Integral Forms of Hall Algebras and their Limits

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Abstract

In this thesis we tell the story of how two isomorphic algebras – quantized enveloping algebras and Bridgeland-Hall algebras – are simultaneous deformations of two simpler algebras: the universal enveloping algebra of a Lie algebra and the coordinate algebra of a Poisson-Lie group.

We will also explain how a similar deformation picture holds for Hall algebras, of which Bridgeland-Hall algebras are a generalization, and a subalgebra of the quantized enveloping algebra called its positive part.

Our particular contribution to this story is to establish the precise way in which Bridgeland-Hall algebras deform coordinate algebras of Poisson-Lie groups. We will give a calculation of the Hall algebraic structure of the resulting Poisson-Lie groups and also explain the relationship with how quantized enveloping algebras deform coordinate algebras of Poisson-Lie groups.

Using the Bridgeland-Hall algebra approach to Poisson-Lie groups we will give a new way to extract simple Lie algebras from Bridgeland-Hall algebras and in addition provide a computation of the Hall algebraic structure of these Lie algebras.

Finally we provide a new, more direct proof of an old but tricky to prove theorem due to De Concini and Procesi that quantized enveloping algebras are deformations of the coordinate algebra of a particular Poisson-Lie group called the standard dual Poisson-Lie group.

Acknowledgements

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Introduction

Thesis Overview

This thesis is concerned with telling the story of how two isomorphic algebras – quantized enveloping algebras and Bridgeland-Hall algebras – are simultaneous deformations of two simpler algebras. One of these more elementary algebras is the universal enveloping algebra of a Lie algebra while the other is the coordinate algebra of a Poisson-Lie group. A Poisson-Lie group is an algebraic group which is also a Poisson variety in a compatible way.

The exact narrative we will give has not been told before although a number of the results can be found scattered throughout the literature, particularly in the works of Deng and Chen [CD15] and Ringel [Rin90b]. Our contribution is to work out the details of how Bridgeland-Hall algebras are deformations of coordinate algebras of Poisson-Lie groups. In addition we establish the relationship with how quantized enveloping algebras deform coordinate algebras of Poisson-Lie groups.

Quantized enveloping algebras are a type of algebra originally introduced by Drinfel'd and Jimbo to deform universal enveloping algebras of Lie algebras. Bridgeland-Hall algebras are a kind of algebra which one can assign to certain Abelian categories. Bridgeland [Bri13] showed that for categories of finite dimensional representations of a simply-laced quiver the resulting Bridgeland-Hall algebra is isomorphic to the quantized enveloping algebra of a simple Lie algebra.

Bridgeland-Hall algebras are a generalization of Hall algebras which are algebras similarly associated to certain Abelian categories. These were originally introduced by Ringel [Rin90a] based on the work of Hall [Hal59]. Ringel proved that for categories of simply-laced quiver representations the resulting Hall algebra is isomorphic to a subalgebra of the quantized enveloping algebra called its positive part. Bridgeland's motivation for his algebras was the question of how to find a Hall algebraic way to recover the whole quantized enveloping algebra and not only its positive part.

The key notion for how algebras may be simultaneous deformations of different algebras is that of an integral form. An integral form is a special $\mathbb{C}[t, t^{-1}]$-subalgebra of a $\mathbb{C}(t)$-algebra which allows one to formally set t to be certain values. A $\mathbb{C}(t)$-algebra can have many different integral forms with each degenerating to *different* algebras on specializing t at a certain value.

The algebras we have mentioned are either $\mathbb{C}(t)$-algebras or in the case of our (Bridgeland-)Hall algebras can naturally be upgraded to $\mathbb{C}(t)$-algebras. These algebras will each have two special integral forms which respectively have a universal enveloping algebra and a coordinate algebra as their $t = 1$ limit. Note, in this thesis we shall use the terms specialization at $t = 1$ and $t = 1$ limit interchangeably.

During the course of this thesis we will explain the simultaneous deformation picture of each of the above algebras independently. This involves discussing each type of algebra's integral forms and $t = 1$ limits. Although isomorphic, the languages of Bridgeland-Hall algebras and quantized enveloping algebras are quite different. As such we will stress not only the deformation story of each type of algebra independently but also how the two pictures match up with each other.

Once we have explained the simultaneous deformation story, another goal of this thesis is to use Bridgeland-Hall algebras to investigate the structure of Poisson-Lie groups and related objects. One example of the results we obtain is that we can give a very explicit calculation of the Hall algebraic structure of Poisson-Lie groups arising from Bridgeland-Hall algebras

Another example comes from an interesting question regarding how Lie algebras arise from Hall algebras. A general observation is that Hall algebras of Abelian categories often have some kind of associated Lie algebra. Unfortunately these Lie algebras are only ever positive parts of a bigger Lie algebra, just as a nilpotent subalgebra is only a part of a simple Lie algebra. An important question then, is how can one realize full Lie algebras in a Hall algebraic way.

Deng and Chen [CD15] were the first to do this by using the universal enveloping algebra component of the deformation story to extract simple Lie algebras from Bridgeland-Hall algebras of categories of simply-laced quiver representations. In this thesis we will explain an alternative way to extract simple Lie algebras from Bridgeland-Hall algebras using the Poisson-Lie group picture.

We will in fact obtain more than a Lie algebra, we will obtain a Lie bialgebra. Lie bialgebras are the infinitesimal analogues of Poisson-Lie groups in the same way that Lie algebras are the infinitesimal analogues of Lie groups. An upshot of our approach is that we will be able to give a very explicit calculation of the Hall algebraic structure of the Lie (bi)algebras we obtain.

Our final goal in this thesis will be to use our results to give a simplification of a difficult proof from the theory of quantized enveloping algebras. In particular, in [DCP93] De Concini and Procesi gave an isomorphism between the Poisson-Lie group arising from quantized enveloping algebras and a particular type of Poisson-Lie group $\boldsymbol{G}^\vee$ called the standard dual Poisson-Lie group. Poisson-Lie groups satisfy a type of duality for which $\boldsymbol{G}^\vee$ can be considered dual to the simple Lie group $\boldsymbol{G}$ with tangent Lie algebra $\mathfrak{g}$.

An unfortunate feature of De Concini and Procesi's proof, however, is that it involves a lengthy case-by-case analysis. As we will explain, using the Bridgeland-Hall algebra approach to Poisson-Lie groups we can give a new, more direct proof of this fact.

In the remainder of this introduction we will give a more in-depth discussion of the various concepts and results we have just outlined.

Integral Forms

The notion of integral forms gives a precise way to say how a $\mathbb{C}(t)$-algebra may simultaneously deform several different algebras.

Suppose that we have a $\mathbb{C}(t)$-algebra B. The algebra B depends on a parameter t and one would often like to set t to be a particular value, for example $t = 1$, in order to study the resulting algebra. A problem arises however: the existence of elements with poles at $t = 1$ makes setting $t = 1$ behave badly.

To get around this problem, one instead takes certain $\mathbb{C}[t,t^{-1}]$-subalgebras $Z \subseteq B$ called integral forms. The defining property of such subalgebras is that the multiplication map induces a $\mathbb{C}(t)$-algebra isomorphism of the following form.

$$\mathbb{C}(t) \otimes_{\mathbb{C}[t,t^{-1}]} Z \to B$$

For an integral form $Z \subseteq B$, its $t=1$ limit is defined to be the quotient algebra of Z by the ideal $(t-1)$. There can be several choices of integral forms of B and *each* can have a different algebra as its $t=1$ limit. In this way a $\mathbb{C}(t)$-algebra may deform several quite different algebras at once.

Quantized Enveloping Algebras: Integral Forms and $t=1$ Limits

Quantized enveloping algebras are $\mathbb{C}(t)$-algebras which were originally introduced by Drinfel'd and Jimbo to deform universal enveloping algebras. Let us give an idea of how they are defined and then say a little bit about their integral forms and $t=1$ limits.

Any complex simple Lie algebra $\mathfrak{g}$ has an associated $\mathbb{C}$-algebra $\mathrm{U}(\mathfrak{g})$ called its universal enveloping algebra. Simple Lie algebras have a well-known generators and relations description. The *same* generators and relations – viewed instead as generating an associative $\mathbb{C}$-algebra – also define $\mathrm{U}(\mathfrak{g})$. The quantized enveloping algebra $\mathrm{U}_t(\mathfrak{g})$ then, is defined by modifying the generators and relations description of $\mathrm{U}(\mathfrak{g})$ to obtain a $\mathbb{C}(t)$-algebra.

As one might hope, in light of its name, one is able to recover $\mathrm{U}(\mathfrak{g})$ from $\mathrm{U}_t(\mathfrak{g})$. In particular in [Lus90a] Lusztig introduced an integral form $\mathrm{U}_t^{Res}(\mathfrak{g}) \subset \mathrm{U}_t(\mathfrak{g})$ called the restricted integral form and proved the following theorem.

Theorem (Lusztig)**.** The $t=1$ limit of $\mathrm{U}_t^{Res}(\mathfrak{g})$ is isomorphic to $\mathrm{U}(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$.

The interesting thing is that there is another natural integral form of $\mathrm{U}_t(\mathfrak{g})$ whose $t=1$ limit is the algebra of functions on a Poisson-Lie group. A Poisson-Lie group is an algebraic group which is also a Poisson variety in a compatible way. In the affine case a Poisson variety is one whose coordinate algebra is endowed with a Poisson bracket.

We will mainly be concerned with a particular type of Poisson-Lie group $\boldsymbol{G}^\vee$ called the standard dual Poisson-Lie group. Poisson-Lie groups satisfy a kind of duality for which $\boldsymbol{G}^\vee$ can be considered dual to the simple Lie group $\boldsymbol{G}$ with tangent Lie algebra $\mathfrak{g}$.

In [DCP93] De Concini and Procesi introduced an integral form of $\mathrm{U}_t(\mathfrak{g})$ which we will denote by $\mathrm{U}_t^{Poiss}(\mathfrak{g})$. We will refer to this integral form as the Poisson integral form, although our terminology here is non-standard. De Concini and Procesi then proved the following non-trivial theorem via a lengthy case-by-case proof.

Theorem (De Concini, Procesi)**.** The $t=1$ limit of $\mathrm{U}_t^{Poiss}(\mathfrak{g})$ is isomorphic to the coordinate algebra of the standard dual Poisson Lie group $\boldsymbol{G}^\vee$.

