Generalised Sorkin-Johnston and Brum-Fredenhagen States for Quantum Fields on Curved Spacetimes

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Only those who will risk going too
far can possibly find out how far one can go.
Do I dare disturb the universe?
T. S. Eliot.

Abstract

The presented work contains a new construction of a class of distinguished quasifree states for the scalar field and Proca field on globally hyperbolic spacetimes. Our idea is based on the axiomatic construction of the Sorkin-Johnston (SJ) state [58]; we call these states *generalised SJ states.* We give a concrete application of this framework with the construction of the 'thermal' SJ state. By slightly modifying the construction of generalised SJ states, we also introduce a new class of Hadamard states, which we call *generalised SJ states with softened boundaries.* We show when these states satisfy the Hadamard condition and compute the Wick polynomials. Finally we construct the SJ and Brum-Fredenhagen (BF) states for the Proca field on ultrastatic slabs with compact spatial sections. We show that the SJ state construction fails for the Proca field, yet the BF state is well defined and, moreover, satisfies the Hadamard condition.

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Authors Declaration

I declare that this thesis is a presentation of original work and I am the sole author. This work has not previously been presented for an award at this, or any other, University. All sources are acknowledged as References. The proof of Theorem 4.7 in Chapter 4 is based on [23, Theorem 3.5]. The contents of Chapter 6 and 7 are entirely my own. Chapter 8 and chapter 9 is work done partly in collaboration with Professor Chris Fewster and Dr Kasia Rejzner.

Chapter 1

Introduction

The development and progression in modern physics stems from the interplay between theory and experiment; a theory is developed and an experiment is performed to verify such a theory. Conversely, new phenomena are discovered from experiments which then generates a driving force to understand these results in a theoretical framework. This has been extremely fruitful, culminating in the construction of the two pillars of modern physics: Einstein's general relativity and quantum mechanics. One of the greatest tasks of modern theoretical physics is the reconciliation of these two theories into a coherent framework, which is largely known as quantum gravity.

Of particular interest to us is the causal set theory approach to quantum gravity and the framework known as quantum field theory in curved spacetime. Causal set theory is a theory of quantum gravity based on the idea that, fundamentally, spacetime is discrete [59, 18, 57, 56]. The theory incorporates two central components of modern physics, the spacetime causal ordering from general relativity and the path-integral from quantum theory. Quantum field theory in curved spacetime is a theory in which the fields involved are treated fully quantum mechanically, but the gravitational 'back-reaction' is treated classically or semi-classically, in accordance to the principles of general relativity. Since quantum field theory in curved spacetime only treats the gravitational field semi-classically at best, it can only serve as an approximation to a full quantum theory of gravity. However, it has produced some remarkable insights; notably so is the particle creation by black holes discovered by Hawking [37] and the Unruh effect [61].

Quantum field theory in curved spacetime is a natural extension of quantum field theory on flat spacetime; however, there are profound conceptual and mathematical distinctions between the two. Quantum field theory on flat spacetime allows for the construction of a distinguished maximally symmetric, minimal energy state known as the vacuum state, which is unique up to a phase factor. The vacuum state then forms the basis of the theory; the Poincaré invariance of the vacuum state allows for the identification of single particle states via Wigner's analysis and the notion of particles has a clear interpretation.

One may then be tempted to seek out a construction of a distinguished state for a general curved spacetime. However, such efforts have been met with failure - and the existence of a distinguished state with physical properties has been firmly dismissed and presented as a no-go theorem [27]. These results have then shifted the efforts of constructing a distinguished state to the construction of a class of states that are in some sense 'physical'; such a class of states are known as *Hadamard states*. Hadamard states have a short distance behaviour that approximates that of states in flat spacetimes with finite energy. A precise definition of a Hadamard state was first given by Kay and Wald in [44] and an elegant reformation of the Hadamard condition was given by Radzikowski in [52]. Hadamard states are considered to be the largest class of states that are physically reasonable, they permit the computation of Wick polynomials such as the stress-energy tensor and give finite results and fluctuations. Fredenhagen and Brunetti showed that the Wick polynomials evaluated in a Hadamard state are finite [12], and a partial converse to this result is given by Fewster and Verch [29]. On a general spacetime, Hadamard states for a spin zero field are known to exist via a deformation argument presented by Fulling, Narcowich and Wald [32]. However, the argument is indirect and does not give any information on how to explicitly construct Hadamard states. Therefore, seeking explicit constructions of Hadamard states is an important question for quantum field theory in curved spacetime. It is the purpose of this thesis to present an explicit construction of a family of quasifree Hadamard states for both the free spin zero field and massive spin one fields.

A particular construction of a class of Hadamard states stems from the development of propagators in causal set theory undertaken by Johnston [42]. Johnston uses the causal set analogue of the Pauli-Jordan function for a free scalar field to construct a unique vacuum state. This construction was then applied to a free scalar field over a continuum spacetime by Afshordi, Aslanbeigi and Sorkin, which culminated in a distinguished pure quasifree state known as the SJ state. The SJ state for quantum field theories on continuum spacetimes is a construction for a pure quasifree state using only the field equations and a bounded region of a globally hyperbolic spacetime. The construction does not rely on any symmetries of the spacetime, and is well defined provided the bounded region in which the SJ state is constructed obeys some technical conditions, which was shown by Fewster and Verch in [28]. The SJ state, however, suffers from severe pathologies, which we expand upon in Chapter 5. In particular, Fewster and Verch showed that the SJ state constructed on an ultrastatic slab with compact spatial sections fails to be Hadamard [28]. There is, however, a modification to the SJ state construction due to Brum and Fredenhagen that yields a class of Hadamard states, known as BF states [11]. The BF states have been constructed on static slab spacetimes with compact spatial section and on expanding slab spacetimes and are shown in both cases to satisfy the Hadamard condition. We review both the SJ and BF state constructions in Chapter 5. The SJ construction essentially is constructing the 'positive part' of the commutator function (also known as the Pauli-Jordan function) when the commutator function extends to a bounded self-adjoint operator on a Hilbert space. The positive part is then used to construct a two-point function, and from the two-point function a quasifree state can be constructed. We review various aspects of quasifree states and Hadamard states in Chapter 4.

The SJ and BF state constructions have also been applied for the free Dirac field on ultrastatic slab spacetimes with compact spatial sections [23]. The idea is to use the fermionic projector (FP) construction of Finster and Reintjes [30], and apply a similar construction to the SJ and BF state constructions. This construction yielded a pure quasifree 'FP' state and a softened FP state for the Dirac field. On ultrastatic slabs, the FP state fails to be Hadamard whereas the softened FP state (based on the BF state construction) satisfies the Hadamard condition.

In this thesis we present a new construction of quasifree states that is based on the axiomatic construction of the SJ state due to Sorkin [58]. We call these quasifree states *generalised SJ states*. The construction is well defined whenever the construction of the SJ state is well defined. The original SJ state construction, due to Afshordi, Aslanbeigi and Sorkin, is based on the observation that, for a suitable spacetime $(\mathcal{M}, g)$, the commutator function $A = i\mathbb{E}$, where $\mathbb{E}$ is the advanced-minus-retarded operator for the Klein Gordon operator, extends to a bounded self-adjoint operator on the Hilbert space $L^2(\mathcal{M},\ \mathbf{d}\mathrm{vol}_g)$. The 'positive part' of the commutator function $A = i\mathbb{E}$ is given by the operator,

$$A^+ = \frac{1}{2}(A + \sqrt{A^2}), \tag{1.1}$$

and is used to construct a two-point function by the prescription,

$$W_{SJ}(f, g) = \langle \overline{f} \mid A^+ g \rangle \qquad \forall f, g \in C_0^\infty(\mathcal{M}). \tag{1.2}$$

Our construction of generalised SJ states uses the spectral theory for bounded self-adjoint operators over a Hilbert space; a generalised SJ state has the two-point function,

$$W_{SJ_\psi}(f,g) = \langle \overline{f} \mid A^+_\psi g\rangle \qquad \forall f,g \in C_0^\infty(\mathcal{M}), \tag{1.3}$$

where A^+_ψ is the unique solution to a set of axioms for a suitable continuous function ψ. We give a complete description of the types of functions ψ so that (1.3) satisfies the field equations for the scalar field, is of positive type, and has the correct antisymmetric part. Furthermore, we give a concrete application of the generalised SJ state construction by constructing the SJ 'thermal' state on an ultrastatic slab spacetime with compact spatial sections. We also present a proof giving sufficient conditions on when summing smooth sections converges, which is presented in Chapter 4. This result will be used throughout the entire thesis.

Using the generalised SJ state construction, we develop a new construction of Hadamard states based on an observation of Sorkin [58]. Sorkin conjectured that one may 'soften the boundary' of the SJ vacuum state to obtain a Hadamard state [58], by modifying the volume form. The volume form $\mathbf{d}\mathrm{vol}_g$ is modified to $\frac{1}{\rho}\,\mathbf{d}\mathrm{vol}_g$ for a suitable smooth compactly supported function ρ. We use the generalised SJ state construction and the idea of Sorkin to construct a class of Hadamard states which we call *generalised SJ states with softened boundaries*. We give sufficient conditions on when a generalised SJ state with softened boundaries satisfies the Hadamard condition on ultrastatic slabs with compact spatial sections. We also construct the thermal SJ state with softened boundaries on ultrastatic slab spacetimes and show that it satisfies the Hadamard condition. We also compute the Wick square of the field evaluated in the thermal SJ state with softened boundaries. Furthermore, on ultrastatic slabs with a spatial section of a three sphere, we calculate the Wick square of the $n-$th derivative of the field evaluated in the SJ vacuum with softened boundaries. We present numerical evidence that the Wick square of the unsoftened SJ state (5.29) diverges in the interior of the ultrastatic slab. Furthermore, we construct the thermal SJ state with softened boundaries on ultrastatic slabs and show that it satisfies the Hadamard condition.

Finally, we examine the SJ and BF state construction for a free massive spin one field on an ultrastatic slab spacetime. The massive spin one field, or Proca field, is most elegantly described in terms of differential forms. However, the inner product on the space of differential forms induced by a Lorentzian metric is actually *indefinite*. The commutator function one would use to construct a SJ or BF state is not an operator on a Hilbert space, but an operator on a Krein space. Krein spaces are essentially complete indefinite inner products that decompose into a direct sum of a positive definite and negative definite subspaces. Analysis in Krein spaces is significantly harder than in Hilbert spaces; it turns out the commutator function extends to a unbounded operator in both the SJ and BF state constructions. We show that, on ultrastatic slab spacetimes with compact spatial sections, one cannot construct the SJ state for the Proca field. A precise statement of this is beyond the scope of this thesis, but we refer the reader to the papers [26, 25] for a complete rigorous construction. However, one can construct a BF state for the Proca field on ultrastatic slab spacetimes for a suitable choice of softening function, and we give an explicit construction to show that this is possible. Furthermore, whenever a BF state on an ultrastatic slab with compact spatial section is well defined, we show that it satisfies the Hadamard conditon. The non-existence of the SJ state for the Proca field and the BF state construction are both rigorously done in [26, 25].

Thesis Layout

The layout of this thesis is as follows. Chapter 2 will include preliminaries required by the reader to understand this thesis. Chapter 3 will review the algebraic quantisation of the free scalar field on globally hyperbolic spacetimes, and is largely standard. Chapter 4 will review quasifree states and the Hadamard condition, which again is largely standard. However, in section 4.3 we present a new result, which gives sufficient conditions for when the summation of smooth sections converges. This result is based on the work appearing in [23] and will be the basis for proving if a state satisfies the Hadamard condition. Chapter 5 will review the SJ vacuum state for the free scalar field on continuum spacetimes and discrete spacetimes. Since our concern is for quantum field theories on continuum spacetimes, the review of the SJ vacuum on discrete spacetimes is pedagogical in nature. Our focus will be on the construction and properties of the SJ vacuum on continuum spacetimes. Furthermore, we review a modification of the SJ vacuum due to Brum and Fredenhagen, which we call BF states.

Chapter 6 will present a new construction of states for a free scalar field on globally hyperbolic spacetimes which we call ‘generalised SJ states’. This construction is based on the axiomatic approach to the SJ state construction given by Sorkin in [58]. Chapter 7 will give an application of the generalised SJ state construction, where we construct a SJ ‘thermal’ state on an ultrastatic slab spacetime. Chapter 8 will present a new construction of a class of states that is based on a observation of Sorkin in [58], which we call generalised SJ states with softened boundaries. We show that the SJ vacuum with softened boundaries constructed on ultrastatic slab spacetimes with compact spatial sections satisfies the Hadmard condition. Furthermore, we give sufficient conditions on when any generalised SJ state with softened boundaries on an ultrastatic slab is Hadamard. We calculate the Wick square of the SJ vacuum with softened boundaires using a construction of compactly supported functions given by [21, 22]. We give indications that the Wick square for the unsoftened SJ vacuum on ultrastatic slabs with compact spatial sections diverges in the interior. Using the construction of the thermal SJ state in Chapter 7, we construct the thermal SJ state with softened boundaries on ultrastatic slab spacetimes and show that it satisfies the Hadamard condition. Finally, in Chapter 9 we extend the SJ state construction to a massive spin-one field on an ultrastatic slab spacetime. We show that one cannot construct the SJ vacuum for the massive spin one field. However, the BF state construction for the massive spin-one field for a suitably chosen softening functions is well defined. Moreover, we show that when the BF states are well defined on ultrastatic slabs they also satisfy the Hadamard condition. This thesis presents contributions to the explicit construction of a large family of quasifree states for both the free scalar field and free Proca field which is valid in a large class of spacetimes. We conclude the thesis with a brief summary, as well as various open problems and directions for future research.

Chapter 2

Mathematical Preliminaries

2.1 Notation

We now briefly review the notation that we shall use throughout this thesis. The space of smooth $\mathbb{K}-$valued functions over a space $\mathcal{M}$ will be denoted $C^{\infty}(\mathcal{M}, \mathbb{K})$, and the subspace of smooth functions with compact support will be denoted $C_0^{\infty}(\mathcal{M}, \mathbb{K})$. Whenever we write $C^{\infty}(\mathcal{M})$ or $C_0^{\infty}(\mathcal{M})$ we implicitly mean the spaces $C^{\infty}(\mathcal{M}, \mathbb{C})$ or $C_0^{\infty}(\mathcal{M}, \mathbb{C})$. We will explicitly write $C^{\infty}(\mathcal{M}, \mathbb{R})$ and $C_0^{\infty}(\mathcal{M}, \mathbb{R})$ as the space of smooth real valued functions and the space of smooth real valued functions with compact support. We use units such that $\hbar = c = 1$. Let $f \in C^{\infty}(\mathbb{K})$ be a smooth function. The Fourier transform of f is defined as,

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} f(t) \mathbf{d}t \tag{2.1}$$

2.2 Functional Analysis

In this section we will review various elements of functional analysis used throughout this thesis. We will concentrate on the analysis of Hilbert spaces and operators on Hilbert spaces and leave analysis of Krein spaces until Chapter 9. Whenever we write a Hilbert space $\mathcal{H}$ we implicitly mean the pair $(\mathcal{H}, \langle \cdot \mid \cdot \rangle)$ where $\langle \cdot \mid \cdot \rangle$ is a positive definite inner product. The space of bounded operators on a Hilbert space $\mathcal{H}$ will be denoted $B(\mathcal{H})$, where each element $T : \mathcal{H} \to \mathcal{H}$ is a bounded operator. We shall be interested in the convergence of operators in various topologies on the space of bounded operators

$B(\mathcal{H})$. Before defining these topologies, we first define the dual space of $\mathcal{H}$,

Definition 2.2.1. Let $\mathcal{H}$ be a Hilbert space. The *dual space* of $\mathcal{H}$, denoted $\mathcal{H}^*$ is the space of all continuous linear functionals over $\mathcal{H}$, where each element $l \in \mathcal{H}^*$ is the map,

$$l : \mathcal{H} \to \mathbb{C}. \tag{2.2}$$

We are now ready to define the required topologies on the space of bounded operator $B(\mathcal{H})$:

Definition 2.2.2. Let $\mathcal{H}$ be a Hilbert space and $B(\mathcal{H})$ be the normed space of bounded operators on $\mathcal{H}$. We can now define two operator topologies on the space $B(\mathcal{H})$,

i The *norm operator topology* on $B(\mathcal{H})$ is the metric topology induced by the norm $\|\cdot\|$. A sequence $(T_n)_{n\in\mathbb{N}}$ converges to T, written as $T_n \to T$, if and only if $\|T_n - T\|_{B(\mathcal{H})} = 0$. This is equivalent to saying that $T_n \to T$ in the norm topology of $B(\mathcal{H})$ if and only if $\|T_n f - Tf\| \to 0$ for every $f \in \mathcal{H}$.

ii The *weak operator topology* on $B(\mathcal{H})$ is the weakest topology such that the maps,

$$\begin{aligned} E_{f,l} : B(\mathcal{H}) &\to \mathbb{C} \\ T &\mapsto l(Tf), \end{aligned} \tag{2.3}$$

are continuous for all $f \in \mathcal{H}$ and all $l \in \mathcal{H}^*$. We draw attention to the following theorem:

Theorem 2.1. *Let $\mathcal{H}$ be a Hilbert space. Let $\{T_n\}_{n\in\mathbb{N}}$ be a sequence of bounded operators and suppose that, for every $f, g \in \mathcal{H}$, $\langle T_n f \mid g\rangle$ converges as $n \to \infty$. Then there exists a $T \in B(\mathcal{H})$ such that $\{T_n\}_{n\in\mathbb{N}}$ converges to T in the weak operator topology.*

Proof. See [53, Theorem VI.1]. ■

Definition 2.2.3. Let $(H, \langle\cdot \mid \cdot\rangle)$ be a Hilbert space. An operator $U \in B(\mathcal{H})$ is *positive* if for all $f \in \mathcal{H}$ we have $\langle f \mid Tf\rangle \geq 0$. If $U \in B(\mathcal{H})$ is a positive operator then we shall write $U \geq 0$. Due to the positivity of the norm induced by the inner product $\langle\cdot \mid \cdot\rangle$, for any $T \in B(\mathcal{H})$, we have $T^*T \geq 0$.

A particularly nice class of operators to deal with are called *compact operators*, which we now define:

Definition 2.2.4. The operator $A \in B(\mathcal{H})$ is *compact* if A maps bounded subsets $U \subset \mathcal{H}$ into precompact sets $AU \in \mathcal{H}$. Equivalently $A \in B(\mathcal{H})$ is compact if and only if every bounded sequence $\{f_n\}_{n\in\mathbb{N}} \subset H$ is mapped into a sequence $\{Af_n\}_{n\in\mathbb{N}}$ with a convergent subsequence.

Compact operators have many useful features, two of which we shall use during this thesis:

i The *Hilbert-Schmidt* theorem for self-adjoint compact operators states the following,

Theorem 2.2. *Let $T \in B(\mathcal{H})$ be a self-adjoint compact operator. Then there is a complete orthonormal basis $\{\psi_n\}_{n\in\mathbb{N}}$ consisting of eigenvectors of T, i.e. $T\psi_n = \lambda_n\psi_n$ where λ_n are eigenvalues that obey $\lambda_n \to 0$ as $n \to \infty$.*

Proof. See [53, Theorem VI.16]. ■

ii The norm limit of compact operators is a compact operator:

Theorem 2.3. *Let $\{T_n\}_{n\in\mathbb{N}}$ be a sequence of compact operators. If $T_n \to T$ in norm topology, then T is compact.*

Proof. See [53, Theorem VI.12]. ■

We will also use the fact that finite rank operators (i.e. operators whose image is finite dimensional) are compact. The type of Hilbert space solely considered in this thesis are *separable*, and we shall use the following theorem,

Theorem 2.4. *A Hilbert space $\mathcal{H}$ is separable if and only if there exists a countable orthonormal basis.*

Proof. See [53, Theorem II.7]. ■

Another class of operators that will be used in Chapter 5 are *Hilbert-Schmidt* operators, to define these, we first need to define the *trace* as a linear functional on $B(\mathcal{H})$ where $\mathcal{H}$ is a separable Hilbert space,

Definition 2.2.5. Let $(\mathcal{H}, \langle\cdot \mid \cdot\rangle$ be a separable Hilbert space and $\{\phi_n\}_{n\in\mathbb{N}}$ be a orthonormal basis. Let $U \in B(\mathcal{H})$ be a positive operator. The *trace* of U is defined as,

$$tr(U) = \sum_{j\in\mathbb{N}} \langle\phi_n \mid U\phi_n\rangle. \tag{2.4}$$

Definition 2.2.6. Let $\mathcal{H}$ be a Hilbert space. An operator $T \in B(\mathcal{H})$ is called *Hilbert Schmidt* if and only if $tr(T^*T) < \infty$.

An important result for operators on Hilbert space is the *spectral theorem for bounded self-adjoint operators*. This theorem will form the basis for the construction of generalised SJ states. We now collect the necessary prerequisites to state the spectral theorem.

Definition 2.2.7. Let A be a bounded operator over a Hilbert space $\mathcal{H}$. The *spectrum* of A, denoted $\sigma(A)$ is the set of all $\lambda \in \mathbb{C}$ such that $A - \lambda \mathbb{1}$ is not invertible, where $\mathbb{1} \in B(\mathcal{H})$ is the identity operator. The *resolvent* is defined as $\rho(A) = \mathbb{C} \setminus \sigma(A)$.

Definition 2.2.8. The space of continuous real-valued functions over the spectrum of a self-adjoint bounded A is denoted by $C(\sigma(A))$, where the norm is given by the sup-norm defined by,

$$\|f\|_\infty = \sup_{\lambda \in \sigma(A)} |f(\lambda)|, \tag{2.5}$$

for all $f \in C(\sigma(A))$.

Theorem 2.5 (Spectral theorem - continuous functional calculus version)**.** *Let A be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$. Then there is a* unique *map $\Theta : C(\sigma(A)) \to B(\mathcal{H})$ that obeys the following:*

i)

$$\begin{aligned} \Theta(fg) &= \Theta(f)\Theta(g) \\ \Theta(\alpha f) &= \alpha\Theta(f) \qquad \forall \alpha \in \mathbb{C}, \\ \Theta(1) &= \mathbb{1}_{B(\mathcal{H})} \\ \Theta(\overline{f}) &= \Theta(f)^*, \end{aligned} \tag{2.6}$$

for all $f, g \in C(\sigma(A))$ where $1/\mathbb{1}_{B(\mathcal{H})}$, denotes the identity on $C(\sigma(A))$ and $B(\mathcal{H})$ respectively, $\overline{f}$ denotes the complex-conjugate of $f \in C(\sigma(A))$ and $\Theta(f)^*$ denotes the adjoint of $\Theta(f) \in B(\mathcal{H})$. These properties entail that Θ is an algebraic $*-$homomorphism.

ii) $\|\Theta(f)\|_{B(\mathcal{H})} = \|f\|_\infty$.

iii) Let $f \in C(\sigma(A))$ be defined by $f(\lambda) = \lambda$; then $\Theta(f) = A$.

iv) If $A\Psi = \lambda\Psi$ then $\Theta(f)\Psi = f(\lambda)\Psi$

v) $\sigma(\Theta(f)) = \{f(\lambda) \mid \lambda \in \sigma(A)\}$

vi) *If $f \geq 0$ then $\Theta(f) \geq 0$.*

vii) *If there exists a $B \in B(\mathcal{H})$ such that $AB = BA$ then $\Theta(f)B = B\Theta(f)$.*

viii) *If a sequence $(f_n)_{n\in\mathbb{N}}$ converges point wise to $f \in C(\sigma(A))$ and $\|f_n\|_\infty$ is bounded, then $\Theta(f_n) \to \Theta(f)$ in the strong operator topology.*

Proof. See [53, Theorem VII.1] for $i - v$ and [53, Theorem VII.2] for $vi - vii$. ■

Throughout this thesis, we write $\Theta(f) = f(A)$ to emphasis the dependence on A.

2.3 Partial Differential Operators

We review some elementary facts about partial differential operators on smooth manifolds. The resources for this section are [62, 3, 46].

Definition 2.3.1. A *smooth $\mathbb{K}-$vector bundle* of order n is a triplet $(\pi, F, \mathcal{M})$ where F and $\mathcal{M}$ are topologcial spaces and $\pi : F \to \mathcal{M}$ is a smooth continuous surjective map. The map π is required to satisfy the following,

1. For every $x \in \mathcal{M}$ the space generated by the preimage of x with respect to π, i.e $\pi^{-1}(x)$, is a vector space isomorphic to $\mathbb{K}^n$. The preimages $\pi^{-1}(x)$ are called the fibres of the bundle.

2. For every $x \in \mathcal{M}$ there exists an open neighbourhood V_x of x and smooth map,
$$\varphi : \pi^{-1}(V_x) \to V_x \times \mathbb{K}^n, \tag{2.7}$$
called a *local trivialisation*.

In the following we will abbreviate a vector bundle $(\pi, F, \mathcal{M})$ as simply the map $\pi : F \to \mathcal{M}$.

Vector bundles then look, locally, like a topological space with a vector space attached in a smoothly varying manner. An important example of a vector bundle is the *tangent bundle* of a smooth manifold $\mathcal{M}$, denote $T\mathcal{M}$, whose fibres are the tangent spaces $T_x\mathcal{M}$, $x \in \mathcal{M}$. The dual bundle to the tangent bundle $T\mathcal{M}$ is the *cotangent bundle*, denoted $T^*\mathcal{M}$, whose fibres are the cotangent spaces $T_x^*\mathcal{M}$, $x \in \mathcal{M}$.

Definition 2.3.2. Let $\pi : F \to \mathcal{M}$ be a vector bundle. The *dual bundle* is the vector bundle $\pi^* : F^* \to \mathcal{M}$ whose fibres ${\pi^*}^{-1}(x)$ at $x \in \mathcal{M}$ are the dual vector space to the fibre $\pi^{-1}(x)$.

Definition 2.3.3. A *smooth section* of a vector bundle $\pi : F \to \mathcal{M}$ is a smooth map $s : \mathcal{M} \to F$ such that $\pi \circ s = id_{\mathcal{M}}$. A section can be thought of as a one-sided inverse of a vector bundle. The space of smooth sections is denoted by $\Gamma^\infty(\mathcal{M}, F)$ and the space of smooth compactly supported sections in F will be denoted $\mathscr{D}^\infty(\mathcal{M}, F)$.

Definition 2.3.4. Let V be a vector space and $b : V \times V \to \mathbb{R}$ be a symmetric bilinear form. The *index* of b is the dimension of the largest subspace $W \subset V$ such that $b\big|_W$ is *negative definite.*

Definition 2.3.5. Let $\mathcal{M}$ be a smooth manifold. A *metric tensor*, or more simply just a *metric*, is smooth section of the second symmetric power of the cotangent bundle $T^*\mathcal{M}$,

$$g \in \Gamma\Big(S^2(T^*\mathcal{M})\Big), \tag{2.8}$$

which is everywhere symmetric, non-degenerate and has a constant index. The pair $(\mathcal{M}, g)$ is called a *Riemannian manifold* if $\text{index}(g) = 0$, i.e. the metric is everywhere positive definite. If $\dim(\mathcal{M}) \geq 2$ and $\text{index}(g) = \dim(\mathcal{M}) - 1$ then $(\mathcal{M}, g)$ is called a *Lorentzian manifold.* The metric defines, for every $p \in \mathcal{M}$, a non-degenerate bilinear form g_p over the tangent space $T_p\mathcal{M}$. In the case of a Riemannian/Lorentzian manifold, the bilinear form g_p is definite/indefinite respectively for each $p \in \mathcal{M}$.

Definition 2.3.6. Let $\pi : F \to \mathcal{M}$ be a $\mathbb{K}-$vector bundle over a smooth manifold $\mathcal{M}$. A *connection* on F is a $\mathbb{K}$ bilinear map,

$$\begin{aligned} \nabla : \Gamma^\infty(\mathcal{M}, T\mathcal{M}) \times \Gamma^\infty(\mathcal{M}, F) &\to \Gamma^\infty(\mathcal{M}, F) \\ (X, s) &\mapsto \nabla_X s, \end{aligned} \tag{2.9}$$

such that, for all $f \in C^\infty(\mathcal{M})$, $X \in \Gamma^\infty(\mathcal{M}, T\mathcal{M})$ and all $s \in \Gamma^\infty(\mathcal{M}, F)$ the following holds:

i It satisfies the *Leibniz rule*,

$$\nabla_X(fs) = \partial_X f \cdot s + f \cdot \nabla_X s. \tag{2.10}$$

ii The map ∇ is $C^\infty(\mathcal{M})-$linear in the sense that,

$$\nabla_{fX} s = f \nabla_X s. \tag{2.11}$$

A connection is said to be compatible with a metric g if $\nabla g = 0$.

Definition 2.3.7. Let $\mathcal{M}$ be a smooth manifold and let ∇ be an arbitrary connection. The *torsion tensor* is defined as,

$$\begin{aligned} \mathcal{T} : \Gamma^\infty(\mathcal{M}, T\mathcal{M}) \times \Gamma^\infty(\mathcal{M}, T\mathcal{M}) &\to \Gamma^\infty(\mathcal{M}, T\mathcal{M}) \\ (X, Y) &\mapsto \nabla_X Y - \nabla_Y X - [X, Y]. \end{aligned} \tag{2.12}$$

The connection ∇ is said to be torsion-free if $\mathcal{T} = 0$. From now on, we assume that all connections are torsion-free.

Definition 2.3.8. Let $\mathcal{M}$ be a smooth $d-$dimensional manifold and let H and F be two $\mathbb{K}-$vector bundles over $\mathcal{M}$ of order k and m respectively. A *linear partial differential operator of order* d is a linear map $P : \Gamma^\infty(\mathcal{M}, H) \to \Gamma^\infty(\mathcal{M}, F)$, subject to the following conditions: For each $p \in \mathcal{M}$ there exists an open neighbourhood U_p with coordinates $(x_1, ..., x_d)$ on which the bundles H, F are trivialised and there are smooth maps,

$$a_\alpha : U_p \to \mathrm{Hom}(\mathbb{K}^k, \mathbb{K}^m), \tag{2.13}$$

such that on U_p,

$$(Pf)(x) = \sum_{|\alpha| \leq k} a_\alpha(x) \frac{\partial^{|\alpha|} f}{\partial x^\alpha}. \tag{2.14}$$

Here, the summation is taken over all multi-indices $\alpha = (\alpha_1, ...\alpha_d) \in \mathbb{N}^d$ and $\mid \alpha \mid = \alpha_1 + \alpha_2 + ... + \alpha_d$.

Definition 2.3.9. [62, 1.3.74] Let $\pi : H \to \mathcal{M}$ and $\pi' : F \to \mathcal{N}$ be two vector bundles over smooth manifolds $\mathcal{M}$ and $\mathcal{N}$ respectively. The *external tensor product* of the bundles $\pi : H \to \mathcal{M}$ and $\pi' : F \to \mathcal{M}$ is a vector bundle over the Cartesian product $\mathcal{M} \times \mathcal{N}$ defined by,

$$H \boxtimes F \to \mathrm{pr}^\star_{\mathcal{M}}(H) \times \mathrm{pr}^\star_{\mathcal{N}}(F), \tag{2.15}$$

where $\mathrm{pr}^\star_{\mathcal{M}}(H) \to \mathcal{M} \times \mathcal{N}$ and $\mathrm{pr}^\star_{\mathcal{N}}(F) \to \mathcal{M} \times \mathcal{N}$ are pullbacks of the projections $\mathrm{pr}_{\mathcal{M}} : \mathcal{M} \times \mathcal{N} \to \mathcal{M}$ and $\mathrm{pr}_{\mathcal{N}} : \mathcal{M} \times \mathcal{N} \to \mathcal{N}$. The vector bundle $H \boxtimes F$ has fibres $H_x \otimes F_y$ for $(x, y) \in \mathcal{M} \times \mathcal{N}$.

Definition 2.3.10. A *hermitian metric* on a complex fibre bundle $\pi : F \to \mathcal{M}$ is a smooth section in $F^* \otimes F^*$ which is a hermitian scalar product on each fibre.

Definition 2.3.11. Let $\mathcal{M}$ be a smooth Riemannian manifold and $\pi : F \to \mathcal{M}$ be a vector bundle with a hermitian fibre metric. The inner product on F is the map,

$$\begin{aligned} \langle \cdot \mid \cdot \rangle : F \boxtimes F &\to \mathbb{C} \\ f \otimes g &\mapsto \int_{\mathcal{M}} (f, g)\big|_p \mathbf{d}\mathrm{vol}(p), \end{aligned} \tag{2.16}$$

where at each point $p \in \mathcal{M}$, the pairing $(f,g)\big|_p$ is constructed from the fibre metric h via,

$$(f,g)\big|_p = \overline{f}^{\alpha}(p) g^{\beta}(p) h_{\alpha\beta}(p) \tag{2.17}$$

Let $\pi : F \to \mathcal{M}, \pi' : H \to \mathcal{M}$ be two $\mathbb{K}-$vector bundle over a smooth manifold $\mathcal{M}$ and let $P : \Gamma^{\infty}(\mathcal{M}, H) \to \Gamma^{\infty}(\mathcal{M}, F)$ be a linear partial differential operator. Then there is a unique linear partial differential operator $P^* : \Gamma^{\infty}(\mathcal{M}, H^*) \to \Gamma^{\infty}(\mathcal{M}, F^*)$ called the *formal adjoint* of P that satisfies, for all $f, g \in \mathscr{D}^{\infty}(\mathcal{M}, F)$,

$$\langle f \mid Pg \rangle = \langle P^* f \mid g \rangle. \tag{2.18}$$

Definition 2.3.12. Let H, F be two $\mathbb{K}-$vector bundle of order k and m respectively over a smooth $d-$dimensional manifold $\mathcal{M}$ and let $P : C^{\infty}(\mathcal{M}, H) \to C^{\infty}(\mathcal{M}, F)$ be a linear partial differential operator of order $n \in \mathbb{N}$. The *principal symbol*, denoted σ_P is defined globally as the map,

$$\sigma_P : T^*\mathcal{M} \to \mathrm{Hom}(H, F). \tag{2.19}$$

Locally, in the coordinate chart of a point $x \in \mathcal{M}$ the operator P takes the form,

$$P = \sum_{|\alpha|=n} a_{\alpha} \frac{\partial^{|\alpha|}}{\partial x^{\alpha}}, \tag{2.20}$$

and the vector bundle F can be trivialised to identify $\mathrm{Hom}(H, F)$ with $\mathrm{Hom}(\mathbb{K}^k, \mathbb{K}^m)$. Then, for every $\xi \in T_x^*\mathcal{M}$ we have,

$$\sigma_P(\xi) = \sum_{|\alpha|=n} \xi^{\alpha} a_{\alpha}(x), \tag{2.21}$$

where $\xi = \Sigma_{i=1}^{d} \xi_i dx^i$, $\xi^{\alpha} := \xi_1^{\alpha_1} \xi_2^{\alpha_2} ... \xi_d^{\alpha_d}$.

We are now able to define a particular class of partial differential operators called normally hyperbolic operators. Normally hyperbolic operators have many useful properties which we will use throughout this thesis.

Definition 2.3.13. Let $\mathcal{M}$ be a d-dimensional Lorentzian manifold and F be $\mathbb{K}-$vector bundle over $\mathcal{M}$. A *normally hyperbolic partial differential operator* is a linear partial differential operator $P : \Gamma^{\infty}(\mathcal{M}, F) \to \Gamma^{\infty}(\mathcal{M}, F)$ of order two with the principal symbol,

$$\sigma_P(\xi) = -g(\xi, \xi) id_{F_x} \tag{2.22}$$

for every $x \in \mathcal{M}$ and every $\xi \in T_x^*\mathcal{M}$. Using local coordinates $(x^1, ..., x^d)$ on $\mathcal{M}$ and a trivialisation of F, a normally hyperbolic partial differential operator may be written as,

$$P = -g^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} + A^j(x)\frac{\partial}{\partial x^j} + B(x) \tag{2.23}$$

where g^{ij} is the inverse of the metric g_{ij}, A^j and B are smooth functions on $\mathcal{M}$ and the summation over repeated indices is understood.

Definition 2.3.14. Let $\mathcal{M}$ be a smooth Lorentzian manifold and $L : \Gamma^\infty(\mathcal{M}, F) \to \Gamma^\infty(\mathcal{M}, F)$ be a partial differential operator. Then L is said to be *elliptic* if its principal symbol $\sigma_L(\xi)$ is invertible for all non-zero $\xi \in T_x^*\mathcal{M}$ and all $x \in \mathcal{M}$.

Elliptic operators have many important properties which we will use throughout this thesis, for example elliptic partial differential operators have smooth eigenvectors [20, Theorem 3 Section 6.3]. An important space that will be used in Chapter 4 are *Sobolev spaces*.

Definition 2.3.15. Let $\pi : F \to X$ be a vector bundle with a connection ∇ on a Riemannian manifold X. For a given $u \in \Gamma^\infty(X, F)$ we have $\nabla u \in \Gamma^\infty(X, T^* \otimes F)$. Using the tensor product, we can apply the connection on u arbitrarily many times; $\nabla^j u \in \Gamma(X, (T^*X \otimes F)^{\otimes j})$ where,

$$(T^*X \otimes F)^{\otimes j} = T^*X \otimes F \underbrace{\otimes \cdots \otimes}_{\text{j times}} T^*X \otimes F \tag{2.24}$$

The *basic Sobolev k-norm* on $\Gamma^\infty(\mathcal{M}, F)$ is defined as,

$$\|u\|_k^2 = \sum_{j=0}^{k} \int_{\mathcal{M}} |\underbrace{\nabla\nabla \cdots \nabla u}_{\text{j times}}|^2. \tag{2.25}$$

The Sobolev basic k-norm is independent of the choice of connection. The *Sobolev space* of order k is then defined as the completion of $\Gamma^\infty(\mathcal{M}, F)$ with respect to the norm topology induced by (2.25) and is denoted by $L_s^2(\mathcal{M}, F)$.

Sobolev spaces will only feature in Chapter 4. W shall also use the following result,

Theorem 2.6. *Let $D : \Gamma^\infty(X, F) \to \Gamma^\infty(X, F)$ be an elliptic operator of order m where X is a* compact *manifold. Then for each $s \in \mathbb{N}_0$ there exists a $C_s > 0$ such that,*

$$\|u\|_s \leq C_s(\|u\|_{s-m} + \|Du\|_{s-m}), \tag{2.26}$$

for all $u \in L_s^2(X, F)$. Hence the norms $\|\cdot\|_s$ and $\|\cdot\|_{s-m} + \|D \cdot\|_{s-m}$ are equivalent.

Proof. [46, Theorem 5.2]. ∎

We shall also require the following results for Chapter 4.

Proposition 2.7. *Let $\pi : H \to \mathcal{M}$ and $\pi' : F \to \mathcal{M}$ be two vector bundles over $\mathcal{M}$. Then,*

$$\Gamma^\infty(\mathcal{M}, H) \otimes \Gamma^\infty(\mathcal{M}, F) \ni f \otimes g \mapsto f \boxtimes g \in \Gamma^\infty(\mathcal{M} \otimes \mathcal{M}, H \boxtimes F), \tag{2.27}$$

defines an injective smooth $C^\infty(\mathcal{M}) \otimes C^\infty(\mathcal{M})-$module morphism with dense image in the $C^\infty-$topology.

Proof. See [62, Theorem 1.3.35]. ∎

2.4 Differential Geometry

We review various elements of differential geometry which will be required to define a spacetime. Our primary resources used are [50, 47]. Over a $n-$dimensional Lorentzian manifold $(\mathcal{M}, g)$ we define the space of smooth complex-valued differential k-forms as smooth sections of the k-th exterior power of the cotangent bundle,

$$\Omega^k(\mathcal{M}, \mathbb{C}) = \Gamma\left(\bigwedge^k T^*\mathcal{M}\right) \otimes \mathbb{C}, \tag{2.28}$$

similarly $\Omega_0^k(\mathcal{M}, \mathbb{C})$ will denote k-forms with compact support. For brevity we supress the $\mathbb{C}$ in $\Omega^k(\mathcal{M}, \mathbb{C})$. Each k-form may be regarded as an antisymmetric covariant k-tensor field over $\mathbb{C}$, hence we shall make use of index notation when it is beneficial. The *exterior product* between two k-forms $\omega, \nu \in \Omega^k(\mathcal{M})$ is the map,

$$\begin{aligned} \wedge : \Omega^p(\mathcal{M}) \times \Omega^q(\mathcal{M}) &\to \Omega^{p+q}(\mathcal{M}) \\ (\omega \wedge \nu)_{a_1 \dots a_{p+q}} &= \frac{(p+q)!}{p!q!} \omega_{[a_1 \dots a_p} \nu_{a_{p+1} \dots a_{p+q}]}, \end{aligned} \tag{2.29}$$

and the *exterior derivative* is the map,

$$\begin{aligned} \mathbf{d} : \Omega^p(\mathcal{M}) &\to \Omega^{p+1}(\mathcal{M}) \\ (\mathbf{d}\omega)_{a_1 \dots a_{p+1}} &= (p+1)\nabla_{[a_1} \omega_{a_2 \dots a_{p+1}]}, \end{aligned} \tag{2.30}$$

where ∇ is any connection on $\mathcal{M}$ and the square brackets denote antisymmetrisation over the indices. By virtue of the antisymmetrisation, the exterior derivative is independent of the choice of connection.

Next, let $\{x_1, ..., x_n\}$ be a local coordinate chart of $(\mathcal{M}, g)$. If there exists a nowhere vanishing $n-$form defined by,

$$\mathbf{d}\text{vol}_g = \sqrt{|g|}dx_1 \wedge ... \wedge dx_n, \tag{2.31}$$

where $|g| = \det(g_{\mu\nu})$ then there is the unique $C^\infty(\mathcal{M})-$module isomorphism called the *Hodge star*,

$$\begin{aligned} \star_{\mathcal{M}} : \Omega^p(\mathcal{M}) &\to \Omega^{n-p}(\mathcal{M}) \\ \omega \wedge \star_{\mathcal{M}} \nu &\mapsto \frac{1}{p!}\omega_{a_1 \ldots a_p} \nu^{a_1 \ldots a_p} \; \mathbf{d}\text{vol}_g. \end{aligned} \tag{2.32}$$

The form $\mathbf{d}\text{vol}_g$ is called a *volume form* associated to the metric g. The property,

$$(\star)^2 \omega = (-1)^{p(n-p)+\text{index(g)}} \omega, \tag{2.33}$$

holds for all p-forms $\omega \in \Omega^p(\mathcal{M})$ over an $n-$dimensional manifold $\mathcal{M}$. Finally, the *coderivative* on $(\mathcal{M}, g)$ is the map,

$$\begin{aligned} \boldsymbol{\delta}_{\mathcal{M}} : \Omega^k(\mathcal{M}) &\to \Omega^{k-1}(\mathcal{M}) \\ \omega &\mapsto (-1)^{p(n-p)+s+1} \star_{\mathcal{M}} \mathbf{d} \star_{\mathcal{M}} \omega \end{aligned} \tag{2.34}$$

By convention the coderivative annihilates zero-forms and, since $\star_{\mathcal{M}}^2 = (-1)^{p(n-p)+s}$ and $\mathbf{d}^2 = 0$, we have $\boldsymbol{\delta}_{\mathcal{M}}^2 = 0$. For future use, we shall denote ker $\boldsymbol{\delta}\big|_{\Omega_0^1(\mathcal{M})}$ as the linear space of compactly supported one forms $f \in \Omega_0^1(\mathcal{M})$ such that $\boldsymbol{\delta}_{\mathcal{M}} f = 0$. Such forms are referred to as coclosed on $\mathcal{M}$.

Suppose (Σ, h) is a Riemannian manifold. The operations above allow us to endow the space $\Omega^k(\Sigma)$ with a positive definite inner product,

$$\langle f \mid g \rangle = \int_\Sigma \overline{f} \wedge \star_\Sigma g. \tag{2.35}$$

Under the norm topology of this inner product, we can form the Hilbert space of square integrable k-forms, which we denote $\Lambda^k(\Sigma) = \overline{\Omega^k(\Sigma)}$. We note here that the inner product (2.35) is positive definite if and only if the manifold $\mathcal{M}$ is Riemannian. In the case of a Lorentzian manifold $\mathcal{M}$, the inner product (2.35) is indefinite, and the completion of $\Omega^k(\mathcal{M})$ yields a *Krein space* of square integrable k-forms. This will be expanded upon in Chapter 9.

Chapter 3

Algebraic Quantisation of Spin Zero Fields on Globally Hyperbolic Spacetimes

In this chapter we will develop the CCR algebra for the free massive spin zero field on globally hyperbolic spacetimes. We begin by detailing some important features of globally hyperbolic spacetimes, and we also define ultrastatic slab spacetimes. The algebraic quantisation of spin one fields on globally hyperbolic spacetimes will be treated separately in Chapter 9.

3.1 Globally Hyperbolic Spacetimes

Suppose $(\mathcal{M}, g)$ is a $n-$dimensional smooth Lorentzian manifold equipped with a smooth metric of index $s = n - 1$ (this implies that we are using the mostly minus sign convention, $+ - \cdots -$). Let $x \in \mathcal{M}$. A vector $\nu \in T_x\mathcal{M}$ is said to be timelike if $g(\nu, \nu) > 0$, spacelike if $g(\nu, \nu) < 0$ and null if $g(\nu, \nu) = 0$. The vector $\nu \in T_x\mathcal{M}$ is said to be causal if it is either timelike or null. A vector field $\xi \in \Gamma^\infty(\mathcal{M}, T\mathcal{M})$ is said to be timelike/causal/ null if $\xi(x) \in T_x\mathcal{M}$ is timelike/causal/null for all $x \in \mathcal{M}$. A C^1 curve in a Lorentzian manifold is a map $\gamma : [a, b] \to \mathcal{M}$ with $-\infty < a < b < \infty$. The curve $\gamma : [a, b] \to \mathcal{M}$ is said to connect points $p \in \mathcal{M}$ and $q \in \mathcal{M}$ if $\gamma(a) = p$ and $\gamma(b) = q$.

A curve is said to be a *timelike curve* if all tangent vectors to the curve are everywhere timelike. A *causal* and *spacelike* curved is defined in a similar way. A closed causal curve is a causal curve whose end points coincide with each other. A Lorentzian manifold is time-orientable if and only if there exists a continuous nowhere vanishing global timelike vector field $\mathfrak{t} \in \Gamma^\infty(\mathcal{M}, T\mathcal{M})$ [50, Lemma 32]. A Lorentzian manifold is orientable if and only if there exists a nowhere vanishing form of maximal degree on $\mathcal{M}$ [50, Lemma 20]; such a form is called a volume form. The volume form generated by the metric g will be denoted $\mathbf{d}\mathrm{vol}_g$.

Definition 3.1.1. A *spacetime* of dimension $n \in \mathbb{N}$ is a collection $(\mathcal{M}, g, \mathfrak{o}, \mathfrak{t})$, where $(\mathcal{M}, g)$ is a $n-$dimensional smooth Lorentzian manifold which is connected,, $\mathfrak{o}$ is a orientation and $\mathfrak{t}$ is a time-orientation. We shall assume a orientation and time-orientation have been chosen, and omit them for brevity. Similarly, when it is clear, we shall omit the metric and simply refer to the spacetime as the manifold $\mathcal{M}$.

The *causal future* $(+)/$ *past* $(-)$ of a point $p \in \mathcal{M}$, denoted $J^\pm(p)$ respectively, is defined to be the set of all points $q \in \mathcal{M}$ such that there exists a future(+)/past(-) directed causal curve $\gamma : [a, b] \to \mathbb{R}$ connecting p and q, i.e. $\gamma(a) = p$ and $\gamma(b) = q$. Similarly, the future/past of a subset $S \subset \mathcal{M}$ of a spacetime $(\mathcal{M}, g)$ is defined as,

$$J^\pm(S) = \bigcup_{x \in S} J^\pm(x). \tag{3.1}$$

A spacetime $(\mathcal{M}, g)$ is *globally hyperbolic* if for all $p, q \in \mathcal{M}$ the set $J^+(p) \cap J^-(q)$ is compact and it satisfies the *causality condition*, i.e. there are no closed causal curves [55, Definition 2.17].[a] Equivalently, a spacetime $(\mathcal{M}, g)$ is globally hyperbolic if it admits smooth foliations into smooth co-dimension one submanifolds $\Sigma \subset \mathcal{M}$ intersected exactly once by every inextensible causal curve; the submanifolds $\{t\} \times \Sigma$ for all $t \in \mathbb{R}$ are called *Cauchy surfaces*. Let $(\mathcal{M}, g)$ be a globally hyperbolic spacetime, then by [4, Theorem 1.1] the spacetime $(\mathcal{M}, g)$ is isometric to the spacetime $(\mathbb{R} \times \Sigma, g = \beta \mathbb{1} \oplus -h)$ where $\beta : \mathbb{R} \times \Sigma \to (0, \infty)$ is a smooth function, $\{t\} \times \Sigma$ is a Cauchy surface for all $t \in \mathbb{R}$ and (Σ, h) is a smooth Riemannian manifold. A particular class of spacetimes used throughout this thesis are *ultrastatic slab spacetimes*. A spacetime $(\mathcal{M}, g)$ is *ultrastatic* if,

$$(\mathcal{M}, g) = (\mathbb{R} \times \Sigma, \pi_1^*(dt) \otimes \pi_1^*(dt) - \pi_2^*(h)), \tag{3.2}$$

[a] The definition 2.17 appearing in [55] defines global hyperbolicity as the compactness of $J^+(p) \cap J^-(q)$ for all $p, q \in \mathcal{M}$ and that $(\mathcal{M}, g)$ is strongly causal. However, the work of Bernal and Sánchez in [5] shows that the strong causality condition is equivalent to the causality condition under the assumption that $J^+(p) \cap J^-(q)$ is compact.

where π_1^*, π_2^* are the pull-backs of the projection maps,

$$\begin{aligned} \pi_1 &: \mathbb{R} \times \Sigma \to \mathbb{R} \\ \pi_2 &: \mathbb{R} \times \Sigma \to \Sigma, \end{aligned} \tag{3.3}$$

respectively and (Σ, h) is a smooth Riemannian manifold. If (Σ, h) is a complete metric space (or if (Σ, h) is compact) then the ultrastatic spacetime (3.2) is globally hyperbolic [43, Proposition 5.2]. An*ultrastatic slab spacetime* is an ultrastatic spacetime of the form $(I \times \Sigma, g = \mathbb{1} \oplus -h)$ where $I \subset \mathbb{R}$ is a relatively compact open interval; in the following we choose, without loss of generality, $I = (-\tau, \tau)$ where $\tau > 0$.

3.2 Wave Equations on Globally Hyperbolic Spacetimes

Let $(\mathcal{M}, g)$ be a globally hyperbolic spacetime and let $P : \Gamma^\infty(\mathcal{M}, F) \to \Gamma^\infty(\mathcal{M}, F)$ be a normally hyperbolic operator. A *wave equation* is an equation of the form $Pu = f$, where f is given and u is to be determined. In this thesis we are interested in wave equations of the form $Pu = 0$. There exist unique maps $\mathbb{E}^\pm : \mathscr{D}(\mathcal{M}, F) \to \Gamma^\infty(\mathcal{M}, F)$ called the advanced(-)/retarded(+) Green's operators that obey [3, Theorem 3.3.1,Corollary 3.4.3],

i) $P \circ \mathbb{E}^\pm = \mathbb{1}_{\mathscr{D}(\mathcal{M},F)}$,

ii) $\mathbb{E}^\pm \circ P\big|_{\mathscr{D}(\mathcal{M},F)} = \mathbb{1}_{\Gamma^\infty(\mathcal{M},F)}$

iii) $\mathrm{supp}(\mathbb{E}^\pm f) \subset J^\pm(\mathrm{supp}(f))$.

Using the retarded and advanced Green's operators $\mathbb{E}^\pm$ one can construct the *advanced-minus-retarded* operator $\mathbb{E} = \mathbb{E}^- - \mathbb{E}^+$, which obeys,

$$P \circ \mathbb{E} = 0 = \mathbb{E} \circ P\big|_{\mathscr{D}(\mathcal{M},F)}. \tag{3.4}$$

One can also construct the *causal propagator* as the map,

$$\begin{aligned} E : \mathscr{D}(\mathcal{M}, F) \times \mathscr{D}(\mathcal{M}, F) &\to \mathbb{C} \\ (f, h) &\mapsto \langle \Gamma f \mid \mathbb{E} h \rangle, \end{aligned} \tag{3.5}$$

where $\Gamma : f \mapsto \overline{f}$ is the complex conjugation map.

