GRAPH PRODUCTS OF GROUPS

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Now unto the King eternal, immortal, invisible, the only wise God, be honour and glory for ever and ever. Amen.

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SUMMARY

In the 1970's Baudisch introduced the idea of the semifree group, that is, a group in which the only relators are commutators of generators. Baudisch was mainly concerned with subgroup problems, employing length arguments on the elements of these groups. More recently Droms and Servatius have continued the study of semifree, or graph groups, as they call them. They answer some of the questions left open by the work of Baudisch. It is possible to take the graph analogy a level higher and study *graph products* of groups, which not only generalise graph groups, but also free and direct products. In this thesis we seek to explore the properties of graph products of groups.

After some preliminaries in chapter 1, chapter 2 quotes the main results from the work of Baudisch, Droms and Servatius on graph groups, and includes a few elementary results. In chapter 3 we show that many of the well known results about free products and direct products will generalise to graph products. We also extend some of the results on graph groups and give a counter example to a plausible conjecture. We develop a normal form for elements in graph products and, with the use of a generalised free product representation, show solvability of the word and conjugacy problems.

In chapter 4 we examine the concept of *graphological indecomposability.*

Having disposed of an obvious conjecture by way of a counter example we present a number of isomorphism theorems.

Chapter 5 is devoted to residual properties of graph products. Much work has been done by Stebe, Allenby and others on residual finiteness, conjugacy separability and potency of free groups and free products. We generalise some of these results.

Finally, in chapter 6 we return to graph groups for a look at the Freiheitssatz. Various subclasses have been covered by Pride, Baumslag and Howie, and we seek to extend their results.

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INTRODUCTION

In the 1970's Andreas Baudisch [3], introduced the idea of the semifree group, that is, a group in which the only relators are commutators of generators. Baudisch was mainly concerned with subgroup problems, employing length arguments on the elements of these groups. In the last few years Carl Droms and Herman Servatius have continued the study of semifree, or graph groups, as they call them. Here the commutator relators are determined by those pairs of adjacent vertices in an associated finite simple graph. With a new name and a new vision they keep the graphological concepts close to the heart of their work and answer some of the questions left open by the work of Baudisch. It is possible to take the graph analogy a level higher and study *graph products* of groups, which not only generalise graph groups, but also free and direct products. In this thesis we seek to explore the properties of graph products of groups.

After some preliminaries in chapter 1, chapter 2 quotes the main results from the work of Baudisch, Droms and Servatius on graph groups, and includes a few elementary results. In chapter 3 we show that many of the well known results about free products and direct products will generalise to graph products. We also extend some of the results on graph groups and give a counter example to a plausible conjecture. We develop a normal form for elements in graph products and, with the use of a generalised free product representation, show solvability of the word and conjugacy problems.

In chapter 4 we examine the concept of *graphological indecomposability.* We pose the question *Is a representation of a group as a graph product of graphologically indecomposable groups essentially unique?* Although the answer is *no*, we can obtain an *isomorphism theorem* in certain special cases.

Chapter 5 is devoted to residual properties of graph products. Much work has been done by Stebe, Allenby and others on residual finiteness, conjugacy separability and potency of free groups and free products. We generalise some of these results.

Finally, in chapter 6 we return to graph groups for a look at the Freiheitssatz. Various subclasses have been covered by Pride, Baumslag and Howie, and we seek to extend their results.

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CHAPTER 1 NOTATION AND PRELIMINARIES

In this chapter we set out the basic definitions required in the thesis. We will introduce notation which will be used later without reference. Firstly we make the following graph theoretic definitions, mainly from [31]:

Definitions 1.1

A *finite simple graph* G is a pair $(V(G), E(G))$, where $V(G)$ is a non empty finite set of elements called *vertices* and $E(G)$ is a finite set of unordered pairs of distinct elements of $V(G)$ called *edges*. We will refer to $V(G)$ as the *vertex set* and $E(G)$ as the *edge set*. (We specify that $E(G)$ is a set so that there can never be more than one edge joining a given pair of vertices). Henceforth a graph will be assumed to be finite and simple.

Two vertices v and w of a graph G are said to be *adjacent* if there is an edge joining them, that is, if $\{v, w\} \in E(G)$; the vertices v and w are then said to be *incident* to such an edge. Similarly, two distinct edges of G are said to be *adjacent* if they have a vertex in common. An edge is said to be incident to a vertex v if v is incident to that edge. The *degree* of a vertex v of G is the number of edges which are incident to v. Any vertex of degree zero is called an *isolated vertex*. A graph in which every vertex has the same degree is called a *regular graph*. If the degree of each of the vertices is r we say the graph is *regular of degree r*.

Two graphs G_1 and G_2 are *isomorphic* if there is a one-one correspondence between the vertices of G_1 and those of G_2, with the property that the number of edges joining any two vertices of G_1 is equal to the number of edges joining the corresponding vertices of G_2.

A *subgraph* of a graph G is a graph such that all of its vertices belong to $V(G)$ and all of its edges belong to $E(G)$. A graph whose edge set is empty is called a *discrete graph.* A graph in which every pair of distinct vertices are adjacent is called a *complete graph.*

There are several ways of joining two graphs together to make a larger one; we shall use one of these methods.

Definitions 1.2

Let

$$G_1 = (V(G_1), E(G_2)), \qquad G_2 = (V(G_2), E(G_2)),$$

be two graphs where $V(G_1)$ and $V(G_2)$ are assumed to be disjoint, then their *union* $G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$.

A graph is *connected* if it cannot be expressed as the union of two subgraphs, otherwise it is *disconnected.* It is clear that any disconnected graph G can be expressed as the union of a finite number of uniquely determined connected subgraphs. Each of these connected subgraphs is called a *connected component* of G.

A connected graph which is regular of degree two is called a *circuit graph*, the circuit graph on n vertices being denoted by C_n. A graph which has two vertices of degree one and the rest of degree two is called a *line graph.* The line graph with m edges is denoted L_m.

We define the *rank* of a graph G to be the number of vertices in the vertex set, and we denote it by $|G|$.

If a graph G has no subgraph which is a circuit graph then we call G a *forest.* If G is a connected forest we call G a *tree.* As all of our graphs are assumed to be finite, a forest is thus a finite collection of trees.

Let G be the graph defined by the pair $(V(G), E(G))$. Let $V(H)$ be a subset of $V(G)$ and $E(H)$ a subset of $E(G)$ such that

$$\forall v, w \in V(H) \qquad \{v, w\} \in E(G) \Leftrightarrow \{v, w\} \in E(H).$$

We call the subgraph $H = (V(H), E(H))$ a *full subgraph* of G.

We will assume the reader is familiar with the definitions of:

group, (normal) subgroup, homomorphism, isomorphism, order of an element, order of a group, set of generators, relator, direct product, free group, abelian group, free abelian group. We will write $G = \langle A; R \rangle$ where G is the group generated by the set A, and with a set of relators R. We will denote the direct product of the groups A, B by $A \times B$.

We make the following well known definitions:

Definitions 1.3

Let N be a subgroup of a group G. We define the *normal closure* of

N in G, denoted N^G to be the smallest normal subgroup of G containing N.

Let a, b be elements of a group G. We define the *commutator* of a and b, denoted $[a, b]$ to be the element $a^{-1}b^{-1}ab$. If $\exists v \in G$ such that $v^{-1}av = b$ we say that a and b are *conjugate* in G and write $a \sim b$.

A *word* in the generators of $G = \langle A; B \rangle$ is merely a sequence of letters from A, and their inverses. Such a word is said to be *freely reduced* if it contains no subsequence aa^{-1} nor $a^{-1}a$, where $a \in A$.

Let G be a group with normal subgroup N. We write $N \lhd G$. If $\frac{G}{N}$ is finite we say that N is of *finite index* in G, and G is a *finite extension* of N. In general if $N \lhd G$ and $\frac{G}{N} \equiv H$, we say that G is an *extension* of N by H.

A group is said to be *torsion free* if every non identity element of the group has infinite order. A group is said to have *torsion* if it is not torsion free.

A *simple group* is a group in which no proper non trivial subgroup is normal. A *p-group*, (p prime), is a finite group in which the order of each element is some power of p.

Let P be a property appertaining to groups. A group G is said to be locally P if every finitely generated subgroup of G has P.

We now state the decision problems posed by Max Dehn in 1911.

Definitions 1.4

Word Problem. Let G be a group with given presentation. For an arbitrary word W in the generators, decide in a finite number of steps whether or not W defines the identity element of G. If this can be done we say that the given presentation of G has a *solvable word problem.*

The word problem has been solved for certain special cases, for example free groups and 1-relator groups, but it is known, [23], that there is no procedure for solving the word problem which will work for every presentation.

Conjugacy Problem. Let G be a group with given presentation. For two given words W_1, W_2 in the generators of G, decide in a finite number of steps whether or not they define conjugate elements of G. If this can be done we say that the given presentation of G has a *solvable conjugacy problem.*

Setting W_1 to the empty word this reduces to the word problem, thus the conjugacy problem is known to be solvable only for a more restricted class of groups. This incudes free groups and some other special classes, but it is unknown whether the conjugacy problem is solvable for 1-relator groups.

Isomorphism Problem. Let G be a group with given presentation. For an arbitrary group G' defined by means of another presentation, decide in a finite number of steps whether or not G is isomorphic to G'. If this can be done we say that the presentation of the group has a *solvable isomorphism problem.*

This is very difficult. Even if G has presentation $\langle a; a\rangle$, clearly the trivial group, the isomorphism problem is unsolvable. The problem is solvable for presentations without relators, and other special classes.

We now define various products and extensions of groups:

Definitions 1.5

The *free product* of the groups

$$A = \langle a_1, a_2, \ldots, a_n; r_1, r_2, \ldots, r_p\rangle, \quad B = \langle b_1, b_2, \ldots, b_m; s_1, s_2, \ldots, s_q\rangle$$

is defined as

$$A * B = \langle a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m; r_1, r_2, \ldots, r_p, s_1, s_2, \ldots, s_q\rangle.$$

A and B are known as *free factors* of $A * B$. If G is the group with presentation

$$\langle a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m; r_1, r_2, \ldots, r_p, s_1, s_2, \ldots, s_q, u_1(a_\nu) = v_1(b_\mu), \ldots\rangle,$$

and A is the subgroup generated by $a_1, a_2, \ldots, a_n$, B is the subgroup generated by $b_1, b_2, \ldots, b_m$, H is the subgroup of A generated by $u_1(a_\nu), \ldots$ and K is the subgroup of B generated by $v_1(b_\mu), \ldots$, such that $H \equiv K$, then G is called the *free product of A and B with the subgroups H and K amalgamated under the mapping $u_i(a_\nu) \to v_i(b_\mu)$*. We also refer to G as a *generalised free product.*

Let G be a group with subgroups A and B such that $\phi : A \to B$ is an isomorphism. The *HNN extension of G relative to A, B and ϕ* is the group

$$G^* = \left\langle G, t; t^{-1}at = \phi(a),\ a \in A \right\rangle .$$

G is called the *base* of G^*, t the *stable letter* and A and B are called the associated subgroups.

Finally we state the definitions of some residual properties which we will consider later:

Definitions 1.6

Let P be a property applicable to groups. The group G is said to have the property P *residually* if $\forall x \in G\ \exists$ a normal subgroup N_x such that $x \notin N_x$ and $\frac{G}{N_x}$ has the property P. It is clear that G is *residually finite* if and only if $\forall g \in G\ \exists$ a finite homomorphic image of G such that the image of g is not the identity.

A group G is said to be *(locally) extended residually finite* or (L)ERF if for each (finitely) generated subgroup H of G, and for each $g \in G \backslash H\ \exists$ a homomorphism ϕ of G onto a finite group such that $\phi(g) \notin \phi(H)$, Burns, [6].

A group G is said to be π_c if for each cyclic subgroup H of G, and for each $g \in G \backslash H\ \exists$ a homomorphism ϕ of G onto a finite group such that $\phi(g) \notin \phi(H)$, Stebe, [27]. So clearly all LERF groups are π_c, or briefly LERF $\subseteq \pi_c$.

A group is said to be *indicable* if it has an infinite cyclic homomorphic image. Such groups were studied by Higman, [15], in connection with the zero divisor and unit problems for group rings.

An element a of a group G is called *conjugacy distinguished* in G if, given any element b of G, either $a \sim b$ or $\exists$ a finite homomorphic image H of G such that the images of a and b are not conjugate in H, Stebe [28]. The group G is said to be *conjugacy separable*, Mostowski [22], if all of its elements are conjugacy distinguished.

We call a group G *potent* if, given $g \in G$ and given $z \in \mathbb{Z}$, $z > 0$, such that z divides the order of g, then there exists a finite homomorphic image of G in which the image of g has order precisely z, Allenby [1].

The concepts of residual finiteness and conjugacy separability are useful in connection with the word and conjugacy problems. Namely, from [20] and [22] respectively, we have

THEOREM 1.7. *Finitely presented residually finite groups have a solvable word problem.*

THEOREM 1.8. *Conjugacy separable groups have a solvable conjugacy problem.*

CHAPTER 2 GRAPH GROUPS

Definition 2.1

Let G be a group with presentation $\langle A; R\rangle$ where A is a set of generators and R, the set of relators, only contains words which are commutators of generators. We call G a *Graph Group.*

An alternative definition is :-

Let Γ be a graph with vertex set V and edge set E. Define $G\Gamma$, the *Graph Group* corresponding to the graph Γ to be the group given by $\langle V; R\rangle$ where $R = \{[v_i, v_j] : \{v_i, v_j\} \in E\}$. Clearly the two definitions are equivalent.

There are two extreme cases for R:

(i) R is the empty set, (that is the graph Γ is null). In this case G is the free group generated by the set A.

(ii) R contains the commutators of all pairs of generators of G, (that is the graph Γ is complete). In this case G is the free abelian group with basis A.

From (i) and (ii) we see that graph groups fall between free and free abelian groups, and for this reason they are called *Semifree Groups* by Andreas Baudisch, [3].

In [3], Baudisch proved the following:

THEOREM 2.2. *Let G be a graph group. Let $u, v \in G$ such that $[u,v] \neq 1$. Then $\{u,v\}$ is a basis of a free group of rank 2.*

Now for $u, v \in G$ with $[u, v] = 1$, $\{u, v\}$ clearly is a basis of a free abelian group of rank 2. Thus Baudisch has shown that all 2 generator subgroups of graph groups are again graph groups.

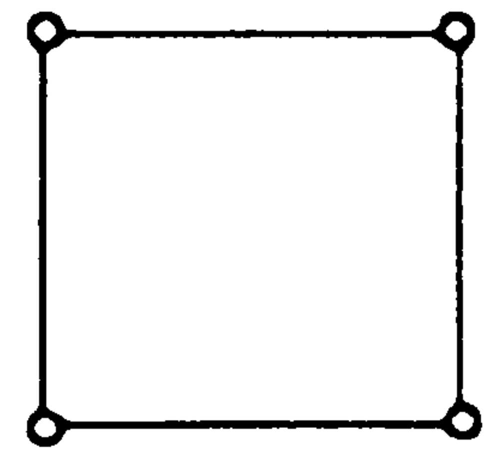
Baudisch [3] also found a normal form for elements $u, v \in G$ such that $[u, v] = 1$, and proved:

THEOREM 2.3. *Every abelian subgroup of a graph group is free abelian.*

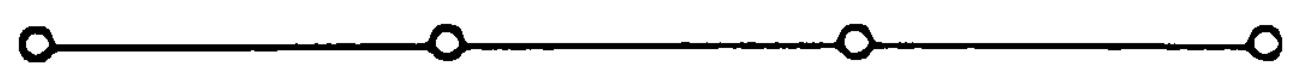
It is well known that subgroups of free groups are free, and subgroups of free abelian groups are free abelian. Can this be generalised to subgroups of graph groups are graph groups? Baudisch [3] answered this question with examples of 3 generator and 4 generator subgroups of graph groups which are not themselves graph groups.

However, the question was more fully dealt with by Carl Droms [7] who obtained the following result.

THEOREM 2.4. *Let X be a graph and GX a corresponding graph group. Every finitely generated subgroup of GX is a graph group $\iff$ X has no full subgraph of either of the two following forms:*



C_4



L_3

In the proof of this theorem Droms used the following lemma which can be exploited further in the study of graph groups whose graphs have no full subgraph of either of the forms C_4, L_3. Following Droms and Servatius, [9], we call these particular graph groups *Special Assembly.* We will also use this lemma in the study of graph products.

LEMMA 2.5. *Let X be a connected graph with no full forbidden subgraphs, that is, no full subgraphs of the forms C_4 or L_3. Then $\exists$ at least one vertex in the vertex set of X which is joined to every other vertex in X.*

The obvious question concerning isomorphisms of graph groups and graphs was resolved by Droms [8] in proving:

THEOREM 2.6. *Let X and Y be finite simple graphs such that $GX \equiv GY$. Then $X \equiv Y$.*

In [9] Droms and Servatius proved the following three results:

THEOREM 2.7. *Let Δ be a subgraph of a finite simple graph Γ. Then the normal closure of $G\Delta$ in $G\Gamma$ is a graph group.*

THEOREM 2.8. *Let T be a finite tree. Then GT is a three manifold group,*

together with a characterisation of all graph subgroups (i.e. subgoups given by full subgraphs of the associated graph of the group) of all tree graph groups.

Using the Power Series Ring method of Magnus, Droms also established the following [10]:

THEOREM 2.9. *Graph Groups are Residually(Finitely Generated Torsion Free Nilpotent).*

From [13] we have the following immediate corollary:

COROLLARY 2.10. *Graph groups are residually finite p, for each prime p.*

Interestingly enough this corollary can be obtained directly, as we will show in a more general setting later. The following result is stated in [9].

PROPOSITION 2.11. *The class of finitely generated special assembly graph groups contains 1 and is closed under the operations $- * -$ and $- \times \mathbb{Z}$, and is the smallest such class.*

PROOF: Let GX be a special assembly graph group with graph X. If X has just one vertex then GX is $\mathbb{Z}$ and thus belongs to the class.

We require to show that every special assembly graph group is *decomposable* into a product of the operations stated above.

We proceed by induction on the number of generators of G, n, that is the number of verticies of X.

Suppose that G has $n > 1$ generators and that all special assembly graph groups with fewer than n generators are decomposable, as claimed. Suppose X is not connected, with connected components $X_1, X_2, \ldots, X_t$. Then clearly we have $GX \equiv GX_1 * GX_2 * \ldots * GX_t$. Now $GX_1, GX_2, \ldots, GX_t$ are special assembly for if any X_i had a full subgraph of either of the forms C_4 or L_3 then this subgraph would also be a full subgraph of X. So, by induction, the GX_i are decomposable, and so also is G.

Now suppose X is connected. Then by Lemma 2.5 $\exists$ at least one vertex z which is joined to every other vertex of X. Let z be such a vertex. Let $X(z)$ denote the subgraph of X with z and all the edges of which z is an end point omitted.

Then $GX \equiv GX(z) \times \mathbb{Z}$.

Now $X(z)$ is a full subgraph of X and so any full subgraph of $X(z)$ is also a full subgraph of X. Thus $GX(z)$ is special assembly, and by induction, decomposable. Thus GX is decomposable, as desired. Hence we have the result by induction.

So, special assembly graph groups are decomposable into products of the stated kind, and clearly form the smallest such class. □

COROLLARY 2.12. *Special Assembly graph groups are Locally Extended Residually Finite (LERF).*

PROOF: We use the following result of [2]:

Let $G = NM$, where $N \cap M = 1$ be a splitting extension of the normal finitely generated subgroup N by the group M. If N is Extended Residually Finite (ERF) and M is LERF, then G is LERF.

The infinite cycle $\mathbb{Z}$ is certainly LERF, and also clearly ERF. For G, the direct product of a LERF group with a copy of $\mathbb{Z}$, we see by the above result that G is LERF since $\mathbb{Z}$ is ERF. The free product of two LERF groups is again LERF by [18].

Hence result by Proposition 2.11. □

Counter Example 2.13

We cannot extend the result above to all graph groups since $GC_4 \equiv F_2 \times F_2$ is not LERF, where F_2 is LERF, as shown in [2].

By contrast we will show later that *all* graph groups are π_c.

We can also glean some information on the conjugacy problem. First we have:

LEMMA 2.14. *The direct product of two Conjugacy Separable groups is again Conjugacy Separable.*

PROOF: Let $G \equiv H \times K$ where H, K are conjugacy separable. Let $x, y \in G$ such that x is not conjugate to y in G. Then $x = (x_1, x_2)$, $y = (y_1, y_2)$, for some $x_1, y_1 \in H$, $x_2, y_2 \in K$ such that there does not exist $(u_1, u_2) \in G$ with $(u_1, u_2)^{-1}(x_1, x_2)(u_1, u_2) = (y_1, y_2)$. So either there does not exist a u_1 with $u_1^{-1} x_1 u_1 = y_1$ in H or there does not exist a u_2 with $u_2^{-1} x_2 u_2 = y_2$ in K.

Suppose without loss of generality, the former. Then, since H is conjugacy separable there exists a finite homomorphic image, H' of H such that the images x_1', y_1' of x_1, y_1 respectively, are not conjugate in H'. We can define a homomorphism $G \to H' \times 1$ in the obvious way, and the images of x and y are not conjugate in this finite group. Thus G is conjugacy separable. □

COROLLARY 2.15. *Special assembly graph groups are conjugacy separable.*

PROOF: Free groups, in particular $\mathbb{Z}$, and free products of conjugacy separable groups are conjugacy separable, by [4]. Further, the direct product of a conjugacy separable group with $\mathbb{Z}$ is conjugacy separable by the comments above, and Lemma 2.14. Hence result by Proposition 2.11. $\square$

We will later see that Corollary 2.15 can be extended somewhat.

Theorem 2.9 allows us to go in several different ways: one in excluding elements from subgroups, one in specifying orders of elements, and one describing homomorphic images of subgroups.

The first of these results is:

THEOREM 2.16. *Let G be a R(fgtfn) group. Then G is Π_c. Thus graph groups are Π_c.*

PROOF: Let G be a R(fgtfn) group. Let $a, u \in G$ such that $a \notin \langle u \rangle$. Since G is R(fgtfn) $\exists M$, a normal subgroup of G such that $a \notin M$ and G/M is fgtfn, and $\exists N$, a normal subgroup of G such that $u \notin N$ and G/N is fgtfn.

Let $L = M \cap N$. As is well known, G/L is isomorphic to a subgroup of $G/M \times G/N$ which is a fgtfn group. Hence G/L is fgtfn, [14].

Let $\bar{a}$ and $\bar{u}$ denote the images of a and u respectively in G/L. If $\bar{a} \notin \langle \bar{u} \rangle$ we are finished, since fgtfn groups are Π_c by [13].

Suppose then, $\bar{a} = \bar{u}^{\alpha}$ for some $\alpha \in \mathbb{Z}$. Then $a^{-1}u^{\alpha} \in L$. Let $1 \neq y \in G$, then $\exists X \lhd G$ such that $y \notin X$ and G/X is fgtfn.

Then $\bar{\bar{G}} = G/(L \cap X)$ is fgtfn, by an embedding argument as above.

Again, if $\bar{\bar{a}} \notin \langle \bar{\bar{u}} \rangle$ we are finished. So, suppose $\bar{\bar{a}} = \bar{\bar{u}}^{\beta}$ for some $\beta \in \mathbb{Z}$. Then $a^{-1}u^{\beta} \in L \cap X \subseteq L$. So $a^{-1}u^{\alpha}$ and $a^{-1}u^{\beta} \in L$. Therefore $u^{\beta-\alpha} \in L$. But $u \notin L$, by choice of N, and since G/L is torsion free, no power of u can belong to L. Thus $u^{\beta-\alpha} = 1$ and so $u^{\beta} = u^{\alpha}$.

Thus $a^{-1}u^{\alpha} \in L \cap X$, for all X such that G/X is fgtfn. Therefore $a^{-1}u^{\alpha} \in L \cap (\cap X)$, where $\cap X$ is the intersection of all possible such X's.

But $\cap X = \langle 1 \rangle$, since G is R(fgtfn). Therefore $a^{-1}u^{\alpha} \in L \cap \langle 1 \rangle = \langle 1 \rangle$, which implies that $a = u^{\alpha}$. Contradiction.

