *Green's relations* [21].

The next proposition gives an alternative characterisation of Green's relations.

Proposition 1.7. *[25] Let S be a semigroup and $a, b \in S$. Then*

(i) $a \,\mathcal{L}\, b$ if and only if there exist $x, y \in S^1$ such that $xa = b$, $yb = a$;

(ii) $a \,\mathcal{R}\, b$ if and only if there exist $u, v \in S^1$ such that $au = b$, $bv = a$;

(iii) $a \,\mathcal{J}\, b$ if and only if there exist $x, y, u, v \in S^1$ such that $xay = b$, $ubv = a$.

The following two lemmas give important properties of $\mathcal{L}$ and $\mathcal{R}$.

Lemma 1.8. *[26] The relation $\mathcal{L}$ is a right congruence and $\mathcal{R}$ is a left congruence.*

Lemma 1.9. *[26] The relations $\mathcal{L}$ and $\mathcal{R}$ commute.*

In view of Lemma 1.2 and Lemma 1.9, we have

$$\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}.$$

We remark that in a group G, we have that

$$\mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{D} = \mathcal{J} = G \times G$$

and in a commutative semigroup, we have that

$$\mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{D} = \mathcal{J}.$$

Before closing this section, we introduce partial orders amongst Green's equivalence classes. First, it is worth making a statement to simplify the notation of equivalence classes. We will denote the $\mathcal{L}$-class (resp. $\mathcal{R}$-class, $\mathcal{H}$-class, $\mathcal{D}$-class, $\mathcal{J}$-class) containing the element a by L_a (resp. R_a, H_a, D_a, J_a). Since $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{J}$ are defined in terms of ideals, a partial order amongst the equivalence classes is induced by the inclusion order amongst these ideals.

Let S be a semigroup and $a, b \in S$. Then

$$\begin{aligned} L_a &\leq L_b \text{ if and only if } S^1 a \subseteq S^1 b, \\ R_a &\leq R_b \text{ if and only if } aS^1 \subseteq bS^1, \\ J_a &\leq J_b \text{ if and only if } S^1 a S^1 \subseteq S^1 b S^1. \end{aligned}$$

Hence, $S/\mathcal{L}$, $S/\mathcal{R}$ and $S/\mathcal{J}$ may be regarded as partially ordered sets.

If T is a subsemigroup of S and $\mathcal{K}$ is a relation on S, then $\mathcal{K}(T)$ is the relation on T and $\mathcal{K}(S)$ is the relation on S.

1.3.3 Regular semigroups

The notion of regularity in a semigroup is derived from von Neumann's definition of a regular ring [42].

An element a of a semigroup S is said to be *regular* if there exists x in S such that $axa = a$. A semigroup S is called *regular* if all its elements are regular. Groups are of course regular semigroups, but the class of regular semigroups is vastly more extensive than the class of groups. As an analogue of the group inverse, we have the notion of a semigroup inverse of an element. If a is an element of a semigroup S, we say that a' is an *inverse* of a if

$$aa'a = a, \quad a'aa' = a'.$$

Notice that an element with an inverse is necessarily regular. Conversely, every regular element has an inverse. Since if $axa = a$ then we put $a' = xax$ and it is routine to verify that a' is an inverse of a.

We remark that an element a may have more than one inverse, say, in rectangular bands every element is an inverse of every other element. We will use $V(a)$ to denote the set of all inverses of a.

Let S be a regular semigroup and $a \in S$. Then there exists $x \in S$ such that $a = axa$, from which it follows that $a \in Sa$ (resp. aS, SaS). In that case, Green's relations can be restated for regular semigroups as follows: for any $a, b \in S$,

$$a \ \mathcal{L} \ b \text{ if and only if } Sa = Sb,$$
$$a \ \mathcal{R} \ b \text{ if and only if } aS = bS,$$
$$a \ \mathcal{J} \ b \text{ if and only if } SaS = SbS.$$

The following theorem is very useful. Here, a $\mathcal{D}$-class is *regular* if all its elements are regular, or equivalently, it contains at least one regular element [26].

Theorem 1.10. *[26] Let a be an element of a regular $\mathcal{D}$-class D in a semigroup S.*

(i) If a' is an inverse of a then $a' \in D$ and the two $\mathcal{H}$-classes $R_a \cap L_{a'}$, $L_a \cap R_{a'}$ contain respectively the idempotents aa' and $a'a$.

(ii) If $b \in D$ is such that $R_a \cap L_b$, $L_a \cap R_b$ contains idempotents e, f, respectively, then H_b contains an inverse a^* of a such that $aa^* = e$, $a^*a = f$.

(iii) No $\mathcal{H}$-class contains more than one inverse of a.

The next proposition is an immediate consequence of Theorem 1.10, which will be of considerable use in Chapter 7.

Proposition 1.11. *[26] Let e, f be idempotents in a semigroup S. Then $e \ \mathcal{D} \ f$ if and only if there exists an element a in S and an inverse a' of a such that $aa' = e$ and $a'a = f$.*

If S is a semigroup with set of idempotents $E(S)$ we shall say that a congruence ρ on S is *idempotent separating* if it has the property that

$$\rho \cap (E(S) \times E(S)) = 1_{E(S)},$$

that is, no ρ-class contains more than one idempotent. In fact, it is of interest to recall the result, as Lallement [31] showed: on a regular semigroup a congruence is idempotent separating if and only if it is contained in $\mathcal{H}$.

A semigroup is *fundamental* if the largest idempotent separating congruence is trivial.

1.3.4 Inverse semigroups

The aim of this section is to cover the basic ideas and facts about inverse semigroups, which may be regarded as the class of semigroups closest to groups.

A semigroup S is said to be *inverse* if every a in S has a unique inverse. We will denote the unique inverse of a in S by a^{-1}. Such a semigroup is certainly regular, but the converse need not be true.

There are some equivalent formulation for inverse semigroups.

Theorem 1.12. *[26] Let S be a semigroup with set of idempotents $E(S)$. Then the following statements are equivalent:*

(i) *S is an inverse semigroup;*

(ii) *S is regular and $E(S)$ is a semilattice;*

(iii) *every $\mathcal{L}$-class and every $\mathcal{R}$-class contains exactly one idempotent.*

It is convenient to recall from [26] the following elementary properties of inverse semigroups.

Proposition 1.13. *[26] Let S be an inverse semigroup with semilattice of idempotents $E(S)$. Then*

(i) $(a^{-1})^{-1} = a$ *for all* $a \in S$*;*

(ii) $e^{-1} = e$ *for all* $e \in E(S)$*;*

(iii) $(ab)^{-1} = b^{-1}a^{-1}$ *for all* $a, b \in S$*;*

(iv) $aea^{-1} \in E(S)$, $a^{-1}ea \in E(S)$ *for all* $a \in S$ *and* $e \in E(S)$*.*

A significant feature of inverse semigroups is the natural partial order relation. Let S be an inverse semigroup with semilattice of idempotents $E(S)$. Given $a, b \in S$, we define

$$a \ \leq \ b \text{ if and only if } (\exists e \in E(S))a = eb.$$

It is routine to verify that this relation $\leq$ is a partial order.

The following list gives several alternative characterisations of $\leq$.

Proposition 1.14. *[26] Let S be an inverse semigroup with semilattice of idempotents $E(S)$ and $a, b \in S$. Then the following statements are equivalent:*

(i) $a \leq b$;	(ii) $(\exists\, e \in E(S))\ a = be$;
(iii) $aa^{-1} = ba^{-1}$;	(iv) $aa^{-1} = ab^{-1}$;
(v) $a^{-1}a = b^{-1}a$;	(vi) $a^{-1}a = a^{-1}b$;
(vii) $a = ab^{-1}a$;	$(viii)$ $a = aa^{-1}b$.

We note that the restriction of $\leq$ to the semilattice $E(S)$ of idempotents of an inverse semigroup S is the natural semilattice order on $E(S)$:

$$e \leq f \text{ if and only if } ef = e \qquad \big(e, f \in E(S)\big).$$

Another crucial property of $\leq$ is that the order relation on any inverse semigroup S is compatible with the operations of multiplication and inverse. Since if $a, b \in S$ with $a \leq b$, then there exists $e \in E(S)$, the semilattice of idempotents of S, such that $a = eb$, and so $ac = ebc$, so that $ac \leq bc$. The left compatibility follows dually from Proposition 1.14(ii).

Usually, we call $\leq$ the *natural partial order* on an inverse semigroup. In particular, the partial order $\leq$ is the identity relation on a group.

Another important notion for inverse semigroups is that of an idempotent separating congruence. Howie [27] gave an alternative description of this congruence:

Theorem 1.15. *[27] If S is an inverse semigroup with semilattice of idempotents $E(S)$, then the relation*

$$\mu = \{(a, b) \in S \times S : (\forall e \in E(S))\, a^{-1}ea = b^{-1}eb\}$$

is the greatest idempotent separating congruence on S.

As any inverse semigroup is regular, it follows from Lallement's result in Section 1.3.3 that μ is the largest congruence contained in $\mathcal{H}$.

1.3.5 Orthodox semigroups

An *orthodox semigroup* is a regular semigroup in which the set of idempotents forms a band.

Due to Reilly and Scheiblich [43], we have:

Proposition 1.16. *[43] If S is a regular semigroup, then the following statements are equivalent:*

(i) S is orthodox;

(ii) for any a, b in S, if a' is an inverse of a and b' is an inverse of b, then $b'a'$ is an inverse of ab;

(iii) if e is idempotent then every inverse of e is idempotent;

(iv) for all $a, b \in S$, if $V(a) \cap V(b) \neq \emptyset$, then $V(a) = V(b)$.

Built on Proposition 1.16 (iv), Yamada [52] considered the equivalence relation

$$\gamma = \{(x, y) \in S \times S : V(x) = V(y)\}$$

on an orthodox semigroup S in which the band B of idempotents is normal, in the sense, $xyzt = xzyt$ for all $x, y, z, t \in B$. A further study in the general orthodox case by Hall [23] showed that:

Lemma 1.17. *If S is an orthodox semigroup with band of idempotents B, then the relation γ defined above is a congruence on S such that $\gamma \cap (B \times B) = \mathcal{J}(B)$. Moreover, it is the least inverse semigroup congruence on S.*

1.3.6 Completely regular semigroups

A semigroup S is said to be *completely regular* if there exists a unary operation $a \mapsto a^{-1}$ such that

$$(a^{-1})^{-1} = a, \quad aa^{-1}a = a, \quad aa^{-1} = a^{-1}a.$$

The following result presents two alternative descriptions of completely regular semigroups.

Lemma 1.18. *[26] Let S be a semigroup. Then the following statements are equivalent:*

(i) S is completely regular;

(ii) every element of S lies in a subgroup of S;

(iii) every $\mathcal{H}$-class in S is a group.

In view of Lemma 1.18 (*iii*), a completely regular semigroup is often referred to as a *union of groups*.

A crucial observation about completely regular semigroups is that:

Lemma 1.19. *[26] Let S be a completely regular semigroup. Then $\mathcal{D} = \mathcal{J}$ and $\mathcal{J}$ is a semilattice congruence on S.*

Let S be a completely regular semigroup. We denote the semilattice $S/\mathcal{J}$ by Y, and for each $\alpha \in Y$ we denote $\alpha(\mathcal{J}^{\natural})^{-1}$ by S_α. Each S_α is a $\mathcal{J}$-class of S and a completely simple semigroup, in the sense that, it has no proper ideals and contains a primitive idempotent (by which we mean an idempotent which is minimal within the set of idempotents, here, for any $e, f \in E(S)$, $e \leq f$ if and only if $ef = fe = e$).

To sum up, we have:

Theorem 1.20. *[26] Every completely regular semigroup is a semilattice of completely simple semigroups.*

Here, we need to mention a special subclass of the class of completely regular semigroups, that is, the class of Clifford semigroups. A semigroup is a Clifford semigroup if it is a semilattice of groups. These are precisely inverse semigroups S with central idempotents, that is, in which $ea = ae$ for all $a \in S$ and $e \in E(S)$.

Notice that any band B is a completely regular semigroup so that $\mathcal{J}$ is a semilattice congruence on B. As each $\mathcal{J}$-class of B is a rectangular band, we obtain that every band is semilattice of rectangular bands, that is, $B = \bigcup_{\alpha \in Y} B_\alpha$, where Y is the index set of the $\mathcal{J}$-classes of B and each B_α is a $\mathcal{J}$-class.

For convenience, we will sometimes denote the B_α containing e by $E(e)$. Then $E(e)$ is a rectangular band and hence $E(e) \subseteq V(e)$. By Theorem 1.10, $V(e) \subseteq E(e)$, and so $V(e) = E(e)$.

In what follows the reader should bear in mind that for a band B, two elements are $\mathcal{D}$-related if and only if they are mutually inverse.

To close this section, we present some results from [26], which will be useful subsequently.

Lemma 1.21. *[26] Let e, f be elements of a band B such that $e \, \mathcal{D} \, f$. Then the maps*

$$\theta_f : x \mapsto fxf, \qquad \theta_e : y \mapsto eye$$

are mutually inverse isomorphisms from $\langle e\rangle$ *onto* $\langle f\rangle$ *and from* $\langle f\rangle$ *onto* $\langle e\rangle$*, respectively.*

It is worth remarking that for a band B and element e of B, $\langle e\rangle = \{x \in B : xe = ex = e\}$. If $e\,\mathcal{D}\,f$, then for any $x \in \langle e\rangle$ and any $y \in \langle f\rangle$, we have that $x\,\mathcal{D}\,fxf$ and $y\,\mathcal{D}\,eye$. To make the domain of the map θ_f clear, we shall use the symbol $\theta_f|_{\langle e\rangle}$ instead of simply θ_f. Notice that the inverse of $\theta_f|_{\langle e\rangle}$ is $\theta_e|_{\langle f\rangle}$.

Lemma 1.22. *[26] If* x*,* e*,* f *are elements of a band* B *such that* $e\ \mathcal{D}\ f$*, then* $L_{exf} = L_{xf}$ *and* $R_{exf} = R_{ex}$*.*

1.4 Biordered sets

Let E be a partial algebra, that is, a set E together with a partial binary operation $\cdot$ on E. Usually, we omit the symbol '$\cdot$', say, if $e, f \in E$ and $e \cdot f$ exists in E, then we write ef for $e \cdot f$. We will express the term '$e \cdot f$ exists in E' as '$\exists\ ef$'. Set

$$D_E = \{(e, f) \in E \times E : \exists\ ef\},$$

that is, D_E is the domain of the partial binary operation. On E we define

$$\omega^r = \{(e, f) : fe = e\}, \quad \omega^l = \{(e, f) : ef = e\},$$
$$\mathcal{R} = \omega^r \cap (\omega^r)^{-1}, \quad \mathcal{L} = \omega^l \cap (\omega^l)^{-1} \text{ and } \omega = \omega^r \cap \omega^l.$$

In addition, for any $e \in E$, we define

$$\omega^r(e) = \{f \in E : f\ \omega^r\ e\}$$

and similarly for $\omega^l(e)$ and $\omega(e)$.

Definition 1.23. Let E be a partial algebra as above. Then E is a *biordered set* if it satisfies axioms (B1), (B21), (B22), (B31), (B32), (B4) and their duals, where e, f, g, h denote arbitrary elements of E.

(B1) ω^r and ω^l are pre-orders on E and $D_E = (\omega^r \cup \omega^l) \cup (\omega^r \cup \omega^l)^{-1}$;

(B21) $f \in \omega^r(e) \Rightarrow f\,\mathcal{R}\ fe\,\omega\,e$;

(B22) $g\ \omega^l\ f$ and $f, g \in \omega^r(e) \Rightarrow ge\,\omega^l\,fe$;

(B31) $g \; \omega^r \; f \, \omega^r \, e \Rightarrow gf = (ge)f$;

(B32) $g \; \omega^l \; f$ and $f, g \in \omega^r(e) \Rightarrow (fg)e = (fe)(ge)$.

Let $M(e,f)$ denote the set $\omega^l(e) \cap \omega^r(f)$ for all $e, f \in E$. We define a relation $\prec$ on $M(e,f)$ by the rule that for any $g, h \in M(e,f)$,

$$g \prec h \text{ if and only if } eg \; \omega^r \; eh, \;\; gf \; \omega^l \; hf.$$

It is easy to see that $\prec$ is a pre-order on $M(e,f)$. Since for any $g \in M(e,f)$, we have that $g \prec g$ and so it is reflexive. To show that $\prec$ is transitive, suppose that $g, h, k \in M(e,f)$ with $g \prec h$, $h \prec k$. Then

$$eg \; \omega^r \; eh, \qquad gf \; \omega^l \; hf,$$

$$eh \; \omega^r \; ek \text{ and } hf \; \omega^l \; kf.$$

As ω^r and ω^l are pre-orders on E, we obtain that

$$eg \; \omega^r \; ek \text{ and } gf \; \omega^l \; kf,$$

that is, $g \prec k$. Hence, $\prec$ is transitive, so that $\prec$ is a pre-order.

We shall denote the set $M(e,f)$, together with the pre-order $\prec$, by $\mathcal{M}(e,f)$.

Then the set $S(e,f) = \{h \in \mathcal{M}(e,f) : g \prec h, \; \forall g \in \mathcal{M}(e,f)\}$ is called the *sandwich set* of e and f.

In particular, for any $e \in E$, $S(e,e) = \{e\}$. Since if $h \in S(e,e)$, then $h \; \omega^l \; e$ and $h \; \omega^r \; e$ so that $h \; \omega \; e$. As $e \in \mathcal{M}(e,e)$, we have that $e \prec h$, that is, $e \; \omega^r \; eh = h$ and $e \; \omega^l \; he = h$, and so $e \; \omega \; h$. Consequently, $e = h$.

(B4) $f, g \in \omega^r(e) \Rightarrow S(f,g)e = S(fe, ge)$.

The biordered set E is *regular* if it also satisfies

(R) $S(e,f) \neq \emptyset, \forall e, f \in E$.

We pause to make some comments on the above axioms. By Axiom (B1), D_E is symmetric. As the axioms for a biordered set are self-dual, the dual of any true proposition is also true. If $f \; \omega^r \; e$, then by (B1), $(f,e) \in D_E$, and so $\exists \; fe$. In (B31), if $g \; \omega^r \; f \; \omega^r \; e$, then by (B1), $\exists \; ge$ and by (B21), $ge \; \mathcal{R} \; g$, and so $ge \; \omega^r \; f$, so that by (B1), $\exists\ (ge)f$. In (B32), if $g\ \omega^l\ f$ and $f \in \omega^r(e)$, then by the dual of (B21), $\exists\ fg$ and $fg\ \omega\ f$, so that $fg\ \omega^r\ e$, which implies that $\exists\ (fg)e$ by (B1). As $g\ \omega^l\ f$ and $f, g \in \omega^r(e)$, by (B22), $ge\ \omega^l\ fe$, and so $\exists\ (fe)(ge)$ by (B1). In (B4), if $h \in S(f,g)$, then $h\ \omega^r\ g$. As $g\ \omega^r\ e$ and ω^r is a pre-order, we obtain that $h\ \omega^r\ e$, and so $\exists\ he$, so that $S(f,g)e$ is well-defined.

A *biordered subset* E' of a biordered set E is a biordered set which is a partial subalgebra in the usual sense. It is easy to see that for any $e \in E$, $\omega(e)$ is a biordered subset of E. In addition, if E is regular, then for any $e \in E$, $\omega(e)$ is regular. Since if $g, k \in \omega(e)$ and $h \in S(g,k)$, then $h\ \omega^l\ g$ and $h\ \omega^r\ k$, and so $h \in \omega(e)$ so that $S(g,k) \subseteq \omega(e)$.

Lemma 1.24. *Let S be a semigroup with non-empty set of idempotents $E(S)$. Then $E(S)$ is a biordered set with respect to the pair of pre-orders $\leq_{\mathcal{R}}$ and $\leq_{\mathcal{L}}$ on S.*

Proof. For (B1), let $D_{E(S)} = (\leq_{\mathcal{R}} \cup \leq_{\mathcal{L}}) \cup (\leq_{\mathcal{R}} \cup \leq_{\mathcal{L}})^{-1}$. Now we show that $E(S)$ forms a partial algebra with respect to $D_{E(S)}$. Suppose that $e, f \in E(S)$. If $e \leq_{\mathcal{R}} f$, then by Lemma 1.6(*i*), we have $fe = e \in E(S)$ and $(ef)^2 = efef = e(fe)f = ef \in E(S)$. Dually, if $e \leq_{\mathcal{L}} f$, then $ef = e \in E(S)$ and $fe \in E(S)$. Hence, $E(S)$ forms a partial algebra.

From the associativity of multiplication in S and the idempotent property of $E(S)$, axioms (B21), (B22),(B31) and (B32) hold.

Finally, we show that (B4) holds. Suppose that $e, f, g \in E(S)$ are such that $f, g \leq_{\mathcal{R}} e$. We first verify that $S(f,g)e \subseteq S(fe, ge)$. Assume that $h \in S(f,g)$. Then $h \leq_{\mathcal{L}} f$ and $h \leq_{\mathcal{R}} g \leq_{\mathcal{R}} e$, and so $he \in E(S)$ by Lemma 1.6. Put $h' = he$. As

$$h'fe = (he)fe = h(ef)e = hfe = he = h'$$

and

$$geh' = ge(he) = g(eh)e = ghe = he = h',$$

we have that $h' \in \mathcal{M}(fe, ge)$. Further, if $k' \in \mathcal{M}(fe, ge)$ and $k = k'f$, then

$$\begin{aligned} k^2 &= k'fk'f \\ &= k'f(ek')f && \big(k' \leq_{\mathcal{R}} ge \ \leq \ e\big) \\ &= k'(fe)k'f \\ &= k'f && \big(k' \leq_{\mathcal{L}} fe\big) \\ &= k \end{aligned}$$

and $gk = gk'f = g(ge)k'f = (ge)k'f = k'f = k$. Hence $k \in \mathcal{M}(f, g)$, so that $k \ \prec \ h$. Also,

$$\begin{aligned} h'f &= hef \\ &= hf && \big(f \leq_{\mathcal{R}} e\big) \\ &= h && \big(h \leq_{\mathcal{L}} f\big) \end{aligned}$$

and $ke = k'fe = k'$ since $k' \leq_{\mathcal{L}} fe$. Therefore,

$$\begin{aligned} ((fe)h')((fe)k') &= fe(h'f)ek' \\ &= fehek' && \big(h'f = h\big) \\ &= f(eh)ek' \\ &= fhek' && \big(h \leq_{\mathcal{R}} g \leq_{\mathcal{R}} e\big) \\ &= fheke && \big(k' = ke\big) \\ &= fhe(gk)e && \big(gk = k\big) \\ &= fh(eg)ke \\ &= fhgke && \big(g \leq_{\mathcal{R}} e\big) \\ &= fhke && \big(k = gk\big) \\ &= f(hf)ke && \big(h \leq_{\mathcal{L}} f\big) \\ &= (fh)(fk)e \\ &= fke && \big(k \ \prec \ h\big) \\ &= fk' = (fe)k' && \big(k' \leq_{\mathcal{R}} ge \ \leq \ e\big), \end{aligned}$$

and so $(fe)k' \leq_{\mathcal{R}} (fe)h'$. Also, we have $k'(ge) \leq_{\mathcal{L}} h'(ge)$ as

$$\begin{aligned}
k'(ge)h'(ge) &= k'g(eh')ge \\
&= k'gh'ge && \big(h' = he \;\mathcal{R}\; h \leq_{\mathcal{R}} e\big) \\
&= k'ghege && \big(h' = he\big) \\
&= k'ghge && \big(g \leq_{\mathcal{R}} e\big) \\
&= keghge && \big(k' = ke\big) \\
&= kghge && \big(g \leq_{\mathcal{R}} e\big) \\
&= kge && \big(k \prec h\big) \\
&= kege && \big(g \leq_{\mathcal{R}} e\big) \\
&= k'(ge) && \big(k' = ke\big).
\end{aligned}$$

So $k' \prec h'$ in $\mathcal{M}(fe, ge)$. Hence, $h' \in S(fe, ge)$, and so $S(f, g)e \subseteq S(fe, ge)$.

To see the converse, suppose that $u' \in S(fe, ge)$. Then $u' \leq_{\mathcal{R}} ge \leq e$ and $u' \leq_{\mathcal{L}} fe \leq e$, and so $u' \leq e$. Put $u = u'f$. Notice that

$$\begin{aligned}
u^2 &= u'fu'f \\
&= u'feu'f && \big(u' \leq e\big) \\
&= u'f && \big(u' \leq_{\mathcal{L}} fe,\ u'^2 = u'\big) \\
&= u.
\end{aligned}$$

Also,

$$\begin{aligned}
gu &= gu'f \\
&= geu'f && \big(u' \leq e\big) \\
&= u'f && \big(u' \leq_{\mathcal{R}} ge\big) \\
&= u.
\end{aligned}$$

Together with $u \leq_{\mathcal{L}} f$, we obtain that $u \in \mathcal{M}(f, g)$. If $v' \in \mathcal{M}(f, g)$ and $v = v'e$, then

$$\begin{aligned}
v^2 &= v'ev'e \\
&= v'e && \big(v' \leq_{\mathcal{R}} g \leq_{\mathcal{R}} e\big) \\
&= v.
\end{aligned}$$

In addition,

$$\begin{aligned} vfe &= v'efe \\ &= v'fe && \big(f \leq_{\mathcal{R}} e\big) \\ &= v'e && \big(v' \leq_{\mathcal{L}} f\big) \\ &= v \end{aligned}$$

and

$$\begin{aligned} gev &= gev'e \\ &= v'e && \big(v' \leq_{\mathcal{R}} g \leq_{\mathcal{R}} e\big) \\ &= v. \end{aligned}$$

Thus, $v \in \mathcal{M}(fe, ge)$, and so $v \prec u'$. Since $vf = v'ef = v'$, we have that

$$\begin{aligned} (fu)(fv') &= (fu'f)fv' \\ &= fu'fv' \\ &= fu'fvf \\ &= (feu')(fev)f && \big(u', v \leq_{\mathcal{R}} ge \leq e\big) \\ &= (fev)f && \big(v' \prec u\big) \\ &= fv'f && \big(v \leq_{\mathcal{R}} g \leq e\big) \\ &= fv', \end{aligned}$$

so that $fv' \leq_{\mathcal{R}} fu$. In addition, we have $v'g \leq_{\mathcal{L}} ug$ as

$$\begin{aligned} (v'g)(ug) &= v'(gu)g \\ &= v'ug && \big(gu = u\big) \\ &= vfug && \big(v' = vf\big) \\ &= vfeug && \big(u \leq_{\mathcal{R}} g \leq_{\mathcal{R}} e\big) \\ &= vug && \big(vfe = v\big) \\ &= vgeug && \big(u \leq_{\mathcal{R}} g \leq_{\mathcal{R}} e\big) \\ &= vgeu'fg && \big(u = u'f\big) \\ &= vgeu'feg && \big(g \leq_{\mathcal{R}} e\big) \\ &= vgeu'g && \big(u' \ \omega^l \ fe\big) \end{aligned}$$

$$
\begin{aligned}
&= vgeu'geg && \big(g \leq_{\mathcal{R}} e\big)\\
&= vgeg && \big(v \prec u' \text{ in } \mathcal{M}(fe, ge)\big)\\
&= vg && \big(g \leq_{\mathcal{R}} e\big)\\
&= vfeg && \big(v = vfe\big)\\
&= v'g && \big(v' = vf \text{ and } g \leq_{\mathcal{R}} e\big).
\end{aligned}
$$

Hence $v' \prec u$. Thus, $u \in S(f,g)$. Observe that $ue = u'fe = u'$ as $u' \leq_{\mathcal{L}} fe$, so that $S(fe, ge) \subseteq S(f,g)e$. □

To the converse of Lemma 1.24, it is shown in [3] that every biordered set E is the set of idempotents of some semigroup. In particular, the set of idempotents of any regular semigroup is a regular biordered set [38]; conversely, every regular biordered set is the set of idempotents of some regular semigroup. For further details the reader is referred to [38].

Definition 1.25. Let E and E' be biordered sets, and $\theta : E \to E'$ a mapping. Then θ is a *morphism* if it satisfies

(M) $(e,f) \in D_E \Rightarrow (e\theta, f\theta) \in D_{E'}$ and $(ef)\theta = (e\theta)(f\theta)$.

θ is a *regular morphism* if it also satisfies

(RM1) $S(e,f)\theta \subseteq S'(e\theta, f\theta)$;

(RM2) $S(e,f) \neq \emptyset \Leftrightarrow S'(e\theta, f\theta) \neq \emptyset, \forall e, f \in E$,

where $S'(e\theta, f\theta)$ denotes the sandwich set of $e\theta$ and $f\theta$ in E'.

It is easy to see that if $e \; \omega^r \; f$ then $e\theta \; \omega^r \; f\theta$ and dually for ω^l.

We remark that if E is a regular biordered set, then a morphism $\theta : E \to E'$ is regular provided only that it satisfies (RM1), since in this case (RM2) follows automatically.

Here, we state that if $\theta_1 : E_1 \to E_2$ and $\theta_2 : E_2 \to E_3$ are two morphisms, then it is clear that $\theta_1\theta_2 : E_1 \to E_3$ is also a morphism and if θ_1 and θ_2 are regular, so is $\theta_1\theta_2$.

A (regular) morphism is said to be a *(regular) isomorphism* if it is bijective and so is the inverse.

Lemma 1.26. *(c.f. Corollary 2.15 [38]) A bijective morphism is an isomorphism if and only if it is regular.*

In the following, we list some necessary properties of (regular) biordered sets which will be used in Chapter 10.

Lemma 1.27. *(c.f. Proposition 2.3 [38]) Let E be a biordered set and $e, f \in E$ be such that $f\ \omega^r\ e$, then for any $g\ \omega^r\ f$,*

$$(gf)e = g(fe) = (ge)(fe).$$

Lemma 1.28. *(c.f. Proposition 2.5 [38]) Let E be a biordered set and $e, e', f, f' \in E$ with $e\,\mathcal{L}\,e'$ and $f\,\mathcal{R}\,f'$. Then $\mathcal{M}(e, f) = \mathcal{M}(e', f')$. In particular, $S(e, f) = S(e', f')$.*

Lemma 1.29. *(c.f. Proposition 2.2 [38]) Let E be a biordered set and $(e, f) \in D_E$, then $ef \in S(f, e)$.*

Lemma 1.30. *(c.f. Proposition 2.12 [38]) Let E be a biordered set and let $e, f, g, h \in E$ be such that $g \in S(e, f)$ and $h\ \omega^r\ f$. Then $S(g, h) \subseteq S(e, h)$; further, $S(g, h) \neq \emptyset$ if and only if $S(e, h) \neq \emptyset$. Dually, if $g \in S(f, e)$ and $h\ \omega^l\ f$, then $S(h, g) \subseteq S(h, e)$, further, $S(h, g) \neq \emptyset$ if and only if $S(h, e) \neq \emptyset$.*

Lemma 1.31. *(c.f. Proposition 2.13 [38]) Let E be a biordered set and let $e, g \in E$ with $\alpha : \omega(f) \to \omega(f')$ being an isomorphism. If $h_1 \in S(e, f)$, $h_2 \in S(f', g)$, $h_1' = (h_1 f)\alpha$ and $h_2' = (f' h_2)\alpha^{-1}$, then $(S(h_1, h_2')f)\alpha = f' S(h_1', h_2)$.*

Lemma 1.32. *(c.f. Corollary 2.8 [38]) Let E be a biordered set and $e, f \in E$ be such that either $e\,\mathcal{R}\,f$ or $e\,\mathcal{L}\,f$. Then the map $\tau(e, f) : \omega(e) \to \omega(f)$, defined by the rule that for all $g \in \omega(e)$,*

$$g\tau(e,f) = \begin{cases} gf & \text{if } e\,\mathcal{R}\,f \\ fg & \text{if } e\,\mathcal{L}\,f, \end{cases}$$

is an isomorphism such that if either $e\,\mathcal{R}\,f\,\mathcal{R}\,g$ or $e\,\mathcal{L}\,f\,\mathcal{L}\,g$, then

$$\tau(e, f)\tau(f, g) = \tau(e, g)$$

and

$$\tau(e,f) = (\tau(f,e))^{-1}.$$

Further, if $g, g' \in \omega^r(e)$ *and* $g \,\mathcal{L}\, g'$*, then*

$$\tau(g,g')\tau(g',ge) = \tau(g,ge)\tau(ge,g'e).$$

At the end of this section, we should recall a notion which is related to biordered sets. In [38], an *E-square* is a 2×2 matrix $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$, where $e,f,g,h \in E$, a biordered set, with $e \,\mathcal{R}\, f$, $g \,\mathcal{R}\, h$, $e \,\mathcal{L}\, g$ and $f \,\mathcal{L}\, h$. An $E-$square is said to be *singular* if it has one of the following two forms:

(*i*) $\begin{pmatrix} g & h \\ eg & eh \end{pmatrix}$ where $g,h \in \omega^l(e)$ and $g\mathcal{R}h$ - row-singular;

(*ii*) $\begin{pmatrix} g & ge \\ h & he \end{pmatrix}$ where $g,h \in \omega^r(e)$ and $g\mathcal{L}h$ - column-singular.

An E-square $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$ is τ-*commutative* if the following diagram commutes:

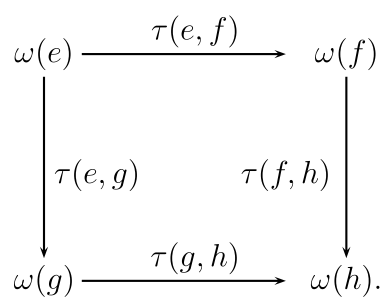

Figure 1.1: The τ-commutative condition

We note that every singular E-square is τ-commutative.

Lemma 1.33. *(c.f. Proposition 2.9 [38]) Let* $g,h \in \omega^r(e)$ *and* $ge \,\mathcal{L}\, he$*. Then there exists a unique E-square* $G = \begin{pmatrix} g & u \\ v & h \end{pmatrix}$ *satisfying the following conditions:*

(*i*) $ge = ue$ *and* $ve = he$*;*

(*ii*) G *is* τ*-commutative, and so* $h(ku) = (vk)h$ *for all* $k \in \omega(g)$*.*

Dually, we have:

Lemma 1.34. *Let $g, h \in \omega^l(e)$ and $eg \,\mathcal{R}\, eh$. Then there exists a unique E-square $G = \begin{pmatrix} g & u \\ v & h \end{pmatrix}$ satisfying the following conditions:*

(i) $eg = ev$ and $eu = eh$;

(ii) G is τ-commutative, and so $h(ku) = (vk)h$ for all $k \in \omega(g)$.

Lemma 1.35. *Let $g, h \in S(e, f)$. Then there exists a unique E-square $G = \begin{pmatrix} g & u \\ v & h \end{pmatrix}$, where $u, v \in S(e, f)$.*

Proof. Suppose that $g, h \in S(e, f)$. Then $g, h \in \omega^l(e) \cap \omega^r(f)$, $eg \,\mathcal{R}\, eh$ and $gf \,\mathcal{L}\, hf$. Since $g, h \in \omega^r(f)$ and $gf \,\mathcal{L}\, hf$, it follows from Lemma 1.33 that there exists a unique E-square $G = \begin{pmatrix} g & u \\ v & h \end{pmatrix}$ such that $gf = uf$, $vf = hf$ and G is τ-commutative.

As $g, h \in \omega^l(e)$ and $eg \,\mathcal{R}\, eh$, therefore by Lemma 1.34, there exists a unique E-square $K = \begin{pmatrix} g & i \\ j & h \end{pmatrix}$ such that $eg = ej$, $ei = eh$ and K is τ-commutative. Now, we claim that $G = K$. Since $u \,\mathcal{L}\, h$, we have that $eu \,\mathcal{L}\, eh$. As $eu \,\mathcal{R}\, eg \,\mathcal{R}\, eh$, we obtain that $eu = eh$, and so $ev \,\mathcal{R}\, eh = eu \,\mathcal{R}\, eg$. Since $g \,\mathcal{L}\, v$, we have that $eg = ev$. Hence, G satisfies the conditions in Lemma 1.34, so that $G = K$.

Finally, we show that $u, v \in S(e, f)$. As $G = \begin{pmatrix} g & u \\ v & h \end{pmatrix}$ is an E-square and $g, h \in \omega^l(e) \cap \omega^r(f)$, we obtain that $u, v \in \omega^l(e) \cap \omega^r(f)$. Notice that $eu = eh$, $uf \,\mathcal{L}\, hf$, then $h \prec u$. As $h \in S(e, f)$, we have that for any $t \in \mathcal{M}(e, f)$, $t \prec h$. Since $\prec$ is a pre-order, we succeed in obtaining that $t \prec u$. Thus, $u \in S(e, f)$. Similarly, we have that $v \in S(e, f)$. $\square$

We make a short comment on Lemma 1.35 that if $S(e, f) \neq \emptyset$ for all $e, f \in E$, then for any $g, h \in S(e, f)$, there exists $k \in S(e, f)$ such that $g \,\mathcal{R}\, k \,\mathcal{L}\, h$.

1.5 Categories

Category theory was first introduced by Eilenberg and Maclane in 1945 [5]. The key idea of category theory is to provide a fundamental and abstract way to describe mathematical entities and their relationships via 'objects' that are linked by 'arrows' (or 'morphisms'). Here, we recall some basic definitions and properties of a category [28] and [48].

Definition 1.36. A category P consists of

(C1) a class $\mathrm{Ob}(P)$ of objects;

(C2) a class $\mathrm{Mor}(P)$ of morphisms (or arrows) between the objects. Each morphism f has a unique domain $\mathbf{d}(f) \in \mathrm{Ob}(P)$ and codomain $\mathbf{r}(f) \in \mathrm{Ob}(P)$. Denote the Mor-class of all morphisms from $A \in \mathrm{Ob}(P)$ to $B \in \mathrm{Ob}(P)$ by $\mathrm{Mor}(A, B)$;

(C3) if $A, B, C, D \in \mathrm{Ob}(P)$, then there is a binary operation

$$\mathrm{Mor}(A, B) \ \times \ \mathrm{Mor}(B, C) \to \mathrm{Mor}(A, C), \ (f, g) \mapsto f \circ g,$$

called *composition* of morphisms such that if $f \in \mathrm{Mor}(A, B)$, $g \in \mathrm{Mor}(B, C)$ and $h \in \mathrm{Mor}(C, D)$, then $(f \circ g) \circ h = f \circ (g \circ h)$;

(C4) for each $A \in \mathrm{Ob}(P)$, there exists a morphism $1_A \in \mathrm{Mor}(A, A)$ such that if $B \in \mathrm{Ob}(P)$, and $f \in \mathrm{Mor}(A, B)$, then $1_A \circ f = f$ and $f \circ 1_B = f$.

A simple but very accessible example of a category is the category $\mathcal{S}et$ of sets, whose objects are sets and whose morphisms are functions from one set to another. Here, we should mention that the objects of a category need not be sets nor the morphisms functions. Particularly, a category is said to be 'concrete' if all objects are (structured) sets, morphisms from A to B are (structure preserving) mappings from A to B, composition of morphisms is composition of mappings, and the identities are the identity mappings. In addition, a category P is called *small* if both $\mathrm{Ob}(P)$ and $\mathrm{Mor}(P)$ are actually sets (small classes) and *large* otherwise. The category $\mathcal{S}et$ is a large category. In this thesis, the large categories will all be concrete, say, the category $\mathcal{S}emigp$, consisting of all semigroups and semigroup morphisms, is a large category since the collection of all semigroups is not a set. Also, $\mathcal{S}emigp$ is a concrete category.

We now introduce the notion of a functor that is a special mapping between categories.

Definition 1.37. Let P_1 and P_2 be categories. A *functor* F from P_1 to P_2 is a pair of maps, both denoted F, from $\mathrm{Ob}(P_1)$ to $\mathrm{Ob}(P_2)$ and from $\mathrm{Mor}(P_1)$ to $\mathrm{Mor}(P_2)$, such that the following conditions hold:

(F1) for any $f \in P_1$, $\mathbf{d}(f)F = \mathbf{d}(fF)$ and $\mathbf{r}(f)F = \mathbf{r}(fF)$;

(F2) if $\exists\ g \circ f$ in P_1, then $(g \circ f)F = gF \circ fF$;

(F3) for $A \in \mathrm{Ob}(P_1)$, $1_A F = 1_{AF}$.

We note that in (F2), if $\exists\ g \circ f$ in P_1, then $\mathbf{r}(g) = \mathbf{d}(f)$. By (F1), we have that $\mathbf{r}(gF) = \mathbf{r}(g)F = \mathbf{d}(f)F = \mathbf{d}(fF)$, and so $\exists\ gF \circ fF$ in P_2.

For any category P, we will use I_P to denote the identity functor, which assigns each object and morphism to itself.

We view small categories as a generalisation of monoids. Clearly, a small category with one object is a monoid. In a small category C, we often identity $e \in \mathrm{Ob}(C)$ with 1_e.

Let G be a group with identity 1_G and I be a non-empty set. We construct a small category $B(G, I)$ as follows:

$$\mathrm{Ob}(B(G, I)) = I$$

and for any $e, f \in I$,

$$\mathrm{Mor}(e, f) = \{(e, g, f) : g \in G\}.$$

We define a partial binary operation on $B(G, I)$ by

$$(e, g, f)(f, h, v) = (e, gh, v).$$

Clearly, the operation is well-defined and associative when it is defined.

For any $e \in \mathrm{Ob}(B(G, I))$, there exists a unique identity $(e, 1_G, e)$ associated to e.

To sum up, $B(G, I)$ forms a category.

A semigroup S with zero is *categorical at zero* or is *C-semigroup* if for all $a, b, c \in S$, if $ab \neq 0$ and $bc \neq 0$ then $abc \neq 0$.

To construct a C-semigroup from $B(G, I)$, we adjoin a zero element 0 to $B(G, I)$. Then we define

$$0x = x0 = 0$$

and for any $x, y \in B(G, I)$, if $x \cdot y$ does not exist in $B(G, I)$, then $x \cdot y = 0$, otherwise, $x \cdot y = xy$, where xy is the product in $B(G, I)$. It is routine to verify that $B(G, I) \cup \{0\}$ becomes a C-semigroup, denoted by $B^0(G, I)$.

A *cancellative category* is a small category in which we have both right and left cancellation for morphisms. Note that any subcategory of a cancellative category is cancellative.

A *groupoid* G is a small category whose morphisms are all invertible, that is, for any $e, f \in \mathrm{Ob}(G)$ and $x \in \mathrm{Mor}(e, f)$, there exists $x^{-1} \in \mathrm{Mor}(f, e)$ such that $x \cdot x^{-1} = 1_e$ and $x \cdot x^{-1} = 1_f$. Any groupoid G is cancellative. Since for any $x, y, z \in G$, if $\exists x \cdot y$, $\exists x \cdot z$ and $x \cdot y = x \cdot z$, then $\mathbf{r}(x) = \mathbf{d}(y) = \mathbf{d}(z)$ and $x^{-1} \cdot x \cdot y = x^{-1} \cdot x \cdot z$, that is, $1_{\mathbf{r}(x)} \cdot y = 1_{\mathbf{r}(x)} \cdot z$, and so, $y = z$. Thus, G is left cancellative. Dually, G is right cancellative. One of the simplest examples of a groupoid is a group or $B(G, I)$.

For the reader's convenience, we will simplify the notation from $\mathrm{Mor}(P)$ to P. In addition, for the term '$x \cdot y$ exists' we may use the expression '$\exists x \cdot y$' or '$x \cdot y$ is defined'.

It is a good position to mention the notion of ordered category. There exist two ways to define an ordered category. We list both in the following. In order to avoid ambiguity, we use subscripts.

Definition 1.38. A category P with a partial order $\leq$ is *ordered*$_1$ if it satisfies the following conditions:

(OC1) if $x, y \in P$ with $x \leq y$, then $\mathbf{r}(x) \leq \mathbf{r}(y)$ and $\mathbf{d}(x) \leq \mathbf{d}(y)$;
(OC2) if $\mathbf{r}(x) = \mathbf{r}(y)$, $\mathbf{d}(x) = \mathbf{d}(y)$ and $x \leq y$, then $x = y$;
(OC3) if $x' \leq x$, $y' \leq y$ and both $x' \cdot y'$ and $x \cdot y$ exist, then $x' \cdot y' \leq x \cdot y$.

An alternative description of ordered category is that:

Definition 1.39. A category P with a partial order $\leq$ is *ordered*$_2$ if it satisfies Conditions (OC1), (OC3) and (OC4):

(OC4) (*i*) for any $x \in P$ and $e \in \mathrm{P}$ with $e \leq \mathbf{d}(x)$, there exists a unique element ${}_e|x$ such that ${}_e|x \leq x$ and $\mathbf{d}({}_e|x) = e$;

(*ii*) for any $x \in P$ and $f \in \mathrm{P}$ with $f \leq \mathbf{r}(x)$, there exists a unique element $x|_f$ such that $x|_f \leq x$ and $\mathbf{r}(x|_f) = f$.

We pause to make a short comment on ordered$_1$ and ordered$_2$ categories. It is easy to see that (OC4) implies (OC2). Since if P is an ordered$_2$ category and $x, y \in P$ with $\mathbf{r}(x) = \mathbf{r}(y)$, $\mathbf{d}(x) = \mathbf{d}(y)$ and $x \leq y$, then by (OC4), $x = {}_{\mathbf{d}(x)}|y = {}_{\mathbf{d}(y)}|y = y$. So an ordered$_2$ category is an ordered$_1$ category. In (OC4), the unique element ${}_e|x$ is called the *restriction* of x to e and dually, the unique element $x|_f$ is called the *co-restriction* of x to f.

A parallel definition of an ordered$_2$ groupoid is that:

Definition 1.40. An ordered groupoid G is a small ordered$_2$ category in which Condition (G) holds:

(G) if $x, y \in G$ and $x \leq y$, then $x^{-1} \ \leq \ y^{-1}$.

If in addition,

(IG) E is a semilattice.

Then $G = (G, \cdot, \leq)$ is called an *inductive*$_1$ *groupoid.*

The subscript is used to distinguish this meaning of the word 'inductive' from both Ehresmann's use and a generalised definition which will occur in Chapter 6.

Let G_1 and G_2 be two inductive$_1$ groupoids. An *inductive*$_1$ *functor* F is a functor $F : G_1 \to G_2$ preserving the order, restrictions and co-restrictions.

Next, we present a useful property of ordered$_2$ categories, which follows from the uniqueness of restrictions and co-restrictions.

Proposition 1.41. *[1] Let P be an ordered$_2$ category. Then for $x \in P$, $e \ \leq \ \mathbf{d}(x)$ and $f \ \leq \ \mathbf{r}(x)$, we have that $f = \mathbf{r}({}_e|x)$ if and only if $e = \mathbf{d}(x|_f)$; moreover, ${}_e|x = x|_f$.*

Further:

Lemma 1.42. *[1] Let P be an ordered$_2$ cancellative category. Then if $x, y \in P$ are such that $\mathbf{r}(x) = \mathbf{d}(y)$ and $e \ \leq \ \mathbf{d}(x)$, then ${}_e|(x \cdot y) = ({}_e|x) \cdot ({}_{\mathbf{r}({}_e|x)}|y)$.*

Now, we introduce the notion of two categories being isomorphic.

Let C and D be two categories. We say that C is *isomorphic* to D if there exist functors: $F : C \to D$ and $G : D \to C$ such that $GF = I_D$ and $FG = I_C$, where I_D and I_C are the identity functors associated to D and C, respectively.

Observe that two isomorphic categories are identical and differ only in the notation of their objects and morphisms. Sometimes this property is too strong, and so we need to introduce a weaker notion of equivalence.

Definition 1.43. Let C and D be categories and let F and G be functors from C to D. Then a *natural transformation* η from F to G consists of morphisms $\eta_X : XF \to XG$ for all $X \in C$, such that for every morphism $f : X \to Y$ in C we have $fF \circ \eta_Y = \eta_X \circ fG$.

For ease of understand $fF \circ \eta_Y = \eta_X \circ fG$, we use the following commutative diagram: If, for all $X \in \mathrm{Ob}(C)$, the morphism η_X is an isomorphism in D, then

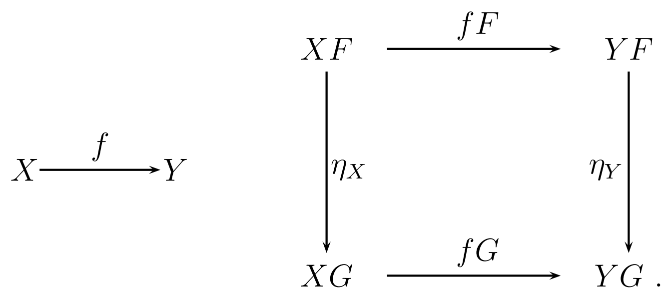

Figure 1.2: The natural transformation property

η is said to be a *natural isomorphism*.

Let C and D be two categories. We say that C and D are *equivalent* if there exists a functor $F : C \to D$, a functor $G : D \to C$, and two natural isomorphisms $\varepsilon : GF \to I_D$ and $\eta : FG \to I_C$.

Clearly, for an equivalence of categories, it is not necessary to require $XGF = X$, but only XGF is isomorphic to X in the category D, and the same for YFG and Y in the category C.

To close this section, we cite from [53] an example of categories below, which are equivalent but not isomorphic.

Let C be a category consisting of a single object a and a single morphism 1_a. Let D be a category with $\mathrm{Ob}(D) = \{b, c\}$ and $\mathrm{Mor}(D) = \{1_b, 1_c, \alpha, \beta\}$, where $\alpha : b \to c$ and $\beta : c \to b$ are isomorphisms. Now, we define a map

$$F : C \to D \ \text{ by } a \mapsto b \ \text{ and } \ 1_a \mapsto 1_b.$$

In addition, we define a map $G : D \to C$ by $XG = a$ and $fG = 1_a$ for all $X \in \mathrm{Ob}(D)$ and $f \in \mathrm{Mor}(D)$. It is routine to verify that F and G are functors. Furthermore, C and D are equivalent but not isomorphic.

Chapter 2

Basic Theory II

This chapter briefly recalls basic definitions and properties of abundant semigroups and weakly U-abundant semigroups that generalise the definitions and properties of regular semigroups we have introduced in Chapter 1.

2.1 Abundant semigroups

2.1.1 Relations $\mathcal{L}^*$, $\mathcal{R}^*$

Let S be a semigroup. We define relations $\leq_{\mathcal{L}^*}$ and $\leq_{\mathcal{R}^*}$ on S by the rule that for any $a, b \in S$,

$$a \leq_{\mathcal{L}^*} b \text{ if and only if } a \leq_{\mathcal{L}} b \text{ in some over semigroup of } S$$

and

$$a \leq_{\mathcal{R}^*} b \text{ if and only if } a \leq_{\mathcal{R}} b \text{ in some over semigroup of } S,$$

where $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{R}}$ are defined in Section 1.3.2. As $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{R}}$ are pre-orders, we have $\leq_{\mathcal{L}^*}$ and $\leq_{\mathcal{R}^*}$ are pre-orders, respectively. Since $\leq_{\mathcal{L}}$ is right compatible and $\leq_{\mathcal{R}}$ is left compatible, we have $\leq_{\mathcal{L}^*}$ is right compatible and $\leq_{\mathcal{R}^*}$ is left compatible.

Now, we denote the associated equivalences by $\mathcal{L}^*$ and $\mathcal{R}^*$, respectively. So, for any $a, b \in S$, $a \; \mathcal{L}^* \; b$ if and only if $a \; \mathcal{L} \; b$ in some oversemigroup of S. The relation $\mathcal{R}^*$ is defined dually. As usual, the intersection of the equivalence relations $\mathcal{L}^*$ and $\mathcal{R}^*$ is denoted by $\mathcal{H}^*$ and their join by $\mathcal{D}^*$.

For ease of description, the $\mathcal{L}^*$-class containing the element a of a semigroup

S will be denoted by L^*_a. Then corresponding notation will be used for the classes of the other relations.

The following basic lemma is from [10].

Lemma 2.1. *[10] Let S be a semigroup and let $a, b \in S$. Then the following statements are equivalent:*

(i) $(a, b) \in \mathcal{L}^*$;

(ii) for all $x, y \in S^1$, $ax = ay$ if and only if $bx = by$.

Clearly, an idempotent e of S acts as a right identity within its $\mathcal{L}^*$-class. In that case, we have:

Lemma 2.2. *If e is an idempotent of a semigroup S, then the following statements are equivalent for $a \in S$:*

(i) $(e, a) \in \mathcal{L}^*$;

(ii) $ae = a$ and for all $x, y \in S^1$, $ax = ay$ implies $ex = ey$.

In view of its definition, $\mathcal{L}^*$ is a right congruence, and dually, $\mathcal{R}^*$ is a left congruence.

We pause to mention that $\mathcal{L} \subseteq \mathcal{L}^*$ on any semigroup S. For any regular elements $a, b \in S$, $(a, b) \in \mathcal{L}^*$ if and only if $(a, b) \in \mathcal{L}$. In particular, if S is a regular semigroup, then $\mathcal{L}^* = \mathcal{L}$.

Another way to define relations $\mathcal{L}^*$ and $\mathcal{R}^*$ is using certain ideals. We now define a left (resp. right) ideal I of a semigroup S to be a *left* (resp. *right) *-ideal* of S if $L^*_a \subseteq I$ (resp. $R^*_a \subseteq I$) for all $a \in I$. A subset I of S is a **-ideal* of S if it is both a left *-ideal and a right *-ideal. In particular, if S is a regular semigroup, then every left (resp. right, two-sided) ideal of S is a left (resp. right, two-sided) *-ideal. Observe that for any element a of a semigroup S, S is a *-ideal of itself. Here, there exists a smallest *-ideal containing a, a smallest left *-ideal containing a and a smallest right *-ideal containing a. We will denote them by $J^*(a)$, $L^*(a)$ and $R^*(a)$, respectively.

Lemma 2.3. *[10] Let S be a semigroup. For any $a, b \in S$,*

(i) $a \mathcal{L}^* b$ if and only if $L^*(a) = L^*(b)$;

(ii) $a \mathcal{R}^* b$ if and only if $R^*(a) = R^*(b)$.

Finally, we define $\mathcal{J}^*$ by analogy with the characterisations of $\mathcal{L}^*$ and $\mathcal{R}^*$ given in Lemma 2.3, by saying that for any $a, b \in S$, $a \mathcal{J}^* b$ if $J^*(a) = J^*(b)$.

2.1.2 Abundant semigroups

This section is concerned with a class of non-regular semigroups built using the relations $\mathcal{L}^*$ and $\mathcal{R}^*$.

We say that a semigroup S is *abundant* if every $\mathcal{L}^*$-class and every $\mathcal{R}^*$-class contains at least one idempotent. If S is a such semigroup and $a \in S$, then we denote idempotents in the L^*_a and R^*_a by a^* and $a^\dagger$, respectively. Note that a^* and $a^\dagger$ need not be unique.

As an analogue of orthodox semigroups in the class of abundant semigroups, we have quasi-adequate semigroups [7]. A *quasi-adequate* semigroup is an abundant semigroup whose set of idempotents forms a subsemigroup. In particular, if the set of idempotents of a quasi-adequate semigroup becomes a semilattice, then it is called an *adequate* semigroup [9]. Note that if S is an adequate semigroup, then for any $a \in S$, a^* and $a^\dagger$ are unique. Since if a° is another idempotent in the $\mathcal{L}^*$-class of a, then we have $aa^\circ = a$, and so $a^*a^\circ = a^*$ by Lemma 2.1, that is, $a^\circ a^* = a^*$ as $E(S)$ is a semilattice. According to the comments succeeding Lemma 2.1, we have $a^\circ = a^*$. Dually, we show that $a^\dagger$ is unique. Thus, in an adequate semigroup S, we have unary operations $a \mapsto a^*$ and $a \mapsto a^\dagger$ for any $a \in S$. So adequate semigroups provide an abundant analogue of inverse semigroups, but see below.

An adequate semigroup S with semilattice of idempotents $E(S)$ is said to be an *ample* semigroup or *a type A semigroup* if it satisfies for all $a \in S$ and $e \in E(S)$:

(i) $ae = (ae)^\dagger a$;

(ii) $ea = a(ea)^*$.

In particular, an inverse semigroup is ample, where $a^\dagger = aa^{-1}$ and $a^* = a^{-1}a$. Ample semigroups are usually thought of as the appropriate abundant analogue of inverse semigroups.

It is easy to see that morphisms between semigroups preserve Green's relations. They need not, however, preserve $\mathcal{L}^*$ and $\mathcal{R}^*$. With this in mind we define the notion of good morphisms.

Let S and T be semigroups and let $\phi : S \to T$ be a morphism. Then ϕ is said to be *good* if for any $a, b \in S$,

$$a \,\mathcal{L}^*\, b \text{ implies } a\phi \,\mathcal{L}^*\, b\phi,$$

$$a\,\mathcal{R}^*\,b \text{ implies } a\phi\,\mathcal{R}^*\,b\phi.$$

2.2 Weakly U-abundant semigroups

For ease of reference we gather together in this section some basic definitions and elementary observations concerning weakly U-abundant semigroups.

2.2.1 Relations $\widetilde{\mathcal{L}}_U$, $\widetilde{\mathcal{R}}_U$

Let S be a semigroup. We denote as usual its set of idempotents by $E(S)$. Consider a non-empty subset $U \subseteq E(S)$; we will call it the set of *distinguished idempotents*. The relation $\leq_{\widetilde{\mathcal{L}}_U}$ on S is defined by the rule that for all $a, b \in S$, $a \leq_{\widetilde{\mathcal{L}}_U} b$ if and only if

$$\{e \in U : be = b\} \subseteq \{e \in U : ae = a\}.$$

It is clear that $\leq_{\widetilde{\mathcal{L}}_U}$ is a pre-order. We denote the associated equivalence relation by $\widetilde{\mathcal{L}}_U$, so that for $a, b \in S$, $a\ \widetilde{\mathcal{L}}_U\ b$ if and only if

$$\{e \in U : ae = a\} = \{e \in U : be = b\}.$$

It is easy to see that $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \widetilde{\mathcal{L}}_U$. In particular, we have:

Lemma 2.4. *[46] Let $E(S)$ be the set of all idempotents of S. If a, b are regular, then $a\,\widetilde{\mathcal{L}}_{E(S)}\,b$ if and only if $a\,\mathcal{L}\,b$.*

It follows that if S is regular and $U = E(S)$, then $\mathcal{L} = \mathcal{L}^* = \widetilde{\mathcal{L}}_U$. Although $\mathcal{L}$ and $\mathcal{L}^*$ are always right compatible, the same need not be true for $\widetilde{\mathcal{L}}_U$. The last fact is shown by a very simple example: the null semigroup of two elements with an adjoined identity, where the distinguished set $U = \{0, 1\}$.

Notice that for $e, f \in U$, $e \leq_{\widetilde{\mathcal{L}}_U} f$ if and only if $e \leq_{\mathcal{L}} f$, so that $e\,\widetilde{\mathcal{L}}_U\,f$ if and only if $e\,\mathcal{L}\,f$. Another useful observation is that :

Lemma 2.5. *[32] If $a \in S$ and $e \in U$, then $a\,\widetilde{\mathcal{L}}_U\,e$ if and only if $ae = a$ and for all $f \in U, af = a$ implies $ef = e$.*

We observe that for any distinguished idempotent a^* in the $\widetilde{\mathcal{L}}_U$-class of a, we have $aa^* = a$.

The relations $\leq_{\widetilde{\mathcal{R}}_U}$ and $\widetilde{\mathcal{R}}_U$ are the left-right duals of $\leq_{\widetilde{\mathcal{L}}_U}$ and $\widetilde{\mathcal{L}}_U$. Also, if $a \in S$ and $a^\dagger \in U$ are such that $a \ \widetilde{\mathcal{R}}_U \ a^\dagger$, then we have $a^\dagger a = a$.

According to Lawson [33], there is another description of $\widetilde{\mathcal{L}}_U$ and $\widetilde{\mathcal{R}}_U$. The $\widetilde{\mathcal{L}}_U$-class and $\widetilde{\mathcal{R}}_U$-class containing a will be denoted by $\widetilde{L}_{U,a}$ and $\widetilde{R}_{U,a}$, respectively, abbreviated as $\widetilde{L}_a$ and $\widetilde{R}_a$, where U is clear. A left ideal I of a semigroup S is said to be a *U-admissible left ideal* if for every $a \in I$, $\widetilde{L}_a \subseteq I$. If a is an element of S, then we define the *principal U-admissible left ideal containing a* to be the intersection of all U-admissible left ideals containing a, and we denote it by $\widetilde{L}(a)$. In particular, for any $e \in U$, $\widetilde{L}(e) = Se$. Dually, we define the *principal U-admissible right ideal containing a* and we denote it by $\widetilde{R}(a)$. Following the above terminology, in [44], an ideal I of S is called *U-admissible* if I is both a U-admissible right ideal and a U-admissible left ideal of S and the principal U-admissible ideal containing a is defined to be the intersection of all U-admissible ideals of S containing a, denoted by $\widetilde{J}(a)$. Clearly, $\widetilde{L}(a) \subseteq \widetilde{J}(a)$ and $\widetilde{R}(a) \subseteq \widetilde{J}(a)$ for all a in S. The following lemma concerning $\widetilde{\mathcal{L}}_U$, $\widetilde{\mathcal{R}}_U$ and $\widetilde{\mathcal{J}}_U$ is extracted from [44].

Lemma 2.6. *[44] Let S be a semigroup and U be a non-empty subset of $E(S)$. Then for any $a, b \in S$,*

$$(a,b) \in \widetilde{\mathcal{L}}_U \quad \text{if and only if} \quad \widetilde{L}(a) = \widetilde{L}(b);$$

$$(a,b) \in \widetilde{\mathcal{R}}_U \quad \text{if and only if} \quad \widetilde{R}(a) = \widetilde{R}(b).$$

Consequently, we define the relation $\widetilde{\mathcal{J}}_U$ by the rule that

$$(a,b) \in \widetilde{\mathcal{J}}_U \quad \text{if and only if} \quad \widetilde{J}(a) = \widetilde{J}(b).$$

To close this section, we define $\widetilde{\mathcal{H}}_U$ and $\widetilde{\mathcal{D}}_U$ as the intersection and the join of $\widetilde{\mathcal{L}}_U$ and $\widetilde{\mathcal{R}}_U$, respectively. Note that we do not always have that $\widetilde{\mathcal{D}}_U = \widetilde{\mathcal{L}}_U \circ \widetilde{\mathcal{R}}_U$.

2.2.2 Weakly U-abundant semigroups

In a manner analogous to the definition of an abundant semigroup, S is said to be *weakly U-abundant* if every $\widetilde{\mathcal{L}}_U$-class and every $\widetilde{\mathcal{R}}_U$-class contains an idempotent of U. If S is a such semigroup and $a \in S$, then we follow usual practice and denote idempotents in the $\widetilde{\mathcal{L}}_U$-class and $\widetilde{\mathcal{R}}_U$-class of a by a^* and $a^\dagger$, respectively. Note that there need not be a unique choice for a^* and $a^\dagger$ unless U is a semilattice.

We make some comments that if the distinguished set of idempotents of a weakly U-abundant semigroup S is the whole set of idempotents $E(S)$, then usually, we call it a *weakly abundant semigroup*. Another point is that if we talk of a *particular* weakly U-abundant semigroup, then we are referring to a *particular* set of idempotents U; on the other hand, if we are talking of the *class* of all weakly U-abundant semigroups, the U varies over *all possible* set of idempotents.

We will be interested in semigroups S in which the relation $\widetilde{\mathcal{L}}_U$ is a right congruence and $\widetilde{\mathcal{R}}_U$ is a left congruence. In this case, we say that S satisfies *Congruence Condition (C)* (with respect to U). Indeed it seem very little theory can be developed if we do not assume the Congruence Condition. If S is weakly U-abundant with (C), then

$$xy \;\widetilde{\mathcal{L}}_U\; (xy)^* \;\widetilde{\mathcal{L}}_U\; (x^*y)^* \;\widetilde{\mathcal{L}}_U\; x^*y,$$

for any $x^*, (xy)^*$ and $(x^*y)^*$. Dually, we have

$$xy \;\widetilde{\mathcal{R}}_U\; (xy)^\dagger \;\widetilde{\mathcal{R}}_U\; (xy^\dagger)^\dagger \;\widetilde{\mathcal{R}}_U\; xy^\dagger,$$

for any $y^\dagger, (xy)^\dagger$ and $(xy^\dagger)^\dagger$.

The next lemma gives an equivalent description of weakly U-abundant semigroups.

Lemma 2.7. *A semigroup S is weakly U-abundant if and only if for any $a \in S$, there exist $e, f \in U$ such that $\widetilde{L}(a) = Se$ and $\widetilde{R}(a) = fS$.*

Proof. For any distinguished idempotent e in U, $\widetilde{L}(e) = Se$. In fact, for any $a, b \in S$, $a \;\widetilde{\mathcal{L}}_U\; b$ if and only if $\widetilde{L}(a) = \widetilde{L}(b)$. Thus, a semigroup is weakly U-abundant if and only if for any $a \in S$, there exists $e \in U$ such that $a \;\widetilde{\mathcal{L}}_U\; e$, that is, $\widetilde{L}(a) = \widetilde{L}(e)$, if and only if $\widetilde{L}(a) = Se$. Dually, $\widetilde{R}(a) = fS$. □

For convenience, we give the next lemma.

Lemma 2.8. *Let S be a weakly U-abundant semigroup. For any $x, y \in S$ we have $(yx)^* \leq_{\mathcal{L}} x^*$ and $(xy)^\dagger \leq_{\mathcal{R}} x^\dagger$. Further, $(xy)^\dagger x^\dagger \;\mathcal{R}\; (xy)^\dagger$ and $y^*(xy)^* \;\mathcal{L}\; (xy)^*$*

Proof. Let $x, y \in S$. Certainly, $yxx^* = yx$, and so $(yx)^*x^* = (yx)^*$ by definition of $(yx)^*$ so that $(yx)^* \leq_{\mathcal{L}} x^*$. Dually, we obtain that $(xy)^\dagger \leq_{\mathcal{R}} x^\dagger$.

In view of Lemma 1.24, the set $E(S)$ of idempotents of S is a biordered set, and so by (B21) and its dual in Section 1.4, $(xy)^\dagger x^\dagger \mathrel{\mathcal{R}} (xy)^\dagger$ and $y^*(xy)^* \mathrel{\mathcal{L}} (xy)^*$ in S. □

Observe that morphisms between semigroups need not preserve $\mathcal{L}^*$ and $\mathcal{R}^*$ as mentioned in Section 2.1.2, nor $\widetilde{\mathcal{L}}_U$ and $\widetilde{\mathcal{R}}_U$. With this in mind we define the notion of admissible morphisms.

Let S and T be semigroups with distinguished subsets of idempotents U and V, respectively, and let $\phi : S \to T$ be a morphism. Then ϕ is said to be (U, V)-*admissible* if for any $a, b \in S$,

$$a \mathrel{\widetilde{\mathcal{L}}_U} b \text{ implies } a\phi \mathrel{\widetilde{\mathcal{L}}_V} b\phi,$$

$$a \mathrel{\widetilde{\mathcal{R}}_U} b \text{ implies } a\phi \mathrel{\widetilde{\mathcal{R}}_V} b\phi,$$

and $U\phi \subseteq V$. Briefly, we will refer to the notion of being (U, V)-admissible as *admissible*, where no ambiguity can occur.

Moreover, ϕ is said to be *strongly admissible* [17] if for any $a, b \in S$,

$$a \mathrel{\widetilde{\mathcal{L}}_U} b \text{ if and only if } a\phi \mathrel{\widetilde{\mathcal{L}}_{U\phi}} b\phi$$

and

$$a \mathrel{\widetilde{\mathcal{R}}_U} b \text{ if and only if } a\phi \mathrel{\widetilde{\mathcal{R}}_{U\phi}} b\phi.$$

Naturally, a congruence ρ on S is said to be *admissible* if the natural morphism $\rho^\natural\colon S \to S/\rho$ is admissible, that is $(U, U\rho)$-admissible.

For admissible morphisms and congruences, the following lemmas are easy to see, making use of Lemma 2.5.

Lemma 2.9. *[46] Let S, T be semigroups with distinguished subsets of idempotents U, V respectively. Suppose that S is weakly U-abundant, and let $\phi : S \to T$ be a morphism. Then ϕ is admissible if and only if $U\phi \subseteq V$ and for any $a \in S$ there exist idempotents $f \in \widetilde{L}_a \cap U$ and $e \in \widetilde{R}_a \cap U$ such that $a\phi \mathrel{\widetilde{\mathcal{L}}_V} f\phi$ and $a\phi \mathrel{\widetilde{\mathcal{R}}_V} e\phi$.*

Lemma 2.10. *[6] Let S be a semigroup with subset of idempotents U and let $\phi : S \to T$ be an admissible surjective morphism. If S is weakly U-abundant, then T is weakly $U\phi$-abundant.*

Lemma 2.11. *If S is a weakly U-abundant semigroup with (C) and ρ is an admissible congruence on S, then S/ρ has (C) with respect to U/ρ.*

Proof. To see that S/ρ satisfies the Congruence Condition, we assume that $a\rho$ and $b\rho$ are elements of S/ρ such that $a\rho\,\widetilde{\mathcal{L}}_{U/\rho}\,b\rho$. Since S is a weakly U-abundant semigroup and ρ is an admissible congruence on S, there exist $e \in \widetilde{L}_a \cap U$ and $f \in \widetilde{L}_b \cap U$ such that $a\rho\,\widetilde{\mathcal{L}}_{U/\rho}\,e\rho$ and $b\rho\,\widetilde{\mathcal{L}}_{U/\rho}\,f\rho$, respectively. It follows that $e\rho\,\widetilde{\mathcal{L}}_{U/\rho}\,f\rho$, that is, $e\rho\,\mathcal{L}\,f\rho$. Clearly, for any $c\rho \in S/\rho$, we have $e\rho\,c\rho\,\mathcal{L}\,f\rho\,c\rho$ because $\mathcal{L}$ is a right congruence. So $(ec)\rho\,\mathcal{L}\,(fc)\rho$. Observe that $a\,\widetilde{\mathcal{L}}_U\,e$ and $b\,\widetilde{\mathcal{L}}_U\,f$. It follows that $ac\,\widetilde{\mathcal{L}}_U\,ec$ and $bc\,\widetilde{\mathcal{L}}_U\,fc$ since $\widetilde{\mathcal{L}}_U$ is a right congruence on S. Also, $(ac)\rho\,\widetilde{\mathcal{L}}_{U/\rho}\,(ec)\rho$ and $(bc)\rho\,\widetilde{\mathcal{L}}_{U/\rho}\,(fc)\rho$ since ρ is an admissible congruence. Thus $a\rho\,c\rho\,\widetilde{\mathcal{L}}_{U/\rho}\,b\rho\,c\rho$. Hence, $\widetilde{\mathcal{L}}_{U/\rho}$ is a right congruence on S/ρ. Dually, we can verify that $\widetilde{\mathcal{R}}_{U/\rho}$ is a left congruence on S/ρ. Consequently, S/ρ satisfies the Congruence Condition (C). $\square$

A weakly U-abundant semigroup is *U-fundamental* if the largest admissible congruence contained in $\widetilde{\mathcal{H}}_U$ is trivial. For convenience we shall sometimes simplify the term 'U-fundamental weakly U-abundant semigroup' to 'fundamental weakly U-abundant semigroup'.

It is easy to see that if S is a weakly U-abundant semigroup with (C) and T is a U-full subsemigroup of S, in the sense that $U \subseteq T$, then T satisfies (C). In addition, if S is U-fundamental, then T is U-fundamental.

2.2.3 Weakly U-regular semigroups

In this section we list some properties of weakly U-abundant semigroups with (C), where U is a regular biordered set.

We first recall that if U is a biordered set, then $\mathcal{M}(e, f)$ denote the preordered set $(\omega^l(e) \cap \omega^r(f), \prec)$, where

$$g \prec h \Leftrightarrow eg\,\omega^r\,eh,\ gf\,\omega^l\,hf.$$

The set $S(e,f) = \{h \in \mathcal{M}(e,f) : g \prec h, \ \forall g \in \mathcal{M}(e,f)\}$ is called the *sandwich set* of e and f. In particular, if U is regular, then $S(e,f) \neq \emptyset$ for all $e, f \in U$.

Lemma 2.12. *Let S be a weakly U-abundant semigroup, where U is a regular biordered set. For $e, f \in U$, we define*

$$S_1(e,f) = \{h \in \mathcal{M}(e,f) : ehf = ef\},$$

and

$$S_2(e,f) = \{h \in \mathcal{M}(e,f) : h(ef)h = h \text{ and } (ef)h(ef) = ef\}.$$

Then $S_1(e,f) = S_2(e,f) \subseteq S(e,f)$.

Proof. To show that $S_1(e,f) = S_2(e,f)$, we assume that $h \in \mathcal{M}(e,f)$. Then $he = h = fh$, and so

$$h(ef)h = (he)(fh) = h \text{ and } (ef)h(ef) = e(fhe)f = ehf.$$

Obviously, $ehf = ef$ if and only if $(ef)h(ef) = ef$. Thus, $S_1(e,f) = S_2(e,f)$. We now turn to show that $S_1(e,f) \subseteq S(e,f)$. Suppose that $h \in S_1(e,f)$ and $g \in \mathcal{M}(e,f)$. Then

$$(eh)(eg) = e(he)g = ehg = eh(fg) = (ehf)g = efg = eg,$$

and so $eg \ \omega^r \ eh$. Dually, $gf \ \omega^l \ hf$. Thus $g \prec h$ so that $h \in S(e,f)$. □

Let U be a set of idempotents of a semigroup S. We will use $\langle U \rangle$ to denote the semigroup generated by U. A weakly U-abundant semigroup with (C) is said to be a *weakly U-regular semigroup* if $\langle U \rangle$ is a regular semigroup whose set of idempotents is U. We remind the reader that this terminology, based on existing convention, needs to be viewed with care: if we talk of a *particular* weakly U-regular semigroup, then we are referring to a *particular* set of idempotents U; on the other hand, if we are talking of the *class* of all weakly U-regular semigroups, the U varies over *all possible* sets of idempotents. It is clear that the collection of weakly U-regular semigroups and admissible morphisms forms a category, which we denote by $\mathcal{WRS}$.

Lemma 2.13. *Let S be a weakly U-regular semigroup. Then $S_1(e,f) = S(e,f) \neq \emptyset$ for all $e, f \in U$.*

Proof. Suppose that $e, f \in U$. Then ef is a regular element in $\langle U \rangle$. If a is an inverse of ef in $\langle U \rangle$ and $h = fae$, then $h \in \langle U \rangle$ and $h^2 = (fae)(fae) = f(aefa)e = fae = h$. Together with $E(\langle U \rangle) = U$, we obtain that $h \in U$. Since, $ehf = efaef = ef$, it follows that $h \in S_1(e, f)$, and so $S_1(e, f) \neq \emptyset$.

By Lemma 2.12, $S_1(e, f) \subseteq S(e, f)$. To show the converse, we assume that $g \in S(e, f)$. Let $h \in S_1(e, f)$. Then by Lemma 2.12, $h \in S(e, f)$, and so $eg \mathrel{\mathcal{R}} eh$, $gf \mathrel{\mathcal{L}} hf$. Thus, $egf = egef = (eg)(ehf) = (eg)(eh)f = ehf = ef$ so that $g \in S_1(e, f)$. Hence $S(e, f) = S_1(e, f)$. □

2.2.4 Weakly B-orthodox semigroups

We recall that an *orthodox semigroup* is a regular semigroup S such that $E(S)$ is a band. Consequently, a weakly B-abundant semigroup is said to be *weakly B-orthodox* if it has (C) and B is a band. Whenever we talk of a *particular* weakly B-orthodox semigroup, then we are referring to a *particular* band B; on the other hand, if we are talking of the *class* of all weakly B-orthodox semigroups, the B varies over *all possible* bands. It is clear that the collection of weakly B-orthodox semigroups and admissible morphisms forms a category, which we denote by $\mathcal{WO}$.

Lemma 2.14. *Let S be a weakly B-orthodox semigroup. For any $x \in S$ and $e, f, g, h \in B$,*

(i) if $e \leq_{\mathcal{R}}$ (resp. $\leq_{\mathcal{L}}$) $g \mathrel{\mathcal{R}} x^{\dagger}$, then $ex \mathrel{\widetilde{\mathcal{R}}_B} e$;

(ii) if $f \leq_{\mathcal{L}}$ (resp. $\leq_{\mathcal{R}}$) $h \mathrel{\mathcal{L}} x^{*}$, then $xf \mathrel{\widetilde{\mathcal{L}}_B} f$.

Proof. To prove (i), suppose that $e \leq_{\mathcal{R}} g \mathrel{\mathcal{R}} x^{\dagger}$, then $ex \mathrel{\widetilde{\mathcal{R}}_B} ex^{\dagger} \mathrel{\widetilde{\mathcal{R}}_B} eg \mathrel{\mathcal{R}} e$, otherwise, $e \leq_{\mathcal{L}} g$, and so $ex \mathrel{\widetilde{\mathcal{R}}_B} eg = e$. By a similar argument, we can show that (ii) holds. □

2.2.5 Ehresmann semigroups

Let S be weakly E-abundant with (C). We say that S is an *Ehresmann semigroup* (with distinguished semilattice E) if E is a semilattice. It is straightforward to see that if S is Ehresmann, then, for any $a \in S$, the elements a^* and $a^{\dagger}$ are the unique elements of E in the $\widetilde{\mathcal{L}}_E$-class and the $\widetilde{\mathcal{R}}_E$-class of a, respectively. We regard Ehresmann semigroups as algebras with signature $(2, 1, 1)$; as such, they form a variety $\mathcal{E}$. Indeed, $\mathcal{E}$ is the variety generated by $\mathcal{A}$, where $\mathcal{A}$ is the quasi-variety of adequate semigroups [29]. The corresponding result is the one-side case may be found in [16] or [30].

An important property of Ehresmann semigroups is given below.

Lemma 2.15. *[32] Let S be an Ehresmann semigroup with semilattice of distinguished idempotents E. Then*

(i) for all $x, y \in S$, $(xy)^* \leq y^*$ and $(xy)^\dagger \leq x^\dagger$;

(ii) for all $e \in E$, $e^* = e$ and $e^\dagger = e$;

(iii) for all $x, y \in S$, $x \widetilde{\mathcal{L}}_E\, y \Leftrightarrow x^* = y^*$ and $x \widetilde{\mathcal{R}}_E\, y \Leftrightarrow x^\dagger = y^\dagger$;

(iv) for all $x, y \in S$, $(xy)^* = (x^*y)^*$ and $(xy)^\dagger = (xy^\dagger)^\dagger$.

We introduce the notion of restriction semigroups as an analogue of ample semigroups. Consequently, a restriction semigroup has also been called a *weakly E-ample semigroup*, where E is the distinguished semilattice of idempotents. There are four ways to define restriction semigroups: as varieties of algebras, representation by (partial) mappings, using generalised Green's relations $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$, and inductive constellations. Here we define restriction semigroups by using $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$.

An Ehresmann semigroup S is *left restriction* with distinguished semilattice E if it satisfies the *left ample* condition (AL).

(AL) $(\forall a \in S,\ e \in E)\ ae = (ae)^\dagger a$.

Similarly, an Ehresmann semigroup S with distinguished semilattice of idempotents E is *right restriction* if it satisfies the *right ample* condition (AR).

(AR) $(\forall a \in S,\ e \in E)\ ea = a(ea)^*$.

An Ehresmann semigroup is a *restriction semigroup* if it is both a left restriction and a right restriction semigroup.

2.3 The idempotent connected condition

We focus on the idempotent connected condition in this section. A fuller version of some of the ideas we present here is contained in [6], [8] and [48]. Essentially, all of the idempotent connected and ample (formely, type A) conditions extant give some control over the position of idempotents in products, usually facilitating results for abundant or weakly abundant semigroups reminiscent of those in the regular case.

2.3.1 (WIC), (IC) and (PIC)

Let S be a weakly U-abundant semigroup. For any $e \in U$, we put

$$\langle e \rangle = \langle eue : u \in \langle U \rangle \text{ and } (eue)^2 = eue \rangle$$

or equivalently,

$$\langle e \rangle = \{u_1 \cdots u_n : u_i \in \langle U \rangle, u_i^2 = u_i \text{ and } u_i \leq e\}.$$

Clearly, $\langle e \rangle$ is a subsemigroup with identity e.

We say that a weakly U-abundant semigroup S satisfies the *weak idempotent connected* condition (WIC) (with respect to U) if for any $a \in S$ and some a^*, $a^\dagger$, if $x \in \langle a^\dagger \rangle$, then there exists $y \in \langle a^* \rangle$ with $xa = ay$; and dually, if $z \in \langle a^* \rangle$ then there exists $t \in \langle a^\dagger \rangle$ with $ta = az$.

We pause here to make some comments on Condition (WIC). The phrase 'for some $a^\dagger$, a^*' may be replaced by 'for any $a^\dagger$, a^*'. For suppose that S has (WIC), $a \in S$, $a^\dagger$ is the chosen idempotent of U in the $\widetilde{\mathcal{R}}_U$-class of a, and a^+ is another element of U in the same $\widetilde{\mathcal{R}}_U$-class. If $v \in \langle a^+ \rangle$, then

$$v = u_1 \cdots u_n \qquad \Big(u_i \in \langle U \rangle,\ u_i = u_i^2,\ u_i \leq a^+\Big).$$

So

$$a^\dagger v a^\dagger = v a^\dagger = (u_1 a^\dagger) \cdots (u_n a^\dagger).$$

Certainly, $u_i a^\dagger \in \langle U \rangle$, $u_i a^\dagger \leq a^\dagger$ and $(u_i a^\dagger)^2 = u_i a^\dagger u_i a^\dagger = u_i a^\dagger$. Thus, $va^\dagger \in \langle a^\dagger \rangle$, and so by (WIC), $va = va^\dagger a = ak$ for some $k \in \langle a^* \rangle$. Then $k = w_1 \cdots w_m$ for some $w_i \in \langle U \rangle$, $w_i^2 = w_i$ and $w_i \leq a^*$. As above, for any $a^\circ \in U$ lying in the $\widetilde{\mathcal{L}}_U$-class of a, we have $a^\circ k = (a^\circ w_1) \cdots (a^\circ w_m) \in \langle a^\circ \rangle$, and so $va = ak = a(a^\circ k)$. Thus in the condition of (WIC) we may choose the y lie in any given $\langle a^\circ \rangle$, and dually, the t to lie in any given $\langle a^\dagger \rangle$.

The following lemma provides an equivalent statement of Condition (WIC).

Lemma 2.16. *Let S be a weakly U-abundant semigroup. Then S satisfies (WIC) if and only if for any $a \in S$ and some (any) a^*, $a^\dagger$, if $u \in \langle U \rangle, u^2 = u$ and $u \leq a^\dagger$ then there exists $y \in \langle a^* \rangle$ with $ua = ay$; and dually, if $v \in \langle U \rangle, v^2 = v$ and $v \leq a^*$ then there exists $t \in \langle a^\dagger \rangle$ with $ta = av$.*

Proof. The necessity is clear. We now show the sufficiency. If $x \in \langle a^\dagger \rangle$, then $x = e_1 \cdots e_n$, where $e_i \in \langle U \rangle$, $e_i^2 = e_i$ and $e_i \leq a^\dagger$. Then we have

$$e_i a = a y_i \qquad \big(y_i \in \langle a^* \rangle\big),$$

and so

$$xa = e_1 \cdots e_n a = e_1 \cdots e_{n-1}(e_n a) = e_1 \cdots e_{n-1} a y_n = \cdots = a y_1 \cdots y_n,$$

where $y_1 \cdots y_n \in \langle a^* \rangle$. Dually, we show the second part holds. □

We now present two stronger versions of Condition (WIC) which have been investigated in [6] and [8].

A weakly U-abundant semigroup S satisfies Condition (IC) if for any $a \in S$ and for some $a^\dagger, a^*$, there exists an order isomorphism $\alpha : \langle a^\dagger \rangle \to \langle a^* \rangle$ such that for all $x \in \langle a^\dagger \rangle$,

$$xa = a(x\alpha).$$

The order isomorphism given above is said to be a *connecting order isomorphism.*

Notice that we can replace 'some' in (IC) by 'any'. For suppose that $a \in S$, $a^\dagger$, a^+ are idempotents of U in the $\widetilde{\mathcal{R}}_U$-class of a, then the map

$$\rho_{a^\dagger} : \langle a^+ \rangle \to \langle a^\dagger \rangle$$

given by

$$x\rho_{a^\dagger} = xa^\dagger$$

for all $x \in \langle a^+ \rangle$, is an isomorphism. Since if $x \in \langle a^+ \rangle$, then

$$x = u_1 \cdots u_n \qquad \big(u_i \in \langle U \rangle,\ u_i = u_i^2,\ u_i \leq a^+\big),$$

and so

$$xa^\dagger = (u_1 a^\dagger) \cdots (u_n a^\dagger),$$

where $u_i a^\dagger \in \langle U \rangle$, $u_i a^\dagger \leq a^\dagger$ and $(u_i a^\dagger)^2 = u_i a^\dagger$, so that $xa^\dagger \in \langle a^\dagger \rangle$. Thus, $\rho_{a^\dagger}$ is well-defined.

Let y be another element in $\langle a^+\rangle$. Then we have

$$(xy)\rho_{a^\dagger} = xya^\dagger = xa^\dagger ya^\dagger = x\rho_{a^\dagger} y\rho_{a^\dagger},$$

and so $\rho_{a^\dagger}$ is a morphism.

Certainly, $\rho_{a^+} : \langle a^\dagger\rangle \to \langle a^+\rangle$ is a morphism. Further, for any $x \in \langle a^+\rangle$, we have

$$x\rho_{a^\dagger}\rho_{a^+} = xa^\dagger a^+ = xa^+ = x$$

and similarly, for any $z \in \langle a^\dagger\rangle$, we have

$$z\rho_{a^+}\rho_{a^\dagger} = z.$$

Hence, ρ_{a^+} is an isomorphism.

Dually, if a^*, a° are idempotent of U in the $\widetilde{\mathcal{L}}_U$-class of a, then the map

$$\lambda_{a^\circ} : \langle a^*\rangle \to \langle a^\circ\rangle$$

given by

$$x\lambda_{a^\circ} = a^\circ x$$

for any $x \in \langle a^*\rangle$, is an isomorphism.

Let α be an order isomorphism from $\langle a^\dagger\rangle \to \langle a^*\rangle$ such that $xa = a(x\alpha)$ for all $x \in \langle a^\dagger\rangle$. Then certainly, $\rho_{a^\dagger}\alpha\lambda_{a^\circ}$ is an order isomorphism from $\langle a^+\rangle$ to $\langle a^\circ\rangle$ and also, for any $z \in \langle a^+\rangle$, we have

$$za = za^\dagger a = a((za^\dagger)\alpha) = aa^\circ((za^\dagger)\alpha) = a(z\rho_{a^\dagger}\alpha\lambda_{a^\circ}).$$

Consequently, $\rho_{a^\dagger}\alpha\lambda_{a^\circ} : \langle a^+\rangle \to \langle a^\circ\rangle$ is an order isomorphism as required in Condition (IC).

Yet another version of Condition (WIC) is Condition (PIC). We say that a weakly U-abundant semigroup S satisfies (PIC) if for all $a \in S$ and for some $a^\dagger, a^*$, there exists an isomorphism $\alpha : \langle a^\dagger\rangle \to \langle a^*\rangle$ such that for all $x \in \langle a^\dagger\rangle$,

$$xa = a(x\alpha).$$

The isomorphism given above is said to be a *connecting isomorphism.* As with the definition of (WIC) (resp. (IC)), we can use the same method as that for Condition (IC) to replace 'some' by 'any', so we omit it.

The following Lemma is cited from [6].

Lemma 2.17. *Let T be a full subsemigroup of a weakly U-abundant semigroup S. If S satisfies (WIC) (resp. (IC), (PIC)), then so does T.*

2.3.2 Special cases

In this section we concentrate on some special kinds of weakly U-abundant semigroup with (WIC) (resp. (IC), (PIC)).

Let S be a weakly U-regular semigroup. Since $E(\langle U\rangle) = U$, it follows that for any $e \in U$,

$$\langle e\rangle = \langle eue : u \in \langle U\rangle \text{ and } eue \in U\rangle,$$

that is,

$$\langle e\rangle = \langle v : v \in U \text{ and } v \ \leq \ e\rangle,$$

or equivalently,

$$\langle e\rangle = \{v_1 \cdots v_n : v_i \in U \text{ and } v_i \ \leq \ e\}.$$

For a band B and element e of B, we have

$$\langle e\rangle = \langle eue : u \in B\rangle$$

so that if $x = (eu_1e)\cdots(eu_ne) \in \langle e\rangle$, then $x \in B$ and $x \ \leq \ e$. Conversely, if $y \in B$ and $y \ \leq \ e$, then $y = eye \in \langle e\rangle$. Thus $\langle e\rangle$ is the principal order ideal generated by e, that is

$$\langle e\rangle = \{x \in B : x \ \leq \ e\} = \{x \in B : ex = xe = e\}.$$

A weakly B-orthodox semigroup with (WIC) (resp. (IC), (PIC)) has been mentioned variously in [6], [11], [14], [15], [16], [17] and [19].

We now describe an important connection between Condition (WIC) and Conditions (AL) and (AR) on an Ehresmann semigroup below.

Lemma 2.18. *Let S be an Ehresmann semigroup with distinguished semilattice of idempotents E. Then S has (WIC) if and only if it satisfies Conditions (AL) and (AR), that is, S is a restriction semigroup.*

Proof. Clearly, if S satisfies Conditions (AL) and (AR), then it has (WIC).

Conversely, suppose that S is an Ehresmann semigroup with distinguished semilattice of idempotents E and satisfying (WIC). For any $a \in S$ and $e \in E$, we have $ae = aa^*ea^* = fa$ for some $f \in E$, from which it follows that $fae = ae$, and so $f(ae)^\dagger = (ae)^\dagger$. Thus,

$$(ae)^\dagger a = f(ae)^\dagger a = (ae)^\dagger fa = (ae)^\dagger ae = ae.$$

Dually, $ea = a(ea)^*$. Hence, S satisfies Conditions (AL) and (AR). $\square$

If S is abundant, we replace the distinguished set of idempotents U by the whole set of idempotents $E(S)$. For an element e of $E(S)$,

$$\begin{aligned}\langle e\rangle &= \{(eu_1e)\cdots(eu_ne) : u_i \in \langle E(S)\rangle, eu_ie \in E(S)\}\\ &= \{v_1\cdots v_n : v_i \in E(S), v_i \leq e\}\\ &= \langle f : f \in E(S), f \leq e\rangle.\end{aligned}$$

Thus, $\langle e\rangle$ is generated by all idempotents f satisfying $f \leq e$.

In [8], El-Qallali and Fountain introduced the notion of Condition (IC) in the abundant case, as we describe below.

An abundant semigroup S with set of idempotents $E(S)$ is *idempotent connected* (IC) if for any $a \in S$, and for some $a^\dagger$, a^*, there exists a bijection $\alpha : \langle a^\dagger\rangle \to \langle a^*\rangle$ such that $xa = a(x\alpha)$ for all $x \in \langle a^\dagger\rangle$.

It is easy to see that the bijection α in Condition (IC) must be an isomorphism. Since if $x, y \in \langle a^\dagger\rangle$, then $xy \in \langle a^\dagger\rangle$ and $(xy)a = a(xy)\alpha$. But also $xya = xa(y\alpha) = a(x\alpha)(y\alpha)$. So $a(xy)\alpha = a(x\alpha)(y\alpha)$. Since $(xy)\alpha, (x\alpha)(y\alpha) \in \langle a^*\rangle$ and $a\,\mathcal{L}^*\,a^*$, we have that $a^*(xy)\alpha = a^*(x\alpha)(y\alpha)$, and so $(xy)\alpha = (x\alpha)(y\alpha)$. Usually, we call such α a *connecting isomorphism.* Thus the notion of (IC) from [8] coincides with the notion of (IC) in Section 2.3.1. A further point is that the connecting isomorphism $\alpha : \langle a^\dagger\rangle \to \langle a^*\rangle$ is unique. Since if $\beta : \langle a^\dagger\rangle \to \langle a^*\rangle$ is another connecting isomorphism, then for any $x \in \langle a^\dagger\rangle$, we have $xa = a(x\alpha) = a(x\beta)$. As $a\ \mathcal{L}^*\ a^*$, we have that $a^*(x\alpha) = a^*(x\beta)$, and so $x\alpha = y\beta$ so that $\alpha = \beta$. Finally, we note that the notion of Condition (IC) defined by El-Qallali and Fountain coincides with Condition (PIC); in addition, Condition (WIC) and Condition (IC) coincide in the abundant case, as the following lemma demonstrates.

Lemma 2.19. *[8] Let S be an abundant semigroup with set of idempotents $E(S)$. Then the following statements are equivalent:*

(i) *S satisfies (IC);*

(ii) *for each $a \in S$, the following two conditions hold:*

(a) *for each $e \leq a^{\dagger}$, there exists $f \leq a^*$ (resp. $f \in S$) such that $ea = af$;*

(b) *for each $g \leq a^*$, there exists $h \leq a^{\dagger}$ (resp. $h \in S$) such that $ha = ag$.*

We make a short comment on Lemma 2.19. In part (a) of (ii), f is unique. Since if $k \leq a^*$ and $ea = ak$, then we have $ak = af$. As $a^* \mathcal{L}^* a$, we obtain that $a^*k = a^*f$, that is, $k = f$. Dually, in part (b) of (ii), h is unique.

We say that an abundant semigroup is a *concordant semigroup* if it satisfies (IC) and the set of idempotents forms a regular biordered set. An abundant semigroup is a *Type W* semigroup if it satisfies (IC) and the set of idempotents forms a band. In view of Lemma 2.18, it is easy to see that an adequate semigroup S has (IC) if and only if it is an ample semigroup.

At the end of this section, we turn our attention to the regular case. If S is a regular semigroup, then for any $a \in S$ and any inverse a' of a, there exists an isomorphism

$$\alpha : \langle aa' \rangle \to \langle a'a \rangle$$

defined by

$$x\alpha = a'xa$$

for all $x \in \langle aa' \rangle$. For suppose that if $x \in \langle aa' \rangle$, then

$$x = v_1 \cdots v_n \qquad \big(v_i \in E(S),\ v_i \leq aa'\big),$$

and so

$$\begin{aligned} a'xa &= a'v_1 \cdots v_n a \\ &= a'v_1 aa' v_2 \cdots aa' v_n a \\ &= w_1 \cdots w_n \qquad \big(w_i = a'v_i a\big). \end{aligned}$$

Now ,

$$\begin{aligned} w_i^2 &= (a'v_ia)(a'v_ia) \\ &= a'(v_iaa'v_i)a \\ &= a'v_i^2a \qquad \big(v_i \ \leq \ aa'\big) \\ &= a'v_ia \qquad \big(v_i \in E(S)\big) \\ &= w_i \end{aligned}$$

and clearly, $w_i \ \leq \ a'a$, so that $x\alpha \in \langle a'a \rangle$. Thus, α is well-defined. If also $y \in \langle aa' \rangle$, then

$$(xy)\alpha = a'xya = a'xaa'ya = (x\alpha)(y\alpha),$$

and so α is a morphism. Similarly, $\beta : \langle a'a \rangle \to \langle aa' \rangle$ given by

$$x\beta = axa' \qquad \big(x \in \langle a'a \rangle\big)$$

is a morphism. Moreover,

$$\alpha\beta = 1_{\langle aa' \rangle} \text{ and } \beta\alpha = 1_{\langle a'a \rangle}$$

so that α is an isomorphism. In addition, for any $x \in \langle aa' \rangle$, we have $xa = aa'xa = a(x\alpha)$ and consequently, S has (IC).

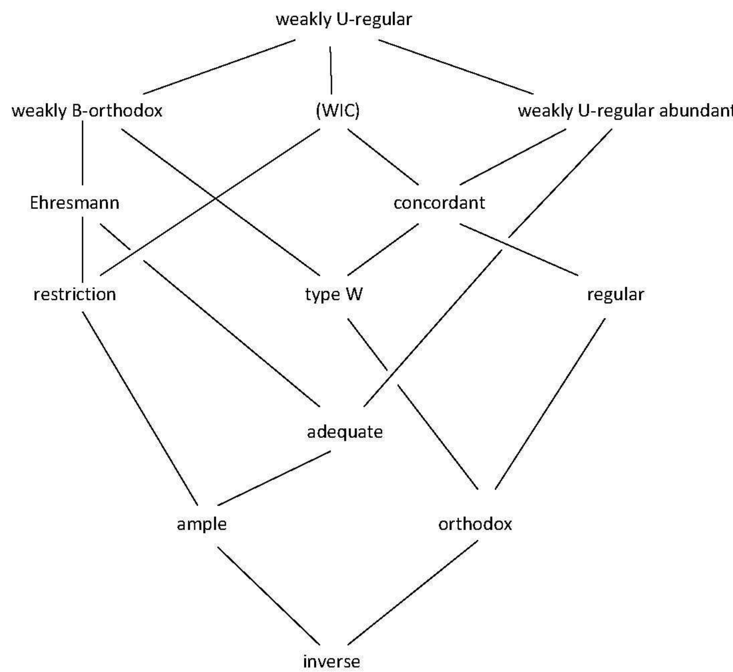

Figure 2.1: Classes of semigroups

In the above picture, (WIC) denotes weakly U-regular semigroups with (WIC).

2.4 An analogue of the least inverse congruence

We denote by γ_B the analogue for a weakly B-orthodox semigroup S of the least inverse congruence on an orthodox semigroup, that is, γ_B is the smallest admissible congruence on S such that S/γ_B is Ehresmann with respect to $\overline{B} = B/\gamma_B$. Since if $\{\rho_i : i \in I\}$ is a non-empty family of admissible congruences on S, then it is easy to see that the intersection $\bigcap\{\rho_i : i \in I\}$ is again an admissible congruence. We use γ_B to denote the admissible congruence generated by

$$\{(e, f) : e \mathcal{D} f \text{ in } B\}.$$

Then S/γ_B is weakly B/γ_B-abundant with (C) by Lemma 2.11. If $\bar{e}, \bar{f} \in B/\gamma_B$, then

$$\bar{e}\ \bar{f} = \overline{ef} = \overline{fe} = \bar{f}\ \bar{e} \qquad \big(\text{as } ef \mathcal{D} fe\big),$$

and so S/γ_B is an Ehresmann semigroup with respect to B/γ_B.

In addition, suppose that ρ is an admissible congruence on S such that S/ρ is an Ehresmann semigroup with respect to B/ρ. Let $e, f \in B$ be such that $e \mathcal{D} f$. Then

$$\begin{aligned} e\rho = (efe)\rho = e\rho f\rho e\rho = e\rho e\rho f\rho = e\rho f\rho \\ = e\rho f\rho f\rho = f\rho e\rho f\rho = (fef)\rho = f\rho, \end{aligned}$$

and so $e \; \rho \; f$ so that

$$\{(e, f) : e \mathcal{D} f\} \subseteq \rho.$$

Thus, $\gamma_B \subseteq \rho$. Hence, γ_B is the smallest admissible congruence on S such that S/γ_B is an Ehresmann semigroup with respect to B/γ_B.

We wish to find a closed form for γ_B. These ideas have been investigated in [6], [7], [14] and [46].

Let S be a weakly B-orthodox semigroup with (WIC) (resp. (IC), (PIC)). The relation δ_B is defined on S as follows:

$$a \, \delta_B \, b \text{ if and only if } a = ebf, b = gah \quad \text{for some } e, f, g, h \in B.$$

Here we remind the reader that given a band B, $E(e)$ denotes the $\mathcal{D}$-class of B containing e.

Lemma 2.20. *[14] (cf. [45], Lemma 3.5) Let S be a weakly B-abundant semigroup. The following conditions are equivalent:*

(i) $a \; \delta_B \; b$*;*

(ii) $a = ebf$ *and* $b = gah$ *for some* $e \in E(b^\dagger)$, $f \in E(b^*)$, $g \in E(a^\dagger)$ *and* $h \in E(a^*)$*;*

(iii) $E(a^\dagger)aE(a^*) = E(b^\dagger)bE(b^*)$.

Moreover, if $a \; \delta_B \; b$, then

$$E(a^\dagger) = E(b^\dagger) \text{ and } E(a^*) = E(b^*).$$

Further:

Lemma 2.21. *[14] Let S be a weakly B-abundant semigroup. For any $e, f \in B$,*

$$e \; \delta_B \; f \text{ if and only if } e \mathcal{D} f.$$

Proof. Let $e, f \in B$. If $e \mathcal{D} f$, then $e = efe$ and $f = fef$, so that $e \delta_B f$.

Conversely, if $e \delta_B f$, then by Lemma 2.20, $E(e^\dagger) = E(f^\dagger)$ and so $e \mathcal{D} f$. $\square$

In view of these equivalent descriptions of δ_B, we now show that if S has (WIC), then it is a congruence on S, arguing as in [6], [14] and [22]. Here $\mathcal{D}$ refers to the band B.

