Infrared behaviour of propagators in cosmological spacetimes

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Abstract

In this thesis, we study the infrared behaviour of propagators in Friedmann-Lemaître-Robertson-Walker (FLRW) spacetimes, of which de Sitter is a special case. For the most part, we are interested in the infrared behaviour of the graviton two-point function, in FLRW spacetime. Naively, it is thought that the two-point function requires an infrared cut-off in order to be well-defined. However, we find a gauge transformation such that the two-point function can be rendered IR finite, for a large class of FLRW spacetimes. The graviton two-point function also experiences an infrared divergence when the separation between the two points is taken to be large. In de Sitter spacetime, in the Landau gauge, this divergence is found to be logarithmic. However, it is found that this logarithmic divergence can also be removed by means of a suitable non-covariant gauge transformation. In finding the large-distance behaviour of the graviton two-point function, we initially found it useful to find the large-distance behaviour of the covariant massless vector propagator, in de Sitter spacetime. Through this calculation, we were able to find a method which could be extended to the more computationally complex case of the graviton two-point function.

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Structure of thesis

This thesis is a study of the infrared behaviour of propagators in cosmological spacetimes. We are concerned with the infrared behaviour of graviton propagators in FLRW spacetimes, with the main focus being their infrared behaviour in de Sitter spacetime. Throughout this thesis, two-point function and propagator are used synonymously.

In Chapter 1, we motivate our interest in the study of propagators in curved spacetimes. The relevant background material underlying the calculations for the rest of the thesis is also introduced. First, General Relativity is briefly recapped. Quantum field theory in flat spacetimes is then introduced, and the key differences between the flat spacetime theory and curved spacetime theory are highlighted. We study the main features of the cosmological spacetimes in which we work, before giving a background on the theory of inflation, which is described by some cosmological spacetimes. The chapter is concluded with a discussion of linearised gravity, in which we give a summary of quantisation in this theory, before reviewing the gauge freedom used in later chapters.

In Chapter 2, we investigate the nature of infrared divergences for the free graviton and inflaton two-point functions in flat FLRW spacetime. The graviton propagator in this chapter is in the transverse, traceless, synchronous (TTS) gauge. These divergences arise because the momentum integral for these two-point functions diverges in the infrared. It is straightforward to see that the power of the momentum in the integrand can be increased by 2 in the infrared using large gauge transformations. This is sufficient for rendering these two-point functions infrared finite for slow-roll inflation. In other words, if, in the infrared, the integrand of the momentum integral for these two-point functions behaves like $p^{-2\nu}$, where p is the momentum, then it can be made to behave like $p^{-2\nu+2}$, by the use of large gauge transformations. On the other hand, it is known that if one smears these two-point functions in a gauge-invariant manner, the power of the momentum in the integrand is changed from $p^{-2\nu}$ to $p^{-2\nu+4}$. This fact suggests that the power of the momentum in the integrand for these two-point functions can be increased by 4 using large gauge transformations. In this chapter, we show that this is indeed the case. Thus, the two-point functions for the graviton and inflaton fields can be made finite by large gauge transformations for a large class of potentials and states in single-field inflation.

In Chapter 3, we study the large-distance behaviour of the covariant massless vector propagator in the covariant gauge in n-dimensional de Sitter spacetime, where $n \geq 4$. Specifically, the large-distance limit of the massless limit of the vector propagator in the Stueckelberg theory - an extension of the Proca theory, with an additional gauge-fixing term - is found. We work to leading order in the de Sitter-invariant Z, as the large-Z limit corresponds to the large-distance limit of the propagator. In this limit, it is shown that this propagator tends to a gauge-dependent constant, where the gauge worked in is described by the Stueckelberg parameter ξ. In the Landau gauge, where $\xi = 0$, this constant is found to be 0. This result is in agreement with the 4 dimensional case discussed in [1].

In Chapter 4, the method described in the previous chapter is applied to a different propagator: the large-distance behaviour of the graviton propagator in n-dimensional de Sitter spacetime is found. For the remainder of the thesis, the graviton propagator is in the covariant gauge. We start from the propagator found in [2], which is written in terms of two gauge parameters α and β, and find the large-distance limit in the case when $\beta > 0$. The propagator is then expanded in terms of the de Sitter invariant Z, which is a measure of the spacetime distance between two points, as stated above. The expected large-distance behaviour, discussed in [2], is found: in the Landau gauge, when $\alpha = 0$, the propagator has a logarithmic divergence, and, when $\alpha = \frac{n+1}{n-1}$, the divergence is linear. Additionally, for $n = 4$ our result reduces to that found in [2] and [3].

In Chapter 5, the logarithmic divergence present in the large-distance limit of the propagator, in the Landau gauge, is studied. We show that a covariant gauge transformation can not be used to remove this logarithmic divergence. Instead, it is found that this logarithmic divergence can only be traded for a linear divergence identical to the one present when $\alpha = \frac{n+1}{n-1}$. However, if one uses a non-covariant transformation, this logarithmic divergence can be gauged away, as is expected from the conclusions of [4] and [5].

Finally, in Chapter 6, we summarise the results of the thesis, as well as discussing some open problems in the area.

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Declaration

I declare that this thesis is a presentation of original work and I am the sole author. This work has been carried out under the supervision of Prof. Atsushi Higuchi and has not previously been presented for an award at this, or any other, University. All sources are acknowledged as References.

Chapter 1 is a review of the backgound material for the rest of the thesis. Chapter 2 is based on a paper in collaboration Atsushi Higuchi [6]. Chapter 3 consists of the work presented in [7]. Chapters 4 and 5 are devoted to work which will form a joint paper with Atsushi Higuchi.

Chapter 1

Preliminaries

In this chapter we motivate our interest in the study of propagators on cosmological spacetimes. Additionally, we provide the mathematical background behind the calculations in this thesis. We first review General Relativity, before introducing quantum field theory in curved spacetimes. Specifically, we highlight the differences found, and problems encountered, when we no longer work in a flat spacetime. In the majority of the thesis we work in de Sitter spacetime, so we give an overview of the historical importance, and mathematical significance, of this spacetime. In this section we also discuss a wider class of spacetimes in which we work, namely FLRW spacetimes, of which the conformally flat chart of de Sitter is an example. Next, we provide a summary of inflation. We then turn our attention to the theory of the quantum fields propagating on such cosmological spacetimes. We give an overview of linearised gravity, the theory behind most chapters. The theory of massless vector fields, the other propagator studied, is left to be introduced in Chapter 3. The infrared divergences of all relevant propagators are then stated. We outline the gauge freedom present in linearised gravity, before briefly explaining how gauge transformations can be used to remove infrared divergences. Finally, a summary of the work presented in this thesis is given.

In what follows, we will use the mostly plus metric convention, as well as natural units, so $\hbar = c = 1$. We also use the standard index notation, so that brackets around indices denote symmetry, and square brackets denote antisymmetry. The indices a, b, c, d, e denote spatial and time indices, whereas the indices i, j, k are purely spatial.

1.1 Motivations for studying propagators on cosmological spacetimes

As General Relativity is not a renormalisable theory [8], it has proven hard to find a full quantum theory of gravity. A natural step towards a full quantum theory of gravity is to consider a semiclassical theory: quantum field theory (QFT) in curved spacetimes (CST). In this theory, we combine the curved spacetimes used in General Relativity with the successful QFT used to describe the other three fundamental forces. This is a semiclassical theory in which we treat the background spacetime classically and quantise perturbations about it; gravitational perturbation is incorporated as a perturbation around a fixed background. We also ignore the backreaction of these perturbations on the background.

While this does not provide a full theory of quantum gravity, it is still of interest to study. First, it is a theory which helps us understand what is fundamental in QFT, as well as learn about the interaction between QFT and gravity. We also note that we can describe the early universe and extreme astrophysical environments without a full theory of quantum gravity, so QFT in CST can be a useful tool. Additionally, other quantum gravity theories, such as string theory, require knowledge of the study of quantum fields on CST, under certain conditions.

Throughout this thesis, we study propagators in de Sitter and the larger class of FLRW spacetimes, some of which are inflationary. This is of physical interest as our universe is believed to have experienced an inflationary period in its early stages of development. Inflation as a theory of the early universe was proposed as a solution to the flatness and horizon problems of the standard Big Bang model. Such a theory was first considered independently by Guth and Sato, in [9] and [10], respectively, before being modified by the authors of [11] and [12], amongst others. We will give more detail in later sections of this introduction. The background spacetime for the inflationary model is a spatially flat FLRW spacetime that expands exponentially. If the expansion is exactly exponential, the spacetime is de Sitter spacetime. Studying the behaviour of propagators on these backgrounds is therefore of interest in developing QFT in the early universe.

1.2 General Relativity

In this section, we give a brief review of General Relativity. We start by looking at the physical reasoning behind the theory, before giving a brief introduction to its mathematical description. We describe the effect of a curved background on the transport of vectors on a manifold, which leads to the definition of a covariant derivative. Next, we define the Riemann tensor, which quantifies the curvature of the spacetime, and some relevant contractions of this tensor. Then, we state Einstein's equations, and discuss some interesting consequences arising from them. We conclude with a short discussion on geodesics. For more detail on the concepts discussed in this chapter, we direct the reader to Carroll [13], amongst others.

Einstein proposed the theory of special relativity in 1905 [14]. The two postulates of special relativity are that:

1. The laws of physics are the same for observers in different inertial reference frames,

2. The speed of light c is constant in all inertial reference frames.

The next step was to try to incorporate accelerating frames into the theory. General Relativity was proposed by Einstein in 1907 [15] and then from 1911 onwards in [16], amongst other, to accomplish this. It is based on the equivalence principle: the effects of a gravitational field are indistinguishable from the acceleration of a local inertial reference frame. Hence, there is no way to experimentally find whether we are uniformly accelerating in free space or feeling the effects of a gravitational field.

Mathematically, we describe General Relativity using the pair $(\mathcal{M}, g_{ab})$. A manifold $\mathcal{M}$ is a space that locally "looks like" $\mathbb{R}^n$, and a metric tensor g_{ab} describes the geometry of the manifold. At each point p on the manifold we define the tangent space T_p as the space of directional derivative operators along curves through p. It is necessary to compare vectors in T_p to those in T_q, for two points p and q. To this end, we introduce parallel transport: move $\xi \in T_p$ from p to q along a curve γ such that the norm of ξ is constant, and it is oriented in the same direction. For flat space, if a vector is transported from a point p along a closed curve back to the point p, it will be brought back to the same vector. However, on a curved manifold, this is not necessarily the case. For a pictorial representation of parallel transport, we direct the reader to Figure 1.1.

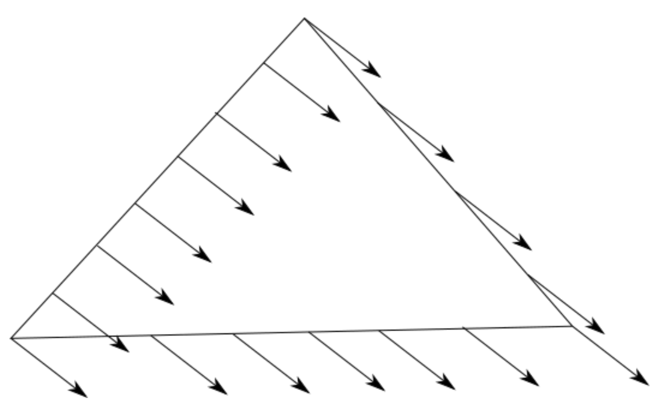

Figure 1.1: Parallel transport of a vector in flat spacetime along a triangular path.

We will define curvature of a manifold through the action of derivatives. First, we must define a derivative, on a curved manifold. In the following, v^b etc. are components of a vector in a particular coordinate system. A derivative operator ∇_a is a map from the space of all (k, l) tensor fields to $(k, l+1)$ tensor fields satisfying the following conditions:

1. Linearity: $\nabla_a \left(\alpha v^b + \beta w^c\right) = \alpha \nabla_c v^a + \beta \nabla_a w^b$ for $\alpha, \beta \in \mathbb{R}$,
2. Leibniz rule: $\nabla_a \left(v^b w^c\right) = \nabla_a v^b \cdot w^c + v^b \cdot \nabla_a w^c$,
3. Torsion free derivative: $\nabla_{[a}\nabla_{b]} f = 0$ for a smooth scalar field f,
4. Consistency with the notion of tangent vectors as directional derivatives acting on scalar fields: $v(f) = v^a \nabla_a f$, where $v(f) = \frac{\mathrm{d}f}{\mathrm{d}\lambda}$, and v^a is tangent to a curve parametrised by λ. As an aside, a derivative operator gives rise to the notion of parallel transport: if u^a is the tangent vector to the curve, then $u^a \nabla_a V^b = 0$ implies that V^b is parallelly transported along the curve.
5. Consistency with tensor contraction: $\nabla_c T^a{}_a = \nabla_c \left(T^a{}_b \delta^b_a\right) = \left(\nabla_c T^a{}_b\right) \delta^b_a$.

Due to the curvature of the manifold, we have lost the notion of a unique derivative operator. Two derivative operators will agree on functions, by which we mean that

$$\nabla_a f = \partial_a f \,, \tag{1.2.1}$$

for a smooth function f, and where the right-hand side is the partial derivative with respect to the coordinates. This agreement is not necessarily true for a general tensor $T^{a_1 \cdots a_m}{}_{b_1 \cdots b_n}$. We define the derivative operator on vectors and covectors by giving the connection C^b_{ac}:

$$\nabla_a w_b = \partial_a w_b - C^c_{ab} w_c \,, \tag{1.2.2}$$
$$\nabla_a w^b = \partial_a w^b + C^b_{ac} w^c \,. \tag{1.2.3}$$

Equation (1.2.3) follows from equation (1.2.2) by using property 2 of the derivative operator, and equation (1.2.1). Due to the symmetry under exchange of the lower two indices, which follows from the torsion free condition on the derivative, this connection has $\frac{n^2(n+1)}{2}$ components. From these two relations, the generalisation to higher order tensors is trivial:

$$\begin{aligned} \nabla_a T^{b_1 b_2 \cdots b_k}{}_{c_1 c_2 \cdots c_l} =& \partial_a T^{b_1 b_2 \cdots b_k}{}_{c_1 c_2 \cdots c_l} + C^{b_1}_{ad} T^{d b_2 \cdots b_k}{}_{c_1 c_2 \cdots c_l} + \cdots \\ & - C^d_{a c_1} T^{b_1 b_2 \cdots b_k}{}_{d c_2 \cdots c_l} - \cdots \,. \end{aligned} \tag{1.2.4}$$

In order to specify a unique derivative, we require the metric-compatibility, i.e.

$$\nabla_c g_{ab} = 0\,. \tag{1.2.5}$$

We quantify the curvature of the manifold by introducing the Riemann tensor. This can be defined as

$$2\nabla_{[a}\nabla_{b]}w_c = R_{abcd}w^d\,. \tag{1.2.6}$$

This tensor is antisymmetric under the exchange of $a \leftrightarrow b$ and $c \leftrightarrow d$, and is symmetric under the exchange $ab \leftrightarrow cd$. Additionally, the Riemann tensor obeys the first Bianchi identity:

$$R_{a[bcd]} = 0\,. \tag{1.2.7}$$

For completeness, we also mention the second Bianchi identity

$$\nabla_{[e}R_{ab]cd} = 0\,, \tag{1.2.8}$$

which we use later to find some relevant conservation equations.

In a coordinate basis, the Riemann tensor is

$$R^a{}_{bcd} = \partial_c\Gamma^a_{db} - \partial_d\Gamma^a_{cb} + \Gamma^a_{ce}\Gamma^e_{db} - \Gamma^a_{de}\Gamma^e_{cb}\,, \tag{1.2.9}$$

where the Christoffel symbols Γ^a_{bc} are components of the connection. These components are defined as

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}\left[\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}\right]\,. \tag{1.2.10}$$

From the Riemann tensor, we can form an additional tensor and scalar. The Ricci tensor is the contraction

$$R_{ab} = R^c{}_{acb}\,, \tag{1.2.11}$$

and the Ricci scalar is

$$R = g^{ab}R_{ab}\,. \tag{1.2.12}$$

We can use the Ricci tensor and scalar to write the Riemann tensor as a traceless tensor with all the same symmetries of the Riemann tensor. This is the Weyl tensor, and it is given by

$$C_{abcd} = R_{abcd} - \frac{2}{n-2}\left[g_{a[c}R_{d]b} - g_{b[c}R_{d]a}\right] + \frac{2}{(n-1)(n-2)}g_{a[c}g_{d]b}R\,. \tag{1.2.13}$$

In Chapter 2, we will use results relating to the linearised Weyl tensor.

Einstein's equations, which describe how the curvature of the spacetime and matter are related, are

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}\,, \tag{1.2.14}$$

where G is the gravitational constant, Λ is the cosmological constant, g_{ab} is the symmetric metric tensor, and T_{ab} is the stress-energy tensor, which describes the matter. Due to the symmetry of all other tensors in Einstein's equations, this tensor is symmetric. It is also conserved:

$$\nabla^a T_{ab} = 0\,. \tag{1.2.15}$$

This can be seen by the following contraction of the second Bianchi identity, given by equation (1.2.8):

$$\begin{aligned} g^{db} g^{ce} \left[\nabla_e R_{abcd} + \nabla_b R_{eacd} + \nabla_a R_{becd}\right] = 0\,, \\ \Rightarrow\; 2\nabla^c R_{ac} - \nabla_a R = 0\,. \end{aligned} \tag{1.2.16}$$

In our statement of Einstein's equations we have included the cosmological constant term, which gives the vacuum energy of the universe. This cosmological constant term was introduced as Einstein's equations initially admitted no static solution, which he believed would be necessary to accurately describe our universe [17]. However, as it was later observed that the universe was expanding, there was no longer a need for this term [18,19]. For this reason, Einstein considered his inclusion of the cosmological constant as his 'biggest blunder'[1]. However, the current accelerated expansion of the universe implies that Λ is in fact non-zero and positive [20,21]. The de Sitter solution is a solution of the vacuum Einstein equations with a cosmological constant $\Lambda > 0$, with a physical interpretation, as will be discussed in later sections of the introduction. As a final remark on this subject, let us point out that the cosmological constant problem is an open problem in cosmology. This is a fine-tuning problem, where observations of the vacuum energy of the universe, where Λ is of the order 1 cm^{-2} [22], are found to be hugely inconsistent with the theoretical value predicted by QFT, where $\Lambda \approx 10^{-55}$cm^{-2} [23].

We conclude this section with a brief discussion on geodesics. These are curves of zero acceleration, and timelike geodesics map out the path a freely falling body would follow under no external forces. The geodesic distance between two points is the extremised distance between them. Mathematically, a geodesic is a curve whose tangent vector, ξ^a, satisfies

$$\xi^a \nabla_a \xi^b = \alpha \xi^b\,, \tag{1.2.17}$$

for a constant of proportionality α. In the case of an affinely parametrised geodesic

[1] According to phycisist George Gamow

$\alpha = 0$. Geodesics can also be related to curvature, as a geodesic is a curve whose tangent vectors remain parallel to each other if they are parallel transported along it. Indeed, another way of describing curvature is through geodesic deviation: two initially parallel geodesics can bend towards or away from each other as they propagate through the spacetime.

1.3 Quantum field theory in curved spacetimes

In this section, we highlight the key points of QFT in CST. This theory combines QFT in flat spacetime with the curved spacetimes considered in General Relativity. We highlight the main differences between QFT in flat and curved spacetimes, before concluding with a brief summary of correlation functions. For a review of QFT in CST, we direct the reader to, for example, [24, 25]. This section provides a summary of key facts which can be found in [24, 25], along with flat space results that can be found in [26].

In order to extend QFT in flat spacetime to a CST background, we use the minimal coupling prescription, so that

$$\partial_a \to \nabla_a \,, \tag{1.3.1}$$

$$\eta^{ab} \to g^{ab} \,, \tag{1.3.2}$$

$$\mathrm{d}^n x \to \sqrt{-g}\,\mathrm{d}^n x \,, \tag{1.3.3}$$

where η_{ab} is the Minkowski metric, and g is the determinant of the metric. Using this new volume element, we define the action S to be

$$S = \int \mathcal{L}\sqrt{-g}\,\mathrm{d}^n x \,, \tag{1.3.4}$$

for a Lagrangian $\mathcal{L}$. For a set of fields $\phi^{(i)}$ propagating in a curved spacetime with the line element

$$\mathrm{d}s^2 = g_{ab}(x)\mathrm{d}x^a \mathrm{d}x^b \,, \tag{1.3.5}$$

we consider an action invariant under diffeomorphisms of the manifold:

$$S[\phi'(x'), \nabla'\phi'(x'), g'_{ab}(x')] = S[\phi(x), \nabla\phi(x), g_{ab}(x)] \,, \tag{1.3.6}$$

for two coordinate systems x and x'.

The majority of this thesis is concerned with the infrared (IR) behaviour of the graviton two-point function, so for the remainder of this section we give a brief recap of important features of two-point functions.

Correlation functions, of which the two-point function is a particular example, are the vacuum expectation values of products of field operators Φ about a set of spacetime points x_i. In the following we state some interesting two-point functions for a field Φ, with arbitrary spin. We will specialise to the graviton two-point function at the end of this section.

The two simplest two-point functions are the Wightman functions:

$$G^{+}(x,x') = \left\langle 0|\Phi(x)\Phi(x')|0\right\rangle \,, \tag{1.3.7}$$
$$G^{-}(x,x') = \left\langle 0|\Phi(x')\Phi(x)|0\right\rangle \,. \tag{1.3.8}$$

The Feynman propagator is the following time-ordered product of fields:

$$iG_F(x,x') = \left\langle 0|T\left(\Phi(x)\Phi(x')\right)|0\right\rangle = \Theta(t-t')G^{+}(x,x') + \Theta(t'-t)G^{-}(x,x')\,, \tag{1.3.9}$$

where the Heaviside step function is

$$\Theta(t) = \begin{cases} 1 & \text{for } t > 0\,, \\ 0 & \text{for } t < 0\,. \end{cases} \tag{1.3.10}$$

The notation $G(x,x')$ is used as these are Green's functions for an operator $P(x)$, which acts in the following way

$$P(x)\Phi(x) = 0\,, \tag{1.3.11}$$

where the latter equation is the field equation for $\Phi(x)$. We have

$$P(x)G^{\pm}(x,x') = 0\,, \tag{1.3.12}$$
$$P(x)G_F(x,x') = -\delta^{(4)}(x-x')\,. \tag{1.3.13}$$

For the majority of this thesis, we consider the graviton two-point function, which is the following Wightman function:

$$G_{ab:a'b'}(x,x') = \left\langle 0|h_{ab}(x)h_{a'b'}(x')|0\right\rangle \,, \tag{1.3.14}$$

for a metric perturbation h_{ab} about a classical spacetime. In Chapter 3, we find the large-distance behaviour of the covariant massless vector propagator. This is the following Wightman function:

$$\left\langle 0|A_a(x)A_{a'}(x')|0\right\rangle \,, \tag{1.3.15}$$

where $A_a(x)$ is the vector potential.

We conclude by noting that propagator (Green's function) and two-point function (satisfying the homogeneous field equation without the delta function) have been used synonymously throughout this thesis because the large-distance behaviour and the IR divergences are the same for these two quantities.

1.4 Cosmological spacetimes

In this section, we will introduce the two spacetimes in which we work for the rest of the thesis: de Sitter and FLRW. Here we note that the conformally-flat chart of de Sitter spacetime is an FLRW spacetime. The majority of this section focuses on de Sitter, as all chapters, except for Chapter 2, use this spacetime. First, we discuss the historical and physical significance of de Sitter spacetime. We then see how it is described mathematically, and mention a few subtleties with definitions of coordinate systems. We finish with a brief introduction of FLRW spacetime, and state a few relations relevant to cosmology.

We begin by mentioning the physical and historical significance of de Sitter spacetime, which is the generalisation of Minkowski spacetime. Mathematically, de Sitter is the maximally symmetric[2] solution to the vacuum Einstein equations, with a positive cosmological constant [27, 28]. As such, it is of special interest to mathematical physicists. This spacetime is of physical interest as it is an inflationary spacetime. Additionally, some spacetimes in the more general FLRW class are inflationary. This will be expanded on in the next section, so here we merely state that in de Sitter, inflation is driven by the cosmological constant Λ, which motivates its inclusion in this chapter.

As we are now working in a curved spacetime, we have lost the concept of a unique vacuum state. The $SO(4,1)$ symmetry of de Sitter spacetime, along with the Hadamard condition, is such that we are able to define a natural vacuum state. Hadamard states are generally taken to be physical states for linearised quantum fields on curved spacetimes, for more detail see, for example, [29, 30]. This state is called the Bunch-Davies vacuum [31], or the Euclidean vacuum, as it is the natural extension of the flat space vacuum to de Sitter in that the Feynmann propagator in this vacuum state is obtained by analytic continuation of the Green's functions on the sphere S^n.

We now look at de Sitter spacetime in more mathematical detail. As previously mentioned, it is the solution to the vacuum Einstein equations:

$$R_{ab} - \frac{1}{2}g_{ab}R + \Lambda g_{ab} = 0\,, \tag{1.4.1}$$

which is equation (1.2.14), with the stress-energy tensor $T_{ab} = 0$.

We can think of de Sitter as an embedding in $\mathbb{R}^{n+1}$. In this embedding space, the

[2] A maximally symmetric spacetime admits the maximum number of Killing vectors, which, for a n dimensional space, is $\frac{n(n+1)}{2}$. Killing vectors satisfy the equation $\nabla_a\xi_b + \nabla_b\xi_a = 0$ [13].

metric, inherited by n-dimensional de Sitter spacetime, is

$$\mathrm{d}s^2 = -\mathrm{d}X_1^2 + \mathrm{d}X_2^2 + \cdots + \mathrm{d}X_{n+1}^2\,, \tag{1.4.2}$$

subject to the condition that

$$X \cdot X = \frac{1}{H^2}\,, \tag{1.4.3}$$

for $X = (X^1, X^2, \cdots, X^{n+1})$. This describes a hyperboloid in $(n+1)$-dimensional Minkowski spacetime. The Hubble constant H gives the rate at which the universe is expanding, and is defined by

$$H = \frac{v}{D}\,, \tag{1.4.4}$$

where v is the recessional velocity - the speed at which an object moves from the observer - and D is the proper distance from the object to the observer. This ratio is constant in de Sitter spacetime, but the same is not true in FLRW spacetime.

This spacetime can be described by multiple coordinate systems, the most common of such being the static, global, and conformal coordinates. These coordinate systems, and their corresponding metrics, are valid on different patches of de Sitter spacetime. Here, we briefly review the key features of these coordinate systems, using the notation of [32]. In the latter paper $H = 1$, so for the following discussion we use this scaling. We start from Euclidean de Sitter, which is the n-dimensional sphere S^n. This is represented by the metric

$$\mathrm{d}s^2_{S^n} = \mathrm{d}\Omega_n^2 = \mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\Omega_{n-1}^2\,, \tag{1.4.5}$$

for $\vartheta \in [0, \pi]$, and where $\mathrm{d}\Omega_n^2$ is the metric of the n-dimensional sphere.

From condition (1.4.3), we can define three further coordinate systems to describe de Sitter spacetime. We perform a Wick rotation on the azimuthal angle, given by

$$\tan\phi = \frac{X^1}{X^2}\,, \tag{1.4.6}$$

such that

$$t = i\phi\,, \tag{1.4.7}$$

for $t \in \mathbb{R}$. From this, we find the metric in static coordinates:

$$\mathrm{d}s^2_{\text{static}} = -\cos^2\theta \mathrm{d}t^2 + \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\Omega_{n-2}^2\,, \tag{1.4.8}$$

where

$$\tan\theta = \sqrt{\frac{(X^3)^2 + ... + (X^{n+1})^2}{(X^1)^2 + (X^2)^2}}\,. \tag{1.4.9}$$

This metric is valid for $\theta \in \left[0, \frac{\pi}{2}\right)$, along with the earlier condition that $t \in \mathbb{R}$.

Alternatively, we could have performed a Wick rotation of the polar angle, such that

$$\Theta = i\left(\vartheta - \frac{\pi}{2}\right) , \tag{1.4.10}$$

for $\Theta \in \mathbb{R}$. Additionally, one defines

$$\tan T = \sinh\Theta\,, \tag{1.4.11}$$

to give the metric in global coordinates:

$$\mathrm{d}s^2_{\text{global}} = \sec^2 T(-\mathrm{d}T^2 + \mathrm{d}\chi^2 + \sin^2\chi \mathrm{d}\Omega^2_{n-2})\,, \tag{1.4.12}$$

valid for $T \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. In writing the metric in this form, we have written the metric of the $(n-1)$-sphere in terms of the coordinate χ as

$$\mathrm{d}\Omega^2_{n-1} = \mathrm{d}\chi^2 + \sin^2\chi \mathrm{d}\Omega^2_{n-2}\,, \tag{1.4.13}$$

for $\chi \in (0, \pi)$.

Finally, we consider the metric in conformal coordinates. This is the system that will be used for the remainder of this thesis. Here, the metric is

$$\mathrm{d}s^2_{\text{conformal}} = \frac{1}{(-\eta)^2}\left(-\mathrm{d}\eta^2 + \mathrm{d}x_1^2 + \cdots \mathrm{d}x^2_{n-1}\right) , \tag{1.4.14}$$

where the conformal scaling factor depends on the conformal time $\eta \in (-\infty, 0)$. We can relate this coordinate system back to the global coordinate system by means of the transformation

$$\eta = -\frac{\cos T}{\sin T + \cos\chi}\,, \tag{1.4.15}$$

$$x_i = \frac{\sin\chi}{\sin T + \cos\chi}\hat{X}_i\,, \tag{1.4.16}$$

with

$$\hat{X}_i = \frac{X_{i+2}}{\sqrt{(X_3)^2 + ... + (X_{n+1})^2}}\,. \tag{1.4.17}$$

Figure 1.2, below, gives the Penrose diagram of de Sitter spacetime, which summarises the coordinate range where each system is valid.

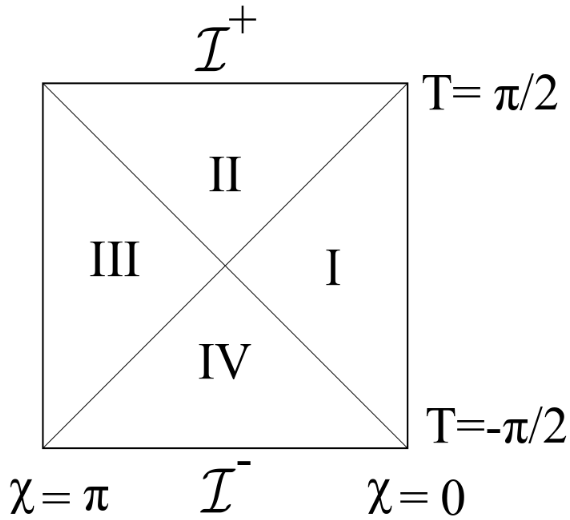

Figure 1.2: Penrose diagram of de Sitter spacetime, using global coordinates. The static patch is region I, and the conformal patch is made up of regions I and II. The future and past spacelike infinities $\mathcal{I}^{\pm}$ are at the $T = \pm\frac{\pi}{2}$, and the cosmological horizons are $T = \pm\left(\chi - \frac{\pi}{2}\right)$.

Region I is the static patch, and this is the only region in which static coordinates can be used. In regions I and II, known as the Poincaré or conformal patch, conformal coordinates are valid. Finally, as the name would suggest, global coordinates are valid everywhere. Also of interest are the horizons in this diagram. The future and past spacelike infinities, $\mathcal{I}^{\pm}$, are given by $T = \pm\frac{\pi}{2}$. Also of interest are the past and future cosmological horizons. These are given by the lines $T = \pm\left(\chi - \frac{\pi}{2}\right)$, defined by the observer at $\theta = 0$. These horizons coincide with the coordinate singularity at $\theta = \frac{\pi}{2}$. For the remainder of the thesis, we work with $H =$ constant, not necessarily equal to 1, in de Sitter spacetime.

For the remainder of the section, we give a brief introduction to FLRW spacetime. We follow the notation of [13], where a more detailed explanation of FLRW spacetime can be found. The idea behind such a spacetime follows from the Copernican principle: that the universe essentially looks the same everywhere. This principle is related to the concepts of homogeneity and isotropy. If a spacetime is homogeneous then it is the same everywhere, so is translation invariant. If it is isotropic at a point, then, for an observer at a particular point in spacetime, the spacetime will look identical in all directions. If a spacetime has both of these properties, then it obeys the Copernican principle. Finally, we note that if a spacetime is both homogeneous and isotropic it is maximally symmetric at constant t, which means that it has the maximum number of Killing vectors. As FLRW spacetimes are homogeneous and isotropic, the fact that they are also maximally symmetric follows trivially.

For the remainder of this section, we work in $n = 4$ dimensions. The class of FLRW spacetimes are the solution to Einstein's equations where the cosmological constant term can be included in the stress-energy tensor, such that

$$R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi G T_{ab}\,, \tag{1.4.18}$$

There are three different cases of FLRW spacetime. To represent these, we use the parameter k, which takes three discrete values: $0, \pm 1$. It relates to the curvature of the universe as follows:

$$k = \begin{cases} +1 & \text{for a closed universe with positive curvature}\,, \\ 0 & \text{for a flat universe}\,, \\ -1 & \text{for an open universe with negative curvature}\,. \end{cases} \tag{1.4.19}$$

For the flat space case, we have a metric of the following form

$$g_{ab}^{\text{FLRW}} = a^2(\eta)\eta_{ab}\,. \tag{1.4.20}$$

We note here that the de Sitter metric in conformal coordinates, given by equation (1.4.14), had this form, where

$$a^2(\eta) = \frac{1}{(-\eta H)^2}\,. \tag{1.4.21}$$

Other interesting values of this scaling factor are

$$a_{\text{matter}}(t) \propto t^{\frac{2}{3}}\,, \tag{1.4.22}$$

$$a_{\text{radiation}}(t) \propto t^{\frac{1}{2}}\,, \tag{1.4.23}$$

for the time coordinate t, related to the conformal time coordinate η, by

$$\eta = \int \frac{\mathrm{d}t}{a(t)}\,. \tag{1.4.24}$$

Indeed, for the remainder of this section, it is more convenient to work in terms of the time coordinate t. The corresponding metric is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}\boldsymbol{x}^2\,. \tag{1.4.25}$$

We conclude by introducing the Friedmann equations, which will be of use in the next section. To find these, we write Einstein's equations in the following form:

$$R_{ab} = 8\pi G \left(T_{ab} - \frac{1}{2} g_{ab} T \right) , \tag{1.4.26}$$

where $T = g^{ab} T_{ab}$. As we consider a FLRW solution, we use the stress tensor of a perfect fluid,

$$T_{ab} = (\rho + p) u_a u_b + p g_{ab} , \tag{1.4.27}$$

where ρ is the energy density, p is the pressure, and u_a is the fluid 4-velocity. We consider a comoving fluid, so that $u^a = (1, 0, 0, 0)$. As this satisfies the condition $u^a u_a = -1$, the trace of the stress-energy tensor is

$$T = -\rho + 3p . \tag{1.4.28}$$

The first Friedmann equation is found from setting $a = b = 0$ in Einstein's equations, so that

$$H^2 \equiv \left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2} , \tag{1.4.29}$$

where a dot represents the derivative with respect to the time coordinate t. To find the second Friedmann equation, we must first find the continuity equation. This is found from the conservation of the stress tensor T_{ab}, and states that

$$\dot{\rho} + \frac{3\dot{a}}{a} (\rho + p) = 0 . \tag{1.4.30}$$

By multiplying equation (1.4.29) by $a^2(t)$, differentiating the resulting formula and substituting equation (1.4.30), this gives the second Friedmann equation:

$$\frac{\ddot{a}}{a} + \frac{4\pi G}{3} (\rho + 3p) = 0 . \tag{1.4.31}$$

We use these relations at the end of the next section, in our discussion of a specific inflationary theory.