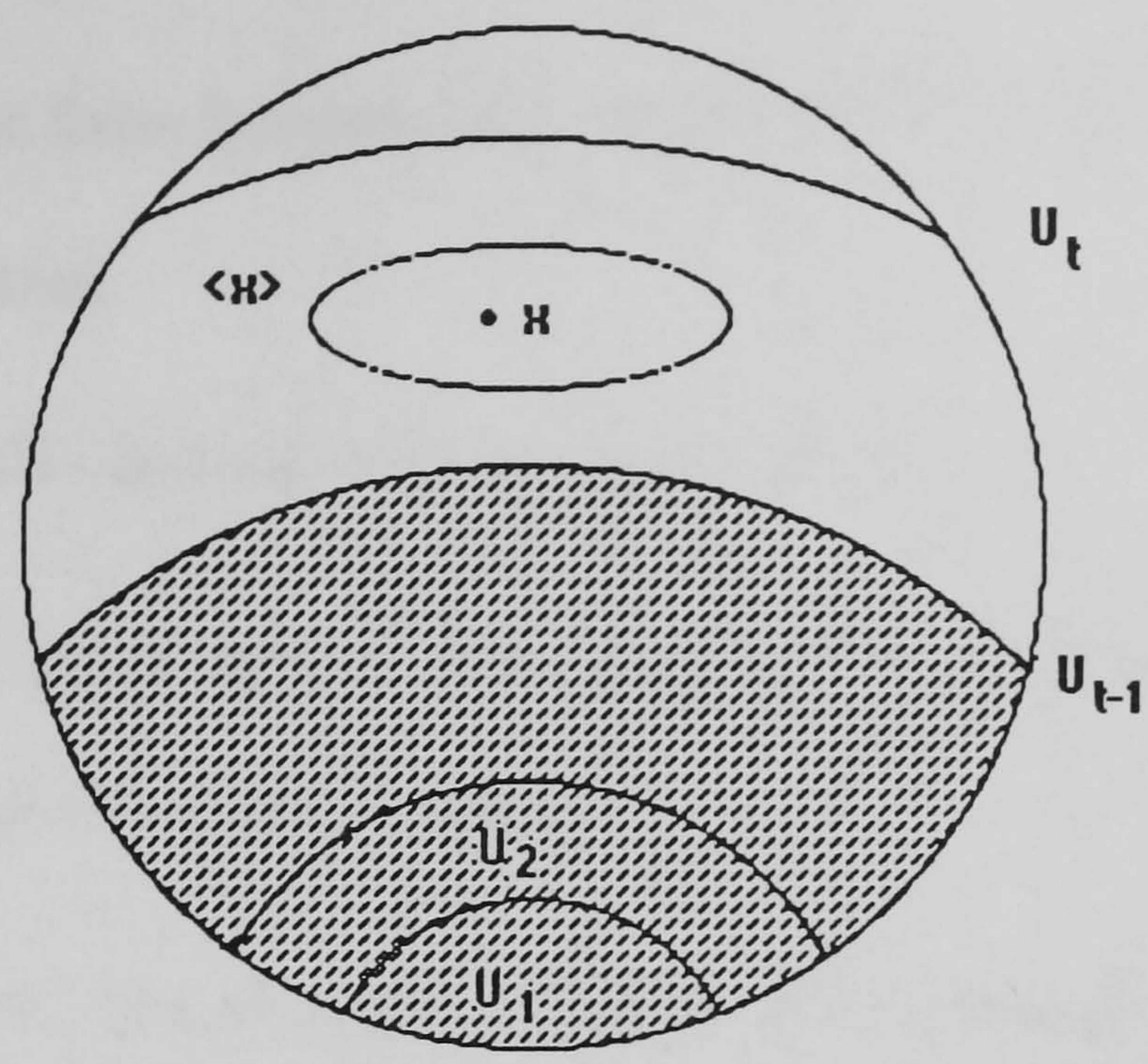
Thus $\bar{\bar{a}} \notin \langle \bar{\bar{u}} \rangle$. So G is Π_c. □

The second result following from Theorem 2.9 is:

THEOREM 2.17. *Let G be a R(fgtfn) group. Then G is potent. Thus graph groups are potent.*

PROOF: Let G be a R(fgtfn) group. Let $1 \neq x \in G$ and $z \in \mathbb{Z}$ be given. (Now x has infinite order since its non trivial image in some factor group of G has infinite order.) We can map G onto $\bar{G}$ such that $\bar{x} \neq 1$ and $\bar{G}$ is fgtfn. Consider the terms of the upper central series of $\bar{G}$,

$1 = U_0, U_1, U_2, \ldots, U_s$, where s is the nilpotency class of $\bar{G}$, U_1 its centre, and U_{k+1}/U_k is the centre of $\bar{G}/U_k$. Since $\bar{G}$ is fgtfn, $\bar{G}/U_k$ is fgtfn, for all k, [25]. Let $t \in \mathbb{Z}$ be such that $\bar{x} \in U_t$ and $\bar{x} \notin U_{t-1}$.



Since $\bar{G}/U_{t-1}$ is torsion free, $\langle \bar{x} \rangle \cap U_{t-1} = \langle 1 \rangle$. Since $\bar{G}/U_t$ is torsion free, $\langle \bar{x} \rangle \cap U_t = \langle \bar{x} \rangle$. So, we can factor out by U_{t-1}. Now $\bar{\bar{x}}$ is in the centre of $\bar{\bar{G}} = \bar{G}/U_{t-1}$. Since $\bar{\bar{G}}$ is fgtfn, C, the centre of $\bar{\bar{G}}$ is a finitely generated torsion free abelian group, and thus free abelian. Let $a_1, a_2, ..., a_r$ be a set of free generators of C, with

$\bar{\bar{x}} = a_1^{\alpha_1} a_2^{\alpha_2} .... a_r^{\alpha_r}$, say.

We can easily choose $\beta_1, \beta_2, ..., \beta_r \in \mathbb{Z}$ such that $\bar{\bar{x}}$ is a z^{th} root of $a = a_1^{\beta_1} a_2^{\beta_2} .... a_r^{\beta_r}$.

Let H be the subgroup of $\bar{\bar{G}}$ generated by a. Then H is a normal subgroup of $\bar{\bar{G}}$. Define $\bar{\bar{G}}'$ as $\bar{\bar{G}}/H$. Then $\bar{\bar{x}}'$ has order n in $\bar{\bar{G}}'$. If $\bar{\bar{G}}'$ is finite we have finished. Suppose not. Then, $\bar{\bar{G}}'$ is fgn and by [13] it is therefore residually finite.

So, for $1 \leq i \leq z-1$

$\exists$ N_i a normal subgroup of finite index in $\bar{\bar{G}}'$, such that $\bar{\bar{x}}' \notin N_i$. But $N = \cap N_i$ is a normal subgroup of finite index in $\bar{\bar{G}}'$. Hence the image of x

has order z in the finite homomorphic image, $\bar{\bar{G}}'/N$, of G.

Thus G is potent. □

The third result obtainable from theorem 2.9 is

THEOREM 2.18. *Let G be a R(fgtfn) group. Then G is locally indicable. Thus graph groups are locally indicable.*

PROOF: Let G be a R(fgtfn) group, and let H be a finitely generated subgroup of G. Let $h \in H$ such that $h \neq 1$. Then $\exists\ \bar{G}$, a fgtfn homomorphic image of G, such that $\bar{h} \neq 1$. Then $\bar{H}$, the image of H in $\bar{G}$, is a non trivial subgroup of $\bar{G}$, thus $\bar{H}$ is fgtfn. Hence if H is nilpotent of class n and $K = \zeta(H)$, then H/K is finitely generated torsion free abelian. Thus G is locally indicable. □

Theorems 2.16 and 2.17 can be extended in a more general setting, as we will see in Chapter 5.

The number of generators of a graph group yields easily to analysis.

PROPOSITION 2.19. *Let X be a graph with a finite number of vertices, n. Let GX be the graph group given by X. Then GX cannot be generated by fewer than n elements.*

PROOF: Let $x_1, x_2, ..., x_n$ denote the vertices of X. So we have a generating set $\{a_1, a_2, ..., a_n\}$ for GX. Then GX/GX' is free abelian on n generators, and no fewer. Hence GX needs at least n elements in any generating set.

□

CHAPTER 3 PRELIMINARY RESULTS ON GRAPH PRODUCTS

Definition 3.1

Let Γ be a graph, with vertex set $V = \{ v_i \; ; \; i = 1, 2, \ldots, n\}$. Let A_k be groups, $k = 1, 2, \ldots, n$, and let $\phi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ be a permutation.

Let $G\Gamma = \langle A_k; \; [A_{\phi(i)}, A_{\phi(j)}], \; \forall \; \{v_i, v_j\} \in E \rangle$, where E is the edge set of Γ. Thus $G\Gamma$ is the quotient of the free product of the A_k by the normal closures of the appropriate commutator subgroups.

We call $G\Gamma_\phi$ the *graph product* of the A_k given by Γ and ϕ. We call Γ the *underlying graph*, ϕ the *group vertex assignment*, and the A_k *generating groups* of $G\Gamma_\phi$. When ϕ is understood we simply denote the $A_{\phi(i)}$ by A_i.

Note 3.2

The above concept generalises that of direct and free products, these two products correspond to the cases of Γ being complete and null, respectively.

We also note that if $A_k \equiv \mathbb{Z}$ for $k = 1, 2, \ldots, n$, then $G\Gamma$ is the graph group corresponding to the graph Γ. Thus graph groups are special graph products.

All three of these special cases are irrespective of group vertex assignment.

LEMMA 3.3. *Let Γ be a finite simple graph on n vertices; $A_1, A_2, \ldots, A_n$ groups and ϕ a group vertex assignment. Then the graph product $G\Gamma$ of* *the A_k given by Γ and ϕ is uniquely determined by $A_1, A_2, \ldots, A_n$, Γ and ϕ. That is, it is independent of choice of presentation of $A_1, A_2, \ldots, A_n$.*

Also $G\Gamma$ is generated by subgroups B_i where $B_i \equiv A_i$ for $i = 1, 2, \ldots, n$ and $B_i \cap gp\{B_j : i \neq j\} = \langle 1 \rangle \quad \forall i$.

PROOF: Let $C_1, C_2, \ldots, C_n$ and $D_1, D_2, \ldots, D_n$ be alternative presentations for $A_1, A_2, ..., A_n$ respectively. Let $\alpha_k : C_k \to D_k$ be isomorphisms from the groups given by the C_k to those given by the D_k. Let $H\Gamma$, $J\Gamma$ be the graph products of the C_k, D_k respectively, given by Γ and ϕ.

Define $\alpha : H\Gamma \to J\Gamma$ by

$\alpha(c_k) = \alpha_k(c_k)$, where $c_k \in C_k$ for $k = 1, 2, \ldots, n$;

α is clearly a homomrphism as it takes relators to relators. Similarly if we define

$\bar{\alpha} : J\Gamma \to H\Gamma$ by

$\bar{\alpha}(d_k) = \alpha_k^{-1}(d_k)$ where $d_k \in D_k \quad k = 1, 2, \ldots, n$;

this also is a homomorphism and clearly α is one-one and onto with $\bar{\alpha} = \alpha^{-1}$. Thus $H\Gamma \equiv J\Gamma$.

Let B_i be the subgroup of $G\Gamma$ generated by the set $\{a_i : a_i \in A_i\}$, for $i = 1, 2, \ldots, n$. Certainly $G\Gamma$ is generated by the B_i. To show $B_i \equiv A_i$ consider the mappings $\beta_i : A_i \to B_i$ defined by

$\beta_i(a_i) = a_i$ for $i = 1, 2, \ldots, n$.

The projection Π_{A_i} of $G\Gamma$ onto A_i defined by

$\Pi_{A_i}(a_i) = a_i$ for $a_i \in A_i$ and $\Pi_{A_i}(a_j) = 1$ for $a_j \in A_j$, $j \neq i$, is such that $\Pi_{A_i}\beta_i$ is the identity on A_i. Hence $A_i \equiv B_i$.

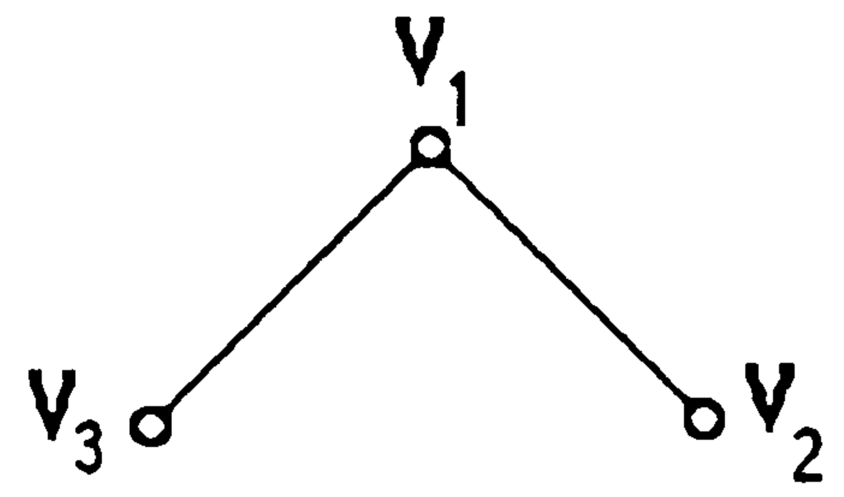
Since $\Pi_{A_i}\beta_i$ maps all elements of B_j $j \neq i$ to 1, we have $B_i \cap gp\{B_j : i \neq j\} = \langle 1 \rangle \quad \forall i \neq j$.

Thus we identify the A_i with the B_i and consider the A_i as subgroups of $G\Gamma$. □

We see very easily, by the following example, that the choice of group vertex assignment affects the graph product.

Example 3.4

Let Γ be the graph on three vertices as shown below.



Recall $C(G)$ denotes the centre of the group G. Let $A_1 = F_1$; $A_2 = F_2$; $A_3 = F_3$; Let ϕ, ψ be group vertex assignments with

$\phi(1) = 1,\ \phi(2) = 2,\ \phi(3) = 3;$

$\psi(1) = 2,\ \psi(2) = 3,\ \psi(3) = 1.$

Then $G\Gamma_\phi$ is

$\langle F_1, F_2, F_3; [F_1, F_2],\ [F_1, F_3] \rangle \ \equiv\ F_1 \times (F_2 * F_3)\ \equiv\ F_1 \times F_5.$

And $G\Gamma_\psi = \langle F_1, F_2, F_3; [F_2, F_3],\ [F_2, F_1] \rangle \ \equiv\ F_2 \times (F_3 * F_1)\ \equiv\ F_2 \times F_4.$

And $G\Gamma_\psi = \langle F_1, F_2, F_3; [F_2, F_3],\ [F_2, F_1]\rangle \equiv F_2 \times (F_3 * F_1) \equiv F_2 \times F_4$.

Now $G\Gamma_\phi \not\equiv G\Gamma_\psi$ since $C(G\Gamma_\phi) \equiv \mathbb{Z}$, but $C(G\Gamma_\psi)$ is trivial.

In order to develop a normal form for elements in a graph product, as an analogue of a reduced sequence in a free product, we introduce a reduced sequence in a graph product.

Definition 3.5

Let $G\Gamma$ be a graph product of the groups $A_1, A_2, \ldots, A_n$. A sequence $g_1, g_2, \ldots, g_m$ of elements from $G\Gamma$ is called *reduced* if

(i) $g_i \in A_{i_s}$ for $i = 1, 2, ...n$.

(ii) $g_i \neq 1$ for $i = 1, 2, ...n$.

(iii) $\forall g_i, g_j$ with $i < j$ for which $\exists k\ \ i \leq k < j$ with

$[g_i, g_{i+1}] = [g_i, g_{i+2}] = \cdots = [g_i, g_k] = 1$ and

$[g_{k+1}, g_j] = [g_{k+2}, g_j] = \ldots [g_{j-1}, g_j] = 1$, g_i and g_j are in different generating groups.

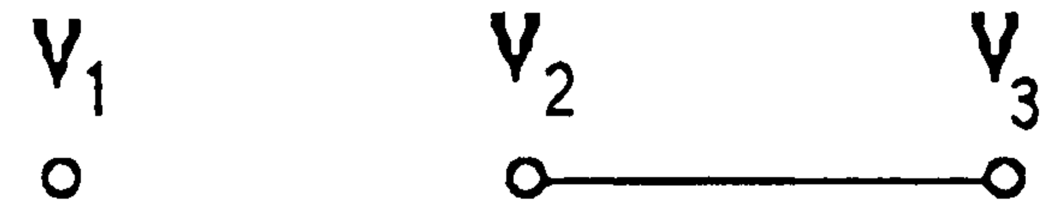
Part (iii) means that repeatedly swapping elements from generating groups whose corresponding vertices are adjacent in Γ cannot bring together two terms from the same generating group. We will refer to this swapping of commuting terms as *syllable shuffling* in this and other contexts. In particular, (iii) includes the corresponding condition for free products, that $\forall i,\ g_i, g_{i+1}$ are not from the same free factor; here $k = i$.

We introduce an equivalence for reduced sequences, denoted $\equiv_E$, generated by

$g_1, g_2, \ldots, g_i, g_{i+1}, \ldots, g_m \equiv_E g_1, g_2, \ldots, g_{i-1}, g_{i+1}, g_i, g_{i+2}, \ldots, g_m \iff$ $[g_i, g_{i+1}] = 1$, that is, g_i, g_{i+1} belong to generating groups whose corresponding vertices are adjacent in the graph Γ. We note that g_i, g_{i+1} cannot belong to the same generating group since the sequence is reduced. Thus two reduced sequences are equivalent if and only if one can be obtained from the other by repeated syllable shuffling.

Example 3.6

Let Γ be the graph



and A, B, C the groups with presentation $A = \langle a; a^2 \rangle$; $B = \langle b; b^3 \rangle$; $C = \langle c; c^7 \rangle$. We have a group vertex assignment ϕ, such that A, B, C are assigned to v_1, v_2, v_3 respectively.

So, $G\Gamma_\phi = \langle a, b, c; a^2, b^3, c^7, [b, c] \rangle$.

The sequence c^{-1}, a, b, c, b is not reduced since the last syllable b will shuffle forward giving c^{-1}, a, b, b, c.

However, the sequence c^{-1}, a, b^2, c *is* reduced as adjacent terms are from different generating groups and the terms are non-identity. Also shuffling c forward of the b^2 term, we notice that a, c are from different generating groups. We cannot bring c^{-1} and c together as c and a do not commute. We note that $c^{-1}, a, b^2, c \equiv_E c^{-1}, a, c, b^2$.

Definition 3.7

Let $G\Gamma$ be a graph product of the groups $A_1, A_2, \ldots, A_n$. Let W be a word in the generators of the A_i.

If $W = W_1 W_2 \ldots W_r$ where each W_j is a word in the generators of only one of the generating groups, no W_j is the empty word, and W_j, W_{j+1} are not in the same generating group for $j = 1, 2, \ldots, r-1$, then the *syllable length*, $\lambda(W)$ of W is r, and $W_1, W_2, \ldots, W_r$ are called the *syllables* of W. The *syllable length* $\lambda(g)$ of an element g in $G\Gamma$ is the minimum syllable length of a word defining g.

Thus the element 1 of $G\Gamma$ has syllable length 0, an element g in one of the generating groups has syllable length 1, where $g \neq 1$, and an element h not in a generating group has syllable length ≥ 2.

We note that syllable shuffling in words may effect a reduction in syllable length. We say that a word W is *reduced* if no word representing the same group element has lesser syllable length. So a word is reduced if and only if the sequence obtained from the group elements represented by its syllables is a reduced sequence.

Example 3.8

We continue example 3.6. Let W_1 be the word $abcb^3$. This has syllable length 4, but represnts the same element as the word $ab^4c = abc$ which has syllable length 3. The group element defined by W_1 is clearly also of syllable length 3.

Let W_2 be the word ab^2ac^3b. This has syllable length 5, and so does the group element it represents, as no amount of syllable shuffling can bring two syllables from the same generating group together.

We formalise these ideas in the following theorem.

THEOREM 3.9 THE NORMAL FORM THEOREM FOR GRAPH PRODUCTS.

Let $G\Gamma$ be a graph product of the groups $A_1, A_2, \ldots, A_n$. Each element $g \neq 1$ of $G\Gamma$ can be uniquely expressed as a product

$$g = g_1 g_2 \ldots g_r \qquad \text{where } g_1, g_2, \ldots, g_r \text{ is a reduced sequence.}$$

PROOF: Since $G\Gamma$ is generated by $A_1, A_2, \ldots, A_n$, any element g can be expressed as a product $h_1 h_2 \ldots h_m$ where each $h_i \in$ a generating group.

Let $g_1 g_2 \ldots g_r$ be such a product, expressing g with the smallest number r of terms. Then $g_1, g_2, \ldots, g_r$ is a reduced sequence; for if $g_i = 1$, for some i, then we can surely express g as a product of $< r$ terms. Also, if syllable shuffling can bring together two elements from the same generating group, g_i, g_j, say, then

$$g_1 g_2 \cdots g_{i-1} g_{i+1} \cdots g_k (g_i g_j) g_{k+1} \cdots g_{j-1} g_{j+1} \cdots g_r$$

is a product expressing g with fewer than r terms. Hence every g can be expressed in the reduced form.

It remains to show uniqueness, up to syllable shuffling, that is, up to equivalence of the reduced sequences concerned. To do this we introduce a process ρ to reduce any word W to a sequence and thence to a reduced sequence. We can adapt to our case the process used in proving the same result in free products.

Definition 3.10

(As before we will use the symbol $\equiv_E$ to denote *equality up to syllable shuffling.*) The *reduced form* $\rho(g_1, g_2, \ldots, g_R)$ of a sequence of elements $g_1, g_2, \ldots, g_R$ where each g_i is in a generating group, is defined inductively as follows:

ρ(empty sequence) = empty sequence

$$\rho(g_1) = \begin{cases} \text{empty sequence}, & \text{if } g_1 = 1; \\ g_1, & \text{if } g_1 \neq 1. \end{cases}$$

If $g_1, g_2, \ldots, g_m$ has reduced form $h_1, h_2, \ldots, h_r$, that is

$\rho(g_1, g_2, \ldots, g_m) = h_1, h_2, \ldots, h_r$, then

$\rho(g_1, g_2, \ldots, g_m, g_{m+1})$

$$= \begin{cases} h_1, h_2, \ldots, h_r, & \text{if } g_{m+1} = 1; \\ h_1, h_2, \ldots, h_{j-1}, h_{j+1}, \ldots h_r, & \text{if } \exists j \text{ such that } h_j \text{ shuffles} \\ & \quad \text{to the end and } g_{m+1} = h_j^{-1}; \\ h_1, h_2, \ldots, h_{j-1}, h_{j+1}, \ldots h_r, (h_j g_{m+1}), & \text{if } \exists j \text{ such that } h_j \text{ shuffles} \\ & \quad \text{to the end and } g_{m+1} \neq h_j^{-1} \\ & \quad \text{but } g_{m+1} \text{ and } h_j \text{ are in the} \\ & \quad \text{same generating group;} \\ h_1, h_2, \ldots, h_r, g_{m+1}, & \text{if } g_{m+1} \text{ is in a different} \\ & \quad \text{generating group from any} \\ & \quad \text{term which can shuffle to the} \\ & \quad \text{end.} \end{cases}$$

So ρ is defined for words in generators of the generating groups as follows:

If $W_1, W_2, \ldots, W_m$ are the syllables of W, and g_i is the element defined in a generating group by W_i for $i = 1, 2, \ldots, m$, then

$\rho(W_1 W_2 \ldots W_m) = \rho(g_1, g_2, \ldots, g_m)$.

We can now extend our equality, $\equiv_E$, up to syllable shuffling, to all sequences, and indeed, to words. Let s_1 and s_2 be sequences of elements, then

$$s_1 \equiv_E s_2 \iff \rho(s_1) \equiv_E \rho(s_2).$$

Similarly for words w_1, w_2,

$$w_1 \equiv_E w_2 \iff \rho(w_1) \equiv_E \rho(w_2).$$

Example 3.11

We continue example 3.6. Let $W = a^3 b^{-1} b^2 c^6 b b^{-2} a^{-1} a c^2$. To compute $\rho(W)$ we break W into its syllables,

$W = a^3.b^{-1}b^2.c^6.bb^{-2}.a^{-1}a.c^2$ and obtain the sequence $a, b, c^6, b^{-1}, 1, c^2$.

We then compute successively:

$\rho(W_1) = \rho(a) = a$

$\rho(W_1 W_2) = \rho(a, b) = a, b$

$\rho(W_1 W_2 W_3) = \rho(a, b, c^6) = a, b, c^6$

$\rho(W_1 W_2 W_3 W_4) = \rho(a, b, c^6, b^{-1}) = a, c^6$

$\rho(W_1 W_2 W_3 W_4 W_5) = \rho(a, b, c^6, b^{-1}, 1) = a, c^6$

$\rho(W_1 W_2 W_3 W_4 W_5 W_6) = \rho(a, b, c^6, b^{-1}, 1, c^2) = a, c^6 c^2 = a, c$.

The theorem will easily follow once we establish the following properties of ρ:

(a) $\rho(g_1, g_2, \ldots, g_m)$ is a reduced sequence of length at most m.

(b) If $\rho(g_1, g_2, \ldots, g_m) \equiv_E h_1, h_2, \ldots, h_r$ then $g_1 g_2 \ldots g_m \equiv_E h_1 h_2 \ldots h_r$.

(c) If $g_1, g_2, \ldots, g_m$ is a reduced sequence, then $\rho(g_1, g_2, \ldots, g_m)$

$= g_1, g_2, \ldots, g_m$.

(d) $\rho(g_1, g_2, \ldots g_k, g_{k+1}, \ldots, g_m) = \rho(\rho(g_1, g_2, \ldots, g_k), g_{k+1}, \ldots, g_m)$.

(e) $\rho(g_1, g_2, \ldots, g_m, 1) = \rho(g_1, g_2, \ldots, g_m)$.

(f) $\rho(g_1, g_2, \ldots, g_i, 1, g_{i+1}, \ldots, g_m) = \rho(g_1, g_2, \ldots, g_i, g_{i+1}, \ldots, g_m)$.

(g) $a \equiv_E b$ implies $\rho(a, c) \equiv_E \rho(b, c)$, for sequences a, b and c.

(h) If g_j, g_{m+1} are in the same generating group, for $1 \leq j \leq m$, and g_j shuffles to the end, that is, $[g_j, g_{j+1}] = [g_j, g_{j+2}] = \cdots = [g_j, g_m] = 1$, then $\rho(g_1, g_2, \ldots, g_m, g_{m+1}) \equiv_E \rho(g_1, g_2, \ldots g_{j-1}, g_{j+1}, \ldots, g_m, g_j \cdot g_{m+1})$.

(i) If $\exists\ i \leq k < j$ such that g_i, g_j are in the same generating group, and $[g_i, g_{i+1}] = [g_i, g_{i+2}] = \cdots = [g_i, g_k] = [g_j, g_{k+1}] = [g_j, g_{k+2}] = \cdots = [g_j, g_{j-1}] = 1$ then

$\rho(g_1, g_2, \ldots, g_{i-1}, g_i, g_{i+1}, \ldots, g_{j-1}, g_j, g_{j+1}, \ldots, g_m) \equiv_E$

$\rho(g_1, g_2, \ldots, g_{i-1}, g_{i+1}, \ldots, g_k, g_i \cdot g_j, g_{k+1}, \ldots, g_{j-1}, g_{j+1}, \ldots, g_m)$.

As in the case of free products the properties (a), (b) and (c) follow immediatley from the definition of ρ using induction on m, (see [19]). Property (d) follows from the inductive definition of ρ and (c), property (e) follows immediately from the definition of ρ; property (f) follows from (d) and (e). Property (g) follows from induction on the sequence length of c. Property (i) follows from (d), (f), (g) and (h), so we require to prove (h).

In order to prove (h) we must first establish the following:

(*) When $[g_i, g_{i+1}] = 1$,

$$\rho(g_1, g_2, \ldots, g_i, g_{i+1}, \ldots, g_m) = \rho(g_1, g_2, \ldots, g_{i+1}, g_i, \ldots, g_m).$$

To show (*), by (d) we need only show $\rho(g_1, g_2, \ldots, g_i, g_{i+1}) = \rho(g_1, g_2, ..., g_{i+1}, g_i)$ where $[g_i, g_{i+1}] = 1$. There are two possible cases:

Case 1 $g_i, g_{i+1} \in$ same generating group.