Lemma 2.22. *Let S be a weakly B-orthodox semigroup with (WIC). Then the relation δ_B is the least $B/\mathcal{D}$-ample congruence on S.*

Proof. In view of Lemma 2.20 (iii), it is easy to see that δ_B is an equivalence. Suppose now that $a, b, c \in S$ with $a \delta_B b$, and $e, f, g, h \in B$ are such that $e, g \in E(a^\dagger) = E(b^\dagger)$, $f, h \in E(a^*) = E(b^*)$ satisfying that $a = ebf$ and $b = gah$. Notice that for any $b^\dagger$ we have that $eb^\dagger \mathcal{D} b^\dagger$ in B, and as $\mathcal{D}$ is a semilattice congruence on B, $c^*eb^\dagger \mathcal{D} c^*b^\dagger$ for any c^*. Consequently,

$$\begin{aligned}
ca &= cebf \\
&= cc^*cb^\dagger bf \\
&= c(c^*eb^\dagger)(c^*b^\dagger)(c^*eb^\dagger)bf \\
&= c(c^*eb^\dagger c^*)(b^\dagger c^*eb^\dagger)be \\
&= (xc)(by)f \\
&= x(cb)yf
\end{aligned}$$

for some $x, y \in B$, using (WIC). Similarly, $cb = u(ca)v$ for some $u, v \in B$. It follows that $ca \delta_B cb$ so that δ_B is a left congruence. Dually, δ_B is a right congruence.

Now, we show that δ_B is an admissible congruence. Suppose that $a \in S$ and $e, f, g \in B$ with $g \widetilde{\mathcal{R}}_B a \widetilde{\mathcal{L}}_B e$ in S. If $a\delta_B f\delta_B = a\delta_B$, then $a = gafh$ for some $g \in E((af)^\dagger)$ and $h \in E((af)^*)$. As $fh \in B$, and so $afh = a$, so that $efh = e$ as $a \widetilde{\mathcal{L}}_B e$. From $a \widetilde{\mathcal{L}}_B e$, we obtain that $af \widetilde{\mathcal{L}}_B ef$, and so $h \in E((ef)^*) = E(ef)$. Thus $e \delta_B ef$, that is, $e\delta_B f\delta_B = e\delta_B$. Hence $a\delta_B \widetilde{\mathcal{L}}_{B/\mathcal{D}} e\delta_B$. Dually, $a\delta_B \widetilde{\mathcal{R}}_{B/\mathcal{D}} g\delta_B$.

To see that δ_B is a $B/\mathcal{D}$-ample congruence, we suppose that $a \in S$, $e \in B$ with $a\delta_B \in S/\delta_B$ and $e\delta_B \in B/\delta_B$ are such that $e\delta_B \leq a^\dagger\delta_B$. Then $e\delta_B a\delta_B = a^\dagger\delta_B e\delta_B a\delta_B = (a^\dagger ea)\delta_B = (a^\dagger ea^\dagger a)\delta_B = (af)\delta_B = a\delta_B f\delta_B$ as $a^\dagger ea^\dagger \leq a^\dagger$ and S satisfies (WIC). As $B/\mathcal{D}$ is a semilattice, it follows from Lemma 2.18 that S/δ_B is a $B/\mathcal{D}$-ample semigroup.

Finally, if τ is an arbitrary $B/\mathcal{D}$-ample congruence on S, then $\tau|_B$ is a semilattice congruence on B. Since $\mathcal{D}$ is the least semilattice congruence on B, it follows that $\mathcal{D} \subseteq \tau|_B$. For any $a, b \in S$ and $a \ \delta_B \ b$, we have $b = eaf$ for some $e \in E(a^\dagger)$ and $f \in E(a^*)$. As $e \ \mathcal{D} \ a^\dagger$ and $f \ \mathcal{D} \ a^*$, we obtain that $e \ \tau \ a^\dagger$ and $f \ \tau \ a^*$, and so $a^\dagger a a^* \ \tau \ eaf$, that is, $a \ \tau \ b$. Thus, $\delta_B \subseteq \tau$, and hence δ_B is the least $B/\mathcal{D}$-ample congruence on S. □

2.5 Orders

Our purpose in this section is to describe certain pre-orders and partial orders on a weakly U-abundant semigroup. The results present here are necessary for Chapters 7, 9 and 10.

2.5.1 The weakly U-abundant case

The aim of this section is to present two pairs of relations on a weakly U-abundant semigroup.

Let S be a weakly U-abundant semigroup. We define relations $\leq_r$ and $\leq_l$ by the rule that for any $x, y \in S$,

$$x \leq_r y \ \text{ if and only if } \ x = uy \text{ for some } u \in \langle U \rangle$$

and

$$x \leq_l y \ \text{ if and only if } \ x = yv \text{ for some } v \in \langle U \rangle.$$

The next lemma is immediate.

Lemma 2.23. *On a weakly U-abundant semigroup S, the relations $\leq_r$ and $\leq_l$ given above are pre-orders.*

In view of Lemma 2.8, we have:

Lemma 2.24. *Let S be a weakly U-abundant semigroup. Then for any $x, y \in S$,*
(i) if $x \leq_r y$, then $x^* \leq_{\mathcal{L}} y^*$;
(ii) if $x \leq_l y$, then $x^\dagger \leq_{\mathcal{R}} y^\dagger$.

Further, a weakly U-abundant semigroup S possesses a pair of relations $\leq_r'$ and $\leq_l'$ as follows: for any $x, y \in S$,

$$x \ \leq_r' \ y \text{ if and only if } x = ey \text{ for some } e \in U \text{ and } \ e \ \leq \ y^\dagger$$

and

$$x \ \leq_l' \ y \text{ if and only if } x = yf \text{ for some } f \in U \text{ and } f \ \leq \ y^*.$$

We remark that if $x \ \leq_r' \ y$ then $x^\dagger \leq_{\mathcal{R}} y^\dagger$. For suppose that $x \ \leq_r' \ y$, then there exists $e \in U$ and $y^\dagger$ such that $e \ \leq \ y^\dagger$ and $x = ey$. Thus,

$$x^\dagger \ \widetilde{\mathcal{R}}_U \ x = ey \ \widetilde{\mathcal{R}}_U \ ey^\dagger = e \ \leq \ y^\dagger,$$

and so $x^\dagger \leq_{\mathcal{R}} y^\dagger$. Dually, if $x \ \leq_l' \ y$ then $x^* \leq_{\mathcal{L}} y^*$.

Lemma 2.25. *On a weakly U-abundant semigroup S, the relations $\leq_r'$ and $\leq_l'$ given above are reflexive and anti-symmetric.*

Proof. It is easy to see that $\leq_r'$ is reflexive as for any $x \in S$, $x = x^\dagger x$. To show that $\leq_r'$ is anti-symmetric, we suppose that $x \ \leq_r' \ y \ \leq_r' \ x$. Then $x = ey$ and $y = fx$, where $e, f \in U$ and $e \ \leq \ y^\dagger$, $f \ \leq \ x^\dagger$, and so by Lemma 2.8 or the comment above, we have

$$x^\dagger \leq_{\mathcal{R}} e \ \leq \ y^\dagger \leq_{\mathcal{R}} f \ \leq \ x^\dagger$$

so that $x^\dagger \ \mathcal{R} \ y^\dagger \ \mathcal{R} \ e \ \mathcal{R} \ f$ and consequently, $x = ey = y$. □

It is useful to make a short comment on the pair of relations $\leq_r'$ and $\leq_l'$. On a weakly U-abundant semigroup, $\leq_r'$ is not transitive. Since if $x \leq_r' y$ and $y \leq_r' z$, then there exist $e, f \in U$ such that $e \ \leq \ y^\dagger$, $f \ \leq \ z^\dagger$, and $x = ey$, $y = fz$. Thus, $x = efz$. As $y^\dagger \ \widetilde{\mathcal{R}}_U \ y = fz \ \widetilde{\mathcal{R}}_U \ fz^\dagger = f$ and $e \ \leq \ y^\dagger$, we obtain that $e \leq_{\mathcal{R}} f$, and so by (B21) in Section 1.4, ef is an idempotent. But, we can not guarantee that $ef \in U$, and so $\leq_r'$ is not transitive. Dually, $\leq_l'$ is not transitive. Now, we use ${\leq_r'}^t$ and ${\leq_l'}^t$ to denote the transitive closures of $\leq_r'$ and $\leq_l'$, respectively.

Lemma 2.26. *Let S be a weakly U-abundant semigroup. Then relations ${\leq_r'}^t$ and ${\leq_l'}^t$ are partial orders on S.*

We call $\leq_r$ and $\leq_l$ (resp. $\leq_r'$ and $\leq_l'$, ${\leq_r'}^t$ and ${\leq_l'}^t$) the *natural pre-orders* (resp. *natural relaions*, *natural partial orders*) of a weakly U-abundant semigroup S.

2.5.2 The weakly U-regular case

Here we remind the reader that the natural pre-orders of a weakly U-abundant semigroup will be pre-orders on a weakly U-regular semigroup. Further the natural relations of a weakly U-abundant semigroup will become partial orders on a weakly U-regular semigroup, as we now demonstrate.

Lemma 2.27. *On a weakly U-regular semigroup S, the relations $\leq'_r$ and $\leq'_l$ given in Section 2.5.1 are partial orders.*

Proof. In view of Lemma 2.25, it is sufficient to show that $\leq'_r$ is transitive and dually, $\leq'_l$ is transitive. Suppose that $x \leq'_r y$ and $y \leq'_r z$. Then there exist $e, f \in U$ such that $e \leq y^\dagger$, $f \leq z^\dagger$, and $x = ey$, $y = fz$. So $x = efz$. Referring to the comment succeeding Lemma 2.25, we have that $e \leq_{\mathcal{R}} f$ in U. As U is a regular biordered set, we obtain that $ef \in U$ by (B1), and so $x \leq'_r y$. □

We call $\leq'_r$ and $\leq'_l$ the *natural partial orders* of a weakly U-regular semigroup.

The next lemma presents an equivalent statement for $\leq'_r$ and $\leq'_l$ on a weakly U-regular semigroup.

Lemma 2.28. *Let S be a weakly U-regular semigroup. For any $x, y \in S$, we have*

(*i*) *$x \leq'_r y$ if and only if there exists $e \in U$ such that $x = ey$ and $x^\dagger \leq_{\mathcal{R}} y^\dagger$;*

(*ii*) *$x \leq'_l y$ if and only if there exists $f \in U$ such that $x = yf$ and $x^* \leq_{\mathcal{L}} y^*$.*

Proof. We first show that part (i) holds and dually, part (ii) holds. Suppose that $x, y \in S$. if $x \leq'_r y$, then by the comment succeeding the definition of $\leq'_r$ in Section 2.5.1, we have $x = ey$ for some $e \in U$ and $x^\dagger \leq_{\mathcal{R}} y^\dagger$.

Conversely, if $x = ey$ and $x^\dagger \leq_{\mathcal{R}} y^\dagger$, then by Lemma 2.8, we have $(ey)^\dagger \leq_{\mathcal{R}} e$, and so $x^\dagger \leq_{\mathcal{R}} e$. Note that

$$(x^\dagger e y^\dagger)(x^\dagger e y^\dagger) = x^\dagger e (y^\dagger x^\dagger) e y^\dagger = x^\dagger e x^\dagger e y^\dagger = x^\dagger x^\dagger e y^\dagger = x^\dagger e y^\dagger.$$

Since S is weakly U-regular, we have that $x^\dagger e y^\dagger \in U$. Clearly, $x^\dagger e y^\dagger \leq y^\dagger$. Also, we have

$$x = x^\dagger x = x^\dagger e y = x^\dagger e y^\dagger y.$$

Thus, $x \leq'_r y$. □

2.5.3 The weakly B-orthodox case

Observe that bands are regular biordered sets, and so a weakly B-orthodox semigroup is a special kind of weakly U-regular semigroup. Consequently, the natural pre-orders (resp. natural relations) defined in Section 2.5.1 are the natural pre-orders (resp. natural partial orders) of a weakly B-orthodox semigroup S. To be easily referred, we present these results in a lemma below.

Lemma 2.29. *Let S be a weakly B-orthodox semigroup. Then $\leq_r$ and $\leq_l$ are pre-orders; $\leq_r'$ and $\leq_l'$ are partial orders on S. If in addition S satisfies Condition (WIC), then $\leq_l' = \leq_r'$.*

Proof. Suppose now that S has (WIC) and $x \leq_r' y$. Then $x = ey$ for some $e \in B$ and $e \leq y^\dagger$. By the remark before Lemma 2.25 we have $x^\dagger \leq_{\mathcal{R}} y^\dagger$, and so $x = y^\dagger x = y^\dagger e y^\dagger y = yf$ for some $f \in B$, since $y^\dagger e y^\dagger \leq y^\dagger$. Clearly as $x = ey$ we have $x^* \leq_{\mathcal{L}} y^*$. Hence $x \leq_l' y$. Dually, $\leq_l' \subseteq \leq_r'$, so that the two relations coincide. □

2.5.4 The Ehresmann case

We remark that if E is a semilattice, then a weakly E-orthodox semigroup is an *Ehresmann semigroup* (with distinguished semilattice E).

Lemma 2.30. *Let S be an Ehresmann semigroup with distinguished semilattice E. Then $\leq_r = \leq_r'$ and $\leq_l = \leq_l'$, so that $\leq_r$ and $\leq_l$ are partial orders.*

Proof. In view of Lemma 2.29, $\leq_r'$ and $\leq_l'$ are partial orders. Further, we notice that if $x \leq_r y$, where $x = ey$ for some $e \in E$, then $x = ey^\dagger y$. As $ey^\dagger \leq y^\dagger$, we have that $x \leq_r' y$; dually, $\leq_l = \leq_l'$. □

Let S be an Ehresmann semigroup with distinguished semilattice of idempotents E. In [32], Lawson introduced a partial order on S as follows, for all $x, y \in S$,

$$x \leq_e y \text{ if and only if } x = eyf \qquad \text{for some } e, f \in E.$$

Further, in an Ehresmann semigroup S,

$$\leq_l \circ \leq_r = \leq_e = \leq_r \circ \leq_l .$$

The following lemma describes some properties of partial orders given above.

Lemma 2.31. *[32] Let S be an Ehresmann semigroup with semilattice of distinguished idempotents E.*

(i) if $x \leq_r y$, then $x^* \leq y^*$ and $x^\dagger \leq y^\dagger$;

(ii) if $x \leq_l y$, then $x^* \leq y^*$ and $x^\dagger \leq y^\dagger$;

(iii) if $x \leq_e y$, then $x^* \leq y^*$ and $x^\dagger \leq y^\dagger$;

(iv) if $x \leq_r y$, $u \leq_r v$, $x^* = u^\dagger$ and $y^* = v^\dagger$, then $xu \leq_r yv$;

(v) if $x \leq_l y$, $u \leq_l v$, $x^* = u^\dagger$ and $y^* = v^\dagger$, then $xu \leq_l yv$.

Lemma 2.32. *[32] Let S be an Ehresmann semigroup with semilattice of distinguished idempotents E. Then*

(i) if $e \leq x^\dagger$, then there exists a unique element $y \in S$ such that $y^\dagger = e$ and $y \leq_r x$;

(ii) if $e \leq x^*$, then there exists a unique element $y \in S$ such that $y^* = e$ and $y \leq_l x$.

We remark that in view of Lemma 2.18, Lemma 2.29 and Lemma 2.30, relations $\leq_r'$, $\leq_l'$, $\leq_r$ and $\leq_l$ coincide on a restriction semigroup, and so we use $\leq$ to denote the natural partial order on a restriction semigroup. In particular, on an inverse semigroup S, for all $a, b \in S$, we have

$$a \leq b \text{ if and only if } a = eb \qquad \Big(e \in E(S)\Big).$$

2.6 Examples

This section is concerned with two examples We first show how a weakly B-orthodox semigroup may be naturally obtained from a monoid acting via morphisms on the left and right of a band with identity. This construction is reminiscent of that underlying the free ample monoid [13], and we believe will be of subsequent use.

Let B be a band with 1 and let T be a monoid acting on the left and right of B by $\cdot$ and $\circ$ via morphisms such that

$$(t \cdot g) \circ t = (1 \circ t)g \text{ and } t \cdot (g \circ t) = g(t \cdot 1),$$

for all $g \in B$ and $t \in T$.

We note that as T acts by morphisms, if $e, f \in B$ with $e \leq_{\mathcal{L}} f$, then for any $t \in T$, $t \cdot e = t \cdot ef = (t \cdot e)(t \cdot f) \leq_{\mathcal{L}} t \cdot f$, so that $\cdot$ preserves $\leq_{\mathcal{L}}$. Dually, $\circ$ preserves $\leq_{\mathcal{R}}$.

Let $S = B *_1 T = \{(e, t) : e \leq_{\mathcal{L}} t \cdot 1\} \subseteq B \times T$ with semidirect product multiplication, i.e.

$$(e, t)(f, s) = (e(t \cdot f), ts).$$

Now if $e \leq_{\mathcal{L}} t \cdot 1$ and $f \leq_{\mathcal{L}} s \cdot 1$, then $t \cdot f \leq_{\mathcal{L}} t \cdot (s \cdot 1) = ts \cdot 1$, and so $e(t \cdot f) \leq_{\mathcal{L}} ts \cdot 1$. Thus S is closed, and consequently, it is a semigroup.

We pause here to make a short comment on the above construction. We required the monoid T acting on both sides of the band B, but when we constructed the semigroup S we only used the action of T on the left of B. The action of T on the right of B is helpful to show that each $\widetilde{\mathcal{L}}_{\overline{B}}$-class of S contains an idempotent which appears below. Here $\overline{B} = \{(e, 1) : e \in B\}$.

We now obtain a series of lemmas to verify that S constructed above is a weakly $\overline{B}$-orthodox semigroup.

Lemma 2.33. *The set* $\overline{B} = \{(e, 1) : e \in B\}$ *is isomorphic to* B.

Proof. Let $e, f \in B$. Then $e \leq_{\mathcal{L}} 1_B = 1_T \cdot 1_B$ and $(e, 1)(f, 1) = (e(1 \cdot f), 1) = (ef, 1)$, whence it follows that $\overline{B}$ is a band isomorphic to B. □

Lemma 2.34. *For any* $(e, t) \in S$, $(e, t) \,\widetilde{\mathcal{R}}_{\overline{B}}\, (e, 1)$.

Proof. Let $(e, t) \in S$. Then $(e, 1)(e, t) = (e(1 \cdot e), t) = (e, t)$ and if $(f, 1)(e, t) = (e, t)$, then $(fe, t) = (e, t)$, so $fe = e$ and $(f, 1)(e, 1) = (e, 1)$. Thus, $(e, t) \,\widetilde{\mathcal{R}}_{\overline{B}}\, (e, 1)$. □

Let $(e, t), (f, s) \in S$. By Lemmas 2.33 and 2.34,

$$(e, t) \,\widetilde{\mathcal{R}}_{\overline{B}}\, (f, s) \Leftrightarrow e \,\mathcal{R}\, f.$$

Lemma 2.35. *For any* $(e, t) \in S$, $(e, t) \,\widetilde{\mathcal{L}}_{\overline{B}}\, (e \circ t, 1)$.

Proof. Let $(e, t) \in S$. Then

$$\begin{aligned}(e, t)(e \circ t, 1) &= (e(t \cdot (e \circ t)), t) \\ &= (e(e(t \cdot 1)), t) \\ &= (e, t) \qquad \Big(e \leq_{\mathcal{L}} t \cdot 1\Big).\end{aligned}$$

Further, if $(e,t)(f,1) = (e,t)$, then $e(t \cdot f) = e$. Now

$$\begin{aligned} e \circ t &= (e(t \cdot f)) \circ t = (e \circ t)((t \cdot f) \circ t) \\ &= (e \circ t)(1 \circ t)f = ((e1) \circ t)f \\ &= (e \circ t)f, \end{aligned}$$

so $(e \circ t, 1)(f,1) = (e \circ t, 1)$. Thus $(e,t)\ \widetilde{\mathcal{L}}_{\overline{B}}\ (e \circ t, 1)$. ☐

Again by Lemma 2.33, $(e,t)\ \widetilde{\mathcal{L}}_{\overline{B}}\ (f,s)$ if and only if $e \circ t\ \mathcal{L}\ f \circ s$ in B.

Lemma 2.36. *The semigroup S is weakly $\overline{B}$-orthodox, where $\overline{B} = \{(e,1) : e \in B\}$.*

Proof. In view of Lemma 2.33, 2.34 and 2.35, it is sufficient to show that S has (C). Suppose that $(e,t)\ \widetilde{\mathcal{R}}_{\overline{B}}\ (f,s)$ and $(g,u) \in S$. Then $(g,u)(e,t) = (g(u \cdot e), ut)$ and $(g,u)(f,s) = (g(u \cdot f), us)$. As $e\ \mathcal{R}\ f$ we have $u \cdot e\ \mathcal{R}\ u \cdot f$ and then $g(u \cdot e)\ \mathcal{R}\ g(u \cdot f)$, so that $\widetilde{\mathcal{R}}_{\overline{B}}$ is a left congruence.

Now let $(e,t)\ \widetilde{\mathcal{L}}_{\overline{B}}\ (f,s)$ and $(g,u) \in S$. Then $(e,t)(g,u) = (e(t \cdot g), tu)$ and $(f,s)(g,u) = (f(s \cdot g), su)$. We have

$$\begin{aligned} (e(t \cdot g)) \circ t &= (e \circ t)((t \cdot g) \circ t) \\ &= (e \circ t)(1 \circ t)g \\ &= (e \circ t)g \\ &\mathcal{L}\ (f \circ s)g = (f(s \cdot g) \circ s), \end{aligned}$$

so that $(e(t \cdot g)) \circ tu\ \mathcal{L}\ (f(s \cdot g)) \circ su$. Thus $\widetilde{\mathcal{L}}_{\overline{B}}$ is a right congruence. Hence, S is weakly $\overline{B}$-orthodox. ☐

We now present another example which is that of a weakly U-regular semigroup, that is not necessarily weakly B-orthodox.

Let M be a monoid and I, Λ be non-empty sets. Let $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix with entries being unit elements in M.

Let $S = I \times M \times \Lambda$ and define a composition on S by

$$(i, a, \lambda)(j, b, \mu) = (i, ap_{\lambda j}b, \mu).$$

Then, we have:

Lemma 2.37. *The set S forms a weakly U-abundant semigroup under the operation defined above, where $U = \{(i, p_{\lambda i}^{-1}, \lambda) : i \in I, \lambda \in \Lambda\}$.*

Proof. Clearly, S forms a semigroup.

For any $(i, p_{\lambda i}^{-1}, \lambda) \in U$, we have

$$(i, p_{\lambda i}^{-1}, \lambda)(i, p_{\lambda i}^{-1}, \lambda) = (i, p_{\lambda i}^{-1} p_{\lambda i} p_{\lambda i}^{-1}, \lambda) = (i, p_{\lambda i}^{-1}, \lambda),$$

and so $U \subseteq E(S)$.

To show that S is weakly U-abundant, we first show that for any $(i, a, \lambda) \in S$ and $(j, p_{\mu j}^{-1}, \mu) \in U$,

$$(i, a, \lambda) \ \widetilde{\mathcal{R}}_U \ (j, p_{\mu j}^{-1}, \mu) \text{ if and only if } i = j,$$

and dually,

$$(i, a, \lambda) \ \widetilde{\mathcal{L}}_U \ (j, p_{\mu j}^{-1}, \mu) \text{ if and only if } \lambda = \mu.$$

Suppose that $(i, a, \lambda) \ \widetilde{\mathcal{R}}_U \ (j, p_{\mu j}^{-1}, \mu)$, then we have

$$(j, p_{\mu j}^{-1}, \mu)(i, a, \lambda) = (i, a, \lambda),$$

that is,

$$(j, p_{\mu j}^{-1} p_{\mu i} a, \lambda) = (i, a, \lambda),$$

and so $i = j$.

Conversely, if $i = j$, then

$$(i, p_{\mu i}^{-1}, \mu)(i, a, \lambda) = (i, p_{\mu i}^{-1} p_{\mu i} a, \lambda) = (i, a, \lambda).$$

Suppose that $(k, p_{\gamma k}^{-1}, \gamma) \in U$ is such that $(k, p_{\gamma k}^{-1}, \gamma)(i, a, \lambda) = (i, a, \lambda)$. Then we obtain that $(k, p_{\gamma k}^{-1} p_{\gamma i} a, \lambda) = (i, a, \lambda)$, and so we must have $k = i$ so that

$$\begin{aligned}(k, p_{\gamma k}^{-1}, \gamma)(i, p_{\mu i}^{-1}, \mu) &= (k, p_{\gamma k}^{-1} p_{\gamma i} p_{\mu i}^{-1}, \mu) \\ &= (i, p_{\gamma i}^{-1} p_{\gamma i} p_{\mu i}^{-1}, \mu) \qquad \big(k = i\big) \\ &= (i, p_{\mu i}^{-1}, \mu).\end{aligned}$$

Thus, $(i, a, \lambda) \ \widetilde{\mathcal{R}}_U \ (i, p_{\mu i}^{-1}, \mu)$. Hence, S is a weakly U-abundant semigroup. $\square$

In view of the proof of Lemma 2.37, the next lemma is immediate.

Lemma 2.38. *For any* (i, a, λ), $(j, b, \mu) \in S$, *we have*

(i) $(i, a, \lambda)\ \widetilde{\mathcal{R}}_U\ (j, b, \mu)$ *if and only if* $i = j$;

(ii) $(i, a, \lambda)\ \widetilde{\mathcal{L}}_U\ (j, b, \mu)$ *if and only if* $\lambda = \mu$.

Lemma 2.39. *The semigroup* S *satisfies (C).*

Proof. Suppose that $(i, a, \lambda), (j, b, \mu), (k, c, \gamma) \in S$ are such that $(i, a, \lambda)\ \widetilde{\mathcal{R}}_U\ (j, b, \mu)$. Then $i = j$. Observe that

$$\begin{aligned}(k, c, \gamma)(i, a, \lambda) &= (k, cp_{\gamma i}a, \lambda) \\ &\widetilde{\mathcal{R}}_U\ (k, cp_{\gamma i}b, \mu) \qquad \big(\text{Lemma 2.38}\big) \\ &= (k, c, \gamma)(i, b, \mu) \\ &= (k, c, \gamma)(j, b, \mu) \qquad \big(i = j\big)\end{aligned}$$

so that $\widetilde{\mathcal{R}}_U$ is a left congruence. Dually, we show that $\widetilde{\mathcal{L}}_U$ is a right congruence. □

Lemma 2.40. *If* $e, f \in U$, *then* $e\ \mathcal{R}\ ef\ \mathcal{L}\ f$ *in* $\langle U \rangle$.

Proof. Let $e = (i, p_{\lambda i}^{-1}, \lambda)$ and $f = (j, p_{\mu j}^{-1}, \mu) \in U$. Then

$$ef = (i, p_{\lambda i}^{-1}, \lambda)(j, p_{\mu j}^{-1}, \mu) = (i, p_{\lambda i}^{-1} p_{\lambda j} p_{\mu j}^{-1}, \mu).$$

Take $a = (j, p_{\lambda j}^{-1}, \lambda)(i, p_{\lambda i}^{-1}, \lambda) \in \langle U \rangle$. Then we have

$$\begin{aligned}efa &= (i, p_{\lambda i}^{-1} p_{\lambda j} p_{\mu j}^{-1}, \mu)(j, p_{\lambda j}^{-1}, \lambda)(i, p_{\lambda i}^{-1}, \lambda) \\ &= (i, p_{\lambda i}^{-1} p_{\lambda j} p_{\mu j}^{-1} p_{\mu j} p_{\lambda j}^{-1} p_{\lambda i} p_{\lambda i}^{-1}, \lambda) \\ &= (i, p_{\lambda i}^{-1}, \lambda) \\ &= e\end{aligned}$$

and $eef = ef$ so that $ef\ \mathcal{R}\ e$ in $\langle U \rangle$. Dually, we have $ef\ \mathcal{L}\ f$ in $\langle U \rangle$. □

Lemma 2.41. *The semigroup* $\langle U \rangle$ *generated by* U *is regular and* $E(\langle U \rangle) = U$.

Proof. Let $(i, a, \lambda) \in \langle U \rangle$. Then

$$(i, a, \lambda) = e_1 \cdots e_n \qquad \big(e_1, \cdots, e_n \in U\big).$$

We now show that $(i, a, \lambda)\ \mathcal{R}\ e_1$ in $\langle U \rangle$ and dually, we have $a\ \mathcal{L}\ e_n$ in $\langle U \rangle$. By Lemma 2.40, if $n = 2$, then certainly the statement holds. We assume that $e_2 \cdots e_n\ \mathcal{R}\ e_2$ in $\langle U \rangle$. Then

$$\begin{aligned} (i, a, \lambda) = e_1(e_2 \cdots e_n)\ &\mathcal{R}\ e_1 e_2 \\ &\mathcal{R}\ e_1 \qquad \big(\text{Lemma 2.40}\big). \end{aligned}$$

Hence, $\langle U \rangle$ is a regular semigroup.

With (i, a, λ) as in the statement of this lemma, we have $(i, a, \lambda)\ \mathcal{R}\ (i, p_{\mu i}^{-1}, \mu)$ for any $\mu \in \Lambda$, and if $(k, p_{\lambda k}^{-1}, \lambda) \in U$, then we have $(i, a, \lambda)\ \mathcal{L}\ (k, p_{\lambda k}^{-1}, \lambda)$.

Now, we show that $E(\langle U \rangle) = U$. Clearly $U \subseteq E(\langle U \rangle)$. To show that $E(\langle U \rangle) \subseteq U$, we suppose that $(i, a, \lambda) \in E(\langle U \rangle)$. Then $(i, a, \lambda)\ \mathcal{H}\ (i, p_{\lambda i}^{-1}, \lambda)$. Since both (i, a, λ) and $(i, p_{\lambda i}^{-1}, \lambda)$ are idempotent and each $\mathcal{H}$-class contains at most one idempotent, we must have that $(i, a, \lambda) = (i, p_{\lambda i}^{-1}, \lambda)$ so that $E(\langle U \rangle) \subseteq U$. □

To sum up, we have :

Theorem 2.42. *The semigroup S is a weakly U-regular semigroup, where $U = \{(i, p_{\lambda i}^{-1}, \lambda) : i \in I, \lambda \in \Lambda\}$.*

Chapter 3

Weakly U-superabundant semigroups with (C)

A weakly U-superabundant semigroup is a weakly U-abundant semigroup in which every $\widetilde{\mathcal{H}}_U$-class contains a distinguished idempotent of U. Such semigroups are analogous to completely regular semigroups. The purpose of this chapter is to build a complete construction modulo the semilattice decomposition for weakly U-superabundant semigroups with (C).

We make the convention that B will always denote a band. Green's relation $\mathcal{D}$ will always refer to B, unless stated otherwise. To avoid ambiguity, if $\mathcal{K}$ is a relation on a semigroup S, then we will use $\mathcal{K}(S)$ to denote the relation on S in some places.

3.1 Weakly U-superabundant semigroups with (C)

In this section, we are concerned with some properties of weakly U-superabundant semigroups with (C), which broadly determine their structure.

Lemma 3.1. *[33] If S is a weakly U-superabundant semigroup, then*

$$\widetilde{\mathcal{D}}_U = \widetilde{\mathcal{L}}_U \circ \widetilde{\mathcal{R}}_U = \widetilde{\mathcal{R}}_U \circ \widetilde{\mathcal{L}}_U.$$

Further:

Lemma 3.2. *Let S be a weakly U-superabundant semigroup. If distinguished idempotents $e, f \in U$ are $\widetilde{\mathcal{D}}_U$-related, then there exists $h \in U$ such that $e \,\mathcal{L}\, h \,\mathcal{R}\, f$.*

Proof. Let $e, f \in U$ be such that $e \;\widetilde{\mathcal{D}}_U\; f$. By Lemma 3.1, there exists $x \in S$ such that $e \,\widetilde{\mathcal{L}}_U\, x \,\widetilde{\mathcal{R}}_U\, f$. As S is weakly U-superabundant, there exists $h \in U$ such that $h \,\widetilde{\mathcal{H}}_U\, x$, and so $e \,\widetilde{\mathcal{L}}_U\, h \,\widetilde{\mathcal{R}}_U\, f$. Thus, $e \,\mathcal{L}\, h \,\mathcal{R}\, f$. □

As an analogue of the fact that $\mathcal{J} = \mathcal{D}$ on a completely regular semigroup, we have:

Lemma 3.3. *[44] If S is a weakly U-superabundant semigroup with (C), then $\widetilde{\mathcal{J}}_U = \widetilde{\mathcal{D}}_U$, and $\widetilde{\mathcal{J}}_U$ is a semilattice congruence on S.*

From Lemma 3.3, if S is a weakly U-superabundant semigroup with (C), then each $\widetilde{\mathcal{D}}_U$-class forms a semigroup. We can say more.

Lemma 3.4. *If a, b are $\widetilde{\mathcal{D}}_U$-equivalent elements in a weakly U-superabundant semigroup S with (C), then $a \,\widetilde{\mathcal{R}}_U\, ab \,\widetilde{\mathcal{L}}_U\, b$.*

Proof. Suppose that $a \;\widetilde{\mathcal{D}}_U\; b$ in S and $e, f \in U$ with $e \;\widetilde{\mathcal{H}}_U\; a$, $f \;\widetilde{\mathcal{H}}_U\; b$. Then $e \;\widetilde{\mathcal{D}}_U\; f$. By Lemma 3.2, there exists $k \in U$ such that $e \;\mathcal{L}\; k \;\mathcal{R}\; f$, and so

$$ab \;\widetilde{\mathcal{R}}_U\; af \;\widetilde{\mathcal{R}}_U\; ak = a \;\text{ and }\; ab \;\widetilde{\mathcal{L}}_U\; eb \;\widetilde{\mathcal{L}}_U\; kb = b.$$

Hence, $a \,\widetilde{\mathcal{R}}_U\, ab \,\widetilde{\mathcal{L}}_U b$. □

It is useful to mention the next lemma.

Lemma 3.5. *[33] If a weakly U-abundant semigroup S satisfies (C) and $e \in U$, then $\widetilde{H}_e$ is a monoid with respect to e.*

Lemma 3.6. *Let S be a weakly U-superabundant semigroup with (C) and let $a \in S$ and $e, f, h \in U$ be such that $h \,\widetilde{\mathcal{H}}_U\, a$ and $e \,\widetilde{\mathcal{L}}_U\, a \,\widetilde{\mathcal{R}}_U\, f$. Then the right translations $\rho_f|_{\widetilde{L}_a}$, $\rho_h|_{\widetilde{L}_f}$ are mutually inverse $\widetilde{\mathcal{R}}_U$-class preserving bijections from $\widetilde{L}_a$ onto $\widetilde{L}_f$ and $\widetilde{L}_f$ onto $\widetilde{L}_a$, respectively.*

Proof. It is easy to see that $\rho_f|_{\widetilde{L}_a}$ is a map from $\widetilde{L}_a$ to $\widetilde{L}_f$ since for any $x \in \widetilde{L}_a$, $x\rho_f = xf \,\widetilde{\mathcal{L}}_U\, hf = f$. Similarly, we can show that $\rho_h|_{\widetilde{L}_f}$ is a map from $\widetilde{L}_f$ to $\widetilde{L}_a$. And we deduce that for any $x \in \widetilde{L}_a$ and $y \in \widetilde{L}_f$,

$$x\rho_f\rho_h = xfh = x(fh) = xh = x \;\text{ and }\; y\rho_h\rho_f = yhf = y(hf) = yf = y.$$

Thus the right translations $\rho_f|_{\widetilde{L}_a}$, $\rho_h|_{\widetilde{L}_f}$ are mutually inverse bijections from $\widetilde{L}_a$ onto $\widetilde{L}_f$ and $\widetilde{L}_f$ onto $\widetilde{L}_a$, respectively. Since $f \,\widetilde{\mathcal{R}}_U\, h$ and $\widetilde{\mathcal{R}}_U$ is a left congruence it follows that for any $x \in \widetilde{L}_a$, $xf \,\widetilde{\mathcal{R}}_U\, xh = x$, that is, the right translation $\rho_f|_{\widetilde{L}_a}$ preserves the $\widetilde{\mathcal{R}}_U$-class. By a similar argument, we have the right translation $\rho_h|_{\widetilde{L}_f}$ preserves the $\widetilde{\mathcal{R}}_U$-class. □

We have a left-right dual featuring:

Lemma 3.7. *Let S be a weakly U-superabundant semigroup with (C) and let $a \in S$ and $e, f, h \in U$ be such that $h \,\widetilde{\mathcal{H}}_U\, a$ and $e \,\widetilde{\mathcal{L}}_U\, a \,\widetilde{\mathcal{R}}_U\, f$. Then the left translations $\lambda_e|_{\widetilde{R}_a}$, $\lambda_h|_{\widetilde{R}_e}$ are mutually inverse $\widetilde{\mathcal{L}}_U$-class preserving bijections from $\widetilde{R}_a$ onto $\widetilde{R}_e$ and $\widetilde{R}_e$ onto $\widetilde{R}_a$, respectively.*

Lemma 3.8. *If a, b are $\widetilde{\mathcal{D}}_U$-equivalent elements in a weakly U-superabundant semigroup S with (C), then $\widetilde{H}_a$ is isomorphic to $\widetilde{H}_b$.*

Proof. Suppose that $a \,\widetilde{\mathcal{D}}_U\, b$ in S and $e, f \in U$ with $e \,\widetilde{\mathcal{H}}_U\, a$, $f \,\widetilde{\mathcal{H}}_U\, b$. By Lemma 3.1, there exists $c \in S$ such that $a \,\widetilde{\mathcal{R}}_U\, c \,\widetilde{\mathcal{L}}_U\, b$. Since S is a weakly U-superabundant semigroup it follows that there is a distinguished idempotent $h \in U$ such that $h \,\widetilde{\mathcal{H}}_U\, c$. Due to Lemma 3.6, $\rho_h|_{\widetilde{H}_a}$ and $\rho_e|_{\widetilde{H}_c}$ are mutually inverse bijections from $\widetilde{H}_a$ onto $\widetilde{H}_c$ and from $\widetilde{H}_c$ onto $\widetilde{H}_a$, respectively. By Lemma 3.7, $\lambda_f|_{\widetilde{H}_c}$ and $\lambda_h|_{\widetilde{H}_b}$ are mutually inverse bijections from $\widetilde{H}_c$ onto $\widetilde{H}_b$ and from $\widetilde{H}_b$ onto $\widetilde{H}_c$, respectively. So we have that $\rho_h|_{\widetilde{H}_a}\lambda_f|_{\widetilde{H}_c}$ and $\lambda_h|_{\widetilde{H}_b}\rho_e|_{\widetilde{H}_c}$ are mutually inverse bijections from $\widetilde{H}_a$ onto $\widetilde{H}_b$ and $\widetilde{H}_b$ onto $\widetilde{H}_a$, respectively.

We still need to show that $\rho_h|_{\widetilde{H}_a}\lambda_f|_{\widetilde{H}_c}$ and $\lambda_h|_{\widetilde{H}_b}\rho_e|_{\widetilde{H}_c}$ are morphisms. To show that $\rho_h|_{\widetilde{H}_a}\lambda_f|_{\widetilde{H}_c}$ is a morphism, it is sufficient to prove that both $\lambda_f|_{\widetilde{H}_c}$ and $\rho_h|_{\widetilde{H}_a}$ are morphisms. If $x, y \in \widetilde{H}_a$, then $x \,\widetilde{\mathcal{R}}_U\, h \,\widetilde{\mathcal{R}}_U\, y$, and so $(xy)\rho_h = xyh = x(hy)h = (xh)(yh) = (x\rho_h)(y\rho_h)$. Thus $\rho_h|_{\widetilde{H}_a}$ is a morphism. Dually, we can show that $\lambda_f|_{\widetilde{H}_c}$ is a morphism. Hence $\rho_h|_{\widetilde{H}_a}\lambda_f|_{\widetilde{H}_c}$ is a morphism as required. Similarly, we can show that the composition $\lambda_h|_{\widetilde{H}_b}\rho_e|_{\widetilde{H}_c}$ is a morphism. So $\widetilde{H}_a$ is isomorphic to $\widetilde{H}_b$ as required. □

In view of Lemma 3.1, we can use an egg-box picture to depict each $\widetilde{\mathcal{D}}_U$-class of a weakly U-superabundant semigroup S. Let D denote a typical $\widetilde{\mathcal{D}}_U$-class of S. We denote the set of $\widetilde{\mathcal{R}}_U$-classes of S in D by I and the set of $\widetilde{\mathcal{L}}_U$-classes of S in D by Λ. As a matter of notation we shall treat I and Λ as index sets and write the $\widetilde{\mathcal{R}}_U$-classes as $\widetilde{R}_i$ $(i \in I)$ and the $\widetilde{\mathcal{L}}_U$-classes as $\widetilde{L}_\lambda$ $(\lambda \in \Lambda)$. The $\widetilde{\mathcal{H}}_U$-class $\widetilde{R}_i \cap \widetilde{L}_\lambda$ is denoted by $\widetilde{H}_{i\lambda}$.

Lemma 3.9. *If S is a weakly U-superabundant semigroup with (C), then each $\widetilde{\mathcal{D}}_U$-class of S is a rectangular band of its $\widetilde{\mathcal{H}}_U$-classes, which are isomorphic monoids.*

Proof. Let D denote a typical $\widetilde{\mathcal{D}}_U$-class of S. In view of Lemma 3.5, each $\widetilde{\mathcal{H}}_U$-class is a monoid, and so $a \;\widetilde{\mathcal{H}}_U\; a^2$, for all $a \in S$. Let $i, j \in I$ and $\mu, \lambda \in \Lambda$. We assume that $a \in \widetilde{H}_{i\lambda}$ and $b \in \widetilde{H}_{j\mu}$. By Lemma 3.4, we have $a \;\widetilde{\mathcal{R}}_U\; ab \;\widetilde{\mathcal{L}}_U\; b$, that is, $ab \in \widetilde{H}_{i\mu}$, or equivalently , $\widetilde{H}_{i\lambda}\widetilde{H}_{j\mu} \subseteq \widetilde{H}_{i\mu}$. Thus each $\widetilde{\mathcal{D}}_U$-class is a rectangular band of monoids, which are isomorphic by Lemma 3.8. □

Next, we present an equivalent statement for a weakly B-superabundant semigroup, where B is a band.

Lemma 3.10. *Let S be a weakly B-abundant semigroup. For any $e, f \in B$,*

$$e\,\widetilde{\mathcal{D}}_B\,f \Leftrightarrow e\,\mathcal{D}\,f$$

if and only if S is a weakly B-superabundant semigroup.

Proof. In view of Lemma 3.2, it is sufficient to show the necessity. Suppose that $x \in S$. Certainly, we have that $x^\dagger \;\widetilde{\mathcal{R}}_B\; x \;\widetilde{\mathcal{L}}_B\; x^*$ for some $x^\dagger, x^* \in B$. It follows that $x^\dagger \;\widetilde{\mathcal{D}}_B\; x^*$. By the hypothesis, we get $x^\dagger \;\mathcal{D}\; x^*$. So $x^\dagger x^* \;\mathcal{L}\; x^* \;\widetilde{\mathcal{L}}_B\; x$ and $x^\dagger x^* \;\mathcal{R}\; x^\dagger \;\widetilde{\mathcal{R}}_B\, x$. Thus $x^\dagger x^* \;\widetilde{\mathcal{H}}_B\; x$. Hence S is a weakly B-superabundant semigroup. □

As an immediate consequence of Lemma 3.10, if S is a weakly B-superabundant and B is a band, then $\mathcal{D}(B) = \widetilde{\mathcal{D}}_B(B)$.

Lemma 3.11. *A weakly B-orthodox semigroup S is B-superabundant if and only if S/γ is weakly $B/\mathcal{D}$-superabundant, where γ is any admissible congruence on S such that $\gamma \cap (B \times B) = \mathcal{D}$.*

Proof. It is easy to see that the necessity holds. It remains to show the converse is true. Suppose that S is a weakly B-orthodox semigroup and S/γ is weakly $B/\mathcal{D}$-superabundant. Then for any $a \in S$, there exists an idempotent $e \in B$ such that $a\gamma \;\widetilde{\mathcal{H}}_{B/\mathcal{D}}\; e\gamma$. Of course, we have $a^\dagger\gamma \;\widetilde{\mathcal{R}}_{B/\mathcal{D}}\; a\gamma \;\widetilde{\mathcal{L}}_{B/\mathcal{D}}\; a^*\gamma$ for any $a^\dagger \in \widetilde{R}_a \cap B$, $a^* \in \widetilde{L}_a \cap B$. It follows that $a^\dagger\gamma \;\widetilde{\mathcal{R}}_{B/\mathcal{D}}\; e\gamma \;\widetilde{\mathcal{L}}_{B/\mathcal{D}}\; a^*\gamma$, that is, $a^\dagger\gamma \;\mathcal{R}_{B/\mathcal{D}}\; e\gamma \;\mathcal{L}_{B/\mathcal{D}}\; a^*\gamma$, and so $a^\dagger\gamma = e\gamma = a^*\gamma$ since $B/\gamma = B/\mathcal{D}$ is a semilattice. Therefore, $a^\dagger \;\mathcal{D}\; e \;\mathcal{D}\; a^*$, which implies that $a^\dagger \ \mathcal{R} \ a^\dagger a^* \ \mathcal{L} \ a^*$, and so $a^\dagger a^* \ \widetilde{\mathcal{H}}_B \ a$. Hence, S is weakly B-superabundant. $\square$

In Chapter 2, given a weakly B-orthodox semigroup S with (WIC) (resp. (IC)), a relation δ_B defined by the rule that for any $a, b \in S$, $a \ \delta_B \ b$ if and only if $a = ebf$, $b = gah$ for some $e, f, g, h \in B$, is an admissible congruence on S with the property that $\delta_B \cap (B \times B) = \mathcal{D}$. It follows from Lemma 3.11 that a weakly B-orthodox semigroup S with (WIC) (resp. (IC)) is weakly B-superabundant if and only if S/δ_B is weakly $B/\mathcal{D}$-superabundant.

3.2 Completely $\widetilde{\mathcal{J}}_U$-simple semigroups

A semigroup S is called *$\widetilde{\mathcal{J}}_U$-simple* if $\widetilde{\mathcal{J}}_U$ is the universal relation on S. A weakly U-abundant semigroup S is called *completely $\widetilde{\mathcal{J}}_U$-simple* if S is a weakly U-superabundant semigroup with (C) and is $\widetilde{\mathcal{J}}_U$-simple.

In [44], Ren, Shum and Quo built the analogue for weakly U-superabundant semigroups with (C) of the structure theorem for superabundant semigroups as follows.

Theorem 3.12. *[44] A semigroup S is a weakly U-superabundant semigroup with (C) if and only if S is a semilattice Y of completely $\widetilde{\mathcal{J}}_U$-simple semigroups $S_\alpha (\alpha \in Y)$ such that for all $\alpha, \beta \in Y$, the following statements hold:*

(i) for each $a \in S_\alpha, \widetilde{L}_a(S) = \widetilde{L}_a(S_\alpha)$ and $\widetilde{R}_a(S) = \widetilde{R}_a(S_\alpha)$;

(ii) for all $a, b \in S_\alpha$ and $x \in S_\beta$, $(a, b) \in \widetilde{\mathcal{L}}_U(S_\alpha)$ implies $(ax, bx) \in \widetilde{\mathcal{L}}_U(S_{\alpha\beta})$ and $(a, b) \in \widetilde{\mathcal{R}}_U(S_\alpha)$ implies $(xa, xb) \in \widetilde{\mathcal{R}}_U(S_{\alpha\beta})$.

In the following, we construct a completely $\widetilde{\mathcal{J}}_U$-simple semigroup from a set of monoids. Together with Theorem 3.12, we succeed in obtaining a complete construction for weakly U-superabundant semigroups with (C).

Let I, Λ be non-empty sets. For each $(i, \lambda) \in I \times \Lambda$, let $M_{i\lambda}$ be a monoid with identity $e_{i\lambda}$. We denote a rectangular band of monoids $M_{i\lambda}$ $(i \in I, \lambda \in \Lambda)$ by T and denote by U the set of $\{e_{i\lambda} : i \in I, \lambda \in \Lambda\}$. In order to consider the relations $\widetilde{\mathcal{R}}_U$ and $\widetilde{\mathcal{L}}_U$ on T, we make a convention that for $i, j \in I$ and $\lambda, \mu \in \Lambda$,

(R) if $i = j$ then $e_{i\lambda} e_{j\mu} = e_{j\mu}$ and $e_{j\mu} e_{i\lambda} = e_{i\lambda}$;

(L) if $\lambda = \mu$ then $e_{i\lambda} e_{j\mu} = e_{i\lambda}$ and $e_{j\mu} e_{i\lambda} = e_{j\mu}$.

The next lemma follows immediately.

Lemma 3.13. *If the rectangular band T of monoids $M_{i\lambda}$ $(i \in I, \lambda \in \Lambda)$ satisfies Conditions (R) and (L), then for any $a, b \in T$ with $a \in M_{i\lambda}$ and $b \in M_{j\mu}$,*

$$a \ \widetilde{\mathcal{R}}_U \ b \Leftrightarrow \ i = j,$$

$$a \ \widetilde{\mathcal{L}}_U \ b \Leftrightarrow \ \lambda = \mu,$$

and consequently,

$$a \ \widetilde{\mathcal{H}}_U \ b \Leftrightarrow \ i = j \text{ and } \lambda = \mu.$$

Proof. We prove $\widetilde{\mathcal{R}}_U$ case. Dually, the $\widetilde{\mathcal{L}}_U$ case holds, and the $\widetilde{\mathcal{H}}_U$ case follows from the result for $\widetilde{\mathcal{R}}_U$ and $\widetilde{\mathcal{L}}_U$. Let $a \in M_{i\lambda}$ and $b \in M_{j\mu}$ be such that $a \ \widetilde{\mathcal{R}}_U \ b$. Suppose that $f \in U$ with $fa = a$. Then $f = e_{i\nu} \in M_{i\nu}$ for some $\nu \in \Lambda$. Since $a \ \widetilde{\mathcal{R}}_U \ b$, it follows that $fb = b$, which leads to $i = j$.

Conversely, suppose that $i = j$. Then by Condition (R), $e_{i\lambda}e_{i\mu} = e_{i\mu}$ and $e_{i\mu}e_{i\lambda} = e_{i\lambda}$, that is, $e_{i\lambda} \ \mathcal{R} \ e_{i\mu}$. Next, we shall claim that $a \ \widetilde{\mathcal{R}}_U \ e_{i\lambda}$ and $b \ \widetilde{\mathcal{R}}_U \ e_{i\mu}$. Certainly, we have $e_{i\lambda}a = a$. Suppose that $e_{k\varepsilon} \in U$ with $e_{k\varepsilon}a = a$. Then $k = i$ and by Condition (R), we have $e_{k\varepsilon}e_{i\lambda} = e_{i\lambda}$. So $a \ \widetilde{\mathcal{R}}_U \ e_{i\lambda}$. Similarly, we could deduce that $b \ \widetilde{\mathcal{R}}_U \ e_{i\mu}$. Together with $e_{i\lambda} \ \mathcal{R} \ e_{i\mu}$, we have $a \ \widetilde{\mathcal{R}}_U \ b$. □

Furthermore, we can get the next result.

Lemma 3.14. *If the rectangular band T of monoids $M_{i\lambda}$ $(i \in I, \lambda \in \Lambda)$ satisfies Conditions (R) and (L), then T is a completely $\widetilde{\mathcal{J}}_U$-simple semigroup.*

Proof. In view of Lemma 3.13, it is easy to see that T is a weakly U-superabundant semigroup. We now claim that T satisfies the Congruence Condition. Suppose that $a, b, c \in T$ with $a \ \widetilde{\mathcal{R}}_U \ b$ and $c \in M_{k\nu}$. By Lemma 3.13, $a \in M_{i\lambda}$ and $b \in M_{i\mu}$ for some $i \in I$, $\lambda, \mu \in \Lambda$. Clearly, $ca \in M_{k\lambda}$ and $cb \in M_{k\mu}$. Again by Lemma 3.13, $ca \ \widetilde{\mathcal{R}}_U \ cb$. Thus, $\widetilde{\mathcal{R}}_U$ is a left congruence. Similarly, we have that $\widetilde{\mathcal{L}}_U$ is a right congruence.

We still need to show that $\widetilde{\mathcal{D}}_U$ is a universal relation on T. Let $a, b \in T$ with $a \in M_{i\lambda}$ and $b \in M_{j\mu}$. Since there exists an element $x \in M_{i\mu}$, it immediately follows from Lemma 3.13 that $a \ \widetilde{\mathcal{R}} \ x \ \widetilde{\mathcal{L}}_U \ b$, that is, $a \ \widetilde{\mathcal{D}}_U \ b$. Hence T is a completely $\widetilde{\mathcal{J}}_U$-simple semigroup. □

In a summary, we have the following structure theorem for completely $\widetilde{\mathcal{J}}_U$-simple semigroups.

Theorem 3.15. *A semigroup S is completely $\widetilde{\mathcal{J}}_U$-simple if and only if it is a rectangular band of monoids $M_{i\lambda}$ $(i \in I,\ \lambda \in \Lambda)$ and satisfies Conditions (R) and (L), where the monoids $M_{i\lambda}$ must be isomorphic.*

Proof. In view of Lemma 3.14, it is sufficient to show that a completely $\widetilde{\mathcal{J}}_U$-simple semigroup is a rectangular band of monoids and satisfies Conditions (R) and (L). Clearly, by Lemma 3.9, a completely $\widetilde{\mathcal{J}}_U$-simple semigroup S is a rectangular band of monoids $M_{i\lambda}$ $(i \in I,\ \lambda \in \Lambda)$ which are isomorphic. Since the set of $\widetilde{\mathcal{R}}_U$-classes of S is denoted by I and the set of $\widetilde{\mathcal{L}}_U$-classes of S is denoted by Λ it follows that Conditions (R) and (L) hold. □

Finally, we consider a special case. If the set U of identities of each $M_{i\lambda}$ forms a band, then the next lemma is immediate.

Lemma 3.16. *Let T be the rectangular band of monoids $M_{i\lambda}$ with identities $e_{i\lambda}$ $(i \in I,\ \lambda \in \Lambda)$. If $U = \{e_{i\lambda} : i \in I, \lambda \in \Lambda\}$ forms a band, then T satisfies Conditions (R) and (L).*

So, we have:

Corollary 3.17. *A semigroup S is completely $\widetilde{\mathcal{J}}_B$-simple if and only if it is a rectangular band of monoids $M_{i\lambda}$ with identities $e_{i\lambda}$ $(i \in I,\ \lambda \in \Lambda)$, where $B = \{e_{i\lambda} : i \in I, \lambda \in \Lambda\}$ forms a band and the monoids $M_{i\lambda}$ must be isomorphic.*

Chapter 4

Representations for generalised orthogroups

In this chapter we begin the study of fundamental semigroups and their analogues in the class of generalised regular semigroups. Precisely, we mainly describe orthogroups in the Hall semigroup W_B and weakly B-superabundant subsemigroups with (C) of V_B (resp. U_B, S_B), which is analogous to W_B.

4.1 Fundamental inverse semigroups

The results in this section are basic but important in the study of inverse semigroups. To make our discussion in the following sections easy to understand, we list them here. The details are referred to [26].

We recall that an inverse semigroup S is *fundamental* if the maximum idempotent separating congruence μ is the identity congruence on S. Such inverse semigroups do exist, since S/μ is fundamental inverse for any inverse semigroup S whatsoever. Specially, every semilattice and every symmetric inverse semigroup $\mathcal{I}(X)$ is fundamental.

Observe that every element a in an inverse semigroup S determines an isomorphism α_a from the principal ideal $E_{aa^{-1}}$ of E onto the principal ideal $E_{a^{-1}a}$. The isomorphism α_a is defined by

$$e\alpha_a = a^{-1}ea \qquad (e \in E_{aa^{-1}}).$$

Built on the above observation, Munn [37] constructed a fundamental inverse semigroup from any semilattice E, as follows.

Let E be a semilattice and $\mathcal{U}$ be the equivalence relation on E given by

$$\mathcal{U} = \{(e, f) \in E \times E : E_e \simeq E_f\}.$$

If $(e, f) \in \mathcal{U}$ let $T_{e,f}$ be the set of all isomorphisms from E_e onto E_f. Let

$$T_E = \bigcup_{(e,f)\in\mathcal{U}} T_{e,f}.$$

Then T_E is an inverse subsemigroup of $\mathcal{I}_E$ and is fundamental. We shall call it *the Munn semigroup* of the semilattice E.

The crucial fact concerning the Munn semigroup is that:

Theorem 4.1. *[26] If S is an inverse semigroup with semilattice of idempotents E, then there is a morphism $\phi : S \to T_E$ whose kernel is μ, the maximum idempotent separating congruence on S. The morphism ϕ is defined by*

$$a\phi = \alpha_a \qquad (a \in S),$$

where α_a is given above.

We pause to mention that the Munn semigroup T_E is determined by its semilattice of idempotents. In view of this, it is natural to be concerned with the influence of the properties of the idempotents of an inverse semigroup on the structure of the inverse semigroup as a whole. Keeping this in mind, we recall that an inverse semigroup S is said to be a *Clifford semigroup* if the idempotents are central, that is, $ex = xe$ for every idempotent e and every x in S, or equivalently, a semilattice of groups.

Theorem 4.1 built a concrete morphism $\phi : S \to T_E$. If S is a Clifford semigroup, then the morphism ϕ has kernel $\mu = \mathcal{H}$.

Theorem 4.2. *If S is a Clifford semigroup with semilattice of idempotents E, then $\mu = \mathcal{H}$ and S/μ is a semilattice, which must be embedded in T_E.*

Proof. For every element a in a Clifford semigroup S, there exists an inverse a^{-1} of a such that $a \mathrel{\mathcal{H}} a^{-1}$, and so $a^{-1}a = aa^{-1} = g$, where g is the idempotent in

H_a. Then α_a is a morphism from E_g to E_g. Since S is a Clifford semigroup, the idempotents are central. Then for any $e \in E_g$,

$$e\alpha_a = a^{-1}ea = a^{-1}ae = ge = e.$$

So α_a is the identity map on E_g, from which it follows that if $a \,\mathcal{H}\, b$ in S, then $\alpha_a = \alpha_b$, that is, $a\phi = b\phi$. Thus, $\mathcal{H} \subseteq \mathrm{Ker}\phi$. Certainly, $\mathrm{Ker}\phi \subseteq \mathcal{H}$. Hence, $\mathrm{Ker}\phi = \mathcal{H}$. By Theorem 4.1, $\mu = \mathcal{H}$. □

4.2 A fundamental orthogroup of W_B

In Section 4.1 we were able to find a morphism ϕ from an inverse semigroup with semilattice of idempotents E to the Munn semigroup T_E. Moreover, if S is a Clifford semigroup, then the image of S in T_E is E (hence is particular also Clifford). If we are to find a generalisation of this to an orthodox semigroup (a regular semigroup whose set of idempotents forms a band), we need begin by recalling the appropriate analogue of the Hall semigroup [26].

Let B be a band. We denote by $\langle e\rangle$ the principal order ideal generated by e for all $e \in B$. We define

$$\mathcal{U} = \{(e,f) \in B \times B : \langle e\rangle \simeq \langle f\rangle\}$$

and write $W_{e,f}$ for the set of all isomorphisms from $\langle e\rangle$ onto $\langle f\rangle$. If $(e,f) \in \mathcal{U}$ and $\alpha \in W_{e,f}$, we may define $\alpha_l \in \mathcal{T}(B/\mathcal{L})$ and $\alpha_r \in \mathcal{T}^*(B/\mathcal{R})$ by the rule that

$$L_x\alpha_l = L_{x\alpha}, \qquad R_x\alpha_r = R_{x\alpha} \qquad (x \in \langle e\rangle).$$

It is routine to verify that $(\alpha_l)^{-1} = (\alpha^{-1})_l$ and $(\alpha_r)^{-1} = (\alpha^{-1})_r$. In this case, we may use the notation α_l^{-1}, α_r^{-1} without ambiguity.

Now, we put

$$W_B = \{(\rho_e\alpha_l, \lambda_f\alpha_r^{-1}) : \alpha \in W_{e,f}, (e,f) \in \mathcal{U}\},$$

where for any $x \in B$,

$$L_x\rho_e = L_{exe}, \qquad R_x\lambda_f = R_{fxf}.$$

In fact, W_B is precisely the analogue of T_E and it is a fundamental orthodox subsemigroup of $\mathcal{T}(B/\mathcal{L}) \times \mathcal{T}^*(B/\mathcal{R})$. We shall call it the *Hall semigroup* of the band B.

It is useful and convenient to present the following result.

Lemma 4.3. *[26] If e, f, g are elements of a band B with $(e, f) \in \mathcal{U}$ and $g \in \langle e \rangle$, then $\langle g \rangle \alpha = \langle g\alpha \rangle$, where $\alpha \in W_{e,f}$.*

In the case of an inverse semigroup, the key idea of conjugates of idempotents guarantees that there exists a representation which provides more useful information about the structure of the semigroup. We note that this idea is still available in the orthodox case, but it is necessary to take a new technique to deal with the inverse of every element, since it is not unique.

Observe that if a is an element in an orthodox semigroup S with band of idempotents B and a^*, a' are inverses of a, then $a^*xa \,\mathcal{L}\, a'xa$ and $axa^* \,\mathcal{R}\, axa'$, where $x \in B$. Built on this observation, we have the following maps.

Let S be an orthodox semigroup with band of idempotents B. For each a in S, a mapping $\rho_a : B/\mathcal{L} \to B/\mathcal{L}$ is defined by

$$L_x\rho_a = L_{a'xa} \qquad \big(x \in B\big),$$

where a' is an arbitrary chosen inverse of a. By dual arguments we can define $\lambda_a : B/\mathcal{R} \to B/\mathcal{R}$ by

$$R_x\lambda_a = R_{axa'} \qquad \big(x \in B\big),$$

where a' is an arbitrary chosen inverse of a.

By Proposition 1.16, $b'a'$ is an inverse of ab, and so $\rho_{ab} = \rho_a\rho_b$, for all a, b in S. Dually, we have that $\lambda_{ab} = \lambda_b\lambda_a$. Moreover:

Theorem 4.4. *[26] Let S be an orthodox semigroup with band of idempotents B, and let ψ be the mapping from S into $\mathcal{T}(B/\mathcal{L}) \times \mathcal{T}^*(B/\mathcal{R})$ defined by*

$$a\psi = (\rho_a, \lambda_a),$$

where ρ_a, λ_a are given as above. Then ψ is a morphism whose kernel is the maximum idempotent separating congruence μ on S.

The result we have achieved presents a representation in $\mathcal{T}(B/\mathcal{L}) \times \mathcal{T}^*(B/\mathcal{R})$ rather than T_E. We have seen that W_B is an analogue of T_E and an orthodox subsemigroup of $\mathcal{T}(B/\mathcal{L}) \times \mathcal{T}^*(B/\mathcal{R})$, having band of idempotents isomorphic to B. So to obtain an exact analogue of Theorem 4.1, we must rewrite the maps ρ_a and λ_a in the form used to define W_B.

Now let S be an orthodox semigroup with band of idempotents B. If $a \in S$ and a' is an inverse of a, then denoting aa' by e and $a'a$ by f, we obtain that

$$(\rho_a, \lambda_a) = (\rho_e \theta_l, \lambda_f \theta_r^{-1}),$$

where θ is the mapping in $W_{e,f}$ given by

$$x\theta = a'xa \qquad \big(x \in \langle e \rangle\big).$$

In this case, the range of the mapping $\psi : a \mapsto (\rho_a, \lambda_a)$ is thus contained in the Hall semigroup

$$W_B = \{(\rho_e \alpha_l, \lambda_f \alpha_r^{-1}) : \alpha \in W_{e,f}, (e,f) \in \mathcal{U}\}.$$

Before moving on we pause to confine ourselves to a consideration of a special kind of orthodox semigroups analogous to Clifford semigroups. We have mentioned in Chapter 1 that a semigroup is *completely regular* if each of its elements is contained in some subgroup of S. An orthodox semigroup is an *orthogroup* if it is completely regular. Obviously, every Clifford semigroup is an orthogroup.

At the end of the previous section, we mentioned that there exists a representation from a Clifford semigroup S to T_E and the image of S is the semilattice of idempotents of T_E. Certainly, every semilattice is a Clifford semigroup. So there exactly exists a representation from a Clifford semigroup to a Clifford subsemigroup of T_E. At a certain stage it becomes natural to ask whether there exists such a representation of an orthogroup to an orthogroup contained in the Hall semigroup W_B.

As a first step what we have to do is to find a subsemigroup of W_B which is an orthogroup. In this case, it is necessary to make use of the statement [49] that an orthodox semigroup is completely regular if and only if its greatest inverse semigroup homomorphic image is completely regular, i.e. a Clifford semigroup.

If B is a band, then it is a semilattice Y of rectangular bands B_α $(\alpha \in Y)$. It is not hard to verify that if $e, x \in B$ with $x \in \langle e \rangle$, then $e \in B_\alpha$ and $x \in B_\xi$

for some $\alpha, \xi \in Y$ with $\xi \leq \alpha$. In addition, if $e, f \in B$ with $e \in B_\alpha$, $f \in B_\beta$ ($\alpha, \beta \in Y$) and $\iota : \langle e \rangle \to \langle f \rangle$ is an isomorphism, then there is an isomorphism $\iota' : \alpha Y \to \beta Y$ corresponding to ι, defined by the property that

$$x\iota \in B_{\xi\iota'} \qquad (\xi \in \alpha Y,\ x \in \langle e \rangle \cap B_\xi).$$

As Y is a semilattice, the Munn semigroup T_Y certainly exists. Then for every element $(\rho_e\theta_l, \lambda_f\theta_r^{-1})$ in W_B, θ is an isomorphism from $\langle e \rangle$ onto $\langle f \rangle$. Due to the above analysis, there is an isomorphism $\theta' : \alpha Y \to \beta Y$ corresponding to θ in T_Y, where $e \in B_\alpha$ and $f \in B_\beta$. Hence, there is a map ν from W_B to T_Y defined by the rule that

$$(\rho_e\theta_l, \lambda_f\theta_r^{-1})\nu = \theta' \qquad ((\rho_e\theta_l, \lambda_f\theta_r^{-1}) \in W_B).$$

Lemma 4.5. *The map ν defined above is a morphism from W_B to T_Y.*

Proof. We first show that ν is well-defined. Let $(\rho_e\theta_l, \lambda_f\theta_r^{-1}), (\rho_h\sigma_l, \lambda_k\sigma_r^{-1}) \in W_B$ be such that $(\rho_e\theta_l, \lambda_f\theta_r^{-1}) = (\rho_h\sigma_l, \lambda_k\sigma_r^{-1})$. Then for any $x \in B$, we have that $L_x\rho_e\theta_l = L_x\rho_h\sigma_l$, that is $(exe)\theta \mathrel{\mathcal{L}} (hxh)\sigma$. Choose $x = e$. We obtain that $e\theta \mathrel{\mathcal{L}} (heh)\sigma$, and so $f \mathrel{\mathcal{L}} (heh)\sigma$ as $e\theta = f$. Since $\mathcal{L}$ is a right congruence and $(heh)\sigma \leq k$, we succeed in obtaining that $fk \mathrel{\mathcal{L}} (heh)\sigma$. By Lemma 1.22, we have that $kfk \mathrel{\mathcal{L}} (heh)\sigma$. As $\sigma : \langle h \rangle \to \langle k \rangle$ is an isomorphism, we have that $(kfk)\sigma^{-1} \mathrel{\mathcal{L}} heh$. Also, for any $x \in B$, we have that $R_x\lambda_f\theta_r^{-1} = R_x\lambda_k\sigma_r^{-1}$, that is, $(fxf)\theta^{-1} \mathrel{\mathcal{R}} (kxk)\sigma^{-1}$. Take $x = f$, we obtain that $f\theta^{-1} \mathrel{\mathcal{R}} (kfk)\sigma^{-1}$. As $f\theta^{-1} = e$, we get that $e \mathrel{\mathcal{R}} (kfk)\sigma^{-1}$. Together with $(kfk)\sigma^{-1} \mathrel{\mathcal{L}} heh$, we have that $e \mathrel{\mathcal{D}} heh$. Similarly, we obtain that $h \mathrel{\mathcal{D}} ehe$. Certainly, $ehe \mathrel{\mathcal{D}} heh$. Thus $e \mathrel{\mathcal{D}} h$. Dually, we have that $f \mathrel{\mathcal{D}} k$. Hence,

$$\operatorname{dom}(\theta') = \operatorname{dom}(\sigma') = \alpha Y \text{ and } \operatorname{im}(\theta') = \operatorname{im}(\sigma') = \beta Y,$$

where $e, h \in B_\alpha$ and $f, k \in B_\beta$.

We still need show that for any $\xi \leq \alpha$, $\xi\theta' = \xi\sigma'$. To do this, we assume that $x \leq e$ and $x \in B_\xi$. Then there exists $y \in B$ such that $x = eye$. Since $e \mathrel{\mathcal{D}} h$, we have that $x = eye \mathrel{\mathcal{D}} hyh \leq h$, and so $hyh \in B_\xi$. As $x\theta = (eye)\theta \mathrel{\mathcal{L}} (hyh)\sigma$ so that

$x\theta\,\mathcal{D}\,(hyh)\sigma$. By the properties of θ' and σ', we have that

$$x\theta \in B_{\xi\theta'} \text{ and } (hyh)\sigma \in B_{\xi\sigma'}.$$

Thus $\xi\theta' = \xi\sigma'$. Consequently, ν is well-defined.

Next, we show that ν is a morphism. Let $(\rho_e\eta_l, \lambda_f\eta_r^{-1}), (\rho_g\sigma_l, \lambda_h\sigma_r^{-1}) \in W_B$ and let $e \in B_\alpha$, $f \in B_\beta$, $g \in B_\gamma$ and $h \in B_\delta$. According to [Chapter VI, Theorem 2.17, [26]], we have that

$$(\rho_e\eta_l, \lambda_f\eta_r^{-1})(\rho_g\sigma_l, \lambda_h\sigma_r^{-1}) = (\rho_i\tau_l, \lambda_j\tau_r^{-1}),$$

where $i = (fgf)\eta^{-1}$, $j = (gfg)\sigma$ and $\tau = (\eta|_{\langle i\rangle})(\theta_{gfg}|_{\langle fgf\rangle})(\sigma|_{\langle gfg\rangle})$. In the following, we show that $\eta' \circ \sigma' = \tau'$.

As $fgf \in B_{\beta\gamma}$, we have that $(fgf)\eta^{-1} \in B_{(\beta\gamma)\eta'^{-1}}$, and so $\operatorname{dom}\tau' = (\beta\gamma)\eta'^{-1}Y$. Similarly, $\operatorname{im}\tau' = (\beta\gamma)\sigma'Y$. Observe that $\operatorname{im}\eta' = \beta Y$ and $\operatorname{dom}\sigma' = \gamma Y$. Thus

$$\begin{aligned}
\operatorname{dom}(\eta' \circ \sigma') &= (\operatorname{im}\eta' \cap \operatorname{dom}\sigma')\eta'^{-1} \\
&= (\beta Y \cap \gamma Y)\eta'^{-1} \\
&= (\beta\gamma)Y\eta'^{-1} \\
&= (\beta\gamma)\eta'^{-1}Y \qquad \big(\text{Lemma 4.3}\big) \\
&= \operatorname{dom}\tau'.
\end{aligned}$$

Similarly, $\operatorname{im}(\eta' \circ \sigma') = \operatorname{im}\tau'$.

Let $\xi \in \operatorname{dom}(\eta' \circ \sigma')$ and $x \in B_\xi \cap \langle i\rangle$. Then

$$x\tau = x(\eta|_{\langle i\rangle})(\theta_{gfg}|_{\langle fgf\rangle})(\sigma|_{\langle gfg\rangle}) \in B_\mu \qquad \text{for some } \mu \in Y,$$

and so $\xi\tau' = \mu$. Since we have remarked that $\theta_{gfg}|_{\langle fgf\rangle}$ fixes $\mathcal{D}$-classes succeeding Lemma 1.21, it follows that $x(\eta|_{\langle i\rangle})\ \mathcal{D}\ x(\eta|_{\langle i\rangle})(\theta_{gfg}|_{\langle fgf\rangle})$ so that $\xi\eta' = \omega$ and $\omega\sigma' = \mu$, where $x(\eta|_{\langle i\rangle}), x(\eta|_{\langle i\rangle})(\theta_{gfg}|_{\langle fgf\rangle}) \in B_\omega$. Hence, $\xi(\eta' \circ \sigma') = \mu$, and so $\eta' \circ \sigma' = \tau'$. □

We now pause to mention that if the image θ' of the element $(\rho_e\theta_l, \lambda_f\theta_r^{-1})$ of W_B under ν is an idempotent, then e, f are $\mathcal{D}$-related and θ' is an identity map on αY, where $e \in B_\alpha$. Hence, the elements of W_B, whose images under ν are idempotent, induce a partial identity mapping on Y. Moreover, we have:

Lemma 4.6. *[49] The elements of W_B whose images under ν are idempotent form an orthogroup in W_B. We denote this orthogroup by OG_1.*

Proof. We first show that OG_1 is closed. Suppose that $(\rho_e\iota_l, \lambda_f\iota_r^{-1})$ and $(\rho_g\tau_l, \lambda_h\tau_r^{-1})$ are in OG_1. Then $\iota', \tau' \in E(T_Y)$. Since $E(T_Y)$ is a semilattice, we have that $\iota'\tau' \in E(T_Y)$. As $OG_1 \subseteq W_B$, we have that the product $(\rho_e\iota_l, \lambda_f\iota_r^{-1})(\rho_g\tau_l, \lambda_h\tau_r^{-1})$ is in W_B. Also,

$$\Big((\rho_e\iota_l, \lambda_f\iota_r^{-1})(\rho_g\tau_l, \lambda_h\tau_r^{-1})\Big)\nu = (\rho_e\iota_l, \lambda_f\iota_r^{-1})\nu(\rho_g\tau_l, \lambda_h\tau_r^{-1})\nu = \iota' \circ \tau' \in E(T_Y).$$

Thus, $(\rho_e\iota_l, \lambda_f\iota_r^{-1})(\rho_g\tau_l, \lambda_h\tau_r^{-1}) \in OG_1$, and so OG_1 is closed.

Clearly, $\overline{B} = \{(\rho_e, \lambda e) : e \in B\}$ is contained in OG_1 and so OG_1 is an orthodox semigroup.

Finally, we claim that OG_1 is completely regular. Let $(\rho_e\iota_l, \lambda_f\iota_r^{-1}) \in OG_1$, $e \in B_\alpha$ and $f \in B_\beta (\alpha, \beta \in Y)$. Then $\iota' : \alpha Y \to \beta Y$ is idempotent. In that case, we must have that $e \,\mathcal{D}\, f$, and so $e \,\mathcal{R}\, ef \,\mathcal{L}\, f$ so that $(\rho_e, \lambda_e) \,\mathcal{R}\, (\rho_{ef}, \lambda_{ef}) \,\mathcal{L}\, (\rho_f, \lambda_f)$. As $(\rho_e, \lambda_e) \,\mathcal{R}\, (\rho_e\iota_l, \lambda_f\iota_r^{-1}) \,\mathcal{L}\, (\rho_f, \lambda_f)$, we obtain that $(\rho_e\iota_l, \lambda_f\iota_r^{-1}) \,\mathcal{H}\, (\rho_{ef}, \lambda_{ef})$. Thus, OG_1 is an orthogroup. □

Lemma 4.6 gives us an abstract description of an orthogroup in W_B. Next, we shall find a closed form for such an orthogroup in W_B which coincides with OG_1.

For any $e \in B$, we write

$$A_e = \{\alpha \in W_{e,e} : \text{for all } x \in \langle e \rangle, x\alpha \,\mathcal{D}\, x\}$$

and put

$$OG_2 = \bigcup_{e \in B} W_e,$$

where $W_e = \{(\rho_e\alpha_l, \lambda_e\alpha_r^{-1}) \in W_B : \alpha \in A_e\}$.

Lemma 4.7. *For any $e \in B$, the set A_e forms a group.*

Proof. Suppose that $\alpha \in A_e$. We claim that $\alpha^{-1} \in A_e$. For any $x \in \langle e \rangle$, certainly, $x\alpha^{-1} \in \langle e \rangle$. Since $\alpha \in A_e$, we have that $x = (x\alpha^{-1})\alpha \,\mathcal{D}\, x\alpha^{-1}$. This implies that $\alpha^{-1} \in A_e$. Obviously, the identity map $1_{\langle e \rangle}$ is the identity of A_e and A_e is closed. Hence, A_e forms a group. □

Further:

Lemma 4.8. *For any $e \in B$, the set W_e forms a subgroup of W_B with identity (ρ_e, λ_e).*

Proof. Clearly, for any $e \in B$, $(\rho_e, \lambda_e) \in W_e$ and it is the identity of W_e. We now show that W_e is closed. Suppose that $(\rho_e\alpha_l, \lambda_e\alpha_r^{-1})$ and $(\rho_e\beta_l, \lambda_e\beta_r^{-1})$ are in W_e. We first consider the product of $\rho_e\alpha_l$ and $\rho_e\beta_l$ because dually, we obtain the similar result for the product of $\lambda_e\alpha_r^{-1}$ and $\lambda_e\beta_r^{-1}$. For any $x \in B$, we have that

$$L_x\rho_e\alpha_l\rho_e\beta_l = L_{(e[(exe)\alpha]e)\beta} = L_{((exe)\alpha)\beta} = L_{(exe)\alpha\beta} = L_x\rho_e(\alpha\beta)_l.$$

Similarly, we obtain that $R_x\lambda_e\beta_r^{-1}\lambda_e\alpha_r^{-1} = R_x\lambda_e(\alpha\beta)_r^{-1}$. As $\alpha, \beta \in A_e$, the composition $\alpha\beta$ certainly belongs to A_e by Lemma 4.7. Thus, $(\rho_e(\alpha\beta)_l, \lambda_e(\alpha\beta)_r^{-1}) \in W_e$, and so W_e is closed.

Next, we show that the group inverse of $(\rho_e\alpha_l, \lambda_e\alpha_r^{-1})$ exists and lies in W_e. To do this, we assume that $(\rho_e\alpha_l, \lambda_e\alpha_r^{-1}) \in W_e$. Then $\alpha \in A_e$. By Lemma 4.7, we have that $\alpha^{-1} \in A_e$, and so $(\rho_e\alpha_l^{-1}, \lambda_e\alpha_r) \in W_e$. In addition, it is routine to check that $(\rho_e\alpha_l, \lambda_e\alpha_r^{-1})$ and $(\rho_e\alpha_l^{-1}, \lambda_e\alpha_r)$ are mutually group inverse in W_e. $\square$

Returning now to the set OG_2 constructed above, we have:

Lemma 4.9. *The set OG_2 forms an orthogroup with band of idempotents $\overline{B} = \{(\rho_e, \lambda_e) : e \in B\}$.*

Proof. In view of Lemma 4.8, it is sufficient to show that for any $e, f \in B$, there exists $h \in B$ such that $W_eW_f \subseteq W_h$. Suppose that $(\rho_e\alpha_l, \lambda_e\alpha_r^{-1}) \in W_e$ and $(\rho_f\beta_l, \lambda_f\beta_r^{-1}) \in W_f$. Then $\alpha \in A_e$, $\beta \in A_f$, and so $(efe)\alpha^{-1} \mathcal{D}\ efe$ and $(fef)\beta\ \mathcal{D}\ fef$ so that $(efe)\alpha^{-1}\,\mathcal{D}\,(fef)\beta$ as $efe\,\mathcal{D}\,fef$. Put

$$\gamma = \Big(\theta_{(efe)\alpha^{-1}}|_{\langle(efe)\alpha^{-1}(fef)\beta\rangle}\Big)\Big(\alpha|_{\langle(efe)\alpha^{-1}\rangle}\Big)\cdot \Big(\theta_{fef}|_{\langle efe\rangle}\Big)\Big(\beta|_{\langle fef\rangle}\Big)\Big(\theta_{(efe)\alpha^{-1}(fef)\beta}|_{\langle(fef)\beta\rangle}\Big).$$

By Lemma 1.21, $\gamma \in W_{(efe)\alpha^{-1}(fef)\beta,(efe)\alpha^{-1}(fef)\beta}$. Again by the remark succeeding Lemma 1.21 and α, β fixing $\mathcal{D}$-classes, we have that γ fixes $\mathcal{D}$-classes, and so

$$\gamma \in A_{(efe)\alpha^{-1}(fef)\beta}.$$

Thus, $(\rho_{(efe)\alpha^{-1}(fef)\beta}\gamma_l, \lambda_{(efe)\alpha^{-1}(fef)\beta}\gamma_r^{-1})$ belongs to $W_{(efe)\alpha^{-1}(fef)\beta}$.

We now show that

$$(\rho_e\alpha_l, \lambda_e\alpha_r^{-1})(\rho_f\beta_l, \lambda_f\beta_r^{-1}) = (\rho_{(efe)\alpha^{-1}(fef)\beta}\gamma_l, \lambda_{(efe)\alpha^{-1}(fef)\beta}\gamma_r^{-1}).$$

For any $x \in B$,

$$\begin{aligned}
&L_x\rho_{(efe)\alpha^{-1}(fef)\beta}\gamma_l \\
&= L_{(efe)\alpha^{-1}(fef)\beta\, x\, (efe)\alpha^{-1}(fef)\beta}\Big(\Big(\theta_{(efe)\alpha^{-1}}|_{\langle(efe)\alpha^{-1}(fef)\beta\rangle}\Big)\cdot \\
&\qquad \Big(\alpha|_{\langle(efe)\alpha^{-1}\rangle}\Big)\Big(\theta_{fef}|_{\langle efe\rangle}\Big)\Big(\beta|_{\langle fef\rangle}\Big)\Big(\theta_{(efe)\alpha^{-1}(fef)\beta}|_{\langle(fef)\beta\rangle}\Big)\Big)_l \\
&= L_{\Big((efe)\alpha^{-1}\cdot(efe)\alpha^{-1}(fef)\beta\, x\, (efe)\alpha^{-1}(fef)\beta\cdot(efe)\alpha^{-1}\Big)\Big(\alpha|_{\langle(efe)\alpha^{-1}\rangle}\Big)\cdot} \\
&\qquad {}_{\Big(\theta_{fef}|_{\langle efe\rangle}\Big)\Big(\beta|_{\langle fef\rangle}\Big)\Big(\theta_{(efe)\alpha^{-1}(fef)\beta}|_{\langle(fef)\beta\rangle}\Big)} \\
&= L_{\Big((efe)\alpha^{-1}(fef)\beta\, x\, (efe)\alpha^{-1}\Big)\Big(\alpha|_{\langle(efe)\alpha^{-1}\rangle}\Big)\Big(\theta_{fef}|_{\langle efe\rangle}\Big)\Big(\beta|_{\langle fef\rangle}\Big)\Big(\theta_{(efe)\alpha^{-1}(fef)\beta}|_{\langle(fef)\beta\rangle}\Big)} \\
&\qquad \Big((efe)\alpha^{-1}\,\mathcal{D}\,(fef)\beta\Big) \\
&= L_{\Big((efe)\alpha^{-1}(fef)\beta\, x\, (efe)\alpha^{-1}\Big)}\Big(\Big(\alpha|_{\langle(efe)\alpha^{-1}\rangle}\Big)\Big(\theta_{fef}|_{\langle efe\rangle}\Big)\Big(\beta|_{\langle fef\rangle}\Big)\cdot \\
&\qquad \Big(\theta_{(efe)\alpha^{-1}(fef)\beta}|_{\langle(fef)\beta\rangle}\Big)\Big)_l \\
&= L_{\Big(x\,(efe)\alpha^{-1}\Big)}\Big(\Big(\alpha|_{\langle(efe)\alpha^{-1}\rangle}\Big)\Big(\theta_{fef}|_{\langle efe\rangle}\Big)\Big(\beta|_{\langle fef\rangle}\Big)\Big(\theta_{(efe)\alpha^{-1}(fef)\beta}|_{\langle(fef)\beta\rangle}\Big)\Big)_l \\
&\qquad \Big(\text{Lemma 1.22}\Big) \\
&= L_{\Big((efe)\alpha^{-1}\, x\, (efe)\alpha^{-1}\Big)\Big(\alpha|_{\langle(efe)\alpha^{-1}\rangle}\Big)\Big(\theta_{fef}|_{\langle efe\rangle}\Big)\Big(\beta|_{\langle fef\rangle}\Big)\Big(\theta_{(efe)\alpha^{-1}(fef)\beta}|_{\langle(fef)\beta\rangle}\Big)} \\
&\qquad \Big(\text{Lemma 1.22}\Big) \\
&= L_{\Big((efe)\alpha^{-1}\, exe\, (efe)\alpha^{-1}\Big)\Big(\alpha|_{\langle(efe)\alpha^{-1}\rangle}\Big)\Big(\theta_{fef}|_{\langle efe\rangle}\Big)\Big(\beta|_{\langle fef\rangle}\Big)\Big(\theta_{(efe)\alpha^{-1}(fef)\beta}|_{\langle(fef)\beta\rangle}\Big)} \\
&\qquad \Big((exe)\alpha^{-1} \in \langle e\rangle\Big) \\
&= L_{\Big((efe)\alpha^{-1}\alpha\,(exe)\alpha\,(efe)\alpha^{-1}\alpha\Big)\Big(\theta_{fef}|_{\langle efe\rangle}\Big)\Big(\beta|_{\langle fef\rangle}\Big)\Big(\theta_{(efe)\alpha^{-1}(fef)\beta}|_{\langle(fef)\beta\rangle}\Big)} \\
&\qquad \Big((efe)\alpha^{-1},\ exe \in \langle e\rangle\Big) \\
&= L_{\Big((efe)(exe)\alpha(efe)\Big)\Big(\theta_{fef}|_{\langle efe\rangle}\Big)\Big(\beta|_{\langle fef\rangle}\Big)\Big(\theta_{(efe)\alpha^{-1}(fef)\beta}|_{\langle(fef)\beta\rangle}\Big)} \\
&= L_{\Big((fef)(efe)(exe)\alpha(efe)(fef)\Big)\Big(\beta|\langle fef\rangle\Big)\Big(\theta_{(efe)\alpha^{-1}(fef)\beta}|_{\langle(fef)\beta\rangle}\Big)} \qquad \Big(\text{Lemma 1.21}\Big)
\end{aligned}$$

$$
\begin{aligned}
&= L_{\left((fefef)(e\cdot(exe)\alpha\cdot e)(fefef)\right)\left(\beta|_{\langle fef\rangle}\right)\left(\theta_{(efe)\alpha^{-1}(fef)\beta}|_{\langle(fef)\beta\rangle}\right)} \\
&= L_{\left((fef)(exe)\alpha(fef)\right)\left(\beta|_{\langle fef\rangle}\right)\left(\theta_{(efe)\alpha^{-1}(fef)\beta}|_{\langle(fef)\beta\rangle}\right)} && \big((exe)\alpha \in \langle e\rangle\big) \\
&= L_{\left(((fef)(exe)\alpha(fef))\beta\right)\left(\theta_{(efe)\alpha^{-1}(fef)\beta}|_{\langle(fef)\beta\rangle}\right)} \\
&= L_{\left(((fef)f((exe)\alpha)f(fef))\beta\right)\left(\theta_{(efe)\alpha^{-1}(fef)\beta}|_{\langle(fef)\beta\rangle}\right)} \\
&= L_{\left((fef)\beta(f(exe)\alpha f)\beta(fef)\beta\right)\left(\theta_{(efe)\alpha^{-1}(fef)\beta}|_{\langle(fef)\beta\rangle}\right)} && \big(fef, f(exe)\alpha f \in \langle f\rangle\big) \\
&= L_{(efe)\alpha^{-1}(fef)\beta\cdot(fef)\beta(f(exe)\alpha f)\beta(fef))\beta\cdot(efe)\alpha^{-1}(fef)\beta} && \big(\text{Lemma 1.21}\big) \\
&= L_{(f(exe)\alpha f)\beta(fef))\beta\cdot(efe)\alpha^{-1}(fef)\beta} && \big((fef)\beta\ \mathcal{D}\ (efe)\alpha^{-1},\ \ \text{Lemma 1.22}\big) \\
&= L_{(f(exe)\alpha f)\beta(fef))\beta} && \big((fef)\beta\ \mathcal{D}(efe)\alpha^{-1}\big) \\
&= L_{(f(exe)\alpha f\cdot fef)\beta} \\
&= L_{(f(exe)\alpha fef)\beta} \\
&= L_{(f(exe)\alpha\cdot e\cdot fef)\beta} && \big((exe)\alpha \in \langle e\rangle\big) \\
&= L_{(f(exe)\alpha ef)\beta} \\
&= L_{(f(exe)\alpha f)\beta} && \big((exe)\alpha \in \langle e\rangle\big) \\
&= L_x\rho_e\alpha_l\rho_f\beta_l.
\end{aligned}
$$

Thus, $\rho_{(efe)\alpha^{-1}(fef)\beta}\gamma_l = \rho_e\alpha_l \cdot \rho_f\alpha_l$. Also, we deduce that

$$
\begin{aligned}
&R_x\lambda_{(efe)\alpha^{-1}(fef)\beta}\gamma_r^{-1} \\
&= R_{(efe)\alpha^{-1}(fef)\beta\, x\, (efe)\alpha^{-1}(fef)\beta}\Big(\left(\theta_{(efe)\alpha^{-1}}|_{\langle(efe)\alpha^{-1}(fef)\beta\rangle}\right)\cdot \\
&\qquad \left(\alpha|_{\langle(efe)\alpha^{-1}\rangle}\right)\left(\theta_{fef}|_{\langle efe\rangle}\right)\left(\beta|_{\langle fef\rangle}\right)\left(\theta_{(efe)\alpha^{-1}(fef)\beta}|_{\langle(fef)\beta\rangle}\right)\Big)_r^{-1} \\
&= R_{\left((efe)\alpha^{-1}(fef)\beta\, x\, (efe)\alpha^{-1}(fef)\beta\right)\left(\theta_{(efe)\alpha^{-1}(fef)\beta}|_{\langle(fef)\beta\rangle}\right)^{-1}\left(\beta|_{\langle fef\rangle}\right)^{-1}\cdot \left(\theta_{fef}|_{\langle efe\rangle}\right)^{-1}\left(\alpha|_{\langle(efe)\alpha^{-1}\rangle}\right)^{-1}\left(\theta_{(efe)\alpha^{-1}}|_{\langle(efe)\alpha^{-1}(fef)\beta\rangle}\right)^{-1}} \\
&= R_{\left((efe)\alpha^{-1}(fef)\beta\, x\, (efe)\alpha^{-1}(fef)\beta\right)\left(\theta_{(fef)\beta}|_{\langle(efe)\alpha^{-1}(fef)\beta\rangle}\right)\left(\beta^{-1}|_{\langle(fef)\beta\rangle}\right)\cdot \left(\theta_{efe}|_{\langle fef\rangle}\right)\left(\alpha^{-1}|_{\langle efe\rangle}\right)\left(\theta_{(efe)\alpha^{-1}(fef)\beta}|_{\langle(efe)\alpha^{-1}\rangle}\right)} && \big(\text{Lemma 1.21}\big)
\end{aligned}
$$

$$= R_{\left((fef)\beta\cdot(efe)\alpha^{-1}(fef)\beta\, x\,(efe)\alpha^{-1}(fef)\beta\cdot(fef)\beta\right)\left(\beta^{-1}|_{\langle (fef)\beta\rangle}\right)\left(\theta_{efe}|_{\langle fef\rangle}\right)\cdot \left(\alpha^{-1}|_{\langle efe\rangle}\right)\left(\theta_{(efe)\alpha^{-1}(fef)\beta}|_{\langle (efe)\alpha^{-1}\rangle}\right)} \qquad \big(\text{Lemma 1.21}\big)$$

$$= R_{\left((fef)\beta\, x\,(efe)\alpha^{-1}(fef)\beta\right)\left(\beta^{-1}|_{\langle (fef)\beta\rangle}\right)\left(\theta_{efe}|_{\langle fef\rangle}\right)\left(\alpha^{-1}|_{\langle efe\rangle}\right)\left(\theta_{(efe)\alpha^{-1}(fef)\beta}|_{\langle (efe)\alpha^{-1}\rangle}\right)} \qquad \big((efe)\alpha^{-1}\,\mathcal{D}\,(fef)\beta\big)$$

$$= R_{\left((fef)\beta\, x\,(efe)\alpha^{-1}(fef)\beta\right)\left(\left(\beta^{-1}|_{\langle (fef)\beta\rangle}\right)\left(\theta_{efe}|_{\langle fef\rangle}\right)\left(\alpha^{-1}|_{\langle efe\rangle}\right)\cdot \left(\theta_{(efe)\alpha^{-1}(fef)\beta}|_{\langle (efe)\alpha^{-1}\rangle}\right)\right)_r}$$

$$= R_{\left((fef)\beta\, x\right)\left(\left(\beta^{-1}|_{\langle (fef)\beta\rangle}\right)\left(\theta_{efe}|_{\langle fef\rangle}\right)\left(\alpha^{-1}|_{\langle efe\rangle}\right)\left(\theta_{(efe)\alpha^{-1}(fef)\beta}|_{\langle (efe)\alpha^{-1}\rangle}\right)\right)_r} \qquad \big(\text{Lemma 1.21}\big)$$

$$= R_{\left((fef)\beta\, x\,(fef)\beta\right)\left(\beta^{-1}|_{\langle (fef)\beta\rangle}\right)\left(\theta_{efe}|_{\langle fef\rangle}\right)\left(\alpha^{-1}|_{\langle efe\rangle}\right)\left(\theta_{(efe)\alpha^{-1}(fef)\beta}|_{\langle (efe)\alpha^{-1}\rangle}\right)} \qquad \big(\text{Lemma 1.22}\big)$$

$$= R_{\left((fef)\beta\, fxf\,(fef)\beta\right)\left(\beta^{-1}|_{\langle (fef)\beta\rangle}\right)\left(\theta_{efe}|_{\langle fef\rangle}\right)\left(\alpha^{-1}|_{\langle efe\rangle}\right)\left(\theta_{(efe)\alpha^{-1}(fef)\beta}|_{\langle (efe)\alpha^{-1}\rangle}\right)} \qquad \big((fef)\beta \in \langle f\rangle\big)$$

$$= R_{\left(((fef)\beta\, fxf\,(fef)\beta)\beta^{-1}\right)\left(\theta_{efe}|_{\langle fef\rangle}\right)\left(\alpha^{-1}|_{\langle efe\rangle}\right)\left(\theta_{(efe)\alpha^{-1}(fef)\beta}|_{\langle (efe)\alpha^{-1}\rangle}\right)}$$

$$= R_{\left((fef)\beta\beta^{-1}\,(fxf)\beta^{-1}\,(fef)\beta\beta^{-1}\right)\left(\theta_{efe}|_{\langle fef\rangle}\right)\left(\alpha^{-1}|_{\langle efe\rangle}\right)\left(\theta_{(efe)\alpha^{-1}(fef)\beta}|_{\langle (efe)\alpha^{-1}\rangle}\right)} \qquad \big((fef)\beta, fxf \in \langle f\rangle\big)$$

$$= R_{\left((fef)\,(fxf)\beta^{-1}\,(fef)\right)\left(\theta_{efe}|_{\langle fef\rangle}\right)\left(\alpha^{-1}|_{\langle efe\rangle}\right)\left(\theta_{(efe)\alpha^{-1}(fef)\beta}|_{\langle (efe)\alpha^{-1}\rangle}\right)} \qquad \big(\text{Lemma 1.21}\big)$$

$$= R_{\left((efe)(fef)\,(fxf)\beta^{-1}\,(fef)(efe)\right)\left(\alpha^{-1}|_{\langle efe\rangle}\right)\left(\theta_{(efe)\alpha^{-1}(fef)\beta}|_{\langle (efe)\alpha^{-1}\rangle}\right)}$$

$$= R_{\left((efefe)(f\cdot(fxf)\beta^{-1}\cdot f)(efefe)\right)\left(\alpha^{-1}|_{\langle efe\rangle}\right)\left(\theta_{(efe)\alpha^{-1}(fef)\beta}|_{\langle (efe)\alpha^{-1}\rangle}\right)}$$

$$= R_{\left((efe)\,(fxf)\beta^{-1}\,(efe)\right)\left(\alpha^{-1}|_{\langle efe\rangle}\right)\left(\theta_{(efe)\alpha^{-1}(fef)\beta}|_{\langle (efe)\alpha^{-1}\rangle}\right)}$$

$$= R_{\left(((efe)\,(fxf)\beta^{-1}\,(efe))\alpha^{-1}\right)\left(\theta_{(efe)\alpha^{-1}(fef)\beta}|_{\langle (efe)\alpha^{-1}\rangle}\right)}$$

$$= R_{\left(((efe)\,e(fxf)\beta^{-1}e\,(efe))\alpha^{-1}\right)\left(\theta_{(efe)\alpha^{-1}(fef)\beta}|_{\langle (efe)\alpha^{-1}\rangle}\right)}$$

$$= R_{\left((efe)\alpha^{-1}\,(e(fxf)\beta^{-1}e)\alpha^{-1}\,(efe)\alpha^{-1}\right)\left(\theta_{(efe)\alpha^{-1}(fef)\beta}|_{\langle (efe)\alpha^{-1}\rangle}\right)} \qquad \big(efe, e(fxf)\beta^{-1}e \in \langle e\rangle\big)$$

$$= R_{(efe)\alpha^{-1}(fef)\beta\cdot(efe)\alpha^{-1}\,(e(fxf)\beta^{-1}e)\alpha^{-1}\,(efe)\alpha^{-1}\cdot(efe)\alpha^{-1}(fef)\beta} \qquad \big(\text{Lemma 1.21}\big)$$
$$= R_{(efe)\alpha^{-1}\,(e(fxf)\beta^{-1}e)\alpha^{-1}\,(efe)\alpha^{-1}(fef)\beta} \qquad \big((fef)\beta\ \mathcal{D}\ (efe)\alpha^{-1}\big)$$
$$= R_{(efe)\alpha^{-1}\,(e(fxf)\beta^{-1}e)\alpha^{-1}} \qquad \big((fef)\beta\ \mathcal{D}\ (efe)\alpha^{-1} \text{ and Lemma 1.22}\big)$$
$$= R_{(efe\cdot e(fxf)\beta^{-1}e)\alpha^{-1}}$$
$$= R_{(efe\cdot ef(fxf)\beta^{-1}e)\alpha^{-1}} \qquad \big((fxf)\beta^{-1} \in \langle f\rangle\big)$$
$$= R_{(ef(fxf)\beta^{-1}e)\alpha^{-1}}$$
$$= R_{(e(fxf)\beta^{-1}e)\alpha^{-1}} \qquad \big((fxf)\beta^{-1} \in \langle f\rangle\big)$$
$$= R_x\lambda_f\beta_r^{-1}\lambda_e\alpha_r^{-1}.$$

Thus, $\lambda_{(efe)\alpha^{-1}(fef)\beta}\gamma_r^{-1} = \lambda_f\beta_r^{-1}\cdot\lambda_e\alpha_r^{-1}$. Hence, OG_2 is a subsemigroup of W_B. By Lemma 4.8, $\overline{B} = \{(\rho_e, \lambda_e) : e \in B\}$ is contained in OG_2. □

Observe that for any element $(\rho_e\theta_l, \lambda_f\theta_r^{-1}) \in OG_1$, we have that $e\,\mathcal{D}\,f$ according to the comments following Lemma 4.5. Whereas, if $(\rho_e\theta_l, \lambda_f\theta_r^{-1}) \in OG_2$, then $e = f$. In that case, we next show that OG_1 is equal to OG_2 by replacing $(\rho_e\theta_l, \lambda_f\theta_r^{-1})$ with $(\rho_g\gamma_l, \lambda_g\gamma_r^{-1})$ in OG_1 for some $g \in B$.

Lemma 4.10. *The semigroup OG_1 coincides with OG_2.*

Proof. We first show that $OG_2 \subseteq OG_1$. Suppose that $(\rho_e\iota_l, \lambda_e\iota_r^{-1}) \in OG_2$. Since B is a semilattice Y of rectangular bands B_α $(\alpha \in Y)$, we assume that $e \in B_\alpha$. According to the construction of OG_2, we have that $b\iota\ \mathcal{D}\ b$ for any $b \in \langle e\rangle$. Specifically, if $b \in \langle e\rangle \cap B_\xi$, where $\xi \in \alpha Y$, then $b\iota\,\mathcal{D}\,b$. As $b\iota \in B_{\xi\iota'}$ and $b \in B_\xi$, we must have that $\xi\iota' = \xi$, which implies that ι' is the identity map on αY. So, $(\rho_e\iota_l, \lambda_e\iota_r^{-1}) \in OG_1$.