Thus $\mathrm{U}_t(\mathfrak{g})$ can be viewed as simultaneously deforming both $\mathrm{U}(\mathfrak{g})$ and $\mathbb{C}[\boldsymbol{G}^\vee]$. It will be convenient to use the terminology quasi-classical and semi-classical limit to differentiate between these two $t = 1$ limits of $\mathrm{U}_t(\mathfrak{g})$ respectively. We may package all of this diagrammatically as follows.

$$\begin{array}{ccc} & \mathrm{U}_t(\mathfrak{g}) & \\ {\scriptstyle \text{quasi-classical limit } t \to 1} \swarrow & & \searrow {\scriptstyle \text{semi-classical limit } t \to 1} \\ \mathrm{U}(\mathfrak{g}) & & \mathbb{C}[\boldsymbol{G}^\vee] \end{array} \tag{1}$$

There is a special case of this story which will be important when we link things up with Hall algebras. In particular there is a certain subalgebra $\mathrm{U}_t(\mathfrak{n}_+)$ of $\mathrm{U}_t(\mathfrak{g})$ called the positive part of the quantized enveloping algebra. This subalgebra is the analogue the positive nilpotent subalgebra $\mathfrak{n}_+ \subset \mathfrak{g}$.

The Poisson and restricted integral forms of $\mathrm{U}_t(\mathfrak{g})$ descend to two natural integral forms $\mathrm{U}_t^{Poiss}(\mathfrak{n}_+)$ and $\mathrm{U}_t^{Res}(\mathfrak{n}_+)$ of the positive part respectively. The $t = 1$ limit of $\mathrm{U}_t^{Res}(\mathfrak{n}_+)$ gives the universal enveloping algebra of $\mathfrak{n}_+$ while that of $\mathrm{U}_t^{Poiss}(\mathfrak{n}_+)$ gives a Poisson subalgebra $\mathbb{C}[\boldsymbol{N}_+] \subset \mathbb{C}[\boldsymbol{G}^\vee]$. Here $\boldsymbol{N}_+$ is the underlying variety of the positive unipotent subgroup of the simple Lie group $\boldsymbol{G}$. Again we may represent these remarks diagrammatically as follows.

$$\begin{array}{ccc} & \mathrm{U}_t(\mathfrak{n}_+) & \\ {\scriptstyle \text{quasi-classical limit } t \to 1} \swarrow & & \searrow {\scriptstyle \text{semi-classical limit } t \to 1} \\ \mathrm{U}(\mathfrak{n}_+) & & \mathbb{C}[\boldsymbol{N}_+] \end{array} \tag{2}$$

We will now introduce (generic) Hall algebras and Bridgeland-Hall algebras of categories of quiver representations. The former will be isomorphic to $\mathrm{U}_t(\mathfrak{n}_+)$ while the later is isomorphic to $\mathrm{U}_t(\mathfrak{g})$. In particular we will be interested in understanding what the pictures (1) and (2) look like from a Hall algebraic perspective.

Hall Algebras: An Overview

Hall algebras are a type of associative algebra which one can assign to any finitary Abelian category. A finitary Abelian category is a small Abelian category such that all Hom and Ext^1 groups have only finitely many elements.

For the purposes of this thesis, the only finitary Abelian categories we will consider are the categories $\mathcal{A}_q$ of representations of a simply-laced quiver $\vec{Q}$ over a finite field $\mathbb{F}_q$. Here q is the number of elements in $\mathbb{F}_q$ and a quiver is simply-laced if on forgetting the direction of arrows of $\vec{Q}$ then one is left with a simply-laced Dynkin diagram. To any such quiver can be assigned a simple Lie algebra $\mathfrak{g}$ of corresponding Dynkin type. In this thesis we will fix $\mathfrak{g}$ to be the Lie algebra associated to $\vec{Q}$ in this way.

The basic idea for Hall algebras is to form a vector space whose elements are linear combinations of (isomorphism classes of) objects in one's chosen category.

$$\mathrm{H}_q := \bigoplus_{L \in \mathrm{Iso}(\mathcal{A}_q)} \mathbb{C} \cdot E_L \tag{3}$$

One then places the following product on H_q which roughly speaking counts extensions of objects.

$$E_M E_N = q^{1/2\langle \hat{M}, \hat{N}\rangle} \sum_{L \in \mathrm{Iso}(\mathcal{A}_q)} \frac{|\mathrm{Ext}^1_{\mathcal{A}_q}(M,N)_L|}{|\mathrm{Hom}_{\mathcal{A}_q}(M,N)|} E_L \tag{4}$$

Here $\langle \hat{M}, \hat{N}\rangle := \dim_{\mathbb{F}_q} \mathrm{Hom}_{\mathcal{A}_q}(M,N) - \dim_{\mathbb{F}_q} \mathrm{Ext}^1_{\mathcal{A}_q}(M,N)$ is what is called the Euler form of $\mathcal{A}_q$ and $\mathrm{Ext}^1_{\mathcal{A}_q}(M,N)_L$ is the set of extensions of M by N whose middle term is isomorphic to L.

A remark which will become important with regards integral forms of Hall algebras is that there is an equivalent product that one may place on H_q. If we define $F^L_{M,N}$ to be the set of subobjects $N \subseteq L$ with quotient object M then with respect to the alternative set of basis vectors $X_L := E_L/|\mathrm{Aut}(L)|$ one can show that Equation (4) is given by the following formula. Here we have denoted the group of automorphisms of L by $\mathrm{Aut}(L)$.

$$X_M X_N = \sum_{L \in \mathrm{Iso}(\mathcal{A}_q)} F^L_{M,N} X_L \tag{5}$$

So far these algebras are $\mathbb{C}$-algebras as opposed to $\mathbb{C}(t)$-algebras, as was the case for quantized enveloping algebras. However note that we have actually defined a whole family of algebras – one for each q a prime power. Ringel observed that the structure constants of H_q are Laurent polynomials in $q^{1/2}$. By this we mean there exist elements of $\mathbb{C}[t, t^{-1}]$ such that when we set $t = q^{1/2}$ one recovers the structure constants of Equation (4) and Equation (5).

The upshot is that one may define a new Hall algebra by replacing $\mathbb{C}$ in Equation (3) with $\mathbb{C}(t)$ and swapping the structure constants of Equation (4) or equivalently Equation (5) for their Laurent polynomial versions. One then obtains a $\mathbb{C}(t)$-algebra called the generic Hall algebra which we will denote by H. Either product Equation (4) or Equation (5) results in the same generic Hall algebra over $\mathbb{C}(t)$.

Although the generic Hall algebra is defined purely in terms of homological data of the categories $\mathcal{A}_q$ a wonderful theorem due to Ringel relates them to positive parts of quantized enveloping algebras.

Theorem (Ringel)**.** There is an isomorphism of $\mathbb{C}(t)$-algebras $\mathrm{U}_t(\mathfrak{n}_+) \to \mathrm{H}$.

Quantized enveloping algebras are defined via generators and relations whereas Hall algebras have a basis indexed essentially by quiver representations and moreover have very concrete product formulas. A consequence of this is that many results involving quantized enveloping algebras have quite unpleasant proofs whereas their analogues for Hall algebras are very explicit and have a homological flavour.

We know that the positive part $\mathrm{U}_t(\mathfrak{n}_+)$ has two integral forms with one specializing to the universal enveloping algebra of a Lie algebra and the other to a Poisson algebra. By Ringel's theorem this must also be true for the generic Hall algebra. A natural question to ask is what do these integral forms and their $t = 1$ limits look like on the Hall algebra side of things.

Hall Algebras: Integral Forms and $t = 1$ Limits

The way that integral forms of generic Hall algebras arise is via the two formulas for the product on H given in Equation (4) and Equation (5). Each formula gives the structure constants of the product on H in a different basis and taking the $\mathbb{C}[t,t^{-1}]$-span of either set of basis vectors results in an integral form. These two integral forms are *not* isomorphic as $\mathbb{C}[t,t^{-1}]$-algebras but *are* isomorphic on base changing to work over $\mathbb{C}(t)$.

We thus define H_{ex} to be the $\mathbb{C}[t,t^{-1}]$-subalgebra of H spanned by the basis vectors E_L. We call this the extension counting integral form of H as its product roughly speaking counts extensions. We will call the $t=1$ limit of H_{ex} the semi-classical Hall algebra and denote it by H_{sc}. In Chapter 9 of this thesis we will prove the following Proposition by mimicking the proof of a similar result due to Bridgeland [Bri12] for a different flavour of Hall algebras called motivic Hall algebras.

Proposition 9.2.1. The semi-classical Hall algebra H_{sc} is a commutative Poisson algebra.

We similarly define H_{fl} to be the $\mathbb{C}[t,t^{-1}]$-subalgebra of H spanned by the elements of the form X_L. This is called the flag counting integral form of H since its product counts flags of subobjects. We call the $t=1$ limit of H_{fl} the quasi-classical Hall algebra and denote it by H_{qc}. In [Rin90b] Ringel proved the following result for H_{qc} .

Proposition (Ringel)**.** The quasi-classical Hall algebra H_{qc} is the universal enveloping algebra of a Lie algebra.

Homological features of the categories $\mathcal{A}_q$ manifest themselves algebraically in H_{sc} and H_{qc}. For example the Hall product on H_{sc} is given by taking direct sums of quiver representations. Moreover as an algebra H_{sc} is the polynomial algebra in the basis vectors corresponding to indecomposable quiver representations. For H_{qc} the Lie algebra sitting inside H_{qc} is the span of the basis vector corresponding to indecomposable quiver representations.

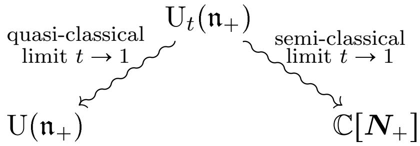
Qualitatively then, the two integral forms H_{ex} and H_{fl} of H have the same kind of $t=1$ limits as the integral forms $\mathrm{U}_t^{Poiss}(\mathfrak{n}_+)$ and $\mathrm{U}_t^{Res}(\mathfrak{n}_+)$ of $\mathrm{U}_t(\mathfrak{n}_+)$. A priori, however, one does not know that the two sides match up. In [Rin95] Ringel partially resolved this with the following theorem.

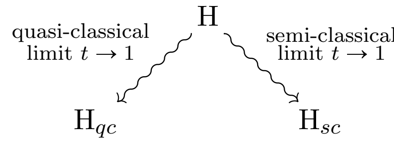
Theorem (Ringel)**.** The $\mathbb{C}(t)$-algebra isomorphism $\mathrm{U}_t(\mathfrak{n}_+) \to \mathrm{H}$ restricts to an isomorphism of integral forms between $\mathrm{U}_t^{Res}(\mathfrak{n}_+)$ and H_{fl}.

The proof of this result is not that difficult whereas the proof that $\mathrm{U}_t^{Poiss}(\mathfrak{n}_+) \cong \mathrm{H}_{ex}$ is much more involved. This is essentially a consequence of the fact that the definition of $\mathrm{U}_t^{Poiss}(\mathfrak{n}_+)$ is quite non-trivial in comparison to that of $\mathrm{U}_t^{Res}(\mathfrak{n}_+)$. In Chapter 16 we will complete the picture by establishing the following result.

Theorem 16.2.5. The $\mathbb{C}(t)$-algebra isomorphism $\mathrm{U}_t(\mathfrak{n}_+) \to \mathrm{H}$ restricts to an isomorphism of integral forms between $\mathrm{U}_t^{Poiss}(\mathfrak{n}_+)$ and H_{ex}.