We note that if $g_i = g_{i+1}$ there is nothing to prove, so we may assume $g_i \neq g_{i+1}$. Suppose, firstly, that g_i or $g_{i+1} = 1$. Without loss of generality take g_i. Then $\rho(g_1, g_2, \ldots, g_i, g_{i+1})$

$= \rho(g_1, g_2, \ldots, g_{i-1}, g_{i+1})$, by (f)

$= \rho(g_1, g_2, \ldots, g_{i-1}, g_{i+1}, 1)$, by (e)

$= \rho(g_1, g_2, \ldots, g_{i+1}, g_i)$ since $g_i = 1$.

So suppose $g_i, g_{i+1} \neq 1$. Let $\rho(g_1, g_2, \ldots, g_{i-1}) = h_1, h_2, \ldots, h_r$. We have three subcases:

(A) $\exists h_j$ which shuffles to the end of $h_1, h_2, \ldots, h_r$ such that $h_j = g_i^{-1}$.

(B) $\exists h_j$ which shuffles to the end of $h_1, h_2, \ldots, h_r$ such that $h_j \neq g_i^{-1}$, but $h_j, g_i \in$ same generating group.

(C) g_i is in a different generating group from any of the h_j's which can shuffle to the end.

Consider subcase (A):

$$\rho(g_1, g_2, \ldots, g_i) = h_1, h_2, \ldots, h_{j-1}, h_{j+1}, \ldots, h_r.$$

So $\rho(g_1, g_2, \ldots, g_i, g_{i+1}) = \rho(h_1, h_2, \ldots, h_{j-1}, h_{j+1}, \ldots, h_r, g_{i+1})$

$= h_1, h_2, \ldots, h_{j-1}, h_{j+1}, \ldots, h_r, g_{i+1}$ for no other h_k which can commute to the end may belong to the same generating group as g_{i+1}. If it did it would belong to the same generating group as g_i, and also h_j, and $h_1, h_2, \ldots, h_r$ would not be reduced.

Now $\rho(g_1, g_2, \ldots, g_{i-1}, g_{i+1}, g_i)$

$= \rho(h_1, h_2, \ldots, h_r, g_{i+1}, g_i)$

$= \rho(\rho(h_1, h_2, \ldots, h_r, g_{i+1}), g_i)$

$= \rho(h_1, h_2, \ldots, h_{j-1}, h_{j+1}, \ldots, h_r, (h_j g_{i+1}), g_i)$

$= h_1, h_2, \ldots, h_{j-1}, h_{j+1}, \ldots, h_r, (h_j g_{i+1} g_i)$

$= h_1, h_2, \ldots, h_{j-1}, h_{j+1}, \ldots, h_r, (h_j g_i g_{i+1})$

$= h_1, h_2, \ldots, h_{j-1}, h_{j+1}, \ldots, h_r, g_{i+1}.$

Thus $\rho(g_1, g_2, \ldots, g_i, g_{i+1}) = \rho(g_1, g_2, \ldots, g_{i-1}, g_i, g_{i+1})$.

Subcase (B):

$$\rho(g_1, g_2, \ldots, g_i) = h_1, h_2, \ldots, h_{j-1}, h_{j+1}, \ldots, h_r, (h_j g_i).$$

Thus $\rho(g_1, g_2, \ldots, g_i, g_{i+1}) = h_1, h_2, \ldots, h_{j-1}, h_{j+1}, \ldots, h_r, (h_j g_i), g_{i+1} =$ $h_1, h_2, \ldots, h_{j-1}, h_{j+1}, \ldots, h_r, (h_j g_i g_{i+1})$, by definition of ρ,

$= h_1, h_2, \ldots, h_{j-1}, h_{j+1}, \ldots, h_r, (h_j g_{i+1} g_i)$, since g_i, g_{i+1} commute,

$= \rho(h_1, h_2, \ldots, h_{j-1}, h_{j+1}, \ldots, h_r, (h_j g_{i+1}), g_i)$, by definition of ρ.

But $h_1, h_2, \ldots, h_{j-1}, h_{j+1}, \ldots, h_r, h_j g_{i+1} = \rho(h_1, h_2, \ldots, h_r, g_{i+1})$, by definition of ρ, so using (d) we have

$$\rho(g_1, g_2, \ldots, g_i, g_{i+1}) = \rho(h_1, h_2, \ldots, h_r, g_{i+1}, g_i) = \rho(g_1, g_2, ..., g_{i-1}, g_{i+1}, g_i).$$

Subcase (C):

If $g_i = g_{i+1}^{-1}$ the result is obvious, since $\rho(g_1, g_2, \ldots, g_i, g_{i+1}) = \rho(g_1, g_2, \ldots, g_{i-1}) = \rho(g_1, g_2, \ldots, g_{i+1}, g_i)$. Assume $g_1 \neq g_{i+1}^{-1}$.

Then $\rho(g_1, g_2, \ldots, g_i) = h_1, h_2, \ldots h_r, g_i$. Thus

$\rho(g_1, g_2, \ldots, g_i, g_{i+1}) = h_1, h_2, \ldots, h_r, g_i g_{i+1}$, by definition of ρ,

$= h_1, h_2, \ldots, h_r, g_{i+1} g_i$, since g_i, g_{i+1} commute,

$= \rho(g_1, g_2, \ldots, g_{i-1}, g_{i+1}, g_i)$, by symmetry.

Case 2 $\quad g_i, g_{i+1} \notin$ same generating group.

Again we have subcases A,B and C.

As above let $\rho(g_1, g_2, \ldots, g_{i-1}) = h_1, h_2, \ldots, h_r$.

Subcase (A):

In this case $\exists j$ such that h_j shuffles to the end of h_r and $h_j = g_i^{-1}$. Then $\rho(h_1, h_2, \ldots, h_r, g_i, g_{i+1})$

$= \rho(\rho(h_1, h_2, \ldots, h_r, g_i), g_{i+1})$ by (d),

$= \rho(h_1, h_2, \ldots, h_{j-1}, h_{j+1}, \ldots, h_r, g_{i+1})$, by definition of ρ,

$$= \begin{cases} h_1, h_2, \ldots, h_{k-1}, h_{k+1}, \ldots, h_{j-1}, h_{j+1}, \ldots, h_r, & \text{if } \exists k \text{ such that } h_k \text{ shuffles to the end and } h_k = g_{i+1}^{-1}; \\ h_1, h_2, \ldots, h_{k-1}, h_{k+1}, \ldots, h_{j-1}, h_{j+1}, \ldots, h_r, h_k g_{i+1} & \text{if } \exists k \text{ such that } h_k \text{ shuffles to the end and } h_k \neq g_{i+1}^{-1}, \text{ but } h_k, g_{i+1} \in \text{the same generating group}; \\ h_1, h_2, \ldots, h_{j-1}, h_{j+1}, \ldots, h_r, g_{i+1} & \text{if } g_{i+1} \text{ is in a different generating group from any term which can shuffle to the end.} \end{cases}$$

$= \rho(h_1, h_2, \ldots, h_{j-1}, h_j, h_{j+1}, \ldots, h_r, g_{i+1}, g_i)$, by definition of ρ, since $[g_i, g_{i+1}] = 1 \Rightarrow [h_j, g_{i+1}] = 1$, as $h_j = g_i^{-1}$.

Subcase (B):

That is $\exists j$ such that h_j shuffles to the end of h_r and $h_j \neq g_i^{-1}$, but $h_j, g_i \in$ the same generating group.

Then $\rho(\rho(h_1, h_2, \ldots, h_r, g_i), g_{i+1}) = \rho(h_1, h_2, \ldots, h_{j-1}, h_{j+1}, \ldots, h_r, h_j g_i, g_{i+1})$, by (d) and the definition of ρ,

$$= \begin{cases} h_1, h_2, \ldots, h_{k-1}, h_{k+1}, \ldots, h_{j-1}, h_{j+1}, \ldots, h_r, h_j g_i & \text{if } \exists k \text{ such that } h_k \text{ shuffles to the end and } h_k = g_{i+1}^{-1}; \\ h_1, h_2, \ldots, h_{k-1}, h_{k+1}, \ldots, h_{j-1}, h_{j+1}, \ldots, h_r, h_j g_i, h_k g_{i+1} & \text{if } \exists k \text{ such that } h_k \text{ shuffles to the end and } h_k \neq g_{i+1}^{-1}, \text{ but } h_k, g_{i+1} \in \text{the same generating group}; \\ h_1, h_2, \ldots, h_{j-1}, h_{j+1}, \ldots, h_r, h_j g_i, g_{i+1} & \text{if } g_{i+1} \text{ is in a different generating group from any term which can shuffle to the end.} \end{cases}$$

We note that k may be less than or greater than j, and that h_jg_i and g_{i+1} are in different generating groups.

Now, $\rho(h_1, h_2, \ldots, h_r, g_{i+1}, g_i) = \rho(h_1, h_2, \ldots h_{j-1}, h_{j+1}, \ldots, h_r, g_{i+1}, h_jg_i)$, by the definition of ρ, and since g_i and g_{i+1} are in different generating groups, g_i and h_j are in the same generating group, and $[g_i, g_{i+1}] = 1 \Rightarrow [h_j, g_{i+1}] = 1$,

$$= \begin{cases} h_1, h_2, \ldots, h_{k-1}, h_{k+1}, \ldots, h_{j-1}, h_{j+1}, \ldots, h_r, h_jg_i & \text{if } \exists k \text{ such that } h_k \text{ shuffles to the end and } h_k = g_{i+1}^{-1}; \\ h_1, h_2, \ldots, h_{k-1}, h_{k+1}, \ldots, h_{j-1}, h_{j+1}, \ldots, h_r, h_kg_{i+1}, h_jg_i & \text{if } \exists k \text{ such that } h_k \text{ shuffles to the end and } h_k \neq g_{i+1}^{-1}, \text{ but } h_k, g_{i+1} \in \text{the same generating group}; \\ h_1, h_2, \ldots, h_{j-1}, h_{j+1}, \ldots, h_r, h_jg_i, g_{i+1} & \text{if } g_{i+1} \text{ is in a different generating group from any term which can shuffle to the end.} \end{cases}$$

Thus, since the reduced sequences

$$h_1, h_2, \ldots, h_{k-1}, h_{k+1}, \ldots, h_{j-1}, h_{j+1}, \ldots, h_r, h_jg_i, h_kg_{i+1}$$

and

$$h_1, h_2, \ldots, h_{k-1}, h_{k+1}, \ldots, h_{j-1}, h_{j+1}, \ldots, h_r, h_kg_{i+1}, h_jg_i$$

are equivalent, as $[h_kg_{i+1}, h_jg_i] = 1$, and the reduced sequences

$$h_1, h_2, \ldots, h_{j-1}, h_{j+1}, \ldots, h_r, h_jg_i, g_{i+1}$$

and

$$h_1, h_2, \ldots, h_{j-1}, h_{j+1}, \ldots, h_r, g_{i+1}, h_jg_i$$

are equivalent, as $[g_{i+1}, h_j g_i] = 1$,

$$\rho(h_1, h_2, \ldots, h_r, g_i, g_{i+1}) \equiv_E \rho(h_1, h_2, \ldots, h_r, g_{i+1}, g_i)$$

Subcase C:

That is, for this case any h_j which shuffles to the end of h_r belongs to a different generating group from g_i and no h_k which shuffles to the end of g_i belongs to the same generating group as g_{i+1}. For, if any such h_k existed, the case would have been dealt with by Case 2, subcases A and B, by symmetry.

Thus $\rho(h_1, h_2, \ldots, g_i, g_{i+1}) = h_1, h_2, \ldots, g_i, g_{i+1}$

$\equiv_E h_1, h_2, ..., g_{i+1}, g_i$, since g_i, g_{i+1} commute

$= \rho(h_1, h_2, \ldots, h_r, g_{i+1}, g_i)$.

Hence (*) is proved.

We now proceed to prove (h) by induction on m.

If $m = 1$

We have $\rho(g_1, g_2)$, where $g_1, g_2 \in$ the same generating group and we want to show that $\rho(g_1, g_2) = \rho(g_1 g_2)$. Now from (d) we have $\rho(g_1, g_2) = \rho(\rho(g_1), g_2)$

$$= \begin{cases} \rho(g_2) & \text{if } g_1 = 1 \\ \rho(g_1 g_2) & \text{if } g_1 \neq 1 \end{cases}$$

$= \rho(g_1 g_2)$.

So the result holds for $m = 1$.

Assume inductively that the result holds for sequences of length $< m$.

If $g_{m+1} = 1$, then

$$\rho(g_1, g_2, \ldots, g_m, g_{m+1}) \equiv_E \rho(g_1, g_2, \ldots, g_{m+1}, g_m)$$

since $[g_m, g_{m+1}] = 1$. Any g_i which commutes to the end can be relabelled as g_m, and we have equivalence by (*).

If the g_j in question $= 1$, then by (*),

$\rho(g_1, g_2, \ldots g_j, \ldots, g_m, g_{m+1})$

$= \rho(g_1, g_2, \ldots, g_{j-1}, g_{j+1}, \ldots, g_m)$, by (f),

$= \rho(g_1, g_2, \ldots, g_{j-1}, g_{j+1}, \ldots, g_j g_m)$, since $g_j = 1$.

So we may assume $g_j, g_{m+1} \neq 1$.

The following two cases arise:

Case 1 $j = m$.

Case 2 $j \neq m$.

We will consider case 2 first. If $j \neq m$, then we have $[g_j, g_{j+1}] = [g_j, g_{j+2}] = \cdots = [g_j, g_m] = 1$. So by (*)

$\rho(g_1, g_2, \ldots, g_m, g_{m+1}) \equiv_E \rho(g_1, g_2, \ldots, g_{j-1}, g_{j+1}, \ldots, g_m, g_j, g_{m+1})$. We can now relabel this and reduce it to case 1.

Consider case 1. Let $\rho(g_1, g_2, \ldots, g_{m-1}) = h_1, h_2, \ldots, h_r$. By (d) and (g) it is sufficient to show that

$$\rho(h_1, h_2, \ldots, h_r, g_m, g_{m+1}) = \rho(h_1, h_2, \ldots, h_r, g_m g_{m+1}).$$

Two subcases arise: either $\exists h_k$ such that h_k shuffles to the end and $h_k, g_m \in$ the same generating group, or not. Suppose not, then

$$\rho(h_1, h_2, \ldots, h_r, g_m) = h_1, h_2, \ldots, h_r, g_m$$

and

$$\rho(h_1, h_2, \ldots, h_r, g_m, g_{m+1}) = \begin{cases} h_1, h_2, \ldots, h_r, & \text{if } g_{m+1} = g_m^{-1}; \\ h_1, h_2, \ldots, h_r, g_m g_{m+1}, & \text{if } g_{m+1} \neq g_m^{-1}; \end{cases}$$

by definition of ρ. Similarly

$$\rho(h_1, h_2, \ldots, h_r, g_m g_{m+1}) = \begin{cases} h_1, h_2, \ldots, h_r, & \text{if } g_m = g_{m+1}^{-1}; \\ h_1, h_2, \ldots, h_r, g_m g_{m+1}, & \text{if } g_m \neq g_{m+1}^{-1}; \end{cases}$$

by definition of ρ. So our result holds.

Suppose, then, $\exists$ an h_k as described above. We note there can be only one such h_k as $h_1, h_2, \ldots, h_r$ is reduced, and by (*) we can relabel and take $k = r$.

Now $\rho(h_1, h_2, \ldots, h_r, g_m, g_{m+1}) = \rho(\rho(h_1, h_2, \ldots, h_r, g_m), g_{m+1})$, by (d). Since $r \leq m-1$ by (a), the inductive hypothesis yields successively:

$$\begin{aligned} & \rho(\rho(h_1, h_2, \ldots, h_r, g_m), g_{m+1}) \\ = \quad & \rho(\rho(h_1, h_2, \ldots, h_r g_m), g_{m+1}) \\ = \quad & \rho(h_1, h_2, \ldots, h_r g_m, g_{m+1}) \\ = \quad & \rho(h_1, h_2, \ldots, h_r g_m g_{m+1}) \\ = \quad & \rho(h_1, h_2, \ldots, h_r, g_m g_{m+1}) \\ = \quad & \rho(h_1, h_2, \ldots, h_r, g_{m+1} g_m) \\ = \quad & \rho(h_1, h_2, \ldots, h_r, g_{m+1}, g_m) \end{aligned}$$

Thus the property (h) holds.

We now continue with the proof of the theorem 3.9.

Suppose that $g_1, g_2, \ldots, g_r$ and $h_1, h_2, \ldots, h_s$ are reduced sequences such that $g_1 g_2 \ldots g_r$ and $h_1 h_2 \ldots h_s$ each define the same element, g, of $G\Gamma$. Now g_i and h_j are defined by words of one syllable, U_i and V_j respectively. So

$$U = U_1 U_2 \ldots U_r$$

and

$$V = V_1 V_2 \ldots V_s$$

are words which define the same element in $G\Gamma$. Thus we can proceed from U to V by means of insertions and deletions of the defining relators $R(a_1), R(a_2), \ldots, R(a_n), [A_i, A_j]$ and the trivial relators $a_i a_i^{-1}$ and $a_i^{-1} a_i$. We shall show that under the insertion or deletion of a relator, the reduced form, ρ, of a word remains unchanged. We have already dealt with the syllable shuffling in (*), so we need only consider relators of one syllable. It is sufficient to consider the case of insertion of a relator P of one syllable, since if P is deleted from X to get Y, P can be inserted in Y to get X.

Let $X = X_1 X_2 \ldots X_m$ be a word in $G\Gamma$ where $X_1, X_2, \ldots, X_m$ are the syllables of X, and let k_i be the element of $G\Gamma$ defined by X_i. Then if Y is obtained from X by the insertion of the one syllable relator P, P is inserted either at the beginning or end of X, in between consecutive syllables X_i, X_{i+1}, or in between symbols in some syllable X_i.

If P has a syllable of X on its immediate left or right which is in the same generating group as P, or if P is inserted into such a syllable of X, then the sequence of elements defined by the syllables of Y is the same as the sequence of elements defined by the syllables of X. Hence $\rho(Y) = \rho(k_1, k_2, \ldots, k_m) = \rho(X)$.

Otherwise there are four possibilities:

Case 1 P is inserted in front of X_1 and $X_1, P \in$ different generating groups.

Case 2 P is inserted after X_n and $P, X_n \in$ different generating groups.

Case 3 P is inserted between X_i and X_{i+1}, neither of which is in the same generating group as P.

Case 4 P is inserted between X_i' and X_i'' where $X_i = X_i'X_i''$ and $P \in$ different generating groups.

Case 1. $\rho(Y)$

$= \rho(PX_1X_2 \ldots X_m)$

$= \rho(1, k_1, k_2, \ldots, k_m)$

$= \rho(k_1, k_2, \ldots, k_m)$, by (f)

$= \rho(X)$.

Case 2. $\rho(Y)$

$= \rho(X_1X_2 \ldots X_mP)$

$= \rho(k_1, k_2, \ldots, k_m, 1)$

$= \rho(k_1, k_2, \ldots, k_m)$, by (f)

$= \rho(X)$.

Case 3. $\rho(Y)$

$= \rho(X_1 X_2 \ldots X_i P X_{i+1} X_{i+2} \ldots X_m)$

$= \rho(k_1, k_2, \ldots, k_i, 1, k_{i+1}, k_{i+2}, \ldots, k_m)$

$= \rho(k_1, k_2, \ldots, k_m)$, by (f)

$= \rho(X)$.

Case 4. Let the group elements represented by the words X_i', X_i'' be k_i', k_i'' respectively. Then $k_i = k_i' k_i''$ and we have $\rho(Y)$

$= \rho(X_1 X_2 \ldots X_{i-1} X_i' P X_i'' X_{i+1} X_{i+2} \ldots X_m)$

$= \rho(k_1, k_2, \ldots, k_{i-1}, k_i', 1, k_i'', k_{i+1}, k_{i+2}, \ldots, k_m)$

$= \rho(k_1, k_2, \ldots, k_i', k_i'', \ldots, k_m)$, by (f)

$= \rho(k_1, k_2, \ldots, k_i' k_i'', \ldots, k_m)$, by (d) and (i)

$= \rho(k_1, k_2, \ldots, k_m)$

$= \rho(X)$.

Thus $\rho(Y) = \rho(X)$.

So, returning to U and V, we must have $\rho(U) \equiv_E \rho(V)$, for there is a sequence

$$U, \ldots, X, Y, \ldots V$$

such that consecutive terms differ by the insertion or deletion of a one syllable relator or by syllable shuffling, and so

$$\rho(U) \equiv_E \cdots \equiv_E \rho(X) \equiv_E \rho(Y) \equiv_E \cdots \equiv_E \rho(V)$$

But $\rho(U) \equiv_E \rho(U_1 U_2 \dots U_r) \equiv_E \rho(g_1, g_2, \dots, g_r) \equiv_E g_1, g_2, \dots, g_r$, by (c) and $\rho(V) \equiv_E \rho(V_1 V_2 \dots V_s) \equiv_E \rho(h_1, h_2, \dots, h_s) \equiv_E h_1, h_2, \dots, h_s$, by (c).

Hence $g_1, g_2, \dots, g_r \equiv_E h_1, h_2, \dots, h_s$.

Thus the proof of Theorem 3.9 is complete. □

As an immediate corollary to the Normal Form Theorem we have:

COROLLARY 3.12. *If $A_1, A_2, \dots, A_n$ are finitely generated groups with a solvable word problem and $G\Gamma$ is a graph product of the A_i, then $G\Gamma$ has a solvable word problem.*

Later we will prove the Normal Form Theorem by another means, but the above method directly provides an algorithm for the word problem in practice, namely the reduction process ρ.

We also have the following:

COROLLARY 3.13. *If $g_1, g_2, \dots, g_r$ is a reduced sequence in $G\Gamma$ then $\lambda(g_1 g_2 \dots g_r) = r$.*

PROOF: If W defines $g_1 g_2 \dots g_r$, then the sequence $\rho(W)$ has no more terms than the number of syllables in W, from (a). But, if U_i is a word of one syllable defining g_i, then

$$\rho(U_1 U_2 \dots U_r) \equiv_E \rho(g_1, g_2, \dots, g_r) \equiv_E g_1, g_2, \dots, g_r.$$

Since $U_1 U_2 \dots U_r$ and W define the same element of $G\Gamma$,

$$\rho(W) \equiv_E \rho(U_1 U_2 \dots U_r), \text{ and so } \lambda(W) \geq r.$$

Recall that syllable shuffling in a reduced sequence or word cannot bring two syllables from the same generating group together. Thus r is the minimal syllable length of any word defining $g_1 g_2 \dots g_r$. □

Definition 3.14

Let $W = W_1 W_2 \dots W_k$ be a reduced word in $G\Gamma$ with the W_i its syllables.

If $k = 1$, or

if $\forall r, s$ such that W_r shuffles to the front of W, and W_s shuffles to the back of W, $W_r \neq W_s^{-1}$,

then we say that W is *cyclically reduced.*

We say an element g of $G\Gamma$ is *cyclically reduced* if any, and necessarily all, of the reduced words representing it are cyclically reduced.

Definition 3.15

We define W as above.

If $k = 1$, or

if $\forall r, s$ such that W_r shuffles to the front of W, and W_s shuffles to the back of W, W_r and W_s represent elements from different generating groups,

then we say that W is *proper cyclically reduced.*

We say that an element g of $G\Gamma$ is *proper cyclically reduced* if the reduced words representing it are proper cyclically reduced. Clearly proper cyclically reduced $\Rightarrow$ cyclically reduced.

We say that the empty word, and thus the identity element of $G\Gamma$, is cyclically, and proper cyclically reduced.

LEMMA 3.16. *Let $g \in G\Gamma$, where $G\Gamma$ is a graph product of groups. Then g is a conjugate of a proper cyclically reduced element of $G\Gamma$.*

PROOF: Let g be represented by a reduced word W with syllables $W_1, W_2, \ldots, W_t$. Proceed by induction on t.

If $t = 1$, then $W = W_1$ and g is proper cyclically reduced by definition.

Suppose inductively that all elements of $G\Gamma$ with syllable length $< t$ are conjugates of proper cyclically reduced elements of $G\Gamma$.

If W is proper cyclically reduced, then so is g. So, we suppose that W is not proper cyclically reduced. Then $\exists r, s$ such that

$$W \equiv_E W_r W_1 W_2 \ldots W_{r-1} W_{r+1} \ldots W_{s-1} W_{s+1} \ldots W_t W_s$$

where W_r, W_s represent the elements g_r, g_s, which are in the same generating group.

Consider $g_r^{-1} g g_r = h \in G\Gamma$. This group element is represented by the word

$$
\begin{aligned}
& W_r^{-1} W W_r \\
&= W_r^{-1} W_r W_1 W_2 \ldots W_{r-1} W_{r+1} \ldots W_{s-1} W_{s+1} \ldots W_t W_s W_r \\
&= W_1 W_2 \ldots W_{r-1} W_{r+1} \ldots W_{s-1} W_{s+1} \ldots W_t (W_s W_r).
\end{aligned}
$$

So, h has syllable length $\leq t-1$, thus, by the inductive hypothesis,

$\exists u, v \in G\Gamma$ such that v is proper cyclically reduced and $h = uvu^{-1}$.

Now,

$$
\begin{aligned}
g &= g_r h g_r^{-1} \\
&= (g_r u) v (g_r u)^{-1}.
\end{aligned}
$$

That is, g is a conjugate of a proper cyclically reduced element of $G\Gamma$. □

Example 3.17

Continuing Example 3.6 again, we consider the element

$$g = bab^{-1}c^3.$$

This is not cyclically reduced, for

$$bab^{-1}c^3 \equiv_E bac^3 b^{-1}.$$

However,

$$h = babc^3$$

is cyclically reduced as $b \neq b^{-1}$, but h is not proper cyclically reduced, since

$$babc^3 = bac^3 b.$$

On the other hand,

$$h = babc^3$$
$$= b(ac^3b^2)b^{-1},$$

where ac^3b is proper cyclically reduced.