Conversely, suppose that $\iota \in W_{e,f}$ and here ι' is an identity in $T_Y = T_{B/\mathcal{D}}$. Then, $e\,\mathcal{D}\,f$, and so $\theta_e|_{\langle ef\rangle}$ and $\theta_{ef}|_{\langle f\rangle}$ are well-defined isomorphisms from $\langle ef\rangle$ onto $\langle e\rangle$ and from $\langle f\rangle$ onto $\langle ef\rangle$, respectively. As $\iota \in W_{e,f}$, we have that $\gamma = (\theta_e|_{\langle ef\rangle})\iota(\theta_{ef}|_{\langle f\rangle})$ is an automorphism of $\langle ef\rangle$. We now show that $(\rho_e\iota_l, \lambda_f\iota_r^{-1}) = (\rho_{ef}\gamma_l, \lambda_{ef}\gamma_r^{-1})$. For any $x \in B$,

$$\begin{aligned} L_x\rho_{ef}\gamma_l &= L_{xef}\gamma_l \\ &= L_{(efxef)(\theta_e|_{\langle ef\rangle})\iota(\theta_{ef}|_{\langle f\rangle})} \\ &= L_{(e\cdot efxef\cdot e)\iota(\theta_{ef}|_{\langle f\rangle})} \\ &= L_{(e\cdot efxef\cdot e)}(\iota(\theta_{ef}|_{\langle f\rangle}))_l \end{aligned}$$

$$
\begin{aligned}
&= L_{exe}(\iota(\theta_{ef}|_{\langle f\rangle}))_l && \big(\text{Lemma 1.22}\big)\\
&= L_{(exe)\iota(\theta_{ef}|_{\langle f\rangle})}\\
&= L_{ef\cdot(exe)\iota\cdot ef}\\
&= L_{(exe)\iota\cdot ef}\\
&= L_{\big((exe)\iota\big)f\cdot ef} && ((exe)\iota \in \langle f\rangle)\\
&= L_{\big((exe)\iota\big)f} && (e\,\mathcal{D}\,f)\\
&= L_{(exe)\iota}\\
&= L_x\rho_e\iota_l.
\end{aligned}
$$

Thus, $\rho_e\iota_l = \rho_{ef}\gamma_l$. Dually, $\lambda_f\iota_r^{-1} = \lambda_{ef}\gamma_r^{-1}$. Hence, $(\rho_e\iota_l, \lambda_f\iota_r^{-1}) = (\rho_{ef}\gamma_r, \lambda_{ef}\gamma_r^{-1})$.

Next, we show that for any $x \in \langle ef\rangle$, $x\gamma\,\mathcal{D}\,x$. Suppose that $e, f \in B_\alpha$, where $\alpha \in Y$. According to the construction of OG_1, ι' is the identity map on αY, that is, for any $\xi \in \alpha Y$ and $b \in \langle e\rangle \cap B_\xi$, we have that $b\iota \in B_{\xi\iota'} = B_\xi$. Thus, $b\iota\,\mathcal{D}\,b$. By Lemma 1.21 and the remark following it, $\theta_e|_{\langle ef\rangle}$ and $\theta_{ef}|_{\langle f\rangle}$ are isomorphisms fixing $\mathcal{D}$-classes. Thus, γ is an automorphism of $\langle ef\rangle$ such that for all $x \in \langle ef\rangle$, $x\gamma\,\mathcal{D}\,x$. So, $(\rho_{ef}\gamma_l, \lambda_{ef}\gamma_r^{-1}) \in OG_2$. Hence, $OG_1 \subseteq OG_2$. □

We return to our question of establishing a representation from an orthogroup to OG_2(resp. OG_1) as an analogue of Theorem 4.2, built on Theorem 4.4.

Theorem 4.11. *If S is an orthogroup with band of idempotents B, then there exists a representation $\psi : S \to OG_2$ whose kernel is μ, the maximal idempotent separating congruence on S.*

Proof. In view of Theorem 4.4, there exists a representation ψ from an orthogroup S to W_B. We now need to show that the image of S under ψ is contained in OG_2. Suppose that $a \in S$ and a^{-1} is the inverse of a in H_a. Then by Theorem 4.4 and the comments succeeding it, we have that

$$a\psi = (\rho_a, \lambda_a) = (\rho_e\theta_l, \lambda_f\theta_r^{-1}),$$

where $e = aa^{-1}$, $f = a^{-1}a$ and θ is an isomorphism in $W_{e,f}$ given by

$$x\theta = a^{-1}xa \qquad (x \in \langle e\rangle).$$

Since $a\,\mathcal{H}\,a^{-1}$ and every $\mathcal{H}$-class of an orthogroup is a group, we have that $aa^{-1} = a^{-1}a$, and so $e = f\,\mathcal{H}\,a$. Then, for any $x \in \langle e\rangle$, $x = exe\,\mathcal{D}\,a^{-1}xa$, that is, $x\,\mathcal{D}\,x\theta$, and so $\theta \in A_e$. Hence, $a\psi = (\rho_e\theta_l, \lambda_e\theta_r^{-1}) \in OG_2$ so that $S\theta \subseteq OG_2$, as required. □

4.3 A fundamental weakly $\overline{B}$-superabundant subsemigroup of V_B

The aim in this section is to move away from the regular case and consider a fundamental weakly B-superabundant semigroup S with (C) and (IC). Here B is a band. Recall that a weakly B-abundant semigroup is said to be *weakly B-superabundant* if every $\widetilde{\mathcal{H}}_B$-class contains a distinguished idempotent in B.

In [6], El-Qallali, Fountain and Gould constructed a fundamental weakly B-orthodox semigroup with (IC), namely, V_B, in a manner analogous to the Hall semigroup W_B. A brief description of the construction is necessary before we build a weakly $\overline{B}$-superabundant subsemigroup of V_B, where $\overline{B}$ is isomorphic to B.

For any $e, f \in B$ we define $V_{e,f}$ to be the set of all order isomorphisms α from $\langle e\rangle$ to $\langle f\rangle$ such that

$$x\alpha y\alpha\,\mathcal{L}\,(xy)\alpha \text{ and } u\alpha^{-1}v\alpha^{-1}\;\mathcal{R}\;(uv)\alpha^{-1},$$

for all $x, y \in \langle e\rangle$ and $u, v \in \langle f\rangle$. For any $\alpha \in V_{e,f}$ we can define partial maps of $B/\mathcal{L}$ and $B/\mathcal{R}$ by

$$L_x\alpha_l = L_{x\alpha} \text{ and } R_y\alpha_r^{-1} = R_{y\alpha^{-1}}.$$

Due to [6], we have that if $e, f \in B$ and $\alpha \in V_{e,f}$, then for all $x, x' \in \langle e\rangle$ and $y, y' \in \langle f\rangle$,

$$x \leq_{\mathcal{L}} x' \text{ implies that } x\alpha \leq_{\mathcal{L}} x'\alpha,$$

$$y \leq_{\mathcal{R}} y' \text{ implies that } y\alpha^{-1} \leq_{\mathcal{R}} y'\alpha^{-1}.$$

This fact is hard evidence showing that α_l and α_r^{-1} are well-defined and order preserving.

Now, we put

$$V_B = \{(\rho_e\alpha_l, \lambda_f\alpha_r^{-1}) : e, f \in B, \alpha \in V_{e,f}\}.$$

Lemma 4.12. *[17] The set V_B is a fundamental weakly $\overline{B}$-orthodox semigroup with (IC), where $\overline{B} = \{(\rho_e, \lambda_e) : e \in B\}$.*

We remark that for any $(\rho_e\alpha_l, \lambda_f\alpha_r^{-1}) \in V_B$, we have that

$$(\rho_f, \lambda_f)\, \widetilde{\mathcal{L}}_{\overline{B}}\, (\rho_e\alpha_l, \lambda_f\alpha_r^{-1})\, \widetilde{\mathcal{R}}_{\overline{B}}\, (\rho_e, \lambda_e).$$

Considering the fact that V_B is an analogue of the Hall semigroup W_B, we can extend the recipe in Section 4.2 from the Hall semigroup W_B to V_B to find a fundamental weakly $\overline{B}$-superabundant subsemigroup of V_B as follows.

Let B be a band. Then it is a semilattice Y of rectangular bands $B_\alpha (\alpha \in Y)$. For every element $(\rho_e\theta_l, \lambda_f\theta_r^{-1})$ in V_B, θ is an order isomorphism from $\langle e \rangle$ to $\langle f \rangle$. Referring to the statement before Lemma 4.5, if $e \in B_\alpha$ and $f \in B_\beta (\alpha, \beta \in Y)$, then there is an order isomorphism $\theta' : \alpha Y \to \beta Y$ corresponding to θ, defined by the property that

$$x\theta \in B_{\xi\theta'} \qquad (\xi \in \alpha Y,\, x \in \langle e \rangle \cap B_\xi).$$

Since an order isomorphism of a semilattice is an isomorphism it follows that $\theta' \in T_Y$.

Observe that for any $(\rho_e\theta_l, \lambda_f\theta_r^{-1}) \in V_B$, if θ' is idempotent, then it induces a partial identity mapping on Y. We put

$$K_1 = \{(\rho_e\theta_l, \lambda_f\theta_r^{-1}) \in V_B : \theta'^2 = \theta'\}.$$

The next lemma is an immediate consequence of Lemma 4.15, so we omit its proof.

Lemma 4.13. *The set K_1 is a weakly $\overline{B}$-superabundant subsemigroup of V_B with (C) and (IC).*

In Section 4.2 we gave a closed form for a subsemigroup of the Hall semigroup W_B that is a particular orthogroup. In the following, We focus on a closed form for a fundamental weakly $\overline{B}$-superabundant subsemigroup of V_B as an analogue of OG_2, beginning as follows.

For any $e \in B$, let

$$OA_e = \{\alpha \in V_{e,e} : \text{ for all } x \in \langle e \rangle, x \mathcal{D} x\alpha\}$$

and

$$K_2 = \bigcup_{e \in B} V_e,$$

where $V_e = \{(\rho_e \alpha_l, \lambda_e \alpha_r^{-1}) \in V_B : e \in B, \alpha \in OA_e\}$.

The proof of the following lemma is similar to that of Lemma 4.9, and so we omit it. Here we remark that the steps using α etc. being a morphism can be replaced by the particular condition for α to lie in $V_{e,f}$.

Lemma 4.14. *The set K_2 is a full subsemigroup of V_B. Consequently, K_2 is a fundamental weakly $\overline{B}$-superabundant semigroup with (C) and (IC).*

The next lemma presents a relationship between K_1 and K_2.

Lemma 4.15. *The semigroup K_2 coincides with K_1.*

Proof. We first show that $K_2 \subseteq K_1$. Suppose that $(\rho_e \iota_l, \lambda_e \iota_r^{-1}) \in K_2$. Since B is a semilattice Y of rectangular bands B_α $(\alpha \in Y)$, we assume that $e \in B_\alpha$. According to the construction of K_2, we have that $b\iota \mathcal{D} b$ for any $b \in \langle e \rangle$. Specifically, if $b \in \langle e \rangle \cap B_\xi$, where $\xi \in \alpha Y$, then $b\iota \mathcal{D} b$. As $b\iota \in B_{\xi\iota'}$ and $b \in B_\xi$, we must have that $\xi\iota' = \xi$, which implies that ι' is the identity map on αY. So, $(\rho_e \iota_l, \lambda_e \iota_r^{-1}) \in K_1$.

Conversely, suppose that $\iota \in V_{e,f}$ and here ι' is an identity in $T_Y = T_{B/\mathcal{D}}$. Then, $e \mathcal{D} f$, and so $\theta_e|_{\langle ef \rangle}$ and $\theta_{ef}|_{\langle f \rangle}$ are well-defined isomorphisms from $\langle ef \rangle$ onto $\langle e \rangle$ and from $\langle f \rangle$ onto $\langle ef \rangle$, respectively. As $\iota \in V_{e,f}$, we have that $\gamma = (\theta_e|_{\langle ef \rangle})\iota(\theta_{ef}|_{\langle f \rangle})$ is an order automorphism of $\langle ef \rangle$. We now show that $(\rho_e \iota_l, \lambda_f \iota_r^{-1}) = (\rho_{ef} \gamma_l, \lambda_{ef} \gamma_r^{-1})$. For any $x \in B$,

$$\begin{aligned} L_x \rho_{ef} \gamma_l &= L_{xef} \gamma_l \\ &= L_{(efxef)(\theta_e|_{\langle ef \rangle})\iota(\theta_{ef}|_{\langle f \rangle})} \quad \big(\text{Lemma 1.22}\big) \\ &= L_{(e \cdot efxef \cdot e)\iota(\theta_{ef}|_{\langle f \rangle})} \\ &= L_{(e \cdot efxef \cdot e)} (\iota(\theta_{ef}|_{\langle f \rangle}))_l \end{aligned}$$

$$= L_{exe}(\iota(\theta_{ef}|_{\langle f\rangle}))_l \qquad \big(\text{Lemma 1.22}\big)$$
$$= L_{(exe)\iota(\theta_{ef}|_{\langle f\rangle})}$$
$$= L_{ef\cdot(exe)\iota\cdot ef}$$
$$= L_{(exe)\iota\cdot ef}$$
$$= L_{\big((exe)\iota\big)f\cdot ef} \qquad ((exe)\iota \in \langle f\rangle)$$
$$= L_{\big((exe)\iota\big)f} \qquad (e\,\mathcal{D}\,f)$$
$$= L_{(exe)\iota}$$
$$= L_x\rho_e\iota_l.$$

Thus, $\rho_e\iota_l = \rho_{ef}\gamma_l$. Dually, $\lambda_f\iota_r^{-1} = \lambda_{ef}\gamma_r^{-1}$. Hence, $(\rho_e\iota_l, \lambda_f\iota_r^{-1}) = (\rho_{ef}\gamma_r, \lambda_{ef}\gamma_r^{-1})$.

Next, we show that for any $x \in \langle ef\rangle$, $x\gamma\,\mathcal{D}\,x$. Suppose that $e, f \in B_\alpha$, where $\alpha \in Y$. According to the construction of K_1, ι' is the identity map on αY, that is, for any $\xi \in \alpha Y$ and $b \in \langle e\rangle \cap B_\xi$, we have that $b\iota \in B_{\xi\iota'} = B_\xi$. Thus, $b\iota\,\mathcal{D}\,b$. By Lemma 1.21 and the remark succeeding it, $\theta_e|_{\langle ef\rangle}$ and $\theta_{ef}|_{\langle f\rangle}$ are isomorphisms preserving $\mathcal{D}$-classes. Thus, γ is an order automorphism of $\langle ef\rangle$ such that for all $x \in \langle ef\rangle$, $x\gamma\,\mathcal{D}\,x$. So, $(\rho_{ef}\gamma_r, \lambda_{ef}\gamma_r^{-1}) \in K_2$. Hence, $K_1 \subseteq K_2$. □

We end this section with a representation of a weakly B-superabundant semigroup with (C) and (IC), which is analogous to Theorem 4.2 and Theorem 4.11. We first explain how a weakly B-orthodox semigroup with (IC) is represented in V_B.

Let S be a weakly B-orthodox semigroup. For any $a \in S$, we define

$$\alpha_a : B/\mathcal{L} \to B/\mathcal{L} \text{ and } \beta_a : B/\mathcal{R} \to B/\mathcal{R}$$

by

$$L_x\alpha_a = L_{(xa)^*} \text{ and } R_x\beta_a = R_{(ax)^\dagger}.$$

Clearly, α_a and β_a are well-defined. We note that for any $e \in B$,

$$(\alpha_e, \beta_e) = (\rho_e, \lambda_e),$$

where for any $x \in B$,

$$L_x\rho_e = L_{xe} \text{ and } R_x\lambda_e = R_{ex}.$$

Lemma 4.16. *[17] Let S be a weakly B-orthodox semigroup with (IC). The map*

$$\phi : S \to V_B$$

given by

$$a\phi = (\alpha_a, \beta_a),$$

is a strongly B-admissible morphism with kernel μ_B. Moreover, putting $\overline{B} = \{(\rho_e, \lambda_e) : e \in B\}$, we have that $\theta|_B : B \to \overline{B}$ is an isomorphism.

We remark that for any $a \in S$, choose $a^\dagger, a^*$ and let $\alpha : \langle a^\dagger \rangle \to \langle a^* \rangle$ be an order isomorphism such that for all $x \in \langle a^\dagger \rangle$,

$$xa = a(x\alpha),$$

then $a\phi = (\alpha_a, \beta_a) = (\rho_{a^\dagger}\alpha_l, \lambda_{a^*}\alpha_r^{-1})$.

The specialisation of Lemma 4.16 to the case of weakly B-superabundant semigroups S with (C) and (IC) is of special interest here, since the image of S under ϕ is contained in the fundamental weakly $\overline{B}$-superabundant semigroup K_2 with (C) and (IC).

Theorem 4.17. *If S is a weakly B-superabundant semigroup with (C) and (IC), then the image of S under the map ϕ given in Lemma 4.16, is contained in K_2.*

Proof. In view of Lemma 4.16, we show that the image of S under ϕ is contained in K_2. By the remark succeeding Lemma 4.16, we have that for any $a \in S$,

$$a\phi = (\rho_e\alpha_l, \lambda_e\alpha_r^{-1}),$$

where $e \in B$, $e\,\widetilde{\mathcal{H}}_B\,a$ in S and $\alpha : \langle e \rangle \to \langle e \rangle$ is an ordered isomorphism such that $xa = a(x\alpha)$ for all $x \in \langle e \rangle$. For any $x \in \langle e \rangle$, we have that

$$x = xe\,\widetilde{\mathcal{R}}_B\,xa = a(x\alpha)\,\widetilde{\mathcal{L}}_B\,e(x\alpha) = x\alpha.$$

Thus, $x\,\widetilde{\mathcal{D}}_B\,x\alpha$. By Lemma 3.10, we have that $x\,\mathcal{D}\,x\alpha$. Hence, $\alpha \in OA_e$. Consequently, $a\phi = (\rho_e\alpha_l, \lambda_e\alpha_r^{-1}) \in K_2$. □

4.4 A fundamental weakly $\overline{B}$-superabundant subsemigroup of U_B

In Chapter 2 we introduced the weak idempotent connected condition (WIC), that coincides with (IC), defined by El-Qallali and Fountain, for abundant semigroups, but not for weakly B-abundant semigroups. Starting with a band B, El-Qallali, Fountain and Gould construct a weakly B-orthodox subsemigroup U_B of $\mathcal{OP}(B/\mathcal{L}) \times \mathcal{OP}^*(B/\mathcal{R})$ satisfying (WIC). The semigroup U_B plays the role of W_B for the class of weakly B-abundant semigroups having a band of idempotents B. Our purpose here is to find a fundamental weakly $\overline{B}$-superabundant subsemigroup of U_B with (C) and (WIC).

We refer the reader to [17] for more details, but for convenience we sketch the construction of U_B as follows.

Let B be a band. For any $e, f \in B$, we commonly denote a relation from $\langle e \rangle$ to $\langle f \rangle$, that is, a subset of $\langle e \rangle \times \langle f \rangle$, by $I^{e,f}$. We say that $I^{e,f}$ is *connecting* if $I^{e,f}$ is a subsemigroup of $\langle e \rangle \times \langle f \rangle$ and for every $(x, x'), (y, y') \in I^{e,f}$ we have that

$$x \leqslant_{\mathcal{L}} y \text{ implies that } x' \leqslant_{\mathcal{L}} y'$$

and

$$x' \leqslant_{\mathcal{R}} y' \text{ implies that } x \leqslant_{\mathcal{R}} y.$$

Let A, B be sets and $R \subseteq A \times B$ be a relation. Then R is *full* if both projection maps are onto.

Lemma 4.18. *[17] Let $I^{e,f}$ be full connecting. Then for any (x, y), $(z, t) \in I^{e,f}$,*

$$x \leqslant_{\mathcal{D}} z \textit{ if and only if } y \leqslant_{\mathcal{D}} t.$$

Set

$$U_B = \{(\rho_e I_l^{e,f}, \lambda_f I_r^{e,f}) : e, f \in B, I^{e,f} \in \langle e \rangle \times \langle f \rangle \text{ is full connecting}\},$$

where $I_l^{e,f}$ is defined by

$$L_x I_l^{e,f} = L_y \qquad (x, y) \in I^{e,f},$$

and $I_r^{e,f}$ is defined by

$$R_y I_r^{e,f} = R_x \qquad (x,y) \in I^{e,f}.$$

Note that for any $e \in B$,

$$\iota^{e,e} = \{(x,x) : x \leq e\}$$

is full connecting, and

$$(\rho_e \iota_l^{e,e}, \lambda_e \iota_l^{e,e}) = (\rho_e, \lambda_e),$$

so that $\overline{B} = \{(\rho_e, \lambda_e) : e \in B\} \subseteq U_B$.

Lemma 4.19. *[17] The set U_B is a fundamental weakly $\overline{B}$-orthodox semigroup of $\mathcal{OP}(B/\mathcal{L}) \times \mathcal{OP}^*(B/\mathcal{R})$, with (WIC).*

We remark that for any $(\rho_e I_l^{e,f}, \lambda_f I_r^{e,f}) \in U_B$, we have that

$$(\rho_f, \lambda_f)\, \widetilde{\mathcal{L}}_{\overline{B}}\, (\rho_e I_l^{e,f}, \lambda_f I_r^{e,f})\, \widetilde{\mathcal{R}}_{\overline{B}}\, (\rho_e, \lambda_e).$$

Since Condition (WIC) gives us a very loose control over the position of idempotents, but does not impose artificially the existence of order isomorphisms, the idea used in previous sections to construct OG_2 and K_2 still works here, however, we need carefully deal with more complicated proofs. We first look at a concrete construction for a subsemigroup of U_B as follows.

For any $e \in B$, we put

$$Q_1 = \bigcup_{e \in B} U_e,$$

where, $U_e = \{(\rho_e I_l^{e,e}, \lambda_e I_r^{e,e}) \in U_B : \text{for all } (x,y) \in I^{e,e}, x\ \mathcal{D}\ y\}$.

Lemma 4.20. *The set Q_1 forms a fundamental weakly $\overline{B}$-superabundant subsemigroup of U_B with (C) and (WIC).*

Proof. We first show that Q_1 is a semigroup. Let $(\rho_e I_l^{e,e}, \lambda_e I_r^{e,e}), (\rho_f J_l^{f,f}, \lambda_f J_r^{f,f}) \in Q_1$. Since

$$efe \leq e \text{ and } fef \leq f,$$

and $I^{e,e}$, $J^{f,f}$ are full connecting, there exist $(z, efe) \in I^{e,e}$ and $(fef, w) \in J^{f,f}$.

By the construction of Q_1, $z \mathcal{D} efe \mathcal{D} fef \mathcal{D} w$. If follows from Lemma 1.21 that

$$\theta_z|_{\langle zw\rangle} : \langle zw\rangle \to \langle z\rangle, \qquad \theta_{fef}|_{\langle efe\rangle} : \langle efe\rangle \to \langle fef\rangle \ \text{ and } \ \theta_{zw}|_{\langle w\rangle} : \langle w\rangle \to \langle zw\rangle$$

are $\mathcal{D}$-class preserving isomorphisms. We claim that $K^{zw,zw}$ is full connecting, where

$$K^{zw,zw} = \Big(\theta_z|_{\langle zw\rangle}\Big)\Big(I^{e,e}|_{\langle z\rangle}\Big)\Big(\theta_{fef}|_{\langle efe\rangle}\Big)\Big(J^{f,f}|_{\langle fef\rangle}\Big)\Big(\theta_{zw}|_{\langle w\rangle}\Big).$$

To show that the projection maps to $\langle zw\rangle$ are onto, assume that $g \in B$ with $g \leq zw$. Since $\theta_z|_{\langle zw\rangle}$ is an isomorphism from $\langle zw\rangle$ onto $\langle z\rangle$, we have that $g\theta_z|_{\langle zw\rangle} \leq z$. As $z \leq e$ and $I^{e,e}$ is full connecting, there exists an element $(g\theta_z|_{\langle zw\rangle}, t) \in I^{e,e}$. Now $g\theta_z|_{\langle zw\rangle} = zgz$ and $I^{e,e}$ is a semigroup, so that

$$(g\theta_z|_{\langle zw\rangle}, efetefe) = (z, efe)(g\theta_z|_{\langle zw\rangle}, t)(z, efe) \in I^{e,e}.$$

Clearly $efetefe \in \langle efe\rangle$, so that

$$(efetefe, f(efetefe)f) \in \theta_{fef}|_{\langle efe\rangle},$$

that is,

$$(efetefe, (fef)fetef(fef)) \in \theta_{fef}|_{\langle efe\rangle}.$$

Now $fetef \in \langle f\rangle$; as $J^{f,f}$ is full connecting, there exists an element $(fetef, k) \in J^{f,f}$. Consequently,

$$(fef, w)(fetef, k)(fef, w) = ((fef)fetef(fef), wkw) \in J^{f,f}.$$

Certainly, $wkw \in \langle w\rangle$, so that

$$(wkw, zw(wkw)zw) \in \theta_{zw}|_{\langle w\rangle}.$$

It follows that

$$(g, zw(wkw)zw) \in K^{zw,zw}.$$

Dually, the projection of $K^{zw,zw}$ to the second coordinate is onto.

Since each of $\theta_z|_{\langle zw\rangle}$, $I^{e,e}$, $\theta_{fef}|_{\langle efe\rangle}$, $J^{f,f}$ and $\theta_{zw}|_{\langle w\rangle}$ is a subsemigroup of $B \times B$, it follows that the same is true of the composition, hence of $K^{zw,zw}$. As $\theta_z|_{\langle zw\rangle}$ and $\theta_{zw}|_{\langle w\rangle}$ are isomorphisms so that they preserve the partial order $\leq_{\mathcal{L}}$

and $\leq_{\mathcal{R}}$. Thus, $K^{zw,zw}$ is full connecting. Finally, since each of the relations concerned fixes the same $\mathcal{D}$-class, the same is clearly true of the composition, we have that $(\rho_{zw}K_l^{zw,zw}, \lambda_{zw}K_r^{zw,zw}) \in Q_1$.

Let $(\rho_e I_l^{e,e}, \lambda_e I_r^{e,e})$, $(\rho_f J_l^{f,f}, \lambda_f J_r^{f,f}) \in Q_1$. We claim that

$$(\rho_e I_l^{e,e}, \lambda_e I_r^{e,e})(\rho_f J_l^{f,f}, \lambda_f J_r^{f,f}) = (\rho_{zw}K_l^{zw,zw}, \lambda_{zw}K_r^{zw,zw}).$$

To see this, let $x \in B$. we have that

$$\begin{aligned} L_x \rho_e I_l^{e,e} \rho_f J_l^{f,f} &= L_{exe} I_l^{e,e} \rho_f J_l^{f,f} \\ &= L_u \rho_f J_l^{f,f} && \text{where } (exe, u) \in I^{e,e} \\ &= L_{fuf} J_l^{f,f} \\ &= L_v && \text{where } (fuf, v) \in J^{f,f}. \end{aligned}$$

On the other hand,

$$\begin{aligned} & L_x \rho_{zw} K_l^{zw,zw} \\ &= L_{zwxzw}\Big(\big(\theta_z|_{\langle zw\rangle}\big)\big(I^{e,e}|_{\langle z\rangle}\big)\big(\theta_{fef}|_{\langle efe\rangle}\big)\big(J^{f,f}|_{\langle fef\rangle}\big)\big(\theta_{zw}|_{\langle w\rangle}\big)\Big)_l \\ &= L_{(zwxzw)\left(\left(\theta_z|_{\langle zw\rangle}\right)\left(I^{e,e}|_{\langle z\rangle}\right)\left(\theta_{fef}|_{\langle efe\rangle}\right)\left(J^{f,f}|_{\langle fef\rangle}\right)\left(\theta_{zw}|_{\langle w\rangle}\right)\right)} \\ &= L_{(z\cdot zwxzw\cdot z)\left(\left(I^{e,e}|_{\langle z\rangle}\right)\left(\theta_{fef}|_{\langle efe\rangle}\right)\left(J^{f,f}|_{\langle fef\rangle}\right)\left(\theta_{zw}|_{\langle w\rangle}\right)\right)} \\ &= L_{(zwxz)\left(\left(I^{e,e}|_{\langle z\rangle}\right)\left(\theta_{fef}|_{\langle efe\rangle}\right)\left(J^{f,f}|_{\langle fef\rangle}\right)\left(\theta_{zw}|_{\langle w\rangle}\right)\right)} && (z \mathrel{\mathcal{D}} efe \mathrel{\mathcal{D}} fef \mathrel{\mathcal{D}} w) \\ &= L_{(zxz)\left(\left(I^{e,e}|_{\langle z\rangle}\right)\left(\theta_{fef}|_{\langle efe\rangle}\right)\left(J^{f,f}|_{\langle fef\rangle}\right)\left(\theta_{zw}|_{\langle w\rangle}\right)\right)} && (\text{Lemma 1.22}) \\ &= L_{(zexez)\left(\left(I^{e,e}|_{\langle z\rangle}\right)\left(\theta_{fef}|_{\langle efe\rangle}\right)\left(J^{f,f}|_{\langle fef\rangle}\right)\left(\theta_{zw}|_{\langle w\rangle}\right)\right)} && \big(z \leq e\big) \\ &= L_{(efeuefe)\left(\left(\theta_{fef}|_{\langle efe\rangle}\right)\left(J^{f,f}|_{\langle fef\rangle}\right)\left(\theta_{zw}|_{\langle w\rangle}\right)\right)} && ((z, efe), (exe, u) \in I^{e,e}) \\ &= L_{(fef\cdot efeuefe\cdot fef)\left(\left(J^{f,f}|_{\langle fef\rangle}\right)\left(\theta_{zw}|_{\langle w\rangle}\right)\right)} \\ &= L_{(fef\cdot fuf\cdot fef)\left(\left(J^{f,f}|_{\langle fef\rangle}\right)\left(\theta_{zw}|_{\langle w\rangle}\right)\right)} && (u \in \langle e\rangle) \\ &= L_{(wvw)\left(\theta_{zw}|_{\langle w\rangle}\right)} && ((fef, w), (fuf, v) \in J^{f,f}) \\ &= L_{zwwvwzw} \end{aligned}$$

$$= L_{zwvw} \qquad (z\ \mathcal{D}\ w)$$
$$= L_{wvw} \qquad \big(\text{Lemma 1.22}\big).$$

Note that $(fuf, v), (fef, w) \in J^{f,f}$, we obtain that $(fefufef, wvw) \in J^{f,f}$ as $J^{f,f}$ is a subsemigroup. As $u \in \langle e \rangle$, and so $fefufef = fefeuefef = feuef = fuf$ so that $(fuf, wvw) \in J^{f,f}$. Since $J^{f,f}$ is full connecting and $(fuf, v) \in J^{f,f}$, it follows that $L_v = L_{wvw}$. Thus, $\rho_{zw} K_l^{zw,zw} = \rho_e I_l^{e,e} \rho_f J_l^{f,f}$. Dually, we obtain that $\lambda_{zw} K_r^{zw,zw} = \lambda_f I_r^{f,f} \lambda_e J_r^{e,e}$, so that

$$(\rho_e I_l^{e,e}, \lambda_e I_r^{e,e})(\rho_f J_l^{f,f}, \lambda_f J_r^{f,f}) = (\rho_{zw} K_l^{zw,zw}, \lambda_{zw} K_r^{zw,zw}).$$

Hence, Q_1 forms a semigroup. In view of the remark following Lemma 4.19, we have that $(\rho_e I_l^{e,e}, \lambda_e I_r^{e,e})\, \widetilde{\mathcal{H}}_{\overline{B}}\, (\rho_e, \lambda_e)$, and so Q_1 is weakly $\overline{B}$-superabundant.

As $\overline{B} \subseteq Q_1$, we have that Q_1 is a full subsemigroup of U_B, and so Q_1 is a fundamental weakly $\overline{B}$-superabundant subsemigroup of U_B with (C) and (WIC). □

Another alternative characterisation of a fundamental weakly $\overline{B}$-superabundant subsemigroup Q_2 of U_B, satisfying (C) and (WIC), is available. The following construction is closely analogous to that of K_1 and OG_1.

Before describing the construction of Q_2, we mention an important fact.

Now, let B be a band. Obviously, it is a semilattice Y of rectangular bands B_α $(\alpha \in Y)$. In addition, there is the Munn semigroup T_Y corresponding to Y. If $e, f \in B$ with $e \in B_\alpha$, $f \in B_\beta (\alpha, \beta \in Y)$, and $I^{e,f}$ is a full connecting relation in $\langle e \rangle \times \langle f \rangle$, then it follows from Lemma 4.18 that there exists a mapping $\overline{I^{e,f}}$ from αY onto βY with the property that

$$(u, v) \in I^{e,f} \text{ implies that } (u, v) \in B_\xi \times B_{\xi \overline{I^{e,f}}},$$

where $u \in \langle e \rangle \cap B_\xi$. In fact, $\overline{I^{e,f}}$ must be a bijection.

We define a map

$$\chi : U_B \to T_Y$$

by

$$(\rho_e I_l^{e,f}, \lambda_f I_r^{e,f})\chi = \overline{I^{e,f}},$$

for any $(\rho_e I_l^{e,f}, \lambda_f I_r^{e,f}) \in U_B$

Lemma 4.21. *The mapping χ is a morphism.*

Proof. We first claim that χ is well-defined. To do this, let $(\rho_e I_l^{e,f}, \lambda_f I_r^{e,f}) \in U_B$. We first show that $\overline{I^{e,f}}$ is an isomorphism from αY onto βY,where $e \in B_\alpha$ and $f \in B_\beta$.

It is easy to see that $\overline{I^{e,f}}$ is well-defined since if $(x,y),(z,t) \in I^{e,f}$ with $x \mathcal{D} z$, then by Lemma 4.18, we have that $y \mathcal{D} t$. Again by Lemma 4.18, we have that $\overline{I^{e,f}}$ is injective. As $I^{e,f}$ is full, it is certainly true that $\overline{I^{e,f}}$ is surjective. In addition, $\overline{I^{e,f}}$ is a morphism because $I^{e,f}$ is a subsemigroup of $\langle e\rangle \times \langle f\rangle$.

If $(\rho_g J_l^{g,h}, \lambda_h J_r^{g,h}) \in U_B$ and $(\rho_e I_l^{e,f}, \lambda_f I_r^{e,f}) = (\rho_g J_l^{g,h}, \lambda_h J_r^{g,h})$, then we show that $\overline{I^{e,f}} = \overline{J^{g,h}}$. We have remarked that

$$(\rho_e, \lambda_e)\, \widetilde{\mathcal{R}}_{\overline{B}}\, (\rho_e I_l^{e,f}, \lambda_f I_r^{e,f})\, \widetilde{\mathcal{L}}_{\overline{B}}\, (\rho_f, \lambda_f).$$

Similarly, $(\rho_g, \lambda_g)\, \widetilde{\mathcal{R}}_{\overline{B}}\, (\rho_g J_l^{g,h}, \lambda_h J_r^{g,h})\, \widetilde{\mathcal{L}}_{\overline{B}}\, (\rho_h, \lambda_h)$. Thus $(\rho_e, \lambda_e)\, \mathcal{R}\, (\rho_g, \lambda_g)$. Since B is isomorphic to $\overline{B}$, we have that $e \,\mathcal{R}\, g$, and so $e \,\mathcal{D}\, g$. Similarly, $f \,\mathcal{D}\, h$. Hence,

$$\mathrm{dom}\overline{I^{e,f}} = \mathrm{dom}\overline{J^{g,h}} = \alpha Y, \qquad \mathrm{im}\overline{I^{e,f}} = \mathrm{im}\overline{J^{g,h}} = \beta Y,$$

where $e, g \in B_\alpha$ and $f, h \in B_\beta$.

Let $\xi \in \mathrm{dom}\overline{I^{e,f}}$ and $x \in B_\xi \cap \langle e\rangle$. Then there exists $y \in B$ such that $x = eye$. Since $e \,\mathcal{D}\, g$, we have that $x = eye \,\mathcal{D}\, gyg \leq g$, and so $gyg \in B_\xi$. Observe that

$$L_y \rho_e I_l^{e,f} = L_{eye} I_l^{e,f} = L_z \qquad \big((eye, z) \in I^{e,f}\big)$$

and

$$L_y \rho_g J_l^{g,h} = L_{gyg} J_l^{g,h} = L_u \qquad \big((gyg, u) \in J^{g,h}\big).$$

As $\rho_e I^{e,f} = \rho_g J^{g,h}$, we have that $z \,\mathcal{L}\, u$. Then $\xi\overline{I^{e,f}} = \xi\overline{J^{g,h}} = \eta$, where $z, u \in B_\eta$. Thus $\overline{I^{e,f}} = \overline{J^{e,f}}$, and so χ is well-defined.

In the following, we show that χ is a morphism. According to Lemma 4.3 of [6], we have that

$$(\rho_e I_l^{e,f}, \lambda_f I_r^{e,f})(\rho_g J_l^{g,h}, \lambda_h J_r^{g,h}) = (\rho_z K_l^{z,w}, \lambda_w K_r^{z,w}),$$

where $(z, fgf) \in I^{e,f}$, $(gfg, w) \in J^{g,h}$ and

$$K^{z,w} = (I^{e,f}(\theta_{gfg}|_{\langle fgf \rangle})J^{g,h}) \cap (\langle z \rangle \times \langle w \rangle).$$

Hence, it is sufficient to show that

$$\overline{I^{e,f}} \circ \overline{J^{g,h}} = \overline{K^{z,w}}.$$

We have remarked that $\text{dom}\overline{I^{e,f}} = \alpha Y$, $\text{im}\overline{I^{e,f}} = \beta Y$. Similarly, $\text{dom}\overline{J^{g,h}} = \gamma Y$, $\text{im}\overline{J^{e,f}} = \delta Y$, where $g \in B_\gamma$, $h \in B_\delta$. Thus

$$\begin{aligned}
\xi \in \quad & \text{dom}\overline{I^{e,f}} \circ \overline{J^{g,h}} \\
\Leftrightarrow \xi \in & \ (\text{ im}\overline{I^{e,f}} \cap \text{ dom}\overline{J^{g,h}})\overline{I^{e,f}}^{-1} \\
\Leftrightarrow \xi \in & \ (\beta\gamma)Y\overline{I^{e,f}}^{-1} \\
\Leftrightarrow \exists x \in & \ B_\xi \cap \langle e \rangle, \ (x, y) \in I^{e,f}, y \in B_\mu, \mu \leq \beta\gamma \\
\Leftrightarrow \xi \leq & \ \tau \text{ where } z \in B_\xi \text{ since } (z, fgf) \in I^{e,f} \\
\Leftrightarrow \xi \in & \ \text{ dom}\,\overline{K^{z,w}}.
\end{aligned}$$

Returning to the above, let $\xi \in \text{ dom}(\overline{I^{e,f}} \circ \overline{J^{g,h}})$ and $x \in B_\xi$, so that $y \in B_{\xi\overline{I^{e,f}}=\mu}$. Now $(gfg)gfyfg(gfg) \in B_\mu$ (as $\mu \leq \beta\gamma$) and $((gfg)gfyfg(gfg), whw) \in J^{g,h}$, so $\mu\overline{J^{g,h}} = \nu$, where $whw \in B_\gamma$. Thus $\xi\overline{I^{e,f}} \circ \overline{J^{g,h}} = \nu = \xi\overline{K^{z,w}}$, as $(x, whw) \in K^{z,w}$. So

$$\overline{I^{e,f}} \circ \overline{J^{g,h}} = \overline{K^{z,w}}.$$

□

We omit the proof of the next lemma as it follows from Lemma 4.23.

Lemma 4.22. *The set of the elements of U_B, whose images under χ are idempotent, forms a fundamental weakly $\overline{B}$-superabundant subsemigroup of U_B with (C) and (WIC). We denote it by Q_2.*

Lemma 4.23. *The semigroup Q_2 coincides with Q_1.*

Proof. It is easy to see that $Q_1 \subseteq Q_2$. We show that $Q_2 \subseteq Q_1$. Suppose that $(\rho_e I^{e,f}, \lambda_f I_r^{e,f}) \in Q_2$. Then $e \mathcal{D} f$ and $\overline{I^{e,f}}$ is the identity map from αY to αY, where $e \in B_\alpha$. So for any $(u, v) \in I^{e,f}$, $u \mathcal{D} v$. Now let $J^{ef,ef} =$

$(\theta_e|_{\langle ef\rangle})I^{e,f}(\theta_{ef}|_{\langle f\rangle})$. We claim that $(\rho_{ef}J_l^{ef,ef}, \lambda_{ef}J_r^{ef,ef}) = (\rho_e I^{e,f}, \lambda_f I_r^{e,f})$. For any $x \in B$, we have that

$$\begin{aligned}
L_x\rho_{ef}J_l^{ef,ef} &= L_{(efxef)J^{ef,ef}} \\
&= L_{(efxef)(\theta_e|_{\langle ef\rangle})I^{e,f}(\theta_{ef}|_{\langle f\rangle})} \\
&= L_{(e\cdot efxef\cdot e)I^{e,f}(\theta_{ef}|_{\langle f\rangle})} \\
&= L_{(e\cdot efxef\cdot e)}\Big(I^{e,f}(\theta_{ef}|_{\langle f\rangle})\Big)_l \\
&= L_{exe}\Big(I^{e,f}(\theta_{ef}|_{\langle f\rangle})\Big)_l && (e\ \mathcal{D}\ f,\ \text{Lemma 1.22}) \\
&= L_{(exe)I^{e,f}(\theta_{ef}|_{\langle f\rangle})} \\
&= L_{y\theta_{ef}|_{\langle f\rangle}} && ((exe,y)\in I^{e,f}) \\
&= L_{efyef} \\
&= L_{yef} \\
&= L_{y\cdot f\cdot ef} && (y\in\langle f\rangle) \\
&= L_{yf} && (f\ \mathcal{D}\ e) \\
&= L_y \\
&= L_{exe}I_l^{e,f} && \Big((exe,y)\in I^{e,f}\Big) \\
&= L_x\rho_e I_l^{e,f}.
\end{aligned}$$

Thus, $\rho_{ef}J_l^{ef,ef} = \rho_e I_l^{e,f}$. Dually, we obtain that $\lambda_{ef}J_r^{ef,ef} = \lambda_f I_r^{e,f}$. Hence $(\rho_{ef}J_l^{ef,ef}, \lambda_{ef}J_r^{ef,ef}) = (\rho_e I_l^{e,f}, \lambda_f I_r^{e,f})$. In addition, by the proof of Lemma 4.21, we have $\overline{J^{ef,ef}} = \overline{I^{e,f}}$. Consequently, $Q_2 \subseteq Q_1$. □

Lemma 4.24. *[17] If S is a weakly B-orthodox semigroup with (WIC), then the map $\theta : S \to U_B$ given by $a\theta = (\alpha_a, \beta_a)$,where α_a and β_a are defined as Section 4.3, is a strongly B-admissible morphism with kernel μ_B. Moreover, $\theta|_B : B \to \overline{B}$ is an isomorphism.*

Consequently, we have:

Theorem 4.25. *If S is a weakly B-superabundant semigroup with (WIC) and (C), then the map $\phi : S \to U_B$ in Lemma 4.24 has image contained in Q_1.*

Proof. Suppose that $a \in S$ and $e \in \widetilde{H}_a \cap B$. In view of [6], we know $a\phi = (\alpha_a, \beta_a) = (\rho_{a^\dagger}I_l^{a^\dagger,a^*}, \lambda_{a^*}I_r^{a^\dagger,a^*})$. Specially, we can use e instead of $a^\dagger$ and a^* so that we obtain $a\phi = (\rho_e I_l^{e,e}, \lambda_e I_r^{e,e})$. In addition, for any $x \in \langle e \rangle$, we have that $xa = ay$, where $(x,y) \in I^{e,e}$. As

$$x = xe \,\widetilde{\mathcal{R}}_B\, xa = ay \,\widetilde{\mathcal{L}}_B\, ey = y.$$

We have that $x \,\widetilde{\mathcal{D}}_B\, y$. It follows from Lemma 3.10 that $x \,\mathcal{D}\, y$ in B. Thus $a\phi = (\rho_e I_l^{e,e}, \lambda_e I_r^{e,e})$ is contained in Q_1. □

4.5 A fundamental weakly $\overline{B}$-superabundant subsemigroup of S_B

The aim of this section is to remove the idempotent connected condition from the results of previous sections. We stress that to do so Gomes and Gould [17] used a completely fresh technique to construct from a band B a weakly B-orthodox subsemigroup S_B of $\mathcal{OP}(B^1/\mathcal{L}) \times \mathcal{OP}^*(B^1/\mathcal{R})$, with the property that any fundamental weakly $\overline{B}$-orthodox semigroup is a subsemigroup of S_B, where $\overline{B}$ is isomorphic to B. As a consequence, the fundamental semigroup U_B constructed in last section, satisfying (C) and (WIC) is embedded into S_B.

To define S_B, we give some notation. For a set X, an equivalence κ on X and $\gamma : X/\kappa \to X/\kappa$, the relation $\bar{\gamma}$ is defined by

$$\bar{\gamma} = \{(x,y) \in X \times X : y \in [x]\gamma\},$$

where $[x]$ is the equivalence class containing x.

We put

$$S_B = \{(\alpha,\beta) \in \mathcal{O}^1(B) : \text{for all } x \in B^1,\ x\bar{\alpha} \in L_x\alpha \text{ and } x\bar{\beta} \in R_x\beta,$$

$$\text{we have } \beta\lambda_x = \lambda_{x\bar{\alpha}}\beta\lambda_x \text{ and } \alpha\rho_x = \rho_{x\bar{\beta}}\alpha\rho_x\},$$

where

$$\mathcal{O}^1(B) = \{(\alpha,\beta) \in \mathcal{OP}(B^1/\mathcal{L}) \times \mathcal{OP}^*(B^1/\mathcal{R}) : \operatorname{Im}\alpha \subseteq B/\mathcal{L}, \operatorname{Im}\beta \subseteq B/\mathcal{R}\}.$$

Lemma 4.26. *[17] The set S_B is weakly $\overline{B}$-orthodox and is $\overline{B}$-fundamental, where $\overline{B} = \{(\rho_e,\lambda_e) : e \in B\}$.*

We remark that if $(\alpha,\beta)\in S_B$, $u\in L_1\alpha$ and $v\in R_1\beta$, then

$$(\alpha,\beta)\ \widetilde{\mathcal{L}}_{\overline{B}}\ (\rho_u,\lambda_u) \text{ and } (\alpha,\beta)\ \widetilde{\mathcal{R}}_{\overline{B}}\ (\rho_v,\lambda_v).$$

In the following we will construct a weakly $\overline{B}$-superabundant subsemigroup of S_B, that is fundamental and satisfies (C). Put

$$N_B=\{(\alpha,\beta)\in S_B : \text{there exists } e\in L_1\alpha\cap R_1\beta\cap B \text{ such that for all } x\in B^1, x\bar{\alpha}\in L_x\alpha \text{ and } x\bar{\beta}\in R_x\beta, \text{ we have } x\bar{\alpha}\ \mathcal{D}\ exe\ \mathcal{D}\ x\bar{\beta}\}.$$

Lemma 4.27. *The set N_B is a subsemigroup of S_B containing the band of idempotents $\overline{B}=\{(\rho_e,\lambda_e): e\in B\}$.*

Proof. To show that N_B is a subsemigroup of S_B, it is sufficient to show that N_B is closed under the multiplication. We suppose that (α,β), $(\gamma,\delta)\in N_B$. Then there exists $e\in L_1\alpha\cap R_1\beta\cap B$ such that for all $x\in B^1, x\bar{\alpha}\in L_x\alpha$ and $x\bar{\beta}\in R_x\beta$, we have that $x\bar{\alpha}\ \mathcal{D}\ exe\ \mathcal{D}\ x\bar{\beta}$. Also, there exists $f\in L_1\gamma\cap R_1\delta\cap B$ such that for all $x\in B^1$, $x\bar{\gamma}\in L_x\gamma$ and $x\bar{\delta}\in R_x\delta$, we have that $x\bar{\gamma}\ \mathcal{D}\ fxf\ \mathcal{D}\ x\bar{\delta}$. We consider the product of (α,β) and (γ,δ). Observe that

$$L_1\alpha\gamma=L_e\gamma=L_{e\bar{\gamma}}, \qquad e\bar{\gamma}\ \mathcal{D}\ fef\ \mathcal{D}\ e\bar{\delta},$$

and

$$R_1\delta\beta=R_f\beta=R_{f\bar{\beta}}, \qquad f\bar{\beta}\ \mathcal{D}\ efe\ \mathcal{D}\ f\bar{\alpha}.$$

Since $efe\ \mathcal{D}\ fef$, we have that

$$L_1\alpha\gamma=L_{(f\bar{\beta})(e\bar{\gamma})}, \quad R_1\delta\beta=R_{(f\bar{\beta})(e\bar{\gamma})}, \text{ and } (f\bar{\beta})(e\bar{\gamma})\ \mathcal{D}\ ef.$$

So, $(f\bar{\beta})(e\bar{\gamma})\in L_1\alpha\gamma\cap R_1\delta\beta$. Now, we fix the choices of $f\bar{\beta}$ and $e\bar{\gamma}$.

For all $x\in B^1$, we have that $x\overline{\alpha\gamma}\in L_x\alpha\gamma$ and $x\overline{\delta\beta}\in R_x\delta\beta$. Hence,

$$x\overline{\alpha\gamma}\ \mathcal{L}\ (x\bar{\alpha})\bar{\gamma}\ \mathcal{D}\ f(x\bar{\alpha})f\ \mathcal{D}\ fexef$$

and

$$x\overline{\delta\beta}\ \mathcal{R}\ (x\bar{\delta})\bar{\beta}\ \mathcal{D}\ e(x\bar{\delta})e\ \mathcal{D}\ efxef.$$

As $(f\bar{\beta})(e\bar{\gamma})\ \mathcal{D}\ ef$, we obtain that

$$x\overline{\alpha\gamma}\ \mathcal{D}\ (f\bar{\beta})(e\bar{\gamma})x(f\bar{\beta})(e\bar{\gamma})\ \mathcal{D}\ x\overline{\delta\beta}.$$

Thus, $(\alpha\gamma, \delta\beta) \in N_B$, and so N_B forms a semigroup.

Finally, we should show that N_B contains the band of idempotents $\overline{B}$. For any $e \in B$, $(\rho_e, \lambda_e) \in \overline{B}$. Clearly, $e \in L_1\rho_e \cap R_1\lambda_e \cap B$. For all $x \in B^1$, $x\overline{\rho_e} \in L_x\rho_e$ and $x\overline{\lambda_e} \in R_x\lambda_e$, we have that $x\overline{\rho_e}\ \mathcal{L}\ xe\ \mathcal{D}\ ex\ \mathcal{R}\ x\overline{\lambda_e}$. It follows that $x\overline{\rho_e}\ \mathcal{D}\ exe\ \mathcal{D}\ x\overline{\lambda_e}$. Hence, $(\rho_e, \lambda_e) \in N_B$. Consequently, $\overline{B} \subseteq N_B$. □

Further information about S_B is obtained from the following result.

Theorem 4.28. *The semigroup N_B is a fundamental weakly $\overline{B}$-superabundant subsemigroup of S_B with (C).*

Proof. For any $(\alpha, \beta) \in N_B$, there exists $e \in L_1\alpha \cap R_1\beta \cap B$. In view of the remark following Lemma 4.26, we have that $(\alpha, \beta)\ \widetilde{\mathcal{H}}_{\overline{B}}\ (\rho_e, \lambda_e)$. Hence, N_B is a weakly $\overline{B}$-superabundant subsemigroup of S_B.

Indeed N_B is $\overline{B}$-fundamental with (C) as $\overline{B} \subseteq N_B$. □

We now want to make full use of the approach of OG_1 to determine a subsemigroup of S_B that is fundamental weakly $\overline{B}$-superabundant with (C). But in view of the fact we no longer have any idempotent connected condition, we can not find a useful mapping from S_B to the Munn semigroup T_Y to get this result, where B is a semilattice Y of rectangular bands B_α $(\alpha \in Y)$. With this in mind, we decide to consider a mapping from S_B to C_Y, where C_Y is a fundamental Ehresmann semigroup with (C). We refer the reader to [18] to acquire more details about C_Y. Here we provide some general facts about C_Y. We have

$$C_Y = \{(\alpha, \beta) \in \mathcal{O}_1(Y^1) \times \mathcal{O}_1^*(Y^1) : \operatorname{Im}\alpha, \operatorname{Im}\beta \subseteq Y \text{ and } \forall x \in Y^1,$$

$$\rho_{x\alpha} \leqslant \beta\rho_x\alpha \text{ and } \rho_{x\beta} \leqslant \alpha\rho_x\beta\}$$

where for any $x \in Y$, $\rho\colon Y^1 \to Y$ is the order preserving map given by $\xi\rho_x = \xi x = x\xi$ and for $\gamma, \delta \in \mathcal{OP}(Y^1), \gamma \leqslant \delta$ means that $y\gamma \leqslant y\delta$ for all $y \in Y^1$. From [18], C_Y is a fundamental Ehresmann semigroup with (C) and is isomorphic to $\mathcal{S}_Y$, where

$$\mathcal{S}_Y = \{(\alpha, \beta) \in \mathcal{O}_1(Y^1) \times \mathcal{O}_1^*(Y^1) : \operatorname{Im}\alpha, \operatorname{Im}\beta \subseteq Y \text{ and } \forall x \in Y^1,$$

$$\beta\rho_x = \rho_{x\alpha}\beta\rho_x \text{ and } \alpha\rho_x = \rho_{x\beta}\alpha\rho_x\}$$

and the set of distinguished idempotents of $\mathcal{S}_Y$ is $\overline{Y} = \{(\rho_\xi, \lambda_\xi) : \xi \in Y\}$.

We define a mapping $\phi\colon S_B \to \mathcal{S}_Y$ by the rule that for any $(\alpha, \beta) \in S_B$

$$(\alpha, \beta)\phi = (\alpha', \beta'),$$

where (α', β') is a pair of mappings from Y^1 to Y having the property that for any $x \in B^1$ with $x \in B^1_\xi$, $x\bar{\alpha} \in L_x\alpha$ and $x\bar{\beta} \in R_x\beta$, we have that $x\bar{\alpha} \in B^1_{\xi\alpha'}$ and $x\bar{\beta} \in B^1_{\xi\beta'}$.

Lemma 4.29. *The mapping ϕ is a morphism.*

Proof. Certainly, for any $(\alpha, \beta) \in S_B$, we have that Im α', Im $\beta' \subseteq Y$. To show that ϕ is well-defined, suppose that $\xi \in Y^1$ and $x \in B^1_\xi$. Then for any $x\bar{\alpha} \in L_x\alpha$ and $x\bar{\beta} \in R_x\beta$, we have that $\beta\lambda_x = \lambda_{x\bar{\alpha}}\beta\lambda_x$ and $\alpha\rho_x = \rho_{x\bar{\beta}}\alpha\rho_x$. For any $\kappa \in Y^1$ and $y \in B^1_\kappa$, we have that

$$R_y\beta\lambda_x = R_y\lambda_{x\bar{\alpha}}\beta\lambda_x \quad \text{ and } \quad L_y\alpha\rho_x = L_y\rho_{x\bar{\beta}}\alpha\rho_x,$$

which means $R_y\beta\lambda_x$ and $R_y\lambda_{x\bar{\alpha}}\beta\lambda_x$ are in the same $\mathcal{D}$-class and the same is true for $L_y\alpha\rho_x$ and $L_y\rho_{x\bar{\beta}}\alpha\rho_x$. Thus, $\kappa\beta'\rho_\xi = \kappa\rho_{\xi\alpha'}\beta'\rho_\xi$ and $\kappa\alpha'\rho_\xi = \kappa\rho_{\xi\beta'}\alpha'\rho_\xi$. Hence $(\alpha', \beta') \in S_Y$.

Clearly, ϕ is a morphism. □

Lemma 4.30. *The set of the elements of S_B whose images under ϕ are distinguished idempotents of $\mathcal{S}_Y$, forms a fundamental weakly $\overline{B}$-superabundant subsemigroup of S_B with (C). We denote it by N'_B.*

Proof. Since ϕ is a morphism and the distinguished idempotents of $\mathcal{S}_Y$ form a semilattice, it follows that N'_B is closed. Also, it is easy to see that $\overline{B} \subseteq N'_B$, which implies that N'_B has (C) and is fundamental.

It remains to show that N'_B is weakly $\overline{B}$-superabundant. We assume that $(\alpha, \beta) \in N'_B$. Then $(\alpha', \beta') \in \overline{E}(\mathcal{S}_Y)$. It follows that $\alpha' = \beta'$. Specially, $1\alpha' = 1\beta'$, and so if $1\bar{\alpha} \in B_{1\alpha'}$ and $1\bar{\beta} \in B_{1\beta'}$, then, $1\bar{\alpha} \;\mathcal{D}\; 1\bar{\beta}$. Therefore, there exists an idempotent $e \in L_{1\bar{\alpha}} \cap R_{1\bar{\beta}}$, that is, $e \in L_1\alpha \cap R_1\beta$. By the remark following Lemma 4.26, we obtain that $(\alpha, \beta) \;\widetilde{\mathcal{H}}_B\; (\rho_e, \lambda_e)$. Consequently, N'_B is weakly $\overline{B}$-superabundant. □

To reveal the relationship between N_B and N'_B, we have:

Lemma 4.31. *The semigroup N_B coincides with N'_B.*

Proof. We begin by showing that $N_B \subseteq N'_B$. Suppose that $(\alpha, \beta) \in N_B$. Then there exists $e \in L_1\alpha \cap R_1\beta \cap B$ such that for all $x \in B^1, x\bar{\alpha} \in L_x\alpha$ and $x\bar{\beta} \in R_x\beta$, we have that $x\bar{\alpha}\ \mathcal{D}\ exe\ \mathcal{D}\ x\bar{\beta}$. It follows that $\alpha' = \beta'$ and for any $\varepsilon \in Y^1$, $\varepsilon\alpha' = (1\alpha')\varepsilon$, that is, $\alpha' = \rho_{1\alpha'}$. Hence, $\alpha' = (\alpha')^2$, and so (α', β') is an idempotent of $\mathcal{S}_Y$. Since there exists an idempotent $e \in L_1\alpha \cap R_1\beta$ with $e \in B_\xi$, it follows that $\varepsilon\alpha' = \rho_\xi\varepsilon = \varepsilon\rho_\xi$. Therefore, $(\alpha', \beta') = (\rho_\xi, \lambda_\xi) \in \bar{E}(\mathcal{S}_Y)$. Consequently, $N_B \subseteq N'_B$.

Conversely, suppose that $(\alpha, \beta) \in N'_B$. Then there exists an idempotent $(\rho_\tau, \lambda_\tau)$ such that $(\rho_\tau, \lambda_\tau) = (\alpha', \beta')$ which implies that $\alpha' = \beta'$ and there exists $e \in L_1\alpha \cap R_1\beta$ with $e \in B_\tau$. For any $x \in B^1$ with $x \in B_\xi$, $x\bar{\alpha} \in L_x\alpha$ and $x\bar{\beta} \in R_x\beta$, we have that $x\bar{\alpha}, x\bar{\beta} \in B_{\xi\rho_\tau}$. Thus, $x\bar{\alpha}\ \mathcal{D}ex\ \mathcal{D}\ x\bar{\beta}$. Hence $N'_B \subseteq N_B$. ☐

Lemma 4.32. *[17] Let S be a weakly B-orthodox semigroup. Then $\theta : S \to S_B$ given by*

$$a\theta = (\alpha_a, \beta_a),$$

where α_a and β_a are defined in Section 4.3, is a strongly admissible morphism with kernel μ_B. Moreover, $\theta|_B : B \to \overline{B}$ is an isomorphism.

Corollary 4.33. *Let S be a weakly B-orthodox semigroup and let K be a subsemigroup of S_B containing $\overline{B}$. Then K is a weakly $\overline{B}$-orthodox semigroup. If K contains the image of S under θ given as in Lemma 4.32, then $\theta : S \to K$ is a strongly admissible morphism with kernel μ_B. Moreover, $\theta|_B : B \to \overline{B}$ is an isomorphism.*

Lemma 4.32 stated that there exists a strongly B-admissible morphism $\theta : S \to S_B$. Consequently, we can improve on this fact to get a similar result for weakly B-superabundant semigroup with (C) as follows.

Theorem 4.34. *If S is a weakly B-superabundant semigroup with (C), then the map $\theta : S \to S_B$ in Lemma 4.32 has the image contained in N_B.*

Proof. Suppose that $a \in S$. Firstly, we need to find an idempotent which belongs to $L_1\alpha_a$ and $R_1\beta_a$. Of course, we have $L_1\alpha_a = L_{a^*}$ and $R_1\beta_a = R_{a^\dagger}$. Also, we have $a^* \mathcal{L} a^\dagger a^* \mathcal{R} a^\dagger$ since S is weakly B-superabundant. It follows that

$$a^\dagger a^* \in L_1\alpha_a \cap R_1\beta_a.$$

According to definitions of α_a and β_a, $L_x\alpha_a = L_{(xa)^*}$ and $R_x\beta_a = R_{(ax)^\dagger}$ for all $x \in B^1$. So $x\overline{\alpha_a} \mathcal{L} (xa)^*$ and $x\overline{\beta_a} \mathcal{R} (ax)^\dagger$. Then

$$x\overline{\alpha_a}\, \mathcal{L}\, (xa)^*\, \widetilde{\mathcal{L}}_B\, xa\, \widetilde{\mathcal{R}}_B\, xa^\dagger\, \widetilde{\mathcal{R}}_B\, xa^\dagger a^*\, \mathcal{D}\, a^\dagger a^* x a^\dagger a^*\, \mathcal{D}\, a^\dagger a^* x\, \widetilde{\mathcal{L}}_B\, ax\, \widetilde{\mathcal{R}}_B\, (ax)^\dagger\, \mathcal{R}\, x\overline{\beta_a}.$$

It follows that $x\overline{\alpha_a}\ \widetilde{\mathcal{D}}_B\ a^\dagger a^* x a^\dagger a^*\ \widetilde{\mathcal{D}}_B\ x\overline{\beta_a}$. By Lemma 3.10, we have

$$x\overline{\alpha_a}\ \mathcal{D}\ a^\dagger a^* x a^\dagger a^*\ \mathcal{D}\ x\overline{\beta_a} \text{ in } B.$$

Hence $a\theta = (\alpha_a, \beta_a) \in N_B$. □

Chapter 5

Structure theorems for weakly B-orthodox semigroups

The goal of this chapter is to provide structure theorems for weakly B-orthodox semigroups, where B is a band. We shall focus on providing a description of a weakly B-orthodox semigroup S as a spined product of a weakly B-orthodox semigroup S_B and S/γ_B, where S_B is the fundamental weakly $\overline{B}$-orthodox semigroup constructed in Chapter 4 and γ_B is the analogue of the least inverse congruence on an orthodox semigroup. This result is analogous to the Hall-Yamada theorem for orthodox semigroups.

Throughout this chapter Green's relation $\mathcal{D}$ always refer to B. Here B denotes a band. To avoid ambiguity, if $\mathcal{K}$ is a relation on a semigroup S, then we will sometimes use $\mathcal{K}(S)$ to denote the relation on S.

5.1 The least admissible Ehresmann congruence

In Chapter 1 we mentioned that there exists the least inverse congruence γ on any orthodox semigroup. As an analogue of the least inverse congruence, many articles have discussed the least $B/\mathcal{D}$-ample congruence δ_B on any weakly B-orthodox semigroup with certain idempotent connected condition, as mentioned in Chapter 2. Here, we concentrate on a correspondence congruence γ_B on weakly B-orthodox semigroups. Such semigroups do not satisfy any idempotent connected condition.

The aim of this section is to find a closed form for γ_B, where γ_B is the least admissible Ehresmann congruence on a weakly B-orthodox semigroup S. For any $a, b \in S$, we define

$$a \, \delta_B \, b \text{ if and only if } a = a^\dagger b a^* \text{ and } b = b^\dagger a b^*,$$

for some $a^\dagger, a^*, b^\dagger, b^* \in B$ with $a^\dagger \, \widetilde{\mathcal{R}}_B \, a \, \widetilde{\mathcal{L}}_B \, a^*$, $b^\dagger \, \widetilde{\mathcal{R}}_B \, b \, \widetilde{\mathcal{L}}_B \, b^*$.

Lemma 5.1. *If $a \, \delta_B \, b$ in a weakly B-orthodox semigroup S, then $a^\dagger \; \mathcal{D} \; b^\dagger$, $a^* \; \mathcal{D} \; b^*$, for any $a^\dagger$, $b^\dagger$, a^* and b^*. In particular, for any $e, f \in B$, $e \, \delta_B \, f$ if and only if $e \; \mathcal{D} \; f$.*

Proof. Suppose that $a, b \in S$ are such that $a \, \delta_B \, b$. Then $a = a^\dagger b a^*$, $b = b^\dagger a b^*$, for some $a^\dagger \in \widetilde{R}_a \cap B$, $a^* \in \widetilde{L}_a \cap B$, $b^\dagger \in \widetilde{R}_b \cap B$ and $b^* \in \widetilde{L}_b \cap B$. From $a = a^\dagger b a^*$, we can deduce that

$$a^\dagger b^\dagger a = a^\dagger b^\dagger a^\dagger b a^* = a^\dagger b^\dagger a^\dagger \cdot b^\dagger \cdot b a^* = a^\dagger b^\dagger b a^* = a^\dagger b a^* = a.$$

Since $a \, \widetilde{\mathcal{R}}_B \, a^\dagger$, we have $a^\dagger b^\dagger a^\dagger = a^\dagger$. By a similar argument, we get $b^\dagger a^\dagger b^\dagger = b^\dagger$. Thus $a^\dagger \, \mathcal{D} \, b^\dagger$. Dually, we have $a^* \, \mathcal{D} \, b^*$. If $a^\diamond$ is another idempotent in $\widetilde{R}_a \cap B$ and $b^\diamond$ is another idempotent in $\widetilde{R}_b \cap B$, then $a^\dagger \; \mathcal{R} \; a^\diamond$ and $b^\dagger \; \mathcal{R} \; b^\diamond$. Together with $a^\dagger \, \mathcal{D} \, b^\dagger$, we obtain that $a^\diamond \, \mathcal{D} \, b^\diamond$. Dually, we have $a^\circ \, \mathcal{D} \, b^\circ$ for any $a^\circ \in \widetilde{L}_a \cap B$ and $b^\circ \in \widetilde{L}_b \cap B$. □

Lemma 5.2. *Let S be a weakly B-orthodox semigroup and let $\theta : S \to T$ be an admissible morphism, where T is an Ehresmann semigroup with respect to $B\theta$. Then $\delta_B \subseteq Ker \; \theta$.*

Proof. Suppose that $a, b \in S$ with $a \, \delta_B \, b$. Then $a = a^\dagger b a^*$ and $b = b^\dagger a b^*$, for some $a^\dagger, a^*, b^\dagger, b^* \in B$ with $a^\dagger \, \widetilde{\mathcal{R}}_B \, a \, \widetilde{\mathcal{L}}_B \, a^*$, $b^\dagger \, \widetilde{\mathcal{R}}_B \, b \, \widetilde{\mathcal{L}}_B \, b^*$. According to Lemma 5.1, we have $a^*\theta \; \mathcal{D}(B\theta) \; b^*\theta$ and $a^\dagger\theta \; \mathcal{D}(B\theta) \; b^\dagger\theta$ since θ is an admissible morphism. But, $B\theta$ is a semilattice, $a^*\theta = b^*\theta$ and $a^\dagger\theta = b^\dagger\theta$ so that $a\theta = (a^\dagger b a^*)\theta = a^\dagger\theta b\theta a^*\theta = b^\dagger\theta b\theta b^*\theta = (b^\dagger b b^*)\theta = b\theta$. Thus, $(a, b) \in \mathrm{Ker}\theta$. □

Corollary 5.3. *If S is a weakly B-orthodox semigroup, then $\delta_B \subseteq \gamma_B$.*

Our aim is to show that, under certain conditions, $\delta_B = \gamma_B$ holds on weakly B-orthodox semigroups.

Lemma 5.4. *Let S be a weakly B-orthodox semigroup. Then the relation δ_B defined above is an equivalence relation.*

Proof. Clearly, δ_B is reflexive and symmetric. To show that δ_B is transitive we assume that $a, b, c \in S$ such that $a\,\delta_B\,b$ and $b\,\delta_B\,c$. Then $a = a^\dagger b a^*$, $b = b^\dagger a b^*$, $b = b^\diamond c b^\circ$ and $c = c^\dagger b c^*$, for some $a^\dagger \in \widetilde{R}_a \cap B$, $a^* \in \widetilde{L}_a \cap B$, $b^\dagger, b^\diamond \in \widetilde{R}_b \cap B$, $b^*, b^\circ \in \widetilde{L}_b \cap B$, $c^\dagger \in \widetilde{R}_c \cap B$ and $c^* \in \widetilde{L}_c \cap B$. By Lemma 5.1, we obtain that

$$a^\dagger\,\mathcal{D}\,b^\dagger\,\mathcal{D}\,b^\diamond\,\mathcal{D}\,c^\dagger \text{ and } a^*\,\mathcal{D}\,b^*\,\mathcal{D}\,b^\circ\mathcal{D}\,c^*.$$

Now

$$\begin{aligned} a = a^\dagger b a^* &= a^\dagger b^\diamond c b^\circ a^* && (b = b^\diamond c b^\circ) \\ &= a^\dagger b^\diamond \cdot c^\dagger \cdot c \cdot c^* \cdot b^\circ a^* \\ &= a^\dagger c^\dagger c c^* a^* && (a^\dagger\,\mathcal{D}\,b^\diamond\,\mathcal{D}\,c^\dagger,\ a^*\,\mathcal{D}\,b^\circ\,\mathcal{D}\,c^*) \\ &= a^\dagger c a^*. \end{aligned}$$

Similarly, we get $c = c^\dagger a c^*$, and so $a\,\delta_B\,c$. Hence, δ_B is transitive. □

Lemma 5.5. *Let S be a weakly B-orthodox semigroup. If the equivalence relation δ_B defined above is a congruence on S, then it is an admissible congruence on S.*

Proof. Suppose that the relation δ_B is a congruence. We show that δ_B is an admissible congruence on S. Assume that $a \in S$, $e, f \in B$ are such that $e\,\widetilde{\mathcal{L}}_B\,a\,\widetilde{\mathcal{R}}_B\,f$ and $(a\delta_B)(k\delta_B) = a\delta_B$ for some $k \in B$. We want to show that $(e\delta_B)(k\delta_B) = e\delta_B$. From $(ak)\delta_B = a\delta_B$ and Lemma 5.1, we get $(ak)^*\,\mathcal{D}\,a^*$. Since $\widetilde{\mathcal{L}}_B$ is a right congruence, $a^*k\,\widetilde{\mathcal{L}}_B\,(ak)^*$. Again, due to $a\,\widetilde{\mathcal{L}}_B\,e$, we obtain that $e\,\mathcal{D}\,a^*$. As $\mathcal{D}$ is a congruence on B, we have that $ek\,\mathcal{D}\,a^*k$. Thus $e\,\mathcal{D}\,a^*\,\mathcal{D}\,a^*k\,\mathcal{D}\,ek$. It immediately follows from Lemma 5.1 that $e\,\delta_B\,ek$, that is, $(e\delta_B)(k\delta_B) = e\delta_B$. Hence $a\delta\,\widetilde{\mathcal{L}}_{B\delta}\,e\delta$.

An argument that is completely dual gives that $a\delta_B\,\widetilde{\mathcal{R}}_{B\delta}\,f\delta_B$. Consequently, according to Lemma 2.9, δ_B is an admissible congruence on S. □

Corollary 5.6. *If ρ is an admissible congruence on a weakly B-orthodox semigroup S satisfying that B/ρ is a semilattice, then S/ρ is an Ehresmann semigroup.*

The following lemma is an immediate consequence of Lemma 5.5, Corollary 5.6 and Corollary 5.3.

Lemma 5.7. *Let S be a weakly B-orthodox semigroup. If the equivalence relation δ_B defined above is a congruence on S, then it is an admissible Ehresmann congruence on S. Moreover, $\delta_B = \gamma_B$.*

Lemma 5.8. *If S is a weakly B-orthodox semigroup, then $\widetilde{\mathcal{H}}_B \cap \delta_B = \iota$.*

Proof. Suppose that $a, b \in S$ and $(a, b) \in \widetilde{\mathcal{H}}_B \cap \delta_B$. Then $a = a^\dagger b a^*$ for some $a^\dagger \in \widetilde{R}_a \cap B$, $a^* \in \widetilde{L}_a \cap B$. Since $a \,\widetilde{\mathcal{H}}_B\, b$, we have $a^\dagger \,\widetilde{\mathcal{R}}_B\, a \,\widetilde{\mathcal{R}}_B\, b$ and $a^* \,\widetilde{\mathcal{L}}_B\, a \,\widetilde{\mathcal{L}}_B\, b$. Thus $a = a^\dagger b a^* = b$ as required. □

5.2 A structure theorem for weakly B-orthodox semigroups

The Hall-Yamada theorem presents a construction for orthodox semigroups, that is, any orthodox semigroup S with band of idempotents B is isomorphic to the spined product of S/γ and the Hall semigroup W_B, where γ is the least inverse congruence on S. So far, many articles have extended the Hall-Yamada theorem to weakly B-orthodox semigroups satisfying certain idempotent connected conditions [6], [46]. In Chapter 4, given a band B, we built a fundamental weakly $\overline{B}$-orthodox semigroup S_B, which is an analogue of the Hall semigroup W_B, where $\overline{B} = \{(\rho_e, \lambda_e) : e \in B\}$. In view of Lemma 4.32, Gomes and Gould found a representation from a weakly B-orthodox semigroup to S_B.

We stress that they did not attempt to produce an analogue of the Hall-Yamada structure theorem for orthodox semigroups. With this in mind we give the following lemmas for weakly B-orthodox semigroups, from which we shall obtain a very general structure theorem for weakly B-orthodox semigroups. We then proceed to show this result can be applied in a number of cases of interest.

We begin by reminding the reader that if we have semigroups S, T, H and morphisms ϕ and ψ as follows,

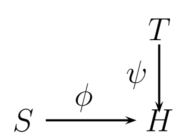

Figure 5.1: The spined product

then the *spined product* $\mathcal{S} = \mathcal{S}(S, T, \phi, \psi)$ of S and T with respect to H, ϕ and ψ is

$$\mathcal{S} = \{(s, t) \in S \times T : s\phi = t\psi\}.$$

From now on, S will be a weakly B-orthodox semigroup, γ_B is the least admissible Ehresmann congruence on S and $\gamma_{\overline{B}}$ is the least admissible Ehresmann congruence on S_B. Define a map $\psi : S/\gamma_B \to S_B/\gamma_{\overline{B}}$ by the rule that $s\gamma_B\psi = s\theta\gamma_{\overline{B}}$ for any $s \in S$, where θ is defined as in Lemma 4.32.

Lemma 5.9. *The mapping $\psi : S/\gamma_B \to S_B/\gamma_{\overline{B}}$ defined above is an admissible morphism such that $\psi|_{B/\gamma_B} : B/\gamma_B \to \overline{B}/\gamma_{\overline{B}}$ is an isomorphism, if and only if for any $a, b \in S$, $a\,\gamma_B\,b$ implies $a\theta\,\gamma_{\overline{B}}\,b\theta$ and $e\,\gamma_B\,f$ if and only if $e\theta\,\gamma_{\overline{B}}\,f\theta$ for any $e, f \in B$, where θ is defined as in Lemma 4.32.*

Proof. Necessity. Suppose that $a, b \in S$ and $a\,\gamma_B\,b$. It immediately follows that $a\gamma_B\psi = b\gamma_B\psi$. By the definition of ψ, we have $a\theta\gamma_{\overline{B}} = a\gamma_B\psi = b\gamma_B\psi = b\theta\gamma_{\overline{B}}$, that is, $a\theta\,\gamma_{\overline{B}}\,b\theta$. Consequently, $e\,\gamma_B\,f$ implies $e\theta\,\gamma_{\overline{B}}\,f\theta$ for any $e, f \in B$.
Conversely, if $e\theta\,\gamma_{\overline{B}}\,f\theta$ then by the definition of ψ, we have $e\gamma_B\psi = e\theta\gamma_{\overline{B}} = f\theta\gamma_{\overline{B}} = f\gamma_B\psi$. Since $\psi|_{B/\gamma_B}$ is an isomorphism it follows that $e\gamma_B = f\gamma_B$, that is, $e\,\gamma_B\,f$.

Sufficiency. According to the hypothesis, it is easy to see that the map $\psi : S/\gamma_B \to S_B/\gamma_{\overline{B}}$ defined by $s\gamma_B\psi = s\theta\gamma_{\overline{B}}$ for any $s \in S$ is well-defined and maps B/γ_B to $\overline{B}/\gamma_{\overline{B}}$. Since θ, $\gamma_B^\natural$ and $\gamma_{\overline{B}}^\natural$ are morphisms it follows that ψ is a morphism.

Next, we claim that ψ is an admissible morphism. Suppose that $a, b \in S$ and $a\gamma_B\ \widetilde{\mathcal{R}}_{B/\gamma_B}\ b\gamma_B$. Since $\gamma_B^\natural$ is admissible, we have

$$a^\dagger\gamma_B\ \widetilde{\mathcal{R}}_{B/\gamma_B}\ a\gamma_B\ \widetilde{\mathcal{R}}_{B/\gamma_B}\ b\gamma_B\ \widetilde{\mathcal{R}}_{B/\gamma_B}\ b^\dagger\gamma_B,$$

for any $a^\dagger \in \widetilde{R}_a \cap B$, $b^\dagger \in \widetilde{R}_b \cap B$. Certainly, $a^\dagger\gamma_B\ \mathcal{R}\ b^\dagger\gamma_B$. As B/γ_B is a semilattice, we get $a^\dagger\gamma_B = b^\dagger\gamma_B$. By the hypothesis, we obtain that $a^\dagger\theta\ \gamma_{\overline{B}}\ b^\dagger\theta$. Since θ and $\gamma_{\overline{B}}^\natural$ are admissible morphisms, we have that

$$a\gamma_B\psi = a\theta\gamma_{\overline{B}}\ \widetilde{\mathcal{R}}_{\overline{B}/\gamma_{\overline{B}}}\ (a\theta\gamma_{\overline{B}})^\dagger = a^\dagger\theta\gamma_{\overline{B}} = b^\dagger\theta\gamma_{\overline{B}} = (b\theta\gamma_{\overline{B}})^\dagger\ \widetilde{\mathcal{R}}_{\overline{B}/\gamma_{\overline{B}}}\ b\theta\gamma_{\overline{B}} = b\gamma_B\psi,$$

and so ψ preserves $\widetilde{\mathcal{R}}_{B/\gamma_B}$. Dually, ψ preserves $\widetilde{\mathcal{L}}_{B/\gamma_B}$. Thus ψ is admissible.

It remains to show that $\psi|_{B/\gamma_B} : B/\gamma_B \to \overline{B}/\gamma_{\overline{B}}$ is bijective. Certainly, $\psi|_{B/\gamma_B}$ is onto. It is sufficient to check that $\psi|_{B/\gamma_B}$ is injective. Suppose that $e\gamma_B\psi = f\gamma_B\psi$ for any $e\gamma_B$, $f\gamma_B \in B/\gamma_B$. Then by the definition of ψ we have that $e\theta\gamma_{\overline{B}} = f\theta\gamma_{\overline{B}}$. Again by the hypothesis, we succeed in obtaining $e\ \gamma_B\ f$. □

Theorem 5.10. *Let S be a weakly B-orthodox semigroup and let $S_B * S/\gamma_B$ be the spined product of S_B and S/γ_B with respect to $S_B/\gamma_{\overline{B}}$, $\gamma^{\natural}_{\overline{B}}$ and ψ, where ψ is defined above and satisfies the conditions in Lemma 5.9. Then the mapping $\phi : a \mapsto (a\theta, a\gamma_B)$ is a monomorphism from S to $S_B * S/\gamma_B$ if and only if $\mu_B \cap \gamma_B = \iota$, where $\mu_B = ker\,\theta$, and θ is defined as in Lemma 4.32.*

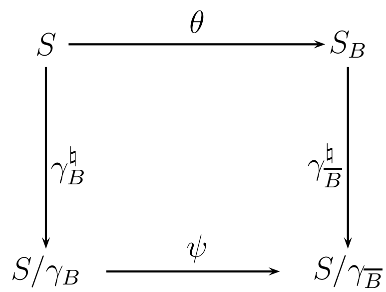

Figure 5.2: The structure of weakly B-orthodox semigroups

Proof. Clearly, ϕ is a morphism, since θ and $\gamma^{\natural}_B$ are admissible morphisms. Then ϕ is injective if and only if, for any $a, b \in S$ we have that $a\phi = b\phi$ if and only if $a = b$. That is, ϕ is injective if and only if $\ker\theta \cap \ker\gamma^{\natural}_B = \mu_B \cap \gamma_B = \iota$. $\square$

The next result is immediate.

Lemma 5.11. *Let $S_B * S/\gamma_B$ be the spined product of S_B and S/γ_B with respect to $S_B/\gamma_{\overline{B}}$, $\gamma^{\natural}_{\overline{B}}$ and ψ, where ψ is defined above and satisfies the conditions in Lemma 5.9. Then the mapping $\phi : a \mapsto (a\theta, a\gamma_B)$ is an epimorphism from a weakly B-orthodox semigroup S to $S_B * S/\gamma_B$ if and only if, if $x \in S_B$ and $(x, s\gamma_B) \in S_B * S/\gamma_B$ for some $s \in S$ then there exists $t \in S$ such that $x = t\theta$ and $t\gamma_B = s\gamma_B$, where θ is defined as in Lemma 4.32.*

We now present the main result of this section. We will see later that in some special cases, it can be significantly simplified. We make a statement that $\mu_B = \ker\theta$, where μ_B is the largest congruence contained in $\widetilde{\mathcal{H}}_B$ and θ is defined as in Lemma 4.32.

Theorem 5.12. *Let S be a weakly B-orthodox semigroup. The mapping $\phi : a \mapsto (a\theta, a\gamma_B)$ is an isomorphism from S to the spined product $S_B * S/\gamma_B$ of S_B and*

S/γ_B with respect to $S_B/\gamma_{\overline{B}}$, $\gamma_{\overline{B}}^{\natural}$ and ψ, where $\psi : S/\gamma_B \to S_B/\gamma_{\overline{B}}$ defined by $s\gamma_B\psi = s\theta\gamma_{\overline{B}}$ for any $s \in S$ is an admissible morphism and $\psi|_{B/\gamma_B} : B/\gamma_B \to \overline{B}/\gamma_{\overline{B}}$ is an isomorphism, if and only if:

(i) for any $a, b \in S$, $a\,\gamma_B\,b$ implies $a\theta\,\gamma_{\overline{B}}\,b\theta$ and $e\,\gamma_B\,f$ if and only if $e\theta\,\gamma_{\overline{B}}\,f\theta$ for any $e, f \in B$;

(ii) $\gamma_B \cap \mu_B = \iota$;

(iii) if $(x, s\gamma_B) \in S_B * S/\gamma_B$ for some $x \in S_B$ and $s \in S$, then there exists $t \in S$ such that $x = t\theta$ and $t\gamma_B = s\gamma_B$.

According to Corollary 4.33, if a weakly B-orthodox semigroup S is embedded into a subsemigroup K of S_B containing $\overline{B}$ under θ given in Lemma 4.32, then the following result is immediate.

Corollary 5.13. *Let S be a weakly B-orthodox semigroup and let K be a $\overline{B}$-full subsemigroup of S_B such that $Im\theta \subseteq K$. Let $\gamma_{\overline{B}}$ denote the least Ehresmann congruence on K. Then the mapping $\phi : a \mapsto (a\theta, a\gamma_B)$ is an isomorphism from S to the spined product $K * S/\gamma_B$ of K and S/γ_B with respect to $K/\gamma_{\overline{B}}$, $\gamma_{\overline{B}}^{\natural}$ and ψ, where $\psi : S/\gamma_B \to K/\gamma_{\overline{B}}$ defined by $s\gamma_B\psi = s\theta\gamma_{\overline{B}}$ for any $s \in S$ is an admissible morphism and $\psi|_{B/\gamma_B} : B/\gamma_B \to \overline{B}/\gamma_{\overline{B}}$ is an isomorphism, if and only if :*

(i) for any $a, b \in S$, $a\,\gamma_B\,b$ implies $a\theta\,\gamma_{\overline{B}}\,b\theta$ and $e\,\gamma_B\,f$ if and only if $e\theta\,\gamma_{\overline{B}}\,f\theta$ for any $e, f \in B$;

(ii) $\gamma_B \cap \mu_B = \iota$;

(iii) if $(x, s\gamma_B) \in K * S/\gamma_B$ for some $x \in K$ and $s \in S$, then there exists $t \in S$ such that $x = t\theta$ and $t\gamma_B = s\gamma_B$.

Finally, we shall present an important and useful lemma for the following work. In Section 5.1, we defined an equivalence relation δ_B on a weakly B-orthodox semigroup S and showed that if δ_B is a congruence then it is the least admissible Ehresmann congruence on S. In this case, i.e. when $\delta_B = \gamma_B$ and $\delta_{\overline{B}} = \gamma_{\overline{B}}$, Theorem 5.12 and Corollary 5.13 immediately simplify.

Lemma 5.14. *Let S be a weakly B-orthodox semigroup. If δ_B and $\delta_{\overline{B}}$ are congruences, then δ_B satisfies Conditions (i), (ii) and (iii) in Theorem 5.12 (resp. Corollary 5.13).*

Proof. To prove (i), suppose that $a, b \in S$ are such that $a\delta_B = b\delta_B$. By the definition of δ_B we have that $a = a^{\dagger}ba^{*}$ and $b = b^{\dagger}ab^{*}$ for some $a^{\dagger}, a^{*}, b^{\dagger}, b^{*} \in B$

with $a^\dagger\,\widetilde{\mathcal{R}}_B\, a\,\widetilde{\mathcal{L}}_B\, a^*$, $b^\dagger\,\widetilde{\mathcal{R}}_B\, b\,\widetilde{\mathcal{L}}_B\, b^*$. It follows that $a\theta = (a^\dagger b a^*)\theta = (a^\dagger\theta)(b\theta)(a^*\theta)$ and $b\theta = (b^\dagger a b^*)\theta = (b^\dagger\theta)(a\theta)(b^*\theta)$. Since $\theta : S \to S_B$ is a strongly admissible morphism, we obtain that $a^\dagger\theta\ \widetilde{\mathcal{R}}_{\overline{B}}\ a\theta\ \widetilde{\mathcal{L}}_{\overline{B}}\ a^*\theta$ and $b^\dagger\theta\ \widetilde{\mathcal{R}}_{\overline{B}}\ b\theta\ \widetilde{\mathcal{L}}_{\overline{B}}\ b^*\theta$. Thus $a\theta\ \delta_{\overline{B}}\ b\theta$.

According to Lemma 5.1 we know that $\delta_B|_B = \mathcal{D}$, and since $\theta|_B : B \to \overline{B}$ is an isomorphism in Lemma 4.32 it follows that $e\,\delta_B\, f$ if and only if $e\theta\,\delta_{\overline{B}}\, f\theta$ for any $e, f \in B$. Hence, Condition (i) holds.

Next, we show that δ_B satisfies Condition (ii). Suppose that $a, b \in S$ are such that $a\ (\delta_B \cap \mu_B)\ b$. Since $\mu_B \subseteq \widetilde{\mathcal{H}}_B$, it follows that $a\ (\delta_B \cap \widetilde{\mathcal{H}}_B)\ b$. By Lemma 5.8, we have that $a = b$, that is, $\delta_B \cap \mu_B = \iota$.

Finally, we claim that δ_B satisfies Condition (iii). Suppose that $(x, s\delta_B) \in S_B * S/\delta_B$. Then $x\delta_{\overline{B}} = s\delta_B\psi = s\theta\delta_{\overline{B}}$. Thus $x = \alpha \cdot s\theta \cdot \beta$, $s\theta = \epsilon \cdot x \cdot \varepsilon$, where $\alpha, \beta, \epsilon, \varepsilon \in \overline{B}$, $\alpha\,\widetilde{\mathcal{R}}_{\overline{B}}\, x\,\widetilde{\mathcal{L}}_{\overline{B}}\,\beta$, $\epsilon\,\widetilde{\mathcal{R}}_{\overline{B}}\, s\theta\,\widetilde{\mathcal{L}}_{\overline{B}}\,\varepsilon$. Since θ is admissible, we can take $\epsilon = s^\dagger\theta$, $\varepsilon = s^*\theta$ for some $s^\dagger \in B \cap \widetilde{R}_s$, $s^* \in B \cap \widetilde{L}_s$. Let $\alpha = g\theta$, $\beta = h\theta$ for some $g, h \in B$. Then $x = (gsh)\theta$. By Lemma 5.1, we get $g\theta\,\mathcal{D}(\overline{B})\, s^\dagger\theta$ and $h\theta\,\mathcal{D}(\overline{B})\, s^*\theta$ in $\overline{B}$. Since $\theta|_B : B \to \overline{B}$ is an isomorphism, it follows that $g\,\mathcal{D}\, s^\dagger$ and $h\,\mathcal{D}\, s^*$ in B. Thus $g\ \delta_B\ s^\dagger$ and $h\ \delta_B\ s^*$ so that $s = s^\dagger s s^*\ \delta_B\ gsh$. Hence, $(gsh)\theta = x$ and $(gsh)\delta_B = s\delta_B$. Condition (iii) holds. □

5.3 Weakly B-orthodox semigroups with (N)

We recall that a band B is called *normal* if $xyzt = xzyt$ for all $x, y, z, t \in B$. Let S be a weakly B-orthodox semigroup with (N), that is, B is a normal band. We show that the least admissible Ehresmann congruence γ_B has the closed form δ_B given in Section 5.1. We can then apply Theorem 5.12 and Lemma 5.14 to give a structure theorem.

Lemma 5.15. *Let S be a weakly B-orthodox semigroup with (N). Then the relation δ_B defined in Section 5.1 is a congruence on S.*

Proof. In view of Lemma 5.4, it remains to show that the relation δ_B defined in Section 5.1 is compatible. It is sufficient to prove that δ_B is left compatible, because dually right compatibility will hold. Suppose that $a, b, c \in S$ and $a\,\delta_B\, b$. Then $a = a^\dagger b a^*$ and $b = b^\dagger a b^*$ for some $a^\dagger \in \widetilde{R}_a \cap B, a^* \in \widetilde{L}_a \cap B, b^\dagger \in \widetilde{R}_b \cap B$ and $b^* \in \widetilde{L}_b \cap B$. According to Lemma 5.1, we have that $a^\dagger\,\mathcal{D}\, b^\dagger$ and $a^*\,\mathcal{D}\, b^*$. Now, we can deduce that

$$\begin{aligned} ca = (ca)^\dagger ca(ca)^* &= (ca)^\dagger c(a^\dagger ba^*)(ca)^* && (a = a^\dagger ba^*) \\ &= (ca)^\dagger cc^*(a^\dagger b^\dagger a^\dagger)ba^*(ca)^* && (a^\dagger \, \mathcal{D} \, b^\dagger) \\ &= (ca)^\dagger c(c^* a^\dagger b^\dagger a^\dagger)ba^*(ca)^* \\ &= (ca)^\dagger c(c^* b^\dagger a^\dagger)ba^*(ca)^* && (\text{by } (N),\ c^* a^\dagger b^\dagger a^\dagger = c^* b^\dagger a^\dagger) \end{aligned}$$

$$\begin{aligned} &= (ca)^\dagger cb^\dagger a^\dagger ba^*(ca)^* \\ &= (ca)^\dagger cb^\dagger a^\dagger b^\dagger ba^*(ca)^* \\ &= (ca)^\dagger cb^\dagger ba^*(ca)^* && (a^\dagger \, \mathcal{D} \, b^\dagger) \\ &= (ca)^\dagger cba^*(ca)^*. \end{aligned}$$

By Lemma 2.8, $a^*(ca)^* \ \mathcal{L} \ (ca)^*$, so we get $a^*(ca)^* \in \widetilde{L}_{ca} \cap B$.

Similarly, we have that $cb = (cb)^\dagger cab^*(cb)^*$, where $b^*(cb)^* \in \widetilde{L}_{cb} \cap B$. Thus $ca \, \delta_B \, cb$, that is, δ_B is left compatible. Consequently, δ_B is a congruence on S. □

Due to Lemma 5.15 and Lemma 5.7, we immediately obtain the following result.

Lemma 5.16. *Let S be a weakly B-orthodox semigroup with (N). Then the relation δ_B defined in Section 5.1 is the least admissible Ehresmann congruence on S.*

Since S_B is constructed from a band B and $\overline{B}$ is isomorphic to B, it follows that if B is a normal band, so is $\overline{B}$. In this case, S_B is a weakly $\overline{B}$-orthodox semigroup with (N), so the relation δ_B given in Section 5.1 is also the least admissible Ehresmann congruence on S_B. We will denote it by $\delta_{\overline{B}}$.

Finally, according to Lemma 5.16, Lemma 5.14 and Theorem 5.12, we obtain a structure theorem for weakly B-orthodox semigroups with (N).

Theorem 5.17. *A weakly B-orthodox semigroup S with (N) is isomorphic to the spined product $S_B * S/\delta_B$ of S_B and S/δ_B with respect to $S_B/\delta_{\overline{B}}$, $\delta_{\overline{B}}^\natural$ and ψ, where $\psi : S/\delta_B \to S_B/\delta_{\overline{B}}$ defined by $s\delta_B\psi = s\theta\delta_{\overline{B}}$ for any $s \in S$ is an admissible morphism, and $\psi|_{B/\delta_B} : B/\delta_B \to \overline{B}/\delta_{\overline{B}}$ is an isomorphism.*

Theorem 5.17 describes S as a spined product of S_B (an analogue of the Hall semigroup) and the greatest admissible Ehresmann image of S (an analogue of the greatest inverse image of an orthodox semigroup).

5.4 Weakly B-superabundant semigroups with (C)

We first make full use of Lemma 3.10 to check the equivalence relation δ_B defined in Section 5.1 is the least admissible Ehresmann congruence on a weakly B-superabundant semigroup S with (C).

Lemma 5.18. *Let S be a weakly B-superabundant semigroup with (C). Then the relation δ_B on S defined in Section 5.1 is the least admissible Ehresmann congruence on S.*

Proof. In view of Lemma 5.4, it remains to show that the relation δ_B is compatible. We first claim that δ_B is right compatible. Dually, left compatibility holds. Suppose that $a, b, c \in S$ and $a \,\delta_B\, b$. Then $a = a^\dagger b a^*$ and $b = b^\dagger a b^*$ for some $a^\dagger \in \widetilde{R}_a \cap B, a^* \in \widetilde{L}_a \cap B, b^\dagger \in \widetilde{R}_b \cap B$ and $b^* \in \widetilde{L}_b \cap B$. By Lemma 5.1, we have that $a^\dagger \,\mathcal{D}\, b^\dagger$ and $a^* \,\mathcal{D}\, b^*$. Since S is a weakly B-superabundant semigroup, it follows from Lemma 3.10 that $a^\dagger \;\mathcal{D}\; a^*$ and $b^* \;\mathcal{D}\; b^\dagger$, so that $a^\dagger \;\mathcal{D}\; a^* \;\mathcal{D}\; b^* \;\mathcal{D}\; b^\dagger$. Obviously, we have that

$$ac = (ac)^\dagger ac(ac)^* = (ac)^\dagger (a^\dagger b a^*) c (ac)^* \qquad\qquad (a = a^\dagger b a^*).$$

By Lemma 2.8, we obtain that $(ac)^\dagger a^\dagger \;\mathcal{R}\; (ac)^\dagger$, and so $(ac)^\dagger a^\dagger \;\widetilde{\mathcal{R}}_B\; ac$. Now we just need to remove a^* from the right side of $ac = (ac)^\dagger a^\dagger b a^* c (ac)^*$. Observe that

$$\begin{aligned} ac = (ac)^\dagger a^\dagger b a^* c (ac)^* &= (ac)^\dagger a^\dagger b a^* c^\dagger c (ac)^* \\ &= (ac)^\dagger a^\dagger b ((ac)^\dagger a^\dagger b)^* (a^* c^\dagger)(c(ac)^*)^\dagger c(ac)^*. \end{aligned}$$

Next, we show that $((ac)^\dagger a^\dagger b)^* \;\mathcal{D}\; a^* c^\dagger \;\mathcal{D}\; (c(ac)^*)^\dagger$. By the Congruence Condition, we get $(ac)^\dagger \;\widetilde{\mathcal{R}}_B\; ac \;\widetilde{\mathcal{L}}_B\; a^* c \;\widetilde{\mathcal{R}}_B\; a^* c^\dagger$, and $(ac)^\dagger \;\widetilde{\mathcal{R}}_B\; ac \;\widetilde{\mathcal{L}}_B\; a^* c \;\widetilde{\mathcal{L}}_B\; (a^* c)^*$, that is, $(ac)^\dagger \;\widetilde{\mathcal{D}}_B\; a^* c^\dagger \;\widetilde{\mathcal{D}}_B\; (a^* c)^*$. It follows from Lemma 3.10 that

$$(ac)^\dagger \;\mathcal{D}\; a^* c^\dagger \;\mathcal{D}\; (a^* c)^*.$$

As $\mathcal{D}$ is a congruence on B, we have that $(ac)^\dagger a^\dagger b^\dagger \,\mathcal{D}\, a^* c^\dagger a^\dagger b^\dagger$, and so

$$((ac)^\dagger a^\dagger b)^* \;\widetilde{\mathcal{L}}_B\; (ac)^\dagger a^\dagger b \;\widetilde{\mathcal{R}}_B\; (ac)^\dagger a^\dagger b^\dagger \;\mathcal{D}\; a^* c^\dagger a^\dagger b^\dagger.$$

Since $a^* \;\mathcal{D}\; a^\dagger \;\mathcal{D}\; b^\dagger$, it follows from Lemma 1.22 that $a^* c^\dagger a^\dagger b^\dagger \;\mathcal{R}\; a^* c^\dagger$. Thus, $((ac)^\dagger a^\dagger b)^* \,\widetilde{\mathcal{D}}_B\, a^* c^\dagger$, that is, $((ac)^\dagger a^\dagger b)^* \,\mathcal{D}\, a^* c^\dagger$ by Lemma 3.10. Since $(a^* c)^* \,\mathcal{D}\, (a^* c)^\dagger$, we have that

$$(c(ac)^*)^\dagger \ \widetilde{\mathcal{R}}_B \ c(ac)^* \ \widetilde{\mathcal{L}}_B \ c^*(ac)^* \ \mathcal{D} \ c^*(a^*c)^* \ \mathcal{D} \ c^*(a^*c)^\dagger \ \mathcal{R} \ c^*a^*c^\dagger.$$

By Lemma 1.22, we obtain that $c^*a^*c^* \ \mathcal{L} \ a^*c^*$, so that $(c(ac)^*)^\dagger \ \widetilde{\mathcal{D}}_B \ a^*c^\dagger$, that is, $(c(ac)^*)^\dagger \ \mathcal{D} \ a^*c^\dagger$. Thus

$$\begin{aligned} ac &= (ac)^\dagger a^\dagger b((ac)^\dagger a^\dagger b)^*(a^*c^\dagger)(c(ac)^*)^\dagger c(ac)^* \\ &= (ac)^\dagger a^\dagger b((ac)^\dagger a^\dagger b)^*(c(ac)^*)^\dagger c(ac)^* && \Big(((ac)^\dagger a^\dagger b)^* \ \mathcal{D} \ a^*c^\dagger \ \mathcal{D} \ (c(ac)^*)^\dagger\Big) \\ &= (ac)^\dagger a^\dagger bc(ac)^* && \Big((ac)^\dagger a^\dagger \ \widetilde{\mathcal{R}}_B \ ac\Big). \end{aligned}$$

Similarly, we have that $bc = (bc)^\dagger ac(bc)^*$ for some $(bc)^\dagger \ \widetilde{\mathcal{R}}_B \ bc \ \widetilde{\mathcal{L}}_B \ (bc)^*$. Hence $ac \ \delta_B \ bc$ and so δ_B is a congruence on S. Again by Lemma 5.7, we obtain that δ_B is the least admissible Ehresmann congruence on S as required. □

At the end of this section we want to build a structure theorem for weakly B-superabundant semigroups with (C) as a spined product. But we can not use S_B to get this result because we can not ensure that S_B is a weakly $\overline{B}$-superabundant semigroup with (C) and therefore that $\gamma_{\overline{B}} = \delta_{\overline{B}}$ on S_B. With this in mind, we will make full use of the weakly $\overline{B}$-superabundant subsemigroup N_B of S_B constructed in Chapter 4.

According to Lemma 5.18, Lemma 5.14, Theorem 4.34 and Corollary 5.13, we build a structure theorem for weakly B-superabundant semigroups with (C) as follows.

Theorem 5.19. *A weakly B-superabundant semigroup S with (C) is isomorphic to the spined product $N_B * S/\delta_B$ of N_B and S/δ_B with respect to $N_B/\delta_{\overline{B}}$, $\delta_{\overline{B}}^\natural$ and ψ, where θ is defined in Lemma 4.32, $\psi : S/\delta_B \to N_B/\delta_{\overline{B}}$, defined by $s\delta\psi = s\theta\delta_{\overline{B}}$ for any $s \in S$, is an admissible morphism, and $\psi|_{B/\delta_B} : B/\delta_B \to \overline{B}/\delta_{\overline{B}}$ is an isomorphism.*

5.5 Examples

We now present a number of examples, allowing us to show that the weak idempotent connected condition (WIC) and the band of distinguished idempotents B being a normal band are not equivalent in a weakly B-orthodox semigroup S.

Let S be a weakly B-orthodox semigroup. For any element e of B we denote by $\langle e\rangle$ the principal order ideal generated by e. We recall that S satisfies the weakly idempotent connected condition (WIC) (with respect to B) if for any $a \in S$ and some $a^\dagger$, a^*, if $x \in \langle a^\dagger\rangle$ then there exists $y \in B$ with $xa = ay$; dually, if $z \in \langle a^*\rangle$ then there exists $t \in B$ with $ta = az$.

Example 5.20.

We begin by citing an example [17] of a weakly B-orthodox semigroup with (WIC). Consider the three element band $B = \{1, a, b\}$ which is a two-element right zero band with an identity adjoined. We have $U_B = \{1, a, b, c\}$ and have table

$*$	1	c	a	b
1	1	c	a	b
c	c	1	a	b
a	a	b	a	b
b	b	a	a	$b.$

Also, we can calculate that $1 * a * b * 1 = b$ and $1 * b * a * 1 = a$, so B is not a normal band. From its very construction, U_B is weakly B-orthodox with (WIC).