1.5 Inflation

In this section, we introduce the theory of inflation. First, we motivate the theory by explaining the problems it was introduced to solve. We then give an overview of the initial theory of inflation, where a first order phase transition provided the mechanism for the exponential expansion. There are some issues with this theory, so we look at a modified theory: new inflation, which is a theory with a second order phase transition. This is studied in the slow-roll approximation. We conclude with a brief discussion of the role of the cosmological constant in inflation in de Sitter.

The theory of inflation came around in the late 1970s, and was proposed independently by both Guth [9] and Sato [10]. It was introduced to solve the following problems with the big bang model of the universe:

1. Horizon problem,
2. Flatness problem,
3. Monopole problem.

The horizon problem is that our universe is homogeneous and isotropic: there are causally disconnected regions of space with the same approximate temperature. The flatness problem is a fine-tuning problem. The density of our universe is approximately that of the critical density required for a flat universe. Such a value is very unstable, so a small deviation leads to a universe very dissimilar to our own. A small deviation leads to either an open universe, which would result in a large decrease in density, or a closed universe, which would have reached its maximum size in a period of time on the order of the Planck time. Initial parameters, such as the Hubble constant, must be very finely tuned for the universe we live in to have existed for this period of time. Finally, the monopole problem is simply that there is no experimental evidence to support the existence of magnetic monopoles. A physical theory should, therefore, be able to explain why they have not yet been discovered.

We now discuss the mechanism by which inflation occurs, in the initial model, discussed in [9] and [10]. Briefly, inflation states that there was a period of exponential expansion in the early universe, which stops when a first order phase transition, through a critical temperature T_C, occurs. At this temperature, one might expect that the universe would transition from the false vacuum state, to the true vacuum state. Intead, the universe supercools through this temperature to, say, a temperature T_S, orders of magnitude lower than T_C. This is a metastable false vacuum state. Bubbles of true vacuum then nucleate and expand at the speed of light. Because this transition occurs at the temperature T_S, instead of T_C, there is a huge change of entropy, so the process is not adiabatic. The energy release associated with this phase transition is such that the universe reheats to a temperature T_R, which is of the same order as T_C. The inhomogeneities necessary for the formation of stars and galaxies are explained by quantum and thermal fluctuations. Observations of such inhomogeneities in the cosmic microwave background (CMB) can give information about the fluctuations predicted by the inflationary model [33, 34].

We now see how inflation solves the problems stated at the beginning of this chapter. First, inflation explains the homogeneity of the universe, as all regions in the observable universe would have had contact with each other before the epoch of inflation. The flatness problem is solved as, due to the rapid expansion of the universe, the density post inflation approaches the critical density of a flat universe. For example, even if the universe is closed, post-inflation this universe will be locally indistinguishable from flat space. The solution to the monopole problem is that, before inflation, monopoles would form. However, due to the rapid expansion of the space after inflation, monopoles will be very spread out, meaning that it is not unlikely that one will not be present in the observable universe.

There are, however, problems with this model. The main one is known as the graceful exit problem, which is associated with the end of inflation [9]. Either inflation does not end, or after reheating it gives a universe which does not describe the one we observe. This is the case as, in order for the bubbles of true vacuum to thermalise, they must collide with other bubbles of true vacuum. In this initial theory, there would not be enough collisions to evolve into the universe that we observe today. In summary, this model leaves the universe either too inhomogeneous, or too empty. Inflation also fails to completely remove the fine tuning problem; in order for inflation to occur we still need some exact initial conditions. In an attempt to solve these problems, we move away from this first order phase transition theory of inflation, known as a false vacuum theory, to that of a second order phase transition, as was considered in, for example, [11] and [12].

We now discuss a specific theory of inflation, with a second order phase transition. There are many such theories, but, for simplicity, we consider single-field inflation [11, 12]. For a more detailed treatment of the theory, we direct the reader to, for example, [35, 36]. We work in a 4-dimensional FLRW metric, which can be written as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\Omega^2\right] , \tag{1.5.1}$$

for a scaling factor $a(t)$, where $\mathrm{d}\Omega^2$ is the metric of the 2-sphere, and where $k = -1, 0, 1$ for open, flat, and closed universes, respectively, as was stated at the end of Section 1.4. In this section, we work with the time coordinate t, as opposed to the conformal time η used for the rest of the thesis. The two coordinates are related through equation (1.4.24).

In this model of inflation, the expansion is driven by a single scalar field, called the inflaton field. The scalar field is minimally coupled to Einstein gravity, resulting in the following action:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{M_{pl}^2}{2}R - \frac{1}{2}g^{ab}\nabla_a\phi\nabla_b\phi - V(\phi)\right] , \tag{1.5.2}$$

where

$$M_{pl} = \frac{1}{\sqrt{8\pi G}} \tag{1.5.3}$$

is the Planck mass, and we consider a potential $V(\phi)$ such as the one shown in Figure 1.3.

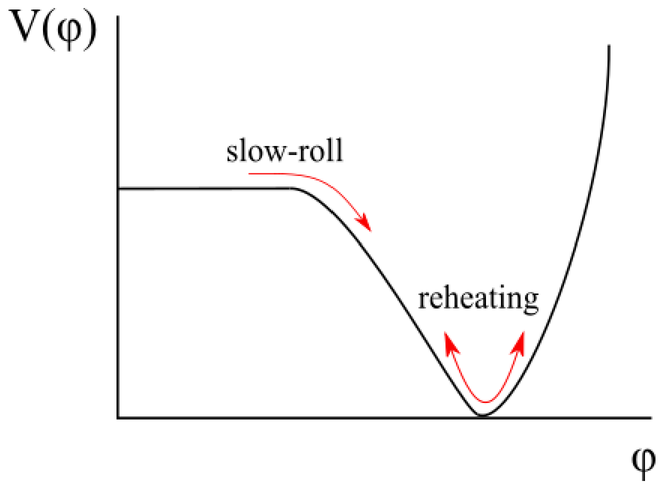

Figure 1.3: The potential of a scalar field which drives inflation. Inflation occurs when the scalar field slowly rolls down the slope, and reheating occurs around the global minimum, which is the true vacuum of the system.

As the scalar field slowly rolls down the slope of the potential, the exponential expansion occurs. Finally, as the potential reaches its minimum, there is a period of reheating, where the energy released during inflation thermalises the vacuum.

In order to discuss the dynamics of this system, we introduce the parameters

$$H \equiv \frac{\dot{a}}{a}\,, \tag{1.5.4}$$

$$\epsilon \equiv -\frac{\dot{H}}{H^2}\,, \tag{1.5.5}$$

$$\delta \equiv \frac{\dot{\epsilon}}{H\epsilon}\,, \tag{1.5.6}$$

$$\eta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}}\,, \tag{1.5.7}$$

where a dot denotes a derivative with respect to the time coordinate t. As mentioned in the previous section, H is the Hubble parameter. Physically, ϵ is related to the slope of the potential, as will be seen later in equation (1.5.25), and δ is related to its curvature. For the rest of this section, η is defined above by equation (1.5.7), as opposed to denoting the conformal time. In order for inflation to occur, we require that

$$\epsilon \ll 1\,, \tag{1.5.8}$$

$$|\delta| \ll 1\,, \tag{1.5.9}$$

$$|\eta| \ll 1\,. \tag{1.5.10}$$

These approximations will be discussed in more detail in the following calculation.

We will work in the slow-roll approximation, which enforces conditions (1.5.8) - (1.5.10). For the rest of this section, we will find the equations of motion, before showing that slow-roll inflation is a valid inflationary model.

The equation of motion for the inflaton field $\phi(t)$, found from varying the action given in equation (1.5.2), is

$$\nabla^a \nabla_a \phi + V'(\phi) = 0\,, \tag{1.5.11}$$

which, as the inflaton field depends on time only, becomes

$$\ddot{\phi} + 3H\dot{\phi} = -V'(\phi)\,. \tag{1.5.12}$$

At the end of the previous section, we found the Friedmann equations, given by equations (1.4.29) and (1.4.31). When $k = 0$, and by using

$$\frac{\ddot{a}}{a} = \dot{H} + H^2\,, \tag{1.5.13}$$

which can be seen by differentiating H, given by equation (1.5.4), it can be seen that the Friedmann equations are

$$H^2 = \frac{\rho}{3M_{pl}^2}\,, \tag{1.5.14}$$

$$\dot{H} + H^2 = -\frac{1}{6M_{pl}^2}(\rho + 3p)\,, \tag{1.5.15}$$

where ρ and p are the energy density and pressure of a perfect fluid. They can be expressed as follows:

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,, \tag{1.5.16}$$

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi)\,, \tag{1.5.17}$$

using the stress-energy tensor of a scalar field:

$$T_{ab} = \nabla_a \phi \nabla_b \phi - g_{ab} \left[\frac{1}{2} \nabla^c \phi \nabla_c \phi + V(\phi) \right] \tag{1.5.18}$$

and that of a perfect fluid, given in Section 1.4 by equation (1.4.27).

From the first Friedmann equation, we see that

$$H^2 = \frac{1}{3M_{pl}^2} \left[\frac{1}{2} \dot{\phi}^2 + V(\phi) \right] . \tag{1.5.19}$$

By differentiating this equation, and using the field equation (1.5.12), it can be seen that

$$\dot{H} = -\frac{\dot{\phi}^2}{2M_{pl}^2} , \tag{1.5.20}$$

so, from the definition given in equation (1.5.5), we have that

$$\epsilon = \frac{\dot{\phi}^2}{2M_{pl}^2 H^2} . \tag{1.5.21}$$

Finally, taking the derivative of the logarithm of this equation, and using the definitions given in equations (1.5.5) - (1.5.7), it can be seen that

$$\delta = 2(\epsilon - \eta) . \tag{1.5.22}$$

This relation can be used to allow us to work in terms of ϵ and δ, and eliminate η, as will also be the case in Chapter 2.

We now use the slow-roll approximation to simplify these equations. From the condition $\eta \ll 1$, we have $\ddot{\phi} \ll H\dot{\phi}$, so the equation of motion, from equation (1.5.12), becomes

$$3H\dot{\phi} \approx -V'(\phi) . \tag{1.5.23}$$

The other necessary condition for slow-roll approximation, that $\epsilon \ll 1$, is used to simplify the first Friedmann equation. Applying this condition to the expression found in equation (1.5.21) means we neglect the term in equation (1.5.19) corresponding to the kinetic energy, so that

$$H^2 \approx \frac{V(\phi)}{3M_{pl}^2} . \tag{1.5.24}$$

We also note that by substituting the simplified equation of motion, given by equation (1.5.23), into equation (1.5.21), we can write the slow-roll parameter

$$\epsilon \approx \frac{V'^2(\phi)}{18M_{pl}^2 H^4} \propto \left(\frac{V'(\phi)}{V(\phi)}\right)^2 . \tag{1.5.25}$$

The slow-roll approximation also allows us to find a simple expression for the length of time over which the period of inflation occurred. In order to find the extent of inflation, we use an e-fold, which is the expansion of space by a factor of e. The number N of e-folds is given by

$$N \equiv \int_{a_i}^{a_f} \mathrm{d}\ln a = \int_{t_i}^{t_f} H(t)\mathrm{d}t\,, \tag{1.5.26}$$

which can be seen from the definition of H, given in equation (1.5.4). The boundary values a_i and a_f are the values of the conformal scaling factor at the start and end of the inflationary period, at times t_i and t_f respectively. As stated earlier, inflation occurs when $\epsilon \ll 1$, so we use these values of ϵ as our boundary values. Rearranging equation (1.5.21):

$$H = \frac{\dot{\phi}}{\sqrt{2\epsilon}M_{pl}}\,, \tag{1.5.27}$$

we see that this integral becomes

$$N = \int_{\phi_i}^{\phi_f} \frac{1}{\sqrt{2\epsilon}} \frac{\mathrm{d}\phi}{M_{pl}}\,, \tag{1.5.28}$$

for ϕ_i and ϕ_f corresponding to the values of the field at the start and end of the inflationary period. In order to solve the horizon problem, it is known that we need a scalar field ϕ such that $N_{\mathrm{CMB}} \approx 40 - 60$ [37].

We conclude by returning to de Sitter spacetime, and the role of the cosmological constant. In de Sitter, $\dot{\phi} = 0$, so $V(\phi) = $ constant, hence H is constant in time also. As mentioned previously, inflation occurs in de Sitter spacetime, and is related to the cosmological constant. The cosmological constant is the energy density of the vacuum. In the scalar single-field model of inflation, this is the energy released during the second order phase transition, which thermalises the true vacuum.

1.6 Linearised Gravity

As linearised gravity is the theory in which we work in most chapters, in this section we study linearised gravity in FLRW spacetimes. The exception is Chapter 3, where we study a massless vector field. We find the field equations for perturbations about a background FLRW spacetime, and look at how the solutions of these equations are quantised. All results in this section also apply to de Sitter, when we take a specific choice of the conformal scaling factor, and we look at the vacuum Einstein equations.

1.6.1 Field equations

We find the field equations for a perturbation about a background FLRW spacetime, in the transverse, traceless, synchronous gauge. All results in this section can also be applied to de Sitter spacetime, and any parts of the calculation that differ for each will be highlighted. This calculation follows the method of [38].

We consider a perturbation about a FLRW background, g_{ab}, such that

$$\tilde{g}_{ab} = g_{ab} + h_{ab}\,, \tag{1.6.1}$$

for a symmetric perturbation h_{ab}. We also assume that

$$\delta\rho = \delta p = \delta u_a = 0\,. \tag{1.6.2}$$

We want to write the perturbation as transverse, traceless, and synchronous:

$$\nabla^a h_{ab} = 0\,, \tag{1.6.3}$$

$$h = 0\,, \tag{1.6.4}$$

$$u^a h_{ab} = h_{0a} = 0\,, \tag{1.6.5}$$

where the last line follows since we consider a comoving fluid $u^a = (-1, 0, 0, 0)$.

We start from Einstein's equations, given by

$$\tilde{R}_{ab} - \frac{1}{2}\tilde{g}_{ab}\tilde{R} = \frac{\kappa^2}{2}\tilde{T}_{ab}\,, \tag{1.6.6}$$

where $\kappa^2 = 16\pi G$, and a tilde represents a quantity calculated using the full metric. In this thesis, we consider two distinct cases of the stress-energy tensor:

$$\tilde{T}_{ab\ \mathrm{dS}} = -\Lambda\tilde{g}_{ab}\,, \tag{1.6.7}$$

$$\tilde{T}_{ab\ \mathrm{FLRW}} = (\rho + p)u_a u_b + p\tilde{g}_{ab}\,, \tag{1.6.8}$$

where the first case corresponds to de Sitter, and we use the second for FLRW. Here, we use the stress-energy tensor for an ideal fluid. For the rest of this chapter, we take the FLRW stress-energy tensor, as the de Sitter case clearly follows by setting $\rho = -p = \Lambda$. We therefore drop the FLRW subscript for the remainder of this discussion.

We will work to linear order in the perturbation h_{ab} in what follows. To first order in the perturbation h_{ab}, we can expand to see

$$\tilde{g}^{ab} = g^{ab} - h^{ab} + \mathcal{O}(h^2)\,, \tag{1.6.9}$$

$$\sqrt{-\tilde{g}} = \sqrt{-g}\left[1 + \frac{1}{2}g^{ab}h_{ab}\right] + \mathcal{O}(h^2)\,, \tag{1.6.10}$$

and, additionally, in the transverse, traceless, synchronous (TTS) gauge,

$$\tilde{T}_{ab} = T_{ab} + ph_{ab}\,, \tag{1.6.11}$$

$$\tilde{T}^a_a = T^a_a + \mathcal{O}(h^2)\,, \tag{1.6.12}$$

where

$$T_{ab} = (\rho + p)u_a u_b + pg_{ab}\,. \tag{1.6.13}$$

The Riemann tensor is

$$\tilde{R}^a{}_{bcd} = \nabla_c \tilde{\Gamma}^a_{db} - \nabla_d \tilde{\Gamma}^a_{cb} + \mathcal{O}(h^2)\,, \tag{1.6.14}$$

where

$$\tilde{\Gamma}^a_{bc} = \frac{1}{2}\tilde{g}^{ad}\left[\nabla_b \tilde{g}_{cd} + \nabla_c \tilde{g}_{bd} - \nabla_d \tilde{g}_{bc}\right]\,, \tag{1.6.15}$$

and ∇_a is the covariant derivative with respect to the background metric. In general (unless otherwise specified) indices will be raised / lowered with respect to the FLRW background metric, which is consistent with the linear order to which we work. The Riemann tensor is therefore given by

$$\begin{aligned}\tilde{R}^a{}_{bcd} =& R^a{}_{bcd} + \frac{1}{2}\left[\nabla_c\nabla_d h^a_b + \nabla_c\nabla_b h^a_d - \nabla_c\nabla^a h_{db} - \nabla_d\nabla_c h^a_b - \nabla_d\nabla_b h^a_c + \nabla_d\nabla^a h_{cb}\right] \\ &+ \mathcal{O}(h^2)\,.\end{aligned} \tag{1.6.16}$$

The Ricci tensor is defined to be $\tilde{R}_{ab} = \tilde{R}^c{}_{acb}$. Calculating this gives

$$\tilde{R}_{ab} = R_{ab} + \frac{1}{2}\left[\nabla_c\nabla_b h^c_a + \nabla_c\nabla_a h^c_b - \Box h_{ab}\right] + \mathcal{O}(h^2)\,, \tag{1.6.17}$$

where we have used the symmetry of the perturbation, and the fact that it is traceless, in order to simplify the expression. Note also that covariant derivatives commute on a scalar field. We must now compute the Ricci scalar which is given by:

$$\tilde{R} = \tilde{g}^{ab}\tilde{R}_{ab}\,. \tag{1.6.18}$$

Note that in this case we must use the full metric, $\tilde{g}_{ab}$, to contract indices as $\tilde{R}$ contains a term to order h^0 in the perturbation. Calculating this and implementing the gauge conditions given by equations (1.6.3) - (1.6.5) gives:

$$\tilde{R} = R - h^{ab} R_{ab} + \mathcal{O}(h^2)\,. \tag{1.6.19}$$

Using these expansions, Einstein's equations (1.6.6) become

$$\frac{1}{2}\nabla_c \nabla_a h^c_b + \frac{1}{2}\nabla_c \nabla_b h^c_a - \frac{1}{2}\Box h_{ab} - \frac{1}{2} R h_{ab} + \frac{1}{2} g_{ab} h^{cd} R_{cd} = \frac{\kappa^2}{2} p h_{ab}\,. \tag{1.6.20}$$

In order to simplify this equation, we make use of the following expressions for the Riemann tensor, Ricci tensor, and Ricci scalar, which here are quoted for general n for later convenience:

$$\begin{aligned} R^a{}_{bcd} &= H^2 \left[g^a_c g_{bd} - g^a_d g_{bc} + \epsilon (u^a u_c g_{bd} + u_b u_d g^a_c - u^a u_d g_{bc} - u_b u_c g^a_d) \right] \,, && (1.6.21) \\ R_{ab} &= H^2 \left[(n-1-\epsilon) g_{ab} + \epsilon (n-2) u_a u_b \right] \,, && (1.6.22) \\ R &= H^2 (n-1)(n-2\epsilon)\,. && (1.6.23) \end{aligned}$$

By evaluating the commutator of covariant derivatives, and imposing the TTS gauge conditions, it can be seen that

$$\begin{aligned} \nabla_a \nabla_c h^a_b &= R_{ac} h^a_b - R^d{}_{bac} h^a_d\,, \\ &= H^2 (4-\epsilon) h_{bc}\,, \end{aligned} \tag{1.6.24}$$

where the second line follows by setting $n = 4$ in equations (1.6.21) and (1.6.22). Additionally, in terms of the Hubble constant H and the slow-roll parameter ϵ, in n dimensions the pressure of the perfect fluid is

$$\kappa^2 p = 2H^2 \epsilon (n-2) - H^2 (n-1)(n-2)\,, \tag{1.6.25}$$

which is found the following component of Einstein's equations:

$$R_{ii} - \frac{1}{2} g_{ii} R = \frac{\kappa^2}{2} T_{ii}\,, \tag{1.6.26}$$

using equations (1.6.22), (1.6.23), and where $T_{ii} = p g_{ii}$, as can be seen from equation (1.6.13).

Using these relations, in $n = 4$ dimensions the field equations for the perturbation, given by equation (1.6.20), can be written as

$$\left(\Box - 2H^2 \right) h_{ab} = 0\,. \tag{1.6.27}$$

In the next section, we see how we quantise the solutions to these field equations.

1.6.2 Quantisation scheme

In this section, we briefly review the symplectic method of quantisation detailed in [39]. We explicitly look at how the method is used for the tensor perturbations of linearised gravity: the solutions to the field equation (1.6.27). This method of quantisation is used in Chapter 2 to normalise the mode functions of both the tensor perturbations, and the scalar field. It is also used in the construction of the graviton two-point function in Chapter 4. We conclude by briefly mentioning how this is applied to solutions of the Klein-Gordon scalar field.

We start from the action for a bosonic field ϕ_I, where the index I may represent spacetime indices corresponding to, for example, a scalar field or linearised gravity. This action is

$$S = \int \sqrt{-g}\mathcal{L}\mathrm{d}^n x\,, \tag{1.6.28}$$

for a quadratic Lagrangian

$$\mathcal{L} = \frac{1}{2}K^{aIbJ}\nabla_a\phi_I\nabla_b\phi_J + \frac{1}{2}S^{IJ}\phi_I\phi_J\,, \tag{1.6.29}$$

where $K^{aIbJ} = K^{bJaI}$ and $S^{IJ} = S^{JI}$. The conjugate momentum is

$$\begin{aligned} \pi^{cI} &= \frac{\partial\mathcal{L}}{\partial\left(\nabla_c\phi_I\right)}\,, && (1.6.30)\\ &= K^{cIbJ}\nabla_b\phi_J\,, && (1.6.31) \end{aligned}$$

for the Lagrangian above. The Euler-Lagrange equation of motion is

$$\nabla_c\left(\frac{\partial\mathcal{L}}{\nabla_c\phi_I}\right) - \frac{\partial\mathcal{L}}{\partial\phi_I} = 0\,, \tag{1.6.32}$$

which, for the Lagrangian given by equation (1.6.29), becomes

$$\nabla_c\pi^{cI} - S^{IJ}\phi_J = 0\,, \tag{1.6.33}$$

where the definition of the conjugate momentum, equation (1.6.30), has been used.

We assume that

$$(\phi_I, \pi^{cJ}) = (f_I^{(i)}, p^{(i)cJ})\,, \tag{1.6.34}$$

where $i = 1, 2$, are solutions to the Euler-Lagrange equations. For these solutions, we define the symplectic current to be

$$j^{(1,2)c} = f_I^{(1)} p^{(2)cI} - f_I^{(2)} p^{(1)cI} \,. \tag{1.6.35}$$

From the definition of the momentum, given by equation (1.6.30), and the equation of motion, given by equation (1.6.33) it can be seen that this current is conserved:

$$\begin{aligned}\nabla_c j^{(1,2)c} &= \nabla_c f_I^{(1)} \cdot p^{(2)cI} + f_I^{(1)} \cdot \nabla_c p^{(2)cI} - \nabla_c f_I^{(2)} \cdot p^{(1)cI} - f_I^{(2)} \cdot \nabla_c p^{(1)cI} \,, \\ &= \left(K^{cIbJ} - K^{bJcI}\right) \nabla_a f_I^{(1)} \nabla_b f_J^{(2)} + \left(S^{IJ} - S^{JI}\right) f_I^{(1)} f_J^{(2)} \,, \\ &= 0 \,, \end{aligned} \tag{1.6.36}$$

where the last line follows from the symmetries of the tensors K^{aIbJ} and S^{IJ}.

We define the symplectic product

$$\left(f^{(1)}, f^{(2)}\right)_{\text{symp}} = i \int_\Sigma \mathrm{d}\Sigma_c \left[\overline{f_I^{(1)}} p^{(2)cI} - f_I^{(2)} \overline{p^{(1)cI}}\right] \,, \tag{1.6.37}$$

where a bar denotes complex conjugation, for a Cauchy surface Σ_c, which is a subset of the spacetime which is intersected exactly once by every inextendible timelike curve. As the integrand of this is the conserved current $j^{(1,2)c}$, the symplectic product is independent of the choice of Cauchy surface Σ_c. We therefore make the choice $\Sigma_0 = \mathrm{d}^3x$.

For a coordinate system where the metric component $g_{0i} = 0$, we have the standard equal-time commutation relations for the components of the tensor perturbation and their covariant conjugate momenta. These are defined in [40], and we state here for completeness:

$$\left[\phi_I(t, \mathbf{x}), \phi_J(t, \mathbf{x}')\right] = \left[\pi^{0I}(t, \mathbf{x}), \pi^{0J}(t, \mathbf{x}')\right] = 0 \,, \tag{1.6.38}$$

$$\left[\phi_I(t, \mathbf{x}), \pi^{0J}(t, \mathbf{x}')\right] = i\delta_I^J \delta^{(n-1)}(\mathbf{x}, \mathbf{x}') \,, \tag{1.6.39}$$

where the delta function is defined such that

$$\int f(\mathbf{x}')\delta^{(n-1)}(\mathbf{x}, \mathbf{x}')\sqrt{g^{(n-1)}}\mathrm{d}^{n-1}\mathbf{x} = f(\mathbf{x}) \,, \tag{1.6.40}$$

where $g^{(n-1)}$ is the determinant of the spatial metric.

For the remainder of this section, we relate the symplectic product to the equal-time commutation relations defined above, in order to find a new quantisation condition. In what follows, we assume that $p^{0I} \neq 0$ for any I. At the end of this section we discuss the case $\pi^{0I} = 0$.

To find such a relation, we first show that

$$\Big[(f^{(1)},\phi_I)_{\rm symp},(\phi_J,f^{(2)})_{\rm symp}\Big] = (f^{(1)},f^{(2)})_{\rm symp}\,, \tag{1.6.41}$$

under the assumption that ϕ_I is a Hermitian field. This follows from the definition of the symplectic product, given in equation (1.6.37), and the commutation relations, given by equations (1.6.38) - (1.6.39):

$$\begin{aligned}
\text{LHS} &= \Big[(f^{(1)},\phi_I)_{\rm symp},(\phi_J,f^{(2)})_{\rm symp}\Big]\,,\\
&= -\int \mathrm{d}^{(n-1)}\mathbf{x}\sqrt{g^{(n-1)}}(t,\mathbf{x})\int \mathrm{d}^{(n-1)}\mathbf{x}'\sqrt{g^{(n-1)}}(t,\mathbf{x}')\times\\
&\qquad\qquad \Big\{\overline{f_I^{(1)}(t,\mathbf{x})}\left[\pi^{0I}(t,\mathbf{x}),\phi_J(t,\mathbf{x}')\right]p^{(2)0J}(t,\mathbf{x}')\\
&\qquad\qquad +\overline{p^{(1)0I}(t,\mathbf{x})}\left[\phi_I(t,\mathbf{x}),\pi^{0J}(t,\mathbf{x}')\right]f_J^{(2)}(t,\mathbf{x}')\Big\}\,,\\
&= i\int_\Sigma \mathrm{d}\Sigma_c\left[\overline{f_I^{(1)}}p^{(2)cI}-f_I^{(2)}\overline{p^{(1)cI}}\right]\,,\\
&= (f^{(1)},f^{(2)})_{\rm symp}\,,\\
&= \text{RHS}\,.
\end{aligned} \tag{1.6.42}$$

We now take a more schematic approach to find another expression for the commutator $\left[(f^{(1)},\phi_I)_{\rm symp},(\phi_J,f^{(2)})_{\rm symp}\right]$. We write the following mode sum:

$$\phi_I(x) = \sum_\sigma A_\sigma f_I^{(\sigma)}(x) = \sum_\sigma A_\sigma^\dagger \overline{f_I^{(\sigma)}(x)}\,, \tag{1.6.43}$$

where the second equality holds as we consider a Hermitian field. Using this mode expansion, the commutator becomes

$$\begin{aligned}
\Big[(f^{(\sigma)},\phi_I)_{\rm symp},(\phi_J,f^{(\rho)})_{\rm symp}\Big] &= \sum_\alpha\sum_\beta\left[A_\alpha\left(f^{(\sigma)},f^{(\alpha)}\right)_{\rm symp},A_\beta^\dagger\left(f^{(\beta)},f^{(\rho)}\right)_{\rm symp}\right]\,,\\
&= M^{\sigma\alpha}\left[A_\sigma,A_\rho^\dagger\right]M^{\beta\rho}\,,
\end{aligned} \tag{1.6.44}$$

where

$$M^{\sigma\rho} = \left(f^{(\sigma)},f^{(\rho)}\right)_{\rm symp}\,. \tag{1.6.45}$$

By combining the two equivalent expressions for the commutator, given by equations (1.6.41) and (1.6.44), we have that

$$M^{\sigma\rho} = M^{\sigma\alpha}\left[A_\alpha,A_\beta^\dagger\right]M^{\beta\rho}\,, \tag{1.6.46}$$

which, if the matrix $M^{\sigma\rho}$ is invertible, implies that

$$\left[A_\sigma, A_\rho^\dagger\right] = \left(M^{-1}\right)_{\sigma\rho} . \tag{1.6.47}$$

In summary, we have derived relation (1.6.47) from the equal-time commutation relations, and we now adopt this expression as our commutation relation. In practice, we expand our fields as the mode sum given in equation (1.6.43), calculate the matrix $M^{\sigma\rho}$ given by equation (1.6.45), before imposing condition (1.6.47).

In Chapter 2, we will calculate expression (1.6.47) and use the resulting relation to normalise mode functions. To demonstrate that this method gives the standard commutation relations, we take the example of a Klein Gordon scalar field. For a scalar field, we have, from evaluating the symplectic product,

$$\left(f^{(k)}, f^{(l)}\right)_{\text{symp}} = \delta^{kl} , \tag{1.6.48}$$

for the modes $f^{(k)}$ defined in equation (1.6.43), which gives the expected commutation relations

$$\left[a_k, a_l^\dagger\right] = \delta_{kl} . \tag{1.6.49}$$

We also look at the scalar field in more detail in Chapter 2, as here we will normalise the mode functions for a massless, minimally coupled (MMC) scalar field.

We conclude this section with a discussion on the case when $\pi^{0k} = 0$, when the matrix $M^{\sigma\rho}$ is not invertible. This is the case in linearised gravity, where the mode

$$f^{(g)}_{ab} = \nabla_a \tilde{\xi}_b + \nabla_b \tilde{\xi}_a \tag{1.6.50}$$

is symplectically orthogonal to all modes, meaning that the symplectic product is degenerate:

$$\left(f^{(g)}, f^{(l)}\right)_{\text{symp}} = 0 . \tag{1.6.51}$$

To use the method of quantisation outlined in this section, we must impose a gauge condition to make the matrix $M^{\sigma\rho}$ invertible. In the case of linearised gravity in FLRW spacetime, imposing the TTS gauge condition is sufficient. Alternatively, a gauge-fixing term can be added to the Lagrangian, as is the case in Chapter 4.

1.7 *Problematic behaviour of propagators*

In this thesis we are interested in two kinds of infrared divergence: the requirement of an IR cut-off to ensure that a propagator is well-defined, and the divergence which occurs when separation between points grows. Both of these issues tend to manifest themselves in the same way, as a term of the form $\log(\alpha r)$, where r is the separation between points, and α is an IR cut-off.

In this section, we clarify which of the propagators studied exhibit which type of IR divergence, and finish by detailing where in the thesis each type of divergence is studied. Throughout this thesis, we are mainly concerned with the IR divergences of the graviton propagator, but we briefly discuss the MMC scalar field, and the covariant massless vector propagator, for reasons we detail below.

We start with the MMC scalar field. Our main interest in this theory is due to its relevance for our later study of the graviton field. It can be seen that, in the TTS gauge, the graviton field satisfies the same field equation as the MMC scalar field in FLRW spacetime. The MMC scalar field in de Sitter space, and other spatially flat FLRW spacetimes, has a two-point function which is divergent in the IR [38]. Hence, if the scale factor and the state are such that the MMC scalar field has an IR-divergent two-point function, then the graviton field will have one also [41]. The MMC scalar field exhibits both IR problems discussed in this section [42], which naively suggests that the graviton two-point function, in the TTS gauge, will also suffer from both [43]. Indeed, for the single-field inflationary model, the two-point functions for the scalar and tensor perturbations are IR-divergent in a similar manner.

We now briefly discuss the covariant massless vector propagator. We are interested in this propagator as the study of its IR divergences will help with our calculations of the IR divergences of the covariant graviton two-point function. The covariant massless vector propagator does not require an IR cut-off to be well-defined. As points approach infinite separation, this propagator approaches a constant in the non-Landau gauge and it falls off in the Landau gauge [44]. Finding the large-distance behaviour of the covariant massless vector propagator will be seen to be useful in finding the large-distance behaviour of the graviton propagator.

We now explore the IR divergences of the graviton propagator, which was the main aim of this section. In this thesis, we work with two different kinds of graviton two-point functions. First, in Chapter 2, we study the propagator in the TTS gauge, where the gauge degrees of freedom are totally fixed, and we have $\frac{(n-2)(n-1)-2}{2}$ polarisation states. In Chapters 4 and 5 we study the graviton two-point function in the covariant gauge, where a gauge fixing term is added in the Lagrangian, so that there are $\frac{n(n+1)}{2}$ polarisation states. The covariant graviton propagator does not require an IR cutoff. From the aforementioned correspondence between the field equation for the MMC scalar field and that of the graviton field in the TTS gauge, one might initially assume that the graviton propagator suffers from both kinds of IR divergence, in the TTS gauge. This, however, is found to not be the case. In de Sitter spacetime, it was shown that there is, in fact, no need for an IR cut-off. Instead, a gauge transformation can be found to remove this IR divergence, suggesting that this divergence is merely a gauge effect [43]. In this thesis, we find a gauge transformation to show this result is true in FLRW spacetime also. As for the second type of IR divergence, the covariant graviton propagator does suffer from large-distance growth [2]. However, in de Sitter spacetime it will be seen that this divergence can also be removed, by means of a suitable gauge transformation, as was previously seen for the $n = 4$ dimensional case [4].

