On hereditary graph classes defined by forbidding Truemper configurations:

Recognition and combinatorial optimization algorithms, and χ-boundedness results

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The candidate confirms that the work submitted is his/her own, except where work which has formed part of a jointly authored publication has been included. The contribution of the candidate and the other authors to this work has been explicitly indicated below. The candidate confirms that appropriate credit has been given within the thesis where reference has been made to the work of others.

Chapters 2 and 3 have been included in the following articles (which have now been accepted for publication, but have not been published yet):

- V. Boncompagni, I. Penev, K. Vušković
  Clique cutsets beyond chordal graphs
  To appear in Journal of Graph Theory.
- V. Boncompagni, M. Radovanović, K. Vušković
  The structure of (theta, pyramid, 1-wheel, 3-wheel)-free graphs
  To appear in Journal of Graph Theory.

Most of the structural results of the first article were written by myself, and most of algorithmic results were written by coauthors. The final outcomes of our research can be considered a joint work. The third author supervised the whole project, and contributed suggestions and ideas throughout the research process.

I wrote most of the second article myself. Again, the final outcomes of our research can be considered a joint work, and the third author supervised the whole project, and contributed suggestions and ideas throughout the research process.

Ciao Marta!

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Abstract

Truemper configurations are four types of graphs that helped us understand the structure of several well-known hereditary graph classes. The most famous examples are perhaps the class of perfect graphs and the class of even-hole-free graphs: for both of them, some Truemper configurations are excluded (as induced subgraphs), and this fact appeared to be useful, and played some role in the proof of the known decomposition theorems for these classes.

The main goal of this thesis is to contribute to the systematic exploration of hereditary graph classes defined by forbidding Truemper configurations. We study many of these classes, and we investigate their structure by applying the decomposition method. We then use our structural results to analyze the complexity of the maximum clique, maximum stable set and optimal coloring problems restricted to these classes. Finally, we provide polynomial-time recognition algorithms for all of these classes, and we obtain χ-boundedness results.

Contents

Chapter 1

Introduction

For some basic graph-theoretic definitions, we refer the reader to [49].

All graphs in this thesis are finite and simple. Given two graphs F and G, we say that F is an *induced subgraph* of G if F can be obtained from G just by deleting vertices. Also, we say that G *contains* F if F is isomorphic to an induced subgraph of G, and that G is *F-free* if G does not contain F. For a (possibly infinite) family of graphs $\mathcal{F}$, we say that G is *$\mathcal{F}$-free* if G is F-free for every $F \in \mathcal{F}$. This thesis is about *hereditary graph classes*, which are classes of graphs that are closed under taking induced subgraphs. This means that, given a hereditary graph class $\mathcal{G}$ and a graph $G \in \mathcal{G}$, all (isomorphic copies of) induced subgraphs of G belong to $\mathcal{G}$. It is easy to see that every hereditary graph class admits a characterization in terms of forbidden induced subgraphs, and a graph class that is defined by a set of forbidden induced subgraphs is obviously hereditary. So, given any hereditary graph class $\mathcal{G}$, $\mathcal{G}$ is equivalent to the class of graphs that are $\mathcal{F}$-free for some suitable choice of $\mathcal{F}$.

Let G be a graph. A *clique* (resp. *stable set*) of G is a set of pairwise adjacent (resp. non-adjacent) vertices of G. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number of colors needed to color the vertices of G so that no adjacent vertices receive the same color.

The main goal of this thesis is to develop efficient algorithms for solving optimization problems on graphs, such as finding a largest clique or stable set, or the value of the chromatic number of a given input graph. All of these problems have countless practical applications. For instance, suppose that you want to throw a party, and so you invite a certain number of guests. Now, let us make things simple and let us assume that, given any two of your guests, they either like each other or they do not. You want to assign people to tables in such a way that you do not create conflicts. In other words, if two guests do not like each other, then you do not want them to be assigned to the same table. Clearly, you may assign each person to a different table, and to do this you would need as many tables as the number of guests. However, in most cases this is not the optimal solution. So, what is the minimum number of tables needed? This problem can be easily mapped into a graph problem, which is solved by computing the chromatic number of a graph. This is done by considering a graph whose vertices represent guests, and by assuming that any two vertices are adjacent if and only if the corresponding guests do not like each other.

Unfortunately, for all the optimization problems mentioned above, polynomial-time algorithms do not appear to exist in general, and hence it is highly unlikely that these problems can be solved efficiently by a computer. So, one possible approach to deal with this is to find classes of graphs for which these problems can be solved in polynomial time. This usually happens when some configurations are excluded (for example as induced subgraphs), and hence some structure is imposed on the input graph. However, sometimes they remain hard even when considerable restrictions are applied. This happens, for instance, for the problem of deciding whether a given triangle-free graph (that is, a graph that does not contain complete graphs on three vertices) admits a coloring that uses at most three colors, even when the maximum degree of the graph is four [29]. If a difficult optimization problem can be solved in polynomial time for a given class, then this class must have some "strong" structure. Understanding structural properties that allow the design of efficient algorithms is our primary interest. In this thesis, we will limit our attention to hereditary graph classes, as many interesting graph classes can in fact be characterized as being $\mathcal{F}$-free for some family of graphs $\mathcal{F}$.

Another algorithmic problem we will be interested in throughout this thesis is the *recognition problem*, which is the problem of deciding whether an input graph belongs to a given class. This problem does not seem to admit a polynomial-time algorithm for many hereditary graph classes (we will encounter some of them later on in this thesis; see Chapter 3), and hence, again, it is of interest to identify classes of graphs for which efficient algorithms are available.

A graph whose vertex set is a clique is called a *complete graph*. Given a graph G, we denote by $\omega(G)$ the *clique number* of G, that is, the size of a largest clique of G. Clearly, $\chi(G) \geqslant \omega(G)$ for every graph G. What can we say about the graphs G that satisfy $\chi(G) = \omega(G)$? Do they have an interesting structure? The answer turns out to be negative. Indeed, let G_1 and G_2 be graphs on disjoint vertex sets; in particular, let G_1 be any graph and let G_2 be complete, and assume that $|V(G_1)| = |V(G_2)|$. Now, consider the graph G such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_1)$. Then $\omega(G) = \omega(G_2)$, $\chi(G_1) \leqslant |V(G_1)| = \omega(G_2)$, $\chi(G_2) = \omega(G_2)$, and hence $\chi(G) = \max\{\chi(G_1), \chi(G_2)\} = \omega(G)$, but our graph G does not have any particular structure! This observation led Berge to define in the 60's the class of *perfect graphs* as follows: a graph G is perfect if, for every induced subgraph F of G, $\chi(F) = \omega(F)$. Clearly, the class of perfect graphs is hereditary, and therefore it admits a characterization in terms of (minimal) forbidden induced subgraphs; finding such a characterization has been a long-standing open problem, settled in 2006 with the Strong Perfect Graph Theorem [9]. If we define a *hole* to be a chordless cycle of length at least four, and an *odd hole* (resp. *even hole*) to be a hole of odd (resp. even) length, then we say that a graph G is *Berge* if G contains neither an odd hole nor the complement of an odd hole, and the Strong Perfect Graph Theorem states that a graph is perfect if and only if it is Berge. Let $k > 1$ be an integer, and let C_{2k+1} (resp. $\overline{C_{2k+1}}$) denote a chordless cycle (resp. the complement of a chordless cycle) of length $2k+1$. Then $\omega(C_{2k+1}) = 2$ and $\chi(C_{2k+1}) = 3$, and $\omega(\overline{C_{2k+1}}) = k$ and $\chi(\overline{C_{2k+1}}) = k+1$. This proves that every perfect graph is Berge. Showing that every Berge graph is perfect is instead a much harder task, and its proof [9] is very long and technical.

Following Gyárfás [21], we now want to generalize the notion of perfection. A graph class $\mathcal{G}$ is said to be χ-*bounded* provided that there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that, for all $G \in \mathcal{G}$

and for every induced subgraph F of G, $\chi(F) \leqslant f(\omega(F))$. Note that a hereditary graph class $\mathcal{G}$ is χ-bounded if and only if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that every graph $G \in \mathcal{G}$ satisfies $\chi(G) \leqslant f(\omega(G))$. Clearly, the class of perfect graphs is the class of graphs χ-bounded by the identity function. Observe that not all graph classes are χ-bounded; for instance, it is well known [33] that triangle-free graphs can have arbitrarily large chromatic number, even though their clique number is at most two. Therefore, a very natural question to ask is: what choices of forbidden induced subgraphs guarantee that a graph class is χ-bounded? In this thesis, we will provide tighter bounds on the chromatic number of several hereditary graph classes that are known already to be χ-bounded.

The decomposition method.

The way we investigate hereditary graph classes is based on the use of the *decomposition method*, a very powerful technique that has been already successfully applied in the past few decades in order to obtain some important results [9, 12, 13, 14, 15]. For further references see for example [48]. The decomposition method is a *divide and conquer* approach, and allows us to understand complex structures by breaking them down into simpler ones. Once these simpler structures are understood, this knowledge is propagated back to the original structure by understanding how their composition behaves.

Let G be a connected graph. A subset S of vertices and/or edges of G is a cutset if its removal results in a disconnected graph. The cutset S is called a *vertex cutset* (resp. *edge cutset*) if it only consists of vertices (resp. edges) of G. A general decomposition theorem for a given hereditary graph class $\mathcal{G}$ has the following form.

Decomposition Theorem. *If a graph G belongs to class $\mathcal{G}$, then G is either "basic" or admits a cutset S for some $S \in \mathcal{S}$.*

Depending on what one wants to prove about the graph class $\mathcal{G}$ using a given decomposition theorem, basic graphs and cutsets in $\mathcal{S}$ must have adequate properties. For example, the fact that every Berge graph is perfect (and hence, the Strong Perfect Graph Theorem [9]) was proved using a decomposition theorem for Berge graphs, by ensuring that basic graphs were simple in the sense that the theorem could be easily proved for them, and the cutsets in $\mathcal{S}$ had the property that no minimal imperfect graph could contain them (or if it did it would have to be an odd hole or the complement of one).

Suppose now that we want to solve a given optimization problem on $\mathcal{G}$, and assume that some decomposition theorem has already been proved for our class. The general approach will follow the two steps below.

1. We consider any graph $G \in \mathcal{G}$, and, by removing a cutset $S \in \mathcal{S}$ from G, we disconnect the graph in two or more components. From these components *blocks of decomposition* are then constructed in a suitable way, typically by adding some more vertices and edges in such a way that the resulting graphs still belong to class $\mathcal{G}$. So, by repeating the same

procedure, we construct a *decomposition tree* T, whose root is the input graph G, and, for every non-leaf vertex H of T, the children of H in T are the blocks of decomposition of H w.r.t. some cutset in $\mathcal{S}$. Also, the leaves L of T are such that they do not admit any of the cutsets in $\mathcal{S}$, and hence, by the decomposition theorem, they are basic graphs.

2. Assuming that, for every leaf L of T, we can solve efficiently our optimization problem, through a bottom-up approach we now want to construct a polynomial-time algorithm that solves the same problem on G. Hopefully, this will be made possible by exploiting the structure of the cutsets involved in the decomposition theorem. In order for such an algorithm to have polynomial running time, observe that we also need to ensure that T can be constructed in polynomial time (this means that we must be able to find the cutsets in polynomial time, and that the decomposition tree must be polynomial in size).

By following a very similar approach, we can also attempt to solve a recognition problem for $\mathcal{G}$, provided that the decompositions are *class-preserving*. In other words, we need that G belongs to $\mathcal{G}$ if and only if all the blocks of decomposition of G w.r.t. any of the cutsets in $\mathcal{S}$ belong to $\mathcal{G}$. If this holds, then the problem of deciding whether G belongs to $\mathcal{G}$ reduces to checking whether the leaves of the decomposition tree are basic graphs, which is typically easier. Unfortunately, decompositions that are class-preserving are not often available. Later on in this thesis we will give several polynomial-time recognition algorithms; some of them are decomposition-based, some others are just direct algorithms that detect the obstructions (if any).

The decomposition method can also help us prove χ-boundedness results for a given hereditary graph class $\mathcal{G}$. We will have to show that all graphs in $\mathcal{G}$ are either in some well understood basic class (in the sense that the bound on the chromatic number can be easily proved for them directly), or can be cut into pieces by some appropriate decomposition that in a sense preserves the bound that is to be proven.

Clearly, when applying the decomposition method, the key is finding a compromise between dealing with sufficiently simple basic graphs and sufficiently strong structured cutsets. Also, note that this is just an ideal scenario, which works very well only for a few graph classes, for example for *chordal graphs*, i.e. hole-free graphs. Indeed, in order to keep the basic graphs simple, we often need to decompose a graph with the help of really exotic cutsets. Unfortunately, it is not clear how to use some of them to successfully complete the decomposition paradigm.

Given a graph G, a set $S \subset V(G)$ is a *clique cutset* of G if S is a clique and a cutset of G. Clique cutsets will appear several times in this thesis. These cutsets are particularly simple to deal with, and their strong structure is very useful when trying to design decomposition-based algorithms [43]; we refer the reader to Chapter 2 for further details. The reason why the decomposition method is so successful when applied to the class of chordal graphs is basically due to the fact that a decomposition theorem involving only clique cutsets is available for such a class. This was first proved in [18], and states that a chordal graph is either a complete graph or it admits a clique cutset.

However, in many other cases it turns out that clique cutsets do not suffice to decompose a given graph down to simple basic graphs, and hence a need for more general cutsets emerges.

Observe that, with clique cutsets, one can only separate vertices that are not contained in a hole. When we need to break a hole, we may try to use a vertex that has neighbors in this hole as a center of a *star cutset*, where a star cutset of a graph G is cutset of G consisting of a vertex (called the *center*) and some of its neighbors. Dealing with star cutsets (and their generalizations) is usually a very hard task, and how to use them is still mostly unknown.

Truemper configurations.

Special types of graphs that are called *Truemper configurations* (or *TC*'s, for short) will play an important role throughout this thesis, so let us introduce them here. We denote by K_n the complete graph on n vertices; a K_3 is also referred to as a *triangle*.

A $3PC(x, y)$ (or a *theta*) is a graph induced by three internally vertex-disjoint chordless paths $P_1 = x, \dots, y$, $P_2 = x, \dots, y$ and $P_3 = x, \dots, y$, of length at least two, and such that no edges exist between the paths except the three edges incident to x and the three edges incident to y.

A $3PC(x_1x_2x_3, y)$ (or a *pyramid*) is a graph induced by three chordless paths $P_1 = x_1, \dots, y$, $P_2 = x_2, \dots, y$ and $P_3 = x_3, \dots, y$, of length at least one, two of which have length at least two, vertex-disjoint except at y, and such that $\{x_1, x_2, x_3\}$ induces a triangle and no edges exist between the paths except those of the triangle and the three edges incident to y.

A $3PC(x_1x_2x_3, y_1y_1y_3)$ (or a *prism*) is a graph induced by three vertex-disjoint chordless paths $P_1 = x_1, \dots, y_1$, $P_2 = x_2, \dots, y_2$ and $P_3 = x_3, \dots, y_3$, of length at least one, and such that $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ both induce a triangle and no edges exist between the paths except those of the two triangles.

A *three-path-configuration* (or *3PC*, for short) is any theta, pyramid or prism.

Finally, a *wheel* is a graph that consists of a hole (called the *rim*) and an additional vertex (called the *center*) that has at least three neighbors in the hole.

A TC is any 3PC or wheel. Truemper configurations are depicted in Figure 1.1, where a solid line denotes an edge and a dashed line denotes a chordless path of length at least one.

They first appeared in a theorem of Truemper [45] that characterizes graphs whose edges can be assigned a $\{0, 1\}$-weight so that the sum of the weights of the edges of every chordless cycle in the graph has a prescribed parity. This theorem states that this can be done for a given graph G if an only if it can be done for all induced subgraphs of G that are thetas, pyramids, prisms, wheels or K_4's.

The configurations that Truemper identified in his theorem ended up playing an important role in understanding the structure of several different graph classes through the decomposition method. In fact, many decomposition theorems for well-known hereditary graph classes are proved by studying how some Truemper configuration contained in the graph attaches to the rest of the graph, and often, the study relies on the fact that some other Truemper configurations are excluded from the class. The most famous examples are perhaps the class of perfect graphs and the class of even-hole-free graphs [9, 12, 15] (see also surveys [44, 48]), but many more exist.

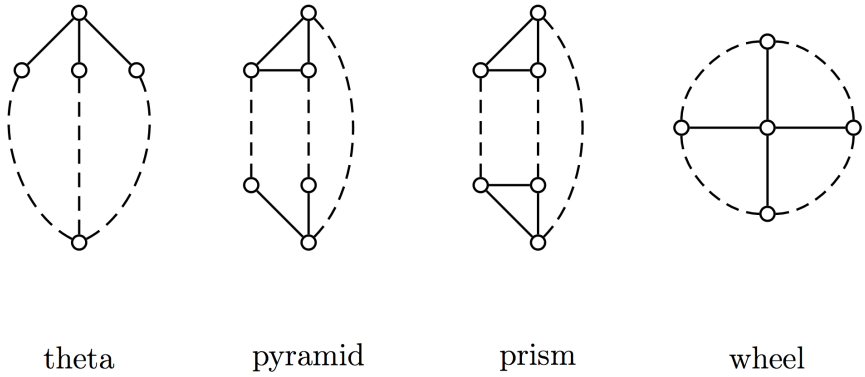

Figure 1.1: Truemper configurations

Pyramids and wheels that contain an odd number of triangles (i.e. *odd wheels*) are excluded structures for perfect graphs, and prisms and wheels that are not odd play a key role in the proof of the decomposition theorem for perfect graphs in the sense that they appear as structures around which the actual decomposition takes place. Similarly, thetas, prisms and wheels whose center has an even number of neighbors in the rim (i.e. *even wheels*) are excluded structures for even-hole-free graphs, and pyramids appear as basic graphs in the decomposition theorem for even-hole-free graphs. Both the decomposition theorems for these classes use star cutsets and *2-joins* (where a 2-join is an edge cutset which we know how to use in algorithms), as well as different generalizations of these. As we mentioned already, it is not clear how to deal with star cutsets in algorithms, and therefore many problems still remain unsolved. In particular, the complexity of the maximum stable set and optimal coloring problems is still not known for even-hole-free graphs, and also it is not known how to solve these problems (together with the maximum clique problem) for perfect graphs by a purely graph-theoretic algorithm (it is known that they can be solved in polynomial time for perfect graphs using the ellipsoid method [20]).

The aim of this thesis is to contribute to the systematic exploration of hereditary graph classes defined by forbidding Truemper configurations. In light of the examples given above, this study might provide insight into well-known hereditary graph classes, besides being interesting in itself. As every Truemper configuration contains a hole, all these classes will be generalizations of the class of chordal graphs. A class for which a decomposition theorem has been known for many years is the one of *universally signable graphs* [14], which is the class where all Truemper configurations are excluded. These graphs are either complete graphs, or holes, or they admit a clique cutset. Such a decomposition theorem led to polynomial-time algorithms to solve the recognition problem, as well as all the optimization problems we are interested in. A few other classes have been studied very recently [16, 39, 40], and a few other decomposition theorems established.

Structure of the thesis and main results.

We first need a few more definitions. Consider a wheel. A subpath of its rim, of length at least one, whose endpoints are adjacent to the center but whose interior vertices are not, is called a *sector*. A sector is *short* if it is of length one, and *long* otherwise. A *twin wheel* is a wheel that has two short sectors and one long sector, and a *universal wheel* is a wheel that has only short sectors. Finally, an *alternating wheel* is a wheel with an even number of sectors, and such that these sectors alternate between short and long sectors.

A *claw* is a graph induced by a vertex, say x, and three pairwise non-adjacent neighbors of x. A *diamond* is a graph obtained from a K_4 by removing a single edge.

This thesis consists of two main chapters, Chapter 2 and Chapter 3.

- In Chapter 2 we present [3], written in collaboration with Penev and Vušković. In this chapter, we study the class $\mathcal{G}_{\mathrm{UT}}$ of graphs that may contain universal wheels, twin wheels, and no other Truemper configurations. Also, we consider three proper subclasses of $\mathcal{G}_{\mathrm{UT}}$: $\mathcal{G}_{\mathrm{U}}$ (resp. $\mathcal{G}_{\mathrm{T}}$) is the class of graphs that, out of all Truemper configurations, may only contain universal wheel (resp. twin wheels), and $\mathcal{G}_{\mathrm{UT}}^{\text{cap-free}}$ is the class of all the graphs in $\mathcal{G}_{\mathrm{UT}}$ that are cap-free, where a *cap* is a graph induced by a hole and a vertex that has two neighbors in the hole, and these neighbors are consecutive vertices of the hole. Note that the class of chordal graphs is a subclass of each of our four classes, and the class of universally signable graphs is a subclass of each of $\mathcal{G}_{\mathrm{UT}}, \mathcal{G}_{\mathrm{U}}, \mathcal{G}_{\mathrm{T}}$.

  For all the four classes, we prove decomposition theorems that only use clique cutsets, as it happens for chordal graphs and universally signable graphs. In order to achieve this, however, the classes of basic graphs that appear must be larger, and so they do not contain only complete graphs and holes. We show that our decomposition theorems for classes $\mathcal{G}_{\mathrm{U}}$, $\mathcal{G}_{\mathrm{T}}$ and $\mathcal{G}_{\mathrm{UT}}^{\text{cap-free}}$ allow us to design decomposition-based polynomial-time algorithms that solve (most of) the optimization problems we are interested in (see Chapter 2 for details). The same does not hold for class $\mathcal{G}_{\mathrm{UT}}$, for which the question of whether (most of) these problems can be solved efficiently remains open (again, we refer the reader to Chapter 2 for details). We also provide polynomial-time recognition algorithms, and χ-boundedness results, for all the four classes. All of these results (but the one that gives a χ-bounding function for class $\mathcal{G}_{\mathrm{UT}}$) rely on our decomposition theorems for $\mathcal{G}_{\mathrm{UT}}$, $\mathcal{G}_{\mathrm{U}}$, $\mathcal{G}_{\mathrm{T}}$ and $\mathcal{G}_{\mathrm{UT}}^{\text{cap-free}}$.

- In Chapter 3 we present [4], a joint work with Radovanović and Vušković. Here, we study the class $\mathcal{G}$ of graphs that, out of all Truemper configurations, may only contain prisms and alternating wheels. It is easy to see that all the forbidden Truemper configurations contain a claw or a diamond, and therefore the class of (claw, diamond)-free graphs is a subclass of $\mathcal{G}$. Also, (claw, diamond)-free graphs have been studied already in the past, and some of their properties established. In particular, every (claw, diamond)-free graph G satisfies $\chi(G) \leqslant \omega(G) + 1$ [22] (this is called the *Vizing's bound*); observe that this bound is best possible, as G is not necessarily perfect. We wondered whether the same result extends to graphs in $\mathcal{G}$, and this was the main motivation for our investigation.

So, for a decomposition theorem for class $\mathcal{G}$, $(\text{claw}, \text{diamond})$-free graphs represent a very natural choice for the class of basic graphs. The decomposition theorem we prove for class $\mathcal{G}$ uses clique cutsets and star cutsets, which are needed to decompose any given graph in $\mathcal{G}$ down to $(\text{claw}, \text{diamond})$-free graphs. Although the star cutsets that appear satisfy additional constraints (we refer the reader to Chapter 3 for details), they do not seem to be powerful enough to prove the Vizing's bound for graphs in $\mathcal{G}$, which we leave as an open problem. By considering only K_4-free graphs in $\mathcal{G}$, however, these cutsets simplify even further (they become five-vertex cutsets with a lot of nice properties), and so we are able to prove that $\chi(G) \leqslant \omega(G) + 1$ for every graph $G \in \mathcal{G}$ that does not contain a K_4.

In addition to this, we provide two polynomial-time algorithms that solve the recognition problem for $\mathcal{G}$; none of them, though, rely on our decomposition theorem for class $\mathcal{G}$.

Chapter 2

Clique cutsets beyond chordal graphs

In this chapter we present [3].

Abstract. Truemper configurations (that is, thetas, pyramids, prisms and wheels) have played an important role in the study of complex hereditary graph classes (e.g. the class of perfect graphs and the class of even-hole-free graphs), appearing both as excluded configurations, and as configurations around which graphs can be decomposed. In this chapter, we study the structure of graphs that contain (as induced subgraphs) no Truemper configurations other than (possibly) universal wheels and twin wheels. We also study several subclasses of this class. We use our structural results to analyze the complexity of the recognition, maximum weight clique, maximum weight stable set and optimal coloring problems for these classes. Furthermore, we obtain polynomial χ-bounding functions for these classes.

2.1 Introduction

All graphs in this chapter are finite, simple and non-null. We say that a graph G *contains* a graph F if F is isomorphic to an induced subgraph of G, and G is F*-free* if G does not contain F. For a family of graphs $\mathcal{F}$, we say that G is $\mathcal{F}$*-free* if G is F-free for every $F \in \mathcal{F}$. A class of graphs is *hereditary* if, for every graph G in the class, all (isomorphic copies of) induced subgraphs of G belong to the class. Note that a graph class $\mathcal{G}$ is hereditary if and only if there exists a family $\mathcal{F}$ of graphs such that $\mathcal{G}$ is precisely the class of $\mathcal{F}$-free graphs (the "if" part is obvious; for the "only if" part, we can take $\mathcal{F}$ to be the collection of all graphs that do not belong to $\mathcal{G}$, but all of whose proper induced subgraphs do belong to $\mathcal{G}$).

Configurations known as thetas, pyramids, prisms and wheels (defined below) have played an important role in the study of such diverse (and important) classes as the classes of regular matroids, balanceable matrices, perfect graphs and even-hole-free graphs (for a survey, see [48]). These configurations are also called *Truemper configurations*, as they appear in a theorem due to Truemper [45] that characterizes graphs whose edges can be labeled so that all induced cycles have prescribed parities. In this chapter, we study various classes of graphs that are defined by excluding certain Truemper configurations.

A *hole* is an induced cycle on at least four vertices, and an *antihole* is the complement of a hole. The *length* of a hole or antihole is the number of vertices that it contains. A hole or antihole is *long* if it is of length at least five. A hole or antihole is *odd* (resp. *even*) if its length is odd (resp. even). For an integer $k \geqslant 4$, a k*-hole* (resp. k*-antihole*) is a hole (resp. antihole) of length k.

A *theta* is any subdivision of the complete bipartite graph $K_{2,3}$; in particular, $K_{2,3}$ is a theta. A *pyramid* is any subdivision of the complete graph K_4 in which one triangle remains unsubdivided, and, of the remaining three edges, at least two edges are subdivided at least once. A *prism* is any subdivision of $\overline{C_6}$ (where $\overline{C_6}$ is the complement of C_6) in which the two triangles remain unsubdivided; in particular, $\overline{C_6}$ is a prism. A *three-path-configuration* (or *3PC*, for short) is any theta, pyramid or prism; the three types of 3PC are represented in Figure 1.1.

A *wheel* is a graph that consists of a hole and an additional vertex that has at least three neighbors in the hole. If this additional vertex is adjacent to all the vertices of the hole, then the wheel is said to be a *universal wheel*; if the additional vertex is adjacent to three consecutive vertices of the hole, and to no other vertices of the hole, then the wheel is said to be a *twin wheel*. For $k \geqslant 4$, the universal wheel on $k+1$ vertices is denoted by W_k, and the twin wheel on $k+1$ vertices is denoted by W_k^t. A *proper wheel* is a wheel that is neither a universal wheel nor a twin wheel. Note that every proper wheel has at least six vertices.

A *Truemper configuration* is any 3PC or wheel. Observe that every Truemper configuration contains a hole. Note, furthermore, that every theta or prism contains an even hole, and every pyramid contains an odd hole. Thus, even-hole-free graphs contain no thetas and no prisms, and odd-hole-free graphs contain no pyramids.

As usual, given a graph G, we denote by $\chi(G)$ the *chromatic number* of G, by $\omega(G)$ the *clique number* (i.e. the maximum size of a clique) of G, and by $\alpha(G)$ the *stability number* (i.e. the maximum size of a stable set) of G. A graph G is *perfect* if all its induced subgraphs F satisfy $\chi(F) = \omega(F)$. A graph is *Berge* if it contains no odd holes and no odd antiholes. The famous Strong Perfect Graph Theorem [9] states that a graph is perfect if and only if it is Berge. The main ingredient of the proof of the Strong Perfect Graph Theorem is a decomposition theorem for Berge graphs; wheels play a particularly important role (as configurations around which graphs can be decomposed) in the proof of this decomposition theorem. Since perfect graphs are odd-hole-free, we see that perfect graphs contain no pyramids; in fact, detection of pyramids plays an important role in the polynomial-time recognition algorithm for Berge (or, equivalently, perfect) graphs [7].

A graph is *chordal* if it contains no holes. Clearly, every Truemper configuration contains a hole, and consequently, chordal graphs contain no Truemper configurations. A *clique cutset* of a graph G is a (possibly empty) clique C such that $G \setminus C$ is disconnected.

Theorem 1. **([18])** *If G is a chordal graph, then either G is a complete graph or it admits a clique cutset. Furthermore, chordal graphs are perfect.*

A graph G is *universally signable* if, for every prescription of parities to the holes of G, there exists an assignment of zero or one weights to the edges of G such that, for each hole, the sum of the weights of its edges has the prescribed parity, and, for every triangle, the sum of the weights of its edges is odd. Clearly, every chordal graph is universally signable: we simply assign weight one to each edge. Note, however, that holes are universally signable, and so not all universally signable graphs are chordal, and, moreover, not all universally signable graphs are perfect.

Theorem 2. **([14])** *A graph is universally signable if and only if it contains no Truemper configurations. Furthermore, if G is a universally signable graph, then either G is a complete graph or a hole, or it admits a clique cutset.*

In this chapter, we are interested in a superclass of universally signable graphs. In particular, we study the class of (3PC, proper wheel)-free graphs; we call this class $\mathcal{G}_{\mathrm{UT}}$. Clearly, the only Truemper configurations that graphs in $\mathcal{G}_{\mathrm{UT}}$ may contain are universal wheels and twin wheels. In view of Theorem 2, we see that the class of universally signable graphs is a proper subclass of class $\mathcal{G}_{\mathrm{UT}}$.

We also study three subclasses of $\mathcal{G}_{\mathrm{UT}}$. $\mathcal{G}_{\mathrm{U}}$ is the class of all (3PC, proper wheel, twin wheel)-free graphs, and $\mathcal{G}_{\mathrm{T}}$ is the class of all (3PC, proper wheel, universal wheel)-free graphs. Clearly, the only Truemper configurations that graphs in $\mathcal{G}_{\mathrm{U}}$ may contain are universal wheels, and the only Truemper configurations that graphs in $\mathcal{G}_{\mathrm{T}}$ may contain are twin wheels. A *cap* is a graph that consists of a hole and an additional vertex that is adjacent to two consecutive vertices of the hole and to no other vertices of the hole. $\mathcal{G}_{\mathrm{UT}}^{\text{cap-free}}$ is the class of all (3PC, proper wheel, cap)-free graphs. Clearly, $\mathcal{G}_{\mathrm{U}}, \mathcal{G}_{\mathrm{T}}, \mathcal{G}_{\mathrm{UT}}^{\text{cap-free}}$ are all proper subclasses of $\mathcal{G}_{\mathrm{UT}}$. Furthermore, classes $\mathcal{G}_{\mathrm{U}}, \mathcal{G}_{\mathrm{T}}, \mathcal{G}_{\mathrm{UT}}^{\text{cap-free}}$ are pairwise incomparable, that is, none of the three classes are included in either of the remaining two. Since every Truemper configuration and every cap contains a hole, we see that the class of chordal graphs is a (proper) subclass of each of our four classes (i.e. classes $\mathcal{G}_{\mathrm{UT}}, \mathcal{G}_{\mathrm{U}}, \mathcal{G}_{\mathrm{T}}, \mathcal{G}_{\mathrm{UT}}^{\text{cap-free}}$). Furthermore, by Theorem 2, the class of universally signable graphs is a proper subclass of each of $\mathcal{G}_{\mathrm{UT}}, \mathcal{G}_{\mathrm{U}}, \mathcal{G}_{\mathrm{T}}$. However, the class of universally signable graphs and class $\mathcal{G}_{\mathrm{UT}}^{\text{cap-free}}$ are incomparable, that is, neither is a subclass of the other (indeed, caps are universally signable, but do not belong to $\mathcal{G}_{\mathrm{UT}}^{\text{cap-free}}$; on the other hand, universal wheels and twin wheels belong to $\mathcal{G}_{\mathrm{UT}}^{\text{cap-free}}$, but they are not universally signable).

In Subsection 2.1.1, we describe our structural results, and, in Subsection 2.1.2, we describe our results that involve χ-boundedness and algorithms. Moreover, in Section 2.2, we introduce some terminology and notation (mostly standard) that we use throughout the chapter, and we prove a few simple lemmas. Finally, in Sections 2.3-2.8, we prove the results outlined in Subsections 2.1.1 and 2.1.2.

2.1.1 Results: Decomposition theorems for classes $\mathcal{G}_{\mathbf{UT}}, \mathcal{G}_{\mathbf{U}}, \mathcal{G}_{\mathbf{T}}, \mathcal{G}_{\mathbf{UT}}^{\textbf{cap-free}}$

In this subsection, we state our decomposition theorems for classes $\mathcal{G}_{\mathrm{UT}}, \mathcal{G}_{\mathrm{U}}, \mathcal{G}_{\mathrm{T}}, \mathcal{G}_{\mathrm{UT}}^{\text{cap-free}}$. We first define classes $\mathcal{B}_{\mathrm{UT}}, \mathcal{B}_{\mathrm{U}}, \mathcal{B}_{\mathrm{T}}, \mathcal{B}_{\mathrm{UT}}^{\text{cap-free}}$, which we think of as "basic" classes corresponding to $\mathcal{G}_{\mathrm{UT}}, \mathcal{G}_{\mathrm{U}}, \mathcal{G}_{\mathrm{T}}, \mathcal{G}_{\mathrm{UT}}^{\text{cap-free}}$, respectively. For each of the classes $\mathcal{G}_{\mathrm{UT}}, \mathcal{G}_{\mathrm{U}}, \mathcal{G}_{\mathrm{T}}, \mathcal{G}_{\mathrm{UT}}^{\text{cap-free}}$, we show that every graph in the class either belongs to the corresponding basic class or admits a clique cutset.

We state these theorems in the present subsection, and we prove them in Sections 2.3-2.6.

The complement of a graph G is denoted by $\overline{G}$. As usual, a *component* of G is a maximal connected induced subgraph of G. A graph is *anticonnected* if its complement is connected. An *anticomponent* of a graph G is a maximal anticonnected induced subgraph of G. (Thus, C is an anticomponent of G if and only if $\overline{C}$ is a component of $\overline{G}$.) Note that anticomponents of a graph G are pairwise "complete" to each other in G, that is, all possible edges between each pair of distinct anticomponents of G are present in G. A component or anticomponent is *trivial* if it has just one vertex, and it is *non-trivial* if it has at least two vertices.

Lemma 3. *Let G and F be graphs, and assume that F is anticonnected. Then G is F-free if and only if all anticomponents of G are F-free.*

PROOF. This follows immediately from the appropriate definitions. ■

For an integer $k \geqslant 4$, a *k-hyper-hole* (or a *hyper-hole of length k*) is any graph obtained from a k-hole by blowing up each vertex to a non-empty clique of arbitrary size. Similarly, a *k-hyper-antihole* (or a *hyper-antihole of length k*) is any graph obtained from a k-antihole by blowing up each vertex to a non-empty clique of arbitrary size. A hyper-hole or hyper-antihole is *long* if it is of length at least five.

A *ring* is a graph R whose vertex set can be partitioned into $k \geqslant 4$ non-empty sets, say $X_1, \dots, X_k$ (with subscripts understood to be in $\mathbb{Z}_k$), such that, for every $i \in \mathbb{Z}_k$, X_i can be ordered as $X_i = \{u_1^i, \dots, u_{|X_i|}^i\}$ so that $X_i \subseteq N_R[u_{|X_i|}^i] \subseteq \dots \subseteq N_R[u_1^i] = X_{i-1} \cup X_i \cup X_{i+1}$. Under these circumstances, we say that the ring R is of *length* k, as well as that R is a *k-ring*. A ring is *long* if it is of length at least five. Furthermore, we say that $(X_1, \dots, X_k)$ is a *good partition* of the ring R. We observe that every k-hyper-hole is a k-ring.

Given a graph G and distinct vertices $x, y \in V(G)$, we say that x *dominates* y in G, or that y is *dominated by* x in G, provided that $N_G[y] \subseteq N_G[x]$.

Lemma 4. *Let G be a graph, and let $(X_1, \dots, X_k)$, with $k \geqslant 4$ and subscripts understood to be in $\mathbb{Z}_k$, be a partition of $V(G)$. Then G is a k-ring with good partition $(X_1, \dots, X_k)$ if and only if all the following hold:*

(i) $X_1, \dots, X_k$ are cliques;

(ii) for all $i \in \mathbb{Z}_k$, X_i is anticomplete to $V(G) \setminus (X_{i-1} \cup X_i \cup X_{i+1})$;

(iii) for all $i \in \mathbb{Z}_k$, some vertex of X_i is complete to $X_{i-1} \cup X_{i+1}$;

(iv) for all $i \in \mathbb{Z}_k$ and all distinct $y_i, y_i' \in X_i$, one of y_i, y_i' dominates the other.

PROOF. This readily follows from the definition of a ring. ■

Let $\mathcal{B}_{\mathrm{UT}}$ be the class of all graphs G that satisfy at least one of the following:

- G has exactly one non-trivial anticomponent, and this anticomponent is a long ring;
- G is (long hole, $K_{2,3}, \overline{C_6}$)-free;

- $\alpha(G) = 2$, and every anticomponent of G is either a 5-hyper-hole or a $(C_5, \overline{C_6})$-free graph.

Note $\alpha(K_{2,3}) = 3$, and that holes of length at least six have stability number at least three. Thus, graphs of stability number at most two contain no $K_{2,3}$ and no holes of length at least six; consequently, note that $(C_5, \overline{C_6})$-free graphs of stability number at most two are in fact $(\text{long hole}, K_{2,3}, \overline{C_6})$-free.

Let $\mathcal{B}_{\text{U}}$ be the class of all graphs G that satisfy one of the following:

- G has exactly one non-trivial anticomponent, and this anticomponent is a long hole;
- all non-trivial anticomponents of G are isomorphic to $\overline{K_2}$.

Let $\mathcal{B}_{\text{T}}$ be the class of all complete graphs, rings and 7-hyper-antiholes.

As usual, a graph is *bipartite* if its vertex set can be partitioned into two (possibly empty) stable sets. A graph is *co-bipartite* if its complement is bipartite. A *chordal co-bipartite graph* is a graph that is both chordal and co-bipartite. Let $\mathcal{B}_{\text{UT}}^{\text{cap-free}}$ be the class of all graphs G that satisfy one of the following:

- G has exactly one non-trivial anticomponent, and this anticomponent is a hyper-hole of length at least six;
- every anticomponent of G is either a 5-hyper-hole or a chordal co-bipartite graph.

Note that every anticomponent of a complete graph is a chordal co-bipartite graph. Thus, complete graphs belong to $\mathcal{B}_{\text{UT}}^{\text{cap-free}}$. Furthermore, if a graph G contains exactly one non-trivial anticomponent, and this anticomponent is a long hyper-hole, then $G \in \mathcal{B}_{\text{UT}}^{\text{cap-free}}$.

By Lemma 14(iv) (which will be stated and proved in Section 2.2), rings are (3PC, proper wheel, universal wheel)-free. Consequently, rings belong to $\mathcal{G}_{\text{T}}$ and to $\mathcal{G}_{\text{UT}}$. Using this fact, we easily obtain the following lemma.

Lemma 5. *$\mathcal{B}_{\text{UT}} \subseteq \mathcal{G}_{\text{UT}}$, $\mathcal{B}_{\text{U}} \subseteq \mathcal{G}_{\text{U}}$, $\mathcal{B}_{\text{T}} \subseteq \mathcal{G}_{\text{T}}$, and $\mathcal{B}_{\text{UT}}^{\text{cap-free}} \subseteq \mathcal{G}_{\text{UT}}^{\text{cap-free}}$.*

PROOF (ASSUMING LEMMA 14). It follows from Lemma 14(iv) that rings belong to $\mathcal{G}_{\text{T}}$ and to $\mathcal{G}_{\text{UT}}$. Furthermore, note that the only Truemper configurations that are not anticonnected are the theta $K_{2,3}$, the twin wheel W_4^t, and universal wheels. The result now follows from Lemma 3 and routine checking. ■

We now state our decomposition theorems for classes $\mathcal{G}_{\text{UT}}, \mathcal{G}_{\text{U}}, \mathcal{G}_{\text{T}}, \mathcal{G}_{\text{UT}}^{\text{cap-free}}$. We prove these theorems in Sections 2.3-2.6.

Theorem 6. *Every graph in $\mathcal{G}_{\text{UT}}$ either belongs to $\mathcal{B}_{\text{UT}}$ or admits a clique cutset.*

Theorem 7. *Every graph in $\mathcal{G}_{\text{U}}$ either belongs to $\mathcal{B}_{\text{U}}$ or admits a clique cutset.*

Theorem 8. *Every graph in $\mathcal{G}_{\text{T}}$ either belongs to $\mathcal{B}_{\text{T}}$ or admits a clique cutset.*

Theorem 9. *Every graph in $\mathcal{G}_{\text{UT}}^{\text{cap-free}}$ either belongs to $\mathcal{B}_{\text{UT}}^{\text{cap-free}}$ or admits a clique cutset.*

A *composition theorem* for a given class $\mathcal{G}$ is a theorem that states that every graph in $\mathcal{G}$ either is "basic" (i.e. it belongs to some well-understood subclass of $\mathcal{G}$) or can be built from smaller graphs via one of several "class-preserving compositions", where a class-preserving composition for $\mathcal{G}$ is an operation that, when applied to graphs that belong to $\mathcal{G}$, produces a graph that also belongs to $\mathcal{G}$.

The clique cutset decomposition has a natural reverse operation, namely the operation of "gluing along a clique". Let G_1 and G_2 be graphs, and assume that $C = V(G_1) \cap V(G_2)$ is a (possibly empty) clique. Let G be the graph such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. Under these circumstances, we say that G is obtained by *gluing G_1 and G_2 along the clique C*, or simply that G is obtained from G_1 and G_2 by *gluing along a clique*.

Lemma 10. *Let $\mathcal{F}$ be a family of graphs, none of which admit a clique cutset, and let $\mathcal{G}$ be the class of $\mathcal{F}$-free graphs. Let $\mathcal{B}$ be a subclass of $\mathcal{G}$. Assume that every graph in $\mathcal{G}$ either belongs to $\mathcal{B}$ or admits a clique cutset. Then a graph belongs to $\mathcal{G}$ if and only if it can be obtained from graphs in $\mathcal{B}$ by repeatedly gluing along cliques.*

Proof. This readily follows from appropriate definitions. ■

Since no 3PC and no wheel admits a clique cutset, Lemmas 5 and 10 imply that Theorems 6, 7 and 8 can readily be converted into composition theorems. On the other hand, since every cap admits a clique cutset, the same does not hold for Theorem 9.

2.1.2 Results: χ-Boundedness and algorithms

In Section 2.7, we study χ-boundedness. A graph class $\mathcal{G}$ is said to be χ-*bounded* provided that there exists a function $f : \mathbb{N}^+ \to \mathbb{N}^+$ (called a χ-*bounding function* for $\mathcal{G}$) such that, for all graphs $G \in \mathcal{G}$, all induced subgraphs F of G satisfy $\chi(F) \leqslant f(\omega(F))$. Note that a hereditary graph class $\mathcal{G}$ is χ-bounded if and only if there exists a function $f : \mathbb{N}^+ \to \mathbb{N}^+$ such that every graph $G \in \mathcal{G}$ satisfies $\chi(G) \leqslant f(\omega(G))$. χ-Boundedness was introduced by Gyárfás [21] as a natural generalization of perfection: clearly, the class of perfect graphs is hereditary and χ-bounded by the identity function. It follows from [27] that the class of theta-free graphs is χ-bounded; consequently, our four classes (i.e. classes $\mathcal{G}_{\rm UT}, \mathcal{G}_{\rm U}, \mathcal{G}_{\rm T}, \mathcal{G}_{\rm UT}^{\text{cap-free}}$) are all χ-bounded. Unfortunately, the χ-bounding function from [27] is superexponential. Using our structural results, we obtain polynomial χ-bounding functions for our four classes. In fact, we obtain linear χ-bounding functions for $\mathcal{G}_{\rm U}$, $\mathcal{G}_{\rm T}$ and $\mathcal{G}_{\rm UT}^{\text{cap-free}}$; our χ-bounding function for $\mathcal{G}_{\rm UT}$ is a fourth-degree polynomial function.

Finally, in Section 2.8, we turn to the algorithmic consequences of our structural results. We consider four algorithmic problems:

- the recognition problem, i.e. the problem of determining whether an input graph belongs to a given class;
- the maximum weight stable set problem (MWSSP), i.e. the problem of finding a maximum weight stable set of an input weighted graph (with real weights);

- the maximum weight clique problem (MWCP), i.e. the problem of finding a maximum weight clique of an input weighted graph (with real weights);
- the optimal coloring problem (ColP), i.e. the problem of finding an optimal coloring of an input graph.

We remark that all our algorithms are *robust*, that is, they either produce a correct solution to the problem in question for the input (weighted) graph, or they correctly determine that the graph does not belong to the class under consideration. (If the input graph does not belong to the class under consideration, a robust algorithm may possibly produce a correct solution to the problem in question, rather than determine that the input graph does not belong to the class.)

A summary of our results is given in the table below. As usual, n is the number of vertices and m the number of edges of the input graph. For the sake of compactness, we write $\mathcal{O}(nm)$ and $\mathcal{O}(n^2m)$ instead of $\mathcal{O}(n^2+nm)$ and $\mathcal{O}(n^3+n^2m)$, respectively. Question marks indicate open problems, and not all χ-bounding functions given in the table are optimal (this is discussed in more detail below).

	Recognition	MWSSP	MWCP	ColP	χ-Boundedness
$\mathcal{G}_{\text{UT}}$	$\mathcal{O}(n^6)$	?	NP-hard	?	$\chi \leqslant 2\omega^4$
$\mathcal{G}_{\text{U}}$	$\mathcal{O}(nm)$	$\mathcal{O}(nm)$	$\mathcal{O}(nm)$ [1]	$\mathcal{O}(nm)$	$\chi \leqslant \omega+1$
$\mathcal{G}_{\text{T}}$	$\mathcal{O}(n^3)$	$\mathcal{O}(n^2m)$	$\mathcal{O}(nm)$	?	$\chi \leqslant \lfloor\frac{3}{2}\omega\rfloor$
$\mathcal{G}_{\text{UT}}^{\text{cap-free}}$	$\mathcal{O}(n^5)$	$\mathcal{O}(n^3)$	$\mathcal{O}(n^3)$	$\mathcal{O}(n^3)$	$\chi \leqslant \lfloor\frac{3}{2}\omega\rfloor$

Most of our algorithms rely on Theorems 6, 7, 8 and 9. Since all four theorems involve clique cutsets, most of our algorithms also rely on techniques developed in [43] for handling clique cutsets.

At this time, we do not know whether rings can be optimally colored in polynomial time, and, for this reason, we do not know the complexity of the ColP for class $\mathcal{G}_{\text{T}}$.

As shown in the table, an $\mathcal{O}(n^2+nm)$-time algorithm solving the MWCP for class $\mathcal{G}_{\text{U}}$ was given in [1]; that algorithm relies on LexBFS [41]. In the present chapter, we give a different algorithm that solves the MWCP for class $\mathcal{G}_{\text{U}}$ (our algorithm has the same complexity as the one from [1], but it relies on our structural results for class $\mathcal{G}_{\text{U}}$). Further, we note that the complexity of the ColP for class $\mathcal{G}_{\text{U}}$ was left open in [1]; here, we give a polynomial-time algorithm that solves this problem. Finally, we note that it was shown in [1] that every graph $G \in \mathcal{G}_{\text{U}}$ has a *bisimplicial* vertex, i.e. a vertex whose neighborhood can be partitioned into two (possibly empty) cliques; this result immediately implies that every graph $G \in \mathcal{G}_{\text{U}}$ satisfies $\chi(G) \leqslant 2\omega(G)-1$. Using our structural results, we obtain a better χ-bounding function for class $\mathcal{G}_{\text{U}}$.

We define the *join* of graphs $G_1, \dots, G_\ell$ on pairwise disjoint vertex sets to be a graph G such that $V(G)=\bigcup_{i=1}^{\ell} V(G_i)$ and $E(G)=\bigcup_{i=1}^{\ell} E(G_i) \cup \{x_jx_k : 1 \leqslant j < k \leqslant \ell, x_j \in V(G_j), x_k \in V(G_k)\}$.

Observe that, if G is the join of an odd hole and a complete graph, then we have that $G \in \mathcal{G}_{\text{U}}$ and $\chi(G) = \omega(G)+1$. Further, if G is the join of arbitrarily many copies of C_5, then $G \in \mathcal{G}_{\text{UT}}^{\text{cap-free}}$ and

$\chi(G) = \lfloor \frac{3}{2}\omega(G) \rfloor$. This proves that our χ-bounding functions for $\mathcal{G}_{\mathrm{U}}$ and $\mathcal{G}_{\mathrm{UT}}^{\text{cap-free}}$ are optimal. We do not know whether our χ-bounding function for class $\mathcal{G}_{\mathrm{T}}$ is optimal. In Section 2.7, we show that class $\mathcal{G}_{\mathrm{UT}}$ is χ-bounded by a function of order $\frac{\omega^4}{\log^2 \omega}$, and thus the corresponding χ-bounding function given in the table above is not optimal. In fact, we do not know the order of the optimal χ-bounding function for class $\mathcal{G}_{\mathrm{UT}}$.

2.2 Preliminaries

In this section, we introduce some (mostly standard) terminology and notation that we use throughout the chapter. We also prove a few preliminary results.