Example 5.21.

Now, we consider the normal band $B = \{e, f, 0\}$ which is a two-element right zero band with a zero adjoined having table

$*$	e	f	0
e	e	f	0
f	e	f	0
0	0	0	$0.$

We claim that S_B does not have (WIC).

Lemma 5.22. *[17] Any pair of the form (c_L, d_R) lies in S_B, where c_L and d_R are the constant maps in $\mathcal{OP}(B^1/\mathcal{L})$ and $\mathcal{OP}(B^1/\mathcal{R})$ with images L and R, respectively.*

Remark: In [Theorem 6.2, [17]] we obtain that for any $(\alpha, \beta) \in S_B$,

$$(\alpha, \beta) \ \widetilde{\mathcal{R}}_{\overline{B}} \ (\rho_v, \lambda_v),$$

where $v \in R_1\beta$.

Take $(c_{L_f}, d_{R_f}) \in S_B$. We have $R_1 d_{R_f} = R_f$. So we choose $v = f$,

$$\langle(\rho_v, \lambda_v)\rangle = (\rho_v, \lambda_v)\overline{B}(\rho_v, \lambda_v) = (\rho_f, \lambda_f)\overline{B}(\rho_f, \lambda_f) = \{(\rho_f, \lambda_f),\, (\rho_0, \lambda_0)\},$$

since $\overline{B} = \{(\rho_e, \lambda_e),\, (\rho_f, \lambda_f),\, (\rho_0, \lambda_0)\}$ is isomorphic to B under $b \mapsto (\rho_b, \lambda_b)$. We have

$$(\rho_0, \lambda_0)(c_{L_f}, d_{R_f}) = (c_{L_f}, \lambda_0),$$

and

$$(c_{L_f}, d_{R_f})\overline{B} = \{(c_{L_e}, d_{R_f}),\, (c_{L_f}, d_{R_f}), (\rho_0, d_{R_f})\}.$$

So $(\rho_0, \lambda_0)(c_{L_f}, d_{R_f}) \notin (c_{L_f}, d_{R_f})\overline{B}$. Thus S_B does not have (WIC).

The final example explains that not every weakly B-superabundant semigroup with (C) has (WIC).

Example 5.23.

Let $\langle a \rangle$ be a monogenic monoid generated by a and $X = \{x_i : i \in \mathbb{N}\}$ be a right zero semigroup. Set $S = \langle a \rangle \bigcup X$. We define the operation $*$ as the following table:

$*$	1	a	a^n	x_i
1	1	a	a^n	x_i
a	a	a^2	a^{n+1}	x_i
a^m	a^m	a^{m+1}	a^{m+n}	x_i
x_j	x_j	x_{j+1}	x_{j+n}	x_i.

We can easily check that S is a semigroup. It is easy to see that the $\widetilde{\mathcal{L}}_B$-classes are $\langle a \rangle$ and $\{x_i\}$ for $i \in \mathbb{N}$, and the $\widetilde{\mathcal{R}}_B$-classes are $\langle a \rangle$ and X, and so the $\widetilde{\mathcal{H}}_B$-classes are $\langle a \rangle$ and $\{x_i\}$ for $i \in N$. It follows that S is a weakly B-superabundant semigroup with distinguished band $\{1\} \bigcup X$. Moreover, S satisfies the Congruence Condition. But we yet find that $ax_1 = x_1 \neq ka$ for any $k \in S$, so S fails to have (WIC).

Chapter 6

Correspondence between algebraic structures and ordered structures

Here we survey briefly some of interesting achievements such as the Ehresmann-Schein-Nambooripad (ESN) Theorem, and its many extensions due to Armstrong [1, 2], Lawson [32], Meakin [35, 36] and Nambooripad [38–40]. These results set up a connection between algebraic structures and ordered structures.

6.1 Inverse semigroups and inductive$_1$ groupoids

The correspondence between inverse semigroups and inductive$_1$ groupoids has been widely investigated. Theorem A below is an amalgamation of Ehresmann [4], Nambooripad and Veeramony [41] and Schein [47].

As mentioned in Chapter 1, an inductive$_1$ groupoid is briefly described as a groupoid equipped with a partial order possessing restrictions and co-restrictions, and the set of idempotents forming a semilattice under the partial order. An inductive$_1$ functor is a functor between two inductive$_1$ groupoinds that is order and meet preserving.

Let G be an inductive$_1$ groupoid. We define a product $\otimes$ on G by the rule that

$$a \otimes b = (a|_{\mathbf{r}(a)\wedge\mathbf{d}(b)}) \cdot ({}_{\mathbf{r}(a)\wedge\mathbf{d}(b)}|b).$$

Then, $G\mathbf{S} = (G, \otimes)$ is an inverse semigroup (having the same partial order as G). Commonly, the product $\otimes$ is called *pseudo-product.*

Conversely, let S be an inverse semigroup with semilattice of idempotents E. We construct a category $S\mathbf{C}$ as follows:

$$\text{Ob}\ (S\mathbf{C}) = E, \quad \text{Mor}(S\mathbf{C}) = S.$$

For any $x \in S\mathbf{C}$, we put $\mathbf{d}(x) = xx'$ and $\mathbf{r}(x) = x'x$, where x' is the inverse of x in S. If $x, y \in S\mathbf{C}$ and $\mathbf{r}(x) = \mathbf{d}(y)$, then we define $x \cdot y = xy$ in $S\mathbf{C}$, where xy is the product of x and y in S. Certainly, the operation $\cdot$ is a partial binary operation and associative as a partial binary operation, so that $S\mathbf{C}$ becomes a category. In addition, it is easy to see that the inverse morphism of x in $S\mathbf{C}$ is the inverse x' of x in S so that $S\mathbf{C}$ is a groupoid. We note that there exists a natural partial order in any inverse semigroup S, defined by the rule that for any $a, b \in S$,

$$a \leq b \ \Leftrightarrow \ a = eb \quad \text{for some } e \in E.$$

We make use of the natural partial order $\leq$ on S to set up $S\mathbf{C}$ as an inductive$_1$ groupoid by defining restriction and co-restriction as follows:

$$_e|a = ea, \ \ a|_f = af,$$

where $e, f \in E$ are such that $e \leq \mathbf{d}(a) = aa'$ and $f \leq \mathbf{r}(a) = a'a$. Then $S\mathbf{C}$ is an inductive$_1$ groupoid associated to S.

Further:

Theorem A (ESN Theorem) *The category of inverse semigroups and morphisms is isomorphic to the category of inductive*$_1$ *groupoids and inductive*$_1$ *functors.*

Inverse semigroups are regular semigroups in which the idempotents form a semilattice. Consequently, we can regard the set of idempotents of a regular semigroup as a generalisation of a semilattice. This idea is precisely described in the definition of a regular biordered set, introduced by Nambooripad [38]. In that article, Nambooripad defined an inductive$_2$ groupoid, as now we demonstrate.

An ordered$_2$ groupoid $(G, \leq)$ with a regular biordered set of objects E is said to be *inductive*$_2$ if the following conditions and the duals (IG1)$^\circ$, (IG3)$^\circ$, (IG4)$^\circ$ and (IG5)$^\circ$ of (IG1), (IG3), (IG4) and (IG5) hold:

(IG1) if $e, f \in E$ are such that $e \mathcal{R} f$ or $e \mathcal{L} f$, then there exists a distinguished morphism $[e, f]$ from e to f such that $[e, e] = 1_e$, the identity associated to e;

(IG2) for any $e, f \in E$, $e \ \omega \ f$ if and only if $1_e \leq 1_f$;

(IG3) if $e, f, g \in E$ are such that $e \mathcal{R} f \mathcal{R} g$, then $[e, f] \cdot [f, g] = [e, g]$;

(IG4) if $g, h, e \in E$ are such that $[g, h]$ exists and $e \ \omega \ g$, then $[e, heh]$ exists and $[e, heh] \leq [g, h]$;

(IG5) let $x \in G$, and for $i = 1, 2$, let $e_i, f_i \in E$ be such that $e_i \leq \mathbf{d}(x)$ and $f_i = \mathbf{r}({}_{e_i}|x)$. If $e_1 \ \omega^r \ e_2$, then $f_1 \ \omega^r \ f_2$ and $[e_1, e_1e_2] \cdot ({}_{e_1e_2}|x) = ({}_{e_1}|x) \cdot [f_1, f_1f_2]$;

(IG6) if $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$ is a singular E-square, then $[e, f] \cdot [f, h] = [e, g] \cdot [g, h]$.

This leads to a generalisation of Theorem A from a semilattice to a regular biordered set.

Theorem B (Nambooripad [38]) *The category of regular semigroups and morphisms is equivalent to the category of inductive*$_2$ *groupoids and inductive*$_2$ *functors.*

The definition of an *inductive*$_2$ *functor* is given in the next section in the more general content of inductive$_2$ cancellative categories.

As we will give more details in the next section for the more general case of concordant semigroups, we omit the process of building Theorem B.

Note that for a technical reason, 'isomorphic' in Theorem A has been replaced by 'equivalent' in Theorem B. Of course, Theorem B may be specialised to orthodox semigroups.

6.2 Concordant semigroups and inductive$_2$ cancellative categories

Theorem B was extended by Armstrong [1] from regular to concordant semigroups, replacing ordered$_2$ groupoids by more general kinds of ordered$_2$ categories.

Before giving Armstrong's result, we recall that a *concordant semigroup* is an abundant semigroup with a regular biordered set of idempotents and satisfying the extra condition of being idempotent connected (IC). That is, a concordant semigroup is a weakly U-regular semigroup for which $U = E(S)$, $\mathcal{R}^* = \widetilde{\mathcal{R}}_U$, $\mathcal{L}^* = \widetilde{\mathcal{L}}_U$ and (IC) holds.

An ordered$_2$ cancellative category $(C, \leq)$ with a regular biordered set of objects E is said to be an *inductive$_2$ cancellative category* if the following conditions and the duals (IC1)°, (IC3)°, (IC4)° and (IC5)° of (IC1), (IC3), (IC4) and (IC5) hold:

(IC1) if $e, f \in E$ are such that $e \mathcal{R} f$ or $e \mathcal{L} f$, then there exists a distinguished morphism $[e, f]$ from e to f such that $[e, e] = 1_e$, the identity associated to e;

(IC2) for any $e, f \in E$, $e\ \omega\ f$ if and only if $1_e \leq 1_f$;

(IC3) if $e, f, g \in E$ are such that $e \mathcal{R} f \mathcal{R} g$, then $[e, f] \cdot [f, g] = [e, g]$;

(IC4) if $g, h, e \in E$ are such that $[g, h]$ exists and $e\ \omega\ g$, then $[e, heh]$ exists and $[e, heh] \leq [g, h]$;

(IC5) let $x \in C$, and for $i = 1, 2$, let $e_i, f_i \in E$ be such that $e_i \leq \mathbf{d}(x)$ and $f_i = \mathbf{r}({}_{e_i}|x)$. If $e_1\ \omega^r\ e_2$, then $f_1\ \omega^r\ f_2$ and $[e_1, e_1e_2] \cdot ({}_{e_1e_2}|x) = ({}_{e_1}|x) \cdot [f_1, f_1f_2]$;

(IC6) if $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$ is a singular E-square, then $[e, f] \cdot [f, h] = [e, g] \cdot [g, h]$.

We pause to make a short comment that an inductive$_2$ groupoid $(G, \leq)$ is an inductive$_2$ cancellative category; conversely, an inductive$_2$ cancellative category $(C, \leq)$ becomes an inductive$_2$ groupoid if every morphism has an inverse. It is necessary to claim that Condition (G) holds, that is, if $x \leq y$ in C, then $x^{-1} \leq y^{-1}$. Suppose that $x \leq y$. Then $\mathbf{d}(x)\ \omega\ \mathbf{d}(y)$ and $\mathbf{r}(x)\ \omega\ \mathbf{r}(y)$, that is, $\mathbf{r}(x)\ \omega\ \mathbf{d}(y^{-1})$, and so by (OC4), there exists a unique element ${}_{\mathbf{r}(x)}|y^{-1}$ such that ${}_{\mathbf{r}(x)}|y^{-1} \leq y^{-1}$ and $\mathbf{d}({}_{\mathbf{r}(x)}|y^{-1}) = \mathbf{r}(x)$. So $x \cdot {}_{\mathbf{r}(x)}|y^{-1}$ is defined and by (OC3), $x \cdot {}_{\mathbf{r}(x)}|y^{-1} \leq y \cdot y^{-1} = 1_{\mathbf{d}(y)}$. As $1_{\mathbf{d}(x)} \leq 1_{\mathbf{d}(y)}$ and $\mathbf{d}(1_{\mathbf{d}(x)}) = \mathbf{d}(x) = \mathbf{d}(x \cdot {}_{\mathbf{r}(x)}|y^{-1})$, it follows from (OC4) that $x \cdot {}_{\mathbf{r}(x)}|y^{-1} = 1_{\mathbf{d}(x)}$, which gives that $x^{-1} = {}_{\mathbf{r}(x)}|y^{-1}$. Thus. $x^{-1} \leq y^{-1}$.

Let C and C' be inductive$_2$ cancellative categories over regular biordered sets E and E', respectively, and $\phi : C \to C'$ be an order-preserving functor. Then ϕ is *inductive$_2$* if

(IOF1) the map $\phi : E \to E'$ is a regular morphism (in the sense of Definition 1.25);

(IOF2) if $e, f \in E$ and $[e, f]$ exists, then $[e, f]\phi = [e\phi, f\phi]$.

Now, we give an outline of the correspondence between concordant semigroups and inductive$_2$ cancellative categories. The approach used here is like the one in regular case.

Let C be an inductive$_2$ cancellative category over regular biordered set E.

We define a relation ρ on C by the rule that for any $x, y \in C$,

$$x \,\rho\, y \Leftrightarrow \mathbf{d}(x) \,\mathcal{R}\, \mathbf{d}(y), \mathbf{r}(x) \,\mathcal{L}\, \mathbf{r}(y) \text{ and } x \cdot [\mathbf{r}(x), \mathbf{r}(y)] = [\mathbf{d}(x), \mathbf{d}(y)] \cdot y.$$

It is routine to check that ρ is an equivalence relation on C.

Suppose that $x \in C$, $h \;\omega^r\; \mathbf{d}(x)$ and $k \;\omega^l\; \mathbf{r}(x)$. We define

$$h * x = [h, h\mathbf{d}(x)] \cdot {}_{h\mathbf{d}(x)}|x \text{ and } x \diamond k = x|_{\mathbf{r}(x)k} \cdot [\mathbf{r}(x)k, k].$$

We then define a binary operation $\odot$ on C/ρ by the rule that for all $x, y \in C$, $h \in S(\mathbf{r}(x), \mathbf{d}(y))$,

$$\bar{x} \odot \bar{y} = \overline{(x \otimes y)_h},$$

where $\bar{x}$ denotes the ρ-class of x in C and $(x \otimes y)_h = (x \diamond h) \cdot (h * y)$. It is proved in [1] that the set $C\mathbf{S} = (C/\rho, \odot)$ forms a concordant semigroup.

To see the converse, let S be a concordant semigroup with regular biordered set of idempotents E. We build a category $S\mathbf{C}$ as follows:

$$\text{Ob } (S\mathbf{C}) = E, \;\; \text{Mor } (S\mathbf{C}) = \{(e, a, f) : e \in R_a^* \cap E, f \in L_a^* \cap E\}.$$

For any $(e, a, f) \in \text{Mor } (S\mathbf{C})$, we set

$$\mathbf{d}((e, a, f)) = e \text{ (abbreviated to } \mathbf{d}(e, a, f) = e)$$

and

$$\mathbf{r}((e, a, f)) = f \text{ (abbreviated to } \mathbf{r}(e, a, f) = f).$$

In addition, we define a partial binary operation on $S\mathbf{C}$ by

$$(e, a, f) \cdot (g, b, h) = \begin{cases} (e, ab, h) & \text{if } f = g \\ \text{undefined} & \text{otherwise,} \end{cases}$$

where ab is the product of a and b in S. Then $S\mathbf{C}$ becomes a category with regular biordered set of objects E.

Certainly, $S\mathbf{C}$ is cancellative. Since if $(e, a, f), (g, b, f), (f, c, v) \in S\mathbf{C}$ and $(e, a, f) \cdot (f, c, v) = (g, b, f) \cdot (f, c, v)$, then $(e, ac, v) = (g, bc, v)$ and so $e = g$ and $ac = bc$. As $f \,\mathcal{R}^*\, c$, we have that $af = bf$, that is, $a = b$ as $a, b \;\mathcal{L}^*\; f$. Hence,

$(e, a, f) = (g, b, f)$, so that $S\mathbf{C}$ is right cancellative. Dually, it is left cancellative, and consequently, $S\mathbf{C}$ is cancellative.

Since any concordant semigroup S satisfies (IC), it follows from the statements in Section 2.3.2 that if $(e, a, f) \in S\mathbf{C}$, then there is a unique connecting isomorphism from $\langle e \rangle$ to $\langle f \rangle$. We remind the reader that if $k \in E$, then $\langle k \rangle$ means the subsemigroup of E generated by the idempotents in $k\langle E \rangle k$. We can therefore define a relation on $S\mathbf{C}$ by the rule that for all $(e, a, f), (g, b, h) \in S\mathbf{C}$,

$$(e, a, f) \leq (g, b, h) \Leftrightarrow e \ \omega \ g, a = eb \text{ and } f = e\beta,$$

where $\beta : \langle e \rangle \to \langle f \rangle$ is a connecting isomorphism. It is routine to show that $\leq$ is a partial order on $S\mathbf{C}$.

If $(e, a, f) \in S\mathbf{C}$, $u, v \in E$ and $u \leq e$ and $v \leq f$, then we define the restriction and co-restriction as

$${}_u|(e, a, f) = (u, ua, u\beta) \text{ and } (e, a, f)|_v = (v\beta^{-1}, av, v),$$

where $\beta : \langle e \rangle \to \langle f \rangle$ is a connecting isomorphism. Then $S\mathbf{C}$ becomes an ordered$_2$ cancellative category under $\leq$

Suppose that $e, f \in E$ are such that $e \,\mathcal{L}\, f$ or $e \,\mathcal{R}\, f$. Then we define $[e, f] = (e, ef, f)$. Clearly, $[e, f]$ is well-defined and it belongs to $S\mathbf{C}$.

Now, we obtain that $(S\mathbf{C}, \leq)$ together with the restriction and co-restriction forms an inductive$_2$ cancellative category. Further details of the proof can be found in [1].

To sum up, we have:

Theorem C (Armstrong [1]) *The category of concordant semigroups and good morphisms is equivalent to the category of inductive*$_2$ *cancellative categories and inductive*$_2$ *functors.*

6.3 Ehresmann semigroups and Ehresmann categories

Theorem A was generalised in a different direction to Ehresmann semigroups by Lawson [32]. His use of two partial orders on an Ehresmann semigroup is an important observation for the idea discussed in Chapter 7.

We recall from [32] that an *Ehresmann category* $C = (C, \cdot, \leq_r, \leq_l)$ is a category $(C, \cdot)$ with set of identities E, equipped with two relations $\leq_l$ and $\leq_r$ such that the following conditions, and the duals (E1)$^\circ$ and (E5)$^\circ$ of (E1) and (E5) hold:

(E1) $(C, \cdot, \leq_r)$ is an ordered$_1$ category with restriction;

(E2) if $e, f \in E$, then $e \leq_r f \Leftrightarrow e \leq_l f$;

(E3) E is a meet semilattice under $\leq_r$ (or $\leq_l$);

(E4) $\leq_r \circ \leq_l = \leq_l \circ \leq_r$;

(E5) if $x \leq_r y$ and $f \in E$, then $x|_{\mathbf{r}(x)f} \leq_r y|_{\mathbf{r}(y)f}$.

We note that [32] interchanges the symbols $\mathbf{r}$ and $\mathbf{d}$ and the notion of restriction and co-restriction, from the conventions of this thesis.

We now pause to make a short comment on Ehresmann categories $C = (C, \cdot, \leq_r, \leq_l)$. Condition (E1) implies that if $x \leq_r y$ in C then $\mathbf{d}(x) \leq \mathbf{d}(y)$ and $\mathbf{r}(x) \leq \mathbf{r}(y)$ by (OC1). Since $(C, \cdot, \leq_r, \leq_l)$ is an ordered$_1$ category with restriction and co-restriction, then there exists a unique element ${}_{\mathbf{d}(x)}|y$ such that ${}_{\mathbf{d}(x)}|y \leq_r y$ and $\mathbf{d}({}_{\mathbf{d}(x)}|y) = \mathbf{d}(x)$. Since $x \leq_r y$, and the uniqueness of restriction gives $x = {}_{\mathbf{d}(x)}|y$. To the converse, if $x = {}_e|y$, then by the definition of restriction, we certainly obtain that $x \leq_r y$. Hence, we have:

Lemma 6.1. *Let $C = (C, \cdot, \leq_r, \leq_l)$ be an Ehresmann category over E. Then for any $x, y \in C$,*

(i) $x \leq_r y$ if and only if $x = {}_e|y$ for some $e \in E$;

(ii) $x \leq_l y$ if and only if $x = y|_f$ for some $f \in E$.

Let $\mathbf{C} = (C, \cdot, \leq_r, \leq_l)$ and $\mathbf{D} = (D, \cdot, \leq_r, \leq_l)$ be Ehresmann categories with semilattice E_C and E_D of identities, respectively. We say that a functor $F : \mathbf{C} \to \mathbf{D}$ is *strongly ordered* if it satisfies the following conditions:

(i) if $x \leq_l y$ (resp. $x \leq_r y$), then $xF \leq_l yF$ (resp. $xF \leq_r yF$);

(ii) if $e, f \in E_C$, then $(ef)F = eFfF$.

Given an Ehresmann semigroup S with distinguished semilattice E, we have introduced two partial orders $\leq_r$ and $\leq_l$ on S in Section 2.2.5. Lawson [32] showed that the category, consisting of the set of objects E and the set of morphisms S, forms an Ehresmann category together with the partial binary operation $\cdot$ defined by the rule that for any $x, y \in S$,

$$x \cdot y = \begin{cases} xy & \text{if } \ x^* = y^\dagger \\ \text{undefined} & \text{otherwise,} \end{cases}$$

where $x^*, y^\dagger \in E$ with $x \, \widetilde{\mathcal{L}}_E \, x^*$ and $y \, \widetilde{\mathcal{R}}_E \, y^\dagger$.

Conversely, let $C = (C, \cdot, \leq_r, \leq_l)$ be an Ehresmann category with set of identities E. For any $x, y \in C$, we define

$$x \overline{\otimes} y = x|_{\mathbf{r}(x) \wedge \mathbf{d}(y)} \cdot {}_{\mathbf{r}(x) \wedge \mathbf{d}(y)}|y.$$

Then $(C, \overline{\otimes})$ is an Ehresmann semigroup.

Theorem D (Lawson [32]) *The category of Ehresmann semigroups and admissible morphisms is isomorphic to the category of Ehresmann categories and strongly ordered functors.*

We recall that a restriction semigroup is an Ehresmann semigroup satisfying (WIC) by Lemma 2.18. On such semigroups the orders $\leq_r$, $\leq_l$ and $\leq_e$ coincide and we denote the unique order by $\leq$.

Notice that if $\leq_r = \leq_l$ on an Ehresmann category $(C, \cdot, \leq_r, \leq_l)$, then $(C, \leq)$ becomes an inductive$_1$ category, that is, an ordered$_2$ category, in which the set of identities is a semilattice.

Corollary 6.2. *[24] The category of restriction semigroups and admissible congruences is isomorphic to the category of inductive$_1$ categories and strongly ordered functors.*

We now turn to ample semigroups. We replace the distinguished semilattice of idempotents E by the whole set of idempotents and use relations $\mathcal{R}^*$ and $\mathcal{L}^*$ instead of $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$ in the definition of restriction semigroups. We thus obtain the class of ample semigroups whose set of idempotents forms a semilattice. An admissible morphism in this context is more usually referred to as a *good* morphism.

Corollary 6.3. *[24] The category of ample semigroups and good morphisms is isomorphic to the category of inductive$_1$ cancellative categories and strongly ordered functors.*

Corollary 6.3 is a little difference in Armstrong's paper [2] and [24].

If we use relations $\mathcal{R}$ and $\mathcal{L}$ to replace $\mathcal{R}^*$ and $\mathcal{L}^*$ in the definition of ample semigroups, we then obtain the class of inverse semigroups. A good morphism in this context is a morphism. Consequently, we obtain Theorem A.

Ehresmann semigroups have a semilattice of idempotents, need not be regular or even abundant, need not satisfy an (IC) condition, and indeed need not be restriction semigroups. Lawson overcomes the lack of an (IC) condition by using two partial order relations. Our aim of Chapter 7 is to extend Lawson's result to the class of weakly B-orthodox semigroups, which extend the class of Ehresmann semigroups by replacing semilattices by bands. In Chapter 7 we use a new technique of *generalised categories*. We *could* use triples such as in [1], and this is the approach we take in Chapter 9 and in the more general weakly U-regular case, in Chapter 10.

Chapter 7

Beyond orthodox semigroups I: weakly B-orthodox semigroups and generalised categories

Our purpose of this chapter is to describe a class of weakly B-orthodox semigroups. In doing so we produce a new approach to characterising orthodox semigroups via inductive generalised groupoids. Here B denotes a band of idempotents; we note that if B is a semilattice then a weakly B-orthodox semigroup is exactly an Ehresmann semigroup. We build a correspondence between our work and a result of Lawson for Ehresmann semigroups [32].

For convenience we make the convention that B will always denote a band. Green's relations and their associated pre-orders will always refer to B, unless stated otherwise. In particular, if S is weakly B-orthodox and $e \in B$, then R_e (L_e) denote the $\mathcal{R}$-class ($\mathcal{L}$-class) of e *in* B.

7.1 Inductive generalised categories

Let I, R, L and D be disjoint sets and let p denote a collection of four (well-defined) onto maps:

$$\begin{array}{cccc} I \twoheadrightarrow R, & I \twoheadrightarrow L, & R \twoheadrightarrow D & \text{and} \quad L \twoheadrightarrow D \\ i \mapsto R_i, & i \mapsto L_i, & R_i \mapsto D_i & \quad L_i \mapsto D_i \end{array}$$

such that

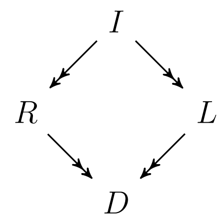

Figure 7.1: Maps

commutes. We denote this configuration by (I, R, L, D, p) and refer to it as a *context*.

We pause to give our motivating example. Let B be a band and p denote the natural maps:

$$B \twoheadrightarrow B/\mathcal{R}, \quad B \twoheadrightarrow B/\mathcal{L}, \quad B/\mathcal{R} \twoheadrightarrow B/\mathcal{D} \text{ and } B/\mathcal{L} \twoheadrightarrow B/\mathcal{D}.$$

Then $(B, B/\mathcal{R}, B/\mathcal{L}, B/\mathcal{D}, p)$ is a context. Of course, if B is a semilattice, then all of Green's relations are trivial and the p-maps are essentially the identity maps.

Definition 7.1. A *generalised category* P over a context (I, R, L, D, p) consists of

(GC1) a class $\mathrm{Ob}(P)$ of objects $R \dot\cup L$;

(GC2) a class $\mathrm{Mor}(P)$ of morphisms between the objects. Each morphism x has a unique domain $\mathbf{d}(x) \in R$ and codomain $\mathbf{r}(x) \in L$. Denote the Mor-class of all morphisms from $R_i \in R$ to $L_j \in L$ by $\mathrm{Mor}(R_i, L_j)$;

(GC3) if $R_i, R_k \in R$ and $L_j, L_h \in L$ with $D_j = D_k$, then there is a binary operation

$$\mathrm{Mor}(R_i, L_j) \ \times \ \mathrm{Mor}(R_k, L_h) \to \mathrm{Mor}(R_i, L_h), \ (x, y) \mapsto x \cdot y$$

called *composition of morphisms* such that if $x \in \mathrm{Mor}(R_i, L_j)$, $y \in \mathrm{Mor}(R_k, L_h)$, and $z \in \mathrm{Mor}(R_m, L_n)$, where $D_j = D_k$ and $D_h = D_m$, then $(x \cdot y) \cdot z = x \cdot (y \cdot z)$;

(GC4) for each $i \in I$, there exists a distinguished morphism, again denoted by i, such that $i \in \mathrm{Mor}(R_i, L_i)$ and if $\mathbf{d}(x) = R_i$ and $\mathbf{r}(y) = L_i$, then $i \cdot x = x$ and $y \cdot i = y$.

Let P be a generalised category over a context (I, R, L, D, p). Following the usual convention when building categories from semigroups, we may identify Mor(P) with P. If B is a band and P is a generalised category over $(B, B/\mathcal{R}, B/\mathcal{L}, B/\mathcal{D}, p)$, where p denote the natural maps, then we say simply P is a *generalised category over* B.

Our notion of a generalised category is motivated by that of the 'trace product' of a weakly B-orthodox semigroup. We explain this in Chapter 8 but comment briefly here on the special case of a band.

We have seen that if B is a band, then $(B, B/\mathcal{R}, B/\mathcal{L}, B/\mathcal{D}, p)$ is a context. Define a generalised category P over B by putting $\text{Mor}(P) = B$ and for $e \in B$, put $\mathbf{d}(e) = R_e$ and $\mathbf{r}(e) = L_e$. Let the partial binary operation be given by $e \cdot f = ef$, where $e \cdot f$ exists. Note the latter is true if and only if $D_e = D_f$. Thus the effect of our generalised category is to restrict the multiplication in B to that within its $\mathcal{D}$-classes.

We now focus on generalised categories over a band B, in general more extensive than the example above, making use of the natural partial order in $B/\mathcal{R}$ and $B/\mathcal{L}$. Note that if $e \in B$ then by (GC4) we have that $e \in \text{Mor}(R_e, L_e)$, so that $\mathbf{d}(e) = R_e$ and $\mathbf{r}(e) = L_e$.

We build on Definition 7.1 to define an inductive generalised category over B, which is an analogue of inductive$_2$ groupoids [38] and inductive$_2$ cancellative categories [1]. We will see that the elements of our inductive generalised category may be pre-ordered or partially ordered, in two ways, reflecting the approach of [32].

Definition 7.2. Let P be a generalised category over a band B. Then P is an *inductive* generalised category if the following conditions and the duals (I1)$^\circ$, (I2)$^\circ$, and (I3)$^\circ$ of (I1), (I2) and (I3) hold:

(I1) if $x \in P$ and $e, u \in B$ with $e \leq_{\mathcal{L}} u \in \mathbf{d}(x)$, then there exists an element ${}_e|x$ in P, called the *restriction* of x to e, such that $e \in \mathbf{d}({}_e|x)$ and $\mathbf{r}({}_e|x) \leq_{\mathcal{L}} \mathbf{r}(x)$; in particular, if $e \in \mathbf{d}(x)$, then ${}_e|x = x$;

(I2) if $x \in P$ and $e, f, g, u \in B$ with $e \leq_{\mathcal{L}} g \;\mathcal{R}\; f \leq_{\mathcal{L}} u \in \mathbf{d}(x)$, then ${}_{ef}|x = {}_e|({}_f|x)$;

(I3) if $x, y \in P$ and $e, u \in B$ with $x \cdot y$ defined in P and $e \leq_{\mathcal{L}} u \in \mathbf{d}(x)$, then ${}_e|(x \cdot y) = ({}_e|x) \cdot ({}_f|y)$, where $f \in \mathbf{r}({}_e|x)$;

(I4) if $x, y \in P$ and $e_1, e_2, f_1, f_2 \in B$ with $e_1, e_2 \in \mathbf{r}(x)$ and $f_1, f_2 \in \mathbf{d}(y)$, then $x|_{e_1 f_1} \cdot {}_{e_1 f_1}|y = x|_{e_2 f_2} \cdot {}_{e_2 f_2}|y$;

(I5) if $x \in P$ and $e, f, u, v, g, h \in B$ with $g \in \mathbf{r}(x)$, $h \in \mathbf{d}(x)$, $u \in \mathbf{d}(x|_{gf})$

and $v \in \mathbf{r}({}_{eh}|x)$, then ${}_{eu}|(x|_{gf}) = ({}_{eh}|x)|_{vf}$;

(I6) if $e, g, h, u, v \in B$ are such that $u \leq_{\mathcal{R}} g \;\mathcal{L}\; e$ and $v \leq_{\mathcal{L}} h \;\mathcal{R}\; e$, then $e|_u = eu$ and ${}_v|e = ve$.

We make some comments on the above definition. In (I3) let $\mathbf{r}(x) = L_v$ and $\mathbf{d}(y) = R_w$. Since there exists $x \cdot y$ we know that $v \;\mathcal{D}\; w$ so we have $\mathbf{r}(x) = L_{wv}$ and $\mathbf{d}(y) = R_{wv}$. Hence by (I1), $f \in \mathbf{r}({}_e|x) \leq_{\mathcal{L}} L_{wv}$ and $wv \in \mathbf{d}(y)$, so that ${}_f|y$ exists and $\mathbf{d}({}_f|y) = R_f$. Hence $({}_e|x) \cdot ({}_f|y)$ exists. To simplify the term '$x \cdot y$ exists' may use the expression '$\exists x \cdot y$' or '$x \cdot y$ is defined'.

Suppose now that P is a generalised category over a band B. We remarked above that if $e \in B$, then $\mathbf{d}(e) = R_e$ and $\mathbf{r}(e) = L_e$, so that if also $f \in B$ then $\exists e \cdot f$ if and only if $e \,\mathcal{D}\, f$. In this case, clearly $e \in \mathbf{d}(e), e \in \mathbf{d}(e \cdot f)$ and by (I1), ${}_e|e = e$, so that $e \in \mathbf{r}({}_e|e)$. Using (I1), (I3), (I6) and (GC4) we have

$$e \cdot f =_e |(e \cdot f) = ({}_e|e) \cdot ({}_e|f) = e \cdot ef = ef.$$

We pause to introduce a pair of pre-orders on an inductive generalised category P over a band B deduced from Definition 7.2. We make use of the restriction and co-restriction of P to define relations $\leq_r$ and $\leq_l$ by the rule that for any $x, y \in P$,

$$x \leq_r y \text{ if and only if } x = {}_e|y \text{ for some } e \in B,$$

and

$$x \leq_l y \text{ if and only if } x = y|_f \text{ for some } f \in B.$$

Lemma 7.3. *The relations $\leq_r$ and $\leq_l$ are pre-orders on P.*

Proof. To prove that $\leq_r$ is a pre-order on P, we first observe that $\leq_r$ is reflexive by (I1). It is necessary to show that $\leq_r$ is transitive. Assume that $x, y, z \in P$ with $x \leq_r y$ and $y \leq_r z$. Then there exist $e, f \in B$ such that $x = {}_e|y$ and $y = {}_f|z$. For ${}_e|y$ and ${}_f|z$ to exist we have $e \leq_{\mathcal{L}} g \in \mathbf{d}(y) = R_f$ and $f \leq_{\mathcal{L}} h \in \mathbf{d}(z)$. From (I2), $x = {}_e|({}_f|z) = {}_{ef}|z$. Hence $x \leq_r z$.

By the dual argument, we show that $\leq_l$ is a pre-order on P. □

The reader might notice that previous authors have used partial orders rather than pre-orders. For our purpose, pre-orders are easier to use, but the partial orders are still there, as we now show.

We define $\leq_r'$ and $\leq_l'$ on P by the rule that

$$x \leq_r' y \text{ if and only if } x = {}_e|y \text{ for some } e \leq u \in \mathbf{d}(y),$$

and

$$x \leq_l' y \text{ if and only if } x = y|_f \text{ for some } f \leq v \in \mathbf{r}(y).$$

Lemma 7.4. *The relations $\leq_r'$ and $\leq_l'$ are partial orders on P.*

Proof. As in Lemma 7.3, $\leq_r'$ is reflexive. If $x \leq_r' y$ and $y \leq_r' z$ then with e, f as in Lemma 7.3, we have $e \leq g$ and $f \leq h$. Certainly, $x = {}_{ef}|z$ and $efh = ef$, as $f \leq h$. Also, $e \leq g \mathcal{R} f \leq h$, so $hef = ef$. Hence $ef \leq h \in \mathbf{d}(z)$.

Finally, suppose that $x \leq_r' y \leq_r' x$. Then $x = {}_e|y$ and $y = {}_f|x$ for some $e \leq u \in \mathbf{d}(y)$ and $f \leq v \in \mathbf{d}(x)$. We have $e \leq u \mathcal{R} f$ and $f \leq v \mathcal{R} e$, so that $e \mathcal{R} f$ and $\mathbf{d}(x) = \mathbf{d}(y)$. Now $x = {}_e|y = y$, by (I1). □

We say that $\leq_r$ and $\leq_l$ are the *natural pre-orders associated with* P and $\leq_r'$ and $\leq_l'$ are the *natural partial orders associated with* P.

We end this section by showing that the class of inductive generalised categories over bands forms a category, together with certain maps referred to as pseudo-functors. They appear in the next definition.

Definition 7.5. Let P_1 and P_2 be inductive generalised categories over bands B_1 and B_2, respectively. A *pseudo-functor* F from P_1 to P_2 is a pair of maps, both denoted F, from B_1 to B_2 and from P_1 to P_2, such that the following conditions and the dual (F2)$^\circ$ of (F2) hold:

(F1) the map F is a morphism from B_1 to B_2;

(F2) if $e \in B_1$ and $e \leq_{\mathcal{L}} u \in \mathbf{d}(x)$ in P_1, then $({}_e|x)F = {}_{eF}|xF$;

(F3) if $\exists x \cdot y$ in P_1 then $\exists xF \cdot yF$ in P_2, and $(x \cdot y)F = xF \cdot yF$.

To see that (F2) makes sense, suppose that $u \in B_1, x \in P_1$ with $u \in \mathbf{d}(x)$. Then $R_u = \mathbf{d}(x)$ so that $\exists u \cdot x$ and $u \cdot x = x$. By (F3), $\exists uF \cdot xF$ and $uF \cdot xF = xF$. Hence $\mathbf{d}(xF) = \mathbf{d}(uF) = R_{uF}$, as $uF \in B_2$. Suppose also that $e \in B_1$, with $e \leq_{\mathcal{L}} u$ in B_1. Using (F1), we have $eF \leq_{\mathcal{L}} uF \in \mathbf{d}(xF)$ in P_2, and so $\exists_{eF}|xF$. Notice that we can define F on $\mathrm{Ob}(P_1)$ by putting $R_e F = R_{eF}$ and $L_e F = L_{eF}$.

From the comments above, it is easy to check that Lemma 7.6 holds.

Lemma 7.6. *Let P_1, P_2 and P_3 be inductive generalised categories over B_1, B_2 and B_3, respectively, and let $F_1 : P_1 \to P_2$ and $F_2 : P_2 \to P_3$ be pseudo-functors. Then $F_1F_2 : P_1 \to P_3$ is a pseudo-functor.*

The next observation follows immediately.

Lemma 7.7. *The class of inductive generalised categories over bands, together with pseudo-functors, forms a category.*

We refer to the category in the above lemma as $\mathcal{IGC}$.

7.2 Construction

Our primary interest in this section will be a construction of a weakly B-orthodox semigroup, built from an inductive generalised category over B.

Let P be an inductive generalised category over a band B. We define a pseudo-product $\otimes$ on P by

$$x \otimes y = (x|_{ef}) \cdot ({}_{ef}|y),$$

where $e \in \mathbf{r}(x)$, $f \in \mathbf{d}(y)$. It follows from (I4) that the pseudo-product is independent of the choices of e and f and thus is well-defined. We will denote the set P, together with the pseudo-product $\otimes$, by $P\mathbf{S}$.

We pause to present our initial idea which follows Armstrong's steps, using the notion of sandwich set, simplifying a little here as our set of idempotents forms a band. We may define a pseudo-product $\otimes'$ on P by the rule that for any $x, y \in P$,

$$x \otimes' y = (x|_{efe}) \cdot ({}_{fef}|y),$$

where $e \in \mathbf{r}(x)$ and $f \in \mathbf{d}(x)$. In that case, Condition (I5) is not enough to guarantee that $\otimes'$ is associative in P. To achieve this it is necessary to add a stronger condition in place of (I5), which effectively says that $e \otimes' (x \otimes' y) = (e \otimes' x) \otimes' y$ for any $x, y \in P$ and $e \in B$. This appears to us too contrived. Keeping this in mind we use the pseudo-product $\otimes$ defined as above.

We now present a series of lemmas related to P, which will help us to show our main result at the end of this section.

Lemma 7.8. *If $x, y \in P$ with $\exists x \cdot y$, then $x \otimes y = x \cdot y$.*

Proof. If $\exists x \cdot y$ then $\mathbf{r}(x) = L_e$ and $\mathbf{d}(y) = R_f$ say, where $e \,\mathcal{D}\, f$. Then $\mathbf{r}(x) = L_{fe}$ and $\mathbf{d}(y) = R_{fe}$, so $x \otimes y = (x|_{fefe}) \cdot ({}_{fefe}|y) = (x|_{fe}) \cdot ({}_{fe}|y) = x \cdot y$ by (I1). $\square$

Lemma 7.9. *If $e, f \in B$ then $e \otimes f = ef$.*

Proof. We have

$$\begin{aligned} e \otimes f &= (e|_{ef}) \cdot ({}_{ef}|f) \\ &= eef \cdot eff && \big(\text{by (I6)}\big) \\ &= ef \cdot ef = ef && \big(\text{by (GC4)}\big). \end{aligned}$$

□

Consequently, B forms the same band under $\otimes$ and the original multiplication.

Lemma 7.10. *If $x \in P$ and $e, f, u \in B$ with $u \mathcal{D} e \leq_{\mathcal{L}} f \in \mathbf{d}(x)$ then $u \cdot ({}_e|x) = {}_{ue}|x$.*

Proof. Since $u \mathcal{D} e$, we deduce that

$$\begin{aligned} u \cdot {}_e|x &= u \otimes {}_e|x && \big(\text{Lemma 7.8}\big) \\ &= (u|_{ue}) \cdot ({}_{ue}|({}_e|x)) \\ &= ue \cdot ({}_{ue}|x) && \big(\text{by (I6), (I2)}\big) \\ &= {}_{ue}|x && \big(\text{by (GC4)}\big). \end{aligned}$$

□

Lemma 7.11. *The set $P\mathbf{S}$ forms a semigroup under the operation $\otimes$.*

Proof. It is sufficient to show that $P\mathbf{S}$ is associative. Suppose that $x, y, z \in P$ with $x^* \in \mathbf{r}(x)$, $y^\dagger \in \mathbf{d}(y)$, $y^* \in \mathbf{r}(y)$ and $z^\dagger \in \mathbf{d}(z)$. Then

$$\begin{aligned} x \otimes (y \otimes z) &= x \otimes \Big((y|_{y^*z^\dagger}) \cdot ({}_{y^*z^\dagger}|z)\Big) \\ &= (x|_{x^*u}) \cdot \Big({}_{x^*u}|\Big((y|_{y^*z^\dagger}) \cdot ({}_{y^*z^\dagger}|z)\Big)\Big) && \Big(u \in \mathbf{d}(y|_{y^*z^\dagger})\Big) \\ &= (x|_{x^*u}) \cdot \Big({}_{x^*u}|(y|_{y^*z^\dagger})\Big) \cdot \Big({}_v|({}_{y^*z^\dagger}|z)\Big) \\ & && \Big(v \in \mathbf{r}({}_{x^*u}|(y|_{y^*z^\dagger})),\ \text{by (I3)}\Big). \end{aligned}$$

Notice, by (I1), that $v \leq_{\mathcal{L}} y^*z^\dagger \in \mathbf{r}(y|_{y^*z^\dagger})$ and by (I5), that

$$ {}_{x^*u}|(y|_{y^*z^\dagger}) = ({}_{x^*y^\dagger}|y)|_{gz^\dagger}, $$

where $g \in \mathbf{r}({}_{x^*y^\dagger}|y)$, and so $v \mathrel{\mathcal{L}} gz^\dagger$ and $x^*u \in \mathbf{d}(({}_{x^*y^\dagger}|y)|_{gz^\dagger})$. Thus

$$\begin{aligned}
x \otimes (y \otimes z) &= (x|_{x^*u}) \cdot \big(({}_{x^*y^\dagger}|y)|_{gz^\dagger}\big) \cdot \big({}_v|({}_{y^*z^\dagger}|z)\big) \\
&= (x|_{x^*u}) \cdot \big(({}_{x^*y^\dagger}|y)|_{gz^\dagger}\big) \cdot ({}_{vy^*z^\dagger}|z) && \big(v \leq_{\mathcal{L}} y^*z^\dagger, \text{ by (I2)}\big) \\
&= (x|_{x^*u}) \cdot \big(({}_{x^*y^\dagger}|y)|_{gz^\dagger}\big) \cdot ({}_v|z) && \big(v \leq_{\mathcal{L}} y^*z^\dagger\big) \\
&= (x|_{x^*u}) \cdot \big(({}_{x^*y^\dagger}|y)|_{gz^\dagger}\big) \cdot ({}_{vgz^\dagger}|z) && \big(v \mathrel{\mathcal{L}} gz^\dagger\big) \\
&= (x|_{x^*u}) \cdot \big(({}_{x^*y^\dagger}|y)|_{gz^\dagger}\big) \cdot (v \cdot ({}_{gz^\dagger}|z)) && \big(\text{Lemma 7.10}\big) \\
&= (x|_{x^*u}) \cdot \big(({}_{x^*y^\dagger}|y)|_{gz^\dagger} \cdot v\big) \cdot ({}_{gz^\dagger}|z) \\
&= (x|_{x^*u}) \cdot \big(({}_{x^*y^\dagger}|y)|_{gz^\dagger}\big) \cdot ({}_{gz^\dagger}|z) && \big(v \mathrel{\mathcal{L}} gz^\dagger, \text{ by (GC4)}\big).
\end{aligned}$$

Due to the dual of (I1), $u \in \mathbf{d}(y|_{y^*z^\dagger}) \leq_{\mathcal{R}} \mathbf{d}(y)$, whence $x^*y^\dagger x^*u = x^*y^\dagger x^*y^\dagger u = x^*y^\dagger u = x^*u$. So

$$\begin{aligned}
x \otimes (y \otimes z) &= \big(x|_{x^*y^\dagger x^*u}\big) \cdot \big(({}_{x^*y^\dagger}|y)|_{gz^\dagger}\big) \cdot ({}_{gz^\dagger}|z) \\
&= \big((x|_{x^*y^\dagger})|_{x^*u}\big) \cdot \big(({}_{x^*y^\dagger}|y)|_{gz^\dagger}\big) \cdot ({}_{gz^\dagger}|z) \\
&\qquad \big(\text{by (I2)}^\circ, \text{ since } x^*u \leq_{\mathcal{R}} x^*y^\dagger\big) \\
&= \big(((x|_{x^*y^\dagger}) \cdot ({}_{x^*y^\dagger}|y))|_{gz^\dagger}\big) \cdot ({}_{gz^\dagger}|z) \\
&\qquad \big(x^*u \in \mathbf{d}(({}_{x^*y^\dagger}|y)|_{gz^\dagger})\big) \\
&= \big((x \otimes y)|_{gz^\dagger}\big) \cdot ({}_{gz^\dagger}|z) \\
&= (x \otimes y) \otimes z.
\end{aligned}$$

□

The following lemma shows that $P\mathbf{S}$ is a weakly B-abundant semigroup.

Lemma 7.12. *Let $x \in P\mathbf{S}$, $e \in \mathbf{r}(x)$ and $g \in \mathbf{d}(x)$. Then $g \mathrel{\widetilde{\mathcal{R}}_B} x \mathrel{\widetilde{\mathcal{L}}_B} e$ in $P\mathbf{S}$.*

Proof. By Lemma 7.8, we obtain that $x \otimes e = x \cdot e = x$. Suppose that $k \in B$ and $x \otimes k = x$. Then

$$\begin{aligned}
x \otimes k &= (x|_{ek}) \cdot ({}_{ek}|k) \\
&= (x|_{ek}) \cdot ek && \big(\text{by (I6)}\big) \\
&= x|_{ek} && \big(\text{by (GC4)}\big).
\end{aligned}$$

Thus $x = x|_{ek}$, which implies that $ek \in \mathbf{r}(x)$, and so $e \mathrel{\mathcal{L}} ek$. It follows that

$$\begin{aligned} e \otimes k &= ek && \big(\text{Lemma 7.9}\big) \\ &= eek \\ &= e && \big(e \mathrel{\mathcal{L}} ek\big). \end{aligned}$$

Consequently, $x \mathrel{\widetilde{\mathcal{L}}_B} e$.

Similarly, we can show that $x \mathrel{\widetilde{\mathcal{R}}_B} g$. □

As an application of Lemma 7.12, we give a concrete description of relations $\leq_{\widetilde{\mathcal{R}}_B}$ and $\leq_{\widetilde{\mathcal{L}}_B}$ on $P\mathbf{S}$ as follows.

Lemma 7.13. *For any $x, y \in P\mathbf{S}$,*

(i) $x \leq_{\widetilde{\mathcal{R}}_B} y$ if and only if $\mathbf{d}(x) \leq_{\mathcal{R}} \mathbf{d}(y)$;

(ii) $x \leq_{\widetilde{\mathcal{L}}_B} y$ if and only if $\mathbf{r}(x) \leq_{\mathcal{L}} \mathbf{r}(y)$.

Proof. We prove (i). Let $x, y \in P$ and let $\mathbf{d}(x) = R_e$ and $\mathbf{d}(y) = R_f$. Then

$$\begin{aligned} x \leq_{\widetilde{\mathcal{R}}_B} y \text{ in } P\mathbf{S} &\Leftrightarrow e \leq_{\widetilde{\mathcal{R}}_B} f \text{ in } P\mathbf{S} && \big(\text{Lemma 7.12}\big) \\ &\Leftrightarrow e \leq_{\mathcal{R}} f \text{ in } B \\ &\Leftrightarrow R_e \leq_{\mathcal{R}} R_f \\ &\Leftrightarrow \mathbf{d}(x) \leq_{\mathcal{R}} \mathbf{d}(y). \end{aligned}$$

□

Now let us sum up results related to $P\mathbf{S}$ in the following theorem:

Theorem 7.14. *If P is an inductive generalised category over B, then $(P\mathbf{S}, \otimes)$ is a weakly B-orthodox semigroup. Further, the natural pre-orders and partial orders in P and $P\mathbf{S}$ coincide.*

Proof. We first show that $(P\mathbf{S}, \otimes)$ has (C). Suppose that $x, y, z \in P\mathbf{S}$ and $x \mathrel{\widetilde{\mathcal{R}}_B} y$. It follows from Lemma 7.13 that $\mathbf{d}(x) = \mathbf{d}(y)$. We deduce that $z \otimes x = (z|_{ve}) \cdot ({}_{ve}|x)$ and $z \otimes y = (z|_{ve}) \cdot ({}_{ve}|y)$, where $v \in \mathbf{r}(z)$ and $e \in \mathbf{d}(x) = \mathbf{d}(y)$. Hence, $\mathbf{d}(z \otimes x) = \mathbf{d}(z|_{ve}) = \mathbf{d}(z \otimes y)$. By Lemma 7.13, $z \otimes x \mathrel{\widetilde{\mathcal{R}}_B} z \otimes y$. Dually, we can show that $\widetilde{\mathcal{L}}_B$ is a right congruence.

Let $x, y \in P$ and suppose that $x \leq_r y$ in P. Then $x = {}_e|y$ for some $e \leq_{\mathcal{L}} u \in \mathbf{d}(y)$. Hence,

$$e \otimes y = e|_{eu} \cdot {}_{eu}|y = eu \cdot {}_{eu}|y = {}_{eu}|y = {}_e|y = x,$$

so that $x \leq_r y$ in $P\mathbf{S}$.

If in addition we have $e \leq u$, so that $x \leq_r' y$ in P, then from $x = {}_e|y$ we have $\mathbf{d}(x) = R_e$ and $x^\dagger \leq_{\mathcal{R}} y^\dagger$, by Lemma 7.13, so $x \leq_r' y$ in $P\mathbf{S}$.

Conversely, if $x \leq_r y$ in $P\mathbf{S}$, then $x = f \otimes y$ for some $f \in B$. Hence,

$$x = f \otimes y = f|_{fy^\dagger} \cdot {}_{fy^\dagger}|y = fy^\dagger \cdot {}_{fy^\dagger}|y = {}_{fy^\dagger}|y,$$

so that $x \leq_r y$ in P.

Further, if $x \leq_r' y$ in $P\mathbf{S}$, then we have $x^\dagger \leq_{\mathcal{R}} y^\dagger$, so that $\mathbf{d}(x) \leq_{\mathcal{R}} \mathbf{d}(y)$, that is, $fy^\dagger \leq_{\mathcal{R}} y^\dagger$. Clearly then $fy^\dagger \leq y^\dagger$, so that $x \leq_r' y$ in P.

The dual result holds for $\leq_l$ and $\leq_l'$. $\square$

We can obtain an admissible morphism between weakly B-orthodox semigroups from a pseudo-functor between inductive generalised categories over bands. This is made more precise in the following lemma.

Lemma 7.15. *Let $F : P_1 \to P_2$ be a pseudo-functor between inductive generalised categories P_1 and P_2, where P_1 and P_2 are over bands B_1 and B_2, respectively. Then the map $F\mathbf{S} : P_1\mathbf{S} \to P_2\mathbf{S}$ defined by the rule that $xF\mathbf{S} = xF$, where $x \in P_1\mathbf{S}$, is an admissible morphism; moreover, if $F_1 : P_1 \to P_2$ and $F_2 : P_2 \to P_3$ are pseudo-functors, then $(F_1F_2)\mathbf{S} = F_1\mathbf{S}F_2\mathbf{S}$.*

Proof. We claim first that $F\mathbf{S}$ is a semigroup morphism. Suppose that $x, y \in P_1\mathbf{S}$. Then by the definition of $F\mathbf{S}$,

$$\begin{aligned}
(x \otimes y)F\mathbf{S} &= (x \otimes y)F \\
&= ((x|_{fu}) \cdot ({}_{fu}|y))F && \big(f \in \mathbf{r}(x), u \in \mathbf{d}(y)\big) \\
&= (x|_{fu})F \cdot ({}_{fu}|y)F && \big(\text{by (F3)}\big) \\
&= (xF|_{(fu)F}) \cdot ({}_{(fu)F}|yF) && \big(\text{by (F2), (F2)}^\circ\big) \\
&= (xF|_{fFuF}) \cdot ({}_{fFuF}|yF) && \big(\text{by } (F1)\big).
\end{aligned}$$

Since $f \in \mathbf{r}(x)$ and $u \in \mathbf{d}(y)$, it follows from the comments succeeding Definition 7.5 that $fF \in \mathbf{r}(xF)$ and $uF \in \mathbf{d}(yF)$. Thus,

$$(x \otimes y)F\mathbf{S} = xF \otimes yF = xF\mathbf{S} \otimes yF\mathbf{S}.$$

We now show that $F\mathbf{S}$ is admissible. Clearly, by (F1), $B_1F\mathbf{S} \subseteq B_2$. For any $e \in \mathbf{r}(x)$, we have $e \ \widetilde{\mathcal{L}}_{B_1} \ x$ and $eF \in \mathbf{r}(xF)$. Thus, $eF \ \widetilde{\mathcal{L}}_{B_2} \ xF$, that is, $eF\mathbf{S} \ \widetilde{\mathcal{L}}_{B_2} \ xF\mathbf{S}$. By a similar argument, we have that for any $k \in \mathbf{d}(x)$, $kF\mathbf{S} \ \widetilde{\mathcal{R}}_{B_2} \ xF\mathbf{S}$. By Lemma 2.9, $F\mathbf{S}$ is an admissible morphism between weakly B-orthodox semigroups $P_1\mathbf{S}$ and $P_2\mathbf{S}$.

The final part of the lemma is clear. □

Theorem 7.14 and Lemma 7.15 show that $\mathbf{S} : \mathcal{IGC} \to \mathcal{WO}$ is a functor.

7.3 Correspondence

In Section 7.2, we start with an inductive generalised category over B and construct a weakly B-orthodox semigroup. Our present aim is to prove a converse to this result and thus provide a correspondence between the class of inductive generalised categories over bands and the class of weakly B-orthodox semigroups, i.e. between $\mathcal{IGC}$ and $\mathcal{WO}$.

Let S be a weakly B-orthodox semigroup. We define $S\mathbf{C}$ to be the set S equipped with the following partial binary operation:

$$x \cdot y = \begin{cases} xy & \text{if } \ x^* \ \mathcal{D} \ y^\dagger \\ \text{undefined} & \text{otherwise,} \end{cases}$$

where xy is the product of x and y in S. This is known as the *trace product* and denoted by $S\mathbf{C} = (S, \cdot)$.

It is an immediate result that if $e, f \in B$ and $x \in S$ are such that $e \ \widetilde{\mathcal{R}}_B \ x \ \widetilde{\mathcal{L}}_B \ f$ then $e \cdot x = x = x \cdot f$.

We now turn to give a number of basic properties of $S\mathbf{C}$, which will be found useful in the sequel.

Lemma 7.16. *If $\exists x \cdot y$ in $S\mathbf{C}$, then $x \ \widetilde{\mathcal{R}}_B \ xy \ \widetilde{\mathcal{L}}_B \ y$ in S.*

Proof. Suppose that x and y are in S such that $x \cdot y$ is defined in $S\mathbf{C}$. Then $x^* \;\mathcal{D}\; y^\dagger$. We assume that $x^* \;\mathcal{L}\; h \;\mathcal{R}\; y^\dagger$, where $h \in B$. Since $\widetilde{\mathcal{R}}_B$ is a left congruence and $\widetilde{\mathcal{L}}_B$ is a right congruence, it follows that $xy \;\widetilde{\mathcal{R}}_B\; xy^\dagger \;\widetilde{\mathcal{R}}_B\; xh = x$ and dually, $xy \;\widetilde{\mathcal{L}}_B\; x^*y \;\widetilde{\mathcal{L}}_B\; hy = y$. So $x \;\widetilde{\mathcal{R}}_B\; xy \;\widetilde{\mathcal{L}}_B\; y$, as required. □

Lemma 7.17. *If S is a weakly B-orthodox semigroup, then $S\mathbf{C}$ is a generalised category over B such that $\mathbf{d}(x) = R_{x^\dagger}$ and $\mathbf{r}(x) = L_{x^*}$.*

Proof. We have $x \in \operatorname{Mor}(R_e, L_f)$ if and only if $x^\dagger \;\mathcal{R}\; e$ and $x^* \;\mathcal{L}\; f$ in B. If in addition $y \in \operatorname{Mor}(R_g, L_h)$, then $\exists x \cdot y$ in $S\mathbf{C}$ if and only if $x^* \;\mathcal{D}\; y^\dagger$, i.e. $D_f = D_g$. Moreover, if $\exists x \cdot y$, then $x \cdot y \in \operatorname{Mor}(R_e, L_h)$ by Lemma 7.16. Clearly Condition (GC3) holds.

For any $e \in B$, we take the distinguished morphism e associated to e to be itself, whose domain is R_e and codomain is L_e. Certainly, if $e \in \mathbf{d}(x)$ $\big($resp. $e \in \mathbf{r}(x)\big)$, then e is a left (resp. right) identity of x. Hence, (GC4) holds. □

We build on the above to show that $S\mathbf{C}$ may be equipped with restrictions and co-restrictions, under which it becomes an inductive generalised category.

For $x \in S$ and $e, f \in B$ with $e \leq_{\mathcal{L}} u \in \mathbf{d}(x)$ and $f \leq_{\mathcal{R}} v \in \mathbf{r}(x)$,

$$_e|x = ex \text{ and } x|_f = xf.$$

Lemma 7.18. *Let S be a weakly B-orthodox semigroup. With the above definition of restriction and co-restriction, $S\mathbf{C}$ becomes an inductive generalised category over B. Further, the natural pre-orders and partial orders in S and $S\mathbf{C}$ coincide.*

Proof. In view of Lemma 7.17, it remains to show that $S\mathbf{C}$ with the restriction and co-restriction defined above satisfies Conditions (I1) to (I6) and the duals (I1)°, (I2)° and (I3)° of (I1), (I2) and (I3).

(I1) If $x \in S$ and $e, u \in B$ with $e \leq_{\mathcal{L}} u \in \mathbf{d}(x)$, then $_e|x = ex$, and so by Lemmas 2.8 and 2.14, Condition (I1) is satisfied.

(I2) Since restriction and co-restriction are given by multiplication in S, it is clear that (I2) and its dual hold.

(I3) Suppose that $x, y \in S$ and $e, u \in B$ with $x \cdot y$ defined in $S\mathbf{C}$, let $e \leq_{\mathcal{L}} u \in \mathbf{d}(x)$ and $f \in \mathbf{r}(_e|x) = L_{(ex)^*}$. Then $_e|(x \cdot y) = exy = exfy = (_e|x) \cdot (_f|y)$.

(I4) It is routine to check that Condition (I4) holds, both products being equal to xy.

(I5) As for (I4) this is again routine, with both sides of the equality we must verify being equal to exf.

(I6) Clearly, it is satisfied by the definitions of the restriction and co-restriction, respectively.

Now, let $x, y \in S$. Then

$$\begin{array}{lll} x \leq_r y \text{ in } S & \Leftrightarrow \; x = ey & \text{some } e \in B \\ & \Leftrightarrow \; x = ey^\dagger y & \text{some } e \in B,\ y^\dagger \in \mathbf{d}(y) \\ & \Leftrightarrow \; x = {}_{ey^\dagger}|y & \text{some } e \in B,\ y^\dagger \in \mathbf{d}(y) \\ & \Leftrightarrow \; x = {}_f|y & \text{some } f \in B \text{ with } f \leq_{\mathcal{L}} u \in \mathbf{d}(y) \\ & \Leftrightarrow \; x \leq_r y \text{ in } S\mathbf{C}. & \end{array}$$

In addition, with notation as above, if $x \leq_r' y$ in S we have that $x^\dagger \leq_{\mathcal{R}} y^\dagger$, so that $x = y^\dagger e y^\dagger y = {}_{y^\dagger e y^\dagger}|y$ and $y^\dagger e y^\dagger \leq y^\dagger$, and so $x \leq_r' y$ in $S\mathbf{C}$. Conversely, if $x \leq_r' y$ in $S\mathbf{C}$, then $x = {}_g|y$, where $g \leq y^\dagger \in \mathbf{d}(y)$. Then $x = gy$ in S, and $x^\dagger \ \mathcal{R} \ g \leq_{\mathcal{R}} y^\dagger$, so that $x \leq_r' y$ in S. □

Proposition 7.19. *Let S be a weakly B-orthodox semigroup and P be an inductive generalised category over B. Then $S\mathbf{CS} = S$ and $P\mathbf{SC} = P$.*

Proof. Let S be a weakly B-orthodox semigroup. It follows from Lemma 7.18 that $S\mathbf{C}$ is an inductive generalised category over B with multiplication a restriction of that in S and $\mathbf{d}(x) = R_{x^\dagger}$, $\mathbf{r}(x) = L_{x^*}$, for any $x \in S$, and if $e \leq_{\mathcal{L}} u \in \mathbf{d}(x)$ and $f \leq_r v \in \mathbf{r}(x)$ then ${}_e|x = ex$ and $x|_f = xf$.

We now construct $S\mathbf{CS}$, which again has underlying set S, by defining the pseudo-product

$$x \otimes y = (x|_{vg}) \cdot ({}_{vg}|y),$$

where $v \in \mathbf{r}(x) = L_{x^*}$ and $g \in \mathbf{d}(y) = R_{y^\dagger}$. Observe that

$$x \otimes y = (x|_{vg}) \cdot ({}_{vg}|y) = xvgvgy = xvgy = xy,$$

so the operations in S and $S\mathbf{CS}$ are the same. Moreover, the distinguished bands of S and $S\mathbf{CS}$ are both B. Hence $S = S\mathbf{CS}$.

We now focus on the converse. Let P be an inductive generalised category over B with partial binary operation $\cdot$. We establish the weakly B-orthodox semigroup $P\mathbf{S}$ by defining the pseudo-product $\otimes$ of Theorem 7.14.

We temporarily use the notation $\odot$ for the partial binary operation in $P\mathbf{SC}$. For any $x, y \in P$ we have

$$\begin{aligned} \exists x \odot y &\Leftrightarrow x^* \, \mathcal{D} \, y^\dagger \text{ in } P\mathbf{S} \\ &\Leftrightarrow e \, \mathcal{D} \, f, \text{ where } \mathbf{r}(x) = L_e \text{ and } \mathbf{d}(y) = R_f \\ &\Leftrightarrow \exists x \cdot y \text{ in } P. \end{aligned}$$

Further, if $\exists x \odot y$, then by Lemma 7.8,

$$x \odot y = x \otimes y = x \cdot y.$$

For $x \in P$ we have that $\mathbf{d}(x) = R_{x^\dagger}$ in $P\mathbf{SC}$, where $x \, \widetilde{\mathcal{R}}_B \, x^\dagger$ in $P\mathbf{S}$. But, the latter holds if and only if $x^\dagger \in \mathbf{d}(x)$ in P, i.e. $\mathbf{d}(x) = R_{x^\dagger}$ in P. Thus $\mathbf{d}$ in P and $P\mathbf{SC}$ coincide, and dually for $\mathbf{r}$.

Clearly, the distinguished morphisms in P and $P\mathbf{SC}$ are the same.

Again as a temporary measure, we use $||$ to denote restriction and co-restriction in $P\mathbf{SC}$.

Let $x \in P$ and let $e, u \in B$ with $e \leq_{\mathcal{L}} u \in \mathbf{d}(x)$. Then in $P\mathbf{SC}$,

$${}_e||x = e \otimes x = e|_{eu} \cdot {}_{eu}|x = {}_{eu}|x = {}_e|x$$

and similarly for co-restrictions. □

We now proceed to establish an isomorphism between $\mathcal{IGC}$ and $\mathcal{WO}$.

The next lemma demonstrates that an admissible morphism between two weakly B-orthodox semigroups gives rise to a pseudo-functor.

Lemma 7.20. *Let S be a weakly B_1-orthodox semigroup and T be a weakly B_2-orthodox semigroup. Suppose that θ is an admissible morphism. Then the map $\theta\mathbf{C} : S\mathbf{C} \to T\mathbf{C}$ given by the rule that $x\theta\mathbf{C} = x\theta$ for $x \in B_1$ and $x \in S$ is a pseudo-functor. Further, if $\theta_1 : S \to T$ and $\theta_2 : T \to Q$ are admissible morphisms, then $(\theta_1\theta_2)\mathbf{C} = \theta_1\mathbf{C}\theta_2\mathbf{C}$.*

Proof. (F1) Since θ is an admissible morphism, it follows that θ is a morphism from B_1 to B_2.

(F2) Suppose that $x \in S$ and $e, f \in B_1$ with $e \leq_{\mathcal{L}} f \in \mathbf{d}(x)$. Then ${}_e|x$ is defined and ${}_e|x = ex$. Since θ is admissible, it follows that $e\theta \leq_{\mathcal{L}} f\theta$ and $f\theta \; \widetilde{\mathcal{R}}_{B_2} \; x\theta$, that is, $f\theta \in \mathbf{d}(x\theta)$, which implies that ${}_{e\theta}|x\theta$ is defined. Then $({}_e|x)\theta\mathbf{C} = (ex)\theta\mathbf{C} = (ex)\theta = e\theta x\theta = {}_{e\theta}|x\theta = {}_{e\theta\mathbf{C}}|x\theta\mathbf{C}$.

(F3) If $\exists x{\cdot}y$ in $S\mathbf{C}$, then $x^* \, \mathcal{D} \, y^\dagger$. Hence there is an $h \in B$ with $x \; \widetilde{\mathcal{L}}_{B_1} \; h \; \widetilde{\mathcal{R}}_{B_1} \; y$. Since θ is admissible, $x\theta \; \widetilde{\mathcal{L}}_{B_2} \; h\theta \; \widetilde{\mathcal{R}}_{B_2} \; y\theta$ and $h\theta \in B_2$. Thus $\exists x\theta \cdot y\theta$ in $T\mathbf{C}$. Clearly, if $x \cdot y$ exists, $(x \cdot y)\theta = (xy)\theta = x\theta y\theta = x\theta \cdot y\theta$, since θ is a morphism.

It is routine to see that $(\theta_1\theta_2)\mathbf{C} = \theta_1\mathbf{C}\theta_2\mathbf{C}$. □

The following result is easy to see, given Lemma 7.15 and 7.20.

Lemma 7.21. *Let $\theta : S \to T$ be an admissible morphism of weakly B-orthodox semigroups, and $F : P_1 \to P_2$ be a pseuo-functor of inductive generalised categories over bands. Then $\theta\mathbf{CS} = \theta$ and $F\mathbf{SC} = F$.*

Lemmas 7.18 and 7.20 show that $\mathbf{C} : \mathcal{WO} \to \mathcal{IGC}$ is a functor and Proposition 7.19 and Lemma 7.21 give that $\mathbf{S}$ and $\mathbf{C}$ are mutually inverse. Hence we deduce our main result.

Theorem 7.22. *The category $\mathcal{WO}$ of weakly B-orthodox semigroups and admissible morphisms is isomorphic to the category $\mathcal{IGC}$ of inductive generalised categories over bands and pseudo-functors.*

7.4 Special cases

In this section, we concentrate on some special kinds of weakly B-orthodox semigroups. We now present a lemma which will be used in our first two cases.

Lemma 7.23. *Let P be an inductive generalised category over B. Suppose that for all $x \in E(P\mathbf{S})$ and $e \in \mathbf{d}(x)$, $f \in \mathbf{r}(x)$ we have $e \, \mathcal{R}^* \, x \, \mathcal{L}^* \, f$ in $P\mathbf{S}$. Then $E(P\mathbf{S}) = B$.*

Proof. Suppose that $x \in P$ and $x \otimes x = x$. Assume that $f \in \mathbf{r}(x)$ and $e \in \mathbf{d}(x)$ with $e \; \mathcal{R}^* \; x \; \mathcal{L}^* \; f$, and so $f \otimes x = f$, that is, $f|_{fe} \cdot {}_{fe}|x = f$, or equivalently, ${}_{fe}|x = f$, which implies that $fe \; \mathcal{R} \; f$. Dually, $fe \; \mathcal{L} \; e$. Thus, $e \; \mathcal{D} \; f$ so that $e \; \mathcal{R} \; ef \; \mathcal{L} \; f$. Since $e \; \mathcal{R}^* \; x \; \mathcal{L}^* \; f$, we have $x \; \mathcal{H}^* \; ef$. An $\mathcal{H}^*$ class contains at most an idempotent so that $x = ef$. □

An (inductive) generalised category P is an *(inductive) generalised groupoid* if for all $x \in P$ with $\mathbf{d}(x) = R_e$ and $\mathbf{r}(x) = L_f$, there exists $y \in P$ with $\mathbf{d}(y) = R_f$ and $\mathbf{r}(y) = L_e$ such that $e = x \cdot y$ and $y \cdot x = f$.

Corollary 7.24. *The category of orthodox semigroups and morphisms is isomorphic to the category of inductive generalised groupoids over bands and pseudo-functors.*

Proof. Let S be an orthodox semigroup with $B = E(S)$. Suppose that $x \in S\mathbf{C}$ with $\mathbf{d}(x) = R_e$ and $\mathbf{r}(x) = L_f$. Since $\mathcal{R} = \widetilde{\mathcal{R}}_B$ and $\mathcal{L} = \widetilde{\mathcal{L}}_B$, we have that $e\,\mathcal{R}\,x\,\mathcal{L}\,f$. It follows from the fact that S is regular that there exists $y \in S$ with $e = xy$ and $yx = f$. We have that $e\,\mathcal{L}\,y\,\mathcal{R}\,f$ and so $\mathbf{d}(y) = R_f$ and $\mathbf{r}(y) = L_e$ and the products $x \cdot y$, $y \cdot x$ exist in $S\mathbf{C}$. Moreover, $x \cdot y = xy = e$ and $y \cdot x = yx = f$.

Conversely, let P be an inductive generalised groupoid over B. Suppose that $x \in P$ and $\mathbf{d}(x) = R_e$, $\mathbf{r}(x) = L_f$. Then there exists $y \in P$ with $\mathbf{d}(y) = R_f$ and $\mathbf{r}(y) = L_e$ such that $f = y \cdot x$ and $e = x \cdot y$. And so $x \otimes y \otimes x = (x \cdot y) \otimes x = e \otimes x = e \cdot x = x$. Thus, $P\mathbf{S}$ is regular. In addition, as $e = x \cdot y = x \otimes y$ and $x = e \otimes x$, we have that $e\,\mathcal{R}\,x$ in $P\mathbf{S}$. Dually, $f\,\mathcal{L}\,x$ in $P\mathbf{S}$. By Lemma 7.23, we have that $E(P\mathbf{S}) = B$. Hence, $P\mathbf{S}$ is an orthodox semigroup. $\square$

Now, we focus on the class of abundant semigroups. We replace the distinguished set of idempotents B by the whole set of idempotents and use relations $\mathcal{R}^*$ and $\mathcal{L}^*$ instead of $\widetilde{\mathcal{R}}_B$ and $\widetilde{\mathcal{L}}_B$ in the definition of weakly B-orthodox semigroups. We thus obtain the class of abundant semigroups whose set of idempotents forms a band. An admissible morphism in this context is more usually referred to as a *good* morphism. We define an inductive generalised category P over a band B to be *abundant* if it satisfies the following condition and its dual (I7)°:

(I7) if $e, f, g \in B$ and $x, y, z \in P$ are such that $e, f \leq_{\mathcal{L}} g \in \mathbf{d}(x)$, $e \in \mathbf{r}(y)$, $f \in \mathbf{r}(z)$ and $y \cdot {}_e|x = z \cdot {}_f|x$, then $y = z$.

Corollary 7.25. *The category of abundant semigroups whose set of idempotents forms a band and good morphisms is isomorphic to the category of abundant inductive generalised categories over bands and pseudo-functors.*

Proof. Let P be an abundant inductive generalised category over a band B. Suppose that $x \in P$, $e \in \mathbf{d}(x)$ and $f \in \mathbf{r}(x)$. We know that $e\,\widetilde{\mathcal{R}}_B\,x$ in $P\mathbf{S}$, so that $e \otimes x = x$. Assume that $y, z \in P$ with $y \otimes x = z \otimes x$, giving that

$(y|_{y^*e}) \cdot ({}_{y^*e}|x) = (z|_{z^*e}) \cdot ({}_{z^*e}|x)$, where $y^* \in \mathbf{r}(y)$ and $z^* \in \mathbf{r}(z)$. By (I7), we obtain that $y|_{y^*e} = z|_{z^*e}$. Thus, $y^*e \,\mathcal{L}\, z^*e$ in B. We have

$$\begin{aligned} y \otimes e &= y|_{y^*e} \cdot {}_{y^*e}|e \\ &= z|_{z^*e} \cdot y^*e && \big(\text{by(I6)}\big) \\ &= z|_{z^*e} && \big(y^*e \,\mathcal{L}\, z^*e\big) \\ &= z|_{z^*e} \cdot {}_{z^*e}|e \\ &= z \otimes e. \end{aligned}$$

This is enough to show that $e \; \mathcal{R}^* \; x$. Dually, we have that $f \; \mathcal{L}^* \; x$.

In view of Lemma 7.23, we have that $E(P\mathbf{S}) = B$.

Conversely, let S be an abundant semigroup with $E(S) = B$. It follows that $\mathcal{R}^* = \widetilde{\mathcal{R}}_B$ and $\mathcal{L}^* = \widetilde{\mathcal{L}}_B$. In view of Lemma 7.18, it is sufficient to claim that $S\mathbf{C}$ satisfies Conditions (I7), and dually, (I7)$^\circ$. Assume that $e, f, g \in B$ and $x, y, z \in P$ are such that $e, f \leq_{\mathcal{L}} g \in \mathbf{d}(x)$, $e \in \mathbf{r}(y)$, $f \in \mathbf{r}(z)$ and $y \cdot {}_e|x = z \cdot {}_f|x$. It follows that $yex = zfx$. Since $g \in \mathbf{d}(x)$, that is, $g \,\mathcal{R}^*\, x$ in S, we have that $yeg = zfg$, that is, $ye = zf$, as $e, f \leq_{\mathcal{L}} g$. Hence, $y = z$, as required. Dually, (I7)$^\circ$ holds. □

We now discuss Ehresmann semigroups. Let S be an Ehresmann semigroup with distinguished semilattice E. We mentioned in Lemma 2.30 that $\leq_r = \leq'_r$ and $\leq_l = \leq'_l$.

Let P be an inductive generalised category over E. The context

$$(E, E/\mathcal{R}, E/\mathcal{L}, E/\mathcal{D}, p)$$

is essentially four copies of E equipped with the identity map. We therefore identify E with $E/\mathcal{R}$, $E/\mathcal{L}$ and $E/\mathcal{D}$ and note that P becomes a category in the usual sense. Notice that as $P = P\mathbf{SC}$, we have that $\leq_r = \leq'_r$, $\leq_l = \leq'_l$ and $\leq_r$ and $\leq_l$ are partial orders on P.

Lemma 7.26. *An inductive generalised category P over a semilattice E with $\leq_r$ forms an ordered$_1$ category with restriction.*

Proof. From comments above P is a category (with the appropriate identifications) and $(P, \leq_r)$ is a poset.

(OC1) Suppose that $x, y \in P$ with $x \leq_r y$. Then there exists $e \in E$ such that $e \leq \mathbf{d}(y)$ and $x = {}_e|y$. Thus, $\mathbf{d}(x) = e \leq \mathbf{d}(y)$ and $\mathbf{r}(x) \leq \mathbf{r}(y)$ by (I1).

(OC2) Suppose that $x, y \in P$ with $\mathbf{r}(x) = \mathbf{r}(y)$, $\mathbf{d}(x) = \mathbf{d}(y)$ and $x \leq_r y$. Then there exists $e \in E$ such that $e \leq \mathbf{d}(y)$ and $x = {}_e|y$. Certainly, $\mathbf{d}(x) = e$ and so $e = \mathbf{d}(y)$, whence from (I1), $x = y$.

(OC3) If $x' \leq_r x$ and $y' \leq_r y$, and both $x' \cdot y'$ and $x \cdot y$ exist, then there exist $e, f \in E$ such that $e \leq \mathbf{d}(x)$, $f \leq \mathbf{d}(y)$, $x' = {}_e|x$ and $y' = {}_f|y$. Thus, we have that $\mathbf{r}({}_e|x) = \mathbf{r}(x') = \mathbf{d}(y') = \mathbf{d}({}_f|y) = f$, and so $x' \cdot y' = ({}_e|x) \cdot ({}_f|y) = {}_e|(x \cdot y)$ by (I3). Hence, $x' \cdot y' \leq_r x \cdot y$.

Finally, we assume that $x \in P$ and $e \in E$ with $e \leq \mathbf{d}(x)$. Then ${}_e|x$ is defined and $\mathbf{d}({}_e|x) = e$. Also, ${}_e|x \leq_r x$ by (I1). Further, ${}_e|x$ is unique since if $z \leq_r x$ and $e = \mathbf{d}(z)$, then there exists $h \in E$ with $h \leq \mathbf{d}(x)$ and $z = {}_h|x$, which gives that $h = \mathbf{d}(z)$. Thus, $e = h$. Hence, $z = {}_e|x$. □

As a dual result of Lemma 7.26, we have the following lemma.

Lemma 7.27. *An inductive generalised category P over a semilattice E with $\leq_l$ forms an ordered$_1$ category with co-restriction.*

Next we show that an inductive generalised category P over a semilattice E is an Ehresmann category as defined in [32] and explained in Chapter 6.

Lemma 7.28. *An inductive generalised category P over a semilattice E with the pair of natural partial orders $(\leq_r, \leq_l)$ forms an Ehresmann category.*

Conversely, an Ehresmann category $(C, \cdot, \leq_r, \leq_l)$ with semilattice of identities E, may be regarded as an inductive generalised category over E with natural partial orders $(\leq_r, \leq_l)$.

Proof. Let P be inductive generalised category over a semilattice E. In view of the above discussion, we have claimed that P is a category with set of identities E. By Lemma 7.26 and Lemma 7.27, Conditions (E1) and (E1)$^\circ$ are satisfied.

(E2) If $e, f \in E$ and $e \leq_r f$, then $e = {}_e|f = ef$ so that we must have $e \leq f$. Then $f|_e$ is defined and $f|_e = fe = e$ so that $e \leq_l f$. Together with the dual, we have that for $e, f \in E$,

$$e \leq_r f \Leftrightarrow e \leq_l f \Leftrightarrow e \leq f,$$

so that in particular, (E2) holds.

(E3) Clearly, E is a semilattice under $\leq_r = \leq_l = \leq$.

(E4) To show that $\leq_r \circ \leq_l \subseteq \leq_l \circ \leq_r$, we assume that $x \leq_r \circ \leq_l y$. Then there exists $z \in P$ such that $x \leq_r z \leq_l y$. And so there exist $e, f \in E$ with $\mathbf{d}(x) = e \leq \mathbf{d}(z) = u$ and $\mathbf{r}(z) = f \leq \mathbf{r}(y) = v$, such that $x = {}_e|z$ and $z = y|_f$. Thus, $x = {}_e|(y|_f) = {}_{eu}|(y|_{vf})$. By (I4), we get that $x = ({}_{eh}|y)|_{gf}$, where $h = \mathbf{d}(y)$ and $g = \mathbf{r}({}_{eh}|y)$. Set $z' = {}_{eh}|y$. Then $x \leq_l z'$ and $z' \leq_r y$. Consequently, $x \leq_l \circ \leq_r y$. With the dual, we obtain (E4).

(E5) Suppose that $x, y \in P$ and $f \in E$ with $x \leq_r y$. Then there exists $k \in E$ with $k \leq \mathbf{d}(y)$ and $x = {}_k|y$. So $x|_{\mathbf{r}(x)f} = ({}_k|y)|_{\mathbf{r}(x)f} = ({}_{k\mathbf{d}(y)}|y)|_{\mathbf{r}(x)f}$. Let $h = \mathbf{d}(y|_{\mathbf{r}(y)f})$. By (I4), we obtain that $({}_{k\mathbf{d}(y)}|y)|_{\mathbf{r}(x)f} = {}_{kh}|(y|_{\mathbf{r}(y)f})$, so that $x|_{\mathbf{r}(x)f} \leq_r y|_{\mathbf{r}(y)f}$.

Conversely, let $C = (C, \cdot, \leq_r, \leq_l)$ be an Ehresmann category with semilattice of identities E. Then $C = (C, \cdot)$ may also be regarded as generalised category over E.

We let $\leq$ denote the restriction of $\leq_r$ (resp. $\leq_l$) to E. It is clear that the first part of (I1) holds, moreover, by uniqueness of restriction, ${}_e|x = x$ if $e = \mathbf{d}(x)$.

For (I2), if $x \in C$ and $e, f, g, u \in E$, with $e \leq_{\mathcal{L}} g \;\mathcal{R}\; f \leq_{\mathcal{L}} u \in \mathbf{d}(x)$, then this simplifies to $e \leq f \leq \mathbf{d}(x)$. Now ${}_{ef}|x = {}_e|x \leq_r x$ and $\mathbf{d}({}_e|x) = e$; also, ${}_e|({}_f|x) \leq_r {}_f|x \leq_r x$ and $\mathbf{d}({}_e|({}_f|x)) = e$. By uniqueness of restriction, ${}_{ef}|x = {}_e|({}_f|x)$.

(I3) If $x, y \in C$ with $\exists x \cdot y$, then $\mathbf{r}(x) = \mathbf{d}(y)$. If $e \leq \mathbf{d}(x)$, then we have

$$ {}_e|(x \cdot y) \leq_r x \cdot y \text{ and } \mathbf{d}({}_e|x \cdot y) = e $$

and also

$$ ({}_e|x) \cdot ({}_f|y) \leq_r x \cdot y \text{ and } \mathbf{d}(({}_e|x) \cdot ({}_f|y)) = e, $$

where $f = \mathbf{r}({}_e|x)$. Hence, ${}_e|(x \cdot y) = ({}_e|x) \cdot ({}_f|y)$.

(I4) This is clear.

(I5) Let $x \in C$ and $e, f, u, v, g, h \in E$ with $g = \mathbf{r}(x)$, $h = \mathbf{d}(x)$, $u = \mathbf{d}(x|_{gf})$ and $v = \mathbf{r}({}_{eh}|x)$. Then $(e \overline{\otimes} x) \overline{\otimes} f = e \overline{\otimes} (x \overline{\otimes} f)$, where $\overline{\otimes}$ is defined [32] and recalled in Chapter 6, by $x \overline{\otimes} y = (x|_k) \cdot ({}_k|y)$, where $k = \mathbf{r}(x)\, \mathbf{d}(y)$. As shown in [32], $\overline{\otimes}$ is associative, hence,

$$\begin{aligned}(e\overline{\otimes}x)\overline{\otimes}f &= ((e|_{eh})\cdot({}_{eh}|x))|_{vf}\cdot({}_{vf}|f)\\ &= (eh\cdot({}_{eh}|x))|_{vf}\cdot vf\\ &= ({}_{eh}|x)|_{vf}\end{aligned}$$

and similarly, $e\overline{\otimes}(x\overline{\otimes}f) = {}_{eu}|(x|_{gf})$, so we obtain that $({}_{eh}|x)|_{vf} = {}_{eu}|(x|_{gf})$.

(I6) Suppose that $e, g, h, u, v \in E$ are such that $u \leq_{\mathcal{R}} g \;\mathcal{L}\; e$ and $v \leq_{\mathcal{L}} h \;\mathcal{R}\; e$, which simplify to $u \leq e$ and $v \leq e$. Clearly, $e|_u = u = eu$ and ${}_v|e = v = ve$. $\square$

Let $\mathbf{C} = (C, \cdot, \leq_r, \leq_l)$ and $\mathbf{D} = (D, \cdot, \leq_r, \leq_l)$ be Ehresmann categories with semilattice E_C and E_D of identities, respectively. We recall from Chapter 6 that a *strongly ordered functor* [32] $F : \mathbf{C} \to \mathbf{D}$ is a functor which preserves $\leq_r, \leq_l$ and the binary operation of the semilattices. Hence F is a morphism $E_C \to E_D$. As shown in [32], F preserves restrictions and co-restrictions. Thus F is a pseudo-functor in the sense of Definition 7.5.

On the other hand, if $G : \mathbf{C} \to \mathbf{D}$ is a pseudo-functor, then from the comments following Definition 7.5, G is a functor, which by (F1) preserves $\wedge$. Suppose now that $x, y \in \mathbf{C}$ with $x \leq_r y$. Then $x = {}_e|y$ for some $e \in E$, and so by (F2), $xG = {}_{eG}|yG$ so that $xG \leq_r yG$. Dually, G preserves $\leq_l$, so that G is a strongly ordered functor. Theorem 7.22, Lemma 7.18 and the comments above now give us Lawson's result from [32], Theorem D.

Corollary 7.29. *[6, Theorem 4.24] The category of Ehresmann semigroups and admissible morphisms is isomorphic to the category of Ehresmann categories and strongly ordered functors.*

We now turn to *weakly B-superabundant semigroups* with (C), which are weakly B-orthodox semigroups such that each $\widetilde{\mathcal{H}}_B$-class contains a distinguished idempotent in B. We say that a generalised category P over a band B is a *super-generalised category* if it is an inductive generalised category and satisfies the following condition:

(I8) if $x \in P$, $e \in \mathbf{d}(x)$ and $f \in \mathbf{r}(x)$, then $e \;\mathcal{D}\; f$.

Corollary 7.30. *The category of weakly B-superabundant semigroups with (C) and admissible morphisms is isomorphic to the category of super-generalised categories over B and pseudo-functors.*

Proof. Let S be a weakly B-superabundant semigroup with (C). It follows from Lemma 7.18 that it is only necessary to show that $S\mathbf{C}$ satisfies Condition (I8). Suppose that $x \in S$, $e \in \mathbf{d}(x)$ and $f \in \mathbf{r}(x)$. Then $e \ \widetilde{\mathcal{R}}_B \ x \ \widetilde{\mathcal{L}}_B \ f$ in S. As S is a weakly B-superabundant semigroup, it follows that there exists $h \in B$ such that $h \ \widetilde{\mathcal{H}}_B \ x$. Thus, $e \ \mathcal{R} \ h \ \mathcal{L} \ f$, which implies that $e \ \mathcal{D} \ f$.

Conversely, let P be a super-generalised category over B. It is sufficient to show that $P\mathbf{S}$ is weakly B-superabundant. Suppose that $x \in P$, $e \in \mathbf{d}(x)$ and $f \in \mathbf{r}(x)$. Then by (I8), $e \ \mathcal{D} \ f$, that is, $e \ \mathcal{R} \ ef \ \mathcal{L} \ f$. As $e \ \widetilde{\mathcal{R}}_B \ x \ \widetilde{\mathcal{L}}_B \ f$ in $P\mathbf{S}$, we get that $x \ \widetilde{\mathcal{H}}_B \ ef$. Hence $P\mathbf{S}$ is a weakly B-superabundant semigroup with (C). □

Finally in this chapter, we discuss the class of weakly B-orthodox semigroups which have Condition (WIC) mentioned in Chapter 2. We define an inductive generalised category P over a band B to be *connected* if it satisfies the following condition and its dual (I9)°:

(I9) if $x \in P$ and $e \leq u \in \mathbf{d}(x)$ then there exists $f \leq v \in \mathbf{r}(x)$ such that ${}_e|x = x|_f$.

Corollary 7.31. *The category of weakly B-orthodox semigroups with (WIC) and admissible morphisms is isomorphic to the category of connected inductive generalised categories over bands and pseudo-functors.*

Proof. Let S be a weakly B-orthodox semigroup with (WIC). In view of Lemma 7.18, it remains to show that $S\mathbf{C}$ satisfies Conditions (I9) and (I9)°. We will show that (I9) holds, dually, (I9)° holds. Suppose that $x \in S$ and $e \leq u \in \mathbf{d}(x)$. Then ${}_e|x = ex$. Since S has (WIC), it follows that there exists $f \in B$ such that $ex = xf$. Then $ex = xvfv$, where $v \in \mathbf{r}(x)$. Thus, $ex = xvfv = x|_{vfv}$.

Conversely, let P be a connected inductive generalised category over a band B. Suppose that $x \in P$ and $e \leq u \in \mathbf{d}(x)$. Then it follows from (I9) that there exists $f \leq v \in \mathbf{r}(x)$ such that ${}_e|x = x|_f$. Thus $e \otimes x = {}_e|x = x|_f = x \otimes f$. Together with the dual argument, we have shown that $P\mathbf{S}$ has (WIC). □

Chapter 8

Trace of weakly B-orthodox semigroups

A weakly B-orthodox semigroup S with zero is *primitive* if for any $e, f \in B$, $e \leq f$ implies that $e = 0$ or $e = f$. Here $e \leq f$ if and only if $ef = fe = e$. The purpose of this chapter is to show that the trace of a weakly B-orthodox semigroup is a primitive weakly B-orthodox semigroup and investigate primitive weakly B-orthodox semigroups via blocked Rees matrix semigroups, which are introduced in [10].

8.1 Preliminaries

In this section our aim is to list some properties of primitive weakly B-orthodox semigroups. Throughout this section we use S to denote a primitive weakly B-orthodox semigroup, unless stated otherwise. Green's relations will always refer to B. For any subset T of a semigroup S, we will use T^* to denote the set of non-zero elements of T.

Lemma 8.1. *Let S be a weakly U-abundant semigroup with zero and x, y be non-zero elements in S with $x \, \widetilde{\mathcal{R}}_U \, e$ and $y \, \widetilde{\mathcal{L}}_U \, f$, where $e, f \in U$. Then $yx = 0$ if and only if $fe = 0$.*

Proof. From $x \, \widetilde{\mathcal{R}}_U \, e$, we have $yx \, \widetilde{\mathcal{R}}_U \, ye$, and so if $yx = 0$, then $ye = 0$. Again, by $y \, \widetilde{\mathcal{L}}_U \, f$, we have $ye \, \widetilde{\mathcal{L}}_U \, fe$, and so $fe = 0$.

Conversely, if $fe = 0$ then $yx = yfex = 0$. □

Lemma 8.2. *If e, f are distinguished idempotents in S and $eS \subseteq fS$ (resp. $Se \subseteq Sf$), then $eS = fS$ (resp. $Se = Sf$) or $e = 0$.*

Proof. Suppose that $e, f \in B$ and $eS \subseteq fS$. Then $e \in fS$, and so $fe = e$, which implies that $e \mathcal{R} ef \leq f$. Since S is primitive, we obtain that $ef = 0$ or $ef = f$. In the former case, we have that $e = ee = efe = 0$, and in the latter case, we obtain that $e \mathcal{R} f$, that is, $eS = fS$. □

Lemma 8.3. *For any $e \in B^*$ and $a \in S^*$, $a \widetilde{\mathcal{L}}_B e$ (resp. $a \widetilde{\mathcal{R}}_B e$) if and only if $a \in Se$ (resp. $a \in eS$) and Se (resp. eS) is contained in every distinguished idempotent-generated left (resp. right) ideal containing a.*

Proof. Suppose that $a \widetilde{\mathcal{L}}_B e$. By Lemma 2.6, $\widetilde{L}(a) = \widetilde{L}(e)$. As $\widetilde{L}(e) = Se$, $a \in Se$. If $a \in Sf$ for some $f \in B$, then $af = a$, and so $ef = e$ giving $Se \subseteq Sf$.

Conversely, suppose that $a \in Se$ and that for any $f \in B$, $a \in Sf$ implies that $Se \subseteq Sf$. Then $\widetilde{L}(a) \subseteq Se$ since Se is a left B- admissible ideal containing a. As S is weakly B-orthodox, it follows from Lemma 2.7 that $\widetilde{L}(a) = Sf$ for some $f \in B$. Hence, by Lemma 8.2, $Se = Sf$, and so $a \widetilde{\mathcal{L}}_B e$. □

The next lemma is an immediate consequence of Lemma 8.2 and Lemma 8.3.

Lemma 8.4. *Let $a \in S$ and $e \in B^*$. Then $a \widetilde{\mathcal{L}}_B e$ (resp. $a \widetilde{\mathcal{R}}_B e$) if and only if $a \neq 0$ and $a \in Se$ (resp. $a \in eS$).*

Lemma 8.5. *If x, y are non-zero elements in S such that $xy \neq 0$, then $xy \in \widetilde{R}_x \cap \widetilde{L}_y$.*

Proof. Suppose that e and f are distinguished idempotents such that $e \widetilde{\mathcal{R}}_B x$ and $f \widetilde{\mathcal{L}}_B y$. Then by Lemma 2.7, $x \in eS$ and $y \in Sf$. Thus, $xy \in eS \cap Sf$. Again by Lemma 8.4, $xy \in \widetilde{R}_e \cap \widetilde{L}_f$, that is, $xy \in \widetilde{R}_x \cap \widetilde{L}_y$. □

8.2 Blocked Rees matrix semigroups

As a tool for the following sections we recall some concepts and results from [10] about blocked Rees matrix semigroups and their structure. Blocked Rees matrix semigroups [10] are a generalisation of Rees matrix semigroups over a monoid. We refer the reader to [10] for more details.

Let I, Λ and Γ be non-empty sets. Suppose that Γ indexes partitions of Λ and I as follows:

$$P(\Lambda) = \{\Lambda_\alpha : \alpha \in \Gamma\}, \quad P(I) = \{I_\beta : \beta \in \Gamma\}.$$

For convenience, i, j, k, h will denote members of I; λ, μ, ν, ρ will denote members of Λ, and $\alpha, \beta, \gamma, \delta$ will denote members of Γ.

We recall from [12] that a non-empty set M is a *partial semigroup* if there is a partial binary operation on M such that for all $a, b, c \in M$, $(ab)c$ is defined if and only if $a(bc)$ is defined, and if $(ab)c$ is defined, then $(ab)c = a(bc)$.

Let $M = \bigcup\{M_{\alpha\beta} : \alpha, \beta \in \Gamma\}$ be a partial semigroup such that for $x, y \in M$, xy is defined if and only if $x \in M_{\alpha\beta}$, $y \in M_{\beta\gamma}$ for some $\alpha, \beta, \gamma \in \Gamma$ and then in this case, $xy \in M_{\alpha\gamma}$. Suppose also that for all $\alpha \in \Gamma$, $M_{\alpha\alpha} = T_\alpha$ is a monoid with identity g_α and for all $\alpha, \beta \in \Gamma$, $M_{\alpha\beta} = \emptyset$ or is a (T_α, T_β)-bisystem, that is, T_α acts on $M_{\alpha\beta}$ on the left, T_β acts on $M_{\alpha\beta}$ on the right and $(tm)t' = t(mt')$ for all $t \in T_\alpha$, $m \in M_{\alpha\beta}$, $t' \in T_\beta$.

We remark that if for any $\alpha, \beta \in \Gamma$, $M_{\alpha\beta}$ is regarded as the set of morphisms from α to β, then M forms a category with set of objects Γ and set of morphisms M.

Let 0 (zero) be a symbol not in any $M_{\alpha\beta}$ and let $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix over $M \cup \{0\}$, where for $\lambda \in \Lambda_\alpha$, $i \in I_\beta$, $p_{\lambda i} \in M_{\alpha\beta} \cup \{0\}$.

Let

$$\begin{aligned} \mathcal{M}^0 &= \mathcal{M}^0(M; I, \Lambda, \Gamma; P) \\ &= \{(i, a, \lambda) : a \in M_{\alpha\beta}, (i, \lambda) \in I_\alpha \times \Lambda_\beta, (\alpha, \beta) \in \Gamma \times \Gamma\} \cup \{0\}. \end{aligned}$$

In order to be able to define a multiplication on $\mathcal{M}^0(M; I, \Lambda, \Gamma; P)$, we say that $0x = x0 = 0$ for every element x of $\mathcal{M}^0$. Now we define a product on non-zero elements of $\mathcal{M}^0(M; I, \Lambda, \Gamma; P)$ by the rule that for any (i, a, λ), $(j, b, \mu) \in \mathcal{M}^0$,

$$(i, a, \lambda)(j, b, \mu) = \begin{cases} (i, ap_{\lambda j}b, \mu) & \text{if } p_{\lambda j} \neq 0 \\ 0 & \text{if } p_{\lambda j} = 0. \end{cases}$$

It is routine to show that this product is associative and is categorical at zero, and so we obtain a semigroup called a *blocked Rees matrix semigroup.* A blocked Rees matrix semigroup $\mathcal{M}^0 = \mathcal{M}^0(M; I, \Lambda, \Gamma; P)$ is called a *weakly orthodox blocked Rees matrix semigroup* or *WO-B Rees matrix semigroup* if the following condition holds:

(B) all non-zero entries of P are in the diagonal blocks, and if $(i, \lambda) \in I_\alpha \times \Lambda_\alpha$ then $p_{\lambda i}$ is the identity g_α of T_α.

Proposition 8.6. *Let $\mathcal{M}^0 = \mathcal{M}^0(M; I, \Lambda, \Gamma; P)$ be a WO-B Rees matrix semigroup and $B = \{(i, g_\alpha, \lambda) : \alpha \in \Gamma, (i, \lambda) \in I_\alpha \times \Lambda_\alpha\} \cup \{0\}$. Then*

(i) for any $(i, g_\alpha, \lambda) \in B$, (i, g_α, λ) is an idempotent;

(ii) the set B forms a band;

(iii) for any $(i, g_\alpha, \lambda) \in B \setminus \{0\}$, (i, g_α, λ) is primitive in B;

(iv) for any non-zero elements (i, a, λ), $(j, b, \mu) \in \mathcal{M}^0$, $(i, a, \lambda)\ \widetilde{\mathcal{R}}_B\ (j, b, \mu)$ if and only if $i = j$;

(v) for any non-zero elements (i, a, λ), $(j, b, \mu) \in \mathcal{M}^0$, $(i, a, \lambda)\ \widetilde{\mathcal{L}}_B\ (j, b, \mu)$ if and only if $\lambda = \mu$.

Proof. (i) If $(i, g_\alpha, \lambda) \in B$, then we have

$$\begin{aligned}(i, g_\alpha, \lambda)(i, g_\alpha, \lambda) &= (i, g_\alpha p_{\lambda i} g_\alpha, \lambda) \\ &= (i, g_\alpha g_\alpha g_\alpha, \lambda) \qquad \Big(\text{since } (i, \lambda) \in I_\alpha \times \Lambda_\alpha,\ p_{\lambda i} = g_\alpha\Big) \\ &= (i, g_\alpha, \lambda).\end{aligned}$$

(ii) Suppose that (i, g_α, λ), $(j, g_\beta, \mu) \in B \setminus \{0\}$. Then

$$(i, g_\alpha, \lambda)(j, g_\beta, \mu) = (i, g_\alpha p_{\lambda j} g_\beta, \mu).$$

If $\alpha = \beta$, then $p_{\lambda j} = g_\alpha = g_\beta$, and so $(i, g_\alpha, \lambda)(i, g_\beta, \mu) = (i, g_\alpha, \mu) \in B$. If $\alpha \neq \beta$, then $p_{\lambda j} = 0$, and so $(i, g_\alpha, \lambda)(j, g_\beta, \mu) = 0 \in B$. Hence, B is closed. Again by (i), B forms a band.

(iii) Suppose that (i, g_α, λ), $(j, g_\beta, \mu) \in B \setminus \{0\}$ are such that $(i, g_\alpha, \lambda) \leq (j, g_\beta, \mu)$. Then

$$(i, g_\alpha, \lambda)(j, g_\beta, \mu) = (i, g_\alpha, \lambda) = (j, g_\beta, \mu)(i, g_\alpha, \lambda),$$

which implies that $i = j$ and $\lambda = \mu$, and so $\alpha = \beta$ so that $g_\alpha = g_\beta$. Hence $(i, g_\alpha, \lambda) = (j, g_\beta, \mu)$.