Definition 3.18

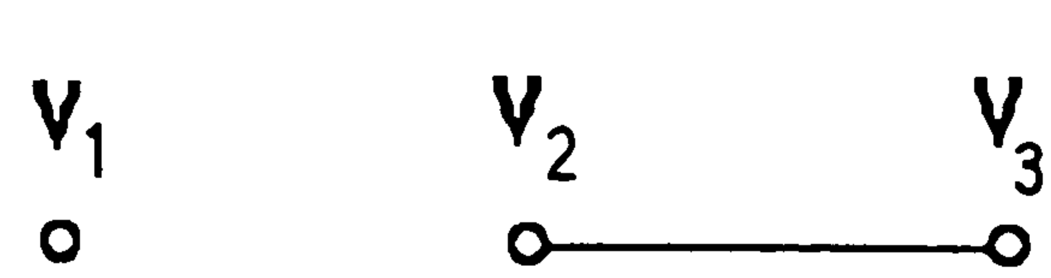
Let $G\Gamma$ be a graph product of groups given by the graph Γ. Let Δ be a full subgraph of Γ. Then $G\Delta$ denotes the graph product of the groups assigned to the verticies of Δ in Γ, given by the graph Δ and the same group vertex assignment as in $G\Gamma$. We say that $G\Delta$ is a *truncated subgroup* of $G\Gamma$.

If Δ is a complete graph, then clearly $G\Delta$ is the direct product of its generating groups. In this case we call $G\Delta$ a *direct truncated subgroup* of $G\Gamma$.

Example 3.19

(i) In a graph product every generating group is a direct truncated subgroup since the full subgraph associated with it is just one vertex which is trivially complete.

(ii) Let Γ be the graph shown:



and let A, B, C be the groups with presentation $A = \langle a; a^2 \rangle; B = \langle b; \ \rangle; C = \langle c; c^7 \rangle$. We have a group vertex assignment ϕ, such that A, B, C are assigned to v_1, v_2, v_3 respectively.

So, $G\Gamma_\phi = \langle a, b, c; a^2, c^7, [b, c] \rangle$.

$G\Gamma$ has four direct truncated subgroups:

$$A, B, C \text{ and } B \times C.$$

We will now introduce another approach to the study of graph products of groups which will prove very helpful as we can exploit results from a well studied area of group theory, the theory of generalised free products.

LEMMA 3.20 REPRESENTING GRAPH PRODUCTS AS GENERALISED FREE PRODUCTS. *Let $G\Gamma$ be a graph product of the groups $A_1, A_2, \ldots A_n$. Then $G\Gamma$ is either*

an ordinary free product,

a direct product, or

a generalised free product,

in each case, of subgroups which are graph products of certain generating groups of $G\Gamma$, given by full subgraphs of Γ and the same group vertex assignment.

PROOF: If Γ has a vertex, v, which is isolated, and A_i is assigned to this vertex, then

$$G\Gamma \equiv A_i * H,$$

where H is the graph product of $A_1, A_2, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n$ given by the full subgraph Δ of Γ, on all the vertices excluding v.

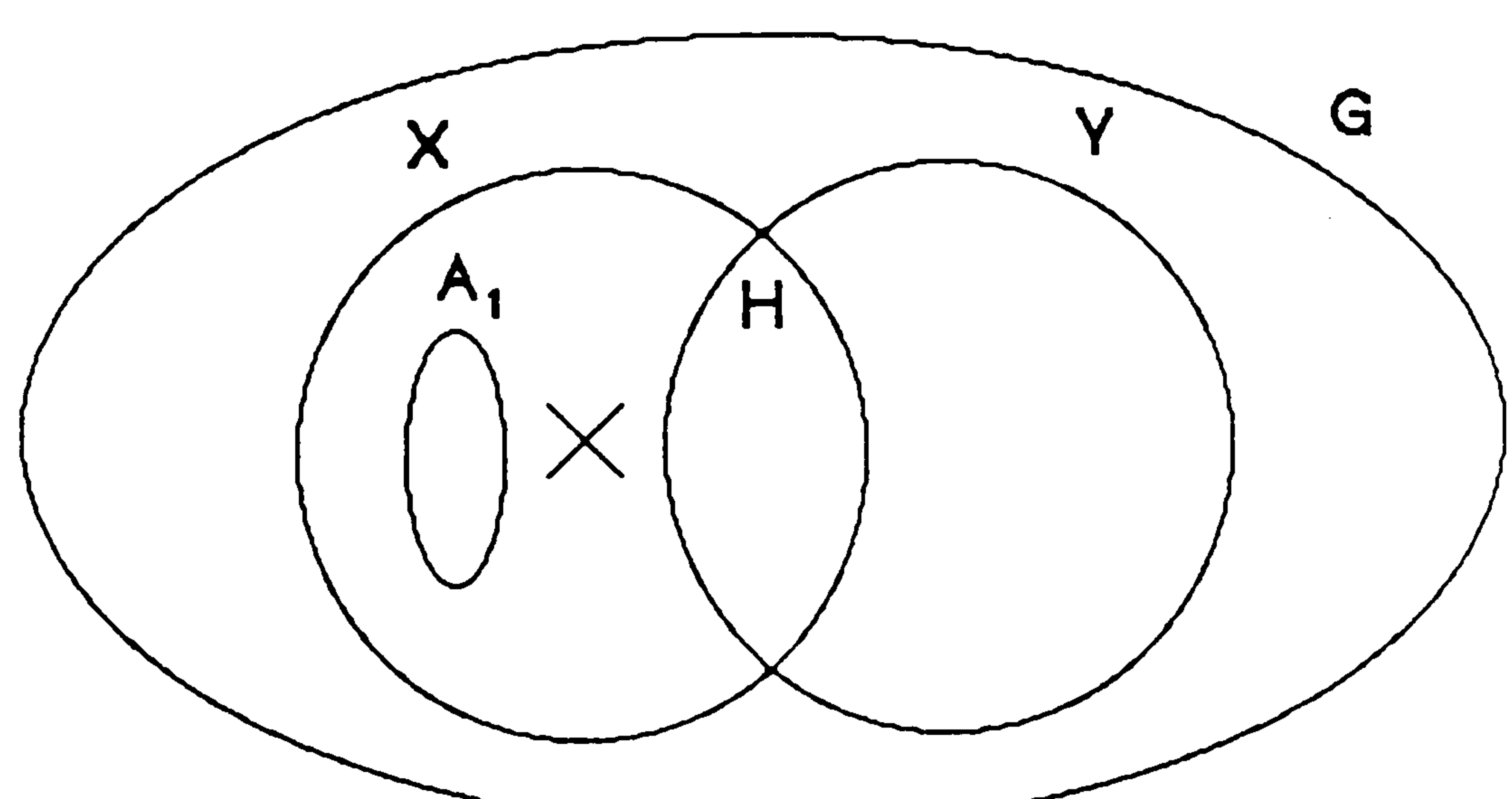
If Γ has a vertex, v, which is joined to every other, and A_i is assigned to this vertex, then

$$G\Gamma \equiv A_i \times H,$$

where H is defined as above.

If neither of the above holds for any vertex in Γ, let v be a given vertex. By relabelling, if neccesary, let A_1 be assigned to v, let $A_2, A_3, \ldots, A_r$ be assigned to the vertices adjacent to A_i, and $A_{r+1}, A_{r+2}, \ldots, A_n$ to the vertices not adjacent to A_1. Denote by H the graph product subgroup of $G\Gamma$ generated by $A_2, A_3, \ldots, A_r$, by X the subgroup generated by A_1 and H, and by Y the subgroup generated by $A_2, A_3, \ldots, A_n$.

Pictorially:



Then, we can easily see that $G\Gamma$ is the generalised free product of X and Y amalgamating H, as shown above. Moreover, X is the direct product of A_1 and H, and it is this fact which makes the representation so amenable. □

As promised we will now offer another proof for Theorem 3.9, the Normal Form Theorem for Graph Products.

PROOF: We proceed by induction on the number of generator groups n.

If $n = 2$, then $G\Gamma$ is either a free or direct product, and the result certainly holds,

Suppose $n > 2$. Let F denote the set of generating groups of $G\Gamma$.

Suppose $\exists A_i \in F$ such that $[A_i, A_j] = 1 \ \forall j \neq i$. Then $G\Gamma \equiv A_i \times \bar{G\Gamma}$, where $\bar{G\Gamma}$ is the subgroup of $G\Gamma$ generated by $A_1, A_2, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n$. Since $\bar{G\Gamma}$ is a graph product of $n-1$ generating groups the result holds in $\bar{G\Gamma}$ by induction, and any element $g \neq 1$ of $G\Gamma$ can be written

$$g = a\bar{g} \quad \text{where } a \in A_i \text{ and } \bar{g} \in \bar{G\Gamma}.$$

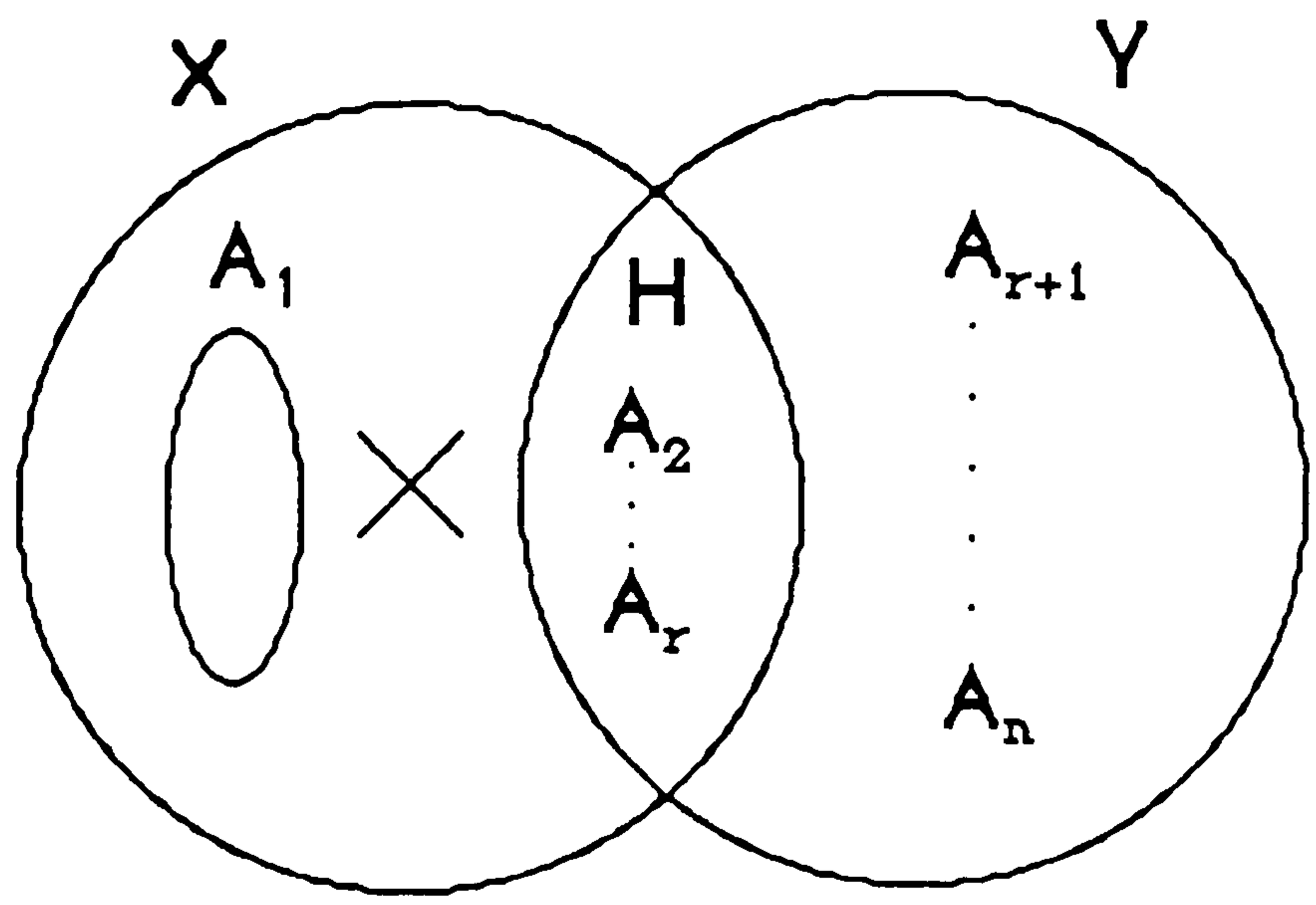
Now $\bar{g}$ is uniquely reduced up to syllable shuffling. Hence g has reduced representation which is unique up to syllable shuffling.

Suppose $\nexists$ such an A_i. Then either $G\Gamma$ is a free product, in which case we achieve the result by the normal form theorem for free products, [19], and induction as above, or, with relabelling if neccesary,

$$[A_1, A_j] = 1 \quad \text{for some } j \text{ and}$$

$$[A_1, A_k] \neq 1 \quad \text{for some } k.$$

We will write $G\Gamma$ with the usual generalised free product representation:



If $g \in X$ or $g \in Y$ we have the result by induction, so assume $g \notin X$ and $g \notin Y$. Suppose g has a reduced product $g_1 g_2 \dots g_m$ representing it. Suppose $g_{i_1}, g_{i_2}, \dots, g_{i_s}$ are the syllables belonging to A_1. Then

$$g_1 g_2 \dots g_m = (b_1) g_{i_1} b_2 g_{i_2}, \dots b_{s-1} g_{i_s} (b_s),$$

where the b's are the group elements given by the intermediate syllables.

Now, since $g_1 g_2 \dots g_m$ is reduced, no b_i can belong to H.

Thus, given a reduced word in terms of graph products, we can uniqely determine a reduced generalised free product representation for the element represented by the word, up to syllable shuffling.

So it is sufficient to show that any two reduced generalised free product representations for the same element are equal, up to syllable shuffling. Let g have the following two generalised free product representations:

$$(a_1) b_1 a_2 b_2 \dots a_k (b_k),$$

$$(c_1) d_1 c_2 d_2 \dots c_k (d_k),$$

where $a_i, c_i \in A_1$ and $b_i, d_i \in B \backslash H$.

We have both words of generalised free product length $2k(-1)(-2)$, by the normal form theorem for generalised free products in [19].

We proceed by induction on the lengths of these words. If length $= 1$, then $g \in A$ or B, in which case we have the result by induction on n. So, suppose length > 1.

Since $b_1, d_1 \in B \backslash H$, a_2 and c_2 cannot commute past them. By the normal form theorem for generalised free products, $a_1 = c_1 h$, where $h \in H$. Now $a_1, c_1 \in A_1$, therefore $c_1^{-1} a_1 \in A_1$, and thus $h \in A_1$, so $h = 1$.

Thus $a_1 = c_1$. But we could have $a_1 = c_1 = 1$, so we cannot yet apply induction. Next consider b_1, d_1.

Again $b_1 = d_1 h_1$. Thus $d_1^{-1} b_1 = h_1 \in H$.

By induction on t, b_1 and d_1 have unique reduced representations, up to syllable shuffling, in terms of the generating groups. Let

$$d_1 = x_1 x_2 \ldots x_i$$

$$b_1 = y_1 y_2 \ldots y_j$$

be such representations. So,

$$d_1^{-1} b_1 = x_i^{-1} x_{i-1}^{-1} \ldots x_2^{-1} x_1^{-1} y_1 y_2 \ldots y_j \in H.$$

All the syllables which do not belong to H must cancel out. Let y_f be the first syllable of b_1 not belonging to H. Now y_f must cancel with a syllable from d_1^{-1}, since b_1 is reduced. Thus y_f must be able to commute to the left to reach this syllable. Putting it another way, y_f must commute to the beginning of b_1, and y_f^{-1} to the end of d_1^{-1}.

By an argument of induction, all the syllables not from H appearing in b_1 must commute to the front of b_1 and their inverses, comprising all the syllables not from H appearing in d_1^{-1}, must commute to the end of d_1^{-1}. So

$$b_1 = wh_b \quad \text{and} \quad d_1 = vh_d,$$

where $h_b, h_d \in H$, and w and v are equal words in $B \backslash H$, up to syllable shuffling. Thus

$$(a_1)b_1a_2b_2 \ldots a_k(b_k)$$

$$(c_1)d_1c_2d_2 \ldots c_k(d_k)$$

can be written as

$$(a_1)wa_2(h_bb_2)a_3b_3 \ldots a_k(b_k)$$

$$(c_1)vc_2(h_dd_2)c_3d_3 \ldots c_k(d_k)$$

since h_b, h_d commute with a_2, c_2 respectively by choice of A_1. Now $(a_1)w$ and $(c_1)v$ are equal up to syllable shuffling and

$$a_2(h_bb_2)a_3b_3 \ldots a_k(b_k)$$

$$c_2(h_dd_2)c_3d_3 \ldots c_k(d_k)$$

represent the same group element. Hence by induction on word length, they are equal up to syllable shuffling.

Thus the two reduced representations of g are equal up to syllable shuffling.

Hence result by induction. □

The Normal Form Theorem gives us solvability of the Word Problem. We can now also tackle the Conjugacy Problem in two ways:

directly using definitions of length and cyclically reduced, in terms of graph products, or

by exploiting the generalised free product representation.

The first method was used by Herman Servatius in [26] for the special case of the conjugacy problem for graph groups. We will now prove the more general result of the solvability of the conjugacy problem being preserved by graph products, using the generalised free product representation method.

We first state without proof the Conjugacy Theorem for Free Products with Amalgamation found in [18]. *Cyclically reduced*, here, is in the sense of generalised free products, as defined in [18].

THEOREM 3.21. *Let P be a generalised free product of G with H amalgamating the subgroup A. Let $u \in P$ be cyclically reduced, where*

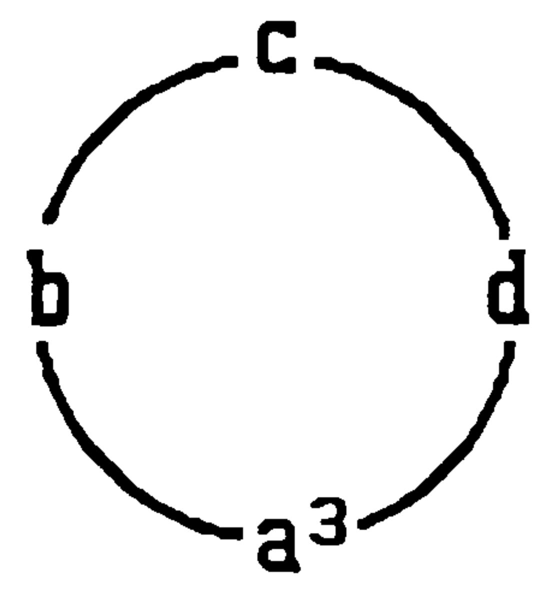
$$u = c_1 c_2 \dots c_r,$$

is a normal form for u. Then every cyclically reduced conjugate of u can be obtained by cyclically permuting $c_1 c_2 \dots c_r$ and then conjugating by an element of the amalgamated part A.

We make the following definition:

Definition 3.22

Let $G\Gamma$ be a graph product. Let w be a word in the generators of $G\Gamma$. We denote the set of all (finitely many) cyclic permutations of w by $w$cycle. We can think of $w$cycle as a circle with the syllables of w written in a clockwise orientation. For example, with $w = a^2bcda$, we can picture w cycle as:



For w_1, w_2 cyclically reduced (in the sense of graph products) words we say that

$$w_1 \,\text{cycle} \equiv_E w_2 \,\text{cycle}$$

if some cyclic permutation of $w_1 \equiv_E$ some cyclic permutation of w_2. We note that extending this concept to non cyclically reduced words would do nothing more than redefine conjugacy.

Example 3.23

In $G = \langle a, b, c, d; [a, b], [c, d] \rangle$ if $w_1 = bcda$ and $w_2 = abcd$ then

$$w_1 \not\equiv_E w_2 \quad \text{but} \quad w_1 cycle \equiv_E w_2 cycle.$$

Finally we make the following observation:

If w_1, w_2 and w_3 are words in the generating symbols of a graph product and $w_1 \equiv_E w_2$ then

$$w_1 w_3 \equiv_E w_2 w_3 \quad \text{and}$$

$$w_3 w_1 \equiv_E w_3 w_2.$$

THEOREM 3.24. *Let $G\Gamma$ be a graph product of the groups $A_1, A_2, \ldots, A_k$, where the A_i's each have solvable conjugacy problem. Then $G\Gamma$ has a solvable conjugacy problem.*

PROOF: The result is obvious if $G\Gamma$ is a direct product. Otherwise, by Lemma 3.16 we can restrict our consideration to proper cyclically reduced words. We will prove the following:

For U, V proper cyclically reduced words representing elements u, v,

$$U \equiv_E \text{ a conjugate of } V \iff U\,\text{cycle} \equiv_E V\,\text{cycle}.$$

The right hand condition is easily decidable in a finite number of steps using any possible syllable shuffling.

We first show $(\Leftarrow)$.

Let $U = U_1U_2$ and $V = V_1V_2$, where

$$U_2U_1 \equiv_E V_2V_1.$$

Then by the above observation

$$U_2U_1(U_2U_2^{-1}) \equiv_E V_2V_1(V_2V_2^{-1}).$$

Thus

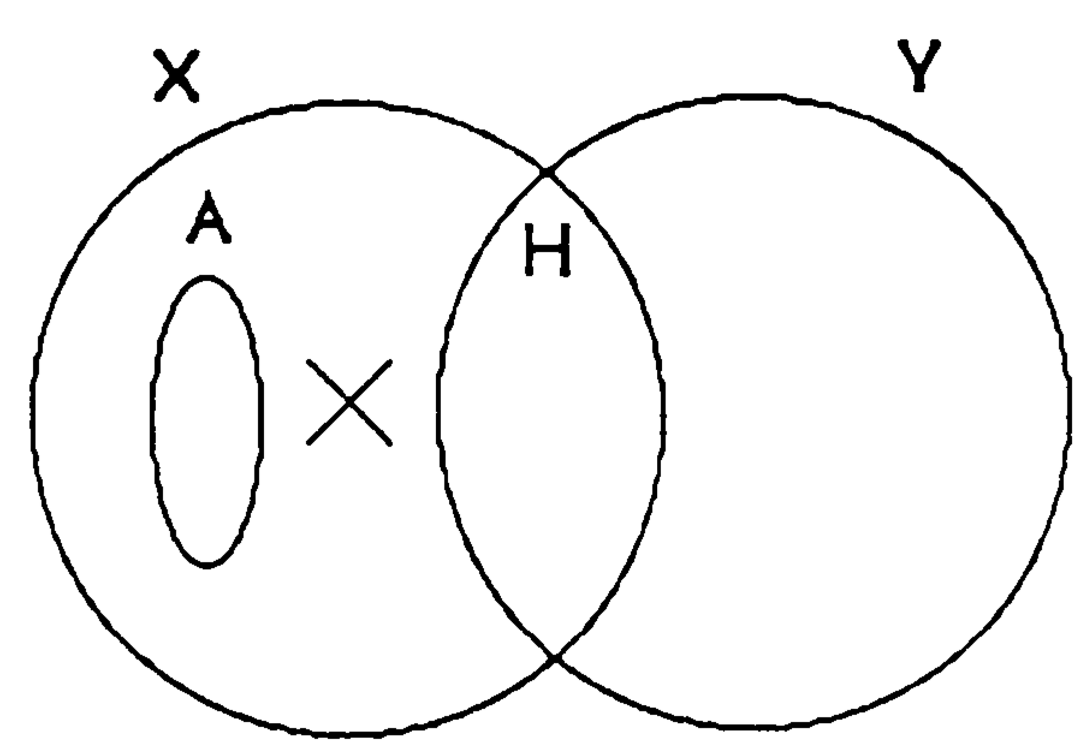
$$U_1U_2 \equiv_E U_2^{-1}V_2(V_1V_2)V_2^{-1}U_2,$$

that is

$$U = (U_2^{-1}V_2)V(U_2^{-1}V_2)^{-1}.$$

Now we show the more difficult $(\Rightarrow)$. $U \equiv_E$ a conjugate of V, and thus $u \equiv_E$ a conjugate of v. By Lemma 3.20 $G\Gamma$ is either a free product, a direct product, or a generalised free product. The first two cases are easily dealt with by induction and using the solvability of the conjugacy problem in free products from [18].

So suppose $G\Gamma$ is a generalised free product as shown:



Let $u = u_1 u_2 \dots u_n$, thus for some i,

$$v = h^{-1} u_i u_{i+1} \dots u_n u_1 u_2 \dots u_{i-1} h,$$

where $h \in H$, and either $[h, u_i] = 1$ or $[u_{i-1}, h] = 1$, since u is cyclically reduced in terms of generalised free products, and so u_i or u_{i-1} must belong to A. Without loss of generality take $[u_i, h] = 1$. Then

$$v = u_i (h^{-1} u_{i+1}) u_{i+2} \dots u_n u_1 u_2 \dots u_{i-1} h.$$

This form of v is cyclically reduced in terms of generalised free products, since u is cyclically reduced. Let u_i have syllables $u_{i_1} u_{i_2} \dots u_{i_{n_i}}$, and $h = h_1 h_2 \dots h_m$ in terms of graph products.

So, writing v in graph product normal form we have

$$v = u_{i_1}u_{i_2}\ldots u_{i_{n_i}}(h_m^{-1}h_{m-1}^{-1}\ldots h_2^{-1}h_1^{-1})(u_{i+1_1}u_{i+1_2}\ldots u_{i+1_{n_{i+1}}})\ldots$$
$$(u_{n_1}u_{n_2}\ldots u_{n_{n_n}})(u_{1_1}u_{1_2}\ldots u_{1_{n_1}})\ldots(u_{i-1_1}u_{i-1_2}\ldots u_{i-1_{n_{i-1}}})h_1h_2\ldots h_m.$$

But $h_m^{-1}h_{m-1}^{-1}\ldots h_2^{-1}h_1^{-1}$ shuffles forwards and

$$v \equiv_E h_m^{-1}h_{m-1}^{-1}\ldots h_2^{-1}h_1^{-1}(u_{i_1}u_{i_2}\ldots u_{i_{n_i}})(u_{i+1_1}u_{i+1_2}\ldots u_{i+1_{n_{i+1}}})\ldots$$
$$(u_{n_1}u_{n_2}\ldots u_{n_{n_n}})(u_{1_1}u_{1_2}\ldots u_{1_{n_1}})\ldots(u_{i-1_1}u_{i-1_2}\ldots u_{i-1_{n_{i-1}}})h_1h_2\ldots h_m.$$

So v cannot be cyclically reduced unless $h = 1$ or h cancels completely at one side. Either way we clearly have

$$u\,\text{cycle} \equiv_E v\,\text{cycle} \quad \text{thus}$$

$$U\,\text{cycle} \equiv_E V\,\text{cycle}.$$

□

THEOREM 3.25 THE TORSION THEOREM FOR GENERALISED FREE PRODUCTS. *Every element of finite order in* $P = \langle G * H; A \equiv B, \phi\rangle$ *is a conjugate of an element of finite order in* G *or* H.