2.2.1 Terminology and notation

The set of non-negative integers is denoted by $\mathbb{N}$, and the set of positive integers by $\mathbb{N}^+$. A *singleton* is a one-element set.

The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. When no confusion is possible, we write G instead of $V(G)$.

A graph is *trivial* if it has just one vertex; a graph is *non-trivial* if it has at least two vertices. For a vertex x of a graph G, $N_G(x)$ is the set of all neighbors of x in G, $d_G(x) = |N_G(x)|$ is the *degree* of x in G, and $N_G[x] = N_G(x) \cup \{x\}$. For a set $S \subseteq V(G)$, $N_G(S)$ is the set of all the vertices in $V(G) \setminus S$ that have at least one neighbor in S, and $N_G[S] = N_G(S) \cup S$. The maximum degree of G is denoted by $\Delta(G)$, that is, $\Delta(G) = \max\{d_G(x) : x \in V(G)\}$.

For a graph G and a non-empty set $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S. For a graph G and a set $S \subset V(G)$, we set $G \setminus S = G[V(G) \setminus S]$. (Since we only deal with non-null graphs, if G is trivial and x is the only vertex of G, then $G \setminus \{x\}$ is undefined.)

Given a graph G, a vertex $x \in V(G)$ and a set $Y \subseteq V(G) \setminus \{x\}$, we say that x is *complete* (resp. *anticomplete*) to Y in G provided that x is adjacent (resp. non-adjacent) to every vertex in Y. Given disjoint sets $X, Y \subseteq V(G)$, we say that X is *complete* (resp. *anticomplete*) to Y in G provided that every vertex in X is complete (resp. anticomplete) to Y.

As usual, a *clique* (resp. *stable set*) in a graph G is a (possibly empty) set of pairwise adjacent (resp. non-adjacent) vertices of G. The *clique number* of G, denoted by $\omega(G)$, is the size of a largest clique of G; the *stability number* of G, denoted by $\alpha(G)$, is the size of a largest stable set of G. A *maximum clique* (resp. *maximum stable set*) of G is a clique (resp. stable set) of size $\omega(G)$ (resp. $\alpha(G)$). A *complete graph* is a graph whose vertex set is a clique. The complete graph on n vertices is denoted by K_n; K_3 is also referred to as a *triangle*.

A *weighted graph* is an ordered pair (G, w), where G is a graph and $w : V(G) \to \mathbb{R}$ is a *weight function* for G. For a set $S \subseteq V(G)$, the *weight* of S, denoted by $w(S)$, is the sum of the weights of all the vertices in S, that is, $w(S) = \sum_{x \in S} w(x)$. The *clique number* (resp. *stability number*) of a weighted graph (G, w), denoted by $\omega(G, w)$ (resp. $\alpha(G, w)$), is the maximum weight of a clique (resp. stable set) of G. A *maximum weight clique* (resp. *maximum weight stable set*) of (G, w) is a clique (resp. stable set) of G whose weight is precisely $\omega(G, w)$ (resp. $\alpha(G, w)$).

Clearly, if (G, w) is a weighted graph and F is an induced subgraph of G, then the restriction of w to $V(F)$, denoted by $w \upharpoonright V(F)$, is a weight function for F, and $(F, w \upharpoonright V(F))$ is a weighted graph; to simplify notation, we usually write (F, w) instead of $(F, w \upharpoonright V(F))$.

For a positive integer k, a *k-coloring* of a graph G is a function $c : V(G) \to \{1, \dots, k\}$ such that $c(x) \neq c(y)$ whenever $xy \in E(G)$; elements of $\{1, \dots, k\}$ are called *colors*. A graph is *k-colorable* if it admits a k-coloring. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest integer k such that G is k-colorable.

A *path* is a graph P with vertex set $V(P) = \{x_1, \dots, x_k\}$ (where $k \geqslant 1$) and edge set $E(P) = \{x_1x_2, x_2x_3, \dots, x_{k-1}x_k\}$; under these circumstances, we write that "$P = x_1, \dots, x_k$ is a path", and we say that the *length* of P is $k - 1$ (i.e. the length of a path is the number of edges that it contains), that the *endpoints* of P are x_1 and x_k (if $k = 1$, then the endpoints of P coincide), that $x_2, \dots, x_{k-1}$ are the *interior vertices* of P (note that P has interior vertices if and only if $k \geqslant 3$), and that P is a path *between x_1 and x_k*. A *path* in a graph G is a subgraph of G that is a path. An *induced path* in a graph G is an induced subgraph of G that is a path.

A *cycle* is a graph C with vertex set $V(C) = \{x_1, \dots, x_k\}$ (where $k \geqslant 3$, and subscripts are understood to be in $\mathbb{Z}_k$) and edge set $E(C) = \{x_1x_2, x_2x_3, \dots, x_{k-1}x_k, x_kx_1\}$; under these circumstances, we write that "$C = x_1, \dots, x_k, x_1$ is a cycle", and we say that the *length* of C is k. A *cycle* in a graph G is a subgraph of G that is a cycle. An *induced cycle* in a graph G is an induced subgraph of G that is a cycle. The *girth* of a graph G is the length of a shortest cycle in G, and acyclic graphs are considered to have infinite girth.

A path of length k is denoted by P_{k+1} (note that P_{k+1} has $k + 1$ vertices and k edges), and a cycle of length k is denoted by C_k (note that C_k has k vertices and k edges).

A *hole* in a graph G is an induced cycle of length at least four. An *antihole* in a graph G is an induced subgraph of G whose complement is a hole in $\overline{G}$. The *length* of a hole or antihole is the number of vertices that it contains; a *k-hole* (resp. *k-antihole*) is a hole (resp. antihole) of length k. A hole or antihole is *long* if it is of length at least five. A hole or antihole is *odd* (resp. *even*) if its length is odd (resp. even). Further, consistent with the notation above, we write "$H = x_1, \dots, x_k, x_1$ is a hole", or simply "$x_1, \dots, x_k, x_1$ is a hole" (with $k \geqslant 4$, and subscripts understood to be in $\mathbb{Z}_k$), when $H = x_1, \dots, x_k, x_1$ is an induced cycle. On the other hand, we write that "$A = x_1, \dots, x_k, x_1$ is an antihole", or simply "$x_1, \dots, x_k, x_1$ is an antihole" (with $k \geqslant 4$, and subscripts understood to be in $\mathbb{Z}_k$), when $\overline{A} = x_1, \dots, x_k, x_1$ is a hole.

Let F be an induced subgraph of a graph G. Two distinct vertices $x, y \in V(G)$ are *twins* w.r.t. F if $N_G[x] \cap V(F) = N_G[y] \cap V(F)$. Given a vertex $x \in V(G)$, we denote by $X_x^G(F)$ the set consisting of x and all the twins of x in G w.r.t. F. The set of all the vertices in $V(G) \setminus V(F)$ that are complete to $V(F)$ is denoted by U_F^G. When no confusion is possible, we omit the superscript G in $X_x^G(F)$ and U_F^G, and instead, we simply write $X_x(F)$ and U_F, respectively. Further, we set $F_G^* = G[\bigcup_{x \in V(F)} X_x^G(F)]$; when no confusion is possible, we omit the subscript G and simply write F^*.

A *hyper-hole* is a graph H whose vertex set can be partitioned into $k \geqslant 4$ non-empty cliques, say $X_1, \dots, X_k$ (with subscripts understood to be in $\mathbb{Z}_k$), such that, for every $i \in \mathbb{Z}_k$, X_i is complete to $X_{i-1} \cup X_{i+1}$ and anticomplete to $V(H) \setminus (X_{i-1} \cup X_i \cup X_{i+1})$; under these circumstances, we say that the hyper-hole H is of *length* k, and we also write that "$H = X_1, \dots, X_k, X_1$ is a hyper-hole"; furthermore, we say that $(X_1, \dots, X_k)$ is a *good partition* of the hyper-hole H. A *k-hyper-hole* is a hyper-hole of length k, and a *long hyper-hole* is a hyper-hole of length at least five. Note that, if H is a k-hyper-hole with good partition $(X_1, \dots, X_k)$, then H is a k-ring with good partition $(X_1, \dots, X_k)$.

A *hyper-antihole* is a graph A whose vertex set can be partitioned into $k \geqslant 4$ non-empty cliques, say $X_1, \dots, X_k$ (with subscripts understood to be in $\mathbb{Z}_k$), such that, for every $i \in \mathbb{Z}_k$, X_i is anticomplete to $X_{i-1} \cup X_{i+1}$ and complete to $V(A) \setminus (X_{i-1} \cup X_i \cup X_{i+1})$; under these circumstances, we say that the hyper-antihole A is of *length* k, and we also write that "$A = X_1, \dots, X_k, X_1$ is a hyper-antihole"; furthermore, we say that $(X_1, \dots, X_k)$ is a *good partition* of the hyper-antihole A. A *k-hyper-antihole* is a hyper-antihole of length k, and a *long hyper-antihole* is a hyper-antihole of length at least five. Note that the complement of a hyper-antihole does not need to be a hyper-hole.

A graph is *bipartite* if its vertex set can be partitioned into two (possibly empty) stable sets. A graph is *co-bipartite* if its complement is bipartite. A *complete bipartite graph* is a graph whose vertex set can be partitioned into two (possibly empty) stable sets that are complete to each other; for $n, m \in \mathbb{N}^+$, $K_{n,m}$ is a complete bipartite graph whose vertex set can be partitioned into two stable sets, one of size n and the other one of size m, that are complete to each other.

A *cutset* of a graph G is a (possibly empty) set $C \subset V(G)$ such that $G \setminus C$ is disconnected. A *cut partition* of a graph G is a partition (A, B, C) of $V(G)$ such that A and B are non-empty and anticomplete to each other (the set C may possibly be empty). Clearly, if (A, B, C) is a cut partition of G, then C is a cutset of G; conversely, every cutset of G gives rise to at least one cut partition of G. A *clique cutset* of a graph G is a (possibly empty) clique of G that is also a cutset of G. A *clique cut partition* of a graph G is a cut partition (A, B, C) of G such that C is a clique. Again, if (A, B, C) is a clique cut partition of G, then C is a clique cutset of G, and conversely, every clique cutset of G gives rise to at least one clique cut partition of G.

Let G be a *3PC*. It follows that G contains three paths, say $P_1 = x_1, \dots, y_1$, $P_2 = x_2, \dots, y_2$ and $P_3 = x_3, \dots, y_3$, such that $V(G) = V(P_1) \cup V(P_2) \cup V(P_3)$, $\{x_1, x_2, x_3\} \cap \{y_1, y_2, y_3\} = \varnothing$, $\{x_1, x_2, x_3\}$ either induces a triangle or is a singleton (i.e. $x_1 = x_2 = x_3$), $\{y_1, y_2, y_3\}$ either induces a triangle or is a singleton (i.e. $y_1 = y_2 = y_3$), and $V(P_i) \cup V(P_j)$ induces a hole for all distinct $i, j \in \{1, 2, 3\}$. If $x_1 = x_2 = x_3$ and $y_1 = y_2 = y_3$, then x_1 is non-adjacent to y_1, and we say that G is a $3PC(x_1, y_1)$; in this case, G is a *theta*. If $\{x_1, x_2, x_3\}$ induces a triangle and $y_1 = y_2 = y_3$, then we say that G is a $3PC(x_1x_2x_3, y_1)$; in this case, G is a *pyramid*. Finally, if $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ both induce a triangle, then we say that G is a $3PC(x_1x_2x_3, y_1y_2y_3)$; in this case, G is a *prism*. When we say that "Σ is a 3PC in G", we always assume that Σ is an induced subgraph of a graph G.

A *wheel* (H, x) is a graph that consists of a hole H, called the *rim*, and an additional vertex x, called the *center*, such that x has at least three neighbors in H. A *universal wheel* is a wheel (H, x) in which x is complete to $V(H)$. A *twin wheel* is a wheel (H, x) such that x has precisely three neighbors in H, and those three neighbors are consecutive vertices of H. A wheel that is neither a universal wheel nor a twin wheel is called a *proper wheel*. When we say that "(H, x) is a wheel in G", we always assume that (H, x) is an induced subgraph of a graph G.

2.2.2 A few preliminary lemmas

Let W_5^4 be the six-vertex wheel consisting of a C_5 and a vertex that has precisely four neighbors in the C_5. We remind the reader that, for $k \geqslant 4$, the universal wheel on $k+1$ vertices is denoted by W_k, and the twin wheel on $k+1$ vertices is denoted by W_k^t.

Lemma 11. *No Truemper configuration has a clique cutset. The only Truemper configurations of stability number two are the prism $\overline{C_6}$, the universal wheels W_4 and W_5, the twin wheels W_4^t and W_5^t, and the proper wheel W_5^4; all other Truemper configurations have stability number at least three. The theta $K_{2,3}$, the prism $\overline{C_6}$, the universal wheel W_4, and the twin wheel W_4^t are the only Truemper configurations that do not contain a long hole. The only Truemper configurations that are not anticonnected are the theta $K_{2,3}$, the twin wheel W_4^t, and universal wheels.*

Proof. This follows by routine checking. ■

Lemma 12. *Let G be a $K_{2,3}$-free graph that has at least two non-trivial anticomponents. Then $\alpha(G) = 2$.*

Proof. Let G be a graph that has at least two non-trivial anticomponents, and assume that $\alpha(G) \geqslant 3$. Let $\{a_1, a_2, a_3\}$ be a stable set of size three of G; clearly, a_1, a_2 and a_3 belong to the same anticomponent of G, say A. Let B be a non-trivial anticomponent of G that is different from A, and fix non-adjacent vertices $b_1, b_2 \in V(B)$. Then $G[\{b_1, b_2, a_1, a_2, a_3\}]$ is a $K_{2,3}$, and so G is not $K_{2,3}$-free, a contradiction. ■

The (unique) cap on five vertices is called the *house*. Note that the house is isomorphic to $\overline{P_5}$. Clearly, every cap-free graph is house-free.

Lemma 13. *Let G be a graph such that $\alpha(G) \leqslant 2$, and assume that G admits a clique cutset. Then the following hold:*

(i) G is co-bipartite, and consequently, G contains no long holes;

(ii) if G is house-free, then G is chordal.

Proof. Let (A, B, C) be a clique cut partition of G. Then A is a clique, since otherwise we fix non-adjacent vertices $a_1, a_2 \in A$, we fix any $b \in B$, and we observe that $\{a_1, a_2, b\}$ is a stable set of G of size three, a contradiction. Similarly, B is a clique. Further, every vertex of C is complete to at least one of A and B, since otherwise we fix some $c \in C$ that has a non-neighbor $a \in A$ and a non-neighbor $b \in B$, and we observe that $\{a, b, c\}$ is a stable set of G of size three, a contradiction. Let C_A be the set of all the vertices of C that are complete to A, and set $C_B = C \setminus C_A$; then C_B is complete to B. Now $A \cup C_A$ and $B \cup C_B$ are (disjoint) cliques whose union is $V(G)$, and so it follows that G is co-bipartite. Since no co-bipartite graph contains a long hole, (i) holds.

It remains to prove (ii). We assume that G is house-free, and we show that G is chordal. In view of (i), we just need to prove that G contains no 4-holes. Assume otherwise, and let $H = x_1, x_2, x_3, x_4, x_1$ be a 4-hole in G. Since H contains no clique cutset, we see that either $V(H) \subseteq A \cup C$ or $V(H) \subseteq B \cup C$; by symmetry, w.l.o.g. we may assume that $V(H) \subseteq A \cup C$. Since A and C are cliques, and since H contains no triangles, we see that each of A and C contains at most two vertices of H, and furthermore, if A or C contains precisely two vertices, then those two vertices are adjacent. So, by symmetry, w.l.o.g. we may now assume that $x_1, x_2 \in A$ and $x_3, x_4 \in C$. But then neither x_3 nor x_4 is complete to A, and consequently, x_3 and x_4 are complete to B. Fix $b \in B$. Now $G[\{x_1, x_2, x_3, x_4, b\}]$ is a house, a contradiction. This proves (ii). ■

Lemma 14. *For some integer $k \geqslant 4$, let R be a k-ring with good partition $(X_1, \dots, X_k)$. Then all the following hold:*

(i) every hole in R intersects each of $X_1, \dots, X_k$ in exactly one vertex;

(ii) every hole in R is of length k;

(iii) for all $i \in \mathbb{Z}_k$, $R \setminus X_i$ is chordal;

(iv) R is (3PC, proper wheel, universal wheel)-free;

(v) R is cap-free if and only if R is a k-hyper-hole with good partition $(X_1, \dots, X_k)$.

Proof. Since no vertex in a hole dominates any other vertex of that hole, Lemma 4(iv) guarantees that a hole in R can intersect each of $X_1, \dots, X_k$ in at most one vertex. Statement (i) now follows from Lemma 4(ii). Statements (ii) and (iii) follow immediately from (i).

Next, we prove (iv). Suppose that Σ is a 3PC in R. We know that Σ contains at least three holes, and, by (i), each of those holes contains exactly one vertex from each of $X_1, \dots, X_k$. Thus, some X_i (with $i \in \mathbb{Z}_k$) contains at least two distinct vertices of Σ. But, by the definition of a 3PC, we see that every pair of distinct vertices of Σ belongs to a hole of Σ. Thus, X_i contains at least two vertices of some hole of Σ, contrary to (i). This proves that R is 3PC-free.

Suppose now that (H, x) is a wheel in R; we must show that (H, x) is a twin wheel. Using (i), for each $i \in \mathbb{Z}_k$, we let x_i be the unique vertex in $V(H) \cap X_i$. It follows from Lemma 4(ii) that the hole H is of the form $H = x_1, \dots, x_k, x_1$. By symmetry, w.l.o.g. we may assume that $x \in X_2$. Since x has at least three neighbors in H (because (H, x) is a wheel), Lemma 4(ii) implies that the neighbors of x in H are precisely x_1, x_2, x_3. Thus, (H, x) is a twin wheel, and we deduce that R is (proper wheel, universal wheel)-free. This proves (iv).

It remains to prove (v). The "if" part follows from (i) and routine checking. For the "only if" part, we assume that R is not a k-hyper-hole with good partition $(X_1, \dots, X_k)$, and we show that R is not cap-free. Since R is a k-ring with good partition $(X_1, \dots, X_k)$, but not a k-hyper-hole with good partition $(X_1, \dots, X_k)$, w.l.o.g. we may assume by symmetry that X_1 is not complete to X_2. Fix non-adjacent vertices $y_1 \in X_1$ and $y_2 \in X_2$. By the definition of a ring, for every $i \in \mathbb{Z}_k$, there exists a vertex $x_i \in X_i$ such that $N_R[x_i] = X_{i-1} \cup X_i \cup X_{i+1}$. Since $y_1 y_2 \notin E(R)$, we see that $y_1 \neq x_1$ and $y_2 \neq x_2$. But now $H = y_1, x_2, \dots, x_k, y_1$ is a hole in R, and $N_R(y_2) \cap V(H) = \{x_2, x_3\}$; it follows that $R[\{y_1, y_2, x_2, \dots, x_k\}]$ is a cap, and so R is not cap-free. This proves (v). ■

2.3 A decomposition theorem for class $\mathcal{G}_{\text{UT}}$

In this section, we prove Theorem 6, which states that every graph in $\mathcal{G}_{\text{UT}}$ either belongs to $\mathcal{B}_{\text{UT}}$ or admits a clique cutset. We begin with a few preliminary lemmas, which will be of use to us, not only in this section, but also in subsequent ones.

Lemma 15. *Let $G \in \mathcal{G}_{\text{UT}}$, and let $H = x_1, \dots, x_k, x_1$, $k \geqslant 4$, be a hole in G. For every $x \in V(G) \setminus V(H)$, one of the following holds:*

(i) x is complete to $V(H)$;

(ii) there exists some $i \in \mathbb{Z}_k$ such that $N_G(x) \cap V(H) = \{x_{i-1}, x_i, x_{i+1}\}$ (i.e. x is a twin of x_i w.r.t. H);

(iii) there exists some $i \in \mathbb{Z}_k$ such that $N_G(x) \cap V(H) \subseteq \{x_i, x_{i+1}\}$ (i.e. $N_G(x) \cap V(H)$ is a clique of size at most two).

Proof. If $|N_G(x) \cap V(H)| \leqslant 1$, then (iii) trivially holds. If $|N_G(x) \cap V(H)| = 2$, then (iii) holds, since otherwise $G[V(H) \cup \{x\}]$ is a theta, a contradiction. If $3 \leqslant |N_G(x) \cap V(H)| \leqslant k-1$, then (ii) holds, since otherwise (H, x) is a proper wheel in G, a contradiction. Finally, if $|N_G(x) \cap V(H)| = k$, then (i) holds. ■

Lemma 16. *Let $G \in \mathcal{G}_{\text{UT}}$, and let $H = x_1, \dots, x_k, x_1$, $k \geqslant 4$, be a hole in G. For every $i \in \mathbb{Z}_k$, set $X_i = X_{x_i}(H)$. Then the following hold:*

- *$X_1, \dots, X_k$ are pairwise disjoint cliques;*
- *if $k \geqslant 5$, then H^* is a k-ring with good partition $(X_1, \dots, X_k)$.*

Proof. It is clear that $X_1, \dots, X_k$ are pairwise disjoint, and hence $(X_1, \dots, X_k)$ is a partition of $V(H^*)$. Let us show that $X_1, \dots, X_k$ are cliques. By symmetry, it suffices to prove that X_1 is a clique. Assume not and fix non-adjacent vertices $y_1, y_1' \in X_1$. It follows that the hole $y_1, x_2, \dots, x_k, y_1$ and vertex y_1' contradict Lemma 15. This proves that $X_1, \dots, X_k$ are cliques.

From now on, we assume that $k \geqslant 5$. Our goal is to show that H^* and $(X_1, \dots, X_k)$ satisfy (i)-(iv) of Lemma 4. We already proved that $X_1, \dots, X_k$ satisfy (i). Also, it is clear that x_i is complete to $X_{i-1} \cup X_{i+1}$ for every $i \in \mathbb{Z}_k$; thus, (iii) holds.

We now prove (ii). Assume that (ii) does not hold. By symmetry, w.l.o.g. we may suppose that, for some index $j \in \mathbb{Z}_k \setminus \{k, 1, 2\}$ and some vertices $y_1 \in X_1$ and $y_j \in X_j$, we have that $y_1 y_j \in E(G)$. By construction, x_1 is anticomplete to X_j and x_j is anticomplete to X_1; since $y_1 y_j \in E(G)$, it follows that $y_1 \neq x_1$ and $y_j \neq x_j$. But now the hole $y_1, x_2, \dots, x_k, y_1$ and vertex y_j contradict Lemma 15. Thus, (ii) holds.

It remains to prove (iv); by symmetry, it suffices to prove this for $i = 1$. Let $y_1, y_1' \in X_1$ be distinct vertices; we claim that one of y_1, y_1' dominates the other in H^*. Suppose otherwise.

Since X_1 is a clique that is anticomplete to $V(H^*) \setminus (X_k \cup X_1 \cup X_2)$, it follows that there exist $z, z' \in X_k \cup X_2$ such that $y_1z, y_1'z' \in E(G)$ and $y_1z', y_1'z \notin E(G)$. By symmetry, w.l.o.g. we may assume that either $z \in X_k$ and $z' \in X_2$, or that $z, z' \in X_2$. Suppose first that $z \in X_k$ and $z' \in X_2$. Then $H' = y_1, y_1', z', x_3, \dots, x_{k-1}, z, y_1$ is a hole. Furthermore, since x_2 is complete to X_1, while $z'y_1 \notin E(G)$, it follows that $x_2 \neq z'$. Thus, we see that $x_2 \notin V(H')$, and that x_2 has precisely four neighbors (namely, y_1, y_1', z', x_3) in H'. Hence, (H', x_2) is a proper wheel in G, a contradiction. Assume now that $z, z' \in X_2$. It follows that $G[\{y_1, y_1', z, z', x_3, \dots, x_k\}]$ is a 3PC$(x_ky_1y_1', x_3zz')$, a contradiction. Thus, one of y_1, y_1' dominates the other in H^*, and (iv) holds. Therefore, we have shown that H^* and $(X_1, \dots, X_k)$ satisfy (i)-(iv) of Lemma 4, and hence H^* is a k-ring with good partition $(X_1, \dots, X_k)$. ■

Lemma 17. *Let $G \in \mathcal{G}_{\text{UT}}$, and assume that G contains a long hole. It follows that either some anticomponent of G is a long ring, or G admits a clique cutset.*

Proof. Let $H = x_1, \dots, x_k, x_1$ be a hole of maximum length in G (it follows that $k \geqslant 5$ and G contains no holes of length greater than k), and, subject to that, assume that H is chosen so that $|V(H^*)|$ is maximum. For every $i \in \mathbb{Z}_k$, set $X_i = X_{x_i}(H)$, and set $K = G[V(H^*) \cup U_{H^*}]$. By Lemma 16, H^* is a k-ring with good partition $(X_1, \dots, X_k)$; Lemma 4 now implies that $X_1, \dots, X_k$ are cliques, and that X_i is anticomplete to $V(H^*) \setminus (X_{i-1} \cup X_i \cup X_{i+1})$ for all $i \in \mathbb{Z}_k$. Clearly, H^* is anticonnected. Therefore, the long ring H^* is an anticomponent of K and hence, if $G = K$, then we are done. So, from now on, we assume that $V(K) \subset V(G)$.

(1) $U_{H^*} = U_H$.

Proof of (1). Clearly, $U_{H^*} \subseteq U_H$. Suppose that $U_H \not\subseteq U_{H^*}$, and fix some $x \in U_H \setminus U_{H^*}$. Fix $i \in \mathbb{Z}_k$ and a vertex $y_i \in X_i$ that is not adjacent to x. Clearly, $y_i \neq x_i$. Now, let H' be the hole induced in G by the vertex set $(V(H) \setminus \{x_i\}) \cup \{y_i\}$. Then H' and vertex x contradict Lemma 15. This proves (1). □

(2) For every vertex $x \in V(G) \setminus V(K)$, $N_G(x) \cap V(H^*) \subseteq X_i \cup X_{i+1}$ for some $i \in \mathbb{Z}_k$. Also, $N_G(x) \cap V(H^*)$ is a clique.

Proof of (2). Fix $x \in V(G) \setminus V(K)$. By (1), x is not complete to $V(H)$. Since $x \notin V(K)$, we know that x is not a twin of a vertex of H w.r.t. H. So, Lemma 15 now implies that $N_G(x) \cap V(H)$ is a clique of size at most two.

We first show that there exists some $i \in \mathbb{Z}_k$ such that $N_G(x) \cap V(H^*) \subseteq X_i \cup X_{i+1}$. Suppose otherwise. By symmetry, w.l.o.g. we may assume that there exists some $j \in \mathbb{Z}_k \setminus \{k, 1, 2\}$ such that x has a neighbor both in X_1 and in X_j. For each $i \in \{1, j\}$, if x is adjacent to x_i, then set $y_i = x_i$, and otherwise let y_i be any neighbor of x in X_i. Since X_1 is anticomplete to X_j, we have that $y_1y_j \notin E(G)$, and it follows that $Y = y_1, x_2, \dots, x_{j-1}, y_j, x_{j+1}, \dots, x_k, y_1$ is a k-hole. Since x has at most two neighbors in H, and Y is a long hole, we know that x is not complete to Y. Also, since x is complete to $\{y_1, y_j\}$, Lemma 15 now implies that x is a twin of a vertex of Y w.r.t. Y. It follows that either $j = 3$ and x is a twin of x_2 w.r.t. Y (and in particular, $xx_2 \in E(G)$), or $j = k-1$ and x is a twin of x_k w.r.t. Y (and in particular $xx_k \in E(G)$); by symmetry, w.l.o.g. we may assume that the former holds. Since x is not a twin of x_2 w.r.t. H, we know that x is non-adjacent to at least one of x_1, x_3 (and consequently, either $y_1 \neq x_1$ or $y_3 \neq x_3$). Set $Y_1 = X_{y_1}(Y)$, $Y_3 = X_{y_3}(Y)$, and $Y_i = X_{x_i}(Y)$ for every $i \in \mathbb{Z}_k \setminus \{1, 3\}$. By Lemma 16, Y^* is k-ring with good partition $(Y_1, \dots, Y_k)$. Our goal is to show that $V(H^*) \subset V(Y^*)$, contrary to the maximality of $|V(H^*)|$. To this end, it suffices to prove that $X_i \subseteq Y_i$ for every $i \in \mathbb{Z}_k \setminus \{2\}$, and that $X_2 \subset Y_2$.

First of all, in view of Lemma 15, it is easy to see that $X_i = Y_i$ for all $i \in \mathbb{Z}_k \setminus \{k, 2, 4\}$. Next, we claim that $X_4 \subseteq Y_4$ and $X_k \subseteq Y_k$; by symmetry, it suffices to prove that $X_4 \subseteq Y_4$. Fix $y_4 \in X_4$; we must show that $y_4 \in Y_4$. If $y_4 = x_4$, then we are done. So, assume that $y_4 \neq x_4$. Clearly, it suffices to prove that $y_4 y_3 \in E(G)$. Suppose otherwise. Since $x_3 y_4 \in E(G)$, we see that $y_3 \neq x_3$, and so, by the choice of y_3, it follows that $x x_3 \notin E(G)$. Furthermore, we have that $x y_4 \notin E(G)$, since otherwise the hole $y_1, x_2, x_3, y_4, x_5, \dots, x_k, y_1$ and vertex x contradict Lemma 15. But then $y_1, x, y_3, x_3, y_4, x_5, \dots, x_k, y_1$ is a hole of length $k+1$ in G, which contradicts the fact that G contains no holes of length greater than k. It follows that $X_4 \subseteq Y_4$, and similarly, $X_k \subseteq Y_k$.

It remains to show that $X_2 \subset Y_2$. First of all, we know that $x \in Y_2 \setminus X_2$, and so $X_2 \neq Y_2$. Thus, it suffices to prove that $X_2 \subseteq Y_2$. Suppose otherwise, and fix $z_2 \in X_2 \setminus Y_2$. Then $z_2 \neq x_2$ and z_2 is complete to $\{x_1, x_2, x_3\}$, anticomplete to $V(H) \setminus \{x_1, x_2, x_3\} = V(Y) \setminus \{y_1, x_2, y_3\}$ and non-adjacent to at least one of y_1, y_3. Suppose now that $x z_2 \notin E(G)$. For each $i \in \{1, 3\}$, fix a minimum-length induced path P_i between x and z_2, all of whose internal vertices are in X_i (such a path exists because x is adjacent to $y_i \in X_i$, z_2 is adjacent to $x_i \in X_i$, and either $x_i = y_i$ or $x_i y_i \in E(G)$; clearly, P_i is of length two or three). But now $G[V(P_1) \cup V(P_3) \cup \{x_4, \dots, x_k\}]$ is a 3PC, a contradiction. Therefore, $x z_2 \in E(G)$. Suppose now that $y_1 \neq x_1$ and $y_3 \neq x_3$, so that, by the choice of y_1 and y_3, x is anticomplete to $\{x_1, x_3\}$. We know already that z_2 is non-adjacent to at least one of y_1, y_3; by symmetry, w.l.o.g. we may assume that $z_2 y_3 \notin E(G)$. But now $G[\{x_1, x_3, x_4, \dots, x_k, y_3, z_2, x\}]$ is a 3PC$(x_3 y_3 x_4, z_2)$, a contradiction. Thus, either $y_1 = x_1$ or $y_3 = x_3$; by symmetry, w.l.o.g. we may assume that $y_3 = x_3$, so that $y_1 \neq x_1$. Note that this implies that $z_2 x_1, x y_1 \in E(G)$ and $z_2 y_1, x x_1 \notin E(G)$. But now $G[\{x_1, x_3, x_4, \dots, x_k, y_1, z_2, x\}]$ is a 3PC$(x z_2 x_3, y_1 x_1 x_k)$, a contradiction. It follows that $X_2 \subset Y_2$, and hence $V(H^*) \subset V(Y^*)$, which contradicts our choice of H. So, this proves that there exists some $i \in \mathbb{Z}_k$ such that $N_G(x) \cap V(H^*) \subseteq X_i \cup X_{i+1}$.

By symmetry, w.l.o.g. let $N_G(x) \cap V(H^*) \subseteq X_1 \cup X_2$, and now assume that $N_G(x) \cap V(H^*)$ is not a clique. Since X_1 and X_2 are cliques, it follows that there exist non-adjacent vertices $y_1 \in X_1$ and $y_2 \in X_2$ such that $x y_1, x y_2 \in E(G)$. But then $y_1, x, y_2, x_3, x_4, \dots, x_k, y_1$ is a $(k+1)$-hole in G, which contradicts the fact that G contains no holes of length greater than k. This proves (2). □

Let C be a component of $G \setminus V(K)$. Our goal is to show that $N_G(C) \cap V(K)$ is a clique. Since K is not a complete graph, this would imply that $N_G(C) \cap V(K)$ is a clique cutset of G.

(3) $N_G(C) \cap V(H^*)$ is a clique.

Proof of (3). Assume otherwise. Let P be a minimal connected induced subgraph of C such that $N_G(P) \cap V(H^*)$ is not a clique. Fix $a_1, a_2 \in V(P)$ such that some vertex in $N_G(a_1) \cap V(H^*)$ is non-adjacent to some vertex of $N_G(a_2) \cap V(H^*)$. By (2), $a_1 \neq a_2$. Furthermore, P is a path between a_1 and a_2, since otherwise any induced path in P between a_1 and a_2 would contradict the minimality of P. Let $P = p_1, \dots, p_n$ with $p_1 = a_1$ and $p_n = a_2$.

By the minimality of P, we know that $N_G(P \setminus \{p_1\}) \cap V(H^*)$ and $N_G(P \setminus \{p_n\}) \cap V(H^*)$ are both cliques; consequently, $N_G(P) \cap V(H^*)$ is the union of two cliques. Since, for every clique X of H^*, there exists some $i \in \mathbb{Z}_k$ such that $X \subseteq X_i \cup X_{i+1}$, we deduce that there exist at most four indices $i \in \mathbb{Z}_k$ such that $N_G(P) \cap X_i \neq \varnothing$; also, since $k \geqslant 5$, there exists an index $i \in \mathbb{Z}_k$ such that $N_G(P) \cap X_i = \varnothing$. On the other hand, since each X_i is a clique and $N_G(P) \cap V(H^*)$ is not a clique, we see that there exist at least two indices $i \in \mathbb{Z}_k$ such that $N_G(P) \cap X_i \neq \varnothing$.

Now, let $X_i, X_{i+1}, \dots, X_j$ be a sequence of maximum length having the property that $N_G(P)$ intersects both X_i and X_j, but fails to intersect $X_{i+1} \cup \dots \cup X_{j-1}$. By what we just showed, the length of the sequence $X_i, X_{i+1}, \dots, X_j$ is at least three, and at most k; in particular, $i \neq j$. Furthermore, $N_G(P) \cap V(H^*) \subseteq X_j \cup X_{j+1} \cup X_{i-1} \cup X_i$.

Fix non-adjacent vertices $y_i \in N_G(P) \cap X_i$ and $y_j \in N_G(P) \cap X_j$. (If $i \neq j+1$, then any two vertices $y_i \in N_G(P) \cap X_i$ and $y_j \in N_G(P) \cap X_j$ are non-adjacent. On the other hand, if $i = j+1$, then we have that $N_G(P) \cap V(H^*) \subseteq X_i \cup X_j$, and the existence of y_i and y_j follows from the fact that $N_G(P) \cap V(H^*)$ is not a clique, whereas both X_i and X_j are cliques.) By the minimality of P, all interior vertices of P are anticomplete to $\{y_i, y_j\}$, and either $p_1 y_i, p_n y_j \in E(G)$ and $p_1 y_j, p_n y_i \notin E(G)$, or $p_1 y_j, p_n y_i \in E(G)$ and $p_1 y_i, p_n y_j \notin E(G)$; by symmetry, w.l.o.g. we may assume that the latter holds, that is, that $p_1 y_j, p_n y_i \in E(G)$ and $p_1 y_i, p_n y_j \notin E(G)$. Then $y_i, x_{i+1}, \dots, x_{j-1}, y_j, p_1, \dots, p_n, y_i$ is a hole in G, and its length is the sum of n and the length of the sequence $X_i, \dots, X_j$. Since G contains no holes of length greater than k, we see that the length of the sequence $X_i, \dots, X_j$ is at most $k - n \leqslant k - 2$, and it follows that the cliques $X_j, X_{j+1}, X_{i-1}, X_i$ are pairwise distinct.

Now, recall that $N_G(P) \cap V(H^*) \subseteq X_j \cup X_{j+1} \cup X_{i-1} \cup X_i$, and that $N_G(P \setminus \{p_1\}) \cap V(H^*)$ and $N_G(P \setminus \{p_n\}) \cap V(H^*)$ are both cliques. Since p_1 has a neighbor in X_j and p_n has a neighbor in X_i, we deduce that $N_G(P \setminus \{p_n\}) \cap V(H^*) \subseteq X_j \cup X_{j+1}$ and $N_G(P \setminus \{p_1\}) \cap V(H^*) \subseteq X_{i-1} \cup X_i$, and hence $N_G(P \setminus \{p_1, p_n\}) \cap V(H^*) \subseteq (X_j \cup X_{j+1}) \cap (X_{i-1} \cup X_i) = \varnothing$. Thus, the interior vertices of P are anticomplete to $V(H^*)$. We also know that $N_G(p_1) \cap V(H^*) \subseteq X_j \cup X_{j+1}$ and $N_G(p_n) \cap V(H^*) \subseteq X_{i-1} \cup X_i$. But then $G[V(P) \cup \{y_i, x_{i+1}, \dots, x_{j-1}, y_j, x_{j+1}, \dots, x_{i-1}\}]$ is a 3PC, a contradiction. This proves (3). $\square$

(4) $N_G(C) \cap V(K)$ *is a clique.*

Proof of (4). In view of (3), it suffices to show that $N_G(C) \cap U_{H^*}$ is a clique. Assume otherwise, and fix a minimal connected induced subgraph P of C such that $N_G(P) \cap U_{H^*}$ is not a clique. Fix non-adjacent vertices $u_1, u_2 \in N_G(P) \cap U_{H^*}$, and fix (not necessarily distinct) vertices $a_1, a_2 \in V(P)$ such that $a_1 u_1, a_2 u_2 \in E(G)$. It is clear that P is a path between a_1 and a_2 (if $a_1 = a_2$, then P is a one-vertex path), since otherwise any induced path in P between a_1 and a_2 would contradict the minimality of P. Let $P = p_1, \dots, p_n$ (with $n \geqslant 1$) so that $p_1 = a_1$ and $p_n = a_2$. By the minimality of P, u_1 is anticomplete to $V(P) \setminus \{p_1\}$ and u_2 is anticomplete to $V(P) \setminus \{p_n\}$. Thus, $P' = u_1, p_1, \dots, p_n, u_2$ is an induced path in G.

Since $N_G(C) \cap V(H^*)$ is a clique, we know that there exists some $i \in \mathbb{Z}_k$ such that $N_G(C) \cap$

$V(H^*) \subseteq X_i \cup X_{i+1}$; by symmetry, w.l.o.g. we may assume that $N_G(C) \cap V(H^*) \subseteq X_1 \cup X_2$. But then $G[V(P) \cup \{u_1, u_2, x_3, x_5\}]$ is a 3PC(u_1, u_2), a contradiction. This proves (4). □

This completes the argument. ■

We remind the reader that $\mathcal{B}_{\text{UT}}$ is the class of all graphs G that satisfy at least one of the following:

- G has exactly one non-trivial anticomponent, and this anticomponent is a long ring;
- G is (long hole, $K_{2,3}$, $\overline{C_6}$)-free;
- $\alpha(G) = 2$, and every anticomponent of G is either a 5-hyper-hole or a $(C_5, \overline{C_6})$-free graph.

We are now ready to prove the main result of this section, namely Theorem 6.

Proof of Theorem 6. Fix $G \in \mathcal{G}_{\text{UT}}$. We assume that G does not admit a clique cutset, and we show that $G \in \mathcal{B}_{\text{UT}}$. Clearly, G is $(K_{2,3}, \overline{C_6})$-free. If G contains no long holes, then $G \in \mathcal{B}_{\text{UT}}$ and we are done. So, assume that G contains a long hole. By Lemma 17, some anticomponent of G is a long ring; if this anticomponent is the only non-trivial anticomponent of G, then $G \in \mathcal{B}_{\text{UT}}$. So, assume that G has at least two non-trivial anticomponents. Lemma 12 then implies that $\alpha(G) = 2$. We claim that every anticomponent of G is either a 5-hyper-hole or a $(C_5, \overline{C_6})$-free graph (this will imply that $G \in \mathcal{B}_{\text{UT}}$). Let C be an anticomponent of G. If C contains no long holes, then C is $(C_5, \overline{C_6})$-free, and we are done. Thus, assume that C does contain a long hole. Since $\alpha(C) \leqslant \alpha(G) = 2$, Lemma 13(i) implies that C does not admit a clique cutset, and so, by Lemma 17, C is a long ring. Since $\alpha(C) \leqslant \alpha(G) = 2$, we deduce that C is a 5-hyper-hole (indeed, any long ring other than a 5-hyper-hole contains a stable set of size three). This proves the theorem. ■

2.4 A decomposition theorem for class $\mathcal{G}_{\text{U}}$

Our goal in this section is to prove Theorem 7, which states that every graph in $\mathcal{G}_{\text{U}}$ either belongs to $\mathcal{B}_{\text{U}}$ or admits a clique cutset.

Lemma 18. *Let $G \in \mathcal{G}_{\text{U}}$, and let $H = x_1, x_2, x_3, x_4, x_1$ be a 4-hole in G. Then either $V(G) = V(H) \cup U_H$ or G admits a clique cutset.*

Proof. We may assume that $V(H) \cup U_H \subset V(G)$, since otherwise we are done.

(1) For every vertex $x \in V(G) \setminus (V(H) \cup U_H)$, $N_G(x) \cap V(H) \subseteq \{x_i, x_{i+1}\}$ for some $i \in \mathbb{Z}_4$.

Proof of (1). Fix $x \in V(G) \setminus (V(H) \cup U_H)$. Then there exists some $i \in \mathbb{Z}_4$ such that either $N_G(x) \cap V(H) \subseteq \{x_i, x_{i+1}\}$, or $N_G(x) \cap V(H) = \{x_i, x_{i+2}\}$, or $N_G(x) \cap V(H) = \{x_{i-1}, x_i, x_{i+1}\}$. In the first case, we are clearly done. In the second case, $G[V(H) \cup \{x\}]$ is a 3PC(x_i, x_{i+2}), a contradiction. In the third case, (H, x) is a twin wheel in G, which again gives a contradiction. This proves (1). □

Let C be a component of $G \setminus (V(H) \cup U_H)$.

(2) $N_G(C) \cap V(H)$ is a clique.

Proof of (2). Assume otherwise, and fix a minimal connected induced subgraph P of C such that $N_G(P) \cap V(H)$ is not a clique. Then, for some $i \in \mathbb{Z}_4$, we have that $x_i, x_{i+2} \in N_G(P) \cap V(H)$; by symmetry, w.l.o.g. we may assume that $x_1, x_3 \in N_G(P) \cap V(H)$. Fix $a_1, a_3 \in V(P)$ such that $a_1x_1, a_3x_3 \in E(G)$; by (1), we have that $a_1x_3, a_3x_1 \notin E(G)$, and, in particular, $a_1 \neq a_3$. Clearly, P is a path between a_1 and a_3, since otherwise any induced path in P between a_1 and a_3 would contradict the minimality of P. Further, the minimality of P implies that all interior vertices of P are anticomplete to $\{x_1, x_3\}$. Let $P = p_1, \dots, p_n$ with $p_1 = a_1$ and $p_n = a_3$.

Assume first that both x_2 and x_4 have a neighbor in P. Suppose now that some interior vertex p of P is adjacent to x_2, and let p' be any vertex of P adjacent to x_4. Then the subpath of P between p and p' contradicts the minimality of P. So, no interior vertex of P is adjacent to x_2, and, similarly, no interior vertex of P is adjacent to x_4. Furthermore, by (1), each of p_1, p_n is adjacent to at most one of x_2, x_4; by symmetry, w.l.o.g. we may now assume that $N_G(p_1) \cap V(H) = \{x_1, x_2\}$ and $N_G(p_n) \cap V(H) = \{x_3, x_4\}$. But then $G[V(H) \cup V(P)]$ is a 3PC$(p_1x_1x_2, p_nx_4x_3)$, a contradiction.

So, from now on, we assume that at most one of x_2, x_4 has a neighbor in P; by symmetry, w.l.o.g. we may assume that x_2 is anticomplete to $V(P)$. Now, if x_4 has a neighbor in P, then we observe that $H' = x_3, x_2, x_1, p_1, \dots, p_n, x_3$ is a hole and (H', x_4) a proper wheel in G, a contradiction. On the other hand, if x_4 has no neighbors in P, then $G[V(H) \cup V(P)]$ is a 3PC(x_1, x_3), again a contradiction. This proves (2). $\square$

(3) $N_G(C) \cap (V(H) \cup U_H)$ is a clique.

Proof of (3). In view of (2), we only need to show that $N_G(C) \cap U_H$ is a clique. Suppose otherwise, and let P be a minimal connected induced subgraph of C such that $N_G(P) \cap U_H$ is not a clique. Fix non-adjacent vertices $u_1, u_2 \in N_G(P) \cap U_H$, and fix (not necessarily distinct) vertices $a_1, a_2 \in V(P)$ such that $a_1u_1, a_2u_2 \in E(G)$. Clearly, P is a path between a_1 and a_2 (if $a_1 = a_2$, then P is a one-vertex path), since otherwise any induced path in P between a_1 and a_2 would contradict the minimality of P. Let $P = p_1, \dots, p_n$ with $p_1 = a_1$ and $p_n = a_2$. By the minimality of P, we have that $P' = u_1, p_1, \dots, p_n, u_2$ is an induced path in G. By (2), and by symmetry, w.l.o.g. we may assume that $N_G(C) \cap V(H) \subseteq \{x_3, x_4\}$. Then $H' = x_1, u_1, p_1, \dots, p_n, u_2, x_1$ is a hole in G, and $x_2 \in X_{x_1}(H')$. Thus, (H', x_2) is a twin wheel in G, a contradiction. This proves (3). $\square$

Clearly, (3) implies that $N_G(C) \cap (V(H) \cup U_H)$ is a clique cutset of G. $\blacksquare$

We remind the reader that $\mathcal{B}_{\mathrm{U}}$ is the class of all graphs G that satisfy one of the following:

- G has exactly one non-trivial anticomponent, and this anticomponent is a long hole;
- all non-trivial anticomponents of G are isomorphic to $\overline{K_2}$.

We are now ready to prove Theorem 7.

Proof of Theorem 7. Fix $G \in \mathcal{G}_{\mathrm{U}}$, and assume that G does not admit a clique cutset; we must show that $G \in \mathcal{B}_{\mathrm{U}}$.

(1) If some anticomponent of G contains more than two vertices, then all other anticomponents of G are trivial.

Proof of (1). Assume otherwise. Then G has at least two non-trivial anticomponents, and so, by Lemma 12, $\alpha(G) = 2$. Let C_1 be an anticomponent of G that contains at least three vertices, and let C_2 be some other non-trivial anticomponent of G. Since $\alpha(G) = 2$ and the anticomponents C_1, C_2 are non-trivial, we have that $\alpha(C_1) = \alpha(C_2) = 2$. Since $|V(C_1)| \geqslant 3$, we deduce that C_1 is not edgeless, and so, since C_1 is anticonnected, it follows that there exist pairwise distinct vertices $a, b, c \in V(C_1)$ such that $ab, bc \notin E(G)$ and $ac \in E(G)$. Fix non-adjacent vertices $x, y \in V(C_2)$. But now $H = a, x, b, y, a$ is a hole and (H, c) is a twin wheel in G, a contradiction. This proves (1). □

Suppose first that G contains a 4-hole H. Then, by Lemma 18, $V(G) = V(H) \cup U_H$. H has two anticomponents, both isomorphic to $\overline{K_2}$, and clearly, these anticomponents of H are also anticomponents of G. It now follows from (1) that no anticomponent of G has more than two vertices. Thus, all non-trivial anticomponents of G are isomorphic to $\overline{K_2}$, and hence $G \in \mathcal{B}_{\mathrm{U}}$.

Suppose next that G contains a long hole. Then, by Lemma 17, some anticomponent C of G is a long ring. But then C is a long hole, since otherwise the ring C would contain a twin wheel. By (1), C is the only non-trivial anticomponent of G. Thus, $G \in \mathcal{B}_{\mathrm{U}}$.

It remains to consider the case when G contains no holes. But then, by definition, G is chordal. Since G does not admit a clique cutset, Theorem 1 implies that G is a complete graph, and consequently, $G \in \mathcal{B}_{\mathrm{U}}$. This proves the theorem. ■

2.5 A decomposition theorem for class $\mathcal{G}_{\mathrm{T}}$

In this section, we prove Theorem 8, which states that every graph in $\mathcal{G}_{\mathrm{T}}$ either belongs to $\mathcal{B}_{\mathrm{T}}$ or admits a clique cutset.

Lemma 19. *Let $G \in \mathcal{G}_{\mathrm{T}}$. Then G contains no antiholes of length six, and no antiholes of length greater than seven. Furthermore, if G contains a long hole, then either G is a long ring or it admits a clique cutset.*

Proof. Since $\overline{C_6}$ is a prism, we see that G contains no antiholes of length six. Furthermore, we observe that if $A = x_1, \dots, x_k, x_1$ (with $k \geqslant 8$) is an antihole in G, then $H = x_1, x_4, x_2, x_5, x_1$ is a 4-hole and (H, x_7) is a universal wheel in G, a contradiction. This proves the first statement.