(iv) Suppose that $(i,a,\lambda) \in \mathcal{M}^0 \setminus \{0\}$. Then there exist $\alpha, \beta \in \Gamma$ such that $(i,\lambda) \in I_\alpha \times \Lambda_\beta$ and $a \in M_{\alpha\beta}$. We now claim that for any $\nu \in \Lambda_\alpha$ and $(i, g_\alpha, \nu) \in B$, $(i, g_\alpha, \nu) \ \widetilde{\mathcal{R}}_B \ (i,a,\lambda)$. Clearly,

$$\begin{aligned}(i,g_\alpha,\nu)(i,a,\lambda) &= (i, g_\alpha p_{\nu i} a, \lambda) = (i, g_\alpha g_\alpha a, \lambda) \quad \big((i,\nu) \in I_\alpha \times \Lambda_\alpha,\ p_{\nu i} = g_\alpha\big)\\ &= (i, g_\alpha a, \lambda) = (i,a,\lambda).\end{aligned}$$

Let $\gamma \in \Gamma$, $(k,\rho) \in I_\gamma \times \Lambda_\gamma$ and $(k, g_\gamma, \rho) \in B$ be such that

$$(k, g_\gamma, \rho)(i,a,\lambda) = (i,a,\lambda).$$

Then $k = i$, and so $\alpha = \gamma$. Thus,

$$\begin{aligned}(k,g_\gamma,\rho)(i,g_\alpha,\nu) &= (k, g_\gamma p_{\rho i} g_\alpha, \nu)\\ &= (i, g_\alpha g_\alpha g_\alpha, \nu) \qquad \big(\alpha = \gamma,\ k = i\big)\\ &= (i, g_\alpha, \nu).\end{aligned}$$

Hence $(i, g_\alpha, \nu) \ \widetilde{\mathcal{R}}_B \ (i,a,\lambda)$.

Let $\eta, \delta \in \Gamma$ and $(j,\mu) \in I_\eta \times \Lambda_\delta$. If $(j,b,\mu) \in \mathcal{M}^0 \setminus \{0\}$, then

$$(j,b,\mu) \ \widetilde{\mathcal{R}}_B \ (j, g_\eta, \sigma),$$

where $\sigma \in \Lambda_\eta$ and $(j, g_\eta, \sigma) \in B$. So

$$\begin{aligned}(i,a,\lambda) \ \widetilde{\mathcal{R}}_B \ (j,b,\mu) &\Leftrightarrow (i,g_\alpha,\nu) \ \widetilde{\mathcal{R}}_B \ (j,g_\eta,\sigma)\\ &\Leftrightarrow (i,g_\alpha,\nu) \ \mathcal{R} \ (j,g_\eta,\sigma)\\ &\Leftrightarrow i = j.\end{aligned}$$

(v) It is the dual proof of (iv). □

The following lemma is an immediate consequence of Proposition 8.6.

Lemma 8.7. *Let $\mathcal{M}^0 = \mathcal{M}^0(M; I, \Lambda, \Gamma; P)$ be a WO-B Rees matrix semigroup and $B = \{(g_\alpha)_{i\lambda} : \alpha \in \Gamma, (i,\lambda) \in I_\alpha \times \Lambda_\alpha\} \cup \{0\}$. Then $\mathcal{M}^0$ is a primitive weakly B-orthodox semigroup.*

Proof. In view of (iv) and (v) of Proposition 8.6, it is easy to see that $\mathcal{M}^0$ satisfies the Congruence Condition (C). □

8.3 Primitive weakly B-orthodox semigroups

Throughout this section S denotes a given fixed primitive weakly B-orthodox semigroup with zero. Our aim, achieved in Theorem 8.13, is to show that S is isomorphic to a WO-B Rees matrix semigroup.

Lemma 8.8. *Let $e \in B^*$ and let*

$$X = \bigcup\{\widetilde{L}_f : f \in B^* \text{ and } f \; \mathcal{D} \; e\},$$

$$Y = \bigcup\{\widetilde{R}_g : g \in B^* \text{ and } g \; \mathcal{D} \; e\}$$

and $Z = X \cap Y$. Then

(i) for any $a, b \in Z$, $(a, b) \in \widetilde{\mathcal{L}}_B \circ \widetilde{\mathcal{R}}_B$ and $(a, b) \in \widetilde{\mathcal{R}}_B \circ \widetilde{\mathcal{L}}_B$;

(ii) if $a \in \widetilde{L}_f \cap \widetilde{R}_g \subseteq Z$, then $gf \in \widetilde{H}_a \cap B^*$ is such that $\rho_f : \widetilde{H}_g \to \widetilde{H}_a$, given by $x\rho_f = xf$ for any $x \in \widetilde{H}_g$, and $\lambda_g : \widetilde{H}_f \to \widetilde{H}_a$, given by $y\lambda_g = gy$ for any $y \in \widetilde{H}_f$, are bijective;

(iii) for any $a, b \in Z$, $|\widetilde{H}_a| = |\widetilde{H}_b|$;

(iv) if $g, h \in B^*$ and $\widetilde{H}_g$ and $\widetilde{H}_h$ contained in Z, then $\widetilde{H}_g$ and $\widetilde{H}_h$ are isomorphic monoids.

Proof. (i) Suppose that $a, b \in Z$. Then there exist $g, f \in B^*$ such that $a \; \widetilde{\mathcal{L}}_B \; f$, $b \; \widetilde{\mathcal{R}}_B \; g$ and $g \; \mathcal{D} \; f \; \mathcal{D} \; e$. Since B is a band, we have $f \; \mathcal{L} \; gf \; \mathcal{R} \; g$, and so $a \; \widetilde{\mathcal{L}}_B \; gf \; \widetilde{\mathcal{R}}_B \; b$, that is, $(a, b) \in \widetilde{\mathcal{L}}_B \circ \widetilde{\mathcal{R}}_B$. Similarly, we have $(a, b) \in \widetilde{\mathcal{R}}_B \circ \widetilde{\mathcal{L}}_B$.

(ii) Let $a \in \widetilde{L}_f \cap \widetilde{R}_g \subseteq Z$. Then $f \; \mathcal{D} \; e \; \mathcal{D} \; g$, and so $gf \in B^*$ and $f \; \mathcal{L} \; gf \; \mathcal{R} \; g$ so that $gf \; \widetilde{\mathcal{H}}_B \; a$. We now show that $\rho_f : \widetilde{H}_g \to \widetilde{H}_a$ is a well-defined bijection. For any $x \in \widetilde{H}_g$, we have

$$x\rho_f = xf \; \widetilde{\mathcal{L}}_B \; gf \; \widetilde{\mathcal{H}}_B \; a$$

and

$$x\rho_f = xf = xgf \; \mathcal{R} \; xg = x \; \widetilde{\mathcal{R}}_B \; g \; \widetilde{\mathcal{R}}_B \; a,$$

and so $x\rho_f \in \widetilde{H}_a$ so that ρ_f is well-defined.

It is easy to see that ρ_f is injective. Since if $x_1, x_2 \in \widetilde{H}_g$ and $x_1\rho_f = x_2\rho_f$, then $x_1 f = x_2 f$. We have

$$\begin{aligned} x_1 = x_1 g = x_1 gfg \qquad & \big(f \mathcal{D} g\big) \\ = x_1 fg = x_2 fg = x_2 gfg = x_2 g = x_2. & \end{aligned}$$

To show ρ_f is surjective, we suppose that $z \in \widetilde{H}_a = \widetilde{L}_f \cap \widetilde{R}_g$. Then we have

$$zg = zfg \ \mathcal{R} \ zf = z \ \widetilde{\mathcal{R}}_B \ g \qquad \big(fg \ \mathcal{R} \ f\big)$$

and

$$zg \ \widetilde{\mathcal{L}}_B \ fg \ \mathcal{L} \ g,$$

and so $zg \in \widetilde{H}_g$. Also, $zg\rho_f = zgf = z$ as $gf \ \mathcal{R} \ g \ \widetilde{\mathcal{R}}_B \ z$. Thus, ρ_f is a bijection.

By a similar argument, we show that $\lambda_g : \widetilde{H}_f \to \widetilde{H}_a$ is a well-defined bijection.

(iii) Suppose that $a, b \in Z$. By (i), there exists $c \in Z$ such that $a \ \widetilde{\mathcal{L}}_B \ c \ \widetilde{\mathcal{R}}_B \ b$. Also, there exists $f, g \in B^*$ such that $f \ \mathcal{D} \ e \ \mathcal{D} \ g$ and $c \in \widetilde{L}_f \cap \widetilde{R}_g$. By (ii), we have

$$|\widetilde{H}_a| = |\widetilde{H}_f| = |\widetilde{H}_c| = |\widetilde{H}_g| = |\widetilde{H}_b|.$$

(iv) Suppose that $g, h \in B^*$ are such that $\widetilde{H}_g$ and $\widetilde{H}_h$ are contained in Z. Then by (i), there exists $a \in Z$ such that $g \ \widetilde{\mathcal{L}}_B \ a \ \widetilde{\mathcal{R}}_B \ h$. By (ii), $\rho_g : \widetilde{H}_h \to \widetilde{H}_a$ and $\lambda_h : \widetilde{H}_g \to \widetilde{H}_a$ are bijective. Using the same method as (ii), we can show that $\lambda_g : \widetilde{H}_a \to \widetilde{H}_g$ is the inverse of λ_h. Then $\rho_g\lambda_g : \widetilde{H}_h \to \widetilde{H}_g$ is a bijection. By Lemma 3.5, $\widetilde{H}_h$ and $\widetilde{H}_g$ are monoids. If $x_1, x_2 \in \widetilde{H}_h$, then

$$\begin{aligned} (x_1x_2)\rho_g\lambda_g = gx_1x_2g &= gx_1hx_2g \qquad && \big(h \ \widetilde{\mathcal{R}}_B \ x_2\big) \\ &= gx_1hgghx_2g && \big(h \ \mathcal{D} \ g\big) \\ &= gx_1ggx_2g = (x_1\rho_g\lambda_g)(x_2\rho_g\lambda_g). && \end{aligned}$$

Hence $\rho_g\lambda_g$ is an isomorphism, and so $\widetilde{H}_h$ is isomorphic to $\widetilde{H}_g$. □

We pause to make a short comment on Lemma 8.8. If $e, f \in B^*$ with $e \ \mathcal{D} \ f$, then by (iv), $\rho_f\lambda_f$ is an isomorphism from $\widetilde{H}_e$ onto $\widetilde{H}_f$. It is easy to see that Z is a union of $\widetilde{\mathcal{H}}_B$-classes and due to part (i), Z can be depicted by an egg-box picture.

Lemma 8.9. *If $e, f \in B^*$ and $\widetilde{H} = \widetilde{R}_e \cap \widetilde{L}_f \neq \emptyset$, then $\widetilde{H}$ is an $(\widetilde{H}_e, \widetilde{H}_f)$-bisystem with respect to the multiplication in S.*

Proof. Suppose that $x \in \widetilde{H} = \widetilde{R}_e \cap \widetilde{L}_f$ and $t \in \widetilde{H}_e$. Then

$$tx \; \widetilde{\mathcal{R}}_B \; te = t \; \widetilde{\mathcal{R}}_B \; e$$

and

$$tx \; \widetilde{\mathcal{L}}_B \; ex = x \; \widetilde{\mathcal{L}}_B \; f,$$

and so $tx \in \widetilde{H}$. Dually, we show that for any $u \in \widetilde{H}_f$, $xu \in \widetilde{H}$. Since the multiplication in S is associative, we have $t(xu) = (tx)u$. Thus, $\widetilde{H}$ is an $(\widetilde{H}_e, \widetilde{H}_f)$-bisystem. □

Lemma 8.10. *Suppose that $e, f \in B^*$ are such that $e \; \mathcal{D} \; f$. If $a \in \widetilde{L}_e$ (resp. $\widetilde{R}_e$), then $\widetilde{L}_f \cap \widetilde{R}_a \neq \emptyset$ (resp. $\widetilde{R}_f \cap \widetilde{L}_a \neq \emptyset$) and there exists a bijection $\theta : \widetilde{H}_a \to \widetilde{L}_f \cap \widetilde{R}_a$ (resp. $\theta : \widetilde{H}_a \to \widetilde{R}_f \cap \widetilde{L}_a$) such that $(xt)\theta = (x\theta)(t\rho_f\lambda_f)$ (resp. $(tx)\theta = (t\rho_f\lambda_f)x\theta$), where $x \in \widetilde{H}_a$, $t \in \widetilde{H}_e$. In addition, if $a \in \widetilde{R}_g$ (resp. $\widetilde{L}_g$) for some $g \in B^*$, then $(rx)\theta = r(x\theta)$ (resp. $(xr)\theta = (x\theta)r$) for all $x \in \widetilde{H}_a$, $r \in \widetilde{H}_g$.*

Proof. If $e \; \mathcal{D} \; f$ in B^* and $a \in \widetilde{L}_e$, then $f \; \mathcal{L} \; ef \; \mathcal{R} \; e$, and so $af = aef \; \widetilde{\mathcal{R}}_B \; ae = a$ and $af = aef \; \widetilde{\mathcal{L}}_B \; eef = ef \; \mathcal{L} \; f$ so that $af \in \widetilde{L}_f \cap \widetilde{R}_a$. Thus $\widetilde{L}_f \cap \widetilde{R}_a \neq \emptyset$.

We now claim that $\rho_f : \widetilde{H}_a \to \widetilde{L}_f \cap \widetilde{R}_a$, given by $x\rho_f = xf$ for all $x \in \widetilde{H}_a$, is a bijection. Since if $x \in \widetilde{H}_a$, then $x \in \widetilde{L}_e$, and so by the above statement, we have $x\rho_f \in \widetilde{L}_f \cap \widetilde{R}_a$.

To see that ρ_f is injective, we suppose that $x_1, x_2 \in \widetilde{H}_a$ and $x_1\rho_f = x_2\rho_f$. Then $x_1 f = x_2 f$, and so

$$\begin{aligned} x_1 = x_1 e = x_1 efe & \qquad \big(e \; \mathcal{D} \; f\big) \\ = x_1 fe & \qquad \big(x_1 \in \widetilde{L}_e\big) \\ = x_2 fe = x_2 efe = x_2 e = x_2 & \end{aligned}$$

so that ρ_f is injective.

For any $y \in \widetilde{L}_f \cap \widetilde{R}_a$, we have $ye = yfe \; \widetilde{\mathcal{R}}_B \; yf = y \; \widetilde{\mathcal{R}}_B \; a$ and $ye \; \widetilde{\mathcal{L}}_B \; fe \; \mathcal{L} \; e \; \widetilde{\mathcal{L}}_B \; a$, and so $ye \in \widetilde{H}_a$. Also, $(ye)\rho_f = yef = yfef = yf = y$. Thus, ρ_f is surjective.

Let $x \in \widetilde{H}_a$ and $t \in \widetilde{H}_e$. By Lemma 8.9, we have $xt \in \widetilde{H}_a$. We also have that

$$\begin{aligned}(xt)\rho_f = xtf &= xetf && \big(x \in \widetilde{L}_e\big) \\ &= xefetf && \big(e\ \mathcal{D}\ f\big) \\ &= xftf && \big(x \in \widetilde{L}_e,\ t \in \widetilde{H}_e\big) \\ &= (x\rho_f)(t\rho_f\lambda_f).\end{aligned}$$

Finally, if $g \in B^*$ and $a \in \widetilde{R}_g$, then $\widetilde{H}_a$ and $\widetilde{L}_f \cap \widetilde{R}_a$ are both left $\widetilde{H}_g$-systems and for $x \in \widetilde{H}_a$, $r \in \widetilde{H}_g$ we have $(rx)\rho_f = rxf = r(x\rho_f)$. □

Let I (resp. Λ) index the non-zero $\widetilde{\mathcal{R}}_B$ (resp. $\widetilde{\mathcal{L}}_B$)-classes of S. For $i \in I$ and $\lambda \in \Lambda$, we will denote the intersection of $\widetilde{R}_i$ and $\widetilde{L}_\lambda$ by $\widetilde{H}_{i\lambda}$. Then

$$S \setminus \{0\} = \bigcup\{\widetilde{H}_{i\lambda} : (i, \lambda) \in I \times \Lambda\}.$$

Let Γ index the $\mathcal{D}$-classes of B. For each $\alpha \in \Gamma$, we define

$$I_\alpha = \{i \in I : D_\alpha \cap \widetilde{R}_i \neq \emptyset\}$$

and

$$\Lambda_\alpha = \{\lambda \in \Lambda : D_\alpha \cap \widetilde{L}_\lambda \neq \emptyset\}.$$

We remark that for any $i \in I$, there exists a unique $\alpha \in \Gamma$ such that $i \in I_\alpha$. Since each $\widetilde{\mathcal{R}}_B$-class of S contains at least one idempotent lying in B and Γ indexes the $\mathcal{D}$-classes of B. Dually, for any $\lambda \in \Lambda$, there exists a unique $\alpha \in \Gamma$ such that $\lambda \in \Lambda_\alpha$. Thus, if $\alpha, \beta \in \Gamma$ with $\alpha \neq \beta$, then $I_\alpha \cap I_\beta = \emptyset$ and $\Lambda_\alpha \cap \Lambda_\beta = \emptyset$. Hence, Γ indexes partitions of I and Λ as follows:

$$P(I) = \{I_\alpha : \alpha \in \Gamma\} \text{ and } P(\Lambda) = \{\Lambda_\alpha : \alpha \in \Gamma\}.$$

Obviously, if an $\widetilde{\mathcal{H}}_B$-class $\widetilde{H}$ contains a distinguished idempotent e, then there exists $\alpha \in \Gamma$ and $(i, \lambda) \in I_\alpha \times \Lambda_\alpha$ such that $\widetilde{H} = \widetilde{R}_i \cap \widetilde{L}_\lambda$. Conversely, each $\widetilde{\mathcal{R}}_B$-class and each $\widetilde{\mathcal{L}}_B$-class contains a distinguished idempotent, so for each $\alpha \in \Gamma$ and $e \in D_\alpha$, we can choose a pair $(i(\alpha), \lambda(\alpha)) \in I_\alpha \times \Lambda_\alpha$ such that $\widetilde{H}_{i(\alpha)\lambda(\alpha)} = \widetilde{R}_{i(\alpha)} \cap \widetilde{L}_{\lambda(\alpha)}$ is an $\widetilde{\mathcal{H}}_B$-class containing e. By Lemma 3.5, $\widetilde{H}_{i(\alpha)\lambda(\alpha)}$ is a monoid. We denote it by $M_{\alpha\alpha}$ or T_α.

By Lemma 8.8 (ii), if $e\ \mathcal{D}\ f$ in B^*, then $\widetilde{H}_e$ is isomorphic to $\widetilde{H}_f$, and so for any $\alpha \in \Gamma$, the monoid structure of T_α is independent of the choices for $\widetilde{\mathcal{H}}_B$-classes containing a distinguished idempotent, indexed by a pair $(i, \lambda) \in I_\alpha \times \Lambda_\alpha$. For convenience, we will use g_α to denote the distinguished idempotent contained in T_α. Certainly, g_α is the identity of T_α.

Notice that for any $\alpha \in \Gamma$ and for each pair $(i, \lambda) \in I_\alpha \times \Lambda_\alpha$, $\widetilde{H}_{i\lambda} \neq \emptyset$ by Lemma 8.8 (i). According to Lemma 8.8 (ii), there exist distinguished idempotents r_i^α and q_λ^α in $\widetilde{H}_{i\lambda(\alpha)}$ and $\widetilde{H}_{i(\alpha)\lambda}$, respectively, such that $x \mapsto r_i^\alpha x$ is a bijection from $\widetilde{H}_{i(\alpha)\lambda(\alpha)}$ onto $\widetilde{H}_{i\lambda(\alpha)}$ and $y \mapsto y q_\lambda^\alpha$ is a bijection from $\widetilde{H}_{i\lambda(\alpha)}$ onto $\widetilde{H}_{i\lambda}$. Thus once we have choose $\{r_i^\alpha \in B^* : i \in I_\alpha,\ \alpha \in \Gamma\}$ and $\{q_\lambda^\alpha \in B^* : \lambda \in \Lambda_\alpha,\ \alpha \in \Gamma\}$ we have a unique expression $r_i^\alpha x q_\lambda^\alpha$ $(x \in T_\alpha = \widetilde{H}_{i(\alpha)\lambda(\alpha)})$ for each element a of $\widetilde{H}_{i\lambda}$, where $(i, \lambda) \in I_\alpha \times \Lambda_\alpha$.

For $\alpha, \beta \in \Gamma$ with $\alpha \neq \beta$, we put $M_{\alpha\beta} = \widetilde{H}_{i(\alpha)\lambda(\beta)}$. Notice that for any $(i, \lambda) \in I_\alpha \times \Lambda_\beta$, $\widetilde{H}_{i\lambda} \neq \emptyset$ if and only if $\widetilde{H}_{i(\alpha)\lambda(\beta)} \neq \emptyset$ by Lemma 8.10. Assume that $M_{\alpha\beta} \neq \emptyset$. Then by Lemma 8.9, $M_{\alpha\beta}$ is a (T_α, T_β)-bisystem. Also, if $(i, \lambda) \in I_\alpha \times \Lambda_\beta$ and r_i^α, q_λ^β are distinguished idempotents in $\widetilde{H}_{i\lambda(\alpha)}$ and $\widetilde{H}_{i(\beta)\lambda}$, respectively, then by Lemma 8.10, we have that $x \mapsto x q_\lambda^\beta$ is a bijection from $\widetilde{H}_{i(\alpha)\lambda(\beta)}$ onto $\widetilde{H}_{i(\alpha)\lambda}$ and that $y \mapsto r_i^\alpha y$ is a bijection from $\widetilde{H}_{i(\alpha)\lambda}$ onto $\widetilde{H}_{i\lambda}$. Hence every element a of $\widetilde{H}_{i\lambda}$ with $(i, \lambda) \in I_\alpha \times \Lambda_\beta$ may be written uniquely as $r_i^\alpha m q_\lambda^\beta$, where $m \in M_{\alpha\beta}$. Further, $\widetilde{H}_{i\lambda}$ is a (T_α, T_β)-system under the actions

$$t_\alpha \cdot (r_i^\alpha m q_\lambda^\beta) = r_i^\alpha t_\alpha m q_\lambda^\beta \text{ and } (r_i^\alpha m q_\lambda^\beta) \cdot t_\beta = r_i^\alpha m t_\beta q_\lambda^\beta,$$

and so $\widetilde{H}_{i\lambda}$ is (T_α, T_β)-isomorphic to $M_{\alpha\beta}$, that is, there exists a (T_α, T_β)-isomorphism from $\widetilde{H}_{i\lambda}$ onto $M_{\alpha\beta}$. In addition, it follows from Lemma 8.8 and Lemma 8.10 that the bisystem structure of $M_{\alpha\beta}$ is independent of the possible choices for T_α and T_β.

We now put

$$M = \{M_{\alpha\beta} : (\alpha, \beta) \in \Gamma \times \Gamma\}.$$

Lemma 8.11. *The set M is a partial semigroup.*

Proof. Let $a, b, c \in M \setminus \{0\}$. Then there exist $\alpha, \beta, \gamma, \delta, \eta, \varepsilon \in \Gamma$ such that $a \in M_{\alpha\beta}$, $b \in M_{\gamma\delta}$ and $c \in M_{\eta\varepsilon}$. Note that $a^* \in D_\beta$, $b^\dagger \in D_\gamma$, $b^* \in D_\delta$ and $c^\dagger \in D_\eta$. Then ab is defined if and only if $ab \neq 0$ if and only if $a^* b^\dagger \neq 0$ by Lemma 8.1. As B is primitive, $a^* b^\dagger \neq 0$ if and only if $a^*\ \mathcal{D}\ b^\dagger$, if and only if $\beta = \gamma$. If $(ab)c$ is defined, then $ab \neq 0$ and $(ab)c \neq 0$, and so $\beta = \gamma$. By Lemma 8.5, $ab\ \widetilde{\mathcal{L}}_B\ b$, and so $(ab)^*\ \mathcal{L}\ b^*$ so that $(ab)^* \in D_\delta$. From $(ab)c \neq 0$, we obtain that $(ab)^*\ \mathcal{D}\ c^\dagger$, and so $\delta = \eta$. Thus, bc is defined and by Lemma 8.5, $bc\ \widetilde{\mathcal{R}}_B\ b$ so that $(bc)^\dagger\ \mathcal{R}\ b^\dagger$, and so $(bc)^\dagger \in D_\gamma$. As $\gamma = \beta$, we have that $a(bc)$ is defined. Since the multiplication of S is associative, we have $(ab)c = a(bc)$. Dually, if $a(bc)$ is defined, then we have $(ab)c$ is defined and $(ab)c = a(bc)$. □

We now define P to be the $\Lambda \times I$ matrix $(p_{\lambda i})$, where for $(\lambda, i) \in I_\alpha \times \Lambda_\beta$, $p_{\lambda i} = q^\alpha_\lambda r^\beta_i$. Now $q^\alpha_\lambda \in \widetilde{H}_{i(\alpha)\lambda}$ and $r^\beta_i \in \widetilde{H}_{i\lambda(\beta)}$ so that $q^\alpha_\lambda \in \widetilde{R}_{i(\alpha)}$ and $r^\beta_\lambda \in \widetilde{L}_{\lambda(\beta)}$, and hence either $q^\alpha_\lambda r^\beta_i = 0$ or by Lemma 8.5, $q^\alpha_\lambda r^\beta_i \in \widetilde{R}_{i(\alpha)} \cap \widetilde{L}_{\lambda(\beta)} = \widetilde{H}_{i(\alpha)\lambda(\beta)} = M_{\alpha\beta}$. Consequently, any non-zero entry in (α, β)-block of P is a member of $M_{\alpha\beta}$.

We now have the necessary ingredients to form a blocked Rees matrix semigroup $\mathcal{M}^0 = \mathcal{M}^0(M; I, \Lambda, \Gamma; P)$.

Lemma 8.12. *The blocked Rees matrix semigroup $\mathcal{M}^0 = \mathcal{M}^0(M; I, \Lambda, \Gamma; P)$ constructed above satisfies (B) and consequently, $\mathcal{M}^0(M; I, \Lambda, \Gamma; P)$ forms a WO-B blocked Rees matrix semigroup.*

Proof. It is sufficient to show Condition (B) holds. Let $\alpha, \beta \in \Gamma$ and $(\lambda, i) \in \Lambda_\alpha \times I_\alpha$. Since q^α_λ and r^β_i are distinguished idempotents in B^* and B forms a band, it follows that $p_{\lambda i} = q^\alpha_\lambda r^\beta_i \neq 0$ if and only if $\alpha = \beta$, and so all non-zero entries in P are in the diagonal blocks and each non-zero entry from T_α is the identity g_α of T_α. Furthermore for any $\alpha \in \Gamma$ and $(i, \lambda) \in I_\alpha \times \Lambda_\alpha$, $p_{\lambda i} = q^\alpha_\lambda r^\beta_i = g_\alpha g_\alpha = g_\alpha \neq 0$, and so Condition (B) holds. □

Finally, we have:

Theorem 8.13. *If S is a primitive weakly B-orthodox semigroup, then S is isomorphic to a WO-B Rees matrix semigroup.*

Proof. Observe that $S \setminus \{0\} = \bigcup\{\widetilde{H}_{i\lambda} : (i, \lambda) \in I \times \Lambda\}$ and thus the map $\phi : \mathcal{M}^0 \to S$ defined by $0\phi = 0$ and

$$(i, a, \lambda)\phi = r^\alpha_i a q^\beta_\lambda$$

for $(i, \lambda) \in I_\alpha \times \Lambda_\beta$ and $a \in M_{\alpha\beta}$ is a bijection.

It is routine to show that ϕ is a morphism. □

We now consider the abundant case. An abundant semigroup is called *primitive* if the non-zero idempotents are primitive under the natural partial order on the idempotents. As a special case of Theorem 8.13, the following result cited from [10] can be derived. It is a little different in Fountain's paper.

Theorem 8.14. *[10] A semigroup S is a primitive abundant semigroup with zero, whose set of idempotents forms a band if and only if it is isomorphic to a blocked Rees matrix semigroup $\mathcal{M}^0(M; I, \Lambda, \Gamma; P)$ satisfying (C), (R) and (IB):*

(C) *if $a, a_1, a_2 \in M_{\alpha\beta}$, $b, b_1, b_2 \in M_{\beta\gamma}$, then $ab_1 = ab_2$ implies $b_1 = b_2$; $a_1 b = a_2 b$ implies $a_1 = a_2$;*

(R) *if $M_{\alpha\beta}$, $M_{\beta\alpha}$ are both non-empty where $\alpha \neq \beta$, then $aba \neq a$ for all $a \in M_{\alpha\beta}$, $b \in M_{\beta\alpha}$;*

(IB) *all non-zero entries of P are in the diagonal blocks, and if $(i, \lambda) \in I_\alpha \times \Lambda_\alpha$ then $p_{i\lambda}$ is the identity g_α of T_α.*

We remark that in [10], a blocked Rees matrix semigroup $\mathcal{M}^0(M; I, \Lambda, \Gamma; P)$ is a *PA blocked Rees matrix semigroup* if (C), (R) and (U) hold:

(U) for each $\alpha \in \Gamma$ and each $\lambda \in \Lambda_\alpha$(resp. $i \in I_\alpha$) there is a member i of I_α(resp. λ of Λ) such that $p_{\lambda i}$ is a unit in T_α.

Note that if Condition (IB) holds, then Condition (U) holds.

8.4 Trace of weakly B-orthodox semigroups

First, we define the trace of a weakly B-orthodox semigroup to be $S\mathbf{C} = (S, \cdot)$, as in Chapter 7. Remark that $S\mathbf{C}$ contains $B\mathbf{C} = (B, \cdot)$ as a substructure, where $B\mathbf{C}$ is the band B with multiplication restricted to $\mathcal{D}$-classes.

Now let P be any generalised category over B. Define $\odot$ on $P^0 = P\dot{\cup}\{0\}$ by the rule that

$$x \odot y = \begin{cases} x \cdot y & \text{if } \exists x \cdot y \text{ in } P \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 8.15. *The set $(P^0, \odot)$ is a semigroup containing a band $(B^0, \odot)$ as a subsemigroup, where $(B^0, \odot)$ is the 0-direct union of the $\mathcal{D}$-classes of B. Further, P^0 is primitive weakly B^0-orthodox, in the sense that distinguished idempotents are all primitive in B^0.*

Proof. Let $x, y, z \in P^0$. If any of x, y, z is 0, then clearly $x \odot (y \odot z) = (x \odot y) \odot z = 0$. Suppose that $x, y, z \in P$. Then

$$\begin{aligned} x \odot (y \odot z) &= \begin{cases} x \odot (y \cdot z) & \text{if } \exists y \cdot z \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} x \cdot (y \cdot z) & \text{if } \exists y \cdot z \text{ and } \exists x \cdot (y \cdot z) \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} x \cdot (y \cdot z) & \text{if } \exists y \cdot z \text{ and } \exists x \cdot y \\ 0 & \text{otherwise} \end{cases} \\ &= (x \odot y) \odot z \end{aligned}$$

for reasons of symmetry. Clearly B^0 is a subsemigroup of P^0.

Let $x \in P^0$. If $x = 0$, then $x \; \widetilde{R}_{B^0} \; 0$. If $x \in P$, then choosing $e \in \mathbf{d}(x)$ we have $\exists e \cdot x$ and $e \cdot x = x$, so that $e \odot x = x$. If $f \in B^0$ and $f \odot x = x$, then clearly $f \in B$ and $\exists f \cdot x$ with $f \cdot x = x$. Hence $R_f = \mathbf{d}(x) = R_e$ so that $e \mathrel{\mathcal{R}} f$ and $f \odot e = e$. Hence $x \; \widetilde{\mathcal{R}}_{B^0} \; e$ and it follows that P^0 is weakly B^0-abundant.

Notice that $x \; \widetilde{\mathcal{R}}_{B^0} \; f$ where $f \in B$ if and only if $\mathbf{d}(x) = R_f$. If follows that $x \; \widetilde{\mathcal{R}}_{B^0} \; y$ if and only if $\mathbf{d}(x) = \mathbf{d}(y)$. Thus for any $z \in P$, $z \odot x = 0$ if and only if $z \odot y = 0$, and if $z \odot x \neq 0$, then $\mathbf{d}(z \odot x) = \mathbf{d}(z) = \mathbf{d}(z \odot y)$. It is clear that (C) holds and P^0 is weakly B^0-orthodox. It is immediate that P^0 is primitive. □

Let S be weakly B-orthodox. From Lemma 7.17, $S\mathbf{C} = (S, \cdot)$ is an inductive generalised category over B. Then $S\mathbf{C}^0$ is a primitive weakly B^0-orthodox semigroup; $S\mathbf{C}^0$ is also sometimes called the *trace* of S. From Lemma 7.18, $S\mathbf{C}$, and with a little adjustment, $S\mathbf{C}^0$, can be endowed with an inductive structure from which we can recover S.

The natural partial orders in any primitive weakly B-orthodox semigroup with 0 are trivial, in the following sense:

Lemma 8.16. *Let S be a primitive weakly B-orthodox semigroup with 0, where $0 \in B$. Then B is a 0-disjoint union of $\mathcal{D}$-classes. If $x, y \in S$, then $x \leq'_r y$ if and only if $x = 0$ or $x = y$.*

Proof. We know that B is a semilattice Y of $\mathcal{D}$-classes D_α, $\alpha \in Y$. We must have that Y contains a zero τ and $D_\tau = \{0\}$. If $\tau < \alpha < \beta$, let $e \in D_\alpha$ and

$f \in D_\beta$. Then $fef \in D_\alpha$ and $0 < fef < f$, a contradiction. It follows that B is a 0-disjoint union of its $\mathcal{D}$-classes.

If $x \neq 0$ and $x \leq_r' y$, then $x = ey$ for some $e \in B$ and $x^\dagger \leq_{\mathcal{R}} y^\dagger$. Thus $x^\dagger y^\dagger \leq y^\dagger$ so that $x^\dagger y^\dagger = y^\dagger$. Also, $x^\dagger \leq_{\mathcal{R}} e$ so that similarly, $x^\dagger e = x^\dagger$. Now $x = ey = x^\dagger ey = x^\dagger y = x^\dagger y^\dagger y = y^\dagger y = y$. □

In view of Theorem 8.13, we have:

Theorem 8.17. *If S is a weakly B-orthodox semigroup, then $S\mathbf{C}^0$ is isomorphic to a WO-B Rees matrix semigroup $\mathcal{M}^0(M; I, \Lambda, \Gamma; P)$ satisfying (B).*

Chapter 9

Beyond orthodox semigroups II: weakly B-orthodox semigroups and categories

The aim of this chapter is to construct weakly B-orthodox semigroups via an adjustment of Armstrong's method for concordant semigroups as mentioned in Chapter 6. Our modifications are to allow for the fact that B is a band, and to compensate for the lack of an idempotent connected condition.

For convenience we make the convention that B will always denote a band. Green's relations and their associated pre-orders will always refer to B, unless stated otherwise.

9.1 Weakly orthodox categories

The purpose in this section is to introduce the notions of a band category and a weakly orthodox category over a band B, and to present a pair of pre-orders which are deduced from the definition of band categories.

Let B be a band. A subset K of B is a *representative* of B if maps $\phi : K \to B/\mathcal{L}$ given by $e \mapsto L_e$ and $\psi : K \to B/\mathcal{R}$ given by $e \mapsto R_e$ are bijective. So for any $e \in B$, there exists a unique $k \in K$ such that $e \,\mathcal{L}\, k$ in B and there exists a unique $h \in K$ such that $e \,\mathcal{R}\, h$ in B. For convenience, we will denote k and h by $e^\star$ and e^+, respectively.

Definition 9.1. Let P be a category in which $\mathrm{Ob}(P)$ is the underlying set of a band B, and let K be a representative of B. Suppose that for $e, f \in B$ with $e\,\mathcal{D}\,f$, there exists a distinguished morphism $[e,f]$ from e to f such that $[e,e] = 1_e$, the identity associated to the object e. Then P is a *band category* if the following conditions and their duals $(\mathrm{OB1})^\circ, (\mathrm{OB2})^\circ, (\mathrm{OB3})^\circ$ and $(\mathrm{OB4})^\circ$ of $(\mathrm{OB1}), (\mathrm{OB2}), (\mathrm{OB3})$ and $(\mathrm{OB4})$ hold:

(OB1) if $x \in P$ and $e \in B$ with $e \leq_{\mathcal{L}} \mathbf{d}(x)$, then there exists an element ${}_e|x$ in P, called the *restriction* of x to e, such that $\mathbf{d}({}_e|x) = e$ and $\mathbf{r}({}_e|x) \leq_{\mathcal{L}} \mathbf{r}(x)$; also, if $e = \mathbf{d}(x)$, then $\mathbf{r}({}_e|x)\ \mathcal{L}\ \mathbf{r}(x)$ and ${}_e|x \cdot [\mathbf{r}({}_e|x), \mathbf{r}(x)] = x$;

(OB2) if $x \in P$ and $e, f \in B$ with $e \leq_{\mathcal{L}} f \leq_{\mathcal{L}} \mathbf{d}(x)$, then ${}_e|({}_f|x) = {}_e|x$; moreover, if $e\ \mathcal{L}\ f \leq_{\mathcal{L}} \mathbf{d}(x)$, then $[e,f] \cdot {}_f|x = {}_e|x$;

(OB3) if $x, y \in P$ and $e \in B$ with $\exists x \cdot y$ in P and $e \leq_{\mathcal{L}} \mathbf{d}(x)$, then ${}_e|(x \cdot y) = {}_e|x \cdot {}_f|y$, where $f = \mathbf{r}({}_e|x)$;

(OB4) if $e, f, h \in B$ with $e\ \mathcal{D}\ f$ and $h \leq_{\mathcal{L}} e$, then ${}_h|[e,f] = [h, (hf)^\star]$;

(OB5) if $e, f, g \in B$ are such that $e\ \mathcal{D}\ f\ \mathcal{D}\ g$, then $[e,f] \cdot [f,g] = [e,g]$.

We make some comments on the above definition. In (OB2), since $[e,f]\cdot{}_f|x = {}_e|x$ we obtain that $\mathbf{r}({}_f|x) = \mathbf{r}({}_e|x)$. In (OB3) since $\exists x\cdot y$ we know that $\mathbf{r}(x) = \mathbf{d}(y)$. By (OB1), $f = \mathbf{r}({}_e|x) \leq_{\mathcal{L}} \mathbf{r}(x) = \mathbf{d}(y)$, so that ${}_f|y$ exists and $\mathbf{d}({}_f|y) = f$. Hence ${}_e|x\cdot{}_f|y$ is defined. In (OB4) since $e\ \mathcal{D}\ f$ we obtain that $e\ \mathcal{R}\ ef$. Hence $he\ \mathcal{R}\ hef$, that is, $h\ \mathcal{R}\ hf$ as $h \leq_{\mathcal{L}} e$, so that $h\ \mathcal{R}\ hf\ \mathcal{L}\ (hf)^\star$, and consequently, $h\ \mathcal{D}\ (hf)^\star$. Hence $[h, (hf)^\star]$ exists.

We note that a band category P depends on the choice of the band B which is the set of objects of P. In order to emphasize that the set of objects is a particular band B, we can express the term 'band category' as 'band category over B'.

Let P be a band category over B. Using the technique in [1], we define a relation ρ on P by the rule that for all $x, y \in P$,

$$x\ \rho\ y \ \Leftrightarrow\ \mathbf{d}(x)\ \mathcal{R}\ \mathbf{d}(y),\ \mathbf{r}(x)\ \mathcal{L}\ \mathbf{r}(y) \text{ and } x \cdot [\mathbf{r}(x), \mathbf{r}(y)] = [\mathbf{d}(x), \mathbf{d}(y)] \cdot y,$$

that is equal to

$$x\ \rho\ y \ \Leftrightarrow\ (\exists\ u, v \in B)\ v\ \mathcal{R}\ \mathbf{d}(y),\ \mathbf{r}(x)\ \mathcal{L}\ u \text{ and } x \cdot [\mathbf{r}(x), u] = [v, \mathbf{d}(y)] \cdot y.$$

We note that if $x \,\rho\, y$ in P, then we have that

$$
\begin{aligned}
& x \cdot [\mathbf{r}(x), \mathbf{r}(y)] = [\mathbf{d}(x), \mathbf{d}(y)] \cdot y \\
\Leftrightarrow\; & x \cdot [\mathbf{r}(x), \mathbf{r}(y)] \cdot [\mathbf{r}(y), \mathbf{r}(x)] = [\mathbf{d}(x), \mathbf{d}(y)] \cdot y \cdot [\mathbf{r}(y), \mathbf{r}(x)] \\
\Leftrightarrow\; & x \cdot [\mathbf{r}(x), \mathbf{r}(x)] = [\mathbf{d}(x), \mathbf{d}(y)] \cdot y \cdot [\mathbf{r}(y), \mathbf{r}(x)] \qquad \big(\text{by (OB5)}\big)
\end{aligned}
$$

$$
\Leftrightarrow x = [\mathbf{d}(x), \mathbf{d}(y)] \cdot y \cdot [\mathbf{r}(y), \mathbf{r}(x)] \qquad \big([\mathbf{r}(x), \mathbf{r}(x)] = 1_{\mathbf{r}(x)}\big).
$$

In particular, if $x, y \in P$ are such that $\mathbf{d}(x) = \mathbf{d}(y)$, then $x \;\rho\; y$ if and only if $\mathbf{r}(x) \;\mathcal{L}\; \mathbf{r}(y)$ and

$$x = y \cdot [\mathbf{r}(y), \mathbf{r}(x)] \text{ or indeed } x \cdot [\mathbf{r}(x), \mathbf{r}(y)] = y.$$

Dually, if $\mathbf{r}(x) = \mathbf{r}(y)$, then $x \;\rho\; y$ if and only if $\mathbf{d}(x) \;\mathcal{R}\; \mathbf{d}(y)$ and

$$x = [\mathbf{d}(x), \mathbf{d}(y)] \cdot y \text{ or indeed } y = [\mathbf{d}(y), \mathbf{d}(x)] \cdot x.$$

Built on the above statement, it is easy to see that for any $e \in B$, $[e^+, e] \;\rho\; 1_e$. Since

$$\mathbf{d}([e^+, e]) = e^+ \;\mathcal{R}\; e = \mathbf{d}(1_e), \quad \mathbf{r}([e^+, e]) = e = \mathbf{r}(1_e)$$

and

$$[e, e^+] \cdot [e^+, e] = [e, e] = 1_e,$$

we have that $[e^+, e] \;\rho\; 1_e$. Dually, $[e, e^\star] \;\rho\; 1_e$.

Lemma 9.2. *The relation ρ defined above is an equivalence on P such that if $x, y \in Mor(e, f)$ and $x \;\rho\; y$, then $x = y$. In particular, no two identities of P are ρ-equivalent.*

Proof. Clearly, ρ is reflexive.

In order to show that ρ is symmetric, we assume that $x, y \in P$ with $x \;\rho\; y$. Then $\mathbf{d}(x) \;\mathcal{R}\; \mathbf{d}(y)$, $\mathbf{r}(x) \;\mathcal{L}\; \mathbf{r}(y)$ and $x \cdot [\mathbf{r}(x), \mathbf{r}(y)] = [\mathbf{d}(x), \mathbf{d}(y)] \cdot y$. Since $\mathcal{R}$ and

$\mathcal{L}$ are symmetric, $[\mathbf{r}(y), \mathbf{r}(x)]$ and $[\mathbf{d}(y), \mathbf{d}(x)]$ exist. Thus,

$$\begin{aligned}
&x \cdot [\mathbf{r}(x), \mathbf{r}(y)] = [\mathbf{d}(x), \mathbf{d}(y)] \cdot y \\
\Rightarrow\ & x = [\mathbf{d}(x), \mathbf{d}(y)] \cdot y \cdot [\mathbf{r}(y), \mathbf{r}(x)] \quad \Big(\text{by the statements above Lemma 9.2}\Big) \\
\Rightarrow\ & [\mathbf{d}(y), \mathbf{d}(x)] \cdot x = [\mathbf{d}(y), \mathbf{d}(x)] \cdot [\mathbf{d}(x), \mathbf{d}(y)] \cdot y \cdot [\mathbf{r}(y), \mathbf{r}(x)] \\
\Rightarrow\ & [\mathbf{d}(y), \mathbf{d}(x)] \cdot x = [\mathbf{d}(y), \mathbf{d}(y)] \cdot y \cdot [\mathbf{r}(y), \mathbf{r}(x)] \qquad \Big(\text{by (OB5)}\Big) \\
\Rightarrow\ & [\mathbf{d}(y), \mathbf{d}(x)] \cdot x = y \cdot [\mathbf{r}(y), \mathbf{r}(x)] \qquad \Big([\mathbf{d}(y), \mathbf{d}(y)] = 1_{\mathbf{d}(y)}\Big).
\end{aligned}$$

So, $y \ \rho \ x$.

Finally, if $x \ \rho \ y$ and $y \ \rho \ z$, then $\mathbf{d}(x) \ \mathcal{R} \ \mathbf{d}(z)$ and $\mathbf{r}(x) \ \mathcal{L} \ \mathbf{r}(z)$ as $\mathcal{R}$ and $\mathcal{L}$ are transitive. Hence, $[\mathbf{d}(x), \mathbf{d}(z)]$ and $[\mathbf{r}(x), \mathbf{r}(z)]$ exist. Then we have that

$$\begin{aligned}
x \cdot [\mathbf{r}(x), \mathbf{r}(z)] &= x \cdot [\mathbf{r}(x), \mathbf{r}(y)] \cdot [\mathbf{r}(y), \mathbf{r}(z)] \quad \Big(\text{by (OB5), } \mathbf{r}(x) \ \mathcal{L} \ \mathbf{r}(y) \ \mathcal{L} \ \mathbf{r}(z)\Big) \\
&= [\mathbf{d}(x), \mathbf{d}(y)] \cdot y \cdot [\mathbf{r}(y), \mathbf{r}(z)] \qquad \Big(x \ \rho \ y\Big) \\
&= [\mathbf{d}(x), \mathbf{d}(y)] \cdot [\mathbf{d}(y), \mathbf{d}(z)] \cdot z \qquad \Big(y \ \rho \ z\Big) \\
&= [\mathbf{d}(x), \mathbf{d}(z)] \cdot z \qquad \Big(\text{by (OB5)}\Big).
\end{aligned}$$

Thus, $x \ \rho \ z$.

As $[e, e] = 1_e$ for all $e \in B$, certainly, if $\mathbf{d}(x) = \mathbf{d}(y)$, $\mathbf{r}(x) = \mathbf{r}(y)$ and $x \ \rho \ y$, then $x = y$. □

We now present a pair of pre-orders on a band category over B built on the relation ρ given above.

Let P be a band category over B. We make use of the restriction and co-restriction of P to define relations $\leq_r$ and $\leq_\ell$ by the rule that for all $x, y \in P$,

$$x \leq_r y \text{ if and only if } x \ \rho \ {}_e|y \quad \text{for some } e \in B,$$

and

$$x \leq_\ell y \text{ if and only if } x \ \rho \ y|_f \quad \text{for some } f \in B.$$

Lemma 9.3. *The relations $\leq_r$ and $\leq_\ell$ are pre-orders on P.*

Proof. We first show that $\leq_r$ is a pre-order on P. Notice that for any $x \in P$, if $e = \mathbf{d}(x)$, then $[\mathbf{d}(x), \mathbf{d}({}_e|x)] = [e, e] = 1_e$, and so $\leq_r$ is reflexive by (OB1). It is sufficient to show that $\leq_r$ is transitive. Suppose that $x, y, z \in P$ with

$x \leq_r y$ and $y \leq_r z$. Then there exist $e, f \in B$ such that $x \;\rho\; {}_e|y$ and $y \;\rho\; {}_f|z$. Thus, $\mathbf{d}(x) \,\mathcal{R}\, e \leq_{\mathcal{L}} \mathbf{d}(y) \,\mathcal{R}\, f \leq_{\mathcal{L}} \mathbf{d}(z)$, and so $\mathbf{d}(x) \,\mathcal{R}\, e = e\mathbf{d}(y) \,\mathcal{R}\, ef$. Set $g = ef$. Since $g = ef \leq_{\mathcal{L}} f \leq_{\mathcal{L}} \mathbf{d}(z)$, ${}_g|z$ is well-defined. Now our aim is to show that $x \;\rho\; {}_g|z$. From $y \;\rho\; {}_f|z$, we have $\mathbf{d}(y) \,\mathcal{R}\, f$, $\mathbf{r}(y) \,\mathcal{L}\, \mathbf{r}({}_f|z)$ and $y \cdot [\mathbf{r}(y), \mathbf{r}({}_f|z)] = [\mathbf{d}(y), f] \cdot ({}_f|z)$. Hence, $y = [\mathbf{d}(y), f] \cdot {}_f|z \cdot [\mathbf{r}({}_f|z), \mathbf{r}(y)]$. For ${}_e|y$ and ${}_f|z$ to exist, we have that $e \leq_{\mathcal{L}} \mathbf{d}(y)$ and $f \leq_{\mathcal{L}} \mathbf{d}(z)$, so that ${}_e|[\mathbf{d}(y), f]$ exists and ${}_e|[\mathbf{d}(y), f] = [e, (ef)^\star]$ by (OB4). As $(ef)^\star \;\mathcal{L}\; ef \leq_{\mathcal{L}} f \leq_{\mathcal{L}} \mathbf{d}(z)$, we obtain that ${}_{(ef)^\star}|({}_f|z)$ is defined and ${}_{(ef)^\star}|({}_f|z) = {}_{(ef)^\star}|z$ by (OB2). Then we have that

$$\begin{aligned}
{}_e|y &= {}_e|([\mathbf{d}(y), f] \cdot {}_f|z \cdot [\mathbf{r}({}_f|z), \mathbf{r}(y)]) \\
&= {}_e|[\mathbf{d}(y), f] \cdot {}_h|({}_f|z) \cdot {}_k|[\mathbf{r}({}_f|z), \mathbf{r}(y)] \\
&\qquad \Big(\text{by (OB3)},\ h = \mathbf{r}({}_e|[\mathbf{d}(y), f]),\ k = \mathbf{r}({}_h|({}_f|z))\Big) \\
&= [e, (ef)^\star] \cdot {}_{(ef)^\star}|({}_f|z) \cdot [k, (k\mathbf{r}(y))^\star] \qquad \Big(\text{by (OB4)}\Big) \\
&= [e, (ef)^\star] \cdot {}_{(ef)^\star}|z \cdot [k, k^\star] \\
&\qquad \Big(\text{by (OB2)},\ k = \mathbf{r}({}_h|({}_f|z)) \leq_{\mathcal{L}} \mathbf{r}({}_f|z) \;\mathcal{L}\; \mathbf{r}(y)\Big).
\end{aligned}$$

Hence, $\mathbf{r}({}_e|y) = k^\star$. From $x \;\rho\; {}_e|y$, we have that $\mathbf{r}(x) \,\mathcal{L}\, k^\star$, $\mathbf{d}(x) \,\mathcal{R}\, e$ and

$$\begin{aligned}
x \cdot [\mathbf{r}(x), k^\star] &= [\mathbf{d}(x), e] \cdot {}_e|y \\
&= [\mathbf{d}(x), e] \cdot [e, (ef)^\star] \cdot {}_{(ef)^\star}|z \cdot [k, k^\star] \\
&= [\mathbf{d}(x), (ef)^\star] \cdot {}_{(ef)^\star}|z \cdot [k, k^\star] \qquad \Big(\text{by (OB5)}\Big).
\end{aligned}$$

Thus,

$$\begin{aligned}
x \cdot [\mathbf{r}(x), k] &= x \cdot [\mathbf{r}(x), k^\star] \cdot [k^\star, k] && \Big(\text{by (OB5)}\Big) \\
&= [\mathbf{d}(x), (ef)^\star] \cdot {}_{(ef)^\star}|z \cdot [k, k^\star] \cdot [k^\star, k] \\
&= [\mathbf{d}(x), g] \cdot [g, (ef)^\star] \cdot {}_{(ef)^\star}|z \cdot [k, k] && \Big(\mathbf{d}(x) \;\mathcal{R}\; g \;\mathcal{L}\; (ef)^\star,\ \text{by (OB5)}\Big) \\
&= [\mathbf{d}(x), g] \cdot [g, (ef)^\star] \cdot {}_{(ef)^\star}|z && \Big(k = \mathbf{r}({}_{(ef)^\star}|z)\Big) \\
&= [\mathbf{d}(x), g] \cdot {}_g|z && \Big(\text{by (OB2)}\Big).
\end{aligned}$$

It follows that $k = \mathbf{r}({}_g|z)$. As $\mathbf{r}(x) \;\mathcal{L}\; \mathbf{r}({}_e|y) = k^\star \;\mathcal{L}\; k = \mathbf{r}({}_g|z)$ and $\mathbf{d}(x) \,\mathcal{R}\, g = \mathbf{d}({}_g|z)$, we have that $x \;\rho\; {}_g|z$. Hence, $x \leq_r z$.

By the dual argument, we show that $\leq_\ell$ is a pre-order on P. □

We remark that if B is a semilattice, then the relation ρ on P is precisely the identity relation so that relations $\leq_r$ and $\leq_\ell$ may be expressed as follows: for all $x, y \in P$,

$$x \leq_r y \text{ if and only if } x = {}_e|y \qquad \text{for some } e \in B,$$

and

$$x \leq_\ell y \text{ if and only if } x = y|_f \qquad \text{for some } f \in B.$$

In addition, the relations $\leq_r$ and $\leq_\ell$ become partial orders. Since if $x \leq_r y \leq_r x$, then $x = {}_e|y$ and $y = {}_f|x$ for some $e \leq \mathbf{d}(y)$ and $f \leq \mathbf{d}(x)$. We have that $\mathbf{d}(x) = e \leq \mathbf{d}(y) = f \leq \mathbf{d}(x)$ so $\mathbf{d}(x) = \mathbf{d}(y)$. By (OB1), $\mathbf{r}({}_{\mathbf{d}(y)}|y) \;\mathcal{L}\; \mathbf{r}(y)$, and so $\mathbf{r}({}_{\mathbf{d}(y)}|y) = \mathbf{r}(y)$ as B is a semilattice. Thus, $x = {}_e|y = {}_{\mathbf{d}(y)}|y$. By (OB1), ${}_{\mathbf{d}(y)}|y \cdot [\mathbf{r}({}_{\mathbf{d}(y)}|y), \mathbf{r}(y)] = y$, so that

$$x = {}_e|y = {}_{\mathbf{d}(y)}|y = {}_{\mathbf{d}(y)}|y \cdot [\mathbf{r}({}_{\mathbf{d}(y)}|y), \mathbf{r}(y)] = y \qquad \big(\mathbf{r}({}_{\mathbf{d}(y)}|y) = \mathbf{r}(y)\big).$$

We pause here to make some further comments on Definition 9.1. In (OB1) let $e = \mathbf{d}(x)$. Then $\mathbf{r}({}_e|x)\mathcal{L}\;\mathbf{r}(x)$ and $x = {}_e|x \cdot [\mathbf{r}({}_e|x), \mathbf{r}(x)]$. Hence due to the definition of ρ, we obtain that $x \;\rho\; {}_e|x$. This fact makes it impossible to define a partial order $\leq_r{}'$ on P by the rule that for all $x, y \in P$, $x \leq_r{}' y$ if and only if $x = {}_e|y$ for some $e \leq \mathbf{d}(y)$ because it is not reflexive; even if $x = {}_e|y$ is replaced by $x \;\rho\; {}_e|y$, we still cannot guarantee that $\leq_r{}'$ is a partial order since $\leq_r{}'$ becomes reflexive but not anti-symmetric.

As an analogue of inductive generalised categories over B in Chapter 7, we will make use of the relation ρ given above to define weakly orthodox categories over B, which are built on Definition 9.1.

Definition 9.4. A band category P over B is *weakly orthodox* if for any $x \in P$ and $e, f \in B$, ${}_{eu}|(x|_{gf}) \;\rho\; ({}_{eh}|x)|_{vf}$, where $g = \mathbf{r}(x)$, $h = \mathbf{d}(x)$, $u = \mathbf{d}(x|_{gf})$ and $v = \mathbf{r}({}_{eh}|x)$.

It is worth considering how the class of weakly orthodox categories over bands forms a category, together with certain functors referred to as orthodox functors. They are described in the next definition.

Definition 9.5. Let P_1 and P_2 be weakly orthodox categories over B_1 and B_2, respectively. An *orthodox functor* F from P_1 to P_2 is a functor consisting of a pair of maps, both denoted F, from B_1 to B_2 and from P_1 to P_2, such that the following conditions and the dual (S3)° of (S3) hold:

(S1) the map F is a morphism from B_1 to B_2;

(S2) if $e, f \in B_1$ with $e \mathcal{D} f$, then $[e, f]_{P_1} F = [eF, fF]_{P_2}$;

(S3) if $x \in P_1$ and $e \in B_1$ with $e \leq_{\mathcal{L}} \mathbf{d}(x)$, then $({}_e|x)F \, \rho \, {}_{eF}|xF$.

We pause here to make a short comment on Definition 9.5. In (S2), if $e, f \in B_1$ with $e \mathcal{D} f$, then by (S1), $eF \mathcal{D} fF$, so that both $[e, f]_{P_1}$ and $[eF, fF]_{P_2}$ are defined. In (S3), if $e \leq_{\mathcal{L}} \mathbf{d}(x)$, then $eF \leq_{\mathcal{L}} \mathbf{d}(xF)$ as F is a functor, so that both ${}_e|x$ and ${}_{eF}|xF$ are well-defined. In addition, the fact that $({}_e|x)F \, \rho \, {}_{eF}|xF$ gives in particular that $\mathbf{r}(({}_e|x)F) \, \mathcal{L} \, \mathbf{r}({}_{eF}|xF)$. For $\mathbf{d}$, we have the corresponding result as F is a functor.

The next lemma is useful for Lemma 9.7.

Lemma 9.6. *Let P_1 and P_2 be weakly orthodox categories over B_1 and B_2, respectively and let $F : P_1 \to P_2$ be an orthodox functor. If $x \; \rho \; y$ in P_1, then $xF \; \rho \; yF$ in P_2.*

Proof. Suppose that $x, y \in P_1$ and $x \; \rho \; y$. Then

$$\begin{aligned}
&\mathbf{d}(x) \, \mathcal{R} \, \mathbf{d}(y), \;\; \mathbf{r}(x) \, \mathcal{L} \, \mathbf{r}(y) \text{ and } x \cdot [\mathbf{r}(x), \mathbf{r}(y)] = [\mathbf{d}(x), \mathbf{d}(y)] \cdot y \\
\Rightarrow\; &\mathbf{d}(x)F \, \mathcal{R} \, \mathbf{d}(y)F, \;\; \mathbf{r}(x)F \, \mathcal{L} \, \mathbf{r}(y)F \text{ and} \\
&\qquad xF \cdot [\mathbf{r}(x), \mathbf{r}(y)]F = [\mathbf{d}(x), \mathbf{d}(y)]F \cdot yF \\
\Rightarrow\; &\mathbf{d}(x)F \, \mathcal{R} \, \mathbf{d}(y)F, \;\; \mathbf{r}(x)F \, \mathcal{L} \, \mathbf{r}(y)F \text{ and} \\
&\qquad xF \cdot [\mathbf{r}(x)F, \mathbf{r}(y)F] = [\mathbf{d}(x)F, \mathbf{d}(y)F] \cdot yF \quad \big(\text{by } (S2)\big) \\
\Rightarrow\; &\mathbf{d}(xF) \, \mathcal{R} \, \mathbf{d}(yF), \;\; \mathbf{r}(xF) \, \mathcal{L} \, \mathbf{r}(yF) \text{ and} \\
&\qquad xF \cdot [\mathbf{r}(xF), \mathbf{r}(yF)] = [\mathbf{d}(xF), \mathbf{d}(yF)] \cdot yF \\
&\qquad\qquad \big(\text{by } F \text{ being a functor}\big).
\end{aligned}$$

Hence, $xF \; \rho \; yF$. □

Lemma 9.7. *Let P_1 and P_2 be weakly orthodox categories over B_1 and B_2, respectively, and let $F_1 : P_1 \to P_2$ and $F_2 : P_2 \to P_3$ be orthodox functors. Then $F_1F_2 : P_1 \to P_3$ is an orthodox functor.*

Proof. (S1) Certainly, F_1F_2 is a functor from P_1 to P_3 and a morphism from B_1 to B_3.

(S2) Suppose that $e, f \in B_1$ are such that $e \mathcal{D} f$. Then $[e, f]_{P_1}$ is defined and using (S2) for F_1 and F_2,

$$[e, f]_{P1} F_1 F_2 = ([e, f]_{P1} F_1) F_2 = [eF_1, fF_1]_{P_2} F_2 = [eF_1F_2, fF_1F_2]_{P3}.$$

(S3) Suppose that $x \in P_1$ and $e \in B_1$ with $e \leq_{\mathcal{L}} \mathbf{d}(x)$. According to the comment succeeding Definition 9.5, we have that ${}_e|x$, ${}_{eF_1}|xF_1$ and ${}_{eF_1F_2}|xF_1F_2$ are well-defined. By (S3), $({}_e|x)F_1 \ \rho \ {}_{eF_1}|xF_1$ and $({}_{eF_1}|xF_1)F_2 \ \rho \ {}_{eF_1F_2}|xF_1F_2$. From $({}_e|x)F_1 \ \rho \ {}_{eF_1}|xF_1$, we obtain that $({}_e|x)F_1F_2 \ \rho \ ({}_{eF_1}|xF_1)F_2$ by Lemma 9.6. Hence, $({}_e|x)F_1F_2 \ \rho \ {}_{eF_1F_2}|xF_1F_2$. □

An immediate observation from Lemma 9.7 is that the class of weakly orthodox categories over bands and orthodox functors forms a category. We refer to it as $\mathcal{WOC}$.

9.2 Construction

Our aim in this section is to build a weakly B-orthodox semigroup from a weakly orthodox category over B. This result is analogous to Armstrong's work [1] building a concordant semigroup from an inductive$_2$ cancellative category. Let P be a weakly orthodox category over B. For any $x, y \in P$, we define

$$x \otimes y = x|_{\mathbf{r}(x)\mathbf{d}(y)} \cdot {}_{\mathbf{r}(x)\mathbf{d}(y)}|y.$$

Before we give a list of lemmas which are necessary to prove our main theorem, we make a comment that since our set of idempotents forms a band it is an advantage to use the product $\otimes$ given above to avoid the notion of sandwich set, which is needed in [1].

Lemma 9.8. *If P is a weakly orthodox category over B and $x \ \rho \ x'$, $y \ \rho \ y'$ in P, then $x \otimes y \ \rho \ x' \otimes y'$.*

Proof. Suppose that $x \ \rho \ x'$ and $y \ \rho \ y'$ in P. Then

$$\mathbf{d}(x) \mathcal{R} \ \mathbf{d}(x'), \mathbf{r}(x) \ \mathcal{L} \ \mathbf{r}(x') \text{ and } x \cdot [\mathbf{r}(x), \mathbf{r}(x')] = [\mathbf{d}(x), \mathbf{d}(x')] \cdot x',$$

whence $x = [\mathbf{d}(x), \mathbf{d}(x')] \cdot x' \cdot [\mathbf{r}(x'), \mathbf{r}(x)]$. The same happens to y and y' as follows

$$\mathbf{d}(y) \mathcal{R}\ \mathbf{d}(y'), \mathbf{r}(y)\ \mathcal{L}\ \mathbf{r}(y') \text{ and } y \cdot [\mathbf{r}(y), \mathbf{r}(y')] = [\mathbf{d}(y), \mathbf{d}(y')] \cdot y',$$

whence $y = [\mathbf{d}(y), \mathbf{d}(y')] \cdot y' \cdot [\mathbf{r}(y'), \mathbf{r}(y)]$. So

$$\mathbf{r}(x)\mathbf{d}(y)\ \mathcal{R}\ \mathbf{r}(x)\mathbf{d}(y')\ \mathcal{L}\ \mathbf{r}(x')\mathbf{d}(y')\ \mathcal{R}\ \mathbf{r}(x')\mathbf{d}(y).$$

As $\mathbf{r}(x)\mathbf{d}(y) \leq_{\mathcal{R}} \mathbf{r}(x)$, $x|_{\mathbf{r}(x)\mathbf{d}(y)}$ is defined and we have that

$$\begin{aligned}
&x|_{\mathbf{r}(x)\mathbf{d}(y)} \\
&= ([\mathbf{d}(x), \mathbf{d}(x')] \cdot x' \cdot [\mathbf{r}(x'), \mathbf{r}(x)])|_{\mathbf{r}(x)\mathbf{d}(y)} \\
&= [\mathbf{d}(x), \mathbf{d}(x')]|_k \cdot (x'|_{(\mathbf{r}(x')\mathbf{r}(x)\mathbf{d}(y))^+}) \cdot [(\mathbf{r}(x')\mathbf{r}(x)\mathbf{d}(y))^+, \mathbf{r}(x)\mathbf{d}(y)] \\
&\qquad\qquad \Big(\text{by (OB4)}^\circ, \text{(OB3)}^\circ, k = \mathbf{d}(x'|_{(\mathbf{r}(x')\mathbf{r}(x)\mathbf{d}(y))^+})\Big) \\
&= [(\mathbf{d}(x)k)^+, k] \cdot (x'|_{(\mathbf{r}(x')\mathbf{d}(y))^+}) \cdot [(\mathbf{r}(x')\mathbf{d}(y))^+, \mathbf{r}(x)\mathbf{d}(y)] \\
&\qquad\qquad \Big(\text{by (OB4)}^\circ,\ \mathbf{r}(x)\,\mathcal{L}\,\mathbf{r}(x')\Big) \\
&= [k^+, k] \cdot (x'|_{(\mathbf{r}(x')\mathbf{d}(y))^+}) \cdot [(\mathbf{r}(x')\mathbf{d}(y))^+, \mathbf{r}(x)\mathbf{d}(y)] \\
&\qquad\qquad \Big(\text{by (OB1)}^\circ, k \leq_{\mathcal{R}} \mathbf{d}(x')\ \mathcal{R}\ \mathbf{d}(x)\Big).
\end{aligned}$$

Similarly, we have that

$$_{\mathbf{r}(x)\mathbf{d}(y)}|y = [\mathbf{r}(x)\mathbf{d}(y), (\mathbf{r}(x)\mathbf{d}(y'))^\star] \cdot ({}_{(\mathbf{r}(x)\mathbf{d}(y'))^\star}|y') \cdot [g, g^\star],$$

where $g = \mathbf{r}({}_{(\mathbf{r}(x)\mathbf{d}(y'))^\star}|y')$. So

$$\begin{aligned}
x \otimes y &= x|_{\mathbf{r}(x)\mathbf{d}(y)} \cdot {}_{\mathbf{r}(x)\mathbf{d}(y)}|y \\
&= [k^+, k] \cdot (x'|_{(\mathbf{r}(x')\mathbf{d}(y))^+}) \cdot [(\mathbf{r}(x')\mathbf{d}(y))^+, \mathbf{r}(x)\mathbf{d}(y)] \cdot \\
&\qquad [\mathbf{r}(x)\mathbf{d}(y), (\mathbf{r}(x)\mathbf{d}(y'))^\star] \cdot ({}_{(\mathbf{r}(x)\mathbf{d}(y'))^\star}|y') \cdot [g, g^\star] \\
&= [k^+, k] \cdot (x'|_{(\mathbf{r}(x')\mathbf{d}(y))^+}) \cdot [(\mathbf{r}(x')\mathbf{d}(y))^+, (\mathbf{r}(x)\mathbf{d}(y'))^\star] \cdot \\
&\qquad ({}_{(\mathbf{r}(x)\mathbf{d}(y'))^\star}|y') \cdot [g, g^\star] && \big(\text{by (OB5)}\big) \\
&= [k^+, k] \cdot (x'|_{(\mathbf{r}(x')\mathbf{d}(y))^+}) \cdot [(\mathbf{r}(x')\mathbf{d}(y))^+, \mathbf{r}(x')\mathbf{d}(y')] \cdot \\
&\qquad [\mathbf{r}(x')\mathbf{d}(y'), (\mathbf{r}(x)\mathbf{d}(y'))^\star] \cdot ({}_{(\mathbf{r}(x)\mathbf{d}(y'))^\star}|y') \cdot [g, g^\star] && \big(\text{by (OB5)}\big) \\
&= [k^+, k] \cdot (x'|_{\mathbf{r}(x')\mathbf{d}(y')}) \cdot ({}_{\mathbf{r}(x')\mathbf{d}(y')}|y') \cdot [g, g^\star] && \big(\text{by (OB2), (OB2)}^\circ\big) \\
&= [k^+, k] \cdot (x' \otimes y') \cdot [g, g^\star].
\end{aligned}$$

Obviously, $\mathbf{d}(x \otimes y) = k^+ \,\mathcal{R}\, k = \mathbf{d}(x' \otimes y')$ and $\mathbf{r}(x \otimes y) = g^\star \,\mathcal{L}\, g = \mathbf{r}(x' \otimes y')$. It follows from the observation succeeding the definition of ρ that $x \otimes y \;\rho\; x' \otimes y'$. □

Lemma 9.9. *If $x, y \in P$ with $\exists x \cdot y$ in P, then $x \otimes y \;\rho\; x \cdot y$.*

Proof. If $\exists x \cdot y$ in P, then $\mathbf{r}(x) = \mathbf{d}(y)$. So

$$x \otimes y = x|_{\mathbf{r}(x)\mathbf{d}(y)} \cdot {}_{\mathbf{r}(x)\mathbf{d}(y)}|y = x|_{\mathbf{r}(x)} \cdot {}_{\mathbf{d}(y)}|y.$$

By (OB1) and (OB1)$^\circ$, we have that $[\mathbf{d}(x), k] \cdot (x|_{\mathbf{r}(x)}) = x$, where $\mathbf{d}(x) \,\mathcal{R}\, k = \mathbf{d}(x|_{\mathbf{r}(x)}) = \mathbf{d}(x \otimes y)$ and $({}_{\mathbf{d}(y)}|y) \cdot [g, \mathbf{r}(y)] = y$, where $\mathbf{r}(y) \,\mathcal{L}\, g = \mathbf{r}({}_{\mathbf{d}(y)}|y) = \mathbf{r}(x \otimes y)$. Thus,

$$x|_{\mathbf{r}(x)} = [k, \mathbf{d}(x)] \cdot x \text{ and } {}_{\mathbf{d}(y)}|y = y \cdot [\mathbf{r}(y), g].$$

So, $x \otimes y = [k, \mathbf{d}(x)] \cdot (x \cdot y) \cdot [\mathbf{r}(y), g]$, that is, $(x \otimes y) \cdot [g, \mathbf{r}(y)] = [k, \mathbf{d}(x)] \cdot (x \cdot y)$. Hence, $x \otimes y \;\rho\; x \cdot y$. □

The next lemma is an immediate consequence of Lemma 9.8 and Lemma 9.9.

Lemma 9.10. *Let $x, x', y, y' \in P$ be such that $x \,\rho\, x'$ and $y \,\rho\, y'$. If $x \cdot y$ and $x' \cdot y'$ exist in P, then $x \cdot y \,\rho\, x' \cdot y'$.*

Let P be a weakly orthodox category over B and let ρ be the equivalence given in Section 9.1. We define

$$P\mathbf{S} = P/\rho,$$

and

$$\bar{x} \circ \bar{y} = \overline{x \otimes y},$$

where $x, y \in P$ and $\bar{x}$ denotes the ρ-class of P containing x.

We remark that by Lemma 9.8, the product $\circ$ defined above is well-defined.

Our next task is to show that $P\mathbf{S}$ is a weakly $\overline{B}$-orthodox semigroup, where $\overline{B} = \{\overline{1_e} : e \in B\}$.

Lemma 9.11. *If $e, f \in B$, then $\overline{1_e} \circ \overline{1_f} = \overline{1_{ef}}$. Further, the map $\varphi : B \to \overline{B}$, given by $e\varphi = \overline{1_e}$ for any $e \in B$, is an isomorphism, where $\overline{B} = \{\overline{1_e} : e \in B\}$.*

Proof. If $e, f \in B$, then $\overline{1_e} \circ \overline{1_f} = \overline{1_e \otimes 1_f}$. Notice that

$$\begin{aligned}
1_e \otimes 1_f &= (1_e|_{ef}) \cdot ({}_{ef}|1_f) \\
&= ([e,e]|_{ef}) \cdot ({}_{ef}|[f,f]) \\
&= [(ef)^+, ef] \cdot [ef, (ef)^\star] && \big(\text{by (OB4), (OB4)}^\circ\big) \\
&= [(ef)^+, (ef)^\star] && \big(\text{by (OB5)}\big).
\end{aligned}$$

Since $(ef)^+ \,\mathcal{R}\, ef \,\mathcal{L}\, (ef)^\star$ and $[(ef)^+, ef] \cdot [ef, ef] = [(ef)^+, (ef)^\star] \cdot [(ef)^\star, ef]$, we obtain that $[(ef)^+, (ef)^\star] \;\rho\; [ef, ef]$, that is, $1_e \otimes 1_f \;\rho\; 1_{ef}$. Hence, $\overline{1_e} \circ \overline{1_f} = \overline{1_{ef}}$, and so φ is a morphism.

It follows from Lemma 9.2 that if $e, f \in B$ and $\overline{1_e} = \overline{1_f}$, then $e = f$ so that φ is injective. Clearly, φ is surjective. Consequently, φ is an isomorphism. □

Lemma 9.12. *If P is a weakly orthodox category over B, then $P\mathbf{S}$ is a semigroup.*

Proof. Suppose that $x, y, z \in P$. Then

$$\begin{aligned}
\bar{x} \circ (\bar{y} \circ \bar{z}) &= \bar{x} \circ \overline{y \otimes z} \\
&= \bar{x} \circ \overline{y|_{\mathbf{r}(y)\mathbf{d}(z)} \cdot {}_{\mathbf{r}(y)\mathbf{d}(z)}|z}
\end{aligned}$$

$$
\begin{aligned}
&= \overline{x \otimes (y|_{\mathbf{r}(y)\mathbf{d}(z)} \cdot {}_{\mathbf{r}(y)\mathbf{d}(z)}|z)} \\
&= \overline{x|_{\mathbf{r}(x)u} \cdot {}_{\mathbf{r}(x)u}|(y|_{\mathbf{r}(y)\mathbf{d}(z)} \cdot {}_{\mathbf{r}(y)\mathbf{d}(z)}|z)} \quad \big(u = \mathbf{d}(y|_{\mathbf{r}(y)\mathbf{d}(z)})\big) \\
&= \overline{x|_{\mathbf{r}(x)u} \cdot {}_{\mathbf{r}(x)u}|(y|_{\mathbf{r}(y)\mathbf{d}(z)}) \cdot {}_{v}|({}_{\mathbf{r}(y)\mathbf{d}(z)}|z)} \\
&\qquad\qquad \big(v = \mathbf{r}({}_{\mathbf{r}(x)u}|(y|_{\mathbf{r}(y)\mathbf{d}(z)})), \text{ by (OB3)}\big).
\end{aligned}
$$

Since P is weakly orthodox, it follows that ${}_{\mathbf{r}(x)u}|(y|_{\mathbf{r}(y)\mathbf{d}(z)}) \;\rho\; ({}_{\mathbf{r}(x)\mathbf{d}(y)}|y)|_{g\mathbf{d}(z)}$, where $g = \mathbf{r}({}_{\mathbf{r}(x)\mathbf{d}(y)}|y)$. Hence, if we put $k = \mathbf{d}(({}_{\mathbf{r}(x)\mathbf{d}(y)}|y)|_{g\mathbf{d}(z)})$, then $\mathbf{r}(x)u \;\mathcal{R}\; k$, $v \;\mathcal{L}\; g\mathbf{d}(z)$ and

$$
{}_{\mathbf{r}(x)u}|(y|_{\mathbf{r}(y)\mathbf{d}(z)}) \cdot [v, g\mathbf{d}(z)] = [\mathbf{r}(x)u, k] \cdot ({}_{\mathbf{r}(x)\mathbf{d}(y)}|y)|_{g\mathbf{d}(z)},
$$

whence

$$
{}_{\mathbf{r}(x)u}|(y|_{\mathbf{r}(y)\mathbf{d}(z)}) = [\mathbf{r}(x)u, k] \cdot ({}_{\mathbf{r}(x)\mathbf{d}(y)}|y)|_{g\mathbf{d}(z)} \cdot [g\mathbf{d}(z), v].
$$

So, we go back to the beginning of this proof,

$$
\begin{aligned}
&\bar{x} \circ (\bar{y} \circ \bar{z}) \\
&= \overline{x|_{\mathbf{r}(x)u} \cdot ([\mathbf{r}(x)u, k] \cdot ({}_{\mathbf{r}(x)\mathbf{d}(y)}|y)|_{g\mathbf{d}(z)} \cdot [g\mathbf{d}(z), v]) \cdot {}_{v}|({}_{\mathbf{r}(y)\mathbf{d}(z)}|z)} \\
&= \overline{x|_{\mathbf{r}(x)u} \cdot ([\mathbf{r}(x)u, k] \cdot ({}_{\mathbf{r}(x)\mathbf{d}(y)}|y)|_{g\mathbf{d}(z)} \cdot [g\mathbf{d}(z), v]) \cdot {}_{v}|z} \\
&\qquad \big(v = \mathbf{r}({}_{\mathbf{r}(x)u}|(y|_{\mathbf{r}(y)\mathbf{d}(z)})) \leq_{\mathcal{L}} \mathbf{r}(y)\mathbf{d}(z) \leq_{\mathcal{L}} \mathbf{d}(z), \text{ by (OB2)}\big) \\
&= \overline{(x|_{\mathbf{r}(x)u} \cdot [\mathbf{r}(x)u, k]) \cdot ({}_{\mathbf{r}(x)\mathbf{d}(y)}|y)|_{g\mathbf{d}(z)} \cdot [g\mathbf{d}(z), v] \cdot {}_{v}|z} \\
&= \overline{x|_{k} \cdot ({}_{\mathbf{r}(x)\mathbf{d}(y)}|y)|_{g\mathbf{d}(z)} \cdot ([g\mathbf{d}(z), v] \cdot {}_{v}|z)} \quad \big(k \,\mathcal{R}\, \mathbf{r}(x)u \leq_{\mathcal{R}} \mathbf{r}(x), \text{ by (OB2)}^{\circ}\big) \\
&= \overline{x|_{k} \cdot ({}_{\mathbf{r}(x)\mathbf{d}(y)}|y)|_{g\mathbf{d}(z)} \cdot {}_{g\mathbf{d}(z)}|z} \quad \big(g\mathbf{d}(z) \;\mathcal{L}\; v \leq_{\mathcal{L}} \mathbf{d}(z), \text{ by (OB2)}\big) \\
&= \overline{(x|_{\mathbf{r}(x)\mathbf{d}(y)})|_{k} \cdot ({}_{\mathbf{r}(x)\mathbf{d}(y)}|y)|_{g\mathbf{d}(z)} \cdot {}_{g\mathbf{d}(z)}|z} \\
&\qquad \big(\text{by (OB1)}^{\circ}, \; k = \mathbf{d}(({}_{\mathbf{r}(x)\mathbf{d}(y)}|y)|_{g\mathbf{d}(z)}) \leq_{\mathcal{R}} \mathbf{r}(x)\mathbf{d}(y) \leq_{\mathcal{R}} \mathbf{r}(x), \text{ by (OB2)}^{\circ}\big) \\
&= \overline{(x|_{\mathbf{r}(x)\mathbf{d}(y)} \cdot {}_{\mathbf{r}(x)\mathbf{d}(y)}|y)|_{g\mathbf{d}(z)} \cdot {}_{g\mathbf{d}(z)}|z} \\
&\qquad \big(g\mathbf{d}(z) \leq_{\mathcal{R}} g = \mathbf{r}({}_{\mathbf{r}(x)\mathbf{d}(y)}|y), \; k = \mathbf{d}(({}_{\mathbf{r}(x)\mathbf{d}(y)}|y)|_{g\mathbf{d}(z)}), \text{ by (OB3)}^{\circ}\big) \\
&= \overline{(x \otimes y)|_{g\mathbf{d}(z)} \cdot {}_{g\mathbf{d}(z)}|z} \\
&= \overline{(x \otimes y) \otimes z} \\
&= \overline{(x \otimes y)} \circ \bar{z} \\
&= (\bar{x} \circ \bar{y}) \circ \bar{z}.
\end{aligned}
$$

□

Lemma 9.13. *If P is a weakly orthodox category over B and $x \in P$, then*

$\overline{1_{\mathbf{d}(x)}}\ \widetilde{\mathcal{R}}_{\overline{B}}\ \bar{x}\ \widetilde{\mathcal{L}}_{\overline{B}}\ \overline{1_{\mathbf{r}(x)}}$.

Proof. Let $x \in P$. By Lemma 9.9, we have that

$$\overline{1_{\mathbf{d}(x)}} \circ \bar{x} = \overline{1_{\mathbf{d}(x)} \otimes x} = \overline{1_{\mathbf{d}(x)} \cdot x} = \bar{x}.$$

Suppose that $k \in B$ is such that $\overline{1_k} \circ \bar{x} = \bar{x}$. Then

$$\begin{aligned}\overline{1_k} \circ \bar{x} = \overline{1_k \otimes x} &= \overline{1_k|_{k\mathbf{d}(x)} \cdot {}_{k\mathbf{d}(x)}|x} \\ &= \overline{[k,k]|_{k\mathbf{d}(x)} \cdot {}_{k\mathbf{d}(x)}|x} \\ &= \overline{[(k\mathbf{d}(x))^+, k\mathbf{d}(x)] \cdot {}_{k\mathbf{d}(x)}|x} && \big(\text{by (OB4)}^\circ\big).\end{aligned}$$

So $\mathbf{d}(x)\ \mathcal{R}\ (k\mathbf{d}(x))^+\ \mathcal{R}\ k\mathbf{d}(x)$, which implies that $k\mathbf{d}(x) = \mathbf{d}(x)$. Thus

$$\begin{aligned}\overline{1_k} \circ \overline{1_{\mathbf{d}(x)}} &= \overline{1_{k\mathbf{d}(x)}} && \big(\text{Lemma 9.11}\big) \\ &= \overline{1_{\mathbf{d}(x)}} && \big(k\mathbf{d}(x) = \mathbf{d}(x)\big).\end{aligned}$$

Hence, $\bar{x}\ \widetilde{\mathcal{R}}_{\overline{B}}\ \overline{1_{\mathbf{d}(x)}}$.

By the dual argument, we show that $\bar{x}\ \widetilde{\mathcal{L}}_{\overline{B}}\ \overline{1_{\mathbf{r}(x)}}$. ☐

Now, using Lemma 9.11 and Lemma 9.13, we obtain a criterion for $\widetilde{\mathcal{R}}_{\overline{B}}$ and $\widetilde{\mathcal{L}}_{\overline{B}}$ on $P\mathbf{S}$.

Lemma 9.14. *Let P be a weakly orthodox category over B and $x, y \in P$. Then*

(i) $\bar{x}\,\widetilde{\mathcal{R}}_{\overline{B}}\,\bar{y}$ in $P\mathbf{S}$ if and only if $\mathbf{d}(x)\,\mathcal{R}\,\mathbf{d}(y)$ in B;

(ii) $\bar{x}\,\widetilde{\mathcal{L}}_{\overline{B}}\,\bar{y}$ in $P\mathbf{S}$ if and only if $\mathbf{r}(x)\,\mathcal{L}\,\mathbf{r}(y)$ in B.

Lemma 9.15. *If $x \in P$ and $u, v \in B$ are such that $u\ \mathcal{R}\ v$, then $\mathbf{d}(x|_{\mathbf{r}(x)u}) = \mathbf{d}(x|_{\mathbf{r}(x)v})$.*

Proof. Suppose that $x \in P$ and $u, v \in B$ are such that $u\,\mathcal{R}\,v$. Then $\mathbf{r}(x)u\,\mathcal{R}\,\mathbf{r}(x)v$. So,

$$\begin{aligned}x|_{\mathbf{r}(x)u} &= x|_{\mathbf{r}(x)u} \cdot [\mathbf{r}(x)u, \mathbf{r}(x)u] && \big([\mathbf{r}(x)u, \mathbf{r}(x)u] = 1_{\mathbf{r}(x)u}\big) \\ &= x|_{\mathbf{r}(x)u} \cdot [\mathbf{r}(x)u, \mathbf{r}(x)v] \cdot [\mathbf{r}(x)v, \mathbf{r}(x)u] && \big(\text{by (OB5)}\big) \\ &= x|_{\mathbf{r}(x)v} \cdot [\mathbf{r}(x)v, \mathbf{r}(x)u] && \big(\mathbf{r}(x)u\ \mathcal{R}\ \mathbf{r}(x)v,\ \text{by (OB2)}^\circ\big).\end{aligned}$$

Hence, $\mathbf{d}(x|_{\mathbf{r}(x)u}) = \mathbf{d}(x|_{\mathbf{r}(x)v})$. ☐

It is a convenient position from which to build our main theorem.

Theorem 9.16. *If P is a weakly orthodox category over B, then $(P\mathbf{S}, \circ)$ is a weakly $\overline{B}$-orthodox semigroup, where $\overline{B} = \{\overline{1_e} : e \in B\}$.*

Proof. In view of Lemma 9.12 and Lemma 9.13, it is only necessary to show that $P\mathbf{S}$ has (C). Suppose that $\bar{x} \mathrel{\widetilde{\mathcal{R}}_{\overline{B}}} \bar{y}$ and $\bar{z} \in P\mathbf{S}$. We have that $\bar{z} \circ \bar{x} = \overline{z \otimes x} = \overline{z|_{\mathbf{r}(z)\mathbf{d}(x)} \cdot {}_{\mathbf{r}(z)\mathbf{d}(x)}|x}$. Similarly, $\bar{z} \circ \bar{y} = \overline{z \otimes y} = \overline{z|_{\mathbf{r}(z)\mathbf{d}(y)} \cdot {}_{\mathbf{r}(z)\mathbf{d}(y)}|y}$. As $\bar{x} \mathrel{\widetilde{\mathcal{R}}_{\overline{B}}} \bar{y}$, we obtain that $\mathbf{d}(x) \mathrel{\mathcal{R}} \mathbf{d}(y)$ from Lemma 9.14, and so by Lemma 9.15, $\mathbf{d}(z|_{\mathbf{r}(z)\mathbf{d}(x)}) = \mathbf{d}(z|_{\mathbf{r}(z)\mathbf{d}(y)})$. It follows from Lemma 9.14 that $\bar{z} \circ \bar{y} \mathrel{\widetilde{\mathcal{R}}_{\overline{B}}} \bar{z} \circ \bar{x}$, and consequently, $\widetilde{\mathcal{R}}_{\overline{B}}$ is a left congruence. Dually, $\widetilde{\mathcal{L}}_{\overline{B}}$ is a right congruence. $\square$

We end this section by producing an admissible morphism between weakly B-orthodox semigroups from an orthodox functor. This appears in the next lemma.

Lemma 9.17. *Let P_1 and P_2 be weakly orthodox categories over B_1 and B_2, respectively, and let $F : P_1 \to P_2$ be an orthodox functor. Then the map $F\mathbf{S} : P_1\mathbf{S} \to P_2\mathbf{S}$ defined by the rule that $\bar{x}F\mathbf{S} = \overline{xF}$, where $\bar{x} \in P_1\mathbf{S}$ and $x \in P_1$, is an admissible morphism. Further, if $F_1 : P_1 \to P_2$ and $F_2 : P_2 \to P_3$ are orthodox functors, then $(F_1F_2)\mathbf{S} = F_1\mathbf{S}F_2\mathbf{S}$.*

Proof. In view of Lemma 9.6, if $\bar{x} = \bar{y}$ in $P_1\mathbf{S}$, that is, $x \,\rho\, y$ in P_1, then $xF \,\rho\, yF$ in P_2. Hence, $F\mathbf{S}$ is well-defined.

We now show that $F\mathbf{S}$ is a semigroup morphism. Suppose that $x, y \in P_1$. Then

$$\begin{aligned}
(\bar{x} \circ \bar{y})F\mathbf{S} &= (\overline{x|_{\mathbf{r}(x)\mathbf{d}(y)} \cdot {}_{\mathbf{r}(x)\mathbf{d}(y)}|y})F\mathbf{S} \\
&= \overline{(x|_{\mathbf{r}(x)\mathbf{d}(y)} \cdot {}_{\mathbf{r}(x)\mathbf{d}(y)}|y)F} \\
&= \overline{(x|_{\mathbf{r}(x)\mathbf{d}(y)})F \cdot ({}_{\mathbf{r}(x)\mathbf{d}(y)}|y)F} \\
&= \overline{xF|_{(\mathbf{r}(x)\mathbf{d}(y))F} \cdot {}_{(\mathbf{r}(x)\mathbf{d}(y))F}|yF} && \big(\text{by } (S3) \text{ and Lemma 9.10}\big) \\
&= \overline{xF|_{\mathbf{r}(x)F\mathbf{d}(y)F} \cdot {}_{\mathbf{r}(x)F\mathbf{d}(y)F}|yF} && \big(\text{by } (S1)\big) \\
&= \overline{xF|_{\mathbf{r}(xF)\mathbf{d}(yF)} \cdot {}_{\mathbf{r}(xF)\mathbf{d}(yF)}|yF} \\
&= \overline{xF \otimes yF} \\
&= \overline{xF} \circ \overline{yF} \\
&= \bar{x}F\mathbf{S} \circ \bar{y}F\mathbf{S}.
\end{aligned}$$

Next, we show that $F\mathbf{S}$ is admissible. For any $x \in P_1$, we have that $\overline{1_{\mathbf{d}(x)}}\ \widetilde{\mathcal{R}}_{\overline{B_1}}\ \bar{x}\ \widetilde{\mathcal{L}}_{\overline{B_1}}\ \overline{1_{\mathbf{r}(x)}}$. Then

$$\begin{aligned}
\overline{1_{\mathbf{d}(x)}}F\mathbf{S} = \overline{1_{\mathbf{d}(x)}F} &= \overline{[\mathbf{d}(x), \mathbf{d}(x)]F} \\
&= \overline{[\mathbf{d}(x)F, \mathbf{d}(x)F]} && \big(\text{by } (S2)\big) \\
&= \overline{[\mathbf{d}(xF), \mathbf{d}(xF)]} \\
&= \overline{1_{\mathbf{d}(xF)}}\ \widetilde{\mathcal{R}}_{\overline{B_2}}\ \overline{xF} = \bar{x}F\mathbf{S}.
\end{aligned}$$

Dually, we have that $\overline{1_{\mathbf{r}(x)}}F\mathbf{S}\ \widetilde{\mathcal{L}}_{\overline{B_2}}\ \bar{x}F\mathbf{S}$.

Finally, $\overline{1_e}F\mathbf{S} = \overline{1_eF} = \overline{1_{eF}}$ as F is a functor, so that $\overline{B_1}F\mathbf{S} \subseteq \overline{B_2}$. Since $F : B_1 \to B_2$ is a morphism, by Lemma 9.11, we have that $F\mathbf{S}$ is a morphism from $\overline{B_1}$ to $\overline{B_2}$.

To sum up, we have that $F\mathbf{S}$ is an admissible morphism from $P_1\mathbf{S}$ to $P_2\mathbf{S}$. $\square$

Consequently, $\mathbf{S} : \mathcal{WOC} \to \mathcal{WO}$ is a functor by Theorem 9.16 and Lemma 9.17.

9.3 Correspondence

In this section, our purpose is, starting with a weakly B-orthodox semigroup, to build a converse to Theorem 9.16. These results present a correspondence between weakly orthodox categories over bands and weakly B-orthodox semigroups.

Let S be a weakly B-orthodox semigroup and let K be a representative of B. For any $e \in B$, we will use $e^\star$ and e^+ to denote the elements of K which are $\mathcal{L}$-related to e in B and $\mathcal{R}$-related to e in B, respectively. Set

$$S\mathbf{C} = \{(e, x, f) : e\ \widetilde{\mathcal{R}}_B\ x\ \widetilde{\mathcal{L}}_B\ f, \quad e, f \in B\} \subseteq B \times S \times B.$$

We put

$$\mathbf{d}((e, x, f)) = e \text{ (abbreviated to } \mathbf{d}(e, x, f) = e)$$

and

$$\mathbf{r}((e, x, f)) = f \text{ (abbreviated to } \mathbf{r}(e, x, f) = f)$$

for all $(e, x, f) \in S\mathbf{C}$, and define a partial binary operation $\cdot$ on $S\mathbf{C}$ by the rule that

$$(e, x, f) \cdot (f, y, v) = (e, xy, v),$$

where $(e,x,f),(f,y,v)\in S\mathbf{C}$ and xy is the product of x and y in S. If $e,f\in B$ with $e\ \mathcal{D}\ f$, then we define $[e,f]=(e,ef,f)$. Obviously, $[e,f]\in S\mathbf{C}$. For any $(e,x,f)\in S\mathbf{C}$ and $u,v\in B$ with $u\leq_{\mathcal{L}} e$ and $v\leq_{\mathcal{R}} f$, we define

$$ {}_u|(e,x,f)=(u,ux,(ux)^{\star}) \text{ and } (e,x,f)|_v=((xv)^{+},xv,v). $$

Lemma 9.18. *The set $S\mathbf{C}$ is a weakly orthodox category over B with restriction and co-restriction defined above.*

Proof. It is easy to see that $S\mathbf{C}$ forms a category with set of objects B and morphisms the triples as given. For any $e\in B$, $[e,e]=(e,e,e)$ is the identity map associated to e.

(OB1) Suppose that $(e,x,f)\in S\mathbf{C}$ and $u\leq_{\mathcal{L}} e$ in B. Then ${}_u|(e,x,f)=(u,ux,(ux)^{\star})$ and so by Lemma 2.8, $(ux)^{\star}\leq_{\mathcal{L}} f$. In particular, if $u=e$, then ${}_e|(e,x,f)=(e,x,x^{\star})$ and $\mathbf{r}({}_e|(e,x,f))=x^{\star}\ \widetilde{\mathcal{L}}_B\ x\ \widetilde{\mathcal{R}}_B\ f$, so that

$$ {}_e|(e,x,f)\cdot[x^{\star},f]=(e,x,x^{\star})\cdot(x^{\star},x^{\star},f)=(e,x,f), $$

as required.

(OB2) If $(e,x,f)\in S\mathbf{C}$ and $g,h\in B$ with $g\leq_{\mathcal{L}} h\leq_{\mathcal{L}} e$, then

$$ {}_g|({}_h|(e,x,f))={}_g|(h,hx,(hx)^{\star})=(g,ghx,(ghx)^{\star})=(g,gx,(gx)^{\star})={}_g|(e,x,f). $$

In addition, if $g\ \mathcal{L}\ h\leq_{\mathcal{L}} e$, then $[g,h]$ is defined and $[g,h]=(g,g,h)$. Thus

$$ \begin{aligned} [g,h]\cdot {}_h|(e,x,f) &= (g,g,h)\cdot(h,hx,(hx)^{\star}) \\ &= (g,ghx,(hx)^{\star}) \\ &= (g,gx,(hx)^{\star}) \\ &= (g,gx,(gx)^{\star}) \qquad \big(g\ \mathcal{L}\ h\big) \\ &= {}_g|(e,x,f). \end{aligned} $$

(OB3) If $(e,x,f),(f,y,v)\in S\mathbf{C}$ and $u\in B$ with $u\leq_{\mathcal{L}} e$, then

$$ {}_u|((e,x,f)\cdot(f,y,v))={}_u|(e,xy,v)=(u,uxy,(uxy)^{\star}), $$

$$ {}_u|(e,x,f)=(u,ux,(ux)^{\star}) $$

and by Lemma 2.8, $(ux)^\star \leq_{\mathcal{L}} f$, we have that

$$_{(ux)^\star}|(f,y,v) = ((ux)^\star, (ux)^\star y, ((ux)^\star y)^\star).$$

So

$$\begin{aligned} {}_u|(e,x,f) \cdot {}_{(ux)^\star}|(f,y,v) &= (u, ux, (ux)^\star) \cdot ((ux)^\star, (ux)^\star y, ((ux)^\star y)^\star) \\ &= (u, uxy, ((ux)^\star y)^\star). \end{aligned}$$

Since $ux \; \widetilde{\mathcal{L}}_B \; (ux)^\star$, we have that $uxy \; \widetilde{\mathcal{L}}_B \; (ux)^\star y$. Thus $(uxy)^\star = ((ux)^\star y)^\star$, and so ${}_u|((e,x,f)\cdot(f,y,v)) = {}_u|(e,x,f) \cdot {}_{(ux)^\star}|(f,y,v)$.

(OB4) If $e,f,h \in B$ with $e \,\mathcal{D}\, f$ and $h \leq_{\mathcal{L}} e$, then ${}_h|[e,f]$ exists and

$${}_h|[e,f] = {}_h|(e,ef,f) = (h, hef, (hef)^\star) = (h, hf, (hf)^\star).$$

As $h \leq_{\mathcal{L}} e \,\mathcal{D}\, f$, we have that $h = he \,\mathcal{D}\, hf \,\mathcal{L}\, (hf)^\star$, and so $[h,(hf)^\star]$ exists. In addition, $[h,(hf)^\star] = (h, h(hf)^\star, (hf)^\star)$. We note that

$$\begin{aligned} hf &= hf(hf)^\star && \big(hf \,\mathcal{L}\, (hf)^\star\big) \\ &= hhf(hf)^\star && \big(h^2 = h\big) \\ &= hehf(hf)^\star && \big(h \leq_{\mathcal{L}} e\big) \\ &= he(hf)^\star && \big(he \,\mathcal{D}\, hf \,\mathcal{D}\, (hf)^\star\big) \\ &= h(hf)^\star && \big(h \leq_{\mathcal{L}} e\big), \end{aligned}$$

whence, $(h, hf, (hf)^\star) = (h, h(hf)^\star, (hf)^\star)$, that is, ${}_h|[e,f] = [h,(hf)^\star]$.

(OB5) If $e,f,g \in B$ are such that $e \,\mathcal{D}\, f \,\mathcal{D}\, g$, then $[e,f]$, $[f,g]$ and $[e,g]$ are defined. Further, we have that

$$[e,f]\cdot[f,g] = (e,ef,f)\cdot(f,fg,g) = (e,effg,g) = (e,eg,g) = [e,g].$$

Hence, (OB5) holds.

Finally, we show that $S\mathbf{C}$ is weakly orthodox. Assume that $(e,x,f) \in S\mathbf{C}$ and $g,h \in B$. Then

$$(e,x,f)|_{fh} = ((xfh)^+, xfh, fh) = ((xh)^+, xh, fh),$$

$$
\begin{aligned}
{}_{g(xh)^+}|((e,x,f)|_{fh}) &= {}_{g(xh)^+}|((xh)^+, xh, fh)\\
&= (g(xh)^+, g(xh)^+xh, (g(xh)^+xh)^\star)\\
&= (g(xh)^+, gxh, (gxh)^\star) \qquad \big((xh)^+ \,\widetilde{\mathcal{R}}_B\, xh\big),
\end{aligned}
$$

$$
{}_{ge}|(e,x,f) = (ge, gex, (gex)^\star) = (ge, gx, (gx)^\star)
$$

and

$$
\begin{aligned}
({}_{ge}|(e,x,f))|_{(gx)^\star h} &= (ge, gx, (gx)^\star)|_{(gx)^\star h}\\
&= ((gx(gx)^\star h)^+, gx(gx)^\star h, (gx)^\star h)\\
&= ((gxh)^+, gxh, (gx)^\star h) \qquad \big((gx)^\star \,\widetilde{\mathcal{L}}_B\, gx\big).
\end{aligned}
$$

Notice that

$$
\mathbf{d}({}_{g(xh)^+}|((e,x,f)|_{fh})) = g(xh)^+ \;\widetilde{\mathcal{R}}_B\; gxh \;\widetilde{\mathcal{R}}_B\; (gxh)^+ = \mathbf{d}(({}_{ge}|(e,x,f))|_{(gx)^\star h})
$$

and

$$
\mathbf{r}({}_{g(xh)^+}|((e,x,f)|_{fh})) = (gxh)^\star \;\widetilde{\mathcal{L}}_B\; gxh \;\widetilde{\mathcal{L}}_B\; (gx)^\star h = \mathbf{r}(({}_{ge}|(e,x,f))|_{(gx)^\star h}).
$$

Further,

$$
\begin{aligned}
&[g(xh)^+, (gxh)^+] \cdot ({}_{ge}|(e,x,f))|_{(gx)^\star h}\\
&= (g(xh)^+, (gxh)^+, (gxh)^+) \cdot ((gxh)^+, gxh, (gx)^\star h)\\
&= (g(xh)^+, gxh, (gx)^\star h)\\
&= (g(xh)^+, gxh, (gxh)^\star) \cdot ((gxh)^\star, (gxh)^\star, (gx)^\star h)\\
&= {}_{g(xh)^+}|((e,x,f)|_{fh}) \cdot [(gxh)^\star, (gx)^\star h].
\end{aligned}
$$

Thus, ${}_{g(xh)^+}|((e,x,f)|_{fh}) \;\rho\; ({}_{ge}|(e,x,f))|_{(gx)^\star h}$. □

In Section 9.2, orthodox functors between weakly orthodox categories over bands give rise to admissible morphisms. In the following, we produce a converse to this result and so provide a functor $\mathbf{C} : \mathcal{WO} \to \mathcal{WOC}$.