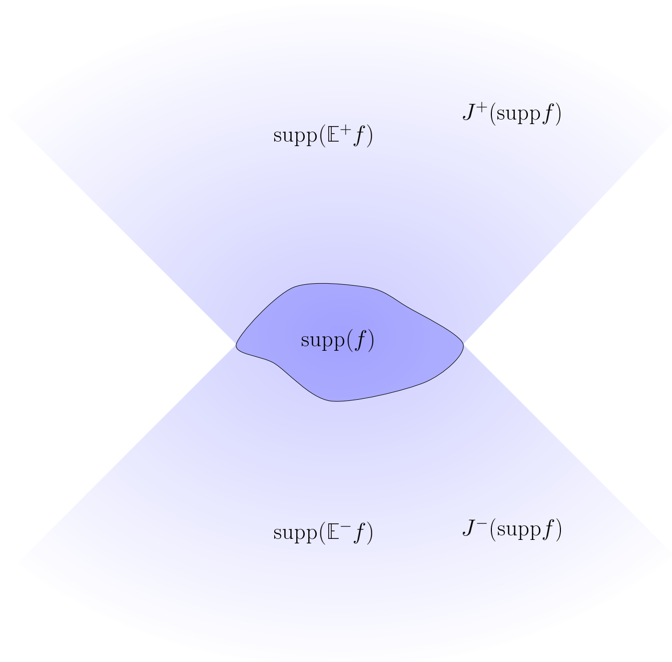


Figure 3.1: Support properties for the advanced/retarded Green's functions $\mathbb{E}^{\pm}$.

3.3 Quantised Scalar Field and The CCR Algebra

Classical Theory

Let $(\mathcal{M}, g)$ be a globally hyperbolic spacetime. The classical free massive scalar field is a function $\phi \in C^\infty(\mathcal{M})$ that obeys the Klein-Gordon equation,

$$(\Box + m^2)\phi = 0, \tag{3.6}$$

where $\Box = \nabla^\alpha \nabla_\alpha$ is the d'Alembertian operator on $(\mathcal{M}, g)$ and $m \geq 0$ is a fixed constant. Since $\mathcal{M}$ is globally hyperbolic and the Klein-Gordon operator $\Box + m^2 : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ is normally hyperbolic, there exist unique advanced(-)/retarded(+) Green's functions,

$$\mathbb{E}^{\pm} : C_0^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}), \tag{3.7}$$

with properties given in Section 3.2. The metric g generates a volume form $\mathbf{d}\mathrm{vol}_g$ which can be used to define a positive definite inner product on the space of test functions $C_0^\infty(\mathcal{M})$ given by,

$$\langle f \mid h \rangle = \int_{\mathcal{M}} \overline{f(t,x)} h(t,x)\ \mathbf{d}\mathrm{vol}_g, \tag{3.8}$$

for all $f, h \in C_0^\infty(\mathcal{M})$.

CCR Algebra of the Free Scalar Field

One starts with a abstract unital $*-$algebra which is generated by the smeared quantum field $\Phi(f)$, labelled by some test function $f \in C_0^\infty(\mathcal{M})$,

$$\mathfrak{A} = \{\Phi(f) \mid f \in C_0^\infty(\mathcal{M})\}. \tag{3.9}$$

The CCR algebra for the free scalar field is a closed, two-sided $*$-ideal of $\mathfrak{A}$ which is generated by taking the quotient of $\mathcal{A}$ with respect to the axioms,

1. $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$ for all $f, g \in C_0^\infty(\mathcal{M})$ and all $\alpha, \beta \in \mathbb{C}$.
2. $\Phi(f)^* = \Phi(\overline{f})$
3. $\Phi(Pg) = 0$ for all $g \in C_0^\infty(\mathcal{M})$ and where $P = \Box + m^2$.
4. $[\Phi(f), \Phi(g)] = i\mathbb{E}(f, g)\mathbb{1}$ where $[\cdot, \cdot]$ is the commutator and $\mathbb{1} \in \mathcal{A}$ is the unit element.

The CCR algebra for the free scalar field is then denoted $\mathfrak{A}(\mathcal{M})$.

Chapter 4

Hadamard States for Quantum Field Theories on Curved Spacetimes

The class of states considered in this thesis are known as 'quasifree' states. In Sections 4.1 we give a brief review of quasifree states over the CCR algebra for the free scalar field and give various characterisation results on proving when a state is quasifree. The work in Section 4.1 largely collects together results appearing in [44, 45, 2, 49]. The work in Section 4.2 collects results appearing in [44, 32, 52]. In section 4.3 we present a new result on the convergence of series whose terms are given by smooth sections; the proof of which is based on Theorem 3.5 in [23].

4.1 Quasifree States

Definition 4.1.1. Let $\mathfrak{A}$ be a unital $*-$algebra over the field of complex numbers $\mathbb{C}$. A *state* is a function $\omega : \mathfrak{A} \to \mathbb{C}$ that satisfies,

- $\omega(\mathcal{A}^*\mathcal{A}) \geq 0$ Positivity.
- $\omega(\alpha\mathcal{A} + \beta\mathcal{A}') = \alpha\omega(\mathcal{A}) + \beta\omega(\mathcal{A}')$ Linearity.
- $\omega(\mathbb{1}) = 1$ Normalisation ,

which hold for all $\mathcal{A}, \mathcal{A}' \in \mathfrak{A}$, all $\alpha, \beta \in \mathbb{C}$ and where $\mathcal{A}^*$ denotes the $*-$adjoint of $\mathcal{A}$ and $\mathbb{1} \in \mathfrak{A}$ denotes the identity.

Definition 4.1.2. Let $\omega : \mathfrak{A}(\mathcal{M}) \to \mathbb{C}$ be a state over the CCR algebra $\mathcal{A}(\mathfrak{M})$. The state ω is *quasifree* if all $n-$point functions can be written as,

$$W_\omega^{(2n-1)}(f_1 \otimes ... \otimes f_{2n-1}) = 0 \tag{4.1}$$

$$W_\omega^{(2n)}(f_1 \otimes ... \otimes f_{2n}) = \sum_\sigma \prod_{j=1}^{n} W_\omega^{(2)}(f_{\sigma(j)} \otimes f_{\sigma(j+n)}), \tag{4.2}$$

where the sum runs over all σ permutations of $\{1, 2, 3, ..., 2n\}$ such that $\sigma(1) < \sigma(2) < ... < \sigma(n)$ and $\sigma(j) < \sigma(j+n)$ for all $j = 1, ..., n$.

Therefore, a quasifree state is completely determined by its two-point function, since all odd $n-$point functions vanish and all even $n-$point functions can be constructed using (4.2). There are some useful characterizations of quasifree states, to which we refer to the work appearing in [45, 49]. Let $P : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ be a formally self adjoint normally hyperbolic operator over a globally hyperbolic spacetime $(\mathcal{M}, g)$ and let $E : C_0^\infty(\mathcal{M}) \times C_0^\infty(\mathcal{M}) \to \mathbb{R}$ be the causal propagator. One observes that if $\mathbb{E}(f - f') = 0$ then $E(f, g) = E(f', g)$ and if $\mathbb{E}(g - g') = 0$ then $E(f, g) = E(f, g')$. Therefore, one can define the real linear vector space,

$$S(\mathcal{M}) = C_0^\infty(\mathcal{M}, \mathbb{R})/P(C_0^\infty(\mathcal{M})), \tag{4.3}$$

of equivalence classes $[f]$ with respect to the equivalence relation,

$$f \sim f' \iff \exists h \in C_0^\infty(\mathcal{M}) \text{ such that } f - f' = Ph \tag{4.4}$$

for all $f, f' \in C_0^\infty(\mathcal{M}, \mathbb{R})$. By [3, Theorem 5.2.1], $\ker(\mathbb{E}) = \mathrm{Im}(P)$ the above equivalence relation is equivalent to,

$$f \sim f' \iff \mathbb{E}(f - f') = 0. \tag{4.5}$$

One may then define,

$$\begin{aligned} \sigma : S(\mathcal{M}) \times S(\mathcal{M}) &\to \mathbb{R} \\ ([f], [g]) &\mapsto E(f, g), \end{aligned} \tag{4.6}$$

which is a well defined symplectic form [45, Proposition 8]. The following result for quasifree states can be found in [45, Theorem 2],

Proposition 4.1. *Let $P = \Box + m^2$ be the Klein Gordon operator and $\mathfrak{A}(\mathcal{M})$ be the algebra of observables for the free massive scalar field. If $\omega : (\mathcal{M}) \to \mathbb{C}$ is a quasifree state then the two point function W_ω satisfies the following,*

i) $W_\omega(f,g) - W_\omega(g,f) = iE(f,g)$

ii) $W_\omega(f,Pg) = W_\omega(Pf,g) = 0,$

iii) $W_\omega(\overline{f},f) \geq 0,$

iv) $Im(W_\omega(f,g)) = \frac{1}{2}E(f,g),$

for all $f,g \in C_0^\infty(\mathcal{M})$.

Proof. i) Since the state ω is normalised and linear, and by the canonical commutation relations we have,

$$\begin{aligned} W_\omega(f,g) - W_\omega(g,f) &= \omega(\phi(f)\phi(g)) - \omega(\phi(g)\phi(f)) \\ &= \omega([\phi(f),\phi(g)]) \\ &= \omega(i\mathbb{E}(f,g)\mathbb{1}) \\ &= i\mathbb{E}(f,g)(\omega(\mathbb{1}) \\ &= i\mathbb{E}(f,g). \end{aligned} \tag{4.7}$$

ii) $W_\omega(f,Pg) = \omega(\phi(f)\phi(Pg)) = 0$ and $W_\omega(Pf,g) = \omega(\phi(Pf)\phi(g)) = 0$ which holds by definition.

iii) Since the state ω is positive, we have, $W_\omega(\overline{f},f) = \omega(\phi(f)^*\phi(f)) \geq 0$.

iv) Follows immediately from the fact that $E(f,g)$ is real.

■

A sufficient condition for a state to be quasifree is given in the following proposition,[a]

Proposition 4.2. *Let W_ω be the two point function given by,*

$$W_\omega(f,g) = \mu_\omega([f],[g]) + \frac{i}{2}\sigma([f],[g]), \tag{4.8}$$

[a]The author would like to thank Chris Fewster for providing this statement and outlining its proof.

where $\mu_\omega : S(\mathcal{M}) \times S(\mathcal{M}) \to \mathbb{R}$ *is a well defined real scalar product and* σ *is defined in* (4.6)*. If there exists a non-negative bilinear form given by,*

$$\begin{aligned} \nu : S(\mathcal{M}) \times S(\mathcal{M}) \to \mathbb{C} \\ \nu(\mathbb{E}f, \mathbb{E}g) = W_\omega(\overline{f}, g), \end{aligned} \tag{4.9}$$

then there exists a quasifree state ω *whose two point function is given by* (4.8)

Proof. By writing the two point function in terms of the symplectic form σ and the scalar product μ_ω as in (4.8), we see that the form defined in (4.9) is clearly bilinear. Now, let $\phi, \psi \in S(\mathcal{M})$ be chosen such that $\phi = \mathbb{E}f$ and $\psi = \mathbb{E}g$ for some $f, g \in C_0^\infty(\mathcal{M}, \mathbb{R})$. The non-negativity of the bilinear form (4.9) then implies,

$$\begin{aligned} 0 \le \nu(\overline{\phi + i\lambda\psi}, \phi + i\lambda\psi) &= W_\omega(f,f) + i\lambda(W_\omega(f,g) - W_\omega(g,f)) + \lambda^2 W_\omega(g,g) \\ &= \mu_\omega([f],[f]) - \lambda\sigma([f],[g]) + \lambda^2\mu_\omega([g],[g]) \end{aligned} \tag{4.10}$$

where we have used $W_\omega(f,f) = \mu_\omega([f],[g])$ and $W_\omega(f,g) - W_\omega(g,f) = i\sigma([f],[g])$. If the inequality (4.10) is to hold for all $\lambda \in \mathbb{R}$, then the discriminant must be at most zero. Requiring the discriminant of (4.10) to be less than or equal to zero then yields,

$$|\sigma([f],[g])|^2 \le 4\mu_\omega([f],[f])\mu_\omega([g],[g]). \tag{4.11}$$

Then, there exists a quasifree state ω with the two point function (4.8) [44, 49]. ■

A quasifree state ω over an algebra $\mathcal{A}$ is *pure* if it is extremal in the convex set of all states, i.e. one can not decompose a pure state as the sum of two other states with positive coefficients. If a state is not pure, then it is said to be *mixed*.

Theorem 4.3. *Let* W_ω *be the two point function for a quasifree state* ω *over* $\mathcal{A}(\mathcal{M})$ *given by* (4.8)

$$W_\omega = \mu_\omega([f],[g]) + \frac{i}{2}\sigma([f],[g]). \tag{4.12}$$

The the state ω *is pure if and only if,*

$$\mu_\omega([f],[f]) = \sup_{0 \ne [g] \in S(\mathcal{M})} \frac{|\sigma([f],[g])|^2}{4\mu_\omega([g],[g])}, \tag{4.13}$$

Proof. Follows from Proposition 3.1 and from the discussion on page 77 in [44]. ■

For a given μ_ω that satisfies (4.11) on can form the completion of $S(\mathcal{M})$ given in (4.3) to form real linear Hilbert space. An equivalent description of pure states is given by the following theorem:

Proposition 4.4. *Let ω be a quasifree state with two point function* (4.12). *The state ω is pure if and only if there exists a bounded self-adjoint operator $R_\mu : S(\mathcal{M}) \to S(\mathcal{M})$ such that $R_\mu^2 = -\mathbb{1}$ and,*

$$\sigma([f],[g]) = 2\mu_\omega([f], R_\mu[g]), \tag{4.14}$$

for all $[f],[g] \in S(\mathcal{M})$.

Proof. See [45, Theorem 2]. ∎

4.2 Hadamard States

Among all the states over the CCR algebra, the class of Hadamard states are states with physically permissible properties, such as having finite expectation values of the stress energy tensor. The Hadamard condition is a criterion on the singular part of the integral kernel of the two-point function. The condition stems from the spectral condition in Minkowski spacetime; it is the requirement that all singularities of the two-point function lie along the future null cone. A rigorous definition of a Hadmard state was first given by Kay and Wald in [44]. A state ω with two-point function W_ω is *Hadamard* if and only if globally, there are no singularities at spacelike separations and for every convex normal neighbourhood $S \subset \mathcal{M}$ the two-point function is of the form, for all $N \in \mathbb{N}$,

$$W_\omega(x,y) = \lim_{\epsilon \to 0^+} \frac{1}{(2\pi)^2}\left[\frac{\Delta^{\frac{1}{2}}}{\sigma_\epsilon(x,y)} + \left(\sum_{n=0}^{N} v_n \sigma_\epsilon(x,y)^n\right)\ln(\sigma_\epsilon(x,y))\right] + R_{N,\omega}(x,y), \tag{4.15}$$

where $R_{N,\omega} \in C^N(\mathcal{M}\times\mathcal{M})$ depends on the state ω,

$$\sigma_\epsilon(x,y) = \sigma(x,y) + 2i\epsilon(T(x) - T(y)) + \epsilon^2, \tag{4.16}$$

where $\sigma(x,y)$ is the (signed) squared geodesic distance between x and y given by,

$$\sigma(x,y) = \pm\left(\int_a^b \left| g_{\mu\nu}(x(\tau))\frac{dx^\mu(\tau)}{d\tau}\frac{dx^\nu(\tau)}{d\tau}\right|^{\frac{1}{2}} d\tau\right)^2, \tag{4.17}$$

where $x(\cdot)$ is a parametrisation of the unique geodesic in S from x to y and the $+/-$ is chosen according to whether $x(\cdot)$ is spacelike or timelike respectively and T is a global time function. Here Δ is the Vleck-Morette determinant and v_n are determined by the Hadamard recursion relations [15, 36].

It is remarkable that the term in the parenthesis in (4.15) is entirely geometric. Furthermore, the term in the parenthesis is independent of the state ω; the state dependence is contained entirely in the function $R_{N,\omega}$. Therefore, if the state ω_H with two-point function W_H is a Hadamard state, one can show that a state ω with two-point function W_ω is Hadamard by showing that,

$$W_\omega - W_H \in C^\infty(\mathcal{M} \times \mathcal{M}). \tag{4.18}$$

Of course, this method of showing that the state ω is Hadamard requires, at the very least, that the Hadamard state ω_H exists. Fortunately, Hadamard states are known to exist in globally hyperbolic spacetimes, which is due to a deformation argument by Fulling, Narcowich and Wald [32, Theorem 2.1], which states the following:

Theorem 4.5. *In an arbitrary globally hyperbolic spacetime $(\mathcal{M}, g)$ there exists a class of states which form a dense subspace of a Hilbert space, whose two-point function has a kernel with singular structure* (4.15).

In order to prove this result, one needs to be able to deform the geometry of various globally hyperbolic spacetimes. Let $(\mathcal{M}_1, g_1)$ and $(\mathcal{M}_2, g_2)$ be two globally hyperbolic spacetimes with spacelike Cauchy surfaces Σ_1 and Σ_2 respectively. One can *smoothly deform the spacetimes* $(\mathcal{M}_1, g_1, \Sigma_1)$ *into* $(\mathcal{M}_2, g_2, \Sigma_2)$ *while preserving global hyperbolicity* if there exists a globally hyperbolic spacetime $(\mathbb{R} \times \Sigma, g)$ which admits the spacelike Cauchy surfaces S_1 and S_2 such that, for $i = 1, 2$, S_i is isometric to Σ_i and a neighbourhood of S_i is isometric to a neighbourhood of Σ_i. Keeping the notation, Fulling et al are then able to prove the following result,

Proposition 4.6. $(\mathcal{M}_1, g_1, \Sigma_1)$ *can be smooth deformed into* $(\mathcal{M}_2, g_2, S_2)$ *while preserving global hyperbolicity.*

Proof. See [33, Proposition C.1]. ■

It was shown in [33] that if a state ω_H satisfies the Hadamard condition in a open neighbourhood of a Cauchy surface Σ in a globally hyperbolic spacetime $(\mathcal{M}, g)$, then it does so globally, i.e. Cauchy evolution preserves the Hadamard singularity structure of the two-point function.

Suppose a globally hyperbolic spacetime $(\mathcal{M}, g)$ is ultrastatic to the past of a Cauchy surface Σ. Since the ultrastatic vacuum has the Hadamard singularity structure (4.15) on a neighbourhood of the cauchy Surface Σ [33], the ultrastatic vacuum with be Hadamard throughout $(\mathcal{M}, g)$.

Now, let $\mathcal{N} = (\mathbb{R} \times \Sigma, g)$ be a globally hyperbolic spacetime. One deforms the spacetime $\mathcal{N}$ to the past of some Cauchy surface S to a spacetime so that it becomes ultrastatic in the past whilst preserving the global hyperbolicity of the deformed spacetime. Let $\widetilde{\mathcal{N}}$ be the deformed spacetime. Then, in the $\widetilde{\mathcal{N}}$ there exist Hadamard states since there is portion which is ultrastatic. One then deforms the geometry of $\tilde{\mathcal{N}}$ back to the spacetime $\mathcal{N}$, and the state is then Hadamard on the whole of the spacetime $\mathcal{N}$.

Whilst it is certainly reassuring that Hadamard states exist in general, the deformation argument is somewhat indirect, and does not give any way of explicitly constructing Hadamard states. This is, in part, the subject of this thesis. This thesis will concern itself with developing new constructions for Hadamard states principally over ultrastatic spacetimes, although much of our results can be generalised to an arbitrary globally hyperbolic spacetime. Since we are mainly interested in constructing Hadamard states on ultrastatic spacetimes, we shall extensively use the ultrastatic vacuum to show that our states satisfy the Hadamard condition. A common method to verify that a state ω is Hadamard on an ultrastatic spacetime is to show that the integral kernel of $W_\omega - W_H$ is smooth, where W_ω is the two-point function of the state ω and W_H is the two-point function for the ultrastatic vacuum. In light of this, in Section 4.3 we shall give a somewhat technical result which will be used extensively throughout the thesis

An equivalent definition of a Hadamard state is that the wavefront set of the two-point function satisfies the *wavefront set spectral condition*. The equivalence was proved by Radzikowski in [52, Theorem 5.1]. In the language of microlocal analysis, a state ω is said to be a *Hadamard state* if the wavefront set of its two-point function ω_2 is given by,

$$WF(\omega_2) = \{(x_1, k_1; x_2, -k_2) \mid (x_1, k_1; x_2, k_2) \in T^*(\mathcal{M} \times \mathcal{M}) \setminus \{0\} \ , \ (x_1, k_1) \sim (x_2, k_2) \ , \ k_1 \in \overline{V_+}\}.$$

Under the equivalence relation $(x_1, k_1) \sim (x_2, k_2)$ two curves are equivalent if there exists a null geodesic with endpoints x_1 and x_2. This geodesic has the cotangent k_1 and k_2 is the vector k_1 parallel transported along the geodesic at x_2. Here, $\overline{V_+}$ is the closure of the forward light cone of $T^*_{x_1}\mathcal{M}$.

4.3 Summing Smooth Sections: A Convergence Result

The main result of this section is Theorem 4.7, which we use extensively to prove that a constructed state satisfies the Hadamard condition or that an operator has a smooth integral kernel. Theorem 4.7

is a relatively straight forward generalisation of the result appearing in [23, Theorem 3.5].

Definition 4.3.1. Let P be a partial differential operator with eigenvalues $\{\omega_j\}_{j\in\mathbb{N}}$ such that $\omega_j \geq 0$ for all $j \in \mathbb{N}$. Suppose that the eigenvalues have finite multiplicity and if $j \leq k \in \mathbb{N}$ then $\omega_j \leq \omega_k$. The *counting function* for P is given by,

$$N(\omega) := \#\{\omega_k \leq \omega \mid \omega_k \text{ is an eigenvalue of } P\}. \tag{4.19}$$

Furthermore, we define the quantity,

$$M(j) := \max\{k \in \mathbb{N} \mid \omega_k = \omega_j\}, \tag{4.20}$$

which exists due to the finite multiplicity of the eigenvalues $\{\omega_j\}_{j\in\mathbb{N}}$.

The following result follows the same strategy as the proof found in [23, Theorem 3.5].

Theorem 4.7. *Let Σ be a compact Riemannian manifold, $I \subset \mathbb{R}$ be a relatively compact interval and define the projection π_Σ by,*

$$\begin{aligned} \pi_\Sigma : I \times \Sigma &\to \Sigma \\ (t, \sigma) &\mapsto \sigma. \end{aligned} \tag{4.21}$$

Let F be a smooth $\mathbb{C}-$vector bundle over Σ with a hermitian metric and a compatible connection. Consider a second order positive elliptic partial differential operator L over $\Gamma^\infty(\Sigma; F)$ which induces a self-adjoint operator on $L^2(\Sigma; F)$ with a discrete spectrum. Let $\{u_n\}_{n\in\mathbb{N}}$ be a orthonormal set in $L^2(\Sigma; F)$ such that each $u_n \in L^2(\Sigma; F)\cap\Gamma^\infty(\Sigma; F)$ is an eigenvector of L with corresponding eigenvalue $\omega_n^2 > 0$.

Suppose, for all $n \in \mathbb{N}$, $W_n \in \Gamma^\infty(\mathcal{M} \times \mathcal{M}; \pi_\Sigma^* F \boxtimes \pi_\Sigma^* F)$ is of the form,

$$W_n = \sum_{\sigma,\sigma'\in\{\pm1\}} \alpha_n^{\sigma\sigma'} (e_{n\sigma} \otimes u_n) \otimes (e_{n\sigma'} \otimes u_n), \tag{4.22}$$

where $e_{n\sigma}(t) = e^{i\sigma\omega_n t}$ and $\alpha_n^{\sigma\sigma'} \in \mathbb{C}$ for all $n \in \mathbb{N}$. Suppose further that,

$$\sum_{n\in\mathbb{N}} \omega_n^p \max_{\sigma,\sigma'\in\{\pm1\}} |\alpha_n^{\sigma\sigma'}|^2 < \infty, \tag{4.23}$$

for all $p \in \mathbb{N}_0$. Then $\sum_{n\in\mathbb{N}} W_n$ converges in $\Gamma^\infty(\mathcal{M} \times \mathcal{M}; \pi_\Sigma^* F \boxtimes \pi_\Sigma^* F)$.

Proof. Without loss of generality let $I = (-\tau, \tau)$ for $\tau > 0$. Consider the smooth embedding,

$$\begin{aligned} \psi = j \times \mathrm{id}_\Sigma : I \times \Sigma &\hookrightarrow S^1 \times \Sigma \doteq X \\ (t, x) &\mapsto (e^{i\pi(t-\tau)/\tau}, x), \end{aligned} \tag{4.24}$$

of the manifold $I \times \Sigma$ into the compact Riemannian manifold $S^1 \times \Sigma$ with metric $\mathbb{1} \oplus h$. Here, we have identified S^1 as the unit circle in $\mathbb{C}$. We have the canonical projection $\widetilde{\pi}_\Sigma : S^1 \times \Sigma \to \Sigma$, $(t, \sigma) \mapsto \sigma$. Fix $\chi \in C_0^\infty(I)$ and define $W_{\chi,n} \in \Gamma^\infty(X \times X; \widetilde{\pi}_\Sigma^* F \boxtimes \widetilde{\pi}_\Sigma^* F)$ by,

$$W_{\chi,n} \doteq (\psi \times \psi)_*((\chi \otimes \chi) W_n), \tag{4.25}$$

where ψ_* denotes the pushforward of ψ. If $\sum_n W_{\chi,n}$ converges in $\Gamma^\infty(X \times X; \widetilde{\pi}_\Sigma^* F \boxtimes \widetilde{\pi}_\Sigma^* F)$ for arbitrary $\chi \in C^\infty(I)$ then $\sum_n W_n$ converges in $\Gamma^\infty(\mathcal{M} \times \mathcal{M}; \pi_\Sigma^* F \boxtimes \pi_\Sigma^* F)$. Since L is assumed to be a second order positive elliptic operator, we may introduce a second order elliptic operator over the space of smooth sections $\Gamma^\infty(X, \widetilde{\pi}_\Sigma^* F)$,

$$\begin{aligned} D : \Gamma^\infty(X, \widetilde{\pi}_\Sigma^* F) &\to \Gamma^\infty(X, \widetilde{\pi}_\Sigma^* F) \\ f &\mapsto (-\partial_t^2 \otimes \mathbb{1}_\Sigma + \mathbb{1}_{S^1} \otimes L) f \end{aligned} \tag{4.26}$$

where $\mathbb{1}_{S^1}, \mathbb{1}_\Sigma$ are identities on S^1, Σ respectively. We now endow the space of smooth sections over the bundle $\widetilde{\pi}_\Sigma^* F \to X$ with the inner product,

$$\langle f \mid g \rangle_{X,s} \doteq \langle f \mid (\mathbb{1} + D^s) g \rangle_{X,0} \tag{4.27}$$

for all $f, g \in \Gamma^\infty(X, \widetilde{\pi}_\Sigma^* F)$ and for all $s \in \mathbb{N}_0$ and where $\langle \cdot \mid \cdot \rangle_{X,0}$ is the fibre wise inner product over the bundles $\widetilde{\pi}_\Sigma^* \to X$ given in definition 2.3.11. For all even numbers $2s \in \mathbb{N}_0$ the inner product (4.27) induces the norm,

$$\|f\|_{X,2s}^2 = \|f\|_{X,0}^2 + \|D^s f\|_{X,0}^2 \tag{4.28}$$

whose topology can be used to complete $\Gamma^\infty(X, \widetilde{\pi}_\Sigma^* F)$ which yields the Sobolev spaces denoted by,

$$L_{2s}^2(X; \widetilde{\pi}_\Sigma^* F) = \overline{\Gamma^\infty(X; \widetilde{\pi}_\Sigma^* F)}^{\|\cdot\|_{X,2s}} \tag{4.29}$$

for all $2s \in \mathbb{N}_0$. We make the identification,

$$L_{2s}^2(X, \widetilde{\pi}_\Sigma^* F) \simeq L_{2s}^2(S^1) \otimes L_{2s}^2(\Sigma, F). \tag{4.30}$$

Since the differential operator (4.26) is elliptic and second order, the Sobolev space (4.29) are equivalent to the space formed by completing $\Gamma^\infty(X;\widetilde{\pi}_\Sigma^* F)$ with respect to the basic Sobolev s-norm (2.25). This is a consequence of Theorem 2.6. We shall also define a norm on the space of smooth sections $\Gamma^\infty(X \times X,;\widetilde{\pi}_\Sigma^* F \boxtimes \widetilde{\pi}_\Sigma^* F)$ defined by,

$$\|f \otimes g\|_{X\times X,2s} = \|f\|_{X,0}^2 \|g\|_{X,0}^2 + \|(\mathbb{1}_X \otimes D^s + D^s \otimes \mathbb{1}_X) f \otimes g\|_{X\times X,0}^2, \tag{4.31}$$

for all even $2s \in \mathbb{N}_0$. The following lemma gives us a useful bound;

Lemma 4.8. *For each $s \in \mathbb{N}_0$, we have the estimate,*

$$\|f \otimes g\|_{X\times X,2s}^2 \leq 2\|f\|_{X,2s}^2 \|g\|_{X,2s}^2, \tag{4.32}$$

for all $f, g \in L^2_{2s}(X;\widetilde{\pi}_\Sigma^* F)$.

Proof. The proof of Lemma 3.8 in [23] is sufficiently similar to the proof required here. ■

Continuation of the proof of Theorem 4.7: Consider now the section $\psi_*(\chi e_{n\sigma} \otimes u_n) \in \Gamma^\infty(X, \widetilde{\pi}_\Sigma^* F)$ for a fixed but arbitrary $\chi \in C_0^\infty(I)$. Observe that, for all $s \in \mathbb{N}_0$,

$$D^s(\psi_*(\chi e_{n\sigma} \otimes u_n)) = \big((-\partial_t^2 + \omega_n^2)^s (j_*(\chi e_{n\sigma})\big) \otimes u_n \doteq L_{\omega_n,s}(j_*(\chi e_{n\sigma})) \otimes u_n, \tag{4.33}$$

where $L_{\omega_n,s}$ is a differential operator on $C^\infty(S^1)$ of order $2s$ with coefficients depending polynomially on ω_n and is given by,

$$L_{\omega_n,s} = \sum_{k=0}^{s} \binom{s}{k} \omega_n^{2k} (-1)^{s-k} \partial_t^{2(s-k)}. \tag{4.34}$$

In the following we implicitly assume that the choices of $\sigma, \sigma' \in \{\pm 1\}$ are independent and arbitrary. By Lemma 4.8 we have,

$$\|\psi_*(\chi e_{n\sigma} \otimes u_n) \otimes \psi_*(\chi e_{n\sigma'} \otimes u_n)\|_{X\times X,2s}^2 \leq 2\|\psi_*(\chi e_{n\sigma} \otimes u_n)\|_{X,2s}^2 \|\psi_*(\chi e_{n\sigma'} \otimes u_n)\|_{X,2s}^2. \tag{4.35}$$

Using (4.33) and the decomposition (4.30) we then obtain,

$$\|\psi_*(\chi e_{n\sigma} \otimes u_n)\|_{X,2s}^2 = \big(\|j_*(\chi e_{n\sigma})\|_{S^1,0}^2 + \|L_{\omega_n,s}(j_*(\chi e_{n\sigma}))\|_{S^1,0}^2\big) \|u_n\|_{\Sigma,0}^2 \tag{4.36}$$

Since $L_{\omega_n,s}$ is a differential operator of order $2s$ whose coefficients depend polynomially on ω_n, it follows that $\|L_{\omega_n,s}(j_*(\chi e_{n,\sigma}))\|_{S^1,0}^2$ defines a polynomial of degree $4s$. Therefore, there exists a $C_{\sigma,2s} > 0$ to be sufficiently large such that,

$$\big(\|j_*(\chi e_{n\sigma})\|_{S^1,0}^2 + \|L_{\omega_n,s}(\chi e_{n\sigma})\|_{S^1,0}^2\big) \|u_n\|_{\Sigma,0}^2 \leq C_{\sigma,2s}\big(1 + \omega_n^{4s}\big), \tag{4.37}$$

whence we immediately obtain,

$$\|\psi_*(\chi e_{n\sigma} \otimes u_n) \otimes \psi_*(\chi e_{n\sigma'} \otimes u_n)\|^2_{X\times X,2s} \leq 2C_{2s}(1+\omega_n^{4s})^2, \tag{4.38}$$

where $C_s = \max_{\sigma,\sigma' \in \{\pm 1\}} C_{\sigma s} C_{\sigma' s}$. Finally, by the Cauchy criterion and Pythagoras' theorem and since u_n are orthonormal, we obtain,

$$\begin{aligned} \|\sum_n W_{\chi,n}\|^2_{X\times X,2s} &= \sum_n \|W_{\chi,n}\|^2_{X\times X,2s} \\ &= \sum_n \sum_{\sigma,\sigma'\in\{\pm1\}} |\alpha_n^{\sigma\sigma'}|^2 \|\psi_*(\chi e_{n\sigma} \otimes u_n) \otimes \psi_*(\chi e_{n\sigma'} \otimes u_n)\|^2_{X\times X,2s} \\ &\leq 2C_{2s} \sum_n \max_{\sigma,\sigma'\in\{\pm1\}} |\alpha_n^{\sigma\sigma'}|^2 (1+\omega_n^{4s})^2 < \infty. \end{aligned} \tag{4.39}$$

Since $\chi \in C_0^\infty(I)$ was arbitrary, we infer that $\sum_n W_n$ converges in $L^2_{2s}(\mathcal{M}\times\mathcal{M}; \pi_\Sigma^* F \boxtimes \pi_\Sigma^* F)$ for all $s \in \mathbb{N}_0$, hence, by the Sobolev embedding theorem, $\sum_n W_n$ converges in $\Gamma^\infty(\mathcal{M}\times\mathcal{M}; \pi_\Sigma^* F \boxtimes \pi_\Sigma^* F)$ [46, Theorem 2.5]. ■

Chapter 5

The Distinguished Sorkin-Johnston State

The Sorkin-Johnston state originated from the causal set approach to quantum gravity. Whilst we give a brief review of the SJ state in the causal set theory framework, our main focus will be on the SJ state for linear quantum field theories on continuum spacetimes. To that end, this chapter will review various properties of the SJ state and the various equivalent methods for constructing the SJ state.

5.1 Causal Set Theory

Causal set theory is a novel approach to developing a quantum theory of gravity [7, 63, 38]. It is based on the idea that, at the Planck scale, spacetime is fundamentally discrete. The path-integral from quantum field theory, causal ordering from general relativity and the discrete spacetime structure form the basis of causal set theory. Quantum field theories defined over causal sets may present a way to regulate the divergences in QFT and the singularities present in general relativity. A causal set is a locally finite partially ordered set $(\mathcal{C}, \preceq)$ where the ordering $\preceq$ is a partial ordering such that, for all $u, v, w \in \mathcal{C}$,

i $u \preceq u$,

ii if $u \preceq v$ and $v \preceq w$ then $u \preceq w$, iii if $u \preceq v$ and $v \preceq u$ then $u = v$,

iv 'locally finite' in the sense that if $u, v \in \mathcal{C}$ are fixed then $|\{w' \in \mathcal{C} \mid u \preceq w' \preceq v\}| < \infty$.

The relation $\preceq$ imposes a causal structure on $\mathcal{C}$; by $x \preceq y$ we mean that x is in the causal past of y, for each $x, y \in \mathcal{C}$. The causal past/future of an element $u \in \mathcal{C}$ is defined as,

$$\begin{aligned} \text{Past}(u) &= \{v \mid v \prec u\} \\ \text{Fut}(u) &= \{v \mid u \prec v\}, \end{aligned} \tag{5.1}$$

respectively. Additionally, we define the relation $x \prec y$ as $x \preceq y$ and $x \neq y$. In order to develop a quantum field theory over a causal set, there are some elementary quantities we need to define. A *link* is a description of an elements nearest neighbour in a causal set. An element $u \in \mathcal{C}$ is linked to $v \in \mathcal{C}$ if $u \prec v$ and there does not exist an element $w \in \mathcal{C}$ such that $u \prec w \prec v$. We write $u \prec_* v$ if $u \in \mathcal{C}$ is linked to $v \in \mathcal{C}$. A *chain* in a causal set $\mathcal{C}$ is a subset $S \subset \mathcal{C}$ for which each pair of elements is related by the relation $\prec$. An *antichain* is a subset $A \subset \mathcal{C}$ in which no pair of elements are related by $\prec$. For a causal set with $p \in \mathbb{N}$ elements, we label the elements $v_1, v_2, ..., v_p$. There are two $p \times p$ adjacency matrices, the *causal matrix* C defined by,

$$C_{mn} \doteq \begin{cases} 1 \text{ if } v_m \prec v_n \\ 0 \text{ otherwise} \end{cases} \tag{5.2}$$

and the *link matrix* L defined by,

$$L_{mn} \doteq \begin{cases} 1 \text{ if } v_m \prec_* \ v_n \\ 0 \text{ otherwise.} \end{cases} \tag{5.3}$$

A *linear extension* of a causal set $(\mathcal{C}, \preceq)$ is a totally ordered set $(\mathcal{C}, \leq)$ which is consistent with the partial ordering, i.e. if $u \preceq v$ then $u \leq v$. A typical method used to compare results for quantum field theories over causal sets and continuum spacetimes is to generate a causal set by 'sprinkling' points into a continuum spacetime. Let $(\mathcal{M}, g)$ be a Lorentzian manifold of dimension n. The causal set $(\mathcal{C}, \preceq)$ is generated by randomly distributing points over $(\mathcal{M}, g)$ using a Possion process with density ρ, which entails that n points are randomly distributed throughout a d-dimensional volume V with probability P given by,

$$\text{Prob(n points in V)} = \frac{(\rho V)^n}{n!} e^{-\rho V}. \tag{5.4}$$

The average numbers of points generated by Possion density with density ρ in some region $R \subset \mathcal{M}$ with volumes V is then equal to ρV. The causal relation $\preceq$ for the causal set $(\mathcal{C}, \preceq)$ is obtained from the Lorentzian spacetime by restriction. One regains the original Lorentzian manifold in the limit $\rho \to \infty$. If the Lorentzian spacetime is, for example, globally hyperbolic, then the causal set is a good approximation of the spactime. One may endow a causal set with a topology by using 'thickened antichains', which can then be used to recover the topology of a globally hyperbolic spacetime from a causal set that faithfully embeds into it at a sufficiently high sprinkling density [48]. An antichain A is 'thickened' by including elements that are in its neighbourhood. A (future volume) *thickend antichain* thickened by a volume v is defined as,

$$T_v(A) \doteq \{u \mid u \in \mathrm{Fut}(A) \cup A \text{ and } |\mathrm{Past}(x) \setminus \mathrm{Past}(A)| \leq v\}, \tag{5.5}$$

where A is an antichain. A past volume thickened antichain is defined similarly. It is hoped that causal sets which approximate the Lorentzian manifolds of general relativity are an emergent feature of causal set theory, that they are selected to be the 'physical' causal sets. A step in this direction is the work done by Gudder [35]. For sufficiently large ρ, one should expect theories defined over causal sets generated by sprinkling to approximate theories over the corresponding continuum spacetime. A schematic picture of sprinkling is shown in figure 5.1.

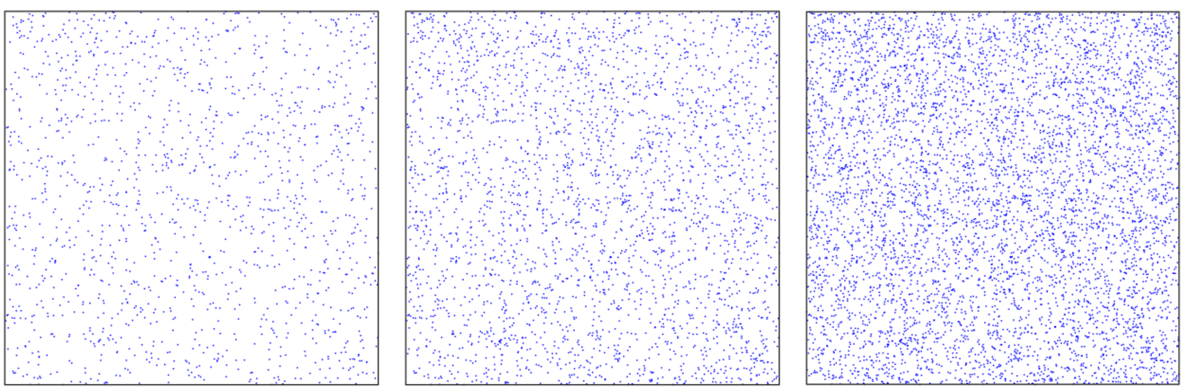


Figure 5.1: Sprinkling over a bounded region of $\mathbb{M}^2$ with the sprinkling density ρ increasing from left to right.

Causal sets generated by sprinkling over Lorentzian spacetimes are good models to use to compare results in the discrete and continuum cases. Of interest to us is the construction of the causal retarded propagator for a massive free scalar field over a causal set generated by sprinkling over Minkowski spacetime [41]. The retarded propagator for a real scalar field of mass m over $d-$dimensional (continuum) Minkowski spacetime $(\mathbb{M}^d, \eta)$, denoted $K_m^{(d)}$, is a Green's function to the Klein-Gordon operator,

$$(\Box + m^2)K_m^{(d)}(x-y) = \delta^d(x-y), \tag{5.6}$$

for all $x, y \in \mathbb{M}^d$ and where δ^d is the $d-$dimensional Dirac delta function and the d'Alembertian is given by,

$$\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - .... - \frac{\partial^2}{\partial x_{d-1}^2}. \tag{5.7}$$

The equation (5.6) can be solved via a Fourier transform, which gives,

$$K_m^d(x-y) \doteq -\frac{1}{(2\pi)^d}\int \mathbf{d}^d p \frac{e^{-ip(x-y)}}{p_0^2 - \underline{p}^2 - m^2}, \tag{5.8}$$

where $p_\mu = (p_0, \underline{p})$ is the four-momentum. The retarded causal propagator is the Greens function (5.8) obtained by avoiding the poles $p_0 = \sqrt{\underline{p}^2 + m^2}$ by two semi-circles in the upper half complex plane. Johnston constructs the retarded causal propagator for a massive free scalar field over causal sets $\mathbb{M}^2$ and $\mathbb{M}^4$ which are generated by sprinkling points over a $1+1$-dimensional and $1+3$-dimensional Minkowski space respectively. Let $(\mathbb{M}^d, \eta)$ be a $d-$dimensional Minkowski spacetime. The $d-$dimensional causal set $(\mathcal{C}^d, \preceq)$ is constructed by sprinkling $p \in \mathbb{N}$ points into $d-$dimensional Minkowski spacetime $(\mathbb{M}^d, \eta)$ with denstiy ρ. The causal retarded propagators for a free massive scalar field for sprinklings into $\mathbb{M}^2$ and $\mathbb{M}^4$ are given by the $p \times p$ matrices,

$$K_R^{(2)} \doteq aC(\mathbb{1} - abC)^{-1} \qquad a = \frac{1}{2}, \;\; b = -\frac{m^2}{\rho} \tag{5.9}$$

and,

$$K_R^{(4)} \doteq aL(\mathbb{1} - abL)^{-1} \qquad a = \frac{\sqrt{\rho}}{2\pi\sqrt{6}}, \;\; b = -\frac{m^2}{\rho}, \tag{5.10}$$

respectively [41], where a, b are chosen to give the correct amplitudes. In each case, the advanced causal propagators are given by the transpose of the retarded propagators, $K_A^{(d)} \doteq (K_R^{(d)})^T$. From this, the real matrix defined by,

$$\Delta \doteq K_R^{(d)} - K_A^{(d)}, \tag{5.11}$$

is the causal set analogue of (minus) the advanced-minus-retarded operator. The matrix $i\Delta$ is a hermitian and skew-symmetric matrix, hence it has an even rank and we assume its rank is non-zero.

The non-zero eigenvalues of $i\Delta$ then come in equal and opposite pairs, and there exists mutually orthogonal normalised eigenvectors u_i, v_i obeying,

$$i\Delta u_i = \lambda_i u_i \qquad\qquad i\Delta v_i = -\lambda_i v_i, \tag{5.12}$$

where $i = 1, ..., s$, where $2s$ is the rank of $i\Delta$. Critically, Johnston defined the $p \times p$ hermitian matrix,

$$Q = \sum_{i=1}^{s} \lambda_i u_i u_i^\dagger, \tag{5.13}$$

and noted that the matrix $i\Delta$ obeys,

$$i\Delta = Q - Q^T. \tag{5.14}$$

The matrix (5.13) is, essentially, the 'positive part' of the matrix $i\Delta$. It is the discrete version of the operator used to construct the SJ state in continuum spacetimes. We shall first see how Johnston defines a vacuum state using (5.13).

Johnston defines an algebra of field operators $\hat{\phi}_x$ acting on a Hilbert space $\mathcal{H}$ to represent a free real scalar field where $x = 1, ..., p$ runs over a causal set $(\mathcal{C}, \preceq)$ with $p \in \mathbb{N}$ elements. For each element $v_x \in \mathcal{C}$ $(x = 1, ..., p)$ we suppose there is a field operator $\hat{\phi}_x$ obeying,

i $\hat{\phi}_x = \hat{\phi}_x^\dagger$

ii $[\hat{\phi}_x, \hat{\phi}_y] = i\Delta_{xy}$

iii $i\Delta w = 0 \implies \sum_{x'=1}^{p} w_{x'} \hat{\phi}_{x'} = 0$.

The first two conditions are direct generalisations of the continuum case, the hermiticity of the field and the canonical commutation relations. The third condition is a causal set version of imposing the Klein Gordon equations on the field operators. From these operators, Jonhston introduced creation and annihilation operators defined as,

$$\hat{a}_i \doteq \sum_{x=1}^{p} (v_i)_x \hat{\phi}_x \qquad\qquad \hat{a}_i^\dagger \doteq \sum_{x=1}^{p} (u_i)_x \hat{\phi}_x, \tag{5.15}$$

for $i = 1, ..., s$, which satisfy,

$$[\hat{a}_i, \hat{a}_j] = 0 \qquad\qquad [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0 \qquad\qquad [\hat{a}_i, \hat{a}_j^\dagger] = \lambda_j u_i^\dagger u_j = \lambda_j \delta_{ij}. \tag{5.16}$$

The field operators $\hat{\phi}_x$ in terms of these creation/annihilation operators is then given by,

$$\hat{\phi}_x = \sum_{i=1}^{s} (u_j)_x \hat{a}_i + (v_i)_x \hat{a}_i^\dagger. \tag{5.17}$$

Using the creation and annihilation operators a vacuum state vector $|0\rangle \in \mathcal{H}$ may be defined by the condition that $\hat{a}_i\,|0\rangle = 0$ for all $i = 1, ..., s$ and $\langle 0 \mid 0\rangle = 1$. Johnston constructs the two point function of the vacuum state vector, which is given by,

$$\begin{aligned} \langle 0 \mid \hat{\phi}_x \hat{\phi}_y \mid 0\rangle &= \sum_{i=1}^{s}\sum_{j=1}^{s} (u_i)_x (v_j)_y \langle 0 \mid \hat{a}_j \hat{a}_j^\dagger \mid 0\rangle \\ &= \sum_{i=1}^{s}\sum_{j=1}^{s} (u_i)_x (v_j)_y \lambda_j \delta_{ij} = Q_{xy}. \end{aligned} \tag{5.18}$$

The Feynman propagator is the time ordered product,

$$(K_F)_{xy} \doteq i\langle 0 \mid T\hat{\phi}_x \hat{\phi}_y \mid 0\rangle = i\big(\bar{A}_{xy} Q_{yx} + \bar{A}_{yx} Q_{xy} + \delta_{xy} Q_{xy}\big), \tag{5.19}$$

where the bar denotes a (non-unique) linear extension of the partial ordering $\preceq$ and δ is the Kronecker-delta symbol. Here the time ordering is taken so that time is increasing from right to left. The Feynman propagator over the causal set is then compared to the continuum Feynman propagator, denoted G_F in two ways; by calculating the average value for different sprinklings over $A \subset \mathbb{M}^d$, $d = 2, 4$, and by taking a continuum limit $\rho \to \infty$. The expectation value of K_F agrees for $\mathbb{M}^2$ case and has the correct continuum limit for $\mathbb{M}^4$ case. For a range of sprinklings, the average value of K_F agrees with G_F provided that $0 \ll m \ll \sqrt{\rho}$. In $\mathbb{M}^2$, there is disagreement between the imaginary parts of K_F and G_F as the massless limit is taken, which is argued to be related to the lack of the massless limit of G_F. There are also numerical errors due to the calculations being performed in a finite region of $\mathbb{M}^2$, these edge effects are not present in the bulk of $A \subset \mathbb{M}^2$.

5.2 The Distinguished SJ State on Curved Spacetimes

The work of Johnston laid the foundations for the development of the SJ state. In [1], Afshordi, Aslanbeigi and Sorkin generalise the construction of the vacuum state (5.18) on a causal set to a free scalar field over globally hyperbolic regions of a general curved spacetime. The resultant state is known as the SJ state. The SJ state constructed on the ultrastatic slab $(-\tau, \tau) \times \Sigma$ is found to converge to the ultrastatic vacuum state in limit $\tau \to \infty$ [28], whence the term 'vacuum like'. We proceed by outlining the SJ state construction given in [1], and deal with the more technical aspects later, in particular the Hadamard condition for the SJ state. Let $(\mathcal{M}, g)$ be a globally hyperbolic spacetime. The classical free massive scalar field is a function $\phi \in C^\infty(\mathcal{M})$ that obeys the Klein-Gordon equation,

$$(\Box + m^2)\phi = 0, \tag{5.20}$$

where $\Box = \nabla^\alpha \nabla_\alpha$ is the d'Alembertian operator on $(\mathcal{M}, g)$ and $m \geq 0$ is a fixed constant. Since $\mathcal{M}$ is globally hyperbolic and the Klein-Gordon operator $\Box + m^2 : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ is normally hyperbolic, there exist unique maps,

$$\mathbb{E}^\pm : C_0^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}), \tag{5.21}$$

with properties given in Chapter 3 Section 3.2. As stated before, we define the advanced-minus-retarded function as $\mathbb{E} = \mathbb{E}^- - \mathbb{E}^+$ which obeys,

$$(\Box + m^2)\mathbb{E}f = 0 = \mathbb{E}(\Box + m^2)f. \tag{5.22}$$

A sufficient condition for the SJ state to be well defined for a free scalar field on a bounded region of a globally hyperbolic is given in the following proposition by Fewster and Verch [28]:

Proposition 5.1. *Let $\mathcal{M}, \mathcal{N}$ be two globally hyperbolic spacetimes in which we assume an orientation and time-orientation have both been chosen. If there exists an isometric embedding $\iota : \mathcal{M} \to \mathcal{N}$, preserving both orientations, such that $\iota(\mathcal{M})$ is a relatively compact, causally convex, subset of $\mathcal{N}$, then the commutator function,*

$$Af = i\mathbb{E}f \qquad\qquad \forall f \in C_0^\infty(\mathcal{M}) \tag{5.23}$$

extends to a bounded self-adjoint operator over the Hilbert space $L^2(\mathcal{M}, \ \boldsymbol{d}vol_g)$.

Proof. See [28, Proposition 3.1]. ■

The key to the SJ state construction is to regard the commutator function (5.23) as a self-adjoint bounded operator over $(\mathcal{M}, g)$. One then takes the 'positive part' of A, denoted A^+, and constructs the two-point function via the prescription,

$$W_{SJ}(f, g) = \langle \overline{f} \mid A^+ g \rangle \qquad\qquad \forall f, g \in C_0^\infty(\mathcal{M}), \tag{5.24}$$

where $\langle \cdot \mid \cdot \rangle$ is the inner product on $L^2(\mathcal{M},\ \mathbf{d}\mathrm{vol}_g)$. There are various methods for constructing the operator A^+, which we will now explore. In the following we assume the spacetime $(\mathcal{M}, g)$ satisfies the conditions given in proposition 5.1 so that the function (5.23) extends to a bounded self-adjoint operator A over $L^2(\mathcal{M},\ \mathbf{d}\mathrm{vol}_g)$.