It remains to prove the second statement. Suppose that G contains a long hole. Then, by Lemma 17, either some anticomponent of G is a ring, or G admits a clique cutset. In the latter case, we are done; so assume that some anticomponent of G, call it R, is a ring. If $U_R \neq \varnothing$, then G contains a universal wheel, a contradiction. Thus, $U_R = \varnothing$, and it follows that $G = R$. So, G is a ring. ■

Lemma 20. *Let $G \in \mathcal{G}_{\mathrm{T}}$, and assume that G contains no long holes but does contain a 7-antihole. Then either G is a 7-hyper-antihole or it admits a clique cutset.*

Proof. Let $A = x_1, x_2, \ldots, x_7, x_1$ be a 7-antihole in G, and, for all $i \in \mathbb{Z}_7$, set $X_i = X_{x_i}(A)$. Thus, $A^* = G[\bigcup_{i \in \mathbb{Z}_7} X_i]$.

(1) A^* is a 7-hyper-antihole with good partition $(X_1, X_2, \ldots, X_7)$.

Proof of (1). By symmetry, it suffices to prove that X_1 is a clique, complete to $X_3 \cup X_4 \cup X_5 \cup X_6$ and anticomplete to $X_2 \cup X_7$.

Suppose that X_1 is not a clique, and fix non-adjacent $y_1, y_1' \in X_1$. By construction, x_1 is complete to $X_1 \setminus \{x_1\}$, and hence $x_1 \notin \{y_1, y_1'\}$. But now $H = y_1, x_3, y_1', x_4, y_1$ is a 4-hole and (H, x_1) is a universal wheel in G, a contradiction.

Next, suppose that X_1 is not anticomplete to $X_2 \cup X_7$; by symmetry, w.l.o.g. we may assume that there exist some $y_1 \in X_1$ and $y_2 \in X_2$ such that y_1y_2 is an edge. It follows that $H = y_2, x_5, x_3, x_6, y_2$ is a 4-hole and (H, y_1) is a universal wheel in G, a contradiction.

Further, suppose that X_1 is not complete to $X_3 \cup X_6$; by symmetry, w.l.o.g. we may assume that some $y_1 \in X_1$ and $y_3 \in X_3$ are non-adjacent. Since x_1 is complete to X_3, we have that $y_1 \neq x_1$. But then $H = y_1, x_5, y_3, x_6, y_1$ is a 4-hole and (H, x_1) a universal wheel in G, a contradiction.

It remains to show that X_1 is complete to $X_4 \cup X_5$. Suppose otherwise; by symmetry, w.l.o.g. we may assume that some $y_1 \in X_1$ and $y_4 \in X_4$ are non-adjacent. But now $y_1, x_5, x_7, y_4, x_6, y_1$ is a 5-hole in G, contrary to the fact that G contains no long holes. This proves (1). □

(2) For all $x \in V(G) \setminus V(A^*)$, and all $i \in \mathbb{Z}_7$, if x has a neighbor both in X_i and in X_{i+1}, then either x is complete to $X_{i-2} \cup X_{i+3}$ and anticomplete to X_{i-3}, or x is complete to X_{i-3} and anticomplete to $X_{i-2} \cup X_{i+3}$.

Proof of (2). Fix $x \in V(G) \setminus V(A^*)$ and assume that, for some $i \in \mathbb{Z}_7$, x has a neighbor both in X_i and in X_{i+1}; by symmetry, w.l.o.g. we may assume that x is adjacent to some $y_1 \in X_1$ and $y_2 \in X_2$. We must show that x is complete to one of $X_4 \cup X_6$ and X_5, and anticomplete to the other.

Fix $j \in \{4, 5\}$, and suppose that x is adjacent to some $y_j \in X_j$ and $y_{j+1} \in X_{j+1}$; then, by (1), $H = y_1, y_j, y_2, y_{j+1}, y_1$ is a 4-hole and (H, x) is a universal wheel in G, a contradiction. Thus, x has a neighbor in at most one of X_j and X_{j+1}. Suppose now that x has a non-neighbor $y_j' \in X_j$ and a non-neighbor $y_{j+1}' \in X_{j+1}$. But then, by (1), $G[\{y_1, y_2, y_j', y_{j+1}', x\}]$ is a $K_{2,3}$, a contradiction. Thus, x has a non-neighbor in at most one of X_j and X_{j+1}. It now follows that x is complete to one of X_j and X_{j+1}, and anticomplete to the other.

We now have that x is complete to one of X_4 and X_5, and anticomplete to the other, and that x is complete to one of X_5 and X_6, and anticomplete to the other. It follows that x is complete to one of $X_4 \cup X_6$ and X_5, and anticomplete to the other. This proves (2). □

(3) For all $x \in V(G) \setminus V(A^*)$, and all $i \in \mathbb{Z}_7$, if x has a neighbor both in X_i and in X_{i+1}, then x is complete to at least one of X_{i-1} and X_{i+2}.

Proof of (3). Suppose otherwise. By symmetry, w.l.o.g. we may assume that some vertex $x \in V(G) \setminus V(A^*)$ has a neighbor both in X_1 and in X_2, and a non-neighbor both in X_3 and in X_7. Fix $y_1 \in X_1$, $y_2 \in X_2$, $y_3 \in X_3$ and $y_7 \in X_7$ such that $xy_1, xy_2 \in E(G)$ and $xy_3, xy_7 \notin E(G)$. But now, by (1), x, y_1, y_3, y_7, y_2, x is a 5-hole in G, which contradicts the fact that G contains no long holes. This proves (3). □

(4) For every $x \in V(G) \setminus V(A^*)$, $N_G(x) \cap V(A^*)$ is a clique.

Proof of (4). Fix $x \in V(G) \setminus V(A^*)$, and suppose that $N_G(x) \cap V(A^*)$ is not a clique. By (1), and by symmetry, w.l.o.g. we may assume that there exist $y_1 \in X_1$ and $y_2 \in X_2$ such that $xy_1, xy_2 \in E(G)$. By (3), with $i = 1$, and by symmetry, w.l.o.g. we may assume that x is complete to X_3. By (2), with $i = 1$, we have that x is complete to one of $X_4 \cup X_6$ and X_5, and anticomplete to the other.

Suppose first that x is complete to $X_4 \cup X_6$ and anticomplete to X_5. By (2), with $i = 2$, we see that x is anticomplete to X_7. By (2), with $i = 3$, x is complete to X_1. By (3), with $i = 3$, x is complete to X_2. Now we have that x is complete to $X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_6$ and anticomplete to $X_5 \cup X_7$. But then x is a twin of x_6 w.r.t. A, and so $x \in X_6$, contrary to the fact that $x \notin V(A^*)$.

Suppose now that x is complete to X_5 and anticomplete to $X_4 \cup X_6$. By (2), with $i = 2$, we see that x is complete to X_7. By (3), with $i = 2$, we see that x is complete to X_1. By (3), with $i = 7$, we see that x is complete to X_2. But now x is complete to $X_1 \cup X_2 \cup X_3 \cup X_5 \cup X_7$ and anticomplete to $X_4 \cup X_6$. It follows that x is a twin of x_5 w.r.t. A, and so $x \in X_5$, which contradicts the fact that $x \notin V(A^*)$. This proves (4). □

(5) For every component C of $G \setminus V(A^*)$, $N_G(C) \cap V(A^*)$ is a clique.

Proof of (5). Suppose otherwise. Fix a minimal connected induced subgraph P of $G \setminus V(A^*)$ such that $N_G(P) \cap V(A^*)$ is not a clique. By (1), and by symmetry, w.l.o.g. we may assume that $N_G(P) \cap X_1 \neq \varnothing$ and $N_G(P) \cap X_2 \neq \varnothing$. Fix $a_1 \in V(P)$ such that a_1 has a neighbor $y_1 \in X_1$, and fix $a_2 \in V(P)$ such that a_2 has a neighbor $y_2 \in X_2$. By (1) and (4), a_1 is anticomplete to $X_2 \cup X_7$, and a_2 is anticomplete to $X_1 \cup X_3$; it follows that $a_1 \neq a_2$. Clearly, P is a path between a_1 and a_2, since otherwise any induced path in P between a_1 and a_2 would contradict the minimality of P. Let $P = p_1, \dots, p_n$ with $p_1 = a_1$ and $p_n = a_2$ (thus, $n \geqslant 2$). By the minimality of P and (1), each interior vertex of P is anticomplete to $X_7 \cup X_1 \cup X_2 \cup X_3$. (Indeed, if some interior vertex p of P had a neighbor in $X_7 \cup X_1 \cup X_2 \cup X_3$, then either the subpath of P between p_1 and p, or the subpath of P between p_n and p, would contradict the minimality of P.) We now observe the following:

(a) if $p_1x_3, p_nx_7 \notin E(G)$, then $y_1, p_1, \dots, p_n, y_2, x_7, x_3, y_1$ is an $(n+4)$-hole in G;

(b) if $p_1x_3 \in E(G)$ and $p_nx_7 \notin E(G)$, then $x_3, p_1, \dots, p_n, y_2, x_7, x_3$ is an $(n+3)$-hole in G;

(c) if $p_1x_3 \notin E(G)$ and $p_nx_7 \in E(G)$, then $y_1, p_1, \dots, p_n, x_7, x_3, y_1$ is an $(n+3)$-hole in G;

(d) if $p_1x_3, p_nx_7 \in E(G)$, then $x_3, p_1, \dots, p_n, x_7, x_3$ is an $(n+2)$-hole in G.

Since $n \geqslant 2$ and G contains no long holes, we deduce that (d) holds with $n = 2$, and hence $P = p_1, p_2$.

Now, if some $x \in \{x_4, x_5, x_6\}$ is anticomplete to $\{p_1, p_2\}$, then x, y_1, p_1, p_2, y_2, x is a 5-hole in G, a contradiction. Therefore, each of x_4, x_5, x_6 has a neighbor in $\{p_1, p_2\}$. By symmetry, w.l.o.g. we may assume that $p_1x_5 \in E(G)$. But then we have that $p_1x_5, p_2x_7 \in E(G)$, and so, since $x_6x_5, x_6x_7 \notin E(G)$, (4) implies that $p_1x_6, p_2x_6 \notin E(G)$, contrary to the fact that x_6 has a neighbor in $\{p_1, p_2\}$. This proves (5). □

Clearly, (1) and (5) together imply that either G is a 7-hyper-antihole, or G admits a clique cutset. ■

Lemma 21. *Let $G \in \mathcal{G}_{\mathrm{T}}$, and let $H = x_1, x_2, x_3, x_4, x_1$ be a 4-hole in G. For every $i \in \mathbb{Z}_4$, set $X_i = X_{x_i}(H)$. Then H^* is a 4-ring with good partition (X_1, X_2, X_3, X_4).*

PROOF. Our goal is to show that H^* and (X_1, X_2, X_3, X_4) satisfy (i)-(iv) from Lemma 4. Clearly, for all $i \in \mathbb{Z}_4$, we have that $N_{H^*}[x_i] = X_{i-1} \cup X_i \cup X_{i+1}$. Therefore, x_i is complete to $X_{i-1} \cup X_{i+1}$, and hence (iii) holds. Further, by Lemma 16, X_1, X_2, X_3, X_4 are cliques, and so (i) holds.

Next, we prove that (ii) holds. By symmetry, it suffices to show that X_1 is anticomplete to X_3. Suppose otherwise, and fix $y_1 \in X_1$ and $y_3 \in X_3$ such that $y_1y_3 \in E(G)$. By construction, x_1 is anticomplete to X_3, and x_3 is anticomplete to X_1, and so we see that $y_1 \neq x_1$ and $y_3 \neq x_3$. But now $H' = y_1, x_2, x_3, x_4, y_1$ is a hole and (H', y_3) is a universal wheel in G, a contradiction. Thus, (ii) holds.

It remains to show that (iv) holds; by symmetry, it suffices to prove (iv) for $i = 1$. Fix distinct $y_1, y_1' \in X_1$; we claim that one of y_1, y_1' dominates the other in H^*. Suppose otherwise. By (i), X_1 is a clique, and, by (ii), both y_1 and y_1' are anticomplete to X_3. Thus, by symmetry, w.l.o.g. we may assume that one of the following holds:

(a) there exist $y_2, y_2' \in X_2$ such that $y_1y_2, y_1'y_2' \in E(G)$ and $y_1y_2', y_1'y_2 \notin E(G)$;

(b) there exist $y_2 \in X_2$ and $y_4' \in X_4$ such that $y_1y_2, y_1'y_4' \in E(G)$ and $y_1y_4', y_1'y_2 \notin E(G)$.

If (a) holds, then $G[\{y_1, y_1', y_2, y_2', x_3, x_4\}]$ is a $3\mathrm{PC}(y_1y_1'x_4, y_2y_2'x_3)$, a contradiction. Therefore, suppose that (b) holds. Since x_1 is complete to $X_2 \cup X_4$, we have that $x_1 \notin \{y_1, y_1'\}$. Using (i) and (ii), we now deduce that $H' = y_1, y_2, x_3, y_4', y_1', y_1$ is a 5-hole in G, and x_1 has precisely four neighbors (namely, y_1, y_1', y_2, y_4') in $V(H')$; thus, (H', x_1) is a proper wheel in G, a contradiction. It follows that one of y_1, y_1' dominates the other in H^*. This proves (iv).

Lemma 4 now implies that H^* is a 4-ring with good partition (X_1, X_2, X_3, X_4). ■

Lemma 22. *Let $G \in \mathcal{G}_{\mathrm{T}}$, assume that G contains no long holes and no 7-antiholes, and let $H = x_1, x_2, x_3, x_4$ be a 4-hole in G, chosen so that $|V(H^*)|$ is maximum. Then, for every $x \in V(G) \setminus V(H^*)$, $N_G(x) \cap V(H^*)$ is a clique.*

PROOF. For each $i \in \mathbb{Z}_4$, set $X_i = X_{x_i}(H)$. By Lemma 21, H^* is a 4-ring with good partition (X_1, X_2, X_3, X_4); in particular, X_1, X_2, X_3, X_4 are cliques, X_1 is anticomplete to X_3, and X_2 is anticomplete to X_4.

Assume that, for some $x \in V(G) \setminus V(H^*)$, $N_G(x) \cap V(H^*)$ is not a clique. First suppose that $N_G(x) \cap V(H^*) \subseteq X_i \cup X_{i+1}$ for some $i \in \mathbb{Z}_4$; by symmetry, w.l.o.g. we may assume that $N_G(x) \cap V(H^*) \subseteq X_1 \cup X_2$. Since $N_G(x) \cap V(H^*)$ is not a clique, there exist non-adjacent vertices $y_1 \in X_1$ and $y_2 \in X_2$ such that $xy_1, xy_2 \in E(G)$. But then x, y_2, x_3, x_4, y_1, x is a 5-hole in G, contrary to the fact that G contains no long holes. It follows that, for some $i \in \mathbb{Z}_4$, x has a neighbor both in X_i and in X_{i+2}.

By symmetry, w.l.o.g. we may assume that x has a neighbor both in X_1 and in X_3. For each $i \in \{1, 3\}$, if $xx_i \in E(G)$, then set $y_i = x_i$, and otherwise let y_i be any neighbor of x in X_i. If x is complete to $\{x_2, x_4\}$, then $H' = y_1, x_2, y_3, x_4, y_1$ is a hole and (H', x) is a universal wheel in G, a contradiction. On the other hand, if x is anticomplete to $\{x_2, x_4\}$, then $G[\{y_1, y_3, x, x_2, x_4\}]$ is a $K_{2,3}$, a contradiction. Thus, x is adjacent to precisely one of x_2, x_4; by symmetry, w.l.o.g. we may assume that x is adjacent to x_2 and non-adjacent to x_4. Further, x is adjacent to at most one of x_1, x_3, since otherwise x would be a twin of x_2 w.r.t. H, and we would have that $x \in X_2$, a contradiction. Thus, either $y_1 \neq x_1$ or $y_3 \neq x_3$. Now, $Y = y_1, x_2, y_3, x_4, y_1$ is a 4-hole in G. Our goal is to show that $V(H^*) \subset V(Y^*)$, which will contradict the maximality of $|V(H^*)|$.

For $i \in \{1, 3\}$, set $Y_i = X_{y_i}(Y)$, and, for $i \in \{2, 4\}$, set $Y_i = X_{x_i}(Y)$. By Lemma 21, Y^* is a 4-ring with good partition (Y_1, Y_2, Y_3, Y_4); in particular, Y_1, Y_2, Y_3, Y_4 are cliques, Y_1 is anticomplete to Y_3, and Y_2 is anticomplete to Y_4. Now, to show that $V(H^*) \subset V(Y^*)$, it suffices to prove that $X_i \subseteq Y_i$ for every $i \in \{1, 3, 4\}$, and that $X_2 \subset Y_2$.

(1) $X_1 = Y_1$ and $X_3 = Y_3$.

Proof of (1). By symmetry, it suffices to show that $X_1 = Y_1$. But this readily follows from the definition of X_1 and Y_1, from the fact that X_1 is a clique, anticomplete to X_3, and from the fact that Y_1 is a clique, anticomplete to Y_3. This proves (1). □

(2) Vertices y_1 and y_3 are complete to X_4, and consequently, $X_4 \subseteq Y_4$.

Proof of (2). Clearly, the first statement implies the second. Suppose that the first statement is false, and fix $y_4 \in X_4$ such that y_4 is non-adjacent to at least one of y_1 and y_3; by symmetry, w.l.o.g. we may assume that y_4 is non-adjacent to y_3, and consequently (since x_3 is complete to X_4, and x_4 is complete to X_3), we have that $y_3 \neq x_3$ and $y_4 \neq x_4$. By the choice of y_3, it follows that $xx_3 \notin E(G)$.

Now suppose that $xy_4 \notin E(G)$. Assume additionally that $y_1y_4 \notin E(G)$; it follows that $y_1 \neq x_1$, and, by the choice of y_1, we see that $xx_1 \notin E(G)$. But then $x_1, y_1, x, y_3, x_3, y_4, x_1$ is a 6-hole in G, contrary to the fact that G contains no long holes. Thus, $y_1y_4 \in E(G)$. But then $y_1, x, y_3, x_3, y_4, y_1$ is a 5-hole in G, again a contradiction. This proves that $xy_4 \in E(G)$.

Next, if $y_1y_4 \in E(G)$, then $y_1, y_3, y_4, x_2, x_4, x, x_3, y_1$ is a 7-antihole in G, a contradiction. This proves that $y_1y_4 \notin E(G)$. Since x_1 is complete to X_4, it follows that $y_1 \neq x_1$, and hence, by the choice of y_1, $xx_1 \notin E(G)$. But then $G[\{x_2, y_4, x, x_1, x_3\}]$ is a $K_{2,3}$, a contradiction. This proves (2). □

(3) Vertex x is complete to X_2.

Proof of (3). Suppose that x has a non-neighbor $y_2 \in X_2$. Since $xx_2 \in E(G)$, we have that $y_2 \neq x_2$. Suppose that y_2 is anticomplete to $\{y_1, y_3\}$. Then $y_1 \neq x_1$ and $y_3 \neq x_3$, and so, by the choice of y_1 and y_3, we have that $xx_1, xx_3 \notin E(G)$. But now $y_2, x_1, y_1, x, y_3, x_3, y_2$ is a 6-hole in G, contrary to the fact that G contains no long holes. Thus, y_2 is adjacent to at least one of y_1, y_3; by symmetry, w.l.o.g. we may assume that $y_1y_2 \in E(G)$. If $y_2y_3 \notin E(G)$, then $y_3 \neq x_3$, and hence, by the choice of y_3, $xx_3 \notin E(G)$. So, we have that $y_2, y_1, x, y_3, x_3, y_2$ is a 5-hole in G, which contradicts the fact that G contains no long holes. Therefore, $y_2y_3 \in E(G)$. But then $H' = y_2, y_1, x, y_3, y_2$ is a 4-hole and (H', x_2) is a universal wheel in G, a contradiction. This proves (3). □

(4) $X_2 \subset Y_2$.

Proof of (4). First of all, we know that $x \in Y_2 \setminus X_2$, and so $X_2 \neq Y_2$. It remains to prove that $X_2 \subseteq Y_2$. Clearly, it suffices to show that y_1 and y_3 are complete to X_2. Suppose otherwise. By symmetry, w.l.o.g. we may assume that y_1 has a non-neighbor $y_2 \in X_2$. Since x_1 is complete to X_2, it follows that $y_1 \neq x_1$, and so, by the choice of y_1, $xx_1 \notin E(G)$. Also, since x_2 is complete to X_1, $y_2 \neq x_2$. By (3), we now have that $xx_2, xy_2 \in E(G)$. But then $H' = x_1, y_1, x, y_2, x_1$ is a 4-hole and (H', x_2) is a universal wheel in G, a contradiction. This proves (4). □

By (1), (2) and (4), we have that $V(H^*) \subset V(Y^*)$, which contradicts our choice of H. ■

Lemma 23. *Let $G \in \mathcal{G}_{\mathrm{T}}$, assume that G contains no long holes and no 7-antiholes, and let $H = x_1, x_2, x_3, x_4$ be a 4-hole in G, chosen so that $|V(H^*)|$ is maximum. Then either $G = H^*$ (and consequently, G is a 4-ring), or G admits a clique cutset.*

Proof. For every $i \in \mathbb{Z}_4$, set $X_i = X_{x_i}(H)$. By Lemma 21, H^* is a 4-ring with good partition (X_1, X_2, X_3, X_4); in particular, X_1, X_2, X_3, X_4 are cliques, X_1 is anticomplete to X_3, and X_2 is anticomplete to X_4. If $G = H^*$, then we are done. So assume that $G \neq H^*$, and let C be a component of $G \setminus V(H^*)$. Our goal is to show that $N_G(C) \cap V(H^*)$ is a clique; since H^* is not complete, this will immediately imply that $N_G(C) \cap V(H^*)$ is a clique cutset of G, which is what we need.

Suppose otherwise, that is, suppose that $N_G(C) \cap V(H^*)$ is not a clique. Let P be a minimal connected induced subgraph of C such that $N_G(P) \cap V(H^*)$ is not a clique. Fix $a, b \in V(P)$ such that some vertex in $N_G(a) \cap V(H^*)$ is non-adjacent to some vertex of $N_G(b) \cap V(H^*)$; by Lemma 22, $a \neq b$. Note that P is a path between a and b, since otherwise any induced path in P between a and b would contradict the minimality of P. Let $P = p_1, \dots, p_n$ with $p_1 = a$ and $p_n = b$ (thus, $n \geqslant 2$).

Now suppose that, for some $i \in \mathbb{Z}_4$, $\{p_1, p_n\}$ is anticomplete to $X_i \cup X_{i+1}$; by symmetry, w.l.o.g. we may assume that $\{p_1, p_n\}$ is anticomplete to $X_3 \cup X_4$, so that $N_G(\{p_1, p_n\}) \cap V(H^*) \subseteq X_1 \cup X_2$. As some vertex in $N_G(p_1) \cap V(H^*)$ is non-adjacent to some vertex of $N_G(p_n) \cap V(H^*)$, w.l.o.g. we may assume by symmetry that there exist non-adjacent vertices $z_1 \in X_1$ and $z_2 \in X_2$ such that $p_1z_1, p_nz_2 \in E(G)$. By Lemma 22, we know that $p_1z_2, p_nz_1 \notin E(G)$. Next, we claim that all interior vertices of P are anticomplete to $\{z_1, z_2\} \cup X_3 \cup X_4$. Assume otherwise, and suppose that some interior vertex p of P has a neighbor in $\{z_1, z_2\} \cup X_3 \cup X_4$. By symmetry, w.l.o.g. we may assume that p has a neighbor $z' \in \{z_2\} \cup X_3$. But then $z_1 z' \notin E(G)$, and we see that the subpath of P between p_1 and p contradicts the minimality of P. This proves our claim. But now $z_1, p_1, \dots, p_n, z_2, x_3, x_4, z_1$ is a long hole in G, a contradiction.

By symmetry, w.l.o.g. we may now assume that $\{p_1, p_n\}$ is anticomplete neither to X_1 nor to X_3. We know that X_1 is anticomplete to X_3; by Lemma 22 and by symmetry, w.l.o.g. we may now assume that p_1 has a neighbor $y_1 \in X_1$, that p_n has a neighbor $y_3 \in X_3$, and that $p_1 y_3, p_n y_1 \notin E(G)$. Note that x_2 has a neighbor in P, since otherwise $y_1, p_1, \dots, p_n, y_3, x_2, y_1$ is a long hole in G, a contradiction. Similarly, x_4 has a neighbor in P.

Now, we claim that interior vertices of P are anticomplete to $\{x_2, x_4\}$. Suppose otherwise. By symmetry, w.l.o.g. we may assume that some interior vertex p of P is adjacent to x_2, and let $p' \in V(P)$ be such that $p' x_4 \in E(G)$. It follows that the subpath of P between p and p' contradicts the minimality of P, and this proves our claim. Since (by the minimality of P) the interior vertices of P are anticomplete to $\{y_1, y_3\}$, we deduce that the interior vertices of P are in fact anticomplete to $\{y_1, x_2, y_3, x_4\}$. It follows that each of x_2, x_4 has a neighbor in $\{p_1, p_n\}$. By Lemma 22, and by symmetry, w.l.o.g. we may assume that $p_1 x_2, p_n x_4 \in E(G)$ and $p_1 x_4, p_n x_2 \notin E(G)$. But then $G[\{y_1, x_2, y_3, x_4\} \cup V(P)]$ is a 3PC$(y_1 x_2 p_1, x_4 y_3 p_n)$, a contradiction. ■

We remind the reader that $\mathcal{B}_{\mathrm{T}}$ is the class of all complete graphs, rings and 7-hyper-antiholes. We are now ready to prove Theorem 8.

Proof of Theorem 8. Fix $G \in \mathcal{G}_{\mathrm{T}}$. If G contains a long hole, then we are done by Lemma 19. So, assume that G contains no long holes. If G contains a 7-antihole, then we are done by Lemma 20. So, assume that G contains no 7-antiholes. If G contains a 4-hole, then we are done by Lemma 23. So, assume that G contains no 4-holes. We now have that G contains no holes, and hence, by definition, G is chordal. But then, by Theorem 1, either G is a complete graph or it admits a clique cutset. This proves the theorem. ■

2.6 A decomposition theorem for class $\mathcal{G}_{\mathrm{UT}}^{\text{cap-free}}$

In this section, we prove Theorem 9, which states that every graph in $\mathcal{G}_{\mathrm{UT}}^{\text{cap-free}}$ either belongs to $\mathcal{B}_{\mathrm{UT}}^{\text{cap-free}}$ or admits a clique cutset. We remind the reader that the *house* is the (unique) cap on five vertices; note that the house is isomorphic to $\overline{P_5}$. Clearly, every cap-free graph is house-free.

Lemma 24. *Let $G \in \mathcal{G}_{\mathrm{UT}}^{\text{cap-free}}$, and assume that G contains a long hole. Then either some anticomponent of G is a long hyper-hole, or G admits a clique cutset.*

Proof. This follows immediately from Lemmas 17 and 14(v). ■

A *domino* is a six-vertex graph D with vertex set $V(D) = \{a_1, a_2, a_3, b_1, b_2, b_3\}$ and edge set $E(D) = \{a_1a_2, a_2a_3, b_1b_2, b_2b_3, a_1b_1, a_2b_2, a_3b_3\}$; under these circumstances, we write that "$D = (a_1, a_2, a_3; b_1, b_2, b_3)$ is a domino".

Lemma 25. *Let $G \in \mathcal{G}_{\mathrm{UT}}^{\text{cap-free}}$, and assume that G contains no long holes but does contain a domino. Then G admits a clique cutset.*

Proof. Let $D = (a_1, a_2, a_3; b_1, b_2, b_3)$ be an induced domino in G. Let S be the set of all the vertices in $V(G) \setminus V(D)$ that are complete to $\{a_2, b_2\}$. Our goal is to show that $\{a_2, b_2\} \cup S$ is a clique cutset of G.

(1) Every vertex in S has a neighbor both in $\{a_1, b_1\}$ and in $\{a_3, b_3\}$.

Proof of (1). Fix $x \in S$ and $i \in \{1, 3\}$. If x is anticomplete to $\{a_i, b_i\}$, then $G[\{a_i, b_i, a_2, b_2, x\}]$ is a house, contrary to the fact that G is cap-free. This proves (1). $\square$

(2) $\{a_2, b_2\} \cup S$ is a clique.

Proof of (2). Since $a_2b_2 \in E(G)$, and since S is complete to $\{a_2, b_2\}$, it suffices to show that S is a clique. Assume otherwise, and fix non-adjacent vertices $x, y \in S$. By (1), each of x, y has a neighbor both in $\{a_1, b_1\}$ and in $\{a_3, b_3\}$. Further, x, y have a common neighbor in each of $\{a_1, b_1\}$ and $\{a_3, b_3\}$, since otherwise it is easy to see that $G[\{a_1, b_1, a_3, b_3, x, y\}]$ contains either a 5-hole or a 6-hole, which contradicts the fact that G contains no long holes. Now, $\{x, y\}$ is not complete to $\{a_1, a_3\}$, since otherwise $G[\{x, y, a_1, b_2, a_3\}]$ would be a $K_{2,3}$, a contradiction. Similarly, $\{x, y\}$ is not complete to $\{b_1, b_3\}$. By symmetry, w.l.o.g. we may now assume that $\{x, y\}$ is complete to $\{a_1, b_3\}$, and that y is non-adjacent to a_3. But then $G[\{a_1, a_2, a_3, b_3, y\}]$ is a house, contrary to the fact that G is cap-free. This proves (2). $\square$

By (2), it remains to show that $\{a_2, b_2\} \cup S$ is a cutset of G. Assume otherwise. Since $\{a_1, b_1\}$ is anticomplete to $\{a_3, b_3\}$, it follows that there exists an induced path $P = p_1, \dots, p_n$ in $G \setminus (V(D) \cup S)$ such that p_1 has a neighbor in $\{a_1, b_1\}$, and p_n has a neighbor in $\{a_3, b_3\}$; in particular, let P be such a path with shortest length. Note that the minimality of P implies that all interior vertices of P are anticomplete to $\{a_1, b_1, a_3, b_3\}$.

(3) At most one of a_2, b_2 has a neighbor in $V(P)$.

Proof of (3). Suppose not. Fix $i, j \in \{1, \dots, n\}$ such that $p_ia_2, p_jb_2 \in E(G)$, and, subject to that, such that $|i - j|$ is minimum. By symmetry, w.l.o.g. we may assume that $i \leqslant j$. If $i = j$, then $p_i = p_j$ belongs to S, a contradiction; thus, $i < j$. If $i+1 < j$, then $p_i, p_{i+1}, \dots, p_j, b_2, a_2, p_i$ is a long hole in G, a contradiction; thus, $j = i + 1$.

Next, we claim that b_2 is anticomplete to $\{p_1, \dots, p_{i-1}\}$. Suppose otherwise, and fix a maximum $\ell \in \{1, \dots, i - 1\}$ such that $p_\ell b_2 \in E(G)$. Then $b_2, p_\ell, \dots, p_i, p_{i+1}, b_2$ is a hole in G; since G contains no long holes, it follows that $\ell = i - 1$. Since $p_\ell \notin S$, we see that $p_\ell a_2 \notin E(G)$, and hence $G[\{p_i, b_2, p_{i-1}, a_2, p_{i+1}\}]$ is a $K_{2,3}$, a contradiction. This proves our claim.

Now, if $p_1b_1 \notin E(G)$, then $p_1a_1 \in E(G)$, and $p_1, \dots, p_{i+1}, b_2, b_1, a_1, p_1$ is a long hole in G, a contradiction. Thus, $p_1b_1 \in E(G)$, and we have that $H = p_1, \dots, p_{i+1}, b_2, b_1, p_1$ is a hole in G. Since G contains no long holes, it follows that H is a 4-hole, and consequently, $i = 1$. But then $G[\{p_1, b_2, b_1, a_2, p_2\}]$ is a $K_{2,3}$, a contradiction. This proves (3). $\square$

By (3), and by symmetry, w.l.o.g. we may assume that a_2 is anticomplete to $V(P)$. Then $n = 1$ (i.e. P is a trivial path) and p_1 is complete to $\{a_1, a_3\}$, since otherwise $G[V(P) \cup (V(D) \setminus \{b_2\})]$

contains a long hole, a contradiction. Now, p_1, a_1, a_2, a_3, p_1 is a 4-hole, and p_1 is anticomplete to $\{b_1, b_3\}$ since G is house-free. Then $p_1b_2 \in E(G)$, since otherwise $p_1, a_1, b_1, b_2, b_3, a_3, p_1$ is a 6-hole in G, a contradiction. But now $G[\{p_1, a_2, a_1, b_2, a_3\}]$ is a $K_{2,3}$, a contradiction. ■

Lemma 26. *Let $G \in \mathcal{G}_{\text{UT}}^{\text{cap-free}}$. Assume that G contains a 4-hole, contains no long holes, and does not admit a clique cutset. Then G has at least two non-trivial anticomponents.*

PROOF. Let $H = x_1, x_2, x_3, x_4, x_1$ be a 4-hole in G, and, for all $i \in \mathbb{Z}_4$, set $X_i = X_{x_i}(H)$. Thus, (X_1, X_2, X_3, X_4) is a partition of $V(H^*)$.

(1) For every $i \in \mathbb{Z}_4$, X_i is a clique that is complete to $X_{i-1} \cup X_{i+1}$.

Proof of (1). By Lemma 16, X_1, X_2, X_3, X_4 are cliques. By symmetry, it now suffices to show that X_1 is complete to X_2. Assume otherwise, and fix non-adjacent vertices $y_1 \in X_1$ and $y_2 \in X_2$; since x_1 is complete to X_2, we have that $y_1 \neq x_1$. But then $G[\{y_1, x_1, y_2, x_3, x_4\}]$ is a house, a contradiction. This proves (1). □

(2) For every $x \in V(G) \setminus (V(H^*) \cup U_H)$, there exists some $i \in \mathbb{Z}_4$ such that $N_G(x) \cap V(H^*) \subseteq X_i \cup X_{i+2}$.

Proof of (2). Suppose otherwise. By symmetry, w.l.o.g. we may assume that there exist some $x \in V(G) \setminus (V(H^*) \cup U_H)$, $y_1 \in X_1$ and $y_2 \in X_2$ such that $xy_1, xy_2 \in E(G)$. By (1), we have that $y_1y_2 \in E(G)$.

Since $x \notin V(H^*) \cup U_H$, x has at most two neighbors in $V(H)$. If x has precisely two neighbors in $V(H)$, then $G[V(H) \cup \{x\}]$ is either a house or a $K_{2,3}$, a contradiction. Thus, x has at most one neighbor in $V(H)$.

If x is anticomplete to $\{x_3, x_4\}$, then $G[\{x, y_1, y_2, x_3, x_4\}]$ is a house, a contradiction. By symmetry, w.l.o.g. we may now assume that $xx_3 \in E(G)$. Since x has at most one neighbor in $V(H)$, we deduce that x_3 is the unique neighbor of x in $V(H)$, and hence $y_1 \neq x_1$. But now $G[\{x_1, y_2, x_3, x_4, x\}]$ is a house, a contradiction. This proves (2). □

(3) For all components C of $G \setminus (V(H^*) \cup U_H)$, there exists some $i \in \mathbb{Z}_4$ such that $N_G(C) \cap V(H^*) \subseteq X_i \cup X_{i+2}$.

Proof of (3). Assume otherwise, and let C be a component of $G \setminus (V(H^*) \cup U_H)$ that contradicts (3). Then, for some $i \in \mathbb{Z}_4$, $N_G(C)$ intersects both X_i and X_{i+1}. Let P be a minimal connected induced subgraph of C such that there exists some $i \in \mathbb{Z}_4$ such that $N_G(P)$ intersects both X_i and X_{i+1}; by symmetry, w.l.o.g. we may assume that $N_G(P)$ intersects both X_1 and X_2. Let $a_1, a_2 \in V(P)$ be such that a_1 has a neighbor $y_1 \in X_1$, and a_2 has a neighbor $y_2 \in X_2$. By (1), $y_1y_2 \in E(G)$. By (2), $N_G(a_1) \cap V(H^*) \subseteq X_1 \cup X_3$ and $N_G(a_2) \cap V(H^*) \subseteq X_2 \cup X_4$; in particular, $a_1 \neq a_2$. Clearly, P is a path between a_1 and a_2, since otherwise any induced path in P between a_1 and a_2 would contradict the minimality of P. Furthermore, by the minimality of P, all interior vertices of P are anticomplete to $X_1 \cup X_2$. Also, P is of length one, since otherwise $G[V(P) \cup \{y_1, y_2\}]$ would be a long hole in G, a contradiction; in particular, $a_1a_2 \in E(G)$.

Next, we have that $a_1x_3 \notin E(G)$, since otherwise the graph $G[\{y_1, x_3, a_1, y_2, x_4\}]$ would be a $K_{2,3}$, a contradiction. Similarly, $a_2x_4 \notin E(G)$. But then $G[\{a_1, a_2, y_1, y_2, x_3, x_4\}]$ is a domino, and so, by Lemma 25, G admits a clique cutset, a contradiction. This proves (3). □

(4) For all components C of $G \setminus (V(H^*) \cup U_H)$, $N_G(C) \cap (V(H^*) \cup U_H)$ is a clique.

Proof of (4). Assume otherwise, and let C be a component of $G \setminus (V(H^*) \cup U_H)$ such that $N_G(C) \cap (V(H^*) \cup U_H)$ is not a clique. Let P be a minimal connected induced subgraph of C such that $N_G(P) \cap (V(H^*) \cup U_H)$ is not a clique. By (3), and by symmetry, w.l.o.g. we may assume that $N_G(P) \cap V(H^*) \subseteq X_1 \cup X_3$. Now, fix non-adjacent $y, y' \in N_G(P) \cap (V(H^*) \cup U_H)$, and fix (not necessarily distinct) $a, a' \in V(P)$ such that $ay, a'y' \in E(G)$. Clearly, P is a path between a and a' (if $a = a'$, then we simply have that P is a one-vertex path), since otherwise any induced path in P between a and a' would contradict the minimality of P. Let $P = p_1, \dots, p_n$ with $p_1 = a$ and $p_n = a'$, and, by the minimality of P, note that $P' = y, p_1, \dots, p_n, y'$ is an induced path in G. Now, since $N_G(P) \cap V(H^*) \subseteq X_1 \cup X_3$, we see that x_2 and x_4 are anticomplete to $V(P)$. Since $\{x_2, x_4\}$ is complete to $X_1 \cup X_3 \cup U_H$, we deduce that $\{x_2, x_4\}$ is complete to $\{y, y'\}$. But then $G[V(P) \cup \{y, y', x_2, x_4\}]$ is a 3PC(y, y'), a contradiction. This proves (4). □

(5) $V(G) = V(H^*) \cup U_H$.

Proof of (5). Assume not, and let C be a component of $G \setminus (V(H^*) \cup U_H)$. It follows from (4) that $N_G(C) \cap (V(H^*) \cup U_H)$ is a clique cutset of G, a contradiction. This proves (5). □

(6) Every vertex in U_H is complete to at least three of the sets X_1, X_2, X_3, X_4.

Proof of (6). Let $x \in U_H$. By symmetry, it suffices to show that, if x has a non-neighbor in X_1, then x is complete to $X_2 \cup X_3 \cup X_4$. So, suppose that x is non-adjacent to some $y_1 \in X_1$. Suppose also that x has a non-neighbor $y_2 \in X_2$. By (1), we have that $y_1y_2 \in E(G)$, and hence $G[\{x, y_1, y_2, x_3, x_4\}]$ is a house, a contradiction. Thus, x is complete to X_2, and similarly, x is complete to X_4. Now suppose that x has a non-neighbor $y_3 \in X_3$. If $y_1y_3 \notin E(G)$, then $G[\{x_2, x_4, y_1, y_3, x\}]$ is a $K_{2,3}$, a contradiction. Therefore, $y_1y_3 \in E(G)$. Since $x \in U_H$, we know that $xx_1, xx_3 \in E(G)$; since $xy_1, xy_3 \notin E(G)$, it follows that $y_1 \neq x_1$ and $y_3 \neq x_3$. But then, by (1), x, x_1, y_1, y_3, x_3, x is a 5-hole in G, a contradiction. This proves (6). □

(7) Every non-trivial anticomponent of $G[U_H]$ is complete to $V(H^*)$.

Proof of (7). Assume otherwise, and let C be the vertex set of a non-trivial anticomponent of $G[U_H]$ such that C is not complete to $V(H^*)$. Fix $x \in C$ such that x has a non-neighbor in $V(H^*)$, and let $y \in C$ be a non-neighbor of x (y exists because $G[C]$ is anticonnected and has at least two vertices). By symmetry, w.l.o.g. we may assume that x has a non-neighbor $y_1 \in X_1$ (clearly, $y_1 \neq x_1$). But now, if $yy_1 \notin E(G)$, then $G[\{x_2, x_4, x, y, y_1\}]$ is a $K_{2,3}$, a contradiction, and, if $yy_1 \in E(G)$, then $G[\{x, y, y_1, x_1, x_3\}]$ is a house, which is again a contradiction. This proves (7). □

Suppose first that U_H is not a clique, and let C be the vertex set of a non-trivial anticomponent of $G[U_H]$. By (5) and (7), we have that C is the vertex set of a non-trivial anticomponent of G. Since $C \cap V(H) = \varnothing$, we see that some other anticomponent of G (for example, the one containing x_1 and x_3) is also non-trivial, and it follows that G contains at least two non-trivial anticomponents, which is what we needed to show.

So, from now on, we assume that U_H is a clique. Let Y be the set of all the vertices in U_H that are complete to $V(H^*)$, and, for all $i \in \mathbb{Z}_4$, let Y_i be the set of all the vertices in U_H that have a non-neighbor in X_i. Clearly, $U_H = Y \cup Y_1 \cup Y_2 \cup Y_3 \cup Y_4$. By (6), we have that Y_i is complete to $X_{i+1} \cup X_{i+2} \cup X_{i+3}$ for all $i \in \mathbb{Z}_4$, and so we deduce that Y, Y_1, Y_2, Y_3, Y_4 are pairwise disjoint. By (5), we know that $V(G) = (X_1 \cup X_3 \cup Y_1 \cup Y_3) \cup (X_2 \cup X_4 \cup Y_2 \cup Y_4) \cup Y$. By (1), $X_1 \cup X_3$ is complete to $X_2 \cup X_4$, and hence the sets $X_1 \cup X_3 \cup Y_1 \cup Y_3$, $X_2 \cup X_4 \cup Y_2 \cup Y_4$ and Y are pairwise complete to each other. Since $x_1x_3 \notin E(G)$, the graph $G[X_1 \cup X_3 \cup Y_1 \cup Y_3]$ contains at least one non-trivial anticomponent, and, since $x_2x_4 \notin E(G)$, the graph $G[X_2 \cup X_4 \cup Y_2 \cup Y_4]$ contains at least one non-trivial anticomponent. It follows that G contains at least two non-trivial anticomponents, and we are done. ■

We remind the reader that $\mathcal{B}_{\text{UT}}^{\text{cap-free}}$ is the class of all graphs G that satisfy one of the following:

- G has exactly one non-trivial anticomponent, and this anticomponent is a hyper-hole of length at least six;
- each anticomponent of G is either a 5-hyper-hole or a chordal co-bipartite graph.

We are now ready to prove Theorem 9.

Proof of Theorem 9. Let $G \in \mathcal{G}_{\text{UT}}^{\text{cap-free}}$, and assume that G does not admit a clique cutset; we must show that $G \in \mathcal{B}_{\text{UT}}^{\text{cap-free}}$.

(1) Every anticomponent of G is either a long hyper-hole or a chordal co-bipartite graph.

Proof of (1). Let C be an anticomponent of G. We want to prove that C is either a long hyper-hole or a chordal co-bipartite graph.

Suppose first that C admits a clique cutset S. Clearly, $\alpha(C) \geqslant 2$. If U_C is a (possibly empty) clique, then $S \cup U_C$ is a clique cutset of G, a contradiction. Thus, U_C is not a clique, and we deduce that G has at least two non-trivial anticomponents. Lemma 12 now implies that $\alpha(G) = 2$; since $\alpha(C) \geqslant 2$, it follows that $\alpha(C) = 2$, and so, by Lemma 13, C is a chordal co-bipartite graph, and we are done. So, from now on, we assume that C does not admit a clique cutset.

Suppose that C contains a long hole. Then, by Lemma 24, C is a long hyper-hole, and we are done. So, from now on, we assume that C contains no long holes. Since C is anticonnected, Lemma 26 implies that C contains no 4-holes. Thus, C contains no holes, and so, by definition, C is chordal. Since C does not admit a clique cutset, Theorem 1 now implies that C is a complete graph (in fact, since C is anticonnected, C is isomorphic to K_1), and, in particular, C is a chordal co-bipartite graph. This proves (1). □

If G contains at most one non-trivial anticomponent, then (1) implies that $G \in \mathcal{B}_{\text{UT}}^{\text{cap-free}}$, and we are done. So, assume that G has at least two non-trivial anticomponents; by Lemma 13, it follows that $\alpha(G) = 2$. Since every hyper-hole of length greater than five contains a stable set of size three, (1) now implies that every anticomponent of G is either a 5-hyper-hole or a chordal co-bipartite graph, and so $G \in \mathcal{B}_{\text{UT}}^{\text{cap-free}}$. This proves the theorem. ■

2.7 χ-Boundedness

In this section, we obtain polynomial χ-bounding functions for the classes $\mathcal{G}_{\text{UT}}, \mathcal{G}_{\text{U}}, \mathcal{G}_{\text{T}}, \mathcal{G}_{\text{UT}}^{\text{cap-free}}$.

In Subsection 2.7.1, we deal with classes $\mathcal{G}_{\text{U}}, \mathcal{G}_{\text{T}}, \mathcal{G}_{\text{UT}}^{\text{cap-free}}$. For each of the three classes, we obtain a linear χ-bounding function; the proofs rely on our decomposition theorems for these classes (i.e. Theorems 7, 8 and 9), as well as on a result from [23].

In Subsection 2.7.2, we obtain a fourth-degree polynomial χ-bounding function for class $\mathcal{G}_{\text{UT}}$. Instead of relying on Theorem 6 (the decomposition theorem for $\mathcal{G}_{\text{UT}}$ that we stated in the Introduction and proved in Section 2.3), we prove a new decomposition theorem for class $\mathcal{G}_{\text{UT}}$, one that "decomposes" graphs in $\mathcal{G}_{\text{UT}}$ into "basic" cap-free induced subgraphs via "double star cutsets" that are "small" relative to the clique number of the graph. We then rely on Theorem 35 (which states that the class $\mathcal{G}_{\text{UT}}^{\text{cap-free}}$ is χ-bounded by a linear function), as well as on a result of [37], to obtain a polynomial χ-bounding function for class $\mathcal{G}_{\text{UT}}$.

2.7.1 Classes $\mathcal{G}_{\mathbf{U}}, \mathcal{G}_{\mathbf{T}}, \mathcal{G}_{\mathbf{UT}}^{\textbf{cap-free}}$

We begin with a very easy lemma, which states that clique cutsets "preserve χ-boundedness" (by the same χ-bounding function).

Lemma 27. *Let $\mathcal{G}$ be a hereditary graph class, and let $f : \mathbb{N}^+ \to \mathbb{N}^+$ be a non-decreasing function. Assume that every graph $G \in \mathcal{G}$ either satisfies $\chi(G) \leqslant f(\omega(G))$ or admits a clique cutset. Then every graph $G \in \mathcal{G}$ satisfies $\chi(G) \leqslant f(\omega(G))$.*

PROOF. Clearly, if (A, B, C) is a clique cut partition of a graph G, then

$$\chi(G) = \max\left\{\chi(G[A \cup C]), \chi(G[B \cup C])\right\}.$$

The result now follows by an easy induction. ■

A function $f : \mathbb{N}^+ \to \mathbb{N}^+$ is *superadditive* if, for all $n, m \in \mathbb{N}^+$, we have that $f(n) + f(m) \leqslant f(n+m)$. Note that every superadditive function is non-decreasing.

Lemma 28. *Let $f : \mathbb{N}^+ \to \mathbb{N}^+$ be a superadditive function, let G be a graph and assume that all anticomponents C of G satisfy $\chi(C) \leqslant f(\omega(C))$. Then $\chi(G) \leqslant f(\omega(G))$.*

PROOF. Let $G_1, \dots, G_t$ be the anticomponents of G. Clearly, we have that $\omega(G) = \sum_{i=1}^{t} \omega(G_i)$ and $\chi(G) = \sum_{i=1}^{t} \chi(G_i)$. By hypothesis, $\chi(G_i) \leqslant f(\omega(G_i))$ for every $i \in \{1, \dots, t\}$. Since f is superadditive, it follows that

$$\chi(G) = \sum_{i=1}^{t} \chi(G_i) \leqslant \sum_{i=1}^{t} f\big(\omega(G_i)\big) \leqslant f\Big(\sum_{i=1}^{t} \omega(G_i)\Big) = f\big(\omega(G)\big),$$

which proves the lemma. ■

A graph G is called a *circular-arc graph* if there exists a one-to-one correspondence between the vertices of G and a set of arcs on a circle such that two distinct vertices are adjacent in

G if and only if the corresponding arcs intersect. The following characterization of circular-arc graphs is due to Tucker [46].

Theorem 29. **([46])** *A graph G on n vertices is a circular-arc graph if and only if there exists a circular ordering $v_1, \dots, v_n$ of its vertices, such that, for $i < j$, if $v_iv_j \in E(G)$ then either $v_{i+1}, \dots, v_j \in N_G(v_i)$ or $v_{j+1}, \dots, v_i \in N_G(v_j)$.*

An easy corollary of Theorem 29 is the fact that rings are circular-arc graphs.

Lemma 30. *Let R be a ring. Then R is a circular-arc graph.*

PROOF. Let $(X_1, \dots, X_k)$, with $k \geqslant 4$, be a good partition of the ring R. For every $i \in \mathbb{Z}_k$, let $X_i = \{u_1^i, \dots, u_{|X_i|}^i\}$, and let $X_i \subseteq N_R[u_{|X_i|}^i] \subseteq \dots \subseteq N_R[u_1^i] = X_{i-1} \cup X_i \cup X_{i+1}$. Now consider the circular ordering $v_1, \dots, v_{|V(R)|} = v_{|\bigcup_{i=1}^k X_i|}$ of the vertices of R defined as follows: let $v_1 = u_1^1, \dots, v_{|X_1|} = u_{|X_1|}^1$, and, for every $j \in \{1, \dots, k-1\}$, let

$$v_{|\bigcup_{i=1}^j X_i|+1} = u_1^{j+1}, \dots, v_{|\bigcup_{i=1}^{j+1} X_i|} = u_{|X_{j+1}|}^{j+1}.$$

Clearly, if $i < j$ and $v_iv_j \in E(R)$, then either $v_{i+1}, \dots, v_j \in N_R(v_i)$ or $v_{j+1}, \dots, v_i \in N_R(v_j)$. The result now follows from Theorem 29. ■

Karapetyan [24] showed that every circular-arc graph (and hence, by Lemma 30, every ring) G satisfies $\chi(G) \leqslant \lfloor \frac{3}{2}\omega(G) \rfloor$, which was first conjectured by Tucker [47]. For completeness, we now give an independent proof of this bound for rings.