Lemma 9.19. *Let S be a weakly B_1-orthodox semigroup and T be a weakly B_2-orthodox semigroup. Suppose that $\theta : S \to T$ is an admissible morphism. Then the map $\theta\mathbf{C} : S\mathbf{C} \to T\mathbf{C}$ given by the rule that $e\theta\mathbf{C} = e\theta$ and $(e,x,f)\theta\mathbf{C} = (e\theta, x\theta, f\theta)$ is an orthodox functor. Further, if $\theta_1 : S \to T$ and $\theta_2 : T \to Q$ are admissible morphisms, then $(\theta_1\theta_2)\mathbf{C} = \theta_1\mathbf{C}\theta_2\mathbf{C}$.*

Proof. Clearly, $\theta\mathbf{C}$ is a morphism from B_1 to B_2 and it is a functor as it preserves products, identities and the domain and co-domain of any morphism in $S\mathbf{C}$.

(S2) Suppose that $e, f \in B_1$ are such that $e \,\mathcal{D}\, f$. Then $[e, f] = (e, ef, f)$, and so

$$\begin{aligned}[e, f]\theta\mathbf{C} &= (e, ef, f)\theta\mathbf{C} = (e\theta, (ef)\theta, f\theta)\\ &= (e\theta, e\theta f\theta, f\theta)\\ &= [e\theta, f\theta]\\ &= [e\theta\mathbf{C}, f\theta\mathbf{C}].\end{aligned}$$

(S3) If $(e, x, f) \in S\mathbf{C}$ and $u \in B_1$ with $u \leq_{\mathcal{L}} e$, then ${}_u|(e, x, f) = (u, ux, (ux)^\star)$, and so $({}_u|(e, x, f))\theta\mathbf{C} = (u\theta, (ux)\theta, (ux)^\star\theta)$. In addition,

$$\begin{aligned}{}_{u\theta\mathbf{C}}|(e, x, f)\theta\mathbf{C} &= {}_{u\theta}|(e\theta, x\theta, f\theta)\\ &= (u\theta, u\theta x\theta, (u\theta x\theta)^\star)\\ &= (u\theta, (ux)\theta, ((ux)\theta)^\star).\end{aligned}$$

Since θ is admissible, $((ux)\theta)^\star \,\mathcal{L}\, (ux)^\star\theta$ and

$$\begin{aligned}(u\theta, (ux)\theta, (ux)^\star\theta) &= (u\theta, (ux)\theta, ((ux)\theta)^\star) \cdot (((ux)\theta)^\star, (ux)^\star\theta, (ux)^\star\theta)\\ &= (u\theta, (ux)\theta, ((ux)\theta)^\star) \cdot [((ux)\theta)^\star, (ux)^\star\theta].\end{aligned}$$

Hence, $({}_u|(e, x, f))\theta\mathbf{C} \;\rho\; {}_{u\theta\mathbf{C}}|(e, x, f)\theta\mathbf{C}$.

It is routine to see that $(\theta_1\theta_2)\mathbf{C} = \theta_1\mathbf{C}\theta_2\mathbf{C}$. □

We close this section by establishing a correspondence between the category of weakly B-orthodox semigroups and the category of weakly orthodox categories over bands.

Lemma 9.20. *If S is a weakly B-orthodox semigroup, then there exists an isomorphism η_S from S to $S\mathbf{CS}$.*

Proof. Let $x \in S$ and $e, f \in B$ with $e \,\widetilde{\mathcal{R}}_B\, x \,\widetilde{\mathcal{L}}_B\, f$. Then $(e, x, f) \in S\mathbf{C}$. We define a mapping $\eta_S : S \to S\mathbf{CS}$ by $x \mapsto \overline{(e, x, f)}$. If $u \,\widetilde{\mathcal{R}}_B\, x \,\widetilde{\mathcal{L}}_B\, v$, then $u \,\mathcal{R}\, e$, $v \,\mathcal{L}\, f$ and $[u, e] \cdot (e, x, f) = (u, e, e) \cdot (e, x, f) = (u, x, f) = (u, x, v) \cdot [v, f]$. Thus, $(e, x, f) \;\rho\; (u, x, v)$. Hence, η_S is well-defined.

Clearly, η_S is surjective. To show that η_S is injective, we suppose that $x, y \in S$ with $x\eta_S = y\eta_S$. Then $\overline{(e, x, f)} = \overline{(g, y, h)}$, where $e \,\widetilde{\mathcal{R}}_B\, x \,\widetilde{\mathcal{L}}_B\, f$ and $g \,\widetilde{\mathcal{R}}_B\, y \,\widetilde{\mathcal{L}}_B\, h$.

So $e \mathcal{R} g$, $f \mathcal{L} h$ and $[e,g] \cdot (g,y,h) = (e,x,f) \cdot [f,h]$, that is, $(e,y,h) = (e,x,h)$. Obviously, $x = y$.

We now show that η_S is a morphism. Assume that $x, y \in S$ are such that $e \widetilde{\mathcal{R}}_B\ x\ \widetilde{\mathcal{L}}_B\ f$ and $g \widetilde{\mathcal{R}}_B\ y\ \widetilde{\mathcal{L}}_B\ h$. Then $xg \widetilde{\mathcal{R}}_B\ xy$, and so $(xg)^+ = (xy)^+$. Dually, $(fy)^\star = (xy)^\star$. Thus,

$$\begin{aligned}
x\eta_S y\eta_S &= \overline{(e,x,f)} \circ \overline{(g,y,h)} = \overline{(e,x,f) \otimes (g,y,h)} \\
&= \overline{(e,x,f)|_{fg} \cdot {}_{fg}|(g,y,h)} \\
&= \overline{((xg)^+, xg, fg) \cdot (fg, fy, (fy)^\star)} \\
&= \overline{((xg)^+, xgfy, (fy)^\star)} \quad \big(xgfy = xfgfgy = xfgy = xy\big) \\
&= \overline{((xg)^+, xy, (fy)^\star)} \\
&= \overline{((xy)^+, xy, (xy)^\star)} \quad \big((xg)^+ = (xy)^+,\ (fy)^\star = (xy)^\star\big) \\
&= (xy)\eta_S.
\end{aligned}$$

Finally, we note that η_S preserves the distinguished band as

$$e\eta_S = \overline{(e,e,e)} = \overline{1_e}$$

for all $e \in B$. □

Conversely, we have:

Lemma 9.21. *Let P be a weakly orthodox category over B. Then there exists an isomorphism τ_P from P to $P\mathbf{SC}$, where an isomorphism between two weakly orthodox categories means a bijective orthodox functor.*

Proof. We define a map $\tau_P : P \to P\mathbf{SC}$ by the rule that $e\tau_P = \overline{1_e}$ and $x\tau_P = (\overline{1_{\mathbf{d}(x)}}, \bar{x}, \overline{1_{\mathbf{r}(x)}})$ for all $e \in B = \mathrm{Ob}(P)$ and $x \in P = \mathrm{Mor}(P)$. Clearly, τ_P maps P into $P\mathbf{SC}$.

Notice that the distinguished band of $P\mathbf{S}$ is $\overline{B}$, which is the set of objects of $P\mathbf{SC}$. By Lemma 9.11, $\tau_P : B \to \overline{B} : e \mapsto \overline{1_e}$ is an isomorphism.

Now, we show that τ_P preserves $\mathbf{d}$ and $\mathbf{r}$. Suppose that $x \in P$. Then by the definition of τ_P,

$$\mathbf{d}(x)\tau_P = \overline{1_{\mathbf{d}(x)}},\ \mathbf{r}(x)\tau_P = \overline{1_{\mathbf{r}(x)}}$$

and

$$x\tau_P = (\overline{1_{\mathbf{d}(x)}}, \bar{x}, \overline{1_{\mathbf{r}(x)}}),$$

so that $\mathbf{d}(x\tau_P) = \overline{1_{\mathbf{d}(x)}} = \mathbf{d}(x)\tau_P$ and dually for $\mathbf{r}$. Thus, τ_P preserves $\mathbf{d}$ and $\mathbf{r}$.

If $x, y \in P$ with $x \cdot y$ defined in P, then $\mathbf{r}(x) = \mathbf{d}(y)$ and so $x\tau_P y\tau_P$ is defined in $P\mathbf{SC}$ and

$$\begin{aligned} x\tau_P y\tau_P &= (\overline{1_{\mathbf{d}(x)}}, \bar{x}, \overline{1_{\mathbf{r}(x)}}) \cdot (\overline{1_{\mathbf{d}(y)}}, \bar{y}, \overline{1_{\mathbf{r}(y)}}) \\ &= (\overline{1_{\mathbf{d}(x)}}, \bar{x} \circ \bar{y}, \overline{1_{\mathbf{r}(y)}}) \\ &= (\overline{1_{\mathbf{d}(x)}}, \overline{x \otimes y}, \overline{1_{\mathbf{r}(y)}} \\ &= (\overline{1_{\mathbf{d}(x)}}, \overline{x \cdot y}, \overline{1_{\mathbf{r}(y)}}) \qquad \big(\mathbf{r}(x) = \mathbf{d}(y), \text{ Lemma 9.9}\big) \\ &= (\overline{1_{\mathbf{d}(x \cdot y)}}, \overline{x \cdot y}, \overline{1_{\mathbf{r}(x \cdot y)}}) \\ &= (x \cdot y)\tau_P \end{aligned}$$

which implies that τ_P preserves products. Also, τ_P preserves identities since $1_e\tau_P = (\overline{1_e}, \overline{1_e}, \overline{1_e}) = 1_{\overline{1_e}} = 1_{e\tau_P}$. Thus, τ_P is a functor.

Let $e, f \in B$ be such that $e \;\mathcal{D}\; f$. Then $[e, f]\tau_P = (\overline{1_e}, \overline{[e, f]}, \overline{1_f})$. As $e \;\mathcal{R}\; ef \;\mathcal{L}\; f$ and $[e, ef] \cdot 1_{ef} = [e, ef] = [e, f] \cdot [f, ef]$, we have that $\overline{[e, f]} = \overline{1_{ef}}$. Thus,

$$\begin{aligned} [e, f]\tau_P &= (\overline{1_e}, \overline{1_{ef}}, \overline{1_f}) \\ &= (\overline{1_e}, \overline{1_e} \circ \overline{1_f}, \overline{1_f}) \qquad \big(\text{Lemma 9.11}\big) \\ &= [\overline{1_e}, \overline{1_f}] \\ &= [e\tau_P, f\tau_P]. \end{aligned}$$

Hence, τ_P satisfies Condition (S2).

To show that (S3) holds, we assume that $x \in P$ and $e \in B$ with $e \leq_{\mathcal{L}} \mathbf{d}(x)$. Then $e\tau_P \leq_{\mathcal{L}} \mathbf{d}(x)\tau_P = \mathbf{d}(x\tau_P)$ as τ_P is an isomorphism from B to $\overline{B}$ shown above. Hence, ${}_e|x$ and ${}_{e\tau_P}|x\tau_P$ are defined. Observe that $({}_e|x)\tau_P = (\overline{1_e}, \overline{{}_e|x}, \overline{1_{\mathbf{r}({}_e|x)}})$ and

$$\begin{aligned} {}_{e\tau_P}|x\tau_P &= {}_{\overline{1_e}}|(\overline{1_{\mathbf{d}(x)}}, \bar{x}, \overline{1_{\mathbf{r}(x)}}) \\ &= (\overline{1_e}, \overline{1_e} \circ \bar{x}, (\overline{1_e} \circ \bar{x})^\star) \\ &= (\overline{1_e}, \overline{1_e \otimes x}, (\overline{1_e \otimes x})^\star) \\ &= (\overline{1_e}, \overline{[e, e]|_{e\mathbf{d}(x)} \cdot {}_{e\mathbf{d}(x)}|x}, (\overline{[e, e]|_{e\mathbf{d}(x)} \cdot {}_{e\mathbf{d}(x)}|x})^\star) \\ &= (\overline{1_e}, \overline{[e, e]|_e \cdot {}_e|x}, (\overline{[e, e]|_e \cdot {}_e|x})^\star) \qquad \big(e \leq_{\mathcal{L}} \mathbf{d}(x)\big) \\ &= (\overline{1_e}, \overline{[e^+, e] \cdot {}_e|x}, (\overline{[e^+, e] \cdot {}_e|x})^\star) \qquad \big(\text{by (OB4)}^\circ\big) \\ &= (\overline{1_e}, \overline{[e, e] \cdot {}_e|x}, (\overline{[e, e] \cdot {}_e|x})^\star) \qquad \big([e^+, e] \,\rho\, 1_e = [e, e], \text{ Lemma 9.10}\big) \end{aligned}$$

$$= (\overline{1_e}, \overline{{}_e|x}, (\overline{{}_e|x})^\star) \qquad \Big([e,e] = 1_e\Big).$$

Clearly,

$$\mathbf{d}(({}_e|x)\tau_P) = \overline{1_e} = \mathbf{d}({}_{e\tau_P}|x\tau_P),$$

$$\mathbf{r}(({}_e|x)\tau_P) = \overline{1_{\mathbf{r}({}_e|x)}} \ \widetilde{\mathcal{L}}_{\overline{B}} \ \overline{{}_e|x} \ \widetilde{\mathcal{L}}_{\overline{B}} \ (\overline{{}_e|x})^\star = \mathbf{r}({}_{e\tau_P}|x\tau_P)$$

and

$$\begin{aligned}
&({}_e|x)\tau_P \cdot [\mathbf{r}(({}_e|x)\tau_P), \mathbf{r}({}_{e\tau_P}|x\tau_P)] \\
&= (\overline{1_e}, \overline{{}_e|x}, \overline{1_{\mathbf{r}({}_e|x)}}) \cdot [\overline{1_{\mathbf{r}({}_e|x)}}, (\overline{{}_e|x})^\star] \\
&= (\overline{1_e}, \overline{{}_e|x}, \overline{1_{\mathbf{r}({}_e|x)}}) \cdot (\overline{1_{\mathbf{r}({}_e|x)}}, \overline{1_{\mathbf{r}({}_e|x)}} \circ (\overline{{}_e|x})^\star, (\overline{{}_e|x})^\star) \\
&= (\overline{1_e}, \overline{{}_e|x}, \overline{1_{\mathbf{r}({}_e|x)}}) \cdot (\overline{1_{\mathbf{r}({}_e|x)}}, \overline{1_{\mathbf{r}({}_e|x)}}, (\overline{{}_e|x})^\star) \qquad \Big(\overline{1_{\mathbf{r}({}_e|x)}} \ \mathcal{L} \ (\overline{{}_e|x})^\star\Big) \\
&= (\overline{1_e}, \overline{{}_e|x} \circ \overline{1_{\mathbf{r}({}_e|x)}}, (\overline{{}_e|x})^\star) \\
&= (\overline{1_e}, \overline{{}_e|x}, (\overline{{}_e|x})^\star) \qquad \Big(\overline{1_{\mathbf{r}({}_e|x)}} \ \widetilde{\mathcal{L}}_{\overline{B}} \ \overline{{}_e|x}\Big) \\
&= {}_{e\tau_P}|x\tau_P,
\end{aligned}$$

so that $({}_e|x)\tau_P \, \rho \, {}_{e\tau_P}|x\tau_P$ and (S3) holds.

Next, suppose that $x, y \in P$ with $x\tau_P = y\tau_P$. Then $(\overline{1_{\mathbf{d}(x)}}, \bar{x}, \overline{1_{\mathbf{r}(x)}}) = (\overline{1_{\mathbf{d}(y)}}, \bar{y}, \overline{1_{\mathbf{r}(y)}})$, which implies that $\bar{x} = \bar{y}$, and also $\mathbf{d}(x) = \mathbf{d}(y)$, $\mathbf{r}(x) = \mathbf{r}(y)$ by Lemma 9.11. Further, by Lemma 9.2, $x = y$.

We now show that τ_P is surjective. Let $(\overline{1_e}, \bar{x}, \overline{1_f})$ be in $P\mathbf{SC}$. Then $\overline{1_e} \ \widetilde{\mathcal{R}}_{\overline{B}} \ \bar{x} \ \widetilde{\mathcal{R}}_{\overline{B}} \ \overline{1_{\mathbf{d}(x)}}$ and $\overline{1_f} \ \widetilde{\mathcal{L}}_{\overline{B}} \ \bar{x} \ \widetilde{\mathcal{L}}_{\overline{B}} \ \overline{1_{\mathbf{r}(x)}}$, that is, $\overline{1_e} \ \mathcal{R} \ \overline{1_{\mathbf{d}(x)}}$ and $\overline{1_f} \ \mathcal{L} \ \overline{1_{\mathbf{r}(x)}}$ so that by Lemma 9.11, $e \, \mathcal{R} \, \mathbf{d}(x)$ and $f \, \mathcal{L} \, \mathbf{r}(x)$. Put $x' = [e, \mathbf{d}(x)] \cdot x \cdot [\mathbf{r}(x), f]$. Certainly, $x' \ \rho \ x$, that is, $\overline{x'} = \bar{x}$. Thus, $x'\tau_P = (\overline{1_e}, \overline{x'}, \overline{1_f}) = (\overline{1_e}, \bar{x}, \overline{1_f})$, and consequently, τ_P is surjective. □

Lemma 9.22. *For any $S \in Ob(\mathcal{WO})$, we define $S\eta = \eta_S$, where η_S is defined in Lemma 9.20. Then η is a natural equivalence of the functors $I_{\mathcal{WO}}$ and* **CS**.

Proof. Let $\theta : S_1 \to S_2$ in $\mathcal{WO}$, where S_1 and S_2 are over B_1 and B_2, respectively. Then for any $x \in S_1$, we have by the definition of η_S in Lemma 9.20 that

$$\begin{aligned}
(x\eta_{S_1})\theta\mathbf{CS} &= \overline{(e, x, f)}\theta\mathbf{CS} \qquad \Big(e \ \widetilde{\mathcal{R}}_{B_1} \ x \ \widetilde{\mathcal{L}}_{B_1} \ f\Big) \\
&= \overline{(e, x, f)\theta\mathbf{C}} \\
&= \overline{(e\theta, x\theta, f\theta)} \\
&= (x\theta)\eta_{S_2} \qquad \Big(e\theta \ \widetilde{\mathcal{R}}_{B_2} \ x\theta \ \widetilde{\mathcal{L}}_{B_2} \ f\theta\Big).
\end{aligned}$$

Thus the diagram below commutes, and so $\eta = (\eta_S)$ is a natural morphism of

$I_{\mathcal{WO}}$ and **CS**. $\square$

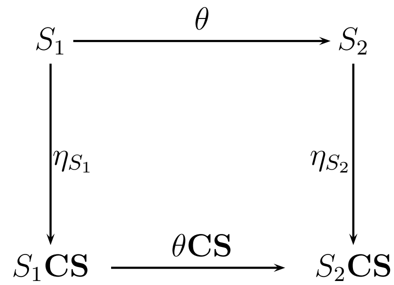

Figure 9.1: A natural transformation of $I_{\mathcal{WO}}$ and **CS**

Similarly, we have:

Lemma 9.23. *For any $P \in Ob(\mathcal{WOC})$, we define $P\tau = \tau_P$, where τ_P is defined in Lemma 9.21. Then τ is a natural equivalence of the functors $I_{\mathcal{WOC}}$ and* **SC**.

Proof. Let $F : P_1 \to P_2$ in $\mathcal{WOC}$, where P_1 and P_2 are over B_1 and B_2, respectively. Then for any $x \in P_1$, we have by the definition of τ_P in Lemma 9.21 that

$$\begin{aligned}
(x\tau_{P_1})F\mathbf{SC} &= (\overline{1_{\mathbf{d}(x)}}, \bar{x}, \overline{1_{\mathbf{r}(x)}})F\mathbf{SC} \\
&= (\overline{1_{\mathbf{d}(x)}}F\mathbf{S}, \bar{x}F\mathbf{S}, \overline{1_{\mathbf{r}(x)}}F\mathbf{S}) \\
&= (\overline{1_{\mathbf{d}(x)}F}, \overline{xF}, \overline{1_{\mathbf{r}(x)}F}) \\
&= (\overline{1_{\mathbf{d}(xF)}}, \overline{xF}, \overline{1_{\mathbf{d}(xF)}}) \\
&= (xF)\tau_{P_2}
\end{aligned}$$

and

$$\begin{aligned}
(e\tau_{P_1})F\mathbf{SC} &= \overline{1_e}F\mathbf{SC} \\
&= \overline{1_e}F\mathbf{S} \\
&= \overline{1_eF} \\
&= \overline{1_{eF}} \\
&= (eF)\tau_{P_2}.
\end{aligned}$$

Thus the diagram below commutes, and so $\tau = (\tau_P)$ is a natural morphism of $I_{\mathcal{WOC}}$ and **SC**.

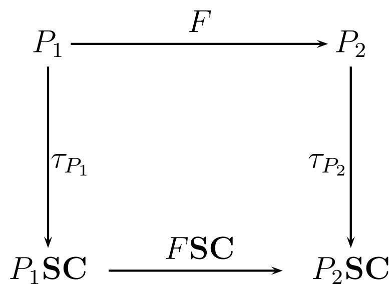

Figure 9.2: A natural transformation of $I_{\mathcal{WOC}}$ and $\mathbf{SC}$

$\square$

To sum up, we have:

Theorem 9.24. *The category $\mathcal{WO}$ of weakly B-orthodox semigroups and admissible morphisms is equivalent to the category $\mathcal{WOC}$ of weakly orthodox categories over bands and orthodox functors.*

9.4 Special cases

Our purpose in this section is to investigate a certain kinds of weakly B-orthodox semigroups.

Lemma 9.25. *Let P be a weakly orthodox category over B. Suppose that for all $\bar{x} \in E(P\mathbf{S})$ we have that $\overline{1_{\mathbf{d}(x)}}\, \mathcal{R}^* \,\bar{x}\, \mathcal{L}^* \,\overline{1_{\mathbf{r}(x)}}$ in $P\mathbf{S}$. Then $E(P\mathbf{S}) = \overline{B}$.*

Proof. Suppose that $x \in P$ and $\bar{x} \circ \bar{x} = \bar{x}$. As $\bar{x}\, \mathcal{L}^* \,\overline{1_{\mathbf{r}(x)}}$, we have that $\overline{1_{\mathbf{r}(x)}} \circ \bar{x} = \overline{1_{\mathbf{r}(x)}}$. Thus

$$\begin{aligned}
&\overline{1_{\mathbf{r}(x)} \otimes x} = \overline{1_{\mathbf{r}(x)}} \\
\Rightarrow\ &\overline{1_{\mathbf{r}(x)}|_{\mathbf{r}(x)\mathbf{d}(x)} \cdot {}_{\mathbf{r}(x)\mathbf{d}(x)}|x} = \overline{1_{\mathbf{r}(x)}} \\
\Rightarrow\ &\overline{[(\mathbf{r}(x)\mathbf{d}(x))^+, \mathbf{r}(x)\mathbf{d}(x)] \cdot {}_{\mathbf{r}(x)\mathbf{d}(x)}|x} = \overline{[\mathbf{r}(x), \mathbf{r}(x)]} \\
\Rightarrow\ &(\mathbf{r}(x)\mathbf{d}(x))^+ \ \mathcal{R}\ \mathbf{r}(x) \\
\Rightarrow\ &\mathbf{r}(x)\mathbf{d}(x)\ \mathcal{R}\ \mathbf{r}(x).
\end{aligned}$$

Dually, $\mathbf{r}(x)\mathbf{d}(x)\ \mathcal{L}\ \mathbf{d}(x)$. Hence, $\mathbf{r}(x)\ \mathcal{D}\ \mathbf{d}(x)$, and so $\mathbf{d}(x)\ \mathcal{R}\ \mathbf{d}(x)\mathbf{r}(x)\ \mathcal{L}\ \mathbf{r}(x)$. By Lemma 9.11, $\overline{1_{\mathbf{d}(x)}}\ \mathcal{R}\ \overline{1_{\mathbf{d}(x)\mathbf{r}(x)}}\ \mathcal{L}\ \overline{1_{\mathbf{r}(x)}}$. Again by $\overline{1_{\mathbf{d}(x)}}\ \mathcal{R}^*\ \bar{x}\ \mathcal{L}^*\ \overline{1_{\mathbf{r}(x)}}$, we have that $\bar{x}\ \mathcal{H}^*\ \overline{1_{\mathbf{d}(x)\mathbf{r}(x)}}$. Since $\mathcal{H}^*$-class contains at most one idempotent, we have that $\bar{x} = \overline{1_{\mathbf{d}(x)\mathbf{r}(x)}}$. □

A weakly orthodox category P over B is an *orthodox groupoid* over B if for all $x \in P$, there exists $y \in P$ with $\mathbf{d}(y) = \mathbf{r}(x)$ and $\mathbf{r}(y) = \mathbf{d}(x)$ such that $1_{\mathbf{d}(x)} = x \cdot y$ and $y \cdot x = 1_{\mathbf{r}(x)}$.

Corollary 9.26. *The category of orthodox semigroups and morphisms is equivalent to the category of orthodox groupoids over bands and orthodox functors.*

Proof. Let S be an orthodox semigroup with $B = E(S)$. Suppose that $(e, x, f) \in S\mathbf{C}$. Since $\mathcal{R} = \widetilde{\mathcal{R}}_B$ and $\mathcal{L} = \widetilde{\mathcal{L}}_B$, we have that $e\,\mathcal{R}\,x\,\mathcal{L}\,f$. It follows from the fact that S is regular that there exists $y \in S$ with $e\,\mathcal{L}\,y\,\mathcal{R}\,f$, $e = xy$ and $yx = f$. Then $(f, y, e) \in S\mathbf{C}$ and the products $(e, x, f) \cdot (f, y, e)$, $(f, y, e) \cdot (e, x, f)$ exist in $S\mathbf{C}$. Moreover, $(e, x, f) \cdot (f, y, e) = (e, xy, e) = (e, e, e) = [e, e] = 1_e$ and similarly, $(f, y, e) \cdot (e, x, f) = 1_f$.

Conversely, let P be an orthodox groupoid over B. Suppose that $x \in P$. Then there exists $y \in P$ with $\mathbf{d}(y) = \mathbf{r}(x)$ and $\mathbf{r}(y) = \mathbf{d}(x)$ such that $1_{\mathbf{r}(x)} = y \cdot x$ and $1_{\mathbf{d}(x)} = x \cdot y$. So $\overline{1_{\mathbf{d}(x)}} = \overline{x \cdot y} = \overline{x \otimes y} = \bar{x} \circ \bar{y}$. Hence, $\bar{x} \circ \bar{y} \circ x = (\bar{x} \circ \bar{y}) \circ \bar{x} = \overline{1_{\mathbf{d}(x)}} \circ \bar{x} = \bar{x}$ so that $P\mathbf{S}$ is regular. In addition, as $\overline{1_{\mathbf{d}(x)}} = \bar{x} \circ \bar{y}$ and $\bar{x} = \overline{1_{\mathbf{d}(x)}} \circ \bar{x}$, we have that $\overline{1_{\mathbf{d}(x)}}\,\mathcal{R}\,\bar{x}$ in $P\mathbf{S}$. Dually, $\overline{1_{\mathbf{r}(x)}}\,\mathcal{L}\,\bar{x}$ in $P\mathbf{S}$. By Lemma 9.25, we have that $E(P\mathbf{S}) = \overline{B}$. Hence, $P\mathbf{S}$ is an orthodox semigroup. □

We now concentrate on the class of abundant semigroups. We replace the distinguished set of idempotents B by the whole set of idempotents and use relations $\mathcal{R}^*$ and $\mathcal{L}^*$ instead of $\widetilde{\mathcal{R}}_B$ and $\widetilde{\mathcal{L}}_B$. In addition, an admissible morphism in this context is more usually referred to as a good morphism. We define a weakly orthodox category P over B to be **-orthodox* if it satisfies Condition (OB6) and its dual (OB6)°:

(OB6) if $y \otimes x\ \rho\ z \otimes x$, then $y|_{\mathbf{r}(y)\mathbf{d}(x)}\ \rho\ z|_{\mathbf{r}(z)\mathbf{d}(x)}$.

Corollary 9.27. *The category of abundant semigroups whose set of idempotents forms a band and good morphisms is equivalent to the category of *-orthodox categories over bands and orthodox functors.*

Proof. Let P be a *-orthodox category over a band B. Suppose that $\bar{x} \in P\mathbf{S}$ and $x \in P$. We know that $\overline{1_{\mathbf{d}(x)}}\,\widetilde{\mathcal{R}}_{\overline{B}}\,\bar{x}$, so that $\overline{1_{\mathbf{d}(x)}} \otimes \bar{x} = \bar{x}$. Assume that $y, z \in P$ with

$\bar{y} \circ \bar{x} = \bar{z} \circ \bar{x}$. Then $y \otimes x \;\rho\; z \otimes x$. By (OB6), we obtain that $y|_{\mathbf{r}(y)\mathbf{d}(x)} \;\rho\; z|_{\mathbf{r}(z)\mathbf{d}(x)}$, and so $\mathbf{r}(y)\mathbf{d}(x)\,\mathcal{L}\,\mathbf{r}(z)\mathbf{d}(x)$, $\mathbf{d}(y|_{\mathbf{r}(y)\mathbf{d}(x)})\,\mathcal{R}\,\mathbf{d}(z|_{\mathbf{r}(z)\mathbf{d}(x)})$, and we have

$$\begin{aligned}
&(y|_{\mathbf{r}(y)\mathbf{d}(x)}) \cdot [\mathbf{r}(y)\mathbf{d}(x), \mathbf{r}(z)\mathbf{d}(x)] = [\mathbf{d}(y|_{\mathbf{r}(y)\mathbf{d}(x)}), \mathbf{d}(z|_{\mathbf{r}(z)\mathbf{d}(x)})] \cdot (z|_{\mathbf{r}(z)\mathbf{d}(x)})\\
\Rightarrow\; &(y|_{\mathbf{r}(y)\mathbf{d}(x)}) \cdot [\mathbf{r}(y)\mathbf{d}(x), \mathbf{r}(z)\mathbf{d}(x)] \cdot ({}_{\mathbf{r}(z)\mathbf{d}(x)}|1_{\mathbf{d}(x)})\\
&\qquad = [\mathbf{d}(y|_{\mathbf{r}(y)\mathbf{d}(x)}), \mathbf{d}(z|_{\mathbf{r}(z)\mathbf{d}(x)})] \cdot (z|_{\mathbf{r}(z)\mathbf{d}(x)}) \cdot ({}_{\mathbf{r}(z)\mathbf{d}(x)}|1_{\mathbf{d}(x)})\\
\Rightarrow\; &(y|_{\mathbf{r}(y)\mathbf{d}(x)}) \cdot ({}_{\mathbf{r}(y)\mathbf{d}(x)}|1_{\mathbf{d}(x)}) = [\mathbf{d}(y|_{\mathbf{r}(y)\mathbf{d}(x)}), \mathbf{d}(z|_{\mathbf{r}(z)\mathbf{d}(x)})] \cdot (z \otimes 1_{\mathbf{d}(x)})\\
&\qquad\qquad \Big(\mathbf{r}(y)\mathbf{d}(x) \;\mathcal{L}\; \mathbf{r}(z)\mathbf{d}(x),\; \text{by(OB2)}\Big)\\
\Rightarrow\; &y \otimes 1_{\mathbf{d}(x)} = [\mathbf{d}(y|_{\mathbf{r}(y)\mathbf{d}(x)}), \mathbf{d}(z|_{\mathbf{r}(z)\mathbf{d}(x)})] \cdot (z \otimes 1_{\mathbf{d}(x)}).
\end{aligned}$$

Hence, $y \otimes 1_{\mathbf{d}(x)} \;\rho\; z \otimes 1_{\mathbf{d}(x)}$, that is, $\bar{y} \circ \overline{1_{\mathbf{d}(x)}} = \bar{z} \circ \overline{1_{\mathbf{d}(x)}}$.

Now, let $x \in P$. Then $\overline{1_{\mathbf{d}(x)}}\,\mathcal{R}^*\,\bar{x}\,\mathcal{L}^*\;\overline{1_{\mathbf{r}(x)}}$, and so by Lemma 9.25, we have that $E(P\mathbf{S}) = B$. Hence, $P\mathbf{S}$ is an abundant semigroup whose set of idempotents forms a band.

Conversely, in view of Lemma 9.18, it is necessary to show that $S\mathbf{C}$ satisfies Condition (OB6) and its dual. Assume that $(e,x,f),(u,y,v),(g,z,h) \in S\mathbf{C}$ such that $(u,y,v) \otimes (e,x,f) \;\rho\; (g,z,h) \otimes (e,x,f)$. Then

$$(u,y,v)|_{ve} \;\cdot\; {}_{ve}(e,x,f) \;\rho\; (g,z,h)|_{he} \cdot {}_{he}|(e,x,f),$$

that is,

$$((ye)^+, ye, ve) \cdot (ve, vx, (vx)^\star) \;\rho\; ((ze)^+, ze, he) \cdot (he, hx, (hx)^\star)$$

or equivalently,

$$((ye)^+, yx, (vx)^\star) \;\rho\; ((ze)^+, zx, (hx)^\star),$$

as $yevx = yvevex = yvex = yx$ and similarly, $zehx = zx$.

Thus, $(ye)^+\,\mathcal{R}\,(ze)^+$ and $(vx)^\star\,\mathcal{L}\,(hx)^\star$, that is, $(ye)^+ = (ze)^+$ and $(vx)^\star = (hx)^\star$ since they are unique. By Lemma 9.2, we have that $((ye)^+, yx, (vx)^\star) = ((ze)^+, zx, (hx)^\star)$, and so $yx = zx$. As $e\;\mathcal{R}^*\;x$ in S, we have that $ye = ze$, and so

$ve\ \mathcal{L}^*\ ye = ze\ \mathcal{L}^*\ he$. Thus,

$$\begin{aligned}(u,y,v)|_{ve} &= ((ye)^+, ye, ve)\\ &= ((ye)^+, ze, ve) && \big(ye = ze\big)\\ &= [(ye)^+, (ze)^+]\cdot((ze)^+, ze, ve)\end{aligned}$$

$$\begin{aligned}&= [(ye)^+, (ze)^+]\cdot((ze)^+, ze, he)\cdot[he, ve] && \big(ve\,\mathcal{L}\,he\big)\\ &= [(ye)^+, (ze)^+]\cdot(g,z,h)|_{he}\cdot[he, ve].\end{aligned}$$

So $(u,y,v)|_{ve}\,\rho\,(g,z,h)|_{he}$. Hence, (OB6) holds. □

Next, we discuss Ehresmann semigroups.

Let P be a weakly orthodox category over a semilattice E. According to the remark succeeding Lemma 9.3, the relation ρ is the identity on P and $\leq_r$ and $\leq_\ell$ are partial orders on P. Then $P\mathbf{S} = P$, and so we will identify $\bar{x}$ with x for all $x \in P$. In that case, for any $x, y \in P$,

$$x \circ y = x \otimes y = x|_{\mathbf{r}(x)\mathbf{d}(y)} \cdot {}_{\mathbf{r}(x)\mathbf{d}(y)}|y.$$

Lemma 9.28. *A weakly orthodox category P over a semilattice E with $\leq_r$ forms an ordered$_1$ category with restriction.*

Proof. Certainly, in view of the comments in Section 9.1, a weakly orthodox category P over a semilattice E forms a poset under $\leq_r$.

(OC1) Suppose that $x, y \in P$ with $x \leq_r y$. Then there exists $e \in E$ such that $e \leq \mathbf{d}(y)$ and $x = {}_e|y$. Thus, $\mathbf{d}(x) = e \leq \mathbf{d}(y)$ and $\mathbf{r}(x) \leq \mathbf{r}(y)$ by (OB1).

(OC2) Suppose that $x, y \in P$ with $\mathbf{r}(x) = \mathbf{r}(y)$, $\mathbf{d}(x) = \mathbf{d}(y)$ and $x \leq_r y$. Then there exists $e \in E$ such that $e \leq \mathbf{d}(y)$ and $x = {}_e|y$. Certainly, $\mathbf{d}(x) = e$, and so $e = \mathbf{d}(y)$, whence from (OB1),

$$\begin{aligned}y &= {}_e|y \cdot [\mathbf{r}({}_e|y), \mathbf{r}(y)]\\ &= {}_e|y \cdot [\mathbf{r}({}_e|y), \mathbf{r}({}_e|y)] && \big(\mathbf{r}(y) = \mathbf{r}(x) = \mathbf{r}({}_e|y)\big)\\ &= {}_e|y = x.\end{aligned}$$

(OC3) If $x' \leq_r x$, $y' \leq_r y$, and both $x' \cdot y'$ and $x \cdot y$ exist, then there exist $e, f \in E$ such that $e \leq \mathbf{d}(x)$, $f \leq \mathbf{d}(y)$, $x' = {}_e|x$ and $y' = {}_f|y$. Thus, we have that

$\mathbf{r}({}_e|x) = \mathbf{r}(x') = \mathbf{d}(y') = \mathbf{d}({}_f|y) = f$, and so $x' \cdot y' = {}_e|x \cdot {}_f|y = {}_e|(x \cdot y)$ by (OB3). Hence, $x' \cdot y' \leq_r x \cdot y$.

Finally, we assume that $x \in P$ and $e \in E$ with $e \leq \mathbf{d}(x)$. Then ${}_e|x$ is defined and $\mathbf{d}({}_e|x) = e$. Also, ${}_e|x \leq_r x$. Further, ${}_e|x$ is unique since if $z \leq_r x$ and $e = \mathbf{d}(z)$, then there exists $h \in E$ with $h \leq \mathbf{d}(x)$ and $z = {}_h|x$, which gives that $h = \mathbf{d}(z)$. Thus, $e = h$, and so $z = {}_e|x$. □

As a dual result to Lemma 9.28, we have the following lemma.

Lemma 9.29. *A weakly orthodox category P over a semilattice E with $\leq_\ell$ forms an ordered$_1$ category with co-restriction.*

Next we show that a weakly orthodox category P over a semilattice E is an Ehresmann category as mentioned in Chapter 6.

Lemma 9.30. *A weakly orthodox category P over a semilattice E with the pair of natural partial orders $(\leq_r, \leq_\ell)$ defined in Section 9.1 forms an Ehresmann category $(P, \cdot, \leq_r, \leq_\ell)$.*

Conversely, an Ehresmann category $(C, \cdot, \leq_r, \leq_l)$ with semilattice of identities E, may be regarded as a weakly orthodox category over E with natural partial orders $(\leq_r, \leq_l)$.

Proof. Let P be a weakly orthodox category over a semilattice. In view of Lemma 9.28 and Lemma 9.29, Conditions (E1) and (E1)$^\circ$ are satisfied. Now, we identify e with 1_e for all $e \in E$.

(E2) If $e, f \in E$ and $e \leq_r f$, then $e = {}_e|f$ so that we must have $e \leq f$. Then $f|_e$ is defined and $f|_e = [f, f]|_e = [(fe)^+, e] = [e, e] = e$ so that $e \leq_\ell f$. Together with the dual, (E2) holds.

(E3) Clearly, E is a semilattice under $\leq_r = \leq_\ell = \leq$.

(E4) To show that $\leq_r \circ \leq_\ell \;\subseteq\; \leq_\ell \circ \leq_r$, we assume that $x \leq_r \circ \leq_\ell y$. Then there exists $z \in P$ such that $x \leq_r z \leq_\ell y$. So there exist $e, f \in E$ with $\mathbf{d}(x) = e \leq \mathbf{d}(z) = u$ and $\mathbf{r}(z) = f \leq \mathbf{r}(y) = v$, such that $x = {}_e|z$ and $z = y|_f$. Thus, $x = {}_e|(y|_f) = {}_{eu}|(y|_{vf})$. Since P is weakly orthodox, we obtain that $x \,\rho\, ({}_{eh}|y)|_{gf}$, where $h = \mathbf{d}(y)$ and $g = \mathbf{r}({}_{eh}|y)$. As ρ is the identity by the remark following Lemma 9.3, we have that $x = ({}_{eh}|y)|_{gf}$. Set $z' = {}_{eh}|y$. Then $x \leq_\ell z'$ and $z' \leq_r y$. Consequently, $x \leq_\ell \circ \leq_r y$.

(E5) Suppose that $x, y \in P$ and $f \in E$ with $x \leq_r y$. Then there exists $k \in E$ with $k \leq \mathbf{d}(y)$ and $x = {}_k|y$. So $x|_{\mathbf{r}(x)\wedge f} = ({}_k|y)|_{\mathbf{r}(x)\wedge f}$. As P is weakly orthodox and ρ is the identity, we obtain that $({}_k|y)|_{\mathbf{r}(x)\wedge f} = {}_{nk}|(y|_{\mathbf{r}(y)\wedge f})$, where $n = \mathbf{d}(y|_{\mathbf{r}(y)\wedge f})$, that is, $x|_{\mathbf{r}(x)\wedge f} = {}_{nk}|(y|_{\mathbf{r}(y)\wedge f})$. Thus, $x|_{\mathbf{r}(x)\wedge f} \leq_r y|_{\mathbf{r}(y)\wedge f}$.

Conversely, let $C = (C, \cdot, \leq_r, \leq_l)$ be an Ehresmann category with semilattice of identities E. If $e \;\mathcal{D}\; f$ in E, then $e = f$ and we put $[e, e] = e$ (E is identified with the set of identities at E).

From (E2) and (E3), $\leq_r$ and $\leq_l$ coincide on E, making E a semilattice: we let $\leq$ denote the restriction of $\leq_r$ (resp. $\leq_l$) to E. It is clear that the first part of (OB1) holds, moreover, by uniqueness of restriction, ${}_e|x = x$ if $e = \mathbf{d}(x)$, so that the second part of (OB1) holds.

For (OB2), if $x \in C$ and $e, f \in E$, with $e \leq_{\mathcal{L}} f \leq_{\mathcal{L}} \mathbf{d}(x)$, then $e \leq f \leq \mathbf{d}(x)$. Now ${}_{ef}|x = {}_e|x \leq_r x$ and $\mathbf{d}({}_e|x) = e$; also, ${}_e|({}_f|x) \leq_r {}_f|x \leq_r x$ and $\mathbf{d}({}_e|({}_f|x)) = e$. By uniqueness of restriction, ${}_{ef}|x = {}_e|({}_f|x)$, that is, ${}_e|x =_e |({}_f|x)$. In particular, if $e \;\mathcal{L}\; f$, then $e = f$ and obviously, $[e, f] \cdot {}_f|x = [f, f] \cdot {}_f|x = {}_f|x = {}_e|x$.

(OB3) If $x, y \in C$ with $\exists x \cdot y$, then $\mathbf{r}(x) = \mathbf{d}(y)$. If $e \leq \mathbf{d}(x)$, $\mathbf{r}({}_e|x) \leq \mathbf{r}(x) = \mathbf{d}(y)$ and we have that

$$ {}_e|(x \cdot y) \leq_r x \cdot y \text{ and } \mathbf{d}({}_e|x \cdot y) = e $$

and also

$$ {}_e|x \cdot {}_f|y \leq_r x \cdot y \text{ and } \mathbf{d}({}_e|x \cdot {}_f|y) = e, $$

where $f = \mathbf{r}({}_e|x)$. Hence, ${}_e|(x \cdot y) = {}_e|x \cdot {}_f|y$, by uniqueness of restriction.

It is easy to see that (OB4) and (OB5) hold.

Finally, we show that C is weakly orthodox. Let $x \in C$ and $e, f, u, v, g, h \in E$ with $g = \mathbf{r}(x)$, $h = \mathbf{d}(x)$, $u = \mathbf{d}(x|_{gf})$ and $v = \mathbf{r}({}_{eh}|x)$. Then $(e \overline{\otimes} x) \overline{\otimes} f = e \overline{\otimes} (x \overline{\otimes} f)$, where $\overline{\otimes}$ is defined [32], by $x \overline{\otimes} y = x|_k \cdot {}_k|y$, where $k = \mathbf{r}(x) \wedge \mathbf{d}(x)$. As shown in [32], $\overline{\otimes}$ is associative. We have

$$ \begin{aligned} (e \overline{\otimes} x) \overline{\otimes} f &= (e|_{eh} \cdot {}_{eh}|x)|_{vf} \cdot {}_{vf}|f \\ &= (eh \cdot ({}_{eh}|x))|_{vf} \cdot vf \\ &= ({}_{eh}|x)|_{vf} \end{aligned} $$

and similarly, $e \overline{\otimes} (x \overline{\otimes} f) = {}_{eu}|(x|_{gf})$, so by associativity we obtain that $({}_{eh}|x)|_{vf} =$

${}_{eu}|(x|_{gf})$.

In view of Lemma 6.1, the partial order $\leq_r$ and $\leq_\ell$ defined in a weakly orthodox category over a semilattice coincide with $\leq_r$ and $\leq_l$ defined in an Ehresmann category, respectively. $\square$

Let $\mathbf{C} = (C, \cdot, \leq_r, \leq_l)$ and $\mathbf{D} = (D, \cdot, \leq_r, \leq_l)$ be Ehresmann categories with semilattice E_C and E_D of identities, respectively. A *strongly ordered functor* [32] $F : \mathbf{C} \to \mathbf{D}$ is a functor which preserves $\leq_r, \leq_l$ and $\wedge$. Since F preserves $\wedge$, F is a morphism $E_C \to E_D$. As shown in [32], F preserves restrictions and co-restrictions. Thus F is an orthodox functor in the sense of Definition 9.5.

On the other hand, if $G : \mathbf{C} \to \mathbf{D}$ is an orthodox functor, then by (S1), it preserves $\wedge$. Suppose now that $x, y \in \mathbf{C}$ with $x \leq_r y$. Then $x = {}_e|y$ for some $e \in E$, so that by (S3), $xG = {}_{eG}|yG$ so that $xG \leq_r yG$. Dually, G preserves $\leq_l$, so that G is a strongly ordered functor. Theorem 9.24 and the comments above now give us

Corollary 9.31. *The category of Ehresmann categories and strongly ordered functors is isomorphic to the category of weakly orthodox categories over semilattices and orthodox functors.*

Let S be an Ehresmann semigroup with distinguished semilattice of idempotents E. Indeed for any $x \in S$, there exists a unique $e \in E$ such that $e \,\widetilde{\mathcal{R}}_E\, x$ and a unique $f \in E$ such that $x \,\widetilde{\mathcal{L}}_E\, f$. Thus the map $\mathbf{C} : S \to S\mathbf{C}$ given by $x \mapsto (e, x, f)$ is bijective. In that case, we identify $S\mathbf{C}$ with $(S, \cdot)$ and so the partial binary operation $\cdot$ on $S\mathbf{C}$ is slightly modified to

$$x \cdot y = xy,$$

where $x, y \in S$ satisfying $x^* = y^\dagger$ and xy is the product of x and y in S. Then

Lemma 9.32. *If S is an Ehresmann semigroup with distinguished semilattice of idempotents E and P is a weakly orthodox category over E. Then $S\mathbf{CS} = S$ and $P\mathbf{SC} = P$.*

Proof. Let S be an Ehresmann semigroup over E. It follows from Lemma 9.18 that S is a weakly orthodox category over E with a restriction of that in S and $\mathbf{d}(x) = x^\dagger$, $\mathbf{r}(x) = x^*$, for any $x \in S$, and if $e \leq x^\dagger$ and $f \leq x^*$ then ${}_e|x = ex$ and $x|_f = xf$.

We now construct $S\mathbf{CS}$, which again has underlying set S as the relation ρ on a weakly orthodox category over a semilattice E is trivial, by defining a product

$$x \circ y = x \otimes y = x|_{x^*y^\dagger} \cdot {}_{x^*y^\dagger}|y.$$

Observe that

$$\begin{aligned} x \circ y &= x|_{x^*y^\dagger} \cdot {}_{x^*y^\dagger}|y \\ &= xx^*y^\dagger \cdot x^*y^\dagger y \\ &= xx^*y^\dagger y \\ &= xy, \end{aligned}$$

so the operation in S and $S\mathbf{CS}$ are the same. Moreover, the distinguished semilattices of S and $S\mathbf{CS}$ are both E. Hence $S = S\mathbf{CS}$.

Conversely, let P be a weakly orthodox category over a semilattice E with partial binary operation $\cdot$. We build the Ehresmann semigroup $P\mathbf{S}$ by modifying the product $\circ$ in Theorem 9.16 as

$$x \circ y = x|_{x^*y^\dagger} \cdot {}_{x^*y^\dagger}|y.$$

We temporarily use the notation $\odot$ for the partial binary operation in $P\mathbf{SC}$. For any $x, y \in P$, we have

$$\begin{aligned} \exists x \odot y &\Leftrightarrow x^* = y^\dagger \text{ in } P\mathbf{S} \\ &\Leftrightarrow \exists x \cdot y \text{ in } P. \end{aligned}$$

Further, if $\exists x \odot y$, then by Lemma 9.9,

$$x \odot y = x \otimes y = x \cdot y.$$

For any $x \in P$ we have that $\mathbf{d}(x) = x^\dagger$ in $P\mathbf{SC}$, where $x \,\widetilde{\mathcal{R}}_B\, x^\dagger$ in $P\mathbf{S}$. But the latter holds if and only if $x^\dagger = \mathbf{d}(x)$ in P. Thus $\mathbf{d}$ in P and $P\mathbf{SC}$ coincide, and dually for $\mathbf{r}$.

Clearly, the distinguished morphisms in P and $P\mathbf{SC}$ are the same.

Again as a temporary measure, we use $||$ to denote restriction and co-restriction in $P\mathbf{SC}$.

Let $x \in P$ and $e \in B$ with $e \leq \mathbf{d}(x)$. Then in $P\mathbf{SC}$,

$$ {}_e||x = e \circ x = e|_{e\mathbf{d}(x)} \cdot {}_{e\mathbf{d}(x)}|x = {}_e|x $$

and similarly for co-restriction. □

The following result is easy to see, given Lemma 9.17 and Lemma 9.19.

Lemma 9.33. *Let $\theta : S \to T$ be an admissible morphism of Ehresmann semigroups, and $F : P_1 \to P_2$ be an orthodox functor of weakly orthodox categories over semilattices. Then $\theta\mathbf{CS} = \theta$ and $F\mathbf{SC} = F$.*

As a immediate consequence of Lemma 9.32 and Lemma 9.33, we have that $\mathbf{SC} = I_{\mathcal{WOC}}$ and $\mathbf{CS} = I_{\mathcal{WO}}$, so that $\mathbf{S}$ and $\mathbf{C}$ are mutually inverse. Hence we have:

Corollary 9.34. *The category of Ehresmann semigroups and admissible morphisms is isomorphic to the category of weakly orthodox categories over semilattices and orthodox functors.*

In view of Corollary 9.31 and Corollary 9.34, we succeed in obtaining Lawson's result [32].

Corollary 9.35. *[Theorem 4.24, [32]] The category of Ehresmann semigroups and admissible morphisms is isomorphic to the category of Ehresmann categories and strongly ordered functors.*

We now look at weakly B-superabundant semigroups with (C), which are weakly B-orthodox semigroups such that each $\widetilde{\mathcal{H}}_B$-class contains a distinguished idempotent in B. We say that a weakly orthodox category over B is *super-orthodox* if it satisfies the following condition:

(OB7) if $x \in P$, then $\mathbf{d}(x)\ \mathcal{D}\ \mathbf{r}(x)$.

Corollary 9.36. *The category of weakly B-superabundant semigroups with(C) and admissible morphisms is equivalent to the category of weakly super-orthodox categories over B and orthodox functors.*

Proof. Let S be a weakly B-superorthodox semigroup. It follows from Lemma 9.18 that it is sufficient to show that $S\mathbf{C}$ satisfies Condition (OB7). Suppose that $\alpha = (e, x, f) \in S\mathbf{C}$. Then $e \ \widetilde{\mathcal{R}}_B \ x \ \widetilde{\mathcal{L}}_B \ f$. As S is a weakly B-superabundant semigroup, there exists $h \in B$ such that $h \ \widetilde{\mathcal{H}}_B \ x$. Thus, $e \ \mathcal{R} \ h \ \mathcal{L} \ f$, which implies that $e \ \mathcal{D} \ f$, that is, $\mathbf{d}(\alpha) \ \mathcal{D} \ \mathbf{r}(\alpha)$.

Conversely, let P be a weakly super-orthodox category over B. It is necessary to show that each $\widetilde{\mathcal{H}}_{\overline{B}}$-class of $P\mathbf{S}$ has a distinguished idempotent belonging to $\overline{B}$. Suppose that $x \in P$. By (OB7), $\mathbf{d}(x) \ \mathcal{D} \ \mathbf{r}(x)$. Then, $\mathbf{d}(x) \ \mathcal{R} \ \mathbf{d}(x)\mathbf{r}(x) \ \mathcal{L} \ \mathbf{r}(x)$. By Lemma 9.11, we have that $\overline{1_{\mathbf{d}(x)}} \ \mathcal{R} \ \overline{1_{\mathbf{d}(x)\mathbf{r}(x)}} \ \mathcal{L} \ \overline{1_{\mathbf{r}(x)}}$. As $\overline{1_{\mathbf{d}(x)}} \ \widetilde{\mathcal{R}}_{\overline{B}} \ \bar{x} \ \widetilde{\mathcal{L}}_{\overline{B}} \ \overline{1_{\mathbf{r}(x)}}$, we obtain that $\bar{x} \ \widetilde{\mathcal{H}}_{\overline{B}} \ \overline{1_{\mathbf{d}(x)\mathbf{r}(x)}}$. Hence, $P\mathbf{S}$ is a weakly $\overline{B}$-superabundant semigroup with (C). □

We now turn to the class of weakly B-orthodox semigroups, which have (WIC) mentioned in Chapter 2. We define a weakly orthodox category over B to be *weakly connected* if it satisfies the following condition and its dual (OB8)°:

(OB8) if $x \in P$ and $e \leq \mathbf{d}(x)$, then there exists $f \leq \mathbf{r}(x)$ such that ${}_e|x \ \rho \ x|_f$.

Corollary 9.37. *The category of weakly B-orthodox semigroups with (WIC) and admissible morphisms is equivalent to the category of weakly connected categories over B and orthodox functors.*

Proof. Starting with a weakly B-orthodox semigroup S with (WIC), we show that $S\mathbf{C}$ satisfies Condition (OB8) and its dual. We show that (OB8) holds. Suppose that $(e, x, f) \in S\mathbf{C}$ and $u \leq e$. Then ${}_u|(e, x, f) = (u, ux, (ux)^\star)$. Since S has (WIC) it follows that there exists $v \in B$ such that $ux = xv$ and we can choose $v \leq f$. Then $(ux)^\star \ \widetilde{\mathcal{L}}_B \ xv \ \widetilde{\mathcal{L}}_B \ fv = v$ and $u = ue \ \widetilde{\mathcal{R}}_B \ ux \ \widetilde{\mathcal{R}}_B \ (xv)^+$. In addition,

$$\begin{aligned} {}_u|(e, x, f) \cdot [(ux)^\star, v] &= (u, ux, (ux)^\star) \cdot ((ux)^\star, (ux)^\star, v) \\ &= (u, ux, v) \\ &= [u, (xv)^+] \cdot ((xv)^+, xv, v) \\ &= [u, (xv)^+] \cdot (e, x, f)|_v. \end{aligned}$$

Thus, ${}_u|(e, x, f) \ \rho \ (e, x, f)|_v$.

Conversely, let P be a weakly connected category over B. Suppose that $\bar{x} \in P\mathbf{S}$ and $\overline{1_e} \ \leq \ \overline{1_{\mathbf{d}(x)}}$. Then $\bar{x} \ \widetilde{\mathcal{R}}_{\overline{B}} \ \overline{1_{\mathbf{d}(x)}}$. By (OB8), there exists $f \leq \mathbf{r}(x)$ such that ${}_e|x \;\rho\; x|_f$. Thus,

$$\overline{1_e} \circ \bar{x} = \overline{1_e \otimes x} = \overline{1_e \cdot {}_e|x} = \overline{{}_e|x} = \overline{x|_f} = \overline{x|_f \cdot 1_f} = \bar{x} \circ \overline{1_f}.$$

Together with the dual argument, we obtain that $P\mathbf{S}$ has (WIC). $\square$

Chapter 10

Weakly U-regular semigroups

A weakly U-regular semigroup is a weakly U-abundant semigroup with (C) and U generating a regular subsemigroup whose set of idempotents is U. The purpose of this chapter is to investigate a correspondence between weakly U-regular semigroups and certain categories, by using the techniques introduced in Chapter 9.

10.1 Weakly regular categories

The goal of this section is to develop the idea of weakly orthodox categories over a band constructed in Chapter 9 to introduce a category with set of objects a regular biordered set U.

Let U be a regular biordered set. A subset K of U is a *representative* of U if maps $\phi : K \to U/\mathcal{L}$ given by $e \mapsto L_e$ and $\psi : K \to U/\mathcal{R}$ given by $e \mapsto R_e$ are bijective. So for any $e \in U$, there exists a unique $k \in K$ such that $e \mathrel{\mathcal{L}} k$ in U and there exists a unique $h \in K$ such that $e \mathrel{\mathcal{R}} h$ in U. For convenience, we will denote k and h by $e^\star$ and e^+, respectively.

Definition 10.1. Let P be a category in which $\mathrm{Ob}(P)$ is the underlying set of a regular biordered set U, and let K be a representative of U. Suppose that for $e, f \in U$ satisfying $e \mathrel{\mathcal{R}} f$ or $e \mathrel{\mathcal{L}} f$, there exists a distinguished morphism $[e, f]$ from e to f, such that $[e, e] = 1_e$, the identity associated to e. Then P is an *RBS category* if the following conditions and the duals (P2)$^\circ$, (P3)$^\circ$, (P4)$^\circ$, and (P5)$^\circ$ of (P2), (P3), (P4) and (P5) hold:

(P1) if $e \mathrel{\mathcal{R}} f \mathrel{\mathcal{R}} g$ or $e \mathrel{\mathcal{L}} f \mathrel{\mathcal{L}} g$, then $[e, f] \cdot [f, g] = [e, g]$;

(P2) if $x \in P$, $h \in U$ and $h\ \omega^l\ \mathbf{d}(x)$, then there exists an element ${}_h|x$ in P, called the *restriction* of x to h, such that $\mathbf{d}({}_h|x) = h$ and $\mathbf{r}({}_h|x)\ \omega^l\ \mathbf{r}(x)$; also, if $h = \mathbf{d}(x)$, then $\mathbf{r}({}_h|x)\ \mathcal{L}\ \mathbf{r}(x)$ and ${}_h|x \cdot [\mathbf{r}({}_h|x), \mathbf{r}(x)] = x$;

(P3) if $g\ \omega\ e$ and $e\,\mathcal{R}\,f$ or $e\,\mathcal{L}\,f$, then ${}_g|[e, f] = [g, gf] \cdot [gf, (gf)^\star]$; and if $g\ \omega^l\ e$ and $e\,\mathcal{L}\,f$, then ${}_g|[e, f] = [g, g^\star]$;

(P4) if $x \in P$ and $e, f \in U$ with $e\ \omega^l\ f\ \omega^l\ \mathbf{d}(x)$, then ${}_e|({}_f|x) = {}_e|x$; also, if $e\,\mathcal{L}\,f\ \omega^l\ \mathbf{d}(x)$, then $[e, f] \cdot {}_f|x = {}_e|x$;

(P5) if $x, y \in P$ and $h \in U$ with $h\ \omega^l\ \mathbf{d}(x)$ and $\exists x \cdot y$ in P, then ${}_h|(x \cdot y) = {}_h|x \cdot {}_g|y$, where $g = \mathbf{r}({}_h|x)$;

(P6) if $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$ is a singular U-square, then $[e, f] \cdot [f, h] = [e, g] \cdot [g, h]$.

Let us pause to make some simple but necessary comments on Definition 10.1. In (P3) since $g\,\omega\,e$, we know that if $e\,\mathcal{R}\,f$ (resp. $e\,\mathcal{L}\,f$), then $g\,\omega^r\,f$ (resp. $g\,\omega^l\,f$), and so by (B21) $g\,\mathcal{R}\,gf$ (resp. $g = gf$). Thus $[g, gf]$ exists. In (P5) since $\exists x \cdot y$ we know that $\mathbf{r}(x) = \mathbf{d}(y)$. By (P2), $g = \mathbf{r}({}_h|x)\,\omega^l\,\mathbf{r}(x) = \mathbf{d}(y)$, so that ${}_g|y$ is defined and $\mathbf{d}({}_g|y) = g$. Hence, ${}_h|x \cdot {}_g|y$ is defined. Condition (P6) implies that

$$[g, e] \cdot [e, f] \cdot [f, h] \cdot [h, f] = [g, e] \cdot [e, g] \cdot [g, h] \cdot [h, f],$$

that is,

$$[g, e] \cdot [e, f] \cdot [f, f] = [g, g] \cdot [g, h] \cdot [h, f]$$

by (P1). Thus,

$$[g, e] \cdot [e, f] = [g, h] \cdot [h, f].$$

Comparing with Definition 9.1, we note that Conditions (P1) and (P6) correspond to Condition (OB5); Conditions (P2), (P4) and (P5) are similar to Conditions (OB1), (OB2) and (OB3), respectively; Condition (P3) is a generalisation of Condition (OB4).

Notice that an RBS category P depends on the choice of the regular biordered set U which is the set of objects of P. In order to avoid the ambiguity, we will express the term 'RBS category' as 'RBS category over U'.

Let P be an RBS category over a regular biordered set U. We define a relation ρ on P by the rule that for any $x, y \in P$,

$x\,\rho\,y$ if and only if

$$\mathbf{d}(x)\ \mathcal{R}\ \mathbf{d}(y), \mathbf{r}(x)\ \mathcal{L}\ \mathbf{r}(y) \text{ and } x \cdot [\mathbf{r}(x), \mathbf{r}(y)] = [\mathbf{d}(x), \mathbf{d}(y)] \cdot y.$$

It is worth making the point that

$$x \cdot [\mathbf{r}(x, \mathbf{r}(y)] = [\mathbf{d}(x), \mathbf{d}(y)] \cdot y$$
$$\Leftrightarrow [\mathbf{d}(y), \mathbf{d}(x)] \cdot x \cdot [\mathbf{r}(x), \mathbf{r}(y)] = y$$

$$\Leftrightarrow [\mathbf{d}(y), \mathbf{d}(x)] \cdot x = y \cdot [\mathbf{r}(y), \mathbf{r}(x)]$$
$$\Leftrightarrow x = [\mathbf{d}(x), \mathbf{d}(y)] \cdot y \cdot [\mathbf{r}(y), \mathbf{r}(x)],$$

so that if $\mathbf{d}(x) = \mathbf{d}(y)$, then $x\ \rho\ y$ if and only if $\mathbf{r}(x)\ \mathcal{L}\ \mathbf{r}(y)$ and

$$x = y \cdot [\mathbf{r}(y), \mathbf{r}(x)] \text{ or indeed } x \cdot [\mathbf{r}(x), \mathbf{r}(y)] = y.$$

Dually, if $\mathbf{r}(x) = \mathbf{r}(y)$, then $x\ \rho\ y$ if and only if $\mathbf{d}(x)\ \mathcal{R}\ \mathbf{d}(y)$ and

$$x = [\mathbf{d}(x), \mathbf{d}(y)] \cdot y \text{ or indeed } y = [\mathbf{d}(y), \mathbf{d}(x)] \cdot x.$$

Further, we have:

Lemma 10.2. *Let $e, f, g \in U$ be such that $e\ \mathcal{R}\ f\ \mathcal{L}\ g$. Then $[e, f]\ \rho\ 1_f\ \rho\ [f, g]$.*

Proof. As $e\ \mathcal{R}\ f$, we have that

$$\mathbf{d}([e, f]) = e\ \mathcal{R}\ f = \mathbf{d}(1_f),$$

$$\mathbf{r}([e, f]) = f = \mathbf{r}(1_f)$$

and

$$[e, f] \cdot 1_f = [e, f]$$

so that $[e, f]\ \rho\ 1_f$. Dually, $[f, g]\ \rho\ 1_f$. □

The proof of the next lemma is the same as that of Lemma 9.2 so we omit it.

Lemma 10.3. *The relation ρ defined above is an equivalence on P such that if $x, y \in Mor(e, f)$ and $x\,\rho\,y$, then $x = y$. In particular, no two identities of P are ρ-equivalent.*

We now present a pair of pre-orders on an RBS category over U built on the relation ρ given above.

Let P be an RBS category over U. We make use of the restriction and co-restriction of P to define relations $\leq_r{}'$ and $\leq_\ell{}'$ by the rule that for all $x, y \in P$,

$$x \leq_r{}' y \text{ if and only if } x \;\rho\; {}_e|y \quad \text{for some } e \in U \text{ and } e \;\omega\; \mathbf{d}(y)$$

and

$$x \leq_\ell{}' y \text{ if and only if } x \;\rho\; y|_f \quad \text{for some } f \in U \text{ and } f \;\omega\; \mathbf{r}(y).$$

Lemma 10.4. *The relations $\leq_r{}'$ and $\leq_\ell{}'$ are pre-orders on P.*

Proof. We first show that $\leq_r{}'$ is a pre-order on P. Notice that for any $x \in P$, if $e = \mathbf{d}(x)$, then $[\mathbf{d}(x), \mathbf{d}({}_e|x)] = [e, e] = 1_e$, and so $\leq_r{}'$ is reflexive by (P2). It is sufficient to show that $\leq_r{}'$ is transitive. Suppose that $x, y, z \in P$ with $x \leq_r{}' y$ and $y \leq_r{}' z$. Then there exist $e, f \in U$ such that $e \;\omega\; \mathbf{d}(y)$, $f \;\omega\; \mathbf{d}(z)$, and $x \;\rho\; {}_e|y$ and $y \;\rho\; {}_f|z$. Thus, $\mathbf{d}(y) \;\mathcal{R}\; f$, $\mathbf{r}(y) \;\mathcal{L}\; \mathbf{r}({}_f|z)$ and $y \cdot [\mathbf{r}(y), \mathbf{r}({}_f|z)] = [\mathbf{d}(y), f] \cdot ({}_f|z)$. Hence, $y = [\mathbf{d}(y), f] \cdot {}_f|z \cdot [\mathbf{r}({}_f|z), \mathbf{r}(y)]$. As $e \;\omega\; \mathbf{d}(y) \;\mathcal{R}\; f$, we have that $e \;\mathcal{R}\; ef \;\omega\; f$. In addition, ${}_e|[\mathbf{d}(y), f]$ exists and ${}_e|[\mathbf{d}(y), f] = [e, ef] \cdot [ef, (ef)^\star]$ by (P3). Since $(ef)^\star \;\mathcal{L}\; ef \;\omega\; f \;\omega\; \mathbf{d}(z)$, we obtain that ${}_{(ef)^\star}|({}_f|z)$ is defined and ${}_{(ef)^\star}|({}_f|z) = {}_{(ef)^\star}|z$ by (P4). Then we have that

$$\begin{aligned}
{}_e|y &= {}_e|([\mathbf{d}(y), f] \cdot {}_f|z \cdot [\mathbf{r}({}_f|z), \mathbf{r}(y)]) \\
&= {}_e|[\mathbf{d}(y), f] \cdot {}_h|({}_f|z) \cdot {}_k|[\mathbf{r}({}_f|z), \mathbf{r}(y)] \\
&\qquad\qquad \Big(\text{by (P5)},\ h = \mathbf{r}({}_e|[\mathbf{d}(y), f]),\ k = \mathbf{r}({}_h|({}_f|z))\Big) \\
&= [e, ef] \cdot [ef, (ef)^\star] \cdot {}_{(ef)^\star}|({}_f|z) \cdot [k, k^\star] \\
&\qquad\qquad \Big(\text{as } k = \mathbf{r}({}_h|({}_f|z)) \;\omega^l\; \mathbf{r}({}_f|z) \;\mathcal{L}\; \mathbf{r}(y),\ \text{by (P3)}\Big) \\
&= [e, ef] \cdot [ef, (ef)^\star] \cdot {}_{(ef)^\star}|z \cdot [k, k^\star] \qquad \Big(\text{by (P4)},\ (ef)^\star \;\mathcal{L}\; ef \;\omega\; f \;\omega\; \mathbf{d}(z)\Big).
\end{aligned}$$

Hence, $\mathbf{r}({}_e|y) = k^\star$. From $x \;\rho\; {}_e|y$, we have that $\mathbf{r}(x) \,\mathcal{L}\, k^\star$, $\mathbf{d}(x) \,\mathcal{R}\, e$ and

$$\begin{aligned}
x \cdot [\mathbf{r}(x), k^\star] &= [\mathbf{d}(x), e] \cdot {}_e|y \\
&= [\mathbf{d}(x), e] \cdot [e, ef] \cdot [ef, (ef)^\star] \cdot {}_{(ef)^\star}|z \cdot [k, k^\star] \\
&= [\mathbf{d}(x), ef] \cdot [ef, (ef)^\star] \cdot {}_{(ef)^\star}|z \cdot [k, k^\star] \qquad \Big(\mathbf{d}(x) \;\mathcal{R}\; e\; ef,\ \text{by (P1)}\Big) \\
&= [\mathbf{d}(x), ef] \cdot {}_{ef}|z \cdot [k, k^\star] \qquad \Big(\text{by (P4)},\ ef \;\mathcal{L}\; (ef)^\star \;\omega^l\; \mathbf{d}(z)\Big).
\end{aligned}$$

Thus,

$$\begin{aligned} x \cdot [\mathbf{r}(x), k] &= x \cdot [\mathbf{r}(x), k^\star] \cdot [k^\star, k] && \big(\text{by (P1)}\big) \\ &= [\mathbf{d}(x), ef] \cdot {}_{ef}|z \cdot [k, k^\star] \cdot [k^\star, k] \\ &= [\mathbf{d}(x), ef] \cdot {}_{ef}|z \cdot [k, k] && \big(k \mathrel{\mathcal{L}} k^\star\big) \\ &= [\mathbf{d}(x), ef] \cdot {}_{ef}|z && \big(k = \mathbf{r}({}_{ef}|z)\big). \end{aligned}$$

It follows that $k = \mathbf{r}({}_{ef}|z)$. As $\mathbf{r}(x) \mathrel{\mathcal{L}} \mathbf{r}({}_{e}|y) = k^\star \mathrel{\mathcal{L}} k = \mathbf{r}({}_{ef}|z)$ and $\mathbf{d}(x) \mathrel{\mathcal{R}} ef = \mathbf{d}({}_{ef}|z)$, we have that $x \ \rho \ {}_{ef}|z$. Together with $ef \ \omega \ f \ \omega \ \mathbf{d}(z)$, we obtain that $x \leq_r{}' z$.

By the dual argument, we show that $\leq_\ell{}'$ is a pre-order on P. □

In addition, there exists another way to define a pair of pre-orders on an RBS category P over U.

Let P be an RBS category over U. For any $x, y \in P$, we define

$$x \ \leq_r \ y \text{ if and only if } x \ \rho \ {}_{e}|y \text{ for some } e \in U$$

and

$$x \ \leq_l \ y \text{ if and only if } x \ \rho \ y|_{f} \text{ for some } f \in U.$$

By (P2) and its dual, relations $\leq_r$ and $\leq_l$ are reflexive, but in general they are not transitive and symmetric. We recall that the transitive closure of a relation θ is denoted by θ^t. We have:

Lemma 10.5. *Let P be an RBS category over U. Then relations $\leq_r^t$ and $\leq_l^t$ are pre-orders on P.*

As the comments succeeding Lemma 9.3, it is impossible to define a pair of partial orders on P.

Let P be an RBS category over a regular biordered set U. By (B1), ω^r and ω^l are pre-orders on U, and $D_U = (\omega^r \cup \omega^l) \cup (\omega^r \cup \omega^l)^{-1}$. Suppose that $x \in P$ and $h \ \omega^r \ \mathbf{d}(x)$. We define

$$h * x = [h, h\mathbf{d}(x)] \cdot {}_{h\mathbf{d}(x)}|x.$$

Clearly, $\mathbf{d}(h * x) = h$. In particular, if $h \ \omega \ \mathbf{d}(x)$, then $h * x = {}_{h}|x$.

Observe that if $h\ \omega^r\ \mathbf{d}(x)$, then by (B1), $h\mathbf{d}(x)$ exists in U. Again by (B21), $h\ \mathcal{R}\ h\mathbf{d}(x)\ \omega\ \mathbf{d}(x)$, so that $[h, h\mathbf{d}(x)]$ exists and by (P2), ${}_{h\mathbf{d}(x)}|x$ is well-defined.

Dually, if $k\ \omega^l\ \mathbf{r}(x)$, then we define

$$x \diamond k = x|_{\mathbf{r}(x)k} \cdot [\mathbf{r}(x)k, k].$$

Note that $\mathbf{r}(x \diamond k) = k$. In particular, if $k\ \omega\ \mathbf{r}(x)$, then $x \diamond k = x|_k$.

We stop here to make some comments on the above operations $*$ and $\diamond$. The initial idea of defining operations $*$ and $\diamond$ is to define a binary operation on an RBS category or a weakly regular category introduced below, via the sandwich set. Let P be an RBS category. Suppose that $x, y \in P$ and $h \in S(\mathbf{r}(x), \mathbf{d}(y))$. Then $h\ \omega^l\ \mathbf{r}(x)$ and $h\ \omega^r\ \mathbf{d}(y)$. Our purpose is to use restrictions and co-restrictions to define a product. Note that $h\ \mathcal{L}\ \mathbf{r}(x)h\ \omega\ \mathbf{r}(x)$ and $h\ \mathcal{R}\ h\mathbf{d}(y)\ \omega\ \mathbf{d}(y)$ so that we can define a binary operation on P by the rule that

$$x\ y = x|_{\mathbf{r}(x)h} \cdot [\mathbf{r}(x)h, h] \cdot [h, h\mathbf{d}(y)] \cdot {}_{h\mathbf{d}(y)}|y.$$

Certainly, it is well defined. For convenience, we defined $*$ and $\diamond$ above.

To maintain the analogy with weakly orthodox categories, we have weakly regular categories described as follows:

Definition 10.6. An RBS category P over a regular biordered set U is *weakly regular* if it satisfies the following condition:

(P7) for $x \in P$, $e, f \in U$, $h_1 \in S(e, \mathbf{d}(x))$ and $h_2 \in S(\mathbf{r}(x), f)$, we put $h_1' = \mathbf{r}({}_{h_1\mathbf{d}(x)}|x)$ and $h_2' = \mathbf{d}(x|_{\mathbf{r}(x)h_2})$. Then there exist $h \in S(h_1, h_2')$ and $h' \in S(h_1', h_2)$ such that

$$((h_1 * x) \diamond h') \cdot [h', h'h_2]\ \rho\ [h_1h, h] \cdot (h * (x \diamond h_2)).$$

It is a good place to make some necessary comments on Definition 10.6. For any $e, f \in U$, $S(e, f)$ denotes the sandwich set of e and f. In (P7), $h_1 \in S(e, \mathbf{d}(x))$ implies that $h_1\ \omega^r\ \mathbf{d}(x)$, and so by (B21) in Section 1.4, $h_1\ \mathcal{R}\ h_1\mathbf{d}(x)\ \omega\ \mathbf{d}(x)$ so that ${}_{h_1\mathbf{d}(x)}|x$ is well-defined and $h_1 * x$ exists. Put $h_1' = \mathbf{r}({}_{h_1\mathbf{d}(x)}|x)$. Similarly, $x|_{\mathbf{r}(x)h_2}$ and $x \diamond h_2$ are well-defined. We put $h_2' = \mathbf{d}(x|_{\mathbf{r}(x)h_2})$. Since U is regular, we obtain that $S(h_1, h_2') \neq \emptyset$ and $S(h_1', h_2) \neq \emptyset$. If $h' \in S(h_1', h_2)$, then $h'\ \omega^l\ h_1' = \mathbf{r}({}_{h_1\mathbf{d}(x)}|x) = \mathbf{r}(h_1 * x)$ and $h'\ \omega^r\ h_2$, so that $(h_1 * x) \diamond h'$ is defined and by (B21), $h'h_2\ \mathcal{R}\ h'$. Hence $[h', h'h_2]$ exists. By a similar argument, $h * (x \diamond h_2)$ is defined and $h_1h\ \mathcal{L}\ h$ so that $[h_1h, h]$ exists.

We are now ready to say that the class of weakly regular categories over regular biordered sets forms a category, together with certain functors, namely RBS functors, which appear in the next definition.

Definition 10.7. Let P_1 and P_2 be weakly regular categories over regular biordered sets U_1 and U_2, respectively, and $F : P_1 \to P_2$ be a functor. Then F is said to be *RBS* if

(PF1) $F : U_1 \to U_2$ is a regular morphism;

(PF2) if $e \mathcal{R} f$ or $e \mathcal{L} f$ in U, then $[e, f]_{P_1} F = [eF, fF]_{P_2}$;

(PF3) if $x \in P_1$ and $h, k \in U_1$ with $h \ \omega^l \ \mathbf{d}(x)$ and $k \ \omega^r \ \mathbf{r}(x)$, then $({}_h|x)F \, \rho \, {}_{hF}|xF$ and $(x|_k)F \, \rho \, xF|_{kF}$.

We pause here to make a short comment on Definition 10.7. In (PF3), if $h \ \omega^l \ \mathbf{d}(x)$, then $hF \ \omega^l \ \mathbf{d}(xF)$ as F is a biordered set morphism and is a functor, so that both ${}_h|x$ and ${}_{hF}|xF$ are well-defined. In addition, the fact that $({}_h|x)F \, \rho \, {}_{hF}|xF$ gives in particular that $\mathbf{r}({}_h|x)F \, \mathcal{L} \, \mathbf{r}({}_{hF}|xF)$. For $\mathbf{d}$, as F is a functor, we have that $\mathbf{d}(({}_h|x)F) = \mathbf{d}({}_h|x)F = hF = \mathbf{d}({}_{hF}|xF)$.

The next lemma is useful for Lemma 10.9.

Lemma 10.8. *Let P_1 and P_2 be weakly regular categories over U_1 and U_2, respectively and let $F : P_1 \to P_2$ be an RBS functor. If $x \ \rho \ y$ in P_1, then $xF \ \rho \ yF$ in P_2.*

Proof. Suppose that $x, y \in P_1$ and $x \ \rho \ y$. Then

$$\mathbf{d}(x) \, \mathcal{R} \, \mathbf{d}(y), \ \mathbf{r}(x) \, \mathcal{L} \, \mathbf{r}(y) \text{ and } x \cdot [\mathbf{r}(x), \mathbf{r}(y)] = [\mathbf{d}(x), \mathbf{d}(y)] \cdot y.$$

Thus,

$$\mathbf{d}(x)F \, \mathcal{R} \, \mathbf{d}(y)F, \ \mathbf{r}(x)F \, \mathcal{L} \, \mathbf{r}(y)F \text{ and } xF \cdot [\mathbf{r}(x), \mathbf{r}(y)]F = [\mathbf{d}(x), \mathbf{d}(y)]F \cdot yF.$$

This gives

$$\mathbf{d}(x)F \, \mathcal{R} \, \mathbf{d}(y)F, \ \mathbf{r}(x)F \, \mathcal{L} \, \mathbf{r}(y)F \text{ and } xF \cdot [\mathbf{r}(x)F, \mathbf{r}(y)F] = [\mathbf{d}(x)F, \mathbf{d}(y)F] \cdot yF,$$

by (PF2), and so

$$\mathbf{d}(xF) \, \mathcal{R} \, \mathbf{d}(yF), \ \mathbf{r}(xF) \, \mathcal{L} \, \mathbf{r}(yF) \text{ and } xF \cdot [\mathbf{r}(xF), \mathbf{r}(yF)] = [\mathbf{d}(xF), \mathbf{d}(yF)] \cdot yF.$$

Hence, $xF\ \rho\ yF$. □

Lemma 10.9. *Let P_1 and P_2 be weakly regular categories over U_1 and U_2, respectively, and let $F_1 : P_1 \to P_2$ and $F_2 : P_2 \to P_3$ be RBS functors. Then $F_1F_2 : P_1 \to P_3$ is an RBS functor.*

Proof. (PF1) Certainly, F_1F_2 is a functor from P_1 to P_3 and a regular morphism from U_1 to U_3.

(PF2) Suppose that $e, f \in U_1$ are such that $e\ \mathcal{R}\ f$. Then $[e, f]_{P_1}$ is defined, and $eF_1\ \mathcal{R}\ fF_1$ and $eF_1F_2\ \mathcal{R}\ fF_1F_2$ by the comment succeeding the definition of regular morphism in Chapter 1. Using (PF2) for F_1 and F_2, we have that

$$[e, f]_{P1}F_1F_2 = ([e, f]_{P1}F_1)F_2 = [eF_1, fF_1]_{P_2}F_2 = [eF_1F_2, fF_1F_2]_{P3}.$$

Dually, if $e\ \mathcal{L}\ f$, then $[e, f]_{P1}F_1F_2 = [eF_1F_2, fF_1F_2]_{P3}$.

(PF3) Assume that $x \in P_1$ and $e \in U_1$ with $e\ \omega^l\ \mathbf{d}(x)$. According to the comment succeeding Definition 10.7, we have that ${}_e|x$, ${}_{eF_1}|xF_1$ and ${}_{eF_1F_2}|xF_1F_2$ are well-defined. By (PF3), $({}_e|x)F_1\ \rho\ {}_{eF_1}|xF_1$ and $({}_{eF_1}|xF_1)F_2\ \rho\ {}_{eF_1F_2}|xF_1F_2$. From $({}_e|x)F_1\ \rho\ {}_{eF_1}|xF_1$, we obtain that $({}_e|x)F_1F_2\ \rho\ ({}_{eF_1}|xF_1)F_2$ by Lemma 10.8. Hence, $({}_e|x)F_1F_2\ \rho\ {}_{eF_1F_2}|xF_1F_2$. □

An immediate observation from Lemma 10.9 is that the class of weakly regular categories over regular biordered sets and RBS functors forms a category. We refer to it as $\mathcal{WRC}$.

We close this section with an important property of RBS fuctors.

Lemma 10.10. *If P_1 and P_2 are weakly regular categories over regular biordered sets U_1 and U_2, respectively, and $F : P_1 \to P_2$ is an RBS functor. Then for any $h, k \in U_1$ and $x \in P_1$, we have*

(i) if $h\ \omega^r\ \mathbf{d}(x)$, then $(h * x)F\ \rho\ hF * xF$;

(ii) if $k\ \omega^l\ \mathbf{r}(x)$, then $(x \diamond k)F\ \rho\ xF \diamond kF$.

Proof. To prove (i), suppose that $x \in P_1$, $h \in U_1$ and $h\,\omega^r\,\mathbf{d}(x)$. Then

$$h * x = [h, h\mathbf{d}(x)] \cdot ({}_{h\mathbf{d}(x)}|x).$$

Since F is RBS, it follows that $F : U_1 \to U_2$ is a regular morphism, and so

$hF \; \omega^r \; \mathbf{d}(x)F$ so that $hF * \mathbf{d}(x)F$ exists. In addition, $(h\mathbf{d}(x))F = hF\mathbf{d}(x)F$ and

$$[h, h\mathbf{d}(x)]F = [hF, (h\mathbf{d}(x))F] = [hF, hF\mathbf{d}(x)F].$$

According to the comments following Definition 10.7, we have that

$$\mathbf{d}(({}_{h\mathbf{d}(x)}|x)F) = (h\mathbf{d}(x))F = hF\mathbf{d}(x)F = \mathbf{d}({}_{(hF\mathbf{d}(x)F}|xF).$$

By (PF3),

$$({}_{h\mathbf{d}(x)}|x)F \; \rho \; {}_{(h\mathbf{d}(x))F}|xF = {}_{hF\mathbf{d}(x)F}|xF.$$

Thus, $\mathbf{r}(({}_{h\mathbf{d}(x)}|x)F) \, \mathcal{L} \, \mathbf{r}({}_{hF\mathbf{d}(x)F}|xF)$ and

$$({}_{h\mathbf{d}(x)}|x)F = {}_{hF\mathbf{d}(x)F}|xF \cdot [\mathbf{r}({}_{hF\mathbf{d}(x)F}|xF), \mathbf{r}(({}_{h\mathbf{d}(x)}|x)F)],$$

so that

$$\begin{aligned}
&(h * x)F \\
&= ([h, h\mathbf{d}(x)] \cdot {}_{h\mathbf{d}(x)}|x)F \\
&= [h, h\mathbf{d}(x)]F \cdot ({}_{h\mathbf{d}(x)}|x)F \\
&= [hF, hF\mathbf{d}(x)F] \cdot ({}_{hF\mathbf{d}(x)F}|xF) \cdot [\mathbf{r}({}_{hF\mathbf{d}(x)F}|xF), \mathbf{r}(({}_{h\mathbf{d}(x)}|x)F)] \\
&= [hF, hF\mathbf{d}(xF)] \cdot ({}_{hF\mathbf{d}(xF)}|xF) \cdot [\mathbf{r}({}_{hF\mathbf{d}(xF)}|xF), \mathbf{r}(({}_{h\mathbf{d}(x)}|x)F)] \\
&= (hF * xF) \cdot [\mathbf{r}({}_{hF\mathbf{d}(xF)}|xF), \mathbf{r}(({}_{h\mathbf{d}(x)}|x)F)] \\
&= (hF * xF) \cdot [\mathbf{r}(hF * xF), \mathbf{r}((h * x)F)].
\end{aligned}$$

Consequently, $(h * x)F \; \rho \; hF * xF$. Similarly, part (ii) holds. ☐

10.2 Structure theorems

In preparation for the main theorem at the end of this section, we have to list some necessary lemmas concerning a weakly regular category over a regular biordered set. Throughout this section, we will use P to denote a weakly regular category over a regular biordered set U.

Lemma 10.11. *Let $x \in P$ and $g, h \in U$ be such that $g \, \omega^r \, h \, \omega^r \, \mathbf{d}(x)$ and $g\mathbf{d}(x) \, \omega \, h\mathbf{d}(x)$. Then $g * (h * x) = g * x$.*

Proof. By the hypothesis, $g * (h * x)$ and $g * x$ are well-defined. By (B21),

$gh\,\omega\,h\,\mathcal{R}\,h\mathbf{d}(x)$. So,

$$\begin{aligned}
& g * (h * x) \\
&= [g, gh] \cdot {}_{gh}|(h * x) \\
&= [g, gh] \cdot {}_{gh}|([h, h\mathbf{d}(x)] \cdot {}_{h\mathbf{d}(x)}|x) \\
&= [g, gh] \cdot {}_{gh}|[h, h\mathbf{d}(x)] \cdot {}_{k}|({}_{h\mathbf{d}(x)}|x) && \Big(\text{by (P5)},\ k = \mathbf{r}({}_{gh}|[h, h\mathbf{d}(x)])\Big) \\
&= [g, gh] \cdot {}_{gh}|[h, h\mathbf{d}(x)] \cdot {}_{k}|x && \Big(\text{by (P2)},\ k\,\omega^{l}\,h\mathbf{d}(x),\ \text{now use (P4)}\Big) \\
&= [g, gh] \cdot [gh, u] \cdot [u, u^{\star}] \cdot {}_{u^{\star}}|x && \Big(\text{by (P3)},\ u = (gh)(h\mathbf{d}(x)), k = u^{*}\Big) \\
&= [g, u] \cdot [u, u^{\star}] \cdot {}_{u^{\star}}|x && \Big(g\,\mathcal{R}\,gh\,\mathcal{R}\,u,\ \text{by (P1)}\Big) \\
&= [g, u] \cdot {}_{u}|x && \Big(\text{by (P4)}\Big).
\end{aligned}$$

As

$$\begin{aligned}
u &= (gh)(h\mathbf{d}(x)) \\
&= g(h(h\mathbf{d}(x))) && \Big(g\,\omega^{r}\,h\,\mathcal{R}\,h\mathbf{d}(x),\ \text{Lemma 1.27}\Big) \\
&= g(h\mathbf{d}(x)) && \Big(h\mathbf{d}(x)\ \mathcal{R}\ h\Big)
\end{aligned}$$

$$\begin{aligned}
&= (g\mathbf{d}(x))(h\mathbf{d}(x)) && \Big(\text{by Lemma 1.27}\Big) \\
&= g\mathbf{d}(x) && \Big(g\mathbf{d}(x)\ \omega\ h\mathbf{d}(x)\Big),
\end{aligned}$$

we obtain that $g * (h * x) = [g, g\mathbf{d}(x)] \cdot {}_{g\mathbf{d}(x)}|x = g * x$. □

Dually, we have:

Lemma 10.12. *If $x \in P$ and $g, h \in U$ with $g\,\omega^{l}\,h\,\omega^{l}\,\mathbf{r}(x)$ and $\mathbf{r}(x)g\,\omega\,\mathbf{r}(x)h$, then $(x \diamond h) \diamond g = x \diamond g$.*

Now, let $x, y \in P$ and $h \in S(\mathbf{r}(x), \mathbf{d}(y))$. We define

$$(x \otimes y)_h = (x \diamond h) \cdot (h * y).$$

Since $h \in S(\mathbf{r}(x), \mathbf{d}(y))$, it follows that $h\,\omega^{l}\,\mathbf{r}(x)$ and $h\,\omega^{r}\,\mathbf{d}(y)$ so that $x \diamond h$ and $h * y$ exist. As $\mathbf{r}(x \diamond h) = h$ and $\mathbf{d}(h * y) = h$, $(x \otimes y)_h = (x \diamond h) \cdot (h * y)$ is well-defined.

Lemma 10.13. *Let $x, y, z \in P$, $h_1 \in S(\mathbf{r}(x), \mathbf{d}(y))$ and $h_2 \in S(\mathbf{r}(y), \mathbf{d}(z))$. Set $h_1' = \mathbf{r}(h_1 * y)$ and $h_2' = \mathbf{d}(y \diamond h_2)$. Then there exist $h \in S(\mathbf{r}(x), h_2')$ and*

$h' \in S(h_1', \mathbf{d}(z))$ *such that* $((x \otimes y)_{h_1} \otimes z)_{h'} = (x \otimes (y \otimes z)_{h_2})_h$.

Proof. Since $h_1' = \mathbf{r}(h_1 * y) = \mathbf{r}({}_{h_1\mathbf{d}(y)}|y)\ \omega^l\ \mathbf{r}(y)$ and $h_2 \in S(\mathbf{r}(y), \mathbf{d}(z))$, it follows from Lemma 1.30 that $S(h_1', h_2) \subseteq S(h_1', \mathbf{d}(z))$. Similarly, as $h_2' = \mathbf{d}(y \diamond h_2) = \mathbf{d}(y|_{\mathbf{r}(y)h_2})\ \omega^r\ \mathbf{d}(y)$ and $h_1 \in S(\mathbf{r}(x), \mathbf{d}(y))$, by the same result, we have that $S(h_1, h_2') \subseteq S(\mathbf{r}(x), h_2')$.

Since $h_1 \in S(\mathbf{r}(x), \mathbf{d}(y))$, $h_2 \in S(\mathbf{r}(y), \mathbf{d}(z))$, $h_1' = \mathbf{r}(h_1 * y)$ and $h_2' = \mathbf{d}(y \diamond h_2)$, by (P7), there exist $h' \in S(h_1', h_2)$ and $h \in S(h_1, h_2')$ such that

$$((h_1 * y) \diamond h') \cdot [h', h'h_2]\ \rho\ [h_1 h, h] \cdot (h * (y \diamond h_2)).$$

Put $g = \mathbf{d}((h_1 * y) \diamond h')$ and $k = \mathbf{r}(h * (y \diamond h_2))$. We obtain that $g\,\mathcal{R}\,h_1 h$, $h'h_2\ \mathcal{L}\ k$, and

$$[h_1 h, g] \cdot ((h_1 * y) \diamond h') \cdot [h', h'h_2] = [h_1 h, h] \cdot (h * (y \diamond h_2)) \cdot [k, h'h_2].$$

Now, we deduce that

$$\begin{aligned}
&((x \otimes y)_{h_1} \otimes z)_{h'} \\
&= ((x \otimes y)_{h_1} \diamond h') \cdot (h' * z) \qquad \Big(h_1' = \mathbf{r}(h_1 * y),\ h' \in S(h_1', \mathbf{d}(z))\Big) \\
&= (((x \diamond h_1) \cdot (h_1 * y)) \diamond h') \cdot (h' * z) \\
&= ((x \diamond h_1) \cdot (h_1 * y))|_{h_1'h'} \cdot [h_1'h', h'] \cdot (h' * z) \\
&= (x \diamond h_1)|_g \cdot (h_1 * y)|_{h_1'h'} \cdot [h_1'h', h'] \cdot (h' * z) \\
&\qquad \Big(\text{by (P5)}^\circ, g = \mathbf{d}((h_1 * y) \diamond h') = \mathbf{d}((h_1 * y)|_{h_1'h'})\Big) \\
&= (x \diamond h_1)|_g \cdot ((h_1 * y) \diamond h') \cdot (h' * z) \\
&= (x \diamond h_1)|_g \cdot ((h_1 * y) \diamond h') \cdot [h', h'] \cdot (h' * z) \\
&= (x \diamond h_1)|_g \cdot ((h_1 * y) \diamond h') \cdot [h', h'h_2] \cdot [h'h_2, h'] \cdot (h' * z) \\
&\qquad \Big(h'\ \mathcal{R}\ h'h_2, \quad \text{by (P1)}\Big) \\
&= (x \diamond h_1)|g \cdot [g, g] \cdot ((h_1 * y) \diamond h') \cdot [h', h'h_2] \cdot [h'h_2, h'] \cdot (h' * z) \\
&= (x \diamond h_1)|g \cdot [g, h_1 h] \cdot [h_1 h, g] \cdot ((h_1 * y) \diamond h') \cdot [h', h'h_2] \cdot [h'h_2, h'] \cdot (h' * z) \\
&\qquad \Big(g\ \mathcal{R}\ h_1 h, \quad \text{by (P1)}\Big) \\
&= (x \diamond h_1)|g \cdot [g, h_1 h] \cdot [h_1 h, h] \cdot (h * (y \diamond h_2)) \cdot [k, h'h_2] \cdot [h'h_2, h'] \cdot (h' * z) \\
&\qquad \Big(\text{as } [h_1 h, g] \cdot ((h_1 * y) \diamond h') \cdot [h', h'h_2] = [h_1 h, h] \cdot (h * (y \diamond h_2)) \cdot [k, h'h_2]\Big)
\end{aligned}$$

$$= (x \diamond h_1)|_{h_1 h} \cdot [h_1 h, h] \cdot (h * (y \diamond h_2)) \cdot [k, h'h_2] \cdot [h'h_2, h'] \cdot (h' * z) \qquad \big(g \ \mathcal{R} \ h_1 h, \text{ by (P4)}^\circ\big)$$

$$= ((x \diamond h_1) \diamond h) \cdot (h * (y \diamond h_2)) \cdot [k, h'h_2] \cdot [h'h_2, h'] \cdot (h' * z) \qquad \big(h \ \omega^l \ h_1\big).$$

From $h \in S(h_1, h_2')$, we have that $h \, \omega^l \, h_1$ and $h \, \omega^r \, h_2'$. As $h_1 \in S(\mathbf{r}(x), \mathbf{d}(y))$ and $h_2' = \mathbf{d}(y \diamond h_2)$, we have that $h_1 \, \omega^l \, \mathbf{r}(x)$ and $h_2' \, \omega^r \, \mathbf{d}(y)$. Thus $h \ \omega^l \ h_1 \ \omega^l \ \mathbf{r}(x)$ and $h \, \omega^r \, h_2 \, \omega^r \, \mathbf{d}(y)$, so that $h \in \mathcal{M}(\mathbf{r}(x), \mathbf{d}(y))$. Since $h_1 \in S(\mathbf{r}(x), \mathbf{d}(y))$, it follows that $\mathbf{r}(x)h \ \omega^r \ \mathbf{r}(x)h_1$. Again by $\mathbf{r}(x)h \, \mathcal{L} \, h \, \omega^l \, h_1 \, \mathcal{L} \, \mathbf{r}(x)h_1$, we have that $\mathbf{r}(x)h \ \omega \ \mathbf{r}(x)h_1$. By Lemma 10.12, $(x \diamond h_1) \diamond h = x \diamond h$. Hence,

$$((x \otimes y)_{h_1} \otimes z)_{h'} = (x \diamond h) \cdot (h * (y \diamond h_2)) \cdot [k, h'h_2] \cdot [h'h_2, h'] \cdot (h' * z).$$

Also,

$$\begin{aligned}
&(x \otimes (y \otimes z)_{h_2})_h \\
&= (x \diamond h) \cdot (h * (y \otimes z)_{h_2}) \\
&= (x \diamond h) \cdot (h * ((y \diamond h_2) \cdot (h_2 * z))) && \big(h_2 \in S(\mathbf{r}(y), \mathbf{d}(z))\big) \\
&= (x \diamond h) \cdot [h, hh_2'] \cdot {}_{hh_2'}|\big((y \diamond h_2) \cdot (h_2 * z)\big) \\
&= (x \diamond h) \cdot [h, hh_2'] \cdot {}_{hh_2'}|(y \diamond h_2) \cdot {}_k|(h_2 * z)) \\
& && \big(\text{by (P5)}, k = \mathbf{r}({}_{hh_2'}|(y \diamond h_2)) = \mathbf{r}(h * (y \diamond h_2))\big) \\
&= (x \diamond h) \cdot (h * (y \diamond h_2)) \cdot {}_k|(h_2 * z) && \big(h \ \omega^r \ h_2' = \mathbf{d}(y \diamond h_2), \ k = \mathbf{r}(h * (y \diamond h_2))\big) \\
&= (x \diamond h) \cdot (h * (y \diamond h_2)) \cdot [k, k] \cdot {}_k|(h_2 * z) \\
&= (x \diamond h) \cdot (h * (y \diamond h_2)) \cdot [k, h'h_2] \cdot [h'h_2, k] \cdot {}_k|(h_2 * z) && \big(k \ \mathcal{L} \ h'h_2, \text{ by (P1)}\big) \\
&= (x \diamond h) \cdot (h * (y \diamond h_2)) \cdot [k, h'h_2] \cdot {}_{h'h_2}|(h_2 * z) && \big(k \ \mathcal{L} \ h'h_2, \text{ by (P4)}\big) \\
&= (x \diamond h) \cdot (h * (y \diamond h_2)) \cdot [k, h'h_2] \cdot [h'h_2, h'h_2] \cdot {}_{h'h_2}|(h_2 * z) \\
&= (x \diamond h) \cdot (h * (y \diamond h_2)) \cdot [k, h'h_2] \cdot [h'h_2, h'] \cdot [h', h'h_2] \cdot {}_{h'h_2}|(h_2 * z) \\
& && \big(h' \ \mathcal{R} \ h'h_2, \text{ by (P1)}\big) \\
&= (x \diamond h) \cdot (h * (y \diamond h_2)) \cdot [k, h'h_2] \cdot [h'h_2, h'] \cdot (h' * (h_2 * z)) \\
& && \big(h' \ \omega^r \ h_2 = \mathbf{d}(h_2 * z)\big).
\end{aligned}$$

Since $h' \ \omega^r \ h_2 \ \omega^r \ \mathbf{d}(z)$, it follows that $h'\mathbf{d}(z) \ \mathcal{R} \ h' \ \omega^r \ h_2 \ \mathcal{R} \ h_2\mathbf{d}(z)$. As

$$h' \in \mathcal{M}(\mathbf{r}(y), \mathbf{d}(z)) \text{ and } h_2 \in S(\mathbf{r}(y), \mathbf{d}(z)),$$

we have that $h'\mathbf{d}(z)\,\omega^l\,h_2\mathbf{d}(z)$. So $h'\mathbf{d}(z)\,\omega\,h_2\mathbf{d}(z)$. By Lemma 10.11, $h'*(h_2*z) = h'*z$. Thus,

$$(x\otimes(y\otimes z)_{h_2})_h = (x\diamond h)\cdot(h*(y\diamond h_2))\cdot[k,h'h_2]\cdot[h'h_2,h']\cdot(h'*z).$$

Hence, $((x\otimes y)_{h_1}\otimes z)_{h'} = (x\otimes(y\otimes z)_{h_2})_h$. □

Lemma 10.14. *If $x, y \in P$ and $x\,\rho\,y$, then $h*x\,\rho\,h*y$ and $x\diamond k\,\rho\,y\diamond k$, where $h\,\omega^r\,\mathbf{d}(x)$ and $k\,\omega^l\,\mathbf{r}(x)$.*

Proof. We first show that $x\,\rho\,y$ and $h\,\omega^r\,\mathbf{d}(x)$ imply $h*x\,\rho\,h*y$. Dually, the second part holds. Clearly, if $x\ \rho\ y$, then $\mathbf{d}(x)\,\mathcal{R}\,\mathbf{d}(y)$ and $\mathbf{r}(x)\,\mathcal{L}\,\mathbf{r}(y)$. Write $h_1 = \mathbf{r}(h*x) = \mathbf{r}({}_{h\mathbf{d}(x)}|x)$. We deduce that

$$\begin{aligned} h\mathbf{d}(x)*(x\cdot[\mathbf{r}(x),\mathbf{r}(y)]) &= {}_{h\mathbf{d}(x)}|(x\cdot[\mathbf{r}(x),\mathbf{r}(y)]) && \big(h\mathbf{d}(x)\ \omega\ \mathbf{d}(x)\big)\\ &= {}_{h\mathbf{d}(x)}|x\cdot{}_{h_1}|[\mathbf{r}(x),\mathbf{r}(y)] && \big(h_1=\mathbf{r}({}_{h\mathbf{d}(x)}|x),\ \text{by (P5)}\big)\\ &= {}_{h\mathbf{d}(x)}|x\cdot[h_1,h_1^\star] && \big(h_1\ \omega^l\ \mathbf{r}(x)\ \mathcal{L}\ \mathbf{r}(y),\ \text{by (P3)}\big)\end{aligned}$$

and

$$\begin{aligned} & h\mathbf{d}(x)*([\mathbf{d}(x),\mathbf{d}(y)]\cdot y)\\ &= {}_{h\mathbf{d}(x)}|([\mathbf{d}(x),\mathbf{d}(y)]\cdot y) && \big(h\mathbf{d}(x)\ \omega\ \mathbf{d}(x)\big)\\ &= {}_{h\mathbf{d}(x)}|[\mathbf{d}(x),\mathbf{d}(y)]\cdot{}_g|y && \big(\text{by (P5)}, g=\mathbf{r}({}_{h\mathbf{d}(x)}|[\mathbf{d}(x),\mathbf{d}(y)])\big)\\ &= [h\mathbf{d}(x),(h\mathbf{d}(x))\mathbf{d}(y)]\cdot[(h\mathbf{d}(x))\mathbf{d}(y),((h\mathbf{d}(x))\mathbf{d}(y))^\star]\cdot{}_g|y\\ &&& \big(h\mathbf{d}(x)\ \omega\ \mathbf{d}(x)\ \mathcal{R}\ \mathbf{d}(y),\ \text{by (P3)}\big)\\ &= [h\mathbf{d}(x),h\mathbf{d}(y)]\cdot[h\mathbf{d}(y),(h\mathbf{d}(y))^\star]\cdot{}_g|y\\ &&& \big(h\ \omega^r\ \mathbf{d}(x)\ \mathcal{R}\ \mathbf{d}(y),\ \text{by(B31)},\ \text{so } g=(h\mathbf{d}(y))^\star\big)\\ &= [h\mathbf{d}(x),h\mathbf{d}(y)]\cdot{}_{h\mathbf{d}(y)}|y && \big(\text{by (P4)}\big).\end{aligned}$$

Since $x\,\rho\,y$, we have that $x\cdot[\mathbf{r}(x),\mathbf{r}(y)] = [\mathbf{d}(x),\mathbf{d}(y)]\cdot y$. Thus,

$$h\mathbf{d}(x)*(x\cdot[\mathbf{r}(x),\mathbf{r}(y)]) = h\mathbf{d}(x)*([\mathbf{d}(x),\mathbf{d}(y)]\cdot y).$$

So ${}_{h\mathbf{d}(x)}|x \cdot [h_1, h_1^\star] = [h\mathbf{d}(x), h\mathbf{d}(y)] \cdot {}_{h\mathbf{d}(y)}|y$. Hence,

$$\begin{aligned}
&(h * x) \cdot [h_1, h_1^\star] \\
&= [h, h\mathbf{d}(x)] \cdot {}_{h\mathbf{d}(x)}|x \cdot [h_1, h_1^\star] \\
&= [h, h\mathbf{d}(x)] \cdot [h\mathbf{d}(x), h\mathbf{d}(y)] \cdot {}_{h\mathbf{d}(y)}|y \\
&= [h, h\mathbf{d}(y)] \cdot {}_{h\mathbf{d}(y)}|y \qquad \big(h \mathcal{R}\ h\mathbf{d}(x)\ \mathcal{R}\ h\mathbf{d}(y),\ \text{by (P1)}\big) \\
&= h * y,
\end{aligned}$$

which implies that $h_1^\star = \mathbf{r}(h * y)$, so that

$$\begin{aligned}
(h * x) \cdot [h_1, h_1^\star] &= (h * x) \cdot [\mathbf{r}(h * x), \mathbf{r}(h * y)] \\
&= h * y \\
&= [h, h] \cdot (h * y) \\
&= [\mathbf{d}(h * x), \mathbf{d}(h * y)] \cdot (h * y).
\end{aligned}$$

Consequently, $h * x \,\rho\, h * y$. □

Lemma 10.15. *If $x \,\rho\, x'$, $y \,\rho\, y'$ and $h \in S(\mathbf{r}(x), \mathbf{d}(y))$, then $(x \otimes y)_h \,\rho\, (x' \otimes y')_h$.*

Proof. Since $x \,\rho\, x'$ and $y \,\rho\, y'$, we have that $\mathbf{r}(x) \,\mathcal{L}\, \mathbf{r}(x')$ and $\mathbf{d}(y) \,\mathcal{R}\, \mathbf{d}(y')$, which gives from Lemma 1.28 that $S(\mathbf{r}(x), \mathbf{d}(y)) = S(\mathbf{r}(x'), \mathbf{d}(y'))$. Thus, $(x' \otimes y')_h$ is well-defined.