Positive Part via the Spectral Theorem

If the commutator function $A = i\mathbb{E}$ extends to a bounded self-adjoint operator on $L^2(\mathcal{M},\ \mathbf{d}\mathrm{vol}_g)$ then it possesses a spectral valued measure $\mathbf{d}P_A$. The operator A^+ is then constructed by using the positive spectral projection of A, given by,

$$P^+ = \int_{[0,\|A\|]} \mathbf{d}P_A(\lambda). \tag{5.25}$$

The projection operator was used in [28] to construct the positive part of A,

$$A^+ = P^+ A = \int_{[0,\|A\|]} \lambda \mathbf{d}P_A(\lambda). \tag{5.26}$$

An equivalent construction of the operator A^+, uses the continuous spectral calculus for self-adjoint bounded operators. Let $f : \sigma(A) \to \mathbb{R}$ be a continuous function defined by,

$$\begin{aligned} f : \sigma(A) &\to \mathbb{R} \\ \lambda &\mapsto \lambda\theta(\lambda), \end{aligned} \tag{5.27}$$

where θ is the Heaviside function defined by,

$$\theta(\lambda) = \begin{cases} 0 & \text{if } \lambda < 0 \\ 1 & \text{if } \lambda \geq 0. \end{cases} \tag{5.28}$$

The function (5.27) then yields the operator,

$$A^+ = f(A) = A\theta(A) \tag{5.29}$$

To elucidate these results, we can explicitly construct the operator (5.29) when A is a self-adjoint compact operator on $L^2(\mathcal{M},\ \mathbf{d}\mathrm{vol}_g)$. In the case when the operator (5.23) is compact there exists a countable orthonormal basis $\{\Psi_j^{\pm}\}_{j\in\mathbb{N}}$ of $L^2(\mathcal{M},\ \mathbf{d}\mathrm{vol}_g)$ consisting of eigenvectors of A with corresponding eigenvalues $\pm\lambda_j$ where $\lambda_j \geq 0$ for all $j \in \mathbb{N}$. Expressing the operator (5.23) in this basis yields the decomposition into 'positive' and 'negative' parts,

$$A = \sum_{j\in\mathbb{N}} \lambda_j \big[|\Psi_j^+\rangle \langle\Psi_j^+| - |\Psi_j^-\rangle \langle\Psi_j^-| \big], \tag{5.30}$$

where the 'positive part' is simply read off,

$$A^+ = \sum_{j\in\mathbb{N}} \lambda_j \, |\Psi_j^+\rangle \langle\Psi_j^+| \,. \tag{5.31}$$

We can directly verify that in the case when A is compact, the function (5.27) yields the correct positive part,

$$\begin{aligned} f(A) &= f\left(\sum_{j\in\mathbb{N}} \lambda_j \, |\Psi_j^+\rangle \langle\Psi_j^+| - \lambda_j \, |\Psi_j^-\rangle \langle\Psi_j^-| \right) \\ &= \sum_{j\in\mathbb{N}} f(\lambda_j) \, |\Psi_j^+\rangle \langle\Psi_j^+| + f(-\lambda_j) \, |\Psi_j^-\rangle \langle\Psi_j^-| \\ &= \sum_{j\in\mathbb{N}} \lambda_j \, |\Psi_j^+\rangle \langle\Psi_j^+| \\ &= A^+. \end{aligned} \tag{5.32}$$

An equivalent construction of the positive part (5.31) is given by the function,

$$\begin{aligned} g : \sigma(A) &\to \mathbb{R} \\ \lambda &\mapsto \frac{1}{2}(\lambda + |\lambda|), \end{aligned} \tag{5.33}$$

where the corresponding operator is given by $g(A) = \frac{1}{2}(A + |A|)$ and where $|A| = \sqrt{A^2}$. Moreover, in the case when the operator (5.30) is defined on a Hilbert space, the operator A^2 is positive and hence admits a unique, positive square root. The advantage of using the function (5.33) to construct the positive part (5.31) is that one only needs to calculate $|A|$. Since the eigenvectors $\{\Psi_j^\pm\}_{j\in\mathbb{N}}$ appearing in (5.30) are orthonormal, and since $\langle \Psi_j^+ \mid \Psi_k^- \rangle = 0$ for all $j, k \in \mathbb{N}$, we find,

$$A^2 = \sum_{j\in\mathbb{N}} \lambda_j^2 \big[|\Psi_j^+\rangle \langle\Psi_j^+| + |\Psi_j^-\rangle \langle\Psi_j^-| \big], \tag{5.34}$$

hence,

$$|A| = \sqrt{A^2} = \sum_{j\in\mathbb{N}} \lambda_j \big[|\Psi_j^+\rangle \langle\Psi_j^+| + |\Psi_j^-\rangle \langle\Psi_j^-| \big], \tag{5.35}$$

which implies the following,

$$
\begin{aligned}
g(A) = \frac{1}{2}(A + |A|) = \frac{1}{2}\left(\sum_j \lambda_j \left[|\Psi_j^+\rangle\langle\Psi_j^+| - |\Psi_j^-\rangle\langle\Psi_j^-|\right] + \lambda_j\left[|\Psi_j^+\rangle\langle\Psi_j^+| + |\Psi_j^-\rangle\langle\Psi_j^-|\right]\right) \\
= \sum_{j\in\mathbb{N}} \lambda_j |\Psi_j^+\rangle\langle\Psi_j^+| \\
= A^+,
\end{aligned}
\tag{5.36}
$$

which shows that the operator $g(A)$ coincides with the operator (5.31).

The SJ Axioms

Alternatively, and we remark here the significance of the following, Sorkin remarked that the operator (5.29) may be viewed as being the unique solution to the set of axioms [58],

$SJ1$) $A^+ - \Gamma A^+ \Gamma = A$ CCRs.

$SJ2$) $A^+ \geq 0$ Positivity.

$SJ3$) $A^+ \Gamma A^+ \Gamma = 0$ Ground state condition

where $\Gamma : L^2(\mathcal{M}, \ \mathbf{d}\text{vol}_g) \to L^2(\mathcal{M}, \ \mathbf{d}\text{vol}_g)$ is the antilinear involutive complex conjugation map defined by $\Gamma(f) = \overline{f}$. The motivation of these axioms is seen in the following,

$SJ1-$ This axiom is essentially a reformulation of the canonical commutation relations; the antisymmetric part of two point function of the SJ state is seen to coincide with the commutator function,

$$
\begin{aligned}
\langle \Gamma f \mid Ag \rangle = W_{SJ}(f,g) - W_{SJ}(g,f) = \langle \Gamma f \mid A^+ g\rangle - \langle \Gamma g \mid A^+ f\rangle \\
= \langle \Gamma f \mid A^+ g\rangle - \langle A^+ \Gamma g \mid f\rangle \\
= \langle \Gamma f \mid A^+ g\rangle - \langle \Gamma f \mid \Gamma A^+ \Gamma g\rangle \\
= \langle \Gamma f \mid (A^+ - \Gamma A^+ \Gamma) g\rangle
\end{aligned}
\tag{5.37}
$$

for all $f, g \in C_0^\infty(\mathcal{M})$. This implies the operator A^+ must obey the decomposition,

$$
A = A^+ - \Gamma A^+ \Gamma. \tag{5.38}
$$

Therefore, axiom $SJ1$ ensures that the SJ two point function has the correct antisymmetric part.

$SJ2-$ The second axiom ensures that the SJ state is positive. Since the SJ state is quasifree [28, Proposition 3.2], all odd $n-$point functions vanish and all even $n-$point function can be expressed in terms of the two point function. The SJ state is positive if the two point function (5.24) is of positive type, i.e. $W_{SJ}(\overline{f}, f) \geq 0$ for all $f \in C_0^\infty(\mathcal{M})$. We see that the positivity of the operator A^+ directly implies that the two point function is of postive type,

$$\omega_{SJ}(\phi(f)^* \phi(f)) = W_{SJ}(\overline{f}, f) = \langle f \mid A^+ f \rangle \geq 0. \tag{5.39}$$

$SJ3-$ The third axiom is what Sorkin dubs as the 'ground-state condition'. Furthermore, the axiom ensures that the SJ state constructed on the ultrastatic slab $((-\tau, \tau) \times \Sigma, g = \mathbb{1} \oplus -h)$ converges to the ultrastatic vacuum state in the limit $\tau \to \infty$ [28, Section 4.3]. The third axiom is verified by the observation,

$$A^+ \Gamma A^+ \Gamma = \frac{-1}{4}(A + |A|)(A - |A|) = \frac{-1}{4}\left(A^2 - A^2\right) = 0, \tag{5.40}$$

In [58], Sorkin demonstrates that the operator (5.29) is in fact the unique solution to the set of axioms $SJ1 - SJ3$. We proceed to sketch a proof of the solution to these axioms.[a] The first axiom $SJ1$, as we shall prove later, is solved by the operator,

$$A^+ = \frac{A}{2} + R, \tag{5.41}$$

where R is a self-adjoint bounded operator over $L^2(\mathcal{M}, \ \mathbf{d}\mathrm{vol}_g)$. By substituting (5.41) into axiom $SJ1$ and by using $A = -\Gamma A \Gamma$ we see,

$$A = A^+ - \Gamma A^+ \Gamma = \frac{A}{2} + R - \frac{\Gamma A \Gamma}{2} - \Gamma R \Gamma = A + R - \Gamma R \Gamma, \tag{5.42}$$

which implies $R = \Gamma R \Gamma$. Substituting the solution (5.41) into axiom $SJ3$ yields the following,

$$\begin{aligned} 0 = A^+ \Gamma A^+ \Gamma &= \left(\frac{A}{2} + R\right)\left(-\frac{A}{2} + R\right) \\ &= -\frac{A^2}{4} + \frac{1}{2}[A, R] + R^2, \end{aligned} \tag{5.43}$$

taking the $\Gamma-$variant and $\Gamma-$invariant parts of (5.43) then gives,

$$R^2 - \frac{A^2}{4} = 0 \qquad\qquad [A, R] = 0. \tag{5.44}$$

[a] A detailed proof of these axioms can be found as the trivial case of the generalised SJ state construction given in Chapter 6.

Since A is a self-adjoint bounded operator over a Hilbert space, it follows that A^2 is a non-negative bounded operator. Therefore, by [53, Theorem VI.9], there exists a unique non-negative R such that $R^2 = \dfrac{A^2}{4}$, which is given by,

$$R = \frac{\sqrt{A^2}}{2}. \tag{5.45}$$

However, the operator $-R$ also solves (5.44). To solve the final ambiguity, the axiom $SJ2$ requires the operator A^+ to be non-negative, and so $-R$ is excluded. Therefore the unique solution to axioms $SJ1-3$ is the operator,

$$A^+ = \frac{A}{2} + \frac{|A|}{2}, \tag{5.46}$$

as desired.

Positive Part via the Polar Decomposition

Finally, we may use the polar decomposition of A; since A is a self-adjoint bounded operator there is a partial isometry U such that $U^2 = \mathbb{1}$ on $\mathrm{Im}(|A|)$ and such that,

$$A = U|A|. \tag{5.47}$$

Since A is self-adjoint, U commutes with $|A|$, which implies,

$$AU = U|A|U = |A|U^2 = |A|. \tag{5.48}$$

The positive part of A, denoted A^+, may be the defined as,

$$A^+ = \frac{A}{2}(\mathbb{1} + U). \tag{5.49}$$

One can observe that since $\mathbb{E} : C_0^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ maps real functions to real functions, it commutes with Γ. Therefore A anticommutes with Γ which implies A^2 commutes with Γ. Therefore $|A| = \sqrt{A^2}$ commutes with Γ, which follows from Theorem 2.5[vii]. This then implies that U anticommutes with

Γ. We can verify that (5.49) does indeed satisfy $SJ1$,

$$\begin{aligned}
A^+ - \Gamma A^+ \Gamma &= \frac{A}{2}(\mathbb{1} + U) - \frac{\Gamma A}{2}(\Gamma + U\Gamma) \\
&= \frac{A}{2}(\mathbb{1} + U) + \frac{A\Gamma}{2}(\Gamma - \Gamma U) \\
&= \frac{A}{2}(\mathbb{1} + U) + \frac{A\Gamma^2}{2}(\mathbb{1} - U) \\
&= \frac{A}{2} + \frac{AU}{2} + \frac{A}{2} - \frac{AU}{2} \\
&= A
\end{aligned} \tag{5.50}$$

and the operator A^+ in (5.49) is seen to be non-negative since $A \leq |A|$. Finally, we see that (5.49) satisfies $SJ3$,

$$\begin{aligned}
A^+ \Gamma A^+ \Gamma &= \frac{A}{2}(\mathbb{1} + U)\frac{\Gamma A}{2}(\mathbb{1} + U)\Gamma \\
&= \frac{A}{2}(\mathbb{1} + U)\frac{\Gamma A \Gamma}{2}(\mathbb{1} - U) \\
&= -\frac{A^2}{4}(\mathbb{1} + U)(\mathbb{1} - U) \\
&= -\frac{A^2}{4}(\mathbb{1} - U^2) \\
&= 0,
\end{aligned} \tag{5.51}$$

which holds since $A^2U^2 = |A|(|A|U^2) = (|A|)^2 = A^2$. As we have shown, there are various ways of constructing the 'positive part' of the commutator function $A = i\mathbb{E}$. In this thesis, particularly in chapter 6, we shall consider generalisations of the SJ state construction, with a particular focus on modifying the SJ axioms $SJ1 - 3$. In order to do such a generalisation, we will rely almost exclusively on the continuous functional calculus for self-adjoint bounded operators. Essentially, the SJ state is constructed by taking functions of the commutator function $A = i\mathbb{E}$. In chapter 6 we show that there are in fact *other* functions of the operator $A = i\mathbb{E}$ that can be used to construct a well defined quasifree state.

5.3 Purity of the SJ Vacuum

Fewster and Verch showed that the SJ vacuum on a bounded region of a globally hyperbolic region is pure [28]. Let $(\mathcal{M}, g)$ be a globally hyperbolic spacetime such that the commutator function $A = i\mathbb{E}$ extends to a self-adjoint bounded operator on $L^2(\mathcal{M})$. The SJ state on the CCR algebra $\mathcal{A}(\mathcal{M})$ is constructed by setting its two-point function to be,

$$
\begin{aligned}
W_{SJ}(f,g) &= \langle \overline{f} \mid \frac{1}{2}(A+|A|)g\rangle \\
&= \frac{1}{2}\langle \overline{f} \mid Ag\rangle + \frac{1}{2}\langle \overline{f} \mid |A|g\rangle .
\end{aligned}
\tag{5.52}
$$

As before, we define $S(\mathcal{M}) = C_0^\infty(\mathcal{M},\mathbb{R})/\ker(\mathbb{E})$ to be the real-linear vector space of equivalence classes $[f]$ with respect to the equivalence relation,

$$
f \sim f' \iff \mathbb{E}(f-f') = 0 \qquad \forall f,g \in C^\infty(\mathcal{M},\mathbb{R}) \tag{5.53}
$$

Using the space $S(\mathcal{M})$, we define the following maps,

$$
\begin{aligned}
\sigma : S(\mathcal{M}) \times S(\mathcal{M}) &\to \mathbb{R} && (5.54)\\
([f],[g]) &\mapsto \langle f \mid \mathbb{E}g\rangle, \\
\mu_{SJ} : S(\mathcal{M}) \times S(\mathcal{M}) &\to \mathbb{R} && (5.55)\\
([f],[g]) &\mapsto \langle f \mid Rg\rangle,
\end{aligned}
$$

where $\sigma(\cdot,\cdot)$ is a well defined symplectic form on $S(\mathcal{M})$ by [45, Proposition 8] and $\mu_{SJ}(\cdot,\cdot)$ is a well defined real-linear scalar product on $S(\mathcal{M})$ [28, Section 3]. We now find,

$$
W_{SJ}(f,g) = \mu_{SJ}([f],[g]) + \frac{i}{2}\sigma([f],[g]), \tag{5.56}
$$

for all $f,g \in C_0^\infty(\mathcal{M},\mathbb{R})$. The SJ state with two-point function (5.52) is pure if the symplectic form $\sigma(\cdot,\cdot)$ has the following *saturation property* with respect to the scalar product $\mu_{SJ}(\cdot,\cdot)$,

$$
\mu_{SJ}([f],[f]) = \sup_{0\neq[h]\in S(\mathcal{M})} \frac{|\sigma([f],[h])|^2}{4\mu_{SJ}([h],[h])} \qquad \forall [f] \in S(\mathcal{M}). \tag{5.57}
$$

Using these definitions, we now detail the proof that the SJ state is pure [28, Proposition 3.2],

Proposition 5.2. *Let $(\mathcal{M},g)$ be a globally hyperbolic spacetime such that the commutator function $A = i\mathbb{E}$ extends to a self-adjoint bounded operator on $L^2(\mathcal{M})$. Then the SJ state with two-point function* (5.52) *is pure.*

Proof. Since A is a bounded self-adjoint operator on $L^2(\mathcal{M})$, there exists a unique partial isometry such that $A = U|A| = |A|U^*$. We observe,

$$
\begin{aligned}
[|A|, U] &= 0, \\
[|A|, \Gamma] &= 0, \\
\{A, \Gamma\} &= 0, \\
\{U, \Gamma\} &= 0,
\end{aligned}
\tag{5.58}
$$

where $[\cdot,\cdot]$ is the commutator and $\{\cdot,\cdot\}$ is the anti-commutator. Using these relations, we can infer that the operator $iU : L^2(\mathcal{M}) \to L^2(\mathcal{M})$ and Γ commute. Let $(h_n)_{n\in\mathbb{N}}$ be a sequence in $C_0^\infty(\mathcal{M},\mathbb{R})$ that converges to $iUf \in L^2(\mathcal{M},\mathbb{R})$. We now have, for all $[f] \in S(\mathcal{M})$,

$$
\begin{aligned}
\lim_{n\to\infty} \frac{|\sigma([f],[h_n])|^2}{4\mu_{SJ}([h_n],[h_n])} &= \lim_{n\to\infty} \frac{|\langle f \mid Ah_n\rangle|^2}{2\langle h_n \mid |A|h_n\rangle} \\
&= \frac{|\langle f \mid iAUf\rangle|^2}{2\langle iUf \mid i|A|Uf\rangle} \\
&= \frac{|\langle f \mid |A|U^*Uf\rangle|^2}{2\langle f \mid |A|U^*Uf\rangle} \\
&= \frac{|\langle f \mid |A|f\rangle|^2}{2\langle f \mid |A|f\rangle} \\
&= \frac{1}{2}\langle f \mid |A|f\rangle \\
&= \mu_{SJ}([f],[f]),
\end{aligned}
\tag{5.59}
$$

where we have used the fact that $U^*U = \mathbb{1}$ on range$(|A|)$. This establishes the saturation property of the scalar product $\mu_{SJ}(\cdot,\cdot)$ with respect to the symplectic form $\sigma(\cdot,\cdot)$. Hence the SJ vacuum is pure. ■

5.4 Local Covariance of the SJ State

Let $(\mathcal{O}, g)$ be a globally hyperbolic spacetime. Consider bounded globally hyperbolic regions $\mathcal{N} \subset \mathcal{M} \subset \mathcal{O}$ such that there exist isometric embedding $\iota : \mathcal{N} \hookrightarrow \mathcal{M}$, $\kappa : \mathcal{N} \hookrightarrow \mathcal{O}$ which preserve orientations and so that the images are causally convex relatively compact subsets of $\mathcal{O}$. Then the SJ state is well defined on $(\mathcal{N}, g)$ and on $(\mathcal{M}, g)$ by proposition 5.1. Let $W_{SJ_\mathcal{N}}$ and $W_{SJ_\mathcal{M}}$ be the SJ states over $(\mathcal{N}, g)$ and $(\mathcal{M}, g)$ respectively. Suppose now that an observable with support properties supp$(f) \subset \mathcal{N} \subset \mathcal{M}$. The expectation value of this observable can then be evaluated in both the states $W_{SJ_\mathcal{N}}$ and $W_{SJ_\mathcal{M}}$. However, since the SJ state construction depends upon the global properties of the chosen spacetime, the expectation values of the observable will be different in both states. It is then unclear as to which is the 'correct' expectation value; the SJ state is certainly distinguished over a spacetime $(\mathcal{M}, g)$, but there is no notion of a 'distinguished' choice of spacetime region. Therefore, it is unclear what physical relevance the SJ state has. To summarise, the SJ state fails to be local covariant because it depends

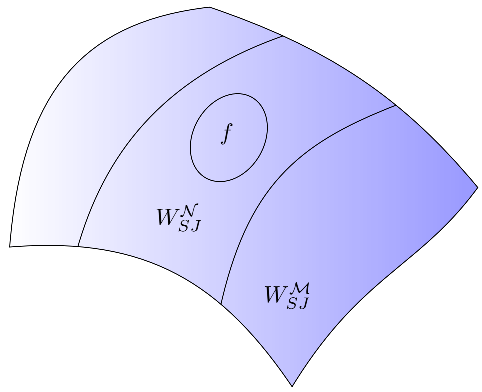


Figure 5.2: Failure of local covariant property for SJ state.

upon an arbitrary choice of underlying spacetime.

5.5 The Hadamard Condition of the SJ State

To elucidate the drawbacks of the SJ state construction, we follow the work of Fewster and Verch by constructing the SJ state on the ultrastatic slab $(-\tau, \tau) \times \Sigma$ [28]. Let $((-\tau, \tau) \times \Sigma, g = \mathbb{1} \oplus -h)$ be a globally hyperbolic ultrastatic slab spacetime where (Σ, h) is a smooth compact Riemannian manifold and $\tau > 0$. In such a spacetime, the Klein Gordon operator becomes,

$$\Box + m^2 = \partial_t^2 + K, \tag{5.60}$$

where $t \in (-\tau, \tau)$ is the ultrastatic time parameter and $K = -\Delta + m^2$, where Δ is the Laplacian on (Σ, h). Then K is essentially self-adjoint on $C_0^\infty(\Sigma) \subset L^2(\Sigma)$, and so admits a self-adjoint extension onto $L^2(\Sigma)$ [60, Chapter 8]. There exists a complete orthonormal basis of $L^2(\Sigma, h)$ consisting of eigenvectors of K, which we denote by ξ_j $(j \in \mathbb{N})$, with corresponding eigenvalues ω_j^2 [20, Theorem 1 Section 6.3]. We make assumption that for each $j \in \mathbb{N}$, $\overline{\xi_j}$ is also a basis element of $L^2(\Sigma, h)$, and we may choose the labelling such that $\overline{\xi_j} = \xi_{\bar{j}}$. Throughout this thesis we assume that $\omega_j \leq \omega_{j+1}$ for all $j \in \mathbb{N}$. The norm on $L^2(-\tau, \tau)$ will be denoted $\|\cdot\|_\tau$. The Hilbert space is given by $L^2(\mathcal{M}) = L^2(-\tau, \tau) \otimes L^2(\Sigma)$.

The commutator function for the Klein Gordon field over the ultrastatic slab $((-\tau,\tau)\times\Sigma, g = \mathbb{1}\oplus -h)$ extends to the self-adjoint, bounded operator,

$$A = i\mathbb{E} = \sum_{j\in\mathbb{N}} \frac{1}{2\omega_j}\big(|\mathcal{A}_j\rangle\langle\mathcal{A}_j| - |\overline{\mathcal{A}_j}\rangle\langle\overline{\mathcal{A}_j}|\big), \tag{5.61}$$

on $L^2(\mathcal{M},\ \mathbf{d}\mathrm{vol}_g)$ where $\mathcal{A}_j(t,x) = e^{-i\omega_j t}\xi_j(x)$. The eigenvectors of (5.61) are given by,

$$\Psi_j^{\pm}(t,x) = \left(C_j(t) \mp i\frac{\|C_j\|_\tau}{\|S_j\|_\tau}S_j(t)\right)\xi_j(x), \tag{5.62}$$

where $C_j(t) = \cos(\omega_j t)$ and $S_j(t) = \sin(\omega_j t)$ and the norms $\|\cdot\|_\tau$ is the norm induced by the inner product on $L^2(-\tau,\tau)$. The corresponding eigenvalues are given by,

$$\lambda_j^{\pm} = \pm\lambda_j = \pm\frac{\|C_j\|_\tau\|S_j\|_\tau}{\omega_j} \tag{5.63}$$

where,

$$\begin{aligned}
\|S_j\|_\tau^2 &= \tau(1-\mathrm{sinc}(2\omega_j\tau)) \\
\|C_j\|_\tau^2 &= \tau(1+\mathrm{sinc}(2\omega_j\tau)) \\
\langle S_j \mid C_j\rangle_\tau &= 0 \\
\|\Psi_j^{\pm}\|^2 &= 2\|C_j\|_\tau^2,
\end{aligned} \tag{5.64}$$

where $\|\cdot\|_\tau$ is the norm on $L^2(-\tau,\tau)$. In the eigenvector basis (5.62) the operator (5.61) is,

$$\begin{aligned}
A &= \sum_{j\in\mathbb{N}} \frac{\lambda_j}{\|\Psi^+\|^2}\left[|\Psi_j^+\rangle\langle\Psi_j^+| - |\Psi_j^-\rangle\langle\Psi_j^-|\right] \\
&= \sum_{j\in\mathbb{N}} \frac{\|S_j\|_\tau}{2\omega_j\|C_j\|_\tau}\left[|\Psi_j^+\rangle\langle\Psi_j^+| - |\Psi_j^-\rangle\langle\Psi_j^-|\right]
\end{aligned} \tag{5.65}$$

which has the positive part,

$$A^+ = \sum_{j\in\mathbb{N}} \frac{\|S_j\|_\tau}{2\omega_j\|C_j\|_\tau}|\Psi_j^+\rangle\langle\Psi_j^+|. \tag{5.66}$$

The integral kernel of the SJ two point function is then given by,

$$W_{SJ}(t,x;t',x') = \sum_{j\in\mathbb{N}} \frac{\|S_j\|_\tau}{2\omega_j\|C_j\|_\tau}\big(e^{-i\omega_j t} + i\delta_j\sin(\omega_j t)\big)\Big(e^{-i\omega_j t'} + i\delta_j\sin(\omega_j t')\Big)\xi_j(x)\overline{\xi_j(x')}, \tag{5.67}$$

where,

$$\delta_j = 1 - \frac{\|C_j\|_\tau}{\|S_j\|_\tau}. \tag{5.68}$$

We shall now use the SJ state defined by (5.67) and summarise the pathologies shown in [28]. As discussed in Chapter 4 section 4.2, in order to prove that the SJ state constructed on a spacetime $\mathcal{M}$ with two point function (5.67) is Hadamard, one uses a reference Hadamard state W_H on $\mathcal{M}$ and shows that the integral kernel of $: W_{SJ} :\doteq W_{SJ} - W_H$ is smooth on $\mathcal{M} \times \mathcal{M}$. For the SJ state constructed on the ultrastatic slab $((-\tau, \tau) \times \Sigma, g = \mathbb{1} \oplus -h)$ with the two point function (5.67), Fewster and Verch gave a necessary condition on when the SJ state satisfies the Hadamard condition. We shall proceed by sketching the proof of the following theorem, given in [28, Theorem 4.2],

Theorem 5.3. *The set of τ for which the SJ state with two point function* (5.67) *is Hadamard on $(-\tau, \tau) \times \Sigma$ is contained in a set of measure zero.*

Proof. Let W_H be a two-point function for the ultrastatic vacuum state,

$$W_H = \sum_{j \in \mathbb{N}} \frac{1}{2\omega_j} e^{-i\omega_j (t-t')} \xi_j(x) \xi_j(x'), \tag{5.69}$$

where ξ_j are eigenvectors with corresponding eigenvalues ω_j^2 of the operator $K = -\Delta + m^2$ as before. The ultrastatic vacuum state is, in particular, Hadamard [33]. Therefore, the SJ state (5.67) is Hadamard if and only if the normal ordered two point function $: W_{SJ} := W_{SJ} - W_H$ has a kernel which is smooth on $\mathcal{M} \times \mathcal{M}$. The normal ordered SJ two point function has the integral kernel,

$$\begin{aligned} : W_{SJ} : (t, x; t', x') &= (W_{SJ} - W_H)(t, x; t', x') \\ &= \sum_{j \in \mathbb{N}} \frac{\delta_j}{4\omega_j (1 - \delta_j)} \Big(\delta_j \cos(\omega_j (t - t')) + (2 - \delta_j) \cos(\omega_j (t + t')) \Big) \xi_j(x) \overline{\xi_j(x')}. \end{aligned} \tag{5.70}$$

If (5.70) is smooth on $\mathcal{M} \times \mathcal{M}$ then in particular it must be C^2. Furthermore, if $: W_{SJ} : (\cdot, \cdot) \in C^\infty(\mathcal{M} \times \mathcal{M})$ then all derivatives may be assumed to be bounded on the subset $\mathcal{M}' \times \mathcal{M}'$ where $\mathcal{M}' = (-\tau', \tau') \times \Sigma$ and $0 < \tau' < \tau$. Therefore, the function,

$$F(t, x; t', x') = \frac{\partial^2}{\partial t \partial t'} : W_{SJ} : (t, x; t', x'), \tag{5.71}$$

must be square-integrable on $\mathcal{M}' \times \mathcal{M}'$. By [53, Theorem VI.23] the function (5.71) is the kernel of the Hilbert-Schmidt operator,

$$\begin{aligned} (Tg)(t, x) &= \int_{\mathcal{M}'} F(t, x; t', x') g(t', x') \, \mathbf{d}\mathrm{vol}_g(t', x') \\ &= \sum_{j \in \mathbb{N}} \frac{\omega_j \delta_j}{2} \left(\frac{1}{1 - \delta_j} S_j(t) \langle S_j \mid g \rangle - C_j(t) \langle C_j \mid g \rangle \right) \xi_j(x), \end{aligned} \tag{5.72}$$

for all $(t,x) \in (-\tau', \tau') \times \Sigma$ and where the inner products are on $L^2((-\tau',\tau') \times \Sigma)$. If the operator (5.72) is Hilbert-Schmidt on $\mathcal{M}'$, the its eigenvalues are square summable. The eigenvalues of (5.72) are found to be,

$$\Big\{ \frac{-\omega_j \delta_j \|C_j\|^2_{\tau'}}{2} \mid j \in \mathbb{N} \Big\} \cup \Big\{ \frac{\omega_j \delta_j \|S_j\|^2_{\tau'}}{2(1-\delta_j)} \mid j \in \mathbb{N} \Big\} \cup \Big\{ 0 \Big\}, \tag{5.73}$$

where the norms are taken in $L^2(-\tau', \tau')$. Since (5.72) is assumed to be Hilbert-Schmidt, the eigenvalues (5.73) are square-summable, implying,

$$\sum_{j \in \mathbb{N}} \omega_j^2 \delta_j^2 \|C_j\|^4_{\tau'} < \infty \qquad \sum_{j \in \mathbb{N}} \frac{\omega_j^2 \delta_j^2 \|S_j\|^4_{\tau'}}{(1-\delta_j)^2} < \infty. \tag{5.74}$$

In the limit $j \to \infty$ we have $\omega_j \to \infty$ and therefore $\operatorname{sinc}(2\omega_j \tau) \to 0$, thus we have $\|C_j\|_{\tau'} \to \sqrt{\tau'}$ and $\|S_j\|_{\tau'} \to \sqrt{\tau'}$. Therefore if (5.74) holds then we must have $\lim_{j\to\infty} \omega_j \delta_j = 0$. Furthermore, since $\omega_j \delta_j \sim (2\tau)^{-1} \sin(2\omega_j \tau)$, the following must hold,

$$\lim_{j \to \infty} \sin(2\omega_j \tau) = 0. \tag{5.75}$$

Let V be the set of all $\tau \in \mathbb{R}$ such that (5.75) holds. Then V is a Borel subset of $\mathbb{R}$, which is assumed to have a nonzero finite measure for the sake of contradiction.[b] Then the characteristic function on V, denoted χ_V, is an L^1 function such that,

$$\lim_{j \to \infty} \chi_V(\tau) \sin^2(2\omega_j \tau) = 0 \qquad \forall \tau \in V. \tag{5.76}$$

We observe the following,

$$\begin{aligned} \chi_V(\tau) \sin^2(2\omega_j \tau) &\leq \chi_V(\tau), \\ \int \chi_V(\tau) &< \infty, \\ \lim_{j \to \infty} \chi_V(\tau) \sin^2(2\omega_j \tau) &= 0, \end{aligned} \tag{5.77}$$

which then implies, by the dominated convergence theorem,

$$\lim_{j \to \infty} \int_{(0,\infty)} \chi_V(\tau) \sin^2(2\omega_j \tau) \mathbf{d}\tau = 0. \tag{5.78}$$

where $\mathbf{d}\tau$ is the measure on V. Using standard trigonometric identities we obtain,

$$\int_{(0,\infty)} \chi_V(\tau) \sin^2(2\omega_j \tau) \mathbf{d}\tau = \frac{1}{2} \int_{(0,\infty)} \chi_V(\tau)(1 - \cos(4\omega_j \tau)) \mathbf{d}\tau, \tag{5.79}$$

[b] The *Borel* sets of $\mathbb{R}$ is defined as the smallest collection of subsets of $\mathbb{R}$ such that the family is closed under complements, closed under countable unions and so that the family contains each open interval.

and the Riemann-Lebesgue lemma yields,

$$\lim_{j\to\infty} \frac{1}{2}\int_{(0,\infty)} \chi_V(\tau)\cos(4\omega_j\tau)\mathbf{d}\tau = 0, \tag{5.80}$$

Therefore, we obtain,

$$\int_{(0,\infty)} \chi_V(\tau)\mathbf{d}\tau = 0, \tag{5.81}$$

thus the Lebesgue measure of V vanishes, which contradicts the assumption that V has a nonzero finite measure. Therefore the SJ state with two point function (5.67) is Hadamard on a measure zero subset of $(-\tau,\tau)\times\Sigma$. ■

To give concrete examples of how the SJ state fails to be Hadamard, Fewster and Verch constructed the SJ state on the ultrastatic slab $(-\tau,\tau)\times\Sigma$ where Σ is either a flat 3-torus of a round 3-sphere. We now review these results:

SJ State on the Flat 3-Torus

Let $((-\tau,\tau)\times\mathbb{T}^3, g = \mathbb{1}\oplus -h)$ be a ultrastatic slab where $\mathbb{T}^3 = \mathbb{R}^3/(L\mathbb{Z})^3$ is a flat 3-torus with common periodicity length $L > 0$. Letting $m > 0$, the eigenvalues of $K = -\Delta_{\mathbb{T}^3} + m^2$, where $-\Delta_{\mathbb{T}^3}$ is the Laplacian on $\mathbb{T}^3$ are given by,

$$\omega_{\mathbf{k}} = \sqrt{\left(\frac{2\pi\|\mathbf{k}\|}{L}\right)^2 + m^2} \tag{5.82}$$

where $\mathbf{k}\in\mathbb{Z}^3$. The subsequences $\omega_{r,0,0}$, $\omega_{r,r,0}$ obey $\omega_{r,0,0}\sim 2\pi r/L$ and $\omega_{r,r,0}\sim 2\pi r\sqrt{2}/L$ respectively as $r\to\infty$. Fewster and Verch showed that the SJ state is Hadamard only if both,

$$\begin{aligned} \sin(4\pi\tau r/L) &\to 0 \\ \sin\left(4\sqrt{2}\pi r\tau/L\right) &\to 0, \end{aligned} \tag{5.83}$$

hold as $r\to\infty$. However the first only holds if $4\tau/L\in\mathbb{Z}$ while the second only holds if $4\sqrt{2}\tau/L\in\mathbb{Z}$, hence there is no $\tau > 0$ that satisfies these conditions and therefore the SJ state on $(-\tau,\tau)\times\mathbb{T}^3$ is not Hadamard.

SJ State on the Round 3-Sphere

Let $((-\tau,\tau)\times S^3, g = \mathbb{1}\oplus -h)$ be an ultrastatic slab where S^3 is a round 3-sphere. Letting $m > 0$, the eigenvalues of $K = -\Delta_{S^3} + m^2$, where Δ_{S^3} is the Laplacian on the round 3-sphere of radius $R > 0$, are given by,

$$\omega_j = \sqrt{\frac{j(j+2)}{R^2} + m^2}, \tag{5.84}$$

and occur with multiplicity $(1+j)^2$ for all $j \in \mathbb{N}_0$. Since $\sin(2\omega_j\tau) \to 0$ only if $2\tau/(\pi R) \in \mathbb{N}$, Fewster and Verch showed that the SJ state on $((-\tau,\tau)\times S^3, g = \mathbb{1} \oplus -h)$ is Hadamard only if $2\tau = k\pi R$ for some $k \in \mathbb{N}$. Letting $2\tau = k\pi R$, they find,

$$\sin^2(2\omega_j\tau) \sim \left(\frac{((mR)^2-1)\pi k}{2j}\right), \tag{5.85}$$

whereby the sums (5.73) are seen to diverge owing to the multiplicities of ω_j. Therefore, there is no value of $\tau > 0$ that ensures the series (5.73) converge and therefore the SJ state on $((-\tau,\tau)\times S^3, g = \mathbb{1} \oplus -h)$ is not Hadamard.

5.6 Brum-Fredenhagen States on Curved Spacetimes

The construction of a Brum-Fredenhagen state, or BF state, is based on the original SJ state construction. However, the BF states satisfy the Hadamard condition in all case with no known exceptions [11]. We shall outline the construction of a BF state over the Weyl algebra for the free scalar field on a static spacetime and show how the BF state satisfies the Hadamard condition.

Let $\mathcal{N} = (\mathbb{R}\times\Sigma, g = \mathbb{1}\oplus -h)$ be a globally hyperbolic spacetime and let $\mathcal{M} = ((-\tau,\tau)\times\Sigma, g = \mathbb{1}\oplus -h)$, $\tau > 0$, be a globally hyperbolic slab spacetime. There is the isometric embedding,

$$\begin{aligned} \iota : \mathcal{M} &\hookrightarrow \mathcal{N} \\ (t,x) &\mapsto (t,x) \end{aligned} \tag{5.86}$$

As before, the Klein-Gordon operator $\Box_{\mathcal{N}} + m^2$ on $\mathcal{N}$ admits unique advanced/retarded fundamental solutions $\mathbb{E}^{\pm}_{\mathcal{N}}$ from which we construct the advanced-minus-retarded operator $\mathbb{E}_{\mathcal{N}} = \mathbb{E}^{-}_{\mathcal{N}} - \mathbb{E}^{+}_{\mathcal{N}}$. Since the advanced-minus-retarded operator is unique on $\mathcal{N}$, the embedding ι induces the advanced-minus-retarded operator on $\mathcal{M}$ via,

$$\mathbb{E}_{\mathcal{M}} = \iota^*\mathbb{E}_{\mathcal{N}}\iota_*, \tag{5.87}$$

where ι^*, ι_* are the pull-back, push-forward of ι respectively. Let $f \in C_0^\infty(\mathcal{N})$ be a real-valued test function which is identically one when restricted to $\iota(\mathcal{M}) \subset \mathcal{N}$, i.e $f\big|_{\iota(\mathcal{M})} \equiv 1$. From this, Brum and Fredenhagen define the self-adjoint bounded operator [10, Theorem 4.1.0.1],

$$A_f \doteq f(-i\mathbb{E}_{\mathcal{M}})f. \tag{5.88}$$

The positive part of A_f is given by,

$$A_f^+ = \frac{1}{2}(A_f + |A_f|), \tag{5.89}$$

The BF state is then constructed in an analogous way to the SJ state, whereby the two point function is defined by,

$$W_{BF_f}(h,g) = \langle \overline{h} \mid A_f^+ g \rangle \qquad \forall h, g \in C_0^\infty(\mathcal{M}). \tag{5.90}$$

Under such a prescription, Brum and Fredenhagen show (5.90) is the two point function for a pure quasifree state over the Weyl algebra $\mathfrak{W}(S_\mathcal{N}, \sigma)$ where $S_\mathcal{N} = C_0^\infty(\mathcal{N}, \mathbb{R})/\ker \mathbb{E}_\mathcal{N} f$ [11, Theorem 3.0.1]. Here, the ***Weyl algebra*** $\mathfrak{W}(S_\mathcal{N}, \sigma)$ is a C^* algebra that is formed from elements of $S_\mathcal{N}$ by introducing symbols $W(f)$, labelled by $f \in S_\mathcal{N}$, that satisfy,[c]

i) $W(0) = \mathbb{1}$

ii) $W(-f) = W(f)^*$

iii) $W(f)W(g) = e^{\frac{-i}{2}\sigma(f,g)}W(f+g)$ for all $f, g \in S_\mathcal{N}$.

The symbols $W(f)$ then generate the Weyl algebra $\mathfrak{W}(S_\mathcal{N}, \sigma)$. For further details see [16, 31]. We now turn our attention to the construction of a BF state over a static slab spacetime. Let $\mathcal{N} = \mathbb{R} \times \Sigma$ be a static spacetime with metric $g = a^2\mathbb{1} \oplus -h$, where $a \in C^\infty(\Sigma)$ is everywhere positive. The BF state constructed over the static spacetime $(-\tau, \tau) \times \Sigma$ is shown to be,

$$W_{BF_f}(t,x;t',x') = \sum_{j\in\mathbb{N}} \frac{1}{2\omega_j}\left(\frac{1}{1-\delta_j}C_j(t) - iS_j(t)\right)(C_j(t') + i(1-\delta_j)S_j(t'))\xi_j(x)\overline{\xi_j(x')}, \tag{5.91}$$

where ξ_j are eigenvectors of $K = a(-\Delta + m^2)$ and,

$$\begin{aligned} S_j(t) &= f(t)\sin(\omega_j t - \theta_j) \\ C_j(t) &= f(t)\cos(\omega_j t - \theta_j) \\ \delta_j &= 1 - \frac{\|C_j\|_\tau}{\|S_j\|_\tau}, \\ \|C_j\|_\tau^2 &= \int_{(-\tau,\tau)} C_j(t)^2 \mathbf{d}t \\ \|S_j\|_\tau^2 &= \int_{(-\tau,\tau)} S_j(t)^2 \mathbf{d}t, \end{aligned} \tag{5.92}$$

[c] A C^* algebra $\mathfrak{A}$ is a normed $*-$algebra such that $\|A^*A\| = \|A\|^2$ for all $A \in \mathfrak{A}$.

and where the angle θ_j is chosen such that,

$$\int f(t)^2 \cos(\omega_j t - \theta_j) \sin(\omega_j t - \theta_j) \mathbf{d}t = 0, \tag{5.93}$$

for each $j \in \mathbb{N}$. The integral kernel of the two-point function of the static vacuum state on $\mathcal{M}$ is given by,

$$\begin{aligned} W_H(t, x; t', x') &= \sum_{j \in \mathbb{N}} \frac{1}{2\omega_j} e^{-i\omega_j (t-t')} \xi_j(x) \overline{\xi_j(x')} \\ &= \sum_{j \in \mathbb{N}} \frac{1}{2\omega_j} f(t) e^{-i\omega_j (t-t')} f(t') \xi_j(x) \overline{\xi_j(x')}, \end{aligned} \tag{5.94}$$

where the second equality holds since $f \equiv 1$ on $\mathcal{M}$. The static vacuum state (5.94) defines a Hadamard state [33]. The proof that (5.91) defines a Hadamard state amounts to proving that the normal ordered BF state, $: W_{BF_f} := W_{BF_f} - W_H$ has a smooth integral kernel. A direct computation shows that the kernel of $: W_{BF_f} :$ is,

$$: W_{BF_f} : (t, x; t', x') = \sum_{j \in \mathbb{N}} \frac{\delta_j}{2\omega_j} \left(\frac{1}{1 - \delta_j} C_j(t') C_j(t) - S_j(t') S_j(t) \right) \xi_j(x) \overline{\xi_j(x')}. \tag{5.95}$$

The integral kernel (5.95) converges in $C^\infty(\mathcal{N} \times \mathcal{N})$ if the following holds,

$$\sum_{j \in \mathbb{N}} \omega_j^p \delta_j < \infty, \tag{5.96}$$

for all $p \in \mathbb{N}$. For each $p \in \mathbb{N}$, the convergence of (5.96) implies that the kernel of the normal ordered BF state (5.95) converges in a Sobolev space of order $p \in \mathbb{N}$ and, by the Sobolev embedding theorem [46, Theorem 2.5], converges in $C^p(\mathcal{M} \times \mathcal{M})$. Hence if (5.96) holds for all $p \in \mathbb{N}$, then the integral kernel of (5.95) is smooth. The convergence of (5.96) for each $p \in \mathbb{N}$ is guaranteed since δ_j decays faster than any polynomial, which, roughly speaking, means that each summand in (5.96) can be made arbitrarily small. This stems from the fact that one may bound δ_j by the Fourier transform of f^2. Therefore, the BF state satisfies the Hadamard condition precisely because f is smooth and compactly supported. Since the BF state depends on the choice of function $f \in C_0^\infty(\mathcal{N})$ it is no longer uniquely specified from the field equations and spacetime geometry. Whilst we have somewhat loosely remarked on the Hadamard condition for the BF state, we shall return to the convergence of series similar to (5.95) later.

Chapter 6

Generalised SJ States for a Quantum Field on Globally Hyperbolic Spacetimes

The axiomatic formulation of the SJ state detailed in Chapter 5 allows the 'positive part' of the commutator function to be viewed as being the unique solution to a set of axioms. The CCR and positivity axioms ($SJ1$ and $SJ2$ respectively) incorporate features that are generic to all quasifree states; namely that the state is positive and that it satisfies the canonical commutation relations. Changing either one of these axioms will then violate these properties, so clearly any generalisation of the SJ vacuum must leave these axioms alone. However the 'ground state condition' ($SJ3$) is not a requirement for a general quasifree state, but a property that is unique to the SJ vacuum. Clearly then, by changing this axiom one can still construct the two-point function that is of positive type with the right antisymmetric part, but has differing properties to the SJ vacuum. Our goal in this chapter is precisely this, to construct a set of axioms with a unique solution which can be used to construct a quasifree state. Such a generalisation will yield a family of quasifree states, which we call generalised SJ states. We shall begin with an outline of this construction, state and motivate the axioms used and show the unique solution to these axioms and how to construct the two-point function for a quasifree generalised SJ state.

Suppose the commutator function $A = i\mathbb{E}$ extends to a self-adjoint bounded operator on $L^2(\mathcal{M})$. Let $\psi : \sigma(A) \to \mathbb{R}$ be a continuous function on the spectrum of A that obeys,

$$\begin{aligned} \psi(-\lambda) &= \psi(\lambda) && \text{CCRs} \\ \psi(\lambda) &\geq 0 && \text{Positivity} \\ \psi(0) &= 0 && \text{Field equations.} \end{aligned} \tag{6.1}$$

Our aim is to construct a quasifree state ω_{SJ_ψ} by setting its two-point function to be,

$$W_{SJ_\psi}(f, g) = \langle \overline{f} \mid A^+_\psi g \rangle \qquad \forall f, g \in C_0^\infty(\mathcal{M}), \tag{6.2}$$

where A^+_ψ is a self-adjoint bounded operator that is a solution to,

$SJ_\psi 1)$ $A^+_\psi - \Gamma A^+_\psi \Gamma = A$ CCRs.

$SJ_\psi 2)$ $A^+_\psi \geq 0$ Positivity.

$SJ_\psi 3)$ $A^+_\psi \Gamma A^+_\psi \Gamma = \psi(A)$.

The axioms $SJ_\psi 1 - 3$ are a generalisation of the axioms $SJ1 - 3$ appearing in Chapter 5. As we will show, the solution to the axioms $SJ_\psi 1 - 3$ is uniquely given by the self-adjoint bounded operator,

$$A^+_\psi = \frac{A}{2} + \sqrt{\frac{A^2}{4} + \psi(A)}. \tag{6.3}$$

We shall show that the axiom $SJ_\psi 1$ ensures the two-point function (6.2) has the correct antisymmetric part and $SJ_\psi 2$ implies that the state ω_{SJ_ψ} is positive. These axioms are, in a straightforward way, analogous to $SJ1$ and $SJ2$. Furthermore, we shall prove that if ψ is even then (6.3) satisfies $SJ_\psi 1$ and that if ψ is non-negative then (6.3) satisfies $SJ_\psi 2$. Finally we shall show if $\psi(0) = 0$ then the two-point function (6.2) satisfies the Klein-Gordon equation. Furthermore, we prove that the quadratic form defined in (6.2) is in fact the two-point function for a quasifree state ω_{SJ_ψ} over the algebra of observables for the free scalar field. Finally, we will address the question as to whether any of the states with two-point function (6.2) satisfy the Hadamard condition. Whilst we do not give a rigorous proof that all generalised SJ states will fail to be Hadamard, we note that a generalised SJ state with two-point function (6.2) will likely fail to be Hadamard because the SJ vacuum generically fails to be Hadamard. However, we continue to show that there are SJ states with two-point function (6.2) that have 'good' properties *relative to the SJ vacuum*, which is expanded upon this in section 6.2. We shall now set out the construction of a generalised SJ state described above for a free scalar field on a curved spacetime.

6.1 Generalised SJ States for a Free Scalar Field on a Curved Spacetime

Let $\mathcal{M}, \mathcal{N}$ be two globally hyperbolic spacetimes. Suppose that there exists an isometric embedding $\iota : \mathcal{M} \to \mathcal{N}$, preserving both orientations, so that $\iota(\mathcal{M})$ is relatively compact, causally convex subset of $\mathcal{N}$. Then, by proposition 5.1 the commutator function $A = i\mathbb{E}$ extends to a bounded self-adjoint operator on $L^2(\mathcal{M})$. Let $\Gamma : L^2(\mathcal{M}) \to L^2(\mathcal{M})$ denote the antilinear complex conjugation map defined by $\Gamma f = \overline{f}$. The complex conjugation map has the following properties,

$$[\Box + m^2, \Gamma] = 0, \qquad [\mathbb{E}^{\pm}, \Gamma] \quad = 0, \qquad \{A, \Gamma\} = 0. \tag{6.4}$$

and, since Γ is an involution, this implies $\Gamma \mathbb{E}^{\pm} \Gamma = \mathbb{E}^{\pm}$ and $\Gamma A \Gamma = -A$. We shall also use the following results, which follow from the spectral theorem given in [53, Theorem VII.2]:

Lemma 6.1. *Let U be a bounded self-adjoint operator on $L^2(\mathcal{M})$. If $\psi : \sigma(U) \to \mathbb{C}$ is a continuous function such that $\psi(0) = 0$, then $\ker U \subset \ker(\psi(U))$.*

Proof. For all $f \in \ker(U)$ we have $Uf = 0$. Therefore f is an eigenvector of U with eigenvalue $\lambda = 0$. The map ψ has the following property [53, Theorem VII.2 (e)],

$$Uf = \lambda f \implies \psi(U)f = \psi(\lambda)f, \tag{6.5}$$

and therefore if $Uf = 0$ we obtain,

$$\psi(U)f = \psi(\lambda)f = \psi(0)f = 0. \tag{6.6}$$

■

Lemma 6.2. *Let $U \in \mathcal{B}(L^2(\mathcal{M}))$ be a bounded self-adjoint operator that obeys $\{U, \Gamma\} = 0$ and $\psi : \sigma(U) \to \mathbb{R}$ be a continuous function. If ψ is even, that is, for all $\lambda \in \sigma(U)$ we have $\psi(\lambda) = \psi(-\lambda)$, then the operator $\psi(U)$ commutes with Γ.*

Proof. Since U is a bounded self-adjoint operator, the spectrum of U is a non-empty compact subset of $\mathbb{R}$. Therefore, by the Weierstraß approximation theorem, the algebra of polynomials on $\sigma(U)$, denoted $P(\sigma(U))$, is dense in the algebra of continuous functions on $\sigma(U)$, denoted $C(\sigma(U))$. Let $f \in C(\sigma(U))$ be an even function. Then for all $\epsilon > 0$ there exists a polynomial $p \in P(\sigma(U))$ such that,

$$\|p - f\|_{\infty} = \sup_{\lambda \in \sigma(U)} |p(\lambda) - f(\lambda)| < \epsilon. \tag{6.7}$$

If $q \in P(\sigma(U))$ is a polynomial defined by,

$$q(\lambda) = \frac{1}{2}(p(\lambda) + p(-\lambda)), \tag{6.8}$$

then the following holds,

$$\begin{aligned}
\|q - f\|_\infty &= \sup_{\lambda \in \sigma(A)} |q(\lambda) - f(\lambda)| \\
&= \sup_{\lambda \in \sigma(A)} \left| \frac{1}{2}(p(\lambda) + p(-\lambda)) - \frac{1}{2}(f(\lambda) + f(-\lambda)) \right| \\
&= \sup_{\lambda \in \sigma(A)} \left| \frac{1}{2}(p(\lambda) - f(\lambda)) + \frac{1}{2}(p(-\lambda) - f(-\lambda)) \right| \\
&\leq \frac{1}{2} \sup_{\lambda \in \sigma(A)} |p(\lambda) - f(\lambda)| + \frac{1}{2} \sup_{\lambda \in \sigma(A)} |p(-\lambda) - f(-\lambda)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned} \tag{6.9}$$

Therefore, if p approximates the even function f to arbitrary precision in sup norm, then f can also be approximated by the polynomial q given in (6.10). Since q and f are both even, it follows that the algebra of even polynomials is dense in the algebra of even continuous functions.