Lemma 31. *Every ring R satisfies $\chi(R) \leqslant \lfloor \frac{3}{2}\omega(R) \rfloor$. In particular, every hyper-hole H satisfies $\chi(H) \leqslant \lfloor \frac{3}{2}\omega(H) \rfloor$.*

PROOF. Since every hyper-hole is a ring, the second statement follows from the first. To prove the first statement, we let R be a ring, and we assume inductively that every ring R' on fewer than $|V(R)|$ vertices satisfies $\chi(R') \leqslant \lfloor \frac{3}{2}\omega(R') \rfloor$. We must show that $\chi(R) \leqslant \lfloor \frac{3}{2}\omega(R) \rfloor$.

Let $(X_1, \dots, X_k)$, with $k \geqslant 4$, be a good partition of the ring R. By symmetry, w.l.o.g. we may assume that $|X_2| = \max\{|X_i| : i \in \mathbb{Z}_k\}$. If $|X_2| = 1$, then R is a hole, we deduce that $\omega(R) = 2$ and $\chi(R) \leqslant 3$, and the result follows. So, from now on, we assume that $|X_2| \geqslant 2$. Further, it follows from Lemma 4(ii) that $\omega(R) \leqslant \max\{|X_i| + |X_{i+1}| : i \in \mathbb{Z}_k\}$, and so the maximality of $|X_2|$ implies that $|X_2| \geqslant \frac{1}{2}\omega(R)$.

Let $x_2 \in X_2$ be such that, for every $x_2' \in X_2$, $N_R[x_2] \subseteq N_R[x_2']$ (the existence of the vertex x_2 follows from the definition of a ring). Set $Y_1 = N_R(x_2) \cap X_1$ and $Y_3 = N_R(x_2) \cap X_3$; then $N_R(x_2) = Y_1 \cup (X_2 \setminus \{x_2\}) \cup Y_3$, and it follows that $d_R(x_2) = |Y_1| + |X_2| + |Y_3| - 1$. Now, the choice of x_2 guarantees that X_2 is complete to $Y_1 \cup Y_3$, which in turn implies that $\max\{|Y_1| + |X_2|, |Y_3| + |X_2|\} \leqslant \omega(R)$. It follows that $|Y_1| + |X_2| + |Y_3| \leqslant 2\omega(R) - |X_2|$, and so

$$d_R(x_2) = |Y_1| + |X_2| + |Y_3| - 1 \leqslant 2\omega(R) - |X_2| - 1 \leqslant \frac{3}{2}\omega(R) - 1.$$

Since $d_R(x_2)$ is an integer, it follows that $d_R(x_2) \leqslant \lfloor \frac{3}{2}\omega(R) \rfloor - 1$.

Now, since $|X_2| \geqslant 2$, the choice of x_2 guarantees that $R \setminus \{x_2\}$ is a ring. By the induction hypothesis, we have that $\chi(R\setminus\{x_2\}) \leqslant \lfloor \frac{3}{2}\omega(R\setminus\{x_2\})\rfloor \leqslant \lfloor \frac{3}{2}\omega(R)\rfloor$. Since $d_R(x_2) \leqslant \lfloor \frac{3}{2}\omega(R)\rfloor - 1$, it follows that $\chi(R) \leqslant \lfloor \frac{3}{2}\omega(R)\rfloor$. ■

A graph is *weakly chordal* (or *weakly triangulated*) if it contains no long holes and no long antiholes. It was shown in [23] that weakly chordal graphs are perfect (note that this can also be deduced from the Strong Perfect Graph Theorem [9]).

Lemma 32. *Every 7-hyper-antihole G satisfies $\chi(G) \leqslant \lfloor \frac{4}{3}\omega(G)\rfloor$.*

Proof. Let $G = X_1, X_2, \dots, X_7, X_1$ be a 7-hyper-antihole. By symmetry, we may assume that $|X_7| = \min\{|X_i| : i \in \mathbb{Z}_7\}$. Since $X_7 \cup X_2 \cup X_4$ is a clique, the minimality of $|X_7|$ implies that $|X_7| \leqslant \frac{1}{3}\omega(G)$. Now, note that $G \setminus X_7$ is weakly chordal, and therefore (by [23]) perfect. Thus, $\chi(G \setminus X_7) = \omega(G \setminus X_7) \leqslant \omega(G)$. Since $|X_7| \leqslant \frac{1}{3}\omega(G)$, it follows that $\chi(G) \leqslant \frac{4}{3}\omega(G)$. The result now follows from the fact that $\chi(G)$ is an integer. ■

We are now ready to prove that each of the classes $\mathcal{G}_{\rm U}, \mathcal{G}_{\rm T}, \mathcal{G}_{\rm UT}^{\text{cap-free}}$ is χ-bounded by a linear function.

Theorem 33. *Every graph in $\mathcal{G}_{\rm U}$ satisfies $\chi(G) \leqslant \omega(G) + 1$.*

Proof. By Theorem 7 and Lemma 27, it suffices to show that every graph $G \in \mathcal{B}_{\rm U}$ satisfies $\chi(G) \leqslant \omega(G) + 1$. But this easily follows from the definition of $\mathcal{B}_{\rm U}$. ■

Theorem 34. *Every graph in $\mathcal{G}_{\rm T}$ satisfies $\chi(G) \leqslant \lfloor \frac{3}{2}\omega(G)\rfloor$.*

Proof. By Theorem 8 and Lemma 27, it suffices to show that every graph $G \in \mathcal{B}_{\rm T}$ satisfies $\chi(G) \leqslant \lfloor \frac{3}{2}\omega(G)\rfloor$. But this easily follows from the definition of $\mathcal{B}_{\rm T}$, and from Lemma 31 and Lemma 32. ■

Theorem 35. *Every graph in $\mathcal{G}_{\rm UT}^{\text{cap-free}}$ satisfies $\chi(G) \leqslant \lfloor \frac{3}{2}\omega(G)\rfloor$.*

Proof. By Theorem 9 and Lemma 27, it suffices to show that every graph $G \in \mathcal{B}_{\rm UT}^{\text{cap-free}}$ satisfies $\chi(G) \leqslant \lfloor \frac{3}{2}\omega(G)\rfloor$. But this easily follows from the definition of $\mathcal{B}_{\rm UT}^{\text{cap-free}}$, from Lemmas 28 and 31, and from the fact that chordal graphs are perfect [18]. ■

2.7.2 Class $\mathcal{G}_{\rm UT}$

It was proved in [2] that every graph of "large" chromatic number contains a "highly connected" induced subgraph of "large" chromatic number. The bound from [2] was subsequently improved in [8], and it was further improved in [37]. We state the result from [37] below.

Theorem 36. ([37]) *Let k be a positive and c a non-negative integer, and let G be a graph such that $\chi(G) > \max\{c + 2k - 2, 2k^2\}$. Then G contains a $(k+1)$-connected induced subgraph of chromatic number greater than c.*

Our next result is an easy corollary of Theorem 36.

Theorem 37. *Let $\mathcal{G}$ and $\mathcal{G}^*$ be hereditary graph classes, and let $f, h : \mathbb{N}^+ \to \mathbb{N}^+$ be non-decreasing functions. Assume that $\mathcal{G}$ is χ-bounded by f, and assume that every graph $G \in \mathcal{G}^*$ either belongs to $\mathcal{G}$ or admits a cutset of size at most $h(\omega(G))$. Then $\mathcal{G}^*$ is χ-bounded by the function $g : \mathbb{N}^+ \to \mathbb{N}^+$ given by $g(n) = \max\{f(n) + 2h(n) - 2, 2h(n)^2\}$ for all $n \in \mathbb{N}^+$.*

PROOF. Assume otherwise. Fix $G \in \mathcal{G}^*$ such that $\chi(G) > g(\omega(G))$. Set $k = h(\omega(G))$ and $c = f(\omega(G))$; then $\chi(G) > \max\{c + 2k - 2, 2k^2\}$, and so, by Theorem 36, G contains a $(k+1)$-connected induced subgraph F such that $\chi(F) > c$. Since $\mathcal{G}^*$ is hereditary, we know that $F \in \mathcal{G}^*$. Since f is non-decreasing, we have that $\chi(F) > c = f(\omega(G)) \geqslant f(\omega(F))$; since $\mathcal{G}$ is χ-bounded by f, it follows that $F \notin \mathcal{G}$. Since $F \in \mathcal{G}^*$ and $F \notin \mathcal{G}$, it follows that F has a cutset of size at most $h(\omega(F))$. But since h is non-decreasing, we have that $h(\omega(F)) \leqslant h(\omega(G)) = k$, and so F has a cutset of size at most k, which contradicts the fact that F is $(k+1)$-connected. This proves that $\mathcal{G}^*$ is χ-bounded by g. ■

Given $k, \ell \in \mathbb{N}^+$, the *Ramsey number* $R(k, \ell)$ is the smallest integer such that all graphs on $R(k, \ell)$ vertices contain a clique of size k or a stable set of size ℓ (see, for instance, Chapter 8.3 of [49]). A *double star cutset* of a graph G is a cutset C of G such that there exist two adjacent vertices $x, y \in C$ (called the *centers* of the double star cutset C) such that $C \subseteq N_G[x] \cup N_G[y]$.

Theorem 38. *Every graph $G \in \mathcal{G}_{\text{UT}}$ satisfies at least one of the following:*

- *G is cap-free (and so $G \in \mathcal{G}_{\text{UT}}^{\text{cap-free}}$);*
- *$\omega(G) \geqslant 3$, and G admits a double star cutset of size at most $R(\omega(G) - 1, 3) + 4\omega(G) - 7$.*

PROOF. Fix $G \in \mathcal{G}_{\text{UT}}$. We may assume that G is not cap-free, since otherwise we are done. Since every cap contains a triangle, this implies that $\omega(G) \geqslant 3$. Since G contains a cap, we know that there exists a hole $H = x, y, x_1, \dots, x_h, x$ (with $h \geqslant 2$) in G, and a vertex $c \in V(G) \setminus V(H)$ such that $N_G(c) \cap V(H) = \{x, y\}$. (Thus, $G[V(H) \cup \{c\}]$ is a cap.) For all $v \in V(H)$, let T_v be the set of all the twins of v w.r.t. H, that is, let $T_v = X_v(H) \setminus \{v\}$. Set $C = \{x, y\} \cup T_x \cup T_y \cup T_{x_1} \cup T_{x_h} \cup U_H$. Our goal is to show that C is a double star cutset (with centers x, y), and $|C| \leqslant R(\omega(G) - 1, 3) + 4\omega(G) - 7$.

(1) $|C| \leqslant R(\omega(G) - 1, 3) + 4\omega(G) - 7$.

Proof of (1). By Lemma 16, T_v is a clique for every $v \in V(H)$. Furthermore, for all $v \in V(H)$, both v and its two neighbors in H are complete to T_v; consequently, $|T_v| \leqslant \omega(G) - 2$ for all $v \in V(H)$, and it follows that $|C \setminus U_H| = |\{x, y\} \cup T_x \cup T_y \cup T_{x_1} \cup T_{x_h}| \leqslant 4\omega(G) - 6$.

It remains to show that $|U_H| \leqslant R(\omega(G)-1, 3)-1$. Since U_H is complete to the clique $\{x, y\}$, we know that $\omega(G[U_H]) \leqslant \omega(G)-2$. Next, note that x and x_1 are non-adjacent and complete to U_H, and so Lemma 12 applied to $G[\{x, x_1\} \cup U_H]$ implies that $\alpha(G[U_H]) \leqslant 2$. Thus, $G[U_H]$ contains neither a clique of size $\omega(G)-1$ nor a stable set of size three, and so $|U_H| \leqslant R(\omega(G)-1, 3)-1$. This proves (1). □

It remains to prove that C is a double star cutset with centers x and y. First of all, it is clear that x and y are adjacent, that x is complete to $T_x \cup T_{x_h}$, that y is complete to $T_y \cup T_{x_1}$, and that $\{x, y\}$ is complete to U_H. Thus, it suffices to show that C is a cutset of G separating c from $\{x_1, \dots, x_h\}$. Clearly, we may now assume that $T_x \cup T_y \cup T_{x_1} \cup T_{x_h} \cup U_H = \varnothing$, and we just need to prove that $\{x, y\}$ is a cutset of G that separates c from $\{x_1, \dots, x_h\}$.

Assume otherwise, that is, suppose that $\{x, y\}$ does not separate c from $\{x_1, \dots, x_h\}$ in G. Fix a shortest induced path P in $G \setminus \{x, y\}$ such that one endpoint of P is c, and the other endpoint of P belongs to $\{x_1, \dots, x_h\}$. Since c is anticomplete to $\{x_1, \dots, x_h\}$, we know that P is of length at least two. Set $P = p_0, p_1, \dots, p_{n+1}$ (with $n \geqslant 1$), so that $p_0 = c$ and $p_{n+1} \in \{x_1, \dots, x_h\}$. By the minimality of P, we know that c is anticomplete to $\{p_2, \dots, p_{n+1}\}$, and that vertices $p_0, \dots, p_{n-1}$ are anticomplete to $\{x_1, \dots, x_h\}$.

(2) $N_G(p_n) \cap V(H)$ is a clique of size at most two.

Proof of (2). Assume not. Since $U_H = \varnothing$, Lemma 15 implies that p_n is a twin of some vertex of H w.r.t. H. Since $T_x \cup T_y \cup T_{x_1} \cup T_{x_h} = \varnothing$, we see that $p_n \in T_{x_i}$ for some $i \in \{2, \dots, h-1\}$ (and, in particular, $h \geqslant 3$). Note that $p_0 = c$ is adjacent to $x, y \in V(H)$; let $j \in \{0, \dots, n-1\}$ be maximal with the property that p_j has a neighbor in $V(H)$. We know that p_j is anticomplete to $\{x_1, \dots, x_h\}$, and so $N_G(p_j) \cap V(H) \subseteq \{x, y\}$, and at least one of x, y is adjacent to p_j. If p_j is complete to $\{x, y\}$, then $G[(V(H) \setminus \{x_i\}) \cup \{p_j, \dots, p_n\}]$ is a 3PC($p_j xy, p_n$), a contradiction. Thus, p_j is adjacent to exactly one of x, y; by symmetry, w.l.o.g. we may assume that p_j is adjacent to x and non-adjacent to y. But now $G[(V(H) \setminus \{x_i\}) \cup \{p_j, \dots, p_n\}]$ is a 3PC(x, p_n), a contradiction. This proves (2). □

(3) Vertex p_n is anticomplete to $\{x, y\}$.

Proof of (3). Assume otherwise. Since p_n has a neighbor (namely, p_{n+1}) in $\{x_1, \dots, x_h\}$, (2) implies that either $N_G(p_n) \cap V(H) = \{y, x_1\}$ or $N_G(p_n) \cap V(H) = \{x, x_h\}$; by symmetry, w.l.o.g. we may assume that $N_G(p_n) \cap V(H) = \{y, x_1\}$. Now, we know that x is adjacent to $p_0 = c$; let $j \in \{0, \dots, n-1\}$ be maximal with the property that $p_j x \in E(G)$. But now it follows that $Y = x, p_j, \dots, p_n, x_1, \dots, x_h, x$ is a hole and (Y, y) is a proper wheel in G, a contradiction. This proves (3). □

We know that p_n has a neighbor (namely, p_{n+1}) in $\{x_1, \dots, x_h\}$. Also, we may assume that p_n has a neighbor in $\{x_1, \dots, x_{h-1}\}$, since the case when x_h is the only neighbor of p_n in $\{x_1, \dots, x_h\}$ is symmetric to the case when x_1 is the only neighbor of p_n in $\{x_1, \dots, x_h\}$. Now, let $i \in \{1, \dots, h-1\}$ be minimal with the property that $p_n x_i \in E(G)$; it follows from (2) and (3) that $x_i \in N_G(p_n) \cap V(H) \subseteq \{x_i, x_{i+1}\}$.

Recall that $p_0 = c$ is adjacent to $x, y \in V(H)$; let $j \in \{0, \dots, n-1\}$ be maximal with the property that p_j has a neighbor in $V(H)$. We know that p_j is anticomplete to $\{x_1, \dots, x_h\}$, and so we have that $N_G(p_j) \cap V(H) \subseteq \{x, y\}$, and that p_j is adjacent to at least one of x, y. Set $K = G[V(H) \cup \{p_j, \dots, p_n\}]$. It then follows from routine checking that y is the only neighbor of p_j in $V(H)$, and that x_1 is the only neighbor of p_n in $V(H)$, since otherwise K is a 3PC, a contradiction. Note that we now have that x is anticomplete to $\{p_j, \dots, p_n\}$. Since x is adjacent to $p_0 = c$, we know that $j \geqslant 1$; let $\ell \in \{0, \dots, j-1\}$ be maximal with the property that $p_\ell x \in E(G)$. But now $Y = x, p_\ell, \dots, p_n, x_1, \dots, x_h, x$ is a hole and (Y, y) is a proper wheel in G, a contradiction. This completes the argument. ■

Theorem 39. *The class $\mathcal{G}_{\text{UT}}$ is χ-bounded by the function $g : \mathbb{N}^+ \to \mathbb{N}^+$ given by $g(1) = 1$, $g(2) = 3$ and $g(n) = 2(R(n-1,3)+4n-7)^2$ for $n \geqslant 3$.*

Proof. Let $f : \mathbb{N}^+ \to \mathbb{N}^+$ be given by $f(n) = \lfloor \frac{3}{2}n \rfloor$, and let $h : \mathbb{N}^+ \to \mathbb{N}^+$ be given by $h(1) = h(2) = 1$ and $h(n) = R(n-1,3)+4n-7$ for $n \geqslant 3$. Define $\widetilde{g} : \mathbb{N}^+ \to \mathbb{N}^+$ by setting $\widetilde{g}(n) = \max\{f(n)+2h(n)-2, 2h(n)^2\}$. By Theorem 35, $\mathcal{G}_{\text{UT}}^{\text{cap-free}}$ is χ-bounded by f. On the other hand, Theorem 38 guarantees that every graph $G \in \mathcal{G}_{\text{UT}}$ either belongs to $\mathcal{G}_{\text{UT}}^{\text{cap-free}}$ or admits a cutset of size at most $h(\omega(G))$. Therefore, by Theorem 37, we have that $\mathcal{G}_{\text{UT}}$ is χ-bounded by $\widetilde{g}$. Now, to show that $\mathcal{G}_{\text{UT}}$ is in fact χ-bounded by g, we fix $G \in \mathcal{G}_{\text{UT}}$, we set $\omega = \omega(G)$, and we prove that $\chi(G) \leqslant g(\omega)$. If $\omega = 1$, then the result is immediate. Next, suppose that $\omega = 2$. Since every cap contains a triangle, it follows that G is cap-free. Therefore, $G \in \mathcal{G}_{\text{UT}}^{\text{cap-free}}$, and so $\chi(G) \leqslant f(2) = 3 = g(2)$. From now on, we assume that $\omega \geqslant 3$. Since $\mathcal{G}_{\text{UT}}$ is χ-bounded by $\widetilde{g}$, we just need to prove that $g(\omega) = \widetilde{g}(\omega)$. By the definition of g and $\widetilde{g}$, and by an easy calculation, we get the following:

$$\begin{aligned}
\widetilde{g}(\omega) &= \max\left\{f(\omega)+2h(\omega)-2, 2h(\omega)^2\right\} \\
&= \max\left\{\lfloor \tfrac{3}{2}\omega \rfloor + 2R(\omega-1,3)+8\omega-14-2, 2(R(\omega-1,3)+4\omega-7)^2\right\} \\
&= 2\big(R(\omega-1,3)+4\omega-7\big)^2 \\
&= g(\omega).
\end{aligned}$$

This completes the argument. ■

Since $R(k,3)$ is of order $k^2/\log k$ [25], Theorem 39 implies that there exists a constant $c > 0$ such that every graph $G \in \mathcal{G}_{\text{UT}}$ that has at least one edge satisfies $\chi(G) \leqslant c\frac{\omega(G)^4}{\log^2 \omega(G)}$. We also have the following corollary of Theorem 39.

Theorem 40. *Every graph $G \in \mathcal{G}_{\text{UT}}$ satisfies $\chi(G) \leqslant 2\omega(G)^4$.*

Proof. Let $\omega = \omega(G)$. If $\omega \leqslant 2$, then the result follows immediately from Theorem 39. So, assume that $\omega \geqslant 3$. In view of Theorem 39, we only need to show that $2(R(\omega-1,3)+4\omega-7)^2 \leqslant 2\omega^4$, which is, in turn, equivalent to showing that $R(\omega-1,3)+4\omega-7 \leqslant \omega^2$. By the Erdős-Szekeres upper bound for Ramsey numbers (see [49]), we know that $R(k,\ell) \leqslant \binom{k+\ell-2}{\ell-1}$ for all $k,\ell \in \mathbb{N}^+$; thus, $R(\omega-1,3) \leqslant \binom{\omega}{2}$, and consequently, $R(\omega-1,3)+4\omega-7 \leqslant \binom{\omega}{2}+4\omega-7$. A simple calculation now shows that $\binom{\omega}{2}+4\omega-7 \leqslant \omega^2$, and the result follows. ■

2.8 Algorithms

Unless stated otherwise, in all our algorithms n denotes the number of vertices and m denotes the number of edges of the input graph.

We remark that our algorithms are *robust*, that is, they either produce a correct solution to the problem in question for the input (weighted) graph, or they correctly determine that the graph does not belong to the class under consideration.

Our decomposition theorems for classes $\mathcal{G}_{\text{UT}}, \mathcal{G}_{\text{U}}, \mathcal{G}_{\text{T}}, \mathcal{G}_{\text{UT}}^{\text{cap-free}}$ all involve clique cutsets, and for this reason, the algorithmic tools developed in [43] for handling clique cutsets will be used extensively in this section. Our next subsection (Subsection 2.8.1) heavily borrows from [43].

2.8.1 Clique cutset decomposition tree

A function $f : \mathbb{N}^p \to \mathbb{N}$ is said to be *non-decreasing* if it satisfies the property that, for all $n_1, \dots, n_p, n'_1, \dots, n'_p \in \mathbb{N}$ such that $n_1 \leqslant n'_1, \dots, n_p \leqslant n'_p$, we have that $f(n_1, \dots, n_p) \leqslant f(n'_1, \dots, n'_p)$; f is said to be *superadditive* if, for all $n_1, \dots, n_p, n'_1, \dots, n'_p \in \mathbb{N}$, we have that $f(n_1, \dots, n_p) + f(n'_1, \dots, n'_p) \leqslant f(n_1 + n'_1, \dots, n_p + n'_p)$. Clearly, any superadditive function is non-decreasing. Note also that any polynomial function, all of whose coefficients are non-negative, and whose free coefficient is zero, is superadditive.

A *rooted tree* is an ordered pair (T, r), where T is a tree and r is a node of T called the *root*. If T has at least two nodes, then the *leaves* of (T, r) are the nodes in $V(T) \setminus \{r\}$ that are of degree one in T; and, if $V(T) = \{r\}$, then we consider the root r to be a *leaf* of T. The set of all the leaves of (T, r) is denoted by $\mathcal{L}(T, r)$. The *internal nodes* of (T, r) are the nodes in $V(T) \setminus \mathcal{L}(T, r)$. If $u, v \in V(T)$, then we say that v is a *descendant* of u, and that u is an *ancestor* of v in (T, r), provided that $u \neq v$ and u belongs to the unique path between r and v in T. Given $u, v \in V(T)$, we say that v is a *child* of u, and that u is the *parent* of v in (T, r), provided that v is a descendant of u in (T, r), and $uv \in E(T)$. Clearly, every node other than the root has a unique parent in (T, r), leaves have no children in (T, r), and all the internal nodes have at least one child in (T, r). If $u \in V(T)$, then the *subtree of* (T, r) *rooted at* u is the rooted tree (T_u, u), where T_u is the subtree of T induced by u and all the descendants of u in (T, r).

A clique cut partition (A, B, C) of a graph G is *extreme* if $G[A \cup C]$ admits no clique cutset. It is easy to see that, if G admits a clique cutset, then G admits an extreme clique cut partition. (To see this, suppose that G admits a clique cutset. Choose a clique cut partition (A, B, C) of G such that $|A \cup C|$ is as small as possible. Then (A, B, C) is readily seen to be an extreme clique cut partition of G.) This implies that, for every graph G, there exists a *clique cutset decomposition tree* of G, which is a rooted tree (T_G, r) equipped with an associated family $\{V^u\}_{u \in V(T_G)}$ of subsets of $V(G)$, having the following properties:

- if G admits no clique cutset, then $V(T_G) = \{r\}$ and $V^r = V(G)$;
- if G does admit a clique cutset, then there exists an extreme clique cut partition (A, B, C) of G such that $V^r = C$, r has precisely two children in (T_G, r), one of them (call it x) is a leaf of (T_G, r) and satisfies $V^x = A \cup C$, and the subtree of (T_G, r) rooted at the other child of r is a clique cutset decomposition tree of $G[B \cup C]$.

If (T_G, r) is a clique cutset decomposition tree of a graph G, then $|V(T_G)| \leqslant 2|V(G)| - 1$ and $|\mathcal{L}(T_G, r)| \leqslant |V(G)|$. It was shown in [43] that a clique cutset decomposition tree of an arbitrary input graph can be computed in $\mathcal{O}(nm)$ time. We remark that a clique cutset decomposition tree of a given graph does not need to be unique.

If G is a graph, (T_G, r) a clique cutset decomposition tree of G and u a node of T_G, then we set

$$G^u = G\Big[\bigcup\big\{V^x : x = u \text{ or } x \text{ is a descendant of } u \text{ in } (T_G, r)\big\}\Big].$$

Note that the family $\{G^u\}_{u \in V(T_G)}$ can be computed in $\mathcal{O}(n^2 + nm)$ time. We also remark that, for all $u \in V(T_G)$, if u is a leaf of (T_G, r), then G^u admits no clique cutset, and, if u is an internal node of (T_G, r), then V^u is a clique cutset of G^u.

The following simple lemma will be used repeatedly.

Lemma 41. *Let $\mathcal{B}$ and $\mathcal{G}$ be hereditary graph classes, and assume that every graph in $\mathcal{G}$ either belongs to $\mathcal{B}$ or admits a clique cutset. Let $G \in \mathcal{G}$, let (T_G, r) be a clique cutset decomposition tree of G and let $\{G^u\}_{u \in V(T_G)}$ be the associated family of induced subgraphs of G. Then all graphs in the family $\{G^u\}_{u \in \mathcal{L}(T_G, r)}$, and all their induced subgraphs, belong to $\mathcal{B}$.*

PROOF. Since $\mathcal{G}$ is hereditary and $G \in \mathcal{G}$, we know that all induced subgraphs of G belong to $\mathcal{G}$; in particular, each graph in $\{G^u\}_{u \in \mathcal{L}(T_G, r)}$ belongs to $\mathcal{G}$. By the definition of a clique cutset decomposition tree, no graph in $\{G^u\}_{u \in \mathcal{L}(T_G, r)}$ admits a clique cutset. Since (by hypothesis) all graphs in $\mathcal{G}$ that do not admit a clique cutset belong to $\mathcal{B}$, we deduce that all graphs in $\{G^u\}_{u \in \mathcal{L}(T_G, r)}$ belong to $\mathcal{B}$. The result now follows from the fact that $\mathcal{B}$ is hereditary. ■

Our next lemma can be seen as a partial converse of Lemma 41.

Lemma 42. *Let G and F be graphs, let (T_G, r) be a clique cutset decomposition tree of G and let $\{G^u\}_{u \in V(T_G)}$ be the associated family of induced subgraphs of G. Assume that F does not admit a clique cutset, and assume that, for all $u \in \mathcal{L}(T_G, r)$, G^u is F-free. Then G is F-free.*

PROOF. If (A, B, C) is a clique cut partition of a graph G, then the fact that F admits no clique cutset implies that G is F-free if and only if both $G[A \cup C]$ and $G[B \cup C]$ are F-free. The result now easily follows from the definition of a clique cutset decomposition tree. ■

We now show how a clique cutset decomposition tree can be used to solve the optimal coloring problem, as well as the maximum weight clique and maximum weight stable set problems, in certain classes of graphs. Lemmas 43 and 44 (which deal with the optimal coloring and maximum weight clique problems, respectively) and their proofs are very similar to the results and arguments from [43], and we include them here for the sake of completeness. The maximum weight stable set problem is dealt with in a slightly different way than in [43] (see Lemmas 45 and 46, and the discussion that follows them).

Lemma 43. *Let $\mathcal{B}$ and $\mathcal{G}$ be hereditary graph classes, and assume that every graph in $\mathcal{G}$ either belongs to $\mathcal{B}$ or admits a clique cutset. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a non-decreasing function. Assume that there exists an algorithm **A** with the following specifications:*

- *Input: A graph G.*

- *Output: Either an optimal coloring of G, or the true statement that $G \notin \mathcal{B}$.*

- *Running time: At most $f(n, m)$.*

Then there exists an algorithm **B** *with the following specifications:*

- *Input: A graph G.*

- *Output: Either an optimal coloring of G, or the true statement that $G \notin \mathcal{G}$.*

- *Running time: $\mathcal{O}(nf(n, m) + n^2 + nm)$.*

PROOF. Let G be an input graph. We first compute a clique cutset decomposition tree (T_G, r) of G and the associated family $\{G^u\}_{u \in V(T_G)}$ of induced subgraphs of G in $\mathcal{O}(n^2 + nm)$ time. By Lemma 41, if $G \in \mathcal{G}$, then all graphs in the family $\{G^u\}_{u \in \mathcal{L}(T_G, r)}$ belong to $\mathcal{B}$.

Suppose first that T_G has just one node (namely, the root r). In this case, we have that either $G \in \mathcal{B}$ or $G \notin \mathcal{G}$. We now run the algorithm **A** with input G; this takes at most $f(n, m)$ time. If the algorithm **A** returns the answer that $G \notin \mathcal{B}$, then our algorithm **B** returns the answer that $G \notin \mathcal{G}$ and stops. On the other hand, if the algorithm **A** returns an optimal coloring of G, then the algorithm **B** returns this coloring and stops.

Suppose now that T_G has more than one node. Let x and y be the children of the root r in (T_G, r); by symmetry, w.l.o.g. we may assume that $x \in \mathcal{L}(T_G, r)$. We first run the algorithm **A** with input G^x; this takes at most $f(n, m)$ time. If we obtain the answer that $G^x \notin \mathcal{B}$, then the algorithm **B** returns the answer that $G \notin \mathcal{G}$ and stops. Suppose now that the algorithm **A** returned an optimal coloring of G^x. We now recursively either determine that $G^y \notin \mathcal{G}$ or obtain an optimal coloring of G^y. If we obtain the answer that $G^y \notin \mathcal{G}$, then the algorithm **B** returns the answer that $G \notin \mathcal{G}$ and stops. Suppose now that we obtained an optimal coloring of G^y. We then permute and rename the colors used by the colorings of G^x and G^y to ensure that the two colorings agree on V^r, and that the set of colors used on one of G^x, G^y is a subset of the set of colors used on the other; this takes $\mathcal{O}(n)$ time. Finally, we take the union of the colorings of G^x and G^y in $\mathcal{O}(n)$ time, and we obtain an optimal coloring of G; we return this coloring of G and stop.

Clearly, the algorithm is correct; it remains to estimate its running time. We run the algorithm **A** at most $|\mathcal{L}(T_G, r)| \leqslant n$ times, and, each time, the input is an induced subgraph of the graph G; thus, the running time of all the calls to **A** together take at most $nf(n, m)$ time. Further, since $|V(T_G)| \leqslant 2n-1$, it is easy to see that all the other steps of the algorithm take $\mathcal{O}(n^2+nm)$ time. It follows that the total running time of the algorithm is $\mathcal{O}(nf(n, m) + n^2 + nm)$. ■

Lemma 44. *Let $\mathcal{B}$ and $\mathcal{G}$ be hereditary graph classes, and assume that every graph in $\mathcal{G}$ either belongs to $\mathcal{B}$ or admits a clique cutset. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a non-decreasing function, and assume that there exists an algorithm* **A** *with the following specifications:*

- *Input: A weighted graph (G, w).*

- *Output: Either a maximum weight clique C of (G, w), or the true statement that $G \notin \mathcal{B}$.*

- *Running time: At most $f(n, m)$.*

Then there exists an algorithm **B** *with the following specifications:*

- *Input: A weighted graph (G, w).*
- *Output: Either a maximum weight clique C of (G, w), or the true statement that $G \notin \mathcal{G}$.*
- *Running time: $\mathcal{O}(nf(n, m) + n^2 + nm)$.*

PROOF. Let (G, w) be an input weighted graph. We first compute a clique cutset decomposition tree (T_G, r) of G and the associated family $\{G^u\}_{u \in V(T_G)}$ of induced subgraphs of G; this can be done in $\mathcal{O}(n^2 + nm)$ time. Clearly, $\omega(G, w) = \max\{\omega(G^u, w) : u \in \mathcal{L}(T_G, r)\}$. By Lemma 41, we know that, if $G \in \mathcal{G}$, then all graphs in the family $\{G^u\}_{u \in \mathcal{L}(T_G, r)}$ belong to $\mathcal{B}$. For every $u \in \mathcal{L}(T_G, r)$, we call the algorithm **A** with input (G^u, w); since $|\mathcal{L}(T_G, r)| \leqslant n$, we see that running the algorithm **A** for all graphs in the family $\{G^u\}_{u \in \mathcal{L}(T_G, r)}$ takes at most $nf(n, m)$ time. If, for some $u \in \mathcal{L}(T_G, r)$, the algorithm **A** returns the answer that $G^u \notin \mathcal{B}$, then we return the answer that $G \notin \mathcal{G}$ and stop. So, suppose now that, for each $u \in \mathcal{L}(T_G, r)$, the algorithm **A** returned a maximum weight clique C^u of (G^u, w). We now find a node $x \in \mathcal{L}(T_G, r)$ such that $\omega(G^x, w) = \max\{\omega(G^u, w) : u \in \mathcal{L}(T_G, r)\}$; since $|\mathcal{L}(T_G, r)| \leqslant n$, this takes $\mathcal{O}(n^2)$ time. Clearly, C^x is a maximum weight clique of (G, w); we return C^x and stop. It is clear that the algorithm is correct, and that its running time is $\mathcal{O}(nf(n, m) + n^2 + nm)$. ■

Lemma 45. *Let (G, w) be a weighted graph, and let (A, B, C) be a clique cut partition of G. Define the weight function $w_B : B \cup C \to \mathbb{R}$ by setting $w_B \restriction B = w \restriction B$, and, for all $c \in C$, setting $w_B(c) = \alpha(G[A \cup \{c\}], w) - \alpha(G[A], w)$. For every $C' \subseteq C$ such that $|C'| \leqslant 1$, let $S_{A \cup C'}$ be a maximum weight stable set of $(G[A \cup C'], w)$. Let S_B be a maximum weight stable set of $(G[B \cup C], w_B)$, and assume that $w_B(x) > 0$ for all $x \in S_B$. Let $\widetilde{C} = S_B \cap C$. Then $|\widetilde{C}| \leqslant 1$, and $S_{A \cup \widetilde{C}} \cup S_B$ is a maximum weight stable set of (G, w).*

PROOF. Since C is a clique and S_B a stable set of G, we have that $|\widetilde{C}| \leqslant 1$. Set $S = S_{A \cup \widetilde{C}} \cup S_B$. We must show that S is a maximum weight stable set of (G, w).

(1) For all $C' \subseteq C$ such that $|C'| \leqslant 1$, we have that $w_B(C') = \alpha(G[A \cup C'], w) - \alpha(G[A], w)$.

Proof of (1). Fix $C' \subseteq C$ such that $|C'| \leqslant 1$. If $C' = \varnothing$, then $w_B(C') = 0$ and $A \cup C' = A$, and the result is immediate. So assume that $|C'| = 1$, and let c be the unique vertex of C'. Then, by construction,

$$\begin{aligned} w_B(C') &= w_B(c) \\ &= \alpha\big(G[A \cup \{c\}], w\big) - \alpha\big(G[A], w\big) \\ &= \alpha\big(G[A \cup C'], w\big) - \alpha\big(G[A], w\big), \end{aligned}$$

which is what we needed. This proves (1). □

(2) $S_{A \cup \widetilde{C}} \cap C = \widetilde{C}$. Consequently, S is a stable set.

Proof of (2). By construction, $S_B \cap C = \widetilde{C}$. Thus, since $S_{A\cup\widetilde{C}}$ and S_B are stable sets of G, and since A is anticomplete to B in G, the first statement clearly implies the second.

Now we prove that $S_{A\cup\widetilde{C}} \cap C = \widetilde{C}$. By construction, $S_{A\cup\widetilde{C}} \subseteq A \cup \widetilde{C}$; consequently, $S_{A\cup\widetilde{C}} \cap C \subseteq \widetilde{C}$. It remains to show that $\widetilde{C} \subseteq S_{A\cup\widetilde{C}} \cap C$. If $\widetilde{C} = \varnothing$, then this is immediate. So, assume that $\widetilde{C} \neq \varnothing$, so that $|\widetilde{C}| = 1$. Let c be the unique vertex of $\widetilde{C}$. Since $c \in S_B$, we have that $w_B(c) > 0$. By construction, $w_B(c) = \alpha(G[A \cup \{c\}], w) - \alpha(G[A], w)$, and so $\alpha(G[A], w) < \alpha(G[A \cup \{c\}], w)$. Thus, every maximum weight stable set of $(G[A \cup \{c\}], w) = (G[A \cup \widetilde{C}], w)$ contains c; in particular, $c \in S_{A\cup\widetilde{C}}$, and it follows that $\widetilde{C} \subseteq S_{A\cup\widetilde{C}} \cap C$. This proves (2). □

(3) $w(S) = \alpha(G[B \cup C], w_B) + \alpha(G[A], w)$.

Proof of (3). By (2), and by construction, we have that $S_{A\cup\widetilde{C}} \cap C = S_B \cap C = \widetilde{C}$. We know that $|\widetilde{C}| \leqslant 1$, and so, by (1), $w_B(\widetilde{C}) = \alpha(G[A \cup \widetilde{C}], w) - \alpha(G[A], w)$. But now we have that

$$\begin{aligned} w(S) &= w(S_{A\cup\widetilde{C}}) + w(S_B \setminus \widetilde{C}) \\ &= w(S_{A\cup\widetilde{C}}) + w_B(S_B \setminus \widetilde{C}) \\ &= w(S_{A\cup\widetilde{C}}) + w_B(S_B) - w_B(\widetilde{C}) \\ &= \alpha\big(G[A \cup \widetilde{C}], w\big) + \alpha\big(G[B \cup C], w_B\big) - \big(\alpha(G[A \cup \widetilde{C}], w) - \alpha(G[A], w)\big) \\ &= \alpha\big(G[B \cup C], w_B\big) + \alpha\big(G[A], w\big), \end{aligned}$$

which is what we needed. This proves (3). □

(4) Every stable set S' of G satisfies $w(S') \leqslant \alpha(G[B \cup C], w_B) + \alpha(G[A], w)$.

Proof of (4). Fix a stable set S' of G; we must show that $w(S') \leqslant \alpha(G[B \cup C], w_B) + \alpha(G[A], w)$. Set $C' = S' \cap C$; since S' is a stable set and C a clique of G, we have that $|C'| \leqslant 1$. By (1), we have that $w_B(C') = \alpha(G[A \cup C'], w) - \alpha(G[A], w)$. So,

$$\begin{aligned} w(S') &= w\big(S' \cap (A \cup C)\big) + w(S' \cap B) \\ &= w\big((S' \cap A) \cup C'\big) + w_B(S' \cap B) \\ &= w\big((S' \cap A) \cup C'\big) + w_B\big(S' \cap (B \cup C)\big) - w_B(C') \\ &\leqslant \alpha\big(G[A \cup C'], w\big) + \alpha\big(G[B \cup C], w_B\big) - \big(\alpha(G[A \cup C'], w) - \alpha(G[A], w)\big) \\ &= \alpha\big(G[B \cup C], w_B\big) + \alpha\big(G[A], w\big), \end{aligned}$$

which is what we needed. This proves (4). □

Clearly, (2), (3) and (4) imply that S is a maximum weight stable set of (G, w). ■

Lemma 46. *Let $\mathcal{B}$ and $\mathcal{G}$ be hereditary graph classes, and assume that every graph in $\mathcal{G}$ either belongs to $\mathcal{B}$ or admits a clique cutset. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a superadditive polynomial function. Assume that there exists an algorithm* **A** *with the following specifications:*

- *Input: A weighted graph (G, w).*
- *Output: Either a maximum weight stable set of (G, w), or the true statement that $G \notin \mathcal{B}$.*
- *Running time: At most $f(n, m)$.*

Then there exists an algorithm **B** *with the following specifications:*

- *Input: A weighted graph (G, w).*
- *Output: Either a maximum weight stable set of (G, w), or the true statement that $G \notin \mathcal{G}$.*
- *Running time: $\mathcal{O}(nf(n, m) + n^2 + nm)$.*

Proof. Let (G, w) be an input weighted graph. As usual, we begin by computing a clique cutset decomposition tree (T_G, r) of G and the associated family $\{G^u\}_{u \in V(T_G)}$ of induced subgraphs of G in $\mathcal{O}(n^2 + nm)$ time. By Lemma 41, if $G \in \mathcal{G}$, then all graphs in the family $\{G^u\}_{u \in \mathcal{L}(T_G, r)}$, and all their induced subgraphs, belong to $\mathcal{B}$.

Suppose first that T_G has precisely one node (namely, the root r). In this case, we have that either $G \in \mathcal{B}$ or $G \notin \mathcal{G}$. We call the algorithm **A** with input (G, w); this takes at most $f(n, m)$ time. If we obtain the answer that $G \notin \mathcal{B}$, then we return the answer that $G \notin \mathcal{G}$ and stop. Else, **A** returns a maximum weight stable set of (G, w), and we return that stable set and stop.

From now on, we assume that T_G has more than one node; in particular, $r \notin \mathcal{L}(T_G, r)$. For every $u \in \mathcal{L}(T_G, r)$, let $p(u)$ denote the parent of u in (T_G, r). Now, for each $u \in \mathcal{L}(T_G, r)$, we compute the sets $A^u = V^u \setminus V^{p(u)}$, $B^u = V(G^{p(u)}) \setminus V^u$ and $C^u = V^{p(u)}$; clearly, (A^u, B^u, C^u) is an extreme clique cut partition of $G^{p(u)}$, and, since $|\mathcal{L}(T_G, r)| \leqslant n$, computing the families $\{A^u\}_{u \in \mathcal{L}(T_G, r)}$, $\{B^u\}_{u \in \mathcal{L}(T_G, r)}$ and $\{C^u\}_{u \in \mathcal{L}(T_G, r)}$ takes $\mathcal{O}(n^2)$ time. From now on, we will use the following notation: for each $u \in \mathcal{L}(T_G, r)$, we set $n_u = |A^u|$, and we let m_u be the number of edges of G^u, at least one of whose endpoints belongs to A^u. Note that $\sum_{u \in \mathcal{L}(T_G, r)} n_u \leqslant n$ and $\sum_{u \in \mathcal{L}(T_G, r)} m_u \leqslant m$.

Let x and y be the children of the root r in T_G; by symmetry, w.l.o.g. we may assume that $x \in \mathcal{L}(T_G, r)$. We form the graph $G[A^x]$ in $\mathcal{O}(n + m)$ time, and then, for each $c \in C^x$, we form the graph $G[A^x \cup \{c\}]$ in $\mathcal{O}(n_x + m_x)$ time. Clearly, forming the family $\{G[A^x \cup C'] : C' \subseteq C^x, |C'| \leqslant 1\}$ takes $\mathcal{O}(n + m + n(n_x + m_x))$ time. Now, for each $C' \subseteq C^x$ with $|C'| \leqslant 1$, we call the algorithm **A** with input $G[A^x \cup C']$. Clearly, we make $\mathcal{O}(n)$ calls to the algorithm **A**, and each input graph has at most $n_x + 1$ vertices and m_x edges; thus, together, these calls to the algorithm **A** take at most $nf(n_x + 1, m_x)$ time, which is $\mathcal{O}(nf(n_x, m_x))$ time (we use the fact that f is superadditive and polynomial). If, for some $C' \subseteq C$ with $|C'| \leqslant 1$, the algorithm **A** returns the answer that $G[A^x \cup C'] \notin \mathcal{B}$, then we return the answer that $G \notin \mathcal{G}$ and stop. Assume now that, for all $C' \subseteq C$ such that $|C'| \leqslant 1$, the algorithm **A** returned a maximum weight stable set $S_{A^x \cup C'}$ of $(G[A^x \cup C'], w)$. Clearly, for all $C' \subseteq C^x$ with $|C'| \leqslant 1$, we have that

$\alpha(G[A^x \cup C'], w) = w(S_{A^x \cup C'})$, and we see that the family $\{\alpha(G[A^x \cup C'], w) : C' \subseteq C, |C'| \leqslant 1\}$ can be computed in $\mathcal{O}(n_x n)$ time. Next, we form the weight function w_B for $G^y = G[B^x \cup C^x]$ as in Lemma 45; this takes $\mathcal{O}(n)$ time. Then, we recursively either determine that $G^y \notin \mathcal{G}$ or obtain a maximum weight stable set S_B of (G^y, w_B). In the former case, we return the answer that $G \notin \mathcal{G}$ and stop. Suppose now that we obtained a maximum weight stable set S_B of (G^y, w_B). Clearly, $w_B(v) \geqslant 0$ for all $v \in S_B$, and furthermore, we may assume that $w_B(v) > 0$ for all $v \in S_B$, since otherwise we simply delete from S_B all the vertices to which w_B assigns weight zero. Set $C' = C^x \cap S_B$; since C' is a clique and S_B a stable set, we know that $|C'| \leqslant 1$. Set $S = S_{A^x \cup C'} \cup S_B$. By Lemma 45, S is a maximum weight stable set of (G, w). We now return the set S and stop.

It is clear that the algorithm is correct; it remains to estimate its running time. Let u^* be the last leaf of (T_G, r) that our algorithm **B** reaches. With the possible exception of the leaf u^*, for every leaf u of (T_G, r) reached by the algorithm **B**, we call the algorithm **A** on at most n induced subgraphs of G^u, and, as we see from the description of the algorithm, this takes $\mathcal{O}(nf(n_u, m_u))$ time. Furthermore, we may possibly call the algorithm **A** on the graph G^{u^*}; this takes at most $f(n, m)$ time. Thus, the total time that all the calls to the algorithm **A** take is $\mathcal{O}((\sum_{u \in \mathcal{L}(T_G, r)} nf(n_u, m_u)) + f(n, m))$; since $\sum_{u \in \mathcal{L}(T_G, r)} n_u \leqslant n$ and $\sum_{u \in \mathcal{L}(T_G, r)} m_u \leqslant m$, and since f is superadditive and polynomial, this is $\mathcal{O}(nf(n, m))$. Using the fact that $|V(T_G)| \leqslant 2n - 1$, and the fact that $\sum_{u \in \mathcal{L}(T_G, r)} n_u \leqslant n$ and $\sum_{u \in \mathcal{L}(T_G, r)} m_u \leqslant m$, we readily see that all the other steps of the algorithm take $\mathcal{O}(n^2 + nm)$ time. It now follows that the total running time of the algorithm **B** is $\mathcal{O}(nf(n, m) + n^2 + nm)$. ■

Let us now briefly discuss the ways in which Lemmas 45 and 46 differ from their analogs in [43]. First of all, in Lemma 45 (which is used in the proof of Lemma 46), the weight function w_B is defined in a slightly different way than the corresponding weight function from [43]; the advantage of our approach is that we never introduce negative weights, that is to say, if the weight function w assigns only non-negative weights, then so does the weight function w_B. Second of all, one of the hypotheses of Lemma 46 is that the function f is superadditive and polynomial (this hypothesis is absent from [43]); this additional hypothesis, together with a more involved complexity analysis, allows us to obtain a running time that is slightly better than the one from [43]. We remark that, if in the statement of Lemma 46 we replaced the hypothesis that f is superadditive and polynomial with the (weaker) hypothesis that f is non-decreasing, then we would simply obtain a running time of $\mathcal{O}(n^2 f(n, m) + n^2 + nm)$ for the algorithm **B**.

2.8.2 Algorithms for chordal graphs and hyper-holes

A vertex x in a graph G is *simplicial* if $N_G(x)$ is a (possibly empty) clique of G. A *simplicial elimination ordering* of a graph G is an ordering $x_1, \dots, x_n$ of the vertices of G such that x_i is a simplicial vertex of $G[\{x_i, \dots, x_n\}]$ for all $i \in \{1, \dots, n\}$. It is well-known (and easy to show) that a graph is chordal if and only if it has a simplicial elimination ordering. There is an $\mathcal{O}(n + m)$-time algorithm that either produces a simplicial elimination ordering of the input graph, or determines that the graph is not chordal [41]. Clearly, given a chordal graph G

and a simplicial elimination ordering $x_1, \dots, x_n$ of G, an optimal coloring of G can be found in $\mathcal{O}(n+m)$ time (we simply color greedily using the ordering $x_n, \dots, x_1$, that is, the reverse of the input simplicial elimination ordering). Further, there is an $\mathcal{O}(n+m)$-time algorithm that, given a weighted chordal graph (G, w) and a simplicial elimination ordering $x_1, \dots, x_n$ of G, finds a maximum weight stable set of (G, w) [19]. Finally, given a weighted chordal graph (G, w) and a simplicial elimination ordering $x_1, \dots, x_n$ of G, a maximum weight clique of G can be found in $\mathcal{O}(n+m)$ time as follows. First, we may assume that w assigns positive weight to all the vertices of G. (If w assign non-positive weight to every vertex of G, then $\varnothing$ is a maximum weight clique of G. If w assigns positive weight to some, but not all, vertices of G, then we find and delete from G and from the sequence $x_1, \dots, x_n$ all the vertices of G to which w assigns negative or zero weight.) For every $i \in \{1, \dots, n\}$, we form the set $C_i = \{x_j : j \geqslant i, x_j \in N_G[x_i]\}$. We then find an index $i \in \{1, \dots, n\}$ such that $w(C_i) = \max\{w(C_j) : 1 \leqslant j \leqslant n\}$. It is easy to see that C_i is a maximum weight clique of G. For the sake of future reference, we summarize these results in the lemma below.