We should point out that the particular isomorphism $\mathrm{U}_t(\mathfrak{n}_+) \cong \mathrm{H}$ we use to establish the above two theorems is slightly different to the one originally used by Ringel. The upshot of these two theorems is that the following two pictures are then equivalent under our isomorphism $\mathrm{U}_t(\mathfrak{n}_+) \cong \mathrm{H}$.





Bridgeland-Hall Algebras: An Overview

Bridgeland-Hall algebras were introduced by Bridgeland in [Bri13] to solve the problem of finding a suitable category whose Hall algebra would extend Ringel's theorem by recovering the *whole* quantized enveloping algebra. Bridgeland gave a general construction depending on finitary Abelian categories satisfying certain conditions, however in this thesis we will only consider Bridgeland-Hall algebras associated to categories of representations of simply-laced quivers.

Bridgeland's key insight was to replace the category of quiver representations $\mathcal{A}_q$ in the definition of the Hall algebra H_q with the category $\mathcal{C}_q$ of $\mathbb{Z}_2$-graded complexes in projective quiver representations. The objects of $\mathcal{C}_q$ are complexes of the following form where L_1 and L_0 are projective objects in $\mathcal{A}_q$.

$$L_\bullet = L_1 \underset{g}{\overset{f}{\rightleftarrows}} L_0 \,, \qquad f \circ g = g \circ f = 0 \tag{6}$$

Morphisms of $\mathcal{C}_q$ are given by usual morphisms of complexes. Note that the category $\mathcal{C}_q$ has an involution given by the usual shift functor. This sends a complex $L_\bullet$ to the complex $L_\bullet^*$ given by switching L_0 with L_1 and f with g. The resulting Hall algebra of $\mathcal{C}_q$ isn't quite the correct object to recover the quantized enveloping algebra however, and must be modified in two ways.

The first modification is that the factor in Equation (4) involving the Euler form must be altered slightly. The other is that certain relations concerning acyclic complexes must be imposed by hand. With these remarks in mind we make the following provisional definition.

$$\mathrm{H}_q(\mathcal{C}_q) := \bigoplus_{L_\bullet \in \mathrm{Iso}(\mathcal{C}_q)} \mathbb{C} \cdot [L_\bullet]$$

The product on $\mathrm{H}_q(\mathcal{C}_q)$ is given by the following where $\mathrm{Ext}^1(M_\bullet, N_\bullet)_{L_\bullet}$ is the set of extensions of complexes $M_\bullet$ by $N_\bullet$ whose middle term is isomorphic to $L_\bullet$.

$$[M_\bullet][N_\bullet] = q^{1/2\langle \hat{M}_0, \hat{N}_0 \rangle + 1/2 \langle \hat{M}_1, \hat{N}_1 \rangle} \sum_{L_\bullet \in \mathrm{Iso}(\mathcal{C}_q)} \frac{|\mathrm{Ext}^1(M_\bullet, N_\bullet)_{L_\bullet}|}{|\mathrm{Hom}(M_\bullet, N_\bullet)|} [L_\bullet] \tag{7}$$

The extra relation one needs to impose in order to get the correct algebra is to require that each $[L_\bullet]$ is the inverse of $[L_\bullet^*]$ for any acyclic complex $L_\bullet$. The resulting algebra is called the Bridgeland-Hall algebra and is denoted by DH_q.

As was the case for the Hall algebra H_q we actually have obtained a family of algebras depending on q. Deng and Chen [CD15] showed that the structure constants of Equation (7) are Laurent polynomials in $q^{1/2}$ and so there exists a $\mathbb{C}(t)$-algebra called the generic Bridgeland-Hall algebra which we will denote by DH.

Bridgeland then succeeded in proving the following theorem, extended by Deng and Chen to the generic case, which allows one to recover the whole quantized enveloping algebra.

Theorem (Bridgeland, Deng, Chen)**.** There is an isomorphism of $\mathbb{C}(t)$-algebras $\mathrm{U}_t(\mathfrak{g}) \to \mathrm{DH}$.

We can play the same game that we did for the generic Hall algebra H with regards to integral forms of DH. There must be two integral forms whose $t = 1$ limits give the universal enveloping algebra of a Lie algebra and the algebra of functions on a Poisson-Lie group respectively. What do these integral forms look like? What do their $t = 1$ limits look like from a Hall algebraic perspective?

Bridgeland-Hall Algebras: Integral Forms and $t = 1$ Limits

The generic Bridgeland-Hall algebra roughly speaking has analogous integral forms to the generic Hall algebra. One might expect that the integral forms of DH arise in exactly the same way as for H. Unfortunately this is only true for the analogue of the extension counting integral form.

We thus define DH_{ex} to be the $\mathbb{C}[t, t^{-1}]$-subalgebra of DH spanned by the elements of the form $[L_\bullet]$. We call this the extension counting integral form of DH as its product counts extensions of complexes. We will call the $t = 1$ limit of DH_{ex} the semi-classical Bridgeland-Hall algebra and denote it by DH_{sc}.

To our knowledge we are the first to consider the extension counting integral form of the Bridgeland-Hall algebra. In particular in Chapter 17 we establish that DH_{ex} enjoys the following property.

Proposition 17.1.1. The semi-classical Bridgeland-Hall algebra DH_{sc} is the commutative Poisson algebra of functions on a Poisson-Lie group.

This proposition, though not difficult to prove, implies that the spectrum of DH_{sc} is a Poisson-Lie group which we call the semi-classical Poisson-Lie group and which we denote by $\boldsymbol{G}^\vee_{sc}$. It was already known that $\mathrm{U}_t(\mathfrak{g})$ deforms coordinate algebras of Poisson-Lie groups. However the precise mechanics of how this works for Bridgeland-Hall algebras had not been investigated.

Now the structure of $\boldsymbol{G}^\vee_{sc}$ is equivalent to that of its coordinate algebra DH_{sc}. Moreover in the previous section we saw that the Bridgeland-Hall algebra has a very explicit product formula. In Proposition 17.2.1 of Section 17.2 we will derive similarly explicit formulas for the Poisson and algebra structure of DH_{sc} which encode homological features of the categories $\mathcal{A}_q$ and $\mathcal{C}_q$.

We thus have a natural way to assign a very geometric object – a Poisson-Lie group $\boldsymbol{G}^\vee_{sc}$ – to DH and moreover the structure of $\boldsymbol{G}^\vee_{sc}$ is determined by homological data. This allows one to think of Bridgeland-Hall algebras from a geometric point of view and Poisson-Lie groups from a homological perspective.

We now turn to the other integral form of DH, the flag counting integral form DH_{fl}. The definition of DH_{fl} is a little more subtle: one does *not* simply take the $\mathbb{C}[t, t^{-1}]$-subalgebra of DH spanned by elements of the form $[L_\bullet]/|\mathrm{Aut}(L_\bullet)|$.

It is a fact that the generic Bridgeland-Hall algebra has two copies of H_{fl} sitting inside as subalgebras along with a subalgebra generated by $[L_\bullet]$ where $L_\bullet$ is acyclic. Taking the $\mathbb{C}[t, t^{-1}]$-subalgebra of DH generated by all three of these subalgebras yields the correct definition for DH_{fl}.

Deng and Chen [CD15] first considered the flag counting integral form of the Bridgeland-Hall algebra. We will call the $t = 1$ limit of DH_{fl} the quasi-classical Bridgeland-Hall algebra and denote it by DH_{qc}.

Proposition (Deng, Chen)**.** The quasi-classical Bridgeland-Hall algebra DH_{qc} is isomorphic to $\mathrm{U}(\mathfrak{g})$ the universal enveloping algebra of the simple Lie algebra $\mathfrak{g}$.

One of the upshots of Deng and Chen's result is that it gives a way to extract full simple Lie algebras from Bridgeland-Hall algebras. Later in this introduction we will explain an alternative way to obtain full simple Lie algebras from Bridgeland-Hall algebras using the semi-classical Bridgeland-Hall algebra instead.

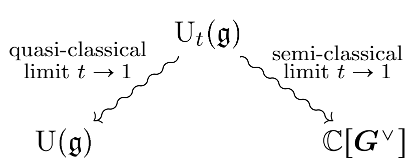
Similar to the case of generic Hall algebras, the two integral forms DH_{ex} and DH_{fl} of DH have the same types of $t = 1$ limits as the integral forms $\mathrm{U}_t^{Poiss}(\mathfrak{g})$ and $\mathrm{U}_t^{Res}(\mathfrak{g})$ of $\mathrm{U}_t(\mathfrak{g})$. Again, however, one does not know a priori that the two sides match up. In [CD15] Deng and Chen established the following theorem.

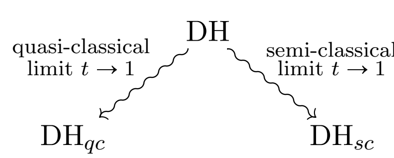
Theorem (Deng, Chen)**.** The $\mathbb{C}(t)$-algebra isomorphism $\mathrm{U}_t(\mathfrak{g}) \to \mathrm{DH}$ descends to an isomorphism of integral forms between $\mathrm{U}_t^{Res}(\mathfrak{g})$ and DH_{fl}.

As was the case for the generic Hall algebra, the proof of this result is not that difficult whereas the proof that $\mathrm{U}_t^{Poiss}(\mathfrak{g}) \cong \mathrm{DH}$ is non-trivial. In Chapter 16 we will complete the picture by establishing the following result.

Theorem 16.2.4. The $\mathbb{C}(t)$-algebra isomorphism $\mathrm{U}_t(\mathfrak{g}) \to \mathrm{DH}$ descends to an isomorphism of integral forms between $\mathrm{U}_t^{Poiss}(\mathfrak{g})$ and DH_{ex}.

We should point out that the particular isomorphism $\mathrm{U}_t(\mathfrak{g}) \cong \mathrm{DH}$ we use to establish the above two theorems in this thesis is slightly different to the one originally used by Deng and Chen [CD15] and Bridgeland [Bri13]. The consequence of these two theorems is that the following two are equivalent pictures under our isomorphism $\mathrm{U}_t(\mathfrak{g}) \cong \mathrm{DH}$.





Lie Algebras and Lie Bialgebras from Bridgeland-Hall Algebras

A general observation is that Hall algebras of Abelian categories often have some kind of associated Lie algebra. Unfortunately these Lie algebras are only ever positive parts of a bigger Lie algebra in the same way that the nilpotent subalgebra $\mathfrak{n}_+$ is only a part of the simple Lie algebra $\mathfrak{g}$.

An important question then is how to realize full Lie algebras in a Hall algebraic way. Moreover if this was possible would there be nice formulas for the Lie bracket in the same way that H and DH have a natural product formula?

Deng and Chen were the first to realize full Lie algebras by considering Bridgeland-Hall algebras of categories of simply-laced quiver representations. In [CD15] they succeeded in recovering the full simple Lie algebra $\mathfrak{g}$ from the quasi-classical Bridgeland-Hall algebra DH_{qc}. Our results give an alternative way to recover $\mathfrak{g}$ from DH via the semi-classical Bridgeland-Hall algebra. The approach we will use will in fact extract more than the simple Lie algebra: we will obtain what is called a Lie bialgebra.