In view of Lemma 10.14, we have that $x \diamond h \,\rho\, x' \diamond h$. So

$$\mathbf{d}((x' \otimes y')_h) = \mathbf{d}(x' \diamond h) \,\mathcal{R}\, \mathbf{d}(x \diamond h) = \mathbf{d}((x \otimes y)_h),$$

$$\mathbf{r}(x' \diamond h) \,\mathcal{L}\, \mathbf{r}(x \diamond h)$$

and

$$(x' \diamond h) \cdot [\mathbf{r}(x' \diamond h), \mathbf{r}(x \diamond h)] = [\mathbf{d}(x' \diamond h), \mathbf{d}(x \diamond h)] \cdot (x \diamond h),$$

that is,

$$(x' \diamond h) \cdot [h, h] = [\mathbf{d}(x' \diamond h), \mathbf{d}(x \diamond h)] \cdot (x \diamond h),$$

so, $x' \diamond h = [\mathbf{d}(x' \diamond h), \mathbf{d}(x \diamond h)] \cdot (x \diamond h)$.

Dually,

$$\mathbf{r}((x' \otimes y')_h) = \mathbf{r}(h * y') \,\mathcal{L}\, \mathbf{r}(h * y) = \mathbf{d}((x \otimes y)_h)$$

and $h * y' = (h * y) \cdot [\mathbf{r}(h * y), \mathbf{r}(h * y')]$. Now, we have that

$$\begin{aligned}(x' \otimes y')_h &= (x' \diamond h) \cdot (h * y') \\ &= [\mathbf{d}(x' \diamond h), \mathbf{d}(x \diamond h)] \cdot (x \diamond h) \cdot (h * y) \cdot [\mathbf{r}(h * y), \mathbf{r}(h * y')] \\ &= [\mathbf{d}(x' \diamond h), \mathbf{d}(x \diamond h)] \cdot (x \otimes y)_h \cdot [\mathbf{r}(h * y), \mathbf{r}(h * y')].\end{aligned}$$

Thus,

$$\begin{aligned}&[\mathbf{d}(x \diamond h), \mathbf{d}(x' \diamond h)] \cdot (x' \otimes y')_h \\ &= [\mathbf{d}(x \diamond h), \mathbf{d}(x' \diamond h)] \cdot [\mathbf{d}(x' \diamond h), \mathbf{d}(x \diamond h)] \cdot (x \otimes y)_h \cdot [\mathbf{r}(h * y), \mathbf{r}(h * y')] \\ &= [\mathbf{d}(x \diamond h), \mathbf{d}(x \diamond h)] \cdot (x \otimes y)_h \cdot [\mathbf{r}(h * y), \mathbf{r}(h * y')] && \big(\text{by (P1)}\big) \\ &= (x \otimes y)_h \cdot [\mathbf{r}(h * y), \mathbf{r}(h * y')],\end{aligned}$$

together with $\mathbf{d}(x' \diamond h) \; \mathcal{R} \; \mathbf{d}(x \diamond h)$ and $\mathbf{r}(h * y') \; \mathcal{L} \; \mathbf{r}(h * y)$, we have that $(x \otimes y)_h \; \rho \; (x' \otimes y')_h$. □

Lemma 10.16. *If $x, y \in P$ and $h, h' \in S(\mathbf{r}(x), \mathbf{d}(y))$, then $(x \otimes y)_h = (x \otimes y)_{h'}$.*

Proof. Since $h, h' \in S(\mathbf{r}(x), \mathbf{d}(y))$, we can set

$$h_1 = \mathbf{d}(x \diamond h) = \mathbf{d}((x \otimes y)_h), \qquad h_2 = \mathbf{r}(h * y) = \mathbf{r}((x \otimes y)_h),$$

$$h_1' = \mathbf{d}(x \diamond h') = \mathbf{d}((x \otimes y)_{h'}), \qquad h_2' = \mathbf{r}(h' * y) = \mathbf{r}((x \otimes y)_{h'}).$$

Suppose that $h \; \mathcal{R} \; h'$. Then $h\mathbf{d}(y) \; \mathcal{R} \; h \; \mathcal{R} \; h' \; \mathcal{R} \; h'\mathbf{d}(y)$. But $h, h' \in S(\mathbf{r}(x), \mathbf{d}(y))$, so that $h \prec h'$ and $h' \prec h$, which imply that $h\mathbf{d}(y) \, \omega^l \, h'\mathbf{d}(y)$ and $h'\mathbf{d}(y) \, \omega^l \, h\mathbf{d}(y)$. Thus $h\mathbf{d}(y) \; \mathcal{L} \; h'\mathbf{d}(y)$. Hence, $h\mathbf{d}(y) = h'\mathbf{d}(y)$, and so ${}_{h\mathbf{d}(y)}|y = {}_{h'\mathbf{d}(y)}|y$, which implies that $h_2 = h_2'$. In addition,

$$\begin{aligned}h' * y &= [h', h'\mathbf{d}(y)] \cdot {}_{h'\mathbf{d}(y)}|y \\ &= [h', h] \cdot [h, h'\mathbf{d}(y)] \cdot {}_{h'\mathbf{d}(y)}|y && \big(h \; \mathcal{R} \; h' \; \mathcal{R} \; h'\mathbf{d}(y), \;\; \text{by (P1)}\big) \\ &= [h', h] \cdot [h, h\mathbf{d}(y)] \cdot {}_{h\mathbf{d}(y)}|y && \big(h'\mathbf{d}(y) = h\mathbf{d}(y)\big) \\ &= [h', h] \cdot (h * y).\end{aligned}$$

Since $h, h' \in S(\mathbf{r}(x), \mathbf{d}(y))$, we have that $h, h' \, \omega^l \, \mathbf{r}(x)$, and so $\mathbf{r}(x)h \, \mathcal{L} \, h$ and

$\mathbf{r}(x)h' \mathcal{L} h'$. As $h \mathcal{R} h'$, by (B32)°, we have that

$$(\mathbf{r}(x)h)(\mathbf{r}(x)h') = \mathbf{r}(x)(hh') = \mathbf{r}(x)h' \text{ and } (\mathbf{r}(x)h')(\mathbf{r}(x)h) = \mathbf{r}(x)h,$$

that is, $\mathbf{r}(x)h \mathcal{R} \mathbf{r}(x)h'$. Now, we obtain a row-singular U-square $\begin{pmatrix} h & h' \\ \mathbf{r}(x)h & \mathbf{r}(x)h' \end{pmatrix}$. By the comments succeeding Definition 10.1, we have

$$[\mathbf{r}(x)h, h] \cdot [h, h'] = [\mathbf{r}(x)h, \mathbf{r}(x)h'] \cdot [\mathbf{r}(x)h', h'],$$

and so

$$\begin{aligned}
(x \diamond h) \cdot [h, h'] &= x|_{\mathbf{r}(x)h} \cdot [\mathbf{r}(x)h, h] \cdot [h, h'] \\
&= x|_{\mathbf{r}(x)h} \cdot [\mathbf{r}(x)h, \mathbf{r}(x)h'] \cdot [\mathbf{r}(x)h', h'] \\
&= x|_{\mathbf{r}(x)h'} \cdot [\mathbf{r}(x)h', h'] \qquad \big(\mathbf{r}(x)h \ \mathcal{R} \ \mathbf{r}(x)h', \text{ by } (P4)^\circ\big) \\
&= x \diamond h'.
\end{aligned}$$

Thus, $\mathbf{d}(x \diamond h) = \mathbf{d}(x \diamond h')$, that is, $h_1' = h_1$. So

$$\begin{aligned}
(x \otimes y)_{h'} &= (x \diamond h') \cdot (h' * y) \\
&= (x \diamond h) \cdot [h, h'] \cdot [h', h] \cdot (h * y) \\
&= (x \diamond h) \cdot [h, h] \cdot (h * y) \qquad \big(h \ \mathcal{R} \ h', \text{ by (P1)}\big) \\
&= (x \diamond h) \cdot (h * y) \\
&= (x \otimes y)_h.
\end{aligned}$$

Dually, if $h \ \mathcal{L} \ h'$ we can show that $(x \otimes y)_h = (x \otimes y)_{h'}$. By the comment succeeding Lemma 1.35, if $h, h' \in S(\mathbf{r}(x), \mathbf{d}(y))$, there exists $k \in S(\mathbf{r}(x), \mathbf{d}(y))$ such that $h \ \mathcal{R} \ k \ \mathcal{L} \ h'$. Thus the lemma holds in all cases. □

Let $P\mathbf{S} = P/\rho$. For $x, y \in P$, $h \in S(\mathbf{r}(x), \mathbf{d}(y))$, we define

$$\bar{x} \odot \bar{y} = \overline{(x \otimes y)_h},$$

where $\bar{x}$ denotes the ρ-class of x in P.

Lemma 10.17. *The set $P\mathbf{S} = P/\rho$ forms a semigroup under $\odot$ defined above.*

Proof. Let $x, x', y, y' \in P$ with $x \,\rho\, x'$ and $y \,\rho\, y'$. If $h, h' \in S(\mathbf{r}(x), \mathbf{d}(y)) = S(\mathbf{r}(x'), \mathbf{d}(y'))$, then by Lemma 10.15 and Lemma 10.16, we have that

$$(x \otimes y)_h \,\rho\, (x' \otimes y')_h = (x' \otimes y')_{h'}.$$

Thus, $\overline{(x \otimes y)_h} = \overline{(x' \otimes y')_{h'}}$, and so the product is well-defined.

In order to show that the operation $\odot$ is associative, we assume that $x, y, z \in P$, $h_1 \in S(\mathbf{r}(x), \mathbf{d}(y))$ and $h_2 \in S(\mathbf{r}(y), \mathbf{d}(z))$. By Lemma 10.13, there exist $h \in S(\mathbf{r}(x), \mathbf{d}((y \otimes z)_{h_2}))$ and $h' \in S(\mathbf{r}((x \otimes y)_{h_1}), \mathbf{d}(z))$ such that $((x \otimes y)_{h_1} \otimes z)_{h'} = (x \otimes (y \otimes z)_{h_2})_h$. Thus

$$\begin{aligned}(\bar{x} \odot \bar{y}) \odot \bar{z} &= \overline{(x \otimes y)_{h_1}} \odot \bar{z} = \overline{((x \otimes y)_{h_1} \otimes z)_{h'}} \\ &= \overline{(x \otimes (y \otimes z)_{h_2})_h} = \bar{x} \odot \overline{(y \otimes z)_{h_2}} = \bar{x} \odot (\bar{y} \odot \bar{z}).\end{aligned}$$

Hence, $P\mathbf{S}$ is a semigroup. □

Lemma 10.18. *If $e, h \in U$ are such that $h \;\omega^r\; e$, then*

$$h * 1_e \;\rho\; 1_{he} \;\rho\; (1_h \otimes 1_e)_h,$$

so that $\overline{1_h} \odot \overline{1_e} = \overline{1_{he}}$.

Proof. Suppose that $e, h \in U$ with $h \,\omega^r\, e$. Then $h \,\mathcal{R}\, he$ and

$$\begin{aligned}[he, h] \cdot (h * 1_e) &= [he, h] \cdot (h * [e, e]) \\ &= [he, h] \cdot [h, he] \cdot [he, (he)^\star] \\ &= [he, he] \cdot [he, (he)^\star] \\ &= 1_{he} \cdot [he, (he)^\star].\end{aligned}$$

Certainly, $\mathbf{r}(h * 1_e) = (he)^\star \,\mathcal{L}\, he$. Thus, $h * 1_e \;\rho\; 1_{he}$.

According to Lemma 1.29, we have that $eh = h \in S(h, e)$, and so $(1_h \otimes 1_e)_h$ is well-defined. Then

$$\begin{aligned}(1_h \otimes 1_e)_h &= ([h, h] \otimes [e, e])_h \\ &= ([h, h] \diamond h) \cdot (h * [e, e]) \\ &= [h^+, h] \cdot [h, he] \cdot [he, (he)^\star] \\ &= [h^+, he] \cdot [he, (he)^\star] \qquad \Big(h^+ \,\mathcal{R}\, h \,\mathcal{R}\, he, \text{ by (P1)}\Big).\end{aligned}$$

Further,

$$[h^+, he] \cdot 1_{he} = [h^+, he] \cdot [he, he]$$
$$= [h^+, he] \cdot [he, (he)^\star] \cdot [(he)^\star, he]$$
$$= (1_h \otimes 1_e)_h \cdot [(he)^\star, he].$$

Hence, $1_{he} \ \rho \ (1_h \otimes 1_e)_h$. □

Dually, we have:

Lemma 10.19. *If $e, h \in U$ are such that $h \ \omega^l \ e$, then*

$$1_e \diamond h \ \rho \ 1_{eh} \ \rho \ (1_e \otimes 1_h)_h$$

so that $\overline{1_e} \odot \overline{1_h} = \overline{1_{eh}}$.

Lemma 10.20. *If P is a weakly regular category over U and $x, y \in P$ with $x \cdot y$ defined, then $\bar{x} \odot \bar{y} = \overline{x \cdot y}$. Moreover, if $e \in U$, then $\overline{1_e} \in E(P\mathbf{S})$.*

Proof. Suppose that $x, y \in P$ are such that $x \cdot y$ is defined. Then $\mathbf{r}(x) = \mathbf{d}(y)$, from which it follows that $S(\mathbf{r}(x), \mathbf{d}(y)) = \{\mathbf{r}(x)\}$. Thus,

$$\bar{x} \odot \bar{y} = \overline{(x \otimes y)_{\mathbf{r}(x)}} = \overline{(x \diamond \mathbf{r}(x)) \cdot (\mathbf{r}(x) * y)} = \overline{x|_{\mathbf{r}(x)} \cdot {}_{\mathbf{r}(x)}|y}.$$

Let $h = \mathbf{d}(y) = \mathbf{r}(x)$. From (P2) and (P2)°, we have $\mathbf{d}(x|_h) \, \mathcal{R} \, \mathbf{d}(x)$, $\mathbf{r}({}_h|y) \, \mathcal{L} \, \mathbf{r}(y)$ and

$${}_h|y \cdot [\mathbf{r}({}_h|y), \mathbf{r}(y)] = y,$$

so

$$x \cdot {}_h|y \cdot [\mathbf{r}({}_h|y), \mathbf{r}(y)] = x \cdot y.$$

But

$$x = [\mathbf{d}(x), \mathbf{d}(x|_h)] \cdot x|_h,$$

so

$$[\mathbf{d}(x), \mathbf{d}(x|_h)] \cdot x|_h \cdot {}_h|y \cdot [\mathbf{r}({}_h|y), \mathbf{r}(y)] = x \cdot y,$$

hence,

$$[\mathbf{d}(x|_h), \mathbf{d}(x)] \cdot [\mathbf{d}(x), \mathbf{d}(x|_h)] \cdot x|_h \cdot {}_h|y \cdot [\mathbf{r}({}_h|y), \mathbf{r}(y)] = [\mathbf{d}(x|_h), \mathbf{d}(x)] \cdot (x \cdot y),$$

that is,

$$[\mathbf{d}(x|_h), \mathbf{d}(x|_h)] \cdot x|_h \cdot {}_h|y \cdot [\mathbf{r}({}_h|y), \mathbf{r}(y)] = [\mathbf{d}(x|_h), \mathbf{d}(x)] \cdot (x \cdot y),$$

or equivalently,

$$x|_h \cdot {}_h|y \cdot [\mathbf{r}({}_h|y), \mathbf{r}(y)] = [\mathbf{d}(x|_h), \mathbf{d}(x)] \cdot (x \cdot y),$$

as $h = \mathbf{d}(y) = \mathbf{r}(x)$, we obtain that

$$(x|_{\mathbf{r}(x)} \cdot {}_{\mathbf{r}(x)}|y) \cdot [\mathbf{r}({}_{\mathbf{r}(x)}|y), \mathbf{r}(y)] = [\mathbf{d}(x|_{\mathbf{r}(x)}), \mathbf{d}(x)] \cdot (x \cdot y).$$

Hence, $x|_{\mathbf{r}(x)} \cdot {}_{\mathbf{r}(x)}|y \ \rho \ x \cdot y$, and so $\bar{x} \odot \bar{y} = \overline{x \cdot y}$.

Since $e \in U$ and $1_e \cdot 1_e = 1_e$, it follows that $\overline{1_e} \in E(P\mathbf{S})$. □

Lemma 10.21. *Let $x \in P$ and $e \in U$ be such that $e \ \omega^l \ \mathbf{d}(x)$. Then $\overline{{}_e|x} = \overline{1_e} \odot \bar{x}$.*

Proof. As $e \ \omega^l \ \mathbf{d}(x)$, we have that $e \ \mathcal{L} \ \mathbf{d}(x)e \ \omega \ \mathbf{d}(x)$, and so

$$\begin{aligned}
\overline{1_e} \odot \bar{x} &= \overline{[e, e]} \odot \bar{x} \\
&= \overline{[e, \mathbf{d}(x)e]} \odot \bar{x} && \big(e \ \mathcal{L} \ \mathbf{d}(x)e, \text{ Lemma 10.2}\big) \\
&= \overline{[e, \mathbf{d}(x)e]} \odot \overline{1_{\mathbf{d}(x)e}} \odot \bar{x}
\end{aligned}$$

$$\begin{aligned}
&= \overline{[e, \mathbf{d}(x)e]} \odot \overline{(1_{\mathbf{d}(x)e} \otimes x)_{\mathbf{d}(x)e}} \\
&\qquad \big(\mathbf{d}(x)e \ \omega \ \mathbf{d}(x), \text{ by Lemma 1.29, } \mathbf{d}(x)e \in S(\mathbf{d}(x)e, \mathbf{d}(x))\big) \\
&= \overline{[e, \mathbf{d}(x)e]} \odot \overline{(1_{\mathbf{d}(x)e} \diamond \mathbf{d}(x)e) \cdot (\mathbf{d}(x)e * x)} \\
&= \overline{[e, \mathbf{d}(x)e]} \odot \overline{1_{\mathbf{d}(x)e} \diamond \mathbf{d}(x)e} \odot \overline{\mathbf{d}(x)e * x} && \big(\text{Lemma 10.20}\big) \\
&= \overline{[e, \mathbf{d}(x)]e} \odot \overline{1_{\mathbf{d}(x)e}} \odot \overline{\mathbf{d}(x)e * x} && \big(\text{Lemma 10.19}\big) \\
&= \overline{[e, \mathbf{d}(x)e]} \odot \overline{\mathbf{d}(x)e * x} \\
&= \overline{[e, \mathbf{d}(x)e]} \odot \overline{[\mathbf{d}(x)e, (\mathbf{d}(x)e)\mathbf{d}(x)] \cdot {}_{(\mathbf{d}(x)e)\mathbf{d}(x)}|x} \\
&= \overline{[e, \mathbf{d}(x)e]} \odot \overline{[\mathbf{d}(x)e, \mathbf{d}(x)e] \cdot {}_{\mathbf{d}(x)e}|x} && \big(\mathbf{d}(x)e \ \omega \ \mathbf{d}(x)\big) \\
&= \overline{[e, \mathbf{d}(x)e]} \odot \overline{{}_{\mathbf{d}(x)e}|x} \\
&= \overline{[e, \mathbf{d}(x)e] \cdot {}_{\mathbf{d}(x)e}|x} && \big(\text{Lemma 10.20}\big) \\
&= \overline{{}_e|x} && \big(e \ \mathcal{L} \ \mathbf{d}(x)e, \text{ by (P4)}\big).
\end{aligned}$$

□

Lemma 10.22. *Let $x, y \in P$ and $e \in U$. If $\bar{x} = \overline{1_e} \odot \bar{y}$, then $x \leq_r h * y$ and $h * y \leq_r y$ in P, where $h \in S(e, \mathbf{d}(y))$.*

Proof. We note that

$$\begin{aligned}
\bar{x} &= \overline{1_e} \odot \bar{y} \\
&= \overline{(1_e \otimes y)_h} && \big(h \in S(e, \mathbf{d}(y))\big) \\
&= \overline{(1_e \diamond h) \cdot (h * y)} \\
&= \overline{1_e \diamond h} \odot \overline{h * y} && \big(\text{Lemma 10.20}\big) \\
&= \overline{1_{eh}} \odot \overline{h * y} && \big(\text{Lemma 10.19}\big) \\
&= \overline{{}_{eh}|(h * y)} && \big(eh \ \mathcal{L} \ h = \mathbf{d}(h * y), \ \text{Lemma 10.21}\big),
\end{aligned}$$

and so $x \ \rho \ {}_{eh}|(h * y)$, that is, $x \leq_r h * y$. Notice that

$$\begin{aligned}
\overline{h * y} &= \overline{[h, h\mathbf{d}(y)] \cdot {}_{h\mathbf{d}(y)}|y} \\
&= \overline{[h, h\mathbf{d}(y)]} \odot \overline{{}_{h\mathbf{d}(y)}|y} && \big(\text{Lemma 10.20}\big) \\
&= \overline{[h\mathbf{d}(y), h\mathbf{d}(y)]} \odot \overline{{}_{h\mathbf{d}(y)}|y} && \big(h \ \mathcal{R} \ h\mathbf{d}(y), \text{Lemma 10.2}\big) \\
&= \overline{{}_{h\mathbf{d}(y)}|y},
\end{aligned}$$

and so $h * y \ \rho \ {}_{h\mathbf{d}(y)}|y$ so that $h * y \leq_r y$. □

Lemma 10.23. *If P is a weakly regular category over U and $x \in P$, then $\overline{1_{\mathbf{d}(x)}} \ \widetilde{\mathcal{R}}_{\overline{U}} \ \bar{x} \ \widetilde{\mathcal{L}}_{\overline{U}} \ \overline{1_{\mathbf{r}(x)}}$, where $\overline{U} = \{\overline{1_e} : e \in U\}$.*

Proof. In view of Lemma 10.20, $\overline{U}$ is a subset of idempotents of $P\mathbf{S}$. Now, we claim that $\bar{x} \ \widetilde{\mathcal{R}}_{\overline{U}} \ \overline{1_{\mathbf{d}(x)}}$, and dually, we have that $\bar{x} \ \widetilde{\mathcal{L}}_{\overline{U}} \ \overline{1_{\mathbf{r}(x)}}$. Clearly, by Lemma 10.20, $\overline{1_{\mathbf{d}(x)}} \odot \bar{x} = \bar{x}$. Assume that $g \in U$ and $\overline{1_g} \odot \bar{x} = \bar{x}$. Then $(1_g \otimes x)_h \ \rho \ x$, where $h \in S(g, \mathbf{d}(x))$, which gives that $(1_g \diamond h) \cdot (h * x) \ \rho \ x$, that is,

$$[(gh)^+, gh] \cdot [gh, h] \cdot (h * x) \ \rho \ x,$$

which implies that $gh \ \mathcal{R} \ (gh)^+ \ \mathcal{R} \ \mathbf{d}(x)$. So

$$\begin{aligned}
\overline{1_g} \odot \overline{1_{\mathbf{d}(x)}} &= \overline{(1_g \otimes 1_{\mathbf{d}(x)})_h} && \big(h \in S(g, \mathbf{d}(x))\big) \\
&= \overline{(1_g \diamond h) \cdot (h * 1_{\mathbf{d}(x)})} \\
&= \overline{1_g \diamond h} \odot \overline{h * 1_{\mathbf{d}(x)}} && \big(\text{Lemma 10.20}\big)
\end{aligned}$$

$$
\begin{aligned}
&= \overline{(1_g \otimes 1_h)_h} \odot \overline{(1_h \otimes 1_{\mathbf{d}(x)})_h} && \big(\text{Lemma 10.18, Lemma 10.19}\big)\\
&= (\overline{1_g} \odot \overline{1_h}) \odot (\overline{1_h} \odot \overline{1_{\mathbf{d}(x)}})\\
&= \overline{1_g} \odot \overline{1_h} \odot \overline{1_{\mathbf{d}(x)}}\\
&= (\overline{1_g} \odot \overline{1_h}) \odot \overline{1_{\mathbf{d}(x)}}\\
&= \overline{(1_g \otimes 1_h)_h} \odot \overline{1_{\mathbf{d}(x)}}\\
&= \overline{1_{gh}} \odot \overline{1_{\mathbf{d}(x)}} && \big(h\ \omega^l\ g,\ \text{Lemma 10.19}\big)\\
&= \overline{(1_{gh} \otimes 1_{\mathbf{d}(x)})_{gh}}\\
&\qquad \big(gh\ \mathcal{R}\ \mathbf{d}(x), \text{so by Lemma 1.29}, gh \in S(gh, \mathbf{d}(x))\big)\\
&= \overline{1_{(gh)\mathbf{d}(x)}} && \big(\text{Lemma 10.18}\big)\\
&= \overline{1_{\mathbf{d}(x)}} && \big(gh\ \mathcal{R}\ \mathbf{d}(x)\big).
\end{aligned}
$$

□

Let $\overline{U} = \{\overline{1_e} : e \in U\}$. For any $\overline{1_e}$, $\overline{1_f} \in \overline{U}$, we have

$$\overline{1_e}\ \omega^r\ \overline{1_f} \text{ if and only if } \overline{1_f} \odot \overline{1_e} = \overline{1_e},$$

and dually,

$$\overline{1_e}\ \omega^l\ \overline{1_f} \text{ if and only if } \overline{1_e} \odot \overline{1_f} = \overline{1_e}.$$

Lemma 10.24. *The map $\chi : U \to \overline{U} : e \mapsto \overline{1_e}$ is a regular isomorphism.*

Proof. Suppose that $(e, f) \in D_U = (\omega^r \cup \omega^l) \cup (\omega^r \cup \omega^l)^{-1}$. If $e\ \omega^r\ f$, then by Lemma 10.18, $\overline{1_e} \odot \overline{1_f} = \overline{1_{ef}}$. Again, by Lemma 1.29, $ef \in S(f, e)$. Thus,

$$
\begin{aligned}
&\overline{1_f} \odot \overline{1_e}\\
&= \overline{(1_f \otimes 1_e)_{ef}}\\
&= \overline{(1_f \diamond ef) \cdot (ef * 1_e)}\\
&= \overline{[f,f]|_{fef} \cdot [fef, ef] \cdot [ef, efe] \cdot {}_{efe}|[e,e]}\\
&= \overline{[f,f]|_{ef} \cdot [ef, ef] \cdot [ef, e] \cdot {}_{e}|[e,e]} && \big(e\ \mathcal{R}\ ef\ \omega\ f\big)\\
&= \overline{[(ef)^+, ef] \cdot [ef, ef] \cdot [ef, e] \cdot [e, e^\star]}\\
&= \overline{[(ef)^+, e] \cdot [e, e^\star]} && \big((ef)^+\ \mathcal{R}\ ef\ \mathcal{R}\ e, \text{by (P1)}\big).
\end{aligned}
$$

Note that $(ef)^+ \mathrel{\mathcal{R}} ef \mathrel{\mathcal{R}} e$, $e^\star \mathrel{\mathcal{L}} e$ and using the above we have

$$\begin{aligned}[(ef)^+, e] \cdot 1_e &= [(ef)^+, e] \cdot [e, e] \\ &= [(ef)^+, e] \cdot [e, e^\star] \cdot [e^\star, e] \\ &= (1_f \otimes 1_e)_{ef} \cdot [e^\star, e].\end{aligned}$$

We have that $\overline{1_f} \odot \overline{1_e} = \overline{1_e} = \overline{1_{fe}}$.

Dually, if $e \,\omega^l\, f$, then $\overline{1_e} \odot \overline{1_f} = \overline{1_{ef}}$ and $\overline{1_f} \odot \overline{1_e} = \overline{1_{fe}}$. Thus χ is a morphism.

To show that χ is a regular morphism, we assume that $h \in S(e, f)$. Then

$$\begin{aligned}\overline{1_e} \odot \overline{1_f} &= \overline{(1_e \otimes 1_f)_h} \\ &= \overline{(1_e \diamond h) \cdot (h * 1_f)} \\ &= \overline{1_e \diamond h} \odot \overline{h * 1_f} && \big(\text{Lemma 10.20}\big) \\ &= \overline{1_{eh}} \odot \overline{1_{hf}} && \big(\text{Lemma 10.18 and Lemma 10.19}\big) \\ &= (\overline{1_e} \odot \overline{1_h}) \odot (\overline{1_h} \odot \overline{1_f}) && \big(\text{Lemma 10.18 and Lemma 10.19}\big) \\ &= \overline{1_e} \odot \overline{1_h} \odot \overline{1_f}.\end{aligned}$$

Thus, by Lemma 10.23 and Lemma 2.12, we have $\overline{1_h} \in S_1(\overline{1_e}, \overline{1_f}) \subseteq S(\overline{1_e}, \overline{1_f})$, and so χ is a regular morphism.

Since no two identities of P are ρ-equivalent by Lemma 10.3, χ is injective. Clearly, χ is surjective. Thus, χ is a bijective regular morphism, and so by Lemma 1.26, we succeed in obtaining that χ is a regular isomorphism. $\square$

The next lemma is useful for Lemma 10.27.

Lemma 10.25. *Let $e, f \in U$ and $h \in S(e, f)$. Then $\overline{1_{eh}} \mathrel{\mathcal{R}} \overline{1_e} \odot \overline{1_f} \mathrel{\mathcal{L}} \overline{1_{hf}}$ in $\langle \overline{U} \rangle$, where $\langle \overline{U} \rangle$ is the semigroup generated by $\overline{U}$.*

Proof. We first show that $\overline{1_{eh}} \mathrel{\mathcal{R}} \overline{1_e} \odot \overline{1_f}$. Notice that

$$\begin{aligned}\overline{1_e} \odot \overline{1_f} &= \overline{(1_e \otimes 1_f)_h} && \big(h \in S(e, f)\big) \\ &= \overline{(1_e \diamond h) \cdot (h * 1_f)} \\ &= \overline{1_e \diamond h} \odot \overline{h * 1_f} && \big(\text{Lemma 10.20}\big) \\ &= \overline{1_{eh}} \odot \overline{1_{hf}} && \big(\text{Lemma 10.18 and Lemma 10.19}\big).\end{aligned}$$

In addition, we have

$$\overline{1_{eh}} \odot (\overline{1_e} \odot \overline{1_f}) = \overline{1_{eh}} \odot \overline{1_{eh}} \odot \overline{1_{hf}} = \overline{1_{eh}} \odot \overline{1_{hf}} = \overline{1_e} \odot \overline{1_f}$$

and

$$\begin{aligned}
(\overline{1_e} \odot \overline{1_f}) \odot \overline{1_h}^2 &= \overline{1_{eh}} \odot \overline{1_{hf}} \odot \overline{1_h} \odot \overline{1_h} \\
&= \overline{[eh, h] \cdot [h, hf]} \odot \overline{[hf, h]} \odot \overline{[h, eh]} \\
&\qquad\qquad\qquad \big(eh \mathrel{\mathcal{L}} h \mathrel{\mathcal{R}} hf,\ \text{Lemma 10.2}\big) \\
&= \overline{[eh, h] \cdot [h, hf] \cdot [hf, h] \cdot [h, eh]} \qquad \big(\text{by Lemma 10.20}\big) \\
&= \overline{[eh, h] \cdot [h, h] \cdot [h, eh]} \qquad \big(\text{by (P1)}\big) \\
&= \overline{[eh, eh]} \qquad \big(\text{by (P1)}\big) \\
&= \overline{1_{eh}}.
\end{aligned}$$

Thus, $\overline{1_{eh}} \mathrel{\mathcal{R}} \overline{1_e} \odot \overline{1_f}$ in $\langle \overline{U} \rangle$. Dually we have that $\overline{1_e} \odot \overline{1_f} \mathrel{\mathcal{L}} \overline{1_h f}$ in $\langle \overline{U} \rangle$. □

We pause here to make a short comment on Lemma 10.25. Suppose that $\bar{x} = \overline{1_e} \odot \overline{1_f}$. As

$$\begin{aligned}
\overline{1_e} \odot \overline{1_f} &= \overline{1_{eh}} \odot \overline{1_{hf}} \\
&= \overline{[eh, h]} \odot \overline{[h, hf]} \qquad \big(eh \mathrel{\mathcal{L}} h \mathrel{\mathcal{R}} hf, \text{Lemma 10.2}\big) \\
&= \overline{[eh, h] \cdot [h, hf]} \qquad \big(\text{Lemma 10.20}\big),
\end{aligned}$$

we have that $\mathbf{d}(x) \mathrel{\mathcal{R}} eh$. By Lemma 10.24, we have that $\overline{1_{\mathbf{d}(x)}} \mathrel{\mathcal{R}} \overline{1_{eh}}$. Since $\overline{1_{eh}} \mathrel{\mathcal{R}} \overline{1_e} \odot \overline{1_f}$ in $\langle \overline{U} \rangle$, we obtain that $\overline{1_{\mathbf{d}(x)}} \mathrel{\mathcal{R}} \overline{1_e} \odot \overline{1_f}$ in $\langle \overline{U} \rangle$, that is, $\overline{1_{\mathbf{d}(x)}} \mathrel{\mathcal{R}} \bar{x}$ in $\langle \overline{U} \rangle$. By a dual argument, we have that $\overline{1_{\mathbf{r}(x)}} \mathrel{\mathcal{L}} \bar{x}$ in $\langle \overline{U} \rangle$.

Lemma 10.26. *Let P be a weakly regular category over U. Suppose that for $x \in E(P\mathbf{S})$, we have $\overline{1_{\mathbf{d}(x)}} \mathrel{\mathcal{R}^*} \bar{x} \mathrel{\mathcal{L}^*} \overline{1_{\mathbf{r}(x)}}$ in $P\mathbf{S}$. Then $x \in \overline{U}$.*

Proof. Suppose that $x \in P$ and $\bar{x} \odot \bar{x} = \bar{x}$. As $\bar{x} \mathrel{\mathcal{L}^*} \overline{1_{\mathbf{r}(x)}}$, we have that $\overline{1_{\mathbf{r}(x)}} \odot \bar{x} = \overline{1_{\mathbf{r}(x)}}$. Let $h \in S(\mathbf{r}(x), \mathbf{d}(x))$. Then, we have

$$\begin{aligned}
&\overline{1_{\mathbf{r}(x)}} \odot \bar{x} = \overline{1_{\mathbf{r}(x)}} \\
\Rightarrow\ &\overline{(1_{\mathbf{r}(x)} \otimes x)_h} = \overline{1_{\mathbf{r}(x)}} \\
\Rightarrow\ &\overline{(1_{\mathbf{r}(x)} \diamond h) \cdot (h * x)} = \overline{1_{\mathbf{r}(x)}}
\end{aligned}$$

$$\Rightarrow \overline{1_{\mathbf{r}(x)}|_{\mathbf{r}(x)h} \cdot [\mathbf{r}(x)h, h] \cdot (h * x)} = \overline{1_{\mathbf{r}(x)}}$$
$$\Rightarrow \overline{[(\mathbf{r}(x)h)^+, \mathbf{r}(x)h] \cdot [\mathbf{r}(x)h, h] \cdot (h * x)} = \overline{1_{\mathbf{r}(x)}}$$
$$\Rightarrow \mathbf{r}(x)h \ \mathcal{R} \ (\mathbf{r}(x)h)^+ \ \mathcal{R} \ \mathbf{r}(x).$$

As $h \ \omega^l \ \mathbf{r}(x)$, we have that $\mathbf{r}(x)h \ \omega \ \mathbf{r}(x)$, and so we must have that $\mathbf{r}(x)h = \mathbf{r}(x)$. Also, we have $h\mathbf{r}(x) = h$, and so $h \ \mathcal{L} \ \mathbf{r}(x)$. By Lemma 10.24, we obtain that $\overline{1_h} \ \mathcal{L} \ \overline{1_{\mathbf{r}(x)}}$. Since $\bar{x} \ \mathcal{L}^* \ \overline{1_{\mathbf{r}(x)}}$, we succeed in obtaining that $\bar{x} \ \mathcal{L}^* \ \overline{1_h}$. Dually, we have that $\bar{x} \ \mathcal{R}^* \ \overline{1_h}$ so that $\bar{x} \ \mathcal{H}^* \ \overline{1_h}$. As $\bar{x}$ is an idempotent and each $\mathcal{H}^*$-class contains at most one idempotent, we must have $\bar{x} = \overline{1_h}$. □

Lemma 10.27. *The set $\overline{U} = \{\overline{1_e} : e \in U\}$ generates a regular subsemigroup $\langle \overline{U} \rangle$ of $P\mathbf{S}$ and satisfies $E(\langle \overline{U} \rangle) = \overline{U}$.*

Proof. To show that $E(\langle \overline{U} \rangle) = \overline{U}$, we first verify that for $\bar{x} \in \langle \overline{U} \rangle$, $\overline{1_{\mathbf{d}(x)}} \ \mathcal{R} \ \bar{x} \ \mathcal{L} \ \overline{1_{\mathbf{r}(x)}}$ by induction. Suppose that $\bar{x} = \overline{1_e} \odot \overline{1_f}$, where $e, f \in U$. Then by the comment succeeding Lemma 10.25, we have that $\overline{1_{\mathbf{d}(x)}} \ \mathcal{R} \ \bar{x}$ in $\langle \overline{U} \rangle$.

Now, we assume that if $\bar{x} = \overline{1_{e_1}} \odot \overline{1_{e_2}} \odot \cdots \odot \overline{1_{e_n}}$, then $\overline{1_{\mathbf{d}(x)}} \ \mathcal{R} \ \bar{x}$ in $\langle \overline{U} \ \rangle$. Let

$$\bar{y} = \overline{1_{e_1}} \odot \overline{1_{e_2}} \odot \cdots \odot \overline{1_{e_n}} \odot \overline{1_{e_{n+1}}}.$$

Put

$$\bar{z} = \overline{1_{e_2}} \odot \overline{1_{e_3}} \odot \cdots \odot \overline{1_{e_{n+1}}}.$$

Then by the hypothesis, $\overline{1_{\mathbf{d}(z)}} \ \mathcal{R} \ \bar{z}$ in $\langle \overline{U} \rangle$, and so there exists $\bar{t} \in \langle \overline{U} \rangle$ such that $\bar{z} \odot \bar{t} = \overline{1_{\mathbf{d}(z)}}$, so that

$$\bar{y} \odot \bar{t} = \overline{1_{e_1}} \odot \bar{z} \odot \bar{t} = \overline{1_{e_1}} \odot \overline{1_{\mathbf{d}(z)}}.$$

If $k \in S(e_1, \mathbf{d}(z))$, by Lemma 10.25, we have that $\overline{1_{e_1}} \odot \overline{1_{\mathbf{d}(z)}} \ \mathcal{R} \ \overline{1_{e_1k}}$ in $\langle \overline{U} \rangle$, that is, $\bar{y} \odot \bar{t} \ \mathcal{R} \ \overline{1_{e_1k}}$ in $\langle \overline{U} \rangle$. Thus, there exists $\bar{u} \in \langle \overline{U} \rangle$ such that

$$\bar{y} \odot \bar{t} \odot \bar{u} = \overline{1_{e_1k}},$$

where $\bar{t} \odot \bar{u} \in \langle \overline{U} \rangle$.

Observe that

$$\begin{aligned}\bar{y} &= \overline{1_{e_1}} \odot \bar{z} \\ &= \overline{(1_{e_1} \otimes z)_k} && \big(k \in S(e_1, \mathbf{d}(z))\big) \\ &= \overline{(1_{e_1} \diamond k) \cdot (k * z)} \\ &= \overline{1_{e_1} \diamond k} \odot \overline{k * z} && \big(\text{Lemma 10.20}\big) \\ &= \overline{1_{e_1k}} \odot \overline{k * z} && \big(\text{Lemma 10.19}\big),\end{aligned}$$

and so $\overline{1_{e_1k}} \odot \bar{y} = \bar{y}$, so that $\overline{1_{e_1k}} \mathrel{\mathcal{R}} \bar{y}$ in $\langle \overline{U} \rangle$. As $e_1k \mathrel{\mathcal{L}} k$, by Lemma 10.2, we have $\overline{1_{e_1k}} = \overline{[e_1k, k]}$, and so

$$\bar{y} = \overline{1_{e_1k}} \odot \overline{k * z} = \overline{[e_1k, k]} \odot \overline{k * z} = \overline{[e_1k, k] \cdot (k * z)}$$

so that $\mathbf{d}(y) \mathrel{\mathcal{R}} e_1k$, and so $\overline{1_{e_1k}} \mathrel{\mathcal{R}} \overline{1_{\mathbf{d}(y)}}$. Hence $\overline{1_{\mathbf{d}(y)}} \mathrel{\mathcal{R}} \bar{y}$ in $\langle \overline{U} \rangle$. Dually, we have that $\bar{y} \mathrel{\mathcal{L}} \overline{1_{\mathbf{r}(y)}}$ in $\langle \overline{U} \rangle$.

For any $\bar{x} \in E(\langle \overline{U} \rangle)$, we have that $\overline{1_{\mathbf{d}(x)}} \mathrel{\mathcal{R}} \bar{x} \mathrel{\mathcal{L}} \overline{1_{\mathbf{r}(x)}}$, and so by Lemma 10.26, we have $\bar{x} \in \overline{U}$. Together with $\overline{U} \subseteq E(\langle \overline{U} \rangle)$, we obtain that $E(\langle \overline{U} \rangle) = \overline{U}$. □

Lemma 10.28. *If P is a weakly regular category over U, then for any $\bar{x}, \bar{y} \in P\mathbf{S}$,*

(*i*) *$\bar{x} \mathrel{\widetilde{\mathcal{L}}_{\overline{U}}} \bar{y}$ if and only if $\mathbf{r}(x) \mathrel{\mathcal{L}} \mathbf{r}(y)$;*

(*ii*) *$\bar{x} \mathrel{\widetilde{\mathcal{R}}_{\overline{U}}} \bar{y}$ if and only if $\mathbf{d}(x) \mathrel{\mathcal{R}} \mathbf{d}(y)$.*

Proof. To prove (*i*), suppose that $\bar{x}, \bar{y} \in P\mathbf{S}$ and $x, y \in P$. Then

$$\begin{aligned}\bar{x} \mathrel{\widetilde{\mathcal{L}}_{\overline{U}}} \bar{y} &\Leftrightarrow \overline{1_{\mathbf{r}(x)}} \mathrel{\widetilde{\mathcal{L}}_{\overline{U}}} \overline{1_{\mathbf{r}(y)}} && \big(\text{by Lemma 10.23}\big) \\ &\Leftrightarrow \overline{1_{\mathbf{r}(x)}} \mathrel{\mathcal{L}} \overline{1_{\mathbf{r}(y)}} \\ &\Leftrightarrow \mathbf{r}(x) \mathrel{\mathcal{L}} \mathbf{r}(y) && \big(\text{by Lemma 10.24}\big).\end{aligned}$$

Similarly, we show that part (*ii*) holds. □

Lemma 10.29. *If P is a weakly regular category over U, then $P\mathbf{S}$ satisfies the Congruence Condition (C).*

Proof. Let $\bar{x}, \bar{y}, \bar{z} \in P\mathbf{S}$ be such that $\bar{x} \mathrel{\widetilde{\mathcal{L}}_{\overline{U}}} \bar{y}$. Then by Lemma 10.28, $\mathbf{r}(x) \mathrel{\mathcal{L}} \mathbf{r}(y)$. According to Lemma 1.28, we have that $S(\mathbf{r}(x), \mathbf{d}(z)) = S(\mathbf{r}(y), \mathbf{d}(z))$. Suppose that $h \in S(\mathbf{r}(x), \mathbf{d}(z))$. Then

$$\bar{x} \odot \bar{z} = \overline{(x \otimes z)_h} = \overline{(x \diamond h) \cdot (h * z)} \text{ and } \bar{y} \odot \bar{z} = \overline{(y \otimes z)_h} = \overline{(y \diamond h) \cdot (h * z)}.$$

As $\mathbf{r}((x \diamond h) \cdot (h * z)) = \mathbf{r}(h * z) = \mathbf{r}((y \diamond h) \cdot (h * z))$, it follows from Lemma 10.28 that $\bar{x} \odot \bar{z} \; \widetilde{\mathcal{L}}_{\overline{E}} \; \bar{y} \odot \bar{z}$, and so $\widetilde{\mathcal{L}}_{\overline{U}}$ is a right congruence on $P\mathbf{S}$. Dually, $\widetilde{\mathcal{R}}_{\overline{U}}$ is a left congruence on $P\mathbf{S}$. □

To sum up, we obtain the following result.

Theorem 10.30. *Let P be a weakly regular category over a regular biordered set U. Then $P\mathbf{S}$ is a weakly $\overline{U}$-regular semigroup, where $\overline{U} = \{\overline{1_e} : e \in U\}$. Moreover, pre-orders $\leq_r'$ and $\leq_\ell{}'$ on P correspond to partial orders $\leq_r'$ and $\leq_l'$ on $P\mathbf{S}$, and pre-orders $\leq_r^t$ and $\leq_l^t$ on P correspond to pre-orders $\leq_r$ and $\leq_l$ on $P\mathbf{S}$.*

Proof. It is sufficient to consider these orders on P and $P\mathbf{S}$. Let $x, y \in P$. We have that

$$\begin{aligned} x \ \leq_r' \ y \text{ in } P &\Leftrightarrow x \ \rho \ {}_e|y \quad \text{for some } e \ \omega \ \mathbf{d}(y) \\ &\Leftrightarrow \bar{x} = \overline{{}_e|y} \text{ in } P\mathbf{S} \quad \text{for some } e \ \omega \ \mathbf{d}(y) \\ &\Leftrightarrow \bar{x} = \overline{1_e} \odot \bar{y} \text{ in } P\mathbf{S} \quad \text{for some } e \ \omega \ \mathbf{d}(y) \quad \big(\text{Lemma 10.21}\big) \\ &\Leftrightarrow \bar{x} \ \leq_r' \ \bar{y} \text{ in } P\mathbf{S}. \end{aligned}$$

Dually, we have that $x \leq_\ell{}' y$ in P if and only if $\bar{x} \ \leq_l' \ \bar{y}$ in $P\mathbf{S}$.

In addition,

$$\begin{aligned} & x \ \leq_r^t \ y \text{ in } P \\ \Rightarrow\ & x = y_0 \leq_r y_1 \leq_r y_2 \leq_r \cdots \leq_r y_n = y \text{ in } P \quad \text{for some } n \ \geq \ 1 \\ \Rightarrow\ & x \ \rho \ {}_{e_1}|y_1, \ y_1 \ \rho \ {}_{e_2}|y_2, \cdots, y_{n-1} \ \rho \ {}_{e_n}|y_n = {}_{e_n}|y \quad \text{for some } e_1, e_2, \cdots, e_n \in U \\ \Rightarrow\ & \bar{x} = \overline{{}_{e_1}|y_1}, \ y_1 = \overline{{}_{e_2}|y_2}, \cdots, y_{n-1} = \overline{{}_{e_n}|y} \text{ in } P\mathbf{S} \\ \Rightarrow\ & \bar{x} = \overline{1_{e_1}} \odot \overline{y_1}, \ \overline{y_1} = \overline{1_{e_2}} \odot \overline{y_2}, \cdots, \overline{y_{n-1}} = \overline{1_{e_n}} \odot \bar{y} \quad \big(\text{Lemma 10.21}\big) \\ \Rightarrow\ & \bar{x} = \overline{1_{e_1}} \odot \overline{1_{e_2}} \odot \cdots \odot \overline{1_{e_n}} \odot \bar{y} \text{ in } P\mathbf{S} \\ \Rightarrow\ & \bar{x} \ \leq_r \ \bar{y} \text{ in } P\mathbf{S}. \end{aligned}$$

Conversely, if $\bar{x} \leq_r \bar{y}$ in $P\mathbf{S}$, then there exist $\overline{1_{e_1}}, \overline{1_{e_2}}, \cdots, \overline{1_{e_n}} \in \overline{U}$ such that

$$\bar{x} = \overline{1_{e_1}} \odot \overline{1_{e_2}} \odot \cdots \odot \overline{1_{e_n}} \odot \bar{y}.$$

Let

$$\overline{y_{n-1}} = \overline{1_{e_n}} \odot \bar{y}, \ \overline{y_{n-2}} = \overline{1_{e_{n-1}}} \odot \overline{y_{n-1}}, \ \cdots, \ \bar{x} = \overline{1_{e_1}} \odot \overline{y_1},$$

and let $h_n \in S(e_n, \mathbf{d}(y)), h_{n-1} \in S(e_{n-1}, \mathbf{d}(y_{n-1})), \cdots, h_1 \in S(e_1, \mathbf{d}(y_1))$. Then by Lemma 10.22, we have that

$$x \le_r h_1 * y_1, \quad h_1 * y_1 \le_r y_1, \ \cdots,$$

$$y_{n-2} \le_r h_{n-1} * y_{n-1}, \quad h_{n-1} * y_{n-1} \le_r y_{n-1},$$

$$y_{n-1} \le_r h_n * y, \quad h_n * y \le_r y,$$

that is, $x \le_r{}^t \ y$ in P.

Similarly, $x \ \le_l^t \ y$ in P if and only if $\bar{x} \ \le_l \ \bar{y}$ in $P\mathbf{S}$. □

We close this section by constructing an admissible morphism between weakly U-regular semigroups from an RBS functor.

Lemma 10.31. *If P_1 and P_2 are weakly regular categories over regular biordered sets U_1 and U_2, respectively, and $F : P_1 \to P_2$ is an RBS functor, then the map $F\mathbf{S} : P_1\mathbf{S} \to P_2\mathbf{S}$ defined by the rule that $\bar{x}F\mathbf{S} = \overline{xF}$ is an admissible morphism; moreover, if $F_1 : P_1 \to P_2$ and $F_2 : P_2 \to P_3$ are RBS functors, then $(F_1F_2)\mathbf{S} = F_1\mathbf{S}F_2\mathbf{S}$.*

Proof. It follows from Lemma 10.8 that if $x, y \in P_1$ and $\bar{x} = \bar{y}$, that is, $x \ \rho \ y$ in P_1, then $xF \ \rho \ yF$, so that $F\mathbf{S}$ is well-defined.

Now, we claim that $F\mathbf{S}$ is a semigroup morphism. Suppose that $\bar{x}, \bar{y} \in P_1\mathbf{S}$ and $h \in S(\mathbf{r}(x), \mathbf{d}(y))$. Then $hF \in S(\mathbf{r}(xF), \mathbf{d}(yF))$, and

$$\begin{aligned}
(\bar{x} \odot \bar{y})F\mathbf{S} &= \overline{(x \otimes y)_h}F\mathbf{S} && \big(h \in S(\mathbf{r}(x), \mathbf{d}(y))\big)\\
&= \overline{(x \diamond h) \cdot (h * y)}F\mathbf{S}\\
&= \overline{((x \diamond h) \cdot (h * y))F}\\
&= \overline{(x \diamond h)F \cdot (h * y)F}\\
&= \overline{(x \diamond h)F} \odot \overline{(h * y)F}\\
&= \overline{xF \diamond hF} \odot \overline{hF * yF} && \big(\text{Lemma 10.10}\big)\\
&= \overline{(xF \diamond hF) \cdot (hF * yF)}\\
&= \overline{(xF \otimes yF)_{hF}} && \big(hF \in S(\mathbf{r}(xF), \mathbf{d}(yF))\big)\\
&= \overline{xF} \odot \overline{yF}\\
&= \bar{x}F\mathbf{S} \cdot \bar{y}F\mathbf{S}.
\end{aligned}$$

Next, we present the proof that $F\mathbf{S}$ is admissible. By Lemma 10.23, for any $x \in P_1$, we have that $\overline{1_{\mathbf{d}(x)}} \ \widetilde{\mathcal{R}}_{\overline{U_1}} \ \bar{x} \ \widetilde{\mathcal{L}}_{\overline{U_1}} \ \overline{1_{\mathbf{r}(x)}}$. Then

$$\begin{aligned}\overline{1_{\mathbf{d}(x)}}F\mathbf{S} &= \overline{1_{\mathbf{d}(x)}F} \\ &= \overline{1_{\mathbf{d}(xF)}} \ \widetilde{\mathcal{R}}_{\overline{U_2}} \ \overline{xF} = \bar{x}F\mathbf{S}.\end{aligned}$$

Dually, we have that $\overline{1_{\mathbf{r}(x)}}F\mathbf{S} \ \widetilde{\mathcal{L}}_{\overline{U_2}} \ \bar{x}F\mathbf{S}$.

Finally, $\overline{1_e}F\mathbf{S} = \overline{1_eF} = \overline{1_{eF}}$ as F is a functor, so that $\overline{U_1}F\mathbf{S} \subseteq \overline{U_2}$.

To sum up, we have that $F\mathbf{S}$ is an admissible morphism from $P_1\mathbf{S}$ to $P_2\mathbf{S}$.

It is routine to show that $(F_1F_2)\mathbf{S} = F_1\mathbf{S}F_2\mathbf{S}$. □

It is an immediate consequence of Theorem 10.30 and Lemma 10.31 that $\mathbf{S} : \mathcal{WRC} \to \mathcal{WRS}$ is a functor.

10.3 Correspondence

The aim of this section is to present a converse to Theorem 10.30.

Let S be a weakly U-regular semigroup and let K be a representative of U. For any $e \in U$, we will use $e^\star$ and e^+ to denote the elements of K which are $\mathcal{L}$-related to e in U and $\mathcal{R}$-related to e in U, respectively. Set

$$S\mathbf{C} = \{(e, x, f) : e \ \widetilde{\mathcal{R}}_U \ x \ \widetilde{\mathcal{L}}_U \ f, \quad e, f \in U\} \subseteq U \times S \times U.$$

We put

$$\mathbf{d}((e, x, f)) = e \ \text{(abbreviated to } \mathbf{d}(e, x, f) = e)$$

and

$$\mathbf{r}((e, x, f)) = f \ \text{(abbreviated to } \mathbf{r}(e, x, f) = f)$$

for all $(e, x, f) \in S\mathbf{C}$, and define a partial binary operation $\cdot$ on $S\mathbf{C}$ by the rule that

$$(e, x, f) \cdot (f, y, v) = (e, xy, v),$$

where $(e, x, f), (f, y, v) \in S\mathbf{C}$ and xy is the product of x and y in S. Since $e \ \widetilde{\mathcal{R}}_U \ x = xf \ \widetilde{\mathcal{R}}_U \ xy$ and $xy \ \widetilde{\mathcal{L}}_U \ fy = y \ \widetilde{\mathcal{L}}_U \ v$, we have that $e \ \widetilde{\mathcal{R}}_U \ xy \ \widetilde{\mathcal{L}}_U \ v$, and so $(e, xy, v) \in S\mathbf{C}$. If $e, f \in U$ with $e \ \mathcal{R} \ f$ or $e \ \mathcal{L} \ f$, then we define $[e, f] = (e, ef, f)$. Obviously, $[e, f] \in S\mathbf{C}$. For any $(e, x, f) \in S\mathbf{C}$ and $u, v \in B$ with $u \leq_{\mathcal{L}} e$ and

$v \leq_{\mathcal{R}} f$, we define

$$_{u}|(e,x,f) = (u, ux, (ux)^{\star}) \text{ and } (e,x,f)|_{v} = ((xv)^{+}, xv, v).$$

Notice that $u = ue \ \widetilde{\mathcal{R}}_U \ ux \ \widetilde{\mathcal{L}}_U \ (ux)^{\star}$ and $(xv)^{+} \ \widetilde{\mathcal{R}}_U \ xv \ \widetilde{\mathcal{L}}_U \ fv = v$. So $(u, ux, (ux)^{\star})$ and $((xv)^{+}, xv, v)$ are in $S\mathbf{C}$, that is, $_{u}|(e,x,f)$ and $(e,x,f)|_{v}$ are well-defined.

Lemma 10.32. *The set $S\mathbf{C}$, together with the restriction, co-restriction and the distinguished morphisms given as above, forms a weakly regular category over U.*

Proof. Clearly, $S\mathbf{C}$ forms a category with set of objects U and morphisms the triples are given as above. For any $e \in U$, $[e,e] = (e,e,e)$ is the identity associated to e. It is necessary to show that $S\mathbf{C}$ satisfies (P1) to (P7) and their duals.

(P1) If $e, f, g \in U$ with $e \ \mathcal{R} \ f \ \mathcal{R} \ g$, then

$$[e,f] \cdot [f,g] = (e,f,f) \cdot (f,g,g) = (e,g,g) = [e,g].$$

Similarly, if $e \, \mathcal{L} \, f \, \mathcal{L} \, g$, then $[e,f] \cdot [f,g] = [e,g]$.

(P2) Suppose that $(e,x,f) \in S\mathbf{C}$, $h \in U$ and $h \leq_{\mathcal{L}} e$. Then $_{h}|(e,x,f) = (h, hx, (hx)^{\star})$ and $\mathbf{d}(_{h}|(e,x,f)) = h$. By Lemma 2.14, $(hx)^{\star} \leq_{\mathcal{L}} f$. In particular, if $h = e$, then $_{e}|(e,x,f) = (e,x,x^{\star})$. Certainly, $x^{\star} \ \mathcal{L} \ f$ and

$$(e,x,x^{\star}) \cdot (x^{\star}, x^{\star}, f) = (e, xx^{\star}, f) = (e,x,f),$$

that is, $_{e}|(e,x,f) \cdot [x^{\star}, f] = (e,x,f)$.

(P3) If $g \leq e$ and $e \ \mathcal{R} \ f$, then $g \leq_{\mathcal{R}} f$, and so $gf \in U$ and

$$\begin{aligned} _{g}|[e,f] &= {}_{g}|(e,f,f) = (g, gf, (gf)^{\star}) \\ &= (g, gf, gf) \cdot (gf, gf, (gf)^{\star}) \\ &= [g, gf] \cdot [gf, (gf)^{\star}]. \end{aligned}$$

If $g \leq e$ and $e \ \mathcal{L} \ f$, then $g \leq_{\mathcal{L}} f$, and so $gf = g$. Thus

$$_{g}|[e,f] = {}_{g}|(e,e,f) = (g,g,g^{\star}) = [g,g^{\star}] \text{ and } [g,gf] \cdot [gf,(gf)^{\star}] = [g,g^{\star}]$$

so that $_{g}|[e,f] = [g,gf] \cdot [gf,(gf)^{\star}]$.

If $g \leq_{\mathcal{L}} e$ and $e \;\mathcal{L}\; f$, then ${}_g|[e,f] = {}_g|(e,e,f) = (g,g,g^\star) = [g,g^\star]$.

(P4) if $(g,x,h) \in S\mathbf{C}$ and $e,f \in U$ with $e \leq_{\mathcal{L}} f \leq_{\mathcal{L}} g$, then

$$\begin{aligned} {}_e|({}_f|(g,x,h)) &= {}_e|(f,fx,(fx)^\star) = (e,efx,(efx)^\star) \\ &= (e,ex,(ex)^\star) = {}_e|(g,x,h). \end{aligned}$$

In particular, if $e \;\mathcal{L}\; f \;\omega^l\; g$, then

$$[e,f] \cdot ({}_f|(g,x,h)) = (e,e,f) \cdot (f,fx,(fx)^\star) \;\; = (e,efx,(fx)^\star) = (e,ex,(fx)^\star)$$

and

$${}_e|(g,x,h) = (e,ex,(ex)^\star).$$

As $(fx)^\star \,\widetilde{\mathcal{L}}_U\, fx \,\widetilde{\mathcal{L}}_U\, ex \,\widetilde{\mathcal{L}}_U\, (ex)^\star$, we have that $(ex)^\star = (fx)^\star$.

(P5) If $(e,x,f),(f,y,k) \in S\mathbf{C}$ and $h \leq_{\mathcal{L}} e$, then

$${}_h|((e,x,f)\cdot(f,y,k)) = {}_h|(e,xy,k) = (h,hxy,(hxy)^\star),$$

$${}_h|(e,x,f) = (h,hx,(hx)^\star)$$

and

$$\begin{aligned} {}_h|(e,x,f) \cdot {}_{(hx)^\star}|(f,y,k) &= (h,hx,(hx)^\star) \cdot ((hx)^\star,(hx)^\star y,((hx)^\star y)^\star) \\ &= (h,hx(hx)^\star y,((hx)^\star y)^\star) \\ &= (h,hxy,((hx)^\star y)^\star). \end{aligned}$$

But, $(hxy)^\star \,\widetilde{\mathcal{L}}_U\, hxy \,\widetilde{\mathcal{L}}_U\, ((hx)^\star y)^\star$, and so $(hxy)^\star = ((hx)^\star y)^\star$. Hence, ${}_h|((e,x,f)\cdot(f,y,k)) = {}_h|(e,x,f) \cdot {}_{(hx)^\star}|(f,y,k)$.

(P6) We know that singular U-squares are of the form:

(a) $\begin{pmatrix} g & h \\ eg & eh \end{pmatrix}$ where, $g,h \in \omega^l(e)$, $(g,h) \in \mathcal{R}$, or

(b) $\begin{pmatrix} g & ge \\ h & he \end{pmatrix}$ where, $g,h \in \omega^r(e), (g,h) \in \mathcal{L}$.

Firstly, consider (a). By $g\,\omega^l\,e$ and $(g,h) \in \mathcal{R}$, we have that

$$[g,h] \cdot [h,eh] = (g,h,h) \cdot (h,h,eh) = (g,h,eh)$$

and

$$[g, eg] \cdot [eg, eh] = (g, g, eg) \cdot (eg, eh, eh) = (g, geh, eh) = (g, gh, eh) = (g, h, eh).$$

Thus $[g, h] \cdot [h, eh] = [g, eg] \cdot [eg, eh]$. Similarly, we prove (b).

(P7) If $(u, x, v) \in S\mathbf{C}$, $h_1 \in S(e, u)$, $h_2 \in S(v, f)$. Then

$${}_{h_1u}|(u, x, v) = (h_1u, h_1x, (h_1x)^\star) \text{ and } (u, x, v)|_{vh_2} = ((xh_2)^+, xh_2, vh_2).$$

Put $h_1' = (h_1x)^\star$ and $h_2' = (xh_2)^+$. Take $h' \in S(h_1', h_2)$ and $h \in S(h_1, h_2')$. Then

$$\begin{aligned}
&((h_1 * (u, x, v)) \diamond h') \cdot [h', h'h_2] \\
&= (([h_1, h_1u] \cdot {}_{h_1u}|(u, x, v)) \diamond h') \cdot [h', h'h_2] \\
&= (([h_1, h_1u] \cdot (h_1u, h_1x, h_1')) \diamond h') \cdot [h', h'h_2] \\
&= (((h_1, h_1u, h_1u) \cdot (h_1u, h_1x, h_1')) \diamond h') \cdot [h', h'h_2] \qquad \big(h_1 \mathrel{\mathcal{R}} h_1u\big) \\
&= ((h_1, h_1x, h_1') \diamond h') \cdot [h', h'h_2] \\
&= (h_1, h_1x, h_1')|_{h_1'h'} \cdot [h_1'h', h'] \cdot [h', h'h_2]
\end{aligned}$$

$$\begin{aligned}
&= ((h_1xh')^+, h_1xh', h_1'h') \cdot (h_1'h', h_1'h', h') \cdot (h', h'h_2, h'h_2) \\
&\qquad\qquad \big(h_1'h' \mathrel{\mathcal{L}} h' \mathrel{\mathcal{R}} h'h_2\big) \\
&= ((h_1xh')^+, h_1xh', h_1'h') \cdot (h_1'h', h_1'h'h_2, h'h_2) \\
&= ((h_1xh')^+, h_1xh'h_1'h'h_2, h'h_2) \\
&= ((h_1xh')^+, h_1xh'h_2, h'h_2) \qquad \big(h' \mathrel{\mathcal{L}} h_1'h'\big) \\
&= ((h_1xh')^+, h_1xh_2, h'h_2) \\
&\qquad \big(h_1xh'h_2 = h_1x(h_1x)^\star h'h_2 = h_1xh_2 \text{ as } h' \mathrel{\mathcal{L}} h_1'h' = (h_1x)^\star h'\big).
\end{aligned}$$

Similarly, we have that

$$[h_1h, h] \cdot (h * ((u, x, v) \diamond h_2)) = (h_1h, h_1xh_2, (h_1xh_2)^\star).$$

Obviously, $(h_1xh')^+ \mathrel{\widetilde{\mathcal{R}}_U} h_1xh_2 \mathrel{\widetilde{\mathcal{R}}_U} h_1h$ and $h'h_2 \mathrel{\widetilde{\mathcal{L}}_U} h_1xh_2 \mathrel{\widetilde{\mathcal{L}}_U} (h_1xh_2)^\star$. Thus, $(h_1xh')^+ \mathrel{\mathcal{R}} h_1h$ and $h'h_2 \mathrel{\mathcal{L}} (hxh_2)^\star$, that is,

$$\mathbf{d}(((h_1 * (u, x, v)) \diamond h') \cdot [h', h'h_2]) \mathrel{\mathcal{R}} h_1h$$

and

$$h'h_2 \mathrel{\mathcal{L}} \mathbf{r}([h_1, h] \cdot (h * ((u, x, v) \diamond h_2))).$$

In addition, we have that

$$\begin{aligned}
&((h_1 * (u, x, v)) \diamond h') \cdot [h', h'h_2] \cdot [h'h_2, (h_1xh_2)^\star] \\
&= ((h_1xh')^+, h_1xh_2, h'h_2) \cdot [h'h_2, (h_1xh_2)^\star] \\
&= ((h_1xh')^+, h_1xh_2, h'h_2) \cdot (h'h_2, h'h_2, (h_1xh_2)^\star) \\
&= ((h_1xh')^+, h_1xh_2, (h_1xh_2)^\star)
\end{aligned}$$

and

$$\begin{aligned}
&[(h_1xh')^+, h_1h] \cdot [h_1h, h] \cdot (h * ((u, x, v) \diamond h_2)) \\
&= ((h_1xh')^+, h_1h, h_1h) \cdot (h_1h, h_1xh_2, (h_1xh_2)^\star) \\
&= ((h_1xh')^+, h_1xh_2, (h_1xh_2)^\star).
\end{aligned}$$

Consequently, $((h_1 * (u, x, v)) \diamond h') \cdot [h', h'h_2] \;\rho\; [h_1, h] \cdot (h * ((u, x, v) \diamond h_2))$. ☐

In Lemma 10.31, we constructed an admissible morphism from an RBS functor. Next, we produce a converse to this result.

Lemma 10.33. *Let S be a weakly U_1-regular semigroup, and let T be a weakly U_2-regular semigroup. If ϕ is an admissible morphism from S to T, then the map $\phi\mathbf{C}$ defined by $e\phi\mathbf{C} = e\phi$ and $(e, x, f)\phi\mathbf{C} = (e\phi, x\phi, f\phi)$ is an RBS functor from $S\mathbf{C}$ to $T\mathbf{C}$. Further, if $\phi_1 : S \to T$ and $\phi_2 : T \to Q$ are admissible morphisms, then $(\phi_1\phi_2)\mathbf{C} = \phi_1\mathbf{C}\phi_2\mathbf{C}$.*

Proof. As ϕ is an admissible morphism, it is clear that ϕ is a regular morphism from U_1 to U_2. Since ϕ preserves products and identities, it is a functor.

To show that (PF2) holds, suppose that $e, f \in U$. If $e \mathrel{\mathcal{R}} f$, then $e\phi \mathrel{\mathcal{R}} f\phi$ as ϕ is an admissible morphism. Thus, $[e, f]\phi\mathbf{C} = (e\phi, f\phi, f\phi) = [e\phi, f\phi] = [e\phi\mathbf{C}, f\phi\mathbf{C}]$. Dually, if $e \mathrel{\mathcal{L}} f$, then $[e, f]\phi\mathbf{C} = [e\phi, f\phi] = [e\phi\mathbf{C}, f\phi\mathbf{C}]$.

Finally, we show that (PF3) holds. Suppose that $(e, x, f) \in S\mathbf{C}$ and $h \in U_1$ with $h \leq_{\mathcal{L}} e$. Then ${}_h|(e, x, f) = (h, hx, (hx)^\star)$, and so

$$\begin{aligned}
({}_h|(e, x, f))\phi\mathbf{C} &= (h, hx, (hx)^\star)\phi\mathbf{C} \\
&= (h\phi, (hx)\phi, (hx)^\star\phi).
\end{aligned}$$

Also, we have that $h\phi \ \omega^l \ e\phi$ and

$$\begin{aligned} {}_{h\phi}|(e\phi, x\phi, f\phi) &= (h\phi, h\phi x\phi, (h\phi x\phi)^\star) \\ &= (h\phi, (hx)\phi, ((hx)\phi)^\star). \end{aligned}$$

Thus, $((hx)\phi)^\star \, \widetilde{\mathcal{L}}_{U_2} \, (hx)\phi \, \widetilde{\mathcal{L}}_{U_2} \, (hx)^\star\phi$, and so $((hx)\phi)^\star \, \mathcal{L} \, (hx)^\star\phi$,

$$\begin{aligned} &({}_h|(e, x, f))\phi\mathbf{C} \cdot [(hx)^\star\phi, ((hx)\phi)^\star] \\ &= (h\phi, (hx)\phi, (hx)^\star\phi) \cdot ((hx)^\star\phi, (hx)^\star\phi, ((hx)\phi)^\star) \\ &= (h\phi, (hx)\phi, ((hx)\phi)^\star) \\ &= {}_{h\phi\mathbf{C}}|(e, x, f)\phi\mathbf{C}. \end{aligned}$$

Hence, $({}_h|(e, x, f))\phi\mathbf{C} \ \rho \ {}_{h\phi\mathbf{C}}|(e, x, f)\phi\mathbf{C}$.

Dually, if $k \in U_1$ and $k \leq_{\mathcal{R}} f$, then $((e, x, f)|_k)\phi\mathbf{C} \ \rho \ (e, x, f)\phi\mathbf{C}|_{k\phi\mathbf{C}}$. Consequently, $\phi\mathbf{C}$ is an RBS functor from $S\mathbf{C}$ to $T\mathbf{C}$.

It is routine to show that $(\phi_1\phi_2)\mathbf{C} = \phi_1\mathbf{C}\phi_2\mathbf{C}$. □

Now, we have that $\mathbf{C} : \mathcal{WRS} \to \mathcal{WRC}$ is a functor.

At the end of this section, we build a correspondence between the category $\mathcal{WRS}$ of weakly U-regular semigroups and the category $\mathcal{WRC}$ of weakly regular categories over regular biordered sets.

Lemma 10.34. *Let S be a weakly U-regular semigroup. Then the mapping $\eta_S : S \to S\mathbf{CS}$ given by $x\eta_S = \overline{(e, x, f)}$, where $e \ \widetilde{\mathcal{R}}_U \ x \ \widetilde{\mathcal{L}}_U \ f$, is an isomorphism.*

Proof. Let $x \in S$, $e, g \in \widetilde{R}_x \cap U$ and $f, h \in \widetilde{L}_x \cap U$. Then $e \ \mathcal{R} \ g$, $f \ \mathcal{L} \ h$ and

$$(e, x, f) \cdot [f, h] = (e, x, f) \cdot (f, f, h) = (e, x, h)$$

and

$$[e, g] \cdot (g, x, h) = (e, g, g) \cdot (g, x, h) = (e, x, h).$$

Thus $(e, x, f) \ \rho \ (g, x, h)$, and so η_S is well-defined.

To show that η_S is injective, we assume that $x\eta_S = y\eta_S$. Then $\overline{(e, x, f)} = \overline{(u, y, v)}$, where $e \ \widetilde{\mathcal{R}}_U \ x \widetilde{\mathcal{L}}_U \ f$ and $u \ \widetilde{\mathcal{R}}_U \ y \ \widetilde{\mathcal{L}}_U \ v$. Thus $e \ \mathcal{R} \ u$ and $f \ \mathcal{L} \ v$. Further,

$$[e, u] \cdot (u, y, v) = (e, x, f) \cdot [f, v],$$

that is, $(e, y, v) = (e, x, v)$, which implies that $x = y$. Hence η_S is injective, as claimed.

Clearly η_S is onto. It remains to show that η_S is a morphism. Suppose that $x, y \in S$, $e \mathrel{\widetilde{\mathcal{R}}_U} x \mathrel{\widetilde{\mathcal{L}}_U} f$, $u \mathrel{\widetilde{\mathcal{R}}_U} y \mathrel{\widetilde{\mathcal{L}}_U} v$ and $h \in S(f, u)$. Then

$$\begin{aligned}
x\eta_S \odot y\eta_S &= \overline{(e, x, f)} \odot \overline{(u, y, v)} \\
&= \overline{((e, x, f) \otimes (u, y, v))_h} \\
&= \overline{((e, x, f) \diamond h) \cdot (h * (u, y, v))} \\
&= \overline{(e, x, f)|_{fh} \cdot [fh, h] \cdot [h, hu] \cdot {}_{hu}|(u, y, v)} \\
&= \overline{((xfh)^+, xfh, fh) \cdot (fh, fh, h) \cdot (h, hu, hu) \cdot (hu, huy, (huy)^\star)} \\
&= \overline{((xh)^+, xh, h) \cdot (h, hy, (hy)^\star)} \qquad \left(f \mathrel{\widetilde{\mathcal{L}}_U} x \text{ and } u \mathrel{\widetilde{\mathcal{R}}_U} y\right) \\
&= \overline{((xh)^+, xhy, (hy)^\star)} \\
&= \overline{((xh)^+, xy, (hy)^\star)} \\
&\qquad\qquad \left(h \in S(f, u) = S_1(f, u), \text{ so } xhy = xfhuy = xy\right) \\
&= (xy)\eta_S.
\end{aligned}$$

In addition, η_S preserves the distinguished set as $e\eta_S = \overline{(e, e, e)} = \overline{1_e}$ for all $e \in U$. Thus η_S is an isomorphism.

Finally, as η_S is an isomorphism, it preserves the pre-orders and partial-orders. □

To the converse, we have:

Lemma 10.35. *Let P be a weakly regular category over U. Then the map $\tau_P : P \to P\mathbf{SC}$ defined by the rule that $e\tau_P = \overline{1_e}$ and $x\tau_P = (\overline{1_{\mathbf{d}(x)}}, \bar{x}, \overline{1_{\mathbf{r}(x)}})$ for all $e \in U = Ob(P)$ and $x \in P = Mor(P)$, is an isomorphism from P to $P\mathbf{SC}$.*

Proof. Note the distinguished subset of $P\mathbf{S}$ is $\overline{U}$, which is the set of objects of $P\mathbf{SC}$. By Lemma 10.24, $\tau_P : U \to \overline{U} : e \mapsto \overline{1_e}$ is a regular isomorphism.

Now, we show that τ_P preserves $\mathbf{d}$ and $\mathbf{r}$. Suppose that $x \in P$. Then by the definition of τ_P,

$$\mathbf{d}(x)\tau_P = \overline{1_{\mathbf{d}(x)}}, \ \mathbf{r}(x)\tau_P = \overline{1_{\mathbf{r}(x)}}$$

and

$$x\tau_P = (\overline{1_{\mathbf{d}(x)}}, \bar{x}, \overline{1_{\mathbf{r}(x)}}).$$

Thus, τ_P preserves $\mathbf{d}$ and $\mathbf{r}$.

If $x, y \in P$ with $x \cdot y$ defined in P, then $\mathbf{r}(x) = \mathbf{d}(y)$ and so $\bar{x} \odot \bar{y} = \overline{x \cdot y}$.

Here, we temporarily use $\circ$ to denote the partial binary operation in $P\mathbf{SC}$. Thus,

$$\begin{aligned} x\tau_P \circ y\tau_P &= (\overline{1_{\mathbf{d}(x)}}, \bar{x}, \overline{1_{\mathbf{r}(x)}}) \circ (\overline{1_{\mathbf{d}(y)}}, \bar{y}, \overline{1_{\mathbf{r}(y)}}) \\ &= (\overline{1_{\mathbf{d}(x)}}, \bar{x} \odot \bar{y}, \overline{1_{\mathbf{r}(y)}}) \\ &= (\overline{1_{\mathbf{d}(x)}}, \overline{x \cdot y}, \overline{1_{\mathbf{r}(y)}}) \qquad \big(\text{Lemma 10.20}\big) \\ &= (\overline{1_{\mathbf{d}(x \cdot y)}}, \overline{x \cdot y}, \overline{1_{\mathbf{r}(x \cdot y)}}) \\ &= (x \cdot y)\tau_P \end{aligned}$$

which implies that τ_P preserves products. Also, τ_P preserves identities since $1_e\tau_P = (\overline{1_e}, \overline{1_e}, \overline{1_e}) = 1_{\overline{1_e}}$. Thus, τ_P is a functor.

Let $e, f \in U$ with $e \mathcal{R} f$ in U. Then $[e, f]\tau_P = (\overline{1_e}, \overline{[e, f]}, \overline{1_f})$. Since $e \mathcal{R} f$, it is easy to see that $[e, f] \; \rho \; [f, f]$. Thus,

$$[e, f]\tau_P = (\overline{1_e}, \overline{1_f}, \overline{1_f}) = (\overline{1_e}, \overline{1_e} \odot \overline{1_f}, \overline{1_f}) = [\overline{1_e}, \overline{1_f}] = [e\tau_P, f\tau_P].$$

Dually, if $e \mathcal{L} f$ in U, then $[e, f]\tau_P = [e\tau_P, f\tau_P]$. Hence, τ_P satisfies Condition (PF2).

To show that (PF3) holds, We assume that $x \in P$ and $e \in U$ with $e \, \omega^l \, \mathbf{d}(x)$. Then $e\tau_P \; \omega^l \; \mathbf{d}(x)\tau_P$ as τ_P is a regular isomorphism from U to $\overline{U}$ shown above. Hence, ${}_e|x$ and ${}_{e\tau_P}|x\tau_P$ are well-defined. Observe that

$$({}_e|x)\tau_P = (\overline{1_e}, \overline{{}_e|x}, \overline{1_{\mathbf{r}({}_e|x)}})$$

and

$$\begin{aligned} &{}_{e\tau_P}|x\tau_P \\ &= {}_{\overline{1_e}}|(\overline{1_{\mathbf{d}(x)}}, \bar{x}, \overline{1_{\mathbf{r}(x)}}) \\ &= (\overline{1_e}, \overline{1_e} \odot \bar{x}, (\overline{1_e} \odot \bar{x})^\star) \\ &= (\overline{1_e}, \overline{{}_e|x}, (\overline{{}_e|x})^\star) \qquad \big(e \; \omega^l \; \mathbf{d}(x), \; \text{Lemma 10.21}\big). \end{aligned}$$

Clearly,

$$\mathbf{d}(({}_e|x)\tau_P) = \mathbf{d}({}_e|x)\tau_P = \overline{1_e} = \mathbf{d}({}_{e\tau_P}|x\tau_P),$$

$$\mathbf{r}(({}_e|x)\tau_P) = \mathbf{r}({}_e|x)\tau_P = \overline{1_{\mathbf{r}({}_e|x)}} \; \widetilde{\mathcal{L}}_{\overline{B}} \; \overline{{}_e|x} \; \widetilde{\mathcal{L}}_{\overline{B}} \; (\overline{{}_e|x})^\star = \mathbf{r}({}_{e\tau_P}|x\tau_P)$$

and

$$\begin{aligned}
&({}_e|x)\tau_P \cdot [\mathbf{r}(({}_e|x)\tau_P), \mathbf{r}({}_{e\tau_P}|x\tau_P)] \\
&= (\overline{1_e}, \overline{{}_e|x}, \overline{1_{\mathbf{r}({}_e|x)}}) \cdot [\overline{1_{\mathbf{r}({}_e|x)}}, (\overline{{}_e|x})^\star] \\
&= (\overline{1_e}, \overline{{}_e|x}, \overline{1_{\mathbf{r}({}_e|x)}}) \cdot (\overline{1_{\mathbf{r}({}_e|x)}}, \overline{1_{\mathbf{r}({}_e|x)}} \odot (\overline{{}_e|x})^\star, (\overline{{}_e|x})^\star) \\
&= (\overline{1_e}, \overline{{}_e|x}, \overline{1_{\mathbf{r}({}_e|x)}}) \cdot (\overline{1_{\mathbf{r}({}_e|x)}}, \overline{1_{\mathbf{r}({}_e|x)}}, (\overline{{}_e|x})^\star) \\
&= (\overline{1_e}, \overline{{}_e|x} \odot \overline{1_{\mathbf{r}({}_e|x)}}, (\overline{{}_e|x})^\star) \\
&= (\overline{1_e}, \overline{{}_e|x}, (\overline{{}_e|x})^\star) \\
&= {}_{e\tau_P}|x\tau_P,
\end{aligned}$$

so that $({}_e|x)\tau_P \,\rho\, {}_{e\tau_P}|x\tau_P$ and (PF3) holds.

Next, suppose that $x, y \in P$ with $x\tau_P = y\tau_P$. Then $(\overline{1_{\mathbf{d}(x)}}, \bar{x}, \overline{1_{\mathbf{r}(x)}}) = (\overline{1_{\mathbf{d}(y)}}, \bar{y}, \overline{1_{\mathbf{r}(y)}})$, which implies that $\bar{x} = \bar{y}$, and also $\mathbf{d}(x) = \mathbf{d}(y)$, $\mathbf{r}(x) = \mathbf{r}(y)$ by Lemma 10.24. Further, by Lemma 10.3, $x = y$.

We now show that τ_P is surjective. Let $(\overline{1_e}, \bar{x}, \overline{1_f})$ be in $P\mathbf{SC}$. Then $\overline{1_e} \;\widetilde{\mathcal{R}}_{\overline{U}}\; \bar{x} \;\widetilde{\mathcal{R}}_{\overline{U}}\; \overline{1_{\mathbf{d}(x)}}$ and $\overline{1_f} \;\widetilde{\mathcal{L}}_{\overline{U}}\; \bar{x} \;\widetilde{\mathcal{L}}_{\overline{U}}\; \overline{1_{\mathbf{r}(x)}}$, that is, $\overline{1_e} \;\mathcal{R}\; \overline{1_{\mathbf{d}(x)}}$ and $\overline{1_f} \;\mathcal{L}\; \overline{1_{\mathbf{r}(x)}}$ so that by Lemma 10.24, $e \,\mathcal{R}\, \mathbf{d}(x)$ and $f \,\mathcal{L}\, \mathbf{r}(x)$. Put $x' = [e, \mathbf{d}(x)] \cdot x \cdot [\mathbf{r}(x), f]$. Certainly, $x' \;\rho\; x$, that is, $\overline{x'} = \bar{x}$. Thus, $\tau_P(x') = (\overline{1_e}, \overline{x'}, \overline{1_f}) = (\overline{1_e}, \bar{x}, \overline{1_f})$, and consequently, τ_P is surjective.

As we have shown τ_P is an RBS functor, we succeed in obtaining that τ_P preserves the two pairs of pre-orders on P by Lemma 10.8. □

Lemma 10.36. *For any $S \in Ob(\mathcal{WRS})$, define $S\eta = \eta_S$, where η_S is defined in Lemma 10.34. Then η is a natural equivalence of the functors $I_{\mathcal{WRS}}$ and* **CS**.

Proof. Let $\theta : S_1 \to S_2$ in $\mathcal{WRS}$, where S_1 and S_2 are over U_1 and U_2, respectively. Then for any $x \in S_1$, we have by the definition of η_S in Lemma 10.34 that

$$\begin{aligned}
(x\eta_{S_1})\theta\mathbf{CS} &= \overline{(e, x, f)}\theta\mathbf{CS} \qquad \left(e \;\widetilde{\mathcal{R}}_{U_1}\; x \;\widetilde{\mathcal{L}}_{U_1}\; f\right) \\
&= \overline{(e, x, f)\theta\mathbf{C}} \\
&= \overline{(e\theta, x\theta, f\theta)} \\
&= (x\theta)\eta_{S_2} \qquad \left(e\theta \;\widetilde{\mathcal{R}}_{U_2}\; x\theta \;\widetilde{\mathcal{L}}_{U_2}\; f\theta\right)
\end{aligned}$$

Thus the diagram below commutes, and so $\eta = (\eta_S)$ is a natural isomorphism between $I_{\mathcal{WRS}}$ and **CS**.

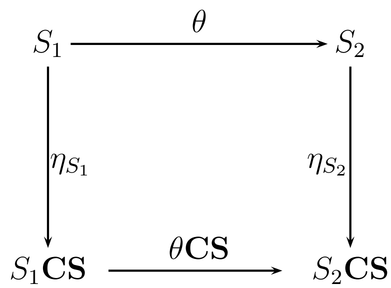

Figure 10.1: A natural transformation of $I_{\mathcal{WRS}}$ and $\mathbf{CS}$

$\square$

Similarly, we have:

Lemma 10.37. *For any $P \in Ob(\mathcal{WRC})$, define $P\tau = \tau_P$, where τ_P is defined in Lemma 10.35. Then τ is a natural equivalence of the functors $I_{\mathcal{WRC}}$ and* $\mathbf{SC}$.

Proof. Let $F : P_1 \to P_2$ in $\mathcal{WRC}$, where P_1 and P_2 are over U_1 and U_2, respectively. Then for any $x \in P_1$, we have by the definition of τ_P in Lemma 10.35 that

$$\begin{aligned}(x\tau_{P_1})F\mathbf{SC} &= (\overline{1_{\mathbf{d}(x)}}, \bar{x}, \overline{1_{\mathbf{r}(x)}})F\mathbf{SC} \\ &= ((\overline{1_{\mathbf{d}(x)}})F\mathbf{S}, (\bar{x})F\mathbf{S}, (\overline{1_{\mathbf{r}(x)}})F\mathbf{S}) \\ &= (\overline{1_{\mathbf{d}(x)}F}, \overline{xF}, \overline{1_{\mathbf{r}(x)}F}) \\ &= (\overline{1_{\mathbf{d}(xF)}}, \overline{xF}, \overline{1_{\mathbf{r}(xF)}}) \\ &= (xF)\tau_{P_2}\end{aligned}$$

and

$$\begin{aligned}(e\tau_{P_1})F\mathbf{SC} &= \overline{1_e}F\mathbf{SC} \\ &= \overline{1_e}F\mathbf{S} \\ &= \overline{1_eF} \\ &= \overline{1_{eF}} \\ &= (eF)\tau_{P_2}.\end{aligned}$$

Thus the diagram below commutes, and so $\tau = (\tau_P)$ is a natural morphism of $I_{\mathcal{WRC}}$ and $\mathbf{SC}$.

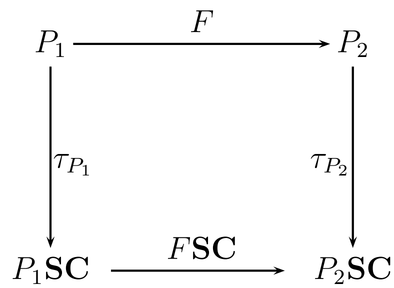

Figure 10.2: A natural transformation of $I_{\mathcal{WRC}}$ and $\mathbf{SC}$

□

To sum up, we have:

Theorem 10.38. *The category $\mathcal{WRS}$ of weakly U-regular semigroups and admissible morphisms is equivalent to the category $\mathcal{WRC}$ of weakly regular categories over regular biordered sets and RBS functors.*

Chapter 11

Special kinds of weakly U-regular semigroups

In this section we focus on some special kinds of weakly U-regular semigroups. We recover Armstrong's result for concordant semigroups and Nambooripad's result for regular semigroups.

11.1 Weakly U-regular semigroups with (WIC)

As mentioned In Chapter 2, a weakly U-regular semigroup satisfies (WIC) (with respect to U) if for any $a \in S$ and some (any) a^*, $a^\dagger$, if $x \in \langle a^\dagger \rangle$, then there exists $y \in \langle a^* \rangle$ with $xa = ay$; and dually, if $z \in \langle a^* \rangle$ then there exists $t \in \langle a^\dagger \rangle$ with $ta = az$.

We say that a weakly regular category P over U has (WIC) if the following condition and its dual (W)° hold:

(W) if $x \in P$ and $u \in U$ with $u \ \omega \ \mathbf{d}(x)$, then there exists $v_1, \cdots, v_n \in U$ such that $v_i \ \omega \ \mathbf{r}(x)$ for $i = 1, \cdots, n$ and $\overline{{}_u|x} = \bar{x} \odot \overline{1_{v_1}} \odot \cdots \odot \overline{1_{v_n}}$.

Corollary 11.1. *The category of weakly U-regular semigroups satisfying (WIC) and admissible morphisms, is equivalent to the category of weakly regular categories with (WIC) and RBS functors.*

Proof. Let S be a weakly U-regular semigroup with (WIC). In view of Lemma 10.32, it is sufficient to show that $S\mathbf{C}$ satisfies Condition (W) and its dual.

Suppose that $(e,x,f) \in S\mathbf{C}$ and $u \in U$ with $u \leq e$. Then ${}_u|(e,x,f) = (u, ux, (ux)^\star)$. Since S satisfies (WIC), it follows that there exist $v_1, \cdots, v_n \in U$ such that $v_i \leq f$ for $i = 1, \cdots, n$ and $ux = xv_1 \cdots v_n$.

By Lemma 10.34, $\eta_S : S \to S\mathbf{CS}$, given by$x \mapsto \overline{(x^\dagger, x, x^*)}$, is an isomorphism. Thus, for any $x, y \in S$, we have $x\eta_S \odot y\eta_S = (xy)\eta_S$, that is,

$$\overline{(x^\dagger, x, x^*)} \odot \overline{(y^\dagger, y, y^*)} = \overline{(xy)^\dagger, xy, (xy)^*},$$

so

$$\overline{(e,x,f)} \odot \overline{1_{v_1}} \odot \cdots \odot \overline{1_{v_n}} = \overline{((xv_1 \cdots v_n)^+, xv_1 \cdots v_n, (xv_1 \cdots v_n)^\star)}.$$

As $ux = xv_1 \cdots v_n$, we have that $u \ \widetilde{\mathcal{R}}_U \ ux = xv_1 \cdots v_n \ \widetilde{\mathcal{R}}_U \ (xv_1 \cdots v_n)^+$ and $(ux)^\star = (xv_1 \cdots v_n)^\star$. In addition, we have

$$\begin{aligned}
&[u, (xv_1 \cdots v_n)^+] \cdot ((xv_1 \cdots v_n)^+, xv_1 \cdots v_n, (xv_1 \cdots v_n)^\star) \\
&= (u, (xv_1 \cdots v_n)^+, (xv_1 \cdots v_n)^+) \cdot ((xv_1 \cdots v_n)^+, xv_1 \cdots v_n, (xv_1 \cdots v_n)^\star) \\
&= (u, xv_1 \cdots v_n, (xv_1 \cdots v_n)^\star)) \\
&= (u, ux, (ux)^\star) \qquad \Big(ux = xv_1 \cdots v_n\Big)
\end{aligned}$$

so that $\overline{{}_u|(e,x,f)} = \overline{(e,x,f)} \odot \overline{1_{v_1}} \odot \cdots \odot \overline{1_{v_n}}$, and so Condtion (W) holds.

Conversely, suppose that P is a weakly regular category over U with (WIC) and $x \in P$. In view of Theorem 10.38, it is sufficient to show that for all $u \in \langle \overline{1_{\mathbf{d}(x)}} \rangle$, there exists $v \in \langle \overline{1_{\mathbf{r}(x)}} \rangle$ satisfying $u \odot \bar{x} = \bar{x} \odot v$. Suppose that $\overline{1_e} \in \langle \overline{1_{\mathbf{d}(x)}} \rangle$. Then $\overline{1_e} = \overline{1_{e_1}} \odot \cdots \odot \overline{1_{e_n}}$, where $e_i \ \omega \ \mathbf{d}(x)$ for $i = 1, \cdots, n$. By (W), for any e_i, there exist $f_{i1}, \cdots, f_{im_i} \in \omega(\mathbf{r}(x))$ such that

$$\overline{{}_{e_i}|x} = \bar{x} \odot \overline{1_{f_{i1}}} \odot \cdots \odot \overline{1_{f_{im_i}}},$$

that is,

$$\overline{1_{e_i}} \odot \bar{x} = \bar{x} \odot \overline{1_{f_{i1}}} \odot \cdots \odot \overline{1_{f_{im_i}}}$$

by Lemma 10.21, and so

$$\begin{aligned}
\overline{1_e} \odot \bar{x} &= \overline{1_{e_1}} \odot \cdots \odot \overline{1_{e_n}} \odot \bar{x} \\
&= \overline{1_{e_1}} \odot \cdots \odot \overline{1_{e_{n-1}}} \odot \bar{x} \odot \overline{1_{f_{n1}}} \odot \cdots \odot \overline{1_{f_{nm_n}}} \\
&= \bar{x} \odot \overline{1_{f_{11}}} \odot \cdots \odot \overline{1_{f_{1m_1}}} \odot \cdots \odot \overline{1_{f_{n1}}} \odot \cdots \odot \overline{1_{f_{nm_n}}},
\end{aligned}$$

where $\overline{1_{f_{11}}} \odot \cdots \odot \overline{1_{f_{1m_1}}} \odot \cdots \odot \overline{1_{f_{n1}}} \odot \cdots \odot \overline{1_{f_{nm_n}}} \in \langle \overline{1_{\mathbf{r}(x)}} \rangle$. Dually, we show that for any $g \in \langle \overline{1_{\mathbf{r}(x)}} \rangle$, there exists $k \in \langle \overline{1_{\mathbf{d}(x)}} \rangle$ satisfying $k \odot \bar{x} = \bar{x} \odot g$. □

11.2 The abundant case

In this section we concentrate on the class of abundant semigroups. We replace the distinguished set of idempotents U by the whole set of idempotents and use relations $\mathcal{R}^*$ and $\mathcal{L}^*$ instead of $\widetilde{\mathcal{R}}_U$ and $\widetilde{\mathcal{L}}_U$ in the definition of weakly U-regular semigroups. We thus obtain the class of abundant semigroups whose set of idempotents generates a regular semigroup. As mentioned in Chapter 7, an admissible morphism in this context is more usually referred to as a *good morphism*.

A weakly regular category P over U is an *abundant category* if it satisfies the following condition and its dual (P8)°:

(P8) if $x, y, z \in P$, $h \in S(\mathbf{r}(x), \mathbf{d}(y))$ and $h' \in S(\mathbf{r}(x), \mathbf{d}(z))$ are such that

$$(x \otimes y)_h \ \rho \ (x \otimes z)_{h'},$$

then $\mathbf{r}(x)h \ \mathcal{R} \ \mathbf{r}(x)h'$.

Corollary 11.2. *The category of abundant semigroups whose set of idempotents generates a regular subsemigroup and good morphisms, is equivalent to the category of abundant cancellative categories and RBS functors.*

Proof. Suppose that S is an abundant semigroup whose set of idempotents generates a regular subsemigroup of S. In view of Lemma 10.32, it is sufficient to show that $S\mathbf{C}$ satisfies (P8). Dually, (P8)° holds. Let $(e, x, f), (u, y, v)$ and $(g, z, k) \in S\mathbf{C}$, and let $h \in S(f, u)$ and $h' \in S(f, g)$ be such that

$$((e, x, f) \otimes (u, y, v))_h \ \rho \ ((e, x, f) \otimes (g, z, k))_{h'}.$$

Notice that

$$\begin{aligned} &((e, x, f) \otimes (u, y, v))_h \\ = &((e, x, f) \diamond h) \cdot (h * (u, y, v)) \end{aligned}$$

$$= (e,x,f)|_{fh} \cdot [fh,h] \cdot [h,hu] \cdot {}_{hu}|(u,y,v)$$
$$= ((xfh)^+, xfh, fh) \cdot (fh, fh, h) \cdot (h, hu, hu) \cdot (hu, huy, (huy)^\star)$$
$$= ((xfh)^+, xfh, h) \cdot (h, huy, (huy)^\star)$$
$$= ((xh)^+, xh, h) \cdot (h, hy, (hy)^\star) \qquad \Big(f \ \mathcal{L}^* \ x, \ u \ \mathcal{R}^* \ y\Big)$$
$$= ((xh)^+, xhy, (hy)^\star)$$
$$= ((xh)^+, xy, (hy)^\star) \qquad \Big(xhy = xfhuy = xfuy = xy\Big),$$

and so, $(xh)^+ \ \mathcal{R}^\star \ xy \ \mathcal{L}^* \ (hy)^\star$. Similarly, we have

$$((e,x,f) \otimes (g,z,k))_{h'} = ((xh')^+, xz, (h'z)^\star)$$

and $(xh')^+ \, \mathcal{R}^\star \, xz \, \mathcal{L}^* \, (h'z)^\star$. Since

$$((e,x,f) \otimes (u,y,v))_h \ \rho \ ((e,x,f) \otimes (g,z,k))_{h'},$$

we have that $(xh')^+ \, \mathcal{R} \, (xh)^+$, $(hy)^\star \, \mathcal{L} \, (h'z)^\star$ and

$$[(xh)^+, (xh')^+] \cdot ((xh')^+, xz, (h'z)^\star) = ((xh)^+, xy, (hy)^\star) \cdot [(hy)^\star, (h'z)^\star].$$

From $(xh')^+ \ \mathcal{R} \ (xh)^+$, we obtain that $(xh')^+ = (xh)^+$ by the uniqueness. Similarly, we have $(hy)^\star = (h'z)^\star$. Thus,

$$((xh')^+, xz, (h'z)^\star) = ((xh)^+, xy, (hy)^\star),$$

and so $xy = xz$. Since $f \ \mathcal{L}^* \ x$, we have that $fy = fz$. As $u \ \mathcal{R}^* \ y$ and $h \ \omega^r \ u$, we obtain that $h \ \mathcal{R} \ hu$, and so $fh \ \mathcal{R} \ fhu \ \mathcal{R}^* \ fhy = fhuy = fuy = fy$. Similarly, $fh' \ \mathcal{R}^* \ fz$, and so $fh \ \mathcal{R}^* \ fy = fz \ \mathcal{R}^* \ fh'$ so that $fh \ \mathcal{R} \ fh'$. Hence, (P8) holds.