For the majority of this thesis, we study the graviton propagator. In Chapter 2, we work in the TTS gauge and use a gauge transformation to show that this propagator does not need an IR cut-off to be well-defined. In Chapter 3, we study the large-distance behaviour of the covariant massless vector propagator. In Chapter 4, we look at the large-distance behaviour of the graviton propagator, in the covariant gauge. In Chapter 5, we use a gauge transformation to show that these non-vanishing large-distance terms can be written in pure gauge form.

1.8 Gauge freedom in Linearised Gravity

In this section we review the gauge freedom in linearised gravity. We discuss the form of this gauge freedom, and discuss the physical relevance of it. We mention that there is debate over whether the IR divergences in the graviton two-point function are physical or merely a gauge artefact, and give some justification of the calculations performed in Chapters 2 and 5.

For the majority of this thesis, we work in the framework of linearised gravity. We start this section by mentioning that we work in the TTS gauge, given by conditions (1.6.3) - (1.6.5) in Section 1.6.1, everywhere except for in the latter half of Chapter 2. Here, we have the additional complication of a scalar (inflaton) field coupled to gravity. In these sections, we work in the gauge where the scalar perturbation is equal to 0. More detail on this gauge will be given in Section 2.5.

In linearised gravity, the gauge freedom is a result of the invariance of the metric under diffeomorphisms of the manifold. If one considers the following infinitesimal change of coordinates, parametrised by ξ^a such that, for two coordinate systems x^a and x'^a,

$$x^a = x'^a - \tilde{\xi}^a \,, \tag{1.8.1}$$

the perturbation h_{ab}, which is the solution of the field equation given in Section 1.6.1, changes like

$$\delta_{\tilde{\xi}} h_{ab} = \tilde{\xi}^c \partial_c g_{ab} + (\partial_a \tilde{\xi}^c) g_{cb} + (\partial_b \tilde{\xi}^c) g_{ac} \,, \tag{1.8.2}$$

which is equivalent to equation (1.6.50). For the specific example of a FLRW metric we used in the previous section, given by equation (1.4.20), we see that, after the following rescaling:

$$h_{ab} = a^2(\eta) H_{ab} \,, \tag{1.8.3}$$
$$\tilde{\xi}_a = a^2(\eta)\xi_a \,, \tag{1.8.4}$$

we have

$$\delta_\xi H_{ab} = \partial_a \xi_b + \partial_b \xi_a - 2Ha\eta_{ab}\xi_0 \,, \tag{1.8.5}$$

where the Hubble parameter H is defined in terms of conformal time η, as

$$H = \frac{a'(\eta)}{a(\eta)^2} \,, \tag{1.8.6}$$

where a prime denotes a derivative with respect to conformal time.

In the TTS gauge, the only non-zero components of the perturbation h_{ab} are those that are purely spatial, so it is natural to set $\xi_0 = 0$, and see that

$$\delta_\xi H_{ij} = \nabla_i \xi_j + \nabla_j \xi_i \,, \tag{1.8.7}$$
$$= \partial_i \xi_j + \partial_j \xi_i \,, \tag{1.8.8}$$

as all Christoffel symbols $\Gamma^\mu_{ij} = 0$ in Minkowski spacetime. This is the familiar form of the gauge freedom for linearised gravity in flat spacetime. We will use this gauge freedom to remove the IR divergence of the graviton two-point function.

As mentioned in the previous section, the physical significance of the IR divergences of the graviton propagator has been studied over the past several years (see, e.g. [45–68]). There is some debate over the use of 'large' gauge transformations [69,70], i.e. gauge transformations that do not become identity at spatial infinity, to remove the IR divergence of the graviton two-point function. In de Sitter spacetime, the IR-divergent piece of the two-point function can be written in pure-gauge form [3, 71]. It was noted that these divergences can be gauged away by linear gauge transformations that correspond to global shear transformations [43], which are large gauge transformations. In this paper, it is argued that it is legitimate to use large non-compactly supported gauge transformations if one is interested only in local physics. Briefly, this is because a large gauge transformation can mathematically be made to be compactly supported, without changing local physics, by multiplication with a smooth compactly supported function which is equal to 1 in the local region of interest and turned off smoothly outside. Then the two-point function will be IR finite if the two points are in the region where this compactly supported function is 1, which is the region of interest, though it is IR divergent elsewhere. In this thesis, we follow the approach of [43] and apply a specific large gauge transformation to each mode function rather the corresponding momentum component of a single gauge transformation. In [69] the Aharonov-Bohm effect is listed as an example where gauge-dependent quantities might play a role. However, since the IR divergences in the graviton (or inflaton) two-point function are not a topological effect, this does not serve as a good example for arguing against using large gauge transformations to gauge away IR divergences. We also point out that the distribution of gravitational fluctuations in momentum space is unchanged by these large gauge transformations; only the mode function for each value of the momentum is modified.

1.9 Summary

In this chapter, we have provided motivation behind our interest in the study of propagators in cosmological spacetimes, along with a mathematical basis for the calculations carried out in the rest of this thesis.

The plan for the rest of the thesis is as follows: in Chapter 2, we work in FLRW spacetime, and find a gauge transformation such that the IR divergence of the graviton two point function can be removed, for certain values of the slow-roll parameter ϵ. In Chapter 3, we work in de Sitter spacetime, as will be the case for the rest of the thesis. We find the long-distance behaviour of the covariant massless vector propagator in n-dimensions. The method involved in this calculation is then used in Chapter 4, where the n-dimensional large-distance behaviour of the graviton propagator is found. In Chapter 5 this large-distance divergence is written in pure gauge form. Finally, in Chapter 6, we discuss open problems related to the work in this thesis.

Chapter 2

Infrared divergences for free quantum fields in cosmological spacetimes

In this chapter, we study the IR divergence of the graviton two-point function in FLRW spacetime, and find a large-coordinate gauge transformation such that this divergence can, for a large class of FLRW spacetimes, be removed. Mathematically, IR divergences in the two-point function arise because, for small p, the power of the momentum p in the integrand of the p-integral is negative and too large for it to converge. Large coordinate gauge transformations, such as global shear transformations and dilation, render the two-point functions IR finite for the tensor and scalar perturbations in slow-roll single-field inflation by increasing the power of p in the IR by 2, from $p^{-2\nu}$, say, to $p^{-2\nu+2}$. It was shown recently that, if one smears the IR-divergent graviton and inflaton two-point functions in a gauge-invariant manner, then the power of p mentioned above is changed from $p^{-2\nu}$ to $p^{-2\nu+4}$ [72, 73]. This suggests that there should be large gauge transformations that change the small-p behaviour of the mode functions of the graviton and inflaton from $p^{-\nu}$ to $p^{-\nu+2}$ so that it changes in the integrands of the two-point functions from $p^{-2\nu}$ to $p^{-2\nu+4}$. In this chapter we find such gauge transformations. We discuss these gauge transformations first for the tensor perturbations, as they are universal for any FLRW spacetime. We then discuss the case of single-field inflation with the emphasis on the scalar perturbations.

2.1 Tensor perturbations of the FLRW metric

We consider the gravitational tensor perturbations around a background FLRW metric in n dimensions. We let $n \geqslant 4$ throughout this chapter. For definiteness we assume that the matter consists of a perfect fluid. There are several actions proposed to describe a perfect fluid in General Relativity [74–76]. The action due to Schutz [75] is

$$I = \int d^n x \, \mathcal{L} \, , \tag{2.1.1}$$

where

$$\mathcal{L} = \frac{1}{\kappa^2} \sqrt{-g} \, R + \sqrt{-g} \, p(\mu, S) \, . \tag{2.1.2}$$

The quantity S is called the specific entropy, and μ and V_a are defined by

$$\mu = \sqrt{-g^{ab} V_a V_b} \, , \tag{2.1.3}$$

$$V_a = \nabla_a \phi + \alpha \nabla_a \beta + \theta \nabla_a S \, . \tag{2.1.4}$$

The positive constant κ is related to Newton's constant G_N by $\kappa^2 = 16\pi G_N$. The independent variables are g^{ab}, α, β, ϕ, θ, and S. The most relevant fact here is that the pressure p depends on the metric through equation (2.1.3). By using the relation [75]

$$\frac{\partial p}{\partial \mu} = \frac{\rho + p}{\mu} \, , \tag{2.1.5}$$

where ρ is the energy density, one readily finds the standard Einstein equations with a perfect fluid:

$$R_{ab} - \frac{1}{2} g_{ab} R = \frac{\kappa^2}{2} \left[(\rho + p) u_a u_b + p g_{ab} \right] \, , \tag{2.1.6}$$

where

$$u_a = \mu^{-1} V_a \, , \tag{2.1.7}$$

where the perfect fluid model is used to model the radiation and matter phases of the early universe.

As is well known, the metric of the form

$$g_{ab} = a^2(\eta) \eta_{ab} \, , \tag{2.1.8}$$

where η_{ab} is the metric of flat spacetime, is a solution of (2.1.6) if $u_a = t_a$, where

$$t^a = a(\eta) \left(\frac{\partial}{\partial \eta} \right)^a \, , \tag{2.1.9}$$

and if

$$\kappa^2\rho = (n-1)(n-2)\left(\frac{a'}{a^2}\right)^2 , \tag{2.1.10}$$

$$\kappa^2 p = -2(n-2)\frac{a''}{a^3} - (n-2)(n-5)\left(\frac{a'}{a^2}\right)^2 . \tag{2.1.11}$$

Equation (2.1.11) was derived in Section 1.6.2, and equation (2.1.10) is found in the same way. A metric of the form in equation (2.1.8) describes a FLRW spacetime.

We consider the tensor perturbation h_{ab}, which is a rescaling of the perturbation considered in Chapter 1 such that $h_{ab} \to a^2(\eta)h_{ab}$. The metric is therefore written as

$$g_{ab} = a^2(\eta)(\eta_{ab} + h_{ab}) , \tag{2.1.12}$$

where h_{ab} is synchronous, transverse, and traceless. That is, we require that h_{ab} have no component in the direction of u^a, i.e. $h_{0a} = 0$, and that its spatial component be transverse, $\partial^j h_{ij} = 0$, and traceless, $\delta^{ij}h_{ij} = 0$, where ∂_j is the spatial derivative operator in flat space, and where the index is raised by Kronecker's delta, δ^{ij}. We write the space components of h_{ab} after choosing this gauge as $h_{ij} = H_{ij}$. From equation (2.1.6) it is clear that the perturbations described by H_{ij} do not mix with perturbations of any other fields at first order. We find that H_{ij} satisfies the following equation to first order [41]:

$$\frac{1}{a^n}\frac{\partial}{\partial\eta}\left(a^{n-2}\frac{\partial}{\partial\eta}H_{ij}\right) - \frac{1}{a^2}\triangle H_{ij} = 0 , \tag{2.1.13}$$

where $\triangle = \delta^{ij}\partial_i\partial_j$ is the Laplacian on flat space. This equation is equivalent to the field equation (1.6.27), derived in Section 1.6.2.

2.2 Quantisation of the tensor perturbation

In order to quantise the field H_{ij} representing the tensor perturbations, we first expand the Lagrangian (2.1.2) to second order in $\tilde{h}_{ab} = a^2 h_{ab}$ with the conditions $\nabla_a\tilde{h}^{ab} = 0$ and $\tilde{h}^a_a = 0$. Thus, we find (up to a total derivative) the quadratic Lagrangian relevant to the tensor perturbations is as follows:

$$\mathcal{L}_T = \frac{\sqrt{-g}}{\kappa^2}\Bigg[-\frac{1}{2}\nabla_a\tilde{h}_{bc}\nabla^a\tilde{h}^{bc} - \frac{\kappa^2}{2(n-2)}(\rho-p)\tilde{h}^{ab}\tilde{h}_{ab} + R_{ab}\tilde{h}^{ac}\tilde{h}^b_c \\ + R^{bdac}\tilde{h}_{ab}\tilde{h}_{cd}\Bigg] , \tag{2.2.1}$$

where ρ, p, R_{ab} and R^{bdac} are the background quantities. By substituting (2.1.10), (2.1.11), and using the formula

$$R_{abcd} = H^2\left[g_{ac}g_{bd} - g_{ad}g_{bc} + \epsilon(t_a t_c g_{bd} + t_b t_d g_{ac} - t_a t_d g_{bc} - t_b t_c g_{ad})\right] , \tag{2.2.2}$$

where, as defined in Chapter 1,

$$H = \frac{a'}{a^2} , \tag{2.2.3}$$

$$\epsilon = -\frac{H'}{H^2 a} , \tag{2.2.4}$$

we can simplify $\mathcal{L}_T$ as

$$\mathcal{L}_T = \frac{1}{4\kappa^2} a^{n-2}\left(H'_{ij}H^{ij\prime} + H_{ij} \triangle H^{ij}\right) , \tag{2.2.5}$$

where H'_{ij} is the partial derivative of H_{ij} with respect to conformal time η and where $\triangle = \partial_k\partial_k$ is the Laplacian on flat space.

The quantisation of the field $H_{ij}(\eta, \mathbf{x})$ is standard, as has been discussed in Section 1.6.2. It is expanded in terms of the mode functions $\gamma_{ij}^{(s,\mathbf{p})}(\eta, \mathbf{x})$ and their complex conjugates as

$$H_{ij}(\eta, \mathbf{x}) = \int \frac{\mathrm{d}^{n-1}\mathbf{p}}{(2\pi)^{n-1}} \sum_s \left[a_s(\mathbf{p})\gamma_{ij}^{(s,\mathbf{p})}(\eta, \mathbf{x}) + a_s^\dagger(\mathbf{p})\gamma_{ij}^{(s,\mathbf{p})*}(\eta, \mathbf{x})\right] . \tag{2.2.6}$$

The mode functions $\gamma_{ij}^{(s,\mathbf{p})}(\eta, \mathbf{x})$ are given by

$$\gamma_{ij}^{(s,\mathbf{p})}(\eta, \mathbf{x}) = \epsilon_{ij}^{(s)}(\mathbf{p}) f_p(\eta) \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}} , \tag{2.2.7}$$

where the polarisation tensors $\epsilon_{ij}^{(s)}(\mathbf{p})$ are traceless, satisfy $\epsilon_{ij}^{(s)}(\mathbf{p})p^j = 0$, and

$$\sum_{ij} \epsilon_{ij}^{(s)}(\mathbf{p})\epsilon_{ij}^{(r)}(\mathbf{p}) = \delta^{sr} . \tag{2.2.8}$$

The functions $f_p(\eta)$, where $p = |\mathbf{p}|$, satisfy

$$\frac{1}{a^{n-2}(\eta)} \frac{\mathrm{d}}{\mathrm{d}\eta}\left[a^{n-2}(\eta) \frac{\mathrm{d}}{\mathrm{d}\eta} f_p(\eta)\right] + p^2 f_p(\eta) = 0 , \tag{2.2.9}$$

which, of course, agrees with (2.1.13). Since the equation of motion (2.2.9) implies

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left\{a^{n-2}(\eta)\left[f_p^*(\eta)\frac{\mathrm{d}f_p(\eta)}{\mathrm{d}\eta} - \frac{\mathrm{d}f_p^*(\eta)}{\mathrm{d}\eta} f_p(\eta)\right]\right\} = 0 , \tag{2.2.10}$$

it is possible to choose the normalisation of $f_p(\eta)$ such that

$$f_p^*(\eta)\frac{\mathrm{d}f_p(\eta)}{\mathrm{d}\eta} - \frac{\mathrm{d}f_p^*(\eta)}{\mathrm{d}\eta}f_p(\eta) = -\frac{2\mathrm{i}\kappa^2}{a^{n-2}(\eta)}\,. \tag{2.2.11}$$

With this choice we find

$$[a_s(\mathbf{p}), a_{s'}^\dagger(\mathbf{p}')] = (2\pi)^{n-1}\delta_{ss'}\delta^{n-1}(\mathbf{p}-\mathbf{p}')\,. \tag{2.2.12}$$

We can reach the normalisation condition, given by equation (2.2.11), by considering the quantisation scheme detailed in Section 1.6.2. We consider the symplectic product given by equation (1.6.37), repeated here for clarity:

$$M_{\sigma,\sigma'} = \left(f^{(\sigma)}, f^{(\sigma')}\right)_{\text{symp}} = i\int \mathrm{d}\mathbf{x}^{n-1}\left[f_{ab}^{(\sigma)*}p^{(\sigma')ab} - p^{(\sigma)ab*}f_{ab}^{(\sigma')}\right]\,. \tag{2.2.13}$$

From Section 1.6.2, we have the following relation between the symplectic form and commutation relations:

$$\left(M^{-1}\right)_{\sigma,\sigma'} = [a_\sigma(\mathbf{p}), a_{\sigma'}^\dagger(\mathbf{p}')]\,, \tag{2.2.14}$$

which we use to impose a normalisation condition on the mode functions. By writing this symplectic form in terms of the mode functions γ_{ij} and their respective conjugate momenta, and using the commutation relation (2.2.12), the relation (2.2.14) can be evaluated:

$$-i\int \mathrm{d}^{n-1}\mathbf{x}\sqrt{-g}\left[\gamma_{ij}^{(s,\mathbf{p})}(\eta,\mathbf{x})^*\partial^0\gamma^{ij(s',\mathbf{p}')}(\eta,\mathbf{x}) - \partial^0\gamma^{ij(s,\mathbf{p})}(\eta,\mathbf{x})^*\cdot\gamma_{ij}^{(s',\mathbf{p}')}(\eta,\mathbf{x})\right]$$
$$= \delta^{ss'}(2\pi)^{n-1}\delta^{(n-1)}(\mathbf{p}-\mathbf{p}')\,. \tag{2.2.15}$$

Computing the right hand side of this expression for the modes (2.2.7) gives the condition

$$W(f_{\mathbf{p}}^*, f_{\mathbf{p}})(\eta) = -\frac{2\mathrm{i}\kappa^2}{a^{n-2}(\eta)}\,, \tag{2.2.16}$$

where the Wronskian is defined as

$$W(f_1, f_2)(\eta) := f_1(\eta)\cdot\frac{\mathrm{d}}{\mathrm{d}\eta}f_2(\eta) - \frac{\mathrm{d}}{\mathrm{d}\eta}f_1(\eta)\cdot f_2(\eta)\,. \tag{2.2.17}$$

This is exactly the relation (2.2.11) found earlier in this section.

As mentioned in Chapter 1, working in a curved spacetime means we lose the notion of a unique vacuum state. In FLRW spacetime however, one can define the vacuum state $|0\rangle$ by requiring that $a_s(\mathbf{p})|0\rangle = 0$ for all s and $\mathbf{p}$. Thus, the choice of the function $f_p(\eta)$ satisfying (2.2.11) determines the vacuum state. The two-point correlation function for $H_{ij}(\eta, \mathbf{x})$ can be found using (2.2.12) as

$$\begin{aligned}\Delta_{ij:i'j'}(\eta, \mathbf{x}; \eta', \mathbf{x}') &:= \langle 0|H_{ij}(\eta, \mathbf{x})H_{i'j'}(\eta', \mathbf{x}')|0\rangle\,, \\ &= \int \frac{\mathrm{d}^{n-1}\mathbf{p}}{(2\pi)^{n-1}} \sum_s \gamma_{ij}^{(s,\mathbf{p})}(\eta, \mathbf{x})\gamma_{i'j'}^{(s,\mathbf{p})*}(\eta', \mathbf{x}')\,. \end{aligned} \tag{2.2.18}$$

It will be useful for later purposes to examine the solution $f_p(\eta)$ for small p. If $p = 0$, equation (2.2.9) becomes

$$\frac{1}{a^{n-2}(\eta)} \frac{\mathrm{d}}{\mathrm{d}\eta} \left[a^{n-2}(\eta) \frac{\mathrm{d}}{\mathrm{d}\eta} f_p(\eta) \right] = 0\,. \tag{2.2.19}$$

Two independent real solutions $f_0(\eta) = F_0^{(1)}(\eta)$ and $F_0^{(2)}(\eta)$, can be chosen as

$$F_0^{(1)}(\eta) = 1\,, \tag{2.2.20}$$

$$F_0^{(2)}(\eta) = \int \frac{\mathrm{d}\eta}{a^{n-2}(\eta)}\,, \tag{2.2.21}$$

where the constant of integration is chosen suitably in (2.2.21). From rescaling equation (2.2.16), we see that two independent real solutions, $F_p^{(1)}(\eta)$ and $F_p^{(2)}(\eta)$, can be found such that

$$F_p^{(1)}(\eta) \frac{\mathrm{d}F_p^{(2)}(\eta)}{\mathrm{d}\eta} - \frac{\mathrm{d}F_p^{(1)}(\eta)}{\mathrm{d}\eta} F_p^{(2)}(\eta) = \frac{1}{a^{n-2}(\eta)}\,, \tag{2.2.22}$$

and that

$$F_p^{(I)}(\eta) = F_0^{(I)}(\eta) + \mathcal{O}(p^2)\,, \tag{2.2.23}$$

for $I = 1, 2$. This is because the p-dependence in (2.2.9) is through p^2. The solutions $f_p(\eta)$ can be expressed as

$$f_p(\eta) = \mathrm{i}A^{(T)}(p)F_p^{(1)}(\eta) + B^{(T)}(p)F_p^{(2)}(\eta)\,. \tag{2.2.24}$$

The functions $F_p^{(1)}(\eta)$ and $F_p^{(2)}(\eta)$ are finite in the limit $p \to 0$, and the source of IR singularities is the singular behaviour of $A^{(T)}(p)$ in this limit. After choosing the real solutions $F^{(1)}(\eta)$ and $F^{(2)}(\eta)$ it is always possible to choose $A^{(T)}(p)$ and $B^{(T)}(p)$ to be real. This is done by first choosing $B^{(T)}(p)$ to be real with adjustment of the phase factor, and then absorbing any imaginary part of $A^{(T)}(p)$ with the redefinition of

$$F^{(2)}(\eta) \to F^{(2)}(\eta) + [\mathrm{Im}\, A^{(T)}(p)/B^{(T)}(p)]F^{(1)}(\eta)\,. \tag{2.2.25}$$

Then, equations (2.2.11) and (2.2.22) imply

$$2A^{(T)}(p)B^{(T)}(p) = \kappa^2 \, . \tag{2.2.26}$$

Changing the exact form of the functions $A^{(T)}(p)$ and $B^{(T)}(p)$ alters the function $f_p(\eta)$, and hence its complex conjugate $f_p^*(\eta)$. This in turn alters the mode functions $\gamma_{ij}^{(s,\mathbf{p})}(\eta,\mathbf{x})$ and $\gamma_{ij}^{(s,\mathbf{p})*}(\eta,\mathbf{x})$, respectively. From their definition in equation (2.2.7), it can be seen that these mode functions determine the positive- and negative-frequency solutions. Hence, the choice of the functions $A^{(T)}(p)$ and $B^{(T)}(p)$ determines the positive- and negative-frequency solutions. In most of important applications, such as slow-roll inflation, the 'positive-frequency' solution $f_p(\eta)$ is chosen such that

$$A^{(T)}(p) \approx \frac{C}{p^\nu} \, , \tag{2.2.27}$$

for $\nu > 0$. Then, by (2.2.26), we find $B^{(T)}(p) \sim p^\nu$ for small p.

We note that there is some freedom in distributing the p-dependence between $A^{(T)}(p)$ and $F_p^{(1)}(\eta)$ and between $B^{(T)}(p)$ and $F_p^{(2)}(\eta)$. We allow this freedom because in many cases there are standard functions to be chosen as $F_p^{(1)}(\eta)$ and $F_p^{(2)}(\eta)$. For example, if $a(\eta) = 1$, which corresponds to flat space, then we can choose

$$F_p^{(1)}(\eta) = \cos p\eta \, , \tag{2.2.28}$$

$$F_p^{(2)}(\eta) = p^{-1}\sin p\eta \, , \tag{2.2.29}$$

with $\eta \in \mathbb{R}$, and

$$f_p^{(\mathrm{flat})}(\eta) = \frac{\mathrm{i}\kappa}{\sqrt{p}} F_p^{(1)}(\eta) + \kappa\sqrt{p}F_p^{(2)}(\eta) \, , \tag{2.2.30}$$

so that $C = \kappa$ and $\nu = 1/2$.

Note that if $\nu \geqslant (n-1)/2$, then the two-point correlation function $\Delta_{iji'j'}(\eta,\mathbf{x};\eta',\mathbf{x}')$ is IR divergent because then the integrand in (2.2.18) will behave like $p^{-2\nu}$, where $2\nu \geqslant n-1$.

In the next section we show that large gauge transformations can be used to make the integrand less singular in the small p limit, so that, in many applications, the IR divergences can be eliminated by large gauge transformations.

2.3 *The gauge transformations for the tensor perturbations*

As discussed in Chapter 1, the linear gauge transformation for $h_{ab} = a^2 H_{ab}$ is

$$\delta_{\tilde{\xi}} h_{ab} = \tilde{\xi}^c \partial_c g_{ab} + (\partial_a \tilde{\xi}^c) g_{cb} + (\partial_b \tilde{\xi}^c) g_{ac} \, , \tag{2.3.1}$$

which can be given, with the definition $\tilde{\xi}_\alpha = a^2(\eta)\xi_\alpha$, as

$$\delta_\xi H_{ab} = \partial_a \xi_b + \partial_b \xi_a - 2Ha\eta_{ab}\xi_0 \,. \tag{2.3.2}$$

We show in this section that one can choose ξ_a for each mode function $\gamma_{ij}^{(s,\mathbf{p})}(\eta,\mathbf{x})$ such that the integrand for the two-point function $\Delta_{iji'j'}(\eta,\mathbf{x};\eta',\mathbf{x}')$ has the power of p reduced by 4 for small p. That is, if

$$\frac{n-1}{2} \leqslant \nu < \frac{n+3}{2} \,, \tag{2.3.3}$$

then, although the graviton two-point function is IR divergent with the mode functions $\gamma_{ij}^{(s,\mathbf{p})}(\eta,\mathbf{x})$ behaving like $p^{-\nu}$ for small p, it will be IR finite with the gauge-transformed mode functions.

The gauge transformation we use is given by

$$\xi_0 = 0 \,, \tag{2.3.4}$$

$$\xi_i = -\frac{\mathrm{i}}{2} A^{(T)}(p) F_0^{(1)}(\eta) \left[\epsilon_{il}^{(s)}(\mathbf{p}) x^l (1 + \mathrm{i}\mathbf{p}\cdot\mathbf{x}) - \frac{\mathrm{i}}{2}\epsilon_{lm}^{(s)}(\mathbf{p}) x^l x^m p_i \right] \mathrm{e}^{-\rho^2 p^2} \,, \tag{2.3.5}$$

where the factor $\mathrm{e}^{-\rho^2 p^2}$, with ρ a positive constant, has been inserted in order not to introduce spurious ultraviolet divergences. The polarisation tensors $\epsilon_{ij}^{(s)}(\mathbf{p})$ have been defined before [see the sentence containing (2.2.8)]. Notice that they depend only on the direction of $\mathbf{p}$ and not on its magnitude. The part of order $\mathbf{p}^0$ inside the square brackets represents the global shear transformation used in [43]. The part of order $\mathbf{p}$ was obtained by determining the coefficients α and β in the general ansatz

$$\alpha \epsilon_{il}^{(s)}(\mathbf{p}) x^l \mathbf{p}\cdot\mathbf{x} + \beta \epsilon_{lm}^{(s)}(\mathbf{p}) x^l x^m p_i \,, \tag{2.3.6}$$

which is linear in $\epsilon_{ij}^{(s)}(\mathbf{p})$ and p_i and quadratic in x^i. This is a large gauge transformation in the sense that ξ_i does not tend to zero as $|\mathbf{x}| \to \infty$. In fact it diverges in this limit.

Now,

$$\begin{aligned} \delta\gamma_{ij}^{(s,\mathbf{p})}(\eta,\mathbf{x}) &= \partial_i \xi_j + \partial_j \xi_i \,, \\ &= -\mathrm{i} A^{(T)}(p) \epsilon_{ij}^{(s)}(\mathbf{p})(1 + \mathrm{i}\mathbf{p}\cdot\mathbf{x}) \mathrm{e}^{-\rho^2 p^2} \,, \end{aligned} \tag{2.3.7}$$

where we used $F_0^{(1)}(\eta) = 1$. Notice that $\delta\gamma_{ij}^{(s,\mathbf{p})}$ is transverse and traceless. Thus, this gauge transformation does not violate the gauge conditions we have imposed, though the transformed field no longer has a non-singular Fourier transform. We find the transformed mode functions as

$$
\begin{aligned}
\tilde{\gamma}_{ij}^{(s,\mathbf{p})}(\eta,\mathbf{x}) &= \epsilon_{ij}^{(s)}(\mathbf{p}) f_p(\mathbf{p}) \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}} - \mathrm{i}A^{(T)}(p)\epsilon_{ij}^{(s)}(\mathbf{p})(1+\mathrm{i}\mathbf{p}\cdot\mathbf{x})\mathrm{e}^{-\rho^2 p^2}\,, \\
&= \mathrm{i}\epsilon_{ij}^{(s)}(\mathbf{p})\Big\{A^{(T)}(p)\left[F_p^{(1)}(\eta) - F_0^{(1)}(\eta)\right](1+\mathrm{i}\mathbf{p}\cdot\mathbf{x}) \\
&\qquad +A^{(T)}(p)F_p^{(1)}(\eta)\left(\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}} - 1 - \mathrm{i}\mathbf{p}\cdot\mathbf{x}\right) \\
&\qquad -A^{(T)}(p)(1+\mathrm{i}\mathbf{p}\cdot\mathbf{x})(\mathrm{e}^{-\rho^2p^2}-1) + B^{(T)}(p)F_p^{(2)}(\eta)\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\Big\}\,, \\
&= \epsilon_{ij}^{(s)}(\mathbf{p})\left[A^{(T)}(p)\times\mathcal{O}(p^2) + B^{(T)}(p)F_p^{(2)}(\eta)\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\right]\,,
\end{aligned}
\tag{2.3.8}
$$

where we have used (2.2.23). Suppose that, in the small p limit, $A^{(T)}(p) \approx Cp^{-\nu}$, for $\nu \geqslant (n-1)/2$, so that the graviton two-point function is IR divergent. As a result, due to (2.2.26), $B^{(T)}(p) \sim p^{\nu}$ for small p. Then, the original mode function $\gamma_{ij}^{(s,\mathbf{p})}(\eta,\mathbf{x})$ behaves like $p^{-\nu}$ whereas the transformed mode function $\tilde{\gamma}_{ij}^{(s,\mathbf{p})}$ behaves like $p^{-\nu+2}$. Thus, if

$$
\frac{n-1}{2} \leqslant \nu < \frac{n+3}{2}\,, \tag{2.3.9}
$$

then the original graviton two-point function $\Delta_{iji'j'}(\eta,\mathbf{x};\eta',\mathbf{x}')$ given by (2.2.18) is IR divergent, whereas the transformed one with $\gamma_{ij}^{(s,\mathbf{p})}(\eta,\mathbf{x})$ replaced by $\tilde{\gamma}_{ij}^{(s,\mathbf{p})}(\eta,\mathbf{x})$ is IR finite. Thus, for this range of ν, the IR divergences of the graviton two-point function can be removed by large gauge transformations.

Below, we apply the general observation described above to the particular case with the conformal scaling factor $a(\eta) = (-\eta/\eta_0)^{-\lambda}$, where η_0 and λ are positive constants, and where η runs from $-\infty$ to 0. In this case the field equation (2.2.9) becomes

$$
f_p''(\eta) + \frac{(n-2)\lambda}{\eta} f_p'(\eta) + p^2 f_p(\eta) = 0\,. \tag{2.3.10}
$$

A solution to this equation is

$$
f_p(\eta) = C^{(T)}(p)(-p\eta)^{\nu} H_{\nu}^{(1)}(-p\eta)\,, \tag{2.3.11}
$$

$$
\nu = \frac{1}{2}\left[1 + (n-2)\lambda\right]\,, \tag{2.3.12}
$$

where $H_{\nu}^{(1)}(z)$ is the Hankel function of the first kind. The constant λ can be related to the slow-roll parameter ϵ defined by (2.2.4) as

$$
\epsilon = 1 - \frac{1}{\lambda}\,. \tag{2.3.13}
$$

We note that both the radiation phase and matter phase correspond to fixed values of ϵ [77]. Since ϵ is time independent, the slow-roll parameter δ, introduced in Section 1.5, is

$$\delta := \frac{\epsilon'}{2Ha\epsilon} = 0 \,. \tag{2.3.14}$$

The mode functions $\gamma_{ij}^{(s,\mathbf{p})}(\eta, \mathbf{x})$ defined by (2.2.7) transform under the scaling $(\eta, \mathbf{x}) \to (\alpha\eta, \alpha\mathbf{x})$, where α is a positive constant, as

$$\gamma_{ij}^{(s,\mathbf{p})}(\alpha\eta, \alpha\mathbf{x}) = f(\alpha)\gamma_{ij}^{(s,\alpha\mathbf{p})}(\eta, \mathbf{x}) \,, \tag{2.3.15}$$

where

$$f(\alpha) = \frac{C^T(p)}{C^T(\alpha p)} \,. \tag{2.3.16}$$

Thus, if one defines the vacuum state $|0\rangle$ in Section 2.2 by adopting the function $f_p(\eta)$ defined by (2.3.11), then it respects the scaling symmetry $(\eta, \mathbf{x}) \to (\alpha\eta, \alpha\mathbf{x})$. This state is the natural vacuum state in this sense and is generally adopted for the slow-roll inflationary models, for example. (The derivatives of the graviton two-point function that are IR finite acquire a constant factor under this scaling for slow-roll inflation [73].) For this reason we study this case below.

The normalisation constant $C^{(T)}(p)$ can readily be found up to an overall phase factor by using the large η limit of (2.2.11) with

$$H_\nu^{(1)}(z) \approx \sqrt{\frac{2}{\pi z}} \mathrm{e}^{\mathrm{i}(z - \frac{\pi}{2}\nu - \frac{\pi}{4})} \,. \tag{2.3.17}$$

We find

$$C^{(T)}(p) = -\kappa \frac{\sqrt{\pi\eta_0}}{\sqrt{2}(p\eta_0)^\nu} \,. \tag{2.3.18}$$

Thus, for small p,

$$C^{(T)}(p) \sim p^{-\nu} = p^{-\frac{1}{2}[1+(n-2)\lambda]} \,. \tag{2.3.19}$$

We can write $f_p(\eta)$ in the form (2.2.24) with

$$F_p^{(1)}(\eta) = -\frac{\pi}{2^\nu\Gamma(\nu)}(-p\eta)^\nu Y_\nu(-p\eta) \,, \tag{2.3.20}$$

$$F_p^{(2)}(\eta) = -\frac{2^{\nu-1}\Gamma(\nu)\eta_0}{(p\eta_0)^{2\nu}}(-p\eta)^\nu J_\nu(-p\eta) \,, \tag{2.3.21}$$

$$A^{(T)}(p) = \kappa\sqrt{\frac{\eta_0}{\pi}}\frac{2^{\nu-\frac{1}{2}}\Gamma(\nu)}{(p\eta_0)^\nu} \,, \tag{2.3.22}$$

$$B^{(T)}(p) = \kappa\sqrt{\frac{\pi}{\eta_0}}\frac{(p\eta_0)^\nu}{2^{\nu-\frac{1}{2}}\Gamma(\nu)} \,, \tag{2.3.23}$$

where $J_\nu(z)$ and $Y_\nu(z)$ are the Bessel functions of the first and second kinds, respectively.