Now, let ψ_N be an even polynomial of degree $2N \in \mathbb{N}$, i.e.,

$$\psi_N(\lambda) = \sum_{n=0}^{N} \alpha_n \lambda^{2n}, \tag{6.10}$$

for all $\lambda \in \sigma(U)$ and all $N \in \mathbb{N}$ and where $\alpha_n \in \mathbb{R}$ for all $n \in \{0, ..., N\}$. From the continuous functional calculus, there exists a unique continuous $*-$homomorphism,

$$\begin{aligned}
\Theta : C(\sigma(U)) &\to \mathcal{B}(L^2(\mathcal{M})) \\
\psi &\mapsto \psi(U)
\end{aligned} \tag{6.11}$$

Using (6.11) there is a unique bounded self-adjoint operator corresponding to (6.10) which is given by,

$$\psi_N(U) = \sum_{n=0}^{N} \alpha_n U^{2n}, \tag{6.12}$$

and since by the assumption that $U = -\Gamma U \Gamma$, we have $U^{2n} = \Gamma U^{2n} \Gamma$ for all $n \in \mathbb{N}$. Therefore $\psi_N(U) = \Gamma \psi_N(U) \Gamma$. Now, let $\psi \in C(\sigma(U))$ be an even continuous function. Then there exists a sequence of even polynomials $(\psi_N)_{N \in \mathbb{N}}$ that converge, in sup norm, to ψ. That is to say,

$$\psi(\lambda) = \lim_{N \to \infty} \psi_N(\lambda), \tag{6.13}$$

for all $\lambda \in \sigma(U)$. Therefore, we obtain,

$$\psi(U) = \Theta(\psi) = \Theta(\lim_{N\to\infty} \psi_N) = \lim_{N\to\infty} \Theta(\psi_N) = \lim_{N\to\infty} \psi_N(U), \tag{6.14}$$

where the second limit is taken in the norm topology of $\mathcal{B}(L^2(\mathcal{M}))$ and holds by [53, Theorem VII.1 (g)]. Therefore,

$$\Gamma\psi(U)\Gamma = \Gamma(\lim_{N\to\infty} \psi_N(U))\Gamma = \lim_{N\to\infty} \Gamma\psi_N(U)\Gamma = \lim_{N\to\infty} \psi_N(U) = \psi(U). \tag{6.15}$$

Therefore, if $\psi : \sigma(U) \to \mathbb{R}$ is a continuous even function then $\psi(U) = \Gamma\psi(U)\Gamma$ and, since $\Gamma^2 = \mathbb{1}$, we have $[\psi(U), \Gamma] = 0$. ■

Our goal is to now show that the operator (6.3) is the unique solution to axioms $SJ_\psi 1 - 3$ where $\psi : \sigma(A) \to \mathbb{R}$ is a contiuous function that obeys (6.1). To accomplish this, we shall use the continuous functional calculus for bounded self-adjoint operators.

Theorem 6.3. *Let A be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$ that obeys $A = -\Gamma A\Gamma$. Suppose $\psi : \sigma(A) \to \mathbb{R}$ is an even, non-negative, continuous function. Then the operator,*

$$A_\psi^+ = \frac{A}{2} + \sqrt{\frac{A^2}{4} + \psi(A)}. \tag{6.16}$$

is the unique solution to the axioms $SJ_\psi 1 - 3$.

Proof. Assume the solution A_ψ^+ to the axioms $SJ_\psi 1 - 3$ exists and define the operator,

$$R \doteq A_\psi^+ - \frac{A}{2}. \tag{6.17}$$

By the assumption that $A = -\Gamma A\Gamma$ we find,

$$\Gamma R\Gamma = \Gamma A_\psi^+\Gamma + \frac{A}{2}, \tag{6.18}$$

and, by $SJ_\psi 1$, this implies,

$$R - \Gamma R\Gamma = A_\psi^+ - \frac{A}{2} - \Gamma A_\psi^+\Gamma - \frac{A}{2} = A - A = 0, \tag{6.19}$$

hence $R = \Gamma R\Gamma$. Furthermore, we have,

$$R + \Gamma R\Gamma = A_\psi^+ - \frac{A}{2} + \Gamma A_\psi^+\Gamma + \frac{A}{2} = A_\psi^+ + \Gamma A_\psi^+\Gamma, \tag{6.20}$$

and since $R = \Gamma R \Gamma$, this implies,

$$R = \frac{1}{2}\Big(A_\psi^+ + \Gamma A_\psi^+ \Gamma\Big), \tag{6.21}$$

and, by $SJ_\psi 2$, this implies that R is a non-negative operator. We also observe that, since ψ is even, by Lemma 6.2, $\psi(A) = \Gamma\psi(A)\Gamma$. Therefore $SJ_\psi 3$ implies the following,

$$0 = \psi(A) - \Gamma\psi(A)\Gamma = A_\psi^+ \Gamma A_\psi^+ \Gamma - \Gamma\Big(A_\psi^+ \Gamma A_\psi^+ \Gamma\Big)\Gamma = A_\psi^+ \Gamma A_\psi^+ \Gamma - \Gamma A_\psi^+ \Gamma A_\psi^+ = [A_\psi^+, \Gamma A_\psi^+ \Gamma], \tag{6.22}$$

which, in conjunction with $SJ_\psi 1$, implies,

$$[A_\psi^+, A] = [A_\psi^+, A_\psi^+ - \Gamma A_\psi^+ \Gamma] = -[A_\psi^+, \Gamma A_\psi^+ \Gamma] = 0. \tag{6.23}$$

We now obtain,

$$\begin{aligned} R^2 = R\Gamma R\Gamma &= \Big(A_\psi^+ - \frac{A}{2}\Big)\Big(\Gamma A_\psi^+ \Gamma + \frac{A}{2}\Big) \\ &= A_\psi^+ \Gamma A_\psi^+ \Gamma - \frac{A^2}{4} + \frac{1}{2}\Big(A_\psi^+ A - A\Gamma A_\psi^+ \Gamma\Big) \end{aligned} \tag{6.24}$$

which, by $SJ_\psi 1$ and $SJ_\psi 3$ reduces to,

$$\begin{aligned} R^2 &= \psi(A) - \frac{A^2}{4} + \frac{1}{2}\Big(A_\psi^+ A - A(A_\psi^+ - A)\Big) \\ &= \psi(A) - \frac{A^2}{4} + \frac{1}{2}\Big([A_\psi^+, A] + A^2\Big) \\ &= \psi(A) + \frac{A^2}{4}. \end{aligned} \tag{6.25}$$

Since $\psi : \sigma(A) \to \mathbb{R}$ is non-negative and continuous, by [53, Theorem VII.1] the operator $\psi(A)$ is bounded and non-negative. Therefore, the operator (6.25) is a non-negative bounded operator and so, by Lemma [53, Theorem VI.9], admits a unique non-negative square root given by,

$$R = \sqrt{\frac{A^2}{4} + \psi(A)}. \tag{6.26}$$

Therefore, if the operator A_ψ^+ exists, then it is uniquely given by,

$$A_\psi^+ = \frac{A}{2} + \sqrt{\frac{A^2}{4} + \psi(A)}. \tag{6.27}$$

We finally show existence. By the continuous functional calculus for bounded self-adjoint operators, the operator (6.27) is uniquely given by the continuous function,

$$\begin{aligned} a_\psi^+ &: \sigma(A) \to \mathbb{R} \\ &\lambda \mapsto \frac{\lambda}{2} + \sqrt{\frac{\lambda^2}{4} + \psi(\lambda)}, \end{aligned} \tag{6.28}$$

by setting $A^+_\psi = a^+_\psi(A)$. We also note that the operator (6.26) is given by the continuous function,

$$\begin{aligned} r &: \sigma(A) \to \mathbb{R} \\ \lambda &\mapsto \sqrt{\frac{\lambda^2}{4} + \psi(\lambda)}, \end{aligned} \tag{6.29}$$

where we set $R = r(A)$. Since, by assumption, $A = -\Gamma A \Gamma$, we see that the operator (6.27) is a solution to $SJ_\psi 1$ if and only if $R = r(A)$ satisfies,

$$R = \Gamma R \Gamma. \tag{6.30}$$

which, by Lemma 6.2 is equivalent to showing that (6.29) is even. Since, by assumption, $\psi(\lambda) = \psi(-\lambda)$ for all $\lambda \in \sigma(A)$ it follows that $r(\lambda) = r(-\lambda)$ for all $\lambda \in \sigma(A)$ and therefore, by Lemma 6.2, the commutation relation (6.30) holds. Therefore (6.27) satisfies $SJ_\psi 1$. Since, by assumption $\psi(\lambda) \geq 0$ for all $\lambda \in \sigma(A)$ we have,

$$\begin{aligned} \psi(\lambda) \geq 0 &\implies \frac{\lambda^2}{4} + \psi(\lambda) \geq \frac{\lambda^2}{4} \\ &\implies \sqrt{\frac{\lambda^2}{4} + \psi(\lambda)} \geq \frac{|\lambda|}{2} \\ &\implies \frac{\lambda}{2} + \sqrt{\frac{\lambda^2}{4} + \psi(\lambda)} \geq \frac{|\lambda|}{2} + \frac{\lambda}{2} \geq 0, \end{aligned} \tag{6.31}$$

and therefore $a^+_\psi(\lambda) \geq 0$ for all $\lambda \in \sigma(A)$. Therefore by [53, Theorem VII.2 (f)], since a^+_ψ is a non-negative function it follows that $A^+_\psi = a^+_\psi(A)$ is non-negative operator. Hence (6.27) satisfies $SJ_\psi 2$. We finally we show that (6.27) solves $SJ_\psi 3$,

$$\begin{aligned} A^+_\psi \Gamma A^+_\psi \Gamma &= \left(\frac{A}{2} + \sqrt{\frac{A^2}{4} + \psi(A)}\right)\left(\frac{-A}{2} + \Gamma\sqrt{\frac{A^2}{4} + \psi(A)}\Gamma\right) \\ &= \left(\frac{A}{2} + \sqrt{\frac{A^2}{4} + \psi(A)}\right)\left(\frac{-A}{2} + \sqrt{\frac{A^2}{4} + \psi(A)}\right) \\ &= -\frac{A^2}{4} + \frac{1}{2}\left[A, \sqrt{\frac{A^2}{4} + \psi(A)}\right] + \frac{A^2}{4} + \psi(A) \\ &= \psi(A), \end{aligned} \tag{6.32}$$

where the commutator appearing in (6.32) holds by [53, Theorem VII.2 (g)]. Therefore, the operator (6.27) exists and is the unique solution to the axioms $SJ_\psi 1 - 3$. ■

Starting from the commutator function $A = i\mathbb{E}$ on the spacetime $\mathcal{M}$ and a non-negative, continuous, even function $\psi : \sigma(A) \to \mathbb{R}$ that obeys $\psi(0) = 0$ we can construct the unique solution A^+_ψ to the axioms

$SJ_\psi 1-3$ given by the operator (6.27). Using the operator (6.27) we can then define the two-point function,

$$W_{SJ_\psi}(f,g) = \langle \overline{f} \mid A_\psi^+ g \rangle, \tag{6.33}$$

for all $f, g \in C_0^\infty(\mathcal{M})$. One may recover the original SJ vacuum by setting $\psi = 0$. We now verify that the two-point function (6.33) has the correct properties:

Positivity: The SJ state ω_{SJ_ψ} is positive if and only if the two-point function satisfies,

$$W_{SJ_\psi}(\overline{f}, f) = \langle f \mid A_\psi^+ f \rangle \geq 0 \qquad\qquad \forall f \in C_0^\infty(\mathcal{M}), \tag{6.34}$$

which clearly holds since $A_\psi^+ \geq 0$.

CCRs: The two-point function (6.33) has the following antisymmetric part,

$$\begin{aligned} W_{SJ_\psi}(f,g) - W_{SJ_\psi}(g,f) &= \langle \overline{f} \mid A_\psi^+ g \rangle - \langle \overline{g} \mid A_\psi^+ f \rangle \\ &= \langle \overline{f} \mid (\frac{A}{2} + R - \frac{1}{2}\Gamma A \Gamma - \Gamma R \Gamma) g \rangle \\ &= \langle \overline{f} \mid A g \rangle, \end{aligned} \tag{6.35}$$

which holds since $A = -\Gamma A \Gamma$ and since (6.29) obeys $r(\lambda) = r(-\lambda)$, by Lemma 6.2 we have $r(A) = R = \Gamma R \Gamma$. Therefore the two-point function (6.33) coincides with the canonical commutation relations.

Field Equations: We now verify, for all $f, g \in C_0^\infty(\mathcal{M})$,

$$W_{SJ_\psi}(Pf, g) = 0 = W_{SJ_\psi}(f, Pg), \tag{6.36}$$

where $P = \Box + m^2$ is the Klein-Gordon operator. Since the Klein-Gordon operator is formally self-adjoint, the condition (6.36) is equivalent to showing,

$$P A_\psi^+ = 0 = A_\psi^+ P. \tag{6.37}$$

Now, since $\mathbb{E}P = 0 = P\mathbb{E}$, it follows that $[A, P] = 0$. Therefore, by [53, Theorem VII.2 (g)], any continuous function of A commutes with P. Hence $[A_\psi^+, P] = [a_\psi^+(A), P] = 0$. Therefore, it is sufficient to check that for all $g \in C_0^\infty(\mathcal{M})$ we have $A_\psi^+(Pg) = 0$. By definition and by (3.4) we have, for every $g \in C_0^\infty(\mathcal{M})$,

$$A(Pg) = i\mathbb{E}(Pg) = 0, \tag{6.38}$$

which shows that $Pg \in \ker(A)$ for all $g \in C_0^\infty(\mathcal{M})$. Therefore, we obtain,

$$\begin{aligned} W_{SJ}(f, Pg) &= \langle \overline{f} \mid A_\psi^+ Pg \rangle \\ &= \langle \overline{f} \mid \frac{1}{2} A(Pg) + R(Pg) \rangle \\ &= 0, \end{aligned} \tag{6.39}$$

where we have used the fact that since $Pg \in \ker(A)$ and the function (6.29) obeys $r(0) = 0$, Lemma 6.1 implies that $Pg \in \ker(R)$.

Before proving that (6.2) defines the two-point function for the quasifree SJ state ω_{SJ_ψ}, we shall use the following. One can rewrite the two-point function (6.2) in the following manner,

$$\begin{aligned} W_{SJ_\psi}(f, g) &= \langle \overline{f} \mid A_\psi^+ g \rangle \\ &= \langle \overline{f} \mid \left(\frac{A}{2} + R \right) g \rangle \\ &= \frac{i}{2} \langle \overline{f} \mid \mathbb{E} g \rangle + \langle \overline{f} \mid R g \rangle \\ &\doteq \frac{i}{2} \sigma([f], [g]) + \mu_\psi([f], [g]), \end{aligned} \tag{6.40}$$

where,

$$\begin{aligned} \sigma : S(\mathcal{M}) \times S(\mathcal{M}) &\to \mathbb{R} \\ ([f], [g]) &\mapsto \langle f \mid \mathbb{E} g \rangle, \end{aligned} \tag{6.41}$$

$$\begin{aligned} \mu_\psi : S(\mathcal{M}) \times S(\mathcal{M}) &\to \mathbb{R} \\ ([f], [g]) &\mapsto \langle f \mid R g \rangle, \end{aligned} \tag{6.42}$$

where the operator $R = r(A)$ is given by the function (6.29) and where,

$$S(\mathcal{M}) = C_0^\infty(\mathcal{M}, \mathbb{R}) / P(C_0^\infty(\mathcal{M}, \mathbb{R})) \tag{6.43}$$

is the real linear vector space of equivalence classes $[f]$ with respect to the equivalence relation,

$$f \sim f' \iff \mathbb{E}(f - f') = 0. \tag{6.44}$$

By Proposition [45, Proposition 8] the map (6.41) is a well defined real valued symplectic form. We now show that the map (6.42) is a well defined real valued scalar product. Choose $g, g' \in C_0^\infty(\mathcal{M}, \mathbb{R})$ such that $g - g' \in \ker(\mathbb{E})$, i.e. two distinct representatives of the same equivalence class. Then

$g - g' \in \ker(A)$, and, moreover, since (6.29) obeys $r(0) = 0$ we have $g - g' \in \ker(R)$. Therefore, we obtain,

$$\mu_\psi([f],[g]) - \mu_\psi([f],[g']) = \langle f \mid Rg\rangle - \langle f \mid Rg'\rangle = \langle f \mid R(g - g')\rangle = 0, \tag{6.45}$$

which implies,

$$\mu_\psi([f],[g]) = \mu_\psi([f],[g']) \tag{6.46}$$

Therefore the scalar product (6.42) is independent of the chosen representative in its second argument. Moreover, since R is self-adjoint, the scalar product (6.42) is symmetric, which implies that, since it is independent of representation in the second slot, it is also independent of representation in the first slot. Therefore (6.42) is a well defined scalar product. Finally, we see that (6.42) is real valued from the following; for all $[f],[g] \in S(\mathcal{M})$ we have,

$$\begin{aligned} \overline{\mu_\psi([f],[g])} &= \overline{\langle f \mid Rg\rangle} \\ &= \langle \Gamma f \mid \Gamma R g\rangle \\ &= \langle \Gamma f \mid R\Gamma g\rangle \\ &= \mu_\psi([f],[g]), \end{aligned} \tag{6.47}$$

which holds since $\Gamma f = f$, $\Gamma g = g$ and $[R,\Gamma] = 0$ holds by Lemma 6.2. We now show that (6.33) is the two-point function for the quasifree SJ state ω_{SJ_ψ}.

Proposition 6.4. *Let $P = \Box + m^2$ be the Klein Gordon operator defined on a globally hyperbolic spacetime $\mathcal{M}$ and let $\mathcal{A}(\mathcal{M})$ be the algebra of observables for the free massive scalar field. Suppose the commutator function $A = i\mathbb{E}$ extends to a self-adjoint bounded operator on $L^2(\mathcal{M})$. Let,*

$$A^+_\psi = \frac{A}{2} + \sqrt{\frac{A^2}{4} + \psi(A)}, \tag{6.48}$$

be the unique solution to $SJ_\psi 1 - 3$ where $\psi : \sigma(A) \to \mathbb{R}$ is an even, non-negative continuous function that obeys $\psi(0) = 0$. Then a quasifree state $\omega_{SJ_\psi} : \mathcal{A}(\mathcal{M}) \to \mathbb{C}$ can be constructed by setting its two-point function to be,

$$W_{SJ_\psi}(f,g) = \langle \overline{f} \mid A^+_\psi g\rangle, \tag{6.49}$$

for all $f,g \in C_0^\infty(\mathcal{M})$.

Proof. By proposition 4.2 it is sufficient to show that,

$$\nu(\mathbb{E}f, \mathbb{E}g) = W_{SJ_\psi}(\overline{f}, g), \tag{6.50}$$

defines a non-negative bilinear form. The two-point function (6.49) is clearly bilinear and, as (6.34) shows, is positive. Therefore (6.50) defines a positive bilinear form and by Proposition 4.2 there exists a quasifree state ω_{SJ_ψ} whose two-point function is given by (6.49). ■

We have shown that, starting with the self-adjoint bounded operator $A = i\mathbb{E}$, a globally hyperbolic spacetime $(\mathcal{M}, g)$ and an even, non-negative, continuous function $\psi(A) : \sigma(A) \to \mathbb{R}$ that obeys $\psi(0) = 0$ one can construct the unique solution A^+_ψ from the axioms $SJ_\psi 1 - 3$ and define the two-point function for a quasifree state via (6.2). We shall now consider the Hadamard condition for the family of SJ states with two-point functions (6.2) and show that there are SJ states that have finite derivatives of the Wick square when normal ordered with respect to the SJ vacuum.

6.2 Wick Polynomials and the Hadamard Condition

Let $\psi : \sigma(A) \to \mathbb{R}$ be an even, non-negative continuous function that obeys $\psi(0) = 0$ and let,

$$A^+_\psi = \frac{A}{2} + \sqrt{\frac{A^2}{4} + \psi(A)}, \tag{6.51}$$

be the unique solution to axioms $SJ_\psi 1 - 3$. By Proposition 6.4 the two-point function for a quasifree state can be defined by the prescription (6.49). Before addressing what properties such a state has, we shall make use of the following parametrisation. The operator (6.51) is uniquely given by the function,

$$\begin{aligned} a^+_\psi &: \sigma(A) \to \mathbb{R} \\ &\lambda \mapsto \frac{\lambda}{2} + \sqrt{\frac{\lambda^2}{4} + \psi(\lambda)}. \end{aligned} \tag{6.52}$$

The function (6.52) then gives the operator A^+_ψ given in (6.51) via $A^+_\psi = a^+_\psi(A)$. We define an even, non-negative, continuous function $\varphi : \sigma(A) \to \mathbb{R}$ by completing the square in (6.52),

$$\frac{\lambda^2}{4} + \psi(\lambda) = \left(\frac{|\lambda|}{2} + \varphi(\lambda)\right)^2, \tag{6.53}$$

rearranging shows that,

$$\varphi(\lambda) = \frac{1}{2}(-|\lambda| + \sqrt{\lambda^2 + 4\psi(\lambda)}) \tag{6.54}$$

from which we see that $\varphi(0) = 0$. Rearranging (6.54) gives,

$$\psi(\lambda) = \varphi(\lambda)(|\lambda| + \varphi(\lambda)), \tag{6.55}$$

for all $\lambda \in \sigma(A)$ and by substituting (6.55) into (6.52) we find,

$$a_\psi^+(\lambda) = \frac{\lambda}{2} + \frac{|\lambda|}{2} + \varphi(\lambda). \tag{6.56}$$

We shall now use the function (6.56) to define the operator $A_\varphi^+ = a_\psi^+(A)$ and the two-point function,

$$\begin{aligned} W_{SJ_\varphi}(f,g) &= \langle \overline{f} \mid a_\psi^+(A) g \rangle \\ &= \langle \overline{f} \mid \frac{1}{2}(A + |A|) g \rangle + \langle \overline{f} \mid \varphi(A) g \rangle \\ &= W_{SJ_0}(f,g) + \langle \overline{f} \mid \varphi(A) g \rangle, \end{aligned} \tag{6.57}$$

for all $f, g \in C_0^\infty(\mathcal{M})$ and where W_{SJ_0} is the two-point function for the vacuum SJ state. We shall now investigate whether there are any functions $\varphi : \sigma(A) \to \mathbb{R}$ that can be used so that (6.57) defines the two-point function for a Hadamard state. Let W_H be a two-point function for a Hadamard state, which exists due to the deformation argument given by Fulling, Narcowich and Wald [32]. Then the two-point function (6.57) defines a Hadamard state if and only if the normal ordered two-point function,

$$\begin{aligned} : W_{SJ_\varphi} : (f,g) &= W_{SJ_0}(f,g) + \langle \overline{f} \mid \varphi(A) g \rangle - W_H(f,g) \\ &\doteq : W_{SJ_0} : (f,g) + \langle \overline{f} \mid \varphi(A) g \rangle \end{aligned} \tag{6.58}$$

has a smooth integral kernel on $\mathcal{M} \times \mathcal{M}$. The SJ vacuum on the ultrastatic slab $(-\tau, \tau) \times \Sigma$ fails to be Hadamard [28, Theorem 4.2], and the authors note that it is likely the SJ vacuum will fail to be Hadamard on a general spacetime. Therefore, if the SJ vacuum fails to be Hadamard on a general spacetime, the integral kernel of $: W_{SJ_0} :$ will fail to be smooth. If we assume that the SJ vacuum fails to be Hadamard, then the generalised SJ state will be Hadamard. Therefore, one must choose φ so that the operator $\varphi(A)$ cancels off the non-smooth contributions coming from the kernel of $: W_{SJ_0} :$. Whilst we do not prove this here, it is unlikely that there exists a continuous, even, non-negative function $\varphi : \sigma(A) \to \mathbb{R}$ so that the integral kernel of (6.58) is smooth. It is reasonable to assume that all generalised SJ states fail to be Hadamard. However, we can observe the following. If one normal orders the SJ state (6.57) with respect to the vacuum SJ state,

$$\begin{aligned} : W_{SJ_\varphi}(f,g) : &= W_{SJ_\varphi}(f,g) - W_{SJ_0}(f,g) \\ &= \langle \overline{f} \mid \varphi(A) g \rangle. \end{aligned} \tag{6.59}$$

and if $\varphi(A)$ is smoothing, that is, it is of the form,

$$(\varphi(A)f)(t,x) = \int_{\mathcal{M}} K_{\varphi}(t,x;t',x')f(t',x')\ \mathbf{d}\mathrm{vol}_g(t',x'), \tag{6.60}$$

where $K_{\varphi} \in C^{\infty}(\mathcal{M}\times\mathcal{M})$ is a smooth integral kernel, then the normal ordered two-point function (6.59) has well defined Wick polynomial. This means that, the derivatives of the Wick square of the normal ordered two-point function (6.59) are finite and are given by,

$$\begin{aligned}\langle :(L\phi)^2:(f)\rangle &= \int_{\mathcal{M}} (L\otimes L(:W_{SJ_{\varphi}}:))(t,x;t,x)f(t,x)\ \mathbf{d}\mathrm{vol}_g(t,x)\\ &= \int_{\mathcal{M}} (L\otimes L(W_{SJ_{\varphi}} - W_{SJ_0}))(t,x;t,x)f(t,x)\ \mathbf{d}\mathrm{vol}_g(t,x)\\ &= \int_{\mathcal{M}} (L\otimes L(W_{SJ_0} + \varphi(A) - W_{SJ_0}))(t,x;t,x)f(t,x)\ \mathbf{d}\mathrm{vol}_g(t,x)\\ &= \int_{\mathcal{M}} (L\otimes L(K_{\varphi}))(t,x;t,x)f(t,x)\ \mathbf{d}\mathrm{vol}_g(t,x),\end{aligned} \tag{6.61}$$

for all $f \in C_0^{\infty}(\mathcal{M})$ and where L is any partial differential operator. Since the integral kernel K_{φ} is assumed to be smooth on $\mathcal{M}\times\mathcal{M}$, all the Wick square of all derivatives of the field exist. Therefore, relative to the SJ vacuum, the two-point function (6.57) well defined Wick polynomials. On the contrary, if $\varphi(A)$ is *not* smoothing, then there will be derivatives of the Wick square which diverge. In this framework it then becomes apparent that if one wants well behaved Wick polynomials for the normal ordered two-point function (6.59) the operator $\varphi(A)$ must be smoothing. This then leads to the natural question,

What conditions on $\varphi : \sigma(A) \to \mathbb{R}$ ensure that $\varphi(A)$ is smoothing?

Whilst we do not answer the question for a general spacetime here, we do, however, give sufficient conditions on the function $\varphi : \sigma(A) \to \mathbb{R}$ such that $\varphi(A)$ is smoothing in the case when the commutator function $A = i\mathbb{E}$ is defined on the ultrastatic slab $(\mathcal{M} = (-\tau,\tau)\times\Sigma, g = \mathbb{1}\oplus -h)$. This will be the subject of the next section.

6.3 Smoothing Operators from the Commutator Function on Ultrastatic Slabs

Let $((-\tau,\tau)\times\Sigma, g=\mathbb{1}\oplus -h)$, $\tau>0$, be an ultrastatic slab spacetime where (Σ,h) is a compact, boundaryless Riemannian manifold. Let $A=i\mathbb{E}$ be a self-adjoint bounded operator and let $\varphi:\sigma(A)\to\mathbb{R}$ be a continuous function.

Lemma 6.5. *Let (Σ,h) be a boundaryless, compact Riemannian manifold of dimension $s\in\mathbb{N}$. Let $K=-\Delta+m^2$, where $m>0$ and Δ is the Laplacian on (Σ,h), be a second order elliptic operator on $L^2(\Sigma)$ with corresponding eigenvalues ω_j^2, $\omega_j>0$, for all $j\in\mathbb{N}$. Then, there exists $\alpha>0$ such that,*

$$j\leq\alpha\omega_j^s. \tag{6.62}$$

Proof. Since (Σ,h) is a boundaryless, compact Riemannian manifold of dimension s, Weyl's law states that the asymptotic distribution of the eigenvalues ω_j^2 are given by [40],

$$N(\omega_j)\sim(2\pi)^{-n}\Omega_n\mathrm{vol}(\Sigma)\omega_j^s, \tag{6.63}$$

where Ω_n is the volume of a $n-$ball. Therefore, there exists a $\alpha>0$ sufficiently large such that $N(\omega_j)\leq\alpha\omega_j^s$. Then, by counting, we have,

$$j\leq M(j)=N(\omega_j)\leq\alpha\omega_j^s. \tag{6.64}$$

■

We now prove the following:

Proposition 6.6. *Let $(\mathcal{M}=(-\tau,\tau)\times\Sigma, g=\mathbb{1}\oplus -h)$ be an ultrastatic slab spacetime where (Σ,h) is a three dimensional, smooth, compact Riemannian manifold. Let $A=i\mathbb{E}$ be the commutator function given by*

$$A=\sum_{j\in\mathbb{N}}\frac{\lambda_j}{\|\Psi^+\|^2}\left[|\Psi_j^+\rangle\langle\Psi_j^+|-|\Psi_j^-\rangle\langle\Psi_j^-|\right], \tag{6.65}$$

where,

$$\begin{aligned}\Psi_j^{\pm}(t,x)&=\left(C_j(t)\mp i\frac{\|C_j\|}{\|S_j\|}S_j(t)\right)\xi_j(x),\\ \lambda_j&=\frac{\|C_j\|\|S_j\|}{\omega_j}=\frac{\tau}{\omega_j}\sqrt{1-\mathrm{sinc}^2(2\omega_j\tau)}\end{aligned} \tag{6.66}$$

and where,

$$
\begin{aligned}
\|S_j\|^2 &= \tau(1 - \operatorname{sinc}(2\omega_j \tau)), \\
\|C_j\|^2 &= \tau(1 + \operatorname{sinc}(2\omega_j \tau)), \\
\|\Psi_j^+\|^2 &= 2\|C_j\|^2.
\end{aligned}
\tag{6.67}
$$

If $\varphi : \sigma(A) \to \mathbb{R}$ is a continuous function such that, for all $N \in \mathbb{N}_0$ there exists a $C_N \geq 0$ such that $|\varphi(\lambda)| \leq C_N |\lambda^N|$, then the operator $\varphi(A)$ is smoothing.

Proof. By the continuous functional calculus for self-adjoint bounded operators, the operator $\varphi(A)$ is given by,

$$
\begin{aligned}
\varphi(A) &= \sum_{j \in \mathbb{N}} \frac{\varphi(\lambda_j)}{\|\Psi^+\|^2} \left[|\Psi_j^+\rangle \langle \Psi_j^+| - |\Psi_j^-\rangle \langle \Psi_j^-| \right] \\
&= \sum_{j \in \mathbb{N}} \frac{\varphi(\lambda_j)}{\|\Psi^+\|^2} \left(\frac{\|C_j\|}{\|S_j\|} \right) \left[|\mathcal{A}_j\rangle \langle \mathcal{A}_j| - |\overline{\mathcal{A}_j}\rangle \langle \overline{\mathcal{A}_j}| \right],
\end{aligned}
\tag{6.68}
$$

where $A_j(t,x) = e^{-i\omega_j t} \xi_j(x)$, with the integral kernel,

$$
\varphi(A)(t,x;t',x') = \sum_{j \in \mathbb{N}} \frac{\varphi(\lambda_j)}{\|\Psi^+\|^2} \left(\frac{\|C_j\|}{\|S_j\|} \right) \left[e^{-i\omega_j (t-t')} - e^{i\omega_j (t-t')} \right] \xi_j(x) \xi_j(x'). \tag{6.69}
$$

By Theorem 4.7, the integral kernel (6.69) converges in $C^\infty(\mathcal{M} \times \mathcal{M})$ if the following holds for all $p \in \mathbb{N}_0$,

$$
\sum_{j \in \mathbb{N}} \omega_j^p \left| \frac{\varphi(\lambda_j)}{\|\Psi^+\|^2} \left(\frac{\|C_j\|}{\|S_j\|} \right) \right|^2 < \infty. \tag{6.70}
$$

We now find the following, for all $p, N \in \mathbb{N}_0$

$$
\begin{aligned}
\sum_{j \in \mathbb{N}} \omega_j^p \left| \frac{\varphi(\lambda_j)}{\|\Psi^+\|^2} \left(\frac{\|C_j\|}{\|S_j\|} \right) \right|^2 &= \sum_{j \in \mathbb{N}} \omega_j^p \left| \frac{\varphi(\lambda_j)}{2\lambda_j \omega_j} \right|^2 \\
&= \sum_{j \in \mathbb{N}} \frac{\omega_j^{p-2}}{4} \left| \frac{\varphi(\lambda_j)}{\lambda_j} \right|^2 \\
&\leq \frac{C_N^2}{4} \sum_{j \in \mathbb{N}} \omega_j^{p-2} \left| \frac{\lambda_j^N}{\lambda_j} \right|^2 \\
&= \frac{\tau^{2(N-1)} C_N^2}{4} \sum_{j \in \mathbb{N}} \omega_j^{p-2} \left| \frac{\sqrt{1 - \operatorname{sinc}^2(2\omega_j \tau)}}{\omega_j} \right|^{2(N-1)} \\
&\leq \frac{\tau^{2(N-1)} C_N^2}{4} \sum_{j \in \mathbb{N}} \frac{1}{\omega_j^{2N-p}}.
\end{aligned}
\tag{6.71}
$$

Now, since (Σ, h) is a three dimensional boundaryless, compact Riemannian manifold, by Lemma 6.5 we have $j \leq \alpha\omega_j^3$ for a sufficiently large $\alpha > 0$. Therefore,

$$\sum_{j\in\mathbb{N}} \omega_j^p \left| \frac{\varphi(\lambda_j)}{\|\Psi^+\|^2} \left(\frac{\|C_j\|}{\|S_j\|} \right) \right|^2 \leq \frac{\alpha^{\frac{1}{3}(2N-p)} \tau^{2(N-1)} C_N^2}{4} \sum_{j\in\mathbb{N}} \frac{1}{j^{(2N-p)/3}}, \tag{6.72}$$

which converges for all $N > \frac{1}{2}(p+3)$. Therefore, by Theorem 4.7, the integral kernel (6.69) converges in $C^\infty(\mathcal{M} \times \mathcal{M})$. Therefore $\varphi(A)$ is smoothing. ■

The previous proposition then prompts the following conjecture for a general globally hyperbolic spacetime $(\mathcal{M}, g)$,

Conjecture 6.7. *Suppose $A = i\mathbb{E}$ extends to a bounded, self-adjoint operator on $L^2(\mathcal{M})$ and let $\varphi : \sigma(A) \to \mathbb{R}$ be a continuous funciton. If, for all $N \in \mathbb{N}_0$ there exists a $C_N \geq 0$ such that $|\varphi(\lambda)| \leq C_N \lambda^N$ then the operator $\varphi(A)$ is smoothing.*

In this chapter we have shown that, under suitable conditions on the function ψ, there exists a unique solution to the axioms $SJ_\psi 1 - 3$ that can be used to construct a two-point function for a quasifree state. This then extends the SJ state construction from a single state to a family of states parameterised by ψ. Whilst it is unlikely that the family of generalised SJ states contain any Hadamard states, we do give a new construction of a class of Hadamard states using the generalised SJ state construction. Before this, we will provide a construction of a generalised SJ state which we call the 'thermal' SJ state in the following chapter.

Chapter 7

Thermal SJ States for the Spin Zero Field on Globally Hyperbolic Spacetimes

The SJ axioms, originally given by Sorkin in [58], can be modified so that they admit a family of solutions, whereby each solution can be used to construct a two point function for a quasifree state. This construction was given, in a general manner, in Chapter 6. It is the purpose of this chapter to give an application of the work in Chapter 6 by constructing a 'thermal' SJ state on ultrastatic slabs with compact spatial sections.

7.1 Thermal SJ States

To elucidate our construction of a thermal SJ state, we draw observations from the SJ vacuum and thermal equilibrium state constructed on the ultrastatic slab spacetime $\mathcal{M} = (-\tau, \tau) \times \Sigma$ with metric $g = \mathbb{1} \oplus -h$. The ultrastatic vacuum state, a thermal equilibrium state at inverse temperature $\beta > 0$

and the SJ vacuum state on the ultrastatic slab $\mathcal{M}$ are given by the two-point functions,

$$\begin{aligned} W_\infty(f,g) &= \langle \overline{f} \mid G_\infty g\rangle, \\ W_\beta(f,g) &= \langle \overline{f} \mid G_\beta g\rangle, \\ W_{SJ_0}(f,g) &= \langle \overline{f} \mid A_0^+ g\rangle \end{aligned} \tag{7.1}$$

for all $f, g \in C_0^\infty(\mathcal{M})$ where the self-adjoint, bounded operators G_∞, G_β and A_0^+ are defined by,

$$\begin{aligned} G_\infty &= \sum_{j\in\mathbb{N}} \frac{1}{2\omega_j} \left|\mathcal{A}_j\right\rangle \left\langle \mathcal{A}_j\right| \\ G_\beta &= \sum_{j\in\mathbb{N}} \frac{1}{2\omega_j} \frac{1}{1-e^{-\beta\omega_j}} \left(\left|\mathcal{A}_j\right\rangle \left\langle \mathcal{A}_j\right| + e^{-\beta\omega_j} \left|\overline{\mathcal{A}_j}\right\rangle \left\langle \overline{\mathcal{A}_j}\right| \right) \\ &= G_\infty + \sum_{j\in\mathbb{N}} \frac{1}{2\omega_j} \frac{e^{-\beta\omega_j}}{1-e^{-\beta\omega_j}} \left(\left|\mathcal{A}_j\right\rangle \left\langle \mathcal{A}_j\right| + \left|\overline{\mathcal{A}_j}\right\rangle \left\langle \overline{\mathcal{A}_j}\right| \right), \\ A_0^+ &= \frac{1}{2}(A + |A|) \end{aligned} \tag{7.2}$$

respectively. As before, we have $\mathcal{A}_j = e^{-i\omega_j t}\xi_j(x)$ and $\{\xi_j\}_{j\in\mathbb{N}}$ are a set of orthonormal eigenvectors of the operator $K = -\Delta + m^2$ that form a basis of $L^2(\Sigma)$ with eigenvalues $\omega_j^2 \geq m^2$ where $m > 0$. We shall demand that the two-point function for a thermal SJ state, denoted W_{SJ_β}, should obey the following limits,

$$\lim_{\tau\to\infty} W_{SJ_\beta}(f,g) = W_\beta(f,g) \tag{7.3}$$

$$\lim_{\beta\to\infty} W_{SJ_\beta}(f,g) = W_{SJ_0}(f,g), \tag{7.4}$$

for all $f, g \in C_0^\infty(\mathcal{M})$ and where the limits are taken in the weak topology on $\mathcal{B}(L^2(\mathcal{M}))$. These limits are summarised in Figure 7.1.

In addition to this, the construction of a thermal SJ state should respect the original ideology of the SJ state construction, namely, the two-point function W_{SJ_β} should be constructed from the geometry of the underlying spacetime and the commutator function $A = i\mathbb{E}$ alone. To accomplish this, we want to construct a continuous function on the spectrum of A so that the corresponding operator can be used to construct a two-point function that obeys the limits (7.3). Suppose $a_\beta^+ : \sigma(A) \to \mathbb{R}$ is such a function and let $A_\beta^+ = a_\beta^+(A)$ be the corresponding self-adjoint, bounded operator on $L^2(\mathcal{M})$. Then if the limits,

$$\lim_{\beta\to\infty} A_\beta^+ = A_0^+ \tag{7.5}$$

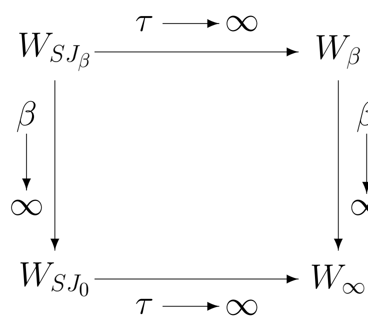


Figure 7.1: Properties of thermal SJ state in the limits $\beta \to \infty$ and $\tau \to \infty$.

$$\lim_{\tau\to\infty} A^+_\beta = G_\beta, \tag{7.6}$$

hold in the norm topology on $\mathcal{B}(L^2(\mathcal{M}))$ then the limits (7.3) will hold in the weak topology on $\mathcal{B}(L^2(\mathcal{M}))$.[a] In order to expand upon this, we shall use the following observations of the eigenvectors and eigenvalues of the commutator function $A = i\mathbb{E}$ on the ultrastatic slab $(\mathcal{M} = (-\tau, \tau) \times \Sigma, g = \mathbb{1} \oplus -h)$ given in Chapter 5 Section 5.5. The (unnormalised) eigenvectors for the operator $A = i\mathbb{E}$ on the ultrastatic slab $(-\tau,\tau)\times\Sigma$ are given by,

$$\Psi^\pm_j(t,x) = \left(C_j(t) \mp i\frac{\|C_j\|}{\|S_j\|}S_j(t)\right)\xi_j(x), \tag{7.7}$$

with corresponding eigenvalues,

$$\lambda^+_j = \frac{\tau}{\omega_j}\sqrt{1-\mathrm{sinc}^2(2\omega_j\tau)}, \tag{7.8}$$

where $S_j(t) = \sin(\omega_j t)$ and $C_j(t) = \cos(\omega_j t)$ for all $t \in (-\tau,\tau)$ with norms,

$$\begin{aligned}\|S_j\| &= \sqrt{\tau(1-\mathrm{sinc}(2\omega_j\tau))}\\ \|C_j\| &= \sqrt{\tau(1+\mathrm{sinc}(2\omega_j\tau))}\end{aligned} \tag{7.9}$$

Therefore, point-wise, the eigenvectors (7.7) have the limits,

$$\begin{aligned}\lim_{\tau\to\infty}\Psi^+_j(t,x) &= (C_j(t) - iS_j(t))\xi_j(x) = e^{-i\omega_j t}\xi_j(x) = \mathcal{A}_j(t,x)\\ \lim_{\tau\to\infty}\Psi^-_j(t,x) &= (C_j(t) + iS_j(t))\xi_j(x) = e^{i\omega_j t}\xi_j(x) = \overline{\mathcal{A}_j}(t,x),\end{aligned} \tag{7.10}$$

[a]This holds since the weak topology is weaker than the norm topology, and so convergenece in the norm topology implies convergence in the weak topology

along with the limit,

$$\lim_{\tau\to\infty}\frac{\lambda_j}{\|\Psi_j^+\|^2}=\lim_{\tau\to\infty}\frac{1}{2\omega_j}\sqrt{\frac{1-\operatorname{sinc}(2\omega_j\tau)}{1+\operatorname{sinc}(2\omega_j\tau)}}=\frac{1}{2\omega_j}. \tag{7.11}$$

Using these observations, we are now ready to define the continuous function $a_\beta^+ : \sigma(A)\to\mathbb{R}$ that will be used to construct the operator A_β^+,

$$\begin{aligned} a_\beta^+ &: \sigma(A)\to\mathbb{R} \\ &\lambda\mapsto\frac{1}{2}(\lambda+|\lambda|)+\varphi_\beta(\lambda), \end{aligned} \tag{7.12}$$

where,

$$\begin{aligned} \varphi_\beta &: \sigma(A)\to\mathbb{R} \\ &\lambda\mapsto|\lambda|\frac{e^{-\beta T|\lambda|^{-1}}}{1-e^{-\beta T|\lambda|^{-1}}}, \end{aligned} \tag{7.13}$$

and where $T>0$ is a timescale. We must introduce a timescale in (7.13) in order to make the argument in the exponential dimensionless and to ensure that the limit (7.6) holds. We define the timescale T by the following condition,

$$\lim_{\tau\to\infty}\frac{\beta T}{|\lambda_j|}=\beta\omega_j, \tag{7.14}$$

for all $j\in\mathbb{N}$. We note that the eigenvalues (7.8) have the asymptotic behaviour $\lambda_j\sim\dfrac{\tau}{\omega_j}$ for all $j\in\mathbb{N}$. Therefore,

$$\frac{\beta T}{|\lambda_j|}\sim\frac{\beta T\omega_j}{\tau}, \tag{7.15}$$

which implies that, in the ultrastatic slab $\mathcal{M}=(-\tau,\tau)\times\Sigma$, the timescale is $T=\tau$, since any other timescale would violate the condition (7.14). Therefore in the ultrastatic slab case the timescale is half the duration of the longest casual curve. Using $T=\tau$, the operator corresponding to the function (7.12) is given by,

$$A_\beta^+=a_\beta^+(A)=A_0^+ + |A|\frac{e^{-\beta\tau|A|^{-1}}}{1-e^{-\beta\tau|A|^{-1}}}. \tag{7.16}$$

In the spirit of the original SJ state construction, the timescale τ must be obtained using the geometry of the spacetime $(\mathcal{M},g)$ and the properties of the commutator function $A=i\mathbb{E}$ alone. In the case of the ultrastatic slab spacetime ($\mathcal{M}=(-\tau,\tau)\times\Sigma, g=\mathbb{1}\oplus -h$) we show how to do this generally, and give an explicit computation in the case when the spatial manifold Σ is a round three sphere. Since $\varphi_\beta : \sigma(A) \to \mathbb{R}$ is an even, non-negative continuous function that obeys $\varphi_\beta(0) = 0$, by Proposition 6.4 and from the discussion in Section 6.2 the two-point function,

$$W_{SJ_\beta}(f, g) = \langle \overline{f} \mid A^+_\beta g \rangle \qquad \forall f, g \in C^\infty_0(\mathcal{M}), \tag{7.17}$$

can be used to construct a quasifree thermal SJ state ω_{SJ_β}. We shall shortly prove that $A^+_\beta \to A^+_0$ in norm topology as $\beta \to \infty$, which then implies that the two-point function (7.17) satisfies the limit (7.4) in the weak topology. Since A^+_0 is independent of β, in order to show that the two-point function has the limit $W_{SJ_\beta} \to W_\beta$ in the weak topology as $\beta \to \infty$, it is sufficient to show that $\varphi_\beta(A) \to 0$ in norm topology as $\beta \to \infty$. Before showing this, we shall require the following result,

Proposition 7.1. *Let A be a bounded self-adjoint operator on $L^2(\mathcal{M})$. Then the operator $\varphi_\beta(A)$ is smoothing.*

Proof. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by,

$$f(x) = x \frac{e^{-\frac{1}{x}}}{1 - e^{-\frac{1}{x}}}, \tag{7.18}$$

where,

$$\varphi_\beta(\lambda) = \beta\tau f\left(\frac{|\lambda|}{\beta\tau}\right). \tag{7.19}$$

In the limit $x \to 0$ we have $e^{-1/x} \to 0$, and so there exists a $c > 1$ such that,

$$\sup \frac{1}{1 - e^{-1/x}} \leq c. \tag{7.20}$$

The numerator in (7.18), $x\exp(-1/x)$, decays rapidly in the limit $x \to 0$. Therefore for all $N \in \mathbb{N}_0$ there exists a $C_N \geq 0$ such that $|f(x)| \leq C_N |x|^N$ it follows that for all $N \in \mathbb{N}_0$ there exists a $C_N \geq 0$ such that $\varphi_\beta(\lambda) \leq C_N |\lambda|^N$. Therefore by Proposition 6.6 the operator $\varphi_\beta(A)$ is smoothing. ∎

We now arrive at the following,

Proposition 7.2. *Let $(\mathcal{M} = (-\tau, \tau) \times \Sigma, \mathbb{1} \oplus -h)$ be an ultrastatic slab spacetime and let $A = i\mathbb{E}$ be a bounded self-adjoint operator on $L^2(\mathcal{M})$. Let $\varphi_\beta : \sigma(A) \to \mathbb{R}$ be defined by,*

$$\varphi_\beta(\lambda) = |\lambda| \frac{e^{-\beta\tau|\lambda|^{-1}}}{1 - e^{-\beta\tau|\lambda|^{-1}}}, \tag{7.21}$$

where $\beta > 0$. Then the operator $\varphi_\beta(A) \to 0$ in norm topology as $\beta \to \infty$.