We now show that $S\mathbf{C}$ is cancellative. Suppose that $(e,x,f), (u,y,f)$ and $(f,z,v) \in S\mathbf{C}$ are such that $(e,x,f)(f,z,v) = (u,y,f)(f,z,v)$. Then $(e,xz,v) = (u,yz,v)$, and so $xz = yz$. As $f \ \mathcal{R}^* \ z$, we have $xf = yf$ and so $x = y$. Thus, $S\mathbf{C}$ is right cancellative, dually, we show that $S\mathbf{C}$ is left cancellative.

Conversely, suppose that P is an abundant cancellative category over U. Due to Theorem 10.30, it is necessary to show that $P\mathbf{S}$ is abundant. For any $x \in P$, we want to show that $\overline{1_{\mathbf{d}(x)}} \ \mathcal{R}^* \ \bar{x} \ \mathcal{L}^* \ \overline{1_{\mathbf{r}(x)}}$. First, we show that $\bar{x} \ \mathcal{L}^* \ \overline{1_{\mathbf{r}(x)}}$. Clearly,

$\bar{x} \odot \overline{1_{\mathbf{r}(x)}} = \bar{x}$. Suppose that $a, b \in P$ and $\bar{x} \odot \bar{a} = \bar{x} \odot \bar{b}$. Notice that

$$\begin{aligned} & \bar{x} \odot \bar{a} = \bar{x} \odot \bar{b} \\ \Rightarrow\ & \overline{(x \otimes a)_h} = \overline{(x \otimes b)_{h'}} \qquad \big(h \in S(\mathbf{r}(x), \mathbf{d}(a)),\ h' \in S(\mathbf{r}(x), \mathbf{d}(b))\big) \\ \Rightarrow\ & \overline{(x \diamond h) \cdot (h * a)} = \overline{(x \diamond h') \cdot (h' * b)} \end{aligned}$$

$$\begin{aligned} \Rightarrow\ & \mathbf{d}(x \diamond h)\ \mathcal{R}\ \mathbf{d}(x \diamond h'),\ \mathbf{r}(h * a)\ \mathcal{L}\ \mathbf{r}(h' * b) \text{ and} \\ & (x \diamond h) \cdot (h * a) \cdot [\mathbf{r}(h * a), \mathbf{r}(h' * b)] = [\mathbf{d}(x \diamond h), \mathbf{d}(x \diamond h')] \cdot (x \diamond h') \cdot (h' * b). \end{aligned} \tag{10.1}$$

Also, we have

$$\begin{aligned} x \diamond h &= x|_{\mathbf{r}(x)h} \cdot [\mathbf{r}(x)h, h] \\ &= x|_{\mathbf{r}(x)h} \cdot [\mathbf{r}(x)h, (\mathbf{r}(x)h)^+] \cdot [(\mathbf{r}(x)h)^+, \mathbf{r}(x)h] \cdot [\mathbf{r}(x)h, h] \quad \big(\text{by } (P1)\big) \\ &= x|_{(\mathbf{r}(x)h)^+} \cdot [(\mathbf{r}(x)h)^+, \mathbf{r}(x)h] \cdot [\mathbf{r}(x)h, h] \qquad \big(\text{by } (\text{P4})^\circ\big) \\ &= x|_{(\mathbf{r}(x)h)^+} \cdot ([\mathbf{r}(x), \mathbf{r}(x)] \diamond h) \\ &= x|_{(\mathbf{r}(x)h)^+} \cdot (1_{\mathbf{r}(x)} \diamond h). \end{aligned}$$

Similarly, $x \diamond h' = x|_{(\mathbf{r}(x)h')^+} \cdot (1_{\mathbf{r}(x)} \diamond h')$. Thus, we can write (10.1) into the following form:

$$\begin{aligned} & x|_{(\mathbf{r}(x)h)^+} \cdot (1_{\mathbf{r}(x)} \diamond h) \cdot (h * a) \cdot [\mathbf{r}(h * a), \mathbf{r}(h' * b)] \\ = {} & [\mathbf{d}(x \diamond h), \mathbf{d}(x \diamond h')] \cdot x|_{(\mathbf{r}(x)h')^+} \cdot (1_{\mathbf{r}(x)} \diamond h') \cdot (h' * b). \end{aligned} \tag{10.2}$$

Since $(x \otimes a)_h\ \rho\ (x \otimes b)_{h'}$, it follows from (P8) that $\mathbf{r}(x)h\ \mathcal{R}\ \mathbf{r}(x)h'$. Thus $(\mathbf{r}(x)h)^+ = (\mathbf{r}(x)h')^+$, which implies that $x|_{(\mathbf{r}(x)h)^+} = x|_{(\mathbf{r}(x)h')^+}$, and so

$$\mathbf{d}(x \diamond h) = \mathbf{d}(x|_{(\mathbf{r}(x)h)^+}) = \mathbf{d}(x|_{(\mathbf{r}(x)h')^+}) = \mathbf{d}(x \diamond h').$$

Thus we can write (10.2) into the following form:

$$x|_{(\mathbf{r}(x)h)^+} \cdot (1_{\mathbf{r}(x)} \diamond h) \cdot (h * a) \cdot [\mathbf{r}(h * a), \mathbf{r}(h' * b)] = x|_{(\mathbf{r}(x)h')^+} \cdot (1_{\mathbf{r}(x)} \diamond h') \cdot (h' * b).$$

As P is cancellative and $x|_{(\mathbf{r}(x)h)^+} = x|_{(\mathbf{r}(x)h')^+}$, we have that

$$(1_{\mathbf{r}(x)} \diamond h) \cdot (h * a) \cdot [\mathbf{r}(h * a), \mathbf{r}(h' * b)] = (1_{\mathbf{r}(x)} \diamond h') \cdot (h' * b).$$

Together with $\mathbf{d}(1_{\mathbf{r}(x)} \diamond h) = (\mathbf{r}(x)h)^+ = (\mathbf{r}(x)h')^+ = \mathbf{d}(1_{\mathbf{r}(x)} \diamond h')$ and $\mathbf{r}(h * a) \mathrel{\mathcal{L}} \mathbf{r}(h' * b)$, we obtain that

$$(1_{\mathbf{r}(x)} \diamond h) \cdot (h * a) \;\rho\; (1_{\mathbf{r}(x)} \diamond h') \cdot (h' * b),$$

that is, $(1_{\mathbf{r}(x)} \otimes a)_h \;\rho\; (1_{\mathbf{r}(x)} \otimes b)_{h'}$, and so $\overline{1_{\mathbf{r}(x)}} \odot \bar{a} = \overline{1_{\mathbf{r}(x)}} \odot \bar{b}$.

Suppose that $c \in P$ is such that $\bar{x} \odot \bar{c} = \bar{x}$, then $\bar{x} \odot \bar{c} = \bar{x} \odot \overline{1_{\mathbf{r}(x)}}$. Using the same method as above, we have that $\overline{1_{\mathbf{r}(x)}} \odot \bar{c} = \overline{1_{\mathbf{r}(x)}}$. Hence, $\bar{x} \mathrel{\mathcal{L}^*} \overline{1_{\mathbf{r}(x)}}$. Dually, we can show that $\bar{x} \mathrel{\mathcal{R}^*} \overline{1_{\mathbf{d}(x)}}$.

For any $\bar{x} \in E(P\mathbf{S})$, we have $\overline{1_{\mathbf{d}(x)}} \mathrel{\mathcal{R}^*} \bar{x} \mathrel{\mathcal{L}^*} \overline{1_{\mathbf{r}(x)}}$. By Lemma 10.26, we obtain that $E(P\mathbf{S}) = \overline{U}$. □

11.3 The concordant case

The aim of this section is to investigate concordant semigroups. We recall that such semigroups satisfy (IC), defined by El-Qallali and Fountain, which coincides with (WIC) in abundant case. Notice that Condition (P7) is a complicated condition, that we would like to omit. To this end, we first define an *IC-RBS* category. The difference between a weakly regular category and an IC-RBS category is that Condition (P7) is replaced by Conditions (PC1), (PC2) and the duals (PC1)°, (PC2)° of (PC1) and (PC2), respectively.

An RBS cancellative category P over U is said to be *IC-RBS* if the following conditions and the duals (PC1)°, (PC2)° of (PC1) and (PC2) hold:

(PC1) if $x \in P$ and $h \in U$ with $h \;\omega\; \mathbf{d}(x)$, then there exists a unique $k \in U$ such that $k \;\omega\; \mathbf{r}(x)$ and ${}_h|x \,\rho\, x|_k$; in particular, if $h = \mathbf{d}(x)$, then ${}_h|x \,\rho\, x$;

(PC2) let $x \in P$ and for $i = 1, 2$, $e_i, f_i \in U$ be such that $e_i \;\omega\; \mathbf{d}(x)$, $f_i \;\omega\; \mathbf{r}(x)$ and ${}_{e_i}|x \,\rho\, x|_{f_i}$. If $e_1 \;\omega^r\; e_2$, then $f_1 \;\omega^r\; f_2$, and ${}_{e_1e_2}|x \,\rho\, x|_{f_1f_2}$. If $e_1 \;\omega^l\; e_2$, then $f_1 \;\omega^l\; f_2$ and ${}_{e_2e_1}|x \,\rho\, x|_{f_2f_1}$.

We pause to make a necessary comment on Condition (PC2). From $e_1 \,\omega\, \mathbf{d}(x)$, $f_i \;\omega\; \mathbf{r}(x)$ and ${}_{e_i}|x \;\rho\; x|_{f_i}$, we obtain that e_i and f_i are unique by (PC1) and its dual (PC1)°. If $f_1, f_2 \,\omega\, \mathbf{r}(x)$ and $f_1 \,\omega^r\, f_2$, then $f_1 f_2 \,\omega\, f_2$, and so $f_1 f_2 \,\omega\, \mathbf{r}(x)$ so that $x|_{f_1 f_2}$ is well-defined. Dually, if $f_1 \;\omega^l\; f_2$, then $x|_{f_2 f_1}$ is well-defined.

There exist two approaches to build a correspondence between concordant semigroups and IC-RBS categories. We could show without mention of semigroups that if P is an *RBS* cancellative category, then P has (W), (W)°, (P7) and (P8) if and only if P has (PC1), (PC2) and the duals (PC1)$^\circ$, (PC2)$^\circ$ of (PC1) and (PC2), respectively. However, we are going to prove the correspondence between concordant semigroups and IC-RBS categories using Theorem 10.38.

Lemma 11.3. *Let P be an IC-RBS category over U. If $x \in P$ and for $i = 1, 2$, $e_i, f_i \in U$ are such that $e_i \; \omega \; \mathbf{d}(x)$, $f_i \; \omega \; \mathbf{r}(x)$, ${}_{e_i}|x \; \rho \; x|_{f_i}$ and $e_1 \; \omega^r \; e_2$, then* $[e_1, e_1e_2] \cdot ({}_{e_1e_2}|x) \cdot [\mathbf{r}({}_{e_1e_2}|x), f_1f_2] = ({}_{e_1}|x) \cdot [\mathbf{r}({}_{e_1}|x), f_1] \cdot [f_1, f_1f_2]$.

Proof. By (PC2), we have that $f_1 \; \omega^r \; f_2$ and ${}_{e_1e_2}|x \; \rho \; x|_{f_1f_2}$. Then $e_1e_2 \; \mathcal{R} \; \mathbf{d}(x|_{f_1f_2})$. Certainly, $[e_1, e_1e_2]$ and $[f_1, f_1f_2]$ exist. As ${}_{e_i}|x \; \rho \; x|_{f_i}$, we obtain that $e_i \; \mathcal{R} \; \mathbf{d}(x|_{f_i})$, $f_i \; \mathcal{L} \; \mathbf{r}({}_{e_i}|x)$ and

$${}_{e_i}|x \cdot [\mathbf{r}({}_{e_i}|x), f_i] = [e_i, \mathbf{d}(x|_{f_i})] \cdot x|_{f_i}.$$

Thus,

$$\begin{aligned}
&{}_{e_1}|x \cdot [\mathbf{r}({}_{e_1}|x), f_1] \cdot [f_1, f_1f_2] \\
&= [e_1, \mathbf{d}(x|_{f_1})] \cdot x|_{f_1} \cdot [f_1, f_1f_2] \\
&= [e_1, \mathbf{d}(x|_{f_1})] \cdot x|_{f_1f_2} && \big(f_1 \; \mathcal{R} \; f_1f_2, \text{ by (P4)}^\circ\big) \\
&= [e_1, \mathbf{d}(x|_{f_1f_2})] \cdot x|_{f_1f_2} && \big(\mathbf{d}(x|_{f_1}) = \mathbf{d}(x|_{f_1f_2})\big) \\
&= [e_1, e_1e_2] \cdot [e_1e_2, \mathbf{d}(x|_{f_1f_2})] \cdot x|_{f_1f_2} && \big(e_1 \; \mathcal{R} \; e_1e_2 \; \mathcal{R} \; \mathbf{d}(x|_{f_1f_2})\big) \\
&= [e_1, e_1e_2] \cdot {}_{e_1e_2}|x \cdot [\mathbf{r}({}_{e_1e_2}|x), f_1f_2] && \big({}_{e_1e_2}|x \, \rho \, x|_{f_1f_2}\big).
\end{aligned}$$

□

The following lemma is necessary for Lemma 11.5. Here we recall from Section 1.4 that if E is a regular biordered set, then for any $e \in E$, $\omega(e)$ is a regular biordered set.

Lemma 11.4. *Let P be an IC-RBS category over U. For any $x \in P$, the map $\sigma_x : \omega(\mathbf{d}(x)) \to \omega(\mathbf{r}(x))$, defined by $e\sigma_x = k$, is an isomorphism, where $k \; \omega \; \mathbf{r}(x)$ and ${}_e|x \; \rho \; x|_k$.*

Proof. Clearly, σ_x does map into $\omega(\mathbf{r}(x))$ by (PC1). Now, we define

$$\tau_x : \omega(\mathbf{r}(x)) \to \omega(\mathbf{d}(x))$$

by $f\tau_x = g$, where $g \; \omega \; \mathbf{d}(x)$ and $x|_f \; \rho \; {}_g|x$. By (PC1)$^\circ$, τ_x is well-defined. If

$e \in \omega(\mathbf{d}(x))$, then it follows from (PC1) and its dual that $e\sigma_x\tau_x = k\tau_x = e$, where $k\,\omega\,\mathbf{r}(x)$ and ${}_e|x\ \rho\ x|_k$. Thus, $\sigma_x = \tau_x^{-1}$, and so σ_x is a bijection.

To show that σ_x is a morphism, suppose that $e_1, e_2 \in \omega(\mathbf{d}(x))$ and $e_1\ \omega^r\ e_2$. Write $f_1 = e_1\sigma_x$ and $f_2 = e_2\sigma_x$. Then, by (PC2), $f_1\ \omega^r\ f_2$ and $x|_{f_1f_2}\ \rho\ {}_{e_1e_2}|x$. Thus, $(e_1e_2)\sigma_x = f_1f_2 = e_1\sigma_x e_2\sigma_x$. If $e_1\ \omega^l\ e_2$, then by (PC2), $f_1\ \omega^l\ f_2$, and so $(e_1e_2)\sigma_x = e_1\sigma_x = f_1 = f_1f_2 = e_1\sigma_x e_2\sigma_x$.

To show that σ_x is regular, we suppose that $e, f \in \omega(\mathbf{d}(x))$, $h \in S(e,f)$ and $k \in S(e\sigma_x, f\sigma_x)$. Then $h\sigma_x \in \mathcal{M}(e\sigma_x, f\sigma_x)$ and $k\tau_x \in \mathcal{M}(e,f)$, and so $k\tau_x \prec h$ in $\mathcal{M}(e,f)$, that is,

$$e(k\tau_x)\ \omega^r\ eh \text{ and } (k\tau_x)f\ \omega^l\ hf.$$

As σ_x is a morphism, we obtain that

$$(e(k\tau_x))\sigma_x\ \omega^r\ (eh)\sigma_x \text{ and } ((k\tau_x)f)\sigma_x\ \omega^l\ (hf)\sigma_x,$$

that is,

$$e\sigma_x(k\tau_x)\sigma_x\ \omega^r\ e\sigma_x h\sigma_x \text{ and } (k\tau_x)\sigma_x f\sigma_x\ \omega^l\ h\sigma_x f\sigma_x,$$

or equivalently,

$$e\sigma_x k\ \omega^r\ e\sigma_x h\sigma_x \text{ and } k(f\sigma_x)\ \omega^l\ h\sigma_x f\sigma_x,$$

and so $h\sigma_x \in S(e\sigma_x, f\sigma_x)$. By Lemma 1.26, σ_x is an isomorphism. □

Before we discuss the relationship amongst IC-RBS categories, concordant semigroups and inductive$_2$ cancellative categories (defined in [1] and mentioned in Chapter 6) we show that:

Lemma 11.5. *If P is an IC-RBS category over U, then it is a weakly regular category.*

Proof. It is sufficient to show that Condition (P7) holds. Suppose that $x \in P$, $e, f \in U$, $h_1 \in S(e, \mathbf{d}(x))$ and $h_2 \in S(\mathbf{r}(x), f)$. We put

$$h_1' = \mathbf{r}({}_{h_1\mathbf{d}(x)}|x) \quad \text{and} \quad h_2' = \mathbf{d}(x|_{\mathbf{r}(x)h_2}).$$

As U is regular, there exists $h \in S(h_1, h_2')$. Since $h_1\ \omega^r\ \mathbf{d}(x)$, by (B21), we have $h_1\mathbf{d}(x)\ \omega\ \mathbf{d}(x)$. Since $h_2\ \omega^l\ \mathbf{r}(x)$, by the dual of (B21), we have that $\mathbf{r}(x)h_2\ \omega\ \mathbf{r}(x)$. By (PC1) and its dual, there exist $u\ \omega\ \mathbf{d}(x)$ and $v\ \omega\ \mathbf{r}(x)$ such that ${}_u|x\ \rho\ x|_{\mathbf{r}(x)h_2}$

and $x|_v \ \rho \ _{h_1\mathbf{d}(x)}|x$. From $_u|x \ \rho \ x|_{\mathbf{r}(x)h_2}$, we obtain that $u \ \mathcal{R} \ h_2'$, $\mathbf{r}(_u|x) \ \mathcal{L} \ \mathbf{r}(x)h_2$ and

$$[u, h_2'] \cdot x|_{\mathbf{r}(x)h_2} = {_u|x} \cdot [\mathbf{r}(_u|x), \mathbf{r}(x)h_2],$$

which implies that $x|_{\mathbf{r}(x)h_2} = [h_2', u] \cdot {_u|x} \cdot [\mathbf{r}(_u|x), \mathbf{r}(x)h_2]$. Thus,

$$\begin{aligned} x \diamond h_2 &= x|_{\mathbf{r}(x)h_2} \cdot [\mathbf{r}(x)h_2, h_2] \\ &= [h_2', u] \cdot {_u|x} \cdot [\mathbf{r}(_u|x), \mathbf{r}(x)h_2] \cdot [\mathbf{r}(x)h_2, h_2] \\ &= [h_2', u] \cdot {_u|x} \cdot [\mathbf{r}(_u|x), h_2] \qquad \Big(\mathbf{r}(_u|x) \ \mathcal{L} \ \mathbf{r}(x)h_2 \ \mathcal{L} \ h_2, \text{ by (P1)}\Big), \end{aligned}$$

and so

$$\begin{aligned} & h * (x \diamond h_2) \\ &= h * ([h_2', u] \cdot {_u|x} \cdot [\mathbf{r}(_u|x), h_2]) \\ &= [h, hh_2'] \cdot {_{hh_2'}|}([h_2', u] \cdot {_u|x} \cdot [\mathbf{r}(_u|x), h_2]) \\ &= [h, hh_2'] \cdot {_{hh_2'}|}[h_2', u] \cdot {_m|}(_u|x) \cdot {_n|}[\mathbf{r}(_u|x), h_2] \\ & \qquad \Big(\text{by (P5)}, \ m = \mathbf{r}(_{hh_2'}|[h_2', u]), n = \mathbf{r}(_m|(_u|x))\Big) \\ &= [h, hh_2'] \cdot [hh_2', (hh_2')u] \cdot [(hh_2')u, ((hh_2')u)^\star] \cdot {_{((hh_2')u)^\star}|}(_u|x) \cdot {_n|}[\mathbf{r}(_u|x), h_2] \\ & \qquad \Big(\text{by (P3)}, \text{ since } h \ \omega^r \ h_2', \text{ we have } hh_2' \ \omega \ h_2' \ \mathcal{R} \ u, \ m = ((hh_2')u)^\star\Big) \\ &= [h, hh_2'] \cdot [hh_2', (hh_2')u] \cdot [(hh_2')u, ((hh_2')u)^\star] \cdot {_{((hh_2')u)^\star}|x} \cdot {_n|}[\mathbf{r}(_u|x), h_2] \\ & \qquad \Big(((hh_2')u)^\star \ \mathcal{L} \ (hh_2')u \ \omega \ u, \text{ by (P4)}\Big) \\ &= [h, hh_2'] \cdot [hh_2', hu] \cdot [hu, (hu)^\star] \cdot {_{(hu)^\star}|x} \cdot {_n|}[\mathbf{r}(_u|x), h_2] \\ & \qquad \Big(\text{by (B31)}, \text{ since } h \ \omega^r \ h_2' \ \mathcal{R} \ u, \text{ we have } h \ \omega^r \ u \ \mathcal{R} \ h_2'\Big) \\ &= [h, hh_2'] \cdot [hh_2', hu] \cdot {_{hu}|x} \cdot {_n|}[\mathbf{r}(_u|x), h_2] \qquad \Big(hu \ \mathcal{L} \ (hu)^\star, \text{ by (P4)}\Big) \\ &= [h, hu] \cdot {_{hu}|x} \cdot {_n|}[\mathbf{r}(_u|x), h_2] \qquad \Big(h \ \mathcal{R} \ hh_2' \ \mathcal{R} \ hu, \text{ by (P1)}\Big) \\ &= [h, hu] \cdot {_{hu}|x} \cdot [\mathbf{r}(_{hu}|x), (\mathbf{r}(_{hu}|x)^\star)] \qquad \Big(n = \mathbf{r}(_{hu}|x) \ \omega^l \ \mathbf{r}(_u|x) \ \mathcal{L} \ h_2, \text{ by (P3)}\Big). \end{aligned}$$

Since $h_2' \ \mathcal{R} \ u$, by Lemma 1.28, we have $S(h_1, h_2') = S(h_1, u)$. From $x|_v \ \rho \ _{h_1\mathbf{d}(x)}|x$, we obtain that $v \ \mathcal{L} \ \mathbf{r}(_{h\mathbf{d}(x)}|x) = h_1'$, and so by Lemma 1.28, $S(h_1', h_2) = S(v, h_2)$. As $h \in S(h_1, h_2')$ and $\sigma_x : \omega(\mathbf{d}(x)) \to \omega(\mathbf{r}(x))$ is an isomorphism, it follows from Lemma 1.31 that there exists $h' \in S(h_1', h_2)$ such that $(h\mathbf{d}(x))\sigma_x = \mathbf{r}(x)h'$. Since $h \ \omega^r \ u \ \omega \ \mathbf{d}(x)$, we have that $h\mathbf{d}(x) \ \mathcal{R} \ h \ \omega^r \ u \ \omega \ \mathbf{d}(x)$ by (B21), and so $(h\mathbf{d}(x))u$ exists and $(h\mathbf{d}(x))u \ \omega \ u \ \omega \ \mathbf{d}(x)$. In addition, by (B31), we have $hu = (h\mathbf{d}(x))u$.

Due to Lemma 11.4, we obtain that

$$ {}_{(h\mathbf{d}(x))u}|x \ \rho \ x|_{((h\mathbf{d}(x))u)\sigma_x} $$

Since $h' \ \omega^r \ h_2$ and $h'_1, h_2 \ \omega^l \ \mathbf{r}(x)$, it follows from (B32)° that

$$ (\mathbf{r}(x)h')(\mathbf{r}(x)h_2) = \mathbf{r}(x)(h'h_2). $$

Observe that $h'h_2 \ \omega \ h_2 \ \omega^l \ \mathbf{r}(x)$. Thus $\mathbf{r}(x)(h'h_2) \ \mathcal{L} \ h'h_2$, and so

$$ \begin{aligned} \mathbf{r}({}_{hu}|x) = \mathbf{r}({}_{(h\mathbf{d}(x))u}|x) \qquad & \big((h\mathbf{d}(x))u = hu\big) \\ \mathcal{L} \ ((h\mathbf{d}(x))u)\sigma_x & \\ = \ (h\mathbf{d}(x))\sigma_x u\sigma_x = (\mathbf{r}(x)h')(\mathbf{r}(x)h_2) = \mathbf{r}(x)(h'h_2) \ \mathcal{L} \ h'h_2. & \end{aligned} $$

Hence, $(\mathbf{r}({}_{hu}|x))^\star \ \mathcal{L} \ \mathbf{r}({}_{hu}|x) \ \mathcal{L} \ h'h_2$, that is, $\mathbf{r}(h * (x \diamond h_2)) \ \mathcal{L} \ h'h_2$.

From ${}_{h_1\mathbf{d}(x)}|x \ \rho \ x|_v$, we obtain that

$$ {}_{h_1\mathbf{d}(x)}|x \cdot [h'_1, v] = [h_1\mathbf{d}(x), \mathbf{d}(x|_v)] \cdot x|_v, $$

which implies that ${}_{h_1\mathbf{d}(x)}|x = [h_1\mathbf{d}(x), \mathbf{d}(x|_v)] \cdot x|_v \cdot [v, h'_1]$. Thus

$$ \begin{aligned} & h_1 * x \\ &= [h_1, h_1\mathbf{d}(x)] \cdot {}_{h_1\mathbf{d}(x)}|x \\ &= [h_1, h_1\mathbf{d}(x)] \cdot [h_1\mathbf{d}(x), \mathbf{d}(x|_v)] \cdot x|_v \cdot [v, h'_1] \\ &= [h_1, \mathbf{d}(x|_v)] \cdot x|_v \cdot [v, h'_1] \quad \big(h_1 \, \mathcal{R} \, h_1\mathbf{d}(x) \, \mathcal{R} \, \mathbf{d}(x|_v), \text{ by (P1)}\big), \end{aligned} $$

and so

$$
\begin{aligned}
&(h_1 * x) \diamond h' \\
&= ([h_1, \mathbf{d}(x|_v)] \cdot x|_v \cdot [v, h_1'])|_{h_1'h'} \cdot [h_1'h', h'] \\
&= [h_1, \mathbf{d}(x|_v)]|_t \cdot (x|_v)|_s \cdot [v, h_1']|_{h_1'h'} \cdot [h_1'h', h'] \\
&\qquad\qquad \Big(\text{by (P5)}^\circ, s = \mathbf{d}([v, h_1']|_{h_1'h'}),\ t = \mathbf{d}((x|_v)|_s)\Big) \\
&= [h_1, \mathbf{d}(x|_v)]|_t \cdot (x|_v)|_{(v(h_1'h'))^+} \cdot [(v(h_1'h'))^+, v(h_1'h')] \cdot [v(h_1'h'), h_1'h'] \cdot [h_1'h', h'] \\
&= [h_1, \mathbf{d}(x|_v)]|_t \cdot (x|_v)|_{(vh')^+} \cdot [(vh')^+, vh'] \cdot [vh', h_1'h'] \cdot [h_1'h', h'] \\
&\qquad\qquad \Big(h' \,\omega^l\, h_1' \,\mathcal{L}\, v,\ \text{by (B31)}^\circ, v(h_1'h') = vh'\Big) \\
&= [h_1, \mathbf{d}(x|_v)]|_t \cdot (x|_v)|_{(vh')^+} \cdot [(vh')^+, vh'] \cdot [vh', h'] \\
&\qquad\qquad \Big(vh' \,\mathcal{L}\, h_1'h' \,\mathcal{L}\, h' \ \text{by (P1)}\Big) \\
&= [h_1, \mathbf{d}(x|_v)]|_t \cdot x|_{(vh')^+} \cdot [(vh')^+, vh'] \cdot [vh', h'] \quad \Big((vh')^+ \ \mathcal{R}\ vh' \ \mathcal{R}\ v,\ \text{by (P4)}^\circ\Big) \\
&= [h_1, \mathbf{d}(x|_v)]|_t \cdot x|_{vh'} \cdot [vh', h'] \quad \Big((vh')^+ \ \mathcal{R}\ vh',\ \text{by (P4)}^\circ\Big) \\
&= [h_1, \mathbf{d}(x|_v)]|_{\mathbf{d}(x|_{vh'})} \cdot x|_{vh'} \cdot [vh', h'] \quad \Big(t = \mathbf{d}(x|_{vh'})\Big) \\
&= [(\mathbf{d}(x|_{vh'}))^+, \mathbf{d}(x|_{vh'})] \cdot x|_{vh'} \cdot [vh', h'] \\
&\qquad\qquad \Big(\mathbf{d}(x|_{vh'})\ \omega^r\ \mathbf{d}(x|_v)\ \mathcal{R}\ h_1,\ \text{by (P3)}^\circ\Big).
\end{aligned}
$$

Since $v\ \mathcal{L}\ h_1'$ and $h'\ \omega^l\ h_1'$, we have that $h'\ \omega^l\ v\ \omega\ \mathbf{r}(x)$, and so by (B31)°, $vh' = v(\mathbf{r}(x)h')$. Also, by (B21)°, $vh'\ \omega\ v\ \omega\ \mathbf{r}(x)$. Thus, $(vh')\tau_x$ exists, and

$$(vh')\tau_x = (v(\mathbf{r}(x)h'))\tau_x = (h_1\mathbf{d}(x))(h\mathbf{d}(x)) = (h_1h)\mathbf{d}(x)$$

so that

$$
\begin{aligned}
(\mathbf{d}(x|_{vh'}))^+ &\ \mathcal{R}\ \mathbf{d}(x|_{vh'}) \\
&\ \mathcal{R}\ (vh')\tau_x \qquad \Big(x|_{vh'}\ \rho\ {}_{(vh')\tau_x}|x\Big) \\
&= (h_1h)\mathbf{d}(x)\ \mathcal{R}\ h_1h,
\end{aligned}
$$

that is, $\mathbf{d}((h_1 * x) \diamond h')\ \mathcal{R}\ h_1h$.

Let $g = \mathbf{d}((h_1 * x) \diamond h') = (\mathbf{d}(x|_{vh'}))^+$ and $k = \mathbf{r}(h * (x \diamond h_2)) = (\mathbf{r}({}_{hu}|x))^\star$. As $g\ \mathcal{R}\ h_1h$ and $k\ \mathcal{L}\ h'h_2$, it follows that $[g, h_1h]$ and $[h'h_2, k]$ are well-defined. Since $h \in S(h_1, h_2')$ and $h_1 \in S(e, \mathbf{d}(x))$, we have $h\ \omega^l\ h_1\ \omega^r\ \mathbf{d}(x)$ and $h\ \omega^r\ h_2'\ \omega^r\ \mathbf{d}(x)$, and so $h\ \omega^r\ \mathbf{d}(x)$ and $h\ \mathcal{L}\ h_1h\ \omega\ h_1\ \omega^r\ \mathbf{d}(x)$ so that $\begin{pmatrix} h & h\mathbf{d}(x) \\ h_1h & (h_1h)\mathbf{d}(x) \end{pmatrix}$ is a column-singular matrix. By the comments succeeding Definition 10.1, we have

$$[h_1h, h] \cdot [h, h\mathbf{d}(x)] = [h_1h, (h_1h)\mathbf{d}(x)] \cdot [(h_1h)\mathbf{d}(x), h\mathbf{d}(x)],$$

which implies that

$$\begin{aligned} &[(h_1h)\mathbf{d}(x), h_1h] \cdot [h_1h, h] \cdot [h, h\mathbf{d}(x)] \\ &= [(h_1h)\mathbf{d}(x), h_1h] \cdot [h_1h, (h_1h)\mathbf{d}(x)] \cdot [(h_1h)\mathbf{d}(x), h\mathbf{d}(x)], \end{aligned}$$

that is,

$$\begin{aligned} &[(h_1h)\mathbf{d}(x), h_1h] \cdot [h_1h, h] \cdot [h, h\mathbf{d}(x)] \\ &= [(h_1h)\mathbf{d}(x), (h_1h)\mathbf{d}(x)] \cdot [(h_1h)\mathbf{d}(x), h\mathbf{d}(x)] \end{aligned}$$

by (P1), that is,

$$[(h_1h)\mathbf{d}(x), h_1h] \cdot [h_1h, h] \cdot [h, h\mathbf{d}(x)] = [(h_1h)\mathbf{d}(x), h\mathbf{d}(x)],$$

from which it follows that

$$\begin{aligned} &[(h_1h)\mathbf{d}(x), h_1h] \cdot [h_1h, h] \cdot [h, h\mathbf{d}(x)] \cdot [h\mathbf{d}(x), h] \\ &= [(h_1h)\mathbf{d}(x), h\mathbf{d}(x)] \cdot [h\mathbf{d}(x), h], \end{aligned}$$

that is,

$$[(h_1h)\mathbf{d}(x), h_1h] \cdot [h_1h, h] \cdot [h, h] = [(h_1h)\mathbf{d}(x), h\mathbf{d}(x)] \cdot [h\mathbf{d}(x), h],$$

or equivalently,

$$[(h_1h)\mathbf{d}(x), h_1h] \cdot [h_1h, h] = [(h_1h)\mathbf{d}(x), h\mathbf{d}(x)] \cdot [h\mathbf{d}(x), h].$$

So,

$$
\begin{aligned}
&[g, h_1h] \cdot [h_1h, h] \cdot (h * (x \diamond h_2)) \\
&= [g, (h_1h)\mathbf{d}(x)] \cdot [(h_1h)\mathbf{d}(x), h_1h] \cdot [h_1h, h] \cdot (h * (x \diamond h_2)) \\
&\qquad\qquad\qquad \Big(g \;\mathcal{R}\; h_1h \;\mathcal{R}\; (h_1h)\mathbf{d}(x), \text{ by (P1)}\Big) \\
&= [g, (h_1h)\mathbf{d}(x)] \cdot [(h_1h)\mathbf{d}(x), h\mathbf{d}(x)] \cdot [h\mathbf{d}(x), h] \cdot (h * (x \diamond h_2)) \\
&= [g, (h_1h)\mathbf{d}(x)] \cdot [(h_1h)\mathbf{d}(x), h\mathbf{d}(x)] \cdot [h\mathbf{d}(x), h] \cdot [h, hu] \cdot {}_{hu}|x \cdot [\mathbf{r}({}_{hu}|x), k] \\
&\qquad\qquad\qquad \Big(k = (\mathbf{r}({}_{hu}|x)^{\star})\Big) \\
&= [g, (h_1h)\mathbf{d}(x)] \cdot [(h_1h)\mathbf{d}(x), h\mathbf{d}(x)] \cdot [h\mathbf{d}(x), hu] \cdot {}_{hu}|x \cdot [\mathbf{r}({}_{hu}|x), k] \\
&\qquad\qquad\qquad \Big(h\mathbf{d}(x) \;\mathcal{R}\; h \;\mathcal{R}\; hu, \text{ by (P1)}\Big) \\
&= [g, (h_1h)\mathbf{d}(x)] \cdot [(h_1h)\mathbf{d}(x), h\mathbf{d}(x)] \cdot [h\mathbf{d}(x), hu] \cdot {}_{hu}|x \cdot \\
&\qquad\qquad\qquad\qquad [\mathbf{r}({}_{hu}|x), \mathbf{r}(x)(h'h_2)] \cdot [\mathbf{r}(x)(h'h_2), k] \\
&\qquad\qquad\qquad \Big(\mathbf{r}({}_{hu}|x) \;\mathcal{L}\; \mathbf{r}(x)(h'h_2) \;\mathcal{L}\; h'h_2 \;\mathcal{L}\; k, \text{ by (P1)}\Big).
\end{aligned}
$$

Since $h \;\omega^r\; u \;\omega\; \mathbf{d}(x)$, we have $h\mathbf{d}(x) \;\mathcal{R}\; h \;\mathcal{R}\; hu$ and $h\mathbf{d}(x), hu \in \omega(\mathbf{d}(x))$. Also, we have

$$(h\mathbf{d}(x))\sigma_x = \mathbf{r}(x)h' \;\omega\; \mathbf{r}(x) \text{ and } (hu)\sigma_x = ((h\mathbf{d}(x))u)\sigma_x = \mathbf{r}(x)(h'h_2) \;\omega\; \mathbf{r}(x).$$

As σ_x is an isomorphism, we obtain that $\mathbf{r}(x)h' \;\mathcal{R}\; \mathbf{r}(x)(h'h_2)$. By Lemma 11.3,

$$
\begin{aligned}
&[h\mathbf{d}(x), hu] \cdot {}_{hu}|x \cdot [\mathbf{r}({}_{hu}|x), \mathbf{r}(x)(h'h_2)] \\
&= {}_{h\mathbf{d}(x)}|x \cdot [\mathbf{r}({}_{h\mathbf{d}(x)}|x), \mathbf{r}(x)h'] \cdot [\mathbf{r}(x)h', \mathbf{r}(x)(h'h_2)].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&[g, h_1h] \cdot [h_1h, h] \cdot (h * (x \diamond h_2)) \\
&= [g, (h_1h)\mathbf{d}(x)] \cdot [(h_1h)\mathbf{d}(x), h\mathbf{d}(x)] \cdot {}_{h\mathbf{d}(x)}|x \cdot [\mathbf{r}({}_{h\mathbf{d}(x)}|x), \mathbf{r}(x)h'] \cdot \\
&\qquad\qquad\qquad\qquad [\mathbf{r}(x)h', \mathbf{r}(x)(h'h_2)] \cdot [\mathbf{r}(x)(h'h_2), k]
\end{aligned}
$$

$$= [g, (h_1h)\mathbf{d}(x)] \cdot {}_{(h_1h)\mathbf{d}(x)}|x \cdot [\mathbf{r}({}_{h\mathbf{d}(x)}|x), \mathbf{r}(x)h'] \cdot [\mathbf{r}(x)h', \mathbf{r}(x)(h'h_2)] \cdot [\mathbf{r}(x)(h'h_2), k]$$
$$\Big((h_1h)\mathbf{d}(x) \ \mathcal{L} \ h\mathbf{d}(x), \text{ by (P4)}\Big)$$
$$= [g, (h_1h)\mathbf{d}(x)] \cdot {}_{(h_1h)\mathbf{d}(x)}|x \cdot [\mathbf{r}({}_{(h_1h)\mathbf{d}(x)}|x), \mathbf{r}(x)h'] \cdot [\mathbf{r}(x)h', \mathbf{r}(x)(h'h_2)] \cdot [\mathbf{r}(x)(h'h_2), k]$$
$$= [g, (h_1h)\mathbf{d}(x)] \cdot {}_{(h_1h)\mathbf{d}(x)}|x \cdot [\mathbf{r}({}_{(h_1h)\mathbf{d}(x)}|x), vh'] \cdot [vh', \mathbf{r}(x)h'] \cdot [\mathbf{r}(x)h', \mathbf{r}(x)(h'h_2)] \cdot [\mathbf{r}(x)(h'h_2), k]$$
$$\Big(vh' \ \mathcal{L} \ \mathbf{r}({}_{(h_1h)\mathbf{d}(x)}|x) \ \mathcal{L} \ \mathbf{r}(x)h'\Big)$$
$$= [g, (h_1h)\mathbf{d}(x)] \cdot [(h_1h)\mathbf{d}(x), \mathbf{d}(x|_{vh'})] \cdot x|_{vh'} \cdot [vh', \mathbf{r}(x)h'] \cdot [\mathbf{r}(x)h', \mathbf{r}(x)(h'h_2)] \cdot [\mathbf{r}(x)(h'h_2), k]$$
$$\Big({}_{(h_1h)\mathbf{d}(x)}|x \ \rho \ x|_{vh'}\Big)$$
$$= [g, \mathbf{d}(x|_{vh'})] \cdot x|_{vh'} \cdot [vh', \mathbf{r}(x)h'] \cdot [\mathbf{r}(x)h', \mathbf{r}(x)(h'h_2)] \cdot [\mathbf{r}(x)(h'h_2), k]$$
$$\Big(g \ \mathcal{R} \ (h_1h)\mathbf{d}(x) \ \mathcal{R} \ \mathbf{d}(x|_{vh'}), \text{ by (P1)}\Big)$$
$$= [g, \mathbf{d}(x|_{vh'})] \cdot x|_{vh'} \cdot [vh', h'] \cdot [h', \mathbf{r}(x)h'] \cdot [\mathbf{r}(x)h', \mathbf{r}(x)(h'h_2)] \cdot [\mathbf{r}(x)(h'h_2), k]$$
$$\Big(vh' \ \mathcal{L} \ h' \ \mathcal{L} \ \mathbf{r}(x)h', \text{ by (P1)}\Big).$$

Since $h' \in S(h_1', h_2)$, we have $h' \ \omega^r \ h_2$ and $h' \ \omega^l \ h_1'$, and so $h' \ \mathcal{R} \ h'h_2 \ \omega \ h_2$. As $h_1', h_2 \ \omega^l \ \mathbf{r}(x)$, we have $h', h'h_2 \ \omega^l \ \mathbf{r}(x)$, it follows that $\begin{pmatrix} h' & h'h_2 \\ \mathbf{r}(x)h' & \mathbf{r}(x)(h'h_2) \end{pmatrix}$ is a row-singular matrix, and so by (P6),

$$[h', \mathbf{r}(x)h'] \cdot [\mathbf{r}(x)h', \mathbf{r}(x)(h'h_2)] = [h', h'h_2] \cdot [h'h_2, \mathbf{r}(x)(h'h_2)].$$

Thus,

$$[g, h_1h] \cdot [h_1h, h] \cdot (h * (x \diamond h_2))$$
$$= [g, \mathbf{d}(x|_{vh'})] \cdot x|_{vh'} \cdot [vh', h'] \cdot [h', h'h_2] \cdot [h'h_2, \mathbf{r}(x)(h'h_2)] \cdot [\mathbf{r}(x)(h'h_2), k]$$
$$= [g, \mathbf{d}(x|_{vh'})] \cdot x|_{vh'} \cdot [vh', h'] \cdot [h', h'h_2] \cdot [h'h_2, k]$$
$$\Big(h'h_2 \ \mathcal{L} \ \mathbf{r}(x)(h'h_2) \ \mathcal{L} \ k, \text{ by (P1)}\Big)$$
$$= [(\mathbf{d}(x|_{vh'}))^+, \mathbf{d}(x|_{vh'})] \cdot x|_{vh'} \cdot [vh', h'] \cdot [h', h'h_2] \cdot [h'h_2, k]$$
$$\Big(g = (\mathbf{d}(x|_{vh'}))^+\Big)$$
$$= ((h_1 * x) \diamond h') \cdot [h', h'h_2] \cdot [h'h_2, k].$$

Together with,

$$\mathbf{d}(((h_1 * x) \diamond h') \cdot [h', h'h_2]) = g \mathrel{\mathcal{R}} h_1 h = \mathbf{d}([h_1 h, h] \cdot (h * (x \diamond h_2)))$$

and

$$\mathbf{r}(((h_1 * x) \diamond h') \cdot [h', h'h_2]) = h'h_2 \mathrel{\mathcal{L}} k = \mathbf{r}([h_1 h, h] \cdot (h * (x \diamond h_2))),$$

we have that $((h_1 * x) \diamond h') \cdot [h', h'h_2] \mathrel{\rho} [h_1 h, h] \cdot (h * (x \diamond h_2))$. □

Now, we turn our attention to concordant semigroups.

Corollary 11.6. *The category of concordant semigroups and good morphisms is equivalent to the category of IC-RBS categories and RBS functors.*

Proof. Suppose that S is a concordant semigroup with set of idempotents $E(S)$. In view of Lemma 10.32, it is sufficient to show that $S\mathbf{C}$ satisfies (PC1) and (PC2).

(PC1) If $(e, x, f) \in S\mathbf{C}$ and $h \leq e$, then $e \mathrel{\mathcal{R}^*} x \mathrel{\mathcal{L}^*} f$. Since S satisfies (IC), it follows from the comments succeeding Lemma 2.19 that there exists a unique $k \in E(S)$ such that $k \leq f$ and $hx = xk$. In addition,

$${}_h|(e, x, f) = (h, hx, (hx)^\star) \text{ and } (e, x, f)|_k = ((xk)^+, xk, k).$$

As $hx = xk$, we obtain that $h \mathrel{\mathcal{R}} (xk)^+$ and $(hx)^\star \mathrel{\mathcal{L}} k$. Further, we have

$$\begin{aligned} {}_h|(e, x, f) \cdot [(hx)^\star, k] &= (h, hx, (hx)^\star) \cdot ((hx)^\star, (hx)^\star, k) \\ &= (h, hx(hx)^\star, k) = (h, hx, k) \end{aligned}$$

and

$$\begin{aligned} [h, (xk)^+] \cdot (e, x, f)|_k &= (h, (xk)^+, (xk)^+) \cdot ((xk)^+, xk, k) \\ &= (h, xk, k) \\ &= (h, hx, k) \qquad \big(xk = hx\big), \end{aligned}$$

so that ${}_h|(e, x, f) \mathrel{\rho} (e, x, f)|_k$.

In particular, if $h = e$, then ${}_e|(e, x, f) = (e, x, x^\star)$. Certainly, $x^\star \mathrel{\mathcal{L}} f$ and

$$(e, x, x^\star) \cdot (x^\star, x^\star, f) = (e, xx^\star, f) = (e, x, f),$$

that is, ${}_e|(e,x,f)\cdot[x^\star,f]=(e,x,f)$, so ${}_e|(e,x,f)\ \rho\ (e,x,f)$.

(PC2) Let $(e,x,f)\in S\mathbf{C}$ and for $i=1,2$, $e_i,f_i\in U$ be such that $e_i\leq e$, $f_i\leq f$ and ${}_{e_i}|(e,x,f)\ \rho\ (e,x,f)|_{f_i}$. Then

$${}_{e_i}|(e,x,f)=(e_i,e_ix,(e_ix)^\star) \text{ and } (e,x,f)|_{f_i}=((xf_i)^+,xf_i,f_i).$$

As ${}_{e_i}|(e,x,f)\ \rho\ (e,x,f)|_{f_i}$, we have that $e_i\ \mathcal{R}\ (xf_i)^+$, $(e_ix)^\star\ \mathcal{L}\ f_i$ and

$${}_{e_i}|(e,x,f)\cdot[(e_ix)^\star,f]=[e_i,(xf_i)^+]\cdot(e,x,f)|_{f_i},$$

that is,

$$(e_i,e_ix,(e_ix)^\star)\cdot((e_ix)^\star,(e_ix)^\star,f_i)=(e_i,(xf_i)^+,(xf_i)^+)\cdot((xf_i)^+,xf_i,f_i),$$

that is,

$$(e_i,e_ix,f_i)=(e_i,xf_i,f_i),$$

and so $e_ix=xf_i$.

If $e_1\leq_{\mathcal{R}} e_2$, then

$$e_1x=e_2e_1x=e_2(e_1x)=e_2xf_1=xf_2f_1.$$

As $e_1x=xf_1$, we get that $xf_2f_1=xf_1$. Since $x\ \mathcal{L}^*\ f$, we have that $ff_2f_1=ff_1$, and so $f_2f_1=f_1$ as $f_1,f_2\leq f$. Thus, $f_1\leq_{\mathcal{R}} f_2$. Note that $e_1\leq_{\mathcal{R}} e_2$ and $f_1\leq_{\mathcal{R}} f_2$, we have that $e_1e_2\leq e_2\leq e$ and $f_1f_2\leq f_2\leq f$, so that ${}_{e_1e_2}|(e,x,f)$ and $(e,x,f)|_{f_1f_2}$ exist. Also, we have

$${}_{e_1e_2}|(e,x,f)=(e_1e_2,e_1e_2x,(e_1e_2x)^\star) \text{ and } (e,x,f)|_{f_1f_2}=((xf_1f_2)^+,xf_1f_2,f_1f_2).$$

As $e_1e_2x=e_1xf_2=xf_1f_2$, we obtain that $e_1e_2\ \mathcal{R}\ (xf_1f_2)^+$, $(e_1e_2x)^\star\ \mathcal{L}\ f_1f_2$, and so $[e_1e_2,(xf_1f_2)^+]$ and $[f_1f_2,(e_1e_2x)^\star]$ exist. Further, we have

$$\begin{aligned}[(xf_1f_2)^+,e_1e_2]\cdot{}_{e_1e_2}|(e,x,f)&=((xf_1f_2)^+,e_1e_2,e_1e_2)\cdot(e_1e_2,e_1e_2x,(e_1e_2x)^\star)\\&=((xf_1f_2)^+,e_1e_2x,(e_1e_2x)^\star)\end{aligned}$$

and

$$\begin{aligned}(e,x,f)|_{f_1f_2}\cdot[f_1f_2,(e_1e_2x)^\star] &= ((xf_1f_2)^+,xf_1f_2,f_1f_2)\cdot(f_1f_2,f_1f_2,(e_1e_2x)^\star)\\ &= ((xf_1f_2)^+,xf_1f_2,(e_1e_2x)^\star)\\ &= ((xf_1f_2)^+,e_1e_2x,(e_1e_2x)^\star) \qquad \big(e_1e_2x = xf_1f_2\big).\end{aligned}$$

So ${}_{e_1e_2}|(e,x,f)\ \rho\ (e,x,f)|_{f_1f_2}$.

Dually, if $e_1 \leq_{\mathcal{L}} e_2$, then $f_1 \leq_{\mathcal{L}} f_2$ and ${}_{e_2e_1}|(e,x,f)\ \rho\ (e,x,f)|_{f_2f_1}$.

Conversely, let P be an IC-RBS category over U. In view of Lemma 11.5 and Lemma 11.2, it is sufficient to show that Condition (P8) holds and $P\mathbf{S}$ satisfies (IC).

(P8) Suppose that $x,y,z \in P$, $h \in S(\mathbf{r}(x),\mathbf{d}(y))$ and $h' \in S(\mathbf{r}(x),\mathbf{d}(z))$ are such that $(x\otimes y)_h\ \rho\ (x\otimes z)_{h'}$. Then $\mathbf{d}((x\otimes y)_h)\ \mathcal{R}\ \mathbf{d}((x\otimes z)_{h'})$, that is, $\mathbf{d}(x\diamond h)\ \mathcal{R}\ \mathbf{d}(x\diamond h')$, or equivalently, $\mathbf{d}(x|_{\mathbf{r}(x)h})\ \mathcal{R}\ \mathbf{d}(x|_{\mathbf{r}(x)h'})$. As $\mathbf{r}(x)h, \mathbf{r}(x)h' \leq \mathbf{r}(x)$, by (PC1), there exist $e_1,e_2 \leq \mathbf{d}(x)$ such that ${}_{e_1}|x\,\rho\,x|_{\mathbf{r}(x)h}$ and ${}_{e_2}|x\,\rho\,x|_{\mathbf{r}(x)h'}$. Thus, $e_1\ \mathcal{R}\ \mathbf{d}(x|_{\mathbf{r}(x)h})\ \mathcal{R}\ \mathbf{d}(x|_{\mathbf{r}(x)h'})\ \mathcal{R}\ e_2$. In addition, from ${}_{e_1}|x\ \rho\ x|_{\mathbf{r}(x)h}$ and ${}_{e_2}|x\ \rho\ x|_{\mathbf{r}(x)h'}$, we obtain that $e_1\sigma_x = \mathbf{r}(x)h$ and $e_2\sigma_x = \mathbf{r}(x)h'$ by Lemma 11.4. As σ_x is an isomorphism and $e_1\ \mathcal{R}\ e_2$, we obtain that $\mathbf{r}(x)h\ \mathcal{R}\ \mathbf{r}(x)h'$. Hence, Condition (P8) holds.

To show that $P\mathbf{S}$ satisfies (IC), we assume that $\bar{x} \in P$ and $\overline{1_e} \in \overline{U}$ with $\overline{1_e} \leq \overline{1_{\mathbf{d}(x)}}$. Then by Lemma 10.24, $e \leq \mathbf{d}(x)$. By (PC1), there exists a unique $k \in U$ such that $k \leq \mathbf{r}(x)$ and ${}_e|x\ \rho\ x|_k$, that is, $\overline{{}_e|x} = \overline{x|_k}$. By Lemma 10.21 and its dual, we have that $\overline{1_e}\odot\bar{x} = \bar{x}\odot\overline{1_k}$. Dually, if $\overline{1_f} \in \overline{U}$ with $\overline{1_f} \leq \overline{1_{\mathbf{r}(x)}}$, then there exists $\overline{1_g} \in \overline{U}$ such that $\overline{1_g} \leq \overline{1_{\mathbf{d}(x)}}$ and $\overline{1_g}\odot\bar{x} = \bar{x}\odot\overline{1_f}$. Thus, $P\mathbf{S}$ has (IC). □

Now, we aim to define new restrictions and co-restrictions on an IC-RBS category to recover the original result of Armstrong. We first show that an IC-RBS category forms an inductive$_2$ cancellative category with respect to the restriction and co-restriction defined below.

Let P be an IC-RBS category over U. If $k,h \in U$, $x \in P$ with $h\ \omega\ \mathbf{d}(x)$, $k\ \omega\ \mathbf{r}(x)$ and ${}_h|x\,\rho\,x|_k$, then we define the restriction and co-restriction by the rule that

$${}_h||x = {}_h|x\cdot[\mathbf{r}({}_h|x),k], \qquad x||_k = [h,\mathbf{d}(x|_k)]\cdot x|_k.$$

Notice that if $h\,\omega\,\mathbf{d}(x)$, $k\,\omega\,\mathbf{r}(x)$ and ${}_h|x\,\rho\,x|_k$, then by (PC1) and Lemma 11.5, we have $h\sigma_x = k$. In partucular, if $h = \mathbf{d}(x)$, then we must have that $k = \mathbf{r}(x)$, and so

$${}_{\mathbf{d}(x)}||x = {}_{\mathbf{d}(x)}|x \cdot [\mathbf{r}({}_{\mathbf{d}(x)}|x), k] = {}_{\mathbf{d}(x)}|x \cdot [\mathbf{r}({}_{\mathbf{d}(x)}|x), \mathbf{r}(x)],$$

that is, ${}_{\mathbf{d}(x)}||x = x$ by (P2). Dually, $x||_{\mathbf{r}(x)} = x$.

We define a relation on P by the rule that for any $x, y \in P$,

$$x \leq y \text{ if and only if } x = {}_e||y \qquad \text{for some } e \in U.$$

Clearly, if $x \leq y$ then there exists $e \in U$ such that $x = {}_e||y$. Then $\mathbf{d}(x) = e$, and so $x = {}_{\mathbf{d}(x)}||y$. For ${}_e||y$ to exist, we have $e\ \omega\ \mathbf{d}(y)$, that is, $\mathbf{d}(x)\ \omega\ \mathbf{d}(y)$. By the definition of the restriction, we obtain that $\mathbf{r}({}_e||y)\ \omega\ \mathbf{r}(y)$, and so $\mathbf{r}(x)\ \omega\ \mathbf{r}(y)$.

Lemma 11.7. *The relation $\leq$ is a partial order on P.*

Proof. It is easy to see that $\leq$ is reflexive by the comments above. Suppose that $x \leq y$ and $y \leq x$. Then $\mathbf{d}(x) = \mathbf{d}(y)$, and so $x = {}_{\mathbf{d}(x)}||y = {}_{\mathbf{d}(y)}||y = y$. To show that $\leq$ is transitive, suppose that $x \leq y$ and $y \leq z$. Then $\mathbf{d}(x)\ \omega\ \mathbf{d}(y)$ and $\mathbf{d}(y)\ \omega\ \mathbf{d}(z)$. Thus, $\mathbf{d}(x)\ \omega\ \mathbf{d}(z)$. Also, we have $y = {}_{\mathbf{d}(y)}||z = {}_{\mathbf{d}(y)}|z \cdot [\mathbf{r}({}_{\mathbf{d}(y)}|z), \mathbf{r}(y)]$. Then

$$\begin{aligned}
{}_{\mathbf{d}(x)}|y &= {}_{\mathbf{d}(x)}|({}_{\mathbf{d}(y)}|z \cdot [\mathbf{r}({}_{\mathbf{d}(y)}|z), \mathbf{r}(y)]) \\
&= {}_{\mathbf{d}(x)}|({}_{\mathbf{d}(y)}|z) \cdot {}_g|[\mathbf{r}({}_{\mathbf{d}(y)}|z), \mathbf{r}(y)] \quad \big(g = \mathbf{r}({}_{\mathbf{d}(x)}|({}_{\mathbf{d}(y)}|z)),\ \text{by (P5)}\big) \\
&= {}_{\mathbf{d}(x)}|({}_{\mathbf{d}(y)}|z) \cdot [g, g^\star] \quad \big(g = \mathbf{r}({}_{\mathbf{d}(x)}|({}_{\mathbf{d}(y)}|z))\ \omega^l\ \mathbf{r}({}_{\mathbf{d}(y)}|z)\ \mathcal{L}\ \mathbf{r}(y),\ \text{by (P3)}\big) \\
&= {}_{\mathbf{d}(x)}|z \cdot [g, g^\star] \quad \big(\mathbf{d}(x)\ \omega\ \mathbf{d}(y),\ \text{by (P4)}\big) \\
&= {}_{\mathbf{d}(x)}|z \cdot [\mathbf{r}({}_{\mathbf{d}(x)}|z), (\mathbf{r}({}_{\mathbf{d}(x)}|z))^\star].
\end{aligned}$$

From $x \leq y$, we have that

$$\begin{aligned}
x &= {}_{\mathbf{d}(x)}||y = {}_{\mathbf{d}(x)}|y \cdot [\mathbf{r}({}_{\mathbf{d}(x)}|y), \mathbf{r}(x)] \\
&= {}_{\mathbf{d}(x)}|z \cdot [\mathbf{r}({}_{\mathbf{d}(x)}|z), (\mathbf{r}({}_{\mathbf{d}(x)}|z))^\star] \cdot [(\mathbf{r}({}_{\mathbf{d}(x)}|z)^\star, \mathbf{r}(x)] \quad \big(\mathbf{r}({}_{\mathbf{d}(x)}|y) = (\mathbf{r}({}_{\mathbf{d}(x)}|z))^\star\big) \\
&= {}_{\mathbf{d}(x)}|z \cdot [\mathbf{r}({}_{\mathbf{d}(x)}|z), \mathbf{r}(x)] \quad \big(\mathbf{r}({}_{\mathbf{d}(x)}|z)\ \mathcal{L}\ (\mathbf{r}({}_{\mathbf{d}(x)}|z))^\star\ \mathcal{L}\ \mathbf{r}(x),\ \text{by (P1)}\big) \\
&= {}_{\mathbf{d}(x)}||z.
\end{aligned}$$

□

Lemma 11.8. *An IC-RBS category over a regular biordered set U with the order defined above forms an ordered$_2$ category.*

Proof. Let P be an IC-RBS category over U.

(OC1) If $x \leq y$, then by the comments before Lemma 11.7, $\mathbf{d}(x)\ \omega\ \mathbf{d}(y)$ and $\mathbf{r}(x)\ \omega\ \mathbf{r}(y)$.

(OC3) Suppose that $x' \leq x$, $y' \leq y$ and $x' \cdot y'$ and $x \cdot y$ are defined. Then $x' = {}_{\mathbf{d}(x')}||x$ and $y' = {}_{\mathbf{d}(y')}||y$. Also ,

$$\begin{aligned}
x' \cdot y' &= {}_{\mathbf{d}(x')}||x \cdot {}_{\mathbf{d}(y')}||y \\
&= {}_{\mathbf{d}(x')}|x \cdot [\mathbf{r}({}_{\mathbf{d}(x')}|x), \mathbf{r}(x')] \cdot {}_{\mathbf{d}(y')}|y \cdot [\mathbf{r}({}_{\mathbf{d}(y')}|y), \mathbf{r}(y')] \\
&= {}_{\mathbf{d}(x')}|x \cdot [\mathbf{r}({}_{\mathbf{d}(x')}|x), \mathbf{r}(x')] \cdot {}_{\mathbf{r}(x')}|y \cdot [\mathbf{r}({}_{\mathbf{r}(x')}|y), \mathbf{r}(y')] \\
&\qquad\qquad \big(\text{since } x' \cdot y' \text{ is defined, } \mathbf{r}(x') = \mathbf{d}(y')\big) \\
&= {}_{\mathbf{d}(x')}|x \cdot \mathbf{r}_{({}_{\mathbf{d}(x')}|x)}|y \cdot [\mathbf{r}({}_{\mathbf{r}(x')}|y), \mathbf{r}(y')] \qquad \big(\mathbf{r}({}_{\mathbf{d}(x')}|x)\ \mathcal{L}\ \mathbf{r}(x'), \text{by (P4)}\big) \\
&= {}_{\mathbf{d}(x')}|(x \cdot y) \cdot [\mathbf{r}({}_{\mathbf{d}(x')}|(x \cdot y)), \mathbf{r}(y')] \qquad \big(\text{by (P5)}\big) \\
&= {}_{\mathbf{d}(x')}||(x \cdot y).
\end{aligned}$$

Thus, $x' \cdot y' \ \leq\ x \cdot y$.

(OC4) For part (i), if $x \in P$ and $e \in U$ are such that $e\ \omega\ \mathbf{d}(x)$, then by the definition of the order, ${}_e||x$ is the unique element satisfying that ${}_e||x \leq x$ and $\mathbf{d}({}_e||x) = e$. Dually, part (ii) holds.

Hence, P is an ordered$_2$ category. □

Lemma 11.9. *An IC-RBS category P over U with respect to the order, restrictions and co-restrictions forms an inductive$_2$ cancellative category.*

Proof. In view of Lemma 11.8, it is sufficient to show that P satisfies Conditions (IC1)-(IC6) mentioned in Chapter 6.

(IC1) Clearly.

(IC2) Suppose that $e, f \in U$ are such that $e\,\omega\,f$. Then ${}_e|1_f = {}_e|[f, f] = [e, e^\star]$ by (P3). Also, by (P3)°, we have $1_f|_e = [e^+, e]$. By Lemma 10.2, we obtain that

$${}_e|1_f = [e, e^\star]\ \rho\ 1_e\ \rho[e^+, e] = 1_f|_e,$$

and so by the definition of the restriction, we have that

$${}_e||1_f = {}_e|1_f \cdot [e^\star, e],$$

that is,

$$ {}_{e}||1_f = [e, e^\star] \cdot [e^\star, e] = [e, e] = 1_e, $$

and so $1_e \leq 1_f$.

Conversely, if $1_e \leq 1_f$, by the comments before Lemma 11.7, we obtain that $e \; \omega \; f$.

(IC3) It follows from (P1).

(IC4) Suppose that $g, h, e \in U$ are such that $e \; \omega \; g \; \mathcal{L} \; h$. Then $[g, h]$ exists and $e \; \mathcal{L} \; he = heh \; \omega \; h$, and so $[e, he]$ exists. By (P3), we have ${}_{e}|[g, h] = [e, e^\star]$. Also, we have

$$ \begin{aligned} [g, h]|_{he} &= [(g(he))^+, g(he)] \cdot [g(he), he] && \big(\text{by (P3)}^\circ\big) \\ &= [e^+, e] \cdot [e, he] && \big(e \; \omega \; g \; \mathcal{L} \; h, \text{ by (B31)}^\circ, g(he) = ge = e\big). \end{aligned} $$

Clearly, $e \; \mathcal{R} \; e^+$, $e^\star \; \mathcal{L} \; e$ and

$$ [e, e^\star] \cdot [e^\star, he] = [e, he] = [e, e^+] \cdot [e^+, e] \cdot [e, eh], $$

that is, ${}_{e}|[g, h] \; \rho \; [g, h]|_{he}$, and so

$$ {}_{e}||[g, h] = {}_{e}|[g, h] \cdot [e^\star, he] = [e, e^\star] \cdot [e^\star, he] = [e, he], $$

so that $[e, he] \leq [g, h]$.

Dually, if $g \mathcal{R} h$, then $[e, eh]$ exists and $[e, eh] \leq [g, h]$.

(IC5) It follows from Lemma 11.3.

(IC6) It follows from (P6).

Thus, P is an inductive$_2$ cancellative category. □

Conversely, let Q be an inductive$_2$ cancellative category with regular biordered set U. For each $\mathcal{L}$-class and $\mathcal{R}$-class, we pick out a special element as its representative. If $h, k \in U$ with $h \; \omega^l \; \mathbf{d}(x)$ and $k \; \omega^r \; \mathbf{r}(x)$, then we define

$$ {}_{h}|x = [h, \mathbf{d}(x)h] \cdot {}_{\mathbf{d}(x)h}|||x \cdot [\mathbf{r}({}_{\mathbf{d}(x)h}|||x), (\mathbf{r}({}_{\mathbf{d}(x)h}|||x))^\star] $$

and

$$ x|_k = [(\mathbf{d}(x|||_{k\mathbf{r}(x)}))^+, \mathbf{d}(x|||_{k\mathbf{r}(x)})] \cdot x|||_{k\mathbf{r}(x)} \cdot [k\mathbf{r}(x), k], $$

where if $e\,\omega\,\mathbf{d}(x)$ and $f\,\omega\,\mathbf{r}(x)$, then we use ${}_e|||x$ and $x|||_f$ to mean the restriction of x to e and the co-restriction of x to f in the sense of inductive$_2$ cancellative categories. So if $e = \mathbf{d}(x)$ and $f = \mathbf{r}(x)$, then ${}_e|||x = x$ and $x|||_f = x$.

In particular, if $h, k \in U$ with $h\ \omega\ \mathbf{d}(x)$ and $k\ \omega\ \mathbf{r}(x)$, then we have that

$${}_h|x = {}_h|||x \cdot [\mathbf{r}({}_h|||x), (\mathbf{r}({}_h|||x))^\star]$$

and

$$x|_k = [(\mathbf{d}(x|||_k))^+, \mathbf{d}(x|||_k)] \cdot x|||_k.$$

In addition, if $h = \mathbf{d}(x)$, then

$${}_h|x = {}_{\mathbf{d}(x)}|||x \cdot [\mathbf{r}({}_{\mathbf{d}(x)}|||x), (\mathbf{r}({}_{\mathbf{d}(x)}|||x))^\star] = x \cdot [\mathbf{r}(x), (\mathbf{r}(x))^\star],$$

and so $\mathbf{r}({}_h|x) = (\mathbf{r}(x))^\star$ and

$${}_h|x \cdot [(\mathbf{r}(x))^\star, \mathbf{r}(x)] = x \cdot [\mathbf{r}(x), (\mathbf{r}(x))^\star] \cdot [(\mathbf{r}(x))^\star, \mathbf{r}(x)],$$

that is,

$${}_h|x \cdot [(\mathbf{r}(x))^\star, \mathbf{r}(x)] = x \cdot [\mathbf{r}(x), \mathbf{r}(x)],$$

by (IC3), or equivalently,

$${}_h|x \cdot [(\mathbf{r}(x))^\star, \mathbf{r}(x)] = x.$$

Dually, if $k = \mathbf{r}(x)$, then $[\mathbf{d}(x), \mathbf{d}(x|_k)] \cdot x|_k = x$.

So, we have:

Lemma 11.10. *An inductive$_2$ cancellative category Q over U forms an IC-RBS category with the restriction and co-restriction defined above.*

Proof. Clearly, Condition (P1) holds by (IC3).

(P2) It follows from the statement before this lemma.

(P3) If $g\ \omega\ e$ and $e\ \mathcal{R}\ f$ or $e\ \mathcal{L}\ f$, then by (IC4),

$${}_g|[e,f] = {}_g|||[e,f] \cdot [\mathbf{r}({}_g|||[e,f]), (\mathbf{r}({}_g|||[e,f]))^\star] = [g, fgf] \cdot [fgf, (fgf)^\star].$$

If $g\ \omega^l\ e$ and $e\ \mathcal{L}\ f$, then $g\ \omega^l\ f$, which implies that $fgf = fg\ \mathcal{L}\ g$, and so

$$\begin{aligned} {}_g|[e,f] &= [g, eg] \cdot {}_{eg}|||[e,f] \cdot [\mathbf{r}({}_{eg}|||[e,f]), (\mathbf{r}({}_{eg}|||[e,f]))^\star] \\ &= [g, eg] \cdot [eg, f(eg)f] \cdot [f(eg)f, (f(eg)f)^\star] \quad \big(\text{by (IC4)}\big) \\ &= [g, eg] \cdot [eg, fg] \cdot [fg, (fg)^\star] \\ &= [g, (fg)^\star] \qquad \big(g\ \mathcal{L}\ eg\ \mathcal{L}\ fg\ \mathcal{L}\ (fg)^\star\big) \\ &= [g, g^\star] \qquad \big(g\ \mathcal{L}\ fg\big). \end{aligned}$$

(P4) If $x \in Q$ and $e, f \in U$ with $e\ \omega^l\ f\ \omega^l\ \mathbf{d}(x)$, then

$${}_f|x = [f, \mathbf{d}(x)f] \cdot ({}_{\mathbf{d}(x)f}|||x) \cdot [\mathbf{r}({}_{\mathbf{d}(x)f}|||x), (\mathbf{r}({}_{\mathbf{d}(x)f}|||x))^\star],$$

and so

$$\begin{aligned} {}_e|({}_f|x) &= {}_e|([f, \mathbf{d}(x)f] \cdot ({}_{\mathbf{d}(x)f}|||x) \cdot [\mathbf{r}({}_{\mathbf{d}(x)f}|||x), (\mathbf{r}({}_{\mathbf{d}(x)f}|||x))^\star]) \\ &= [e, fe] \cdot {}_{fe}|||([f, \mathbf{d}(x)f] \cdot {}_{\mathbf{d}(x)f}|||x \cdot [\mathbf{r}({}_{\mathbf{d}(x)f}|||x), (\mathbf{r}({}_{\mathbf{d}(x)f}|||x))^\star]) \cdot [u, u^\star] \\ &\qquad \Big(u = \mathbf{r}({}_{fe}|||([f, \mathbf{d}(x)f] \cdot {}_{\mathbf{d}(x)f}|||x \cdot [\mathbf{r}({}_{\mathbf{d}(x)f}|||x), (\mathbf{r}({}_{\mathbf{d}(x)f}|||x))^\star]))\Big) \\ &= [e, fe] \cdot [fe, k] \cdot {}_k|||({}_{\mathbf{d}(x)f}|||x) \cdot {}_g|||[\mathbf{r}({}_{\mathbf{d}(x)f}|||x), (\mathbf{r}({}_{\mathbf{d}(x)f}|||x))^\star] \cdot [u, u^\star] \\ &\qquad \Big(k = (\mathbf{d}(x)f)(fe)(\mathbf{d}(x)f),\ g = \mathbf{r}({}_k|||({}_{\mathbf{d}(x)f}|||x))\Big) \\ &= [e, fe] \cdot [fe, k] \cdot {}_k|||({}_{\mathbf{d}(x)f}|||x) \cdot [g, (\mathbf{r}({}_{\mathbf{d}(x)f}|||x))^\star g] \cdot [u, u^\star] \\ &= [e, fe] \cdot [fe, k] \cdot {}_k|||({}_{\mathbf{d}(x)f}|||x) \cdot [g, u^\star] \\ &\qquad \Big(g\ \mathcal{L}\ (\mathbf{r}({}_{\mathbf{d}(x)f}|||x))^\star g = u \mathcal{L}\ u^\star,\ \text{by (IC3)}\Big) \\ &= [e, fe] \cdot [fe, k] \cdot {}_k|||({}_{\mathbf{d}(x)f}|||x) \cdot [g, g^\star] \qquad \big(g\ \mathcal{L}\ u\big). \end{aligned}$$

Since ${}_k|||({}_{\mathbf{d}(x)f}|||x) \leq {}_{\mathbf{d}(x)f}|||x \leq x$ and ${}_k|||x \leq x$, we obtain that ${}_k|||({}_{\mathbf{d}(x)f}|||x) = {}_k|||x$ by (OC4). Thus

$$\begin{aligned} {}_e|({}_f|x) &= [e, fe] \cdot [fe, k] \cdot {}_k|||x \cdot [g, g^\star] \\ &= [e, fe] \cdot [fe, k] \cdot {}_k|||x \cdot [\mathbf{r}({}_k|||x), (\mathbf{r}({}_k|||x))^\star]. \end{aligned}$$

Notice that $e\ \omega^l\ f\ \omega^l\ \mathbf{d}(x)$, by (B21)°, we have that

$$fe\ \mathcal{L}\ e\ \mathcal{L}\ \mathbf{d}(x)e\ \omega\ \mathbf{d}(x) \text{ and } fe\ \omega\ f\ \mathcal{L}\ \mathbf{d}(x)f\ \omega\ \mathbf{d}(x).$$

Thus,

$$k = (\mathbf{d}(x)f)(fe)(\mathbf{d}(x)f) = (\mathbf{d}(x)f)(fe)\ \mathcal{L}\ fe\ \mathcal{L}\ e\ \mathcal{L}\ \mathbf{d}(x)e$$

and also, $k\ \omega\ \mathbf{d}(x)$. In addition, by the dual of (IC5), we have that

$$\mathbf{r}({}_{\mathbf{d}(x)e}|||x)\ \mathcal{L}\ \mathbf{r}({}_{k}|||x)$$

and

$$[\mathbf{d}(x)e, k(\mathbf{d}(x)e)] \cdot {}_{k(\mathbf{d}(x)e)}|||x = {}_{\mathbf{d}(x)e}|||x \cdot [\mathbf{r}({}_{\mathbf{d}(x)e}|||x), \mathbf{r}({}_{k}|||x)\mathbf{r}({}_{\mathbf{d}(x)e}|||x)],$$

that is,

$$[\mathbf{d}(x)e, k] \cdot {}_{k}|||x = {}_{\mathbf{d}(x)e}|||x \cdot [\mathbf{r}({}_{\mathbf{d}(x)e}|||x), \mathbf{r}({}_{k}|||x)],$$

and so, we have

$$\begin{aligned}
&{}_{e}|({}_{f}|x) \\
&= [e,k] \cdot ({}_{k}|||x) \cdot [\mathbf{r}({}_{k}|||x), (\mathbf{r}({}_{k}|||x))^{\star}] \qquad \big(e\ \mathcal{L}\ fe\ \mathcal{L}\ k,\ \text{by (IC3)}\big) \\
&= [e, \mathbf{d}(x)e] \cdot [\mathbf{d}(x)e, k] \cdot {}_{k}|||x \cdot [\mathbf{r}({}_{k}|||x), (\mathbf{r}({}_{k}|||x))^{\star}] \\
&\qquad\qquad\qquad\qquad\qquad\qquad \big(e\ \mathcal{L}\ \mathbf{d}(x)e\ \mathcal{L}\ k,\ \text{by (IC3)}\big) \\
&= [e, \mathbf{d}(x)e] \cdot {}_{\mathbf{d}(x)e}|||x \cdot [\mathbf{r}({}_{\mathbf{d}(x)e}|||x), \mathbf{r}({}_{k}|||x)] \cdot [\mathbf{r}({}_{k}|||x), (\mathbf{r}({}_{k}|||x))^{\star}] \\
&= [e, \mathbf{d}(x)e] \cdot {}_{\mathbf{d}(x)e}|||x \cdot [\mathbf{r}({}_{\mathbf{d}(x)e}|||x), (\mathbf{r}({}_{k}|||x))^{\star}] \quad \big(\text{by (IC3)}\big) \\
&= [e, \mathbf{d}(x)e] \cdot {}_{\mathbf{d}(x)e}|||x \cdot [\mathbf{r}({}_{\mathbf{d}(x)e}|||x), (\mathbf{r}({}_{\mathbf{d}(x)e}|||x))^{\star}] \quad \big(\mathbf{r}({}_{\mathbf{d}(x)e}|||x)\ \mathcal{L}\ \mathbf{r}({}_{k}|||x)\big) \\
&= {}_{e}|x.
\end{aligned}$$

If $e\ \mathcal{L}\ f\ \omega^{l}\ \mathbf{d}(x)$, then $\mathbf{d}(x)e\ \mathcal{L}\ e\ \mathcal{L}\ f\ \mathcal{L}\ \mathbf{d}(x)f$ and $\mathbf{d}(x)e, \mathbf{d}(x)f\ \omega\ \mathbf{d}(x)$. By the dual of (IC5), we have that

$$[\mathbf{d}(x)e, (\mathbf{d}(x)f)(\mathbf{d}(x)e)] \cdot {}_{(\mathbf{d}(x)f)(\mathbf{d}(x)e)}|||x = {}_{\mathbf{d}(x)e}|||x \cdot [\mathbf{r}({}_{\mathbf{d}(x)e}|||x), \mathbf{r}({}_{(\mathbf{d}(x)f)(\mathbf{d}(x)e)}|||x)],$$

that is,

$$[\mathbf{d}(x)e, \mathbf{d}(x)f] \cdot {}_{\mathbf{d}(x)f}|||x = {}_{\mathbf{d}(x)e}|||x \cdot [\mathbf{r}({}_{\mathbf{d}(x)e}|||x), \mathbf{r}({}_{\mathbf{d}(x)f}|||x)]$$

as $\mathbf{d}(x)e \;\mathcal{L}\; \mathbf{d}(x)f$. Then, we have

$$\begin{aligned}
&[e,f]\cdot({}_f|x)\\
&=[e,f]\cdot[f,\mathbf{d}(x)f]\cdot({}_{\mathbf{d}(x)f}|||x)\cdot[\mathbf{r}({}_{\mathbf{d}(x)f}|||x),(\mathbf{r}({}_{\mathbf{d}(x)f}|||x))^{\star}]\\
&=[e,\mathbf{d}(x)f]\cdot({}_{\mathbf{d}(x)f}|||x)\cdot[\mathbf{r}({}_{\mathbf{d}(x)f}|||x),(\mathbf{r}({}_{\mathbf{d}(x)f}|||x))^{\star}]\\
&\qquad\qquad\qquad\qquad\qquad\qquad\Big(e\;\mathcal{L}\;f\;\mathcal{L}\;\mathbf{d}(x)f,\ \text{by (IC3)}\Big)\\
&=[e,\mathbf{d}(x)e]\cdot[\mathbf{d}(x)e,\mathbf{d}(x)f]\cdot({}_{\mathbf{d}(x)f}|||x)\cdot[\mathbf{r}({}_{\mathbf{d}(x)f}|||x),(\mathbf{r}({}_{\mathbf{d}(x)f}|||x))^{\star}]\\
&=[e,\mathbf{d}(x)e]\cdot({}_{\mathbf{d}(x)e}|||x)\cdot[\mathbf{r}({}_{\mathbf{d}(x)e}|||x),\mathbf{r}({}_{\mathbf{d}(x)f}|||x)]\cdot[\mathbf{r}({}_{\mathbf{d}(x)f}|||x),(\mathbf{r}({}_{\mathbf{d}(x)f}|||x))^{\star}]\\
&=[e,\mathbf{d}(x)e]\cdot{}_{\mathbf{d}(x)e}|||x\cdot[\mathbf{r}({}_{\mathbf{d}(x)e}|||x),(\mathbf{r}({}_{\mathbf{d}(x)e}|||x))^{\star}]\qquad\Big(\text{by (IC3)}\Big)\\
&={}_e|x.
\end{aligned}$$

(P5) If $h\;\omega^{l}\;\mathbf{d}(x)$ and $x\cdot y$ is defined. Then

$$\begin{aligned}
&{}_h|(x\cdot y)\\
&=[h,\mathbf{d}(x)h]\cdot{}_{\mathbf{d}(x)h}|||(x\cdot y)\cdot[v,v^{\star}]\qquad\Big(v=\mathbf{r}({}_h|||(x\cdot y))\Big)\\
&=[h,\mathbf{d}(x)h]\cdot{}_{\mathbf{d}(x)h}|||x\cdot{}_g|||y\cdot[v,v^{\star}]\qquad\Big(g=\mathbf{r}({}_{\mathbf{d}(x)h}|||x)\Big)\\
&=[h,\mathbf{d}(x)h]\cdot{}_{\mathbf{d}(x)h}|||x\cdot[g,g]\cdot{}_{\mathbf{r}({}_{\mathbf{d}(x)h}|||x)}|||y\cdot[v,v^{\star}]\\
&=[h,\mathbf{d}(x)h]\cdot{}_{\mathbf{d}(x)h}|||x\cdot[g,g^{\star}]\cdot[g^{\star},g]\cdot{}_g|||y\cdot[v,v^{\star}]\\
&={}_h|x\cdot[g^{\star},g]\cdot{}_g|||y\cdot[v,v^{\star}]\\
&={}_h|x\cdot[g^{\star},\mathbf{d}(y)g^{\star}]\cdot[\mathbf{d}(y)g^{\star},g]\cdot{}_g|||y\cdot[v,v^{\star}]\\
&\qquad\qquad\Big(g^{\star}\;\mathcal{L}\;g\;\omega\;\mathbf{d}(y),\ \text{and so } g^{\star}\;\mathcal{L}\;\ \mathbf{d}(y)g^{\star}\;\omega\;\mathbf{d}(y)\Big).
\end{aligned}$$

As $\mathbf{d}(y)g^{\star}\;\mathcal{L}\;g^{\star}\;\mathcal{L}\;g$ and $\mathbf{d}(y)g^{\star}, g\;\omega\;\mathbf{d}(y)$, by the dual of (IC5), we have that

$$\mathbf{r}({}_{\mathbf{d}(y)g^{\star}}|||y)\;\mathcal{L}\;\mathbf{r}({}_g|||y)$$

and

$$[\mathbf{d}(y)g^{\star},g(\mathbf{d}(y)g^{\star})]\cdot{}_{g(\mathbf{d}(y)g^{\star})}|||y={}_{\mathbf{d}(y)g^{\star}}|||y\cdot[\mathbf{r}({}_{\mathbf{d}(y)g^{\star}}|||y),\mathbf{r}({}_g|||y)\mathbf{r}({}_{\mathbf{d}(y)g^{\star}}|||y)],$$

that is,

$$[\mathbf{d}(y)g^{\star},g]\cdot{}_g|||y={}_{\mathbf{d}(y)g^{\star}}|||y\cdot[\mathbf{r}({}_{\mathbf{d}(y)g^{\star}}|||y),\mathbf{r}({}_g|||y)],$$

and so

$$\begin{aligned}
&{}_h|x \cdot [g^\star, \mathbf{d}(y)g^\star] \cdot [\mathbf{d}(y)g^\star, g] \cdot {}_g|||y \cdot [v, v^\star] \\
&= {}_h|x \cdot [g^\star, \mathbf{d}(y)g^\star] \cdot {}_{\mathbf{d}(y)g^\star}|||y \cdot [\mathbf{r}({}_{\mathbf{d}(y)g^\star}|||y), \mathbf{r}({}_g|||y)] \cdot [v, v^\star] \\
&= {}_h|x \cdot [g^\star, \mathbf{d}(y)g^\star] \cdot {}_{\mathbf{d}(y)g^\star}|||y \cdot [\mathbf{r}({}_{\mathbf{d}(y)g^\star}|||y), v^\star] \\
&\qquad\qquad\qquad \Big(v = \mathbf{r}({}_h|||(x \cdot y)) = \mathbf{r}({}_g|||y), \text{ by (IC3)}\Big) \\
&= {}_h|x \cdot [g^\star, \mathbf{d}(y)g^\star] \cdot {}_{\mathbf{d}(y)g^\star}|||y \cdot [\mathbf{r}({}_{\mathbf{d}(y)g^\star}|||y), (\mathbf{r}({}_{\mathbf{d}(y)g^\star}|||y))^\star] \\
&\qquad \Big(v = \mathbf{r}({}_g|||y) \; \mathcal{L} \; \mathbf{r}({}_{\mathbf{d}(y)g^\star}|||y), \text{ and so } v^\star = (\mathbf{r}({}_{\mathbf{d}(y)g^\star}|||y))^\star\Big) \\
&= {}_h|x \cdot {}_{g^\star}|y \\
&= {}_h|x \cdot {}_{\mathbf{r}({}_h|x)}|y \qquad \Big(g^\star = \mathbf{r}({}_h|x)\Big).
\end{aligned}$$

(P6) It follows from (IC6).

(PC1) Suppose that $x \in Q$, $h \in U$ and $h \,\omega\, \mathbf{d}(x)$. Then by Proposition 1.41, $\mathbf{r}({}_h|||x) \,\omega\, \mathbf{r}(x)$ and ${}_h|||x = x|||_{\mathbf{r}({}_h|||x)}$. Thus, $\mathbf{d}(x|||_{\mathbf{r}({}_h|||x)}) = h$. In addition, we have that

$$ {}_h|x = {}_h|||x \cdot [\mathbf{r}({}_h|||x), (\mathbf{r}({}_h|||x))^\star] $$

and

$$\begin{aligned}
x|_{\mathbf{r}({}_h|||x)} &= [(\mathbf{d}(x|||_{\mathbf{r}({}_h|||x)}))^+, \mathbf{d}(x|||_{\mathbf{r}({}_h|||x)})] \cdot x|||_{\mathbf{r}({}_h|||x)} \\
&= [h^+, h] \cdot x|||_{\mathbf{r}({}_h|||x)}.
\end{aligned}$$

Clearly, we have that

$$\begin{aligned}
&{}_h|x \cdot [(\mathbf{r}({}_h|||x))^\star, \mathbf{r}({}_h|||x)] \\
&= {}_h|||x \cdot [\mathbf{r}({}_h|||x), (\mathbf{r}({}_h|||x))^\star] \cdot [(\mathbf{r}({}_h|||x))^\star, \mathbf{r}({}_h|||x)] \\
&= {}_h|||x \\
&= x|||_{\mathbf{r}({}_h|||x)} \\
&= [h, (\mathbf{d}(x|||_{\mathbf{r}({}_h|||x)}))^+] \cdot x|_{\mathbf{r}({}_h|||x)} \\
&\qquad\qquad \Big(x|_{\mathbf{r}({}_h|||x)} = [(\mathbf{d}(x|||_{\mathbf{r}({}_h|||x)}))^+, h] \cdot x|||_{\mathbf{r}({}_h|||x)}\Big).
\end{aligned}$$

Hence, ${}_h|x \,\rho\, x|_{\mathbf{r}({}_h|||x)}$.

Suppose that $k\,\omega\,\mathbf{r}(x)$ and ${}_h|x\,\rho\,x|_k$. Then

$$\begin{aligned} &{}_h|x\cdot[\mathbf{r}({}_h|x),k]=[h,\mathbf{d}(x|_k)]\cdot x|_k\\ \Rightarrow\ &({}_h|||x)\cdot[\mathbf{r}({}_h|||x),(\mathbf{r}({}_h|||x))^\star]\cdot[(\mathbf{r}({}_h|||x))^\star,k]\\ &\qquad=[h,(\mathbf{d}(x|||_k))^+]\cdot[(\mathbf{d}(x|||_k))^+,\mathbf{d}(x|||_k)]\cdot x|||_k\\ \Rightarrow\ &({}_h|||x)\cdot[\mathbf{r}({}_h|||x),k]=[h,\mathbf{d}(x|||_k)]\cdot x|||_k\quad\big(\text{by (IC3)}\big),\end{aligned}$$

that is, ${}_h|||x\,\rho\,x|||k$ in Q, where ρ is defined in Chapter 6. Since $h\,\omega\,\mathbf{d}(x)$, we have that $h\in S(h,\mathbf{d}(x))$. Similarly, $k\in S(\mathbf{r}(x),k)$. Now, we calculate in Q/ρ

$$\begin{aligned}\overline{1_h}\odot\bar{x}&=\overline{(1_h\diamond h)\cdot(h*x)}\\ &=\overline{(1_h\otimes x)_h}\\ &=\overline{1_h|||_h\cdot[h,h]\cdot[h,h\mathbf{d}(x)]\cdot{}_{h\mathbf{d}(x)}|||x}\\ &=\overline{1_h|||_h\cdot{}_h|||x}\qquad\big(h\ \omega\ \mathbf{d}(x)\big)\\ &=\overline{1_h\cdot{}_h|||x}\\ &=\overline{{}_h|||x}\end{aligned}$$

and dually, we have $\bar{x}\odot\overline{1_k}=\overline{x|||_k}$. As ${}_h|||x\,\rho\,x|||k$, we obtain that $\overline{1_h}\odot\bar{x}=\bar{x}\odot\overline{1_k}$. Since ${}_h|||x=x|||_{\mathbf{r}({}_h|||x)}$, we obtain that $\overline{1_h}\odot\bar{x}=\bar{x}\odot\overline{1_{\mathbf{r}({}_h|||x)}}$. Hence, $\bar{x}\odot\overline{1_k}=\bar{x}\odot\overline{1_{\mathbf{r}({}_h|||x)}}$. Due to $\bar{x}\,\mathcal{L}^*\,\overline{1_{\mathbf{r}(x)}}$ and $k,\mathbf{r}({}_h|||x)\in\omega(\mathbf{r}(x))$, we have that $\overline{1_k}=\overline{1_{\mathbf{r}({}_h|||x)}}$. Since U is isomorphic to $\overline{U}$, it follows that $k=\mathbf{r}({}_h|||x)$, and consequently, the uniqueness holds.

We note that ${}_{\mathbf{d}(x)}|x={}_{\mathbf{d}(x)}|||x\cdot[\mathbf{r}({}_{\mathbf{d}(x)}|||x),(\mathbf{r}({}_{\mathbf{d}(x)}|||x))^\star]=x\cdot[\mathbf{r}(x),(\mathbf{r}(x))^\star]$. Clearly, ${}_{\mathbf{d}(x)}|x\,\rho\,x$. Thus, (PC1) holds.

(PC2) Suppose that $x\in Q$ and for $i=1,2$, $e_i,f_i\in U$ are such that $e_i\,\omega\,\mathbf{d}(x)$, $f_i\,\omega\,\mathbf{r}(x)$ and ${}_{e_i}|x\,\rho\,x|_{f_i}$. In view of the proof of (PC1), we have that $f_i=\mathbf{r}({}_{e_i}|||x)$. If $e_1\,\omega^r\,e_2$, by (IC5), we have that $f_1\,\omega^r\,f_2$ and

$$[e_1,e_1e_2]\cdot({}_{e_1e_2}|||x)=({}_{e_1}|||x)\cdot[f_1,f_1f_2].$$

Thus, $\mathbf{r}({}_{e_1e_2}|||x)=f_1f_2$. Again in view of the proof of (PC1), we have that ${}_{e_1e_2}|x\,\rho\,x|_{f_1f_2}$.

Similarly, the second part holds. □

To sum up, we have:

Corollary 11.11. *An IC-RBS category with respect to the restriction and co-restriction defined before Lemma 11.7 forms an inductive*$_2$ *cancellative category.*

Conversely, an inductive$_2$ *cancellative category with respect to the restriction and co-restriction defined before Lemma 11.10 forms an IC-RBS category.*

Also, we have:

Lemma 11.12. *Let* P_1 *and* P_2 *be IC-RBS categories and* $F : P_1 \to P_2$ *be an RBS functor. Then* F *is an inductive*$_2$ *functor from* P_1 *to* P_2*.*

Conversely, let Q_1 *and* Q_2 *be inductive*$_2$ *cancellative categories over* U_1 *and* U_2*, respectively, and* $\phi : Q_1 \to Q_2$ *be an inductive*$_2$ *functor. Then* ϕ *is an RBS functor from* Q_1 *to* Q_2*.*

Proof. Suppose that P_1 and P_2 are IC-RBS categories and $F : P_1 \to P_2$ is an RBS functor. If $x \leq y$ in P_1, then there exists $e \in U$ such that $e \;\omega\; \mathbf{d}(y)$ and $x = {}_e||y$, that is, $x = {}_e|y \cdot [\mathbf{r}({}_e|y), k]$, where $k \in \mathbf{r}(y)$ and ${}_e|y \;\rho\; y|_k$. Certainly, we have $eF \;\omega\; \mathbf{d}(yF)$ and $kF \;\omega\; \mathbf{r}(yF)$. From ${}_e|y \;\rho\; y|_k$, we have $({}_e|y)F \;\rho\; (y|_k)F$ by Lemma 10.8, that is, ${}_{eF}|yF \;\rho\; yF|_{kF}$ by (PF3). Since $({}_e|y)F \;\rho\; {}_{eF}|yF$ and $\mathbf{d}(({}_e|y)F) = eF = \mathbf{d}({}_{eF}|yF)$, we have that

$$({}_e|y)F = {}_{eF}|yF \cdot [\mathbf{r}({}_{eF}|yF), \mathbf{r}(({}_e|y)F)],$$

and so we have

$$\begin{aligned} xF &= ({}_e|y)F \cdot [\mathbf{r}({}_e|y), k]F \\ &= {}_{eF}|yF \cdot [\mathbf{r}({}_{eF}|yF), \mathbf{r}(({}_e|y)F)] \cdot [\mathbf{r}({}_e|y)F, kF] \qquad \big(\text{by (PF2)}\big) \\ &= {}_{eF}|yF \cdot [\mathbf{r}({}_{eF}|yF), \mathbf{r}(({}_e|y)F)] \cdot [\mathbf{r}(({}_e|y)F), kF] \\ &= {}_{eF}|yF \cdot [\mathbf{r}({}_{eF}|yF), kF] \qquad \big(\mathbf{r}({}_{eF}|yF) \;\mathcal{L}\; \mathbf{r}(({}_e|y)F) \;\mathcal{L}\; kF\big) \\ &= {}_{eF}||yF \end{aligned}$$

so that $xF \leq yF$. Hence F is order-preserving. Together with (PF1) and (PF2), F is an inductive$_2$ functor.

Conversely, suppose that Q_1 and Q_2 are inductive$_2$ cancellative categories over U_1 and U_2, respectively, and $\phi : Q_1 \to Q_2$ is an inductive$_2$ functor. By (IOF1) and (IOF2), (PF1) and (PF2) hold. We now show that Condition (PF3) holds. If $x \in Q_1$ and $h \in U_1$ with $h \ \omega^l \ \mathbf{d}(x)$, then

$$ {}_h|x = [h, \mathbf{d}(x)h] \cdot {}_{\mathbf{d}(x)h}|||x \cdot [\mathbf{r}({}_{\mathbf{d}(x)h}|||x), (\mathbf{r}({}_{\mathbf{d}(x)h}|||x))^{\star}]. $$

Note that ${}_{\mathbf{d}(x)h}|||x \leq x$, and so ${}_{\mathbf{d}(x)h}|||x\phi \leq x\phi$ as ϕ is order-preserving. Since $(\mathbf{d}(x)h)\phi \ \omega \ \mathbf{d}(x)\phi$, ${}_{(\mathbf{d}(x)h)\phi}|||x\phi$ is defined and ${}_{(\mathbf{d}(x)h)\phi}|||x\phi \leq x\phi$. As $\mathbf{d}(({}_{\mathbf{d}(x)h}|||x)\phi) = (\mathbf{d}(x)h)\phi$, we obtain that

$$ ({}_{\mathbf{d}(x)h}|||x)\phi = {}_{(\mathbf{d}(x)h)\phi}|||x\phi $$

by the uniqueness of restrictions. Then

$$
\begin{aligned}
({}_h|x)\phi &= [h, \mathbf{d}(x)h]\phi \cdot ({}_{\mathbf{d}(x)h}|||x)\phi \cdot [\mathbf{r}({}_{\mathbf{d}(x)h}|||x), (\mathbf{r}({}_{\mathbf{d}(x)h}|||x))^{\star}]\phi \\
&= [h\phi, (\mathbf{d}(x)h)\phi] \cdot {}_{(\mathbf{d}(x)h)\phi}|||x\phi \cdot [\mathbf{r}({}_{\mathbf{d}(x)h}|||x)\phi, (\mathbf{r}({}_{\mathbf{d}(x)h}|||x))^{\star}\phi] \quad \big(\text{by (IOF2)}\big) \\
&= [h\phi, \mathbf{d}(x)\phi h\phi] \cdot {}_{\mathbf{d}(x)\phi h\phi}|||x\phi \cdot [\mathbf{r}({}_{\mathbf{d}(x)\phi h\phi}|||x\phi), (\mathbf{r}({}_{\mathbf{d}(x)h}|||x))^{\star}\phi] \\
&= [h\phi, \mathbf{d}(x)\phi h\phi] \cdot {}_{\mathbf{d}(x)\phi h\phi}|||x\phi \cdot [\mathbf{r}({}_{\mathbf{d}(x)\phi h\phi}|||x\phi), (\mathbf{r}({}_{\mathbf{d}(x)\phi h\phi}|||x\phi))^{\star}] \cdot \\
&\qquad\qquad [(\mathbf{r}({}_{\mathbf{d}(x)\phi h\phi}|||x\phi))^{\star}, (\mathbf{r}({}_{\mathbf{d}(x)h}|||x))^{\star}\phi] \\
&= {}_{h\phi}|x\phi \cdot [(\mathbf{r}({}_{\mathbf{d}(x)\phi h\phi}|||x\phi))^{\star}, (\mathbf{r}({}_{\mathbf{d}(x)h}|||x))^{\star}\phi] \\
&= {}_{h\phi}|x\phi \cdot [\mathbf{r}({}_{h\phi}|x\phi), \mathbf{r}(({}_h|x)\phi)]
\end{aligned}
$$

so that $({}_h|x)\phi \ \rho \ {}_{h\phi}|x\phi$ and dually, if $k \ \omega^r \ \mathbf{r}(x)$, we have $(x|_k)\phi \ \rho \ x\phi|_{k\phi}$ so that (PF3) holds. Hence, ϕ is an RBS functor. □

Further, by Corollary 11.6, Corollary 11.11 and Lemma 11.12, we obtain Theorem C mentioned in Chapter 6 as follows:

Theorem C (Armstrong [1]) *The category of concordant semigroups and good morphisms is equivalent to the category of inductive$_2$ cancellative categories and inductive$_2$ functors.*

11.4 The regular case

We focus on regular semigroups in this section. An RBS category P over U is a *regular groupoid* over U if Conditions (RG), (PC1), (PC2) and the duals $(\text{PC1})^{\circ}$ and $(\text{PC2})^{\circ}$ of (PC1) and (PC2) hold:

(RG) for all $x \in P$, there exists $y \in P$ with $\mathbf{d}(y) = \mathbf{r}(x)$ and $\mathbf{r}(y) = \mathbf{d}(x)$ such that $1_{\mathbf{d}(x)} = x \cdot y$ and $y \cdot x = 1_{\mathbf{r}(x)}$.

Notice that in a regular groupoid P, for any $x \in P$, there exists $y \in P$ such that $1_{\mathbf{d}(x)} = x \cdot y$ and $y \cdot x = 1_{\mathbf{r}(x)}$, and so y is the inverse of x so that P is a groupoid. Consequently, P is cancellative and so it is an IC-RBS category. Together with the comments succeeding the definition of inductive$_2$ cancellative category in Chapter 6, we have an immediate consequence of Corollary 11.11 as follows:

Corollary 11.13. *A regular groupoid with respect to the restriction and co-restriction defined before Lemma 11.7 forms an inductive$_2$ groupoid.*

Conversely, an inductive$_2$ groupoid with respect to the restriction and co-restriction defined before Lemma 11.10 forms a regular groupoid.

Since a regular groupoid is an IC-RBS category and a regular semigroup is a special concordant semigroup, it follows from Lemma 11.6 that:

Corollary 11.14. *The category of regular semigroups and morphisms is equivalent to the category of regular groupoids over regular biordered sets and RBS functors.*

Proof. Let S be a regular semigroup with $U = E(S)$. It is sufficient to show that $S\mathbf{C}$ satisfies Condition (RG). Suppose that $(e, x, f) \in S\mathbf{C}$. Since $\mathcal{R} = \widetilde{\mathcal{R}}_U$ and $\mathcal{L} = \widetilde{\mathcal{L}}_U$, we have that $e \; \mathcal{R} \; x \; \mathcal{L} \; f$. It follows from the fact that S is regular that there exists $y \in S$ with $e \; \mathcal{L} \; y \; \mathcal{R} \; f$, $e = xy$ and $yx = f$. Then $(f, y, e) \in S\mathbf{C}$ and the products $(e, x, f) \cdot (f, y, e)$, $(f, y, e) \cdot (e, x, f)$ exist in $S\mathbf{C}$. Moreover, $(e, x, f) \cdot (f, y, e) = (e, xy, e) = (e, e, e) = [e, e] = 1_e$ and similarly, $(f, y, e) \cdot (e, x, f) = 1_f$. Hence, Conditon (RG) holds.

Conversely, let P be a regular groupoid over U. We need to show that $P\mathbf{S}$ is regular. Suppose that $x \in P$. Then there exists $y \in P$ with $\mathbf{d}(y) = \mathbf{r}(x)$ and $\mathbf{r}(y) = \mathbf{d}(x)$ such that $1_{\mathbf{r}(x)} = y \cdot x$ and $1_{\mathbf{d}(x)} = x \cdot y$. So $\overline{1_{\mathbf{d}(x)}} = \overline{x \cdot y} = \overline{x} \odot \overline{y}$. Hence, $\bar{x} \odot \bar{y} \odot \bar{x} = (\bar{x} \odot \bar{y}) \odot \bar{x} = \overline{1_{\mathbf{d}(x)}} \odot \bar{x} = \bar{x}$ so that $P\mathbf{S}$ is regular. ◻

In view of Corollary 11.13 and Corollary 11.14, we have:

Theorem B (Nambooripad [38]) *The category of regular semigroups and morphisms is equivalent to the category of inductive$_2$ groupoids and inductive$_2$ functors.*

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