In Section 2.2, we stated that $F_p^{(1)}(\eta) - F_0^{(1)}(\eta) = \mathcal{O}(p^2)$, if $\nu \geqslant (n-1)/2$, i.e. if the original two-point function is IR divergent. We now show that this is, indeed, the case. We define

$$G_\nu(z) := z^\nu Y_\nu(z)\,, \tag{2.3.24}$$

as, for $z = p\eta$, we have $F_p^{(1)}(\eta) \propto G_\nu(z)$. All we need to show is that $G_\nu(z) - G_\nu(0) = \mathcal{O}(z^2)$ for small z. If ν is not an integer, then we have

$$z^\nu Y_\nu(z) = -\frac{1}{\sin(\pi\nu)} z^\nu J_{-\nu}(z) + \cot(\pi\nu) z^\nu J_\nu(z)\,, \tag{2.3.25}$$

where

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+1+\nu)} z^{2k+\nu}\,. \tag{2.3.26}$$

The second term in (2.3.25) is $\mathcal{O}(z^{2\nu})$. The original two-point function is IR divergent if $\nu \geqslant (n-1)/2$ because the power of p in the integral for small p is $p^{-2\nu}$ whereas the integration measure is $\mathrm{d}p\, p^{n-2}$. Thus, for $n \geqslant 3$, if the original two-point function is IR divergent, then the second term in (2.3.25) is $\mathcal{O}(z^2)$ or higher, which can be neglected. Then since

$$z^\nu J_{-\nu}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+1-\nu)} z^{2k}\,, \tag{2.3.27}$$

it is clear that $G_\mu(z) - G_\mu(0) = \mathcal{O}(z^2)$.

If $\nu = N$ is a positive integer, then

$$\begin{aligned} z^N Y_N(z) =& \frac{2}{\pi}\left[\log\left(\frac{z}{2}\right) + \gamma\right] z^N J_N(z) - \frac{1}{\pi}\sum_{k=0}^{N-1}\left[(N-k-1)!\left(\frac{z}{2}\right)^{2k}\right] \\ & - \frac{1}{\pi}\sum_{k=0}^{\infty}\left[(-1)^k\left(\frac{\phi(k)+\phi(N+k)}{k!(N+k)!}\right)\left(\frac{z}{2}\right)^{2k+2N}\right]\,, \end{aligned} \tag{2.3.28}$$

where $\phi(p) = \sum_{n=1}^{p} \frac{1}{n}$. Therefore, as the power of z increases in increments of 2 in the first sum, the same argument as above holds and we conclude that $G_N(z) - G_N(0) = \mathcal{O}(z^2)$, which concludes our discussion.

Returning to our discussion of slow-roll, from (2.3.9) we find that, if

$$1 \leqslant \lambda < \frac{n+2}{n-2}\,, \tag{2.3.29}$$

then the two-point function (2.2.18) is IR divergent, but that the two-point function formed from the mode functions acted on by the large gauge transformations, given by (2.3.5), is IR finite. In terms of the slow-roll parameter ϵ, this condition can be written as

$$0 \leqslant \epsilon < \frac{4}{n+2}\,, \tag{2.3.30}$$

the case $\epsilon = 0$ being the de Sitter limit. Interestingly, the spacetime with $\epsilon < 0$ $(0 < \lambda < 1)$ (Big Rip spacetime [78]) causes no IR problems.

The case $\lambda = 0$ gives Minkowski space, so we neglect this, and, instead, briefly discuss the case with $\lambda < 0$. Here, we have $a(\eta) = (\eta/\eta_0)^{|\lambda|}$ and the variable η runs from 0 to $+\infty$ in order to have an expanding universe. The function $f_p(\eta)$ that we can adopt in this case is

$$f_p^{(\lambda<0)}(\eta) = -C^{(T)}(p)(p\eta)^{\nu} H_\nu^{(2)}(p\eta)\,, \tag{2.3.31}$$

where $C^{(T)}(p)$ is given by (2.3.18) and where $\nu = \frac{1}{2}\,|(n-2)\lambda + 1|$. In a way similar to the positive λ case, we find that the two-point function (2.2.18) is IR divergent but can be rendered IR finite by the large gauge transformations given by (2.3.5) if

$$2 - \frac{6}{n+4} < \epsilon \leqslant 2 - \frac{2}{n}\,. \tag{2.3.32}$$

Combining this case and (2.3.30) for positive λ, we find that the two-point function (2.2.18) is IR divergent if

$$0 \leqslant \epsilon \leqslant 2 - \frac{2}{n}\,, \tag{2.3.33}$$

but the gauge transformations (2.3.5) render it IR finite unless

$$\frac{4}{n+2} \leqslant \epsilon \leqslant 2 - \frac{6}{n+4}\,. \tag{2.3.34}$$

It is known that the two-point function for the linearised Weyl tensor, which is a local gauge invariant, is also IR divergent if and only if [69]

$$\frac{4}{n+2} \leqslant \epsilon \leqslant 2 - \frac{6}{n+4}\,, \tag{2.3.35}$$

for the vacuum state chosen here. This implies that it is impossible to render the graviton two-point function IR finite by any gauge transformations outside this range of values for ϵ because the linearised Weyl tensor is invariant under any gauge transformation, large or otherwise.

For $0 \leqslant \epsilon \ll 1$, i.e. for slow-roll inflationary FLRW universe, our result clearly shows that the IR divergence of the two-point function for the tensor perturbations can be eliminated by large gauge transformations. In the next section we show that we can do the same for the scalar perturbations in this spacetime.

2.4 Scalar perturbations in single-field inflation

We now consider inflationary FLRW spacetime such that the inflation is driven by a scalar, inflaton, field

$$\tilde{\phi}(\eta, \mathbf{x}) = \phi(\eta) + \psi(\eta, \mathbf{x}) \tag{2.4.1}$$

with the background field $\phi(\eta)$ depending only on conformal time η. The linear gauge transformations are given by (2.3.1) on the gravitational field and

$$\delta\psi = \mathcal{L}_{\tilde{\xi}}\phi = \tilde{\xi}^a \partial_a \phi\,, \tag{2.4.2}$$

is the transformation of the perturbation ψ of the inflaton. We start by considering the parts of the components of the graviton h_{ab} and inflaton ψ invariant under *local* gauge transformations and then write down the action in terms of those gauge invariant variables. It is only after writing down the action that we make a gauge choice.

The components of the gravitational field h_{ab} can be written in terms of 4 gauge invariant quantities. There is one gauge invariant tensor, H_{kl}, which is the transverse traceless part of h_{kl}. Additionally, we have one gauge invariant transverse vector, denoted V_k. We have two gauge invariant scalars, labelled S and Σ; we shall see later that there is only one dynamical gauge invariant scalar, which is a linear combination of S and Σ. (For full details of this decomposition, see [79].) Writing the components of the perturbation h_{ab} in terms of these quantities gives

$$h_{00} = S + 2X_0' + 2HaX_0\,, \tag{2.4.3}$$

$$h_{0k} = V_k + X_k' + \partial_k X_0\,, \tag{2.4.4}$$

$$h_{kl} = H_{kl} + \delta_{kl}\Sigma + 2\partial_{(k}X_{l)} - 2Ha\delta_{kl}X_0\,. \tag{2.4.5}$$

In this form the gauge transformation (2.3.1) can be attributed to that of the fields X_a:

$$\delta X_a = \xi_a\,. \tag{2.4.6}$$

We similarly write the perturbation ψ of the inflaton in terms of this vector X_a and another gauge invariant scalar Ψ as

$$\psi = \Psi - X_0\phi'\,. \tag{2.4.7}$$

In fact, this equation defines the scalar Ψ.

Now, we consider the Einstein-Hilbert action, along with the action for a minimally coupled scalar field $\tilde{\phi}$,

$$I = \frac{1}{\kappa^2}\int \tilde{R}\sqrt{-\tilde{g}}\mathrm{d}^n x - \frac{1}{2}\int \sqrt{-\tilde{g}}\mathrm{d}^n x \left[\tilde{g}^{ab}(\partial_a\tilde{\phi})(\partial_b\tilde{\phi}) + V(\tilde{\phi})\right] , \tag{2.4.8}$$

for some potential $V(\phi)$. One expands this action to second order and substitutes (2.4.3)-(2.4.5) and (2.4.7) into the resulting quadratic action. Varying the action with respect to V_k and S results in the following constraint equations [79]:

$$V_k = 0\,, \tag{2.4.9}$$

$$S = (n-3)\Sigma\,. \tag{2.4.10}$$

Here we are working in the space of functions where the Laplacian $\triangle$ is invertible. Then, by introducing the Sasaki-Mukhanov variable [77],

$$Q \equiv \frac{2Ha}{\phi'}\Psi - \Sigma\,, \tag{2.4.11}$$

which describes the scalar perturbations, and the non-dynamical variables S and V_k through equations (2.4.9) and (2.4.10), one finds the following action [79]:

$$\begin{aligned} I^{(2)} =& \frac{1}{4\kappa^2}\int \left[H'_{kl}H'_{kl} + H_{kl}\triangle H_{kl}\right] a^{n-2}\mathrm{d}^n x + \frac{n-2}{4\kappa^2}\int \left[Q'^2 + Q\triangle Q\right]\epsilon a^{n-2}\mathrm{d}^n x \\ &+ \frac{n-2}{4\kappa^2}\int \frac{\left(\triangle\Sigma - \frac{H'}{H}Q'\right)^2}{(n-1-\epsilon)H^2a^2}a^{n-2}\mathrm{d}^n x\,. \end{aligned} \tag{2.4.12}$$

Lagrange's equation for Σ is the constraint equation:

$$\triangle\Sigma = -\epsilon HaQ'\,. \tag{2.4.13}$$

After this constraint is imposed, the field equation for Q takes the following form:

$$Q'' + (n-2+2\delta)\,HaQ' - \triangle Q = 0\,, \tag{2.4.14}$$

where the slow-roll parameter δ is given by (2.3.14).

The tensor perturbation H_{kl} here can be treated in exactly the same way as in Sections 2.1 and 2.2 and the results obtained there will apply for single-field inflation as well. The Sasaki-Mukhanov variable Q can be quantised in the standard manner. One finds

$$Q(\eta,\mathbf{x}) = \int \frac{\mathrm{d}^{n-1}\mathbf{p}}{(2\pi)^{n-1}}a(\mathbf{p})q_p(\eta)e^{i\mathbf{p}\cdot\mathbf{x}} + h.c.\,, \tag{2.4.15}$$

where the function $q_p(\eta)$ satisfies

$$q_p''(\eta) + (n-2+2\delta)\, Ha q_p'(\eta) + p^2 q_p(\eta) = 0\,. \tag{2.4.16}$$

We now follow the method from Section 2.2. By considering the Wronskian for this equation, we find that we can require

$$q_p^*(\eta) q_p'(\eta) - q_p(\eta) q_p^{*\prime}(\eta) = \frac{2\mathrm{i}\kappa^2}{(n-2)\epsilon(\eta) a^{n-2}(\eta)}\,. \tag{2.4.17}$$

The operators $a(\mathbf{p})$ and $a^\dagger(\mathbf{p})$ then satisfy

$$[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = (2\pi)^{n-1}\delta^{n-1}(\mathbf{p}-\mathbf{p}')\,. \tag{2.4.18}$$

Defining the vacuum state $|0\rangle$ by requiring that $a(\mathbf{p})|0\rangle = 0$ for all $\mathbf{p}$, we find the two-point function for $Q(\eta, \mathbf{x})$ as

$$\begin{aligned} \Delta(\eta, \mathbf{x}; \eta', \mathbf{x}') &:= \langle 0 | Q(\eta, \mathbf{x}) Q(\eta', \mathbf{x}') | 0 \rangle \\ &= \int \frac{\mathrm{d}^{n-1}\mathbf{p}}{(2\pi)^{n-1}} q_p(\eta) q_p^*(\eta') \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}\,. \end{aligned} \tag{2.4.19}$$

Now, we analyse the solutions $q_p(\eta)$ for small p. For $p = 0$, the equation of motion (2.4.16) becomes

$$q_p''(\eta) + (n-2+2\delta)\, Ha q_p'(\eta) = 0\,. \tag{2.4.20}$$

Two independent real solutions $q_0(\eta) = Q_0^{(1)}(\eta)$ and $Q_0^{(2)}(\eta)$ can be chosen as

$$Q_0^{(1)}(\eta) = 1, \tag{2.4.21}$$

$$Q_0^{(2)}(\eta) = \int \frac{\mathrm{d}\eta}{\epsilon(\eta) a^{n-1}(\eta)}\,, \tag{2.4.22}$$

where the constant of integration in (2.4.22) is suitably chosen. As in the tensor case, two independent solutions $Q_p^{(1)}(\eta)$ and $Q_p^{(2)}(\eta)$ can be chosen for nonzero p such that

$$Q_p^{(1)}(\eta)\frac{\mathrm{d}Q_p^{(2)}(\eta)}{\mathrm{d}\eta} - \frac{\mathrm{d}Q_p^{(1)}(\eta)}{\mathrm{d}\eta} Q_p^{(2)}(\eta) = \frac{1}{\epsilon(\eta) a^{n-2}(\eta)}\,, \tag{2.4.23}$$

and

$$Q_p^{(I)}(\eta) = Q_0^{(I)}(\eta) + \mathcal{O}(p^2)\,, \tag{2.4.24}$$

for $I = 1, 2$. Again, in most applications, such as slow-roll inflation, the solutions $q_p(\eta)$

are chosen as

$$q_p(\eta) = \mathrm{i}A^{(S)}(p)Q_p^{(1)}(\eta) + B^{(S)}(p)Q_p^{(2)}(\eta)\,, \tag{2.4.25}$$

such that

$$A^{(S)}(p) \approx \frac{C'}{p^{\nu'}}\,, \tag{2.4.26}$$

where C' and ν' are positive constants, for small p. As in the tensor case, we can let $B^{(S)}(p) \sim p^{\nu'}$ as $p \to 0$. If $\nu' \geqslant (n-1)/2$, then the two-point function $\Delta(\eta, \mathbf{x}; \eta', \mathbf{x}')$ given by (2.4.19) is IR divergent. The IR divergences of the two-point functions for the tensor perturbations and Sasaki-Mukhanov variable are reflected in those for the graviton h_{ab} and the inflaton ψ. In the next section we shall see that these IR divergences can be gauged away by large gauge transformations if $\nu' < (n+3)/2$.

2.5 IR divergences in single-field inflation

In this section we show that, even if the two-point functions for the tensor and scalar perturbations are IR divergent, one can eliminate these divergences by large gauge transformations, as long as they are not very severe. Since the mechanism for the IR-divergence elimination has been discussed already for the tensor perturbations in Section 2.3, here we focus on the scalar perturbations.

We start with the graviton field in the gauge where the perturbation ψ in the scalar field is set to 0. Due to equation (2.4.6), the gauge is fixed by choosing the fields X_a. We choose them as follows:

$$X_0 = \frac{\Psi}{\phi'}\,, \tag{2.5.1}$$

$$X_k = 0\,. \tag{2.5.2}$$

Then we find, from (2.4.3)-(2.4.5), that

$$h_{00} = \frac{1}{Ha}Q'\,, \tag{2.5.3}$$

$$h_{0k} = \frac{1}{2}\partial_k\left(\frac{1}{Ha}Q - \epsilon \triangle^{-1} Q'\right)\,, \tag{2.5.4}$$

$$h_{kl} = H_{kl} - \delta_{kl}Q\,, \tag{2.5.5}$$

$$\psi = 0\,. \tag{2.5.6}$$

We now sketch how the latter expressions are derived. We start from the expressions for the components of the metric perturbation given in terms of quantities that are invariant under local gauge transformations. These are given by equations (2.4.3)-(2.4.7). Using the gauge condition (2.5.1), and the definition of the Sasaki-Mukhanov variable given in equation (2.4.11), we find

$$X_0 = \frac{\Psi}{\phi'} = \frac{1}{2Ha}\left(Q+\Sigma\right) . \tag{2.5.7}$$

Taking the derivative of this with respect to η, we find

$$X_0' = \frac{1}{2Ha}\left(Q+\Sigma\right)' + \frac{1}{2}(\epsilon-1)\left(Q+\Sigma\right) , \tag{2.5.8}$$

and we recall that $X_k = 0$ for all k.

Let us first consider the h_{00} component. Equation (2.4.3), given here again for convenience, is

$$h_{00} = S + 2X_0' + 2HaX_0 . \tag{2.5.9}$$

Using (2.5.7) and (2.5.8) and the equation of motion for the gauge invariant scalars, S and Σ, given by (2.4.10) and (2.4.13), we find

$$\begin{aligned} h_{00} &= -\epsilon Ha(n-3)\,\triangle^{-1} Q' + \frac{1}{Ha}\left(Q - \epsilon Ha\,\triangle^{-1} Q'\right)' + \epsilon\left(Q - \epsilon Ha\,\triangle^{-1} Q'\right) , \\ &= \frac{1}{Ha}Q' - \triangle^{-1}\epsilon\left[Q'' + (n-2+2\delta)\,HaQ' - \triangle Q\right] , \\ &= \frac{1}{Ha}Q' , \end{aligned} \tag{2.5.10}$$

where the quantity in the square brackets vanishes because of the equation of motion (2.4.14) for Q. The component h_{0k} is

$$\begin{aligned} h_{0k} &= V_k + X_k' + \partial_k X_0 , \\ &= \frac{1}{2}\partial_k\left(\frac{1}{Ha}Q - \epsilon\,\triangle^{-1} Q'\right) , \end{aligned} \tag{2.5.11}$$

where the last line follows from the gauge conditions (2.5.1) and (2.5.2), and equations of motion (2.4.9)-(2.4.13). The expression for h_{kl} also readily follows from these conditions:

$$\begin{aligned} h_{kl} &= H_{kl} + \delta_{kl}\left[\Sigma - (\Sigma+Q)\right] , \\ &= H_{kl} - \delta_{kl}Q . \end{aligned} \tag{2.5.12}$$

The expressions for h_{ab} given in equations (2.5.10), (2.5.11), and (2.5.12), are exactly those given by equation (2.5.3) - (2.5.5).

This gauge corresponds to imposing the following conditions on the components of the graviton and scalar fields:

$$\partial^l \left(h_{kl} - \frac{1}{n-1}\delta_{kl}\delta^{ij}h_{ij} \right) = 0\,, \tag{2.5.13}$$

$$\psi = 0\,. \tag{2.5.14}$$

Note that the condition (2.5.13) states that the traceless part of h_{kl} is transverse. Thus, the field components h_{ab} can be expressed as

$$h_{00}(\eta,\mathbf{x}) = \int \frac{\mathrm{d}^{n-1}\mathbf{p}}{(2\pi)^{n-1}} a(\mathbf{p})\gamma_{00}^{(\mathbf{p})}(\eta,\mathbf{x}) + \text{h.c.}\,, \tag{2.5.15}$$

$$h_{0k}(\eta,\mathbf{x}) = \int \frac{\mathrm{d}^{n-1}\mathbf{p}}{(2\pi)^{n-1}} a(\mathbf{p})\gamma_{0k}^{(\mathbf{p})}(\eta,\mathbf{x}) + \text{h.c.}\,, \tag{2.5.16}$$

$$h_{kl}(\eta,\mathbf{x}) = \int \frac{\mathrm{d}^{n-1}\mathbf{p}}{(2\pi)^{n-1}} \left[a(\mathbf{p})\gamma_{kl}^{(\mathbf{p})}(\eta,\mathbf{x}) + \sum_s a_s(\mathbf{p})\gamma_{kl}^{(s,\mathbf{p})}(\eta,\mathbf{x}) \right] \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}} + \text{h.c.}\,, \tag{2.5.17}$$

where $\gamma_{kl}^{(s,\mathbf{p})}(\eta,\mathbf{x})$ are defined by (2.2.7), and where

$$\gamma_{00}^{(\mathbf{p})}(\eta,\mathbf{x}) = \frac{1}{H(\eta)a(\eta)} q_p'(\eta)\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\,, \tag{2.5.18}$$

$$\gamma_{0k}^{(\mathbf{p})}(\eta,\mathbf{x}) = \frac{\mathrm{i}}{2}p_k \left[q_p(\eta) + \frac{\epsilon(\eta)}{p^2}q_p'(\eta) \right] \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\,, \tag{2.5.19}$$

$$\gamma_{kl}^{(\mathbf{p})}(\eta,\mathbf{x}) = -\delta_{kl}q_p(\eta)\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\,. \tag{2.5.20}$$

The space components of the two-point function of h_{ab} are

$$\begin{aligned}\langle 0|h_{kl}(\eta,\mathbf{x})h_{k'l'}(\eta',\mathbf{x}')|0\rangle = \int \frac{\mathrm{d}^{n-1}\mathbf{p}}{(2\pi)^{n-1}} \Bigg[& \gamma_{kl}^{(\mathbf{p})}(\eta,\mathbf{x})\gamma_{k'l'}^{(\mathbf{p})*}(\eta',\mathbf{x}') \\ & + \sum_s \gamma_{kl}^{(s,\mathbf{p})}(\eta,\mathbf{x})\gamma_{k'l'}^{(s,\mathbf{p})*}(\eta',\mathbf{x}') \Bigg]\,.\end{aligned} \tag{2.5.21}$$

The other components are given by

$$\langle 0|h_{ab}(\eta,\mathbf{x})h_{a'b'}(\eta',\mathbf{x}')|0\rangle = \int \frac{\mathrm{d}^{n-1}\mathbf{p}}{(2\pi)^{n-1}} \gamma_{ab}^{(\mathbf{p})}(\eta,\mathbf{x})\gamma_{a'b'}^{(\mathbf{p})}(\eta',\mathbf{x}')\,, \tag{2.5.22}$$

where at least one of the indices a, b, a' and b' is the time index 0. The IR properties of these two-point functions are determined by the small-p behaviour of the integrand for the p-integral.

Assuming the properties of $q_p(\eta)$ stated in Section 2.4, in particular that for small

p one has

$$q_p(\eta) \approx \mathrm{i}A^{(S)}(p)\left[1+\mathcal{O}(p^2)\right]\,, \tag{2.5.23}$$

with

$$A^{(S)}(p) \sim p^{-\nu'}\,, \tag{2.5.24}$$
$$B^{(S)}(p) \sim p^{\nu'}\,, \tag{2.5.25}$$

for $\nu' \geqslant (n-1)/2$, it is clear that the small-p behaviour of the derivative, $q_p'(\eta)$, is better by a factor of p^2, i.e. $q_p'(\eta) \sim 1/p^{\nu'-2}$ for small p. As a result, we find

$$\gamma_{00}^{(\mathbf{p})}(\eta,\mathbf{x}) \sim 1/p^{\nu'-2}\,, \tag{2.5.26}$$
$$\gamma_{0k}^{(\mathbf{p})}(\eta,\mathbf{x}) \sim 1/p^{\nu'-1}\,, \tag{2.5.27}$$

for small p. We now show that large gauge transformations similar to those given by (2.3.5) can be used so that all functions $\gamma_{ab}^{(\mathbf{p})}(\eta,\mathbf{x})$ and $\gamma_{ab}^{(s,\mathbf{p})}(\eta,\mathbf{x})$ are modified to behave like $1/p^{\nu'-2}$ rather than $1/p^{\nu'}$.

Let us define

$$\tilde{Q}_p^{(1)}(\eta) := \frac{1}{p^2}Q_p^{(1)\prime}(\eta)\,, \tag{2.5.28}$$

and we note that the function $\tilde{Q}_0^{(1)}(\eta) := \lim_{p\to 0}\tilde{Q}_p^{(1)}(\eta)$ is well defined because $Q_p^{(1)\prime}(\eta) = \mathcal{O}(p^2)$. Then,

$$\gamma_{00}^{(\mathbf{p})}(\eta,\mathbf{x}) \approx \frac{\mathrm{i}}{H(\eta)a(\eta)}p^2 A^{(S)}(p)\tilde{Q}_p^{(1)}(\eta)\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\,, \tag{2.5.29}$$
$$\gamma_{0k}^{(\mathbf{p})}(\eta,\mathbf{x}) \approx -\frac{1}{2}p_k A^{(S)}(p)\left[Q_p^{(1)}(\eta)+\epsilon(\eta)\tilde{Q}_p^{(1)}(\eta)\right]\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\,, \tag{2.5.30}$$
$$\gamma_{kl}^{(\mathbf{p})}(\eta,\mathbf{x}) \approx -\mathrm{i}\delta_{kl}A^{(S)}(p)Q_p^{(1)}(\eta)\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\,, \tag{2.5.31}$$

for small p. The tensor modes are modified as described in Section 2.3. For the scalar modes, we make the large gauge transformation with $\xi_0 = 0$ and

$$\begin{aligned}\xi_i =& \frac{\mathrm{i}}{2}A^{(S)}(p)Q_0^{(1)}(\eta)\left[(1+\mathrm{i}\mathbf{p}\cdot\mathbf{x})x_i - \frac{\mathrm{i}}{2}p_i x^2\right]e^{-\rho^2p^2}\\ &+\frac{1}{2}A^{(S)}(p)p_i\int \mathrm{d}\eta\left[Q_0^{(1)}(\eta)+\epsilon(\eta)\tilde{Q}_0^{(1)}(\eta)\right]e^{-\rho^2p^2}\,.\end{aligned} \tag{2.5.32}$$

The first line was obtained in a way similar to the method used in Section 2.3. The second line gauges away the $\mathcal{O}(p)$ term in $\gamma_{0k}^{(\mathbf{p})}(\eta,\mathbf{x})$. Then the two-point functions (2.5.21) and (2.5.22) are modified in such a way that the tensor mode functions $\gamma_{kl}^{(s,\mathbf{p})}(\eta,\mathbf{x})$ are replaced by $\tilde{\gamma}_{kl}^{(s,\mathbf{p})}(\eta,\mathbf{x})$ given by (2.3.8) and that the functions $\gamma_{ab}^{(\mathbf{p})}(\eta,\mathbf{x})$, given by

(2.5.18) - (2.5.20), are replaced by

$$\begin{aligned}
\tilde{\gamma}^{(\mathbf{p})}_{00}(\eta,\mathbf{x}) &= \gamma^{(\mathbf{p})}_{00}(\eta,\mathbf{x})\,, \qquad (2.5.33)\\
\tilde{\gamma}^{(\mathbf{p})}_{0k}(\eta,\mathbf{x}) &= -\frac{1}{2}p_k\Big\{A^{(S)}(p)\left[Q^{(1)}_p(\eta)-Q^{(1)}_0(\eta)\right]-\mathrm{i}B^{(S)}(p)Q^{(2)}_p(\eta)\\
&\qquad +\epsilon(\eta)A^{(S)}(p)\left[\tilde{Q}^{(1)}_p(\eta)-\tilde{Q}^{(1)}_0(\eta)\right]\\
&\qquad -\mathrm{i}\epsilon(\eta)p^{-2}B^{(S)}(p)Q^{(2)\prime}_p(\eta)\Big\}\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\\
&\quad -\frac{1}{2}p_kA^{(S)}(p)\left[Q^{(1)}_0(\eta)+\epsilon(\eta)\tilde{Q}^{(1)}_0(\eta)\right](\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}}-\mathrm{e}^{-\rho^2p^2})\,, \qquad (2.5.34)\\
\tilde{\gamma}^{(\mathbf{p})}_{kl}(\eta,\mathbf{x}) &= -\mathrm{i}\delta_{kl}\Big\{A^{(S)}(p)\left[Q^{(1)}_p(\eta)-Q^{(1)}_0(\eta)\right](1+\mathrm{i}\mathbf{p}\cdot\mathbf{x})\\
&\qquad +A^{(S)}(p)Q^{(1)}_p(\eta)\left(\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}}-1-\mathrm{i}\mathbf{p}\cdot\mathbf{x}\right)\\
&\qquad -A^{(S)}(p)(1+\mathrm{i}\mathbf{p}\cdot\mathbf{x})(\mathrm{e}^{-\rho^2p^2}-1)-\mathrm{i}B^{(S)}(p)Q^{(2)}_p(\eta)\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\Big\}\,. \qquad (2.5.35)
\end{aligned}$$

Thus, all components of $\tilde{\gamma}^{(\mathbf{p})}_{ab}(\eta,\mathbf{x})$ behave like $p^{\nu'-2}$ or better for small p. This implies that, although the two-point function for the metric perturbation is IR divergent if

$$\max(\nu,\nu') \geqslant \frac{n-1}{2}\,, \qquad (2.5.36)$$

the two-point function modified by the large gauge transformations given by (2.3.5) and (2.5.32) is IR divergent only if

$$\max(\nu,\nu') \geqslant \frac{n+3}{2}\,. \qquad (2.5.37)$$

We now return to our explicit slow-roll example universe. If the slow-roll parameter $\epsilon\,(>0)$ is constant, i.e. if the scale parameter takes the form $a(\eta)=(-\eta/\eta_0)^{-\lambda}$, then the tensor perturbation H_{ab} satisfies the same equation as in Section 2.3, so the functions $F^{(1)}_p(\eta)$, $F^{(2)}_p(\eta)$, and the constants $A^{(T)}(p)$, $B^{(T)}(p)$, are given by (2.3.20), (2.3.21), (2.3.22) and (2.3.23). As for the scalar perturbation Q, the function $q_p(\eta)$ satisfies the same equation as $f_p(\eta)$. Taking into account the normalisation condition (2.2.11) we find

$$q_p(\eta) = C^{(S)}(p)(-p\eta)^{\nu}H^{(1)}_{\nu}(-p\eta)\,, \qquad (2.5.38)$$

where $\nu=[1+(n-2)\lambda]/2$ as before, and

$$C^{(S)}(p) = -\kappa\frac{\sqrt{\pi\eta_0}}{\sqrt{2(n-2)\epsilon}(p\eta_0)^{\nu}}\,. \qquad (2.5.39)$$

Then we can let

$$Q_p^{(1)}(\eta) = -\frac{\pi}{2^\nu\Gamma(\nu)}(-p\eta)^\nu Y_\nu(-p\eta)\,, \tag{2.5.40}$$

$$Q_p^{(2)}(\eta) = -\frac{2^{\nu-1}\Gamma(\nu)\eta_0}{(p\eta_0)^{2\nu}}(-p\eta)^\nu J_\nu(-p\eta)\,, \tag{2.5.41}$$

$$A^{(S)}(p) = \kappa\sqrt{\frac{\eta_0}{\pi(n-2)\epsilon}}\frac{2^{\nu-\frac{1}{2}}\Gamma(\nu)}{(p\eta_0)^\nu}\,, \tag{2.5.42}$$

$$B^{(S)}(p) = \kappa\sqrt{\frac{\pi}{(n-2)\epsilon\eta_0}}\frac{(p\eta_0)^\nu}{2^{\nu-\frac{1}{2}}\Gamma(\nu)}\,, \tag{2.5.43}$$

As an aside, the discussion that $Q_p^{(1)}(\eta)-Q_0^{(1)}(\eta)=\mathcal{O}(p^2)$ for constant ϵ or for slow-roll inflation is almost identical to the one presented earlier in Section 2.3.

By using the recursion formula for the Bessel functions,

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[z^\nu J_\nu(z)\right] = z^\nu J_{\nu-1}(z)\,, \tag{2.5.44}$$

and similarly for $Y_\nu(z)$, we find

$$\tilde{Q}_p^{(1)}(\eta) = -\frac{\pi\eta}{2^\nu\Gamma(\nu)}(-p\eta)^{\nu-1}Y_{\nu-1}(-p\eta)\,, \tag{2.5.45}$$

$$p^{-2}Q_p^{(2)\prime}(\eta) = -\frac{2^{\nu-1}\Gamma(\nu)\eta_0\eta}{(p\eta_0)^{2\nu}}(-p\eta)^{\nu-1}J_{\nu-1}(-p\eta)\,. \tag{2.5.46}$$

The function $\tilde{Q}_p^{(1)}(\eta)$ is non-singular as $p\to 0$ as concluded from the general discussion. The range of the parameter ϵ for which the IR divergences can be gauged away is the same as that for the tensor perturbations and given by (2.3.29) or, equivalently, by (2.3.30) if $\lambda > 0$. The discussion for the case $\lambda < 0$ is also exactly the same as the tensor-perturbation case.

The scale factor corresponding to the slow-roll inflation can be written as

$$a(\eta) = (-\eta/\eta_0)^{-\frac{1}{1-\epsilon}+\epsilon\delta\ln(-\eta/\eta_0)}\,, \tag{2.5.47}$$

where $\epsilon \geqslant 0$, and $\epsilon, |\delta| \ll 1$, in a range of η where ϵ and δ can be treated as constants[1]. We work to first order in ϵ and δ. One readily finds that the slow-roll parameters agree with the parameters ϵ and δ in (2.5.47) to lowest order. The discussion of the IR divergences for the tensor perturbations will be exactly the same as in Section 2.3 except that here the slow-roll parameter ϵ is assumed to be much smaller than 1. The discussion for the scalar perturbations is changed slightly if $\delta \neq 0$. Noting that

[1] One cannot write $a(\eta)$ in this form over the long range of η for which variations in η and δ need to be taken into account.

$Ha = -[(1-\epsilon)\eta]^{-1}$ to first order, one finds that equation (2.4.16) becomes

$$q_p''(\eta) + \frac{(n-2)(1+\epsilon)+2\delta}{\eta} q_p'(\eta) + p^2 q_p(\eta) = 0\,. \tag{2.5.48}$$

By comparing this equation with (2.2.9), we find that two independent solutions can be chosen as

$$q_p(\eta) \propto (-p\eta)^{\nu'} H_{\nu'}^{(1)}(-p\eta) \tag{2.5.49}$$

and its complex conjugate, where

$$\nu' = \frac{1}{2}[1 + (n-2)(1+\epsilon) + 2\delta]\,. \tag{2.5.50}$$

The normalisation constant can be found from (2.4.17). By noting that we can write

$$\epsilon(\eta) = \epsilon_0(-\eta/\eta_0)^{-2\delta}\,, \tag{2.5.51}$$

to next leading order in ϵ and δ, where ϵ_0 is a constant, we find

$$q_p(\eta) = C^{(S')}(-p\eta)^{\nu'} H_{\nu'}^{(1)}(-p\eta)\,, \tag{2.5.52}$$

where

$$C^{(S')}(p) = -\kappa \frac{\sqrt{\pi\eta_0}}{2(n-2)\epsilon_0(p\eta_0)^{\nu'}}\,. \tag{2.5.53}$$

The functions $Q_p^{(1)}(\eta)$, and $Q_p^{(2)}(\eta)$, and the constants $A^{(S)}(p)$, and $B^{(S)}(p)$, are given by replacing ν by ν' and ϵ by ϵ_0 in (2.5.40)-(2.5.43).