Lemma 47. *Chordal graphs can be recognized and optimally colored in $\mathcal{O}(n+m)$ time. A maximum weight clique and a maximum weight stable set of a weighted chordal graph can be found in $\mathcal{O}(n+m)$ time.*

Given a graph G, two distinct vertices $x, y \in V(G)$ are said to be *true twins* in G if $N_G[x] = N_G[y]$. Clearly, the relation of being a true twin is an equivalence relation; a *true twin class* of G is an equivalence class w.r.t. the true twin relation. Thus, $V(G)$ can be partitioned into true twin classes of G in a unique way, and clearly, every true twin class of G is a clique of G. An exercise from [42] states that, given an input graph G, all true twin classes of G can be found in $\mathcal{O}(n+m)$ time; a detailed proof of this result can be found in [5]. Given a graph G and a partition $\mathcal{P}$ of $V(G)$ into true twin classes of G, we define the graph $G_{\mathcal{P}}$ (called the *quotient graph* of G w.r.t. $\mathcal{P}$) to be the graph whose vertex set is $\mathcal{P}$, and in which distinct $X, Y \in \mathcal{P}$ are adjacent if and only if X and Y are complete to each other in G. Clearly, given G and $\mathcal{P}$, the graph $G_{\mathcal{P}}$ can be found in $\mathcal{O}(n+m)$ time. We summarize these results below for future reference.

Lemma 48. *There exists an algorithm with the following specifications:*

- *Input: A graph G.*
- *Output: The partition $\mathcal{P}$ of $V(G)$ into true twin classes, and the quotient graph $G_{\mathcal{P}}$.*
- *Running time: $\mathcal{O}(n+m)$.*

Clearly, a graph G is a hole (resp. long hole) if and only if G has at least four vertices (resp. at least five vertices), G is connected (this can be checked in $\mathcal{O}(n+m)$ time using, for example, BFS), and all the vertices of G are of degree two. Thus, holes and long holes can be recognized in $\mathcal{O}(n+m)$ time. The proof of our next lemma (Lemma 49) is an easy exercise, and we leave it to the reader.

Lemma 49. *Let G be a graph, and let $\mathcal{P}$ be a partition of $V(G)$ into true twin classes of G. Then G is a hyper-hole (resp. long hyper-hole) if and only if $G_{\mathcal{P}}$ is a hole (resp. long hole). Consequently, there exists an $\mathcal{O}(n+m)$-time recognition algorithm for hyper-holes (resp. long hyper-holes).*

Given a weighted graph (G, w), where w is positive integer valued, a *weighted coloring* of (G, w) is a function c that assigns to each vertex $x \in V(G)$ a set of precisely $w(x)$ colors, and furthermore, satisfies the property that $c(x_1) \cap c(x_2) = \varnothing$ for all adjacent vertices $x_1, x_2 \in V(G)$. An *optimal weighted coloring* of (G, w) is a weighted coloring that uses as few colors as possible. An $\mathcal{O}(n)$-time optimal weighted coloring algorithm for holes was given in [35]. Together with Lemmas 48 and 49, this yields the following result.

Lemma 50. *There exists an algorithm with the following specifications:*

- *Input: A graph G.*
- *Output: Either an optimal coloring of G, or the true statement that G is not a hyper-hole.*
- *Running time: $\mathcal{O}(n+m)$.*

Proof. Let G be an input graph. We first find a partition $\mathcal{P}$ of $V(G)$ into true twin classes of G, and we form the quotient graph $G_{\mathcal{P}}$; by Lemma 48, this can be done in $\mathcal{O}(n+m)$ time. Clearly, all members of $\mathcal{P}$ are cliques of G. Next, we check in $\mathcal{O}(n+m)$ time whether $G_{\mathcal{P}}$ is a hole, and, if not, then we return the answer that G is not a hyper-hole (by Lemma 49, this is correct) and stop. From now on, we assume that $G_{\mathcal{P}}$ is a hole (and consequently, by Lemma 49, G is a hyper-hole). We define $w_{\mathcal{P}} : \mathcal{P} \to \mathbb{N}^+$ by setting $w_{\mathcal{P}}(X) = |X|$ for all $X \in \mathcal{P}$; this takes $\mathcal{O}(n)$ time. Using the algorithm from [35], we then find an optimal weighted coloring c of $(G_{\mathcal{P}}, w_{\mathcal{P}})$; this takes a further $\mathcal{O}(n)$ time. Using the weighted coloring c of $(G_{\mathcal{P}}, w_{\mathcal{P}})$, we easily obtain an optimal coloring of G: for each $X \in \mathcal{P}$, we assign to each vertex of X one of the colors from the set $c(X)$, making sure that each vertex in X gets a different color; this takes $\mathcal{O}(n)$ time. Clearly, the algorithm is correct, and its total running time is $\mathcal{O}(n+m)$. ■

Lemma 51. *There exists an algorithm with the following specifications:*

- *Input: A weighted graph (G, w).*
- *Output: Either a maximum weight clique C and a maximum weight stable set S of (G, w), or the true statement that G is not a hyper-hole.*
- *Running time: $\mathcal{O}(n+m)$.*

Proof. Let (G, w) be an input weighted graph. If w assigns zero or negative weight to all the vertices of G (note that this can be checked in $\mathcal{O}(n)$ time), then $\varnothing$ is both a maximum weight clique and a maximum weight stable set of (G, w), and we are done. Otherwise, we first update (G, w) by deleting all the vertices of G to which w assigns zero or negative weight; this takes $\mathcal{O}(n+m)$ time. Clearly, any induced subgraph of a hyper-hole is either a hyper-hole or a chordal graph. Using Lemma 47, we now check whether G is chordal, and, if so, we find a maximum weight clique C and a maximum weight stable set S of (G, w), and we return C and S and stop; this takes $\mathcal{O}(n+m)$ time. Suppose now that the algorithm from Lemma 47 returned the answer that G is not a chordal graph. We then find a partition $\mathcal{P}$ of $V(G)$ into true twin classes of G, and we form the quotient graph $G_{\mathcal{P}}$; by Lemma 48, this can be done in $\mathcal{O}(n+m)$ time. Clearly, all members of $\mathcal{P}$ are cliques of G. We check in $\mathcal{O}(n+m)$ time whether $G_{\mathcal{P}}$ is a hole; if not, then we return the answer that G is not a hyper-hole (by Lemma 49, this is correct) and stop. So, from now on, we assume that $G_{\mathcal{P}}$ is a hole.

We find a maximum weight clique C of (G, w) as follows. We define $w_{\mathcal{P}} : \mathcal{P} \to \mathbb{R}$ by setting $w_{\mathcal{P}}(X) = \sum_{x \in X} w(x)$ for all $X \in \mathcal{P}$; finding $w_{\mathcal{P}}$ takes $\mathcal{O}(n)$ time. We then find an edge XY of the hole $G_{\mathcal{P}}$ for which the sum of weights (w.r.t. $w_{\mathcal{P}}$) of its endpoints is maximum; this takes $\mathcal{O}(n)$ time. Set $C = X \cup Y$. Clearly, C is a maximum weight clique of (G, w).

We find a maximum weight stable set S of (G, w) as follows. For each $X \in \mathcal{P}$, we find a vertex $x_X \in X$ such that $w(x_X) = \max\{w(x) : x \in X\}$; finding the family $\{x_X\}_{X \in \mathcal{P}}$ takes $\mathcal{O}(n)$ time. We then form the graph $H = G[\{x_X : X \in \mathcal{P}\}]$ in $\mathcal{O}(n+m)$ time. Since G is a hyper-hole, we see that H is a hole. Clearly, $\alpha(G, w) = \alpha(H, w)$, and furthermore, any maximum weight stable set of (H, w) is a maximum weight stable set of (G, w).

We find a maximum weight stable set of (H, w) as follows. Let x be any vertex of H, and let y and z be the two neighbors of x in H. We form induced subgraphs $H \setminus \{x\}$ and $H \setminus \{x, y, z\}$ of H in $\mathcal{O}(n+m)$ time and, using the $\mathcal{O}(n)$-time algorithm from [6], we find a maximum weight stable set S_1 of the weighted path $(H \setminus \{x\}, w)$, and a maximum weight stable set S_2 of the weighted path $(H \setminus \{x, y, z\}, w)$. (Note that we can also find S_1 and S_2 using the algorithm from Lemma 47.) Clearly, $\{x\} \cup S_2$ is a stable set of H. If $w(S_1) \geqslant w(\{x\} \cup S_2)$, then we set $S = S_1$, and otherwise we set $S = \{x\} \cup S_2$. Clearly, S is a maximum weight stable set of (H, w), and therefore of (G, w) as well.

The algorithm now returns the clique C and the stable set S and stops. It is clear that the algorithm is correct, and that its running time is $\mathcal{O}(n+m)$. ■

2.8.3 Class $\mathcal{G}_{\mathbf{UT}}$

In this subsection, we give a polynomial-time recognition algorithm for class $\mathcal{G}_{\mathrm{UT}}$, and we prove that the maximum clique problem is NP-hard for this class. The complexity of the optimal coloring and maximum stable set problems is still open.

Theorem 52. *The maximum clique problem is* NP*-hard for* (*long hole*, $K_{2,3}, \overline{C_6}$)*-free graphs. Consequently, the maximum clique problem is* NP*-hard for class* $\mathcal{G}_{\mathrm{UT}}$.

PROOF. Since every 3PC other than $K_{2,3}$ and $\overline{C_6}$ contains a long hole, as does every proper wheel, we see that every (long hole, $K_{2,3}, \overline{C_6}$)-free graph belongs to $\mathcal{G}_{\mathrm{UT}}$. Therefore, the first statement implies the second.

Let us now prove the first statement. First of all, it is easy to show that the maximum stable set problem is NP-hard for the class of graphs of girth at least nine. To see this, consider the operation of subdividing every edge of a graph G twice (i.e. the operation of replacing each edge by an induced three-edge path); this yields a graph G' of girth at least nine. As observed in [38], $\alpha(G') = \alpha(G) + |E(G)|$, and so computing the stability number of a graph of girth at least nine is as hard as computing it in a general graph. Now, note that if G is a graph of girth at least nine, then $\overline{G}$ is (long hole, $K_{2,3}$, $\overline{C_6}$)-free. Therefore, if we could compute the clique number of a (long hole, $K_{2,3}, \overline{C_6}$)-free graph in polynomial time, then we could also compute the stability number of a graph of girth at least nine in polynomial time. It follows that the problem of computing the clique number of a (long hole, $K_{2,3}, \overline{C_6}$)-free graph is NP-hard. ■

We now turn to the recognition problem for class $\mathcal{G}_{\text{UT}}$. We begin with a corollary of Theorem 6, which is more convenient than Theorem 6 itself for algorithmic purposes.

Lemma 53. *Let G be a graph, let (T_G, r) be a clique cutset decomposition tree of G, and let $\{G^u\}_{u \in V(T_G)}$ be the associated family of induced subgraphs of G. Then the following statements are equivalent:*

(i) $G \in \mathcal{G}_{\text{UT}}$;

(ii) *G is $(K_{2,3}, \overline{C_6}, W_5^4)$-free, and furthermore, for all $u \in \mathcal{L}(T_G, r)$ and all anticomponents C of G^u, either C is a long ring, or C contains no long holes, or $\alpha(C) \leqslant 2$.*

Proof. It is clear that every graph in $\mathcal{G}_{\text{UT}}$ is $(K_{2,3}, \overline{C_6}, W_5^4)$-free. The fact that (i) implies (ii) now follows immediately from Theorem 6.

Now suppose that the graph G satisfies (ii); we must show that G satisfies (i), that is, that G is (3PC, proper wheel)-free. Clearly, no 3PC and no proper wheel admits a clique cutset, and so, by Lemma 42, it suffices to prove that each graph in $\{G^u\}_{u \in \mathcal{L}(T_G, r)}$ is (3PC, proper wheel)-free. Fix $u \in \mathcal{L}(T_G, r)$. Clearly, every 3PC other than $K_{2,3}$ is anticonnected, as is every proper wheel; since G (and therefore G^u as well) is $K_{2,3}$-free, it now suffices to show that every anticomponent of G^u is (3PC, proper wheel)-free. Let C be an anticomponent of G^u; by hypothesis, C is $(K_{2,3}, \overline{C_6}, W_5^4)$-free, and furthermore, either C is a long ring, or C contains no long holes, or $\alpha(C) \leqslant 2$. If C is a long ring, then Lemma 14 implies that C is (3PC, proper wheel)-free. So, assume that C either contains no long holes or satisfies $\alpha(C) \leqslant 2$. Clearly, every 3PC or proper wheel other than $K_{2,3}$ and $\overline{C_6}$ contains a long hole; furthermore, every 3PC or proper wheel other than $\overline{C_6}$ and W_5^4 contains a stable set of size three. Since C is $(K_{2,3}, \overline{C_6}, W_5^4)$-free, it follows that C is (3PC, proper wheel)-free, and we are done. ■

It can be determined in $\mathcal{O}(n + m^2)$ time whether a graph contains a long hole [36]. In view of this, and of Lemma 53, the problem of recognizing graphs in $\mathcal{G}_{\text{UT}}$ essentially reduces to the problem of recognizing long rings.

Lemma 54. *There exists an algorithm with the following specifications:*

- *Input: A graph G.*
- *Output: Either the true statement that G is a ring, together with the length and good partition of the ring, or the true statement that G is not a ring.*
- *Running time: $\mathcal{O}(n^2)$.*

Proof. Consider the following algorithm.

Step 1: We first check in $\mathcal{O}(n+m)$ time whether G is connected; if not, then the algorithm returns the answer that G is not a ring and stops. From now on, we assume that G is connected. Next, we check in $\mathcal{O}(n+m)$ time whether G is chordal (we use Lemma 47); if so, then the algorithm returns the answer that G is not a ring and stops (this is correct because every ring contains a hole). From now on, we assume that G is not chordal, and, in particular, that G is not complete, and we go to Step 2.

Step 2: For every vertex $v \in V(G)$, we compute $d_G(v)$, and we find a vertex $x \in V(G)$ such that $d_G(x) = \Delta(G)$; this takes $\mathcal{O}(n+m)$ time. Next, we let X_1 be the set of all the vertices y of G such that $N_G[y] \subseteq N_G[x]$; computing X_1 takes $\mathcal{O}(n+m)$ time. Set $n_1 = |X_1|$. We order X_1 as $X_1 = \{u_1^1, \dots, u_{n_1}^1\}$ so that $d_G(u_{n_1}^1) \leqslant \dots \leqslant d_G(u_1^1)$; this takes $\mathcal{O}(n_1^2)$ time. Next, we check in $\mathcal{O}(n_1 n)$ time whether $N_G[u_{n_1}^1] \subseteq \dots \subseteq N_G[u_1^1]$; if not, then the algorithm returns the statement that G is not a ring and stops. So, assume that the algorithm found that $N_G[u_{n_1}^1] \subseteq \dots \subseteq N_G[u_1^1]$. (Note that this implies that X_1 is a clique. Since G is not a complete graph, it follows that $X_1 \subset V(G)$. Since G is connected, and since $N_G[u_{n_1}^1] \subseteq \dots \subseteq N_G[u_1^1]$, we see that u_1^1 has a neighbor in $V(G) \setminus X_1$.) Next, we check in $\mathcal{O}(n+m)$ time whether $G \setminus X_1$ is chordal (we use Lemma 47); if not, then the algorithm returns the statement that G is not a ring and stops (this is correct by Lemma 14). So, assume that $G \setminus X_1$ is indeed chordal. Let X_2 be the vertex set of a component of $G[N_G(u_1^1) \setminus X_1]$; clearly, X_2 can be found in $\mathcal{O}(n+m)$ time. Set $n_2 = |X_2|$. We order X_2 as $X_2 = \{u_1^2, \dots, u_{n_2}^2\}$ so that $d_G(u_{n_2}^2) \leqslant \dots \leqslant d_G(u_1^2)$, and then we check whether $N_G[u_{n_2}^2] \subseteq \dots \subseteq N_G[u_1^2]$; this takes $\mathcal{O}(n_2 n)$ time. If it is not the case that $N_G[u_{n_2}^2] \subseteq \dots \subseteq N_G[u_1^2]$, then the algorithm returns the answer that G is not a ring and stops. So, assume that $N_G[u_{n_2}^2] \subseteq \dots \subseteq N_G[u_1^2]$. We now set $k := 2$, and we go to Step 3.

Step 3: Having constructed the ordered sets

$$X_1 = \{u_1^1, \dots, u_{n_1}^1\}, \; X_2 = \{u_1^2, \dots, u_{n_2}^2\}, \; \dots, \; X_k = \{u_1^k, \dots, u_{n_k}^k\},$$

we proceed as follows. We first compute the set $X_{k+1} = N_G(u_1^k) \setminus (X_1 \cup \dots \cup X_k)$; this takes $\mathcal{O}(n)$ time. Set $n_{k+1} = |X_{k+1}|$. If $n_{k+1} = 0$, then we go to Step 4. Therefore, assume that $n_{k+1} \geqslant 1$. In this case, we order X_{k+1} as $X_{k+1} = \{u_1^{k+1}, \dots, u_{n_{k+1}}^{k+1}\}$ so that $d_G(u_{n_{k+1}}^{k+1}) \leqslant \dots \leqslant d_G(u_1^{k+1})$, and then we check whether $N_G[u_{n_{k+1}}^{k+1}] \subseteq \dots \subseteq N_G[u_1^{k+1}]$; this takes $\mathcal{O}(n_{k+1} n)$ time. If it is not the case that $N_G[u_{n_{k+1}}^{k+1}] \subseteq \dots \subseteq N_G[u_1^{k+1}]$, then the algorithm returns the answer that G is not a ring and stops. Otherwise, we update $k := k+1$, and we go back to Step 3.

Step 4: If $k \leqslant 3$, or if $X_1 \cup \dots \cup X_k \subset V(G)$ (this can be checked in $\mathcal{O}(n)$ time), then the algorithm returns the answer that G is not a ring and stops. So, assume that $k \geqslant 4$ and $V(G) = X_1 \cup \dots \cup X_k$. Now we check whether $u_1^1, u_1^2, \dots, u_1^k, u_1^1$ is a hole in G (this takes $\mathcal{O}(n+m)$ time), and, if so, the algorithm returns the statement that G is a ring of length k, together with the good partition $(X_1, \dots, X_k)$ of the ring G; otherwise, the algorithm returns the answer that G is not a ring.

Clearly, the algorithm is correct. The running time of the algorithm is $\mathcal{O}(n^2 + \sum_{i=1}^{k} n_i n)$; since $\sum_{i=1}^{k} n_i \leqslant n$, it follows that the running time of the algorithm is $\mathcal{O}(n^2)$. ■

We are now ready to give a recognition algorithm for class $\mathcal{G}_{\mathrm{UT}}$.

Theorem 55. *There exists an algorithm with the following specifications:*

- *Input: A graph G.*
- *Output: Either the true statement that $G \in \mathcal{G}_{\mathrm{UT}}$, or the true statement that $G \notin \mathcal{G}_{\mathrm{UT}}$.*
- *Running time: $\mathcal{O}(n^6)$.*

Proof. We test for (ii) from Lemma 53. In particular, we first check in $\mathcal{O}(n^6)$ time whether G is $(K_{2,3}, \overline{C_6}, W_5^4)$-free; if not, then the algorithm returns the answer that $G \notin \mathcal{G}_{\mathrm{UT}}$ and stops. So, assume that G is $(K_{2,3}, \overline{C_6}, W_5^4)$-free. We compute a clique cutset decomposition tree (T_G, r) of G, together with the associated family $\{G^u\}_{u \in V(T_G)}$ of induced subgraphs of G; this takes $\mathcal{O}(n^2 + nm)$ time, which is $\mathcal{O}(n^3)$ time. Fix $u \in \mathcal{L}(T_G, r)$. We first compute the anticomponents $G_1^u, \dots, G_t^u$ of G^u in $\mathcal{O}(n^2)$ time (this can be done by first computing $\overline{G^u}$, then, using BFS, computing the components of $\overline{G^u}$, and finally computing the complements of those components). For each $i \in \{1, \dots, t\}$, set $n_i^u = |V(G_i^u)|$; clearly, $\sum_{i=1}^{t} n_i^u = |V(G^u)| \leqslant n$. Now, for each $i \in \{1, \dots, t\}$, we determine in $\mathcal{O}((n_i^u)^4)$ time whether at least one of the following holds:

(a) G_i^u is a long ring (we use the $\mathcal{O}(n^2)$-time algorithm from Lemma 54);

(b) G_i^u contains no long holes (we use the $\mathcal{O}(n + m^2)$-time algorithm from [36]);

(c) $\alpha(G_i^u) \leqslant 2$.

Checking this for all the anticomponents of G^u takes $\mathcal{O}(\sum_{i=1}^{t} (n_i^u)^4)$ time, which is $\mathcal{O}(n^4)$ time; since $|\mathcal{L}(T_G, r)| \leqslant n$, performing this computation for all graphs in the family $\{G^u\}_{u \in \mathcal{L}(T_G, r)}$ takes $\mathcal{O}(n^5)$ time. Now, if every anticomponent of every graph in the family $\{G^u\}_{u \in \mathcal{L}(T_G, r)}$ satisfies (a), (b) or (c), then the algorithm returns the answer that $G \in \mathcal{G}_{\mathrm{UT}}$ and stops; else, the algorithm returns the answer that $G \notin \mathcal{G}_{\mathrm{UT}}$ and stops. The correctness of the algorithm follows from Lemma 53, and clearly, its running time is $\mathcal{O}(n^6)$. ■

2.8.4 Class $\mathcal{G}_{\mathrm{U}}$

In this subsection, we give polynomial-time algorithms that solve the recognition, optimal coloring, maximum weight clique and maximum weight stable set problems for class $\mathcal{G}_{\mathrm{U}}$.

Let $\mathcal{B}_{\mathrm{U}}^{\mathrm{h}}$ be the class of all induced subgraphs of graphs in $\mathcal{B}_{\mathrm{U}}$. Clearly, $\mathcal{B}_{\mathrm{U}} \subseteq \mathcal{B}_{\mathrm{U}}^{\mathrm{h}}$, and $\mathcal{B}_{\mathrm{U}}^{\mathrm{h}}$ is hereditary. Furthermore, a graph G belongs to $\mathcal{B}_{\mathrm{U}}^{\mathrm{h}}$ if and only if one of the following holds:

- every non-trivial anticomponent of G is isomorphic to $\overline{K_2}$;
- G has exactly one non-trivial anticomponent, and this anticomponent is a long hole;

- G has exactly one non-trivial anticomponent, and this anticomponent has at least three vertices and is a disjoint union of paths.

Lemma 56. *The class $\mathcal{B}_{\mathrm{U}}^{\mathrm{h}}$ is hereditary, and $\mathcal{B}_{\mathrm{U}}^{\mathrm{h}} \subseteq \mathcal{G}_{\mathrm{U}}$. Furthermore, every graph in $\mathcal{G}_{\mathrm{U}}$ either belongs to $\mathcal{B}_{\mathrm{U}}^{\mathrm{h}}$ or admits a clique cutset.*

Proof. The fact that $\mathcal{B}_{\mathrm{U}}^{\mathrm{h}}$ is hereditary follows immediately from the definition of $\mathcal{B}_{\mathrm{U}}^{\mathrm{h}}$. Next, by Lemma 5, we have that $\mathcal{B}_{\mathrm{U}} \subseteq \mathcal{G}_{\mathrm{U}}$. By definition, $\mathcal{B}_{\mathrm{U}}^{\mathrm{h}}$ is the class of all induced subgraphs of graphs in $\mathcal{B}_{\mathrm{U}}$; since $\mathcal{G}_{\mathrm{U}}$ is hereditary, it follows that $\mathcal{B}_{\mathrm{U}}^{\mathrm{h}} \subseteq \mathcal{G}_{\mathrm{U}}$.

It remains to show that every graph in $\mathcal{G}_{\mathrm{U}}$ either belongs to $\mathcal{B}_{\mathrm{U}}^{\mathrm{h}}$ or admits a clique cutset. But this follows immediately from Theorem 7, and from the fact that $\mathcal{B}_{\mathrm{U}} \subseteq \mathcal{B}_{\mathrm{U}}^{\mathrm{h}}$. ■

Lemma 57. *Let G be a graph, let (T_G, r) be a clique cutset decomposition tree of G and let $\{G^u\}_{u \in V(T_G)}$ be the associated family of induced subgraphs of G. Then $G \in \mathcal{G}_{\mathrm{U}}$ if and only if all graphs in the family $\{G^u\}_{u \in \mathcal{L}(T_G, r)}$ belong to $\mathcal{B}_{\mathrm{U}}^{\mathrm{h}}$.*

Proof. The "only if" part follows immediately from Lemma 56 (and in particular, the fact that every graph in $\mathcal{G}_{\mathrm{U}}$ either belongs to $\mathcal{B}_{\mathrm{U}}^{\mathrm{h}}$ or admits a clique cutset). The "if" part follows from Lemma 42, from the fact that (by Lemma 56) $\mathcal{B}_{\mathrm{U}}^{\mathrm{h}} \subseteq \mathcal{G}_{\mathrm{U}}$, and from the fact that no 3PC and no wheel admits a clique cutset. ■

Lemma 58. *There exists an algorithm with the following specifications:*

- *Input: A graph G.*
- *Output: Exactly one of the following:*
  - *The true statement that $G \in \mathcal{B}_{\mathrm{U}}^{\mathrm{h}}$, together with the anticomponents $G_1, \dots, G_t$ of G, and, for each $i \in \{1, \dots, t\}$, the correct information whether*
    - *(i) G_i is isomorphic to K_1, or*
    - *(ii) G_i is isomorphic to $\overline{K_2}$, or*
    - *(iii) G_i is an odd long hole, or*
    - *(iv) G_i is an even long hole, or*
    - *(v) G_i has at least three vertices and is a disjoint union of paths.*
  - *The true statement that $G \notin \mathcal{B}_{\mathrm{U}}^{\mathrm{h}}$.*
- *Running time: $\mathcal{O}(n + m)$.*

Proof. We first compute the degree of all the vertices of G; this takes $\mathcal{O}(n+m)$ time. Suppose first that we have $d_G(x) \geqslant n-2$ for all $x \in V(G)$; note that this can be checked in $\mathcal{O}(n)$ time. In this case, we have that $m \geqslant \frac{1}{2}n(n-2)$. We now compute the anticomponents $G_1, \dots, G_t$ of G; this takes $\mathcal{O}(n^2)$ time, which is $\mathcal{O}(n+m)$ time (because $m \geqslant \frac{1}{2}n(n-2)$). We now have that, for each $i \in \{1, \dots, t\}$, G_i is isomorphic to K_1 or $\overline{K_2}$, that is, G_i satisfies (i) or (ii); clearly, we can determine in $\mathcal{O}(n)$ time which G_i's satisfy (i) and which satisfy (ii).

Suppose now that at least one vertex of G is of degree at most $n-3$. We first form the set U of all the vertices of degree $n-1$ in G, and we set $V = V(G) \setminus U$; clearly, computing U and V takes $\mathcal{O}(n)$ time, and futhermore, for all $u \in U$, $G[\{u\}]$ is a trivial anticomponent of G. Note that the vertex of G that is of degree at most $n-3$ must belong to V, and furthermore, all non-neighbors of this vertex belong to V; thus, $|V| \geqslant 3$, and it follows that $G[V]$ satisfies neither (i) nor (ii). Now, we form the graph $G[V]$ and check whether $G[V]$ satisfies (iii), (iv) or (v); this takes $\mathcal{O}(n+m)$ time. If $G[V]$ satisfies none of (iii), (iv) and (v), then the algorithm returns the answer that $G \notin \mathcal{B}_{\mathrm{U}}^{\mathrm{h}}$ and stops. Suppose now that $G[V]$ satisfies (iii), (iv) or (v). Then $G[V]$ is anticonnected unless it is isomorphic to P_3. But if $G[V]$ is isomorphic to P_3, then the (unique) interior vertex of the path $G[V]$ is of degree $n-1$ in G, and consequently, it belongs to U, a contradiction. Thus, $G[V]$ is indeed anticonnected. The algorithm now returns the answer that $G \in \mathcal{B}_{\mathrm{U}}^{\mathrm{h}}$, together with the anticomponents $G[\{u_1\}], \dots, G[\{u_\ell\}], G[V]$, where $U = \{u_1, \dots, u_\ell\}$, and furthermore, the algorithm returns the answer that $G[\{u_1\}], \dots, G[\{u_\ell\}]$ satisfy (i), and that $G[V]$ satisfies (iii), (iv) or (v), as determined by the algorithm. (If $U = \varnothing$, then the algorithm simply returns the anticomponent $G[V] = G$, together with the information that $G[V] = G$ satisfies (iii), (iv) or (v), as determined by the algorithm.)

Clearly, the algorithm is correct, and its running time is $\mathcal{O}(n+m)$. ■

Theorem 59. *There exists an algorithm with the following specifications:*

- *Input: A graph G.*
- *Output: Either the true statement that $G \in \mathcal{G}_{\mathrm{U}}$, or the true statement that $G \notin \mathcal{G}_{\mathrm{U}}$.*
- *Running time: $\mathcal{O}(n^2 + nm)$.*

Proof. First, we compute a clique cutset decomposition tree (T_G, r) of G, together with the associated family $\{G^u\}_{u \in V(T_G)}$ of induced subgraphs of G; this takes $\mathcal{O}(n^2 + nm)$ time. Then, using the $\mathcal{O}(n+m)$-time algorithm from Lemma 58, we check whether all graphs in the family $\{G^u\}_{u \in \mathcal{L}(T_G, r)}$ belong to $\mathcal{B}_{\mathrm{U}}^{\mathrm{h}}$; since $|\mathcal{L}(T_G, t)| \leqslant n$, checking this for the entire family $\{G^u\}_{u \in \mathcal{L}(T_G, r)}$ takes $\mathcal{O}(n^2 + nm)$ time. If all graphs in the family $\{G^u\}_{u \in \mathcal{L}(T_G, r)}$ belong to $\mathcal{B}_{\mathrm{U}}^{\mathrm{h}}$, then the algorithm returns the answer that $G \in \mathcal{G}_{\mathrm{U}}$, and otherwise the algorithm returns the answer that $G \notin \mathcal{G}_{\mathrm{U}}$. The correctness of the algorithm follows from Lemma 57. Clearly, its running time is $\mathcal{O}(n^2 + nm)$. ■

Theorem 60. *There exists an algorithm with the following specifications:*

- *Input: A graph G.*
- *Output: Either an optimal coloring of G, or the true statement that $G \notin \mathcal{G}_{\mathrm{U}}$.*
- *Running time: $\mathcal{O}(n^2 + nm)$.*

Proof. In view of Lemmas 43 and 56, it suffices to show that there exists an algorithm with the following specifications:

- Input: A graph G.

- Output: Either an optimal coloring of G, or the true statement that $G \notin \mathcal{B}_{\mathrm{U}}^{\mathrm{h}}$.
- Running time: $\mathcal{O}(n+m)$.

In view of Lemmas 47, 50 and 58, it is easy to see that such an algorithm exists. ■

Theorem 61. *There exists an algorithm with the following specifications:*

- *Input: A weighted graph (G, w).*
- *Output: Either a maximum weight clique C and a maximum weight stable set S of (G, w), or the true statement that $G \notin \mathcal{G}_{\mathrm{U}}$.*
- *Running time: $\mathcal{O}(n^2 + nm)$.*

PROOF. In view of Lemmas 44, 46 and 56, it suffices to show that there exists an algorithm with the following specifications:

- Input: A weighted graph (G, w).
- Output: Either a maximum weight clique C and a maximum weight stable set S of (G, w), or the true statement that $G \notin \mathcal{B}_{\mathrm{U}}^{\mathrm{h}}$.
- Running time: $\mathcal{O}(n+m)$.

In view of Lemmas 47, 51 and 58, it is easy to see that such an algorithm exists. ■

2.8.5 Class $\mathcal{G}_{\mathbf{T}}$

In this subsection, we give polynomial-time algorithms that solve the recognition, maximum weight clique and maximum weight stable set problems for class $\mathcal{G}_{\mathrm{T}}$. We remark that we do not know whether graphs in $\mathcal{G}_{\mathrm{T}}$ can be optimally colored in polynomial time; this is because we do not know whether rings can be optimally colored in polynomial time. We begin with a corollary of Theorem 8.

Lemma 62. *Let G be a graph, let (T_G, r) be a clique cutset decomposition tree of G, and let $\{G^u\}_{u \in V(T_G)}$ be the associated family of induced subgraphs of G. For all $u \in V(T_G)$, let $\mathcal{P}_u$ be the partition of $V(G^u)$ into true twin classes of G^u. Then the following are equivalent:*

(i) $G \in \mathcal{G}_{\mathrm{T}}$;

(ii) for all $u \in \mathcal{L}(T_G, r)$, the quotient graph $G^u_{\mathcal{P}_u}$ is a ring, a one-vertex graph or a 7-antihole.

PROOF. Since no 3PC and no wheel admits a clique cutset, Lemma 5 (and in particular, the fact that $\mathcal{B}_{\mathrm{T}} \subseteq \mathcal{G}_{\mathrm{T}}$), Theorem 8 and Lemma 42 imply that $G \in \mathcal{G}_{\mathrm{T}}$ if and only if all graphs from the family $\{G^u\}_{u \in \mathcal{L}(T_G, r)}$ belong to $\mathcal{B}_{\mathrm{T}}$. On the other hand, it follows from the definition of $\mathcal{B}_{\mathrm{T}}$ that a graph F belongs to $\mathcal{B}_{\mathrm{T}}$ if and only if the quotient graph $F_{\mathcal{P}}$ (where $\mathcal{P}$ is the partition of $V(F)$ into true twin classes) is either a ring, a one-vertex graph or a 7-antihole. The result is now immediate. ■

Theorem 63. *There exists an algorithm with the following specifications:*

- *Input: A graph G.*
- *Output: Either the true statement that $G \in \mathcal{G}_{\mathrm{T}}$, or the true statement that $G \notin \mathcal{G}_{\mathrm{T}}$.*
- *Running time: $\mathcal{O}(n^3)$.*

Proof. We use Lemma 62. First, we compute a clique cutset decomposition tree (T_G, r) of G, together with the associated family $\{G^u\}_{u \in V(T_G)}$ of induced subgraphs of G; this takes $\mathcal{O}(n^2 + nm)$ time. For every node $u \in \mathcal{L}(T_G, r)$, we compute the partition $\mathcal{P}_u$ of $V(G^u)$ into true twin classes of G^u, and we compute the quotient graph $G^u_{\mathcal{P}_u}$; Lemma 48 and the fact that $|\mathcal{L}(T_G, r)| \leqslant n$ imply that the family $\{G^u_{\mathcal{P}_u}\}_{u \in \mathcal{L}(T_G, r)}$ can be computed in $\mathcal{O}(n^2 + nm)$ time. By Lemma 54, rings can be recognized in $\mathcal{O}(n^2)$ time, and clearly, one can check in $\mathcal{O}(1)$ time whether a graph is trivial (i.e. whether it has just one vertex) or is a 7-antihole. Since $|\mathcal{L}(T_G, r)| \leqslant n$, it follows that it can be checked in $\mathcal{O}(n^3)$ time whether the family $\{G^u_{\mathcal{P}_u}\}_{u \in \mathcal{L}(T_G, r)}$ satisfies condition (ii) of Lemma 62; if so, then the algorithm returns the answer that $G \in \mathcal{G}_{\mathrm{T}}$, and otherwise it returns the answer that $G \notin \mathcal{G}_{\mathrm{T}}$. Clearly, the algorithm is correct, and its running time is $\mathcal{O}(n^3)$. ■

Lemma 64. *Let G be a graph. Then the following are equivalent:*

(i) G contains no universal wheels;

(ii) for all $x \in V(G)$, $G[N_G[x]]$ is chordal.

Proof. First suppose that (i) holds. Fix $x \in V(G)$ and note that, if $G[N_G(x)]$ contains a hole H, then (H, x) is a universal wheel in G, a contradiction. Thus, $G[N_G(x)]$ is chordal. Since x is complete to $N_G(x)$, we deduce that $G[N_G[x]]$ is also chordal. Thus, (ii) holds.

Now assume that (ii) holds. Suppose that G contains a universal wheel, say (H, x). Then H is a hole in $G[N_G[x]]$, contrary to the fact that $G[N_G[x]]$ is chordal. ■

Theorem 65. *There exists an algorithm with the following specifications:*

- *Input: A weighted graph (G, w).*
- *Output: Either a maximum weight clique C of (G, w), or the true statement that G contains a universal wheel (and therefore $G \notin \mathcal{G}_{\mathrm{T}}$).*
- *Running time: $\mathcal{O}(n^2 + nm)$.*

Proof. For every vertex $x \in V(G)$, we form the graph $G_x = G[N_G[x]]$, we check whether G_x is chordal, and, if so, we compute a maximum weight clique C_x of G_x; in view of Lemma 47, for each $x \in V(G)$ individually, we can perform these computations in $\mathcal{O}(n + m)$ time, and so, for all $x \in V(G)$ together, we can perform them in $\mathcal{O}(n^2 + nm)$ time. If, for some $x \in V(G)$, we determined that G_x is not chordal, then the algorithm returns the answer that G contains a universal wheel (this is correct by Lemma 64) and stops. So, assume that the algorithm computed a maximum weight clique C_x for every G_x. Among all the cliques in the family $\{C_x\}_{x\in V(G)}$, the algorithm finds one of maximum weight and it returns that clique and stops. It is clear that the algorithm is correct, and that its running time is $\mathcal{O}(n^2+nm)$. ■

Lemma 66. *Let $G \in \mathcal{G}_{\mathrm{T}}$. Then at least one of the following holds:*

- *for all $x \in V(G)$, $G \setminus N_G(x)$ is chordal;*
- *G admits a clique cutset.*

Proof. Assume that G does not admit a clique cutset. Fix $x \in V(G)$; we must show that $G \setminus N_G(x)$ is chordal. By Theorem 8, G is either a ring, a complete graph or a 7-hyper-antihole. If G is a ring, then the result follows from Lemma 14, and, if G is a complete graph or a 7-hyper-antihole, then the result is immediate. ■

Theorem 67. *There exists an algorithm with the following specifications:*

- *Input: A weighted graph (G,w).*
- *Output: Either a maximum weight stable set S of (G,w), or the true statement that $G \notin \mathcal{G}_{\mathrm{T}}$.*
- *Running time: $\mathcal{O}(n^3+n^2m)$.*

Proof. Let $\mathcal{B}$ be the class of all graphs G such that, for every vertex $x \in V(G)$, we have that $G \setminus N_G(x)$ is chordal. Clearly, $\mathcal{B}$ is a hereditary graph class, and, by Lemma 66, every graph in $\mathcal{G}_{\mathrm{T}}$ either belongs to $\mathcal{B}$ or admits a clique cutset. In view of Lemma 46, it now suffices to show that there exists an algorithm with the following specifications:

- Input: A weighted graph (G,w).
- Output: Either a maximum weight stable set S of (G,w), or the true statement that $G \notin \mathcal{B}$.
- Running time: $\mathcal{O}(n^2+nm)$.

Let (G,w) be an input weighted graph. For every $x \in V(G)$, we form the graph $G_x = G \setminus N_G(x)$, we check whether G_x is chordal, and, if so, we compute a maximum weight stable set S_x of (G_x,w); by Lemma 47, for each $x \in V(G)$ individually, these computations can be performed in $\mathcal{O}(n+m)$ time, and so, for all $x \in V(G)$ together, they can be performed in $\mathcal{O}(n^2+nm)$ time. If the algorithm determined that, for some $x \in V(G)$, G_x is not chordal, then we return the answer that $G \notin \mathcal{B}$ and stop. So, assume that, for every $x \in V(G)$, the algorithm found a maximum weight stable set S_x of (G_x,w). Clearly, $\alpha(G,w) = \max\{w(S_x) : x \in V(G)\}$. We now find a vertex $y \in V(G)$ such that $w(S_y) = \max\{w(S_x) : x \in V(G)\}$; this takes $\mathcal{O}(n^2)$ time. We finally return S_y and stop. Clearly, the algorithm is correct, and its running time is $\mathcal{O}(n^2+nm)$. ■

2.8.6 Class $\mathcal{G}_{\textbf{UT}}^{\textbf{cap-free}}$

In this subsection, we show that the recognition, optimal coloring, maximum weight clique and maximum weight stable set problems can be solved in polynomial time for class $\mathcal{G}_{\text{UT}}^{\text{cap-free}}$.

Let $\mathcal{B}_{\text{C}}^{\text{H}}$ be the class of all graphs G such that every anticomponent of G is either a long hyper-hole or a chordal graph.

Lemma 68. *The class $\mathcal{B}_{\text{C}}^{\text{H}}$ is hereditary. Furthermore, every graph in $\mathcal{G}_{\text{UT}}^{\text{cap-free}}$ either belongs to $\mathcal{B}_{\text{C}}^{\text{H}}$ or admits a clique cutset.*

Proof. Clearly, the class of chordal graphs is hereditary, and every induced subgraph of a long hyper-hole is either a long hyper-hole or a chordal graph; this implies that $\mathcal{B}_{\text{C}}^{\text{H}}$ is hereditary. Also, it is clear that $\mathcal{B}_{\text{UT}}^{\text{cap-free}} \subseteq \mathcal{B}_{\text{C}}^{\text{H}}$. This, together with Theorem 9, implies that every graph in $\mathcal{G}_{\text{UT}}^{\text{cap-free}}$ either belongs to $\mathcal{B}_{\text{C}}^{\text{H}}$ or admits a clique cutset. ■

Lemma 69. *Let G be a graph, let (T_G, r) be a clique cutset decomposition tree of G, and let $\{G^u\}_{u \in V(T_G)}$ be the associated family of induced subgraphs of G. Then the following are equivalent:*

(i) $G \in \mathcal{G}_{\text{UT}}^{\text{cap-free}}$;

(ii) G is $(K_{2,3}$, cap)-free, and all graphs in $\{G^u\}_{u \in \mathcal{L}(T_G, r)}$ belong to $\mathcal{B}_{\text{C}}^{\text{H}}$.

Proof. Clearly, every graph in $\mathcal{G}_{\text{UT}}^{\text{cap-free}}$ is $(K_{2,3}$, cap)-free. The fact that (i) implies (ii) now follows from Lemma 68.

For the converse, we assume (ii) and we prove (i). By (ii), G is cap-free, and so it suffices to show that G is (3PC, proper wheel)-free. No 3PC and no proper wheel admits a clique cutset, and so, by Lemma 42, it suffices to prove that all graphs in the family $\{G^u\}_{u \in \mathcal{L}(T_G, r)}$ are (3PC, proper wheel)-free. Fix a node $u \in \mathcal{L}(T_G, r)$. Note that every 3PC other than $K_{2,3}$ is anticonnected, as is every proper wheel; since G^u is $K_{2,3}$-free (because G is), it now suffices to show that every anticomponent of G^u is (3PC, proper wheel)-free. Let C be an anticomponent of G^u. By (ii), we have that $G^u \in \mathcal{B}_{\text{C}}^{\text{H}}$, and so, by the definition of $\mathcal{B}_{\text{C}}^{\text{H}}$, C is either a chordal graph or a long hyper-hole. In the former case, it is clear that C is (3PC, proper wheel)-free (this is because every 3PC and every wheel contains a hole, and, by definition, chordal graphs contain no holes). So, assume that C is a hyper-hole. Then C is a ring, and so, by Lemma 14, C is (3PC, proper wheel)-free. This proves (i). ■

Theorem 70. *There exists an algorithm with the following specifications:*

- *Input: A graph G.*
- *Output: Either the true statement that $G \in \mathcal{G}_{\text{UT}}^{\text{cap-free}}$, or the true statement that $G \notin \mathcal{G}_{\text{UT}}^{\text{cap-free}}$.*
- *Running time: $\mathcal{O}(n^5)$.*

PROOF. We test for (ii) from Lemma 69. We begin by checking in $\mathcal{O}(n^5)$ time whether G is $(K_{2,3}, \text{cap})$-free (to test whether G is $K_{2,3}$-free, we simply examine all five-tuples of vertices of G, and, to check whether G is cap-free, we use the $\mathcal{O}(n^5)$-time algorithm from [5]). If G is not $(K_{2,3}, \text{cap})$-free, then the algorithm returns the answer that $G \notin \mathcal{G}_{\text{UT}}^{\text{cap-free}}$ and stops. So, assume that G is $(K_{2,3}, \text{cap})$-free. We now compute a clique cutset decomposition tree (T_G, r) of G, together with the associated family $\{G^u\}_{u \in V(T_G)}$ of induced subgraphs of G; this takes $\mathcal{O}(n^2 + nm)$ time. For all $u \in \mathcal{L}(T_G, r)$, we proceed as follows. First, we compute the anticomponents $G_1^u, \dots, G_t^u$ of G^u in $\mathcal{O}(n^2)$ time. For each $i \in \{1, \dots, t\}$, set $n_i^u = |V(G_i^u)|$; clearly, $\sum_{i=1}^{t} n_i^u = |V(G^u)| \leqslant n$. For every $i \in \{1, \dots, t\}$, we test in $\mathcal{O}((n_i^u)^2)$ time whether G_i^u is either a chordal graph or a long hyper-hole (we use Lemmas 47 and 49); testing this for all anticomponents of G^u together takes $\mathcal{O}(\sum_{i=1}^{t}(n_i^u)^2)$ time, which is $\mathcal{O}(n^2)$ time. Since $|\mathcal{L}(T_G, r)| \leqslant n$, performing this computation for all graphs in the family $\{G^u\}_{u \in \mathcal{L}(T_G, r)}$ takes $\mathcal{O}(n^3)$ time. If, for each $u \in \mathcal{L}(T_G, r)$, we determined that every anticomponent of G^u is either a chordal graph or a long hyper-hole, then (by the definition of $\mathcal{B}_{\text{C}}^{\text{H}}$) we have that every graph in the family $\{G^u\}_{u \in \mathcal{L}(T_G, r)}$ belongs to $\mathcal{B}_{\text{C}}^{\text{H}}$, and so, by Lemma 69, we know that $G \in \mathcal{G}_{\text{UT}}^{\text{cap-free}}$, and we return this answer and stop. Otherwise, we return the answer that $G \notin \mathcal{G}_{\text{UT}}^{\text{cap-free}}$ and stop. Clearly, the algorithm is correct, and its running time is $\mathcal{O}(n^5)$. ■

Theorem 71. *There exists an algorithm with the following specifications:*

- *Input: A graph G.*
- *Output: Either an optimal coloring of G, or the true statement that $G \notin \mathcal{G}_{\text{UT}}^{\text{cap-free}}$.*
- *Running time: $\mathcal{O}(n^3)$.*

PROOF. In view of Lemmas 43 and 68, it suffices to show that there exists an algorithm with the following specifications:

- Input: A graph G.
- Output: Either an optimal coloring of G, or the true statement that $G \notin \mathcal{B}_{\text{C}}^{\text{H}}$.
- Running time: $\mathcal{O}(n^2)$.

Let G be an input graph. We begin by computing the anticomponents $G_1, \dots, G_t$ of G in $\mathcal{O}(n^2)$ time. For each $i \in \{1, \dots, t\}$, we set $n_i = |V(G_i)|$, and we proceed as follows. We first check whether G_i is chordal, and, if so, we compute an optimal coloring c_i of G_i; by Lemma 47, this can be done in $\mathcal{O}(n_i^2)$ time. If G_i is not chordal, then we call the algorithm from Lemma 50, and we obtain either an optimal coloring c_i of G_i, or the true statement that G_i is not a hyper-hole; this takes $\mathcal{O}(n_i^2)$ time. If, for some $i \in \{1, \dots, t\}$, we determined that G_i is neither a chordal graph nor a hyper-hole, then the algorithm returns the answer that $G \notin \mathcal{B}_{\text{C}}^{\text{H}}$ and stops. So, assume that, for every $i \in \{1, \dots, t\}$, the algorithm found an optimal coloring c_i of G_i. We then rename the colors used by the colorings $c_1, \dots, c_t$ so that the color sets used by these colorings are pairwise disjoint (this takes $\mathcal{O}(n)$ time), and then we let c be the union of the resulting t colorings. The algorithm now returns the coloring c and stops. Clearly, the algorithm is correct, and its running time is $\mathcal{O}(n^2 + \sum_{i=1}^{t} n_i^2)$, which is $\mathcal{O}(n^2)$. ■

Theorem 72. *There exists an algorithm with the following specifications:*

- *Input: A weighted graph* (G, w)*.*
- *Output: Either a maximum weight clique* C *and a maximum weight stable set* S *of* (G, w)*, or the true statement that* $G \notin \mathcal{G}_{\mathrm{UT}}^{\text{cap-free}}$*.*
- *Running time:* $\mathcal{O}(n^3)$*.*

Proof. In view of Lemmas 44, 46 and 68, it suffices to show that there exists an algorithm with the following specifications:

- Input: A weighted graph (G, w).
- Output: Either a maximum weight clique C and a maximum weight stable set S of (G, w), or the true statement that $G \notin \mathcal{B}_{\mathrm{C}}^{\mathrm{H}}$.
- Running time: $\mathcal{O}(n^2)$.