Lie bialgebras are infinitesimal analogues of Poisson-Lie groups in the same way that Lie algebras are infinitesimal analogues of Lie groups. A Lie bialgebra is a Lie algebra whose dual vector space is also a Lie algebra in a compatible way. The tangent Lie algebra of any Poisson-Lie group is a Lie bialgebra with the additional Lie bracket induced from linearizing the Poisson structure.

We thus have a way to extract Lie bialgebras from Bridgeland-Hall algebras. Indeed in Chapter 18 we will take the tangent Lie bialgebra of the semi-classical Poisson-Lie group $\boldsymbol{G}_{sc}^{\vee}$ which we will denote by $\mathfrak{g}_{sc}^{\vee}$. An interesting feature of finite dimensional Lie bialgebras is that they satisfy a simple duality. The vector space dual of any Lie bialgebra is again a Lie bialgebra. In our case if we take the Lie bialgebra dual of $\mathfrak{g}_{sc}^{\vee}$ we then obtain another Lie bialgebra $\mathfrak{g}_{sc}$.

The machinery of Hall algebras allows us to very explicitly calculate the structure of these bialgebras. Indeed in Chapter 18 we will explain that $\mathfrak{g}_{sc}$ comes equipped with a natural basis involving indecomposable quiver representations. Moreover in Theorem 18.2.1 and Theorem 18.2.2 we will calculate the Hall algebraic structure constants of the Lie brackets of $\mathfrak{g}_{sc}$.

What is the Lie bialgebra $\mathfrak{g}_{sc}$? It is a fact that any simple Lie algebra $\mathfrak{g}$ can be endowed with what is called the standard Lie bialgebra structure. In Section 18.4 we prove the following theorem.

Theorem 18.4.2. There is an isomorphism between the Lie bialgebra $\mathfrak{g}_{sc}$ and the simple Lie algebra $\mathfrak{g}$ endowed with the standard Lie bialgebra structure.

The upshot of this theorem then is that it provides a new way to recover the whole simple Lie algebra from Bridgeland-Hall algebras and a homological perspective on its structure.

New Proofs of Old Results

A general feature of results involving Hall algebras and Bridgeland-Hall algebras is that their proofs are often more straightforward than the corresponding ones for quantized enveloping algebras. A natural question to ask is whether one can use the Bridgeland-Hall algebra approach to Poisson-Lie groups to simplify proofs of old but tricky to prove theorems.

In the introductory section on quantized enveloping algebras we mentioned that in [DCP93] De Concini and Procesi proved that the $t = 1$ limit of the Poisson integral form of the quantized enveloping algebra was isomorphic to the coordinate algebra of the standard dual Poisson-Lie group $\boldsymbol{G}^{\vee}$. We also mentioned that the proof of this theorem involved a long case-by-case analysis.

In Chapter 19, using the machinery of Bridgeland-Hall algebras we will provide a new, more direct proof of this theorem. Recall that we defined the semi-classical Poisson Lie group $\boldsymbol{G}_{sc}^{\vee}$ to be the spectrum of DH_{sc}.

Theorem 19.2.1. There is an isomorphism of Poisson-Lie groups between the semi-classical Poisson Lie group $\boldsymbol{G}_{sc}^{\vee}$ and the standard dual Poisson-Lie group $\boldsymbol{G}^{\vee}$.

The semi-classical Bridgeland-Hall algebra is the coordinate algebra of $\boldsymbol{G}_{sc}^{\vee}$ and moreover, as we have mentioned, in Chapter 17 we can explicitly calculate the algebra and Poisson structure of DH_{sc}. A consequence of the above theorem then is that, not only can we simplify an old proof, but we also obtain a new point of view on the structure of the standard dual Poisson-Lie group $\boldsymbol{G}^{\vee}$.

Additional Remarks

Advice on Reading this Thesis

This thesis is divided into four distinct parts which we feel comprise the major logical divisions of the text. Part I consists of background material, Part II is devoted to generic Hall algebras, Part III is concerned with generic Bridgeland-Hall algebras and finally Part IV deals with semi-classical Bridgeland-Hall algebras. Each of these parts is then subdivided into chapters on the main topics in these areas and each chapter is further subdivided into sections on more specialized concepts.

There is quite a bit of notation littered throughout this thesis. As such we have included an annotated glossary which we encourage the reader to refer to whenever necessary. The entries in the glossary consist of the most frequently used mathematical objects in this thesis and so we have highlighted each of these objects in a red hyperlink on their page of definition. This should enable one to quickly find the place of definition of any unfamiliar objects.

Many readers will already be familiar with the background theory in the various chapters of Part I. A short-cut for this part would be to skim through any sections on known material, glancing only at the red hyperlinks to familiarize oneself with notation.

Other readers may also already be comfortable with Hall algebras of Abelian categories. If one simply wants a quick flavour of how the ideas in this thesis work then we advise reading the 9 pages contained in Chapter 8, Chapter 9 and Chapter 10 of Part II. In particular one should get a good idea of how generic Hall algebras come equipped with two natural integral forms and the flavour of their $t = 1$ limits.

Assumptions and Conventions

The sole assumption we will make is that the only (Bridgeland-)Hall algebras considered in this thesis are those associated to categories of representations of a simply-laced quiver.

By affine variety we mean an irreducible, reduced affine scheme of finite type over $\mathbb{C}$. For us an algebraic group will be an affine algebraic group over $\mathbb{C}$. We will write e for the group identity of various algebraic groups. In each case we will always make clear from context which group is being referred to.

Part I

Background Material

Overview

Part I is devoted to collecting various pieces of background material that will be used in this thesis. A nice feature of the theory of Hall algebras is that it lies at the intersection of a wide variety of rich and interesting topics. We thus hope that the chapters in this part may be of independent interest to those seeking a concise overview of the areas tangentially related to Hall algebras.

We begin with Chapter 1 where we run through the basic theory of simple Lie algebras and simple Lie groups. One of the undercurrents of this thesis is how closely Hall algebras of quiver representations are related to these Lie algebras and Lie groups. We will see that many of the features of these Lie theoretic objects manifest themselves again and again as algebraic properties of Hall algebras.

In Chapter 2 we collect a number of different definitions and results pertaining to quivers, their representations and complexes of their representations. Categories of quiver representations provide the underlying category of the Hall algebras that will be considered in this thesis. Similarly categories of $\mathbb{Z}_2$-graded complexes in projective quiver representations give the underlying category of Bridgeland-Hall algebras

The material in Chapter 3 concerns various topics regarding Poisson-Lie groups and Lie bialgebras. These objects arise from $t = 1$ limits of integral forms of quantized enveloping algebras and Bridgeland-Hall algebras. Lie bialgebras are the infinitesimal analogues of Poisson-Lie groups in the same way that Lie algebras are the infinitesimal analogues of Lie groups. One of the most important things we will do in Chapter 3 is to construct a Poisson-Lie group $\boldsymbol{G}^\vee$ and Lie bialgebra $\mathfrak{g}^\vee$ called the standard dual Poisson-Lie group and standard dual Lie bialgebra respectively.

Chapter 4 is a short chapter where we rigorously define the notion of an integral form of a $\mathbb{C}(t)$-algebra. We also explain the formal requirements and process by which coordinate algebras of Poisson-Lie groups may arise from integral forms.

In Chapter 5 we define quantized enveloping algebras of simple Lie algebras. We also give an overview of two integral forms of the quantized enveloping algebra – the restricted integral form and what we call the Poisson integral form. The restricted integral form has the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ as its $t = 1$ limit while the Poisson integral form has the coordinate algebra $\mathbb{C}[\boldsymbol{G}^\vee]$ as its $t = 1$ limit.

Finally Chapter 6 is an overview of Hall algebras and Bridgeland-Hall algebras associated to categories of quiver representations of simply-laced quivers. Hall algebras of categories of quiver representations were originally defined by Ringel who used them to recover positive parts of quantized enveloping algebras. Bridgeland-Hall algebras were introduced by Bridgeland to extend Ringel's results and recover the *whole* quantized enveloping algebra.

Chapter 1

Lie Algebras and Algebraic Groups

In this chapter we recall various definitions and results regarding Lie algebras and algebraic groups. Section 1.1 is concerned with an overview of simple Lie algebras and simple Lie groups. The story of integral forms of quantized enveloping algebras and Hall algebras makes heavy use of these objects. In Section 1.2 we outline the technicalities of how adjoint actions of algebraic groups on their Lie algebras may be described in the language of the functor of points approach to schemes.

1.1 Simple Lie Algebras and Lie Groups

One of the crowning achievements of the theory of simple Lie algebras is that they are determined by certain graphs called Dynkin diagrams. We will take the approach of defining simple Lie algebras in terms of these diagrams.

In this thesis we will not consider all simple Lie algebras, only those determined by what are called simply-laced Dynkin diagrams. A complete list of the simply-laced Dynkin diagrams can be found at the end of this section.

A simply-laced Dynkin diagram is equivalent to the data of what is called a symmetric Cartan matrix $(a_{ij})_{i,j=1}^r$. Such a matrix is non-degenerate and is determined from the Dynkin diagram via the following formula where m_{ij} is the number of edges between the vertices i and j.

$$a_{ij} := 2\delta_{ij} - m_{ij} \tag{1.1}$$

It is this symmetric Cartan matrix that allows one to define a simple Lie algebra via the following generators and relations description.

Definition 1.1.1. Let $(a_{ij})_{i,j=1}^r$ be a Cartan matrix associated to a simply-laced Dynkin diagram. Define the associated simple Lie algebra $\mathfrak{g}$ with Lie bracket $[-,-]$ to be the complex Lie algebra with generators e_i, f_i and h_i for $1 \leqslant i \leqslant r$ subject to the following relations where $1 \leqslant i, j \leqslant r$.

$$[h_i, h_j] = 0 \qquad\qquad [h_i, e_j] = a_{ij}e_j$$
$$[e_i, f_j] = \delta_{ij}h_j \qquad\qquad [h_i, f_j] = -a_{ij}f_j$$

We also require that the following so-called Serre relations hold for $i \neq j$ where ad denotes the adjoint action.

$$ad_{e_i}^{1-a_{ij}}(e_j) = 0 \qquad\qquad ad_{f_i}^{1-a_{ij}}(f_j) = 0$$

Simple Lie algebras have some special Lie subalgebras that we will make frequent use of. The first is the Cartan subalgebra $\mathfrak{h}$ which is generated by the elements h_i. There are also of the positive and negative nilpotent and Borel subalgebras which are generated as follows.

$$\mathfrak{n}_+ := \langle e_i \rangle \qquad \mathfrak{b}_+ := \langle e_i, h_i \rangle \tag{1.2}$$
$$\mathfrak{n}_- := \langle f_i \rangle \qquad \mathfrak{b}_- := \langle f_i, h_i \rangle$$

In light of the generators and relations definition of $\mathfrak{g}$ there is a vector space triangular decomposition of $\mathfrak{g}$ into these subalgebras.