2.6 Discussion

In this chapter we studied the nature of IR divergences in the free two-point functions for the tensor perturbations in general FLRW spacetime, and the scalar perturbations in single-field inflation. These IR divergences occur because for small momentum p, the mode functions behave like $p^{-\nu}$ with $\nu \geqslant (n-1)/2$, in n dimensions. We pointed out that global shear transformations and dilation can increase the power by 1, i.e. from $p^{-\nu}$ to $p^{-\nu+1}$, and showed that in fact there are large gauge transformations which increase the power of p in the IR limit of the mode functions by 2. This implies that the two-point functions for the tensor and scalar perturbations can be made IR finite by large gauge transformations in a larger set of FLRW spacetimes (for the scale-invariant vacuum state) than previously thought. Our focus was on the slow-roll inflation, but the reduction of the power of p in the IR in the p-integration is valid for any potential $V(\phi)$ including those leading to bouncing cosmologies [80].

Our findings are consistent with the fact that the graviton and inflaton fields smeared in a gauge-invariant manner are equivalent to the linearised Weyl tensor and another tensor whose p-dependence is less singular than the original fields by a factor of p^2 [72, 73] (see also [81]). This is because the latter work indicates that the terms of order p^0, as well as those of order p, are of pure-gauge form, and this is what we have verified.

Unlike the global shear transformations and dilation, we have not found a simple geometric interpretation for the large gauge transformations that gauge away the terms of order p in the mode functions, which is an extension of the global shear transformations and dilation. It would be interesting to find a geometric picture of these gauge transformations. We note that, in this context, that the vectors ξ_i specifying these gauge transformations are hypersurface orthogonal.

It would not be straightforward to incorporate interactions in the method of gauging away IR divergences presented in this chapter, for example, to discuss three-point functions relevant to non-Gaussianities. The obvious drawbacks are its non-locality and lack of manifest translation invariance. It would be interesting to investigate whether these difficulties could be overcome to construct perturbation theory for inflationary models that were manifestly IR finite.

We return to this result in Chapter 5. However, for the next couple of chapters we are concerned with the large-distance behaviour of propagators.

Chapter 3

Large-distance behaviour of the covariant massless vector two-point function in de Sitter spacetime

In this chapter, we study the large-distance behaviour of the covariant massless vector propagator in de Sitter spacetime. To this end, we will consider the massless limit of the Stueckelberg theory [82], a theory of a massive vector field, with an additional gauge-fixing term. A useful summary of this theory is provided by Ruegg and Ruiz-Altaba [83]. The massless limit of the Stueckelberg theory, with finite Stueckelberg parameter ξ, is equivalent to the massless vector theory, with covariant gauge-fixing term.

The large-distance behaviour of the covariant massless vector propagator in de Sitter spacetime is well-known [1, 44]. For points with large (spacelike or timelike) separations, the vector propagator tends to a non-zero constant in 4 dimensions, for $\xi \neq 0$ [1], where we clarify that by large spacelike separation we mean when the conformal spacelike distance between two points becomes large. We show that the propagator behaves in the same way for general n dimensions, for $n \geq 4$, in order to verify that this behaviour is not unique to 4 dimensions. Knowledge of the long-distance behaviour of the vector propagator in n dimensions is also useful in calculations involving dimensional regularisation. We are also interested in establishing a method of finding the large-distance limit of a propagator in de Sitter spacetime so that we can apply it to the case of a graviton propagator.

3.1 Preliminaries

We consider the Stueckelberg theory [82], which is an extension of the Proca theory, with an additional gauge-fixing term. We give a brief description of the Stueckelberg theory in this section, and further details necessary for the calculation will be provided throughout the chapter. The Stueckelberg theory for massive vector fields would usually include a scalar of the same mass, which transforms under a gauge transformation such that we have a gauge symmetry, even in the massive case [84,85]. In the massless case, the vector field alone possesses a gauge symmetry, so we ignore the scalar field in what follows.

The Stueckelberg Lagrangian for a massive vector field is given by

$$\mathcal{L} = -\frac{1}{4}\sqrt{-g}\left[F^{ab}F_{ab} + 2m^2 A_b A^b + \frac{2}{\xi}\left(\nabla_a A^a\right)^2\right] , \tag{3.1.1}$$

where the field strength tensor $F_{ab} = \nabla_a A_b - \nabla_a A_b$, and ξ is the Stueckelberg parameter. Different values of this parameter correspond to different gauges. For example, $\xi = 0$ corresponds to the Landau gauge, and $\xi \to \infty$ gives the unitary gauge. We take the massless limit of this theory, which corresponds to the standard massless vector theory, with a covariant gauge-fixing term. As is the case for the rest of this thesis, we work in de Sitter spacetime, described by the metric

$$\mathrm{d}s^2 = \frac{1}{(-H\eta)^2}\left(-\mathrm{d}\eta^2 + \mathrm{d}\boldsymbol{x}^2\right) , \tag{3.1.2}$$

where H is a positive constant, and the conformal time $\eta \in (-\infty, 0)$.

In order to find the large-distance behaviour of the propagator, we will work in terms of the de Sitter invariant Z. This invariant arises through embedding n-dimensional de Sitter spacetime in $(n+1)$-dimensional Minkowski spacetime. In this way, de Sitter spacetime is thought of as the set of points X^i in Minkowski spacetime such that $X \cdot X = \frac{1}{H^2}$. The de Sitter invariant of two spacelike separated points, x and x', is defined in terms of the geodesic separation of two points, denoted $\sigma(x, x')$, as

$$Z(x, x') = \cos\left[H\sigma(x, x')\right] , \tag{3.1.3}$$

where the geodesic separation is defined to be

$$\sigma(x, x') = \int_0^1 \left[g_{ab}\frac{\partial x^a(\lambda)}{\partial\lambda}\frac{\partial x^b(\lambda)}{\partial\lambda}\right]^{\frac{1}{2}} \mathrm{d}\lambda\,, \tag{3.1.4}$$

where $x(0) = x$ and $x(1) = x'$. For spacelike separated points, $|Z| < 1$. For timelike separated points, where $Z > 1$, it is defined through analytic continuation. There is no geodesic connecting two points in the region where $Z < -1$.

We work in conformal coordinates, so, for two points X and X', Z is defined in this coordinate system as

$$Z = X \cdot X' = 1 - \frac{||\boldsymbol{x} - \boldsymbol{x}'||^2 - (\eta - \eta')^2}{2\eta\eta'} \,. \tag{3.1.5}$$

The parameter Z is a measure of the geodesic distance between two spacetime points X and X', and the large-distance limit corresponds to the limit $|Z| \to \infty$. To see this, we study both the large spacelike and timelike separations, as follows. First, when $||\mathbf{x} - \mathbf{x}'|| \to \infty$ - when the spatial distance between the two points is large - it can be seen from the definition of Z given above that $Z \to -\infty$. Second, when η' is held fixed and $\eta \to 0$, which corresponds to future infinity in our coordinate system, the de Sitter invariant $Z \to \infty$. Hence, it is consistent to take the limit $|Z| \to \infty$ to find the large-distance behaviour of the propagator.

We work in a basis of products of the following tensors: $n_a(x, x')$, $n_{a'}(x, x')$, $g_{ab}(x, x)$, $g_{a'b'}(x'x')$, and $g_{ab'}(x, x')$, which are defined below. In the following, unprimed indices refer to the point x, and primed indices refer to the point x', so we can omit the arguments of these tensors without ambiguity. Indices are raised and lowered using the metric tensors g_{ab} and $g_{a'b'}$, for unprimed and primed indices, respectively. By definition, as can be found in [86], for example, the parallel propagator $g_{ab'}$ is the unique solution to

$$\nabla^c \sigma(x, x') \nabla_c g_{ab'}(x, x') = 0 \,. \tag{3.1.6}$$

Additionally

$$\lim_{x' \to x} g_{ab'}(x, x') = g_{ab}(x, x) \,. \tag{3.1.7}$$

The unit vectors

$$n_a = \nabla_a \sigma(x, x') \,, \tag{3.1.8}$$

$$n_{b'} = \nabla_{b'} \sigma(x, x') \,, \tag{3.1.9}$$

are tangent to the geodesic at the points x and x', respectively. These vectors have opposite direction to each other, so that

$$g^{ab'} n_a = -n^{b'} \,. \tag{3.1.10}$$

The massless limit of the vector two-point function, as found by Fröb and Higuchi

[44], is

$$\lim_{m^2\to 0}\langle 0|A_a(x)A_{b'}(x')|0\rangle = \frac{H^{n-4}}{(4\pi)^{\frac{n}{2}}}\left[\frac{n-2}{n-3}I^{(0)}(Z)\partial_a\partial_{b'}Z + \frac{1}{n-3}I^{(0)\prime}(Z)(\partial_a Z)(\partial_{b'}Z)\right] + \left(\xi - \frac{n-1}{n-3}\right)\partial_a\partial_{b'}\tilde{\triangle}(Z)\,, \tag{3.1.11}$$

where

$$I^{(0)}(Z) = \frac{\Gamma(n-2)}{\Gamma\left(\frac{n}{2}\right)}\ {}_2F_1\left(n-2, 1; \frac{n}{2}; \frac{1+Z}{2}\right)\,, \tag{3.1.12}$$

and ${}_2F_1(a,b;c;z)$ is the hypergeometric function. Additionally, we have defined

$$\tilde{\triangle}(Z) = -\lim_{m^2\to 0}\frac{\partial}{\partial m^2}\left(\triangle_{m^2}(Z) - \triangle_{m^2}(-1)\right)\,, \tag{3.1.13}$$

where

$$\triangle_{m^2}(Z) = \frac{H^{n-2}}{(4\pi)^{\frac{n}{2}}}I_\mu(Z)\,, \tag{3.1.14}$$

and

$$I_\mu(Z) = \frac{\Gamma\left(\frac{n-1}{2}+\mu\right)\Gamma\left(\frac{n-1}{2}-\mu\right)}{\Gamma\left(\frac{n}{2}\right)}\ {}_2F_1\left(\frac{n-1}{2}+\mu, \frac{n-1}{2}-\mu; \frac{n}{2}; \frac{1+Z}{2}\right)\,, \tag{3.1.15}$$

for

$$\mu = \sqrt{\frac{(n-1)^2}{4} - \frac{m^2}{H^2}}\,. \tag{3.1.16}$$

We here note that $\xi = 0$ corresponds to the massless vector two-point function found by Tsamis and Woodard [87], and the massless vector propagator for $\xi = 1$, when $n = 4$, was given by Allen and Jacobson [88].

We now calculate the relevant derivatives of Z to express the propagator in a more convinient form. These derivatives are calculated by Allen and Jacobson [88], and we present them here for later use. No approximations are made in these calculations, so the following results are exact. From the definition of Z, given by equation (3.1.3), one obtains

$$\partial_{b'}Z = -H\sqrt{1-Z^2}\,n_{b'}\,, \tag{3.1.17}$$

so

$$(\partial_a Z)(\partial_{b'}Z) = H^2(1-Z^2)n_a n_{b'}\,, \tag{3.1.18}$$

and

$$\partial_a\partial_{b'}Z = H^2\left[g_{ab'} + (1-Z)n_a n_{b'}\right]\,, \tag{3.1.19}$$

where the results of Allen and Jacboson [88] have been used to evaluate $\partial_{b'}n_a$, which we quote here for completeness:

$$\partial_{b'} n_a = -\frac{H}{\sqrt{1-Z^2}}(g_{ab'} + n_a n_{b'})\,. \tag{3.1.20}$$

The results (3.1.18) and (3.1.19) are frequently referred to in the rest of this chapter.

The final term of the propagator, given by equation (3.1.11), is $\partial_a \partial_{b'} \tilde{\triangle}(Z)$. We want to write this in terms of derivatives of Z. We use the chain rule, along with the results of this section, to express this term as

$$\begin{aligned}\partial_a \partial_{b'} \tilde{\triangle}(Z) &= (\partial_a \partial_{b'} Z)\tilde{\triangle}'(Z) + (\partial_{b'} Z)(\partial_a Z)\tilde{\triangle}''(Z)\,,\\ &= H^2\left[g_{ab'} + (1-Z)n_a n_{b'}\right]\tilde{\triangle}'(Z) + H^2(1-Z^2)n_a n_{b'}\tilde{\triangle}''(Z)\,,\end{aligned} \tag{3.1.21}$$

where the derivatives of the function $\tilde{\triangle}(Z)$ will be found later in the chapter.

Using these expressions, we split the propagator into gauge-independent and gauge-dependent parts, and write it as

$$\lim_{m^2\to 0}\langle 0|A_a(x)A_{b'}(x')|0\rangle = \langle 0|A_a(x)A_{b'}(x')|0\rangle_{GI} + \langle 0|A_a(x)A_{b'}(x')|0\rangle_{GD}\,, \tag{3.1.22}$$

where the first term is made up of the gauge-independent terms of the propagator, and the second term comprises of the gauge-dependent term. Explicitly, from, equation (3.1.11), we have

$$\langle 0|A_a(x)A_{b'}(x')|0\rangle_{GI} = A_{GI}(Z)g_{ab'} + B_{GI}(Z)n_a n_{b'}\,, \tag{3.1.23}$$

$$\langle 0|A_a(x)A_{b'}(x')|0\rangle_{GD} = A_{GD}(Z)g_{ab'} + B_{GD}(Z)n_a n_{b'}\,, \tag{3.1.24}$$

where

$$A_{GI}(Z) = \frac{H^{n-2}}{(4\pi)^{\frac{n}{2}}}\frac{n-2}{n-3}I^{(0)}(Z)\,, \tag{3.1.25}$$

$$A_{GD}(Z) = H^2\left(\xi - \frac{n-1}{n-3}\right)\tilde{\triangle}'(Z)\,, \tag{3.1.26}$$

and

$$B_{GI}(Z) = \frac{H^{n-2}}{(4\pi)^{\frac{n}{2}}}\left[\frac{n-2}{n-3}(1-Z)I^{(0)}(Z) + \frac{1}{n-3}(1-Z^2)I^{(0)\prime}(Z)\right]\,, \tag{3.1.27}$$

$$B_{GD}(Z) = H^2\left(\xi - \frac{n-1}{n-3}\right)\left[(1-Z)\tilde{\triangle}'(Z) + (1-Z^2)\tilde{\triangle}''(Z)\right]\,. \tag{3.1.28}$$

The leading order behaviour of the functions $A_{GI}(Z)$ and $A_{GD}(Z)$ is different to that of $B_{GI}(Z)$ and $B_{GD}(Z)$; we therefore find $A_i(Z)$ and $B_i(Z)$ separately.

As different methods are used to find the large-distance behaviour of the gauge-independent and gauge-dependent terms of the propagator, we find the large-Z behaviour of the gauge-independent and gauge-dependent parts separately. Derivatives of Z appear in all terms of the propagator, so we start by finding an explicit expression for these.

3.2 Large-distance limit of the gauge-independent terms in the propagator

In this section we find the $|Z| \to \infty$ limit of the gauge-independent part of the propagator, given by equation (3.1.23). As we have already found $\partial_a \partial_{b'} Z$ and $(\partial_a Z)(\partial_{b'} Z)$, it only remains to find the large-Z behaviour of $I^{(0)}(Z)$ and $I^{(0)\prime}(Z)$, which is the focus of this section.

First, we find the large-distance behaviour of $I^{(0)}(Z)$, defined in equation (3.1.12), and repeated here for clarity:

$$I^{(0)}(Z) = \frac{\Gamma(n-2)}{\Gamma\left(\frac{n}{2}\right)} \, {}_2F_1\left(n-2, 1; \frac{n}{2}; \frac{1+Z}{2}\right) . \tag{3.2.1}$$

This function is proportional to the hypergeometric function, which can be written as the following series expansion:

$${}_2F_1(a, b; c; z) = 1 + \frac{ab}{c} z + \mathcal{O}(z^2) , \tag{3.2.2}$$

defined for $|z| < 1$. In order to study the behaviour of $I^{(0)}(Z)$ in the limit $|Z| \to \infty$, we use the well known transformation property [89]

$$\begin{aligned} {}_2F_1(a, b; c; z) =& \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} (1-z)^{-a} \, {}_2F_1\left(a, c-b; a-b+1; \frac{1}{1-z}\right) \\ &+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} (1-z)^{-b} \, {}_2F_1\left(b, c-a; b-a+1; \frac{1}{1-z}\right) , \end{aligned} \tag{3.2.3}$$

as we can use the above series expansion of the hypergeometric function to write

$${}_2F_1\left(a, b; c; \frac{1}{1-Z}\right) \to 1 + \mathcal{O}\left(Z^{-1}\right) . \tag{3.2.4}$$

Using the transformation (3.2.3), it can be seen that

$$\begin{aligned} I^{(0)}(Z) =& \frac{\Gamma(n-2)\Gamma(3-n)}{\Gamma\left(2-\frac{n}{2}\right)}\left(\frac{1-Z}{2}\right)^{-(n-2)} {}_2F_1\left(n-2, \frac{n}{2}-1; n-2; \frac{2}{1-Z}\right) \\ &+ \frac{\Gamma(n-3)}{\Gamma\left(\frac{n}{2}-1\right)}\left(\frac{1-Z}{2}\right)^{-1} {}_2F_1\left(1, 2-\frac{n}{2}; 4-n; \frac{2}{1-Z}\right) . \end{aligned} \tag{3.2.5}$$

We note that, despite the apparent singularity due to the integer n case of the $\Gamma(3-n)$ factor in equation (3.2.5), the result of this section is well-defined. This is because the singular terms combine with singular terms which occur at higher orders in the expansion of the hypergeometric function second line of equation (3.2.5), to give a well-defined contribution in the $n \to$ integer limit. This term behaves like $[(1-Z)/2]^{-(n-2)} \log Z$, which is still suppressed in comparison with $[(1-Z)/2]^{-1}$.

We now take the $|Z| \to \infty$ limit of equation (3.2.5), where we work to $\mathcal{O}\left(Z^{-2}\right)$, which is valid as the neglected terms have no effect on the large-distance behaviour of the propagator. This can be seen most clearly from the explicit expressions for $A_{GI}(Z)$ and $B_{GI}(Z)$, given by equations (3.1.25) and (3.1.27), respectively. In $B_{GI}(Z)$, $I^{(0)}(Z)$ appears when multiplied by a factor of $(1-Z)$, and in $A_{GI}(Z)$, $I^{(0)}(Z)$ appears without being multiplied by any function of Z, so terms of order Z^{-2} can be neglected in the $|Z| \to \infty$ limit.

Expanding the hypergeometric function in the first term of the right hand side of equation (3.2.5), it can be seen that, in the large-Z limit, the leading order term is

$$\frac{\Gamma(n-2)\Gamma(3-n)}{\Gamma\left(2-\frac{n}{2}\right)}\left(\frac{1-Z}{2}\right)^{-(n-2)} , \tag{3.2.6}$$

which, for $n > 2$, is of lower order than Z^{-2}. As we consider $n \geqslant 4$, we are free to ignore the first hypergeometric function from the transformation given by equation (3.2.3). Details of $n = 2, 3, 4$ cases are described by Fröb and Higuchi [44].

We therefore need only consider terms originating from the series expansion of the second hypergeometric function in equation (3.2.5). Taking the large-distance limit, it can be seen that

$$I^{(0)}(Z) \to \frac{\Gamma(n-3)}{\Gamma\left(\frac{n}{2}-1\right)}\left(\frac{1-Z}{2}\right)^{-1} + \mathcal{O}\left(Z^{-2}\right) . \tag{3.2.7}$$

Therefore, in the $|Z| \to \infty$ limit, we find,

$$I^{(0)}(Z)\partial_a\partial_{b'}Z \to 2H^2\frac{\Gamma(n-3)}{\Gamma\left(\frac{n}{2}-1\right)} n_a n_{b'} + \mathcal{O}\left(Z^{-1}\right) . \tag{3.2.8}$$

We now find the large-distance behaviour of the function $I^{(0)\prime}(Z)$. In order to calculate $I^{(0)\prime}(Z)$, first note that the derivative of the hypergeometric function is

$$\frac{\mathrm{d}}{\mathrm{d}z}\ {}_2F_1(a,b;c;z) = \frac{ab}{c}\ {}_2F_1(a+1,b+1;c+1;z)\,. \tag{3.2.9}$$

The derivative of $I^{(0)}(Z)$, defined by (3.1.12), is therefore

$$\begin{aligned} I^{(0)\prime}(Z) &= \frac{\mathrm{d}}{\mathrm{d}Z} I^{(0)}(Z)\,, \\ &= \frac{\Gamma(n-1)}{n\Gamma\left(\frac{n}{2}\right)}\ {}_2F_1\left(n-1, 2; \frac{n}{2}+1; \frac{1+Z}{2}\right)\,. \end{aligned} \tag{3.2.10}$$

We now use the same method used above to find the large-distance behaviour of $I^{(0)}(Z)$.

In the following, when taking the large-Z limit of the series expansion of hypergeometric functions, we work to $\mathcal{O}\left(Z^{-3}\right)$. It can be seen that this is valid as $I^{(0)\prime}(Z)$ only appears in the propagator in the function $B_{GI}(Z)$, given by equation (3.1.27). It appears here multiplied by a factor of $(1-Z^2)$, the contribution of $I^{(0)\prime}(Z)$ to $B_{GI}(Z)$ from terms proportional to Z^{-3} and higher are vanishing in the large-Z limit.

Again, using transformation (3.2.3),

$$\begin{aligned} I^{(0)\prime}(Z) =& \frac{\Gamma(n-1)\Gamma(3-n)}{2\Gamma\left(2-\frac{n}{2}\right)}\left(\frac{1-Z}{2}\right)^{-(n-1)}\ {}_2F_1\left(n-1, \frac{n}{2}-1; n-2; \frac{2}{1-Z}\right) \\ &+ \frac{\Gamma(n-3)}{2\Gamma\left(\frac{n}{2}-1\right)}\left(\frac{1-Z}{2}\right)^{-2}\ {}_2F_1\left(2, 2-\frac{n}{2}; 4-n; \frac{2}{1-Z}\right)\,. \end{aligned} \tag{3.2.11}$$

As in the previous calculation, the integer n singularity from the $\Gamma(3-n)$ factor in the following equation is cancelled by higher order contributions from the expansion of the second hypergeometric function in equation (3.2.3).

The leading order term in the expansion of the first line of equation (3.2.11) is

$$\frac{\Gamma(n-1)\Gamma(3-n)}{2\Gamma\left(2-\frac{n}{2}\right)}\left(\frac{1-Z}{2}\right)^{-(n-1)}\,, \tag{3.2.12}$$

which, for $n > 3$, is $\mathcal{O}\left(Z^{-3}\right)$, so we neglect contributions to $I^{(0)\prime}(Z)$ from the first hypergeometric function in equation (3.2.11).

In the large-Z limit, it can be seen that

$$I^{(0)\prime}(Z) \to \frac{\Gamma(n-3)}{2\Gamma\left(\frac{n}{2}-1\right)}\left(\frac{1-Z}{2}\right)^{-2} + \mathcal{O}\left(Z^{-3}\right)\,, \tag{3.2.13}$$

which is found from the leading order contribution of the series expansion of the hypergeometric function on the second line in equation (3.2.11). As quick check we note that the same result for the large-distance of $I^{(0)\prime}(Z)$ can be attained by differentiating the large-Z limit of $I^{(0)}(Z)$, found in equation (3.2.7).

For large-Z, we therefore see that

$$I^{(0)\prime}(Z)(\partial_a Z)(\partial_{b'} Z) \to -H^2 \frac{\Gamma(n-3)}{\Gamma\left(\frac{n}{2}-1\right)} n_a n_{b'} + \mathcal{O}\left(Z^{-1}\right) \,, \tag{3.2.14}$$

as, in the limit $|Z| \to \infty$,

$$\frac{1-Z^2}{(1-Z)^2} = \frac{1+Z}{1-Z} \to -1 + \mathcal{O}\left(Z^{-1}\right) \,. \tag{3.2.15}$$

Finally, we combine the results for the large-distance behaviour of $I^{(0)}(Z)$, given by (3.2.8), and $I^{(0)\prime}(Z)$, given by equation (3.2.14) with the results of Section 3.1, to see that the large-distance behaviour of the gauge-independent terms of the propagator is

$$\langle 0|A_a(x)A_{b'}(x')|0\rangle_{GI} = \frac{H^{n-2}}{(4\pi)^{\frac{n}{2}}} \frac{\Gamma(n-1)}{\Gamma\left(\frac{n}{2}\right)} \frac{1}{n-3} n_a n_{b'} + \mathcal{O}\left(Z^{-1}\right) \,, \tag{3.2.16}$$

where the prefactor involving gamma functions has been written in a slightly different form, for later convenience.

3.3 Large-distance limit of the gauge-dependent terms in the propagator

A different approach must be used to find the large-Z behaviour of the gauge-dependent term of the propagator, given by equation (3.1.24), as it has a different form to the term discussed in the previous section. Specifically, we must find the limit as $|Z| \to \infty$ of

$$\tilde{\triangle}(Z) = -\frac{H^{n-2}}{(4\pi)^{\frac{n}{2}}} \lim_{m^2 \to 0} \frac{\partial}{\partial m^2} \left(I_\mu(Z) - I_\mu(-1)\right) \,, \tag{3.3.1}$$

where

$$I_\mu(Z) = \frac{\Gamma\left(\frac{n-1}{2}+\mu\right)\Gamma\left(\frac{n-1}{2}-\mu\right)}{\Gamma\left(\frac{n}{2}\right)} \, {}_2F_1\left(\frac{n-1}{2}+\mu, \frac{n-1}{2}-\mu; \frac{n}{2}; \frac{1+Z}{2}\right) \,, \tag{3.3.2}$$

$$\mu = \sqrt{\frac{(n-1)^2}{4} - \frac{m^2}{H^2}} \,, \tag{3.3.3}$$

as defined in Section 3.1, repeated here for convenience. Additionally in Section 3.1, we wrote the function $\tilde{\triangle}(Z)$ in terms of the functions $\tilde{\triangle}'(Z)$ and $\tilde{\triangle}''(Z)$, and derivatives of Z. For convenience, we repeat this expression here:

$$\begin{aligned}\partial_a\partial_{b'}\tilde{\triangle}(Z) &= (\partial_a\partial_{b'}Z)\tilde{\triangle}'(Z) + (\partial_{b'}Z)(\partial_a Z)\tilde{\triangle}''(Z)\,,\\ &= H^2\left[g_{ab'} + (1-Z)n_a n_{b'}\right]\tilde{\triangle}'(Z) + H^2(1-Z^2)n_a n_{b'}\tilde{\triangle}''(Z)\,. \end{aligned} \tag{3.3.4}$$

It therefore only remains to find the large-distance behaviour of the functions $\tilde{\triangle}'(Z)$ and $\tilde{\triangle}''(Z)$.

We first calculate the large-distance limit of the first and second derivatives of $\tilde{\triangle}(Z)$. The large-distance behaviour of the functions $\tilde{\triangle}'(Z)$ and $\tilde{\triangle}''(Z)$ are then calculated, using the same method. To find $\tilde{\triangle}'(Z)$ and $\tilde{\triangle}''(Z)$, we first differentiate $\tilde{\triangle}(Z)$, defined in equation (3.3.1), once with respect to Z, to find that

$$\begin{aligned}\tilde{\triangle}'(Z) &= -\frac{H^{n-2}}{(4\pi)^{\frac{n}{2}}}\lim_{m^2\to 0}\frac{\partial}{\partial m^2}\left(\frac{\mathrm{d}}{\mathrm{d}Z}I_\mu(Z) - \frac{\mathrm{d}}{\mathrm{d}Z}I_\mu(-1)\right)\,,\\ &= -\frac{H^{n-2}}{2(4\pi)^{\frac{n}{2}}}\lim_{m^2\to 0}\frac{\partial}{\partial m^2}\left[\frac{\Gamma\left(\frac{n+1}{2}+\mu\right)\Gamma\left(\frac{n+1}{2}-\mu\right)}{\Gamma\left(\frac{n}{2}+1\right)}\right.\\ &\qquad\left.\times\,{}_2F_1\left(\frac{n+1}{2}+\mu, \frac{n+1}{2}-\mu; \frac{n}{2}+1; \frac{1+Z}{2}\right)\right]\,,\end{aligned} \tag{3.3.5}$$

where elementary properties of the gamma function are used to simplify the prefactor. The second term in the first line of the latter equation, proportional to $I_\mu(-1)$, vanishes, as this has no dependence on Z. Similarly, we see that

$$\begin{aligned}\tilde{\triangle}''(Z) &= -\frac{H^{n-2}}{(4\pi)^{\frac{n}{2}}}\lim_{m^2\to 0}\frac{\partial}{\partial m^2}\left(\frac{\mathrm{d}^2}{\mathrm{d}Z^2}I_\mu(Z) - \frac{\mathrm{d}^2}{\mathrm{d}Z^2}I_\mu(-1)\right)\,,\\ &= -\frac{H^{n-2}}{4(4\pi)^{\frac{n}{2}}}\lim_{m^2\to 0}\frac{\partial}{\partial m^2}\left[\frac{\Gamma\left(\frac{n+3}{2}+\mu\right)\Gamma\left(\frac{n+3}{2}-\mu\right)}{\Gamma\left(\frac{n}{2}+2\right)}\right.\\ &\qquad\left.\times\,{}_2F_1\left(\frac{n+3}{2}+\mu, \frac{n+3}{2}-\mu; \frac{n}{2}+2; \frac{1+Z}{2}\right)\right]\,.\end{aligned} \tag{3.3.6}$$

In order to find the large-distance limit of $\tilde{\triangle}'(Z)$ and $\tilde{\triangle}''(Z)$, given by equations (3.3.5) and (3.3.6) respectively, we first find the large-Z limit of the hypergeometric function, which appears as a factor in both functions. We then find the derivative with respect to m^2 of this, before taking the limit $m^2 \to 0$.

We first apply the transformation of the hypergeometric function used in previous sections, given by equation (3.2.3), and use the series expansion of the hypergeometric function, given by equation (3.2.2). This gives

$$\tilde{\triangle}'(Z) \to -\frac{H^{n-2}}{2(4\pi)^{\frac{n}{2}}} \lim_{m^2\to 0} \frac{\partial}{\partial m^2} \frac{\Gamma\left(\frac{n+1}{2}-\mu\right)\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}+\mu\right)} \left(\frac{1-Z}{2}\right)^{-\left(\frac{n+1}{2}-\mu\right)} + \mathcal{O}\left(Z^{-\left(\frac{n+3}{2}-\mu\right)}\right), \tag{3.3.7}$$

which is the leading order term in the expansion of the second hypergeometric function in equation (3.2.3). In this expression, we have neglected terms from the first hypergeometric function in equation (3.2.3), which we now justify. In the limit $m^2 \to 0$, which corresponds to $\mu \to \frac{n-1}{2}$, the large-Z expansion of the first hypergeometric function in equation (3.2.3) has a leading term of order $(1-Z)^{-n}$. As is seen in equation (3.1.21), in the propagator, the function $\triangle'(Z)$ is multiplied by a factor of $\mathcal{O}(Z)$. Any contribution to the propagator will therefore vanish, for $n > 2$, in the large-Z limit.

Similarly,

$$\tilde{\triangle}''(Z) \to -\frac{H^{n-2}}{4(4\pi)^{\frac{n}{2}}} \lim_{m^2\to 0} \frac{\partial}{\partial m^2} \frac{\Gamma\left(\frac{n+3}{2}-\mu\right)\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}+\mu\right)} \left(\frac{1-Z}{2}\right)^{-\left(\frac{n+3}{2}-\mu\right)} + \mathcal{O}\left(Z^{-\left(\frac{n+5}{2}-\mu\right)}\right). \tag{3.3.8}$$

Again, this term is the leading order term in the expansion of the second hypergeometric function in equation (3.2.3). We now justify why we can neglect terms originating from the first hypergeometric function in equation (3.2.3). In the large-Z and $m^2 \to 0$ limits, the expansion of the first hypergeometric function in equation (3.2.3) gives a leading term of order $(1-Z)^{-(n+1)}$. As is seen from equation (3.1.21), in the propagator the function $\triangle''(Z)$ appears when multiplied by a term of $\mathcal{O}(Z^2)$. Any contribution to the propagator will therefore vanish, for $n > 2$, in the large-Z limit. As we look in the case when $n > 4$, we work to the order of Z given in equations (3.3.7) and (3.3.8).