Let (G, w) be an input weighted graph. We begin by computing the anticomponents $G_1, \dots, G_t$ of G in $\mathcal{O}(n^2)$ time. For each $i \in \{1, \dots, t\}$, we set $n_i = |V(G_i)|$. Clearly, $\sum_{i=1}^{t} n_i = |V(G)| = n$. For every $i \in \{1, \dots, t\}$, we proceed as follows. We first check whether G_i is chordal, and, if so, we find a maximum weight clique C_i and a maximum weight stable set S_i of (G_i, w); by Lemma 47, this can be done in $\mathcal{O}(n_i^2)$ time. If G_i is not chordal, then we call the algorithm from Lemma 51, and we obtain either a maximum weight clique C_i and a maximum weight stable set S_i of (G_i, w), or the true statement that G_i is not a hyper-hole; this takes $\mathcal{O}(n_i^2)$ time. If, for some $i \in \{1, \dots, t\}$, we determined that G_i is neither a chordal graph nor a hyper-hole, then the algorithm returns the answer that $G \notin \mathcal{B}_{\mathrm{C}}^{\mathrm{H}}$ and stops. So, assume that, for each $i \in \{1, \dots, t\}$, the algorithm found a maximum weight clique C_i and a maximum weight stable set S_i of (G_i, w). We then form the clique $C = C_1 \cup \dots \cup C_t$, and we find an index $j \in \{1, \dots, t\}$ such that $w(S_j) = \max\{w(S_i) : 1 \leqslant i \leqslant t\}$; clearly, this can be done in $\mathcal{O}(n)$ time. The algorithm now returns the clique C and the stable set S_j and stops. Clearly, the algorithm is correct, and its running time is $\mathcal{O}(n^2 + \sum_{i=1}^{t} n_i^2)$, which is $\mathcal{O}(n^2)$. ■

Chapter 3

The structure of (theta, pyramid, 1-wheel, 3-wheel)-free graphs

In this chapter we present [4].

Abstract. In this chapter, we study the class of graphs $\mathcal{G}$ defined by excluding the following structures as induced subgraphs: thetas, pyramids, 1-wheels and 3-wheels. We describe the structure of graphs in $\mathcal{G}$, and we give a polynomial-time recognition algorithm for this class. We also prove that K_4-free graphs in $\mathcal{G}$ are 4-colorable. We remark that $\mathcal{G}$ includes the class of chordal graphs, as well as the class of line graphs of triangle-free graphs.

3.1 Introduction

Throughout this chapter, all graphs are finite and simple. We say that a graph G *contains* a graph F if F is isomorphic to an induced subgraph of G, and it is F*-free* if it does not contain F. For a (possibly infinite) family of graphs $\mathcal{F}$, we say that G is $\mathcal{F}$*-free* if G is F-free for every $F \in \mathcal{F}$. A *hole* in a graph is a chordless cycle of length at least four, and it is *even* or *odd* depending on the parity of its length.

In 1982 Truemper [45] gave a theorem that characterizes graphs whose edges can be labeled so that all chordless cycles have prescribed parities. The characterization states that this can be done for a given graph G if and only if it can be done for all induced subgraphs of G that are either K_4's or of a few specific types (depicted in Figure 1.1 in Chapter 1), which we will call *Truemper configurations* and will describe precisely later. Truemper was originally motivated by the problem of obtaining a co-NP characterization of bipartite graphs that are signable to be balanced (i.e. bipartite graphs whose vertex-vertex adjacency matrices are balanceable matrices, a class of matrices that have important polyhedral properties).

The configurations that Truemper identified in his theorem later played an important role in understanding the structure of several seemingly diverse classes of objects, such as regular matroids, balanceable matrices, perfect graphs, odd-hole-free and even-hole-free graphs (for a survey see [48]). All these classes were studied using the decomposition method. In these decomposition theorems, Truemper configurations appear both as excluded structures that are convenient to work with, and as structures around which the actual decomposition takes place.

In this chapter, we study the class $\mathcal{G}$ of (theta, pyramid, 1-wheel, 3-wheel)-free graphs, which we formally define in Section 3.1.2. This class contains all chordal graphs and all line graphs of triangle-free graphs (or, equivalently, (claw, diamond)-free graphs [22]). It was first studied in [1], where it was shown that every graph in $\mathcal{G}$ has a vertex whose neighborhood is the disjoint union of two (possibly empty) cliques with no edges between them, and furthermore an ordering of such vertices can be found by running LexBFS. A consequence of this is a linear-time algorithm for the maximum clique problem on $\mathcal{G}$, as well as a linear-time coloring algorithm that colors the vertices of any given graph $G \in \mathcal{G}$ with at most $2\omega(G) - 1$ colors, where $\omega(G)$ denotes the size of a largest clique of G. The optimal coloring problem is NP-hard on line graphs of triangle-free graphs, and in fact it is also NP-hard on (K_4, claw, diamond)-free graphs [26]. The complexity of the maximum stable set problem on $\mathcal{G}$ is open, and in fact it is open even for the subclass of K_4-free graphs in $\mathcal{G}$. On the other hand, the maximum stable set problem is polynomial-time solvable on claw-free graphs [32, 34], and hence on line graphs of triangle-free graphs.

In this chapter, we describe the structure of graphs in $\mathcal{G}$, and, as a consequence, we obtain a series of decomposition theorems that use cutsets that combine star cutsets and 2-joins in the simplest possible ways. These theorems present a good setting for studying various problems, and in particular the maximum stable set problem, restricted to class $\mathcal{G}$. Two much studied hereditary graph classes are even-hole-free graphs and perfect graphs (see for example surveys [48] and [44]). The complexity of the maximum stable set problem on even-hole-free graphs is still not known, and also it is not known how to solve the maximum stable set problem in polynomial time for perfect graphs by a purely graph-theoretic algorithm (it is known that this problem can be solved in polynomial time for perfect graphs using the ellipsoid method [20]). The known decomposition theorems for these classes use star cutsets and 2-joins, as well as different generalizations of these. It is not clear how to make use of star cutsets for the maximum stable set problem (and other problems), and so it would be of interest to understand how very structured star cutsets, like the ones that appear in this chapter, behave in algorithms.

The chapter is organized as follows. In Sections 3.1.1 and 3.1.2, we introduce the terminology and notation that will be used throughout the chapter. In Section 3.1.3, we give an overview of subclasses of $\mathcal{G}$ that were studied in literature. In Section 3.1.4, we recall some well-known results about the complexity of recognizing different Truemper configurations, and, in Section 3.2, we give two polynomial-time recognition algorithms for $\mathcal{G}$. In Section 3.1.5, we describe the structure of graphs in $\mathcal{G}$, which we prove in Sections 3.3 and 3.4. Finally, in Section 3.5, using our decomposition theorem for $\mathcal{G}$, we prove that K_4-free graphs in $\mathcal{G}$ are 4-colorable.

3.1.1 Terminology and notation

Let G be a graph. As usual, the vertex set and edge set of G are denoted by $V(G)$ and $E(G)$, respectively. Sometimes, when clear from context, for notational simplicity we will refer to

$V(G)$ just as G. Given a vertex $x \in V(G)$, $N_G(x)$ denotes the set of all the neighbors of x in G, and $N_G[x] = N_G(x) \cup \{x\}$. Given a set $S \subseteq V(G)$, $G[S]$ is the subgraph of G induced by S, $G \setminus S = G[V(G) \setminus S]$, $N_G(S)$ is the set of all the vertices in $V(G) \setminus S$ with at least one neighbor in S, and $N_G[S] = N_G(S) \cup S$. Note that, if S is empty, then $G[S]$ is the null graph, and $N_G(S) = N_G[S] = \varnothing$.

Let F and G be two graphs. Every time we say that F is a graph in G we mean that F is an induced subgraph of G.

Given a graph G, let A and B be two disjoint subsets of $V(G)$. Then A is *complete* to B if every vertex of A is adjacent to every vertex of B, and A is *anticomplete* to B if no vertex of A is adjacent to a vertex of B. Given a set $A \subset V(G)$ and a vertex $x \in V(G) \setminus A$, we say that x is complete (resp. anticomplete) to A if x is adjacent (resp. non-adjacent) to every vertex of A.

In a graph G, a *clique* is a (possibly empty) subset of $V(G)$ consisting of pairwise adjacent vertices. The size of a largest clique of G is denoted by $\omega(G)$. A *complete graph* is a graph whose vertex set is a clique. A complete graph on n vertices is denoted by K_n, and a K_3 is also referred to as a *triangle*.

A *stable set* of a graph G is a set of vertices of G, no two of which are adjacent. A graph is *bipartite* if its vertex set can be partitioned into two (possibly empty) stable sets. A *complete bipartite graph* is a graph whose vertex set can be partitioned into two (possibly empty) stable sets that are complete to each other.

A *path* (resp. *chordless path*) $P = x_1, \dots, x_k$ is a graph with vertex set $V(P) = \{x_1, \dots, x_k\}$ (where $k \geqslant 1$, and $x_i \neq x_j$ for all $1 \leqslant i < j \leqslant k$) and edge set $E(P) \supseteq \{x_1x_2, x_2x_3, \dots, x_{k-1}x_k\}$ (resp. $E(P) = \{x_1x_2, x_2x_3, \dots, x_{k-1}x_k\}$). The *endpoints* of P are x_1 and x_k (if $k = 1$, then the endpoints of P coincide), and P is said to be an *x_1x_k-path*, or a path *between x_1 and x_k*. The vertices in $V(P)$ that are not endpoints of P are called the *interior vertices* of P; note that P has interior vertices if and only if $k \geqslant 3$. Also, we set the *length* of P to be equal to $k-1$. Let x_i and x_j be any two vertices of P such that $1 \leqslant i \leqslant j \leqslant k$. Then the path $x_i, x_{i+1}, \dots, x_{j-1}, x_j$ is called the *x_ix_j-subpath of* P and is denoted by $P^{x_ix_j}$. Given an x_1x_k-path P and a subset $S \subseteq V(P)$, we say that a vertex $u \in S$ is *closest* to x_1 if $V(P^{x_1u}) \cap S = \{u\}$.

A *cycle* (resp. *chordless cycle*) $C = x_1, \dots, x_k, x_1$ is a graph with vertex set $V(C) = \{x_1, \dots, x_k\}$ (where $k \geqslant 3$, and $x_i \neq x_j$ for all $1 \leqslant i < j \leqslant k$) and edge set $E(C) \supseteq \{x_1x_2, \dots, x_{k-1}x_k, x_kx_1\}$ (resp. $E(C) = \{x_1x_2, \dots, x_{k-1}x_k, x_kx_1\}$). We say that the *length* of the cycle C is k. A *hole* is a chordless cycle of length at least four, and a graph is *chordal* if it is hole-free. A hole is *even* if its length is even, and it is *odd* otherwise.

Given a graph G, a subset S of vertices and/or edges of G is a *cutset* if its removal results in a disconnected graph. A cutset S is a *clique cutset* of G if S is a clique of G. Note that a graph with no clique cutset is connected. A cutset S is a *star cutset* of G if, for some vertex $x \in S$, $S \subseteq N_G[x]$.

A *wheel* (H, x) is a graph that consists of a hole H, called the *rim*, and a vertex x, called the *center*, that has at least three neighbors in H. A *sector* of a wheel is a subpath of the rim, of length at least one, whose endpoints are adjacent to the center but whose interior vertices are not. A sector is said to be *short* if it is of length one, and *long* otherwise.

Throughout the chapter, when we refer to a wheel (H, x), we will use the following associated terminology and notation. Let $x_1, \dots, x_n$ be the neighbors of x on H, appearing in this order when traversing H. For every $1 \leqslant i \leqslant n$, the sector of (H, x) with endpoints x_i and x_{i+1} (we assume that $x_{n+1} = x_1$) will be denoted by S_i (and throughout we will also assume that $S_{n+1} = S_1$). If S_i is a long sector, then we denote by x'_i (resp. x'_{i+1}) the neighbor of x_i (resp. x_{i+1}) in S_i. (We observe that the wheels in the class we will work with in this chapter do not have consecutive long sectors, and hence x'_i and x'_{i+1} are well defined). Also, for a long sector S_i, the hole induced by $V(S_i) \cup \{x\}$ will be denoted by H_i.

For a positive integer k, a *k-coloring* of a graph G is a function $c : V(G) \to \{1, \dots, k\}$ such that $c(x) \neq c(y)$ whenever $xy \in E(G)$; elements of $\{1, \dots, k\}$ are called *colors*. A graph is *k-colorable* if it admits a k-coloring. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest integer k such that G is k-colorable.

3.1.2 Truemper configurations

The first three graphs in Figure 1.1 are referred to as *three-path-configurations* ($3PC$'s). They are graphs consisting of three (chordless) paths, say P_1, P_2 and P_3, such that $V(P_i) \cup V(P_j)$ induces a hole for every $1 \leqslant i < j \leqslant 3$. More specifically, a $3PC(x, y)$ is a graph that consists of three paths that connect two non-adjacent vertices x and y; a $3PC(x_1x_2x_3, y)$, where $\{x_1, x_2, x_3\}$ induces a triangle, is a graph that consists of three paths having endpoints x_1, x_2 and x_3 respectively and a common endpoint $y \notin \{x_1, x_2, x_3\}$; a $3PC(x_1x_2x_3, y_1y_2y_3)$, where $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ induce two vertex-disjoint triangles, is a graph that consists of three paths P_1, P_2 and P_3 such that, for all $1 \leqslant i \leqslant 3$, path P_i has endpoints x_i and y_i. We say that a graph G contains a $3PC(\cdot, \cdot)$ if it contains a 3PC(x, y) for some $x, y \in V(G)$, a $3PC(\triangle, \cdot)$ if it contains a 3PC$(x_1x_2x_3, y)$ for some $x_1, x_2, x_3, y \in V(G)$, and a $3PC(\triangle, \triangle)$ if it contains a 3PC$(x_1x_2x_3, y_1y_2y_3)$ for some $x_1, x_2, x_3, y_1, y_2, y_3 \in V(G)$. Observe that all paths of a 3PC$(\cdot, \cdot)$ have length greater than one. Also, the condition that the vertices of P_i and P_j, for $i \neq j$, induce a hole, implies that at most one path of a 3PC$(\triangle, \cdot)$ has length one. In literature, a 3PC$(\cdot, \cdot)$ is also referred to as a *theta*, a 3PC$(\triangle, \cdot)$ as a *pyramid*, and a 3PC$(\triangle, \triangle)$ as a *prism*.

We refer to three-path-configurations and wheels as *Truemper configurations* (TC's).

A wheel is a *1-wheel* if, for some consecutive vertices x, y, z of the rim, the center is adjacent to y, and non-adjacent to x and z. A wheel is a *2-wheel* if, for some consecutive vertices x, y, z of the rim, the center is adjacent to x and y, and non-adjacent to z. A wheel is a *3-wheel* if, for some consecutive vertices x, y, z of the rim, the center is adjacent to x, y and z. It is easy to see that every wheel is a 1-wheel, a 2-wheel or a 3-wheel. Also, observe that a wheel can simultaneously be a 1-wheel, a 2-wheel and a 3-wheel; all the possibilities are illustrated with examples in the Venn diagram below (see Figure 3.1).

An *alternating wheel* is a wheel with an even number of sectors, and whose sectors alternate between short and long sectors. A *line wheel* is an alternating wheel with exactly two long sectors and two short ones. A *long alternating wheel* is an alternating wheel that is not a line wheel.

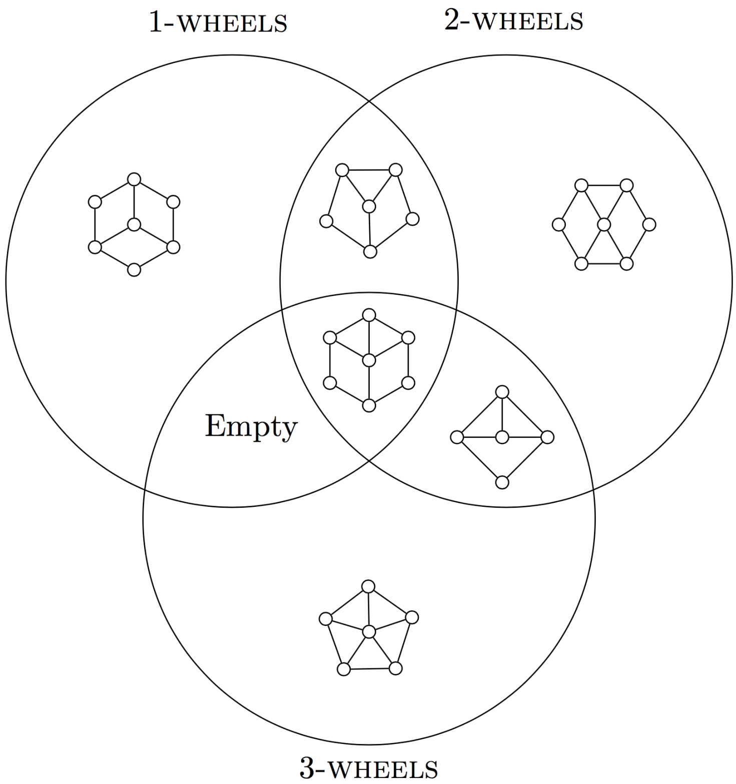

Figure 3.1: 1-wheels, 2-wheels and 3-wheels

From now on, we will denote by $\mathcal{G}$ the class of (theta, pyramid, 1-wheel, 3-wheel)-free graphs. Observe that the only Truemper configurations that these graphs may contain are prisms and alternating wheels.

3.1.3 Some subclasses of $\mathcal{G}$

The class $\mathcal{G}$ clearly contains all chordal graphs. We will now describe some other subclasses of $\mathcal{G}$ that were studied in literature.

Let G be a graph, and x and y two non-adjacent vertices of G. The *separability of x and y* is the minimum cardinality of a set $S \subset V(G)$ such that x and y are in different components of $G \setminus S$. The *separability of G* is the maximum over all separabilities of pairs of non-adjacent vertices of G (unless G is complete, in which case the graph has separability 0). So, the graphs of separability at most k are precisely the graphs in which every two non-adjacent vertices can be separated by removing a set of at most k other vertices. By Menger's Theorem, the separability of G is equal to the maximum number of internally vertex-disjoint paths connecting two non-adjacent vertices. Graphs of separability at most two were studied in [11], where the following characterization is obtained (together with a number of other properties of this class). K_5^- is the graph obtained from a K_5 by removing a single edge.

Theorem 73. **([11])** *A graph is of separability at most two if and only if it is (K_5^-, TC)-free.*

Let γ be a $\{0,1\}$-vector whose entries are in one-to-one correspondence with the holes of a given graph G. Then G is *universally signable* if, for all choices of vector γ, there exists a subset F of the edge set of G such that $|F \cap E(H)| \equiv \gamma_H \pmod 2$ for every hole H in G, and $|F \cap E(T)| \equiv 1 \pmod 2$ for every triangle T in G. By the above mentioned theorem of Truemper [45], it is easy to obtain the following characterization of universally signable graphs in terms of forbidden induced subgraphs.

Theorem 74. **([14])** *A graph is universally signable if and only if it is TC-free.*

This characterization of universally signable graphs can be then used to obtain the following decomposition theorem, which generalizes the classical decomposition of chordal graphs with clique cutsets.

Theorem 75. **([14])** *A universally signable graph is either a complete graph or a hole, or it admits a clique cutset.*

Clique cutsets have been studied extensively in literature [43], and it is well understood how to use them in algorithms. So, in particular, Theorem 75 implies efficient algorithms for recognition of universally signable graphs, and for optimal coloring, maximum clique and maximum stable set problems on this class.

As already observed, the only Truemper configurations that graphs in $\mathcal{G}$ may contain are prisms and alternating wheels. Graphs that may only contain prisms (and no other TC's) were studied in [16], where the following decomposition theorem is obtained. Given a graph G, its *line graph* $L(G)$ is a graph such that each vertex of $L(G)$ represents an edge of G and two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint in G. A graph is *chordless* if all of its cycles are chordless.

Theorem 76. **([16])** *If G is a (theta, pyramid, wheel)-free graph, then G is the line graph of a triangle-free chordless graph or it admits a clique cutset.*

A *claw* is the complete bipartite graph with three vertices on one side of the bipartition and one vertex on the other. A *diamond* is the graph on four vertices that has exactly one pair of non-adjacent vertices. Note that the class of (claw, diamond)-free graphs is a subclass of $\mathcal{G}$.

Theorem 77. **([22])** *A graph G is the line graph of a triangle-free graph if and only if G is (claw, diamond)-free.*

By Theorem 77, the class of line graphs of triangle-free graphs is a subclass of $\mathcal{G}$. The main result in this chapter is to show that graphs in $\mathcal{G}$ that are not line graphs of triangle-free graphs have a particular structure.

3.1.4 Recognizing Truemper configurations

A natural question to ask is whether Truemper configurations can be recognized in polynomial time. These questions in fact arose when studying how to recognize even-hole-free graphs and perfect graphs in polynomial time. Observe that, if a graph contains a prism or a theta, then it must contain an even hole, and, if it contains a pyramid, then it must contain an odd hole. In fact, the class of even-hole-free graphs is included in the class of (theta, prism, even wheel)-free graphs (where an *even wheel* is a wheel with an even number of sectors), and the class of odd-hole-free graphs, and hence of perfect graphs, is included in the class of (pyramid, odd wheel)-free graphs (where an *odd wheel* is a wheel with an odd number of short sectors). We now want to briefly describe different general techniques that were developed when trying to recognize whether a graph contains a particular Truemper configuration.

In [7] it is shown that detecting whether a graph contains a pyramid can be done in $\mathcal{O}(n^9)$ time. This algorithm is based on the *shortest-paths detector* technique developed by Chudnovsky and Seymour. The idea of their algorithm is as follows. If a graph G contains a pyramid, then it contains a pyramid Σ with fewest number of vertices. The algorithm "guesses" some vertices of Σ, and then finds shortest paths in G between the guessed vertices that are joined by a path in Σ. If the graph induced by the union of these paths is a pyramid, then clearly G contains a pyramid. If it is not, then it turns out that G is pyramid-free.

Chudnovsky and Seymour [10] show that detecting whether a graph contains a theta can be done in $\mathcal{O}(n^{11})$ time. For this detection problem, the shortest-paths detector technique does not work. The detection of thetas relies on being able to solve a more general problem called the *three-in-a-tree problem*, which is defined as follows: given a graph G and three specified vertices a, b and c, the question is whether G contains a tree that passes through a, b and c. It is shown in [10] that this problem can be solved in $\mathcal{O}(n^4)$ time. What is interesting is that the algorithm for the three-in-a-tree problem is based on an explicit construction of the cases when the desired tree does not exist, and that this construction can be directly converted into an algorithm. The three-in-a-tree algorithm is quite general, and can be used to solve different detection problems, including the detection of a theta, and of a pyramid (the latter in $\mathcal{O}(n^{10})$ time).

Maffray and Trotignon show that detecting whether a graph contains a prism is NP-complete [30]. Furthermore, detecting whether a graph contains a wheel is NP-complete too, as shown by Diot, Tavenas and Trotignon [17]. In fact they prove that the problem remains NP-complete even when restricted to bipartite graphs. Since all wheels in bipartite graphs are 1-wheels, it follows that recognizing whether a graph is 1-wheel-free is NP-complete. A number of other detection problems related to graph classes defined by excluding combinations of Truemper configurations have been studied in literature. In Section 3.2, we will give a polynomial-time recognition algorithm for $\mathcal{G}$.

3.1.5 The structure of graphs in $\mathcal{G}$

We say that a connected graph G is *structured* if there exists a partition

$$\mathcal{S} = (\{x\}, X_1, X_2, X_3, Y_1, Y_2, Y_3, C_1, C_2, C_3, C_X, C_Y)$$

of $V(G)$ that satisfies the following:

(i) For $1 \leqslant i \leqslant 2$, X_i, Y_i and C_i are all non-empty. There exist $x_1 \in X_1$, $x_2 \in X_2$ such that x_1 is complete to $X_2 \cup X_3$ and x_2 is complete to $X_1 \cup X_3$, and $y_1 \in Y_1$, $y_2 \in Y_2$ such that y_1 is complete to $Y_2 \cup Y_3$ and y_2 is complete to $Y_1 \cup Y_3$.

(ii) Let $X = X_1 \cup X_2 \cup X_3$ and $Y = Y_1 \cup Y_2 \cup Y_3$. Then x is complete to $X \cup Y$, X is anticomplete to Y, and, for all $1 \leqslant i, j \leqslant 3$, every vertex of $X_i \cup Y_i$ has a neighbor in C_j if and only if $i = j$.

(iii) For every $1 \leqslant i \leqslant 3$, X_i and Y_i are both cliques, and X_3 (resp. Y_3) is complete to $X_1 \cup X_2$ (resp. $Y_1 \cup Y_2$).

(iv) C_1, C_2, C_3, C_X and C_Y are pairwise anticomplete to each other.

(v) $N_G(C_X) \subseteq \{x\} \cup X$ and $N_G(C_Y) \subseteq \{x\} \cup Y$.

If G is structured, we also say that $\mathcal{S}$ is a *structured partition* of $V(G)$. We prove the following theorem.

Theorem 78. *If $G \in \mathcal{G}$ is not a line graph of a triangle-free graph and does not admit a clique cutset, then G is structured.*

The decomposition theorems that follow are immediate corollaries of Theorem 78.

A cutset S of a graph G is a *bisimplicial cutset* if, for some vertex $x \in S$, $S \subseteq N_G[x]$ and $S \setminus \{x\}$ is the disjoint union of two cliques of size at least two that are anticomplete to each other.

A *2-amalgam* (K, V_1, V_2) of a connected graph G is a partition of $V(G)$ into subsets V_1, V_2 and K such that, for every $1 \leqslant i \leqslant 2$, W_i and Z_i are disjoint non-empty subsets of V_i and the following hold:

- $V_i \setminus (W_i \cup Z_i) \neq \varnothing$ for every $1 \leqslant i \leqslant 2$.
- W_1 (resp. Z_1) is complete to W_2 (resp. Z_2), and there are no other edges between V_1 and V_2.
- K is a clique that is complete to $W_1 \cup W_2 \cup Z_1 \cup Z_2$.

Observe that the removal of K, together with the edges with one end in V_1 and the other in V_2, disconnects G. A 2-amalgam is called *special* if K consists of a single vertex, W_i and Z_i are cliques for every $1 \leqslant i \leqslant 2$, and W_1 (resp. W_2) is anticomplete to Z_1 (resp. Z_2). A 2-amalgam is *small* if it is special and $|W_i| = |Z_i| = 1$ for every $1 \leqslant i \leqslant 2$. Note that if G admits a special 2-amalgam, then it has a bisimplicial cutset that satisfies additional properties.

Theorem 79. *If $G \in \mathcal{G}$, then G is the line graph of a triangle-free graph or it admits a clique cutset or a bisimplicial cutset.*

Proof. If G is not the line graph of a triangle-free graph and does not admit a clique cutset, then, by Theorem 78, it is structured. First observe that, for $1 \leqslant i \leqslant 2$, $\{x\} \cup X_i \cup Y_i$ is a cutset of G separating C_i from the rest of the graph. Now suppose that $\{x\} \cup X_1 \cup Y_1$ is not a bisimplicial cutset. Then, w.l.o.g. $|X_1| = 1$. If $|Y_1| = 1$ or $|Y_2| = 1$, then $\{x\} \cup X_1 \cup X_2 \cup Y_1 \cup Y_2$

is a bisimplicial cutset. So, we may assume that $|Y_1| \geqslant 2$, $|Y_2| \geqslant 2$. But then $\{x\} \cup X_1 \cup X_2 \cup Y_1$ is a bisimplicial cutset. ■

Theorem 80. *If $G \in \mathcal{G}$ is a K_4-free graph, then G is the line graph of a triangle-free graph or it admits a clique cutset or a small 2-amalgam.*

Proof. Assume otherwise. By Theorem 78, G is structured. If G is K_4-free, then it must be that $|X_1| = |X_2| = |Y_1| = |Y_2| = 1$ and $X_3 \cup Y_3 = \varnothing$. Since $N_G(C_3) \subseteq \{x\}$ and G does not admit a clique cutset, $C_3 = \varnothing$. Also, since $X \cup \{x\}$ and $Y \cup \{x\}$ are both cliques, $C_X = C_Y = \varnothing$. So, if we define $K = \{x\}$, $W_1 = \{x_1\}$, $Z_1 = \{y_1\}$, $W_2 = \{x_2\}$, $Z_2 = \{y_2\}$, $V_1 = W_1 \cup Z_1 \cup C_1$ and $V_2 = W_2 \cup Z_2 \cup C_2$, then (K, V_1, V_2) is a small 2-amalgam of G, a contradiction. ■

Theorem 81. *If $G \in \mathcal{G}$ is a K_5^--free graph, then G is the line graph of a triangle-free graph or it admits a clique cutset or a special 2-amalgam.*

Proof. Assume otherwise. By Theorem 78, we know that G is structured. When G is K_5^--free, X and Y must both be cliques, and therefore $C_X = C_Y = \varnothing$. So, if $K = \{x\}$, $W_1 = X_1$, $Z_1 = Y_1$, $W_2 = X_2 \cup X_3$, $Z_2 = Y_2 \cup Y_3$, $V_1 = W_1 \cup Z_1 \cup C_1$ and $V_2 = W_2 \cup Z_2 \cup C_2 \cup C_3$, then (K, V_1, V_2) is a special 2-amalgam of G, a contradiction. ■

As intermediate results, we also prove the following three theorems. Let (H, x) be a wheel in a graph $G \in \mathcal{G}$. Then we say that a chordless path $P = p_1, \dots, p_k$, $k > 2$, in $G \setminus (V(H) \cup \{x\})$ is an *appendix* of (H, x) that attaches to S_i if, for some long sector S_i of (H, x), the following hold: $N_G(p_1) \cap (V(H) \cup \{x\}) = \{x, x_i\}$, $N_G(p_k) \cap (V(H) \cup \{x\}) = \{x_i, x'_i\}$ and $N_G(p_j) \cap (V(H) \cup \{x\}) \subseteq \{x_i\}$ for every $1 < j < k$.

Theorem 82. *If $G \in \mathcal{G}$ does not contain a wheel with an appendix or a long alternating wheel, then G is the line graph of a triangle-free graph or it admits a clique cutset.*

Theorem 83. *If $G \in \mathcal{G}$ does not contain a wheel with an appendix, then G is the line graph of a triangle-free graph, or it admits a clique cutset or a special 2-amalgam.*

Theorem 84. *If $G \in \mathcal{G}$ contains a wheel with an appendix or a long alternating wheel, then G admits a clique cutset or it is structured.*

Theorem 78 follows directly from Theorem 82 and Theorem 84. Theorem 82 is proved in Section 3.3, and Theorems 83 and 84 are proved in Section 3.4.

Finally, the following result is proved in Section 3.5.

Theorem 85. *If $G \in \mathcal{G}$ is a K_4-free graph, then G is 4-colorable.*

3.2 Recognizing graphs in $\mathcal{G}$

Throughout this section, for a graph G we let $n = |V(G)|$ and $m = |E(G)|$. We now give two polynomial-time algorithms that decide whether an input graph G belongs to $\mathcal{G}$. The first algorithm is obtained by a direct search for certain Truemper configurations, so, although it is slower than the second one, we believe that it is of independent interest. The second algorithm has running time $\mathcal{O}(n^5)$ and is based on the description of the local structure of graphs in $\mathcal{G}$ that is obtained in [1]. Both of these algorithms do not rely on our main decomposition theorem for $\mathcal{G}$ (Theorem 79).

In [31], Maffray, Trotignon and Vušković give an $\mathcal{O}(n^7)$-time algorithm that decides whether a graph contains a theta or a pyramid. Recall that deciding whether a graph contains a 1-wheel is NP-complete [17]. In Lemma 86, we give an $\mathcal{O}(n^6)$-time algorithm that decides whether a graph contains a theta, a pyramid or a 1-wheel. In Lemma 87, we give an $\mathcal{O}(n^6)$-time algorithm that decides whether a graph contains a 3-wheel. These two algorithms together provide our first recognition algorithm for $\mathcal{G}$.

Lemma 86. *There is an algorithm with the following specifications:*

- *Input: A graph G.*
- *Output:* YES *if G contains a theta, a pyramid or a 1-wheel, and* NO *otherwise.*
- *Running time: $\mathcal{O}(n^4m + n^5)$.*

PROOF. Consider the following algorithm.

Step 1: Let $\mathcal{L}$ be the set of all 4-element subsets of $V(G)$.

Step 2: If $\mathcal{L} = \varnothing$, then return NO. Otherwise, take $S \in \mathcal{L}$ and remove S from $\mathcal{L}$.

Step 3: If S does not induce a claw, go to Step 2. Otherwise, let $S = \{u, a, b, c\}$ be such that u is complete to $\{a, b, c\}$.

Step 4: If there exists a component C of $G \setminus N_G[u]$ such that all of a, b and c have a neighbor in C, then return YES. Otherwise, go to Step 2.

Since $\mathcal{L}$ has $\mathcal{O}(n^4)$ elements, and Step 4 takes $\mathcal{O}(n+m)$ time, the running time of this algorithm is $\mathcal{O}(n^4m + n^5)$.

Let us now prove its correctness. First suppose that, for some component C of $G \setminus N_G[u]$ (in Step 4), all of a, b and c have a neighbor in C, and let C' be a minimal connected induced subgraph of C such that all of a, b and c have a neighbor in C'. Let P be a chordless ac-path in the graph induced by $V(C') \cup \{a, c\}$. If b has a neighbor in P, then $V(P) \cup \{u, b\}$ induces a theta or a 1-wheel. Otherwise, let Q be a chordless bv-path in $G[V(C') \cup \{b\}]$ such that v has a neighbor in $P \setminus \{a, c\}$ and no vertex of $Q \setminus \{v\}$ has a neighbor in $P \setminus \{a, c\}$. By the minimality of C', not both a and c can have a neighbor in Q, and v has one or two (adjacent) neighbors in P. So, w.l.o.g. $N_G(c) \cap V(Q) = \varnothing$. Let H be the hole contained in $G[(V(P) \setminus \{a\}) \cup V(Q) \cup \{u\}]$ that contains Q, u and c. If a has at least three neighbors in H, then (H, a) is a 1-wheel. If a has exactly two neighbors in H, then $V(H) \cup \{a\}$ induces a theta. So, we may assume that a has no neighbors in $H \setminus \{u\}$. But then the graph induced by $V(P) \cup V(Q) \cup \{u\}$ is either a theta or a pyramid. It follows that the algorithm correctly returns YES in Step 4.

Let us now assume that the output is NO, but that G contains a theta, a pyramid or a 1-wheel D. Let $\{u, a, b, c\}$ induce a claw contained in D and let u be complete to $\{a, b, c\}$. Additionally, in case D is a 1-wheel, then w.l.o.g. we may assume that b is its center. Clearly, a component of $G \setminus N_G[u]$ has neighbors from all of a, b and c, and hence the algorithm returns YES in Step 4, a contradiction. This proves the lemma. ■

Lemma 87. *There is an algorithm with the following specifications:*

- *Input: A graph G.*
- *Output:* YES *if G contains a 3-wheel, and* NO *otherwise.*
- *Running time: $\mathcal{O}(n^4 m + n^5)$.*

PROOF. Consider the following algorithm.

Step 1: Let $\mathcal{L}$ be the set of all 4-element subsets of $V(G)$.

Step 2: If $\mathcal{L} = \varnothing$, then return NO. Otherwise, take $S \in \mathcal{L}$ and remove S from $\mathcal{L}$.

Step 3: If S does not induce a diamond, go to Step 2. Otherwise, let $S = \{a, b, x, y\}$ be such that ab is not an edge.

Step 4: Let $N_x = N_G[x] \setminus \{a, b\}$ and $N_y = N_G[y] \setminus \{a, b\}$. If a and b are in the same component of $G \setminus N_x$, or in the same component of $G \setminus N_y$, then return YES. Otherwise, go to Step 2.

Since $\mathcal{L}$ has $\mathcal{O}(n^4)$ elements, and Step 4 takes $\mathcal{O}(n+m)$ time, the running time of this algorithm is $\mathcal{O}(n^4 m + n^5)$.

Let us now prove that the algorithm is correct. First, if the output is YES, then G contains a 3-wheel with center x or y. Indeed, if a and b are in the same component C of $G \setminus N_z$, for some $z \in \{x, y\}$, then a shortest path between a and b in C, together with $\{x, y\}$, induces a 3-wheel with center t, where $t \in \{x, y\} \setminus \{z\}$.

Let us now assume that the output is NO, but that G contains a 3-wheel. Let (H, x) be this 3-wheel, and let $a, b, y \in N_G(x) \cap V(H)$ be such that a and b are distinct neighbors of y. Then the vertex set $\{a, b, x, y\}$ induces a diamond, and a and b are in the same component of $G \setminus N_y$. Therefore, the algorithm returns YES in Step 4, a contradiction. This proves the lemma. ■

To describe our second algorithm, we first recall some definitions from [1]. Let $\mathcal{F}$ be a (possibly infinite) family of graphs. A graph G is *locally $\mathcal{F}$-decomposable* if, for every vertex $x \in V(G)$ such that $V(G) \setminus N_G[x] \neq \varnothing$, and every component C of $G \setminus N_G[x]$, if a graph $F \in \mathcal{F}$ is in $G[N_G(x)]$ then there exists a vertex of F that has a non-neighbor in F and no neighbors in C. A class of graphs is said to be *locally $\mathcal{F}$-decomposable* if every graph in the class is locally $\mathcal{F}$-decomposable.

Let S_3 be the graph whose vertex set is a stable set of size three, and let P_3 be the chordless path on three vertices. The following theorem is the key for our second algorithm.

Theorem 88. **([1])** *The class $\mathcal{G}$ is exactly the class of locally (S_3, P_3)-decomposable graphs.*

Theorem 89. *There is an algorithm with the following specifications:*

- *Input: A graph G.*
- *Output:* YES *if $G \in \mathcal{G}$, and* NO *otherwise.*
- *Running time: $\mathcal{O}(n^5)$.*

Proof. Consider the following algorithm.

Step 1: Let $\mathcal{L} = V(G)$.

Step 2: If $\mathcal{L} = \varnothing$, then return YES. Otherwise, take $x \in \mathcal{L}$ and remove x from $\mathcal{L}$. Let $\mathcal{L}_x$ be the set of all 3-element subsets of $N_G(x)$, and let $\mathcal{C}_x$ be the set of all the components of $G \setminus N_G[x]$.

Step 3: If $\mathcal{L}_x = \varnothing$ or $\mathcal{C}_x = \varnothing$, then go to Step 2. Else, take $S \in \mathcal{L}_x$ and remove S from $\mathcal{L}_x$.

Step 4: If S does not induce a S_3 or a P_3, then go to Step 3. Otherwise, for every $C \in \mathcal{C}_x$, compute the sets $N_G(y) \cap V(C)$ and $N_G(y) \cap S$ for all $y \in S$. If the first set is empty and the second is not equal to $S \setminus \{y\}$ for some $y \in S$, then go to Step 3. Else, return NO.

The correctness of the algorithm follows immediately from Theorem 88. Let $x \in \mathcal{L}$. The set $\mathcal{L}_x$ has $\mathcal{O}(n^3)$ elements and $\mathcal{C}_x$ can be found in time $\mathcal{O}(n+m)$, so Step 2 takes $\mathcal{O}(n^2)$ time (for each $x \in \mathcal{L}$). Step 4 takes $\mathcal{O}(n)$ time (since $|\bigcup_{C \in \mathcal{C}_x} V(C)| < n$ and $|S| = 3$), and so the running time of the algorithm is $\mathcal{O}(n \cdot n^3 \cdot n) = \mathcal{O}(n^5)$. ■

3.3 Proof of Theorem 82

The following easy observation will be used throughout the chapter.

Lemma 90. *Let $G \in \mathcal{G}$ and let H be a hole in G. If $x \in V(G) \setminus V(H)$ has at least two non-adjacent neighbors in H, then (H, x) is an alternating wheel.*

Proof. If x has exactly two neighbors in H, and they are not adjacent, then $G[V(H) \cup \{x\}]$ is a theta. So, assume that x has at least three neighbors in H. Then (H, x) is a wheel, and hence an alternating wheel. ■

Theorem 82 immediately follows from Theorem 77 and from the two results below, whose proof is postponed to Section 3.3.1 and Section 3.3.2, respectively.

Theorem 91. *Assume that $G \in \mathcal{G}$ does not contain a wheel with an appendix. If G contains a diamond, then it admits a clique cutset.*

Theorem 92. *Assume that $G \in \mathcal{G}$ is a diamond-free graph that does not contain a long alternating wheel. If G contains a claw, then it admits a clique cutset.*

3.3.1 Proof of Theorem 91

In order to prove Theorem 91, it is convenient to work with *expanded diamonds*. Given a graph G, let $K = \{v_1, \ldots, v_\ell\}$, $\ell \geqslant 2$, be a clique of G of size ℓ. An expanded diamond $D = (K, x, y)$ of G is an induced subgraph of G with vertex set $V(D)$ given by the disjoint union of K and $\{x, y\}$, and such that x and y are distinct, non-adjacent and both complete to K. We say that an expanded diamond D of G is *maximum* if G does not contain an expanded diamond with more vertices. The above terminology and notation will be used throughout. Note that D is a diamond when $\ell = 2$.

Lemma 93. *Let $D = (K, x, y)$ be a maximum expanded diamond of a graph $G \in \mathcal{G}$. Then, for every vertex $u \in V(G) \setminus V(D)$, $N_G(u) \cap V(D)$ is a clique of size at most $\ell + 1$.*

Proof. Assume otherwise. Then $x, y \in N_G(u) \cap V(D)$. Since, for every $1 \leqslant i, j \leqslant \ell$, $i \neq j$, $\{x, y, v_i, v_j, u\}$ cannot induce a 3-wheel, u is complete to $V(D)$. Let $K' = K \cup \{u\}$. Then the expanded diamond induced by $K' \cup \{x, y\}$ contradicts the maximality of D. ■

Proof of Theorem 91. Let $D = (K, x, y)$ be a maximum expanded diamond of G. We prove that K is a clique cutset of G separating x from y. Assume not, and let $Q = q_1, \ldots, q_r$ be a shortest path in $G \setminus V(D)$ such that q_1 (resp. q_r) is adjacent to x (resp. y). By Lemma 93, $r \geqslant 2$. By the minimality of Q, Q is chordless, no vertex of $Q \setminus \{q_1\}$ is adjacent to x, and no vertex of $Q \setminus \{q_r\}$ is adjacent to y, so that $N_G(q_i) \cap V(D) \subseteq K$ for every $1 < i < r$.

Since the graph induced by $V(Q) \cup V(D)$ cannot contain a 3-wheel, v_i has a neighbor in Q for every $1 \leqslant i \leqslant \ell$. Let q_j be the vertex of Q with lowest index that has a neighbor in K. W.l.o.g. let $v_1 q_j \in E(G)$.

(1) q_j is not complete to K.

Proof of (1). Suppose it is. If $j > 1$, then $V(Q^{q_1 q_j}) \cup \{x, v_2\}$ induces a hole H and (H, v_1) is a 3-wheel, a contradiction. So, $j = 1$.

We now prove that q_2 is complete to K. Assume otherwise, and w.l.o.g. suppose that $v_2 q_2$ is not an edge. Since $V(Q) \cup \{y, v_1, v_2\}$ cannot induce a 3-wheel, both v_1 and v_2 have a neighbor in $Q^{q_2 q_r}$. Let q_k (resp. q_h) be the vertex of $Q^{q_2 q_r}$ with lowest index that is adjacent to v_1 (resp. v_2). If $k = h$, then $k > 2$, and hence $V(Q^{q_1 q_k}) \cup \{v_1, v_2\}$ induces a 3-wheel, a contradiction. So w.l.o.g. $k < h$. Let H be the hole induced by $V(Q^{q_1 q_h}) \cup \{v_2\}$. Then, by Lemma 90, (H, v_1) is an alternating wheel, and hence v_1 is not adjacent to q_h. Let q_m be the neighbor of v_1 in $Q^{q_1 q_h}$ with highest index. Observe that $k < m < h$. Suppose that $N_G(v_1) \cap V(Q^{q_{h+1} q_r}) = \varnothing$. Since, by Lemma 90, $V(Q^{q_m q_r}) \cup \{y, v_1, v_2\}$ must induce an alternating wheel with center v_2, we have that $h < r - 1$ and v_2 is adjacent to q_{h+1}. It follows that the chordless path induced by $V(Q^{q_{h+1} q_r}) \cup \{y\}$ is an appendix of (H, v_1), a contradiction. So, let q_s be the neighbor of v_1 in $Q^{q_{h+1} q_r}$ with lowest index and let H' be the hole induced by $V(Q^{q_m q_s}) \cup \{v_1\}$. By Lemma 90, (H', v_2) is an alternating wheel. Also, $s > h + 2$ and q_{h+1} and q_s are both adjacent to v_2. But then $Q^{q_{h+1} q_s}$ is an appendix of (H, v_1), a contradiction. So, q_2 is complete to K.

Now let $K' = K \cup \{q_1\}$. Since q_1 is complete to K, K' is a clique. Further, q_2 is complete to

K', and hence the expanded diamond induced by $K' \cup \{x, q_2\}$ contradicts the maximality of D. This proves (1). $\square$

By (1), w.l.o.g. $v_2 q_j \notin E(G)$. Let q_k be the neighbor of v_2 in $Q^{q_{j+1}q_r}$ with lowest index. Then $V(Q^{q_1 q_k}) \cup \{x, v_2\}$ induces a hole H, and, by Lemma 90, (H, v_1) is an alternating wheel. So, $j > 1$, $k > j+1$ and v_1 is adjacent to q_{j+1} and not adjacent to q_k. For some $j+1 \leqslant h < k$, let q_h be the neighbor of v_1 in Q with highest index.

First suppose that v_1 has no neighbors in $Q^{q_{k+1}q_r}$. By Lemma 90, $V(Q^{q_h q_r}) \cup \{y, v_1, v_2\}$ must induce an alternating wheel with center v_2, and hence $k < r-1$ and v_2 is adjacent to q_{k+1}. But then the chordless path induced by $V(Q^{q_{k+1}q_r}) \cup \{y\}$ is an appendix of (H, v_1), a contradiction.

Therefore, let q_m be the neighbor of v_1 in $Q^{q_{k+1}q_r}$ with lowest index and let H' be the hole induced by $V(Q^{q_h q_m}) \cup \{v_1\}$. By Lemma 90, (H', v_2) is an alternating wheel. So, $m > k+2$ and q_{k+1} and q_m are both adjacent to v_2. But then $Q^{q_{k+1}q_m}$ is an appendix of (H, v_1), a contradiction. This proves the theorem. $\blacksquare$

3.3.2 Proof of Theorem 92

In order to prove Theorem 92, we need some preliminary results. Throughout this section, we assume that $G \in \mathcal{G}$ is a diamond-free graph that does not contain a long alternating wheel.

Extended triangles.

An *extended triangle* $E = (K, S_1, S_2)$ of G is an induced subgraph of G defined as follows: $K = \{u, v_1, v_2\}$ is a clique of G of size three, and $S_1 = x_1, \dots, x_n$ and $S_2 = y_1, \dots, y_m$ are vertex-disjoint chordless paths in $G \setminus K$ such that

- $N_G(x_1) \cap (V(S_2) \cup K) = \{u\}$, $N_G(x_n) \cap (V(S_2) \cup K) = \{v_1\}$ and $N_G(x_i) \cap (V(S_2) \cup K) = \varnothing$ for every $1 < i < n$.
- $N_G(y_1) \cap (V(S_1) \cup K) = \{u\}$, $N_G(y_m) \cap (V(S_1) \cup K) = \{v_2\}$ and $N_G(y_i) \cap (V(S_1) \cup K) = \varnothing$ for every $1 < i < m$.

Note that, since G is diamond-free, $m, n \geqslant 2$. For $1 \leqslant i \leqslant 2$, let H_i be the hole induced by $V(S_i) \cup \{u, v_i\}$. We say that an extended triangle E of G is *minimum* if G does not contain an extended triangle with a smaller number of vertices. This terminology and notation will be used in Lemma 94 and Lemma 95.

Lemma 94. *Let $E = (K, S_1, S_2)$ be a minimum extended triangle of G. For every vertex $z \in V(G) \setminus V(E)$, either z has at most one neighbor in K or z is complete to K. Also, one of the following holds:*

(i) $N_G(z) \cap V(E)$ is a clique of size at most three.

(ii) $N_G(z) \cap V(E) = \{v_i, w_1, w_2\}$, where $1 \leqslant i \leqslant 2$ and w_1, w_2 are adjacent vertices of S_{3-i}.

Proof. Assume otherwise. Since G is diamond-free, either z has at most a single neighbor in K or z is complete to K. W.l.o.g. we may assume that z has a neighbor in S_1. Suppose that $N_G(z) \cap V(E) \subseteq V(H_1)$. Since (i) does not hold, (H_1, z) is a line wheel by Lemma 90. But then $G[V(E) \cup \{z\}]$ contains an extended triangle with fewer vertices than E, a contradiction. It follows that z has a neighbor in $H_2 \setminus \{u\}$.

Suppose that $v_2 \in N_G(z) \cap V(E) \subseteq V(S_1) \cup K$. If z has a single neighbor in H_1, then such a neighbor belongs to S_1 and $V(H_1) \cup \{z, v_2\}$ induces a pyramid. So, since (ii) does not hold, (H_1, z) is a line wheel by Lemma 90. Then z is complete to K, since otherwise the graph induced by $V(H_1) \cup \{z, v_2\}$ contains a 3PC(uv_1v_2, z). Let x_i be the neighbor of z in S_1 with lowest index. Then $(K \setminus \{v_1\}) \cup V(S_1^{x_1x_i}) \cup V(S_2) \cup \{z\}$ induces an extended triangle that contradicts our choice of E.