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$$

Now the Cartan subalgebra acts via adjoint action on $\mathfrak{g}$. That is, there is a map $\mathfrak{h} \to \mathrm{End}(\mathfrak{g})$ given by $h \mapsto ad_h$ where $ad_h(x) = [h, x]$ for any $x \in \mathfrak{g}$. The linear operators ad_h are simultaneously diagonalizable and induce what is called the root space decomposition of $\mathfrak{g}$.

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \text{ a root}} \mathfrak{g}[\alpha]$$

Here a root α is elements of $\mathfrak{h}^*$ such that the subspace $\mathfrak{g}[\alpha] := \{x \in \mathfrak{g} \mid ad_h = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$ is non-zero. The subspaces $\mathfrak{g}[\alpha]$ are called the root spaces of $\mathfrak{g}$. We will denote by $\Phi \subset \mathfrak{h}^*$ the set of roots of $\mathfrak{g}$ which is of finite cardinality.

The $\mathbb{Z}$-span of the roots in $\mathfrak{h}^*$ form a lattice Λ_Φ called the root lattice. One can choose certain minimal sets of generators of Λ_Φ called simple roots. Given the generators and relations description of $\mathfrak{g}$ we have a canonical choice of simple roots α_i which may be defined via the Cartan matrix as follows.

$$\alpha_i(h_j) = a_{ij} \tag{1.3}$$

Throughout this thesis r will be the number of simple roots of $\mathfrak{g}$. In light of Equation (1.3) the elements h_i of the Cartan subalgebra $\mathfrak{h}$ are called the simple coroots of $\mathfrak{g}$. Root lattices have important automorphisms called simple reflections which are given as follows.

$$s_i : \Lambda_\Phi \to \Lambda_\Phi, \qquad \alpha_j \mapsto \alpha_j - a_{ij}\alpha_i$$

The simple reflections generate a group $W \subset \mathrm{Aut}(\mathfrak{h}^*)$ called the Weyl group of $\mathfrak{g}$. The length of an element $w \in W$ is defined to be the minimal number l of simple reflections such that $w = s_{i_1} \cdots s_{i_l}$. It is a fact that there exists a unique element w_0 of longest length in W. In particular the length of w_0 is given by the number N of positive roots Φ^+ of $\mathfrak{g}$.

The simple roots α_i split the set of roots up into a set of positive roots Φ^+ and negative roots Φ^-. The positive roots are those roots which are positive linear combinations of the simple roots while the negative roots are ones given by negative linear combinations.

Simple Lie algebras have a non-degenerate bilinear form $(-,-)_\mathfrak{g}$ on $\mathfrak{g}$ called the Cartan-Killing form. This is given by $(x, y)_\mathfrak{g} := \mathrm{Tr}(ad_x \circ ad_y)$ for any $x, y \in \mathfrak{g}$. We may rescale this form so that for any $1 \leqslant i, j \leqslant r$ we have $(h_i, h_j)_\mathfrak{g} = a_{ij}$ which we do in this thesis.

For simple Lie algebras the root spaces $\mathfrak{g}[\alpha]$ are all one dimensional. A root vector is a basis vector for $\mathfrak{g}[\alpha]$. In general there is no canonical way to choose root vectors for $\mathfrak{g}$. For each positive root α

then, we will make a choice of positive root vector e_α in $\mathfrak{g}[\alpha]$ and negative root vector f_α in $\mathfrak{g}[-\alpha]$. Moreover will require that these satisfy the following condition for any two $\alpha, \beta \in \Phi^+$.

$$(e_\alpha, f_\beta)_\mathfrak{g} = \delta_{\alpha,\beta}$$

We end with some words on simple Lie groups. It is well known that a simple Lie algebra $\mathfrak{g}$ can be integrated to a simple algebraic group $\boldsymbol{G}$. The subalgebras $\mathfrak{n}_\pm$, $\mathfrak{h}$ and $\mathfrak{b}_\pm$ integrate to a choice of what are called positive and negative unipotent subgroups $\boldsymbol{N}_\pm \subset \boldsymbol{G}$, maximal torus $\boldsymbol{T} \subset \boldsymbol{G}$ and positive and negative Borel subgroups $\boldsymbol{B}_\pm \subset \boldsymbol{G}$ respectively. One can show that there is a semi-direct product decomposition $\boldsymbol{B}_\pm = \boldsymbol{N}_\pm \rtimes \boldsymbol{T}$ and that $\boldsymbol{T} = \boldsymbol{B}_- \cap \boldsymbol{B}_+$.

A character of a torus is a group homomorphism $\chi : \boldsymbol{T} \to \mathbb{G}_m$. The set of characters forms a lattice $X^\bullet(\boldsymbol{T})$ called the character lattice. Since $\boldsymbol{T}$ is a complex algebraic torus then the group algebra $\mathbb{C}[X^\bullet(\boldsymbol{T})]$ of the character lattice is canonically isomorphic to the coordinate algebra $\mathbb{C}[\boldsymbol{T}]$. The isomorphism here is given by taking a character and viewing it as a function on $\boldsymbol{T}$ via $\mathbb{G}_m \subset \mathbb{A}^1_\mathbb{C}$.

The reason why we are mentioning character lattices is that in general there are several Lie groups $\boldsymbol{G}$ with the same Lie algebra $\mathfrak{g}$. We will choose $\boldsymbol{G}$ to be what is called of adjoint form. The only consequence of this that we shall need is that the following map induces a canonical isomorphism between the character lattice of $\boldsymbol{T}$ and the root lattice $\Lambda_\Phi \subset \mathfrak{h}^*$ of $\mathfrak{g}$.

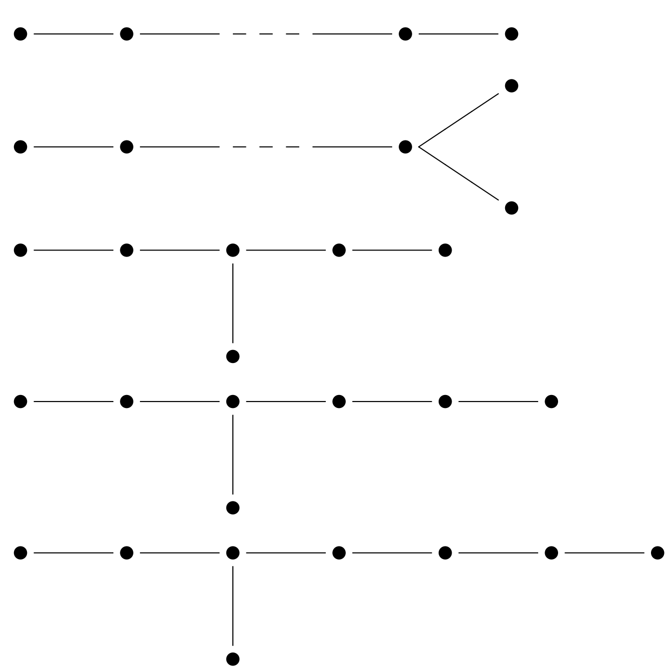
$$X^\bullet(\boldsymbol{T}) \to \Lambda_\Phi, \qquad \chi \mapsto d_e\chi$$

Here by $d_e\chi$ we mean the differential of χ at the torus identity e.

The Simply-Laced Dynkin Diagrams

The following is a complete list of the simply-laced Dynkin diagrams. We have used the letter r to denote the number of vertices.

A_r :
D_r :
E_6 :
E_7 :
E_8 :



1.2 Adjoint Actions of Algebraic Groups

A general feature of a (differentiable, algebraic etc.) Lie group is that it natural acts on its Lie algebra via what is called the adjoint action. In this section we will explain, in the case of algebraic groups, how to compute this adjoint action in terms of algebraic data. This material will be needed in Chapter 19. A reference for the following is Section 3 in Chapter II of [Mil].

We begin by discussing algebraic groups and their Lie algebras in the language of their functor of points before then giving a formula for the adjoint action. In this section we will denote by $\boldsymbol{K}$ a complex affine algebraic group and $\mathfrak{k}$ its Lie algebra. Note that we will view $\mathfrak{k}$ as a scheme as opposed to a complex vector space.

Recall that the algebraic group structure on the underlying variety of $\boldsymbol{K}$ is equivalent to a Hopf algebra structure on the coordinate algebra $\mathbb{C}[\boldsymbol{K}]$. The coproduct Δ is the pullback of functions via the group multiplication map, the antipode S is the pullback of functions via the inverse map and the counit ε picks out the group identity.

From the functor of points perspective the maps Δ, S and ε induce the structure of a group on the B-valued points of $\boldsymbol{K}$ for any $\mathbb{C}$-algebra B. A special case of this is that if $B[\delta]$ is the algebra of dual numbers over B then $\boldsymbol{K}(B[\delta])$ is a group. One should view $\boldsymbol{K}(B[\delta])$ as the set of B-valued points of the tangent bundle of $\boldsymbol{K}$. The functor of points way to write the fact that the $\boldsymbol{K}$ sits inside its tangent bundle as the zero section is given by the following.

$$\boldsymbol{K}(B) \hookrightarrow \boldsymbol{K}(B[\delta]), \qquad y \mapsto y + 0 \cdot \delta$$

Turning to the Lie algebra, the B-valued points of $\mathfrak{k}$ consists of the set of $\mathbb{C}$-derivations of $\mathbb{C}[\boldsymbol{K}]$ in B where B is regarded as a $\mathbb{C}[\boldsymbol{K}]$-module via the counit ε. Explicitly $\mathfrak{k}(B)$ is the set of $\mathbb{C}$-linear maps $X : \mathbb{C}[\boldsymbol{K}] \to B$ such that for any functions $f, g \in \mathbb{C}[\boldsymbol{K}]$ the following holds.

$$X(fg) = \varepsilon(f)X(g) + \varepsilon(g)X(f)$$

Geometrically the Lie algebra $\mathfrak{k}$ sits inside the tangent bundle of $\boldsymbol{K}$ as the tangent space at the group identity. The functor of points way of writing this is given by the following.

$$\mathfrak{k}(B) \hookrightarrow \boldsymbol{K}(B[\delta]), \qquad X \mapsto \varepsilon + X \cdot \delta$$

Using the functor of points descriptions above we now give the adjoint action of $\boldsymbol{K}$ on $\mathfrak{k}$. The idea is to view $\mathfrak{k}(B)$ as sitting inside $\boldsymbol{K}(B[\delta])$ and then perform the adjoint action operations using the group structure on $\boldsymbol{K}(B[\delta])$. Let y be a point in the group $K(B)$ and X a derivation in $\mathfrak{k}(B)$. The following formula then gives the induced map on B-valued points of the adjoint action $Ad : \boldsymbol{K} \times \mathfrak{k} \to \mathfrak{k}$.

$$Ad_y(X) = y * (\varepsilon + X \cdot \delta) * y^{-1} \tag{1.4}$$

Here we have written $*$ and $(-)^{-1}$ for the induced group multiplication and inverse operations on $\boldsymbol{K}(B[\delta])$ respectively. One can check that the expression given in Equation (1.4) does indeed land in $\mathfrak{k}(B) \subset \boldsymbol{K}(B[\delta])$ again.