Equations (3.3.7) and (3.3.8) have the same basic structure. We therefore define

$$\chi(\mu, Z) = \lim_{m^2\to 0} \frac{\partial}{\partial \mu} \left[f(\mu) \left(\frac{1-Z}{2}\right)^{-g(\mu)} \right] \frac{\partial \mu}{\partial m^2}, \tag{3.3.9}$$

where

$$f(\mu) = \frac{\Gamma(l-\mu)\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}+\mu\right)}, \tag{3.3.10}$$

$$g(\mu) = l - \mu. \tag{3.3.11}$$

Equation (3.3.9) gives the leading order behaviour of equations (3.3.7) and (3.3.8), for

$l = \frac{n+1}{2}$ and $l = \frac{n+3}{2}$, respectively, up to an overall multiplicative constant. From the definition of μ, given by (3.1.16),

$$\frac{\partial \mu}{\partial m^2} = -\frac{1}{2H^2\mu}\,, \tag{3.3.12}$$

and we see that, as already mentioned, as $m^2 \to 0$, $\mu \to \frac{n-1}{2}$. The function $\chi(\mu, Z)$, given by equation (3.3.9), becomes

$$\chi(\mu, Z) = -\frac{1}{H^2(n-1)} \lim_{\mu \to \frac{n-1}{2}} \left[\left(f'(\mu) - f(\mu) g'(\mu) \log\left(\frac{1-Z}{2}\right) \right) \left(\frac{1-Z}{2}\right)^{-g(\mu)} \right]. \tag{3.3.13}$$

It now remains to find $f'(\mu)$ and $g'(\mu)$. As $f(\mu)$ consists of gamma functions, we start by noting that the derivative of the gamma function is given by

$$\Gamma'(x) = \Gamma(x)\psi_0(x)\,, \tag{3.3.14}$$

where the digamma function $\psi_0(n)$ has the following value at $x = n \in \mathbb{N}$,

$$\psi_0(n) = -\gamma + \sum_{k=1}^{n-1} \frac{1}{k}\,, \tag{3.3.15}$$

and, for a half-integer argument,

$$\psi_0\left(n + \frac{1}{2}\right) = -\gamma - 2\ln 2 + \sum_{k=1}^{n} \frac{2}{2k-1}\,, \tag{3.3.16}$$

where γ is the Euler-Mascheroni constant. We therefore find that

$$\begin{aligned} f'(\mu) &= \left[\frac{-\Gamma'(l-\mu)\Gamma(2\mu) + 2\Gamma(l-\mu)\Gamma'(2\mu)}{\Gamma\left(\frac{1}{2}+\mu\right)} - \frac{\Gamma(l-\mu)\Gamma(2\mu)\Gamma'\left(\frac{1}{2}+\mu\right)}{\left(\Gamma\left(\frac{1}{2}+\mu\right)\right)^2} \right], \\ &= -f(\mu)Q(l)\,, \end{aligned} \tag{3.3.17}$$

where we define

$$Q(l) = \psi_0\left(l - \frac{n-1}{2}\right) - 2\psi_0(n-1) + \psi_0\left(\frac{n}{2}\right). \tag{3.3.18}$$

We also find, trivially, that

$$g'(\mu) = -1\,. \tag{3.3.19}$$

By evaluating the functions $f(\mu)$, $g(\mu)$, $f'(\mu)$, and $g'(\mu)$, given by equations (3.3.10), (3.3.11), (3.3.17), and (3.3.19), respectively, at $\mu = \frac{n-1}{2}$, and substituting into equation

(3.3.13), we see that

$$\chi(\mu, Z) = \frac{1}{H^2(n-1)} \frac{\Gamma(l - \frac{n-1}{2})\Gamma(n-1)}{\Gamma\left(\frac{n}{2}\right)} \left[Q(l) - \log\left(\frac{1-Z}{2}\right)\right] \left(\frac{1-Z}{2}\right)^{-(l-\frac{n-1}{2})} . \tag{3.3.20}$$

We can now use this expression for $\chi(\mu, Z)$, given by (3.3.20), to find $\tilde{\triangle}'(Z)$ and $\tilde{\triangle}''(Z)$, as defined in (3.3.7) and (3.3.8) respectively. Remembering to include the n-dependent constant prefactor, and to put in the corresponding values of l, we find

$$\tilde{\triangle}'(Z) \to -\frac{H^{n-4}}{(4\pi)^{\frac{n}{2}}(n-1)} \frac{\Gamma(n-1)}{\Gamma\left(\frac{n}{2}\right)} \left[Q\left(\frac{n+1}{2}\right) - \log\left(\frac{1-Z}{2}\right)\right] \frac{1}{1-Z} + \mathcal{O}\left(Z^{-2}\right) , \tag{3.3.21}$$

and

$$\begin{aligned}\tilde{\triangle}''(Z) \to -\frac{H^{n-4}}{(4\pi)^{\frac{n}{2}}(n-1)} \frac{\Gamma(n-1)}{\Gamma\left(\frac{n}{2}\right)} \left[Q\left(\frac{n+3}{2}\right) - \log\left(\frac{1-Z}{2}\right)\right] \frac{1}{(1-Z)^2} \\ + \mathcal{O}\left(Z^{-3}\right) .\end{aligned} \tag{3.3.22}$$

To see why we work to these orders of Z in $\tilde{\triangle}'(Z)$ and $\tilde{\triangle}''(Z)$, we refer back to the gauge-dependent terms in the propagator, given by equation (3.1.24). As the function $\tilde{\triangle}'(Z)$ appears in the propagator through $A_{GD}(Z)$, given by equation (3.1.26), and through $B_{GD}(Z)$, given by equation (3.1.28), when multiplied by $(1-Z)$, all non-leading order contributions to the propagator from equation (3.3.21) vanish, so we are free to neglect these terms. The function $\tilde{\triangle}''(Z)$ only appears in the propagator through $B_{GD}(Z)$, where it is multiplied by $(1-Z^2)$. The only contribution to the propagator from $\tilde{\triangle}''(Z)$ therefore comes from this leading order term, so we are free to neglect terms of order Z^{-3} and higher.

These expressions for the large-Z behaviour of $\tilde{\triangle}'(Z)$, given by equation (3.3.21), and $\tilde{\triangle}''(Z)$, given by equation (3.3.22), can now be combined with the results of Section 3.1 to see that the large-Z behaviour of $\partial_a \partial_{b'} \triangle Z$, given by equation (3.1.21), is

$$\partial_a \partial_{b'} \tilde{\triangle}(Z) \to \frac{H^{n-2}\Gamma(n-1)}{(4\pi)^{\frac{n}{2}}(n-1)\Gamma\left(\frac{n}{2}\right)} n_a n_{b'} + \mathcal{O}\left(Z^{-1}\right) , \tag{3.3.23}$$

where we have used the fact that

$$Q\left(\frac{n+1}{2}\right) - Q\left(\frac{n+3}{2}\right) = \psi_0(1) - \psi_0(2) = -1 . \tag{3.3.24}$$

The latter follows from the definition of $Q(l)$, given by equation (3.3.18), and the representation of the digamma function given by equation (3.3.15).

In the limit that $|Z| \to \infty$, the gauge-dependent term in the propagator, as defined in equation (3.1.24), is therefore

$$\langle 0|A_a(x)A_{b'}(x')|0\rangle_{GD} = \left(\xi - \frac{n-1}{n-3}\right) \frac{H^{n-2}\Gamma(n-1)}{(4\pi)^{\frac{n}{2}}(n-1)\Gamma\left(\frac{n}{2}\right)} n_a n_{b'} + \mathcal{O}\left(Z^{-1}\right) . \quad (3.3.25)$$

We will now combine this with the result of Section 3.2 to find the propagator in the $|Z| \to \infty$ limit.

3.4 *Large-distance behaviour of the propagator*

We now combine the results of the previous sections to find an expression for the $|Z| \to \infty$ limit of the propagator. Using definitions from Section 3.1, it can be seen that

$$\lim_{m^2 \to 0} \langle 0|A_a(x)A_{b'}(x')|0\rangle = [A_{GI}(Z) + A_{GD}(Z)]g_{ab'} + [B_{GI}(Z) + B_{GD}]n_a n_{b'} \, , \quad (3.4.1)$$

for $A_{GI}(Z)$, $A_{GD}(Z)$, $B_{GI}(Z)$, and $B_{GD}(Z)$ defined by equations (3.1.25) - (3.1.28).

As equations (3.2.16) and (3.3.25) both have no terms proportional to $g_{ab'}$, we find that

$$A_{GI}(Z) + A_{GD}(Z) \to \mathcal{O}\left(Z^{-1}\right) \, , \quad (3.4.2)$$

as $|Z| \to \infty$. Additionally, combining terms proportional to $n_a n_{b'}$ in equations (3.2.16) and (3.3.25), we see that

$$B_{GI}(Z) + B_{GD} \to \xi \frac{H^{n-2}\Gamma(n-1)}{(4\pi)^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)(n-1)} + \mathcal{O}\left(Z^{-1}\right) . \quad (3.4.3)$$

as $|Z| \to \infty$.

We therefore see that, in $|Z| \to \infty$ limit, the propagator is proportional to $n_a n_{b'}$, and tends to

$$\lim_{m^2 \to 0} \langle 0|A_a(x)A_{b'}(x')|0\rangle \to \xi \frac{H^{n-2}\Gamma(n-1)}{(4\pi)^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)(n-1)} n_a n_{b'} + \mathcal{O}\left(Z^{-1}\right) \, , \quad (3.4.4)$$

which is a non-zero gauge-dependent constant. Hence, in the limit $|Z| \to \infty$, the two-point function vanishes in the Landau gauge, but tends to a non-zero constant in a general gauge where $\xi \neq 0$. Our result reduces, for $n = 4$, to the result found by Youssef [1], which is that

$$\beta(Z) \to \frac{\xi H^2}{24\pi^2} . \quad (3.4.5)$$

3.5 Discussion

In this chapter, we have found the large-Z behaviour of the covariant massless vector propagator in n-dimensional de Sitter space. As in the $n = 4$ case discussed by Youssef [1], in the large-Z limit, it was found that the propagator tends towards a gauge-dependent constant. It was found that this constant is equal to zero in the Landau gauge, where $\xi = 0$.

A method similar to the one used in this chapter to calculate the long-distance behaviour of the covariant massless vector propagator could be used to find the long-distance behaviour of the graviton two-point function, although it is expected that there will be additional difficulties associated with this calculation. For example, we will have to work to higher orders of the de Sitter-invariant Z, as the graviton two-point function exhibits a linear divergence in Z. Additionally, the graviton two-point function, found by Fröb, Higuchi, and Lima [2], for example, has a larger number of terms than the vector two-point function. Both of these facts mean that the calculation of the large-distance behaviour of the graviton two-point function becomes far more lengthy than the calculation carried out in this chapter. This calculation will form the next chapter of this thesis.

Chapter 4

Large-distance behaviour of the covariant graviton propagator

In this chapter we find the large-distance behaviour of the covariant graviton propagator, in de Sitter spacetime. Although the large-distance behaviour of this propagator is known for the 4-dimensional case [3], for calculations involving dimensional regularisation knowledge of that of the n-dimensional graviton propagator is necessary. It is therefore of interest to study the large-distance behaviour of such a propagator. In the previous chapter, the large-distance behaviour of the covariant massless vector propagator was found, so we use this as a basis for the calculation. We follow the basic method set out in that chapter, but extend this to the more computationally complex case of the graviton propagator. Specifically, it is no longer sufficient to work to just leading order in the series expansions used in the previous chapter, and the graviton propagator has a more complex, tensor, structure than that of its vector counterpart.

4.1 Preliminaries

We consider perturbations h_{ab} about de Sitter spacetime, which can be represented by the metric

$$\bar{g}_{ab} = a^2(\eta)\eta_{ab} + \kappa h_{ab}\,, \tag{4.1.1}$$

where η_{ab} is the Minkowski metric, the conformal time $\eta \in (-\infty, 0)$, and the conformal scale factor $a^2(\eta) = \frac{1}{(-H\eta)^2}$, for H constant. Here we scale the perturbation by the constant $\kappa \equiv \sqrt{16\pi G_N}$. As in previous chapters, we use the mostly plus metric convention.

As can be seen in [2], the Lagrangian density is written as the following sum of terms:

$$\mathcal{L} \approx \mathcal{L}_{\rm inv} + \mathcal{L}_{\rm gf}\,. \tag{4.1.2}$$

The first term of the Lagrangian density is found from the Einstein-Hilbert Lagrangian density:

$$\mathcal{L}_{\text{grav}} \equiv \frac{\sqrt{-\bar{g}}}{\kappa^2}(\bar{R} - 2\Lambda)\,. \tag{4.1.3}$$

This is expanded to second order in the perturbation to give

$$\begin{aligned}\mathcal{L}_{\text{inv}} \equiv -\frac{\sqrt{-g}}{4}\Big[&\nabla_c h_{ab}\nabla^c h^{ab} - \nabla_c\nabla^c h + 2\nabla_a h^{ab}\nabla_b h - 2\nabla^a h_{ab}\nabla_b h^{bc} \\ &+ 2H^2\left(h_{ab}h^{ab} + \frac{n-3}{2}h^2\right)\Big]\,.\end{aligned} \tag{4.1.4}$$

The most general quadratic covariant gauge fixing term involving two derivatives is added to the Lagrangian density. This term has the form

$$\mathcal{L}_{\text{gf}} \equiv -\frac{1}{2\alpha}G_b G^b\sqrt{-g}\,, \tag{4.1.5}$$

for

$$G_b \equiv \nabla^a h_{ab} - \frac{1+\beta}{\beta}\nabla_b h\,. \tag{4.1.6}$$

Different values of the parameters α and β correspond to different gauges. Of particular interest to us is the Landau gauge, which corresponds to the limit $\alpha \to 0$.

We start from the graviton propagator found in [2]. Here, a Fierz-Pauli mass term:

$$\mathcal{L}_{\text{mass}} = -\frac{m^2}{4}\sqrt{-g}\left(h_{ab}h^{ab} - h^2\right)\,, \tag{4.1.7}$$

which was introduced in [2] as an IR regulator [90]. The massless limit was taken after the propagator was found. We will find the large-distance limit of the propagator found after the massless limit has been taken.

As in Chapter 3, we work with a propagator found using the Bunch-Davies vacuum state. In [2], the tensor perturbation is divided into two parts: the tensor-vector and scalar parts, denoted $h_{ab}^{(T+V)}$ and $h_{ab}^{(S)}$, respectively. Following this method, we write the metric perturbation the following sum:

$$h_{ab} = h_{ab}^{(T+V)} + h_{ab}^{(S)}\,. \tag{4.1.8}$$

The propagator formed from these modes contains no cross-terms, i.e. terms of the form $\langle 0|h_{ab}^{(TV)}(x)h_{a'b'}^{(S)}(x')|0\rangle = 0$, so we are free to consider the scalar and tensor-vector parts independently. When $\beta > 0$, it can be seen that, in the large-distance limit, the contribution to the propagator from the scalar modes vanishes. This result is straightforward, as will be seen in Section 4.3. In all other sections of this chapter, we consider the propagator in the tensor-vector sector only, so the remainder of this section will introduce the propagator in this sector.

As in the previous chapter, in order to find the propagator in the large-distance limit, we work in terms of the de Sitter invariant Z. We initially introduced this invariant in equation (3.1.3), which we repeat here for convenience:

$$Z(x,x') = \cos\left[H\sigma(x,x')\right] . \tag{4.1.9}$$

Again, we work in a basis of tensors constructed from g_{ab}, $g_{a'b'}$, $g_{a'b}$, n_a, and $n_{a'}$, which are defined in Section 3.1.

The graviton propagator in the tensor-vector sector, as found in [2], for example, is

$$\begin{aligned}\triangle_{ab:a'b'}(x,x') = \frac{H^{n-2}}{(4\pi)^{\frac{n}{2}}}\Big[&G^{(TV,1)}(Z)g_{ab}g_{a'b'} + G^{(TV,2)}(Z)[g_{ab}n_{a'}n_{b'} + g_{a'b'}n_a n_b] \\ &+ G^{(TV,3)}(Z)n_a n_b n_{a'} n_{b'} + 4G^{(TV,4)}(Z)n_{(a}g_{b)(a'}n_{b')} \\ &+ 2G^{(TV,5)}(Z)g_{a(a'}g_{b')b}\Big] ,\end{aligned} \tag{4.1.10}$$

where the functions $G^{(TV,k)}(Z)$ in equation (4.1.10) are made up of polynomials in Z multiplied by linear combinations of the functions

$$I_\mu^{(k)}(Z) = \frac{\Gamma\left(a_+ + k\right)\Gamma\left(a_- + k\right)}{2^k\Gamma\left(\frac{n}{2}+k\right)}\ {}_2F_1\left(a_+ + k, a_- + k; \frac{n}{2}+k; \frac{1+Z}{2}\right) , \tag{4.1.11}$$

and

$$\tilde{I}_\mu^{(k)}(Z) \equiv -\frac{1}{2\mu}\frac{\partial}{\partial\mu}I_\mu^{(k)}(Z) , \tag{4.1.12}$$

where

$$a_\pm = \frac{n-1}{2} \pm \mu , \tag{4.1.13}$$

with $\mu = \frac{n\pm1}{2}$ in the tensor-vector sector.

Before stating the explicit expressions for the functions $G^{(TV,k)}(Z)$, we quote some results from [2], which are used throughout the calculation. We make use of the following relations between $I_\mu^{(k)}(Z)$ and $\tilde{I}_\mu^{(k)}(Z)$:

$$\begin{aligned}&(1-Z^2)I_\mu^{(k+2)}(Z) - (n+2k)ZI_\mu^{(k+1)}(Z) \\ &\qquad + \left[\mu^2 - \frac{(n-1)^2}{4} - k(n+k-1)\right]I_\mu^{(k)}(Z) = 0 ,\end{aligned} \tag{4.1.14}$$

$$\begin{aligned}&(1-Z^2)\tilde{I}_\mu^{(k+2)}(Z) - (n+2k)Z\tilde{I}_\mu^{(k+1)}(Z) \\ &\qquad + \left[\mu^2 - \frac{(n-1)^2}{4} - k(n+k-1)\right]\tilde{I}_\mu^{(k)}(Z) = I_\mu^{(k)}(Z) .\end{aligned} \tag{4.1.15}$$

In addition, for more specific values of $I_\mu^{(k)}(Z)$,

$$(1-Z^2)I^{(2)}_{\frac{n-1}{2}}(Z) - nZI^{(1)}_{\frac{n-1}{2}}(Z) = \frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)}\,, \tag{4.1.16}$$

$$(1-Z^2)I^{(3)}_{\frac{n+1}{2}}(Z) - (n+2)ZI^{(2)}_{\frac{n+1}{2}}(Z) = \frac{\Gamma(n+2)}{2\Gamma\left(\frac{n}{2}+1\right)}\,. \tag{4.1.17}$$

Using these equations, the functions $I^{(4)}_{\frac{n+1}{2}}(Z)$, $I^{(3)}_{\frac{n+1}{2}}(Z)$, $\tilde{I}^{(4)}_{\frac{n\pm1}{2}}(Z)$, $\tilde{I}^{(3)}_{\frac{n-1}{2}}(Z)$, and $I^{(2)}_{\frac{n-1}{2}}(Z)$ can been eliminated from the expressions for $G^{(TV,3)}(Z)$ and $G^{(TV,4)}(Z)$ given in [2]. The resulting expressions are given below.

For ease of notation, we suppress the argument of the functions $I_\mu^{(k)}(Z)$, $\tilde{I}_\mu^{(k)}(Z)$, and $G^{(TV,k)}(Z)$, so in the following, unless otherwise stated, all functions are assumed to be functions of Z. The functions $G^{(TV,k)}$ are as follows. First, the coefficient of the $g_{ab}g_{a'b'}$ component of the propagator is

$$\begin{aligned} G^{(TV,1)} =& \frac{-2}{n-1}\left[\frac{1}{n-2}ZI^{(2)}_{\frac{n+1}{2}} + Z^2\tilde{I}^{(2)}_{\frac{n-1}{2}} + nZ\tilde{I}^{(1)}_{\frac{n-1}{2}} - \frac{2}{n-2}ZI^{(1)}_{\frac{n-1}{2}} - \frac{n-1}{n-2}\frac{\Gamma(n-1)}{\Gamma\left(\frac{n}{2}\right)}\right] \\ &+ \frac{\alpha}{n-1}\left[\frac{1}{n-1}ZI^{(2)}_{\frac{n+1}{2}} + 2Z\tilde{I}^{(2)}_{\frac{n+1}{2}}\right]\,. \end{aligned} \tag{4.1.18}$$

The coefficient of the $g_{ab}n_{a'}n_{b'} + g_{a'b'}n_an_b$ component of the propagator is found to be

$$\begin{aligned} G^{(TV,2)} =& -\frac{2}{n-1}(1-Z^2)\left[-\frac{1}{n-2}I^{(3)}_{\frac{n+1}{2}} - Z\tilde{I}^{(3)}_{\frac{n-1}{2}} - (n+1)\tilde{I}^{(2)}_{\frac{n-1}{2}} + \frac{1}{n-2}I^{(2)}_{\frac{n-1}{2}}\right] \\ &- \frac{\alpha}{n-1}(1-Z^2)\left[\frac{1}{n-1}I^{(3)}_{\frac{n+1}{2}} + 2\tilde{I}^{(3)}_{\frac{n+1}{2}}\right]\,. \end{aligned} \tag{4.1.19}$$

The coefficient of the $n_a n_b n_{a'} n_{b'}$ component of the propagator is

$$\begin{aligned}
G^{(TV,3)} \;=\; & -\frac{2}{n-1}\Bigg[\frac{1}{n-2}\left[2(n+1)(1-Z)^2+(n+2)(n+4)Z\right] I^{(2)}_{\frac{n+1}{2}} \\
& \quad +\left[(n+1)(n+2)+4(n+1)Z-(n-2)Z^2\right]\tilde{I}^{(2)}_{\frac{n-1}{2}} \\
& \quad +\left[2n(n+1)-n(n-2)Z\right]\tilde{I}^{(1)}_{\frac{n-1}{2}}+2[Z+n+1]I^{(1)}_{\frac{n-1}{2}} \\
& \quad +\frac{n-1}{n-2}\left[2(n+1)Z+n(n+4)\right]\frac{\Gamma(n-1)}{\Gamma\left(\frac{n}{2}\right)}\Bigg] \\
& +\frac{\alpha}{n-1}\Bigg[2(1-Z^2)\left[2Z+(n+2)\right]\tilde{I}^{(3)}_{\frac{n+1}{2}}+4(1-Z)\left[Z+n+1\right]\tilde{I}^{(2)}_{\frac{n+1}{2}} \\
& \quad +\frac{1}{n-1}\left[4Z^2+(n^2+2n+4)Z+4n\right]I^{(2)}_{\frac{n+1}{2}} \\
& \quad +(n+1)(2Z+n+2)\frac{\Gamma(n-1)}{\Gamma\left(\frac{n}{2}\right)}\Bigg].
\end{aligned} \tag{4.1.20}$$

The coefficient of the $n_{(a}g_{b)(a'}n_{b')}$ component of the propagator is

$$\begin{aligned}
G^{(TV,4)} \;=\; & -\frac{2}{n-1}\Bigg[\frac{1}{n-2}\left[\frac{n+1}{2}(1-Z)^2+(n+2)Z\right] I^{(2)}_{\frac{n+1}{2}} \\
& \quad +\left[\frac{n+1}{2}(1+2Z)-\frac{n-1}{2}Z^2\right]\tilde{I}^{(2)}_{\frac{n-1}{2}}\Bigg] \\
& -\frac{2}{n-1}\Bigg[\left[\frac{n(n-1)}{2}(1-Z)+n\right]\tilde{I}^{(1)}_{\frac{n-1}{2}}+\left[\frac{n+1}{2}+\frac{n-1}{n-2}Z\right]I^{(1)}_{\frac{n-1}{2}} \\
& \quad +\frac{n-1}{2(n-2)}\left[(n+1)Z+2n\right]\frac{\Gamma(n-1)}{\Gamma\left(\frac{n}{2}\right)}\Bigg] \\
& +\frac{\alpha}{n-1}\Bigg[(1+Z)(1-Z^2)\tilde{I}^{(3)}_{\frac{n+1}{2}}+(1-Z)\left[n+1+Z\right]\tilde{I}^{(2)}_{\frac{n+1}{2}} \\
& \quad +\frac{1}{n-1}\left[Z(1+Z)+n\right]I^{(2)}_{\frac{n+1}{2}}+\frac{n+1}{2}(Z+1)\frac{\Gamma(n-1)}{\Gamma\left(\frac{n}{2}\right)}\Bigg].
\end{aligned} \tag{4.1.21}$$

Finally, the coefficient of the $g_{a(a'}g_{b')b}$ component of the propagator is

$$\begin{aligned} G^{(TV,5)} &= -\frac{2}{n-1}\Bigg[\frac{1}{n-2}ZI^{(2)}_{\frac{n+1}{2}} + \left[\frac{n+1}{2} - \frac{n-1}{2}Z^2\right]\tilde{I}^{(2)}_{\frac{n-1}{2}} - \frac{n(n-1)}{2}Z\tilde{I}^{(1)}_{\frac{n-1}{2}} \\ &\qquad + \frac{n-1}{n-2}ZI^{(1)}_{\frac{n-1}{2}} + \frac{n(n-1)}{n-2}\frac{\Gamma(n-1)}{\Gamma\left(\frac{n}{2}\right)}\Bigg] \\ &\quad + \frac{\alpha}{n-1}\left[\frac{1}{n-1}ZI^{(2)}_{\frac{n+1}{2}} + (1-Z^2)\tilde{I}^{(3)}_{\frac{n+1}{2}} - nZ\tilde{I}^{(2)}_{\frac{n+1}{2}} + \frac{n+1}{2}\frac{\Gamma(n-1)}{\Gamma\left(\frac{n}{2}\right)}\right]. \end{aligned} \tag{4.1.22}$$

As the tensor-vector sector of the two-point function is traceless, only three of these coefficients are independent: $G^{(TV,3)}$ and $G^{(TV,5)}$ can be written in terms of the other functions $G^{(TV,k)}$ in the following way:

$$G^{(TV,3)} = 4G^{(TV,4)} - nG^{(TV,2)}\,, \tag{4.1.23}$$

$$G^{(TV,5)} = -\frac{n}{2}G^{(TV,1)} - \frac{1}{2}G^{(TV,2)}\,. \tag{4.1.24}$$

The first equation is manifest when equations (4.1.14) - (4.1.17) are used to remove $I^{(3)}_{\frac{n+1}{2}}$, $\tilde{I}^{(3)}_{\frac{n-1}{2}}$, and $\tilde{I}^{(2)}_{\frac{n-1}{2}}$, from $G^{(TV,2)}$ and $G^{(TV,4)}$, and the second can similarly be seen when equations (4.1.14) - (4.1.17) are used to remove $I^{(3)}_{\frac{n+1}{2}}$, $\tilde{I}^{(3)}_{\frac{n-1}{2}}$, and $I^{(2)}_{\frac{n-1}{2}}$, from $G^{(TV,2)}$. Equations (4.1.23) and (4.1.24) can be used later as a consistency check for the large-distance behaviour of the functions $G^{(TV,k)}$.

4.2 Large-distance behaviour of $I_\mu^{(k)}(Z)$ and $\tilde{I}_\mu^{(k)}(Z)$

In this section we find the large-distance behaviour of the functions $I_\mu^{(k)}$ and $\tilde{I}_\mu^{(k)}$, which will then be used in later sections to find the large-distance behaviour of the graviton propagator.

As $I_\mu^{(k)}$ is proportional to the hypergeometric function, we consider the series expansion used in the previous chapter, which states that

$${}_2F_1(a,b;c;z) = 1 + \frac{ab}{c}z + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \mathcal{O}(z^3)\,, \tag{4.2.1}$$

which is defined for $|z| < 1$. In order to find an expression for ${}_2F_1(a,b;c;z)$ which is valid in the large-Z limit, we must first use the well-known expression, see, for example, [89],

$$\begin{aligned} {}_2F_1(a,b;c;z) =& \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(1-z)^{-a}\ {}_2F_1\left(a, c-b; a-b+1; \frac{1}{1-z}\right) \\ &+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(1-z)^{-b}\ {}_2F_1\left(b, c-a; b-a+1; \frac{1}{1-z}\right) , \end{aligned} \tag{4.2.2}$$

which we also made use of in Chapter 3.

From the definition of $I_\mu^{(k)}$, given in equation (4.1.11), we have the following values

$$a = a_+ + k \,, \tag{4.2.3}$$

$$b = a_- + k \,, \tag{4.2.4}$$

$$c = \frac{n}{2} + k \,, \tag{4.2.5}$$

where $a_\pm$ is defined in equation (4.1.13). Singular terms in the first line of equation (4.2.2) arise from a gamma function taking a negative integer value, i.e. as $b - c = -2\mu = -(n \pm 1)$. However, these combine with singular terms, which occur at higher orders in the series expansion in the second line of equation (4.2.2), to give a well-defined contribution in the $n \to$ integer limit. This term behaves like $[(1-Z)/2]^{-(n-1)/2-\mu-k} \log Z$, which is still suppressed in comparison with $[(1-Z)/2]^{-1}$.

As stated in the previous section, the parameter $\mu = \frac{n\pm1}{2}$. For these values of μ, $a_+ > a_-$, where $a_\pm$ is defined in equation (4.1.13). Therefore, after we apply the above transformation, we neglect the first term on the right hand side of equation (4.2.2), which we claim contributes a term to the propagator which vanishes in the limit $|Z| \to \infty$. It can be seen that we are justified in neglecting this term by expanding it as the series given in equation (4.2.1). To leading order, it is proportional to $Z^{-\left(\frac{n-1}{2}+\mu+k\right)}$. From the expressions for the functions $G^{(TV,k)}$ given in equations (4.1.18) - (4.1.22), it can be seen that the factors in $G^{(TV,k)}$ that arise from this neglected hypergeometric function are, at most, of order Z^{-n}. As these vanish in the limit $|Z| \to \infty$, (for $n > 0$), we neglect the first term in the transformation given by equation (4.2.2) for the rest of the calculation.

We write the second term of equation (4.2.2) as the series given by equation (4.2.1), and use this to write $I_\mu^{(k)}$, defined in equation (4.1.11), as

$$I_\mu^{(k)} = J_\mu^{(k)} + J_\mu^{(k+1)} + \mathcal{O}\left(Z^{-g(\mu,k+2)}\right) , \tag{4.2.6}$$

where

$$J_\mu^{(k)} = f(\mu,k)\left(\frac{2}{1-Z}\right)^{g(\mu,k)} , \tag{4.2.7}$$

and

$$f(\mu,k) = \frac{\Gamma\left(\frac{n-1}{2}-\mu+k\right)\Gamma(2\mu)}{2^k\Gamma\left(\mu+\frac{1}{2}\right)}\,, \tag{4.2.8}$$

$$g(\mu,k) = \frac{n-1}{2}-\mu+k\,. \tag{4.2.9}$$

As seen in equation (4.1.12), $\tilde{I}_\mu^{(k)}$ is proportional to the derivative of $I_\mu^{(k)}$, so writing the function $I_\mu^{(k)}$ in the form given by equation (4.2.6) gives a useful expression for $\tilde{I}_\mu^{(k)}$. In order to find this expression, we first find the derivative of $J_\mu^{(k)}$. We use the results of the previous chapter to see that

$$\frac{\partial}{\partial\mu}J_\mu^{(k)} = f(\mu,k)\left[Q(\mu,k)+\log\left(\frac{1-Z}{2}\right)\right]\left(\frac{2}{1-Z}\right)^{g(\mu,k)}\,, \tag{4.2.10}$$

where

$$Q(\mu,k) = -\psi_0\left(\frac{n-1}{2}-\mu+k\right)+2\psi_0(2\mu)-\psi_0\left(\mu+\frac{1}{2}\right)\,, \tag{4.2.11}$$

and $\psi_0(x)$ is the digamma function, the derivative of the logarithm of the gamma function. Series expansions for integer and half-integer arguments of the digamma function are given by equations (3.3.15) and (3.3.16) respectively, in Section 3.3.

To the order presented in equation (4.2.6) we have

$$\begin{aligned}
\frac{\partial}{\partial\mu}I_\mu^{(k)} =& \frac{\partial}{\partial\mu}J_\mu^{(k)}+\frac{\partial}{\partial\mu}J_\mu^{(k+1)}+\mathcal{O}\left(Z^{-g(\mu,k+2)}\right)\,,\\
=& f(\mu,k)\left[Q(\mu,k)+\log\left(\frac{1-Z}{2}\right)\right]\left(\frac{2}{1-Z}\right)^{g(\mu,k)}\\
&+f(\mu,k+1)\left[Q(\mu,k)-\frac{1}{g(\mu,k)}+\log\left(\frac{1-Z}{2}\right)\right]\left(\frac{2}{1-Z}\right)^{g(\mu,k+1)}\\
&+\mathcal{O}\left(Z^{-g(\mu,k+2)}\right)+\mathcal{O}\left(Z^{-g(\mu,k+1)}\log Z\right)\,,\\
=& \left[Q(\mu,k)+\log\left(\frac{1-Z}{2}\right)\right]I_\mu^{(k)}-\frac{f(\mu,k+1)}{g(\mu,k)}\left(\frac{2}{1-Z}\right)^{g(\mu,k+1)}\\
&+\mathcal{O}\left(Z^{-g(\mu,k+2)}\right)+\mathcal{O}\left(Z^{-g(\mu,k+1)}\log Z\right)\,,
\end{aligned} \tag{4.2.12}$$

where we have used the recursion relation

$$Q(\mu,k+1) = Q(\mu,k)-\frac{1}{g(\mu,k)}\,. \tag{4.2.13}$$

which can be seen from the following recursion relation for the digamma function:

$$\psi_0(n+1) = \psi_0(n) + \frac{1}{n}\,, \tag{4.2.14}$$

Combining this expression for the derivative with equation (4.1.12), we see that the large-distance behaviour of $\tilde{I}_\mu^{(k)}$ is

$$\begin{aligned}\tilde{I}_\mu^{(k)} = &-\frac{1}{2\mu}\left[Q(\mu,k) + \log\left(\frac{1-Z}{2}\right)\right] I_\mu^{(k)} + \frac{1}{2\mu}\frac{f(\mu,k+1)}{g(\mu,k)}\left(\frac{2}{1-Z}\right)^{g(\mu,k+1)} \\ &+ \mathcal{O}\left(Z^{-g(\mu,k+2)}\right) + \mathcal{O}\left(Z^{-g(\mu,k+1)}\log Z\right)\,,\end{aligned} \tag{4.2.15}$$

where, in the latter equation, and for the rest of this section, $I_\mu^{(k)}$ is valid up to the order presented in equation (4.2.6). We have now found the $|Z| \to \infty$ limit of $I_\mu^{(k)}$, given by equation (4.2.6), and $\tilde{I}_\mu^{(k)}$, given by equation (4.2.15). We use these expressions to find the large-distance behaviour of the functions $G^{(TV,k)}$.

In most cases, we need only work to leading order in the expansion of $I_\mu^{(k)}$ given in equation (4.2.6), as contributions from higher order terms vanish in the limit $|Z| \to \infty$. Specifically, all factors in $G^{(TV,1)}$, $G^{(TV,2)}$, and $G^{(TV,5)}$ that arise from non-leading order terms in the expansion of $I_\mu^{(k)}$ vanish in the limit $|Z| \to \infty$. In these cases, we can just consider

$$I_\mu^{(k)} = f(\mu,k)\left(\frac{2}{1-Z}\right)^{g(\mu,k)} + \mathcal{O}\left(Z^{-g(\mu,k+1)}\right)\,, \tag{4.2.16}$$

so that

$$\tilde{I}_\mu^{(k)} = -\frac{1}{2\mu}\left[Q(\mu,k) + \log\left(\frac{1-Z}{2}\right)\right] I_\mu^{(k)} + \mathcal{O}\left(Z^{-g(\mu,k+1)}\right)\,. \tag{4.2.17}$$

In order to find an expression for $I_\mu^{(k+1)}$ and $\tilde{I}_\mu^{(k+1)}$, we make use of the following relations:

$$f(\mu,k+1) = \frac{1}{2}g(\mu,k)f(\mu,k)\,, \tag{4.2.18}$$

$$g(\mu,k+1) = g(\mu,k) + 1\,, \tag{4.2.19}$$

which follow from the definitions of $f(\mu,k)$ and $g(\mu,k)$, given by equations (4.2.8) and (4.2.9) respectively. Using these relations, and the recursion relation for $Q(\mu,k)$ found earlier in equation (4.2.13) we find

$$I_\mu^{(k+1)} = \frac{1}{1-Z}g(\mu,k)I_\mu^{(k)} + \mathcal{O}\left(Z^{-g(\mu,k+2)}\right)\,, \tag{4.2.20}$$

and

$$\tilde{I}_\mu^{(k+1)} = -\frac{1}{1-Z}\frac{g(\mu,k)}{2\mu}\left[Q(\mu,k) + \log\left(\frac{1-Z}{2}\right) - \frac{1}{g(\mu,k)}\right] I_\mu^{(k)} + \mathcal{O}\left(Z^{-g(\mu,k+2)}\right), \tag{4.2.21}$$

where the second expression is especially useful in the calculation of $G^{(TV,1)}$, $G^{(TV,2)}$, and $G^{(TV,5)}$.