Proof. From the continuous functional calculus for self-adjoint operators on a Hilbert space, we have [53, Theorem VII.1 (g)],

$$\|\varphi_\beta(A)\|_{\mathcal{B}(L^2(\mathcal{M}))} = \|\varphi_\beta\|_\infty, \tag{7.22}$$

where the sup-norm $\|\cdot\|_\infty$ is taken on the spectrum of A. Therefore, if $\lim_{\beta\to\infty}\|\varphi_\beta\|_\infty = 0$ then,

$$\lim_{\beta\to\infty}\|\varphi_\beta(A)\|_{\mathcal{B}(L^2(\mathcal{M}))} = 0. \tag{7.23}$$

We now show that $\varphi_\beta \to 0$ in sup norm as $\beta \to \infty$. Since, for all $j \in \mathbb{N}$, $\omega_j \geq m$, we have,

$$\lambda_j = \frac{\tau}{\omega_j}\sqrt{1 - \operatorname{sinc}^2(2\omega_j\tau)} \leq \frac{\tau}{m}, \tag{7.24}$$

which implies,

$$\sigma(A) \subset \left[-\frac{\tau}{m}, \frac{\tau}{m}\right]. \tag{7.25}$$

Since, for all $N \in \mathbb{N}_0$ there exists a $C_N \geq 0$ such that the function (7.18) obeys $|f(x)| \leq C_N|x^N|$, we then obtain,

$$\begin{aligned}
\lim_{\beta\to\infty}\|\varphi_\beta\|_\infty &= \lim_{\beta\to\infty}\sup_{\lambda\in\sigma(A)}\varphi_\beta(\lambda)\\
&= \lim_{\beta\to\infty}\beta\tau\sup_{\lambda\in\sigma(A)} f\left(\frac{|\lambda|}{\beta\tau}\right)\\
&\leq \lim_{\beta\to\infty}\beta\tau\sup_{\lambda\in\left[-\frac{\tau}{m},\frac{\tau}{m}\right]} f\left(\frac{|\lambda|}{\beta\tau}\right)\\
&= \lim_{\beta\to\infty}\beta\tau\sup_{x\in\left[-\frac{1}{\beta m},\frac{1}{\beta m}\right]} f(x)\\
&\leq C_N\lim_{\beta\to\infty}\beta\tau\sup_{x\in\left[-\frac{1}{\beta m},\frac{1}{\beta m}\right]} |x^N|\\
&= C_N\lim_{\beta\to\infty}\beta\tau\sup_{x\in\left[0,\frac{1}{\beta m}\right]} x^N\\
&= \frac{\tau C_N}{m^N}\lim_{\beta\to\infty}\beta^{1-N}\\
&= 0,
\end{aligned} \tag{7.26}$$

where the last equality holds for all $N \geq 2$. Since the sup norm is positive semi-definite we therefore obtain $\varphi_\beta(A) \to 0$ as $\beta \to \infty$ in the norm topology on $\mathcal{B}(L^2(\mathcal{M}))$. ■

7.2 Thermal SJ States on Ultrastatic Slab Spacetimes

In this section we shall compute the two-point function of the quasifree thermal SJ state on ultrastatic slabs. Let $\mathcal{M} = (-\tau, \tau) \times \Sigma$, $\tau > 0$ be an ultrastatic slab spacetime with metric $g = \mathbb{1} \oplus -h$ where (Σ, h) is a smooth compact Riemannian manifold. A thermal SJ state on the ultrastatic slab $\mathcal{M} = (-\tau, \tau) \times \Sigma$ has the two-point function,

$$\begin{aligned} W_{SJ_\beta}(f, g) &= \langle \overline{f} \mid A_0^+ g \rangle + \langle \overline{f} \mid \varphi_\beta(A) g \rangle \\ &= W_{SJ_0}(f, g) + \langle \overline{f} \mid \varphi_\beta(A) g \rangle \end{aligned} \tag{7.27}$$

for all $f, g \in C_0^\infty(\mathcal{M})$ where φ_β is defined in (10.2),

$$W_{SJ_0}(f, g) = \langle \overline{f} \mid A_0^+ g \rangle, \tag{7.28}$$

is the two-point function for the vacuum SJ state. On the ultrastatic slab, the Klein Gordon equation is,

$$\Box + m^2 = \partial_t^2 + K, \tag{7.29}$$

where $t \in (-\tau, \tau)$ is the ultrastatic time parameter and $K = -\Delta + m^2$, where Δ is the Laplacian on (Σ, h). There exists a complete orthonormal basis of eigenvectors of K, which we denote by ξ_j for all $j \in \mathbb{N}$ [20, Theorem 1 Section 6.5]. The eigenvalues corresponding to the eigenvectors ξ_j are ω_j^2, where $\omega_j > 0$ for all $j \in \mathbb{N}$. We shall also assume that $\overline{\xi_j}$ is an element of this basis and label the eigenvectors $\{\xi_j\}_{j \in \mathbb{N}}$ such that $\overline{\xi_j} = \xi_{\overline{j}}$ for all $j \in \mathbb{N}$. As before, we shall also assume that $\omega_j \leq \omega_{j+1}$ for all $j \in \mathbb{N}$. The Hilbert space is given by $L^2(\mathcal{M}) = L^2(-\tau, \tau) \otimes L^2(\Sigma)$. The commutator function on the ultrastatic slab $\mathcal{M} = (-\tau, \tau) \times \Sigma$ is given by, [b]

$$\begin{aligned} A &= \sum_{j \in \mathbb{N}} \frac{\lambda_j}{\|\Psi^+\|^2} \left[|\Psi_j^+\rangle \langle \Psi_j^+| - |\Psi_j^-\rangle \langle \Psi_j^-| \right] \\ &= \sum_{j \in \mathbb{N}} \frac{\|S_j\|}{2\omega_j \|C_j\|} \left[|\Psi_j^+\rangle \langle \Psi_j^+| - |\Psi_j^-\rangle \langle \Psi_j^-| \right] \end{aligned}, \tag{7.30}$$

where $\{\Psi_j^\pm\}_{j \in \mathbb{N}}$ are given in (7.7) and obey $\Psi_j^+ = \Gamma \Psi_j^-$. Since the set of eigenvectors $\{\xi_j\}_{j \in \mathbb{N}}$ are orthonormal in $L^2(\Sigma,\ \mathbf{d}\mathrm{vol}_h)$, it follows that $\langle \Psi_j^+ \mid \Psi_k^- \rangle = 0$ for all $j, k \in \mathbb{N}$. A straightforward calculation yields,

$$|A| = \sum_{j \in \mathbb{N}} \frac{\lambda_j}{\|\Psi_j^+\|^2} \left[|\Psi_j^+\rangle \langle \Psi_j^+| + |\Psi_j^-\rangle \langle \Psi_j^-| \right], \tag{7.31}$$

[b] See Chapter 5 Section 5.5 for further details.

and so the operator A_0^+ is given by,

$$A_0^+ = \frac{1}{2}(A + |A|) = \sum_{j\in\mathbb{N}} \frac{\lambda_j}{\|\Psi_j^+\|^2} \left|\Psi_j^+\right\rangle \left\langle\Psi_j^+\right| . \tag{7.32}$$

The eigenvalues corresponding to the eigenvectors $\Psi_j^\pm$ are given by,

$$\lambda_j^\pm = \pm\lambda_j = \pm\frac{\|C_j\|\|S_j\|}{\omega_j}. \tag{7.33}$$

We shall now turn our attention to the construction of the operator $\varphi_\beta(A)$, where $\varphi_\beta : \sigma(A) \to \mathbb{R}$ is given in (7.13). We now obtain,

$$\begin{aligned} \varphi_\beta(A) &= |A| \frac{e^{-\beta T|A|^{-1}}}{1 - e^{-\beta T|A|^{-1}}} \\ &= \sum_{j\in\mathbb{N}} \frac{\|S_j\|}{2\omega_j\|C_j\|} \frac{e^{-\beta\tau|\lambda_j|^{-1}}}{1 - e^{-\beta\tau|\lambda_j|^{-1}}} \left[\left|\Psi_j^+\right\rangle \left\langle\Psi_j^+\right| + \left|\Psi_j^-\right\rangle \left\langle\Psi_j^-\right|\right] \end{aligned} \tag{7.34}$$

Using the operators A_0^+ and $|A|$, we can construct a thermal SJ state over the ultrastatic slab $\mathcal{M} = (-\tau, \tau) \times \Sigma$,

$$\begin{aligned} W_{SJ_\beta}(f, g) &= \langle \overline{f} \mid A_\beta^+ g\rangle \\ &= \langle \overline{f} \mid A_0^+ g\rangle + \langle \overline{f} \mid \varphi_\beta(A) g\rangle \\ &= \sum_{j\in\mathbb{N}} \frac{1}{2\|C_j\|^2} \Big((\lambda_j + \varphi_\beta(\lambda_j))\langle \overline{f} \mid \Psi_j^+\rangle\langle\Psi_j^+ \mid g\rangle + \varphi_\beta(\lambda_j)\langle \overline{f} \mid \Psi_j^-\rangle\langle\Psi_j^- \mid g\rangle \Big) \end{aligned} \tag{7.35}$$

for all $f, g \in C_0^\infty(\mathcal{M})$ and where $T = \tau$ and $\beta > 0$. We can immediately obtain the following:

Proposition 7.3. *The thermal SJ state with two-point function* (7.17) *is not Hadamard.*

Proof. Since the SJ vacuum fails to be Hadamard on the ultrastatic slab [28, Theorem 4.2] and since, by proposition 7.1, the operator $\varphi_\beta(A)$ is smoothing, it then immediately follows that the thermal SJ state with two-point function (7.17) is not Hadamard. ■

The following section demonstrates how one can construct a timescale for a general globally hyperbolic spacetime for the free scalar field using only the commutator function and the dimension of the spatial section.

7.3 Calculating a Time Scale from the Commutator Function

Let $(\mathcal{M} = (-\tau, \tau) \times \Sigma, g)$ where $\tau > 0$ be a bounded region of a globally hyperbolic spacetime such that the commutator function $A = i\mathbb{E}$ defines a bounded self-adjoint operator on $L^2(\mathcal{M}, \ \mathbf{d}\mathrm{vol}_g)$.We assume that $\dim(\Sigma) > 1$ and that Σ is boundaryless. In this section we propose a general way to find a timescale of the spacetime $(\mathcal{M}, g)$, which holds even when $(\mathcal{M}, g)$ is not static. The proposed formulae is given by,

$$T = \left(\frac{2(2\pi)^n}{\mathrm{Vol}(\mathcal{M})\Omega_n} \lim_{\lambda \to 0+} \lambda^n \widetilde{N}(\lambda) \right)^{\frac{1}{n-1}}, \tag{7.36}$$

where $\lambda > 0$ is an eigenvalue of $A = i\mathbb{E}$, $n = \dim(\Sigma)$, $\mathrm{Vol}(\mathcal{M}) = 2\tau \times \mathrm{Vol}(\Sigma)$, Ω_n is the volume of a $n-$dimensional unit sphere in $\mathbb{R}^n$ given by,

$$\Omega_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)}, \tag{7.37}$$

and,

$$\widetilde{N}(\lambda) = \#\{j \in \mathbb{N} \mid \lambda_j \geq \lambda\}, \tag{7.38}$$

is a counting function for the eigenvalues $\{\lambda_j\}_{j \in \mathbb{N}}$ of A. We shall proceed to derive the formulae (7.45) when $(\mathcal{M}, g)$ is an ultrastatic slab spacetime. On the ultrastatic slab $\mathcal{M} = (-\tau, \tau) \times \Sigma$, the positive eigenvalues of the operator $A = i\mathbb{E}$ given in (7.33) are,

$$\lambda_j = \frac{\tau}{\omega_j} \sqrt{1 - \mathrm{sinc}^2(2\omega_j \tau)}, \tag{7.39}$$

which are asymptotically $\lambda_j \sim \dfrac{\tau}{\omega_j}$. Consider now the eigenvalues $\{\omega_j^2\}_{j \in \mathbb{N}}$ of the operator $K = -\Delta + m^2$, where we define the counting function,

$$N(\omega) = \#\{n \in \mathbb{N} \mid \omega_n \leq \omega\}. \tag{7.40}$$

As mentioned before, Weyl's law gives us the asymptotic distribution of eigenvalues for the operator K [64] (or [60, Theorem 3.1])

$$N(\omega) \sim (2\pi)^{-n} \Omega_n \mathrm{Vol}(\Sigma) \omega^n, \tag{7.41}$$

the counting function for the eigenvalues of the operator (7.30),

$$\widetilde{N}(\lambda) = \#\{n \in \mathbb{N} \mid \lambda_n \geq \lambda\}. \tag{7.42}$$

Since $\lambda_j \sim \dfrac{\tau}{\omega_j}$ we infer that $\widetilde{N}(\lambda) \sim N(\frac{\tau}{\lambda}) \sim N(\omega)$. Weyl's law then gives,

$$N(\omega) \sim \widetilde{N}(\lambda) \sim (2\pi)^{-n}\Omega_d \mathrm{Vol}(\Sigma)\omega^n = (2\pi)^{-n}\Omega_n \mathrm{Vol}(\Sigma)\Big(\frac{\tau}{\lambda}\Big)^n. \tag{7.43}$$

Using $\mathrm{Vol}(\mathcal{M}) = 2\tau\mathrm{Vol}(\Sigma)$ we then obtain,

$$\widetilde{N}(\lambda) \sim \frac{(2\pi)^{-n}}{2\lambda^n}\Omega_n \mathrm{Vol}(\mathcal{M})\tau^{n-1}. \tag{7.44}$$

Rearranging and taking an appropriate limit we arrive at,

$$\tau = \left(\frac{2(2\pi)^n}{\mathrm{Vol}(\mathcal{M})\Omega_n}\lim_{\lambda\to 0+}\lambda^n\widetilde{N}(\lambda)\right)^{\frac{1}{n-1}}. \tag{7.45}$$

To further illustrate this, we give a compute the time scale (7.45) when the spatial manifold (Σ, h) is a three sphere. Let $\Sigma = S^3$ be endowed with the usual metric. The spacetime volume is given by,

$$\begin{aligned}\mathrm{Vol}(\mathcal{M}) &= 2\tau \times \mathrm{Vol}(S^3)\\ &= 2\tau\big(2\pi^2R^3\big).\end{aligned} \tag{7.46}$$

In the case of a round three sphere of radius R, the eigenvalues ω_j of the operator $K = -\Delta + m^2$ are given in (5.84) and occur with multiplicity $(1+j)^2$, implying that the eigenvalues λ_j occur with the same multiplicity. Since the eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$ are monotonically decreasing in j, the counting function takes the form,

$$N(\lambda_j) = \sum_{k=0}^{j}(1+k)^2. \tag{7.47}$$

Using $\Omega_3 = \dfrac{4}{3}\pi$, the timescale (7.45) in the case of the round three sphere is then,

$$\begin{aligned}T &= \left(\frac{2(2\pi)^3}{2\tau(2\pi^2R^3)}\left(\frac{3}{4\pi}\right)\lim_{\lambda\to 0^+}\lambda^3\widetilde{N}(\lambda)\right)^{\frac{1}{2}}\\ &= \left(\frac{3}{\tau R^3}\lim_{\lambda\to 0^+}\lambda^3\widetilde{N}(\lambda)\right)^{\frac{1}{2}}\\ &= \left(\frac{3}{\tau R^3}\tau^3\lim_{j\to\infty}\left(\frac{\sqrt{1-\mathrm{sinc}^2(2\omega_j\tau)}}{\omega_j}\right)^3\sum_{k=0}^{j}(1+k)^2\right)^{\frac{1}{2}}\\ &= \tau\left(\frac{3}{R^3}\frac{1}{3R^{-3}}\right)^{\frac{1}{2}}\\ &= \tau\end{aligned} \tag{7.48}$$

The proposed formulae (7.36) is a general way of constructing a timescale from the commutator function on a general bounded region of a globally hyperbolic spacetime. In the case of an ultrastatic spacetime $((-\tau, \tau) \times \Sigma, g = \mathbb{1} \oplus -h)$, the formulae (7.36) coincides with the value of τ, which is the correct time scale needed for (7.14) to hold.

Chapter 8

Softened SJ States: The Hadamard Condition and Wick Polynomials

The SJ vacuum state, originally constructed in [1] and reviewed in Chapter 5, admits pathologies that cast doubt on its relevance as a physically viable state. In particular, the SJ vacuum constructed on ultrastatic slabs, generically fails to be Hadamard [28]; essentially ruling out the physical relevance of the SJ vacuum if one accepts the theory of renormalising the stress-energy tensor and point-splitting methods used in constructing Wick polynomials. There are many reasons why Hadamard states are considered to be the correct states to consider for linear quantum fields defined over continuum spacetimes. Fredenhagen and Brunetti proved in [12] that all Wick polynomials have finite fluctuations if the Wick normal ordering is defined with respect to a Hadamard state. A partial converse to this result was proved by Fewster and Verch in [29]. Therefore, Hadamard states allow one to construct Wick polynomials with finite fluctuations, and so quantities such as the stress-energy tensor can be constructed and meaningful results about the physical properties of the state can be obtained. We emphasise that this analysis only applies to quantum fields over *continuum* spacetimes. Quantum fields on discrete spacetimes, such causal sets, may not view the Hadamard condition in such a significant way. It may be the case that there is a 'discrete' version of the Hadamard condition for quantum fields on causal sets, which could furnish causal set theory with a notion of the 'correct' states to consider. We shall, unfortunately, not explore this avenue here and return firmly to linear quantum field defined over continuum spacetimes.

Hadamard states are considered to be the largest class of physically viable states, and, therefore constructing generalised SJ states that satisifes the Hadamard is an important question. There is a construction, due to Brum and Fredenhagen, that suitably regularises the SJ vacuum as to restore the Hadamard condition [11]. This construction yields the so called BF states which are Hadamard in all cases considered with no known exceptions. Our goal in this chapter is to propose a new construction that yields a family of Hadamard states, which is inspired by an observation of Sorkin. Let $(\mathcal{M}, g)$ be a globally hyperbolic spacetime. Using a suitable function $\rho \in C_0^\infty(\mathcal{M})$, we construct a modified Hilbert space $(\mathcal{H}, \langle \cdot \mid \cdot \rangle_\rho)$ and a modified commutator function A_ρ, which is defined by,

$$\langle \overline{f} \mid A_\rho g \rangle_\rho = iE(f,g) \qquad \forall f, g \in C_0^\infty(\mathcal{M}), \tag{8.1}$$

where $E(f,g)$ is the causal propagator on $\mathcal{M}$. The original idea of Sorkin, appearing in [58], is to modify the volume form $\mathbf{d}\mathrm{vol}_g$ appearing in the $L^2(\mathcal{M})$ inner product to $\rho\, \mathbf{d}\mathrm{vol}_g$. However, we choose to modify the volume form $\mathbf{d}\mathrm{vol}_g$ to $\frac{1}{\rho}\, \mathbf{d}\mathrm{vol}_g$ since this ensures that the modified commutator function A_ρ has the correct properties to construct a Hadamard state, at least in the case of the ultrastatic slab. If one were to modify the volume measure to $\rho\, \mathbf{d}\mathrm{vol}_g$ instead of $\frac{1}{\rho}\, \mathbf{d}\mathrm{vol}_g$, then the corresponding modified commutator function $A_{\frac{1}{\rho}}$ would not define a bounded self-adjoint operator, which stems from the fact that $\frac{1}{\rho}$ is not smooth on the real line. Now, if the modified commutator function A_ρ defines a bounded, self-adjoint operator over $\mathcal{H}_\rho$, then the operator,

$$A^+_{\psi,\rho} = \frac{A_\rho}{2} + \sqrt{\frac{|A_\rho|^2}{4} + \psi(A_\rho)}, \tag{8.2}$$

, where $\psi : \sigma(A_\rho) \to \mathbb{R}$ is a continuous, even, non-negative function obeying $\psi(0) = 0$, is the unique solution to the set of modified SJ axioms,

$SJ_{\psi,\rho}1)$ $A^+_{\psi,\rho} - \Gamma A^+_{\psi,\rho}\Gamma = A_\rho$

$SJ_{\psi,\rho}2)$ $A^+_{\psi,\rho} \geq 0$

$SJ_{\psi,\rho}3)$ $A^+_{\psi,\rho}\Gamma A^+_{\psi,\rho}\Gamma = \psi(A_\rho)$,

by Theorem 6.3. The generalised SJ state with softened boundaries then has the two point function,

$$W_{SJ_{\psi,\rho}}(f,g) = \langle \overline{f} \mid A^+_{\psi,\rho} g \rangle \qquad \forall f, g \in C_0^\infty(\mathcal{M}). \tag{8.3}$$

As we will show, in the case of the ultrastatic slab $(-\tau, \tau) \times \Sigma$ the definition (8.1) yields a *unique* self-adjoint compact operator A_ρ over the Hilbert space $(\mathcal{H}_\rho, \langle \cdot \mid \cdot \rangle_\rho)$. In the first instance, we use the operator $A_\rho \in B(\mathcal{H}_\rho)$ to construct the softened SJ vacuum over $(-\tau, \tau) \times \Sigma$ by using the operator,

$$A^+_{0,\rho} = \frac{1}{2}(A_\rho + |A_\rho|), \tag{8.4}$$

which is recovered from (8.38) by choosing $\psi = 0$. Remarkably so, the SJ vacuum with softened boundaries constructed from the operator (8.4) turns out to be Hadamard. Moreover, for suitably chosen functions $\psi : \sigma(A_\rho) \to \mathbb{R}$, the generalised SJ state with softened boundaries constructed from the operator (8.38) will also turn out to be Hadamard. The choices of function $\psi : \sigma(A_\rho) \to \mathbb{R}$ that ensure the SJ state constructed from (8.38) are all even, non-negative, continuous functions such that $\varphi(A_\rho)$ is smoothing. Therefore, by modifying the Hilbert space in a suitable manner, this then modifies the properties of the commutator function which then changes the properties of a generalised SJ state, such as the Hadamard condition.

8.1 A Rank Two Hilbert Space Operator Toy Model

Let $\mathcal{H}$ be a Hilbert space and $w, \overline{w} \in \mathcal{H}$, where the bar denotes the antilinear complex conjugate. It will be useful to derive the eigenvectors of the rank two operator,

$$A = |w\rangle \langle w| - |\overline{w}\rangle \langle \overline{w}| \,. \tag{8.5}$$

By Cauchy-Schwarz we have $|\langle w \mid \overline{w}\rangle| \leq |\langle w \mid w\rangle|$. Let $\alpha = \langle w \mid \overline{w}\rangle$. We assume that α is real and non-negative since if one multiples w by $e^{-i\theta}$ for some arbitrary phase $\theta \in \mathbb{R}$, we have $\alpha \mapsto e^{2i\theta}\langle w \mid \overline{w}\rangle$. Therefore, one can always choose a phase θ such that α is real an positive. We define the following,

$$e^{-2u} = \sqrt{1 - \frac{|\langle w \mid \overline{w}\rangle|^2}{|\langle w \mid w\rangle|^2}}. \tag{8.6}$$

Since the operator (8.5) is rank-two, a straightforward calculation shows that the unnormalised eigenvectors are,

$$\Psi^{\pm} = \alpha w + \beta^{\pm}\overline{w}, \tag{8.7}$$

where,

$$\begin{aligned} \beta^{\pm} &= -\langle w \mid w\rangle \pm \sqrt{|\langle w \mid w\rangle|^2 - |\langle w \mid \overline{w}\rangle|^2} \\ &= \|w\|^2\left(-1 \pm \sqrt{1 - \frac{|\langle w \mid \overline{w}\rangle|^2}{|\langle w \mid w\rangle|^2}}\right) \\ &= \|w\|^2\left(-1 \pm e^{-2u}\right) \end{aligned} \tag{8.8}$$

with the corresponding eigenvalues,

$$\lambda^{\pm} = \pm\sqrt{|\langle w \mid w\rangle|^2 - |\langle \overline{w} \mid w\rangle|^2} = \pm\|w\|^2\sqrt{1 - \frac{|\langle w \mid \overline{w}\rangle|^2}{|\langle w \mid w\rangle|^2}} = \pm\|w\|^2 e^{-2u}. \tag{8.9}$$

We note the expressions,

$$\alpha = \langle w \mid \overline{w}\rangle = \|w\|^2\sqrt{1 - e^{-4u}} = \|w\|^2\sqrt{(-1 \pm e^{-2u})(-1 \mp e^{-2u})}. \tag{8.10}$$

The norm of the eigenvectors $\Psi^{\pm}$ is shown to be,

$$\langle \Psi^{\pm} \mid \Psi^{\pm}\rangle = 2\|w\|^6 e^{-4u}\big(1 \mp e^{-2u}\big) \tag{8.11}$$

The normalised eigenvectors are then,

$$\Psi^{\pm} = \frac{\alpha}{\sqrt{\langle \Psi^{\pm} \mid \Psi^{\pm}\rangle}} w + \frac{\beta^{\pm}}{\sqrt{\langle \Psi^{\pm} \mid \Psi^{\pm}\rangle}} \overline{w}, \tag{8.12}$$

We now obtain,

$$\begin{aligned}
\frac{\alpha}{\sqrt{\langle \Psi^{\pm} \mid \Psi^{\pm}\rangle}} &= \frac{1}{\sqrt{\langle w \mid w\rangle}} \frac{\sqrt{1 - e^{-4u}}}{\sqrt{2e^{-4u}(1 \mp e^{-2u})}} \\
&= \frac{1}{\sqrt{\langle w \mid w\rangle}} \sqrt{\frac{(1 \mp e^{-2u})(1 \pm e^{-2u})}{2e^{-4u}(1 \mp e^{-2u})}} \\
&= \frac{1}{\sqrt{\langle w \mid w\rangle}} \sqrt{\frac{1 \pm e^{-2u}}{2e^{-4u}}} \\
&= \frac{e^{3u/2}}{\sqrt{\langle w \mid w\rangle}} \sqrt{\frac{1}{2}(e^{u} \pm e^{-u})} \\
&= \begin{cases} \dfrac{e^{3u/2}}{\sqrt{\langle w \mid w\rangle}} \sqrt{\cosh(u)} & + \\ \dfrac{e^{3u/2}}{\sqrt{\langle w \mid w\rangle}} \sqrt{\sinh(u)} & -, \end{cases}
\end{aligned} \tag{8.13}$$

and,

$$
\begin{aligned}
\frac{\beta}{\sqrt{\langle\Psi^{\pm}\mid\Psi^{\pm}\rangle}} &= \frac{-(1\mp e^{-2u})}{\sqrt{\langle w\mid w\rangle}\sqrt{2e^{-4u}(1\mp e^{-2u})}} \\
&= \frac{-1}{\sqrt{\langle w\mid w\rangle}}\sqrt{\frac{1\mp e^{-2u}}{2e^{-4u}}} \\
&= -\frac{e^{3u/2}}{\sqrt{\langle w\mid w\rangle}}\sqrt{\frac{1}{2}(e^{u}\mp e^{-u})} \\
&= \begin{cases} -\dfrac{e^{3u/2}}{\sqrt{\langle w\mid w\rangle}}\sqrt{\sinh(u)} & + \\ -\dfrac{e^{3u/2}}{\sqrt{\langle w\mid w\rangle}}\sqrt{\cosh(u)} & -\,. \end{cases}
\end{aligned}
\tag{8.14}
$$

We note here that the normalised eigenvectors are well defined in the limit $\alpha\to 0$. Therefore, the normalised eigenvectors of (8.5) are given by,

$$
\begin{aligned}
\Psi^{+} &= i\frac{e^{3u/2}}{\sqrt{\langle w\mid w\rangle}}(\sqrt{\cosh(u)}w-\sqrt{\sinh(u)}\overline{w}) \\
\Psi^{-} &= i\frac{e^{3u/2}}{\sqrt{\langle w\mid w\rangle}}(\sqrt{\sinh(u)}w-\sqrt{\cosh(u)}\overline{w}),
\end{aligned}
\tag{8.15}
$$

where we have multiplied through by i so that the eigenvectors obey $\Gamma\Psi^{+}=\Psi^{-}$. We can now express the operator (8.5) in terms of the eigenvectors (8.7). A simple rearrangement yields,

$$
w = ie^{-u/2}\sqrt{\langle w\mid w\rangle}\Big(\sqrt{\sinh(u)}\Psi^{-}-\sqrt{\cosh(u)}\Psi^{+}\Big), \tag{8.16}
$$

hence,

$$
\overline{w} = -ie^{-u/2}\sqrt{\langle w\mid w\rangle}\Big(\sqrt{\sinh(u)}\Psi^{+}-\sqrt{\cosh(u)}\Psi^{-}\Big). \tag{8.17}
$$

We then find,

$$
\begin{aligned}
|w\rangle\langle w| &= e^{-u}\langle w\mid w\rangle\Big[\sinh(u)\,|\Psi^{-}\rangle\langle\Psi^{-}|+\cosh(u)\,|\Psi^{+}\rangle\langle\Psi^{+}|-\sqrt{\cosh(u)\sinh(u)}(|\Psi^{-}\rangle\langle\Psi^{+}|+|\Psi^{+}\rangle\langle\Psi^{-}|)\Big] \\
|\overline{w}\rangle\langle\overline{w}| &= e^{-u}\langle w\mid w\rangle\Big[\sinh(u)\,|\Psi^{+}\rangle\langle\Psi^{+}|+\cosh(u)\,|\Psi^{-}\rangle\langle\Psi^{-}|-\sqrt{\cosh(u)\sinh(u)}(|\Psi^{-}\rangle\langle\Psi^{+}|+|\Psi^{+}\rangle\langle\Psi^{-}|)\Big].
\end{aligned}
\tag{8.18}
$$

Upon which we arrive at,

$$
\begin{aligned}
A &= |w\rangle\langle w|-|\overline{w}\rangle\langle\overline{w}| \\
&= e^{-2u}\langle w\mid w\rangle\big[|\Psi^{+}\rangle\langle\Psi^{+}|-|\Psi^{-}\rangle\langle\Psi^{-}|\big] \\
&= \lambda\big[|\Psi^{+}\rangle\langle\Psi^{+}|-|\Psi^{-}\rangle\langle\Psi^{-}|\big],
\end{aligned}
\tag{8.19}
$$

where $\lambda=e^{-2u}\langle w\mid w\rangle$ is related to the eigenvalues of (8.5) by $\lambda^{\pm}=\pm\lambda$.

8.2 The SJ Vacuum State with Softened Boundaries on Ultrastatic Slabs

Let $(\mathcal{M} = (-\tau, \tau) \times \Sigma, g = \mathbb{1} \oplus -h)$ be an ultrastatic slab spacetime where $\tau > 0$ and (Σ, h) is a smooth compact Riemannian manifold. Let $\mathbb{E}^{\pm}$ be the advanced$(-)$/retarded$(+)$ Green's function for the massive Klein-Gordon operator $P = \Box + m^2$. They exist uniquely by [3, Theorem 3.3.1]. From the Green's functions $\mathbb{E}^{\pm}$, define the advanced-minus-retarded function $\mathbb{E} = \mathbb{E}^- - \mathbb{E}^+$. By Proposition 5.1 the commutator function $A = i\mathbb{E}$ defined over the ultrastatic slab $\mathcal{M}$ extends to a self-adjoint, bounded operator over $L^2(\mathcal{M},\ \mathbf{d}\mathrm{vol}_g)$. Let $\rho \in C_0^\infty(\mathbb{R})$ be a real-valued function that obeys $\rho(t) > 0$ for all $t \in (-\tau, \tau)$ and $\rho(t) \to 0$ as $t \to \pm\tau$.

8.3 The Softened Commutator Function on Ultrastatic Slabs

The modified Hilbert space, denoted $\mathcal{H}_\rho$, will be defined as the completion of $C_0^\infty(\mathcal{M})$ with respect to the volume form $\frac{1}{\rho}\ \mathbf{d}\mathrm{vol}_g$, where the inner product is,

$$\langle f \mid g \rangle_\rho = \int_{\mathcal{M}} \overline{f(t,x)} g(t,x) \frac{\mathbf{d}\mathrm{vol}_g(t,x)}{\rho(t)}. \tag{8.20}$$

The Hilbert space $\mathcal{H}_\rho$ may be decomposed,

$$\mathcal{H}_\rho = L^2(\mathcal{M}, \frac{1}{\rho}\ \mathbf{d}\mathrm{vol}_g) = L((-\tau,\tau), \frac{1}{\rho}\mathbf{d}t) \otimes L^2(\Sigma,\ \mathbf{d}\mathrm{vol}_h). \tag{8.21}$$

In the case of the ultrastatic slab, the advanced-minus-retarded operator $\mathbb{E}$ has the integral kernel [28],

$$E(t,x;t',x') = \sum_{j\in\mathbb{N}} \frac{\sin(\omega_j(t'-t))}{\omega_j} \xi_j(x)\overline{\xi_j(x')} \tag{8.22}$$

where $\{\xi_j\}_{j\in\mathbb{N}}$ form an orthonormal basis of $L^2(\Sigma)$ such that $K\xi_j = \omega_j^2$ where $\omega_j > 0$ for all $j \in \mathbb{N}$ and $K = -\Delta + m^2$ [20, Theorem 1 Section 6.5]. The operator used to construct the SJ vacuum with softened boundaries is then given by,

$$\begin{aligned} A_\rho f &= \sum_{j\in\mathbb{N}} \frac{1}{2\omega_j}\Big(\rho\mathcal{A}_j \langle \rho\mathcal{A}_j \mid g\rangle_\rho - \rho\overline{\mathcal{A}_j}\langle \rho\overline{\mathcal{A}_j} \mid g\rangle_\rho\Big) \\ &\doteq \sum_{j\in\mathbb{N}} A_{\rho,j} f, \end{aligned} \tag{8.23}$$

where,

$$A_{\rho,j} f \doteq \frac{1}{2\omega_j}\Big(\rho\mathcal{A}_j \langle \rho\mathcal{A}_j \mid f\rangle_\rho - \rho\overline{\mathcal{A}_{\overline{j}}}\langle \rho\overline{\mathcal{A}_{\overline{j}}} \mid f\rangle_\rho\Big). \tag{8.24}$$

We calculate some inner products,

$$\begin{aligned} \langle \rho\mathcal{A}_j \mid \rho\mathcal{A}_k\rangle_\rho &= \widehat{\rho}(0)\delta_{jk} \\ \langle \rho\mathcal{A}_j \mid \rho\overline{\mathcal{A}_{\overline{k}}}\rangle_\rho &= \widehat{\rho}(2\omega_j)\delta_{jk}, \end{aligned} \tag{8.25}$$

where the function ρ is extended to zero before the Fourier transform is taken. We now proceed by assuming the operator (8.23) is well defined and start our analysis of the operators $A_{\rho,j}$. We shall prove in Proposition 8.1 that the operator (8.23) does converges, in norm topology, to a compact operator. Setting aside the convergence of (8.23), we now calculate the eigenvectors of the operators $A_{\rho,j}$. First, we make the identification,

$$\begin{aligned} w &\equiv \frac{\rho\mathcal{A}_j}{\sqrt{\widehat{\rho}(0)}} \\ \overline{w} &\equiv \frac{\rho\overline{\mathcal{A}_{\overline{j}}}}{\sqrt{\widehat{\rho}(0)}} \end{aligned} \tag{8.26}$$

and apply the toy model detailed in Section 8.1 to find the normalised eigenvectors and eigenvalues of the operators (8.24). The normalised eigenvectors of the operator (8.24) are then, for each $j \in \mathbb{N}$,

$$\begin{aligned} \Psi^+_{\rho,j} &= i\rho\sqrt{\frac{e^{3u_j}}{\widehat{\rho}(0)}}\Big(\sqrt{\cosh(u_j)}\mathcal{A}_j - \sqrt{\sinh(u_j)}\overline{\mathcal{A}_{\overline{j}}}\Big) \\ \Psi^-_{\rho,j} &= i\rho\sqrt{\frac{e^{3u_j}}{\widehat{\rho}(0)}}\Big(\sqrt{\sinh(u_j)}\mathcal{A}_j - \sqrt{\cosh(u_j)}\overline{\mathcal{A}_{\overline{j}}}\Big) \end{aligned}, \tag{8.27}$$

which obey,

$$\begin{aligned} \langle \Psi^\pm_{\rho,j} \mid \Psi^\pm_{\rho,k}\rangle_\rho &= \delta_{jk} \\ \langle \Psi^\mp_{\rho,j} \mid \Psi^\pm_{\rho,k}\rangle_\rho &= 0, \end{aligned} \tag{8.28}$$

for all $j, k \in \mathbb{N}$ and where,

$$e^{-2u_j} = \sqrt{1 - \left|\frac{\widehat{\rho}(2\omega_j)}{\widehat{\rho}(0)}\right|^2}. \tag{8.29}$$

We also note that,

$$\lim_{j\to\infty} e^{-2u_j} = \lim_{j\to\infty}\sqrt{1 - \left|\frac{\widehat{\rho}(2\omega_j)}{\widehat{\rho}(0)}\right|^2} = 1, \tag{8.30}$$

which implies $\lim_{j\to\infty} u_j = 0$. The eigenvalues corresponding to the eigenvectors $\Psi^{\pm}_{\rho,j}$ are given by,

$$\lambda_j^{\pm} = \frac{\pm\widehat{\rho}(0)e^{-2u_j}}{2\omega_j} \tag{8.31}$$

The operator $A_{\rho,j}$ defined in (8.23) is, in the eigenvector basis (8.27),

$$A_{\rho,j}f = \frac{\widehat{\rho}(0)e^{-2u_j}}{2\omega_j}\big(\Psi^{+}_{\rho,j}\langle\Psi^{+}_{\rho,j} \mid f\rangle_\rho - \Psi^{-}_{\rho,j}\langle\Psi^{-}_{\rho,j} \mid f\rangle_\rho\big), \tag{8.32}$$

whence we obtain the expression,

$$A_\rho f = \sum_{j\in\mathbb{N}} \frac{\widehat{\rho}(0)e^{-2u_j}}{2\omega_j}\big(\Psi^{+}_{\rho,j}\langle\Psi^{+}_{\rho,j} \mid f\rangle_\rho - \Psi^{-}_{\rho,j}\langle\Psi^{-}_{\rho,j} \mid f\rangle_\rho\big). \tag{8.33}$$

We now show that the operator (8.33) is compact and self-adjoint.

Proposition 8.1. *The operator* (8.33) *is a compact self-adjoint operator on* $L^2(\mathcal{M}, \frac{1}{\rho}\ \boldsymbol{dvol_g})$.

Proof. First, we define the operators,

$$A^{\pm}_{\rho} = \sum_{j\in\mathbb{N}} \pm\lambda_j \left|\Psi^{\pm}_{\rho,j}\right\rangle \left\langle\Psi^{\pm}_{\rho,j}\right|, \tag{8.34}$$

where $A_\rho = A^{+}_{\rho} + A^{-}_{\rho}$. Now, define the partial sum operators $A^{\pm}_{\rho,N}$ by setting,

$$A^{\pm}_{\rho,N} = \sum_{j=1}^{N} \pm\lambda_j \left|\Psi^{+}_{\rho,j}\right\rangle \left\langle\Psi^{+}_{\rho,j}\right|. \tag{8.35}$$

Since $\lambda_j \in \mathbb{R}$ for all $j \in \mathbb{N}$, the operator $A^{\pm}_{\rho,N}$ defines a self-adjoint, finite rank operator for all $N \in \mathbb{N}$ and is therefore compact. Without loss of generality let $M \leq N$. We now find,

$$\|A^{+}_{\rho,N} - A^{+}_{\rho,M}\|^2_\rho = \|\sum_{j=M+1}^{N} \lambda_j \left|\Psi^{+}_{\rho,j}\right\rangle \left\langle\Psi^{+}_{\rho,j}\right| \|^2_\rho \leq \sum_{j=M+1}^{N} \lambda_j^2, \tag{8.36}$$

Since $\lim_{j\to\infty}\lambda_j = 0$, it follows that the sum $\sum_{j=M+1}^{N}\lambda_j^2$ can be made arbitrarily small by taking $N, M \in \mathbb{N}$ to be sufficiently large. Therefore, for all $\epsilon > 0$ there exists a sufficiently large $N, M \in \mathbb{N}$ such that the following holds,

$$\|A^{+}_{\rho,N} - A^{+}_{\rho,M}\|^2_\rho \leq \sum_{j=N+1}^{\infty} \lambda_j^2 < \epsilon. \tag{8.37}$$

Therefore the sequence of operators $\{A_{\rho,N}\}_{N\in\mathbb{N}}$ is Cauchy, which converges since $B(\mathcal{H}_\rho)$ is complete. Therefore the operator A^{+}_{ρ} is the norm limit of the self-adjoint compact operators $A^{\pm}_{\rho,N}$. Hence, by [53, Theorem VI.12 (a)], A^{+}_{ρ} is a self-adjoint compact operator. A similar analysis shows that A^{-}_{ρ} is a self-adjoint compact operator. Hence A_ρ is self-adjoint and compact. ∎

Since the operator (8.1) is a self-adjoint compact operator, one can use Theorem 6.3 and show that the operator,

$$A^{+}_{\psi,\rho} = \frac{A_\rho}{2} + \sqrt{\frac{|A_\rho|^2}{4} + \psi(A_\rho)}, \tag{8.38}$$

where $\psi : \sigma(A_\rho) \to \mathbb{R}$ is an even, non-negative, continuous function obeying $\psi(0) = 0$, is the unique solution to the modified SJ axioms $SJ_{\psi,\rho}1-3$. The generalised SJ state with softened boundaries then has the two point function,

$$W_{SJ_{\psi,\rho}}(f,g) = \langle \overline{f} \mid A^{+}_{\psi,\rho} g\rangle, \tag{8.39}$$

for all $f, g \in C_0^\infty(\mathcal{M})$. By virtue of $SJ_{\psi,\rho}1$ and by (8.1), the antisymmetric part of (8.39) coincides with the commutator function (8.22). Our goal now is to construct the softened SJ vacuum over the ultrastatic slab $(-\tau,\tau)\times\Sigma$. The softened SJ vacuum is constructed from the operator (8.38) by setting $\psi = 0$, which yields,

$$A^{+}_{\rho} = \frac{1}{2}(A_\rho + |A_\rho|), \tag{8.40}$$

where A_ρ is given in (8.33). Since the eigenvectors $\{\Psi^{\pm}_{j}\}_{j\in\mathbb{N}}$ obey (8.28) a straightforward calculation gives,

$$|A_\rho| f = \sum_{j\in\mathbb{N}} \frac{\widehat{\rho}(0)e^{-2u_j}}{2\omega_j}\big(\Psi^{+}_{\rho,j}\langle\Psi^{+}_{\rho,j} \mid f\rangle_\rho + \Psi^{-}_{\rho,j}\langle\Psi^{-}_{\rho,j} \mid f\rangle_\rho\big), \tag{8.41}$$

Therefore, the operator (8.40) is given by,

$$A^{+}_{\rho} f = \frac{1}{2}(A_\rho + |A_\rho|)f = \sum_{j\in\mathbb{N}} \frac{\widehat{\rho}(0)e^{-2u_j}}{2\omega_j}\Psi^{+}_{\rho,j}\langle\Psi^{+}_{\rho,j} \mid f\rangle_\rho, \tag{8.42}$$

for all $f \in H_\rho$. The vacuum SJ state with softened boundaries on the ultrastatic slab $(-\tau, \tau \times \Sigma)$ is then given by the two-point function,

$$W_{SJ_{0,\rho}}(f,g) = \langle \overline{f} \mid A^{+}_{\rho} g\rangle_\rho = \sum_{j\in\mathbb{N}} \frac{\widehat{\rho}(0)e^{-2u_j}}{2\omega_j}\langle \overline{f} \mid \Psi^{+}_{\rho,j}\rangle_\rho\langle\Psi^{+}_{\rho,j} \mid g\rangle_\rho. \tag{8.43}$$

We note the following expressions,

$$\begin{aligned}
e^{-4u_j} &= 1 - \left|\frac{\widehat{\rho}(2\omega_j)}{\widehat{\rho}(0)}\right|^2 \\
\sqrt{\cosh(u_j)\sinh(u_j)} &= \sqrt{\frac{1}{2}\sinh(2u_j)} = \sqrt{\frac{1-e^{-4u_j}}{4e^{-2u_j}}} \\
&= \frac{1}{2}e^{u_j}\sqrt{1-e^{-4u_j}} \\
&= \frac{1}{2}e^{u_j}\left|\frac{\widehat{\rho}(2\omega_j)}{\widehat{\rho}(0)}\right| \\
\cosh(u_j) &= e^{-u_j} + \sinh(u_j)
\end{aligned} \tag{8.44}$$

and,

$$\Psi^+_{\rho,j}(t,x) = i\rho(t)\sqrt{\frac{e^{3u_j}}{\widehat{\rho}(0)}}\left(\sqrt{\cosh(u_j)}e^{-i\omega_j t} - \sqrt{\sinh(u_j)}e^{i\omega_j t}\right)\xi_j(x) \tag{8.45}$$

Using these expressions we then find the following,

$$\begin{aligned}
\Psi^+_j(t,x)\overline{\Psi^+_j(t',x')} &= \frac{\rho(t)\rho(t')e^{3u_j}}{|\widehat{\rho}(0)|}\Big(e^{-u_j}e^{-i\omega_j(t-t')} + 2\sinh(u_j)\cos(\omega_j(t-t')) \\
&\quad - 2\sqrt{\cosh(u_j)\sinh(u_j)}\cos(\omega_j(t+t'))\Big)\xi_j(x)\overline{\xi_j(x')} \\
&= \frac{\rho(t)\rho(t')e^{2u_j}}{|\widehat{\rho}(0)|}\Big(e^{-i\omega_j(t-t')} + 2e^{u_j}\sinh(u_j)\cos(\omega_j(t-t')) \\
&\quad - e^{2u_j}\left|\frac{\widehat{\rho}(2\omega_j)}{\widehat{\rho}(0)}\right|\cos(\omega_j(t+t'))\Big)\xi_j(x)\overline{\xi_j(x')},
\end{aligned} \tag{8.46}$$

for all $j \in \mathbb{N}$. The integral kernel of (8.43) is (relative to the volume form $\mathbf{d}\mathrm{vol}_g$) can be shown to be,

$$\begin{aligned}
W_{SJ_{\rho,0}}(t,x;t',x') = \sum_{j\in\mathbb{N}} \frac{1}{2\omega_j}\left(2e^{u_j}\sinh(u_j)\cos(\omega_j(t-t')) - e^{2u_j}\left|\frac{\widehat{\rho}(2\omega_j)}{\widehat{\rho}(0)}\right|\cos(\omega_j(t+t'))\right)\xi_j(x)\overline{\xi_j(x')} \\
+ W_H(t',x';t,x),
\end{aligned} \tag{8.47}$$

where,

$$W_H(t',x';t',x') = \sum_{j\in\mathbb{N}} \frac{1}{2\omega_j}e^{-i\omega_j(t-t')}\xi_j(x)\overline{\xi_j(x')}, \tag{8.48}$$

is the integral kernel of the ultrastatic vacuum state. One can verify that the SJ vacuum (8.43) is a weak bisolution to the Klein-Gordon operator $P = \Box + m^2$,

$$W_{SJ_{0,\rho}}(f,Pg) = \sum_{j\in\mathbb{N}} \frac{e^{-2u_j}}{2\omega_j}\langle\overline{f} \mid \Psi^+_{\rho,j}\rangle_\rho\langle\Psi^+_{\rho,j} \mid Pg\rangle_\rho = 0, \tag{8.49}$$

for all $f, g \in C_0^\infty(\mathcal{M})$ where the last equality holds since,

$$\begin{aligned}\langle \Psi^+_{\rho,j} \mid Pg\rangle_\rho &= -i\sqrt{\frac{e^{3u_j}}{\widehat{\rho}(0)}}\langle \rho\Big(\sqrt{\cosh(u_j)}\mathcal{A}_j - \sqrt{\sinh(u_j)}\overline{\mathcal{A}_{\bar{j}}}\Big) \mid Pg\rangle_\rho \\ &= -i\sqrt{\frac{e^{3u_j}}{\widehat{\rho}(0)}}\langle(\sqrt{\cosh(u_j)}\mathcal{A}_j - \sqrt{\sinh(u_j)}\overline{\mathcal{A}_{\bar{j}}} \mid Pg\rangle \\ &= -i\sqrt{\frac{e^{3u_j}}{\widehat{\rho}(0)}}\langle(\sqrt{\cosh(u_j)}P\mathcal{A}_j - \sqrt{\sinh(u_j)}P\overline{\mathcal{A}_{\bar{j}}} \mid g\rangle \\ &= 0,\end{aligned} \tag{8.50}$$

where we have used $P\mathcal{A}_j(t,x) = (\partial_t^2 + K)e^{i\omega_j t}\xi_j(x) = (\omega_j^2 - \omega_j^2)e^{i\omega_j t}\xi_j(x) = 0$ for all $(t,x) \in \mathcal{M}$ and that P is formally self-adjoint with respect to the $\langle \cdot \mid \cdot \rangle-$inner product. A similar argument shows that $W_{SJ_\rho}(Pf, g) = 0$ for all $f, g \in C_0^\infty(\mathcal{M})$.

Purity of the SJ Vacuum with Softened Boundaries

Before proving that the SJ vacuum with softened boundaries is pure, we shall make use of the following definitions. First, we define,

$$\sigma_\rho : S(\mathcal{M}) \times S(\mathcal{M}) \to \mathbb{R} \tag{8.51}$$

$$([f],[g]) \mapsto \langle f \mid -iA_\rho g\rangle_\rho,$$

$$\mu_{0,\rho} : S(\mathcal{M}) \times S(\mathcal{M}) \to \mathbb{R} \tag{8.52}$$

$$([f],[g]) \mapsto \frac{1}{2}\langle f \mid |A_\rho| g\rangle_\rho,$$

where $S(\mathcal{M}) = C_0^\infty(\mathcal{M}, \mathbb{R})/\ker \mathbb{E}$ is the real-linear vector space of equivalence classes $[f]$ with respect to the equivalence relation $f \sim g$ if and only if $\mathbb{E}(f-g) = 0$. The two-point function (8.43) is then given by,

$$W_{SJ_{0,\rho}}(f,g) = \mu_{0,\rho}([f],[g]) + \frac{i}{2}\sigma_\rho([f],[g]) \tag{8.53}$$

We observe,

$$\sigma_\rho([f],[g]) = \langle f \mid -iA_\rho g\rangle_\rho = -i\langle f \mid Ag\rangle = \langle f \mid \mathbb{E}g\rangle = \sigma([f],[g]), \tag{8.54}$$

for all $[f],[g] \in S(\mathcal{M})$. Therefore, since $\sigma(\cdot,\cdot)$ is a well defined symplectic form by [45, Proposition 8], it follows that (8.51) is a well defined symplectic form on $S(\mathcal{M})$. Since A_ρ is a self-adjoint compact operator on $\mathcal{H}_\rho$, there is the unique partial isometry $U \in B(\mathcal{H}_\rho)$ such that $A_\rho = U|A_\rho| = |A_\rho|U^*$. We also have $A_\rho U = |A_\rho|$ since $U^*U = \mathbb{1}$ on range$(|A_\rho|)$. If $h, h' \in C_0^\infty(\mathcal{M}, \mathbb{R})$ are chosen such that $\mathbb{E}(h - h') = 0$ then we obtain $\langle f \mid |A_\rho|(h - h')\rangle_\rho = \langle U^* f \mid A_\rho(h - h')\rangle_\rho = \langle U^* f \mid A(h - h')\rangle = 0$. Therefore the right-hand slot in (8.52) is independent of the choice of representative. Since $|A_\rho|$ is self-adjoint, it follows that (8.52) is independent of chosen representative in the left-hand argument, and (8.52) is hence well defined. To show that $\mu_{0,\rho}([f],[g]) \in \mathbb{R}$ for all $[f],[g] \in S(\mathcal{M})$ it is sufficient to note that $\Gamma|A_\rho| = |A_\rho|\Gamma$ and that $\Gamma f = f$, $\Gamma g = g$. Using the proof that the *unsoftened* SJ vacuum is pure (originally given in [28, Proposition 3.2] and detailed in Proposition 5.2, we now obitan,

Proposition 8.2. *The SJ vacuum with softened boundaries given by the two-point function* (8.43) *is a pure state on* $\mathcal{A}(\mathcal{M})$.

Proof. It is sufficient to show that, for all $[f] \in S(\mathcal{M})$,

$$\mu_\rho([f],[f]) = \sup_{0 \neq [h] \in S(\mathcal{M})} \frac{|\sigma_\rho([f],[h])|^2}{4\mu_{\rho([h],[h])}}. \tag{8.55}$$

We first observe that the proof given in Proposition 5.2 is independent of the underlying Hilbert space. The saturation property give in (8.55) then follows, *verbatim*, from the original purity proof of the SJ vacuum given in Proposition 5.2. ■

Comparison to the Brum-Fredenhagen Vacuum State

As shown in Chapter 5, Brum and Fredenhagen introduce a method to regularise the SJ vacuum as to obtain a family of Hadamard states, which are known as BF states. Let $\mathcal{M} = ((-\tau,\tau)\times\Sigma), g = \mathbb{1}\oplus -h)$, $\tau > 0$, and $\mathcal{N} = ((\mathbb{R} \times \Sigma), g = \mathbb{1} \oplus -h)$ and let $\iota : \mathcal{M} \hookrightarrow \mathcal{N}$, $(t,x) \mapsto (t,x)$ be an isometric embedding. The BF state is constructed by taking the positive part of the operator,

$$A_f = f(-i\mathbb{E}_{\mathcal{M}})f, \tag{8.56}$$

where $f \in C_0^\infty(\mathcal{N})$ and setting the two-point function to be,

$$W_{BF_f}(h,g) = \langle \overline{h} \mid A_f^+ g\rangle, \tag{8.57}$$

for all $h, g \in C_0^\infty(\mathcal{M})$ and where $A_f^+ = \frac{1}{2}(A_f + |A_f|)$. The SJ vacuum with softened boundaries involves constructing a modified commutator function A_ρ on $\mathcal{M}$ defined by,

$$\langle \overline{h} \mid A_\rho g\rangle_\rho = iE(f,g) \qquad \forall h, g \in C_0^\infty(\mathcal{M}), \tag{8.58}$$

where the $\langle \cdot \mid \cdot \rangle_\rho$ inner product is defined in (8.20) and $\rho \in C_0^\infty(\mathbb{R})$ obeys $\rho(t) > 0$ for all $t \in (-\tau, \tau)$ and $\rho(t) \to 0$ as $t \to \pm\tau$. The two-point function for the SJ vacuum with softened boundaries is,

$$W_{SJ_\rho}(h, g) = \langle \overline{h} \mid A_\rho^+ g \rangle_\rho. \tag{8.59}$$

We shall compare the BF states to the SJ vacuum with softened boundaries on the ultrastatic slab $\mathcal{M} = ((-\tau, \tau) \times \Sigma, g = \mathbb{1} \oplus -h)$. The first observation is that the inner product (8.20) coincides with the standard inner product on $L^2(\mathcal{M}, \ \mathbf{d}\mathrm{vol}_g)$ whenever $\rho(t) = 1$. By letting $\rho(t) = f^2(t)$ on some interval (τ', τ') where $0 < \tau' < \tau$, the two-point function (8.59) coincides with the two-point function (8.57) whenever on the smaller spacetime $(-\tau', \tau') \times \Sigma$. Therefore, the class of softened SJ states contain the class of BF states. However the class of SJ states with softened boundaries is strictly larger than the class of BF states since the function ρ does have to be identically one on some relatively compact subset of $\mathbb{R}$, the CCRs are inbuilt by virtue of (8.1).