THEOREM 3.26 THE TORSION THEOREM FOR GRAPH PRODUCTS. *Let* $G\Gamma$ *be a graph product of* $A_1, A_2, .., A_m$. *An element* u *of finite order in* $G\Gamma$ *belongs to a conjugate of a direct truncated subgroup* $G\Delta$ *of* $G\Gamma$.

Further, if $G\Delta$ *has generating groups* $B_1, B_2, \ldots, B_r$ *then*

$$u = wb_1b_2\ldots b_rw^{-1} \quad \text{where} \quad b_i \in B_i, w \in G\Gamma$$

and the order of $u =$ *the lowest common multiple of the orders of the* b_i.

PROOF: Let u have order n. We will proceed by induction on m.

If $m = 1$ the result is trivial.

Suppose the result holds for graph products of less than m groups.

If $G\Gamma = A_i \times \bar{G}\Gamma$, where $\bar{G}\Gamma$ is the graph product of the groups other than A_i given by Γ, for some A_i, then

$$u = a_i \bar{u},$$

where a_i is of finite order in A_i, and $\bar{u}$ is of finite order in $\bar{G}\Gamma$. By induction, $\bar{u} = g^{-1}vg$ for some $g \in \bar{G}\Gamma$ and $v \in$ a direct truncated subgroup of $\bar{G}\Gamma$. Thus

$$u = a_i g^{-1} v g = g^{-1} a_i v g$$

where $g \in G\Gamma$ and $a_i v \in$ a direct truncated subgroup of $G\Gamma$.

If $G\Gamma$ is a free product of two truncated subgroups we have the result by Theorem 3.25 and induction.

Suppose then, that we have the general case where $G\Gamma$ is the generalised free product of X and Y amalgamating H, and $X = A_i \times H$, for some A_i. By Theorem 3.25

$$u \in g^{-1} X g \quad \text{or} \quad u \in h^{-1} Y h, \qquad g, h \in G\Gamma.$$

Again the result follows easily by induction.

Clearly the order of u is, in each case, as given. Hence result by induction. □

COROLLARY 3.27. *Graph products are periodic* $\Longleftrightarrow$ *they are direct products of periodic groups.*

PROOF: ($\Leftarrow$) Obvious.

($\Rightarrow$) Let $G\Gamma$ be a periodic graph product of the groups $A_1, A_2, \ldots, A_n$. Clearly the A_i's must be periodic as isomorphic copies of the A_i's appear as subgroups of $G\Gamma$.

Consider the element $g = a_1 a_2 \ldots a_n \in G\Gamma$, where $a_i \in A_i$. This has finite order and thus by Theorem 3.26 belongs to a conjugate of a direct truncated subgroup. g is clearly cyclically reduced and thus belongs to a direct truncated subgroup. Now g has syllable length n, thus the direct truncated subgroup contains n generating groups, so $G\Gamma$ is the direct product of its generating groups. □

COROLLARY 3.28. *Graph products are torsion free* $\Longleftrightarrow$ *all their generating groups are torsion free.*

PROOF: ($\Rightarrow$) Obvious, since generating groups are isomorphically contained in the graph product.

($\Leftarrow$) Let $G\Gamma$ be a graph product of the groups $A_1, A_2, ..., A_n$, where the A_i are torsion free. Let $g \in G\Gamma$ be an element of finite order. By theorem 3.26 $g \in$ a conjugate of a direct truncated subgroup, and if

$$g = w b_1 b_2 \ldots b_k w^{-1}$$

then the order of $g =$ the lowest common multiple of the orders of the b_i.

If $k \neq 0$, then the order of g is not finite since the generator groups are torsion free. But $k = 0 \Rightarrow g = 1$. Therefore $G\Gamma$ is torsion free. □

COROLLARY 3.29. *Graph groups are torsion free.*

PROOF: Graph groups are graph products of copies of $\mathbb{Z}$ and $\mathbb{Z}$ is torsion free. □

We now prove an analogue of another result in free and direct products:

PROPOSITION 3.30. *Let $A_1, A_2, \ldots A_n$ be subgroups of a group G such that*

$$A_i \cap gp\{A_j : j \neq i\} = 1, \quad \forall i,$$

and suppose that each element g of G can be written uniquely, up to certain syllable shuffling, as a product

$$g = g_1 g_2 \cdots g_r,$$

where $g_1, g_2, \ldots, g_r$ is a reduced sequence; and that this syllable shuffling is between all $a_i \in A_i$ and all $a_j \in A_j$; for certain pairs $\{A_i, A_j\}$. Denote the set of such pairs R.

Then G is the graph product of the A_i with underlying graph Γ and group vertex assignment ϕ such that Γ has vertices $v_1, v_2, ..., v_n$ and $\{A_i, A_j\} \in R \Longleftrightarrow v_i, v_j$ are adjacent in Γ, with $\phi(i) = i$, for $i = 1, 2, \ldots, n$.

PROOF: Now G is generated by the A_i. Let

$$A_i = \left\langle a_{i_1}, a_{i_2}, \ldots, a_{i_{r_i}}; R(a_{i_v}) \right\rangle$$

be presentations for the A_i. Then G has relators $R(a_{i_v})$, $i = 1, 2, ...n$ and $\forall a_i \in A_i$, $\forall a_j \in A_j$ such that $\{A_i, A_j\} \in R$, G has the relator $[a_i, a_j]$. Denote this set of commutators by R_G.

If G required no other relators then G would clearly be the graph product in question. So, we will show that every relator of G can be derived from the $R(a_{i_v})$ and R_G.

Let

$$W = W_1 W_2 \dots W_s$$

be a relator of G, where $W_1, W_2, \dots, W_s$ are its syllables. We proceed by induction on s.

If $s = 0$, then W is clearly derivable from the $R(a_{i_v})$ and R_G. Suppose inductively, that every relator of G with fewer than s syllables is derivable from the $R(a_{i_v})$ and R_G. Consider a relator $W_1 W_2 \dots W_s$ of G. Let g_i be the group element defined by the word W_i such that $W_1 W_2 \dots W_s$ represents $g_1 g_2 \dots g_s = g$.

Consider the sequence $g_1, g_2, \dots, g_s$.

Suppose $g_i \neq 1 \quad \forall i$.

We may employ syllable shuffling in an attempt to reduce the length of $g_1, g_2, \dots, g_s$.

If $\exists g_i, g_j, g_k$ with $i \leq k < j$ and

$$[g_i, g_{i+1}] = [g_i, g_{i+2}] = \cdots = [g_i, g_k] =$$

$$[g_{k+1}, g_j] = [g_{k+2}, g_j] = \cdots = [g_{j-1}, g_j] = 1$$

and $g_i, g_j \in$ the same A_l, then

$$g_1, g_2, \dots, g_s \equiv_E g_1, g_2, \dots, g_{i-1}, g_{i+1}, \dots, g_k, g_i g_j, g_{k+1}, \dots, g_{j-1}, g_{j+1}, \dots, g_s.$$

But then

$$W_1W_2\dots W_{i-1}W_{i+1}\dots W_k(W_iW_j)W_{k+1}\dots W_{j-1}W_{j+1}\dots W_s$$

is a relator of G with fewer than s syllables and thus derivable from $R(a_{i_v})$ and R_G. Hence $W_1W_2\dots W_s$ is derivable from 1 using the $R(a_{i_v})$ and R_G.

So, if $\nexists g_i, g_j$ with such a g_k, then $g_1, g_2, \dots, g_s$ is reduced and thus, by hypothesis , $g_1g_2\dots g_s$ is the unique product, up to syllable shuffling, which is the reduced form of g. Thus $W_1W_2\dots W_s$ cannot define 1. This is a contradiction.

Now consider the case when $g_i = 1$ for some g_i. Then W_i is a relator from A_i and hence is derivable from 1 using the $R(a_{i_v})$. But then $W_1W_2 \dots W_{i-1}W_{i+1}\dots W_s$ is a relator with $< s$ syllables and so is derivable from 1 using the $R(a_{i_v})$ and R_G. Thus $W_1W_2\dots W_{i-1}W_iW_{i+1}\dots W_s$ is derivable from $R(a_{i_v})$ and R_G. □

Whilst not every subgroup of a graph product is a graph product, we can generalise a theorem for direct and free products.

PROPOSITION 3.31. *Let $G\Gamma$ be the graph product of the groups $A_1, A_2, \dots, A_n$ given by the graph Γ and the group vertex assignment ϕ. Let B_i be a subgroup of A_i for $i = 1, 2, \dots n$. If $H\Gamma$ is the subgroup of $G\Gamma$ generated by $B_1, B_2, \dots, B_n$, then $H\Gamma$ is the graph product of the B_i given by Γ and ϕ.*

PROOF: Let $h \in H$. Then since H is generated by the B_i, $h = g_1g_2\dots g_r$, where each $g_i \in B_{j_i}$, and $g_1, g_2, \dots, g_r$ is a reduced sequence in $G\Gamma$.

(For, given a product equal to h we can reduce it, using ρ, and in so doing we will not introduce any elements not in B_i for some i.)

Then, since $h \in G\Gamma$, this product is unique up to the syllable shuffling determined by Γ and ρ. Since the product is unique in $G\Gamma$ it is certainly unique in $H\Gamma$. Thus H has subgroups $B_1, B_2, \ldots, B_n$ such that

$$\forall i,j \text{ such that } i \neq j \quad B_i \cap B_j \subseteq A_i \cap A_j = 1$$

and every element of $H\Gamma$ can be written uniquely as a reduced product which is unique up to syllable shuffling determined by Γ and ϕ. Thus, by Proposition 3.30, H is the graph product of $B_1, B_2, \ldots, B_n$ given by Γ and ϕ. □

In the following we generalise a result for direct products:

PROPOSITION 3.32. *Let $G\Gamma$ be a graph product of $A_1, A_2, \ldots, A_n$. Let $a \in A_i$, for some i, and let $g \in G\Gamma$, such that*

$$g = g_1 g_2 \ldots g_r, \quad \text{where } g_1, g_2, \ldots, g_r$$

is a reduced sequence. Then

$$gag^{-1} \in A_i \Rightarrow g_j \in A_i \text{ or } [g_j, a] = 1 \quad \forall j = 1, 2, \ldots r.$$

In particular, if $\forall j = 1, 2, \ldots r$ $[g_j, a] \neq 1$ and $g_j \notin A$, then

$$(gA_ig^{-1}) \cap A_i = 1.$$

PROOF: (Assume that $a \neq 1$ for $a = 1 \Rightarrow [g_j, a] = 1 \quad \forall g_j$.)

We proceed by induction on r.

If $r = 0$, the result is trivial. Suppose $r > 0$.

Assume inductively that the result is true $\forall g$, $\forall a$ with $\lambda(g) < r$. Consider $g = g_1 g_2 \dots g_r$. Now $gag^{-1} \in A_i$, thus

$$\lambda(g_1 g_2 \dots g_r a g_r^{-1} \dots g_2^{-1} g_1^{-1}) = 1.$$

Suppose $\forall g_j$ such that $[g_j, g_{j+1}] = [g_j, g_{j+2}] = \cdots = [g_j, g_r] = 1$ we have $g_j \notin A_i$ and $[g_j, a] \neq 1$, (including $j = r$, of course). Then

$$\begin{aligned}
&\rho(gag^{-1}) \\
&= \rho(g_1, g_2, \dots, g_r, a, g_r^{-1}, \dots, g_2^{-1}, g_1^{-1}) \\
&= \rho(\rho(g_1, g_2, \dots, g_r, a), g_r^{-1}, \dots, g_2^{-1}, g_1^{-1}) \quad \text{by (d).}
\end{aligned}$$

Now $\rho(g_1, g_2, \dots, g_r) = g_1, g_2, \dots, g_r$. Since $g_j \notin A_j$ and since $[g_j, a] = 1$ we have, from the definition of ρ:

$$\rho(g_1, g_2, \dots, g_r, a) = g_1, g_2, \dots, g_r, a.$$

Similarly

$$\begin{aligned}
\rho(g_1, g_2, \dots, g_r, a, g_r^{-1}) &= g_1, g_2, \dots, g_r, a, g_r^{-1} \\
\vdots \quad \vdots \quad \vdots \quad &\vdots \quad \vdots \quad \vdots \quad \vdots \\
\rho(g_1, g_2, \dots, g_r, a, g_r^{-1}, \dots, g_2^{-1}) &= g_1, g_2, \dots, g_r, a, g_r^{-1}, \dots, g_2^{-1} \\
\rho(g_1, g_2, \dots, g_r, a, g_r^{-1}, \dots, g_2^{-1}, g_1^{-1}) &= g_1, g_2, \dots, g_r, a, g_r^{-1}, \dots, g_2^{-1}, g_1^{-1}.
\end{aligned}$$

This gives a contradiction since $\lambda(g_1, g_2, \dots, g_r, a, g_r^{-1}, \dots, g_2^{-1}, g_1^{-1}) = 2r + 1$, by Corollary 3.13, $\neq 1$ since $r \neq 0$.

Thus $\exists g_j$ which shuffles to the end of g such that $g_j \in A_i$ or $[g_j, a] = 1$. We can relabel so that g_r has this property. So

$$g_1 g_2 \dots g_r a g_r^{-1} \dots g_2^{-1} g_1^{-1} = g_1 g_2 \dots g_{r-1} \bar{a} g_{r-1}^{-1} \dots g_2^{-1} g_1^{-1}$$

where $\bar{a} = a$ if $[g_r, a] = 1$ and $\bar{a} = g_r a g_r^{-1}$ if $g_r \in A_i$.

In either case, $\bar{a} \in A_i$, so, by the induction hypothesis, $\forall j = 1, 2, \dots, r-1, \quad g_j \in A_i$, or $[g_j, \bar{a}] = 1$.

Let g_j be such that $g_j \notin A_i$. Let $g_j \in A_k$. Then we have

$[g_j, \bar{a}] = 1 \Rightarrow [A_k, A_i] \in$ relators of $G\Gamma \Rightarrow [g_j, a] = 1$.

Hence $\forall j = 1, 2 \dots r, \quad [g_j, a] = 1$ or $g_j \in A_i$

Hence result by induction. □

COROLLARY 3.33. *Let $G\Gamma, g, a$ be as in Proposition 3.32. Then*

$$ga = ag \Rightarrow [g_j, a] = 1 \quad \forall g_j, \quad j = 1, 2, \dots r.$$

PROOF: $ga = ag \Rightarrow gag^{-1} = a \in A_i$. Thus by Proposition 3.32 $[g_j, a] = 1$ or $g_j \in A_i \quad \forall j = 1, 2, \dots r$. If $[g_j, a] = 1 \quad \forall j = 1, 2, \dots r$ then we are finished. So, suppose $\exists j$ such that $[g_j, a] \neq 1$, and choose the greatest such j. So $g_j \in A_i$. Then

$$\begin{aligned} gag^{-1} &= g_1 g_2 \dots g_j \dots g_r a g_r^{-1} \dots g_j^{-1} \dots g_2^{-1} g_1^{-1} \\ &= g_1 g_2 \dots g_j a g_j^{-1} \dots g_2^{-1} g_1^{-1}. \end{aligned}$$

We make the following claim:

CLAIM. $\forall k = 1, 2, \ldots j-1$; $[g_k, g_j] = 1 = [g_k, a]$.

PROOF: Suppose not. Choose k maximal such that $[g_k, g_j] \neq 1$ or $[g_k, a] \neq 1$. If $g_k \in A_i$, then

$$\begin{aligned} g &= g_1 g_2 \cdots g_k \cdots g_j \cdots g_{r-1} g_r \\ &= g_1 g_2 \cdots g_{k-1}(g_k g_j) g_{k+1} \cdots g_{r-1} g_r \quad \text{since } g_k, g_j \in A_i. \end{aligned}$$

But this is a contradiction, since $g_1, g_2, \ldots, g_r$ is a reduced sequence. So $g_k \notin A_i$, but by Proposition 3.32 $[g_k, a] = 1$. Thus, if $g_k \in A_l$, then $[A_l, A_i] \in$ relators of $G\Gamma$. But then $[g_k, g_j] = 1 = [g_k, a]$, since $g_j \in A_i$. This is again a contradiction and hence the claim is true. □

By the claim, $gag^{-1} = g_j a g_j^{-1} = a \Rightarrow [g_j, a] = 1$. We have a contradiction, hence the result of the corollary holds. □

THEOREM 3.34 THE CENTRES THEOREM FOR GRAPH PRODUCTS. *Let $G\Gamma$ be a graph product. Then $G\Gamma$ has non trivial centre $\Longleftrightarrow$ $G\Gamma$ is a direct product of truncated subgroups, at least one of which has non trivial centre.*

PROOF: ($\Leftarrow$) Obvious, since the centre of $G\Gamma$ is the direct product of the centres of its factor groups.

($\Rightarrow$) Suppose $G\Gamma$ is a graph product of the groups $A_1, A_2, \ldots, A_n$, and not a direct product. Suppose $G\Gamma$ has non trivial centre containing the element $g \neq 1$. Let a be a generator of one of the generating groups of $G\Gamma$. Then $ga = ag$, thus by Corollary 3.33, $[g_j, a] = 1$ or $g_j \in A_i$.

Since this is true $\forall$ such a, and g is non trivial, $G\Gamma$ is a direct product of the smallest truncated subgroup containing the centre, with the corresponding graph product of the remaining generating groups. Hence the result is shown. □

Chapter 4 Subgroup and Isomorphism Theorems of Graph Products

In this chapter we wish to determine what we can say about the underlying graphs and generating groups when two graph products are isomorphic. It is clearly false that such an isomorphism implies isomorphisms of the graphs and generating groups in general. However, it is not so obvious whether the result will hold if we insist that none of the generating groups can be written as a graph product.

We make the following definition:

Definition 4.1

Let G be a group such that $\forall$ graph products $H\Gamma$, $G \not\cong H\Gamma$. We say that G is graphologically indecomposable.

Clearly free and direct indecomposability are necessary conditions for graphological indecomposability. In the following example we will show that they are not sufficient.

Example 4.2

The graph group GL_3,

is a graph product of four copies of $\mathbb{Z}$, and we will show it is freely and directly indecomposable. Suppose the generators of the copies of $\mathbb{Z}$ are

a, b, c, d and we have $[a, b] = [b, c] = [c, d] = 1$. Then if $GL_3 \equiv X * Y$, by a result, [19], on commuting elements in a free product, we have reduced forms

$$a = w_1^{-1} \bar{a} w_1$$

$$b = w_1^{-1} \bar{b} w_1,$$

where $\bar{a}$, $\bar{b}$ are either in the same free factor, or are powers of the same element. Similarly we have

$$b = w_2^{-1} \bar{\bar{b}} w_2$$

$$c = w_2^{-1} \bar{c} w_2$$

$$c = w_3^{-1} \bar{\bar{c}} w_3$$

$$d = w_3^{-1} \bar{d} w_3$$

where $\bar{\bar{b}}, \bar{c}$, and $\bar{\bar{c}}, \bar{d}$ are as $\bar{a}, \bar{b}$ above. Suppose, without loss of generality that $\bar{a}, \bar{b} \in X$, then $\bar{c} \in X$, or $\bar{c}$ is a power of the same element as $\bar{b}$, which, by a simple length argument $\Rightarrow \bar{c} \in X$ also. Similarly $\bar{d} \in X$. Therefore $X \equiv GL_3$. But

$$\frac{GL_3}{[GL_3, GL_3]} \equiv \frac{X}{[X, X]}$$

$=$ the free abelian group of rank 4. Thus $Y = \langle 1 \rangle$.

Thus $\bar{a}, \bar{b}$ are not both in X or both in Y. Similarly $\bar{\bar{b}}, \bar{c}$ and $\bar{\bar{c}}, \bar{d}$. Thus we have

$$a = w_1^{-1} x^{\alpha} w_1 \qquad b = w_1^{-1} x^{\beta} w_1,$$

$$b = w_2^{-1} y^{\gamma} w_2 \qquad c = w_2^{-1} y^{\delta} w_2,$$

$$c = w_3^{-1} z^{\epsilon} w_3 \qquad d = w_3^{-1} z^{\eta} w_3,$$

for some $x, y, z \in GL_3$. Now $c^{\gamma} = b^{\delta}$ and $b^{\alpha} = a^{\beta}$ therefore $c^{\gamma\alpha} = b^{\delta\alpha} =$

$a^{\beta\delta}$. Thus

$$w_1^{-1}x^{\alpha\beta\delta}w_1 = w_2^{-1}y^{\delta\alpha\gamma}w_2,$$

but $\langle a,c\rangle$ is a free group, so $w_1 = w_2$. Similarly $w_2 = w_3$.

Also, $x^\beta = y^\gamma$, and similarly y and z are powers of the same element, therefore $GL_3 \equiv \mathbb{Z}$, which is clearly false. Thus GL_3 is not a non-trivial free product.

Now, suppose $GL_3 \equiv N \times M$. Then $a = n_a m_a$, $b = n_b m_b$, $c = n_c m_c$, $d = n_d m_d$ for some $n_a, n_b, n_c, n_d \in N, m_a, m_b, m_c, m_d \in M$. Now $\langle a,c\rangle \equiv F_2$, and

$$\langle n_a m_a, n_c m_c\rangle \subseteq \langle n_a, n_c\rangle \times \langle m_a, m_c\rangle.$$

Thus, without loss of generality,

$$\langle n_a m_a, n_c m_c\rangle \subseteq \langle n_a, n_c\rangle.$$

Therefore $m_a = m_c = 1$ and $a, c \in N$. Similarly $b, d \in N$ or M. Suppose $b \in N$. Then $d \in N$ since $[b,d] \neq 1$. Therefore $G \subseteq N$, and $M = \langle 1\rangle$.

If instead $b \in M$, then $d \notin M$ since $[a,d] \neq 1$, and $d \notin N$ as $[b,d] \neq 1$, which gives a contradiction. So GL_3 is a graph product and is also freely and directly indecomposable.

We present some examples of graphologically indecomposable groups.

Example 4.3

(i) Simple groups are graphologically indecomposable since the normal closure of a generating group is a non-trivial normal subgroup of a graph product.

(ii) Periodic groups are graphologically indecomposable $\Leftrightarrow$ they are directly indecomposable, by Corollary 3.27.

(iii) Finitely generated abelian groups are graphologically indecomposable $\Leftrightarrow$ they are primary or infinite cyclic, since finitely generated abelian groups are direct products of cycles.

(iv) Groups with non-trivial centre are graphologically indecomposable $\Leftrightarrow$ they are directly indecomposable, by Theorem 3.28.

We may now try to rephrase our conjecture. The following seems reasonable: *Decompositions of groups into graph products of graphologically indecomposable groups are essentially unique.*

Unfortunately even this is false: there is a counter example due to Kurosh, [17].

Counter Example 4.4

In [17] it is shown that

$$A = \left\langle a_1, a_2; a_1^2 = a_2^2 \right\rangle,$$

$$B = \left\langle b_1, b_2; b_1^3 = b_2^3 \right\rangle,$$

$$C = \left\langle c_1, c_2, c_3, c_4; c_1^2 = c_2^2 = c_3^3 = c_4^3 \right\rangle,$$

$$D = \langle d \rangle$$

are directly indecomposable, and that $G \equiv A \times B \equiv C \times D$, where $A \not\equiv C$ or D, and $B \not\equiv C$ or D. It is also shown that A, B, C, D

each have non-trivial centre. Thus, by Example 4.3 (iv), A, B, C, D are graphologically indecomposable. So G has two distinct representations as a graph product of graphologically indecomposable groups.

We can, however, obtain a result of the desired kind, in some special cases. In Theorem 2.6 we noted that Droms proved that decompositions of a group into graph products of infinite cycles are essentially unique. It is therefore natural to look next, at the case of finite cycles. Clearly we may have distinct decompositions if non-prime cycles are permitted for $C_6 \equiv C_3 \times C_2$, and so on, so we restrict our consideration to prime cycles.

We first require some preliminaries:

LEMMA 4.5. *Let $G\Gamma$ be a graph product and H a subgroup of $G\Gamma$ of finite order. Then H is a subgroup of a conjugate of a direct truncated subgroup of $G\Gamma$.*

PROOF: If $G\Gamma$ is a free product then the result follows easily by an induction argument, since H must be a subgroup of a conjugate of one of the free factors.

If $G\Gamma = M \times N$, then H is a subgroup of a conjugate of $M_H \times N_H$, where M_H, N_H are finite subgroups of M, N respectively. By induction M_H, N_H are thus subgroups of conjugates of direct truncated subgroups of M, N respectively, and hence the result holds.

In the general case, when $G\Gamma \equiv \langle X, Y; K \rangle$ where $X \equiv A \times K$, for some generating group A, H must certainly be a subgroup of a conjugate of X or Y, and the result follows easily by induction. □

We state the well known:

THEOREM 4.6 KUROSH SUBGROUP THEOREM.

Let $G \equiv \prod^*_{i=1,2,\dots n} A_i$, *and let* H *be a subgroup of* G. *Then* H *is a free product,* $H = F * (\Pi^* H_j)$, *where* F *is a free group, and each* H_j *is the intersection of* H *with a conjugate of some free factor* A_i *of* G.

In particular then, subgroups of free products are either free groups of rank 1, subgroups of the free factors, or non-trivial free products.

In order to obtain an isomorphism theorem we require the following lemma:

LEMMA 4.7. *Let* $G\Gamma$ *be a graph product of the freely indecomposable groups* $A_1, A_2 \dots, A_n$. *Then* $G\Gamma$ *is a non-trivial free product* $\Leftrightarrow$ Γ *is disconnected.*

Also, if Γ *has connected components* $\Gamma_1, \Gamma_2, \dots, \Gamma_s$, *then*

$$G\Gamma \equiv G\Gamma_1 * G\Gamma_2 * \cdots * G\Gamma_s,$$

where the $G\Gamma_i$ *are the graph products given by the subgraph* Γ_i *and the corresponding generating groups from* $G\Gamma$, *and these* $G\Gamma_i$ *are freely indecomposable.*

PROOF: First consider the easy case $(\Leftarrow)$.