Chapter 2

Quivers, Representations and Complexes

In this chapter we collect various definitions and results regarding quivers, quiver representations and complexes of quiver representations.

Section 2.1 is concerned with defining quivers and their Abelian categories of representations. Hall algebras are algebras that one can associate to certain Abelian categories. This section then should be viewed as describing the underlying categories of the Hall algebras used in this thesis.

In Section 2.2 we discuss various properties of categories of quiver representations. These properties will manifest themselves as various algebraic properties of Hall algebras. Important results in this section are the Krull-Schmidt theorem and Gabriel's theorem which give a characterization of all representations of the types of quivers we will consider.

The material in Section 2.3 concerns certain categories of $\mathbb{Z}_2$-graded complexes of quiver representations. These will form the underlying categories of Bridgeland-Hall algebras which in some sense give a doubled version of Hall algebras of categories of quiver representations.

We end this chapter with Section 2.4 on certain functors between categories of quiver representations called BGP reflection functors. Reflection functors are of interest to us as they induce certain isomorphisms between Bridgeland-Hall algebras.

2.1 Quivers and their Representations

In this section we recall the basic facts we will need regarding quivers and their representations. Good introductory references for quivers and their representations are [Bri08] and [Sch09].

We begin by discussing quivers before defining their categories of representations. A quiver $\vec{Q}$ is a finite directed graph. As part of the data of a quiver we have a set of vertices Q_0 and a set of arrows Q_1. We also have the following source and target maps which respectively pick out the source vertex and target vertex of each arrow.

$$s, t : Q_1 \to Q_0$$

We will denote by r the number of vertices of $\vec{Q}$. It will be convenient to choose a total ordering on Q_0 by labelling the vertices $1, \dots, r$.

In this thesis we will only consider simply-laced quivers. By this we mean a quiver such that when one forgets the direction of the arrows one is left with one of the simply-laced Dynkin diagrams that we gave in Section 1.1. There is a deep relationship between quivers and Lie algebras. Indeed a simply-laced quiver $\vec{Q}$ determines a symmetric Cartan matrix of corresponding Dynkin type as follows.

$$(a_{ij})_{i,j=1}^{r} := (2\delta_{ij} - n_{ij} - n_{ji}) \tag{2.1}$$

Here n_{ij} is the number of arrows from i to j. We will assume throughout that our choice of simple Lie algebra $\mathfrak{g}$ is the one determined by $\vec{Q}$ in this way.

Quivers give rise to Abelian categories via their category of representations. We will denote by $\mathcal{A}_q$ the category of finite dimensional representations of $\vec{Q}$ over a finite field $\mathbb{F}_q$. Here q is the prime power giving the number of elements of $\mathbb{F}_q$.

An object L_q in $\mathcal{A}_q$ consists of the data of a finite dimensional $\mathbb{F}_q$-vector space L_i for each $i \in Q_0$ along with an $\mathbb{F}_q$-linear map $L_a : L_{s(a)} \to L_{t(a)}$ for each arrow $a \in Q_1$. We define the dimension of L_q to be the sum of the dimensions of the L_i.

A morphism in $\mathcal{A}_q$ between two quiver representations M_q and N_q is the data of an $\mathbb{F}_q$-linear map $\psi_i : M_i \to N_i$ for each vertex i of $\vec{Q}$ such that for each arrow a the following diagram commutes.

$$\begin{array}{ccc} M_{s(a)} & \xrightarrow{M_a} & M_{t(a)} \\ \downarrow{\scriptstyle \psi_{s(a)}} & & \downarrow{\scriptstyle \psi_{t(a)}} \\ N_{s(a)} & \xrightarrow{N_a} & N_{t(a)} \end{array}$$

It is a standard fact that $\mathcal{A}_q$ is an Abelian category linear over $\mathbb{F}_q$. It is worth pointing out that we have in fact an infinite family of distinct categories $\mathcal{A}_q$ with one for each choice of prime power q. We will see in Section 7.1 however that many features of these categories are 'independent of q'.

An equivalent and often useful way to view quiver representations is as modules over the path algebra of $\vec{Q}$.

Definition 2.1.1. The path algebra of $\vec{Q}$ is the $\mathbb{F}_q$-algebra $\mathbb{F}_q\vec{Q}$ with generators g_i and g_a for $i \in Q_0$ and $a \in Q_1$ subject to the following relations.

$$g_i^2 = g_i \qquad g_i g_j = 0 \quad (i \neq j) \qquad g_{t(a)} g_a = g_a g_{s(a)} = g_a$$

A standard result in the theory of quiver representations is that the category of finite dimensional (left) modules over $\mathbb{F}_q\vec{Q}$ is equivalent to the category of quiver representations $\mathcal{A}_q$. This equivalence takes an $\mathbb{F}_q\vec{Q}$-module V to the quiver representation with vector spaces $g_i V$ at each vertex $i \in Q_0$ and linear maps $g_{s(a)}V \to g_{t(a)}V$ for each $a \in Q_1$ given by multiplication by g_a.

2.2 Properties of Categories of Quiver Representations

In this section we discuss various properties of categories of quiver representations.

We begin by describing the simple objects of $\mathcal{A}_q$. For each vertex i of $\vec{Q}$ there is a simple representation $S_{i,q}$ in $\mathcal{A}_q$ determined by the requirement that the dimension of the vector space assigned to vertex j is one if $j = i$ and zero otherwise. For simply-laced quivers these give all the simple representations up to isomorphism.

We will use the notation $K(\mathcal{A}_q)$ for the Grothendieck group of $\mathcal{A}_q$. Recall that the Grothendieck group is the Abelian group generated by isomorphism classes $\widehat{L}_q$ of objects in $\mathcal{A}_q$ modulo the relations $\widehat{N}_q - \widehat{L}_q + \widehat{M}_q = 0$ for any short exact sequence $0 \to N_q \to L_q \to M_q \to 0$.

In the case of simply-laced quiver representations, $K(\mathcal{A}_q)$ is freely generated by the classes $\widehat{S}_{i,q}$ corresponding to the simple representations. This follows from the fact that $\mathcal{A}_q$ is a finite length category, that is, any object L_q has a finite composition sequence with simple factors.

The Grothendieck group $K(\mathcal{A}_q)$ is canonically isomorphic to the root lattice Λ_Φ of the simple Lie algebra $\mathfrak{g}$ via the following map. Here we recall that $\alpha_i \in \Phi$ is a simple root of $\mathfrak{g}$.

$$K(\mathcal{A}_q) \to \Lambda_\Phi, \qquad \hat{S}_{i,q} \mapsto \alpha_i \tag{2.2}$$

The reason why Equation (2.2) is an isomorphism is that, on the one hand, since $\hat{S}_{i,q}$ form a basis for $K(\mathcal{A}_q)$ then the rank of $K(\mathcal{A}_q)$ is the same as r, the number of vertices of $\vec{Q}$. On the other hand, it is well known that the simple roots α_i form a basis for the root lattice Λ_Φ and that the number of these is given by the number of nodes in the Dynkin diagram of $\mathfrak{g}$, which is r by definition.

Recall from Section 1.1 that we have simple reflections s_i of the root lattice Λ_Φ of $\mathfrak{g}$. Via Equation (2.2) the simple reflections induce automorphisms of the Grothendieck group $K(\mathcal{A}_q)$ which we will also denote by s_i. There is a bilinear form on $K(\mathcal{A}_q)$ called the Euler form given by the following.

$$\langle \hat{M}_q, \hat{N}_q \rangle := \dim_{\mathbb{F}_q} \mathrm{Hom}(M_q, N_q) - \dim_{\mathbb{F}_q} \mathrm{Ext}^1(M_q, N_q) \tag{2.3}$$

We will also need the symmetrization $(-,-)$ and skew-symmetrization $(-,-)_{skew}$ of the Euler form, which are given by the following.

$$(\hat{M}_q, \hat{N}_q) := \langle \hat{M}_q, \hat{N}_q \rangle + \langle \hat{N}_q, \hat{M}_q \rangle \qquad (\hat{M}_q, \hat{N}_q)_{skew} := \langle \hat{M}_q, \hat{N}_q \rangle - \langle \hat{N}_q, \hat{M}_q \rangle \tag{2.4}$$

One can check that in the basis given by the simple representations the symmetrized Euler form gives the Cartan matrix from Equation (2.1).

After simple representations the next most important flavour of quiver representations for us will be the indecomposable ones. These give the building blocks of the objects of $\mathcal{A}_q$ via the Krull-Schmidt theorem.

Theorem 2.2.1 (Krull, Schmidt)**.** Any object L_q in $\mathcal{A}_q$ has a direct sum decomposition into multiples of pairwise non-isomorphic indecomposables $I_1, \ldots, I_k$ as follows.

$$L_q \cong I_1^{n_1} \oplus \ldots \oplus I_k^{n_k}$$

Moreover, up to isomorphism, these indecomposables objects I_k of $\mathcal{A}_q$ and their multiplicities $n_1, \ldots, n_k \in \mathbb{Z}_{\geqslant 0}$ are uniquely determined up to reordering.

Proof. The Krull-Schmidt theorem holds for any finite length Abelian category, that is, every object has a finite composition series with simple factors. A reference for this fact is Theorem 1.5.7 of [EGNO15]. It is well known that $\mathcal{A}_q$ is a finite length category, see for example Section 3.1 of [Sch09]. □

A nice feature of representations of simply-laced quivers is that we can classify all indecomposable representations via the following theorem due to Gabriel. Let $\mathrm{Ind}(\mathcal{A}_q)$ denote the set of isomorphism classes of indecomposable objects in $\mathcal{A}_q$.

Theorem 2.2.2 (Gabriel)**.** The composition of the map $\mathrm{Ind}(\mathcal{A}_q) \to K(\mathcal{A}_q)$, given by $I_q \mapsto \widehat{I}_q$, with $K(\mathcal{A}_q) \cong \Lambda_\Phi$ from Equation (2.2) is a bijection onto the set of positive roots $\Phi^+ \subset \Lambda_\Phi$. In particular the category $\mathcal{A}_q$ has only finitely many indecomposable objects up to isomorphism.

Proof. This is Theorem 3.7 in [Sch09]. □

For each positive root α of $\mathfrak{g}$ we will denote by $I_{\alpha,q}$ a choice of indecomposable representation corresponding to α via Theorem 2.2.2. We will find it convenient to denote by N the number of isomorphism classes of indecomposable representations of $\vec{Q}$. This also coincides with the number of positive roots of $\mathfrak{g}$ by Gabriel's theorem.