8.4 The Hadamard condition for the SJ Vacuum with Softened Boundaries

Our goal of this section is to prove the following:

Proposition 8.3. *Let $\mathcal{M} = ((-\tau, \tau) \times \Sigma, \boldsymbol{dt} \oplus -h)$ be an ultrastatic slab spacetime where (Σ, h) is a compact Riemannian manifold. Let $\rho \in C_0^\infty(\mathbb{R})$ obey $\rho(t) \to 0$ as $t \to \pm\tau$ and have support properties supp$(\rho) = (-\tau, \tau)$. The vacuum SJ state ω_{SJ_ρ} with two-point function* (8.43) *as constructed above is a Hadamard state.*

Proof. Since the ultrastatic vacuum state is Hadamard [33], the SJ state with two-point function (8.47) is Hadamard if and only if the integral kernel,

$$\begin{aligned} :W_{SJ_{0,\rho}}:(t,x;t',x') &= W_{SJ_{0,\rho}}(t,x;t',x') - W_H(t,x;t',x') \\ &= \sum_{j\in\mathbb{N}} \frac{1}{2\omega_j}\Big[e^{u_j}\sinh(u_j)(e^{i\omega_j(t-t')} + e^{i\omega_j(t-t')}) \\ &\qquad - \frac{1}{2}e^{2u_j}\left|\frac{\widehat{\rho}(2\omega_j)}{\widehat{\rho}(0)}\right|(e^{i\omega_j(t+t')} + e^{-i\omega_j(t+t')})\Big]\xi_j(x)\overline{\xi_j(x')} \end{aligned} \tag{8.60}$$

converges as a series in j in $C^\infty(\mathcal{M} \times \mathcal{M})$. Since $K = -\Delta_\Sigma + m^2$ is an elliptic operator, elliptic regularity ensures that the eigenvectors ξ_j are smooth for all $j \in \mathbb{N}$ [20, Theorem 3 Section 6.3]. Hence each term appearing in (8.60) is smooth, so one only needs to check that the limit is smooth. We first observe the following,

$$\begin{aligned} e^{u_j}\sinh(u_j) &= e^{u_j}\left(\frac{1-e^{-2u_j}}{2e^{-u_j}}\right) \\ &= \frac{1}{2}e^{2u_j}(1-e^{-2u_j}) \\ &< \frac{1}{2}e^{2u_j}(1-e^{-2u_j})(1+e^{-2u_j}) \\ &= \frac{1}{2}e^{2u_j}(1-e^{-4u_j}) \\ &= \frac{1}{2}e^{2u_j}\left|\frac{\widehat{\rho}(2\omega_j)}{\widehat{\rho}(0)}\right|^2, \end{aligned} \tag{8.61}$$

where we have used $e^{-4u_j} = 1 - |\widehat{\rho}(2\omega_j)|^2$. We now observe that, since $\lim_{j\to\infty}\widehat{\rho}(\omega_j) = 0$ and since $|\widehat{\rho}(\omega_j)| < |\widehat{\rho}(0)|$, we have,

$$\sup_j e^{2u_j} = \sup_j \frac{1}{\sqrt{1-\frac{|\widehat{\rho}(2\omega_j)|^2}{|\widehat{\rho}(0)|}}} < \infty. \tag{8.62}$$

Therefore the coefficients of the functions $\mathcal{A}_j(t,x) = e^{-i\omega_j t}\xi_j(x)$ and their complex conjugates appearing in the kernel (8.60) may be bounded by,

$$\gamma_j = \frac{1}{2}\sup_j(e^{2u_j})\left(\left|\frac{\widehat{\rho}(2\omega_j)}{\widehat{\rho}(0)}\right|^2 + \left|\frac{\widehat{\rho}(2\omega_j)}{\widehat{\rho}(0)}\right|\right). \tag{8.63}$$

By Theorem 4.7, if,

$$\sum_{j\in\mathbb{N}} \omega_j^p|\gamma_j|^2 < \infty, \tag{8.64}$$

holds for all $p \in \mathbb{N}_0$, then the integral kernel (8.60) converges in $C^\infty(\mathcal{M}\times\mathcal{M})$, and therefore the vacuum SJ state with softened boundaries given by two-point function (8.43) is Hadamard. Since $\rho \in C_0^\infty(\mathcal{M})$, for all $N \in \mathbb{N}_0$ there exists $C_N > 0$ such that,

$$\left|\frac{\widehat{\rho}(2\omega_j)}{\widehat{\rho}(0)}\right|^2 + \left|\frac{\widehat{\rho}(2\omega_j)}{\widehat{\rho}(0)}\right| \leq C_N\omega_j^{-N}, \tag{8.65}$$

for all $j \in \mathbb{N}$. Furthermore, since (Σ, h) is a boundaryless, compact Riemannian manifold of dimension $n = 3$, by Lemma 6.5 there exists a $\alpha > 0$ such that $j \leq \alpha\omega_j^3$ for all $j \in \mathbb{N}$. This then implies,

$$\sum_{j\in\mathbb{N}} \omega_j^p|\gamma_j|^2 \leq \frac{C_N^2(\sup_j e^{2u_j})^2}{4}\sum_{j\in\mathbb{N}}\frac{1}{\omega_j^{2N-p}} \leq \frac{(\sup_j e^{2u_j})^2 C_N^2}{4}\alpha\sum_{j\in\mathbb{N}}\frac{1}{j^{\frac{2N-p}{3}}}, \tag{8.66}$$

which converges for all $N > \frac{1}{2}(p+3)$. ■

8.5 Wick Square for the Softened SJ Vacuum

In this section we calculate the Wick square of the $n-$th derivative of the field normal ordered with respect to the ultrastatic vacuum and evaluated in the SJ vacuum with softened boundraies on the ultrastatic slab $((-\tau,\tau)\times\Sigma, g = \mathbb{1}\oplus -h)$. We also give explicit numerical computations in Maple for the Wick square when the spatial manifold (Σ,h) is a round three sphere. The Wick square of the $n-$th time derivative of the field normed ordered with respect to the ultrastatic vacuum and evaluated in the softened SJ vacuum (8.60) is,

$$\langle :(\partial_t^n\phi)^2:(f)\rangle = \int_{\mathcal{M}} \langle :(\partial_t^n\phi)^2:\rangle(t,x) f(t,x)\ \mathbf{d}\mathrm{vol}_g \tag{8.67}$$

where,

$$\begin{aligned}\langle :(\partial_t^n\phi)^2:\rangle(t,x) &= \lim_{(t',x')\to(t,x)} \partial_t^n\partial_{t'}^n : W_{SJ_{0,\rho}} : (t',x';t,x)\\ &= \sum_{j\in\mathbb{N}} \frac{1}{2}\omega_j^{2n-1} e^{u_j}\left[2\sinh(u_j) - (-1)^n e^{u_j}\left|\frac{\widehat{\rho}(2\omega_j)}{\widehat{\rho}(0)}\right|(\cos(2\omega_j t))\right]|\xi_j(x)|^2\end{aligned} \tag{8.68}$$

In the following we construct a class of smooth, compactly supported functions $\rho\in C_0^\infty(\mathbb{R})$ that decay smoothly to zero at the points $\pm\tau$ (we call them 'plateau functions') and we use them to Wick square (8.68) when the spatial manifold (Σ,h) is a round three sphere. First, we use the work appearing in [21, 22] to construct a smooth compactly supported function with unit integral. Let $f:\mathbb{R}\to\mathbb{R}$ be a function defined by,

$$\phi(x) = \begin{cases} \dfrac{e^{-\frac{1}{4x}}}{2\sqrt{\pi}x^{3/2}} & x>0\\ 0 & x\leq 0. \end{cases} \tag{8.69}$$

From this, we can define a compactly supported smooth function

$$\begin{aligned} H &: \mathbb{R}\to\mathbb{R}\\ &x\mapsto \frac{1}{c}\phi(x+0.5)\phi(0.5-x). \end{aligned} \tag{8.70}$$

where $c>0$ is a constant such that (8.70) has unit integral. From this, we define another smooth compactly supported function with unit integral, which is a function of t with dimensions of inverse time,

$$H_{\tau_s}(t) \doteq \frac{1}{\tau_s} H\left(\frac{t}{\tau_s}\right), \tag{8.71}$$

where $\tau_s > 0$ is a timescale called the 'switch on' time. Since (8.70) is a function of a dimensionless variable, to construct a plateau function we express everything in the units of the switch on time. Using the compactly supported function (8.71), a plateau function can now be defined,

$$\begin{aligned}\rho(t) &\doteq \int_{-\infty}^{t} H_{\tau_s}(t' + \tau_p) - H_{\tau_s}(t' - \tau_p)\ \mathbf{d}t', \\ &= \frac{1}{\tau_s}\int_{-\infty}^{t} H\left(\frac{t' + \tau_p}{\tau_s}\right) - H\left(\frac{t' - \tau_p}{\tau_s}\right)\ \mathbf{d}t',\end{aligned} \tag{8.72}$$

where $\tau_p > 0$ is a timescale called the plateau time and $c > 0$ is a constant chosen so that (8.72) has unit integral. The plateau function has support $\mathrm{supp}(\rho) = [-\frac{\tau_s}{2} - \tau_p, \tau_p + \frac{\tau_s}{2}]$ and obeys $\rho(t) = 1$ for all $t \in [-\tau_p + \frac{\tau_s}{2}, \tau_p - \frac{\tau_s}{2}]$.

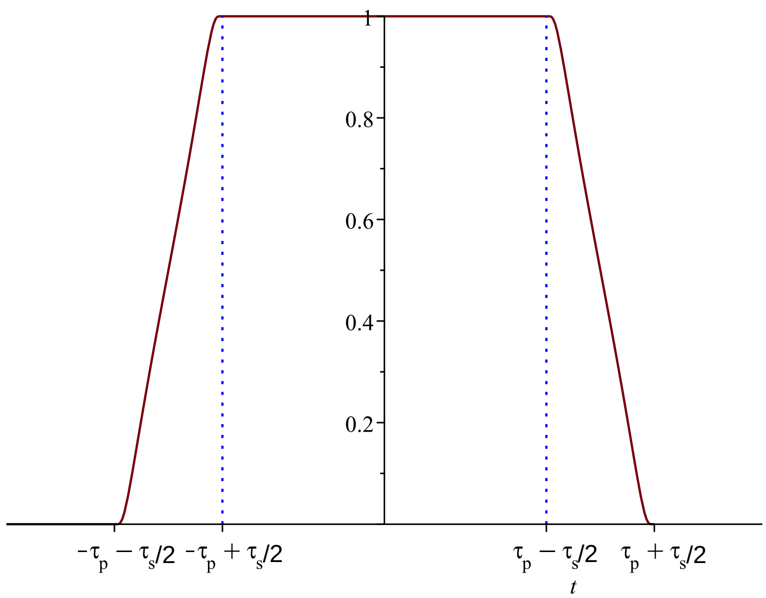


Figure 8.1: Plot of a plateau function will labels showing the switch on time τ_s and plateau time τ_p where $\tau = \tau_p + \dfrac{\tau_s}{2}$.

The Fourier transform of (8.72) is,

$$\begin{aligned}\widehat{\rho}(\omega) &= \frac{1}{i\omega}\left[e^{i\omega\tau_p}\widehat{H}_{\tau_s}(\omega) - e^{-i\omega\tau_p}\widehat{H}_{\tau_s}(\omega)\right] \\ &= 2\tau_p\frac{\sin(\omega\tau_p)}{\omega\tau_p}\widehat{H}_{\tau_s}(\omega) \\ &= 2\tau_p\mathrm{sinc}(\omega\tau_p)\widehat{H}_{\tau_s}(\omega)\end{aligned} \tag{8.73}$$

where,

$$\begin{aligned}\widehat{H_{\tau_s}}(\omega) &= \mathcal{F}\left(\frac{1}{\tau_s}H\left(\frac{t}{\tau_s}\right)\right)(\omega) \\ &= \frac{1}{\tau_s}\left(\tau_s\widehat{H}(\tau_s\omega)\right) \\ &= \widehat{H}(\tau_s\omega),\end{aligned} \tag{8.74}$$

where $\mathcal{F}(\cdot)$ is the Fourier transform. We then obtain,

$$\widehat{\rho}(\omega) = 2\tau_p\mathrm{sinc}(\tau_p\omega)\widehat{H}(\tau_s\omega). \tag{8.75}$$

Our motivation for studying compactly supported plateau functions is that, in the limit $\tau_s \to 0$, the plateau function (8.72) tends to a characteristic function on the interval $[-\tau_p, \tau_p]$. In the limit $\tau_s \to 0$, the Fourier transform of the plateau function (8.72) has the limit,

$$\lim_{\tau_s\to 0}\widehat{\rho}(\omega) = \lim_{\tau_s\to 0} 2\tau_p\mathrm{sinc}(\tau_p\omega)\widehat{H}(\tau_s\omega) = 2\tau_p\mathrm{sinc}(\tau_p\omega)\widehat{H}(0) = 2\tau_p\mathrm{sinc}(\tau_p\omega). \tag{8.76}$$

Therefore, the softened commutator function (8.1) on the slab $(-\tau_p - \frac{1}{2}\tau_s, \tau_p + \frac{1}{2}\tau_s)$ will give the commutator function (8.22) on the ultrastatic slab $(-\tau_p, \tau_p) \times \Sigma$ in the limit $\tau_s \to 0$. This implies that the SJ vacuum with softened boundaries tends to the SJ vacuum on the ultrastatic slab $(-\tau_p, \tau_p) \times \Sigma$ in the limit $\tau_s \to 0$. Since the Wick square of the time derivatives of the SJ vacuum diverges [29], we can expect that the Wick square of the time derivatives of the SJ vacuum with softened boundaries will diverge in the limit $\tau_s \to \infty$. We now use the plateau functions (8.72) to calculate the Wick square of the softened SJ vacuum given in (8.68) and investigate the limit as $\tau_s \to 0$ in the case when the spatial manifold (Σ, h) is a round three-sphere.

Wick Square for the SJ Vacuum with Softened Boundaries on a Three-Sphere

Let $\rho \in C_0^\infty(\mathbb{R})$ be a plateau function given in (8.72) and let (S^3, h) be a round three sphere of radius R. The eigenvalues of the operator $K = -\Delta_{S^3} + m^2$ are given in (5.84) and occur with multiplicity

$(1+j)^2$. Using the Fourier transform (8.77) we obtain,

$$\begin{aligned}\frac{\widehat{\rho}(\omega_j)}{\widehat{\rho}(0)} &= \operatorname{sinc}(\tau_p\omega_j)\widehat{H}(\tau_s\omega_j) \\ &= \operatorname{sinc}(\kappa\sqrt{j(j+2)+(mR)^2})\widehat{H}(\alpha\kappa\sqrt{j(j+2)+(mR)^2})\end{aligned} \tag{8.77}$$

where,

$$\begin{aligned}\alpha &= \frac{\tau_s}{\tau_p} \\ \kappa &= \frac{\tau_p}{R}.\end{aligned} \tag{8.78}$$

A plot of the normalised Fourier transform (8.77) for parameters $\alpha = 0.5, 1$ and $\kappa = 2$ is found in Figure 8.2.

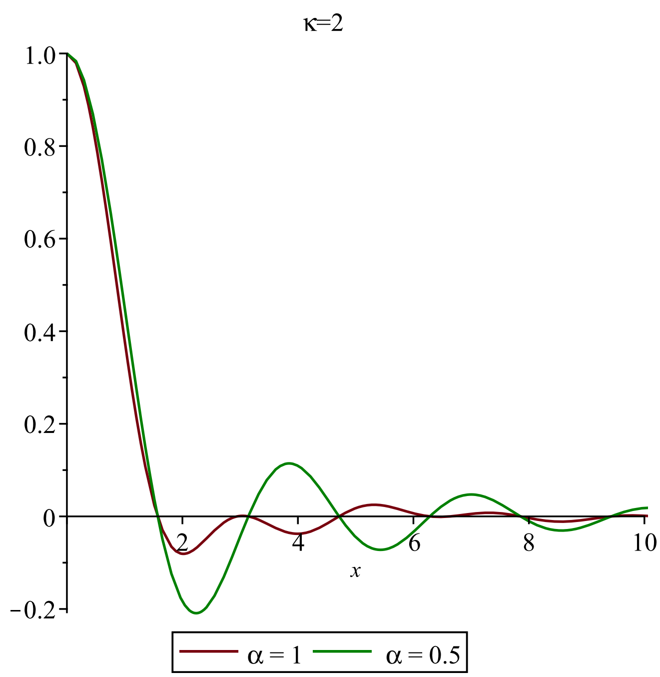


Figure 8.2: Plot of the Fourier transform $\frac{\widehat{\rho}(\omega)}{\widehat{\rho}(0)}$ for the parameters $\alpha = \frac{\tau_s}{\tau_p} = 0.5, 1$.

Since the three-sphere is spherically symmetric, the Wick square evaluated at a point (t, x_0) is equal to the Wick square averaged over the three sphere evaluated at time t. In this section, we plot the first N terms of the Wick square of the $n-th$ time derivative of the field given by, for all $(t,x_0)\in(-\tau,\tau)\times S^3$,

$$\begin{aligned}
\langle : (\partial_t^n\phi)^2 :\rangle(t,x_0) &= \frac{1}{\mathrm{Vol}(S^3)}\int_\Sigma \lim_{(t',x')\to(t,x)} \partial_t^n\partial_{t'}^n : W_{SJ_{0,\rho}} : (t,x;t',x')\ \mathbf{d}\mathrm{vol}_h \\
&= \frac{1}{2\mathrm{Vol}(S^3)}\sum_{j=1}^{N}(j+1)^2\omega_j^{2n-1}e^{u_j}\left[2\sinh(u_j)-(-1)^n e^{u_j}\left|\frac{\widehat{\rho}(2\omega_j)}{\widehat{\rho}(0)}\right|(\cos(2\omega_j t))\right]\int_\Sigma |\xi_j(x)|^2\ \mathbf{d}\mathrm{vol}_h \\
&= \frac{1}{2\mathrm{Vol}(S^3)}\sum_{j=1}^{N}(j+1)^2\omega_j^{2n-1}e^{u_j}\left[2\sinh(u_j)-(-1)^n e^{u_j}\left|\frac{\widehat{\rho}(2\omega_j)}{\widehat{\rho}(0)}\right|(\cos(2\omega_j t))\right] \\
&= \frac{1}{2\mathrm{Vol}(S^3)}\sum_{j=1}^{N}(j+1)^2\omega_j^{2n-1}e^{u_j}\left[2\sinh(u_j)-(-1)^n e^{u_j}\left|\mathrm{sinc}(2\omega_j\tau_p)\widehat{H}(2\omega_j\tau_p\alpha)\right|(\cos(2\omega_j t))\right],
\end{aligned} \tag{8.79}$$

where $\alpha=\frac{\tau_s}{\tau_p}$ is a dimensionless constant. In Figures 8.3, 8.4 and 8.5 we set $N=100$, keep τ_p fixed and vary the switch-on time τ_s. Since the softened SJ state is constructed on the ultrastatic slab $(-\tau_p+\frac{\tau_s}{2},\tau_p+\frac{\tau_s}{2})\times\Sigma$, by varying the value of the switch-on time we are comparing different SJ states on different ultrastatic slabs. However, we are interested in studying the softened SJ states within the interior of the ultrastatic slab, far away from the boundary. Furthermore, in the limit $\alpha\to 0$ the plateau function (8.72) tends to a characteristic function on $[-\tau_p,\tau_p]$ and the SJ vacuum with softened boundaries converges to the unsoftened SJ vacuum in the limit $\alpha\to 0$. The values ω_j are measured with units R^{-1} and the time is measured in units of R, where R is the radius of the three sphere. In Figures 8.3 and 8.4 we plot the first $N=100$ terms of Wick square (8.79) for plateau times $\tau_p=0.5R$ and $\tau_p=2R$ respectively with values $\alpha=0.1,0.3,0.5$. In Figure 8.5 we plot the first $N=100$ terms Wick square of the first derivative (the $n=1$ case of (8.79)) for a plateau time $\tau_p=2R$ and parameters $\alpha=0.3,0.5$. Finally in Figures 8.6 and 8.7 we plot the first $N=25$ and $N=100$ terms of the Wick square (8.79) for a plateau time $\tau_p=2R$ and parameters $\alpha=0.3$ and $\alpha=0$ respectively.

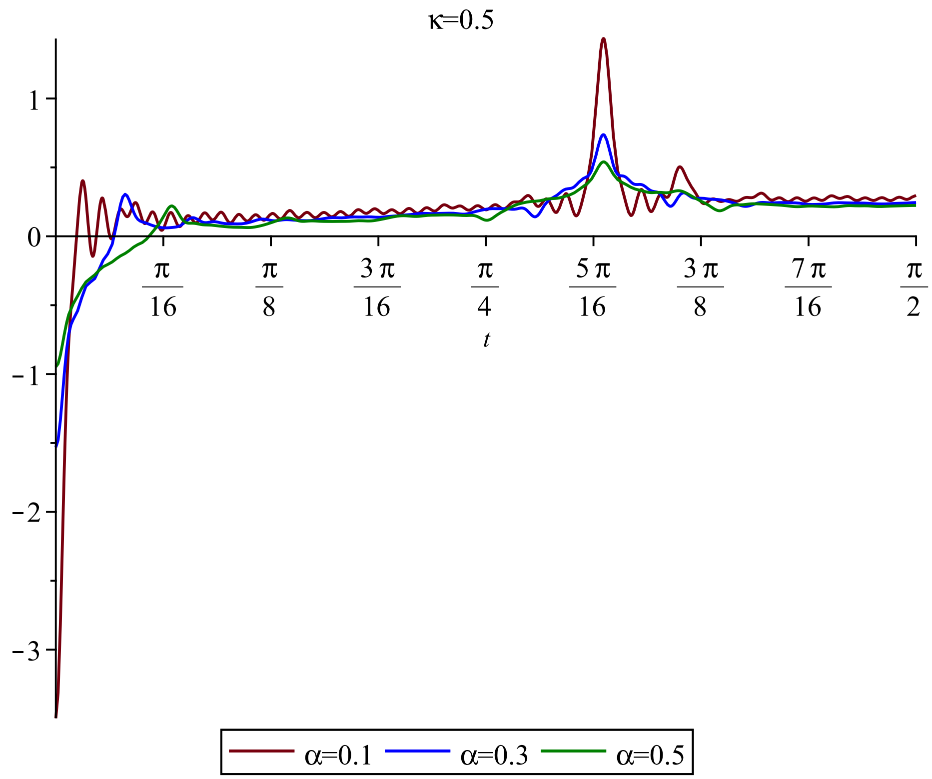


Figure 8.3: Plot of the Wick Square for the SJ vacuum with softened boundaries and with parameters $mR = 1$, $N = 100$, $\kappa = 0.5$ and $\alpha = 0.1, 0.3, 0.5$.

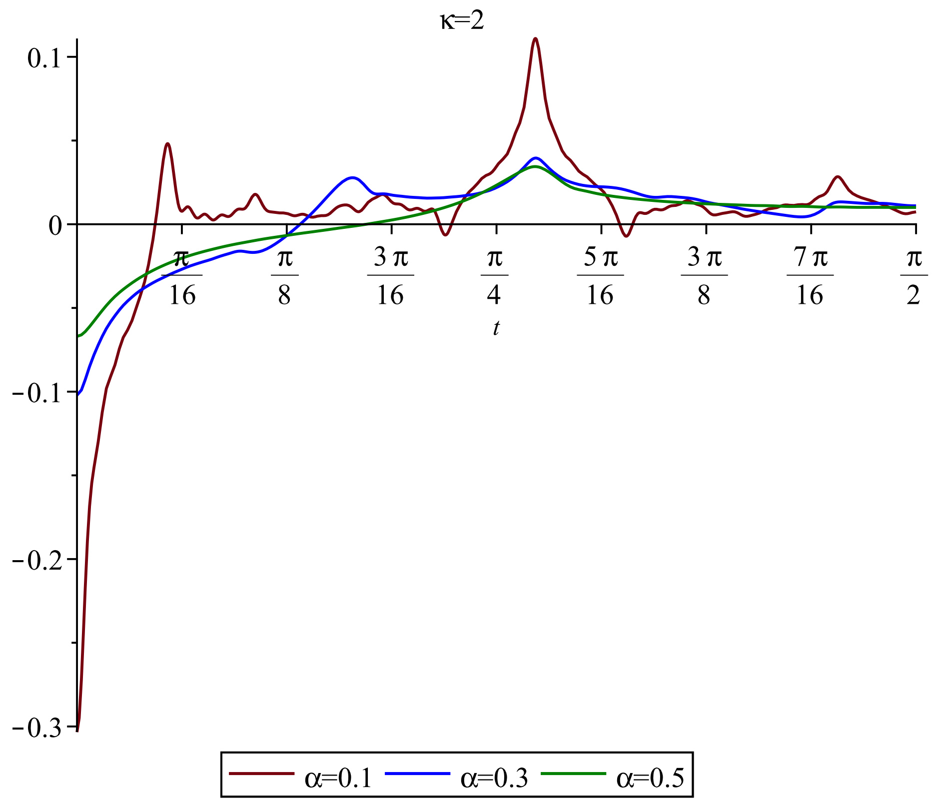


Figure 8.4: Plot of the Wick Square for the SJ vacuum softened boundaries with parameters $mR = 1$, $N = 100$, $\kappa = 2$ and $\alpha = 0.1, 0.3, 0.5$.

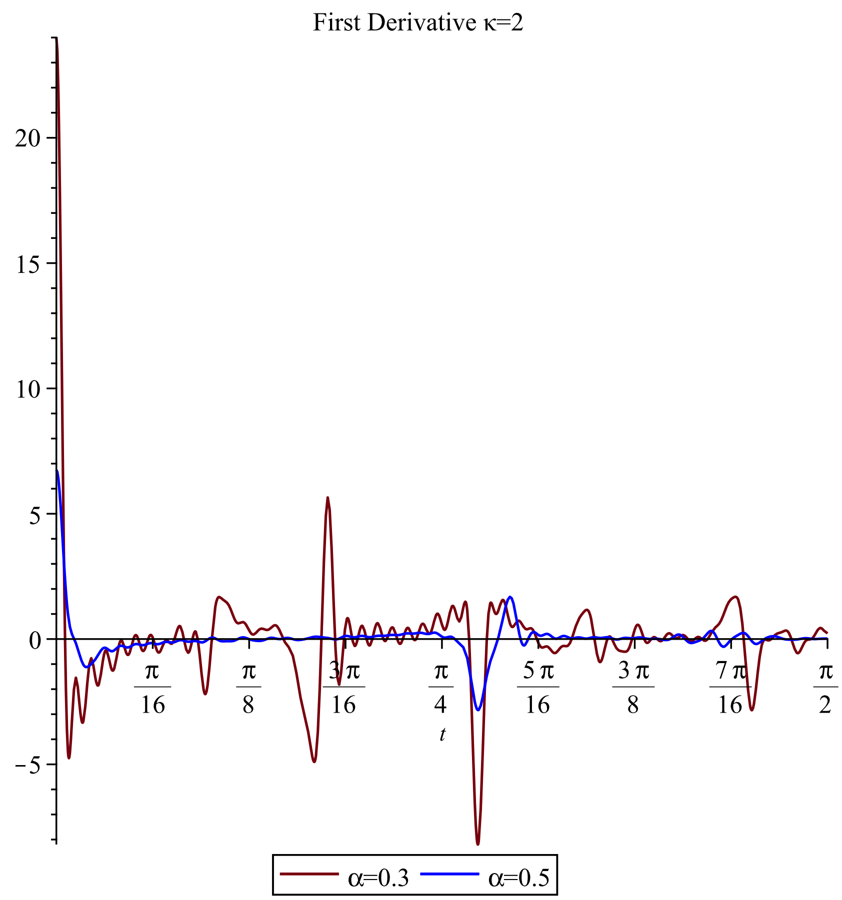


Figure 8.5: Plot of the Wick Square of the first derivative of the field normal ordered in the ultrastatic vacuum and evaluated in the softened SJ vacuum for parameters $mR = 1$, $N = 100$, $\kappa = 2$ and $\alpha = 0.3, 0.5$.

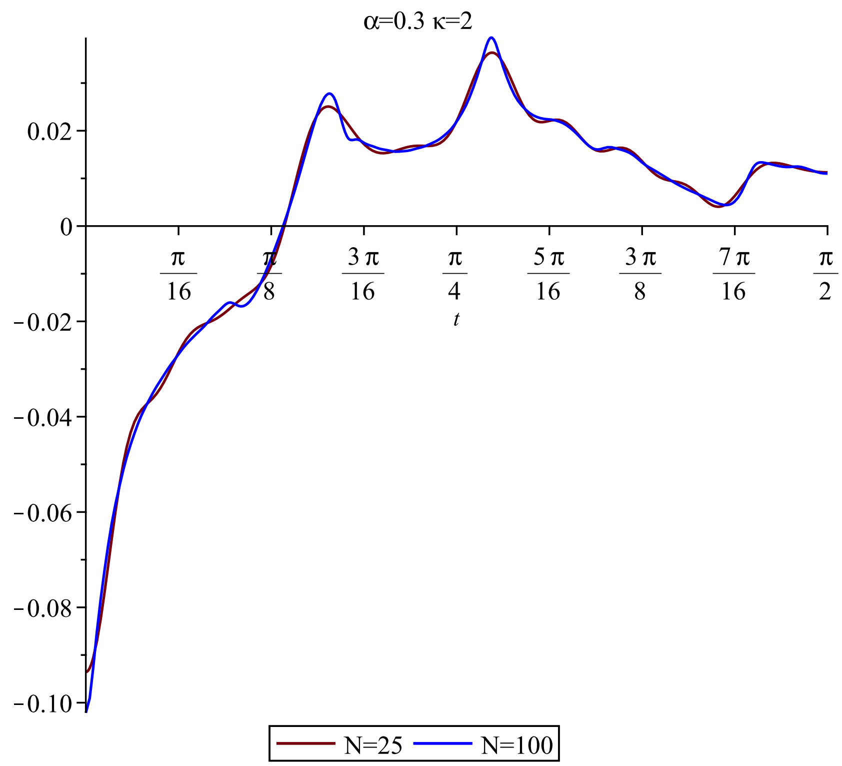


Figure 8.6: Plot of the Wick Square for the SJ vacuum with softened boundaries for parameters $mR = 1$, $N = 25, 100$, $\kappa = \dfrac{\tau_p}{R} = 1$ and $\alpha = 0.3$.

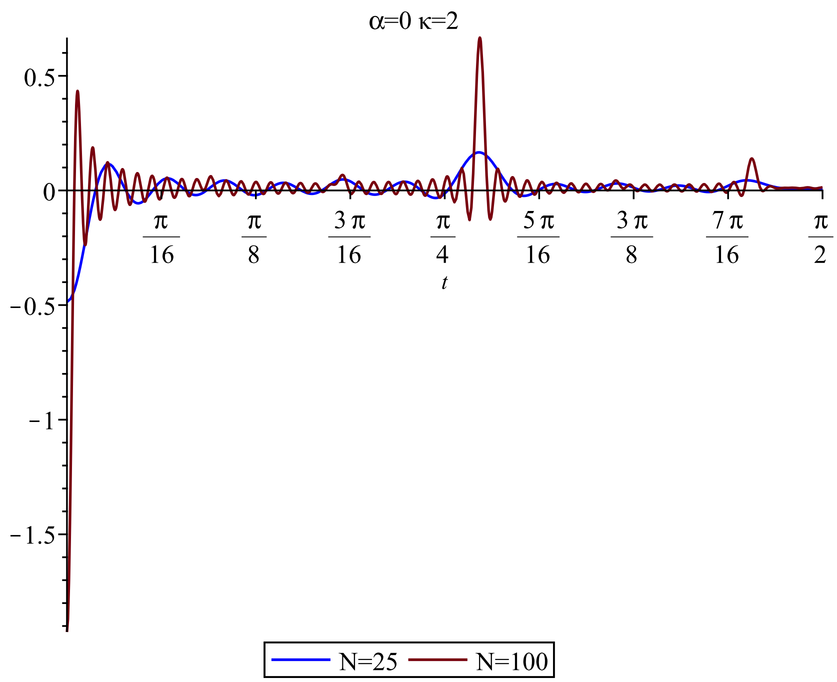


Figure 8.7: Plot of the Wick Square for the *unsoftened* SJ vacuum with parameters $mR = 1$, $N = 25, 100$, $\kappa = \dfrac{\tau_p}{R} = 2$ and $\alpha = 0$.

We draw the following conclusions: For the plots of the Wick square for the SJ vacuum with softened boundaries given in (8.79) and shown in Figure 8.3 and Figure 8.4 we see that at $t = 0$, the value of the Wick square is negative and increases in magnitude as the ratio of the switch-on times to the plateau times decreases. This is an indicator that in the limit $\alpha \to 0$, the Wick square for the field normal ordered with respect to the ultrastatic vacuum on $(-\tau, \tau) \times S^3$ and evaluated in the *unsoftened* SJ diverges at $t = 0$, i.e. in the middle of the interval $(-\tau, \tau)$ far from the boundary. This would indicate that the unsoftened SJ vacuum fails to be Hadamard within the interior of the spacetime as well as the boundary. This is supported by Figure 8.7, which shows the first $N = 25, 100$ terms of the Wick square of the field evaluated in the unsoftened SJ vacuum. As the value of N increases from $N = 25$ to $N = 100$, the values of the Wick square at $t = 0$ increases in magnitude. This indicates that the Wick square for the unsoftened SJ state will diverge in the limit $N \to \infty$ at $t = 0$.

Furthermore, one can also observe in Figure 8.5 that the Wick square of the first derivative of the field increases in magnitude as the value of N increases, indicating that the Wick square of the first derivative diverges in the interior in the limit $N \to \infty$. Therefore, it appears that the Wick square of the first derivatives evaluated in the unsoftend SJ vacuum state diverges in the interior. However, the Wick square for the softened SJ vaccum state does *not* exhibit this behaviour. In Figure 8.6 we plot the Wick square of the softened SJ vacuum state for a plateau time $\kappa = 2$ and set the parameter $\alpha = 0.3$. In Figure 8.6 there is a clear convergence as the value of N increases, indicating that the Wick square for the softened SJ vacuum is finite in the limit $N \to \infty$. Choosing a cutoff of $N = 100$ for the Wick square of the softened SJ vacuum is justified because for terms $N > 100$, the magnitude of the Wick square for the softened SJ vacuum is of order 10^{-6}, and presents a negligible contribution to the Wick square. Therefore we choose a cutoff of $N = 100$, since higher order terms will contribute a insignificant amount. Finally, in Figure 8.5 we see the Wick square of the first derivative of the field normal ordered with respect to the ultrastatic vacuum on $(-\tau, \tau) \times S^3$ and evaluated in the SJ vacuum with softened boundaries. In Figure 8.5, there are rapid oscillations in the Wick square of the first derivative as the value of α decreases. It is then reasonable to infer that as $\alpha \to 0$, the Wick square of the first derivative will not be smooth.

8.6 Generalised SJ States with Softened Boundaries

Having constructed the softened vacuum SJ state, we may now take functions of the operator (8.33) to construct the class of generalised SJ states with softened boundaries. Let $\mathcal{M} = ((-\tau, \tau) \times \Sigma, g = \mathbb{1} \oplus h)$ be an ultrastatic slab spacetime and let A_ρ be the softened commutator function given in (8.33). Let,

$$\begin{aligned} \psi : \sigma(A_\rho) &\to \mathbb{R} \\ \lambda &\mapsto \varphi(\lambda)(|\lambda| + \varphi(\lambda)), \end{aligned} \tag{8.80}$$

be an even, non-negative continuous function where $\varphi : \sigma(A_\rho) \to \mathbb{R}$ is an even, non-negative, continuous function. The class of softened SJ states is then constructed by using the softened vacuum SJ state and the function (8.80). The softened SJ state $\omega_{\varphi,\rho}$ has the two-point function,

$$W_{SJ_{\rho,\varphi}}(f, g) = W_{SJ_{\rho,0}}(f, g) + \langle \overline{f} \mid \varphi(A_\rho) g \rangle_\rho \qquad \forall f, g \in C_0^\infty(\mathcal{M}) \tag{8.81}$$

We have the following result,

Proposition 8.4. *There exists a generalised SJ state with softened boundaries whose two-point function is given by* (8.81).

Proof. First, we show that (8.81) is a weak bisolution to the Klein Gordon operator $P = \Box + m^2$,

$$\begin{aligned} W_{SJ_{\rho,\varphi}}(f, Pg) &= W_{SJ_{0,\rho}}(f, Pg) + \langle \overline{f} \mid \varphi(A_\rho) Pg \rangle \\ &= \langle \overline{f} \mid \varphi(A_\rho) Pg \rangle \\ &= \sum_{j \in \mathbb{N}} \varphi(\lambda_j) \big[\langle \overline{f} \mid \Psi^+_{\rho,j} \rangle_\rho \langle \Psi^+_{\rho,j} \mid Pg \rangle_\rho - \langle \overline{f} \mid \Psi^-_{\rho,j} \rangle_\rho \langle \Psi^-_{\rho,j} \mid Pg \rangle_\rho \big] \\ &= 0 \end{aligned} \tag{8.82}$$

for all $f, g \in C_0^\infty(\mathcal{M})$, which holds since $W_{SJ_{0,\rho}}(\cdot, \cdot)$ is a weak bisolution to the Klein Gordon operator and,

$$\langle \Psi^+_{\rho,j} \mid Pg \rangle_\rho = 0, \tag{8.83}$$

which is shown in (8.50). Similarly,

$$\begin{aligned} \langle \Psi^-_{\rho,j} \mid Pg \rangle_\rho &= \langle \Gamma Pg \mid \Gamma \Psi^-_{\rho,j} \rangle_\rho \\ &= \langle P\Gamma g \mid \Psi^+_{\rho,j} \rangle_\rho \\ &= \langle P\Gamma g \mid \Psi^+_{\rho,j} \rangle_\rho \\ &= 0. \end{aligned} \tag{8.84}$$

A similar argument shows that $W_{SJ_{\rho,\varphi}}(Pf, g) = 0$ for all $f, g \in C_0^\infty(\mathcal{M})$. Next, we show that,

$$w([\mathbb{E}f], [\mathbb{E}g]) = W_{SJ_\varphi}(\overline{f}, g), \tag{8.85}$$

is a well defined positive semi-definite sesquilinear form on $S(\mathcal{M}) \times S(\mathcal{M})$. Since $\ker \mathbb{E} = \mathrm{Ran}(P)$ and since (8.81) is a weak bisolution to the Klein-Gordon operator $P = \Box + m^2$, it follows that (8.85) is independent of chosen representative. Positivity is seen from the following,

$$w([\mathbb{E}f], [\mathbb{E}f]) = W_{SJ_\psi}(\overline{f}, f) = \langle f \mid A^+_\psi f \rangle \geq 0, \tag{8.86}$$

which holds since $A^+_\psi \geq 0$ and sesquilinearity follows from the $\langle \cdot \mid \cdot \rangle$−inner product. Therefore, by Proposition 4.2, there exists a state with the two-point function (8.81). ■

8.7 Hadamard Condition for the Generalised SJ states with Softened Boundaries

Since the vacuum SJ state with softened boundaries is Hadamard, a generalised SJ state with softened boundaries is Hadamard if and only if the integral kernel of the normal ordered two-point function,

$$\begin{aligned} :W_{SJ_{\rho,\varphi}}:(f,g) &= \langle \overline{f} \mid A^{+}_{0,\rho} g\rangle_\rho + \langle \overline{f} \mid \varphi(A_\rho) g\rangle_\rho - \langle \overline{f} \mid A^{+}_{0,\rho} g\rangle_\rho \\ &= \langle \overline{f} \mid \varphi(A_\rho) g\rangle_\rho, \end{aligned} \tag{8.87}$$

is smooth, i.e. that $\varphi(A_\rho)$ is smoothing. Therefore all generalised softened SJ states will be Hadamard if and only if both the operator $\varphi(A_\rho)$ has a smooth integral kernel and if the SJ vacuum with softened boundaries is Hadamard.

Proposition 8.5. *Let $(\mathcal{M} = (-\tau,\tau)\times\Sigma, g = \mathbb{1}\oplus -h)$ be an ultrastatic slab spacetime where (Σ, h) is a smooth compact Riemannian manifold. Let $\rho \in C_0^\infty(\mathcal{M})$ obey $\rho(t) > 0$ for all $t \in (-\tau,\tau)$ and decay smoothly to zero as $t \to \pm\tau$. Let A_ρ be the softened commutator function given in* (8.33). *Suppose $\varphi : \sigma(A_\rho) \to \mathbb{R}$ is a continuous function such that, for all $N \in \mathbb{N}_0$ there exists a $C_N \geq 0$ such that $|\varphi(\lambda)| \leq C_N|\lambda^N|$ for all $\lambda \in \sigma(A_\rho)$ and $\varphi(0) = 0$. Then the operator $\varphi(A_\rho)$ is smoothing.*

Proof. The softened commutator function (8.33) is given by the operator,

$$A_\rho f = \sum_{j\in\mathbb{N}} \lambda_j \big(\Psi^{+}_{\rho,j}\langle \Psi^{+}_{\rho,j} \mid f\rangle_\rho - \Psi^{-}_{\rho,j}\langle \Psi^{-}_{\rho,j} \mid f\rangle_\rho\big), \tag{8.88}$$

where the normalised eigenvectors $\Psi^{\pm}_{\rho,j}$ are given in (8.27) with corresponding eigenvalues $\pm\lambda_j$ given in (8.31). By the continuous functional calculus for bounded self-adjoint operators, the operator $\varphi(A_\rho)$ is given by,

$$\varphi(A_\rho) f = \sum_{j\in\mathbb{N}} \varphi(\lambda_j) \big(\Psi^{+}_{\rho,j}\langle \Psi^{+}_{\rho,j} \mid f\rangle_\rho - \Psi^{-}_{\rho,j}\langle \Psi^{-}_{\rho,j} \mid f\rangle_\rho\big), \tag{8.89}$$

which has the integral kernel,

$$\varphi(A_\rho)(t,x;t',x') = \sum_{j\in\mathbb{N}} \frac{\varphi(\lambda_j)e^{2u_j}}{2\omega_j}\big(e^{-i\omega(t-t')} - e^{i\omega(t-t')}\big)\xi_j(x)\overline{\xi_j(x')}. \tag{8.90}$$

Theorem 4.7 states that if the following holds,

$$\sum_{j\in\mathbb{N}} \omega_j^p |e^{2u_j}\varphi(\lambda_j)|^2 < \infty, \tag{8.91}$$

for all $p \in \mathbb{N}_0$ then the integral kernel (8.90) converges in $C^\infty(\mathcal{M} \times \mathcal{M})$, hence $\varphi(A_\rho)$ is smoothing. We now obtain, for all $p, N \in \mathbb{N}_0$,

$$\begin{aligned}
\sum_{j\in\mathbb{N}} \omega_j^p e^{4u_j} |\varphi(\lambda_j)|^2 &\leq \left(\sup_j (e^{2u_j})\right)^2 \sum_{j\in\mathbb{N}} \omega_j^p |\varphi(\lambda_j)|^2 \\
&\leq \left(\sup_j (e^{2u_j})\right)^2 C_N^2 \sum_{j\in\mathbb{N}} \omega_j^p |\lambda_j|^{2N} \\
&= \gamma \sum_{j\in\mathbb{N}} \omega_j^{p-2N} \left(1 - \left|\frac{\widehat{\rho}(2\omega_j)}{\widehat{\rho}(0)}\right|^2\right)^N \\
&\leq \gamma \sum_{j\in\mathbb{N}} \omega_j^{p-2N},
\end{aligned} \tag{8.92}$$

where,

$$\gamma = \left(\sup_j (e^{2u_j})\right)^2 C_N^2 \left(\frac{\widehat{\rho}(0)}{2}\right)^{2N} \tag{8.93}$$

and where we have used,

$$\lambda_j = \frac{\widehat{\rho}(0)}{2\omega_j} \sqrt{1 - \left|\frac{\widehat{\rho}(2\omega_j)}{\widehat{\rho}(0)}\right|^2} \tag{8.94}$$

for all $j \in \mathbb{N}$. Since (Σ, h) is a boundaryless, compact Riemannian manifold of dimension $n = 3$, Lemma 6.5 states that $j \leq \alpha\omega_j^3$ for a sufficiently large $\alpha > 0$. Therefore, we have,

$$\sum_{j\in\mathbb{N}} \omega_j^p |e^{2u_j} \varphi(\lambda_j)|^2 \leq \gamma \sum_{j\in\mathbb{N}} \omega_j^{p-2N} \leq \gamma \alpha^{\frac{1}{3}(2N-p)} \sum_{j\in\mathbb{N}} \frac{1}{j^{\frac{1}{3}(2N-p)}}, \tag{8.95}$$

which converges for all $N > \frac{1}{2}(p+3)$. Therefore, by Theorem 4.7, the integral kernel (8.90) converges in $C^\infty(\mathcal{M} \times \mathcal{M})$. Therefore $\varphi(A)$ is smoothing. ■

Using the fact that, for suitable chosen $\varphi : \sigma(A_\rho) \to \mathbb{R}$, the operator $\varphi(A_\rho)$ is smoothing and the fact that the SJ vacuum with softened boundaries with two-point function (8.43) is Hadamard, we arrive at the following result:

Proposition 8.6. *Let $\mathcal{M} = (-\tau, \tau) \times \Sigma$ be an ultrastatic slab spacetime with metric $g = \mathbb{1} \oplus -h$, where (Σ, h) is a compact Riemannian manifold. Let $\rho \in C_0^\infty(\mathbb{R})$ be a smooth function obeying supp$(\rho) = [-\tau, \tau]$, $\rho(t) > 0$ for all $\rho \in (-\tau, \tau)$ such that $\rho(t) \to 0$ as $t \to \pm\tau$. Let A_ρ be the softened commutator function for the free scalar field given by* (8.23)*. Let $\varphi : \sigma(A_\rho) \to \mathbb{R}$ be an even, non-negative,* *continuous function such that $\varphi(0) = 0$. If, for all $N \in \mathbb{N}_0$ there exists a $C_N \geq 0$ such that $|\varphi(\lambda)| \leq C_N \lambda^N$, then the SJ state with two-point function,*

$$W_{SJ_{\varphi,\rho}}(f,g) = \langle \overline{f} \mid A^+_{0,\rho} g\rangle_\rho + \langle \overline{f} \mid \varphi(A_\rho) g\rangle_\rho \qquad \forall f,g \in C_0^\infty(\mathcal{M}) \tag{8.96}$$

defines a Hadamard state.

Proof. The SJ vacuum state with softened boundaries given by the two-point function,

$$W_{SJ_{0,\rho}}(f,g) = \langle \overline{f} \mid A^+_{0,\rho} g\rangle_\rho \qquad \forall f,g \in C_0^\infty(\mathcal{M}) \tag{8.97}$$

is, by Proposition 8.3, Hadamard. Therefore, the generalised SJ state with softened boundaries given by the two-point function (8.96) is Hadamard if and only if the normal ordered two-point function,

$$: W_{SJ_{\varphi,\rho}} : (f,g) = W_{SJ_{\varphi,\rho}}(f,g) - W_{SJ_{0,\rho}}(f,g), \tag{8.98}$$

has a smooth integral kernel. A straightforward calculation shows,

$$\begin{aligned} : W_{SJ_{\varphi,\rho}} : (t,x;t',x') &= A^+_{0,\rho}(t,x;t',x') + \varphi(A_\rho)(t,x;t',x') - A^+_{0,\rho}(t,x;t',x') \\ &= \varphi(A_\rho)(t,x;t',x'), \end{aligned} \tag{8.99}$$

where $A^+_{0,\rho}(\cdot,\cdot)$ is the integral kernel of (8.97) and $\varphi(A_\rho)(\cdot,\cdot)$ is the integral kernel of $\varphi(A_\rho)$. Now, by Proposition 8.5, $\varphi(A_\rho)$ is smoothing. Therefore (8.99) is smooth on $\mathcal{M} \times \mathcal{M}$. Therefore (8.96) is a two-point function for a Hadamard state. ■

8.8 Softened SJ Thermal States on the Ultrastatic Slab

In this section we construct the thermal SJ state with softened boundaries using the construction of the thermal SJ state appearing in Chapter 7 and the construction of a generalised SJ state with softened boundaries in Section 8.6. Let $\mathcal{M} = (-\tau,\tau) \times \Sigma$ be an ultrastatic slab spacetime with metric $g = \mathbb{1} \oplus -h$ where (Σ, h) is a smooth compact Riemannian manifold. As before, let $\rho \in C_0^\infty(\mathbb{R})$ obey $\rho(t) > 0$ for all $t \in (-\tau,\tau)$ and $\rho(t) \to 0$ as $t \to \pm\tau$. Let $L^2(\mathcal{M}, \frac{1}{\rho}\,\mathbf{d}\mathrm{vol}_g)$ be the Hilbert space formed by the completion of $C_0^\infty(\mathcal{M})$ with respect to the norm topology induced by the inner product $\langle \cdot \mid \cdot \rangle_\rho$ given in (8.20). Let A_ρ be the softened commutator function (8.23) given by,

$$A_\rho f = \sum_{j\in\mathbb{N}} \lambda_j \big(\Psi^+_{\rho,j} \langle \Psi^+_{\rho,j} \mid f\rangle_\rho - \Psi^-_{\rho,j} \langle \Psi^-_{\rho,j} \mid f\rangle_\rho \big), \tag{8.100}$$

where,

$$\lambda_j = \frac{\widehat{\rho}(0)e^{-2u_j}}{2\omega_j} = \frac{\widehat{\rho}(0)}{2\omega_j}\sqrt{1-\left|\frac{\widehat{\rho}(2\omega_j)}{\widehat{\rho}(0)}\right|^2}. \tag{8.101}$$

Since $\langle \Psi_j^\sigma \mid \Psi_j^{\sigma'}\rangle_\rho = \delta_{\sigma\sigma'}$ where $\sigma, \sigma' \in \{\pm\}$ for all $j \in \mathbb{N}$, the operator $|A_\rho|$ is,

$$|A_\rho| f = \sum_{j\in\mathbb{N}} \frac{\widehat{\rho}(0)e^{-2u_j}}{2\omega_j}\left(\Psi^+_{\rho,j}\langle \Psi^+_{\rho,j} \mid f\rangle_\rho + \Psi^-_{\rho,j}\langle \Psi^-_{\rho,j} \mid f\rangle_\rho\right). \tag{8.102}$$

The thermal SJ state with softened boundaries is then constructed using the operator,

$$A^+_{\rho,\beta} = \frac{1}{2}(A_\rho + |A_\rho|) + \varphi_\beta(A_\rho), \tag{8.103}$$

where,

$$\begin{aligned} \varphi_\beta : \sigma(A_\rho) &\to \mathbb{R} \\ \lambda &\mapsto |\lambda| \frac{e^{-\beta\tau|\lambda|^{-1}}}{1-e^{-\beta\tau|\lambda|^{-1}}}, \end{aligned} \tag{8.104}$$

The thermal SJ states with softened boundaries on the ultrastatic slab $(-\tau, \tau) \times \Sigma$ is given by,

$$\begin{aligned} W_{SJ_{\rho,\beta}}(f,g) &= \langle \overline{f} \mid A^+_{\rho,0} g\rangle_\rho + \langle \overline{f} \mid \varphi_\beta(A_\rho) g\rangle_\rho \\ &= \sum_{j\in\mathbb{N}} \frac{\widehat{\rho}(0)e^{-2u_j}}{2\omega_j}\langle \overline{f} \mid \Psi^+_{\rho,j}\rangle\langle \Psi^+_{\rho,j} \mid g\rangle + \varphi_\beta\left(\frac{\widehat{\rho}(0)e^{-2u_j}}{2\omega_j}\right)\left(\langle \overline{f} \mid \Psi^-_{\rho,j}\rangle\langle \Psi^-_{\rho,j} \mid g\rangle + \langle \overline{f} \mid \Psi^+_{\rho,j}\rangle\langle \Psi^+_{\rho,j} \mid g\rangle\right) \end{aligned} \tag{8.105}$$

We now obtain the following,

Proposition 8.7. *The thermal SJ state with softened boundaries given by the two-point function* (8.105) *is Hadamard.*

Proof. In similar manner to the unsoftened thermal state, we observe that (8.104) may be written as,

$$\varphi_\beta(\lambda) = \beta\tau f\left(\frac{|\lambda|}{\beta\tau}\right), \tag{8.106}$$

where,

$$f(x) = \frac{xe^{-1/x}}{1-e^{-1/x}}. \tag{8.107}$$

Since, for all $N \in \mathbb{N}_0$ there exists a $C_N \geq 0$ such that $|f(x)| \leq C_N|x|^N$, by Proposition 8.5, the operator $\varphi_\beta(A_\rho)$ is smoothing. Furthermore, by Proposition 8.3, the SJ vacuum with softened boundaries is Hadamard. Therefore, by Proposition 8.96 the thermal SJ with two-point function (8.105) is Hadamard. ∎

8.9 Wick Square for the Softened Thermal SJ State

In this section we calculate the Wick square of the $n-th$ derivative of field normal ordered with respect to the SJ vacuum with softened boundaries and evaluated in the softened thermal SJ state. The normal ordered two-point function is given by,

$$:W_{SJ_{\rho,\beta}}:(f,g) = W_{SJ_{\rho,\beta}}(f,g) - W_{SJ_{\rho,0}}(f,g) = \langle \overline{f} \mid \varphi_\beta(A_\rho) g \rangle, \tag{8.108}$$

for all $f, g \in C_0^\infty(\mathcal{M})$. The expectation value of the Wick square for the $n-th$ derivative of the field is then (relative to the volume measure $\mathbf{d}\mathrm{vol}_g$),

$$\langle : (\partial_t^n \phi)^2 : (f) \rangle = \int_{\mathcal{M}} \langle : (\partial_t^n \phi)^2 : \rangle (t,x) f(t,x) \, \mathbf{d}\mathrm{vol}_g, \tag{8.109}$$

where,

$$\begin{aligned}\langle : (\partial_t^n \phi)^2 : \rangle_\beta (t,x) &= \lim_{(t,x)\to(t',x')} \partial_t^n \partial_{t'}^n : W_{SJ_{\rho,\beta}} : (t,x;t',x') \\ &= \sum_{j\in\mathbb{N}} \frac{e^{-\beta\tau|\lambda_j|^{-1}}}{1-e^{-\beta\tau|\lambda_j|^{-1}}} \omega_j^{2n-1} e^{2u_j} \left[1 - (-1)^n \left| \frac{\widehat{\rho}(2\omega_j)}{\widehat{\rho}(0)} \right| \cos(2\omega_j t) \right] |\xi_j(x)|^2 \end{aligned} \tag{8.110}$$

Using the plateau functions constructed in Section 8.5 we now calculate the Wick square (8.110) when the spatial manifold (Σ, h) is a round three sphere. In the case when (Σ, h) is a there sphere the eigenvalues ω_j are given in (5.84) and occur with multiplicity $(1+j)^2$. Using the fact that the sphere is spherically symmetric the Wick square (8.110) evaluated at $(t,x) \in (-\tau,\tau) \times S^3$ is equal to the Wick square averaged over the three sphere evaluated at time t. The first N terms of the Wick square (8.110) evaluated at a point $(t,x_0) \in (-\tau,\tau) \times S^3$ is then,

$$\begin{aligned}\langle : (\partial_t^n \phi)^2 : \rangle_\beta (t,x_0) = \frac{1}{\mathrm{Vol}(S^3)} \sum_{j=1}^{N} (j+1)^2 \frac{e^{-\beta\tau|\lambda_j|^{-1}}}{1-e^{-\beta\tau|\lambda_j|^{-1}}} \omega_j^{2n-1} e^{2u_j} \\ \left[1 - (-1)^n \left| \mathrm{sinc}(2\omega_j \tau_p) \widehat{H}(2\omega_j \tau_p \alpha) \right| \cos(2\omega_j t) \right]\end{aligned} \tag{8.111}$$

In the following plots we measure the temperature in units of R^{-1}, the plateau time τ_p and ultrastatic time t in units of R and set $\alpha = \frac{\tau_s}{\tau_p}$, as before.