Suppose Γ is disconnected, and has connected components $\Gamma_1, \Gamma_2, \ldots, \Gamma_s$. Then clearly from the definition of graph products, the $G\Gamma_i$, as defined above, are free factors of $G\Gamma$, that is

$$G\Gamma \equiv G\Gamma_1 * G\Gamma_2 * \cdots * G\Gamma_s.$$

($\Rightarrow$). Let $G\Gamma \equiv B * C$, and suppose Γ is connected.

Suppose $[A_a, A_b] \in R_{G\Gamma}$. Then, by theorem 4.6, the group $\langle A_a, A_b \rangle \equiv A_a \times A_b$ is a free product

$$\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z} * \Pi^* \underbrace{\{ \qquad\qquad \}}_{\substack{\text{conjugates of} \\ \text{subgroups of } B,C}}.$$

Now, $A_a \times A_b$ is freely indecomposable and not cyclic, hence

$$A_a \times A_b \subseteq g^{-1}Bg,$$

say. In particular, $A_a, A_b \subseteq g^{-1}Bg$.

Let A_i, A_j be given. Then since Γ is connected, $\exists$ a path in Γ from the vertex corresponding to A_i to that corresponding to A_j. That is, $\exists$ a sequence

$$A_i = A_{(0)}, A_{(1)}, \ldots, A_{(t-1)}, A_{(t)} = A_j$$

such that $[A_s, A_{s+1}] = 1$ for $s = 0, 1, \ldots, t-1$. Suppose, without loss of generality, that $A_s \subseteq g^{-1}Bg$. Let $a_s \in A_s, a_{s+1} \in A_{s+1}$ be given. Then $\exists b_s \in B$ such that $a_s = g^{-1}b_s g$.

Now a_s, a_{s+1} commute in $B * C$, so by the result form [19] mentioned earlier

$$a_s = h^{-1}\bar{a_s}h$$

$$a_{s+1} = h^{-1}\bar{a_{s+1}}h$$

such that a_s, a_{s+1} are in the same free factor or are powers of the same element. If a_s, a_{s+1} are in the same free factor then we can choose $h = g$ and we have $a_{s+1} = g^{-1}b_{s+1}g$, for some $b_{s+1} \in B$. If a_s, a_{s+1} are powers of the same element, then by a length argument this element must be in B, b say. Thus we can take $h = g$, and $a_{s+1} = g^{-1}b_{s+1}g$ for some $b_{s+1} \in B$.

So by induction on t, and since A_i, A_j were arbitrary,

$$A_1, A_2, \ldots, A_n \subseteq g^{-1}Bg.$$

Thus $G\Gamma \subseteq g^{-1}Bg$ and $C = \langle 1 \rangle$. This is a contradiction, so Γ is disconnected.

Therefore we see that graph products of freely indecomposable groups given by connected graphs are freely indecomposable. Clearly then, if Γ has connected components $\Gamma_1, \Gamma_2, \ldots, \Gamma_s$ then

$$G\Gamma \equiv G\Gamma_1 * G\Gamma_2 * \cdots * G\Gamma_s,$$

where the $G\Gamma_i$ are freely indecomposable. □

We state the following result from [18]:

THEOREM 4.8 ISOMORPHISM THEOREM FOR FREE PRODUCTS.

Let $G \equiv A_1 * A_2 * \cdots * A_n \equiv B_1 * B_2 * \ldots B_m$,

where A_i, B_j *are finitely generated non-trivial freely indecomposable* *groups, for* $i = 1,2,\ldots,n$: $j = 1,2,\ldots,m$*. Then* $n = m$ *and* $\exists$ *a permutation* ϕ *of* $\{1,2,\ldots,n\}$ *such that* $A_i \equiv B_{\phi(i)}$ *for* $i = 1,2,\ldots,n$*. That is, free decompositions of finitely many freely indecomposable groups are unique up to isomorphisms of free factors.*

We generalise this result to graph products for the following special case:

THEOREM 4.9. *Let* $G\Gamma$ *be the graph product of the distinct, non isomorphic prime cycles* $P_1, P_2, \ldots, P_n$*, given by the graph* Γ *and the group vertex assignment* ϕ*. Let* $H\Delta$ *be a graph product of the prime cyclic groups* $A_1, A_2, \ldots, A_m$*. Then* $G\Gamma \equiv H\Delta \Rightarrow$

(i) $n = m$,

(ii) $\Gamma \equiv \Delta$,

(iii) we can relabel the A_i *such that the same group vertex applies. That is, groups at corresponding vertices are isomorphic under some isomorphism of the graphs.*

PROOF: Let the group P_i be the group generated by an element g_i of order p_i for $i = 1,2,\ldots,n$. Now $\frac{H}{[H,H]} \equiv \frac{G}{[G,G]}$ = the direct product of the generating groups, that is, a cycle of order $p_1 p_2 \ldots p_n$. Thus $n = m$ and the generating groups of $H\Delta$ are of orders $p_1, p_2, \ldots, p_n$, that is, precisely the same prime cycles appear as generating groups in $H\Delta$ as in $G\Gamma$. We can relabel them so that $P_i \equiv A_i$ for $i = 1,2,\ldots,n$. For each g_i $\exists h_i \in H\Delta$ corresponding to g_i under some given isomorphism. So h_i has order p_i and by the torsion theorem for graph products $h_i \in$ a conjugate of a direct truncated subgroup. Thus $h_i = c_i^{-1} z_i c_i$ for $c_i \in H\Delta$, of minimal length, and z_i a generator of A_i, for $i = 1, 2, \ldots, n$.

Suppose Γ is connected.

Now a relator appears in $G\Gamma \Leftrightarrow$ the corresponding relator appears in $H\Gamma$. Thus

$$[g_i, g_j] = 1_{G\Gamma} \Leftrightarrow [c_i^{-1} z_i c_i, c_j^{-1} z_j c_j] = 1_{H\Delta}$$

$$\Leftrightarrow c_1^{-1} z_i c_i c_j^{-1} z_j c_j c_i^{-1} z_i^{-1} c_i c_j^{-1} z_j^{-1} c_j = 1.$$

We wish to show that $[z_i, z_j] = 1$. If $c_i = c_j$ the result is obvious, so suppose not.

We have two cases:

Case 1 $\quad c_j = dc_i$, in reduced form.

Case 2 $\quad$ For all reduced words v, w such that c_j has reduced form $c_j = vw$, $w \neq c_i$.

Firstly consider case 1. There are three subcases:

Subcase A $\quad d = fz_i$ is a reduced form of d, for some f;

Subcase B $\quad d = fz$ is a reduced from of d, for some f, where $z \in P_i$;

Subcase C $\quad$ No syllable which can commute to the end of d is in the same generating group as z_i.

In subcase A

$$
\begin{aligned}
&[c_i^{-1}z_ic_i, c_j^{-1}z_jc_j] \\
&= c_i^{-1}z_ic_ic_j^{-1}z_jc_jc_i^{-1}z_i^{-1}c_ic_j^{-1}z_j^{-1}c_j \\
&= z_id^{-1}z_jdz_i^{-1}d^{-1}z_j^{-1}d \\
&= 1.
\end{aligned}
$$

Now $d = fz_i$, and so we have

$$f^{-1}z_jfz_i^{-1}f^{-1}z_j^{-1}fz_i = 1,$$

and $f \neq 1$ since $c_i \neq c_j$. Now $c_j^{-1}z_jc_j$ is reduced, and therefore $f^{-1}z_jf$ is reduced. fz_i is reduced, thus fz_i^{-1} is also reduced, as any syllable commuting to the end of f to cancel or amalgamate with z_i^{-1} would certainly amalgamate or cancel with z_i. We note that $z_jz_i^{-1}$ is reduced. If z_i commutes with z_j we have finished, so suppose not.

Then $f^{-1}z_jfz_i^{-1}$ is reduced. Now z_if is reduced, therefore $f^{-1}z_i^{-1}$ is reduced. If $[z_i, f] = 1$ then

$$
\begin{aligned}
f^{-1}z_jfz_i^{-1}f^{-1}z_j^{-1}fz_i &= \\
f^{-1}z_jz_i^{-1}z_j^{-1}z_if &= 1 \\
\Rightarrow [z_i, z_j] &= 1
\end{aligned}
$$

which is a contradiction, so z_i and f do not commute, and $f^{-1}z_jfz_i^{-1}f^{-1}$ is reduced. By similar arguments

$$f^{-1}z_jfz_i^{-1}f^{-1}z_j^{-1}$$

and

$$f^{-1}z_jfz_i^{-1}f^{-1}z_j^{-1}f$$

are both reduced. Finally, $f^{-1}z_jfz_i^{-1}f^{-1}z_j^{-1}fz_i$ is reduced, which gives a contradiction since it has length greater than 1.

Thus $[z_i, z_j] = 1$ in this case.

Now consider subcase B. Again we have

$$z_id^{-1}z_jdz_i^{-1}d^{-1}z_j^{-1}d = 1,$$

and $d = fz$, where $z = z_i^{\alpha}$, say. Thus

$$z_i^{1-\alpha}f^{-1}z_jfz_i^{-1}f^{-1}z_j^{-1}fz_i^{\alpha} = 1$$

$$\Rightarrow z_if^{-1}z_jfz_i^{-1}f^{-1}z_j^{-1}f = 1.$$

Again, if $[z_i, f] = 1$ then $[z_i, z_j] = 1$ and we have finished, so suppose not. Now we proceed with a length argument as above and arrive at a contradiction.

In subcase C we repeat the argument of subcase B with f replaced by d.

Case 2 Let the reduced form of $c_ic_j^{-1}$ be u; we know that u contains syllables from c_i. We have

$$c_i^{-1}z_ic_ic_j^{-1}z_jc_jc_i^{-1}z_i^{-1}c_ic_j^{-1}z_j^{-1}c_j =$$

$$c_i^{-1}z_iuz_ju^{-1}z_i^{-1}uz_j^{-1}c_j = 1.$$

Now, if u commutes with z_i or z_j the proof is complete, so suppose not. Now $c_i^{-1}z_i$ is reduced since $c_i^{-1}z_ic_i$ is reduced. We can proceed as in case 1, replacing d by u, and considering the three possible subcases. The arguments are identical.

Thus the graphs are isomorphic and the same group vertex assignment applies.

Now consider the case when Γ is disconnected. Let Γ have connected components $\Gamma_1, \Gamma_2, \ldots, \Gamma_s$. By Lemma 4.7

$$G\Gamma \equiv G\Gamma_1 * G\Gamma_2 * \cdots * G\Gamma_s,$$

where each $G\Gamma_i$ is freely indecomposable, and by Theorem 4.8, since $s \neq 1$, $H\Delta$ is a non-trivial free product of the freely indecomposable groups $K_1, K_2, \ldots, K_s$, where $K_i \equiv G\Gamma_i$ for $i = 1, 2, \ldots, s$.

Now, by Lemma 4.7 Δ must have connected components $\Delta_1, \Delta_2, \ldots, \Delta_s$ such that

$$H\Delta \equiv H\Delta_1 * H\Delta_2 * \cdots * H\Delta_s,$$

where $H\Delta_i \equiv K_i \equiv G\Gamma_i$, for $i = 1, 2, \ldots, s$, by Theorem 4.8. Now apply the argument for Γ connected to each Γ_i, and hence result. □

Note 4.10

Recall Example 3.4. We see that it is not sufficient for the graphs to be isomorphic and the generating groups to be isomorphic in pairs, but an isomorphism of the graphs must take generating groups to their isomorphic copies. At the other extreme we may consider graph products of *isomorphic* prime cycles.

LEMMA 4.11. *Let $G\Gamma$ be a graph product of n generating groups each isomorphic to the prime cycle P of order p. Let $H\Delta$ be a graph product* *of m generating groups each isomorphic to the prime cycle P of order p. Then $G\Gamma \equiv H\Delta \Rightarrow$*

(i) $n = m$,

(ii) $\Gamma \equiv \Delta$.

PROOF: Now $m = n$ since

$$\frac{H}{[H,H]} \equiv \frac{G}{[G,G]} = \underbrace{P \times P \times \cdots \times P}_{n \text{ times}}.$$

Suppose Γ is connected. Label the generating groups of $G\Gamma$, $A_1, A_2, \ldots, A_n$, and of $H\Delta$, $B_1, B_2, \ldots, B_n$. We can relabel the B_j so that, as in Theorem 4.9, $A_i \to c_i^{-1} B_i c_i$, under some isomorphism from $G\Gamma$ to $H\Delta$. Suppose A_1 and A_2 commute (relabelling if necessary). For an isomorphism ϕ we have

$$\phi(a_1) = c_1^{-1} p_1 c_1, \phi(a_2) = c_2^{-1} p_2 c_2,$$

where $p_1 \neq p_2$, since the group generated by A_1 and A_2 in $G\Gamma$ is certainly not isomorphic to a cylic group. We assume that the forms given are reduced as any part of a c_i which commutes through p_i can be cancelled and a relabelling take place. We also have $[a_1, a_2] = 1$ and we want to show that p_1 and p_2 commute. Now

$$[c_1^{-1} p_1 c_1, c_2^{-1} p_2 c_2] = 1.$$

If $c_1 = c_2$ the result is obvious, so suppose not.

We have two cases:

Case 1 $\quad c_2 = dc_1$, in reduced form.

Case 2 For all reduced words v, w such that c_2 has reduced form $c_2 = vw$, $w \neq c_1$.

We proceed as in the proof of Lemma 4.11, and see that

$$[a_1, a_2] = 1 \Leftrightarrow [c_1, c_2] = 1.$$

So $\Gamma \equiv \Delta$.

Now suppose Γ is not connected, but has connected components $\Gamma_1, \Gamma_2, \ldots, \Gamma_s$. Then, as in Theorem 4.9 we can apply Lemma 4.7 and Theorem 4.8 and use the connected case.

Thus the result is shown. □

We will now generalise Theorem 4.9 to graph products of any prime cycles, in the following:

THEOREM 4.12. *Let $G\Gamma$ be the graph product of the prime cycles $P_1, P_2, \ldots, P_n$, given by the graph Γ and the group vertex assignment ϕ. Let $H\Delta$ be a graph product of the prime cyclic groups $A_1, A_2, \ldots A_m$. Then $G\Gamma \equiv H\Delta \Rightarrow$*

(i) $n = m$,

(ii) $\Gamma \equiv \Delta$,

(iii) we can relabel the A_i such that the same group vertex applies. That is, groups at corresponding vertices are isomorphic under some isomorphism of the graphs.

PROOF: Consider a generating group P of $G\Gamma$ generated by an element

g of order p. Consider the truncated subgroup of $G\Gamma$ generated by the generating groups of order p. This truncated subgroup is clearly isomorphic to its counterpart in $H\Delta$, thus by Lemma 4.11 the number of generating groups involved is equal in the two truncated subgroups and their graphs are isomorphic.

Let $G\Gamma$ have distinct prime generating groups of orders $p_1, p_2, \ldots, p_t$ with n_i copies of each P_i. Clearly, by the remarks above $H\Delta$ has exactly n_i copies of each P_i.

As in Lemma 4.11 we can assume Γ is connected, for, if it were not, $G\Gamma$ would be the ordinary free product of the graph products given by the connected components of Γ and we could proceed using Lemma 4.7 and Theorem 4.8.

Under a given isomorphism from $G\Gamma$ to $H\Delta$, using Lemma 4.11, for a prime p_i, the n_i copies of P are denoted $P_{i_1}, P_{i_2}, \ldots, P_{i_{n_i}}$ and map to $c_i^{-1}A_{i_1}c_i, c_i^{-1}A_{i_2}c_i, \ldots, c_i^{-1}A_{i_{n_i}}c_i$ where $A_{i_1}, A_{i_2}, \ldots, A_{i_{n_i}}$ are the n_i copies of P among the generating groups of $H\Delta$.

Now, consider P_{i_a}, P_{j_b} where $i \neq j$.

$$[P_{i_a}, P_{j_b}] = 1_{G\Gamma} \Leftrightarrow [c_i^{-1}A_{i_a}c_i, c_j^{-1}A_{j_b}c_j] = 1_{H\Delta}$$

$$\Leftrightarrow [A_{i_a}, A_{j_b}] = 1_{H\Delta}$$

by an argument identical to that used in Lemma 4.11 Therefore $\Gamma \equiv \Delta$, and the generating groups correspond. □

Chapter 5 Residual Properties of Graph Products

In this chapter we explore the preserving of residual properties by graph products. Many such properties are preserved in free products and we seek to generalise these results. We first turn our attention to residual finiteness.

Lemma 5.1. *Let X be a graph product on n groups. Let R be the subgroup of X generated by all but one of the generating groups. Then any normal subgroup of finite index in R can be extended to a normal subgroup of finite index (f.i.) in X.*

Proof: We proceed by induction on n.

For $n = 2$, we have two possible cases:

Case 1 $X = \langle A, B; \rangle = A * B$.

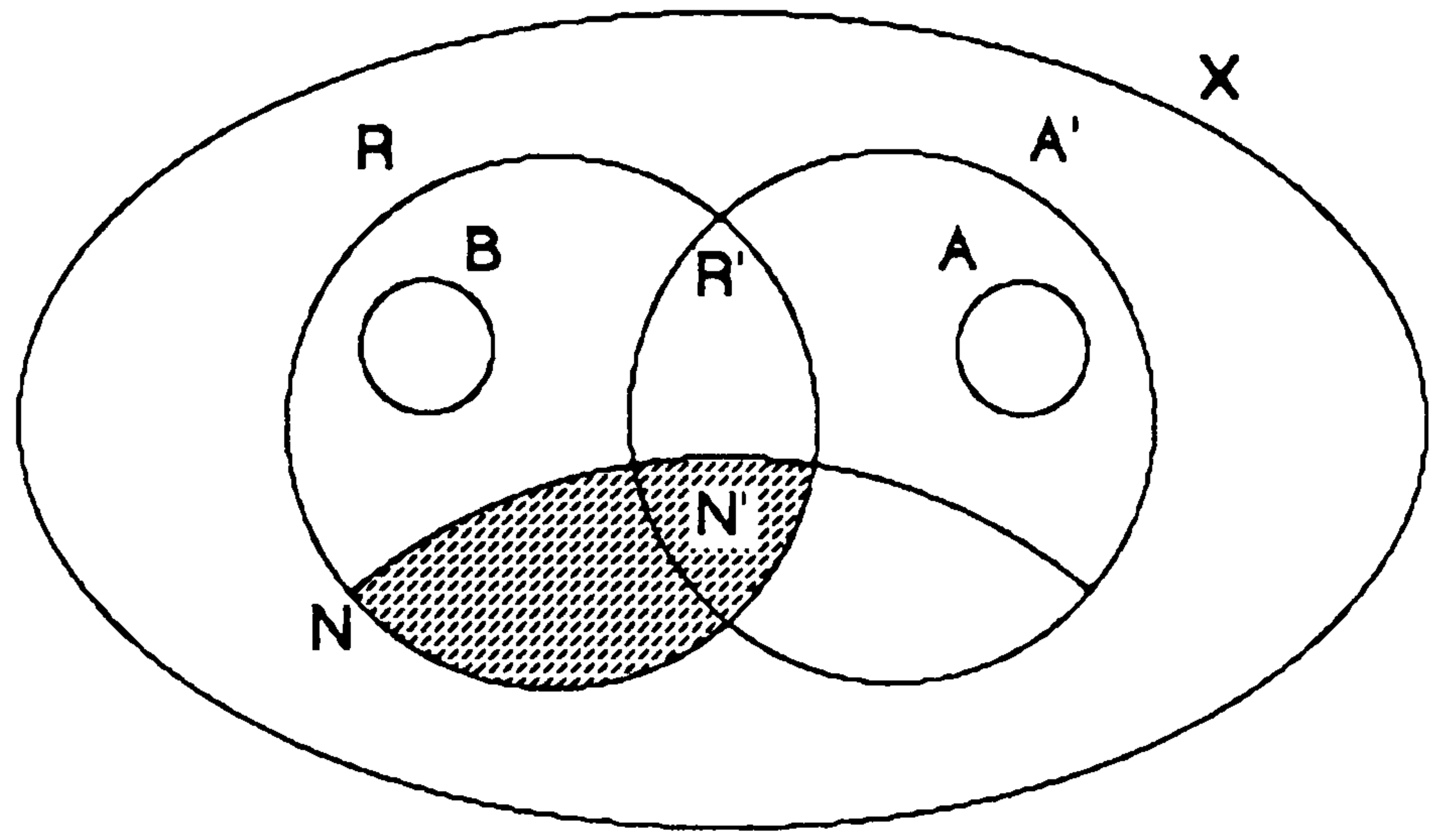
If $N \underset{\text{f.i.}}{\lhd} A$, then factoring out B and N gives the homomorphic image $\frac{A}{N}$ of X, which is finite. So, N can be extended since $(\ker: X \longrightarrow \frac{A}{N}) \cap A = N$.

Case 2 $X = \langle A, B; [A, B] \rangle = A \times B$.

If $N \underset{\text{f.i.}}{\lhd} A$ then $N \times B$ will do.

Suppose inductively that extension is possible in graph products of $\leq n - 1$ generating groups. Let X be a graph product of n groups. Let R be a subgroup as described above, let A be the omitted generator group, and $N \underset{\text{f.i.}}{\lhd} R$ be the given normal subgroup. If A commutes with all of the other generator groups then $X = R \times A$, so N extends to $N \times A$.

Suppose then, $\exists B$ among the generator groups such that A does not commute with B.



(The above picture is merely an illustration, as a normal subgroup of finite index in R would, in general, intersect with B, and similarly for A in A'.)

So $X \equiv R \underset{R'}{*} A'$, that is the generalised free product of R with A' amalgamating R', where A' is the subgroup of X generated by all the generator groups but B, and $R' = R \cap A'$.

Let $N' = N \cap R'$, so $N' \underset{\text{f.i.}}{\triangleleft} R'$.

Thus by induction hypothesis, N' extends to a normal subgroup of finite index in A'. So X maps homomorphically onto a generalised free product of finite groups which is residually finite. So, N infact extends to a normal subgroup of finite index in X.

Hence result by induction. $\square$

Definition 5.2

Let $G\Gamma$ be a graph product of groups. Let H be given such that H is generated by certain of the generating groups of $G\Gamma$. We call H a *truncated subgroup* of $G\Gamma$. Let $g \in G\Gamma$ be given such that $g \notin H$. If for all such H, g $\exists$ a finite homomorphic image $\bar{G\Gamma}$ of $G\Gamma$ such that $\bar{g} \notin \bar{H}$ we say that $G\Gamma$ is *truncated subgroup separable.*

Clearly this is a property akin to LERFness.

THEOREM 5.3. *Graph products of residually finite groups are truncated subgroup separable.*

PROOF: Let G be a graph product of the residually finite groups $A_1, A_2, \ldots A_n$. Let K be a given truncated subgroup and $g \in G \backslash K$, a given element. Let $A_i = A$, say, be one of the generating groups not contained in K. Let H denote the truncated subgroup of G containing all generating groups which commute with A, and let Y denote the truncated subgroup of G containing all generating groups but A.

With $X = A \times H$, as usual we have a generalised representation for G, given by

$$G = \langle X, Y; H \rangle .$$

If $H = Y$, then $G = A \times H$ and thus $g = ah$, for $a \in A,\ h \in H$. If $a = 1$, we obtain the result by the subgroup separability of H and induction. If $a \neq 1$, we factor out H^G and use the residual finiteness of A.

If $H = \langle 1 \rangle$, or if G is a non-trivial generalised free product proceed as follows:

(K may be trivial, the whole of Y, or in between).

Now g has generalised free product normal form,

$$g = (a_1)c_1a_2c_2 \dots a_r(c_r),$$

where we can suppose $c_i \notin H \quad \forall i$, unless $g \in X$.

In the case when $|g| \leq 1$ we have either $g \in X \backslash Y$ or $g \in Y$. If $g \in Y$ the result is immediate, since Y is truncated subgroup separable by induction. If $g \in X \backslash Y$ then $g = ah$, $1 \neq a \in A, h \in H$. We factor out Y^G and use the residual finiteness of A. Now it remains to consider the case when $|g| \geq 2$.

By induction, Y is truncated subgroup separable. Using the truncated subgroup H we find a finite homomorphic image $\bar{Y}$ of Y such that

$$\bar{c}_1 \notin \bar{H}, \bar{c}_2 \notin \bar{H}, \dots (\bar{c}_r \notin \bar{H}).$$

Similarly, there exists a finite homomorphic image $\bar{A}$ of A such that

$$(\bar{a}_1 \neq 1), \bar{a}_2 \neq 1, \dots \bar{a}_r \neq 1.$$

Now we pass to

$$\bar{G} = \langle \bar{A} \times \bar{H}, \bar{Y}; \bar{H} \rangle,$$

which is a generalised free product of finite groups and therefore residually finite. Now $\bar{K} \subseteq \bar{Y}$, and $\bar{g}$ has preserved its generalised free product length, thus $\bar{g} \notin \bar{K}$.

Then there exists a finite homomrphic image $\bar{\bar{G}}$ of G in which $\bar{\bar{g}} \notin \bar{\bar{K}}$.

□

As an immediate corollary we have:

COROLLARY 5.4. *Graph products of residually finite groups are redisually finite.*

We cannot extend this result to LERF because, as we saw in Counter Example 2.13 $F_2 \times F_2$ is not LERF, and F_2 is LERF since it is free.

However, we can extend our results in the direction of residual finite p-ness. We first require another definition.

Definition 5.5

Let $G\Gamma$ be a graph product of groups. Let H be a truncated subgroup of $G\Gamma$. Let $g \in G\Gamma$ be given such that $g \notin H$. If for all such H, g $\exists$ a finite-p group, homomorphic image $\bar{G\Gamma}$ of $G\Gamma$ such that $\bar{g} \notin \bar{H}$ we say that $G\Gamma$ is *truncated subgroup p-separable.*

THEOREM 5.6. *Graph products of residually finite-p groups are truncated subgroup p-separable.*

PROOF: We follow the proof of Theorem 5.3 until we pass to

$$\bar{G} = \left\langle \bar{A} \times \bar{H}; \bar{Y} \right\rangle,$$

which, in this case is a generalised free product of finite-p groups. Now $\bar{A} \times \bar{Y}$ is finite p, thus $\bar{G}$ is free by finite-p, and hence residually finite-p, by [13].