4.3 *The scalar sector*

We now have the information necessary to justify the omission of the graviton propagator in the scalar sector from the rest of this chapter. As in [2], we write the graviton propagator in the scalar sector as

$$\triangle^{(S)}_{ab:a'b'}(x,x') = \frac{H^{n-2}}{(4\pi)^{\frac{n}{2}}}\sum_{k=1}^{5} F^{(S,k)}_{ab:a'b'}, \tag{4.3.1}$$

where

$$F^{(S,1)}_{ab:a'b'} = F^{(S,1)} g_{ab} g_{a'b'}, \tag{4.3.2}$$
$$F^{(S,2)}_{ab:a'b'} = H^{-2} F^{(S,2)} (g_{ab} Z_{;a'} Z_{;b'} + g_{a'b'} Z_{;a} Z_{;b}), \tag{4.3.3}$$
$$F^{(S,3)}_{ab:a'b'} = H^{-4} F^{(S,3)} Z_{;a} Z_{;b} Z_{;a'} Z_{;b'}, \tag{4.3.4}$$
$$F^{(S,4)}_{ab:a'b'} = H^{-4} F^{(S,4)} Z_{;(a} Z_{;b)(a'} Z_{;b')}, \tag{4.3.5}$$
$$F^{(S,5)}_{ab:a'b'} = 2H^{-2} F^{(S,5)} Z_{;a(a'} Z_{;b')b}, \tag{4.3.6}$$

where $\lambda \equiv 2(n-1) - (n-2)\alpha$, and we use the standard notation $Z_{;a} = \nabla_a Z$.

The functions $F^{(S,k)}$ are

$$\begin{aligned} F^{(S,1)} =& \frac{\lambda}{(n-1)^2(n-2)}\left(Z^2 I_S^{(2)} + Z I_S^{(1)}\right) + \frac{2n+\lambda\beta}{(n-1)(n-2)}\left(Z^2 \tilde{I}_S^{(2)} + Z\tilde{I}_S^{(1)}\right) \\ & - \frac{4}{(n-1)(n-2)} Z I_S^{(1)}, \end{aligned} \tag{4.3.7}$$

$$\begin{aligned} F^{(S,2)} =& -\frac{\lambda}{(n-1)^2(n-2)}\left(Z I_S^{(3)} + 2 I_S^{(2)}\right) - \frac{2n+\lambda\beta}{(n-1)(n-2)}\left(Z \tilde{I}_S^{(3)} + 2\tilde{I}_S^{(2)}\right) \\ & + \frac{2}{(n-1)(n-2)} I_S^{(2)}, \end{aligned} \tag{4.3.8}$$

$$F^{(S,3)} = \frac{1}{(n-1)(n-2)}\left[\frac{\lambda}{n-1} I_S^{(4)} + (2n+\lambda\beta)\tilde{I}_S^{(4)}\right], \tag{4.3.9}$$

$$F^{(S,4)} = \frac{1}{(n-1)(n-2)}\left[\frac{\lambda}{n-1}I_S^{(3)} + (2n+\lambda\beta)\tilde{I}_S^{(3)}\right], \qquad (4.3.10)$$

$$F^{(S,5)} = \frac{1}{(n-1)(n-2)}\left[\frac{\lambda}{n-1}I_S^{(2)} + (2n+\lambda\beta)\tilde{I}_S^{(2)}\right]. \qquad (4.3.11)$$

We follow the notation in [2], and define $I_S^{(k)} \equiv I_{\mu_S}^{(k)}$, and $\tilde{I}_S^{(k)} \equiv \tilde{I}_{\mu_S}^{(k)}$, where

$$\mu_S = \sqrt{\frac{(n-1)^2}{4} - (n-1)\beta}\,. \qquad (4.3.12)$$

We now investigate the large-distance behaviour of the components $F_{aba'b'}^{(S,k)}$. The tensor structure consists of combinations of the derivatives of Z, so we first find the large-distance behaviour of these derivatives. From Section 3.1, we have, to leading order in Z,

$$Z_{;a} \propto Z n_a + \mathcal{O}\left(Z^0\right), \qquad (4.3.13)$$

$$Z_{;ab'} \propto Z n_a n_{b'} + \mathcal{O}\left(Z^0\right). \qquad (4.3.14)$$

We now find the large-distance behaviour of the functions $I_S^{(k)}$ and $\tilde{I}_S^{(k)}$. We need only consider leading order behaviour of the expansions $I_\mu^{(k)}$ and $\tilde{I}_\mu^{(k)}$ found at the end of Section 4.2. From equation (4.2.6), we see that

$$I_S^{(k)} \propto Z^{-\frac{n-1}{2}+\mu_S-k} + \mathcal{O}\left(Z^{-\frac{n-1}{2}+\mu_S-k-1}\right), \qquad (4.3.15)$$

and, from equation (4.2.17), we have

$$\tilde{I}_S^{(k)} \propto \log\left(\frac{1-Z}{2}\right) I_S^{(k)} + \mathcal{O}\left(Z^{-\frac{n-1}{2}+\mu_S-k}\right). \qquad (4.3.16)$$

From rearranging equation (4.3.12) for β:

$$(n-1)\beta = \left(\frac{n-1}{2}+\mu_S\right)\cdot\left(\frac{n-1}{2}-\mu_S\right), \qquad (4.3.17)$$

it is clear that if $\beta > 0$ then

$$\mu_S - \frac{n-1}{2} < 0\,. \qquad (4.3.18)$$

From the definitions of $F^{(S,k)}$, given by equations (4.3.7) - (4.3.11), and the large-distance behaviour of the derivatives of Z, given by equations (4.3.13) and (4.3.14), it can be seen that all components $F_{ab:a'b'}^{(S,k)}$ have the same leading order behaviour in the large-distance limit:

$$F^{(S,k)}_{ab:a'b'} \propto Z^{\mu_S-\frac{n-1}{2}} \log\left(\frac{1-Z}{2}\right) C^{(S,k)}_{ab:a'b'} + \mathcal{O}\left(Z^{\mu_S-\frac{n-3}{2}} \log\left(\frac{1-Z}{2}\right)\right) + \mathcal{O}\left(Z^{\mu_S-\frac{n-1}{2}}\right) . \tag{4.3.19}$$

for constant tensors $C^{(S,k)}_{ab:a'b'}$. From inequality (4.3.18) we see that, as $|Z| \to \infty$,

$$F^{(S,k)}_{ab:a'b'} \to C^{(S)}_{ab:a'b'} \times \mathcal{O}\left(Z^{-1}\log(Z)\right) , \tag{4.3.20}$$

for some constant tensor $C^{(S)}_{ab:a'b'}$. This was the result we expected from the conclusions of [2], [4], and [91]. For the remainder of this chapter, we are therefore free to focus on the propagator formed from the tensor-vector modes.

4.4 *Large-distance behaviour of the covariant graviton propagator*

We now find the large-distance behaviour of the propagator. From [2] and [3], where the large-distance behaviour of the graviton propagator was found for $n = 4$ dimensions, we have some idea about the kind of behaviour to expect. It is found that, along with constant terms, in the large-distance limit the propagator has terms proportional to both Z and $\log Z$. In the Landau gauge, $\alpha = 0$, we expect to find no linear divergence. Additionally, the logarithmic divergence is known to vanish when $\alpha = \frac{n+1}{n-1}$. We find the large-distance behaviour of the functions $G^{(TV,k)}$ separately, before combining these results to find the large-distance behaviour of the propagator.

Due to their relative simplicity, we first find the $|Z| \to \infty$ limit of the functions $G^{(TV,1)}$, $G^{(TV,2)}$, and $G^{(TV,5)}$. These functions are given in terms of $I^{(k)}_\mu$ and $\tilde{I}^{(k)}_\mu$ in Section 4.1 by equations (4.1.18), (4.1.19), and (4.1.22), respectively. We use the large-distance expansions of $I^{(k)}_\mu$ and $\tilde{I}^{(k)}_\mu$ given in Section 4.2. As we take the limit $|Z| \to \infty$, we need only work to leading order in $(1-Z)^{-1}$ in the expansions of $I^{(k)}_\mu$ and $\tilde{I}^{(k)}_\mu$. Hence, we use equations (4.2.16) - (4.2.21), as higher order terms present in the more general expansions, given by equation (4.2.6) and equation (4.2.15), give vanishing contributions to $G^{(TV,1)}$, $G^{(TV,2)}$, and $G^{(TV,5)}$ in this large-distance limit.

The method involved in finding each function $G^{(TV,k)}$, for $k = 1, 2, 5$, is the same: we write the relevant function $G^{(TV,k)}$ in terms of the functions $I^{(k)}_\mu$ and $\tilde{I}^{(k)}_\mu$, as is presented in Section 4.1. Using the relations from Section 4.2, given by equations (4.2.17) and (4.2.21), the function $\tilde{I}^{(k)}_\mu$ is eliminated from the expression for the large-Z behaviour of $G^{(TV,k)}$. We then evaluate the remaining terms, involving $I^{(k)}_\mu$ only, in the limit $|Z| \to \infty$.

Using this method, we find that

$$\begin{aligned}
G^{(TV,1)} =& -\frac{2}{n-1}\frac{\Gamma(n-1)}{\Gamma\left(\frac{n}{2}\right)}\left[-\frac{2n-3}{n-1}+Q\left(\frac{n-1}{2},1\right)+\log\left(\frac{1-Z}{2}\right)\right] \\
&+\frac{\alpha}{n+1}\frac{\Gamma(n-1)}{\Gamma\left(\frac{n}{2}\right)}\left[-\frac{n-3}{n-1}+2Q\left(\frac{n-1}{2},1\right)+2\log\left(\frac{1-Z}{2}\right)\right] \\
&+\mathcal{O}(Z^{-1})+\mathcal{O}(Z^{-1}\log Z)\,, \qquad (4.4.1)\\
G^{(TV,2)} =& \frac{2}{n-1}\frac{\Gamma(n-1)}{\Gamma\left(\frac{n}{2}\right)}\left[-\frac{2n-3}{n-1}+Q\left(\frac{n-1}{2},1\right)+\log\left(\frac{1-Z}{2}\right)\right] \\
&+\frac{\alpha}{n+1}\frac{\Gamma(n-1)}{\Gamma\left(\frac{n}{2}\right)}\left[\frac{3n-5}{n-1}-2Q\left(\frac{n-1}{2},1\right)-2\log\left(\frac{1-Z}{2}\right)\right] \\
&+\mathcal{O}(Z^{-1})+\mathcal{O}(Z^{-1}\log Z)\,, \qquad (4.4.2)\\
G^{(TV,5)} =& \frac{\Gamma(n-1)}{\Gamma\left(\frac{n}{2}\right)}\left[-\frac{2n-3}{n-1}+Q\left(\frac{n-1}{2},1\right)+\log\left(\frac{1-Z}{2}\right)\right] \\
&+\alpha\frac{n-1}{n+1}\frac{\Gamma(n-1)}{\Gamma\left(\frac{n}{2}\right)}\left[\frac{n-5}{2(n-1)}-Q\left(\frac{n-1}{2},1\right)-\log\left(\frac{1-Z}{2}\right)\right] \\
&+\mathcal{O}(Z^{-1})+\mathcal{O}(Z^{-1}\log Z)\,, \qquad (4.4.3)
\end{aligned}$$

where the following recurrence relations for $Q(\mu,k)$, defined by equation (4.2.11) in Section 4.2, have been used to simplify the expressions:

$$Q(\mu,k+1)=Q(\mu,k)-\frac{1}{g(\mu,k)}\,, \qquad (4.4.4)$$

$$Q\left(\frac{n+1}{2},k+1\right)=Q\left(\frac{n-1}{2},k\right)+\frac{2}{n-1}\,. \qquad (4.4.5)$$

As a check, it can be seen that the logarithmic divergence does indeed vanish when $\alpha=\frac{n+1}{n-1}$. Additionally, it can be seen that $Q\left(\frac{3}{2},1\right)=2$, which is used to verify that, when $n=4$, our result is in agreement with the result of [2].

The calculation of the large-distance behaviour of the remaining two terms in the propagator is slightly more involved. This is for a couple of reasons. First, as can be seen by equations (4.1.20) and (4.1.21) in Section 4.1, $G^{(TV,3)}$ and $G^{(TV,4)}$ are more complicated than the other three functions. Second, we must work to higher than leading order in the series expansion of the hypergeometric function. This is because the functions $I^{(k)}_\mu$ and $\tilde{I}^{(k)}_\mu$ are multiplied by third and fourth powers of Z, so contributions from the higher order terms in the expansions of $I^{(k)}_\mu$ and $\tilde{I}^{(k)}_\mu$, given by equations (4.2.6) and (4.2.15), respectively, no longer vanish in the limit $|Z|\to\infty$. We therefore use equation (4.2.15), in order to write $\tilde{I}^{(k)}_\mu$ in terms of $I^{(k)}_\mu$, instead of the simpler relations used in the calculation of the large-distance behaviour of $G^{(TV,1)}$, $G^{(TV,2)}$, and $G^{(TV,5)}$.

We follow the same basic method as outlined earlier: we write the relevant function $G^{(TV,k)}$ in terms of the functions $I_\mu^{(k)}$ and $\tilde{I}_\mu^{(k)}$, as is presented in Section 4.1. Equation (4.2.15) is used to rewrite all terms involving $\tilde{I}_\mu^{(k)}$ in terms of $I_\mu^{(k)}$ only. The limit $|Z| \to \infty$ is taken, to see that

$$\begin{aligned} G^{(TV,3)} &= \frac{2(n-2)}{n-1}\frac{\Gamma(n-1)}{\Gamma\left(\frac{n}{2}\right)}\left[-\frac{2n-3}{n-1}+Q\left(\frac{n-1}{2},1\right)+\log\left(\frac{1-Z}{2}\right)\right] \\ &\quad +\frac{2\alpha}{n+1}\frac{\Gamma(n-1)}{\Gamma\left(\frac{n}{2}\right)}\left[(n-1)Z-\frac{n^2+7n-10}{2(n-1)}-(n-2)Q\left(\frac{n-1}{2},1\right)\right. \\ &\qquad \left. -(n-2)\log\left(\frac{1-Z}{2}\right)\right]+\mathcal{O}(Z^{-1})+\mathcal{O}(Z^{-1}\log Z)\,, \end{aligned} \tag{4.4.6}$$

$$\begin{aligned} G^{(TV,4)} &= \frac{\Gamma(n-1)}{\Gamma\left(\frac{n}{2}\right)}\left[-\frac{2n-3}{n-1}+Q\left(\frac{n-1}{2},1\right)+\log\left(\frac{1-Z}{2}\right)\right] \\ &\quad +\alpha\frac{n-1}{n+1}\frac{\Gamma(n-1)}{\Gamma\left(\frac{n}{2}\right)}\left[\frac{1}{2}Z+\frac{n-5}{2(n-1)}-Q\left(\frac{n-1}{2},1\right)-\log\left(\frac{1-Z}{2}\right)\right] \\ &\quad +\mathcal{O}(Z^{-1})+\mathcal{O}(Z^{-1}\log Z)\,, \end{aligned} \tag{4.4.7}$$

where equations (4.4.4) and (4.4.5) have been used to write the large-distance limits in this form. As was found for $G^{(TV,1)}$, $G^{(TV,2)}$, and $G^{(TV,5)}$, the logarithmic divergence vanishes for $\alpha = \frac{n+1}{n-1}$, and the linear divergence vanishes in the Landau gauge, $\alpha = 0$. When $n = 4$, our results agree with [2].

Using the large-distance limits found for all functions $G^{(TV,k)}$, it can be seen that the relations between the functions $G^{(TV,k)}$ in Section 4.1, given by equations (4.1.23) and (4.1.24), also hold in the $|Z| \to \infty$ limit.

In summary, the large-distance behaviour of the propagator is found to be

$$\begin{aligned} \triangle_{ab:a'b'} &= \frac{2\Lambda}{n-1}\left[1-\alpha\frac{n-1}{n+1}\right]\big[-g_{ab}g_{a'b'}+[g_{ab}n_{a'}n_{b'}+g_{a'b'}n_an_b] \\ &\qquad +(n-2)n_an_bn_{a'}n_{b'}+2(n-1)n_{(a}g_{b)(a'}n_{b')} \\ &\qquad +(n-1)g_{a(a'}g_{b')b}\big]\log\left(\frac{1-Z}{2}\right) \\ &\quad +\frac{2\alpha\Lambda(n-1)}{n+1}\left[n_an_bn_{a'}n_{b'}+n_{(a}g_{b)(a'}n_{b')}\right]Z+\Lambda C_{ab:a'b'} \\ &\quad +\mathcal{O}(Z^{-1})+\mathcal{O}(Z^{-1}\log Z)\,, \end{aligned} \tag{4.4.8}$$

where

$$\Lambda = \frac{H^{n-2}}{(4\pi)^{\frac{n}{2}}}\frac{\Gamma(n-1)}{\Gamma\left(\frac{n}{2}\right)}\,, \tag{4.4.9}$$

and the constant tensor

$$\begin{aligned}
C_{ab:a'b'} \quad = \quad & \left[\frac{2n-3}{n-1} - Q\left(\frac{n-1}{2},1\right)\right]\left[\frac{2}{n-1}[g_{ab}g_{a'b'} - [g_{ab}n_{a'}n_{b'} + g_{a'b'}n_a n_b]]\right. \\
& \qquad -\frac{2(n-2)}{n-1}n_a n_b n_{a'} n_{b'} - 4n_{(a}g_{b)(a'}n_{b')} \\
& \qquad \left. -2g_{a(a'}g_{b')b}\right] \\
& + \frac{\alpha}{n+1}\left[\left[-\frac{n-3}{n-1} + 2Q\left(\frac{n-1}{2},1\right)\right] g_{ab}g_{a'b'}\right. \\
& \quad + \left[\frac{3n-5}{n-1} - 2Q\left(\frac{n-1}{2},1\right)\right][g_{ab}n_{a'}n_{b'} + g_{a'b'}n_a n_b] \\
& \quad - \left[\frac{n^2+7n-10}{n-1} + 2(n-2)Q\left(\frac{n-1}{2},1\right)\right] n_a n_b n_{a'} n_{b'} \\
& \quad \left. + \left[(n-5) - 2(n-1)Q\left(\frac{n-1}{2},1\right)\right][2n_{(a}g_{b)(a'}n_{b')} + g_{a(a'}g_{b')b}]\right].
\end{aligned} \tag{4.4.10}$$

In equation (4.4.8), we have grouped terms according to their dependence on Z, which will be of use in the next chapter.

4.5 *Discussion*

In this chapter we have found the large-distance limit of the covariant graviton propagator in de Sitter spacetime. The propagator is found to be linearly divergent in this limit, in a general gauge such that $\alpha \neq 0$. It is logarithmically divergent in the Landau gauge, which corresponds to $\alpha = 0$. From the conclusions of [2], these results were expected. In the next chapter, we look to find a gauge transformation to render the graviton two point function IR finite.

Chapter 5

Pure gauge form of the covariant graviton propagator

In this chapter we focus on the logarithmic divergence of the covariant graviton two-point function in the Landau gauge. As in the previous chapter, we work in de Sitter spacetime. We expect that this divergence can be written in pure gauge form. We attempt to use a covariant gauge transformation to remove this divergence, and although we find this not to be possible, it leads to the interesting conclusion that the logarithmic divergence can only be traded for a linear one. It is, however, possible to remove this divergence if a non-covariant gauge transformation is used, and we find such a transformation at the end of the chapter. From the results of [5], where the physical graviton two-point function was found to be well behaved in the IR, we expect to be able to find such a gauge transformation. In $n = 4$ dimensions, the growing large-distance contribution of the graviton propagator has been written in pure gauge form, in [4]. In order to show that this is not merely a feature of the $n = 4$ dimensional field theory, and is not specific to the TTS gauge, we explicitly find a gauge transformation such that the large-distance growth of the covariant graviton propagator vanishes.

5.1 Preliminaries

In this section we state the large-distance limit of the propagator, in the Landau gauge. Additionally, we review the form that the gauge freedom, originally seen in Chapter 2, takes. As in the previous chapter, unless otherwise stated, all functions are assumed to be functions of Z.

In the Landau gauge, the large-distance limit of the propagator, which is found by setting $\alpha = 0$ in equation (4.4.8), is

$$\begin{aligned}
\triangle_{ab:a'b'} &= \Lambda \log\left(\frac{1-Z}{2}\right)\left[\frac{2}{n-1}\left[-g_{ab}g_{a'b'} + g_{ab}n_{a'}n_{b'} + g_{a'b'}n_a n_b\right]\right. \\
&\qquad \left. + \frac{2(n-2)}{n-1} n_a n_b n_{a'} n_{b'} + 4 n_{(a}g_{b)(a'}n_{b')} + 2 g_{a(a'}g_{b')b}\right] \\
&\quad + \Lambda C^{(1)}_{ab:a'b'} + \mathcal{O}\left(Z^{-1}\right) + \mathcal{O}\left(Z^{-1}\log Z\right) . \qquad (5.1.1)
\end{aligned}$$

The n-dependent constant Λ is defined by equation (4.4.9) in the previous chapter, and is repeated here for completeness:

$$\Lambda = \frac{H^{n-2}}{(4\pi)^{\frac{n}{2}}} \frac{\Gamma(n-1)}{\Gamma\left(\frac{n}{2}\right)} , \qquad (5.1.2)$$

and the constant tensor

$$\begin{aligned}
C^{(1)}_{ab:a'b'} &= \left[\frac{2n-3}{n-1} - Q\left(\frac{n-1}{2}, 1\right)\right]\left[\frac{2}{n-1}\left[g_{ab}g_{a'b'} - \left[g_{ab}n_{a'}n_{b'} + g_{a'b'}n_a n_b\right]\right.\right. \\
&\qquad \left. - n_a n_b n_{a'} n_{b'}\right] - 4 n_{(a}g_{b)(a'}n_{b')} \\
&\qquad \left. - 2 g_{a(a'}g_{b')b}\right] . \qquad (5.1.3)
\end{aligned}$$

As discussed in Chapters 1 and 2, we use a gauge transformation which corresponds to the following change in the quantum operator

$$h_{ab} \to h_{ab} + \nabla_a \xi_b + \nabla_b \xi_a , \qquad (5.1.4)$$

where ξ_a is also a quantum operator. As an aside, we note that, since the change in the gauge alters the mode functions, when multiplied by the annihilation (or creation) operator, as

$$a_I h^{(I)}_{ab} \to a_I h^{(I)}_{ab} + a_I \left(\nabla_a \tilde{\xi}^{(I)}_b + \nabla_b \tilde{\xi}^{(I)}_a\right) , \qquad (5.1.5)$$

the transformation vector ξ_a above is an operator in general. Then the two-point function changes as:

$$\begin{aligned}
\langle 0|h_{ab}(x)h_{a'b'}(x')|0\rangle \to & \langle 0|h_{ab}(x)h_{a'b'}(x')|0\rangle + 2\nabla_{(a}\langle 0|\xi_{b)}(x)h_{a'b'}(x')|0\rangle \\
& + 2\nabla_{(a'}\langle 0|h_{ab}(x')\xi_{b')}(x)|0\rangle + 4\nabla_{(a}\nabla_{(a'}\langle 0|\xi_{b)}(x)\xi_{b')}(x')|0\rangle , \\
= & \langle 0|h_{ab}(x)h_{a'b'}(x')|0\rangle + 2\nabla_{(a}G_{b)a'b'}(x,x') + 2\nabla_{(a'}G_{b')ab}(x,x') , \qquad (5.1.6)
\end{aligned}$$

where

$$G_{aa'b'}(x,x') = \langle 0|\xi_a(x)h_{a'b'}(x')|0\rangle + \nabla_{(a'}\langle 0|\xi_{|a|}(x)\xi_{b')}(x')|0\rangle \,. \tag{5.1.7}$$

This transformation leaves the two-point function of a local gauge-invariant tensor, for example the linearised Weyl tensor, invariant, but leads to the following transformation of the propagator:

$$\tilde{\triangle}_{ab:a'b'} = \triangle_{ab:a'b'} - \mathcal{G}_{ab:a'b'} \,, \tag{5.1.8}$$

where

$$\mathcal{G}_{ab:a'b'} = 2\nabla_{(a}G_{b)a'b'} + 2\nabla_{(a'}G_{b')ab} \,. \tag{5.1.9}$$

The tensor $G_{ba'b'}$ is symmetric under the exchange of $a' \leftrightarrow b'$, and we initially require that $g^{a'b'}G_{aa'b'} = 0$. In the following section, we find the only transformation of this kind that could remove the logarithmic divergence, which leads to some interesting consequences.

We conclude this section with a brief review of the de Sitter invariant Z. This is originally defined in conformal coordinates in equation (3.1.5), which we repeat here for convenience:

$$Z = 1 - \frac{||\boldsymbol{x} - \boldsymbol{x}'||^2 - (\eta - \eta')^2}{2\eta\eta'} \,. \tag{5.1.10}$$

As discussed in Chapter 3, $Z \to -\infty$ when the spacelike distance between two points is large, and at large timelike separations $Z \to \infty$.

5.2 *Covariant gauge transformation to remove logarithmic divergence*

In this section, we show that it is not possible to use a covariant gauge transformation to remove the logarithmic divergence present in the large-distance limit of the propagator, in the Landau gauge. Specifically, we show that it is not possible to find a covariant gauge transformation of the form described in the previous section such that $\tilde{\triangle}_{ab:a'b'}$ tends to, at most, order Z^0 as $|Z| \to \infty$.

The gauge transformation is given by equation (5.1.9), which we repeat here for clarity:

$$\tilde{\triangle}_{ab:a'b'} = \triangle_{ab:a'b'} - \sum_i \mathcal{G}^{(i)}_{ab:a'b'} \,, \tag{5.2.1}$$

where we now consider a sum of terms of the form given in equation (5.1.9). We require that

$$\mathcal{G}^{(i)}_{ab:a'b'} = 2\nabla_{(a}G^{(i)}_{b)a'b'} + 2\nabla_{(a'}G^{(i)}_{b')ab} \,, \tag{5.2.2}$$

where

$$G^{(i)}_{aa'b'} = f^{(i)}(Z)T^{(i)}_{ba'b'} \,, \tag{5.2.3}$$

and $g^{a'b'}T^{(i)}_{ba'b'} = 0$. Additionally, as discussed at the end of the previous section, we require that $G^{(i)}_{ba'b'}$ is symmetric under the exchange of $a' \leftrightarrow b'$. We now show that there is no transformation of this form such that the logarithmic divergence of the graviton propagator is removed.

The only bitensors with the symmetries described in the previous paragraph are $g_{a'b'}n_a$, $n_a n_{a'} n_{b'}$, and $g_{a(a'}n_{b')}$. The most general combination of these is

$$T^{(i)}_{aa'b'} = \alpha_i g_{a'b'} n_a + \beta_i n_a n_{a'} n_{b'} + \gamma_i g_{a(a'}n_{b')} \,, \tag{5.2.4}$$

for α_i, β_i, and γ_i constants. In order to ensure that we have a traceless tensor, we impose the additional condition

$$\alpha_i n + \beta_i - \gamma_i = 0 \,. \tag{5.2.5}$$

From the definition of $\mathcal{G}^{(i)}_{ab:a'b'}$, given in equation (5.2.2), we see that it is useful to denote the fully symmetric combination of the derivative of the tensor $T^{(i)}_{aa'b'}$ to be

$$\mathcal{T}^{(i)}_{ab:a'b'} = 2\nabla_{(a}T^{(i)}_{b)a'b'} + 2\nabla_{(a'}T^{(i)}_{b')ab} \,. \tag{5.2.6}$$

Using $T^{(i)}_{aa'b'}$, as defined in equation (5.2.4), we find this to be

$$\begin{aligned} \mathcal{T}^{(i)}_{ab:a'b'} =& 4\alpha_i A g_{ab} g_{a'b'} + [2A(-\alpha_i + \beta_i - \gamma_i) - 2\gamma_i C] \left[g_{ab} n_{a'} n_{b'} + g_{a'b'} n_a n_b\right] \\ &+ 4\beta_i(2C - A) n_a n_b n_{a'} n_{b'} + [8\beta_i C - 4\gamma_i A] n_{(a} g_{b)(a'} n_{b')} + 4\gamma_i C g_{a(a'} g_{b')b} \,, \end{aligned} \tag{5.2.7}$$

where the following relations from [88] have been used:

$$\nabla_a n_b = A(g_{ab} - n_a n_b) \,, \tag{5.2.8}$$

$$\nabla_a n_{b'} = C(g_{ab'} + n_a n_{b'}) \,, \tag{5.2.9}$$

$$\nabla_a g_{bc'} = -(A + C)(g_{ab} n_{c'} + g_{ac'} n_b) \,, \tag{5.2.10}$$

for

$$A = \frac{HZ}{\sqrt{1 - Z^2}} \,, \tag{5.2.11}$$

$$C = -\frac{H}{\sqrt{1 - Z^2}} \,, \tag{5.2.12}$$

where, again, we suppress the argument of these functions. The gauge transformation, defined by equation (5.2.2), is

$$\mathcal{G}^{(i)}_{ab:a'b'} = -2H\sqrt{1-Z^2} f^{(i)\prime}(Z) \left[n_{(a} T^{(i)}_{b)a'b'} + n_{(a'} T^{(i)}_{b')ab} \right] + \mathcal{T}^{(i)}_{ab:a'b'} f^{(i)}(Z) \,, \qquad (5.2.13)$$

which, from the definition of $T^{(i)}_{aa'b'}$ given by equation (5.2.4), can be seen to be

$$\begin{aligned} \mathcal{G}^{(i)}_{ab:a'b'} = -2H\sqrt{1-Z^2} f^{(i)\prime}(Z) \Big[& \alpha_i [g_{ab} n_{a'} n_{b'} + g_{a'b'} n_a n_b] + 2\beta_i n_a n_b n_{a'} n_{b'} \\ & + 2\gamma_i n_{(a} g_{b)(a'} n_{b')} \Big] + \mathcal{T}^{(i)}_{ab:a'b'} f^{(i)}(Z) \,. \end{aligned} \qquad (5.2.14)$$

Using the expression for $\mathcal{T}^{(i)}_{ab:a'b'}$ given by equation (5.2.7), we find that the gauge transformation, in terms of the functions $f^{(i)}$, and the constants, α_i, β_i, and γ_i, is

$$\begin{aligned} \mathcal{G}^{(i)}_{ab:a'b'} = & - 2 \left[H\sqrt{1-Z^2} \alpha_i f^{(i)\prime}(Z) + [\alpha_i (n+1) A + \gamma_i C] \, f^{(i)} \right] [g_{ab} n_{a'} n_{b'} + g_{a'b'} n_a n_b] \\ & + 4\beta_i \left[-H\sqrt{1-Z^2} f^{(i)\prime}(Z) + (2C - A) f^{(i)} \right] n_a n_b n_{a'} n_{b'} \\ & + 4 \left[-H\sqrt{1-Z^2} \gamma_i f^{(i)\prime}(Z) + [2\beta_i C - \gamma_i A] f^{(i)} \right] n_{(a} g_{b)(a'} n_{b')} \\ & + 4\gamma_i C f^{(i)} g_{a(a'} g_{b')b} + 4\alpha_i A f^{(i)} g_{ab} g_{a'b'} \,. \end{aligned} \qquad (5.2.15)$$

We now have a general transformation of the form in equation (5.2.2). We now find the transformed propagator, as given in equation (5.2.1). We look to find functions $f^{(i)}(Z)$, along with values of the constants α_i, β_i, and γ_i, such that the divergence in the propagator, given in equation (5.1.1), is removed.

It is immediately obvious that any contribution proportional to $g_{ab}g_{a'b'}$ or $g_{a(a'}g_{b')b}$ must come from the final term in equation (5.2.14). We first write the logarithmic divergence of the coefficient of the $g_{a(a'}g_{b')b}$ term in the propagator in pure gauge form. Comparing the coefficient of $g_{a(a'}g_{b')b}$ in the above equation with the coefficient of $g_{a(a'}g_{b')b}$ in equation (5.1.1), we see that we must set

$$f^{(1)}(Z) = -\frac{\Lambda}{2\gamma_1 H}\sqrt{1-Z^2}\log\left(\frac{1-Z}{2}\right) \,, \qquad (5.2.16)$$

$$\alpha_1 = 0 \,, \qquad (5.2.17)$$

$$\beta_1 = 1 \,, \qquad (5.2.18)$$

$$\gamma_1 = 1 \,. \qquad (5.2.19)$$

The constants α_1, β_1, and γ_1 are consistent with the condition, given by equation (5.2.5), imposed to ensure that the transformation $G_{aa'b'}$ is traceless. The transformation is

$$\begin{aligned}\mathcal{G}^{(1)}_{ab:a'b'} =& 4\Lambda\left[-\frac{1+Z}{2}+\log\left(\frac{1-Z}{2}\right)\right]\left[n_a n_b n_{a'} n_{b'} + n_{(a}g_{b)(a'}n_{b')}\right] \\ &+ \Lambda \log\left(\frac{1-Z}{2}\right)\left[-[g_{ab}n_{a'}n_{b'} + g_{a'b'}n_a n_b] + g_{a(a'}g_{b')b}\right] ,\end{aligned} \tag{5.2.20}$$

which changes the propagator as follows:

$$\begin{aligned}\tilde{\triangle}_{ab:a'b'} =& \triangle_{ab:a'b'} - \mathcal{G}^{(1)}_{ab:a'b'} , \\ =& \frac{\Lambda}{n-1}\log\left(\frac{1-Z}{2}\right)\Big[-2g_{ab}g_{a'b'} + (n+1)[g_{ab}n_{a'}n_{b'} + g_{a'b'}n_a n_b]\Big] \\ &+ 2\Lambda\left[(1+Z) - \frac{n}{n-1}\log\left(\frac{1-Z}{2}\right)\right] n_a n_b n_{a'} n_{b'} + 2\Lambda(1+Z) n_{(a}g_{b)(a'}n_{b')} \\ &+ \mathcal{O}\left(Z^{-1}\right) + \mathcal{O}\left(Z^{-1}\log Z\right) .\end{aligned} \tag{5.2.21}$$

We now use the same method to write the logarithmic divergence of the coefficient of the $g_{ab}g_{a'b'}$ term in the propagator in pure gauge form. As the transformation described by equation (5.2.20) has no term proportional to $g_{ab}g_{a'b'}$, we are free to just compare the coefficient of $g_{ab}g_{a'b'}$ in equation (5.2.15) with the coefficient of $g_{ab}g_{a'b'}$ in equation (5.1.1). From this, we see we must set

$$f^{(2)}(Z) = -\frac{\Lambda}{2\alpha_2(n-1)H}\frac{\sqrt{1-Z^2}}{Z}\log\left(\frac{1-Z}{2}\right) , \tag{5.2.22}$$

$$\beta_2 = -n\alpha_2 , \tag{5.2.23}$$

$$\gamma_2 = 0 . \tag{5.2.24}$$

Again, α_1, β_1, and γ_1 are consistent with condition (5.2.5), which imposes the traceless condition. In the limit $|Z| \to \infty$,

$$\begin{aligned}\mathcal{G}^{(2)}_{ab:a'b'} =& \frac{\Lambda}{n-1}\log\left(\frac{1-Z}{2}\right)\Big[-2g_{ab}g_{a'b'} + (n+1)[g_{ab}n_{a'}n_{b'} + g_{a'b'}n_a n_b] \\ &\qquad -2n n_a n_b n_{a'} n_{b'}\Big] \\ &+\Lambda C^{(5)}_{ab:a'b'} + \mathcal{O}(Z^{-1}) + \mathcal{O}\left(Z^{-1}\log Z\right) ,\end{aligned} \tag{5.2.25}$$

and

$$C^{(5)}_{ab:a'b'} = -\frac{1}{n-1}[g_{ab}n_{a'}n_{b'} + g_{a'b'}n_a n_b] + \frac{2n}{(n-1)} n_a n_b n_{a'} n_{b'} . \tag{5.2.26}$$

In the large-distance limit, the transformed propagator then becomes

$$
\begin{aligned}
\tilde{\triangle}_{ab:a'b'} &= \triangle_{ab:a'b'} - \mathcal{G}^{(1)}_{ab;a'b'} - \mathcal{G}^{(2)}_{ab:a'b'} \, , \\
&= 2\Lambda Z[n_a n_b n_{a'} n_{b'} + n_{(a} g_{b)(a'} n_{b')}] + \Lambda C^{(2)}_{ab:a'b'} + \mathcal{O}\left(Z^{-1}\right) + \mathcal{O}\left(Z^{-1}\log Z\right) \, ,
\end{aligned}
\tag{5.2.27}
$$

where

$$
C^{(2)}_{ab:a'b'} = C^{(1)}_{ab:a'b'} - C^{(5)}_{ab:a'b'} \, , \tag{5.2.28}
$$

and the constant tensor $C^{(1)}_{ab:a'b'}$ is defined in equation (5.1.3).