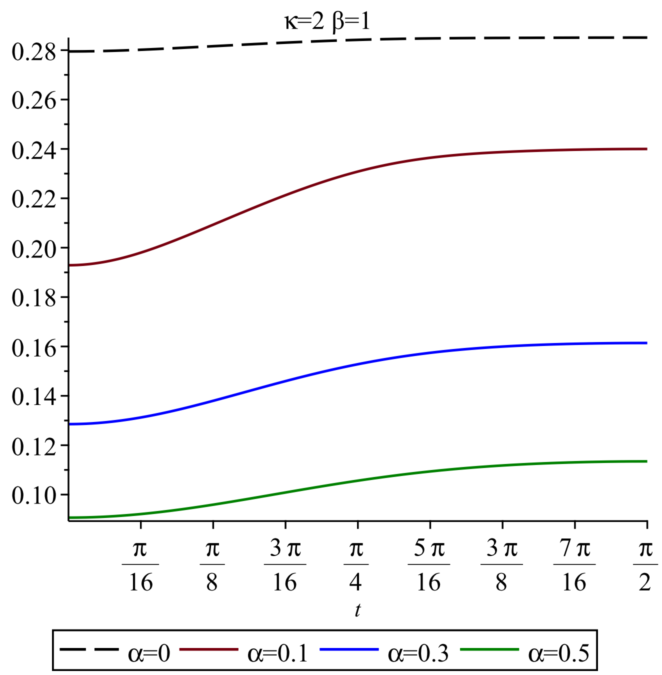


Figure 8.8: Plot of the Wick Square of the field normal ordered with respect to the softened SJ vacuum and evaluated in the softened thermal SJ state for parameters $mR = 1$, $\beta = R$, $\kappa = \dfrac{\tau_p}{R} = 2$ and $\alpha = 0, 0.1, 0.3, 0.5$.

We draw the following conclusion: Following the work of [54, 13] whenever the Wick square is positive, a local temperature can be defined by,

$$T_\omega(t,x) = \sqrt{12(:\phi^2:(t,x))}, \tag{8.112}$$

where the Wick square appearing in (8.112) is normal ordered with respect to a Hadamard state and evaluated in the state ω. We clearly see in Figure 8.8 that the Wick square is positive for all $t \in (0, \frac{\pi}{2})$. This indicates that the softened thermal SJ state (8.105) can be used to define a local temperature, although these results are only preliminary. Secondly, we notice that in the limit $\alpha \to 0$, the Wick square of the softened thermal SJ state does not diverge, which can be traced back to the smoothing properties of φ_β.

Chapter 9

Physical States for the Massive Spin One Field

A natural question to ask is whether the SJ state construction can be applied to higher spin fields. Previous constructions for the SJ and BF states have only been applied to the free spin zero scalar field. In this chapter we consider the SJ and BF state constructions for a massive spin one field, namely the Proca field. The Proca field on a globally hyperbolic spacetime is most elegantly described using the language of differential forms. However, since the metric for a spacetime is indefinite, the natural inner product on the space of differential forms is also indefinite. Therefore, the space that the commutator function acts on is not a Hilbert space but a complete indefinite inner product space, also known as a *Krein space*. The analysis of operators on Krein spaces is drastically different than the analysis of operators on Hilbert spaces. Self-adjoint bounded operators over Hilbert spaces enjoy properties that are essential to the SJ construction; real eigenvalues and the spectral theorem which guarantees the existence and uniqueness of the positive part. The story for operators over a Krein space, however, is very different. The problems when faced considering the SJ and BF state construction are:

- The commutator function for the Proca field extends to a unbounded non-definitisable operator on a Krein space.
- The spectral theory for self-adjoint operators on Krein spaces is only developed for definitisable operators and is significantly harder than in Hilbert spaces.

- The spectrum of the commutator function for the Proca field on an ultrastatic slab with compact spatial section is complex.

We consider the construction of both the SJ and BF states on the ultrastatic slab $(-\tau, \tau) \times \Sigma$. We show that the SJ state construction is ill-defined when applied to the Proca field on an ultrastatic slab with compact spatial section since the commutator has a complex spectrum. However, the BF state construction is well defined but there are still complications that are not present in the scalar field case. We show that, under a suitable choice of softening function, the BF state is well defined. We also show that whenever the BF state is well defined, it also satisfies the Hadamard condition.

9.1 Krein Spaces

To begin with, we recall some standard notions about Krein spaces. For a complete exposition on the subject the reader is referred to [6]. Let $(\mathcal{K}, \langle . \mid . \rangle)$ be a vector space equipped with an indefinite sesquilinear form $\langle \cdot \mid \cdot \rangle$. Then, $(\mathcal{K}, \langle \cdot \mid \cdot \rangle)$ is *Krein space* if there exists a *fundamental decomposition*,

$$\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-, \tag{9.1}$$

which is the direct (orthogonal with respect to $\langle \cdot \mid \cdot \rangle$) sum of two Hilbert spaces $(\mathcal{K}^+, \langle . \mid . \rangle)$ and $(\mathcal{K}^-, -\langle \cdot \mid \cdot \rangle)$. Let $P^{\pm}$ denote projection operators onto $\mathcal{K}^{\pm}$ corresponding to the fundamental decomposition (9.1) and introduce the self-adjoint operator $J = P^+ - P^-$ called the *fundamental symmetry*. Since the decomposition (9.1) is orthogonal, we have $J^2 = \mathbb{1}$. The fundamental symmetry induces a positive definite inner product by setting $\langle \cdot \mid J \cdot \rangle$, and the inner product space $(\mathcal{K}, \langle \cdot \mid J \cdot \rangle)$ then forms a Hilbert space. All topological notions in a Krein space arise from the norm topology induced by the inner-product $\langle \cdot \mid J \cdot \rangle$, and, although there are many decompositions of the form (9.1), all norms induced by the inner-product $\langle \cdot \mid J \cdot \rangle$ are equivalent [19, Corollary 2]. Therefore the norm topology of a Krein space does not depend on the choice of fundamental symmetry. We introduce the space of continuous operators over $\mathcal{K}$,

$$\mathfrak{L}(\mathcal{K}) := \{U : \mathcal{K} \to \mathcal{K} \mid U \text{ is continuous and linear with respect to } \langle \cdot \mid J \cdot \rangle\}. \tag{9.2}$$

We call a vector $f \in \mathcal{K}$ *timelike* if $\langle f \mid f \rangle > 0$, *null* if $\langle f \mid f \rangle = 0$ and *spacelike* if $\langle f \mid f \rangle < 0$. A subspace $S \subset \mathcal{K}$ is timelike/null/spacelike if all elements $f \in S$ are timelike/null/spacelike. We call an operator $B \in \mathfrak{L}(\mathcal{K})$ *positive* if $\langle f \mid Bg \rangle = \langle Bf \mid g \rangle$ for all $f, g \in \mathcal{K}$ and,

$$\langle f \mid Bf \rangle \geq 0, \tag{9.3}$$

for all $f \in \mathcal{K}$. A set of elements $\{\xi_j\}_{j \in \mathbb{N}}$ in a Krein space is called *pseudo-orthonormal* if $\langle \xi_j \mid \xi_k \rangle = \pm\delta_{jk}$. The spectrum and resolvent set for an operator B are the same definitions in Krein spaces as they are in Hilbert spaces. An operator $B \in \mathfrak{L}(\mathcal{K})$ is called *definitisable* if the resolvent set of A is non-empty, $\rho(A) \neq \emptyset$ and there exists a real polynomial p such that $p(A) \geq 0$. The operator $p(A)$ is understood to be the operator,

$$p(A) = \sum_{j=1}^{N} a_j A^N, \tag{9.4}$$

corresponding to the polynomial $p(\lambda) = \sum_{j=1}^{N} a_j \lambda^N$.

A Rank Two Krein Space Operator Toy Model

Let $\mathcal{K}$ be a Krein space and suppose $\langle w \mid w \rangle = \langle v \mid v \rangle \neq 0$ and that w and v are linearly independent. We allow for both possibilities that either $\langle w \mid w \rangle < 0$ or $\langle w \mid w \rangle > 0$. Consider the rank two operator,

$$A = |w\rangle \langle w| - |v\rangle \langle v| . \tag{9.5}$$

The eigenvalues are easily found to be,

$$\lambda^{\pm} = \pm\sqrt{\langle w \mid w \rangle^2 - |\langle w \mid v \rangle|^2}. \tag{9.6}$$

One may not infer that the eigenvalues $\lambda^{\pm}$ are real since the Cauchy-Schwarz inequality does not hold in a Krein space. Using a similar analysis to the analysis done in section 8.1 Chapter 8 a pair pseudo-orthonormal eigenvectors of the operator (9.5) is given by,

$$\begin{aligned} \phi_1 &= \frac{e^{3u_j/2}}{|\langle w \mid w \rangle|^{1/2}} \left(\sqrt{\cosh(u_j)} w - \sqrt{\sinh(u_j)} v \right) \\ \phi_2 &= \frac{e^{3u_j/2}}{|\langle w \mid w \rangle|^{1/2}} \left(\sqrt{\sinh(u_j)} w - \sqrt{\cosh(u_j)} v \right). \end{aligned} \tag{9.7}$$

The eigenvectors $\Psi^{\pm}$ corresponding to the eigenvalues $\lambda^{\pm}$ are given by,

$$\left. \begin{aligned} \Psi^+ &= \phi_1 \\ \Psi^- &= \phi_2 \end{aligned} \right\} \qquad \text{if } \langle w \mid w \rangle > 0 \tag{9.8}$$

$$\left.\begin{aligned} \Psi^+ &= \phi_2 \\ \Psi^- &= \phi_1 \end{aligned}\right\} \qquad \text{if } \langle w \mid w \rangle < 0. \tag{9.9}$$

In either of these cases the eigenvectors $\Psi^{\pm}$ obey,

$$\begin{aligned} \langle \Psi^{\pm} \mid \Psi^{\pm} \rangle &= \operatorname{sgn}(\langle w \mid w \rangle) \\ \langle \Psi^{\pm} \mid \Psi^{\mp} \rangle &= 0, \end{aligned} \tag{9.10}$$

where sgn is the sign function. The operator (9.5) in terms of the eigenvectors (9.8) is then,

$$A = \frac{\lambda^+}{\langle \Psi^+ \mid \Psi^+ \rangle} \left| \Psi^+ \right\rangle \left\langle \Psi^+ \right| + \frac{\lambda^-}{\langle \Psi^- \mid \Psi^- \rangle} \left| \Psi^- \right\rangle \left\langle \Psi^- \right|. \tag{9.11}$$

Suppose further that $v = \Gamma w$ where Γ is a anti-unitary involution. Then one has $A = -\Gamma A \Gamma$, therefore $\Gamma \Psi^+$ is an eigenvector of A with the negative eigenvalue $\lambda^- = -\lambda^+$ and is some scalar multiple of Ψ^-. In this case we have the decomposition,

$$A = \lambda^+ P + \lambda^- \Gamma P \Gamma, \tag{9.12}$$

where P is a rank one projection onto the eigenspace with positive eigenvalue given by,

$$P = \frac{\left| \Psi^+ \right\rangle \left\langle \Psi^+ \right|}{\langle \Psi^+ \mid \Psi^+ \rangle}. \tag{9.13}$$

If we assume that $|\langle w \mid v \rangle| \leq |\langle w \mid w \rangle|$ then the eigenvalues (9.6) are real and one can construct the positive part of the operator (9.12). If $\langle w \mid w \rangle > 0$ then one recovers the same formulae as in the Hilbert space case and the positive part (9.17) is then,

$$A^+ = \lambda^+ P. \tag{9.14}$$

In the case when $\langle w \mid w \rangle < 0$, the positive part of the operator (9.5) is then given by,

$$A^+ = \lambda^- \Gamma P \Gamma, \tag{9.15}$$

which we can see is positive by the following,

$$\langle f \mid A^+ f \rangle = \lambda^- \frac{|\langle \Gamma \Psi^+ \mid f \rangle|^2}{\langle \Psi^+ \mid \Psi^+ \rangle} \geq 0, \tag{9.16}$$

which holds since $\lambda^- \leq 0$ *and* $\langle \Psi^+ \mid \Psi^+ \rangle = \operatorname{sgn}(\langle w \mid w \rangle) = -1$. Hence in general the positive part is,

$$A^+ = \theta(\langle w \mid w \rangle) \lambda^+ P + \theta(-\langle w \mid w \rangle) \lambda^- \Gamma P \Gamma, \tag{9.17}$$

where θ is the Heaviside function. Therefore, in this simple toy model of a rank two operator on a Krein space, we can see that extracting the positive part is not a straightforward generalisation of the Hilbert space case. Moreover, if $|\langle w \mid v\rangle| > |\langle w \mid w\rangle|$ then the eigenvalues (9.6) are pure imaginary. In this case it is not clear how one would construct the positive part of the operator (9.5).

The Krein Space of Differential Forms on Ultrastatic Slab

Let $(\mathcal{M} = (-\tau, \tau) \times \Sigma, g = \mathbb{1} \oplus -h)$ be an ultrastatic slab spacetime where (Σ, h) is a compact smooth three dimensional Riemannian manifold. Let $\Omega^1(\mathcal{M})$ be the space of square integrable one forms over $\mathcal{M}$ and let $\Omega^1_0(\mathcal{M}) \subset \Omega^1(\mathcal{M})$ be the subspace of compactly supported one forms. The exterior derivative will be denoted $\mathbf{d}$, the coderivative on the manifold $(\mathcal{M}, g)$ will be denoted $\delta_{\mathcal{M}} = \star_g \mathbf{d} \star_g$ where $\star_g$ is the Hodge star relative to the metric g. Using the auxiliary Euclidean metric $\tilde{g} = \mathbb{1} \oplus h$, one may equip the space $\Omega^1_0(\mathcal{M})$ with a *positive definite* inner product $(\cdot \mid \cdot)$ defined by,

$$(\psi \mid \phi)_{\tilde{g}} = \int_{\mathcal{M}} \overline{\psi} \wedge *_{\tilde{g}} \phi = \int_{\mathcal{M}} \overline{\psi^\mu(t,x)} \phi^\nu(t,x) \tilde{g}_{\mu\nu} \mathbf{d}\mathrm{vol}_{\tilde{g}}, \tag{9.18}$$

for all $\psi, \phi \in \Omega^1_0(\mathcal{M})$ and where $\mu, \nu = 0, 1, 2, 3$. Completing the space $\Omega^1_0(\mathcal{M})$ with respect to the norm topology induced by the inner product (9.18) yields the Hilbert space of square integrable one-forms over $\mathcal{M}$ denoted by $\Lambda^1(\mathcal{M}, \tilde{g})$. Making a 1+3 decomposition, we may make the following identification,

$$\Lambda^1(\mathcal{M}, \tilde{g}) = L^2(-\tau, \tau) \otimes (\Lambda^0(\Sigma) \oplus \Lambda^1(\Sigma)), \tag{9.19}$$

where $\Lambda^p(\Sigma)$ is the Hilbert space of square integrable $p-$forms over (Σ, h) for each $p \in \mathbb{N}_0$ and the tensor product $\otimes$ is the completed Hilbert space tensor product [53, Section II.4]. The Hilbert spaces $\Lambda^p(\Sigma)$ are constructed by completing $\Omega^p_0(\Sigma)$ with respect to the inner product,

$$(\sigma \mid \xi)_h = \int_\Sigma \overline{\sigma} \wedge *_h \xi = \int_\Sigma \overline{\sigma(x)}^r \xi(x)^s h_{rs} \ \mathbf{d}\mathrm{vol}_h. \tag{9.20}$$

The Krein space of one forms on the ultrastatic slab $(\mathcal{M} = (-\tau, \tau) \times \Sigma, g = \mathbb{1} \oplus -h)$ is then defined to be the Hilbert space $\Lambda^1(\mathcal{M}, \tilde{g})$ equipped with the fundamental symmetry,

$$J = \mathbb{1} \otimes (\mathbb{1} \oplus -\mathbb{1}), \tag{9.21}$$

where we have used the decomposition (9.19) and the identities are understood to be on their respective spaces. We then denote $\mathcal{K} = (\Lambda^1(\mathcal{M}, \tilde{g}), \langle \cdot \mid \cdot \rangle, J)$, where the indefinite inner product is given by,

$$\langle \psi \mid \phi \rangle = \int_{\mathcal{M}} \overline{\psi} \wedge *_g \phi = \int_{\mathcal{M}} \overline{\psi}^\mu \phi^\nu g_{\mu\nu} \ \mathbf{d}\mathrm{vol}_g, \tag{9.22}$$

for all $\psi, \phi \in \mathcal{K}$ where $\mu, \nu = 0, 1, 2, 3$. Finally, we introduce one further Krein space; the Krein space of static one-forms which is defined to be the space of one forms which are independent of the ultrastatic time parameter t. Using the decomposition (9.19) the space of static one forms is given by $\mathcal{K}_{static} = \Lambda^0(\Sigma) \oplus \Lambda^1(\Sigma)$. We equip the space of static one forms with the indefinite inner product,

$$\langle\!\langle \xi \mid \eta \rangle\!\rangle = \int_\Sigma \overline{\xi_\mu(x)} \eta^\mu(x) \, \mathbf{d}\mathrm{vol}_h = (\xi_0 \mid \eta_0)_{\Lambda^0(\Sigma)} - (\xi_\Sigma \mid \eta_\Sigma)_{\Lambda^0(\Sigma)} \tag{9.23}$$

for all $\xi, \eta \in \Lambda^0(\Sigma) \oplus \Lambda^1(\Sigma)$ and where we have made the identification $\xi = \xi_0 \mathbf{d}t + \xi_\Sigma$ for all $\xi \in \Lambda^0(\Sigma) \oplus \Lambda^1(\Sigma)$.

9.2 Quantisation of the Proca Field

We review the classical theory of the massive spin one field over globally hyperbolic spacetimes, which is based on the work appearing in [24].

Classical Theory

Let $(\mathcal{M} = \mathbb{R} \times \Sigma, g)$ be a globally hyperbolic spacetime where (Σ, h) is a compact smooth Riemannian manifold. The classical uncharged spin-one field is a one form $\mathcal{A} \in \Omega^1(\mathcal{M})$ that obeys the Proca equation,

$$(-\boldsymbol{\delta}_{\mathcal{M}} \mathbf{d} + M^2)\mathcal{A} = 0, \tag{9.24}$$

where $M > 0$ is the mass of the field. Applying the coderivative to (9.24) we see that any solution $\mathcal{A}$ to the Proca equation is necessarily coclosed,

$$\boldsymbol{\delta}_{\mathcal{M}} \mathcal{A} = 0 \tag{9.25}$$

Hence, any solution to (9.24) is also a solution of the massive Klein Gordon equation,

$$(\Box + M^2)\mathcal{A} = 0, \tag{9.26}$$

where $\Box = -(\mathbf{d}\boldsymbol{\delta}_{\mathcal{M}} + \boldsymbol{\delta}_{\mathcal{M}}\mathbf{d})$. The advantage of working with the system (9.25, 9.26) is that the operator $\Box + M^2$ is normally hyperbolic [3, Example 1.5.3], therefore there exist unique advanced(-)/retarded(+) Greens operators denoted by $\mathbb{E}^\pm$. One uses Greens operator $\mathbb{E}^\pm$ for the massive Klein Gordon equation to solve the inhomogeneous Proca equation,

$$(-\boldsymbol{\delta}_{\mathcal{M}} \mathbf{d} + M^2)\mathcal{A} = \mathcal{J}, \tag{9.27}$$

with advanced and retarded boundary conditions and where $\mathcal{J} \in \Omega^1_0(\mathcal{M})$. Assuming there exists a solution to (9.27), we apply the coderivative to both sides obtaining $\boldsymbol{\delta}_{\mathcal{M}}\mathcal{A} = M^{-2}\boldsymbol{\delta}_{\mathcal{M}}\mathcal{J}$. We now rewrite (9.27) as,

$$(\Box + M^2)\mathcal{A} = (-\boldsymbol{\delta}_{\mathcal{M}}\mathbf{d} + M^2)\mathcal{A} - \mathbf{d}\boldsymbol{\delta}_{\mathcal{M}}\mathcal{A} = (\mathbb{1} - M^{-2}\mathbf{d}\boldsymbol{\delta}_{\mathcal{M}})\mathcal{J}, \tag{9.28}$$

whereby $\mathcal{A}^{\pm} = \mathbb{E}^{\pm}(\mathbb{1} - M^{-2}\mathbf{d}\boldsymbol{\delta}_{\mathcal{M}})\mathcal{J}$ are unique solutions supported in $J^{\pm}(\mathrm{supp}(\mathcal{J}))$. By [51, Proposition II.1] the operators $\boldsymbol{\delta}_{\mathcal{M}}$ and $\mathbb{E}^{\pm}$ commute, and the one forms $\mathcal{A}^{\pm} = \mathbb{E}^{\pm}(\mathbb{1} - M^{-2}\mathbf{d}\boldsymbol{\delta}_{\mathcal{M}})\mathcal{J}$ are seen to be the solutions to (9.27). We now define,

$$\Delta^{\pm}_M = \mathbb{E}^{\pm}(\mathbb{1} - M^{-2}\mathbf{d}\boldsymbol{\delta}_{\mathcal{M}}), \tag{9.29}$$

which are Green operators for (9.27), along with,

$$\Delta_M = \Delta^-_M - \Delta^+_M = \mathbb{E}(\mathbb{1} - M^{-2}\mathbf{d}\boldsymbol{\delta}_{\mathcal{M}}), \tag{9.30}$$

which will appear in the CCR algebra for the Proca field, as we shall now see.

Algebraic Quantisation of the Proca Field

The algebraic quantisation of the Proca field in globally hyperbolic spacetimes was accomplished by Dimock [17] and Furlani [34]. Here, we use the algebra isomorphic to those constructions given in [24]. The algebra of observables for the Proca field starts by defining a set of abstract objects labelled by compactly supported one forms,

$$\mathcal{A} = \{\mathcal{A}(f) \mid f \in \Omega^1_0(\mathcal{M})\}, \tag{9.31}$$

whereby each object is interpreted as a smeared one form field: $\mathcal{A}(f) = \langle \overline{\mathcal{A}} \mid f \rangle$. This generates a free unital $*-$algebra over $\mathbb{C}$. The CCR algebra for the free massive spin one field over $(\mathcal{M}, g)$, denoted $\mathfrak{A}_M(\mathcal{M}, g)$, is formed by taking the quotient of $\mathcal{A}$ by the following relations,

P1. Linearity - $\mathcal{A}(\alpha f + \beta g) = \alpha\mathcal{A}(f) + \beta\mathcal{A}(g)$ for all $\alpha, \beta \in \mathbb{C}$ and all $f, g \in \Omega^1_0(\mathcal{M})$;

P2. Hermiticity - $\mathcal{A}(f)^* = \mathcal{A}(\overline{f})$ for all $f \in \Omega^1_0(\mathcal{M})$;

P3. Field equations - $\mathcal{A}((-\boldsymbol{\delta}_{\mathcal{M}}\mathbf{d} + M^2)f) = 0$ for all $f \in \Omega^1_0(\mathcal{M})$;

P4. CCRs - $[\mathcal{A}(f), \mathcal{A}(g)] = \langle \overline{f} \mid -i\Delta_M g\rangle\mathbb{1}$ for all $f, g \in \Omega^1_0(\mathcal{M})$ and where $\mathbb{1}$ is the unit and Δ_M is given in (9.30).

9.3 The Commutator Function for the Proca Field on Ultrastatic Slabs

The following section is based on the work undertaken in [24] and we give a detailed review for the benefit of the reader. Let $\mathcal{M} = (-\tau, \tau) \times \Sigma$ be an ultrastatic slab spacetime with metric $g = \mathbb{1} \oplus -h$ where $\tau > 0$ and (Σ, h) is a boundaryless, compact Riemannian manifold. We assume that the homology group $H_1(\Sigma)$ is trivial, which implies that the space of harmonic one forms over Σ is trivial.[a] This is equivalent to assuming that the compact support de Rahm cohomology group $H^3_c(\mathcal{M})$ is trivial.[b] Our construction of the SJ state begins by constructing the advanced-minus-retarded operator $\mathbb{E}$ for the Klein Gordon operator $\Box + M^2 : \Omega^1(\mathcal{M}) \to \Omega^1(\mathcal{M})$. Using the operator $\mathbb{E}$ we can construct the operator,

$$A \doteq -i\Delta_M = -i\mathbb{E}(\mathbb{1} - M^{-2}\mathbf{d}\boldsymbol{\delta}_{\mathcal{M}}) \tag{9.32}$$

which appears in the CCRs for the Proca field. In ultrastatic spacetimes the wave equation $\Box + M^2$ reduces to $\partial_t^2 + K$ where K is the elliptic operator,

$$\begin{aligned} K : \Lambda^0(\Sigma) \oplus \Lambda^1(\Sigma) &\to \Lambda^0(\Sigma) \oplus \Lambda^1(\Sigma) \\ \xi_0 \mathbf{d}t + \xi_\Sigma &\mapsto ((-\Delta^s_\Sigma + M^2)\xi_0)\mathbf{d}t + (-\Delta_\Sigma + M^2)\xi_\Sigma, \end{aligned} \tag{9.33}$$

and where Δ^s_Σ and Δ_Σ denote the scalar and one-form Laplace-Beltrami operators on (Σ, h) respectively. The ansatz $\mathcal{A}(t, x) = e^{-i\omega t}\xi(x)$ is therefore a solution to the Klein Gordon operator $P = \partial_t^2 + K$ if and only if,

$$K\xi = \omega^2 \xi, \tag{9.34}$$

where $\xi \in \Lambda^0(\Sigma) \oplus \Lambda^1(\Sigma)$ is a static one form (i.e. independent of the ultrastatic time parameter t). Since the operator K is positive, the eigenvalues can be expressed as the squares of nonnegative quantities. Furthermore, since the Krein space of static one forms decomposes into the direct sum $\Lambda^0(\Sigma) \oplus \Lambda^1(\Sigma)$, the eigenvalue problem (9.34) splits into two eigenvalue problems,

$$\begin{aligned} (-\Delta^s_\Sigma + M^2)\xi_0 &= \omega^2 \xi_0 \qquad \xi_0 \in \Lambda^0(\Sigma) \\ (-\Delta_\Sigma + M^2)\xi_\Sigma &= \omega^2 \xi_\Sigma \qquad \xi_\Sigma \in \Lambda^1(\Sigma). \end{aligned} \tag{9.35}$$

[a] The a one-form $\omega \in \Omega^1(\Sigma)$ is called harmonic if $(\mathbf{d}\boldsymbol{\delta}_\Sigma + \boldsymbol{\delta}_\Sigma\mathbf{d})\omega = 0$.

[b] See [71] in [24] for further details.

To proceed, we choose φ_j to label a complete basis of $(-\Delta^s_\Sigma + M^2)-$eigenfunctions for $\Lambda^0(\Sigma)$ with the corresponding eigenvalues $\omega^2(S,j)$, and with the countable labelling set $j \in J(S)$, where S stands for scalar. By elliptic regularity, each φ_j is smooth [20, Theorem 3 Section 6.3]. The scalar K-eigenfunctions $\xi(S,j)$ are then given by $\xi(S,j) = \varphi_j \mathbf{d}t$, which are clearly pseudo-orthonormal and timelike in $\mathcal{K}_{\text{static}}$. The corresponding one form 'scalar modes' with positive frequency, denoted by $\mathcal{A}(S,j)$, are, for every $j \in J(S)$,

$$\mathcal{A}(S,j) = \Phi_j \mathbf{d}t, \tag{9.36}$$

where $\Phi_j(t,x) = e^{-i\omega(S,j)t}\varphi_j(x)$. The scalar modes are solutions to the Klein Gordon equation (9.26). The remaining eigenvectors ξ_Σ form a complete basis for the spacelike subspace $\Lambda^1(\Sigma)$. To study the second eigenvalue problem in (9.35) we shall make use of the Hodge decomposition theorem. Since $H_1(\Sigma)$ is trivial, this gives decomposition $\Lambda^1(\Sigma) = \overline{\mathbf{d}_\Sigma\Omega^0(\Sigma)} \oplus \overline{\boldsymbol{\delta}_\Sigma\Omega^2(\Sigma)}$ [46, Corollary 5.6]. The decomposition is orthogonal with respect to the inner product $(.,.)_{\Lambda^1(\Sigma)}$ (and hence with respect to the indefinite inner product $\langle\!\langle . \mid . \rangle\!\rangle$) and the bar denotes closure in the norm topology of $\Lambda^1(\Sigma)$. We observe that the 'longitudinal' subspace $\overline{\mathbf{d}_\Sigma\Omega^0(\Sigma)}$ is spanned by a set of non-zero vectors of the form $\mathbf{d}_\Sigma\xi(S,j)$. This is the case since the eigenvectors $\xi(S,j)$ span the subspace $\Omega^0(\Sigma)$ and the longitudinal subspace is the image of $\Omega^0(\Sigma)$ under the exterior derivative. Owing to the relation $\Delta_\Sigma \mathbf{d}_\Sigma = \mathbf{d}_\Sigma \Delta^s_\Sigma$ the vectors $\mathbf{d}_\Sigma\xi(S,j)$ are eigenvectors of the operator $-\Delta_\Sigma + M^2$ with eigenvalues $\omega(L,j) = \omega(S,j)$. Since the only vanishing vector of this form is the spatially constant mode we choose the labelling set $J(L) = \{j \in J(S) \mid \omega(S,j) > M\}$. Furthermore, by the calculation,

$$\langle\!\langle \mathbf{d}_\Sigma\varphi_j \mid \mathbf{d}_\Sigma\varphi_k \rangle\!\rangle = -(\mathbf{d}_\Sigma\varphi_j, \mathbf{d}_\Sigma\varphi_k)_{\Lambda^1(\Sigma)} = -(\varphi_j, \boldsymbol{\delta}_\Sigma\mathbf{d}_\Sigma\varphi_k)_{\Lambda^1(\Sigma)} = -(\omega^2(S,j) - M^2)\delta_{jk}, \tag{9.37}$$

we see that the correctly normalised eigenvectors are given by,

$$\xi(L,j) = (\omega^2(S,j) - M^2)^{-1/2}\mathbf{d}_\Sigma\varphi_j, \tag{9.38}$$

which are spacelike and pseudo-orthonormal in $\mathcal{K}_{\text{static}}$. The corresponding modes $\mathcal{A}(L,j)$ may be expressed as,

$$\mathcal{A}(L,j) = \frac{\mathbf{d}_\Sigma\Phi_j + i\omega(L,j)\mathcal{A}(S,j)}{\sqrt{\omega(L,j)^2 - M^2}}. \tag{9.39}$$

The 'transverse' modes $\xi(T,j)$ lie in the subspace $\overline{\boldsymbol{\delta}_\Sigma\Omega^2(\Sigma)}$ with labelling set $j \in J(T)$. The corresponding eigenvalues $\omega(T,j)$ are typically distinct from the eigenvalues $\omega(S,j)$ [24, Section VIC].

Since the transverse modes are coexact on Σ, they are necessarily coclosed on Σ. The transverse modes $\xi(T,j)$ form a pseudo-orthonormal basis for the subspace $\overline{\boldsymbol{\delta}_\Sigma \Omega^2(\Sigma)}$ with the countable labelling set $j \in J(T)$. The corresponding transverse Proca modes are given by,

$$\mathcal{A}(T,j) = e^{-i\omega(T,j)t}\xi(T,j), \tag{9.40}$$

for all $j \in J(T)$. Since the operator K commutes with complex conjugation we infer that for any eigenvector $\xi(m,j)$, we also have $\overline{\xi(m,j)}$ as an eigenvector of K, where $m \in \{S,L,T\}$. Accordingly, we can choose a basis for the labelling set $J(m)$ that respects this property, namely that for every $j \in J(m)$ there exists a $\overline{j} \in J(m)$ such that $\overline{\xi(m,j)} = \xi(m,\overline{j})$ for each $m \in \{S,L,T\}$. For the scalar, longitudinal and transverse modes, we have the following inner products,

$$\left.\begin{aligned} \langle \mathcal{A}(m,j) \mid \mathcal{A}(m,k)\rangle &= 2\tau\sigma(m)\delta_{jk} \\ \langle \mathcal{A}(m,j) \mid \overline{\mathcal{A}(m,\overline{k})}\rangle &= 2\tau\sigma(m)\mathrm{sinc}(2\omega(m,j)\tau)\delta_{jk}, \end{aligned}\right\} \qquad \forall m \in \{S,L,T\} \qquad \forall j,k \in J(m) \tag{9.41}$$

where,

$$\sigma(m) := \langle\!\langle \xi(m,j) \mid \xi(m,j)\rangle\!\rangle = \begin{cases} +1 & m = S \\ -1 & m \in \{L,T\}. \end{cases} \tag{9.42}$$

By [24, Theorem IV.1] the modes $\mathcal{A}(m,j)$ ($m \in \{S,L,T\}, j \in J(m)$) are used to construct a Hadamard $(\Box + M^2)-$bisolution W_M on $(\mathcal{M},g)$,

$$W_M(f,g) = -\sum_{m\in\{S,L,T\}}\sum_{j\in J(m)} \frac{1}{2\omega(m,j)}\sigma(m)\langle f \mid \mathcal{A}(m,j)\rangle\langle \mathcal{A}(m,j) \mid g\rangle \qquad f,g \in \Omega^1_0(\mathcal{M}), \tag{9.43}$$

which is invariant under a change of basis in $\mathcal{K}_{\text{static}}$. Furthermore, the antisymmetric part of W_M is given by,

$$W_M(f,g) - W_M(g,f) = -i\mathbb{E}(f,g) \qquad f,g \in \Omega^1_0(\mathcal{M}), \tag{9.44}$$

where $-i\mathbb{E}(f,g) = \langle \overline{f} \mid -i\mathbb{E}g\rangle$. Taking the anti-symmetric part of the bidistribution (9.43) and relabelling $j \to \overline{j}$ in the second term, we obtain,

$$-i\mathbb{E}(f,g) = \sum_{m\in\{S,L,T\}}\sum_{j\in J(m)} \frac{-\sigma(m)}{2\omega(m,j)}\Big[\langle \overline{f} \mid \mathcal{A}(m,j)\rangle\mathcal{A}(m,j) \mid g\rangle - \langle \overline{f} \mid \overline{\mathcal{A}(m,\overline{j})}\rangle\overline{\mathcal{A}(m,\overline{j})} \mid g\rangle\Big] \tag{9.45}$$

where $f, g \in \Omega^1_0(\mathcal{M})$. Since $\Omega^1_0(\mathcal{M})$ is dense in $\Omega^1(\mathcal{M})$ we can uniquely extract the operator corresponding to the bidistribution (9.45),

$$-i\mathbb{E} = \sum_{m\in\{S,L,T\}} \sum_{j\in J(m)} \frac{-\sigma(m)}{2\omega(m,j)} \Big[|\mathcal{A}(m,j)\rangle\langle\mathcal{A}(m,j)| - |\overline{\mathcal{A}(m,\overline{j})}\rangle\, \langle\overline{\mathcal{A}(m,\overline{j})}| \Big]. \tag{9.46}$$

Before constructing the operator $A = -i\mathbb{E}(\mathbb{1} - M^{-2}\mathbf{d}\boldsymbol{\delta}_{\mathcal{M}})$ we observe that the scalar and longitudinal modes are not coclosed,

$$\boldsymbol{\delta}_{\mathcal{M}}\mathcal{A}(S,j) = i\omega(S,j)\Phi_j, \tag{9.47}$$

for all $j \in J(S)$ and,

$$\boldsymbol{\delta}_{\mathcal{M}}\mathcal{A}(L,j) = -\sqrt{\omega(S,j)^2 - M^2}\Phi_j, \tag{9.48}$$

for all $j \in J(L)$. On the other hand the transverse modes $\mathcal{A}(T,j)$ are coclosed for all $j \in J(T)$, which we see from the following (we suppress the (T,j) label for convenience). Let $f \in C^\infty(-\tau,\tau)$ be a function of time and consider the following,

$$\begin{aligned}
\boldsymbol{\delta}_{\mathcal{M}}(f\xi) &= *_{\mathcal{M}}\mathbf{d} *_{\mathcal{M}} (f\xi) \\
&= *_{\mathcal{M}}\mathbf{d}(f *_{\mathcal{M}} \xi) \\
&= *_{\mathcal{M}}\left(\frac{\partial f}{\partial t}\mathbf{d}t \wedge *_{\mathcal{M}}\xi + f\mathbf{d} *_{\mathcal{M}} \xi\right) \\
&= f *_{\mathcal{M}} \mathbf{d} *_{\mathcal{M}} \xi \\
&= f *_{\mathcal{M}} \mathbf{d}(\xi_i *_{\mathcal{M}} \mathbf{d}x^i) \\
&= f *_{\mathcal{M}} \left(\frac{\partial \xi_i}{\partial x^j}\mathbf{d}x^j \wedge *_{\mathcal{M}}\mathbf{d}x^i\right) \\
&= f\frac{\partial \xi_i}{\partial x^i} *_{\mathcal{M}} (\mathbf{d}x^j \wedge *_{\mathcal{M}}\mathbf{d}x^j) \\
&= 0,
\end{aligned} \tag{9.49}$$

where the last equality holds since,

$$\frac{\partial \xi_i}{\partial x^i} = \boldsymbol{\delta}_{\Sigma}\xi = 0, \tag{9.50}$$

and where ξ_i is the $i-$th component of ξ. Since the transverse modes are of the form,

$$\mathcal{A}(T,j)(t,x) = e^{-i\omega(T,j)t}\xi(T,j)(x), \tag{9.51}$$

we see that $\boldsymbol{\delta}_{\mathcal{M}}\mathcal{A}(T,j) = 0$ by setting $f(t) = e^{-i\omega(T,j)t}$ in (9.49). Since the scalar and longituidinal modes are not coclosed, we choose to perform a change of basis of the operator $-i\mathbb{E}$ by using the 'Proca' and 'gradient' modes introduced in [24] respectively,

$$\mathcal{A}(P,j)(t,x) = \frac{e^{-i\omega(S,j)t}}{M}\Big[\omega(S,j)\xi(L,j)(x) - i\sqrt{\omega^2(S,j)-M^2}\xi(S,j)(x)\Big] \tag{9.52}$$

$$\mathcal{A}(G,j)(t,x) = \frac{e^{-i\omega(S,j)t}}{M}\Big[\sqrt{\omega^2(S,j)-M^2}\xi(L,j)(x) - i\omega(S,j)\xi(S,j)(x)\Big], \tag{9.53}$$

with labelling sets $J(P) = J(G) = J(L)$. Since the Proca and gradient modes are linear combinations of the scalar and longitudinal modes, and since the longitudinal and scalar modes are solutions to $\Box + M^2$, it follows that the Proca and gradient modes are solutions to $\Box + M^2$. By [24, Theorem IV.1], the operator (9.46) is invariant under a change of basis. Therefore, in terms of the $\{T, P, G\}$ basis, the operator (9.46) is,

$$-i\mathbb{E} = \sum_{m\in\{T,P,G\}}\sum_{j\in J(m)} \frac{-\sigma(m)}{2\omega(S,j)}\Big[|\mathcal{A}(m,j)\rangle\langle\mathcal{A}(m,j)| - |\overline{\mathcal{A}(m,\bar{j})}\rangle\langle\overline{\mathcal{A}(m,\bar{j})}|\Big], \tag{9.54}$$

where $\sigma(G) = 1$ and $\sigma(T) = \sigma(P) = -1$. We shall use the following expressions for the gradient and Proca modes,

$$\begin{aligned}\mathcal{A}(G,j) &= M^{-1}\mathbf{d}\Phi_j,\\ \mathcal{A}(P,j) &= \frac{iM}{\sqrt{\omega^2(S,j)-M^2}}\Big[\mathcal{A}(S,j) - i\frac{\omega(S,j)}{M^2}\mathbf{d}\Phi_j\Big],\end{aligned} \tag{9.55}$$

which have the following coderivatives,

$$\begin{aligned}\boldsymbol{\delta}_{\mathcal{M}}\mathcal{A}(P,j) &= 0,\\ \boldsymbol{\delta}_{\mathcal{M}}\mathcal{A}(G,j) &= M\Phi_j,\end{aligned} \tag{9.56}$$

for all $j \in J(L)$. Since the transverse and Proca modes are coclosed on $\mathcal{M}$ and are solutions $\Box + M^2$, it follows that they are solutions to the Proca equation (9.24),

$$\big(-\boldsymbol{\delta}_{\mathcal{M}}\mathbf{d} + M^2\big)\mathcal{A}(m,j) = \big(\Box + M^2 + \mathbf{d}\boldsymbol{\delta}_{\mathcal{M}}\big)\mathcal{A}(m,j) = 0, \tag{9.57}$$

for each $m \in \{T, P\}$ and all $j \in J(m)$. Conversely, since the gradient modes (9.53) are not coclosed, it follows that they are not solutions to the Proca equation (9.24). However, we note the following expressions for the transverse, gradient and Proca modes,

$$\begin{aligned}(\mathbb{1} - M^{-2}\mathbf{d}\boldsymbol{\delta}_{\mathcal{M}})\mathcal{A}(G,j) &= M^{-1}\mathbf{d}\Phi_j - M^{-2}\mathbf{d}(M\Phi_j) = 0 &&\forall j\in J(L)\\ (\mathbb{1} - M^{-2}\mathbf{d}\boldsymbol{\delta}_{\mathcal{M}})\mathcal{A}(P,j) &= \mathcal{A}(P,j) &&\forall j\in J(L)\\ (\mathbb{1} - M^{-2}\mathbf{d}\boldsymbol{\delta}_{\mathcal{M}})\mathcal{A}(T,j) &= \mathcal{A}(T,j) &&\forall j\in J(T).\end{aligned} \tag{9.58}$$

Therefore, since the gradient modes are annihilated under the action of the operator $\mathbb{1} - M^{-2}\mathbf{d}\boldsymbol{\delta}_{\mathcal{M}}$, and since the transverse and Proca modes are left invariant, the operator(9.32) is then given by,

$$\begin{aligned} A &= -i\mathbb{E}(\mathbb{1} - M^{-2}\mathbf{d}\boldsymbol{\delta}_{\mathcal{M}}) \\ &= \sum_{m\in\{T,P\}} \sum_{j\in J(m)} \frac{1}{2\omega(m,j)} \Big[|\mathcal{A}(m,j)\rangle\langle\mathcal{A}(m,j)| - |\overline{\mathcal{A}(m,\overline{j})}\rangle \, \overline{\langle\mathcal{A}(m,\overline{j})|} \Big], \\ &\doteq \sum_{m\in\{T,P\}} \sum_{j\in J(m)} A(m,j) \end{aligned} \tag{9.59}$$

where we have used $\sigma(T) = \sigma(P) = -1$ and where,

$$A(m,j) = \frac{1}{2\omega(m,j)} \Big[|\mathcal{A}(m,j)\rangle\langle\mathcal{A}(m,j)| - |\overline{\mathcal{A}(m,\overline{j})}\rangle \, \overline{\langle\mathcal{A}(m,\overline{j})|} \Big], \tag{9.60}$$

A close examination of the Proca and gradient modes reveals the inner products,

$$\begin{aligned} \langle\mathcal{A}(m,j) \mid \mathcal{A}(m,k)\rangle &= 2\tau\sigma(m)\delta_{jk} \\ \langle\overline{\mathcal{A}(m,\overline{j})} \mid \mathcal{A}(m,k)\rangle &= -2\tau\left(2\left(\frac{\omega(S,j)}{M}\right)^2 - 1\right)\mathrm{sinc}(2\omega(S,j)\tau)\delta_{jk} \end{aligned} \tag{9.61}$$

for all $m \in \{P, G\}$ and all $j \in J(m)$. We now turn our attention to the non-existence of the SJ state for the Proca field over the ultrastatic slab $(\mathcal{M} = (-\tau,\tau) \times \Sigma, g = \mathbb{1} \oplus -h)$.

9.4 The Non-Existence of the SJ State for the Proca Field

A rigorous definition of the operator (9.59) requires a careful analysis, since it turns out the operator $-i\Delta$ is an unbounded, non-definitisable operator on the Krein space $\mathcal{K}$ of square integrable one-forms. For a complete rigorous construction of the operator (9.59) the reader is referred to [25, 26]. Due to these technical difficulties, the original SJ axioms presented in Chapter 5 section 5.2 need to be modified to deal with unbounded operators on a Krein space. The construction can be summarised as the following: Let $\mathcal{K}$ be a Krein space and A be a closed symmetric operator with a dense domain $D(A) \subset \mathcal{K}$ and let Γ be an anti-unitary involution so that $A = -\Gamma A \Gamma$. The aim is to axiomatically construct the positive part of the operator A, denoted A^+, where A^+ obeys the axioms,

$SJ1)$ $A^+ - \Gamma A^+ \Gamma = A$ on $D(A)$.

$SJ2)$ $\overline{\mathrm{Im}(A^+)}$ and $\overline{\Gamma \mathrm{Im}(A^+)}$ are orthogonal and non-tangential[c]

[c]Non-tangential subspaces is a generalisation of orthogonality in Krein spaces, more details are given in [25].

$SJ3)$ $A^+ \geq 0$.

The set of closed operators A^+ such that $D(A) \subset D(A^+)$ and such that A^+ obeys $SJ1-3$ will be denoted $SJ(A)$ whereby each operator $A^+ \in SJ(A)$ will be called an SJ operator. If $SJ(A)$ is empty then A does not admit SJ operators. One can solve the axioms $SJ1-3$ in the same spirit as the original SJ axioms due to Sorkin; $SJ1$ implies,

$$A^+ = R + \frac{A}{2}, \tag{9.62}$$

where $R = \Gamma R \Gamma$. $SJ2$ implies,

$$\begin{aligned} R^2 &= \frac{A^2}{4} \\ [A, R] &= 0, \end{aligned} \tag{9.63}$$

and $SJ1$, $SJ3$ imply that $R = \frac{1}{2}(A^+ + \Gamma A^+ \Gamma) \geq 0$. A complete description and motivation for these axioms can be found in [25]. A necessary condition for the operator (9.59) to admit SJ operators is given by the following Lemma [25, Lemma 4.5],

Lemma 9.1. *Let A be a self-adjoint operator on $\mathcal{K}$. Then A admits an SJ operator only if the spectrum of A is real.*

Proof. Suppose $A^+ = R + \frac{1}{2}A \in SJ(A)$ is solution to $SJ1-3$. Therefore $R \geq 0$, and by [6, Theorem VII.1.3], the spectrum of R is real. Therefore, the spectrum of $R^2 = \dfrac{A^2}{4}$ is contained in $[0, \infty)$ which implies the spectrum of A is real. ■

Therefore, for the purpose of showing the non-existence of the SJ state for the Proca field it is sufficient to show that the spectrum of the operator (9.59) is complex. Barring convergence issues, this amounts to checking whether or not the spectrum of the operators $A(m, j)$ is real for each $m \in \{T, P\}$ and all $j \in J(m)$. Using the toy model presented in section 9.1 The eigenvalues of the operator (9.60) are,

$$\lambda^{\pm}(m, j) = \frac{\pm\tau}{2\omega(m, j)} \sqrt{1 - |\beta(m, j)|^2}, \tag{9.64}$$

where,

$$\beta(m, j) = \begin{cases} \left(2\left(\dfrac{\omega(S, j)}{M}\right)^2 - 1\right) \operatorname{sinc}(2\omega(S, j)\tau) & m = P, \forall j \in J(L) \\ \operatorname{sinc}(2\omega(T, j)\tau) & m = T, \forall j \in J(T). \end{cases} \tag{9.65}$$

However, we now show that there exists an operator $A(m,j)$ with a purely *imaginary* eigenvalues. Consider the operator $A(P,j)$, with eigenvalues,

$$\begin{aligned}\lambda^{\pm}(P,j) &= \frac{\pm\tau}{2\omega(P,j)}\sqrt{1-|\beta(P,j)|^2} \\ &= \frac{\pm\tau}{2\omega(P,j)}\sqrt{1-\left|\left(2\left(\frac{\omega(S,j)}{M}\right)-1\right)\mathrm{sinc}(2\omega(S,j)\tau)\right|^2}.\end{aligned} \tag{9.66}$$

Since $\omega(P,j)\to\infty$ in the limit $j\to\infty$, it follows that $\beta(P,j)\to\infty$ as $j\to\infty$, and therefore there exists a $j\in J(P)$ such that $|\beta(P,j)|>1$ and hence for this j we have $\lambda(P,j)\in\mathbb{C}$. Therefore, there exists an operator $A(m,j)$ with a complex spectrum, hence the operator (9.59) has a complex spectrum and by Lemma 9.1, the operator (9.59) does not admit SJ operators.