One can give a very explicit description of all possible projective quiver representations. The indecomposable projective $\mathbb{F}_q\vec{Q}$-modules are all of the form $P(i) = \mathbb{F}_q\vec{Q}g_i$, that is, the left ideals of $\mathbb{F}_q\vec{Q}$ generated by the elements g_i. Using Krull-Schmidt for $\mathcal{A}_q$ and the equivalence of categories between $\mathcal{A}_q$ and the category of $\mathbb{F}_q\vec{Q}$-modules this describes all possible projective quiver representations up to isomorphism.

A standard result about the categories $\mathcal{A}_q$ is that they are at most of global dimension 1. The easiest way to see this is that any $\mathbb{F}_q\vec{Q}$-module V has a projective resolution of the following form called the standard resolution.

$$0 \longrightarrow \bigoplus_{a \in Q_1} P(t(a)) \otimes_{\mathbb{F}_q} g_{s(a)}V \xrightarrow{F} \bigoplus_{i \in Q_0} P(i) \otimes_{\mathbb{F}_q} g_i V \xrightarrow{G} V \longrightarrow 0$$

The maps F and G here are given by $F(g_b \otimes v) = g_b g_a \otimes v - g_b \otimes g_a v$ and $G(g_c \otimes w) = g_c w$ where $g_b \in P(t(a))$, $g_c \in P(i)$, $v \in g_{s(a)}V$ and $w \in g_i V$.

We end this section with the following proposition on split extensions of simply-laced quiver representations.

Proposition 2.2.3. For any simply-laced quiver $\vec{Q}$ the only extension class in $\mathrm{Ext}^1_{\mathcal{A}_q}(M_q, N_q)$ whose middle term is isomorphic to $M_q \oplus N_q$ is the split extension.

Proof. This follows from Theorem 1.2 in [Str05] on restricting to the special case of the zero ideal. □

2.3 $\mathbb{Z}_2$-Graded Complexes

In this section we discuss categories of $\mathbb{Z}_2$-graded complexes in projective quiver representations. The projective part means that we only consider complexes of projective objects. The $\mathbb{Z}_2$-graded part means that our complexes are unbounded complexes which repeat every two steps. We begin with some definitions before giving a Krull-Schmidt type theorem for complexes. We then end this section with a discussion on extensions of certain complexes. A reference for the material which follows is Section 3 of [Bri13] and Section 4 of [CD15].

Define $\mathcal{C}_q$ to be the category of $\mathbb{Z}_2$-graded complexes in projective objects in $\mathcal{A}_q$. The objects $L_{\bullet q}$ of $\mathcal{C}_q$ are complexes of the following form where L_1 and L_0 are projective objects in $\mathcal{A}_q$.

$$L_1 \underset{g}{\overset{f}{\rightleftarrows}} L_0 \ , \qquad f \circ g = g \circ f = 0$$

Morphisms are given by usual morphisms of complexes. A feature of the $\mathbb{Z}_2$-grading is that the usual shift functor for complexes induces an involution $* : \mathcal{C}_q \to \mathcal{C}_q$ which sends a complex $L_{\bullet q}$ to the following shifted complex $L^*_{\bullet q}$.

$$L_0 \underset{f}{\overset{g}{\rightleftarrows}} L_1$$

We now unravel some of the structure of $\mathcal{C}_q$ by describing four special types of complexes and then explain how they form Krull-Schmidt type building blocks of $\mathcal{C}_q$. For any quiver representation L_q we'll first need to fix what is called a minimal projective resolution of L_q as follows.

$$0 \to P_{L_q} \xrightarrow{d} Q_{L_q} \to L_q \to 0 \tag{2.5}$$

Minimal here means that if we decompose $P_{L_q} = \oplus P_j$ and $Q_{L_q} = \oplus Q_j$ using the Krull-Schmidt Theorem 2.2.1 then in the corresponding decomposition $d = (d_{ij})$ we have that none of the d_{ij} are isomorphisms. Any minimal projective resolution is unique up to isomorphism, a fact which is proved in Lemma 4.1 of [Bri13]. Throughout this thesis we will use the notation in Equation (2.5) to denote a minimal projective resolution.

The point is that we can then associate to any representation L_q the following two objects of $\mathcal{C}_q$ which are canonical up to isomorphism and interchanged by the shift functor.

$$C_{L_q} := \ P_{L_q} \underset{0}{\overset{d}{\rightleftarrows}} Q_{L_q} \qquad\qquad C^*_{L_q} := \ Q_{L_q} \underset{d}{\overset{0}{\rightleftarrows}} P_{L_q}$$

In particular these complexes have L_q and 0 as their two homology objects. The homology of C_{L_q} is given by L_q in degree zero while that of $C^*_{L_q}$ is given by L_q in degree one. The other two important types of objects in $\mathcal{C}_q$ are the following acyclic complexes associated to any projective object P_q in $\mathcal{A}_q$. Again these two complexes are interchanged by the shift involution.

$$K_{P_q} := \ P_q \underset{0}{\overset{id}{\rightleftarrows}} P_q \qquad\qquad K^*_{P_q} := \ P_q \underset{id}{\overset{0}{\rightleftarrows}} P_q \tag{2.6}$$

We then have the following Krull-Schmidt type theorem for $\mathbb{Z}_2$-graded complexes in projective quiver representations.

Theorem 2.3.1 (Bridgeland)**.** Any object $L_{\bullet q}$ in $\mathcal{C}_q$ splits as a direct sum of the following form.

$$C_{A_q} \oplus C^*_{B_q} \oplus K_{P_q} \oplus K^*_{Q_q} \tag{2.7}$$

Moreover the quiver representations A_q, B_q, P_q and Q_q are unique up to isomorphism.

Proof. This is Lemma 4.2 in [Bri13]. □

Combining Theorem 2.3.1 with Gabriel's Theorem 2.2.2 it follows that any object $L_{\bullet q}$ in $\mathcal{C}_q$ is determined uniquely up to isomorphism by a map of sets $\Phi \coprod (\mathbb{Z}_2 \times Q_0) \to \mathbb{Z}_{\geqslant 0}$ which determines A_q, B_q, P_q and Q_q in Equation (2.7). By Gabriel's theorem, the $\Phi \to \mathbb{Z}_{\geqslant 0}$ component determines the two quiver representations A_q and B_q. To see that the component $(\mathbb{Z}_2 \times Q_0) \to \mathbb{Z}_{\geqslant 0}$ determines two projectives P_q and Q_q we use the fact that the projective indecomposable representations of $\vec{Q}$ are in bijection with Q_0 .

We end with an interpretation of some extensions of complexes. That the claims in the following discussion hold follows from the comments preceding Lemma 4.3 in [CD15]. We begin by letting M_q and N_q be quiver representations and $L_{\bullet q}$ be a complex fitting into a short exact sequence of complexes of the following form, that is a short exact sequence in the category $\mathcal{C}_q$.

$$0 \to C^*_{N_q} \to L_{\bullet q} \to C_{M_q} \to 0 \tag{2.8}$$

Suppose $L_{\bullet q}$ has homology objects A_q and B_q in degree zero and degree one respectively. Taking the induced long exact sequence on homology we obtain an exact sequence of the following form where δ is the connecting homomorphism.

$$0 \to A_q \to M_q \xrightarrow{\delta} N_q \to B_q \to 0$$

The map we have just described which takes a short exact sequence to δ in fact gives an isomorphism between the following sets.

$$\mathrm{Ext}^1(C_{M_q}, C^*_{N_q}) \to \mathrm{Hom}(M_q, N_q) \tag{2.9}$$

If $L_{\bullet q}$ fits in to an extension of the form Equation (2.8) then it is actually uniquely determined by its homology objects A_q and B_q. In particular one can show that there is an isomorphism of the following form where the acyclic direct summands are uniquely determined up to isomorphism by the requirements that $P_{M_q} \cong P_{MA_q} \oplus P_{A_q}$ and $Q_{N_q} \cong Q_{NB_q} \oplus Q_{B_q}$.

$$L_{\bullet q} \cong C_{A_q} \oplus C^*_{B_q} \oplus K_{P_{MA_q}} \oplus K^*_{Q_{NB_q}} \tag{2.10}$$

Note that we have chosen the notation P_{MA_q} here to signify that this projective representation is determined uniquely (up to isomorphism) by the representations P_{M_q} and P_{A_q}. Similarly Q_{NB_q} is determined uniquely by Q_{N_q} and Q_{B_q} respectively.

Now let $\mathrm{Ext}^1(C_{M_q}, C^*_{N_q})_{L_{\bullet q}}$ denote the subset of extensions with middle term isomorphic to $L_{\bullet q}$. Denote also by $\mathrm{Hom}(M_q, N_q)_{A_q,B_q}$ subset of morphisms with kernel A_q and cokernal B_q. From the above discussion the isomorphism from Equation (2.9) descends an isomorphism between the following sets where $L_{\bullet q}$ is related to A_q and B_q via Equation (2.10).

$$\mathrm{Ext}^1(C_{M_q}, C^*_{N_q})_{L_{\bullet q}} \to \mathrm{Hom}(M_q, N_q)_{A_q,B_q} \tag{2.11}$$

2.4 BGP Reflection Functors

In this section we recall some of the details regarding BGP reflection functors between categories of quiver representations. A reference for the material which follows is Section 1.4 of [DDPW08].

Let's introduce some notation. We will find it convenient to write $\mathcal{A}_q^{\vec{Q}}$ when we need to make the dependence on $\vec{Q}$ explicit. For any vertex $i \in Q_0$ we have the following full subcategory of representations which do not have the simple representation $S_{i,q}$ as a direct summand.

$$\mathcal{A}_q^{\vec{Q}}\langle i\rangle \subset \mathcal{A}_q^{\vec{Q}}$$

Now from any vertex i of $\vec{Q}$ one can form a new quiver $\sigma_i\vec{Q}$ from $\vec{Q}$ by inverting the direction of all arrows incident at i. We define a sink to be a vertex of $\vec{Q}$ which only has arrows pointing into it. Conversely a source is a vertex which only has arrows pointing out of it. With this notation we can state the main theorem of this section.

Theorem 2.4.1 (Bernstein, Gelfand, Ponomarev)**.** Let i be a sink for $\vec{Q}$ with i then being a source for $\sigma_i\vec{Q}$. There exists functors called BGP reflection functors between the following categories.

$$\sigma_i^+ : \mathcal{A}_q^{\vec{Q}} \to \mathcal{A}_q^{\sigma_i\vec{Q}} \qquad \sigma_i^- : \mathcal{A}_q^{\sigma_i\vec{Q}} \to \mathcal{A}_q^{\vec{Q}}$$

Moreover the $\sigma_i^{\pm}$ have the following properties.

1. The functors $\sigma_i^{\pm}$ restrict to mutually inverse equivalences of categories $\sigma_i^+ : \mathcal{A}_q^{\vec{Q}}\langle i\rangle \to \mathcal{A}_q^{\sigma_i\vec{Q}}\langle i\rangle$ and $\sigma_i^- : \mathcal{A}_q^{\sigma_i\vec{Q}}\langle i\rangle \to \mathcal{A}_q^{\vec{Q}}\langle i\rangle$. In particular these restricted functors take indecomposable representations to indecomposables representations.