The proof concludes in the same way as the proof of theorem 5.3. □

From Theorem 5.6 we know that graph groups are truncated subgroup p-separable, and thus residually finite-p. This is a direct proof of Corollary 2.10, as was promised.

We now turn our attention to potency. We will generalise Theorem 2.17 in the setting of graph products.

It is not known whether the free product of two potent groups is again potent but a result can be obtained if we include the additional property of torsion freeness. We will therefore use this property for our case.

LEMMA 5.7. *Direct products of torsion free potent groups are potent.*

PROOF: Let A, B be torsion free potent, and $G = A \times B$. Let $g \in G$, $g \neq 1$, and $n \in \mathbb{Z}$ be given. We can write $g = ab$, $a \in A, b \in B$.

If $a = 1$ we use the fact that B is potent.

If $a \neq 1$, we factor out B and use the fact that A is potent. □

We state without proof the following result from [1]:

LEMMA 5.8. *Free products of torsion free potent groups are (torsion free) potent.*

LEMMA 5.9. *Let G be the generalised free product of the groups X and Y amalgamating the subgroup B such that $X = A \times B$, for some subgroup A. Let $g \in G$ be an element of length ≥ 2 in the sense of generalised free products. If*

$$g = (a_1)y_1 a_2 y_2 \ldots a_r(y_r) \quad \text{where} \quad (a_1)a_2 \ldots a_r = 1 = y_1 y_2 \ldots y_{r-1}(y_r).$$

then $g \in [A, Y]$.

PROOF: Since $G = gp\{A, Y\}$, $[A, Y] \lhd G$. If we factor out $[A, Y]$ then

$$g \longrightarrow (\bar{a_1})\bar{a_2} \ldots \bar{a_r}\bar{y_1}\bar{y_2} \ldots (\bar{y_r}),$$

since $[\bar{A}, \bar{Y}] = 1$. Therefore $g \longrightarrow \bar{1}$. Thus $g \in [A, Y]$. □

From [19], Corollary 4.9.2, we have the following result:

THEOREM 5.10. *Let G be a free product of A, B, C with amalgamation from the factor A, that is, all defining relators either involve one type of generator, or have the form $U(a_\nu) = V(b_\mu)$ or $U(a_\nu) = W(c_\zeta)$. Then any subgroup H of G whose intersection with the conjugates of A, B and C is 1, must be a free group.*

Again from [1] we have:

THEOREM 5.11. *Let G be a group which is a finite extension of a finitely generated free subgroup F. Then for each $1 \neq z \in F$ and for each $n \geq 1$ in $\mathbb{Z}$, $\exists$ a homomorphism θ of G onto a finite group such that $z\theta$ has order exactly n.*

We now proceed to the theorem.

THEOREM 5.12. *Let $G\Gamma$ be a graph product of torsion free potent groups. Then $G\Gamma$ is torsion free potent.*

PROOF: Let $1 \neq g \in G\Gamma$ and $n \in \mathbb{Z}$ be given. Now, as usual $G\Gamma$ is either a direct product, a free product or a generalised free product of certain truncated subgroups. The first two cases are dealt with by Lemmas 5.3 and 5.4 respectively.

Let A be a generating group of $G\Gamma$ and H the truncated subgroup of all generating groups commuting with A. Let Y, C be the truncated subgroups given by all the generating groups excluding A and A, H respectively. Then as usual we have $G = X \underset{H}{*} Y$, where $X = A \times H$.

If $g \in X$ (respectively Y) we factor out C (respectively A) and proceed by induction on the number of generating groups. We have the following case remaining:

$$g = (a_1)y_1a_2y_2\dots a_r(y_r), \qquad \text{where} \quad a_i \in A,\ y_i \in Y\backslash H,$$

and length $g \geq 2$.

If $a_1a_2\dots a_r \neq 1$ we factor out Y and use the potency of A.

If $y_1y_2\dots y_r \neq 1$ we factor out A and proceed by induction. Suppose

$$a_1a_2\dots a_r = y_1y_2\dots y_r = 1.$$

Then by Lemma 5.9 $g \in [A, Y]$. Now, potency $\Rightarrow$ residual finiteness, (take $n \neq 1$), so by Theorem 5.3 X, Y are truncated subgroup separable.

$$\text{Therefore} \quad \exists N_i \underset{\text{f.i.}}{\lhd} X \quad \text{such that} \quad a_i \notin \bar{H} \quad \text{in} \quad \frac{X}{N_i},$$

$$\text{and} \quad \exists M_i \underset{\text{f.i.}}{\lhd} Y \quad \text{such that} \quad y_i \notin \bar{H} \quad \text{in} \quad \frac{Y}{M_i},$$

Now, we can map $\bar{G}$ naturally onto $\bar{A} \times \bar{Y}$, and the kernel of this mapping is $[\bar{A}, \bar{Y}]^{\bar{G}}$.

By Theorem 5.10 $[\bar{A}, \bar{Y}]^{\bar{G}}$ is free since it intersects neither $\bar{X}$ nor $\bar{Y}$, and it is certainly finitely generated since $\bar{A}, \bar{Y}$ are finite.

So, since $\bar{A} \times \bar{Y}$ is finite, $\bar{G}$ is a finite extension of a finitely generated free group, and we have the result by Theorem 5.11. □

In Chapter 3 we showed that a solvable conjugacy problem is preserved by graph products. We will attempt to take this a step further and consider conjugacy separability in graph products. Joan Dyer [11] has a powerful result, namely that free by finite groups are conjugacy separable. It would be convenient to employ this in the case of graph products of conjugacy separable groups, but we will show that such groups are not free by finite in general. We first require the following theorem:

THEOREM 5.13. *Finitely generated free by finite groups are LERF.*

PROOF: Let G be a finitely generated free by finite group with F a free normal subgroup of finite index in G. Let $g \in G$ and $S \subseteq G$ be given such that $g \notin S$ and S is finitely generated. We want to show the existance of a finite homomorphic image $\bar{G}$ of G in which the image of g does not lie in the image of S, that is $\bar{g} \notin \bar{S}$.

Now F is of finite index in the finitely generated group G, thus F is finitely generated. Suppose $g \in F$. Clearly $F \cap S$ is of finite index in S, so $F \cap S$ is finitely generated. Now $g \notin F \cap S \subset F$, and F is LERF, since it is free, so there exists a normal subgroup N, of finite index in F, such that $g \notin N(F \cap S)$.

Let M be the intersection, over the automorphisms θ, of F, of the $N\theta$. Then M is characteristic in F, which is normal in G, therefore M is normal in G.

Now, there are only finitely many normal subgroups of a given finite index in a finitely generated group. Therefore, there are only finitely many $N\theta$ where $\theta \in \mathrm{Aut}\, F$. Thus M is of finite index in G.

$g \notin N(F \cap S)$ so certainly $g \notin MS$. Thus factoring out M we have the required finite homomorphic image.

Now suppose $g \notin F$. Let $g = af$, where $1 \neq a \in G$, $f \in F$. If we factor out F and $\bar{g} \notin \bar{S}$ then we have finished, so suppose $\bar{g} \in \bar{S}$, that is $g = sf_1$, where $s \in S$, $f_1 \in F$, $s \notin F$, $f_1 \notin S$.

As above, we choose M, normal of finite index in G, such that $f_1 \notin M(F \cap S)$. If $g \notin SM$, we have finished, so suppose $g \in SM$. Then $sf_1 = s_1 m$, where $s_1 \in S$, $m \in M$. Thus $f_1 = s^{-1}s_1 m \in (S \cap F)M$. This is a contradiction and hence the result is shown. □

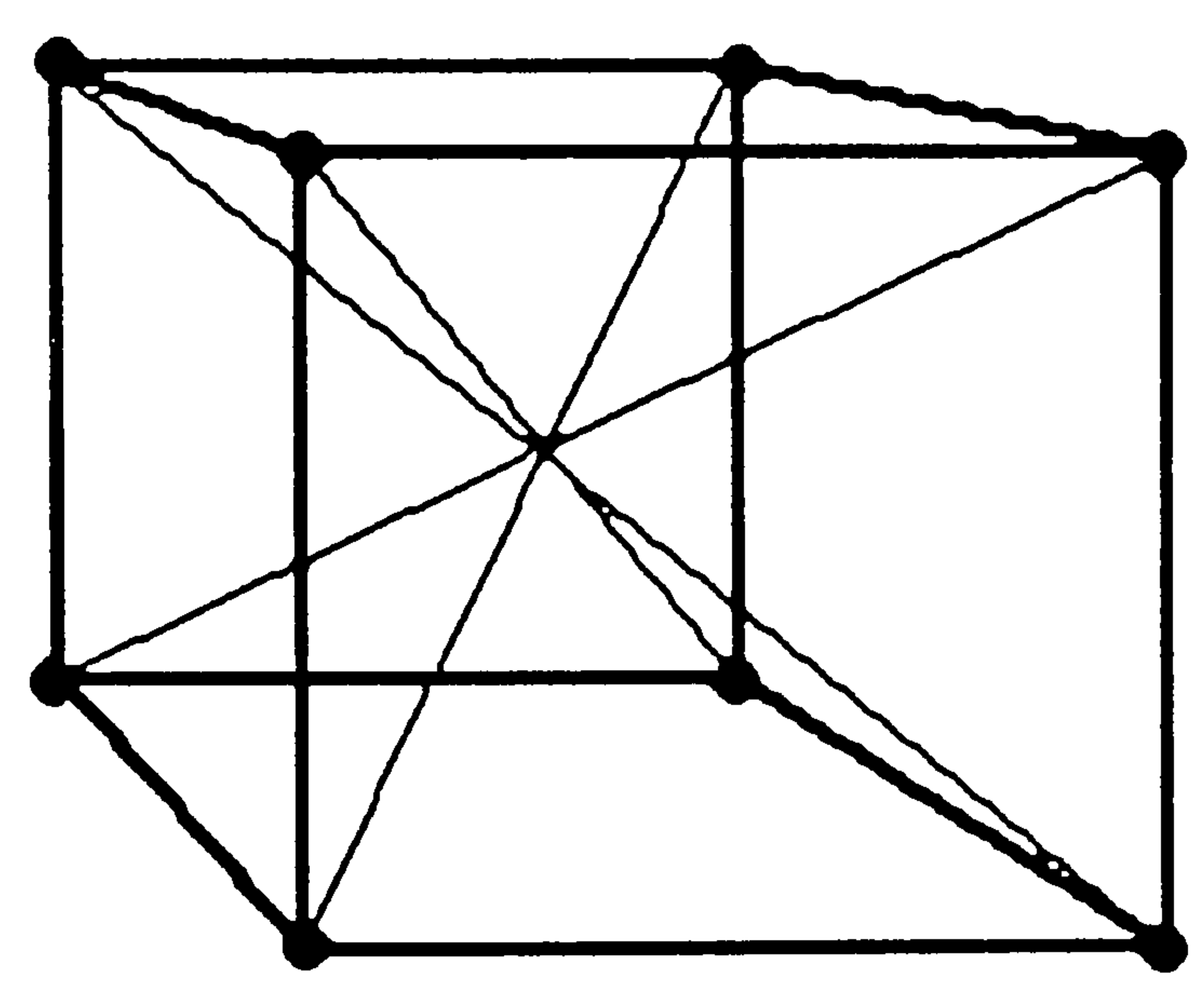
Graph products of finitely generated groups are finitely generated, but are not LERF, in general. Thus by the above theorem they are not free by finite in general. However, we would still be able to use Dyer's result if graph products of finite groups were free by finite. We see in the following example that even this is not true.

Counter Example 5.14

Let A, B, C, D, E, F, G and H be non trivial finite groups. We will show that the graph product

$$G\Gamma = \langle A, B, C, D, E, F, G, H; [A,C], [A,D], [B,C], [B,D], [E,G], [E,H], [F,G], [F,H], [A,E], [A,F], [B,E], [B,F], [C,G], [C,H], [D,G], [D,H] \rangle$$

is not free by finite. The graph Γ is illustrated below.



The upper four vertices are assigned the groups A, C, B, D, and the corresponding lower four vertices, the groups E, G, F, H, respectively.

Let $1 \neq a \in A$, $1 \neq b \in B$, ..., $1 \neq h \in H$.

The element ab is of infinite order in $G\Gamma$, by the torsion theorem. Thus $\langle a\, b \rangle \equiv \mathbb{Z}$. Similarly

$$\langle a\, b \rangle \equiv \langle c\, d \rangle \equiv \langle e\, f \rangle \equiv \langle g\, h \rangle \equiv \mathbb{Z}.$$

Now clearly

$$\langle ab, gh \rangle \equiv \langle cd, ef \rangle \equiv F_2.$$

But $[ab, cd] = [ab, ef] = [gh, cd] = [gh, ef] = 1$, thus

$$\langle ab, cd, ef, gh \rangle \equiv F_2 \times F_2 \subseteq G\Gamma.$$

Now, subgroups of LERF groups are LERF, as is easily seen from the definition, but $F_2 \times F_2$ is not LERF (Counter Example 2.13), therefore $G\Gamma$ is not LERF. However, $G\Gamma$ is finitely generated, so $G\Gamma$ is not free by finite.

We have two obvious methods to use in attempting a conjugacy separability result for graph products of conjugacy separable groups. We can regard graph products as free products with a certain amount of commuting, or as generalised free products of a rather special type.

In the first case we might elect to follow Stebe's work in proving that free products of conjugacy separable groups are conjugacy separable in [28]. Here he maps down to free products of finite groups and then uses their free-by-finiteness. Graph products of finite groups are not free by finite, so, were we to follow Stebe's arguments, we would, at this point, have to revert to our second approach, and regard the graph product of finite groups as a generalised free product. However, this leads to further difficulty as I will demonstrate.

Let $A_1, A_2, \ldots A_n$ be finite groups and $G\Gamma$ a graph product of the A_i's. Let $a, b \in G\Gamma$ be given such that $a \not\sim b$. We can find $g, h \in G\Gamma$, such that g, h are proper cyclically reduced and $g \sim a$, $h \sim b$. So $g \not\sim h$.

Suppose $\lambda(g) = \lambda(h) = 1$. Then $g \in A_i$, $h \in A_j$ for some $i, j \in \{1, 2, \ldots, n\}$.

If $i = j$, we can map $G\Gamma$ naturally onto A_i, and A_i is finite with $\bar{g} \not\sim \bar{h} \Rightarrow \bar{a} \not\sim \bar{b}$.

If $i \neq j$, then either $A_i \times A_j$ or $A_i * A_j$ is a truncated subgroup of $G\Gamma$. It is easy to see that truncated subgroups of graph products are homomorphic images of the graph product concerned. Thus we can map $G\Gamma$ to $\langle A_i, A_j \rangle$.

If $\langle A_i, A_j \rangle = A_i \times A_j$ we are finished as this group is finite and $\bar{g} \not\sim \bar{h} \Rightarrow \bar{a} \not\sim \bar{b}$.

If $\langle A_i, A_j \rangle = A_i * A_j$ we have $\bar{g} \not\sim \bar{h}$. Now, free products of finitely many finite groups are conjugacy separable by Lemma 4 in [28], and hence we have the result.

So, suppose, without loss of generality, that $\lambda(g) > 1$.

Consider A_i for some i.

If A_i is a direct factor of $G\Gamma$ we have the result by induction on the number of generating groups, since the direct product of two conjugacy separable groups is again conjugacy separable by Lemma 2.14.

If A_i is a free factor of $G\Gamma$ we have the result by induction on the number of generating groups, since the free product of two finite groups is again conjugacy separable by Lemma 4 in [28].

So, suppose neither of the above is the case. As usual we have a generalised free product representation. Denote the truncated subgroup given by those generating groups, other than A_i, commuting with A_i as B, and the truncated subgroup given by those generating groups, other than A_i, not commutimg with A_i as C. We note that B and C are non-trivial. Denote the truncated subgroup given by A_i and B, B and C, respectively as X, Y.

Thus $G\Gamma = X \underset{B}{*} Y$ with $X = A_i \times B$. We can assume that g, h are cyclically reduced in the sense of generalised free products by Theorem 2.6 in [18]. We have several possibilities for g, h. They may be contained in:

$$X \backslash B, \qquad Y \backslash B, \qquad B, \qquad G\Gamma \backslash \{X \cup Y\}.$$

Without loss of generality we have the following cases for g, h:

Case 1 $g \in X \backslash B$, $h \in X \backslash B$.

Case 2 $g \in X \backslash B$, $h \in Y \backslash B$.

Case 3 $g \in X \backslash B$, $h \in B$.

Case 4 $g \in Y \backslash B$, $h \in Y \backslash B$.

Case 5 $g \in Y \backslash B$, $h \in B$.

Case 6 $g \in B$, $h \in B$.

Case 7 $g \in G\Gamma \backslash \{X \cup Y\}$. That is, g has generalised free product reduced normal form $c_1 c_2 \dots c_r$, where $r \geq 2$.

We may proceed in our attempted proof of the conjugacy separability of $\bar{G}\Gamma$, by induction on n.

If $n = 1$ the result is trivial.

If $n = 2, 3$ we have two cases, $G\Gamma$ is either a free or a direct product and we have covered these possibilities above.

Suppose then that $n \geq 4$, and suppose inductively that graph products with fewer that n finite generating groups are conjugacy separable.

To deal with Cases 1, 3, 4, 5, 6 we factor out

$$C^{G\Gamma}, \quad C^{G\Gamma}, \quad A_i^{G\Gamma}, \quad A_i^{G\Gamma}, \quad \langle A_i; C\rangle^{G\Gamma}, \quad \text{respectively},$$

and map $G\Gamma$ onto X, X, Y, Y, and B respectively. The forms of g, h are preserved, and $\bar{g} \not\sim \bar{h}$, so we complete the proof using induction.

For Case 2 we have $g \in X \backslash B$. Thus $g = ab$ where $1 \neq a \in A_i$, $b \in B$ and $h = y \in Y \backslash B$. As above we factor out $Y^{G\Gamma}$ and map $G\Gamma$ naturally onto A_i. Clearly $\bar{g} = a \neq 1$, and $\bar{h} = 1$. Thus $\bar{g} \not\sim \bar{h}$, and since A_i is finite we are finished.

Finally we have Case 7. Let $\mu(x)$ denote the length of an element in terms of reduced form for generalised free products.

This is where the argument becomes inadequate. In [19] we have the following:

THEOREM 5.15. *Let $P = \langle A * B; H\rangle$, and let $x \in P$ be of minimal length in its conjugacy class. Suppose $y \in P$, y is cyclically reduced, and $y \sim_P x$.*

(1) If $\mu(x) = 0$, then $\mu(y) \leq 1$ and if $y \in A$, say, there is a sequence $h_1, h_2, \ldots, h_r$ of elements in H such that $y \sim_A h_1 \sim_B h_2 \sim_A \cdots \sim_B h_r = x$.

(2) *If* $\mu(x) = 1$, *then* $\mu(y) = 1$ *and either* $x \in A, y \in A$ *and* $x \sim_A y$ *or else* $x \in B, y \in B$ *and* $x \sim_B y$.

(3) *If* $\mu(x) \geq 2$, *then* $\mu(y) = \mu(x)$ *and* $y \sim_H x*$, *where* $x*$ *is a cyclic permutation of* x.

In [12] Dyer comments that any cyclically reduced element of P is of minimal length in its conjugacy class except for elements of length one that are conjugate to elements in H. For such elements we have:

$$\begin{aligned} & a \in A \backslash H \quad \textit{and} \quad \{a\}^A \cap H \neq \phi \\ \textit{or} \quad & a \in B \backslash H \quad \textit{and} \quad \{a\}^A \cap H \neq \phi \end{aligned}$$

She also observes that if $x = u_1 u_2 \ldots u_k$ and $y = v_1 v_2 \ldots v_k$, where both products are alternating, then $x \sim_H y$ if and only if there exists a finite sequence $h_0, h_1, h_2, \ldots, h_k$ of elements in H such that

$$\begin{aligned} u_1 &= h_0^{-1} v_1 h_1 \\ u_2 &= h_1^{-1} v_2 h_2 \\ u_3 &= h_2^{-1} v_3 h_3 \\ \vdots \quad & \quad \vdots \quad \vdots \\ u_k &= h_{k-1}^{-1} v_k h_k \end{aligned}$$

and

$$h_k = h_0,$$

for these conditions are equivalent to $x = h_0^{-1} y h_0$ with $h_0 \in H$.

In our case, with

$$g = (a_1)y_1a_2y_2\dots a_r(y_r)$$

$$h = (\alpha_1)z_1\alpha_2z_2\dots\alpha_r(z_r)$$

with $a_i, \alpha_i \in A$ and $y_i, z_i \in Y \backslash B$, these conditions become

$$a_i = \alpha_i \qquad i = 1, 2, \dots, r.$$

$$y_1 = b_0^{-1}z_1b_1$$

$$y_2 = b_1^{-1}z_2b_2$$

$$y_3 = b_2^{-1}z_3b_3$$

$$\vdots \quad \vdots \quad \vdots$$

$$y_r = b_{r-1}^{-1}z_rb_r$$

and

$$b_r = b_0,$$

where $b_i \in B$ for $i = 0, 1, \dots, r$.

Dyer goes on to show

THEOREM 5.16. *If A and B are conjugacy separable and H is finite, then $\langle A * B; H\rangle$ is conjugacy separable*

The proof uses the easy fact that, in a conjugacy separable group any finite subgroup can be separated from any (disjoint) conjugacy class by a map onto a finite group. It need not be the case that conjugacy classes and arbitrary subgroups are separable, and this provides an obstruction to the conjugacy separability of free products with amalgamation.

Again in [12] Dyer shows:

Lemma 5.17. *Suppose G contains a subgroup H and an element x such that $\{x\}^G$ and H are disjoint, but are not separated by any homomorphism of G with finite image. Then $\langle G * G; H\rangle$ is not conjugacy separable.*

She goes on to say that Miller, [21], by constructing HNN extensions which may be embedded in amalgamated free products in the usual way, has provided examples of residually finite, finitely presented groups with unsolvable conjugacy problem. Thus the class P of amalgamated free products contains residually finite groups that are not conjugacy separable.

Although Dyer proves that generalised free products of finitely generated nilpotent groups amalgamating a cycle, and generalised free products of free groups amalgamating a cycle are conjugacy separable, nothing positive is known when the amalgam is neither finite nor cyclic, which is the case, in general, in graph products. However, we recall special assembly graph groups (Lemma 2.5), and make the following definition:

Definition 5.18

Let $G\Gamma$ be a graph product of groups where Γ has no full subgraph of either of the forms L_3 or C_4. We call $G\Gamma$ a *special assembly graph product.*

$G\Gamma$ is clearly a free product of direct products of free products, and so on, of its generating groups.

Proposition 5.19. *Special assembly graph products of conjugacy separable groups are conjugacy separable.*

PROOF: By induction on the number of generating groups, using Lemma 2.14, and the fact that free products of conjugacy separable groups are conjugacy separable, Stebe, [28]. □

We can obtain a similar result for another class of graph products and first require the following lemma.

LEMMA 5.20. *Let $G\Gamma$ be a graph product of the residually finite groups $A_1, A_2, \ldots, A_n$. Let $g \in G\Gamma$ be a proper cyclically reduced element which does not belong to a direct truncated subgroup of $G\Gamma$. Then for all $h \in G\Gamma$ there are finite homomorphic images $\bar{A_1}, \bar{A_2}, \ldots, \bar{A_n}$ of $A_1, A_2, \ldots, A_n$ such that $\bar{g} \sim \bar{h}$ in $\bar{G\Gamma} \Rightarrow g \sim h$ in $G\Gamma$.*

PROOF: Let $g = g_1 g_2 \ldots g_r$, $r > 1$ be a reduced form for g. Let $h \in G\Gamma$, with reduced form $h = h_1 h_2 \ldots h_s$, $s \geq 1$, be given, such that $g \not\sim h$ in $G\Gamma$.

(We note that if $s = 0$, that is $h = 1$, we have the result since graph products of residually finite groups are residually finite, by Theorem 5.4.)

Consider the syllables of g, h, respectively, which belong to the generating group A_1, and denote them:

$$g_{i_1}, g_{i_2}, \ldots, g_{i_u} \quad \text{and} \quad h_{j_1}, h_{j_2}, \ldots, h_{j_v}.$$

Since A_1 is residually finite, $\exists$ normal subgroups, M_{i_k} of finite index in A_1 such that $g_{i_k} \not\equiv 1 \mod M_{i_k}$ for $i = 1, 2, \ldots, u$. Similarly $\exists P_{j_l} \underset{\text{f.i.}}{\lhd} A_1$ such that $h_{j_l} \not\equiv 1 \mod P_{j_l}$ for $j = 1, 2, \ldots, v$. Also $\exists N_{k_l} \underset{\text{f.i.}}{\lhd} A_1$ such that

$$\begin{cases} N_{k_l} = A_1, & \text{if } g_{i_k} = h_{k_l}; \\ g_{i_k} \not\equiv h_{j_l} \mod N_{k_l}, & \text{if } g_{i_k} \neq h_{j_l} \end{cases}$$

for $k = 1, 2, \ldots, u$, $l = 1, 2, \ldots, v$.

Let $N_1 = \cap\{M_{i_k}, P_{j_l}, N_{k_l}\}$.

Similarly we define $N_2, N_3 \ldots N_n$, normal subgroups of finite index in $A_2, A_3, \ldots, A_n$, respectively.