Therefore z has a neighbor in both S_1 and S_2. Let x_j (resp. y_k) be the neighbor of z in S_1 (resp. S_2) with highest index. Assume that zu is an edge but z is not complete to K. Then $v_1, v_2 \notin N_G(z)$, and hence, by Lemma 90, $x_1, y_1 \in N_G(z)$. So, $\{u, z, x_1, y_1\}$ induces a diamond, a contradiction. If z is complete to K, then, by Lemma 90, (H_1, z) and (H_2, z) are both line wheels and $(K \setminus \{u\}) \cup V(S_1^{x_jx_n}) \cup V(S_2^{y_ky_m}) \cup \{z\}$ induces an extended triangle with a smaller number of vertices. It follows that z is not adjacent to u. Let x_h (resp. y_ℓ) be the neighbor of z in S_1 (resp. S_2) with lowest index, and let $H = V(H_1) \cup V(S_2^{y_1y_\ell}) \cup \{z\}$. If $V(H_1) \cup \{z\}$ induces a line wheel, then $G[H]$ contains a 3PC(u, z). If z has two neighbors in H_1 and these neighbors are adjacent, then H induces a pyramid. So, by Lemma 90, and by symmetry, x_h and y_ℓ are the only neighbors of z in E. If $h > 1$, then H induces a 3PC(u, x_h). If $h = 1$, then $V(H_1) \cup V(S_2^{y_\ell y_m}) \cup \{z, v_2\}$ induces a 1-wheel with center u, a contradiction. ■

Lemma 95. *If G contains an extended triangle, then it admits a clique cutset.*

Proof. Let $E = (K, S_1, S_2)$ be a minimum extended triangle of G, and let W be the set of vertices of $G \setminus V(E)$ that are complete to K. We prove that $K \cup W$ is a clique cutset of G separating S_1 from S_2. First consider the following claim.

(1) $K \cup W$ is a clique of G.

Proof of (1). Assume not. Then there exist two vertices $w_1, w_2 \in W$, $w_1 \neq w_2$, such that w_1w_2 is not an edge. It follows that $\{v_1, v_2, w_1, w_2\}$ induces a diamond, a contradiction. □

By (1), we only need to show that $K \cup W$ is a cutset of G separating S_1 from S_2. Assume otherwise, and let $Q = q_1, \dots, q_r$ be a shortest path in $G \setminus (K \cup W)$ such that q_1 (resp. q_r) has a neighbor in S_1 (resp. S_2). By Lemma 94, $r \geqslant 2$. By the minimality of Q, Q is chordless and no vertex of $Q \setminus \{q_1\}$ (resp. $Q \setminus \{q_r\}$) has a neighbor in S_1 (resp. S_2), and so $N_G(q_i) \cap V(E) \subseteq K$ for every $1 < i < r$. By the definition of Q and Lemma 94, every vertex of Q has at most one neighbor in K.

(2) q_1 (resp. q_r) has a single neighbor in S_1 (resp. S_2).

Proof of (2). Suppose that q_1 has two adjacent neighbors in S_1 and no other neighbors in H_1. Let y_i be the neighbor of q_r in S_2 with lowest index. If u and v_1 do not have a neighbor in

$Q \setminus \{q_1\}$, then $V(H_1) \cup V(S_2^{y_1 y_i}) \cup V(Q)$ induces a pyramid. So, for some $1 < j \leqslant r$, let q_j be the vertex of Q with lowest index that is adjacent to a vertex of $\{u, v_1\}$. But then $V(H_1) \cup V(Q^{q_1 q_j})$ induces a pyramid, a contradiction. By Lemma 94, it follows that q_1 has a single neighbor in S_1, and, by symmetry, q_r has a single neighbor in S_2. $\square$

By Lemma 94 and (2), $N_G(q_1) \cap V(E) \subset V(H_1)$ and $N_G(q_r) \cap V(E) \subset V(H_2)$ are both cliques of size at most two. Assume that K is not anticomplete to $V(Q)$, and let q_i (resp. q_j) be the vertex of Q with lowest (resp. highest) index that has a neighbor in K.

(3) q_i and q_j are adjacent to u.

Proof of (3). Suppose that q_i is not adjacent to u. If q_i is adjacent to v_2, then $i > 1$, and, by (2), $V(H_1) \cup V(Q^{q_1 q_i}) \cup \{v_2\}$ induces a pyramid, a contradiction. So, q_i is adjacent to v_1. If $x_n q_1$ is not an edge, then, by (2), $V(H_1) \cup V(Q^{q_1 q_i})$ induces a theta. So, q_1 is adjacent to x_n and has no other neighbors in S_1. Let R be a chordless uq_i-path contained in the graph induced by $V(S_2) \cup V(Q^{q_i q_r}) \cup \{u\}$. Then $V(H_1) \cup V(Q^{q_1 q_i}) \cup V(R)$ induces a 1-wheel with center v_1, a contradiction. It follows that q_i is adjacent to u, and, by symmetry, so is q_j. $\square$

Note that $N_G(q_1) \cap V(S_1) = \{x_1\}$, since otherwise, by (2) and (3), $V(H_1) \cup V(Q^{q_1 q_i})$ induces a theta. By symmetry, $N_G(q_r) \cap V(S_2) = \{y_1\}$. Moreover, $i \neq j$, since otherwise the vertex set $V(E) \cup V(Q)$ induces a 1-wheel with center u.

(4) $\{v_1, v_2\}$ is anticomplete to $V(Q)$.

Proof of (4). Assume not, w.l.o.g. suppose that v_1 has a neighbor in the interior of $Q^{q_i q_j}$, and, for some $i < k < j$, let q_k be the vertex of Q with lowest index that is adjacent to v_1. Then $V(H_1) \cup V(Q^{q_1 q_k})$ induces a 1-wheel with center u, a contradiction. $\square$

By (4), $V(E) \cup V(Q)$ induces a 1-wheel, a 3-wheel or a long alternating wheel with center u. It follows that K is anticomplete to $V(Q)$. By (2), q_1 (resp. q_r) has a single neighbor in E, and this neighbor belongs to S_1 (resp. S_2). If $N_G(q_1) \cap V(S_1) = \{x_1\}$ and $N_G(q_r) \cap V(S_2) = \{y_1\}$, then $V(S_1) \cup V(S_2) \cup V(Q) \cup \{v_1, v_2\}$ induces a hole H and (H, u) is a 1-wheel. So, w.l.o.g. x_1 is not the neighbor of q_1 in S_1. But then the graph induced by $(V(E) \setminus \{v_2\}) \cup V(Q)$ contains a theta, a contradiction. ■

Unichord cycles.

A *unichord cycle* $U = (u, v, S_1, S_2)$ of G is an induced subgraph of G defined as follows: u and v are adjacent vertices of G, and, for some $m, n \geqslant 2$, $S_1 = x_1, \dots, x_n$ and $S_2 = y_1, \dots, y_m$ are vertex-disjoint chordless paths in $G \setminus \{u, v\}$ such that

- $N_G(x_1) \cap (V(S_2) \cup \{u, v\}) = \{u\}$, $N_G(x_n) \cap (V(S_2) \cup \{u, v\}) = \{v\}$ and $N_G(x_i) \cap (V(S_2) \cup \{u, v\}) = \varnothing$ for every $1 < i < n$.
- $N_G(y_1) \cap (V(S_1) \cup \{u, v\}) = \{u\}$, $N_G(y_m) \cap (V(S_1) \cup \{u, v\}) = \{v\}$ and $N_G(y_i) \cap (V(S_1) \cup \{u, v\}) = \varnothing$ for every $1 < i < m$.

For $1 \leqslant i \leqslant 2$, let H_i be the hole induced by $V(S_i) \cup \{u, v\}$. We say that a unichord cycle U of G is *minimum* if G does not contain a unichord cycle with a smaller number of vertices. This terminology and notation will be used in Lemma 96 and Lemma 97.

Lemma 96. *Let $U = (u, v, S_1, S_2)$ be a minimum unichord cycle of G. For every vertex $z \in V(G) \setminus V(U)$, one of the following holds:*

(i) $N_G(z) \cap V(U)$ is a clique of size at most two.

(ii) $|N_G(z) \cap V(U)| = 4$, $\{u, v\} \subset N_G(z)$ and (H_i, z) is a line wheel for some $1 \leqslant i \leqslant 2$.

Proof. Assume otherwise. W.l.o.g. z has a neighbor in S_1. If z has non-adjacent neighbors in H_1, then, by Lemma 90, (H_1, z) is a line wheel. Suppose that $N_G(z) \cap V(U) \subseteq V(H_1)$. Since (i) and (ii) do not hold, $V(H_1) \cup \{z\}$ induces a line wheel and z is not complete to $\{u, v\}$. But then $G[V(U) \cup \{z\}]$ contains a unichord cycle that contradicts our choice of U.

It follows that z has a neighbor in both S_1 and S_2. Let x_i (resp. y_j) be the neighbor of z in S_1 (resp. S_2) with highest index. If z is complete to $\{u, v\}$, then, by Lemma 90, (H_1, z) and (H_2, z) are both line wheels and $V(S_1^{x_i x_n}) \cup V(S_2^{y_j y_m}) \cup \{v, z\}$ induces a unichord cycle with a smaller number of vertices. Therefore, w.l.o.g. we may assume that z is not adjacent to v. Let $H = V(H_1) \cup V(S_2^{y_j y_m}) \cup \{z\}$. Suppose that (H_1, z) is a line wheel. If $j > 1$, then $G[H]$ contains a 3PC(v, z). So $j = 1$, and hence either $G[H]$ contains a 3PC(uzy_1, v) (if zu is an edge) or $G[V(H_1) \cup \{y_1, z\}]$ contains a 3PC(u, z) (otherwise). Therefore, (H_1, z) is not a line wheel. If z has two neighbors in H_1, then, by Lemma 90, these neighbors are adjacent, and hence either $G[H]$ is a pyramid (if $j > 1$), or $G[V(H_1) \cup \{y_1, z\}]$ is a pyramid (if $j = 1$ and z is not adjacent to u) or $G[V(U) \cup \{z\}]$ is a 3-wheel with center u (otherwise). So, by symmetry, we may assume that $N_G(z) \cap V(U) = \{x_i, y_j\}$. If $i = j = 1$ (resp. $i = n$ and $j = m$), then $V(U) \cup \{z\}$ induces a 1-wheel with center u (resp. v). So, if $i < n$, then either $G[H]$ is a 3PC(v, x_i) (if $j > 1$) or $G[V(H_1) \cup \{y_1, z\}]$ is a 3PC(u, x_i) (otherwise), and, if $i = n$, then $V(H_1) \cup V(S_2^{y_1 y_j}) \cup \{z\}$ induces a 3PC(u, x_n), a contradiction. ■

Lemma 97. *If G contains a unichord cycle, then it admits a clique cutset.*

Proof. Assume otherwise and let $U = (u, v, S_1, S_2)$ be a minimum unichord cycle of G. By Lemma 95, G does not contain an extended triangle. Since $\{u, v\}$ is not a clique cutset of G separating S_1 from S_2, let $Q = q_1, \dots, q_r$ be a shortest path in $G \setminus \{u, v\}$ such that q_1 (resp. q_r) has a neighbor in S_1 (resp. S_2). By Lemma 96, $r \geqslant 2$. By the minimality of Q, Q is chordless and no vertex of $Q \setminus \{q_1\}$ (resp. $Q \setminus \{q_r\}$) has a neighbor in S_1 (resp. S_2), so that $N_G(q_i) \cap V(U) \subseteq \{u, v\}$ for every $1 < i < r$.

(1) No vertex of Q is complete to $\{u, v\}$.

Proof of (1). Assume otherwise and let q_i be such a vertex with lowest index. Suppose that a vertex of $Q^{q_1 q_{i-1}}$ has a neighbor in $\{u, v\}$, and let q_j be such a vertex with highest index. Then $V(H_2) \cup V(Q^{q_j q_i})$ induces an extended triangle, a contradiction. So, $V(Q^{q_1 q_{i-1}})$ is anticomplete to $\{u, v\}$. If $i = 1$, then, by Lemma 96, $G[V(U) \cup \{q_1\}]$ contains an extended triangle. So $i > 1$, and, by symmetry, $i < r$. Let x_k be the neighbor of q_1 in S_1 with lowest index. If $k = n$, then $V(H_1) \cup V(Q^{q_1 q_i})$ induces a pyramid. So $k < n$, and hence $V(H_2) \cup V(S_1^{x_1 x_k}) \cup V(Q^{q_1 q_i})$ induces an extended triangle, a contradiction. $\square$

(2) $\{u, v\}$ is anticomplete to $V(Q)$.

Proof of (2). Assume not and let q_i be the vertex of Q with lowest index that has a neighbor in $\{u, v\}$. W.l.o.g. suppose that q_i is adjacent to u. By (1), vq_i is not an edge. If $N_G(q_1) \cap V(S_1) \neq \{x_1\}$, then, by Lemma 96, $V(H_1) \cup V(Q^{q_1 q_i})$ either induces a theta or a pyramid. So, q_1 is adjacent to x_1 and has no other neighbors in S_1. Let R be the chordless vq_i-path contained in the graph induced by $V(S_2) \cup V(Q^{q_i q_r}) \cup \{v\}$. Then, by (1), $V(H_1) \cup V(Q^{q_1 q_i}) \cup V(R)$ induces a 1-wheel with center u, a contradiction. $\square$

(3) q_1 (resp. q_r) has a single neighbor in S_1 (resp. S_2).

Proof of (3). Suppose that q_1 has at least two neighbors in S_1. Then, by Lemma 96 and (1), q_1 has two adjacent neighbors in S_1 and no other neighbors in U. Let y_i be the neighbor of q_r in S_2 with lowest index. By (2), u and v do not have a neighbor in Q. W.l.o.g. we may assume that $i < m$. But then $V(H_1) \cup V(S_2^{y_1 y_i}) \cup V(Q)$ induces a pyramid, a contradiction. So, q_1 has a single neighbor in S_1, and, by symmetry, q_r has a single neighbor in S_2. $\square$

By (2) and (3), $N_G(q_1) \cap V(U) = \{x_i\}$ for some $1 \leqslant i \leqslant n$, $N_G(q_r) \cap V(U) = \{y_j\}$ for some $1 \leqslant j \leqslant m$, and no interior vertex of Q has a neighbor in U. If either $i = j = 1$ or $i = n$ and $j = m$, then $V(U) \cup V(Q)$ induces a 1-wheel with center u or v. So w.l.o.g. $1 < i \leqslant n$ and $1 \leqslant j < m$. But then $V(H_1) \cup V(S_2^{y_1 y_j}) \cup V(Q)$ induces a 3PC(u, x_i), a contradiction. ■

Putting things together.

We are now ready to prove Theorem 92.

PROOF OF THEOREM 92. Let C be a claw in G, with vertex set $V(C) = \{u, v_1, v_2, v_3\}$ and edge set $E(C) = \{uv_1, uv_2, uv_3\}$, and assume that G does not admit a clique cutset. Since $\{u\}$ is not a clique cutset of G, there exists a path $Q = q_1, \ldots, q_r$ in $G \setminus \{u, v_1, v_2, v_3\}$ such that q_1 is adjacent to v_i for some $1 \leqslant i \leqslant 3$ and q_r has a neighbor in $\{v_1, v_2, v_3\} \setminus \{v_i\}$. Assume that the claw and the path are chosen so that Q is of shortest length, and w.l.o.g. suppose that q_1 is adjacent to v_1 and q_r is adjacent to v_2. It follows that Q is chordless, no vertex of $Q \setminus \{q_1\}$ is adjacent to v_1, no vertex of $Q \setminus \{q_r\}$ is adjacent to v_2, and v_3 is anticomplete to $V(Q) \setminus \{q_1, q_r\}$. Also, since G is diamond-free, u has no neighbors in Q, and hence $V(Q) \cup \{u, v_1, v_2\}$ induces a hole H. If v_3 is adjacent to q_1 or q_r, then $V(H) \cup \{v_3\}$ either induces a theta or a 1-wheel with center v_3. So, v_3 has no neighbors in Q.

Since $\{u\}$ is not a clique cutset of G, there exists a path $T = t_1, \ldots, t_\ell$ in $G \setminus (V(H) \cup \{v_3\})$ such that t_1 is adjacent to v_3 and t_ℓ has a neighbor in $V(H) \setminus \{u\}$. In particular, let T be such a path of shortest length. Then T is chordless, no vertex of $T \setminus \{t_1\}$ is adjacent to v_3, and no vertex of $T \setminus \{t_\ell\}$ has a neighbor in $V(H) \setminus \{u\}$. If t_ℓ has non-adjacent neighbors in H, then, by Lemma 90, (H, t_ℓ) is a line wheel.

First assume that t_ℓ is not adjacent to u, and let T' be the chordless ut_ℓ-path contained in the graph induced by $V(T) \cup \{u, v_3\}$. Then t_ℓ must have a single neighbor in H, and this vertex must belong to $\{v_1, v_2\}$, since otherwise the graph induced by $V(H) \cup V(T')$ contains a theta or a pyramid. But then $V(H) \cup V(T')$ induces a unichord cycle, contradicting Lemma 97.

Therefore, ut_ℓ is an edge. Then $N_G(t_\ell) \cap \{v_1, v_2\} \neq \varnothing$, since otherwise $V(H) \cup \{t_\ell\}$ either induces a theta or a 1-wheel with center t_ℓ. W.l.o.g. let t_ℓ be adjacent to v_1. Since G is diamond-free, neither v_2t_ℓ nor v_3t_ℓ is an edge, and hence $\ell \geqslant 2$. If (H, t_ℓ) is a line wheel, then the claw induced by $\{u, t_\ell, v_2, v_3\}$ and a proper subpath of Q contradict our choice of C and Q. So, $N_G(t_\ell) \cap V(H) = \{u, v_1\}$. Since G is diamond-free, u is not adjacent to $t_{\ell-1}$, and hence the graph induced by $V(H) \cup V(T) \cup \{v_3\}$ contains an extended triangle, contradicting Lemma 95. This proves the theorem. ■

3.4 Proof of Theorem 84

Throughout this section, we assume that $G \in \mathcal{G}$ contains a wheel with an appendix or a long alternating wheel, but does not admit a clique cutset (and hence G is connected). We want to show that G is structured. By our assumptions, w.l.o.g. G satisfies exactly one of the properties below, and we define a graph $H^\star$ depending on which property is satisfied.

Property 1 : G contains a wheel with an appendix. Let (H, x) be a wheel with an appendix in G with shortest rim, and let $P = p_1, \dots, p_k$ be its appendix with shortest length. Assume that (H, x) has short odd sectors and P is attached to S_2, and let $H^\star = G[V(H) \cup V(P) \cup \{x\}]$.

Property 2 : G does not contain a wheel with an appendix, but it contains a long alternating wheel. Let (H, x) be a long alternating wheel in G with shortest rim, assume that (H, x) has short odd sectors and let $H^\star = G[V(H) \cup \{x\}]$.

Suppose that Property 1 holds and let W be the hole induced by $(V(S_2) \setminus \{x_2\}) \cup V(P) \cup \{x\}$. Then we say that $y \in V(G) \setminus (V(H) \cup V(P) \cup \{x\})$ is a *special vertex* of G if it is complete to $\{x, x_1, x_2\}$, $N_G(y) \cap (V(H) \setminus \{x_1, x_2\}) = \varnothing$, and $\{p_1\} \subset N_G(y) \cap V(P)$ in such a way that $V(W) \cup \{y\}$ induces an alternating wheel.

We prove Theorem 84 by the following sequence of lemmas.

Lemma 98. *For every vertex $y \in V(G) \setminus V(H^\star)$, either $N_G(y) \cap V(H^\star)$ is a clique of size at most three, or G satisfies Property 1 and y is special.*

We postpone the proof of Lemma 98 to Section 3.4.2.

Now, let $M = V(H) \setminus (V(S_2) \cup \{x_1, x_4\})$ and

$$N = \begin{cases} (V(S_2) \setminus \{x_2, x_3\}) \cup V(P) & \text{if Property 1 holds,} \\ V(S_2) \setminus \{x_2, x_3\} & \text{if Property 2 holds.} \end{cases}$$

Furthermore, we denote by A (resp. B) the set of vertices in $V(G) \setminus V(H^\star)$ that are complete to $\{x, x_1, x_2\}$ (resp. $\{x, x_3, x_4\}$). Note that, by Lemma 98, $A \cap B = \varnothing$. In particular, if $u \in A$, then either u is a special vertex of G (when Property 1 holds) or $N_G(u) \cap V(H^\star) = \{x, x_1, x_2\}$. If $u \in B$, then $N_G(u) \cap V(H^\star) = \{x, x_3, x_4\}$.

Lemma 99. *$A \cup B \cup \{x, x_1, x_2, x_3, x_4\}$ is a cutset of G that separates N from M.*

Proof. Assume not. Then $G \setminus (V(H^\star) \cup A \cup B)$ contains a chordless path $T = t_1, \dots, t_m$ such that no vertex in $T \setminus \{t_1, t_m\}$ has a neighbor in $H^\star \setminus \{x, x_1, x_2, x_3, x_4\}$, t_1 has a neighbor in N, and t_m has a neighbor in M. By Lemma 98, $m \geqslant 2$, $N_G(t_1) \cap V(H^\star) \subset N \cup \{x, x_2, x_3\}$ and $N_G(t_m) \cap V(H^\star) \subset M \cup \{x, x_1, x_4\}$.

It suffices to consider the following two cases.

Case 1: (H, x) is a line wheel, and hence Property 1 holds.

We have that $N_G(t_m) \cap V(H^\star) \subset V(S_4)$. Let u (resp. v) be the neighbor of t_m in S_4 that is closest to x_1 (resp. x_4). By Lemma 98, either $u = v$ (and, if that is the case, $u \notin \{x_1, x_4\}$) or $uv \in E(G)$.

(1) At least one of the sets $\{x_1, x_2\}$, $\{x_3, x_4\}$ is anticomplete to $V(T) \setminus \{t_1, t_m\}$.

Proof of (1). Assume otherwise. Then there exists a minimal subpath $T^{t_i t_j}$ of $T \setminus \{t_1, t_m\}$ such that w.l.o.g. t_i is adjacent to a vertex of $\{x_1, x_2\}$ and t_j is adjacent to a vertex of $\{x_3, x_4\}$. Note that no interior vertex of $T^{t_i t_j}$ has a neighbor in H. Also, by Lemma 98, $i \neq j$. So, since $V(T^{t_i t_j}) \cup V(H)$ cannot induce a theta or a pyramid, it follows that $N_G(t_i) \cap V(H) = \{x_1, x_2\}$ and $N_G(t_j) \cap V(H) = \{x_3, x_4\}$, and hence (by the definition of T) neither t_i nor t_j is adjacent to x. But then $V(S_4) \cup V(T^{t_i t_j}) \cup \{x\}$ either induces a theta or a 1-wheel with center x, a contradiction. $\square$

(2) x has a neighbor in $T \setminus \{t_1, t_m\}$.

Proof of (2). Assume not and let R be the chordless $x_1 t_1$-path contained in the graph induced by the ux_1-subpath of S_4 together with $V(T)$. First suppose that t_1 is adjacent to x, so that, by Lemma 98, $N_G(t_1) \cap N = \{p_1\}$. If $x_2 t_1$ is not an edge, let R' be the chordless $x_2 t_1$-path contained in the graph induced by the ux_2-subpath of $H \setminus \{x_2'\}$ together with $V(T)$. Then $V(R') \cup \{p_1\}$ induces a hole H' and (H', x) is a 3-wheel. So, $N_G(t_1) \cap V(H^\star) = \{x, x_2, p_1\}$. But then $V(R) \cup \{x, x_2\}$ induces a 3-wheel with center x_2. It follows that t_1 is not adjacent to x. Let D be the chordless $t_1 p_1$-path contained in the graph induced by $N \cup \{t_1\}$. Then $V(R) \cup V(D) \cup \{x\}$ induces a hole H'' and (H'', x_2) is a 3-wheel, a contradiction. $\square$

By (2), let t_i be the neighbor of x in $T \setminus \{t_1, t_m\}$ with highest index.

(3) Either x_1 or x_4 is adjacent to t_i.

Proof of (3). Assume that (3) does not hold. If x_1 and x_4 have no neighbors in $T^{t_i t_{m-1}}$, then $V(S_4) \cup V(T^{t_i t_m}) \cup \{x\}$ either induces a theta or a pyramid. So, let t_j be the vertex of $T^{t_{i+1} t_{m-1}}$ with highest index that has a neighbor in $\{x_1, x_4\}$. W.l.o.g. let t_j be adjacent to x_1, and hence, by Lemma 98, t_j is not adjacent to x_4. Then it must be that either $u = x_1$ or $u = v = x_1'$, since otherwise $V(S_4) \cup V(T^{t_j t_m}) \cup \{x\}$ induces a theta or a pyramid. Furthermore, by (1),

$N_G(x_4) \cap V(T^{t_{i+1}t_{j-1}}) = \varnothing$. So, let H' be the hole induced by $(V(S_4) \setminus \{x_1\}) \cup V(T^{t_i t_m}) \cup \{x\}$. Then (H', x_1) is a 1-wheel, a contradiction. $\square$

(4) x_1 is adjacent to t_i.

Proof of (4). Assume otherwise. Then, by (3), t_i is adjacent to x_4 and hence, since $t_i \notin B$, not adjacent to x_3. By (1), x_1 and x_2 have no neighbors in the interior of T. Let R be a chordless x_2t_1-path contained in the graph induced by $N \cup \{x_2, t_1\}$. If $N_G(x_4) \cap V(T^{t_2t_{i-1}}) \neq \varnothing$, let t_j be the neighbor of x_4 in $T^{t_2t_{i-1}}$ with lowest index. Then t_j is adjacent to x, since otherwise $V(S_4) \cup V(R) \cup V(T^{t_1t_j}) \cup \{x\}$ induces a 1-wheel with center x. So, x_3t_j is not an edge. Now, let R' be the chordless x_3t_j-path contained in the graph induced by $N \cup V(T^{t_1t_j}) \cup \{x_3\}$. It follows that $V(R') \cup \{x, x_4\}$ induces a 3-wheel with center x, a contradiction. So, $N_G(x_4) \cap V(T^{t_2t_{i-1}}) = \varnothing$. If we denote by D the chordless x_3t_i-path contained in the graph induced by $N \cup V(T^{t_1t_i}) \cup \{x_3\}$, then the vertex set $V(D) \cup \{x_4\}$ induces a hole H' and (H', x) is a 3-wheel, a contradiction. $\square$

(5) $\{x_3, x_4\}$ is anticomplete to $V(T) \setminus \{t_1, t_m\}$.

Proof of (5). It follows from (1) and (4). $\square$

By (4), x_1t_i is an edge. Since $t_i \notin A$, t_i is not adjacent to x_2. Let R be the chordless x_3t_1-path contained in the graph induced by $N \cup \{x_3, t_1\}$. If $N_G(x_1) \cap V(T^{t_2t_{i-1}}) \neq \varnothing$, then let t_j be the neighbor of x_1 in $T^{t_2t_{i-1}}$ with lowest index. Then t_j is adjacent to x, since otherwise, by (5), $V(S_4) \cup V(R) \cup V(T^{t_1t_j}) \cup \{x\}$ induces a 1-wheel with center x. So, x_2t_j is not an edge. Now, let R' be a chordless x_2t_j-path contained in the graph induced by $N \cup V(T^{t_1t_j}) \cup \{x_2\}$. It follows that $V(R') \cup \{x, x_1\}$ induces a 3-wheel with center x, a contradiction. So, x_1 has no neighbors in $T^{t_2t_{i-1}}$. If we denote by D a chordless x_2t_i-path contained in the graph induced by $N \cup V(T^{t_1t_i}) \cup \{x_2\}$, then the vertex set $V(D) \cup \{x_1\}$ induces a hole H' and (H', x) is a 3-wheel, a contradiction.

<u>**Case 2**</u>**:** (H, x) is a long alternating wheel.

First assume that t_1 has a neighbor in $N \setminus V(P)$, and let u (resp. v) be the neighbor of t_1 in S_2 that is closest to x_2 (resp. x_3). By Lemma 98, t_1 is not adjacent to x.

(6) A vertex of $\{x_2, x_3\}$ has a neighbor in $T \setminus \{t_1, t_m\}$.

Proof of (6). Assume not and let R be a chordless xt_1-path contained in the graph induced by $M \cup V(T) \cup \{x\}$. If $u = v$, then $u \notin \{x_2, x_3\}$, and hence $V(S_2) \cup V(R)$ induces a 3PC(x, u). So, by Lemma 98, uv is an edge. But then $V(S_2) \cup V(R)$ induces a 3PC(uvt_1, x), a contradiction. $\square$

By (6), let t_i be the vertex of $T \setminus \{t_1, t_m\}$ with highest index that has a neighbor in $\{x_2, x_3\}$. W.l.o.g. let t_i be adjacent to x_2. Then, by Lemma 98, t_i is anticomplete to $\{x_3, x_4\}$.

(7) t_i is adjacent to x.

Proof of (7). The graph induced by $(V(H) \setminus \{x_1\}) \cup V(T^{t_i t_m})$ contains a hole H' that contains x_4, $V(S_2)$ and t_i. By Lemma 90, (H', x) is an alternating wheel, and hence (7) holds. $\square$

By (7), Lemma 98 and the definition of T, $N_G(t_i) \cap (V(H) \cup \{x\}) = \{x, x_2\}$. Let R be the chordless x_1t_i-path contained in the graph induced by $M \cup V(T^{t_it_m}) \cup \{x_1\}$. By our choice of t_i, $V(R) \cup \{x_2\}$ induces a hole H' and (H', x) is a 3-wheel, a contradiction.

So, $N_G(t_1) \cap N \subseteq V(P)$. Let p_j be the neighbor of t_1 in P with highest index. Instead of T, consider now the chordless path induced by $\{p_k, \ldots, p_j, t_1, \ldots, t_m\}$, and the arguments above still apply. This proves the lemma. ■

Attachments.

Let $u \in A \cup B$ and let $Q = q_1, \ldots, q_r$ be a chordless path in $G \setminus (V(H^\star) \cup A \cup B)$ such that $N_G(u) \cap V(Q) = \{q_1\}$, no vertex in $Q \setminus \{q_r\}$ has a neighbor in $H^\star \setminus \{x, x_1, x_2, x_3, x_4\}$, and q_r has a neighbor in M. Then we say that Q is an *attachment* of u to M. By Lemma 98, $N_G(q_r) \cap V(H^\star) \subset M \cup \{x, x_1, x_4\}$. Also, let $X_1' \subseteq A$ (resp. $Y_1' \subseteq B$) be the set of vertices in A (resp. B) that have an attachment to M, and let $X_1 = X_1' \cup \{x_1\}$ (resp. $Y_1 = Y_1' \cup \{x_4\}$).

Lemma 100. *Let $u \in X_1'$ and let $Q = q_1, \ldots, q_r$ be an attachment of u to M. Then the following hold:*

(i) x_2 and x_3 have no neighbors in Q.

(ii) x_4 has no neighbors in $Q \setminus \{q_r\}$.

Proof. Let v (resp. w) be the neighbor of q_r in $H \setminus V(S_2)$ that is closest to x_1 (resp. x_4). Since $N_G(q_r) \cap V(H^\star) \subset M \cup \{x, x_1, x_4\}$, the following holds.

(1) q_r is anticomplete to $\{x_2, x_3\}$.

(2) x_3 and x_4 have no neighbors in $Q \setminus \{q_r\}$. In particular, (ii) holds.

Proof of (2). Assume not. Let q_i be the lowest indexed vertex of $Q \setminus \{q_r\}$ that has a neighbor in $\{x_3, x_4\}$, and let R be the chordless x_2q_i-path contained in the graph induced by $V(Q^{q_1q_i}) \cup \{x_2, u\}$. First suppose that q_i is adjacent to x_4. Then the graph induced by $(V(H) \setminus V(S_2)) \cup V(Q^{q_1q_i}) \cup \{u\}$ contains a hole H' that contains $V(H) \setminus V(S_2)$ and q_i. Also, by Lemma 90, (H', x) is an alternating wheel. So, q_i is adjacent to x. By the definition of Q, q_i is not adjacent to x_3. But then $V(S_2) \cup V(R) \cup \{x, x_4\}$ induces a 3-wheel with center x, a contradiction. It follows that q_i is adjacent to x_3 and not adjacent to x_4. Then q_i is adjacent to x, since otherwise $V(S_2) \cup V(R) \cup \{x\}$ either induces a theta or a 1-wheel with center x. Furthermore, the graph induced by the vertex set $(V(H) \setminus V(S_2)) \cup V(Q^{q_1q_i}) \cup \{x_3, u\}$ contains a hole H'' that contains $V(H) \setminus V(S_2)$, x_3 and q_i. But then (H'', x) is a 3-wheel, a contradiction. □

(3) x_2 has no neighbors in $Q \setminus \{q_r\}$.

Proof of (3). Assume not and let q_i be the highest indexed vertex of $Q \setminus \{q_r\}$ that is adjacent to x_2. Then q_i must be adjacent to x, since otherwise, by (1) and (2), the x_2w-subpath of $H \setminus \{x_1\}$, together with $V(Q^{q_iq_r}) \cup \{x\}$, induces a 1-wheel with center x. Let R be the chordless x_1q_i-path contained in the graph induced by $V(Q^{q_iq_r})$ together with the x_1v-subpath of $H \setminus \{x_2\}$. Since

$q_i \notin A$, q_i is not adjacent to x_1, and hence $V(R) \cup \{x_2\}$ induces a hole H' that contains x_1, x_2 and q_i, and (H', x) is a 3-wheel, a contradiction. □

By (1), (2) and (3), (i) holds. ■

Analogous arguments prove Lemma 101.

Lemma 101. *Let $u \in Y_1'$ and let $Q = q_1, \dots, q_r$ be an attachment of u to M. Then the following hold:*

(i) x_2 and x_3 have no neighbors in Q.

(ii) x_1 has no neighbors in $Q \setminus \{q_r\}$.

Now let $u \in A \cup B$ be a vertex that is not special, and let $Q = q_1, \dots, q_r$ be a chordless path in $G \setminus (V(H^\star) \cup A \cup B)$ such that $N_G(u) \cap V(Q) = \{q_1\}$, $N_G(q_i) \cap (V(H^\star) \setminus \{x, x_1, x_2, x_3, x_4\}) = \varnothing$ for every $1 \leqslant i < r$, and q_r has a neighbor in N. Then we say that Q is an *attachment* of u to N. By Lemma 98, $N_G(q_r) \cap V(H^\star) \subset N \cup \{x, x_2, x_3\}$. Also, let $X_2' \subseteq A$ be the set of vertices in A that are either special or have an attachment to N, let $Y_2' \subseteq B$ be the set of vertices in B that have an attachment to N, and let $X_2 = X_2' \cup \{x_2\}$ and $Y_2 = Y_2' \cup \{x_3\}$.

Lemma 102. *Let $u \in X_2'$ be a vertex that is not special and let $Q = q_1, \dots, q_r$ be an attachment of u to N. Then the following hold:*

(i) x_1 and x_4 have no neighbors in Q.

(ii) x_3 has no neighbors in $Q \setminus \{q_r\}$.

Proof. Since $N_G(q_r) \cap V(H^\star) \subset N \cup \{x, x_2, x_3\}$, the following holds.

(1) q_r is anticomplete to $\{x_1, x_4\}$.

(2) x_3 and x_4 have no neighbors in $Q \setminus \{q_r\}$. In particular, (ii) holds.

Proof of (2). Assume otherwise. Let q_i be the lowest indexed vertex of $Q \setminus \{q_r\}$ that has a neighbor in $\{x_3, x_4\}$, and let R be the chordless x_1q_i-path contained in the graph induced by $V(Q^{q_1q_i}) \cup \{x_1, u\}$. First suppose that q_i is not adjacent to x_4, so that $x_3q_i \in E(G)$. Then the graph induced by $V(S_2) \cup V(Q^{q_1q_i}) \cup \{u\}$ contains a hole H' that contains $V(S_2)$ and q_i. By Lemma 90, (H', x) is an alternating wheel. Thus, q_i is adjacent to x and $(V(H) \setminus V(S_2)) \cup V(R) \cup \{x, x_3\}$ induces a 3-wheel with center x, a contradiction. So, x_4q_i is an edge. Then q_i is adjacent to x, since otherwise $(V(H) \setminus V(S_2)) \cup V(R) \cup \{x\}$ either induces a theta or a 1-wheel with center x. Since $q_i \notin B$, q_i is not adjacent to x_3. Therefore the graph induced by $V(S_2) \cup V(Q^{q_1q_i}) \cup \{x_4, u\}$ contains a hole H'' that contains $V(S_2)$, x_4 and q_i. But then (H'', x) is a 3-wheel, a contradiction. □

(3) x_1 has no neighbors in $Q \setminus \{q_r\}$.

Proof of (3). Assume not and let q_i be the highest indexed vertex of $Q \setminus \{q_r\}$ that is adjacent to x_1. First suppose that q_r has a neighbor in the interior of S_2, and let v (resp. w) be the neighbor of q_r in S_2 that is closest to x_2 (resp. x_3). Then q_i must be adjacent to x, since otherwise, by (1) and (2), the x_1w-subpath of $H \setminus \{x_2\}$, together with $V(Q^{q_iq_r}) \cup \{x\}$, induces a 1-wheel with center x. Let R be the chordless x_2q_i-path contained in the graph induced by $V(Q^{q_iq_r})$ together with the x_2v-subpath of $H \setminus \{x_1\}$. Since $q_i \notin A$, q_i is not adjacent to x_2, and hence $V(R) \cup \{x_1\}$ induces a hole H' that contains x_1, x_2 and q_i, and (H', x) is a 3-wheel, a contradiction. It follows that q_r has no neighbors in the interior of S_2, and therefore $N_G(q_r) \cap V(P) \neq \varnothing$. Let p_j be the neighbor of q_r in P with highest index, and, by (1) and (2), let H'' be the hole induced by the $x_2'x_1$-subpath of $H \setminus \{x_2\}$ together with $V(Q^{q_iq_r}) \cup V(P^{p_jp_k})$. Then $xq_i \in E(G)$, since otherwise (H'', x) is a 1-wheel. Since q_i cannot be complete to $\{x, x_1, x_2\}$, q_i is not adjacent to x_2. Let R' be the chordless x_2q_i-path contained in the graph induced by $V(Q^{q_iq_r}) \cup V(P^{p_jp_k}) \cup \{x_2\}$. Then $V(R') \cup \{x, x_1\}$ induces a 3-wheel with center x, a contradiction. □

By (1), (2) and (3), (i) holds. ■

Analogous arguments prove Lemma 103.

Lemma 103. *Let $u \in Y_2'$ and let $Q = q_1, \ldots, q_r$ be an attachment of u to N. Then the following hold:*

(i) x_1 and x_4 have no neighbors in Q.

(ii) x_2 has no neighbors in $Q \setminus \{q_r\}$.

Note that, by Lemma 98, $X_i \cap Y_j = \varnothing$ for every $1 \leqslant i, j \leqslant 2$. We can also show the following.

Lemma 104. *$X_1 \cap X_2 = Y_1 \cap Y_2 = \varnothing$.*

Proof. Assume that $X_1 \cap X_2 \neq \varnothing$, and let $u \in A$ be a vertex that is not special and has an attachment $Q = q_1, \ldots, q_r$ to N and an attachment $T = t_1, \ldots, t_m$ to M. By Lemma 99, $V(Q) \cap V(T) = \varnothing$ and $V(Q)$ is anticomplete to $V(T)$. Let v be the neighbor of t_m in $H \setminus V(S_2)$ that is closest to x_4. Suppose that q_r has a neighbor in the interior of S_2, and let w be the neighbor of q_r in S_2 that is closest to x_3.

First assume that x is not adjacent to t_1. By Lemma 102, x_3 (resp. x_4) has no neighbors in $Q \setminus \{q_r\}$ (resp. Q), and, by Lemma 100, x_4 (resp. x_3) has no neighbors in $T \setminus \{t_m\}$ (resp. T). So, the vw-subpath of $H \setminus V(S_1)$, together with $V(Q) \cup V(T) \cup \{u\}$, induces a hole H' that contains x_3, x_4 and u. By Lemma 90, (H', x) is an alternating wheel, and hence x is adjacent to q_1. By Lemma 102, x_1 has no neighbors in Q, and therefore the x_1w-subpath of $H \setminus \{x_2\}$, together with $V(Q) \cup \{x, u\}$, induces a 3-wheel with center x, a contradiction. So, xt_1 is an edge. By Lemma 100, x_2 has no neighbors in T. It follows that the x_2v-subpath of $H \setminus \{x_1\}$, together with $V(T) \cup \{x, u\}$, induces a 3-wheel with center x, a contradiction.

So, q_r has no neighbors in the interior of S_2 but has a neighbor in P. Instead of Q, consider now the chordless q_1p_k-path contained in the graph induced by $V(Q^{q_1q_r}) \cup V(P)$, and the arguments above still apply. The same approach works when u is a special vertex of G. This proves that $X_1 \cap X_2 = \varnothing$. Analogously, it can be shown that $Y_1 \cap Y_2 = \varnothing$. ■

Lemma 105. *X_1, X_2, Y_1 and Y_2 are all cliques of G.*

PROOF. Suppose that there exist $u, v \in X_1$, $u \neq v$, such that uv is not an edge, and let Q (resp. T) be an attachment of u (resp. v) to M. Let R be a chordless uv-path contained in the graph induced by $M \cup V(Q) \cup V(T) \cup \{u, v\}$. By Lemma 100, $V(R) \cup \{x_2\}$ induces a hole H', and hence (H', x) is a 3-wheel, a contradiction. So, X_1 is a clique. Analogous arguments show that X_2, Y_1 and Y_2 are cliques too. ■

Ears.

Let $T = t_1, \ldots, t_m$ be a chordless path in $G \setminus (V(H^\star) \cup A \cup B)$ such that no vertex of T has a neighbor in $H^\star \setminus \{x, x_1, x_2, x_3, x_4\}$, and let $u \in A$ and $v \in B$ be such that $N_G(u) \cap V(T) = \{t_1\}$ and $N_G(v) \cap V(T) = \{t_m\}$. Then we say that T is an *ear* of $H^\star$, while u and v are said to be the *attachments* of T. Also, let X_3 (resp. Y_3) be the set of vertices of $A \setminus (X_1 \cup X_2)$ (resp. $B \setminus (Y_1 \cup Y_2)$) that are attachments of an ear of $H^\star$.

Lemma 106. *If $T = t_1, \ldots, t_m$ is an ear of $H^\star$, then $N_G(t_i) \cap V(H) = \varnothing$ for all $1 \leqslant i \leqslant m$.*

PROOF. By definition, $N_G(t_i) \cap V(H) \subseteq \{x_1, x_2, x_3, x_4\}$ for every $1 \leqslant i \leqslant m$. We now show that t_i is anticomplete to $\{x_1, x_2\}$ for every $1 \leqslant i \leqslant m$. Assume otherwise and let t_j be the vertex of T with highest index that is adjacent to a vertex of $\{x_1, x_2\}$, and let $u \in A$ and $v \in B$ be the attachments of T. By Lemma 98, t_j is anticomplete to $\{x_3, x_4\}$. Furthermore, let R (resp. R') be the chordless $t_j x_4$-path (resp. $t_j x_3$-path) contained in the graph induced by $V(T^{t_j t_m}) \cup \{x_4, v\}$ (resp. $V(T^{t_j t_m}) \cup \{x_3, v\}$). First suppose that $x_1 t_j$ is an edge, and let H' be the hole induced by the $x_1 x_4$-subpath of $H \setminus \{x_2\}$ together with R. Then, by Lemma 90, (H', x) is an alternating wheel, and hence t_j is adjacent to x. Since $t_j \notin A$, it follows that t_j is not adjacent to x_2. Now, let H'' be the hole induced by $V(S_2) \cup V(R') \cup \{x_1\}$. Then (H'', x) is a 3-wheel, a contradiction. Hence, t_j is adjacent to x_2 and $x_1 t_j$ is not an edge. Also, t_j is adjacent to x, since otherwise $V(S_2) \cup V(R') \cup \{x\}$ either induces a theta or a 1-wheel with center x. Therefore the $x_2 x_4$-subpath of $H \setminus \{x_3\}$, together with $V(R) \cup \{x\}$, induces a 3-wheel with center x, a contradiction. So, t_i is anticomplete to $\{x_1, x_2\}$ for every $1 \leqslant i \leqslant m$. Analogously it can be shown that t_i is also anticomplete to $\{x_3, x_4\}$ for every $1 \leqslant i \leqslant m$. ■

Lemma 107. *X_3 (resp. Y_3) is a clique of G that is complete to $X_1 \cup X_2$ (resp. $Y_1 \cup Y_2$).*

PROOF. We only prove that X_3 is a clique of G and that it is complete to $X_1 \cup X_2$, since similar arguments show that Y_3 is a clique of G that is complete to $Y_1 \cup Y_2$.

(1) X_3 is a clique of G.

Proof of (1). Suppose that there exist $u, v \in X_3$, $u \neq v$, such that uv is not an edge. Let $T = t_1, \ldots, t_m$ (resp. $D = d_1, \ldots, d_h$) be an ear of $H^\star$ with attachments $u \in A$ and $w \in B$ (resp. $v \in A$ and $z \in B$). Let R be a chordless uv-path contained in the graph induced by $V(T) \cup V(D) \cup \{x_3, u, v, w, z\}$. By Lemma 106, $V(R) \cup \{x_1\}$ induces a hole H', and hence (H', x) is a 3-wheel, a contradiction. □

(2) X_3 is complete to $X_1 \cup X_2$.

Proof of (2). Assume that X_3 is not complete to X_2. Let $u \in X_3$ and $v \in X_2$ (v not special) be such that $uv \notin E(G)$. Let Q be an attachment of v to N, and let T be an ear of $H^\star$ with attachments $u \in A$ and $w \in B$. Let R be a chordless uv-path contained in the graph induced by $N \cup V(Q) \cup V(T) \cup \{x_3, u, v, w\}$. By Lemma 102 and Lemma 106, $V(R) \cup \{x_1\}$ induces a hole H' and (H', x) is a 3-wheel, a contradiction. A similar argument applies when v is special. So, X_3 is complete to X_2. Analogously it can be shown that X_3 is also complete to X_1. □

This proves the lemma. ■

Let $X = X_1 \cup X_2 \cup X_3$ and $Y = Y_1 \cup Y_2 \cup Y_3$.

Lemma 108. *X and Y are disjoint and anticomplete to each other.*

Proof. Sets X and Y are disjoint by Lemma 98. Now suppose that $u \in X$ and $v \in Y$ are such that uv is an edge. Then $V(S_2) \cup \{x, u, v\}$ induces a 3-wheel with center x, a contradiction. So, X and Y are anticomplete to each other. ■

Putting things together.

We are now ready to conclude the proof of Theorem 84. Let $S = \{x\} \cup X \cup Y$.

Lemma 109. *S is a cutset of G that separates N from M.*

Proof. Assume that S is not a cutset of G that separates N from M. So, there exists a path $Q = q_1, \dots, q_r$ in $G \setminus S$ such that q_1 (resp. q_r) has a neighbor in N (resp. M). In particular, let Q be of shortest length. It follows that Q is chordless and no vertex of $Q \setminus \{q_1, q_r\}$ has a neighbor in $H^\star \setminus \{x, x_1, x_2, x_3, x_4\}$. By Lemma 99, $V(Q) \cap (A \cup B) \neq \varnothing$. So, let q_i be the vertex of $V(Q) \cap (A \cup B)$ with lowest index, and w.l.o.g. assume that $q_i \in A$. But then either $i = 1$ and q_1 is special, or $i > 1$ and $Q^{q_1 q_{i-1}}$ is an attachment of q_i to N, and hence $q_i \in X_2$, a contradiction. This proves the lemma. ■

By Lemma 109, let $\mathcal{C}$ be the set of all the components of $G \setminus S$. The sets C_1, C_2, C_3, C_X, C_Y are defined as follows:

- C_1 (resp. C_2) is the vertex set of the component from $\mathcal{C}$ that contains M (resp. N);
- C_X (resp. C_Y) is the union of the vertex sets of all $C \in \mathcal{C}$ such that $N_G(C) \subseteq \{x\} \cup X$ (resp. $N_G(C) \subseteq \{x\} \cup Y$);
- C_3 is the union of the vertex sets of all $C \in \mathcal{C}$ such that $V(C) \nsubseteq C_1 \cup C_2 \cup C_X \cup C_Y$.

Lemma 110. $C_X \cap C_Y = \varnothing$.

Proof. Assume not. It follows that there exists a component $C \in \mathcal{C}$ such that $N_G(C) \subseteq \{x\}$. But then G admits a clique cutset, a contradiction. ■

Lemma 111. *If a component $C \in \mathcal{C}$ is such that $V(C) \nsubseteq C_1 \cup C_X \cup C_Y$, then x_1 and x_4 have no neighbors in C.*

Proof. First consider the following claim.

(1) x_1 and x_4 have no neighbors in C_2.

Proof of (1). Assume otherwise and let $Q = q_1, \dots, q_r$ be a shortest path in $G[C_2 \setminus N]$ such that q_1 (resp. q_r) has a neighbor in $\{x_1, x_4\}$ (resp. N). Note that (since $q_i \notin S$ for every $1 \leqslant i \leqslant r$) q_r is not special, and hence, by Lemma 98, $r \geqslant 2$. By the minimality of Q, Q is chordless, no vertex of $Q \setminus \{q_1\}$ is adjacent to x_1 or x_4, and no vertex of $Q \setminus \{q_r\}$ has a neighbor in N. Also, no vertex of Q has a neighbor in M.

W.l.o.g. suppose $x_1 q_1 \in E(G)$. Then, by our choice of Q and Lemma 98, x_4 is anticomplete to $V(Q)$. Let R be the chordless $x_3 q_1$-path contained in the graph induced by $N \cup V(Q) \cup \{x_3\}$. Then the $x_1 x_3$-subpath of $H \setminus \{x_2\}$, together with $V(R) \cup \{x\}$, induces a 1-wheel with center x, unless x is adjacent to q_1. So, $x q_1 \in E(G)$. If $x_2 q_1$ is an edge, then $Q^{q_2 q_r}$ is an attachment of q_1 to N, and hence $q_1 \in X_2$, a contradiction. So, $x_2 q_1$ is not an edge. Now, let R' be a chordless $x_2 q_1$-path contained in the graph induced by $N \cup V(Q) \cup \{x_2\}$. Then $V(R') \cup \{x_1\}$ induces a hole H' and (H', x) is a 3-wheel, a contradiction. □

Now suppose that the lemma does not hold. By (1), w.l.o.g. x_1 has a neighbor in a component $C \in \mathcal{C}$ such that $V(C) \nsubseteq C_1 \cup C_2 \cup C_X \cup C_Y$. So, C contains a chordless path $T = t_1, \dots, t_m$ such that t_1 is adjacent to x_1, t_m is adjacent to a vertex of Y, and $N_G(t_i) \cap V(H^\star) \subseteq \{x, x_1, x_2, x_3, x_4\}$ for every $1 \leqslant i \leqslant m$. Let T be such a path of minimum length. Then no vertex of $T \setminus \{t_1\}$ is adjacent to x_1, and no vertex of $T \setminus \{t_m\}$ has a neighbor in Y.