By comparison of coefficients in the logarithmically divergent terms of the propagator with coefficients of equation (5.2.15), we see that the only possible form that the functions $f^{(1)}(Z)$ and $f^{(2)}(Z)$ can take are those given by equations (5.2.16) and (5.2.22). We conclude that, when using a gauge transformation of the form given by equation (5.2.1), we can only trade the logarithmic divergence for a linear one. Interestingly, this is the same linear divergence that appears when one considers the large-distance limit for the propagator when $\alpha = \frac{n+1}{n-1}$. Hence, we can apply any conclusions reached about the large-distance behaviour of the propagator in the Landau gauge to the large-distance behaviour of the propagator in the gauge where $\alpha = \frac{n+1}{n-1}$.

In the next section, we show that there is no covariant gauge transformation of the form described by equation (5.1.8) that removes the linear divergence in the propagator, given in equation (5.2.27), shown in this section to be equivalent to the logarithmic divergence in the propagator, given in equation (5.1.1).

5.3 *Covariant gauge transformation: No-go theorem*

In this section we show that the linear divergence in the transformed propagator, found at the end of Section 5.2, can not be written in pure gauge form by using the covariant gauge transformation outlined in the previous section. This suggests that the linear and logarithmic divergences can be traded, but not simultaneously removed.

We consider a transformation of the form given by equation (5.1.9), where we take the general tensor

$$
G_{aa'b'} = \tilde{\alpha}(Z) g_{a'b'} n_a + \tilde{\beta}(Z) n_a n_{a'} n_{b'} + \tilde{\gamma}(Z) g_{a(a'} n_{b')} \, , \tag{5.3.1}
$$

and no longer require that this tensor is traceless - so we no longer require the functions $\tilde{\alpha}(Z)$, $\tilde{\beta}(Z)$, and $\tilde{\gamma}(Z)$ to satisfy equation (5.2.5). The fully symmetric combination of covariant derivative of this tensor, introduced in equation (5.1.9), is

$$\begin{aligned}
\mathcal{G}_{ab:a'b'} \;=\;& 4A\tilde{\alpha}(Z)g_{ab}g_{a'b'} + \Big[2A(-\tilde{\alpha}(Z)+\tilde{\beta}(Z)-\tilde{\gamma}(Z)) - 2C\tilde{\gamma}(Z) \\
& \qquad -2H\sqrt{1-Z^2}\tilde{\alpha}'(Z)\Big][g_{ab}n_{a'}n_{b'} + g_{a'b'}n_a n_b] \\
& + \Big[4(2C-A)\tilde{\beta}(Z) - 4H\sqrt{1-Z^2}\tilde{\beta}'(Z)\Big]\, n_a n_b n_{a'} n_{b'} \\
& + \Big[8C\tilde{\beta}(Z) - 4A\tilde{\gamma}(Z) - 4H\sqrt{1-Z^2}\tilde{\gamma}'(Z)\Big]\, n_{(a}g_{b)(a'}n_{b')} \\
& +4C\tilde{\gamma}(Z)g_{a(a'}g_{b')b}\,,
\end{aligned} \tag{5.3.2}$$

where the functions A and C are defined in the previous section, by equations (5.2.11) and (5.2.12), respectively.

We now find the transformed propagator

$$\bar{\triangle}_{ab:a'b'} = \tilde{\triangle}_{ab:a'b} - \mathcal{G}_{ab:a'b'}\,, \tag{5.3.3}$$

where $\tilde{\triangle}_{ab'a'b'}$ is given by equation (5.2.27). It is convenient to rescale the functions $\tilde{g}(Z) = \tilde{\alpha}(Z)$, $\tilde{\beta}(Z)$, or $\tilde{\gamma}(Z)$, such that

$$g(Z) = \frac{H}{\sqrt{1-Z^2}}\tilde{g}(Z)\,. \tag{5.3.4}$$

By differentiating the above equation, it can be seen that

$$H\sqrt{1-Z^2}\tilde{g}'(Z) = (1-Z^2)g'(Z) - Zg(Z)\,. \tag{5.3.5}$$

In terms of the scaled functions, equation (5.3.2) becomes

$$\begin{aligned}
\mathcal{G}_{ab:a'b'} =& 4Z\alpha(Z)g_{ab}g_{a'b'} \\
& + 2\left[Z(\beta(Z)-\gamma(Z)) + \gamma(Z) - (1-Z^2)\alpha'(Z)\right][g_{ab}n_{a'}n_{b'} + g_{a'b'}n_a n_b] \\
& - 4\left[2\beta(Z) + (1-Z^2)\beta'(Z)\right] n_a n_b n_{a'} n_{b'} \\
& - 4\left[2\beta(Z) + (1-Z^2)\beta(Z)\right] n_{(a}g_{b)(a'}n_{b')} - 4\gamma(Z)g_{a(a'}g_{b')b}\,.
\end{aligned} \tag{5.3.6}$$

We now show that there are no functions $\alpha(Z)$, $\beta(Z)$, and $\gamma(Z)$ such that the transformed propagator $\bar{\triangle}_{ab:a'b'}$ is, at most, of order Z^0 in the limit $|Z| \to \infty$. Requiring that $\bar{\triangle}_{ab:a'b'}$ is constant in Z is equivalent to the condition that, as $|Z| \to \infty$,

$$\mathcal{G}_{ab:a'b'} \to \tilde{\triangle}_{ab:a'b'} + \mathcal{C}_{ab:a'b'}\,, \tag{5.3.7}$$

for a constant tensor

$$\begin{aligned}\mathcal{C}_{ab:a'b'} =& c_1 g_{ab} g_{a'b'} + c_2[g_{ab} n_{a'} n_{b'} + g_{a'b'} n_a n_b] + c_3 n_a n_b n_{a'} n_{b'} + c_4 n_{(a} g_{b)(a'} n_{b')} \\ &+ c_5 g_{a(a'} g_{b')b} + \mathcal{O}(Z^{-1}) \,,\end{aligned} \tag{5.3.8}$$

where c_i are constants. Looking at equation (5.3.7) component-wise, we see that we must simultaneously satisfy the following equations:

$$Z\alpha(Z) \to c_1 \,, \tag{5.3.9}$$

$$Z(\beta(Z) - \gamma(Z)) + \gamma(Z) - (1 - Z^2)\alpha'(Z) \to c_2 \,, \tag{5.3.10}$$

$$-2\beta(Z) - (1 - Z^2)\beta'(Z) \to \frac{1}{2}\Lambda Z + c_3 \,, \tag{5.3.11}$$

$$-2\beta(Z) - (1 - Z^2)\gamma'(Z) \to \frac{1}{2}\Lambda Z + c_4 \,, \tag{5.3.12}$$

$$\gamma(Z) \to c_5 \,. \tag{5.3.13}$$

We now show that equations (5.3.9) - (5.3.13) can not simultaneously be satisfied.

By subtracting equation (5.3.11) from equation (5.3.12), we see that, as $|Z| \to \infty$,

$$(1 - Z^2)(\beta'(Z) - \gamma'(Z)) \to c_4 - c_3 \,. \tag{5.3.14}$$

Integrating this, before taking the limit $|Z| \to \infty$, gives

$$\beta(Z) - \gamma(Z) \to (c_4 - c_3) \int^Z \frac{\mathrm{d}Z}{1 - Z^2} = \frac{c_4 - c_3}{2} \log\left|\frac{Z+1}{Z-1}\right| + c_6 \to c_6 \,, \tag{5.3.15}$$

where c_6 is a constant of integration. Using the expression given for $\gamma(Z)$ given in equation (5.3.13), we conclude that

$$\beta(Z) \to c_5 + c_6 \,. \tag{5.3.16}$$

We now use this limiting value for the function $\beta(Z)$ in equation (5.3.12) to find a contradiction with equation (5.3.13). Equation (5.3.12) becomes

$$(Z^2 - 1)\gamma'(Z) \to 2\Lambda Z + c_4 + 2(c_5 + c_6) \,. \tag{5.3.17}$$

Integrating this gives

$$\gamma(Z) \to \Lambda \log(Z^2 - 1) + (c_4 + 2(c_5 + c_6)) \int^Z \frac{\mathrm{d}Z}{Z^2 - 1} \to \Lambda \log(Z^2 - 1) + c_7 \,, \tag{5.3.18}$$

where c_7 is another integration constant. This is in contraction with the limiting value of $\gamma(Z)$ as found in equation (5.3.13). We therefore conclude that it is not possible to find a covariant gauge transformation to remove the logarithmic divergence of the propagator. However, in the next section, we show that this divergence can be gauged away non-covariantly, as is expected from the results of [4].

5.4 Non-covariant gauge transformation

In this section we show we can write the logarithmic divergence of the graviton propagator in the Landau gauge in pure gauge form, in a non-covariant manner, for large spacelike and timelike separations separately.

Before we look at such a non-covariant gauge transformation, we write the propagator in a more convenient form. This is achieved by using a covariant gauge transformation of the form given by equation (5.1.8) in Section 5.1. As in the previous section, we relax the traceless condition, so no longer require equation (5.2.5) to hold, and consider

$$G_{aa'b'} = \frac{\Lambda}{2iH(n-1)} \log\left(\frac{1-Z}{2}\right) \left[g_{a'b'}n_a + (2n-3)n_a n_{a'} n_{b'} + 2(n-1)g_{a(a'}n_{b')}\right] , \tag{5.4.1}$$

for Λ defined by equation (4.4.9). In the notation of Section 5.2, to find the gauge transformation $\mathcal{G}_{ab:a'b'}$ we substitute

$$f(Z) = \frac{\Lambda}{2iH(n-1)} \log\left(\frac{1-Z}{2}\right) , \tag{5.4.2}$$

$$\alpha = 1 , \tag{5.4.3}$$

$$\beta = (2n-3) , \tag{5.4.4}$$

$$\gamma = 2(n-1) , \tag{5.4.5}$$

into equation (5.2.15). In the limit $|Z| \to \infty$, this gives

$$\begin{aligned}\mathcal{G}_{ab:a'b'} =& 2\Lambda \left[\frac{1}{n-1}\left[-g_{ab}g_{a'b'} + (2n-3)n_a n_b n_{a'} n_{b'}\right] + 2n_{(a}g_{b)(a'}n_{b')}\right] \log\left(\frac{1-Z}{2}\right) \\ &+ \Lambda C^{(3)}_{ab:a'b'} + \mathcal{O}\left(Z^{-1}\right) + \mathcal{O}\left(Z^{-1}\log Z\right) ,\end{aligned} \tag{5.4.6}$$

where

$$C^{(3)}_{ab:a'b'} = \frac{2}{n-1}[g_{ab}n_{a'}n_{b'} + g_{a'b'}n_a n_b] + \frac{4(2n-3)}{n-1} n_a n_b n_{a'} n_{b'} + 8n_{(a}g_{b)(a'}n_{b')} . \tag{5.4.7}$$

The transformed propagator is

$$\begin{aligned}\tilde{\triangle}_{ab:a'b'} =& \triangle_{ab:a'b'} - \mathcal{G}_{ab:a'b'} \,, \\ =& 2\Lambda[-n_a n_b n_{a'} n_{b'} + g_{a(a'} g_{b')b}] \log\left(\frac{1-Z}{2}\right) + \Lambda C^{(4)}_{ab:a'b'} + \mathcal{O}\left(Z^{-1}\right) \\ & + \mathcal{O}\left(Z^{-1}\log Z\right) \,, \end{aligned} \tag{5.4.8}$$

where $\triangle_{ab:a'b'}$ is the graviton propagator in the Landau gauge, given by equation (5.1.1), and

$$C^{(4)}_{ab:a'b'} = C^{(1)}_{ab:a'b'} - C^{(3)}_{ab:a'b'} \,, \tag{5.4.9}$$

where the constant tensor $C^{(1)}_{ab:a'b'}$ is defined in equation (5.1.3).

We now write this propagator in terms of conformal coordinates, $(\eta, \mathbf{x})$, before finding a non-covariant gauge transformation to remove the logarithmic divergence. In what follows, we neglect terms of order Z^0, as the constant term, $C^{(4)}_{ab:a'b'}$ has no effect on the final result. We therefore neglect it for the rest of the section. Purely spatial indices are denoted by i, j, k, and the index 0 refers to conformal time.

The propagator, given by equation (5.4.8), is composed of the bivectors n_a and $g_{aa'}$, defined in Section 3.1. These can be expressed in terms of Z as follows:

$$n_a(x, x') = -\frac{1}{H\sqrt{1-Z^2}} \partial_a Z(x, x') \,, \tag{5.4.10}$$

$$g_{aa'}(x, x') = \frac{1}{H^2}\left[\partial_a \partial_{a'} Z(x, x') - \frac{1}{1+Z(x,x')} \partial_a Z(x, x') \cdot \partial_{a'} Z(x, x')\right] . \tag{5.4.11}$$

Equation (5.4.10) is found from the definitions of Z and n_a, given by equation (3.1.5) and equation (3.1.8), respectively. The expression for $g_{aa'}(x, x')$ can be found using equation (5.2.9), repeated here for clarity:

$$\nabla_{a'} n_a = -\frac{H}{\sqrt{1-Z^2}} \left[g_{aa'} + n_a n_{a'}\right] . \tag{5.4.12}$$

Combined with the latter result for n_a, this gives equation (5.4.11). We now write these bivectors in terms of the conformal coordinates, and define $r^2 = ||\mathbf{x} - \mathbf{x}'||^2$, $r_i = x_i - x'_i$, and $r_{i'} = x'_i - x_i$, so that $r_{i'} = -r_i$. In terms of these quantities,

$$Z(\eta, r) = 1 - \frac{r^2 - (\eta - \eta')^2}{2\eta\eta'} \,, \tag{5.4.13}$$

and it can be seen that

$$\partial_0 Z(\eta, r) = \frac{r^2 + \eta^2 - \eta'^2}{2\eta^2\eta'}\,, \tag{5.4.14}$$

$$\partial_{0'} Z(\eta, r) = \frac{r^2 - \eta^2 + \eta'^2}{2\eta\eta\prime 2}\,, \tag{5.4.15}$$

$$\partial_i Z(\eta, r) = -\partial_{i'} Z(\eta, r) = -\frac{1}{\eta\eta'} r_i\,. \tag{5.4.16}$$

Additionally,

$$\partial_0\partial_{0'} Z(\eta, r) = -\frac{r^2 + \eta^2 + \eta'^2}{2\eta^2\eta'^2}\,, \tag{5.4.17}$$

$$\partial_0\partial_{i'} Z(\eta, r) = -\frac{1}{\eta^2\eta'} r_i\,, \tag{5.4.18}$$

$$\partial_i\partial_{0'} Z(\eta, r) = \frac{1}{\eta\eta'^2} r_i\,, \tag{5.4.19}$$

$$\partial_i\partial_{i'} Z(\eta, r) = \frac{1}{\eta\eta'} \delta_{ii'}\,. \tag{5.4.20}$$

It can then be seen that the components of the tangent vector are

$$n_0 = -\frac{r^2 + \eta^2 - \eta'^2}{H\eta\sqrt{4\eta^2\eta'^2 - (\eta^2 + \eta'^2 - r^2)^2}}\,, \tag{5.4.21}$$

$$n_{0'} = -\frac{r^2 - \eta^2 + \eta'^2}{H\eta'\sqrt{4\eta^2\eta'^2 - (\eta^2 + \eta'^2 - r^2)^2}}\,, \tag{5.4.22}$$

$$n_i = \frac{2r_i}{H\sqrt{4\eta^2\eta'^2 - (\eta^2 + \eta'^2 - r^2)^2}}\,, \tag{5.4.23}$$

$$n_{i'} = -\frac{2r_i}{H\sqrt{4\eta^2\eta'^2 - (\eta^2 + \eta'^2 - r^2)^2}}\,, \tag{5.4.24}$$

and the components of the parallel propagator are

$$g_{00'} = -\frac{1}{H^2\eta\eta'} \frac{r^2 + (\eta + \eta')^2}{(\eta + \eta')^2 - r^2}\,, \tag{5.4.25}$$

$$g_{ij'} = \frac{2}{H^2\eta\eta'} \frac{r_i r_j}{(\eta + \eta')^2 - r^2} + \frac{1}{H^2\eta\eta'} \delta_{ij'}\,, \tag{5.4.26}$$

$$g_{0i'} = -\frac{2}{H^2\eta\eta'} \frac{\eta + \eta'}{(\eta + \eta')^2 - r^2} r_i\,, \tag{5.4.27}$$

$$g_{i0'} = \frac{2}{H^2\eta\eta'} \frac{\eta + \eta'}{(\eta + \eta')^2 - r^2} r_i\,. \tag{5.4.28}$$

We use these components to find the propagator, defined by equation (5.4.8), in the large-distance limit.

We first scale the graviton mode functions h_{ab} such that

$$h_{ab} \to a^{-2}(\eta) h_{ab} \,, \tag{5.4.29}$$

where, for de Sitter spacetime, $a^2(\eta) = \frac{1}{(-H\eta)^2}$. This rescales the propagator in the following way:

$$\triangle'_{ab:a'b'}(x,x') = H^4 \eta^2 \eta'^2 \, \triangle_{ab:a'b'} \, (x,x')(x,x') \,, \tag{5.4.30}$$

so in the following all scaled propagators will be denoted by a prime. We consider the following diffeomorphism of the rescaled mode functions, as discussed in Section 1.8,

$$\delta_{\xi(x,x')} h_{ab}(x,x') = \partial_a \xi_b(x,x') + \partial_b \xi_a(x,x') - 2Ha\eta_{ab}\xi_0(x,x') \,, \tag{5.4.31}$$

which transforms the rescaled propagator, given by equation (5.4.30), in the following way,

$$\bar{\triangle}'_{ab:a'b'}(x,x') = \tilde{\triangle}'_{ab:a'b'}(x,x') - B_{ab:a'b'}(x,x') \,, \tag{5.4.32}$$

where

$$\begin{aligned} B_{ab:a'b'}(x,x') =& 2\partial_{(a} A_{b)a'b'}(x,x') + 2\partial_{(a'} A_{b')ab}(x,x') + \frac{2}{\eta}\eta_{ab} A_{0a'b'}(x,x') \\ &+ \frac{2}{\eta'}\eta_{a'b'} A_{0'ab}(x,x') \,, \end{aligned} \tag{5.4.33}$$

for

$$A_{ba'b'} = \partial_{(a'} \left\langle 0 | \xi_{|b|} \xi_{b')} | 0 \right\rangle + Ha(\eta') \left\langle 0 | \xi_b \xi_{0'} \eta_{a'b'} | 0 \right\rangle \,, \tag{5.4.34}$$

where $A_{aa'b'} = A_{ab'a'}$. The components $A_{aa'b'}$ will be given later in this section. We note the similarity between this non-covariant gauge transformation, and the one considered in the covariant case, defined by equation (5.1.9).

We first find the transformed propagator in the $Z \to -\infty$ limit. As stated in Section 3.1, this limit corresponds to the limit $r \to \infty$. In the large-r limit, the propagator, given by equation (5.4.8), has only two non-vanishing components: $\tilde{\triangle}'_{0i:0'j'}(x,x')$, and $\tilde{\triangle}'_{ij:k'l'}(x,x')$. By this, we mean that all other components are of order, at most, $\mathcal{O}(r^0)$. These two non-vanishing components are

$$\tilde{\triangle}'_{0i:0'j'}(x,x') = 4\Lambda \frac{r_i r_{j'}}{r^2} \log r + 2\Lambda \delta_{ij'} \log r + \mathcal{O}(r^0) + \mathcal{O}\left(r^{-1}\log(r)\right) \,, \tag{5.4.35}$$

$$\begin{aligned} \tilde{\triangle}'_{ij:k'l'}(x,x') =& 16\Lambda \frac{r_i r_j r_{k'} r_{l'}}{r^4} \log r + 4\Lambda \left[\frac{\delta_{ik'} r_j r_{l'} + \delta_{il'} r_j r_{k'} + \delta_{jk'} r_i r_{l'} + \delta_{jl'} r_i r_{k'}}{r^2} \right] \log r \\ &+ 2\Lambda \left[\delta_{ik'}\delta_{jl'} + \delta_{il'}\delta_{jk'} \right] \log r + \mathcal{O}(r^0) + \mathcal{O}\left(r^{-1}\log(r)\right) \,. \end{aligned} \tag{5.4.36}$$

We now find a transformation $B_{ab:a'b'}$ such that we can write these components in pure gauge form. To do this, we take

$$A_{jk'l'}(x,x') = -2\Lambda \frac{r_j r_{k'} r_{l'}}{r^2} \log r - \frac{1}{2}\Lambda \left[\delta_{jk'} r_{l'} \log r + \delta_{jl'} r_{k'} \log r\right] , \tag{5.4.37}$$

$$A_{jk'0'}(x,x') = -2\Lambda(\eta' - \eta) \frac{r_j r_{k'}}{r^2} \log r - \Lambda(\eta' - \eta)\delta_{jk'} \log r , \tag{5.4.38}$$

$$A_{0k'l'}(x,x') = 2\Lambda\eta \frac{r_{k'} r_{l'}}{r^2} \log r , \tag{5.4.39}$$

$$A_{j0'0'}(x,x') = A_{0k'0'}(x,x') = A_{00'0'}(x,x') = 0 . \tag{5.4.40}$$

It can be seen, from the definition of $B_{aba'b'}$, given by equation (5.4.33), that

$$B_{0i:0'j'}(x,x') = 4\Lambda \frac{r_i r_{j'}}{r^2} \log r + 2\Lambda\delta_{ij'} \log r , \tag{5.4.41}$$

$$\begin{aligned} B_{ij:k'l'}(x,x') =& 16\Lambda \frac{r_i r_j r_{k'} r_{l'}}{r^4} \log r + 4\Lambda \left[\frac{\delta_{ik'} r_j r_{l'} + \delta_{il'} r_j r_{k'} + \delta_{jk'} r_i r_{l'} + \delta_{jl'} r_i r_{k'}}{r^2}\right] \log r \\ &+ 2\Lambda \left[\delta_{ik'}\delta_{jl'} + \delta_{il'}\delta_{jk'}\right] \log r + \Lambda c_{ij:k'l'}(x,x') , \end{aligned} \tag{5.4.42}$$

where

$$c_{ij:k'l'}(x,x') = -8 \frac{r_i r_j r_{k'} r_{l'}}{r^4} - \frac{\delta_{il'} r_j r_{k'} + \delta_{jl'} r_i r_{k'} + \delta_{ik'} r_j r_{l'} + \delta_{jk'} r_i r_{l'}}{r^2} . \tag{5.4.43}$$

Hence

$$B_{0i:0'j'}(x,x') = \tilde{\triangle}'_{0i:0'j'}(x,x') + \mathcal{O}(r^0) + \mathcal{O}\left(r^{-1}\log(r)\right) , \tag{5.4.44}$$

$$B_{ij:k'l'}(x,x') = \tilde{\triangle}'_{ij:k'l'}(x,x') + \mathcal{O}(r^0) + \mathcal{O}\left(r^{-1}\log(r)\right) . \tag{5.4.45}$$

We conclude that we can write the divergent components of the propagator in pure gauge form. We now show a transformation of this form does not introduce additional divergences in other, formerly vanishing, components of the propagator. It can be seen from the definition of $B_{ab:a'b'}$ that

$$B_{00:0'0'}(x,x') = B_{00:0'i'}(x,x') = B_{0i:0'0'}(x,x') = B_{00:i'j'}(x,x') = B_{ij:0'0'}(x,x') = 0 . \tag{5.4.46}$$

The final two components are non-zero, but as

$$B_{0i:i'j'}(x,x') = B_{ij:i'0'}(x,x') = \mathcal{O}\left(r^{-1}\log(r)\right) , \tag{5.4.47}$$

no new divergences have been introduced.

In summary, in the limit $Z \to -\infty$, we find the transformed propagator, from equation (5.4.32), to be

$$\bar{\triangle}'_{ab:a'b'}(x,x') = \tilde{\triangle}'_{ab:a'b'}(x,x') - B_{ab:a'b'}(x,x') \to c^{-}_{ab:a'b'}\,, \tag{5.4.48}$$

for the tensor $c^{-}_{ab:a'b'} = \mathcal{O}(r^0)$, and for the components of $B_{ab:a'b'}(x,x')$ given by equations (5.4.44), (5.4.45), (5.4.46), and (5.4.47).

Finally, we find the transformed propagator in the $Z \to \infty$ limit. As stated in Section 3.1, this limit is equivalent to the limit $\eta \to 0$, for fixed η', which corresponds to future timelike infinity. For simplicity, we take this limit when $x_i \to x_{i'}$, giving the additional condition that $r_i, r_{i'} \to 0$. Without loss of generality, the two points can be arranged so that $x_i = x_{i'}$ using the de Sitter transformation if the two points becomes separated by infinite timelike distance. In this limit, the propagator, defined in equation (5.4.8), has only two logarithmically divergent components:

$$\tilde{\triangle}'_{0i:0'j'}(x,x') = \frac{1}{2}\delta_{ij'}\log\eta + \mathcal{O}(\eta^0) + \mathcal{O}(\eta\log\eta)\,, \tag{5.4.49}$$

$$\tilde{\triangle}'_{ij:k'l'}(x,x') = -\frac{1}{2}\left(\delta_{ik'}\delta_{jl'} + \delta_{il'}\delta_{jk'}\right)\log\eta + \mathcal{O}(\eta^0) + \mathcal{O}(\eta\log\eta)\,. \tag{5.4.50}$$

Again, we find a transformation $B_{ab:a'b'}$ such that we can write these components in pure gauge form. Taking

$$A_{jk'l'}(x,x') = \frac{1}{8}\left[\delta_{jk'}r_{l'} + \delta_{jl'}r_{k'}\right]\log\eta\,, \tag{5.4.51}$$

$$A_{0k'0'}(x,x') = -\frac{1}{4}r_{k'}\log\eta\,, \tag{5.4.52}$$

$$A_{jk'0'}(x,x') = A_{j0'0'}(x,x') = A_{0k'l'}(x,x') = A_{00'0'}(x,x') = 0\,, \tag{5.4.53}$$

we find that

$$B_{0i:0'j'}(x,x') = \tilde{\triangle}'_{0i:0'j'}(x,x') + \mathcal{O}(\eta^0) + \mathcal{O}(\eta\log\eta)\,, \tag{5.4.54}$$

$$B_{ij;k'l'}(x,x') = \tilde{\triangle}'_{ij:k'l'}(x,x') + \mathcal{O}(\eta^0) + \mathcal{O}(\eta\log\eta)\,, \tag{5.4.55}$$

as required. We now check that no other components of $B_{ab:a'b'}$ introduce divergences. As most components of $A_{aa'b'}$ are equal to zero, we immediately see that

$$B_{00:0'0'}(x,x') = B_{00:i'j'}(x,x') = B_{ij:0'0'}(x,x') = 0\,, \tag{5.4.56}$$

identically. The remaining components of $B_{ab:a'b'}$ are

$$B_{00:0'i'}(x,x') = -\frac{1}{2\eta}r_{i'} + \frac{1}{2\eta}r_{i'}\log\eta \to 0\,, \tag{5.4.57}$$

$$B_{0i:0'0'}(x,x') = \frac{1}{2\eta'}r_i\log\eta \to 0\,, \tag{5.4.58}$$

$$B_{0i:j'k'}(x,x') = \frac{1}{8\eta}\left[\delta_{ij'}r_{k'} + \delta_{ik'}r_{j'}\right] - \frac{1}{2\eta'}\delta_{j'k'}r_i\log\eta \to 0\,, \tag{5.4.59}$$

$$B_{ij:k'0'}(x,x') = -\frac{1}{2\eta}\delta_{ij}r_{k'}\log\eta \to 0\,, \tag{5.4.60}$$

where the final equalities are true as $r_i, r_{i'} \to 0$, as we work in the case when $x_i \to x_{i'}$. In conclusion, we have not introduced any additional divergences to the transformed propagator $\bar{\triangle}'_{ab:a'b'}(x,x')$ by using a transformation of this form.

In summary, in the limits $r \to \infty$ and $\eta \to 0$, we find the transformed propagator, from equation (5.4.32), to be

$$\bar{\triangle}'_{ab:a'b'}(x,x') = \tilde{\triangle}'_{ab:a'b'}(x,x') - B_{ab:a'b'}(x,x') \to c^{+}_{ab:a'b'}\,, \tag{5.4.61}$$

for the tensor $c^{+}_{ab:a'b'} = \mathcal{O}(\eta^0)$, and for the components of $B_{ab:a'b'}(x,x')$ given by equations (5.4.54) - (5.4.60).

In this section we have shown that the logarithmic divergence in the Landau gauge, or indeed the linear divergence in the gauge where $\alpha = \frac{n+1}{n-1}$, can be removed using a non-covariant gauge transformation.

5.5 Discussion

We expected that the logarithmic divergence of the graviton propagator, in the Landau gauge, could be written in pure gauge form using a covariant gauge transformation. However, when a transformation of this kind is used, the logarithmic divergence could only be traded for a linear divergence identical to the one present in the large-distance limit of the propagator in the case when $\alpha = \frac{n+1}{n-1}$. It was then shown that this linear divergence could not be written in pure gauge form covariantly. However, as shown in the final section, it is possible to non-covariantly gauge away the logarithmic divergence in the Landau gauge. We note that this chapter provides consistency with the results of [4], where a gauge transformation was found to remove the logarithmic behaviour of the covariant graviton propagator in $n = 4$ dimensions.

Chapter 6

Conclusions and Outlook

In this thesis, we studied the IR divergences of propagators. Two different types of IR divergences were investigated: when the propagator requires an IR regulator to be well-defined, and when the propagator grows at large separations. Both types of IR divergence manifest themselves in the same way, in that they tend to give a term proportional to $\log(\alpha r)$, for a separation r and an IR cut-off α. We summarise below how we treated each type of divergence, before giving more detail on the results found in each chapter of this thesis. Finally, a number of open problems relevant to the work presented in this thesis are discussed.

Although it would appear to be necessary to introduce an IR cut-off to define the graviton propagator in the TTS gauge, a gauge transformation was found such that it was possible to render it IR finite, in a large class of FLRW spacetimes. Such a regulator was therefore found not to be necessary. Indeed, there is no need for an IR cut-off to define the covariant two-point function coming from the Euclidean propagator. To show that the covariant graviton propagator experiences the second type of IR divergence in de Sitter spacetime, whereas the covariant massless vector propagator does not, we found the large-distance limits of these propagators. In the case of the graviton two-point function, a gauge transformation was found to remove this divergence.

In Chapter 2, we studied the IR divergences of the graviton two-point function in FLRW spacetime.In this chapter we initially consider the two point function in the TTS gauge. Later in the chapter, when single-field inflation is incorporated, we work in the gauge where the scalar perturbation vanishes. The two-point function was found from the tensor and scalar perturbations in slow-roll single-field inflation. The tensor and scalar, perturbations were written as a mode sum, where each mode function was proportional to $p^{-\nu}$, in the IR limit. From a power counting argument, the two-point function was found to be IR divergent for $\nu \geqslant \frac{n-1}{2}$, which is the case for most applications. However, due to the equivalence between the graviton two-point function and the linearised Weyl tensor, in FLRW spacetime, it was expected that the terms in the mode functions of order p^0 and p could be written in pure-gauge form. It was shown that this was indeed the case. Using a large-coordinate gauge transformation, we were able to transform our mode functions such that they were proportional to $p^{-\nu+2}$. The resulting two-point function is only IR divergent for $\nu \geqslant \frac{n+3}{2}$, which is the same range of values for which the linearised Weyl tensor is IR divergent. Hence, the graviton two-point function was found to be IR finite for a larger range of FLRW spacetimes originally thought. To give an example of its use, this gauge transformation was applied to a slow-roll inflationary universe. This transformation is equally applicable for an arbitrary potential $V(\phi)$.

In Chapters 3 and 4, we focussed on the IR divergences resulting from the large-distance separation of propagators, in de Sitter spacetime. In Chapter 3, the large-distance behaviour of the covariant massless vector propagator was studied. In this limit, the propagator is equal to a gauge-dependent constant. In the Landau gauge, this constant vanishes. This method was extended, in Chapter 4, to find the large-distance limit of the covariant graviton two-point function. This propagator was found to experience both linear and logarithmic divergences. In the Landau gauge, the divergence was found to be purely logarithmic, and in the case when the parameter $\alpha = \frac{n+1}{n-1}$, the divergence is linear. There is, however, no value of the parameters α, β, such that these divergences simultaneously vanish.

Finally, in Chapter 5, we found a gauge transformation to remove the IR divergence of the graviton propagator. In the previous chapter, it was found that the propagator was, in the large-distance limit, logarithmically divergent, in the Landau gauge. By the use of a covariant gauge transformation, it was found that this logarithmic divergence can be traded for a linear one, but that it could not be removed. However, a non-covariant gauge transformation was found such that the logarithmic divergence could be removed.

We conclude by suggesting a number of open problems which might be of interest to the reader. The first couple of these concern the results of Chapter 2. As mentioned at the end of this chapter, there is no geometric explanation for the gauge transformation used to remove the term proportional to p in the graviton mode functions. Finding such an interpretation might give some reasoning on why our transformation takes the form it does. Finally, it would be of interest to find a physical explanation as to why, in Chapter 5, it was not possible to find a covariant gauge transformation to remove the logarithmic divergence.

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