We can factor out the $N_1, N_2, \ldots, N_n$, thus mapping $G\Gamma$ to $\bar{G}\Gamma$, where $\bar{G}\Gamma$ is the graph product of $\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_n$, given the same graph and group vertex assignment as $G\Gamma$, where $\bar{A}_i = \frac{A_i}{N_i}$ for $i = 1, 2, \ldots, n$.

$$\text{In } \bar{G}\Gamma, \quad \bar{g} = \bar{g_1}\bar{g_2}\ldots\bar{g_r} \qquad (\bar{g_a} \neq 1, \quad a = 1, 2, \ldots, r)$$

$$\bar{h} = \bar{h_1}\bar{h_2}\ldots\bar{h_s} \qquad (\bar{h_b} \neq 1, \quad a = 1, 2, \ldots, s.)$$

Suppose $\bar{g} \sim \bar{h}$ in $\bar{G}\Gamma$. Then by Definition 3.22

$$\bar{g}\,\text{cycle} =_E \bar{h}\,\text{cycle}$$

$$\Rightarrow g\,\text{cycle} =_E h\,\text{cycle}, \qquad \text{by choice of the } N_i,$$

$$\Rightarrow g \sim h \qquad \text{in } G\Gamma.$$

This is a contradiction, thus $\bar{g} \not\sim \bar{h}$. □

Lemma 5.20 can easily be extended to all $g \in G\Gamma$ if we allow the generating groups to be conjugacy separable. We recall the definitions of trees and forests and make the following definition.

Definition 5.21

Let Γ be a forest(tree). Then we call a graph product $G\Gamma$ a *forest(tree) graph product.*

We now have

THEOREM 5.22. *Tree graph products of conjugacy separable groups are conjugacy separable.*

PROOF: Let $G\Gamma$ be a tree graph product of the conjugacy separable groups $A_1, A_2, \ldots, A_n$. If $n = 1, 2$ the result is trivial. If $n \geq 3$, $G\Gamma$ can have no direct truncated subgroup containing more than 2 generating groups. Let $g, h \in G\Gamma$ be given such that $g \not\sim h$.

If $\lambda(g) \leq 2$ we have two cases. If g, h belong to the same direct truncated subgroup, H, of $G\Gamma$, we can map $G\Gamma$ onto H and use the conjugacy separability of H (Lemma 2.14). If g, h do not belong to the same direct truncated subgroup we may map $G\Gamma$ onto a graph product of finite images of the generating groups, as in Lemma 5.20, such that $\bar{g} \not\sim \bar{h}$.

If $\lambda(g) \geq 3$ we can apply Lemma 5.20 to g. We can now assume that the generating groups are finite. Without loss of generality suppose A_1 is assigned to the base of the tree. We have two cases:

Case 1 A_1 commutes with only one other generating group.

Case 2 A_1 commutes with more than one other generating group.

We proceed by induction on n. We first deal with case 1. Suppose the generating group commuting with A_1 is A_j. Then

$$G\Gamma = \langle (A_1 \times A_j) * X; A_j \rangle$$

where X is the truncated subgroup of $G\Gamma$ given by $A_2, A_3, \ldots, A_n$. Now $A_1 \times A_j$ is finite and X is a tree graph products of $n-1$ conjugacy separable groups, so $G\Gamma$ is the generalised free product of two conjugacy separable groups with finite amalgamation, and therefore conjugacy separable, by Theorem 5.16.

Now we look at case 2. Suppose A_1 commutes with $A_{j_1}, A_{j_2}, \ldots, A_{j_k}$, and these groups are assigned to vertices $v_{j_1}, v_{j_2}, \ldots, v_{j_k}$, respectively. Let Γ_1 denote the maximal subtree of Γ containing v_{j_1} but excluding $v_{j_2}, \ldots, v_{j_k}$. Let Γ_2 denote the maximal subtree of Γ containing $v_{j_2}, \ldots, v_{j_k}$ but excluding v_{j_1}. Let $G\Gamma_1$ and $G\Gamma_2$ denote the corresponding truncated subgroups of $G\Gamma$.

By induction $G\Gamma_1$ and $G\Gamma_2$ are conjugacy separable, and clearly

$$G\Gamma \equiv \langle G\Gamma_1 * G\Gamma_2; A_1 \rangle ,$$

and again we have the result by Theorem 5.16 □

COROLLARY 5.23. *Forest graph products of conjugacy separable groups are conjugacy separable.*

PROOF: Using Theorem 5.22 and the fact that free products of cojugacy separable groups are conjugacy separable, [28]. □

We now turn to Stebe's property of Π_c, [27].

THEOREM 5.24. *Graph products of Π_c groups are Π_c.*

PROOF: Let G be a graph product of the Π_c groups $a_1, a_2 \dots a_n$, and let $g_1, g_2 \in G$ be given such that $g_1 \neq g_2^t$ any $t \in \mathbb{Z}$. We wish to show that there exists a finite homomorphic image of G in which the images $\bar{g_1}, \bar{g_2}$ of g_1, g_2 respectively, are such that $\bar{g_1} \neq \bar{g_2}^t$ for any $t \in \mathbb{Z}$.

As usual, G is a free product, a direct product, or it can be written as a generalised free product

$$G = \langle A_k \times H; Y \rangle$$

for some $k = 1, 2, \dots, n$.

The first two cases are dealt with by Stebe in [27]. We therefore assume G is a generalised free product and denote $A_k \times H$ as X. Let $\mu(g)$ denote the generalised free product length of g. We note that $g_1 \neq 1$, since $g_2^0 = 1$.

If $g_2 = 1$ we have the result since $\Pi_c \Rightarrow$ residually finite, and by Theorem 5.3. We have three cases to consider:

Case 1 $\mu(g_1) = 1, \mu(g_2) = 1$.

Case 2 $\mu(g_1) \geq 1, \mu(g_2) \geq 2$.

Case 3 $\mu(g_1) \geq 2, \mu(g_2) = 1$.

First consider Case 1. We have four subcases:

Subcase A: $g_1 \in X, g_2 \in X$.

Subcase B: $g_1 \in X \backslash H, g_2 \in Y \backslash H$.

Subcase C: $g_1 \in Y \backslash H, g_2 \in X \backslash H$.

Subcase D: $g_1 \in Y, g_2 \in Y$.

Subcases A and D: We proceed by induction on the number of generating groups, n of G. X and Y are graph products of fewer than n generating groups, all of which are Π_c.

Subcase B: $g_1 = ah$, where $1 \neq a \in A_k$, $h \in H$. $g_2 \in Y \backslash H$. If we factor out Y and map G naturally onto A_k, we have

$$\bar{g_2} = 1 \quad \text{and} \quad \bar{g_1} = \bar{a} \neq 1.$$

Now A_k is $\pi_c \Rightarrow$ residually finite, so there exists a finite homomorphic image in which $\bar{\bar{g_1}} \neq \bar{\bar{g_2}}^t$, for any $t \in \mathbb{Z}$.

Subcase C: $g_2 = ah$, where $1 \neq a \in A_k$, $h \in H$. $g_1 \in Y \backslash H$. We consider h and g_1. $h^t \in H$ for all t, and $g_1 \notin H$, thus $g_1 \neq h^t$ for any t. We can factor out A mapping G naturally onto Y and proceed by induction.

We now move on to Case 2. Clearly we can assume that g_2 is cyclically reduced, in terms of graph products. There is at least one pair of syllables in g_2 which do not commute with each other, thus, since g_2 is cyclically reduced,

$$\mu(g_2^m) > \mu(g_2^n) \quad \Longleftrightarrow m > n.$$

If $\mu(g_1) = \mu(g_2^t)$, for some t, then consider the element $g_1^{-1} g_2^t$ of G. This is clearly $\neq 1$, by choice of g_1, g_2, so there exists a normal subgroup N of finite index in G, such that $\bar{g_1}^{-1} \bar{g_2}^t \neq 1$ in G/N, since G is residually finite. If $\mu(g_1) \neq \mu(g_2^t)$ for all t, then set $N = 1$.

Now consider the normal forms of g_1, g_2:

$$g_1 = (a_1)y_1a_2y_2\dots a_r(y_r)$$

$$g_2 = (\alpha_1)z_2\alpha_2z_2\dots \alpha_s(z_s)$$

where $a_i, \alpha_i \in A_k$, $y_i, z_i \in Y\backslash H$. (It is possible that $r = 1$ and $y_1 \in H$, but this does not affect the argument.)

We can choose normal subgroups

$$M \underset{\text{f.i.}}{\triangleleft} X, P \underset{\text{f.i.}}{\triangleleft} Y$$

such that $\bar{a_i}, \bar{\alpha_i} \notin \bar{H}$ in X/M, and $\bar{y_i}, \bar{z_i} \notin \bar{H}$ in Y/P, by the subgroup separability of X and Y. Now $M \cap P \underset{\text{f.i.}}{\triangleleft} H$ and thus

$$(M \cap A_k) \times (M \cap P) \underset{\text{f.i.}}{\triangleleft} X,$$

since $X = A_k \times H$.

By Lemma 5.1 we can extend $M \cap P$ to a normal subgroup of finite index in Y, Q, say. So, factoring out $M \cap A_k$ and Q, we have

$$\bar{G} = \langle \bar{A_k} \times \bar{H}, \bar{Y}; \bar{H} \rangle,$$

where $\bar{X} = \bar{a_k} \times \bar{H}$, and $\bar{X}, \bar{Y}$ are finite. Now $\mu(\bar{g_1}) = \mu(g_1)$ and $\mu(\bar{g_2}) = \mu(g_2)$, so

$$\bar{g_1} = \bar{g_2}^t \Rightarrow \mu(\bar{g_1}) = \mu(\bar{g_2}^t) \Rightarrow \mu(g_1) = \mu(g_2^t).$$

But $g_1^{-1}g_2^t \neq 1$, by choice of N. So we have a contradiction, and hence $\bar{g_1} \neq \bar{g_2}^t$ for all t.

Now $\bar{G}$ is clearly free by finite, and free groups, and Π_c by finite groups are Π_c, by [27], thus $\bar{G}$ is Π_c. Hence we have the required homomorphic image of G, and G is Π_c.

Finally we look at Case 3: $g_2 = 1$, so $\lambda(g_2^t) = 1$ for all t, thus, if g_1 preserves its length under a mapping it will not become equal to a power of g_2. As in case 2 we can find normal subgroups of finite index in X, Y, such that on factoring them out we have

$$\bar{G} = \left\langle \bar{A_k} \times \bar{H}, \bar{Y}; \bar{H} \right\rangle,$$

where $\bar{X} = \bar{a_k} \times \bar{H}$, and $\bar{X}, \bar{Y}$ are finite. Now $\lambda(\bar{g_1}) = \lambda(g_1) > 1$. Thus $\bar{g_1} \neq \bar{g_2}^t$ for all t. We now proceed as in Case 2.

Hence the proof is complete. □

We finally consider two further residual properties of graph groups. In fact the results hold for the larger class of R(fgtfn) groups. The first result is an extension of Theorem 2.17 and the second result is a generalisation of Theorem 2.16.

THEOREM 5.25. *Let G be a R(fgtfn) group. Let $g \in G$ and u a positive integer be given. If $g \neq 1$, there exists a finite homomorphic image $\bar{G}$ of G such that the image $\bar{g}$ of g in $\bar{G}$ has order u. If u is a power of a prime q, we may choose $\bar{G}$ to be a q-group.*

PROOF: If u is not a prime power we have the result by Theorem 2.17. Let $u = q^e$ for prime q, $e \neq 0$. Choose $N \lhd G$ such that $\bar{G} = \frac{G}{N}$ is fgtfn and $g \neq N$.

Now $\bar{G}$ is torsion free, and residually finite p, for each prime p. We may choose normal subgroups N_i of $\bar{G}$, $1 \leq i \leq q^e - 1$ such that the $\frac{\bar{G}}{N_i}$ are finite q groups and $\bar{g}, \bar{g}^2, \bar{g}^3, \ldots \bar{g}^{q^e-1}$ respectively, are not equal to the identity in $\frac{\bar{G}}{N_i}$.

Then $\frac{\bar{G}}{\cap N_i}$ is a finite q group in which $\bar{\bar{g}}$ has q power order $\geq q^e$. We may now factor out successive p cycles until the image of g has the required order. □

In order to obtain our second result we require a theorem of Stebe's [30].

THEOREM 5.26. *Let G be a finitely generated nilpotent group. Let S_1 and S_2 be subgroups of G. If $g \in G$, then either $g \in S_1 S_2$ or there is a finite homomorphic image $\bar{G}$ of G in which $\bar{g} \notin \bar{S}_1 \bar{S}_2$.*

Our second result is then as follows.

THEOREM 5.27. *Let G be a R(fgtfn) group. Let $a, b, c \in G$ be such that $a \neq b^r c^s$, for all $r, s \in \mathbb{Z}$. Then there exists a finite homomorphic image $\bar{\bar{G}}$ of G in which $\bar{\bar{a}} \neq \bar{\bar{b}}^r \bar{\bar{c}}^s$ for all $r, s \in \mathbb{Z}$.*

PROOF: Suppose $[b, c] \neq 1$ in G. Since G is R(fgtfn) we may consider a series of normal subgroups

$$G \supset N_1 \supseteq N_2 \supseteq \cdots$$

where $\cap N_i = \langle 1 \rangle$, and where the $\frac{G}{N_i}$ are fgtfn.

We can choose N_k such that $[\bar{b}, \bar{c}] \neq 1$ in $\frac{G}{N_k} = \bar{G}$. Suppose there exists a $0 \neq t \in \mathbb{Z}$ such that $[\bar{b}, \bar{c}^t] = 1$. Let

$$\langle 1 \rangle = \zeta_0(\bar{G}) \subset \zeta_1(\bar{G}) \subset \zeta_2(\bar{G}) \subset \dots$$

be the upper central series of $\bar{G}$. Suppose

$$[\bar{b}, \bar{c}] \in \zeta_i(\bar{G}) \backslash \zeta_{i-1}(\bar{G}).$$

Then in $\bar{\bar{G}} = \frac{\bar{G}}{\zeta_{i-1}(\bar{G})}$, $[\bar{\bar{b}}, \bar{\bar{c}}]$ is central, and $\bar{\bar{G}}$ is torsion free. So $[\bar{\bar{c}}, [\bar{\bar{b}}, \bar{\bar{c}}]] = 1$. Now if $[\bar{b}, \bar{c}^t] = 1$ in $\bar{G}$, then certainly $[\bar{\bar{b}}, \bar{\bar{c}}^t] = 1$ in $\bar{\bar{G}}$. But

$$\begin{aligned}
[\bar{\bar{b}}, \bar{\bar{c}}^t] &= [\bar{\bar{b}}, \bar{\bar{c}}][\bar{\bar{b}}\bar{\bar{c}}, \bar{\bar{c}}^{t-1}] \\
&= [\bar{\bar{b}}, \bar{\bar{c}}][\bar{\bar{b}}, \bar{\bar{c}}^{t-1}]^c \\
&= [\bar{\bar{b}}, \bar{\bar{c}}]\left([\bar{\bar{b}}, \bar{\bar{c}}][\bar{\bar{b}}, \bar{\bar{c}}^{t-1}]^c\right)^c \\
&\vdots \quad \vdots \quad \vdots \\
&= [\bar{\bar{b}}, \bar{\bar{c}}]^t,
\end{aligned}$$

since $[\bar{\bar{c}}, [\bar{\bar{b}}, \bar{\bar{c}}]] = 1$. Therefore $[\bar{\bar{b}}, \bar{\bar{c}}] = 1$, as $\bar{\bar{G}}$ is torsion free, which is a contradiction. So $[\bar{b}, \bar{c}^t] \neq 1$ for all $0 \neq t \in \mathbb{Z}$.

We want to show that there exists N_i such that $\hat{a} \notin \left\langle \hat{b} \right\rangle \langle \hat{c} \rangle$ in $\hat{G} = \frac{G}{N_i}$. Suppose to the contrary that

$$\begin{aligned}
a &\equiv b^{\beta_1} c^{\gamma_1} \quad mod \quad N_1 \\
a &\equiv b^{\beta_2} c^{\gamma_2} \quad mod \quad N_2 \\
a &\equiv b^{\beta_3} c^{\gamma_3} \quad mod \quad N_3 \\
&\vdots \qquad \vdots \qquad \vdots
\end{aligned}$$

Let N_k be as above. Suppose $a \equiv b^u c^v$ mod N_{k+m} and $a \equiv b^w c^x$ mod

N_{k+m+n}. Then $a \equiv b^w c^x \mod N_{k+m}$. Hence $b^u c^v \equiv b^w c^x \mod N_{k+m}$, and thus $b^{u-w} \equiv c^{x-v} \mod N_{k+m}$, that is $\bar{b}^{u-v} = \bar{c}^{x-v}$ in $\frac{G}{N_{k+m}}$.

This implies that $[\bar{b}, \bar{c}^{x-v}] = 1$ in $\frac{G}{N_{k+m}}$, and hence in $\frac{G}{N_k}$, which gives a contradiction, as above, unless $x = v$. By symmetry we also have $u = w$. But then $a^{-1}b^u c^v \in N_i$ for all $i \geq k$, which is itself a contradiction, since $a^{-1}b^u c^v \neq 1$.

Hence we have a fgtfn image of G in which $\bar{a} \neq \bar{b}^r \bar{c}^s$ for all $r, s \in \mathbb{Z}$. The proof is completed using Theorem 5.26 with $S_1 = \langle \bar{b} \rangle, S_2 = \langle \bar{c} \rangle$.

We now suppose that $[b, c] = 1$. Again we suppose that

$$\begin{aligned} a &\equiv b^{\beta_1} c^{\gamma_1} \mod N_1 \\ a &\equiv b^{\beta_2} c^{\gamma_2} \mod N_2 \\ a &\equiv b^{\beta_3} c^{\gamma_3} \mod N_3 \\ &\vdots \end{aligned}$$

Then $b^{\beta_1} c^{\gamma_1} \equiv b^{\beta_2} c^{\gamma_2} \mod N_1$, and so on. Thus

$$\begin{aligned} b^{\beta_1 - \beta_2} &\equiv c^{\gamma_2 - \gamma_1} \mod N_1 \\ b^{\beta_2 - \beta_3} &\equiv c^{\gamma_3 - \gamma_2} \mod N_2 \\ &\vdots \end{aligned}$$

Now either $\beta_i - \beta_{i-1}$ is 0 for all i after some point, in which case we have a contradiction since $\cap N_i = \langle 1 \rangle$ and $a \neq b^r c^s$ for all $r, s \in \mathbb{Z}$, or $\beta_i \neq \beta_{i-1}$ for a subsequence $N_{i_1}, N_{i_2}, \ldots$. We consider this possibility.

Suppose that

$$b^{x_1} \equiv c^{y_1} \mod N_{i_1}$$

$$b^{x_2} \equiv c^{y_2} \mod N_{i_2}$$

$$\vdots \qquad \vdots \qquad \vdots$$

where $x_1, x_2, \ldots$ are the least non-negative such exponents. We note that, for large enough i, (say $i \geq L$), we can ensure that $c \not\equiv 1 \bmod N_i$, which implies that $c^{y_i} \not\equiv 1 \bmod N_i$, since $\frac{G}{N_i}$ is torsion free, which implies that $x_i \neq 0$.

Let x_s be the least positive x_i, over $i_h \geq L$. Then all x_j $(i_j \geq L)$ are multiples of x_s, and if, $(i_t \geq L)$, $x_t = fx_s$, then $y_t = fy_s$ and $b^{fx_s}c^{-fy_s} = (b^{x_s}c^{-y_s})^f \in N_{i_t}$, since $[b,c] = 1$.

Now $\frac{G}{N_{i_t}}$ is torsion free so $f = 1$, by the minimality of the x_i. Thus $b^{x_s}c^{-y_s} \in N_{i_t}$ for all $t \geq s$. This gives a contradiction, since $\cap N_l = \langle 1 \rangle$. Again we have a fgtfn homomorphic image of G in which $\bar{a} \neq \bar{b}^r \bar{c}^s$ for any $r, s \in \mathbb{Z}$, and we complete the proof using Stebe's result as before. □

CHAPTER 6 ONE-RELATOR GRAPH PRODUCTS

In this chapter we explore the Freiheitssatz to see how we can generalise it to the case of graph groups. The following theorem was proved by Magnus in the early 1930's.

THEOREM 6.1 THE FREIHEITSSATZ.

Let $G = \langle t, b, c, \dots; r\rangle$ *where* r *is cyclically reduced. If* L *is a subset of* $\{t, b, c, \dots\}$ *which omits a generator occuring in* r, *then the subgroup* M *generated by* L *is freely generated by* L.

Following Baumslag and Pride in [4] we make the following definition:

Definition 6.2

Let G be the free product of the groups A and B. Let $R \in G$. We shall say that the *Freiheitssatz holds* for $G = A*B$ if the natural homomorphisms of A, B into $G/\langle R\rangle^G$ are injective.

A class of graph groups has been dealt with by B.Baumslag and S.J.Pride in [4], where they showed the following:

THEOREM 6.3. *(Countable) free products of (restricted) direct products of free groups (PDF groups) have a Freiheitssatz.*

It would be interesting to try to extend this result to larger classes of graph groups. We denote the class of free products of direct products of free products of direct products of free groups as FDFDF groups, or $\mathrm{F(DF)}^2$

groups. In this way we consider $\mathrm{F(DF)}^t$ groups, and show in the following example, that these classes are strictly increasing in size.

Example 6.4

Define a set of groups as follows:

$$F(DF_2)^0 = F_2,$$

$$F(DF_2)^{n+1} = [F(DF_2)^n \times F(DF_2)^n] * [F(DF_2)^n \times F(DF_2)^n].$$

Now clearly

$$F(DF_2)^0 \subset F(DF)^1 \subset F(DF)^2 \subset \dots$$

and also

$$F(DF_2)^n \in F(DF)^n \subset F(DF)^{n+1}.$$

We will show that, for all n,

$$F(DF_2)^{n+1} \notin F(DF)^n,$$

and this establishes the strict subset condition.

Proceeding by induction, we first show that $F(DF_2)^2 \notin F(DF)^1$. Clearly FDF$\subseteq$FDFDF, so we need to find an FDFDF group which is not an FDF. Let $G \equiv H \equiv (F_2 \times F_2) * (F_2 \times F_2)$. Then G and H are FDF. So $A = (G \times H) * (G \times H)$ is an FDFDF group. Now suppose A is FDF. F_2 has trivial centre, and thus so has $(F_2 \times F_2)$. Similarly, so has G, and thus so has $G \times H$. Now $\exists B, C$ such that $A \equiv B * C$, where B and C are direct products of free groups. Now $G \times H$ and $B \times C$ are freely indecomposable since they are non trivial direct products, therefore $B \equiv G \times H \equiv C$.

Now B is a direct product of free groups, so $\exists D, E$, free groups, such that $B \equiv D \times E$.

Since centre $D =$ centre $E =$ centre $G =$ centre $H = 1$, and D, E, G, H are directly indecomposable, without loss of generality we have $D \equiv G$ $E \equiv H$. But D, E are free and G, H are not. Thus A is not PDF.

This establishes a basis. Assume the result is true for all $n < k$. So, we require to prove that $F(DF)^{k+1} \notin F(DF)^k$. Well,

$$F(DF_2)^{k+1} = [F(DF_2)^k \times F(DF_2)^k] * [F(DF_2)^k \times F(DF_2)^k].$$

Now, clearly $F(DF_2)^k$ is a non trivial group with trivial centre, thus $F(DF_2)^k \times F(DF_2)^k$ is a non-trivial direct product, and so is freely indecomposable. Now suppose $F(DF_2)^{k+1} \in F(DF)^k$. Proceeding as in the basis step it is easy to obtain a contradiction. Hence we have the result by induction.

By generalising the proof of Baumslag and Pride [4], using induction, we obtain:

THEOREM 6.5. *$F(DF)^t$ groups have a freiheitssatz.*

However, more than this result was shown by Howie [16]:

THEOREM 6.6. *Free products of locally indicable groups have a freiheitssatz.*

Now, graph groups are locally indicable, by Theorem 2.18, so graph groups which are free products have a freiheitssatz.

It is easy to construct a counter example to show that graph products which are direct products do not have isomorphically preserved direct factors under the addition of certain relators.

However, the question remains open as to whether a graph group which is neither a free nor a direct product, has isomorphically preserved truncated subgroups under the addition of relators.

We now mention a result concerning a special type of *1-relator graph group*. This uses, but also generalises, a result of Allenby [1]. We will look at the generalised free product of two graph groups amalgamating an infinite cycle. When Baumslag, [5], looked at the residual finiteness of the free product of two free groups amalgamating an infinite cycle he used the fact that the centraliser of any non-trivial element in a free group is cyclic. This result is certainly not true for graph groups, so we have to alter the approach slightly.

THEOREM 6.7. *Let* A, B *be graph groups, and let* $G = A \underset{\langle h \rangle}{*} B$, *where* $\langle h \rangle$ *is a cyclic subgroup of* A, B. *Then* G *is potent.*

PROOF: Now A, B are R(fgtfn). Let $g = (a_1)b_1a_2b_2 \dots a_r(b_r)$ and $n \in \mathbb{Z}$ be given. If $g \in A$ or B we have the result by $R(fgtfn) \Rightarrow$ potent, Theorem 2.17. Thus we can assume $\mu(g) \geq 2$.

Let $M_1 \supset M_2 \supset \dots$ be a sequence of normal subgroups of A such that $h \notin M_i$, A/M_i is fgtfn and $\cap M_i = \langle 1 \rangle$.

Suppose $a_1 \in HM_1 \cap HM_2 \cap \dots \cap HM_t$. Then

$$a_1 = h^{i_1} m_1 = h^{i_2} m_2 = \ldots .$$

Therefore $h^{i_1 - i_2} \in M_1$, but $M_1 \cap H = \langle 1 \rangle$, and h has infinite order, thus $h^{i_1} = h^{i_2}$.

Similarly $h^{i_2} = h^{i_3} = \ldots, = h^k$, say. Therefore

$$a_1 = h^k m_1 = h^k m_2 = \ldots \qquad \Rightarrow m_1 = m_2 = \ldots, = m,$$

say. Thus $m \in \cap M_i = \langle 1 \rangle$. This is a contradiction. Thus to each a_i we find an M_i such that $a_i \notin HM$.

Hence, by taking intersections we can easily find normal subgroups M, N of finite index in A, B, respectively, such that

$$M \cap H = N \cap H = \langle 1 \rangle,$$

and $A/M, B/N$ are finite nilpotent, and such that g preserves its length when N and M are factored out.

So $\bar{G} = \langle \bar{A}, \bar{B}; \bar{H} \rangle$, where $\bar{A}, \bar{B}$ are fgtfn, and H is cyclic. $\bar{g}$ has infinite order in $\bar{G}$, which is potent, by [1]. Hence result. □

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