(2) t_m is anticomplete to $\{x_3, x_4\}$.

Proof of (2). Assume otherwise and note that, by Lemma 98, $m \geqslant 2$. Now let H' be the hole induced by $(V(H) \setminus V(S_2)) \cup V(T)$, together with x_3 if t_m is not adjacent to x_4. By Lemma 90, (H', x) is an alternating wheel, and hence t_1 is adjacent to x. Suppose that $x_2 t_1 \notin E(G)$, and let R be the chordless $x_2 t_1$-path contained in the graph induced by $V(S_2) \cup V(T)$, together with x_4 if t_m is not adjacent to x_3. Then $V(R) \cup \{x_1\}$ induces a hole H'' and (H'', x) is a 3-wheel, a contradiction. So, $x_2 t_1 \in E(G)$. Let R' be the chordless $x_2 t_m$-path contained in $G[V(T) \cup \{x_2\}]$. First suppose that t_m is adjacent to x_3. Then $x t_m$ is an edge, since otherwise $V(S_2) \cup V(R') \cup \{x\}$ either induces a theta or a 1-wheel with center x. Also, if $x_4 t_m \notin E(G)$ or $m = 2$, then (H', x) is a 3-wheel. So, t_m is adjacent to x_4 and $m > 2$. But then $T^{t_2 t_{m-1}}$ is an ear of $H^\star$ with attachments t_1 and t_m, implying that $t_1 \in X$ and $t_m \in Y$, a contradiction. It follows that $x_3 t_m \notin E(G)$ and $x_4 t_m \in E(G)$. Since (H', x) is an alternating wheel, $x t_m$ is an edge. But then $V(S_2) \cup V(R') \cup \{x, x_4\}$ induces a 3-wheel with center x, a contradiction. □

By (2), t_m has a neighbor $u \in Y \setminus \{x_3, x_4\}$. Observe that $x t_1$ is an edge, since otherwise the vertex set $(V(H) \setminus V(S_2)) \cup V(T) \cup \{x, u\}$ induces a 1-wheel with center x. If $m = 1$, then $(V(H) \setminus V(S_2)) \cup \{u, t_1\}$ induces a hole H' and (H', x) is a 3-wheel. Therefore, $m \geqslant 2$. If t_1 is adjacent to x_2, then $T^{t_2 t_m}$ is an ear of $H^\star$, and hence $t_1 \in X$, a contradiction. So, $x_2 t_1$ is not an edge, and, if R denotes the chordless $x_2 t_1$-path contained in the graph induced by

$V(S_2) \cup V(T) \cup \{u\}$, then $V(R) \cup \{x, x_1\}$ induces a 3-wheel with center x, a contradiction. ■

Analogous arguments prove Lemma 112.

Lemma 112. *If a component $C \in \mathcal{C}$ is such that $V(C) \nsubseteq C_2 \cup C_X \cup C_Y$, then x_2 and x_3 have no neighbors in C.*

Lemma 113. *$X \setminus X_i$ and $Y \setminus Y_i$ are anticomplete to C_i for every $1 \leqslant i \leqslant 2$.*

PROOF. Suppose that a vertex $u \in X \setminus X_1$ has a neighbor in C_1. By Lemma 112, $u \neq x_2$. Also, $G[C_1 \setminus M]$ contains a chordless path $Q = q_1, \dots, q_r$ such that q_1 is adjacent to u and q_r has a neighbor in M, and no vertex of Q has a neighbor in N. If we choose Q to be of shortest length, then no vertex of $Q \setminus \{q_1\}$ is adjacent to u, and no vertex of $Q \setminus \{q_r\}$ has a neighbor in M. So, since $u \notin X_1$ and hence Q is not an attachment of u to M, $V(Q) \cap (A \cup B) \neq \varnothing$, contradicting Lemma 112. It follows that $X \setminus X_1$ is anticomplete to C_1. Similar arguments show that $Y \setminus Y_1$ is anticomplete to C_1, and that $X \setminus X_2$ and $Y \setminus Y_2$ are both anticomplete to C_2. ■

Lemma 114. *X_i and Y_i are anticomplete to C_3 for every $1 \leqslant i \leqslant 2$.*

PROOF. By Lemma 111 and Lemma 112, $\{x_1, x_2, x_3, x_4\}$ is anticomplete to C_3. Now suppose that a vertex $u \in X_1 \setminus \{x_1\}$ has a neighbor in C_3. It follows that $G[C_3]$ contains a chordless path $T = t_1, \dots, t_m$ such that t_1 is adjacent to u, t_m is adjacent to a vertex $v \in Y \setminus \{x_3, x_4\}$, and $N_G(t_i) \cap V(H^\star) \subseteq \{x\}$ for every $1 \leqslant i \leqslant m$. Let T be such a path of minimum length. Then no vertex of $T \setminus \{t_1\}$ is adjacent to u, and no vertex of $T \setminus \{t_m\}$ has a neighbor in $Y \setminus \{x_3, x_4\}$. Note that T is an ear of $H^\star$, with attachments u and v. By Lemma 108, uv is not an edge. Let $Q = q_1, \dots, q_r$ be an attachment of u to M. Since T and Q belong to different components of $G \setminus S$, $V(T) \cap V(Q) = \varnothing$ and $V(T)$ is anticomplete to $V(Q)$. Also, $V(Q) \cap \{v\} = \varnothing$. Now, let R be the chordless uv-path contained in the graph induced by $M \cup V(Q) \cup \{u, v, x_4\}$. Then $V(R) \cup V(T)$ induces a hole H', and, by Lemma 90, (H', x) is an alternating wheel. If x is adjacent to q_1, let R' be the chordless x_3u-path contained in the graph induced by $M \cup V(Q) \cup \{u, x_3, x_4\}$. Then, by Lemma 100, $V(R') \cup V(S_2)$ induces a hole H'' and (H'', x) is a 3-wheel, a contradiction. Hence, x is adjacent to t_1. It follows that the vertex set $V(S_2) \cup V(T) \cup \{u, v, x\}$ induces a 3-wheel with center x, a contradiction. Therefore X_1 is anticomplete to C_3, and, similarly, so is Y_1. Analogous arguments show that $X_2 \cup Y_2$ is anticomplete to C_3. ■

PROOF OF THEOREM 84. By definitions, Lemma 104, Lemma 108, Lemma 109 and Lemma 110, $\mathcal{S} = (\{x\}, X_1, X_2, X_3, Y_1, Y_2, Y_3, C_1, C_2, C_3, C_X, C_Y)$ is a partition of $V(G)$. Moreover, if we set $y_1 = x_4$ and $y_2 = x_3$, $\mathcal{S}$ satisfies (i) by the definition of sets X_i, Y_i and C_i for $1 \leqslant i \leqslant 2$, and X_3, Y_3. By Lemma 108, X is anticomplete to Y, by Lemma 113 and Lemma 114, $X_i \cup Y_i$ is anticomplete to C_j if $i \neq j$, and, by definitions, x is complete to $X \cup Y$ and every vertex of $X_i \cup Y_i$ has a neighbor in C_i for $1 \leqslant i \leqslant 3$. So, $\mathcal{S}$ satisfies (ii). By Lemma 105 and Lemma 107, $\mathcal{S}$ satisfies (iii). Finally, properties (iv) and (v) follow from the definition of sets C_1, C_2, C_3, C_X and C_Y, and Lemma 109. So, G is structured. This proves the theorem. ■

3.4.1 Proof of Theorem 83

In this section we prove Theorem 83, so we further assume that $G \in \mathcal{G}$ does not contain a wheel with an appendix. By Theorem 82, it is enough to consider the case when G contains a long alternating wheel, that is, we may suppose that G satisfies Property 2. So, we keep the same notation as before, and we prove that the structured partition $\mathcal{S}$ of $V(G)$, obtained in Theorem 84, has some additional properties (under the assumption that G does not admit a clique cutset).

Lemma 115. *Let $u \in X_2'$ and let $Q = q_1, \dots, q_r$ be an attachment of u to N. Then x is anticomplete to $V(Q)$. Also, if x_2 is not adjacent to q_r, then it is anticomplete to $V(Q)$.*

Proof. Consider the following claims.

(1) x is anticomplete to $V(Q)$.

Proof of (1). Assume otherwise. Let q_i be the highest indexed vertex of Q that is adjacent to x, and let v be the neighbor of q_r in S_2 that is closest to x_3. By Lemma 98, x is not adjacent to q_r, and hence $i < r$.

First suppose that $x_2v \notin E(G)$. Then, by Lemma 98, x_2 is not adjacent to q_r, and x_2 must have a neighbor in $Q^{q_iq_{r-1}}$, since otherwise, by Lemma 98 and Lemma 102, $V(S_2) \cup V(Q^{q_iq_r}) \cup \{x\}$ either induces a theta or a pyramid. If x_2 has a neighbor in $Q^{q_{i+1}q_{r-1}}$, then the graph induced by $V(S_2) \cup V(Q^{q_{i+1}q_r}) \cup \{x\}$ contains a theta or a pyramid. So, $N_G(x_2) \cap V(Q^{q_iq_r}) = \{q_i\}$. By Lemma 102, the vx_1-subpath of $H \setminus \{x_2\}$, together with $V(Q^{q_iq_r}) \cup \{x, x_2\}$, induces a 3-wheel with center x, a contradiction. It follows that $x_2v \in E(G)$. By Lemma 102, the vx_3-subpath of S_2, together with $V(Q^{q_iq_r}) \cup \{x\}$, induces a hole H', and, by Lemma 90, (H', x_2) is an alternating wheel. So, $i < r - 1$, and x_2 is adjacent to q_i and q_r. But then, by Lemma 102, $Q^{q_iq_r}$ is an appendix of (H, x), a contradiction. □

(2) If x_2 is not adjacent to q_r, then it is anticomplete to $V(Q)$.

Proof of (2). Assume that x_2 is not adjacent to q_r but it is not anticomplete to $V(Q)$. Let q_i be the highest indexed vertex of Q that is adjacent to x_2, and let v be the neighbor of q_r in S_2 that is closest to x_3. By Lemma 102 and (1), the vx_3-subpath of S_2, together with $V(Q) \cup \{x, u\}$, induces a hole H', and therefore x_2v is not an edge, since otherwise (H', x_2) is a 1-wheel. But then, by Lemma 98, Lemma 102 and (1), $V(S_2) \cup V(Q^{q_iq_r}) \cup \{x\}$ either induces a theta or a pyramid, a contradiction. □

The lemma follows from (1) and (2). ■

Lemma 116. *X_1 (resp. Y_1) is complete to X_2 (resp. Y_2).*

Proof. Let $u \in X_1$ and $v \in X_2$, and suppose that $uv \notin E(G)$. It follows that $u \neq x_1$ and $v \neq x_2$. Let $Q = q_1, \dots, q_r$ (resp. $T = t_1, \dots, t_m$) be an attachment of v (resp. u) to N (resp. M). By Lemma 99, $V(Q) \cap V(T) = \varnothing$ and $V(Q)$ is anticomplete to $V(T)$.

Denote by w (resp. z) the neighbor of q_r (resp. t_m) in S_2 (resp. $V(H) \setminus V(S_2)$) that is closest to x_3 (resp. x_4). First suppose that $x_2w \in E(G)$. It follows that $r > 1$ and q_r is adjacent to x_2, since otherwise, by Lemma 102 and Lemma 115, the vertex set $V(S_2) \cup V(Q) \cup \{x, v\}$ induces a 1-wheel or a 3-wheel with center x_2. By Lemma 100, the x_2z-subpath of $H \setminus \{x_1\}$, together with $V(T) \cup \{u\}$, induces a hole H', and, by Lemma 90, (H', x) is an alternating wheel. Also, by Lemma 102 and Lemma 115, the chordless path induced by $V(Q) \cup \{v\}$ is an appendix of (H', x), a contradiction. Therefore x_2w is not an edge, and hence, by Lemma 98, x_2 is not adjacent to q_r. So, by Lemma 100, Lemma 102 and Lemma 115, the wz-subpath of $H \setminus \{x_1\}$, together with $V(Q) \cup V(T) \cup \{x_2, u, v\}$, induces a hole H'', and (H'', x) is a 3-wheel, a contradiction. So, X_1 is complete to X_2, and, by symmetry, Y_1 is complete to Y_2. ■

Proof of Theorem 83. Let $K = \{x\}$, $W_1 = X_1$, $Z_1 = Y_1$, $W_2 = X_2 \cup X_3$, $Z_2 = Y_2 \cup Y_3$, $V_1 = W_1 \cup Z_1 \cup C_1$ and $V_2 = W_2 \cup Z_2 \cup C_2 \cup C_3$. We now show that, if G does not admit a clique cutset, then (K, V_1, V_2) is a special 2-amalgam of G. By Lemma 105, Lemma 107 and Lemma 116, the sets $X_1 \cup X_2 \cup X_3$ and $Y_1 \cup Y_2 \cup Y_3$ are cliques, and hence so are the sets $N_G(C_X)$ and $N_G(C_Y)$. So, $C_X = C_Y = \varnothing$, and therefore (K, V_1, V_2) is a special 2-amalgam of G. ■

3.4.2 Proof of Lemma 98

We prove Lemma 98 by considering Property 1 and Property 2 separately.

The following simple result will be used throughout.

Lemma 117. *Let (H, x) be an alternating wheel in a graph $G \in \mathcal{G}$, and consider a vertex $y \in V(G) \setminus (V(H) \cup \{x\})$ that is adjacent to x. If u and v are consecutive neighbors of y in H, then they cannot belong to the interior of two different long sectors of (H, x).*

Proof. Otherwise the uv-subpath of H that does not contain any other neighbor of y in H, together with $\{x, y\}$, induces a 1-wheel with center x. ■

Property 1 holds.

We first assume that G satisfies Property 1. The wheel (H, x), its appendix P and the associated notation are as in the beginning of Section 3.4. Let y be a vertex of $G \setminus (V(H) \cup V(P) \cup \{x\})$; we set $N = V(S_2) \setminus \{x_2, x_3\}$, $M = V(H) \setminus (V(S_2) \cup \{x_1, x_4\})$ and $N' = N \cup V(P)$.

Lemma 118. *If y is adjacent to x and not adjacent to p_1, then $N_G(y) \cap V(P) = \varnothing$.*

Proof. Assume not and let p_i be the neighbor of y in P with lowest index. If y is adjacent to x_2, then $V(P^{p_1p_i}) \cup \{x, x_2, y\}$ induces a 3-wheel with center x_2. So, x_2y is not an edge. By Lemma 90, (W, y) is an alternating wheel, and so y is adjacent to x_3. If $i = k$, then y is also adjacent to x_2' and hence $\{x, x_2, x_2', y, p_k\}$ induces a 3-wheel with center p_k. So, $i < k$. Let y' be the neighbor of y in S_2 that is closest to x_2, and let R be the x_2y'-subpath of S_2. First assume that $y' \neq x_3$. If $N_G(p_j) \cap \{x_2\} = \varnothing$ for every $1 < j \leqslant i$, $V(R) \cup V(P^{p_1p_i}) \cup \{x, y\}$ induces a 3PC(xx_2p_1, y). Otherwise, let T be the chordless x_2y-path contained in the graph induced by $V(P^{p_2p_i}) \cup \{x_2, y\}$. But then $V(T) \cup V(R) \cup \{x\}$ induces a 3PC(x_2, y), a contradiction. So, $y' = x_3$. If there exists a chordless x_2y-path T contained in the graph induced by $V(P^{p_2p_i}) \cup \{x_2, y\}$, then

$V(S_2) \cup V(T) \cup \{x\}$ induces a 3PC(xx_3y, x_2). So, x_2 has no neighbors in $P^{p_2p_i}$. Let y'' be the neighbor of y in $H \setminus \{x_2\}$ that is closest to x_1. Then the $y''x_1$-subpath of $H \setminus \{x_2\}$, together with $V(P^{p_1p_i}) \cup \{x_2, y\}$, induces a hole H', and (H', x) is a 3-wheel, a contradiction. ■

Lemma 119. *If y is adjacent to p_1, then $N_G(y) \cap (V(H) \cup \{x\}) \subseteq \{x, x_1, x_2\}$.*

Proof. Assume that y is adjacent to p_1 but $N_G(y) \cap (V(H) \cup \{x\}) \nsubseteq \{x, x_1, x_2\}$.

(1) y is adjacent to x_2.

Proof of (1). Assume not and let y' be the neighbor of y in $H \setminus \{x_2\}$ that is closest to x_1. If $y' \neq x_2'$, then the $y'x_1$-subpath of $H \setminus \{x_2\}$, together with $\{x, x_2, y, p_1\}$, induces a 3-wheel with center x. Therefore, $y' = x_2'$. By Lemma 90, (W, y) is an alternating wheel, and hence y is adjacent to p_k. Let H' be the hole induced by $\{x_2, x_2', y, p_1\}$. Then (H', p_k) is a 3-wheel, a contradiction. □

(2) y is adjacent to x.

Proof of (2). Assume otherwise. By (1), x_2y is an edge. It follows that the graph induced by $(V(H) \setminus \{x_1, x_2\}) \cup \{x, y\}$ contains a chordless xy-path R, $V(R) \cup \{p_1\}$ induces a hole H', and (H', x_2) is a 3-wheel, a contradiction. □

(3) y is adjacent to x_1.

Proof of (3). Assume not. By (1) and (2), xy and x_2y are both edges. Let y' be the neighbor of y in $H \setminus \{x_2\}$ that is closest to x_1. First assume that $y' \neq x_2'$, and let R be the $y'x_1$-subpath of $H \setminus \{x_2\}$. Then $V(R) \cup \{x_2, y\}$ induces a hole H' and (H', x) is a 3-wheel. So, $y' = x_2'$. Let H'' be the hole induced by $(V(S_2) \setminus \{x_2\}) \cup \{x, y\}$. Then (H'', x_2) is a 3-wheel, a contradiction. □

By (1), (2) and (3), $\{x, x_1, x_2\} \subseteq N_G(y) \cap (V(H) \cup \{x\})$.

(4) y is anticomplete to $\{x_3, \ldots, x_n\}$.

Proof of (4). It must be that y is not adjacent to x_3 (resp. x_n), since otherwise (H_2, y) (resp. (H_n, y)) is a 3-wheel. Now assume that y is adjacent to x_i for some $3 < i < n$. In particular, let x_i be the neighbor of y in H with lowest index. By Lemma 90, (H, y) is an alternating wheel. If i is even, then (H_i, y) is a 3-wheel. So, i is odd.

Let R be the x_2x_i-subpath of $H \setminus \{x_1\}$. If y does not have a neighbor in $R \setminus \{x_2, x_i\}$, then $V(R) \cup \{x, y\}$ induces a 3-wheel with center x. So, y has a neighbor in $R \setminus \{x_2, \ldots, x_i\}$. Such a neighbor cannot belong to the interior of any long sector S_j of (H, x) for $2 < j < i-1$, since otherwise y and H_j contradict Lemma 90. Also, by Lemma 117, y does not have a neighbor in the interior of S_2 or S_{i-1}. W.l.o.g. assume that y has a neighbor in the interior of S_{i-1}. Then (H_{i-1}, y) is a wheel, and hence an alternating wheel, with appendix given by the x_2x_{i-2}-subpath of R. Since $|V(H_{i-1})| < |V(H)|$, our choice of (H, x) is contradicted. □

(5) y has no neighbors in the interior of any long sector of (H, x) that is not S_2.

Proof of (5). Assume not. If y has a neighbor in the interior of a long sector S_i of (H, x) for some $2 < i < n$, then, by (4), y and H_i contradict Lemma 90. So, y has a neighbor in the interior of S_n

and (H_n, y) is a wheel, and hence an alternating wheel, with rim shorter than H. By Lemma 117 and (4), y is anticomplete to $V(S_2) \setminus \{x_2\}$, and hence $N_G(y) \cap V(H) \subseteq (V(S_n) \setminus \{x_n\}) \cup \{x_2\}$. If $N_G(y) \cap V(P) = \{p_1\}$, then (H_n, y) has an appendix induced by the $x'_2 x_{n-1}$-subpath of $H \setminus \{x_2\}$ together with $V(P)$, contradicting our choice of (H, x). So, y has a neighbor in $P \setminus \{p_1\}$, and let p_j be such a neighbor with highest index. Let y' be the neighbor of y in S_n that is closest to x_n. Then the $x'_2 y'$-subpath of $H \setminus \{x_2\}$, together with $V(P^{p_j p_k}) \cup \{x, y\}$, induces a 1-wheel with center x, a contradiction. □

By (4) and (5), $N_G(y) \cap V(H) \subseteq (V(S_2) \setminus \{x_3\}) \cup \{x_1\}$, and, by our initial assumption, y has a neighbor in the interior of S_2. It follows that (H_2, y) is a wheel and hence an alternating wheel. Let y' be the neighbor of y in S_2 that is closest to x_3, and let R be the $y' x_1$-subpath of $H \setminus \{x_2\}$. Then $V(R) \cup \{y\}$ induces a hole H' and (H', x) is an alternating wheel with appendix induced by $(V(H) \setminus (V(R) \cup \{x_2\})) \cup V(P)$. Since $|V(H')| < |V(H)|$, our choice of (H, x) is contradicted. ■

Lemma 120. *y is anticomplete to at least one of N', M.*

PROOF. Suppose that y has a neighbor in both N' and M. It suffices to consider the following two cases.

Case 1: y has a neighbor in M and a neighbor in N.

(1) (H, y) is an alternating wheel.

Proof of (1). It follows from our assumptions and Lemma 90. □

(2) y is not adjacent to x.

Proof of (2). Assume it is. By Lemmas 118 and 119, $N_G(y) \cap V(P) = \varnothing$. Let u (resp. v) be the neighbor of y in $H \setminus \{x_2\}$ that is closest to x'_2 (resp. x_1). By our assumptions, $u \in N$ and $v \in M \cup \{x_1\}$. Let R be $x_2 u$-subpath of $H \setminus \{x_1\}$. If y is not adjacent to x_2, then the $v x_1$-subpath of $H \setminus \{x_2\}$, together with $V(R) \cup \{y\}$, induces a hole H'. Also, (H', x) is a wheel, and hence an alternating wheel, with appendix P and such that $|V(H')| < |V(H)|$, a contradiction. It follows that $x_2 y$ is an edge. But then $V(P) \cup V(R) \cup \{x, y\}$ induces a 3-wheel with center x_2, a contradiction. □

(3) (H, x) is a long alternating wheel.

Proof of (3). Assume not. So, (H, x) is a line wheel. By (2), xy is not an edge. Let R be the chordless xy-path contained in the graph induced by $N' \cup \{x, y\}$. If y has a single neighbor in S_4, then this neighbor belongs to the interior of S_4, which contradicts (1). If y has two neighbors in S_4, and these neighbors are adjacent, then $V(S_4) \cup V(R)$ induces a pyramid. It follows that y has non-adjacent neighbors in S_4. But then the graph induced by $V(S_4) \cup V(R)$ contains a 3PC(x, y), a contradiction. □

By (2), y is not adjacent to x. By (3), the graph induced by $M \cup \{x, y\}$ contains a chordless xy-path R. If y has non-adjacent neighbors in S_2, then the graph induced by $V(S_2) \cup V(R)$

contains a 3PC(x, y). So, by (1), y has two neighbors in S_2 and these neighbors are adjacent. But then $V(S_2) \cup V(R)$ induces a pyramid, a contradiction.

<u>**Case 2:**</u> y has a neighbor in M, a neighbor in P and no neighbors in N.

(4) y is not adjacent to x.

Proof of (4). Otherwise, by Lemma 118, yp_1 is an edge, and so Lemma 119 is contradicted. □

By (4), xy is not an edge. Let p_i be the neighbor of y in P with lowest index, and let R be the chordless xy-path induced by $V(P^{p_1 p_i}) \cup \{x, y\}$.

(5) y has at least two neighbors in $H \setminus V(S_2)$.

Proof of (5). Suppose that y' is the unique neighbor of y in $H \setminus V(S_2)$. Then $y' \in M$, and hence, by Lemma 90, $N_G(y) \cap V(H) = \{y'\}$. If y' belongs to the interior of a long sector S_i of (H, x) for some $4 \leqslant i \leqslant n$, then $V(S_i) \cup V(R)$ induces a 3PC(x, y'). So, $y' = x_j$ for some $4 < j \leqslant n$. First assume that j is even. Let R' be the chordless $x_2 p_i$-path contained in the graph induced by $V(P^{p_1 p_i}) \cup \{x_2\}$. Then the $x_j x_2$-subpath of $H \setminus \{x_3\}$, together with $V(R') \cup \{y\}$, induces a hole H', and (H', x) is a 1-wheel. So, j is odd. Let p_r be the neighbor of y in P with highest index. Then the $x_2' x_j$-subpath of $H \setminus \{x_2\}$, together with $V(P^{p_r p_k}) \cup \{x, y\}$, induces a 1-wheel with center x, a contradiction. □

(6) y does not have non-adjacent neighbors in $H \setminus V(S_2)$.

Proof of (6). Otherwise the graph induced by the vertex set $(V(H) \setminus V(S_2)) \cup V(R)$ contains a 3PC(x, y), a contradiction. □

By (5) and (6), y has two neighbors in $H \setminus V(S_2)$, say y' and y'', and $y'y''$ is an edge. If they both belong to the same long sector S_i of (H, x) for some $4 \leqslant i \leqslant n$, then $V(S_i) \cup V(R)$ induces a 3PC$(yy'y'', x)$. So, w.l.o.g. $y' = x_j$ and $y'' = x_{j+1}$ for some $4 < j < n$, j odd. By Lemma 90, y is not adjacent to x_2. Let R' be the chordless $x_2 p_i$-path contained in the graph induced by $V(P^{p_1 p_i}) \cup \{x_2\}$. Then the $x_{j+1} x_2$-subpath of $H \setminus \{x_3\}$, together with $V(R') \cup \{y\}$, induces a hole H', and (H', x) is a 1-wheel, a contradiction. ■

Proof of Lemma 98 (under the assumption that Property 1 holds).
Assume otherwise.

(1) y has no neighbors in M.

Proof of (1). Assume it does. By Lemma 120, y has no neighbors in N'. First suppose that y has non-adjacent neighbors in $H \setminus N$. Let y' (resp. y'') be the one that is closest to x_2 (resp. x_3). Then the $y'x_2$-subpath of $H \setminus N$, together with the $x_3 y''$-subpath of $H \setminus N$ and $V(S_2) \cup \{y\}$, induces a hole H'. By Lemma 90, (H', x) is an alternating wheel with appendix P. By Lemma 90, (H, y) is an alternating wheel, and hence $|V(H')| < |V(H)|$. It follows that (H', x) contradicts our choice of (H, x). So, y does not have non-adjacent neighbors in $H \setminus N$. But then y is adjacent to x and has a neighbor that belongs to the interior of a long sector S_i of (H, x) for some $4 \leqslant i \leqslant n$, and y and H_i contradict Lemma 90. □

(2) y is not adjacent to x_1.

Proof of (2). Assume it is. By (1), y has no neighbors in M. Suppose that y has no neighbors in $V(H)\setminus\{x_1, x_2\}$. It follows that y has a neighbor in P, and let p_i be such a neighbor with highest index. Then y is adjacent to x, since otherwise $(V(H) \setminus \{x_2\}) \cup V(P^{p_i p_k}) \cup \{x, y\}$ induces a 1-wheel with center x. So, by Lemma 118, yp_1 is an edge. Since $\{x, x_1, x_2, y, p_1\}$ cannot induce a 3-wheel with center x, y is adjacent to x_2. Therefore, y is complete to $\{x, x_1, x_2, p_1\}$. If $N_G(y) \cap V(P) = \{p_1\}$, then $(V(H) \setminus \{x_2\}) \cup V(P) \cup \{x, y\}$ induces a 3-wheel with center x. So, $\{p_1\} \subset N_G(y) \cap V(P)$. But then, by Lemma 90 applied to W and y, y is a special vertex of G, a contradiction.

It follows that y has a neighbor in $H \setminus (M \cup \{x_1, x_2\})$. Let y' be the one that is closest to x_4, and let H' be the hole induced by the $y'x_1$-subpath of $H \setminus \{x_2\}$ together with y. By Lemma 90, (H', x) is an alternating wheel, and hence $xy \in E(G)$ and $y' \notin \{x_3, x_4\}$. By Lemma 90, (H, y) is an alternating wheel, and so x_2y is an edge. By Lemma 90, (W, y) is an alternating wheel, and hence $yp_1 \in E(G)$, which contradicts Lemma 119. □

(3) y is not adjacent to x_4.

Proof of (3). Assume that y is adjacent to x_4. By (1) and (2), y has no neighbors in $M \cup \{x_1\}$. Suppose that y has a neighbor in P. Then, by Lemmas 118 and 119, y is not adjacent to x. Let R be a chordless x_2y-path contained in the graph induced by $V(P) \cup \{x_2, y\}$, and let H' be the hole induced by $(V(H) \setminus V(S_2)) \cup V(R)$. But then (since xy is not an edge) (H', x) is a 1-wheel, a contradiction. So, y has no neighbors in P. Since $N_G(y) \cap V(H)$ is not a clique, y must have a neighbor in $S_2 \setminus \{x_3\}$, and let y' be the one that is closest to x_2. Also, let H'' be the hole induced by y together with the x_4y'-subpath of $H \setminus \{x_3\}$. By Lemma 90, (H'', x) is an alternating wheel, and hence y is adjacent to x and $y' \neq x_2$. By Lemma 90, (H, y) is an alternating wheel, and so $|V(H'')| < |V(H)|$. Note that P is an appendix of (H'', x), and hence our choice of (H, x) is contradicted. □

(4) y is not adjacent to x_2.

Proof of (4). Assume it is. By (1), (2) and (3), y has no neighbors in $H \setminus V(S_2)$. First suppose that y has no neighbors in N. Then y is not adjacent to x_3, since otherwise $V(S_2) \cup \{x, y\}$ induces a theta or a 3-wheel with center x. Now assume that xy is an edge. By our assumptions, y has a neighbor in $P \setminus \{p_1\}$. So, by Lemma 118, y is adjacent to p_1. Let p_i be the neighbor of y in P with highest index. Since $V(W) \cup \{y\}$ must induce an alternating wheel by Lemma 90, $i > 2$. It follows that the chordless path induced by $V(P^{p_i p_k}) \cup \{y\}$ is an appendix of (H, x) that is shorter than P, a contradiction.

So, y is not adjacent to x and has a neighbor in P. Let p_j (resp. p_r) be the neighbor of y in P with lowest (resp. highest) index. First suppose that $j \neq r$ and p_jp_r is not an edge. By Lemma 90, (W, y) is an alternating wheel, and so $r > j + 3$. Therefore, the chordless path induced by $V(P^{p_1 p_j}) \cup V(P^{p_r p_k}) \cup \{y\}$ is an appendix of (H, x) that is shorter than P, a contradiction. Now assume that $p_jp_r \in E(G)$. Since $N_G(y) \cap (V(P) \cup \{x_2\})$ is not a clique of size three, x_2 is not adjacent to at least one of p_j, p_r, and hence the graph induced by $V(P) \cup \{x_2, y\}$ contains a 1-wheel or a 3-wheel with center y. It follows that $j = r$ and x_2p_j is not an edge. But then the graph induced by $V(P) \cup \{x_2, y\}$ contains a theta, a contradiction.

So, y has a neighbor in N, and let y' be the one that is closest to x_3 on S_2. First assume that $y' = x_2'$. Then y is anticomplete to $\{x, x_3\}$, since otherwise $V(S_2) \cup \{x, y\}$ induces a 1-wheel or a 3-wheel with center y. So, by our assumptions, y has a neighbor in $P \setminus \{p_k\}$, and let p_ℓ be the one with lowest index. By Lemma 90, (W, y) is an alternating wheel, and so $\ell < k - 2$. By Lemma 119, $\ell > 1$, and hence the chordless path induced by $V(P^{p_1p_\ell}) \cup \{y\}$ is an appendix of (H, x) that is shorter than P, a contradiction.

So, $y' \neq x_2'$ and $V(S_2) \cup \{x, y\}$ induces an alternating wheel with center y. If y is adjacent to x, then the $y'x_2$-subpath of $H \setminus \{x_2'\}$, together with $\{x, y\}$, induces a 3-wheel with center x. So, y is adjacent to x_2' and not adjacent to x. Also, (W, y) is an alternating wheel, and hence y is adjacent to p_k. Let p_s be the lowest indexed neighbor of y in P. By Lemma 119, $s > 1$. In particular, $s > 2$ and x_2p_s is an edge, since otherwise the $y'x_3$-subpath of S_2, together with $V(P^{p_1p_s}) \cup \{x, x_2, y\}$, induces a 1-wheel or a 3-wheel with center x_2. Let H' be the hole induced by the $y'x_2$-subpath of $H \setminus \{x_2'\}$ together with y. Then (H', x) is an alternating wheel with appendix $P^{p_1p_s}$ and such that $|V(H')| < |V(H)|$, which contradicts our choice of (H, x). □

(5) y has no neighbors in $V(S_2) \setminus \{x_2\}$.

Proof of (5). Assume otherwise and let y' be the neighbor of y in $V(S_2) \setminus \{x_2\}$ that is closest to x_3. By (1), (2), (3) and (4), y is anticomplete to $M \cup \{x_1, x_2, x_4\}$. If $y' = x_2'$, then y is not adjacent to x (else $V(S_2) \cup \{x, y\}$ induces a theta), and so, since $N_G(y) \cap (V(H) \cup V(P) \cup \{x\})$ is not a clique, y has a neighbor in $P \setminus \{p_k\}$. By Lemma 90, $V(W) \cup \{y\}$ induces an alternating wheel with center y, and therefore y is adjacent to p_k. Now, let p_r be the neighbor of y in $P \setminus \{p_k\}$ with lowest index, and let R be the chordless yx_2-path contained in the graph induced by $V(P^{p_1p_r}) \cup \{x_2, y\}$. It follows that $V(R) \cup \{x_2'\}$ induces a hole H', and (H', p_k) is a 3-wheel, a contradiction.

So, $y' \neq x_2'$. Let R' be the $y'x_2$-subpath of $H \setminus \{x_2'\}$. First suppose that $N_G(y) \cap V(P) \neq \varnothing$. Let p_i (resp. p_j) be the neighbor of y in P with lowest (resp. highest) index. By Lemma 119, $i > 1$. Then, by Lemma 118, xy is not an edge. If $i \neq j$ and p_ip_j is not an edge, then the graph induced by $V(R') \cup V(P^{p_1p_i}) \cup V(P^{p_jp_k}) \cup \{y\}$ contains a 3PC(y, x_2). Now assume that p_ip_j is an edge. If x_2 is not complete to $\{p_i, p_j\}$, then the graph induced by $V(R') \cup V(P) \cup \{y\}$ contains a 3PC(p_ip_jy, x_2). So, x_2 is adjacent to both p_i and p_j. Since (W, x_2) is an alternating wheel, $i > 2$ and p_{i-1} is not adjacent to x_2. It follows that the $y'x_3$-subpath of S_2, together with $V(P^{p_1p_i}) \cup \{x, x_2, y\}$, induces a 1-wheel with center x_2, a contradiction. Therefore, $i = j$. By Lemma 90, (W, y) is an alternating wheel. So, $N_G(y) \cap V(P) = \{p_k\}$, y is adjacent to x_2', and $x_2'y'$ is not an edge. Then the $y'x_3$-subpath of S_2, together with $\{x, x_2, x_2', y, p_k\}$, induces a 3-wheel with center p_k, a contradiction.

It follows that y has no neighbors in P. Now, let y'' be the neighbor of y in $V(S_2) \setminus \{x_2\}$ that is closest to x_2'. If y is adjacent to x, then, by Lemma 90, (W, y) is an alternating wheel, and hence $y' = x_3$, $y' \neq y''$ and $y'y''$ is not an edge. So, the $x_2'y''$-subpath of $H \setminus \{x_2\}$, together with $V(R') \cup \{x, y\}$, induces a 3-wheel with center x. Therefore, y is not adjacent to x. Since

$N_G(y) \cap (V(H) \cup V(P) \cup \{x\})$ is not a clique, $y' \neq y''$, $y'y''$ is not an edge and hence, by Lemma 90, (W, y) is an alternating wheel. It follows that the $x_2'y''$-subpath of $H \setminus \{x_2\}$, together with $V(R') \cup \{y\}$, induces a hole H' that is shorter than H. Also, (H', x) is an alternating wheel with appendix P, a contradiction. $\square$

By (1), (2), (3), (4) and (5), $N_G(y) \cap (V(H) \cup V(P) \cup \{x\}) \subseteq V(P) \cup \{x\}$, and, since $N_G(y) \cap (V(H) \cup V(P) \cup \{x\})$ is not a clique, (W, y) is an alternating wheel by Lemma 90. If y is not adjacent to x, then the graph induced by $V(P) \cup \{y\}$ contains a chordless p_1p_k-path that contains y and is an appendix of (H, x) that is shorter than P, a contradiction. Thus, y is adjacent to x, and hence $\{p_1\} \subset N_G(y) \cap V(P)$. Let p_i be the neighbor of y in P with highest index. Then the graph induced by $V(P^{p_ip_k}) \cup \{x, x_2, y, p_1\}$ contains a 3-wheel with center p_1, a contradiction. This proves the lemma. $\blacksquare$

Property 2 holds.

We now assume that G satisfies Property 2. The wheel (H, x) and the associated notation are as in the beginning of Section 3.4.

PROOF OF LEMMA 98 (under the assumption that Property 2 holds).
Let $y \in V(G) \setminus (V(H) \cup \{x\})$ and assume that $N_G(y) \cap (V(H) \cup \{x\})$ is not a clique. It suffices to consider the following two cases.

Case 1: y is adjacent to x.

(1) For every long sector S_i of (H, x), either $N_G(y) \cap V(S_i) \subseteq \{x_j\}$ for $i \leqslant j \leqslant i+1$ or (H_i, y) is a line wheel.

Proof of (1). Note that y cannot be adjacent to both x_i and x_{i+1}, since otherwise (H_i, y) is a 3-wheel. If y has a neighbor in the interior of S_i, then, by Lemma 90, (H_i, y) is an alternating wheel. In particular, by our choice of (H, x), (H_i, y) is a line wheel and hence (1) holds. $\square$

(2) y is complete to a short sector of (H, x).

Proof of (2). Assume not. W.l.o.g. y has a neighbor in a long sector S_i of (H, x). Therefore, by (1), y has a neighbor in $\{x_i, x_{i+1}\}$. By symmetry, w.l.o.g. y is adjacent to x_i. Let y' be the neighbor of y in $H \setminus \{x_i\}$ that is closest to x_{i-1} (it exists since $N_G(y) \cap (V(H) \cup \{x\})$ is not a clique). By (1), $y' \neq x_i'$. But then the $y'x_i$-subpath of H that contains x_{i-1}, together with $\{x, y\}$, induces a 3-wheel with center x, a contradiction. $\square$

By (2), and by symmetry, w.l.o.g. we may assume that y is complete to S_1. Moreover, by our assumptions, y has a neighbor in $H \setminus \{x_1, x_2\}$.

(3) y has a neighbor in $\{x_3, \dots, x_n\}$.

Proof of (3). Assume otherwise. It follows that y has a neighbor in the interior of a long sector of (H, x), say S_i. Note that $i = 2$ or $i = n$, since otherwise (1) is contradicted. W.l.o.g. let $i = 2$, and let y' be the neighbor of y in the interior of S_2 that is closest to x_3 on S_2. By (1),

$y' \neq x_2'$. By Lemma 117, y has no neighbors in the interior of S_n. But then the $y'x_1$-subpath of $H \setminus \{x_2\}$, together with $\{x, y\}$, induces a long alternating wheel with center x and rim shorter than H, a contradiction. □

(4) y is anticomplete to $\{x_3, x_4, x_{n-1}, x_n\}$.

Proof of (4). Assume not and, by symmetry, w.l.o.g. suppose that y has a neighbor in $\{x_3, x_4\}$. By (1), x_3y is not an edge. But then the graph induced by $V(S_2) \cup \{x, x_4, y\}$ contains a 3-wheel with center x, a contradiction. □

By (3) and (4), y is adjacent to x_i for some $4 < i < n-1$. W.l.o.g. we may assume that y has no neighbors in $\{x_3, x_4, \ldots, x_{i-1}\}$.

(5) i is odd, and x_2 and x_i are not consecutive neighbors of y in H.

Proof of (5). Let R be the x_2x_i-subpath of $H \setminus \{x_1\}$, and let y' be the neighbor of y in $R \setminus \{x_i\}$ that is closest to x_i on R. Now, let R' be the $y'x_i$-subpath of R. If i is even or $y' = x_2$, then $V(R') \cup \{x, y\}$ induces a 3-wheel with center x, a contradiction. □

By (5), y has a neighbor in the interior of a long sector S_j of (H, x) for some $1 < j < i$. By (1), $j = 2$ or $j = i-1$. Thus, by symmetry, w.l.o.g. let $j = 2$, and, by (1), let y' and y'' be the adjacent neighbors of y in the interior of S_2, where y' is closer to x_2 on S_2. By Lemma 117, y'' and x_i are consecutive neighbors of y in H. Also, since (H_2, y) is a line wheel, y' is not adjacent to x_2. Let H' be the hole induced by the $y''x_i$-subpath of $H \setminus \{x_2\}$ together with y. Then (H', x) is an alternating wheel with appendix given by the x_2y'-subpath of S_2, a contradiction.

Case 2: y is not adjacent to x.

(6) (H, y) is an alternating wheel.

Proof of (6). Since $N_G(y) \cap V(H)$ is not a clique, y has at least two non-adjacent neighbors in H, and so, by Lemma 90, (6) holds. □

By symmetry, w.l.o.g. we may assume that y has a neighbor in S_2.

(7) $N_G(y) \cap V(H) \subseteq V(S_2) \cup \{x_1, x_4\}$.

Proof of (7). Assume not. Then the graph induced by $(V(H) \setminus (V(S_2) \cup \{x_1, x_4\})) \cup \{x, y\}$ contains a chordless xy-path R. If y has non-adjacent neighbors in S_2, then the graph induced by $V(S_2) \cup V(R)$ contains a 3PC(x, y). If y has exactly two neighbors in S_2, then, by Lemma 90 applied to y and H_2, these two neighbors are adjacent and hence $V(S_2) \cup V(R)$ induces a pyramid. Therefore, y has a unique neighbor y' in S_2. If $y' \notin \{x_2, x_3\}$, then $V(S_2) \cup V(R)$ induces a 3PC(x, y'). So, w.l.o.g. $y' = x_2$. Let y'' be the neighbor of y in $H \setminus V(S_2)$ that is closest to x_4. Since $y'' \neq x_1$, the x_2y''-subpath of H that contains x_3, together with $\{x, y\}$, induces a 1-wheel with center x, a contradiction. □

Let y' (resp. y'') be the neighbor of y in the path induced by $V(S_2) \cup \{x_1, x_4\}$ that is closest to x_1 (resp. x_4). By (6) and (7), the $y'y''$-subpath of H that contains x_5, together with y, induces a hole H' that is shorter than H. Also, by Lemma 90, (H', x) is an alternating wheel, and so our choice of (H, x) is contradicted. This proves the lemma. ■

3.5 Proof of Theorem 85

Proof of Theorem 85. We prove Theorem 85 by induction on $|V(G)|$. By Theorem 80, G is the line graph of a triangle-free graph or it admits a clique cutset or a small 2-amalgam. Let us consider these three cases separately.

Case 1: G is the line graph of a triangle-free graph.

By Vizing's theorem, $\chi(G) \leqslant \omega(G) + 1 \leqslant 4$, and so the theorem holds.

Case 2: G admits a clique cutset.

Let K be a clique cutset of G, and let $C_1, \ldots, C_k$, $k \geqslant 2$, be the components of $G \setminus K$. For $1 \leqslant i \leqslant k$, let $G_i = G[V(C_i) \cup K]$. By the inductive hypothesis, G_i is 4-colorable for every $1 \leqslant i \leqslant k$. For $1 \leqslant i \leqslant k$, let c_i be a 4-coloring of G_i. Since K is a clique, vertices of K must have different colors in all of these colorings. So, we can permute the colors of the c_i's so that they all agree on the vertices of K, and by putting them together we get a coloring of G that uses at most four colors.

Case 3: G admits a small 2-amalgam.

Let $K = \{x\}$, $W_1 = \{x_1\}$, $Z_1 = \{x_4\}$, $W_2 = \{x_2\}$, $Z_2 = \{x_3\}$, $W_1 \cup Z_1 \subset V_1$ and $W_2 \cup Z_2 \subset V_2$, and let (K, V_1, V_2) be a small 2-amalgam of G. Furthermore, let $G_1 = G[V_1 \cup \{x, x_2, x_3\}]$ and $G_2 = G[V_2 \cup \{x, x_1, x_4\}]$. Since G is K_4-free, $\omega(G) = \omega(G_1) = \omega(G_2) = 3$. Let c_1 (resp. c_2) be a 4-coloring of G_1 (resp. G_2). W.l.o.g. we may assume that c_1 and c_2 agree on $\{x, x_1, x_2\}$. If they also agree on $\{x_3, x_4\}$, then we are done. So, consider the case where they do not agree. W.l.o.g. suppose that $c_1(x) = c_2(x) = 1$, $c_1(x_1) = c_2(x_1) = 2$ and $c_1(x_2) = c_2(x_2) = 3$.

If $c_1(x_4) \neq c_2(x_3)$, then we can obtain a 4-coloring of G by coloring every vertex in $V(G_1) \setminus \{x_3\}$ with the same color as in c_1, and by coloring every vertex in $V(G_2) \setminus \{x_4\}$ with the same color as in c_2.

So, assume that $c_1(x_4) = c_2(x_3)$. First suppose that $c_1(x_4) \in \{2, 3\}$. Then w.l.o.g. we may assume that $c_1(x_4) = 2$. To obtain a 4-coloring c of G we first color every vertex in $V(G_1) \setminus \{x_3\}$ with the same color as in c_1. Then, for every vertex $v \in V_2$, we define $c(v)$ in the following way: $c(v) = c_2(v)$ if $c_2(v) \in \{1, 3\}$, $c(v) = 2$ if $c_2(v) = 4$, and $c(v) = 4$ if $c_2(v) = 2$.

Finally, let $c_1(x_4) = c_2(x_3) = 4$. To obtain a 4-coloring c of G we first color every vertex in $V(G_1) \setminus \{x_3\}$ with the same color as in c_1. Then, for every vertex $v \in V_2$, we define $c(v)$ in the following way: $c(v) = c_2(v)$ if $c_2(v) \in \{1, 3\}$, $c(v) = 2$ if $c_2(v) = 4$, and $c(v) = 4$ if $c_2(v) = 2$. This completes the proof of the theorem. ■

Chapter 4

Conclusions

In this thesis, we applied the decomposition method to investigate the structure of several hereditary graph classes, which generalize the well-known class of chordal graphs. All of these classes were defined by excluding (some) Truemper configurations as induced subgraphs.

In Chapter 2, we studied the class $\mathcal{G}_{\mathrm{UT}}$ of all graphs that contain no Truemper configurations other than (possibly) universal wheels and twin wheels, together with three proper subclasses of $\mathcal{G}_{\mathrm{UT}}$, namely $\mathcal{G}_{\mathrm{U}}$, $\mathcal{G}_{\mathrm{T}}$ and $\mathcal{G}_{\mathrm{UT}}^{\text{cap-free}}$. For each of these classes, we proved a decomposition theorem that involves only clique cutsets, which we later used to show that the recognition problem can be solved in polynomial time for all of them. Our decomposition theorem for class $\mathcal{G}_{\mathrm{UT}}$ does not seem to be powerful enough to allow us to design a decomposition-based polynomial-time algorithm that solves any of the combinatorial optimization problems we are interested in, which thus appear to be out of our reach. We remark that it would not be surprising if no such algorithm existed. On the other hand, we provided decomposition-based polynomial-time algorithms that solve the maximum clique and maximum stable set problems for classes $\mathcal{G}_{\mathrm{U}}$, $\mathcal{G}_{\mathrm{T}}$ and $\mathcal{G}_{\mathrm{UT}}^{\text{cap-free}}$, as well as the optimal coloring problem for classes $\mathcal{G}_{\mathrm{U}}$ and $\mathcal{G}_{\mathrm{UT}}^{\text{cap-free}}$. However, shortly after this thesis was submitted, Maffray, Penev and Vušković [28] proved that rings can be optimally colored in polynomial time, and hence so can all graphs in $\mathcal{G}_{\mathrm{T}}$. This solves one of the problems we left as open in Chapter 2. Finally, we also obtained χ-boundedness results for all of the four classes; we believe that that the bounds we presented for $\mathcal{G}_{\mathrm{UT}}$ and $\mathcal{G}_{\mathrm{T}}$ are not optimal, and hence can be improved.

In Chapter 3, we studied the class $\mathcal{G}$ of all graphs that, out of all Truemper configurations, may only contain prisms and alternating wheels. The decomposition theorem we proved for class $\mathcal{G}$ involves clique cutsets and (very structured) star cutsets, it was used to show that K_4-free graphs in $\mathcal{G}$ are 4-colorable, but had no algorithmic consequences. As we discussed already, it appears that many well-known hereditary graph classes require star cutsets when we want to decompose any given graph in the class down to a sufficiently simple basic graph. However, how to make use of star cutsets in algorithms is still mostly unclear. This is the main obstacle that prevents the decomposition method from being successful when applied to any of these classes. So, in the future, it would be definitely worth trying to develop techniques for handling star cutsets, at least when they satisfy several additional constraints, as they do in this thesis.

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