Discussion

As previously mentioned, there are eigenvalues $\omega(P,j)>M$ such that $|\beta(P,j)|>1$. If $j\in J(P)$ satisfies $|\beta(P,j)|>1$ then the operator $A(P,j)$ will have a pure imaginary spectrum. One can examine precisely which modes $\omega(P,j)$ have the property $|\beta(P,j)|>1$. Let $\widetilde{\omega}$ denote a 'cutoff' frequency which is defined by the property such that $|\beta(P,j)|\leq 1$ for all $\omega(P,j)<\widetilde{\omega}$ and if $|\beta(P,j)|>1$ then $\omega(P,j)>\widetilde{\omega}$. We now given an estimate for $\widetilde{\omega}$ in terms of the parameters M and τ. Since $\mathrm{sinc}(x)\leq\frac{1}{x}$ we obtain,

$$\beta(P,j)=\left(2\left(\frac{\omega(P,j)}{M}\right)^2-1\right)\mathrm{sinc}\left(\frac{\omega(P,j)}{M}2M\tau\right)\leq\frac{\omega(P,j)}{M}\left(\frac{1}{M\tau}\right), \tag{9.67}$$

hence if $|\beta(P,j)|\leq 1$ we obtain,

$$\frac{\omega(P,j)}{M}\leq M\tau. \tag{9.68}$$

We therefore obtain a cutoff frequency $\widetilde{\omega}=M^2\tau$. Typically the cutoff frequency $\widetilde{\omega}$ is very large; W-bosons have mass of order 100 GeV and setting $\tau\approx 10^{40}$ GeV^{-1} to be the lifetime of the Universe gives $\widetilde{\omega}\approx 10^{44}$GeV, well above energy levels of any terrestrial experiment such as the Large Hadron Collider. A plot of the function $\beta(P,j)$ with $M\tau=3$ is shown in Figure 9.1.

We have shown that a straightforward generalisation of the SJ vacuum state for the massive spin one field over ultrastatic slab fails because the commutator function has a complex spectrum. Whilst a complete rigorous statement of this result is beyond the scope of this thesis, we refer the reader to [25, 26] for further details. However, interestingly enough, the BF state construction, when applied to the massive spin one field on an ultrastatic slab, not only gives a state, but this state also satisfies the Hadamard condition. This will be the subject of the following section.

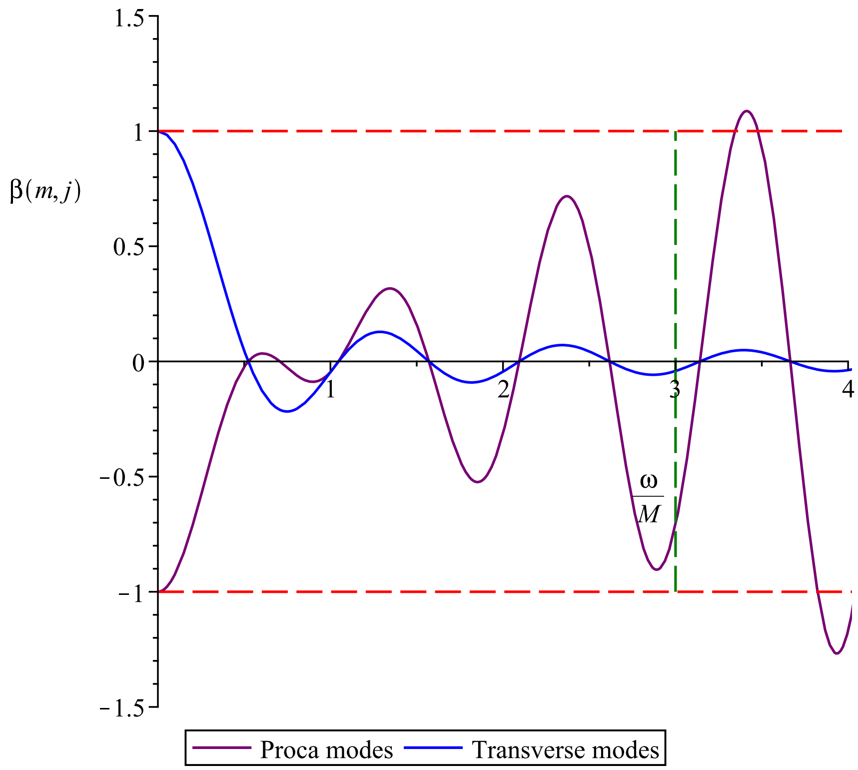


Figure 9.1: Plot of the function $\beta(m,j)$ in units $\frac{\omega(m,j)}{M}$ for a cuttoff frequency $\frac{\tilde{\omega}}{M} = M\tau = 3$ shown as the dashed green line. One observes that for $\omega(P,j) \leq \tilde{\omega}$ we have $|\beta(P,j)| < 1$, and so the eigenvalue $\lambda(P,j)$ real. The transverse modes obey $|\beta(T,j)| \leq 1$ for all $j \in J(T)$, and therefore the eigenvalues $\lambda(T,j)$ are real for all $j \in J(T)$.

9.5 BF States for the Massive Spin One Field on Ultrastatic Slab Spacetimes

Clearly, the SJ state construction is ill defined for the massive spin one field due to the lack of positivity. However, one may consider a modification to the SJ construction originally due to Brum and Fredenhagen [11]. Such a modification yields the so called BF states. The BF states for the free scalar field satisfy the Hadamard property in both static and expanding spacetimes. We adapt this prescription to the massive spin one field and show that, not only does this prescription yield a family of states, but these states also satisfy the Hadamard property.

Let $\mathcal{N} = \mathbb{R}\times\Sigma$ be a globally hyperbolic ultrastatic spacetime with metric $g = \mathbb{1}\oplus -h$ such that (Σ, h) is a complete, compact Riemannian manifold and let $\mathcal{M} = (-\tau,\tau)\times\Sigma$ be an ultrastatic slab spacetime. There is the canonical embedding of $\mathcal{M}$ into $\mathcal{N}$ via the isometry $\iota : \mathcal{M} \hookrightarrow \mathcal{N}$, $(t,x) \mapsto (t,x)$. By the uniqueness of the fundamental solutions the advanced-minus-retarded operator on $\mathcal{M}$ is obtained from the corresponding operator on $\mathcal{N}$,

$$\mathbb{E}_{\mathcal{M}} = \iota^* \mathbb{E}_N \iota_*, \tag{9.69}$$

where ι^*, ι_* are the pull-back and push-forward of ι respectively. Let $f \in C_0^\infty(\mathbb{R})$ be real valued. Using the operator A defined in (9.59) and the decomposition (9.19) one can define the operator on $\mathcal{K}$ as,

$$A_f \doteq f(-i\Delta_M) f, \tag{9.70}$$

where Δ_M is defined in (7). Whenever $f \in C_0^\infty(\mathbb{R})$ *and* $f\big|_{(-\tau,\tau)} = 1$, we shall refer to the operator (9.70) as the softened commutator function and the function f as softening function. The significance of the condition that $f\big|_{(-\tau,\tau)} = 1$ will become clear once we construct the two-point function for the BF state. When f is the characteristic function on $(-\tau,\tau)$ one recovers the commutator function (9.59). Using the mode decomposition of the operator (9.59), we formally arrive at,

$$A_f = \sum_{m\in\{T,P\}} \sum_{j\in J(m)} \frac{1}{2\omega(m,j)} \Big[|f\mathcal{A}(m,j)\rangle\langle f\mathcal{A}(m,j)| - |\overline{f\mathcal{A}(m,\overline{j})}\rangle \, \langle\overline{f\mathcal{A}(m,\overline{j})}| \Big], \tag{9.71}$$

which (formally) reduces to an orthogonal sum of rank-two operators,

$$A_f(m,j) = \frac{1}{2\omega(m,j)} \Big[|f\mathcal{A}(m,j)\rangle\langle f\mathcal{A}(m,j)| - |\overline{f\mathcal{A}(m,\overline{j})}\rangle \, \langle\overline{f\mathcal{A}(m,\overline{j})}| \Big]. \tag{9.72}$$

Whilst a rigorous definition of the operator A_f requires substantially more analysis, since it turns out to be an unbounded non-definitisable operator over a Krein space, we shall only work with the formal expressions as results obtained will be the same as results obtained by more rigorous methods. Our construction for the BF state for the Proca field requires constructing the SJ operator $A_f^+(m,j)$ for each $m \in \{T, P\}$ and all $j \in J(m)$ which is the a solution to the SJ axioms (9.4). One then formally constructs the positive part of the operator (9.71) by summing (modulo convergence issues),

$$A_f^+ \doteq \sum_{m \in \{T,P\}} \sum_{j \in J(m)} A_f^+(m,j). \tag{9.73}$$

The two-point function for the BF state is constructed in an analogous way to the SJ state,

$$W_{BF_f}(h,g) = \langle \overline{h} \mid A_f^+ g \rangle, \tag{9.74}$$

for all $h, g \in \Omega_0^1(\mathcal{M})$. We show that, for a suitably chosen test function $f \in C_0^\infty(\mathbb{R})$ the point spectrum of the operators $A_f(m,j)$ is real for each $m \in \{T, P\}$ and all $j \in J(m)$. We show this with an explicit example by constructing plateau functions, much in the same spirit as in Chapter 8 Section 8.5. We now turn our attention to the operators $A_f(m,j)$ and the construction of the positive part $A_f^+(m,j)$. Calculating some inner products, we find,

$$\begin{aligned} \langle f\mathcal{A}(m,j) \mid f\mathcal{A}(m,j)\rangle &= \langle f\overline{\mathcal{A}(m,\overline{j})} \mid f\overline{\mathcal{A}(m,\overline{j})}\rangle = -\widehat{f^2}(0) \\ \langle f\overline{\mathcal{A}(m,\overline{j})} \mid f\mathcal{A}(m,j)\rangle &= -\beta_f(\omega(m,j))\widehat{f^2}(0) \end{aligned} \tag{9.75}$$

where,

$$\beta_f(\omega(m,j)) \doteq \begin{cases} \dfrac{\widehat{f^2}(2\omega(T,j))}{\widehat{f^2}(0)} & m = T \\ \left(2\left(\dfrac{\omega(S,j)}{M}\right)^2 - 1\right)\dfrac{\widehat{f^2}(2\omega(P,j))}{\widehat{f^2}(0)} & m = P. \end{cases} \tag{9.76}$$

The eigenvectors of the rank two operators $A_f(m,j)$ are for each $m \in \{T, P\}$ and all $j \in J(m)$,

$$\begin{aligned} \Psi_f^-(m,j) &= \frac{fe^{3u(m,j)/2}}{\sqrt{\widehat{f^2}(0)}}\left(\sqrt{\cosh(u(m,j))}\mathcal{A}(m,j) - \sqrt{\sinh(u(m,j))}\overline{\mathcal{A}(m,\overline{j})}\right) \\ \Psi_f^+(m,j) &= \frac{fe^{3u(m,j)/2}}{\sqrt{\widehat{f^2}(0)}}\left(\sqrt{\sinh(u(m,j))}\mathcal{A}(m,j) - \sqrt{\cosh(u(m,j))}\overline{\mathcal{A}(m,\overline{j})}\right), \end{aligned} \tag{9.77}$$

where,

$$e^{-2u(m,j)} = \sqrt{1 - |\beta_f(\omega(m,j))|^2} \tag{9.78}$$

and are pseudo-orthonormal in the sense that,

$$\begin{aligned}\langle \Psi^{\pm}(m,j) \mid \Psi^{\pm}(m,j)\rangle &= -1\\ \langle \Psi^{+}(m,j) \mid \Psi^{-}(m,j)\rangle &= 0.\end{aligned} \tag{9.79}$$

The corresponding eigenvalues are given by,

$$\lambda^{\pm}(m,j) = \pm\frac{\widehat{f^2}(0)e^{-2u(m,j)}}{2\omega(m,j)} \doteq \pm\lambda(m,j) \tag{9.80}$$

The positive part of the (9.72) is formally given by the expression [25, Theorem 6.6],

$$A_f^{+} = \sum_{m\in\{T,P\}}\sum_{j\in J(m)} -\lambda(m,j)\frac{|\Psi^{-}(m,j)\rangle\,\langle\Psi^{-}(m,j)|}{\langle\Psi^{-}(m,j) \mid \Psi^{-}(m,j)\rangle}. \tag{9.81}$$

The two-point function for the BF state on the ultrastatic slab $(-\tau,\tau)\times\Sigma$ then has the two-point function,

$$\begin{aligned}W_{BF_f}(h,g) &= \langle \overline{h} \mid A_f^{+} g\rangle\\ &= \sum_{m\in\{T,P\}}\sum_{j\in J(m)} \lambda(m,j)\langle \overline{h} \mid \Psi^{-}(m,j)\rangle\langle\Psi^{-}(m,j) \mid g\rangle.\end{aligned} \tag{9.82}$$

We see that the the two-point function (9.82) is a weak bisolution of the Proca operator $-\boldsymbol{\delta}_{\mathcal{M}}\mathbf{d}+M^2$ by the following observations: Since both the transverse and Proca modes are coclosed on $\mathcal{M}$, since $f\big|_{\mathcal{M}} = 1$ and since $(\Box + M^2)\mathcal{A}(m,j) = 0$ for each $m \in \{T,P\}$ and all $j \in J(m)$ it can be shown that the eigenvectors $\Psi^{\pm}$ are coclosed on $\mathcal{M}$ and are solutions to the Klein Gordon operator $\Box + M^2$. Therefore we obtain,

$$\begin{aligned}W_{BF_f}((-\boldsymbol{\delta}_{\mathcal{M}}\mathbf{d}+M^2)h,g) &= \sum_{m\in\{T,P\}}\sum_{j\in J(m)} \lambda(m,j)\langle(-\boldsymbol{\delta}_{\mathcal{M}}\mathbf{d}+M^2)\overline{h} \mid \Psi^{-}(m,j)\rangle\langle\Psi^{-}(m,j) \mid g\rangle\\ &= \sum_{m\in\{T,P\}}\sum_{j\in J(m)} \lambda(m,j)\langle\overline{h} \mid (-\boldsymbol{\delta}_{\mathcal{M}}\mathbf{d}+M^2)\Psi^{-}(m,j)\rangle\langle\Psi^{-}(m,j) \mid g\rangle\\ &= \sum_{m\in\{T,P\}}\sum_{j\in J(m)} \lambda(m,j)\langle\overline{h} \mid (\Box+M^2)\Psi^{-}(m,j)\rangle\langle\Psi^{-}(m,j) \mid g\rangle\\ &= 0.\end{aligned} \tag{9.83}$$

The antisymmetric part of the two-point function (9.82) coincides with the operator A_f given in (9.70) which coincides with the commutator function $-i\Delta_M$ given in (9.32) on $(\mathcal{M},g)$ precisely because

$f\big|_{\mathcal{M}} \equiv 1$. The two-point function (9.82) is of positive type if and only if the operator A_f^+ is non-negative in the sense that $\langle g \mid A_f^+ g\rangle \geq 0$ for all $g \in \mathcal{K}$. One observes that the operator (9.81) is non-negative if and only if the eigenvalues $\lambda(m,j)$ are non-negative for all $m \in \{T,P\}$ and all $j \in J(m)$,

$$\langle h \mid A_f^+ h\rangle = \sum_{m\in\{T,P\}} \sum_{j\in J(m)} \lambda(m,j)|\langle\Psi^-(m,j) \mid h\rangle|^2, \tag{9.84}$$

for all $h \in \mathcal{K}$. We now show that there are functions $f \in C_0^\infty(\mathbb{R})$ such that $\lambda(m,j) \in \mathbb{R}$ for all $j \in J(m)$. The explicit construction of such functions will be the subject of the next section.

9.6 Existence of BF states: A scaling argument

A necessary condition for the existence of a BF state is that the eigenvalues of the operator A_f are all real. The eigenvalues of each operator $A_f(m,j)$ are, for $m \in \{T,P\}$ and $j \in J(m)$,

$$\lambda(m,j) = \frac{\widehat{f^2}(0)}{2\omega_j}\sqrt{1-|\beta_f(\omega(m,j))|^2}. \tag{9.85}$$

All these eigenvalues must be real. Hence the function $f \in C_0^\infty(\mathbb{R})$ must be chosen such that,

$$|\beta_f(\omega(m,j))| < 1. \tag{9.86}$$

Dealing with the transverse modes first, we see that $\lambda(T,j)$ is manifestly real for all $j \in J(T)$ from the following proposition:

Lemma 9.2. *Let $f \in L^2(\mathbb{R})$ be positive almost everywhere, then,*

$$|\widehat{f}(\omega)| < \widehat{f}(0), \tag{9.87}$$

for all $\omega \in \mathbb{R}$.

Proof. [26, Lemma 3.1]. ■

However, the above Lemma is not sufficient to show that $\lambda(P,j) \in \mathbb{R}$ for all $j \in J(P)$. We will now show by explicit construction that there exist functions $f \in C_0^\infty(\mathbb{R})$ such that $\lambda(P,j) \in \mathbb{R}$ for all $j \in J(P)$. Let,

$$F(t) = \int_{-\infty}^{t} \mathbf{d}t' H_\tau(t'+\tau_p) - H_\tau(t'-\tau_p), \tag{9.88}$$

be a plateau function with a plateau time $\tau_p > 0$ and switch on time $\tau_s > 0$ as constructed in Chapter 8 section 8.5. The function (9.88) obeys $F \equiv 1$ on $[-\tau_p + \frac{1}{2}\tau_s, \tau_p - \frac{1}{2}\tau_s]$ and has support $\mathrm{supp}(F) = [-\frac{1}{2}\tau_s - \tau_p, \tau_p + \frac{1}{2}\tau_s]$. The function (9.88) admits a smooth square root, which we denote $f = \sqrt{F}$. This is expanded upon in [26]. The square root $f = \sqrt{F}$ is a smooth compactly supported function and obeys $f\big|_{(-\tau_p+\tau_s, \tau_p-\tau_s)} \equiv 1$, hence it is a softening function. The Fourier transform of the function (9.88) is,

$$\widehat{F}(\omega) = \widehat{f^2}(\omega) = 2\tau_p \mathrm{sinc}(\omega\tau_p)\widehat{H}(\omega\tau_p\alpha), \tag{9.89}$$

where $\alpha = \dfrac{\tau_s}{\tau_p}$ is a dimensionless parameter. Using the Fourier transform (9.89) we now give an analytic bound on the parameter α such that $|\beta_f(\omega(P, j))| < 1$ for all $j \in J(L)$. Since $\widehat{H}(0) = 1$, we have $\widehat{F}(0) = 2\tau_p$, we then obtain, using (9.89),

$$\beta_f(\omega) = \left(2\left(\frac{\omega}{M}\right)^2 - 1\right)\mathrm{sinc}(2\omega\tau_p)\widehat{H}(2\omega\tau_p\alpha). \tag{9.90}$$

By varying the dimensionless parameter α, we now show that there exists an $\alpha > 0$ such that $\beta_f(\omega) < 1$ for all $\omega \in \mathbb{R}$. The leading order asymptotic approximation of the Fourier transform $\widehat{H}(\cdot)$ can be shown to be [21],

$$\widehat{H}(\omega) \sim e^{-1/4}\pi^{-1/2}e^{-\sqrt{|\omega|/2}}\cos\left(\frac{|\omega|}{2} - \sqrt{\frac{|\omega|}{2}}\right) \qquad |\omega| \to \infty, \tag{9.91}$$

Therefore, there exists a $k > 0$ such that,

$$|\widehat{H}(\omega\tau_p\alpha)| \leq k e^{-\sqrt{\frac{\omega\tau_p\alpha}{2}}}, \tag{9.92}$$

where,

$$k = \sup_x |\widehat{H}(x)| e^{\sqrt{\frac{x}{2}}} \approx 2.55, \tag{9.93}$$

which is seen in Figure 9.2.

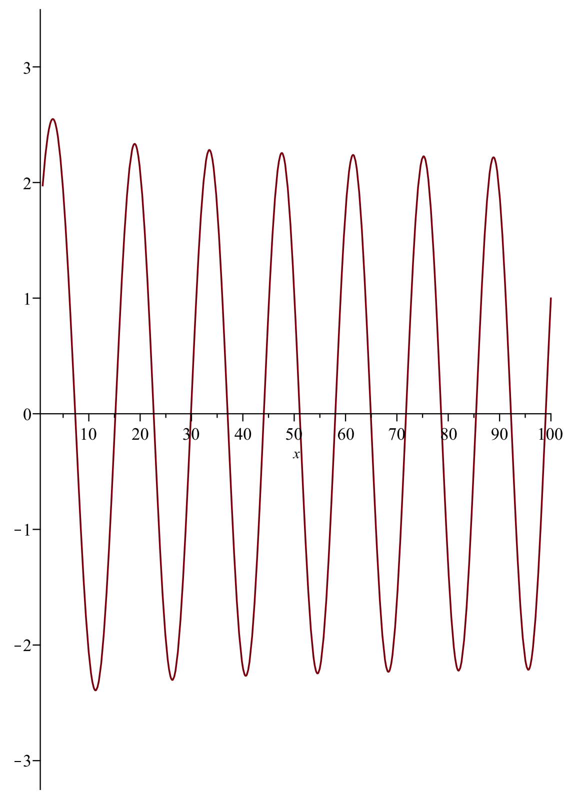


Figure 9.2: Plot of the function $\widehat{H}(x)e^{\sqrt{\frac{x}{2}}}$.

Using the bound (9.92) and the bound $|\text{sinc}(x)| \leq \frac{1}{x}$ we then obtain,

$$
\begin{aligned}
|\beta_f(\omega)| &\leq 2\Big(\frac{\omega}{M}\Big)^2\left(\frac{1}{2\omega\tau_p}\right)|\widehat{H}(2\omega\tau_p\alpha)| \\
&\leq \frac{\omega}{M^2\tau_p}ke^{-\sqrt{\dfrac{\omega\tau_p\alpha}{2}}} \\
&= \frac{k}{\alpha(M\tau_p)^2}\left[\omega\tau_p\alpha e^{-\sqrt{\omega\tau_p\alpha/2}}\right] \\
&\leq \frac{k}{\alpha(M\tau_p)^2}\sup_x xe^{-\sqrt{x/2}} \\
&= \frac{4k}{e^2\alpha(M\tau_p)^2}.
\end{aligned}
\tag{9.94}
$$

Therefore if,

$$\frac{4k}{e^2\alpha(M\tau_p)^2} \leq 1, \tag{9.95}$$

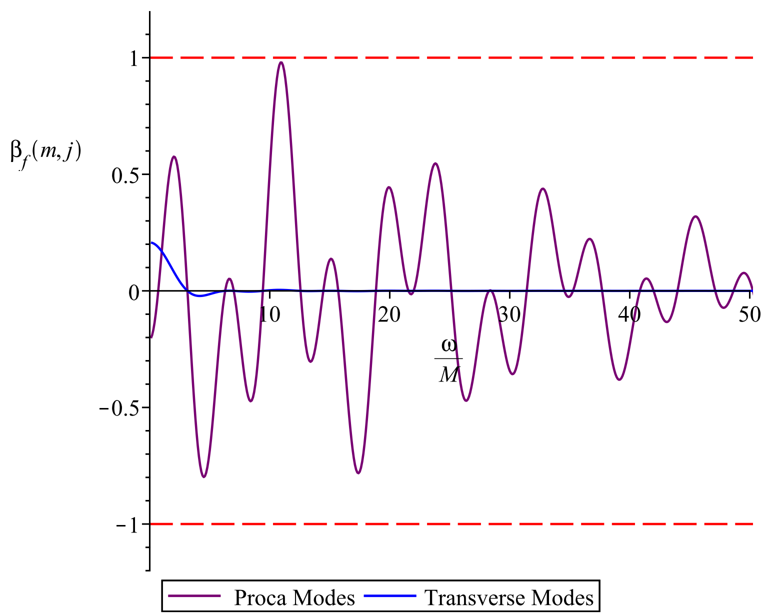
holds then the condition (9.86) holds and the eigenvalues (9.80) are real. Hence the condition,

$$\frac{4k}{e^2\alpha(M\tau_p)^2} \leq 1 \implies \alpha = \frac{\tau_s}{\tau_p} \geq \frac{4k}{e^2(M\tau_p)^2} \approx \frac{1.4}{(M\tau_p)^2}. \tag{9.96}$$

To illustrate how small the switch-on time could be, consider the example of a massive $W-$boson with mass of order 100Gev and $\tau_p \approx 10^{40}\text{Gev}^{-1}$ to be the lifetime of the Universe. The smallest switch-on time allowed is given by,

$$\tau_s = \frac{2\tau_p}{(M\tau_p)^2} = \frac{2}{M(M\tau_p)} \approx \frac{10^{-42}}{M} \approx 10^{-20}s. \tag{9.97}$$

A plot for the functions $\beta_f(m,j)$ is shown in Figure 9.3 for sufficient and insufficient scaling.



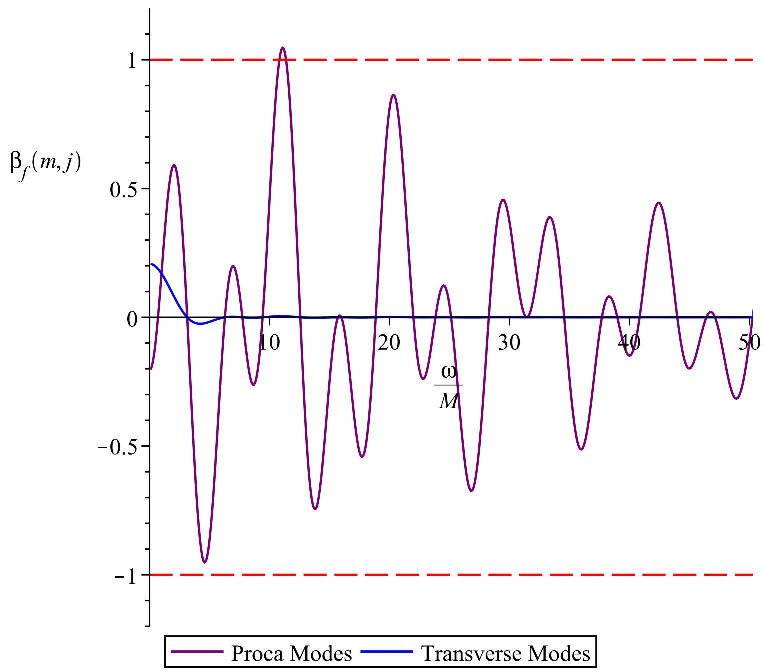


Figure 9.3: Influence of changing the parameter α in (9.90) for $M\tau_p = 1$. The first plot has sufficient scaling whereas the second plot has insufficient scaling.

We have shown with an explicit construction of a compactly supported function F that the eigenvalues of the operators $A_f(m,j)$, where $f = \sqrt{F}$, are real for a sufficiently large switch-on times. Therefore, we have explicitly constructed a smooth compactly supported function such that A_f^+ is an SJ operator of A_f.

9.7 Hadamard Condition for BF States for the Massive Spin One Field on Ultrastatic Slabs

Our goal of this section is to prove the the BF state with two-point function,

$$W_{BF_f}(h,g) = \langle \overline{h} \mid A_f^+ g\rangle = \sum_{m\in\{T,P\}} \sum_{j\in J(m)} \lambda(m,j)\langle h \mid \Psi^-(m,j)\rangle\langle\Psi^-(m,j) \mid g\rangle, \tag{9.98}$$

where,

$$A_f^+ = \sum_{m\in\{T,P\}} \sum_{j\in J(m)} -\lambda(m,j) \frac{|\Psi^-(m,j)\rangle \langle \Psi^-(m,j)|}{\langle \Psi^-(m,j) \mid \Psi^-(m,j)\rangle}, \tag{9.99}$$

is a Hadamard state on the algebra of observables $\mathfrak{A}_M(\mathcal{M}, g)$.

Theorem 9.3. *Let $\mathcal{N} = \mathbb{R} \times \Sigma$ be an ultrastatic spacetime equipped with a metric $g = \mathbb{1} \oplus -h$ where (Σ, h) is a compact Riemannian manifold. Suppose the spacetime $(\mathcal{M} = (-\tau,\tau) \times \Sigma, \mathbb{1} \oplus -h)$ is isometrically embedded into $\mathcal{N}$ and let $f \in C_0^\infty(\mathbb{R})$ be identically 1 when restricted to the interval $(-\tau,\tau)$. If the operator (9.99) has a real spectrum, then the BF state ω_{BF_f} with two-point function (9.98) is a Hadamard state.*

Proof. In ordered to show that the BF state ω_{BF_f} is Hadamard, it is sufficient to show that the normal ordered two-point function $:W_{BF_f} := W_{BF_f} - W_H$ is smooth on $\mathcal{M} \times \mathcal{M}$, where W_H is the two-point function for the ultrastatic vacuum state, which is in particular Hadamard. For a discussion and precise definition of Hadamard states for the Proca field, the reader is refered to [24]. To accomplish this, we define,

$$W_{BF_f}(t,x;t',x') = \sum_{m\in\{T,P\}} \sum_{j\in J(m)} W_{BF_f}(m,j)(t,x;t',x'), \tag{9.100}$$

and express the Proca modes in terms of the scalar and longitudinal modes. The components of the two-point function (9.98) are,

$$\begin{aligned} W_{BF_f}(P,j)(t,x;t',x')_{00} = \frac{e^{u(S,j)}}{2\omega(S,j)} \nu_j^2 \Bigg(& 2\sinh(u(S,j))\cos(\omega(S,j)(t-t')) \\ & + 2\sqrt{\frac{\sinh(2u(S,j))}{2}} \cos(\omega(S,j)(t+t')) + e^{-u(S,j)} e^{-i\omega(S,j)(t-t')} \Bigg) \xi_0(S,j)(x)\overline{\xi_0(S,j)(x')} \end{aligned}$$

$$\begin{aligned} W_{BF_f}(P,j)(t,x;t',x')_{rk} = \frac{e^{u(S,j)}}{2\omega(S,j)} \left(\frac{\omega(S,j)}{M}\right)^2 \Bigg(& 2\sinh(u(S,j))\cos(\omega(S,j)(t-t')) \\ & - 2\sqrt{\frac{\sinh(2u(S,j))}{2}} \cos(\omega(S,j)(t+t')) + e^{-u(S,j)} e^{-i\omega(S,j)(t-t')} \Bigg) \xi_r(L,j)(x)\overline{\xi_k(L,j)(x')} \end{aligned}$$

$$W_{BF_f}(P,j)(t,x;t',x')_{0k} = \frac{-i\nu_j e^{u(S,j)}}{2M} \Bigg(- 2i\sinh(u(S,j))\sin(\omega(S,j)(t-t'))$$

$$+ 2i\sqrt{\frac{\sinh(2u(S,j))}{2}}\sin(\omega(S,j)(t+t')) + e^{-u(S,j)}e^{-i\omega(S,j)(t-t')}\Bigg)\xi_0(S,j)(x)\overline{\xi_k(L,j)(x')}$$

$$W_{BF_f}(P,j)_{k0}(t,x;t',x') = \frac{i\nu_j e^{u(S,j)}}{2M}\Bigg(- 2i\sinh(u(S,j))\sin(\omega(S,j)(t-t'))$$
$$- 2i\sqrt{\frac{\sinh(2u(S,j))}{2}}\sin(\omega(S,j)(t+t')) + e^{-u(S,j)}e^{-i\omega(S,j)(t-t')}\Bigg)\xi_k(L,j)(x)\overline{\xi_0(S,j)(x')}$$

for all $j \in J(P)$ where $r, k = 1, 2, 3$ and where,

$$\nu_j = \sqrt{\left(\frac{\omega(S,j)}{M}\right)^2 - 1}. \tag{9.101}$$

The contributions from the transverse modes is,

$$W_{BF_f}(T,j)(t,x;t',x')_{\mu\nu} = \frac{e^{u(T,j)}}{2\sinh(u(T,j))\omega(T,j)}\Bigg(2\sinh(u(T,j))\cos(\omega(T,j)(t-t'))$$
$$- 2\sqrt{\frac{\sinh(u(T,j))}{2}}\cos(\omega(T,j)(t+t'))$$
$$+ e^{-u(T,j)}e^{-i\omega(T,j)(t-t')}\Bigg),$$

where $\mu, \nu = 0, 1, 2, 3$. The ultrastatic vacuum state for the Proca field on the ultrastatic slab $(-\tau, \tau) \times \Sigma$ is given by,

$$W_H(h,g) = \sum_{m\in\{T,P\}}\sum_{j\in J(m)} \langle h \mid \mathcal{A}(m,j)\rangle\langle\mathcal{A}(m,j) \mid g\rangle, \tag{9.102}$$

with the integral kernel,

$$W_H(t,x;t',x') = \sum_{m\in\{T,P\}}\sum_{j\in J(m)} W_H(m,j)(t,x;t',x'). \tag{9.103}$$

which has components,

$$\begin{aligned}
W_H(P,j)_{00}(t,x;t',x') &= \frac{\nu_j^2}{2\omega(S,j)}e^{-i\omega(S,j)(t-t')}\xi_0(S,j)(x)\overline{\xi_0(S,j)(x')}\\
W_H(P,j)_{0k}(t,x;t',x') &= \frac{-i\nu_j}{M}e^{-i\omega(S,j)(t-t')}\xi_0(S,j)(x)\overline{\xi_k(L,j)(x')}\\
W_H(P,j)_{k0}(t,x;t',x') &= \frac{i\nu_j}{M}e^{-i\omega(S,j)(t-t')}\xi_k(L,j)(x)\overline{\xi_0(S,j)(x')}\\
W_H(P,j)_{rk}(t,x;t',x') &= \frac{1}{2\omega(S,j)}\left(\frac{\omega(S,j)}{M}\right)^2 e^{-i\omega(S,j)(t-t')}\xi_r(L,j)(x)\overline{\xi_k(L,j)(x')}\\
W_H(T,j)_{\mu\nu}(t,x;t',x') &= \frac{1}{2\omega(T,j)}e^{-i\omega(T,j)(t-t')}\xi_\mu(T,j)\overline{\xi_\nu(T,j)(x')}
\end{aligned} \tag{9.104}$$

The ultrastatic vacuum state (9.102) satisfies the Hadamard condition [24]. Therefore, if the normal ordered two-point function $: W_{BF_f} := W_{BF_f} - W_H$ has a smooth integral kernel then the BF state (9.98) satisfies the Hadamard condition. We observe that all terms in the kernel of the normal ordered two-point function $: W_{BF_f} :$ may be written as,

$$: W_{BF_f} : (m, j) = \sum_{m \in \{S,L,T\}} \sum_{j \in J(m)} \sum_{\sigma\sigma' \in \{\pm 1\}} \alpha^{\sigma\sigma'}(\omega(m,j))(e_{j\sigma}(m,j) \otimes \xi(m,j)) \otimes (e_{j\sigma'}(m,j) \otimes \xi(m,j)). \tag{9.105}$$

where $e_{\sigma j}(m,j)(t) = e^{i\sigma\omega(m,j)t)}$ for $\sigma \in \{\pm 1\}$ for each $m \in \{S, L, T\}$ and all $j \in J(m)$ and where $\alpha^{\sigma\sigma'}$ obeys,

$$\begin{aligned} |\alpha^{\sigma\sigma'}(\omega(m,j))| &\leq \frac{\omega(m,j)}{M^2}\left(e^{u(m,j)}\sinh(u(m,j)) + \sqrt{\frac{\sinh(2u(m,j))}{2}}\right) \\ &< \frac{\omega(m,j)e^{2u(m,j)}}{2M^2}\left(|\beta_f(m,j)|^2 + |\beta_f(m,j)|\right), \end{aligned} \tag{9.106}$$

where $\beta_f(m,j)$ is given in (9.76) and where we have used,

$$\begin{aligned} e^{u(m,j)}\sinh(u(m,j)) &< \frac{e^{2u(m,j)}}{2}|\beta_f(m,j)|^2 \\ \sqrt{\frac{\sinh(2u(m,j))}{2}} &= \frac{e^{u(m,j)}}{2}|\beta_f(m,j)|. \end{aligned} \tag{9.107}$$

Since $\omega(m,j) \to \infty$ as $j \to \infty$ we have $\widehat{f^2}(\omega(m,j)) \to 0$ as $j \to \infty$ with rapid decay for each $m \in \{T, P\}$. Furthermore by Lemma 9.2 we have $|\beta_f(m,j)| < 1$ for each $m \in \{S, L, T\}$ and all $j \in J(m)$. We therefore obtain,

$$\lim_{j\to\infty} e^{2u(m,j)} = \lim_{j\to\infty} \frac{1}{\sqrt{1 - |\beta_f(m,j)|^2}} < \infty, \tag{9.108}$$

and hence,

$$e^{2u(m,j)} \leq \sup_j e^{2u(m,j)} = \sup_j \frac{1}{\sqrt{1 - |\beta_f(m,j)|^2}} < \infty. \tag{9.109}$$

Since $f \in C_0^\infty(\mathbb{R})$, it follows that $f^2 \in C_0^\infty(\mathbb{R})$ and therefore the Fourier transform $\widehat{f^2}(\cdot)$ decays rapidly. Hence, for all $N \in \mathbb{N}_0$ there exists a $C_N \geq 0$ such that $|\beta_f(m,j)| \leq c_N \omega^{-N}$ for all $\omega \geq M$. Hence, for all $N \in \mathbb{N}_0$ there exists a $C_N \geq 0$ such that,

$$|\alpha^{\sigma\sigma'}(\omega(m,j))| < C_N \frac{\sup_j(e^{2u(m,j)})}{2M^2} \frac{1}{\omega(m,j)^{N-1}} \tag{9.110}$$

By Theorem 4.7, if the following holds,

$$\sum_{j\in J(m)} \omega^p(m,j) \max_{\sigma,\sigma'\in\{\pm1\}} |\alpha^{\sigma\sigma'}(\omega(m,j))|^2 < \infty, \tag{9.111}$$

for all $p \in \mathbb{N}_0$ and each $m \in \{S,L,T\}$ then the normal ordered two-point function $:W_{BF_f}:$ has a smooth integral kernel and the BF state is therefore Hadamard. Using the bound (9.110) we now obtain,

$$\sum_{j\in J(m)} \omega^p(m,j) \max_{\sigma,\sigma'\in\{\pm1\}} |\alpha^{\sigma\sigma'}(\omega(m,j))|^2 < C_N^2 \frac{\left(\sup_j(e^{2u(m,j)})\right)^2}{4M^4} \sum_{j\in J(m)} \frac{1}{\omega(m,j)^{2N-2-p}}, \tag{9.112}$$

for each $m \in \{S,L,T\}$. To conclude the proof, we use Weyl asymptotics to provide a bound on $N(\omega(m,j))$, where N is the counting function defined by,

$$N(\omega(m,j)) = \left|\{\omega(m,k) \leq \omega(m,j) \mid \omega(m,k) \text{ is an eigenvalue of } P_m + M^2\}\right|, \tag{9.113}$$

and where P_m is $-\Delta_\Sigma^s$ for the Proca modes and $-\Delta_\Sigma$ for the transverse modes. Since (Σ, h) is a compact Riemannian manifold of dimension $n = 3$, the asymptotic behaviour of the eigenvalues $\omega^2(m,j)$ is given by Weyl's law,

$$N(\omega(m,j)) \sim \alpha\omega^3(m,j), \tag{9.114}$$

for some constant $\alpha > 0$ [14]. Now let $M_m(j) \doteq \max\{k \in J(m) \mid \omega(m,k) = \omega(m,j)\}|$, which is well defined [20, Theorem 1 Section 6.5]. By counting, we obtain, for each $m \in \{S,L,T\}$ and all $j \in J(m)$,

$$j \leq M_m(j) = N(\omega(m,j)) \leq \alpha'\omega^3(m,j), \tag{9.115}$$

where we choose $\alpha' > \alpha$ to be sufficiently large so that the inequality holds. This then implies,

$$\sum_{j\in J(m)} \omega^p(m,j) \max_{\sigma,\sigma'\in\{\pm1\}} |\alpha^{\sigma\sigma'}(\omega(m,j))|^2 < C_N^2 \frac{\left(\sup_j(e^{2u(m,j)})\right)^2}{4M^4} \sum_{j\in J(m)} \frac{1}{j^{1/3(2N-p-2)}} \tag{9.116}$$

which converges for all $N > \frac{1}{2}(p+5)$. Therefore, by Theorem 4.7, the integral kernel of the normal ordered two-point function $:W_{BF_f}: := W_{BF_f} - W_H$ converges in $\Omega^1(\mathcal{M}\times\mathcal{M})$. Since W_H is the two-point function of a Hadamard state, we can conclude the the BF state ω_{BF_f} is Hadamard. ■

In this chapter we have shown that there are significant difficulties when one applies the SJ and BF state construction to the spin-one field. In the case when the commutator function for the Proca field is defined on an ultrastatic slab spacetime with compact spatial section, the SJ state construction fails since the commutator function has a complex spectrum. It is reasonable to assume that one would face similar problems in other spacetimes. Furthermore, due to the complex spectrum of the commutator function, it is unlikely that one could use the generalised SJ state construction presented in Chapter 6 to construct a quasifree state for the Proca field. However, we have also shown that, for suitable functions, the BF state is well defined for the Proca field on ultrastatic spacetimes with compact spatial sections. It is reasonable to assume that one may extend the BF state construction for the Proca field to other spacetimes, but this would require a substantial amount of work. It is also reasonable to assume that the SJ vacuum with softened boundaries presented in Chapter 8 would yield a quasifree Hadamard state for the Proca field, for reasons similar to the BF case. Finally, the generalised SJ state with softened boundaries construction in Chapter 8 could be applied to the Proca field, but since the spectral theory is significantly harder for Krein space operators, this would present a serious obstacle.

Chapter 10

Outlook and Summary

What we call the beginning is often the end.
And to make an end is to make a beginning.
The end is where we start from.
T. S. Eliot.

In this thesis we have presented a novel construction of quasifree states for the scalar field based on the SJ vacuum axioms. We call these states *generalised SJ states* and are valid in bounded regions of globally hyperbolic spacetimes. In Chapter 7 we gave an explicit construction of an SJ thermal state on ultrastatic slab spacetimes. We have presented a new construction of Hadamard states for the scalar field based on the generalised SJ state construction and an observation of Sorkin. Finally, we applied the BF and SJ state construction to the Proca field on ultrastatic spacetimes. We showed that, whilst the SJ state is ill-defined, the BF state for the Proca field exists and satisfies the Hadamard condition. We now present various potential avenues of research regarding the various constructions presented in this thesis.

Generalised SJ States for the Free Scalar Field

In Chapter 6 we presented a new construction of quasifree states for the free scalar field valid in bounded regions of globally hyperbolic spacetimes. The construction of a quasifree state ω_{SJ_φ} on the

CCR algebra $\mathfrak{A}(\mathcal{M})$ is achieved by setting its two point function to be,

$$W_{SJ_\varphi}(f,g) = \langle \overline{f} \mid A^+_\varphi g\rangle \qquad \forall f,g \in C_0^\infty(\mathcal{M}), \tag{10.1}$$

where A^+_φ is the unique solution to a set of generalised SJ axioms for a suitable continuous function φ. One could investigate whether there are any continuous, even, non-negative functions φ such that the two point function (10.1) defines a Hadamard state. The choice $\varphi = 0$ yields the unsoftened SJ vacuum and the choice,

$$\varphi_\beta(\lambda) = |\lambda| \frac{e^{-\beta\tau|\lambda|^{-\lambda}}}{1 - e^{-\beta\tau|\lambda|^{-1}}}, \tag{10.2}$$

defines the thermal SJ state. The SJ vacuum state is not Hadamard on the ultrastatic slab spacetimes and we have shown that the thermal SJ state also fails to be Hadamard on ultrastatic slabs. It would be interesting to see if the function (10.2) holds in more general spacetimes. Furthermore, it would be interesting to compute the timescale constructed in Chapter 7 Section 7.3 in other spacetimes, for example a causal diamond or an expanding spacetime.

Now, if $\psi(A)$ is smoothing and the SJ vacuum fails to be Hadamard, then the generalised SJ state with two point function (10.1) will also fail to be Hadamard. However if $\psi(A)$ is *not* smoothing it could be possible that one could construct a function so that the two point function (10.1) defines a Hadamard state. We remark here, however, that this is unlikely - it seems reasonable to assume that all generalised SJ states will fail to be Hadamard. To prove this, one could follow a similar argument to the proof that the SJ vacuum fails to be Hadamard on ultrastatic slab spacetimes [28]. Finally, it would be interesting to see how the generalised SJ state construction presented in Chapter 6 fits into causal set theory. One could follow a suitably modified construction of the SJ vacuum state presented in [42] to construct a generalised SJ state for a free scalar field of a causal set.

Generalised SJ States with Softened Boundaries for the Free Scalar Field

In Chapter 8 we presented a new construction for a class of Hadamard states, which is based on an observation of Sorkin [58] and on the new construction given in Chapter 6. We show that, on ultrastatic slabs, the two point function

$$W_{SJ_{\rho,\varphi}}(f,g) = \langle \overline{f} \mid A^+_{\rho,\varphi} g\rangle \qquad \forall f,g \in C_0^\infty(\mathcal{M}), \tag{10.3}$$

where,

$$A^+_{\rho,\varphi} = \frac{A_\rho}{2} + \frac{|A_\rho|}{2} + \varphi(A_\rho), \tag{10.4}$$

defines a Hadamard state whenever $\varphi(A_\rho)$ is smoothing. It would be interesting to see what role the functions φ and ρ play on the properties of the SJ state with two point function (10.3). In [58], Sorkin remarks that the axiom SJ2 can be thought of as a 'ground state condition'. In our generalised framework, the axiom is modified to $A^+_\varphi \Gamma A^+_\varphi \Gamma = \psi(A)$ for a suitable function ψ, with $\psi = 0$ being the minimum. The normal ordered two point function (10.3) with respect to the SJ vacuum with softened boundaries (constructed from the operator $A^+_{\rho,0}$) gives,

$$: W_{SJ_{\rho,\varphi}} := \varphi(A_\rho). \tag{10.5}$$

Therefore, for the normal ordered two point function above, the energy momentum tensor would depend upon the choices of functions φ and ρ. It would be interesting to see this relationship. Furthermore, it would be interesting to apply the generalised SJ state construction to other fields. The Maxwell field is one possibility, although the reader is warned that the SJ state based on the field strength tensor of the Maxwell field does not exist; one faces similar problems that the SJ vacuum state construction faces for the Proca field. It does however, remain unclear if one can construct the SJ state for the Maxwell field based on the vector potential. One does not encounter imaginary eigenvalues, but a general construction for the vector potential still remains elusive. If one can solve this problem, an ambitious project could be the construction of a generalised SJ state for linearised gravity.

Let $\mathcal{M} = (\mathbb{R} \times \mathbb{R}, g)$ be a globally hyperbolic spacetime and let ω_H be a quasifree Hadamard state constructed on $\mathcal{M}$. It would be interesting to investigate whether there exist functions φ, ρ such that the generalised SJ state with softened boundaries $\omega_{SJ_{\rho,\varphi}}$ constructed on the slab spacetime $((-\tau, \tau) \times \Sigma, g)$ converges to the Hadamard state ω_H as $\tau \to \infty$. In other words, whether any quasifree Hadamard state can be 'approximated' by a generalised SJ state with softened boundaries. In Chapter 9 we constructed the BF states for the Proca field on ultrastatic slab spacetime. One could consider constructing the SJ vacuum with softened boundaries for the Proca field; it would likely face the same complications as the BF state construction but it seems reasonable to assume it would yield a quasifree state that satisfies the Hadamard condition. One could also consider constructing a generalised SJ state with softened boundaries for the Proca field on bounded regions of globally hyperbolic spacetimes, but this would require a substantial amount of work.

For want of a nail the shoe was lost.
For want of a shoe the horse was lost.
For want of a horse the rider was lost.
For want of a rider the message was lost.
For want of a message the battle was lost.
For want of a battle the kingdom was lost.
And all for the want of a horseshoe nail.

Appendices

Appendix A

Functional Analysis

We briefly give various definitions for the functional analysis used in this thesis. For a more complete exposition on the subject, the reader is referred to [53].

Definition A.0.1. An *inner product space* is a vector space V endowed with an inner product,

$$\langle \cdot \mid \cdot \rangle : V \times V \to \mathbb{C}, \tag{A.1}$$

that satisfies, for all $\alpha, \beta \in \mathbb{C}$ and all $f, g, h \in V$,

i) $\langle f \mid f \rangle \geq 0$,

ii) $\langle f \mid f \rangle = 0$ if and only if $f = 0$.

iii) $\langle f \mid \alpha g + \beta h \rangle = \alpha \langle f \mid g \rangle + \beta \langle f \mid h \rangle$

iv) $\langle f \mid g \rangle = \overline{\langle g \mid f \rangle}$.

The inner product $\langle \cdot \mid \cdot \rangle$ then induces a norm, denoted by $\| \cdot \|$ by the prescription $\|f\| = \sqrt{\langle f \mid f \rangle}$ for all $f \in V$ [53, Theorem II.2]. The norm induces a metric $d(\cdot, \cdot)$ by setting $d(f, g) = \|f - g\|$ for all $f, g \in V$.

Definition A.0.2. An *orthonormal basis* of a Hilbert space $\mathcal{H}$ is a set $\{e_j\}_{j \in \mathbb{N}} \subset \mathcal{H}$ such that the linear space of $\{e_j\}_{j \in \mathbb{N}}$ is dense in $\mathcal{H}$ and so that,

$$\langle e_j \mid e_k \rangle = \delta_{jk}, \tag{A.2}$$

where δ_{jk} is the Kronecker delta symbol.

Definition A.0.3. A metric space in which all Cauchy sequences converge is called a *complete* metric space.

Definition A.0.4. A completed normed space $(V, \|\cdot\|)$ is called a *Banach space.*

Definition A.0.5. A complete inner product space $(\mathcal{H}, \langle\cdot \mid \cdot\rangle)$ is called a *Hilbert space.*

Definition A.0.6. A *bounded operator* is a map $A : \mathcal{H} \to \mathcal{H}$ over a Hilbert space $\mathcal{H}$ such that there exists a $c \geq 0$ such that, for each $f \in \mathcal{H}$, $\|Af\| \leq c\|f\|$.

Definition A.0.7. Let $A \in B(\mathcal{H})$ be a bounded operator over a Hilbert space $\mathcal{H}$. The *adjoint* of A is the operator $A^* \in B(\mathcal{H})$ such that $\langle f \mid Ag\rangle = \langle A^* \mid g\rangle$ for all $f, g \in \mathcal{H}$. The operator $A \in B(\mathcal{H})$ is called *self-adjoint* if and only if $A = A^*$.

Definition A.0.8. Let $A \in B(\mathcal{H})$ be a bounded operator over a Hilbert space $\mathcal{H}$. The *adjoint* of A is the operator $A^* \in B(\mathcal{H})$ such that $\langle f \mid Ag\rangle = \langle A^* \mid g\rangle$ for all $f, g \in \mathcal{H}$. The operator $A \in B(\mathcal{H})$ is called *self-adjoint* if and only if $A = A^*$.

Proposition A.1. *The set of all bounded operators over $\mathcal{H}$, denoted by $B(\mathcal{H})$, is a Banach space with norm,*

$$\|A\| = \sup_{0 \neq f \in \mathcal{H}} \frac{\|Af\|}{\|f\|}. \tag{A.3}$$

Proof. See [53, Theorem III.2]. ■

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