2. For any representation L_q in $\mathcal{A}_q^{\vec{Q}}\langle i\rangle$ the Grothendieck group class of $\sigma_i^{\pm}(L_q)$ is $s_i(\hat{L}_q)$ where s_i denotes the simple reflection on $K(\mathcal{A}_{q)}$ from Section 2.2.

Proof. This can be found in Theorem 1.18 and Corollary 1.19 of [DDPW08]. □

We refer the interested reader to Section 1.4 of [DDPW08] for the explicit constructions of the functors $\sigma_i^{\pm}$, although they will not be used in this thesis.

Chapter 3

Poisson-Lie Groups and Lie Bialgebras

In this chapter we give an overview of various results we shall need from the theory of Poisson-Lie groups and Lie bialgebras.

In Section 3.1 we give the definition of a Poisson-Lie group. Poisson-Lie groups are algebraic groups which are at the same time Poisson varieties such that the two structures are compatible. Section 3.2 concerns the theory of Lie bialgebras. These are the infinitesimal analogues of Poisson-Lie groups in the same way that Lie algebras are the infinitesimal versions of Lie groups.

Finally in Section 3.3 we construct the main example of the Poisson-Lie groups and Lie bialgebras that we use in this thesis. In particular we will construct a Poisson-Lie group $\boldsymbol{G}^\vee$ which is dual in some sense to a simple Lie group $\boldsymbol{G}$ with Lie algebra $\mathfrak{g}$.

The importance of the Poisson-Lie group $\boldsymbol{G}^\vee$ for us lies in the fact that its coordinate Hopf algebra appear as the semi-classical limit of quantized enveloping algebras and Bridgeland's Hall algebra. We will say more on this in Chapter 5 and Part IV.

3.1 Poisson-Lie Groups

In this section we define the notion of a Poisson-Lie group. A reference for the following material is Section 6.2 of [CP95].

Recall that the structure of an algebraic group on a complex affine variety $\boldsymbol{K}$ is equivalent to a Hopf algebra structure on its coordinate algebra $\mathbb{C}[\boldsymbol{K}]$. The coproduct is the pullback of functions via the multiplication map, the antipode is the pullback of functions via the inverse map and the counit picks out the group identity.

A Poisson-Lie group is an algebraic group $\boldsymbol{K}$ which is also a Poisson variety such that the Poisson and algebraic group structures are compatible. The requirement on the algebraic side of things is for $\mathbb{C}[\boldsymbol{K}]$ to be a Poisson-Hopf algebra.

Definition 3.1.1. Let $\boldsymbol{K}$ be a complex affine algebraic group which is also a Poisson variety with Poisson bracket $\{-,-\}$. We say that $\boldsymbol{K}$ is a Poisson-Lie group if its coordinate Hopf algebra is

Poisson-Hopf, that is, if the following condition holds for any functions f and g in $\mathbb{C}[\boldsymbol{K}]$.

$$\Delta\{f, g\} = \{f_1, g_1\} \otimes f_2 g_2 + f_1 g_1 \otimes \{f_2, g_2\} \tag{3.1}$$

Here we have written Δ for the coproduct on $\mathbb{C}[\boldsymbol{K}]$ and the subscripts 1 and 2 denote Sweedler's notation, a shorthand way of writing $\Delta(f)$. More explicitly, since there exists $f_1^{(i)}$ and $f_2^{(i)}$ such that $\Delta(f) = \sum_i f_1^{(i)} \otimes f_2^{(i)}$ then one frequently drops the summation and simply writes $f_1 \otimes f_2$ instead of $\Delta(f)$. This notation allows one to more easily manipulate expressions involving coproducts.

Equation (3.1) is an algebraic compatibility condition. There is also an equivalent geometric condition which we will give for $\mathbb{C}$-valued points of $\boldsymbol{K}$. In particular for any $\mathbb{C}$-valued points x and y of $\boldsymbol{K}$ and functions f and g in $\mathbb{C}[\boldsymbol{K}]$ the following holds.

$$\{f, g\}(x \cdot y) = \{l_x f, l_x g\}(y) + \{r_y f, r_y g\}(x) \tag{3.2}$$

In Equation (3.2) the symbol $\cdot$ denotes group multiplication. The notation $l_x f$ denotes the pullback of f by left multiplication by x and $r_y f$ denotes the pullback of f by right multiplication by y.

3.2 Lie Bialgebras, Duality and Manin Triples

In this section we discuss Lie bialgebras and explain how they are the infinitesimal analogues of Poisson-Lie groups. We will describe a kind of duality for Lie bialgebras which allows one to define duality of Poisson-Lie groups. We then explain an equivalent and useful characterization of Lie bialgebras called Manin triples. Finally we end with algebraic Manin triples which are the analogues of Manin triples for algebraic groups.

Our references for the material which follows are Section 2.2 of [ES01] for Lie bialgebras and Section 2.2.3 of [ES01] for duality of bialgebras. For Manin triples we have two references: Section 4.1 of [ES01] and Section 11 of [DCP93].

We begin with the definition of a Lie bialgebra. A Lie bialgebra is a Lie algebra with the additional structure of a compatible Lie algebra on its vector space dual.

Definition 3.2.1. A Lie bialgebra $(\mathfrak{k}, [-,-]_{\mathfrak{k}}, [-,-]_{\mathfrak{k}^*})$ is a finite dimensional complex Lie algebra $(\mathfrak{k}, [-,-]_{\mathfrak{k}})$ along with a Lie bracket $[-,-]_{\mathfrak{k}^*}$ on the dual vector space $\mathfrak{k}^*$ such that if we let the $p : \mathfrak{k} \to \mathfrak{k} \otimes \mathfrak{k}$ be the dual linear map to $[-,-]_{\mathfrak{k}^*}$ then the following compatibility condition is satisfied for any $X, Y \in \mathfrak{k}$.

$$p([X, Y]_{\mathfrak{k}}) = (ad_X \otimes id_{\mathfrak{k}} + id_{\mathfrak{k}} \otimes ad_X)p(Y) - (ad_Y \otimes id_{\mathfrak{k}} + id_{\mathfrak{k}} \otimes ad_Y)p(X)$$

Here ad denotes the adjoint action endomorphism of $\mathfrak{k}$ and $id_{\mathfrak{k}}$ is the identity endomorphism of $\mathfrak{k}$.

We will often simply write $\mathfrak{k}$ in place of $(\mathfrak{k}, [-,-]_{\mathfrak{k}}, [-,-]_{\mathfrak{k}^*})$ for a Lie bialgebra. A morphism of Lie biagebras $\mathfrak{k}_1 \to \mathfrak{k}_2$ is a Lie algebra homomorphism such that the induced map on dual vector spaces $\mathfrak{k}_2^* \to \mathfrak{k}_1^*$ is also a Lie algebra homomorphism.

Lie bialgebras satisfy a very simple duality by switching the roles of the vector spaces $\mathfrak{k}$ and $\mathfrak{k}^*$. If $(\mathfrak{k}, [-,-]_{\mathfrak{k}}, [-,-]_{\mathfrak{k}^*})$ is a Lie bialgebra then, using the canonical isomorphism $\mathfrak{k} \cong (\mathfrak{k}^*)^*$, we have that

$(\mathfrak{k}^*, [-,-]_{\mathfrak{k}^*}, [-,-]_{\mathfrak{k}})$ is again a Lie bialgebra. A proof of this fact can be found in Proposition 2.2 of [ES01]. Performing the procedure twice recovers $\mathfrak{k}$ up to canonical isomorphism of Lie bialgebras.

We will use the notation $\mathfrak{k}^\vee$ as shorthand for $(\mathfrak{k}^*, [-,-]_{\mathfrak{k}^*}, [-,-]_{\mathfrak{k}})$ and refer to it as the dual Lie bialgebra of $\mathfrak{k}$. We will reserve the notation $\mathfrak{k}^*$ for the vector space underlying $\mathfrak{k}^\vee$. More generally we will say that two Lie bialgebras $\mathfrak{a}$ and $\mathfrak{b}$ are dual if $\mathfrak{a}^\vee$ is isomorphic to $\mathfrak{b}$ as a Lie bialgebra.

Let's explain how Lie bialgebras arise from Poisson-Lie groups. First of all, a Poisson-Lie group $\boldsymbol{K}$ is an algebraic group and so has a tangent Lie algebra $(\mathfrak{k}, [-,-]_{\mathfrak{k}})$ i.e. via the usual Lie group-Lie algebra correspondence. In addition, linearizing the Poisson bracket $\{-,-\}$ at the group identity $e \in \boldsymbol{K}$ induces the structure of a Lie algebra on the vector space $\mathfrak{k}^*$ dual to $\mathfrak{k}$. In particular the resulting Lie bracket on $\mathfrak{k}^*$ is given as follows for any functions $f, g \in \mathbb{C}[\boldsymbol{K}]$.

$$[d_e f, d_e g]_{\mathfrak{k}^*} := d_e\{f, g\} \tag{3.3}$$

We shall refer to $[-,-]_{\mathfrak{k}}$ and $[-,-]_{\mathfrak{k}^*}$ as the tangent and cotangent Lie bracket of $\mathfrak{k}$ respectively. One can check, as for example in Proposition 11.34 of [LGPV13], that the Poisson-Lie group compatibility condition from Equation (3.1) induces the Lie bialgebra compatibility condition making $(\mathfrak{k}, [\cdot,\cdot]_{\mathfrak{k}}, [\cdot,\cdot]_{\mathfrak{k}^*})$ a Lie bialgebra. We call $\mathfrak{k}$ the tangent Lie bialgebra of $\boldsymbol{K}$.

The notion of duality for Lie bialgebras also extends to Poisson-Lie groups. If $\boldsymbol{A}$ and $\boldsymbol{A}^\vee$ are Poisson-Lie groups then we say that $\boldsymbol{A}^\vee$ is dual to $\boldsymbol{A}$ if their tangent Lie bialgebras are dual.

In the remainder of this section we will discuss Manin triples and algebraic Manin triples. Manin triples are an equivalent and useful way to package the data of a Lie bialgebra. Algebraic Manin triples are the corresponding analogues for Poisson-Lie groups.

Definition 3.2.2. A Manin triple is a triple of finite dimensional complex Lie algebras $(\mathfrak{k}, \mathfrak{a}, \mathfrak{a}^\vee)$ with the following properties.

(i) The Lie algebras $\mathfrak{a}$ and $\mathfrak{a}^\vee$ are Lie subalgebras of $\mathfrak{k}$ such that $\mathfrak{k} = \mathfrak{a} \oplus \mathfrak{a}^\vee$ as a vector space

(ii) There is a non-degenerate, invariant, symmetric bilinear form on $\mathfrak{k}$ with respect to which $\mathfrak{a}$ and $\mathfrak{a}^\vee$ are isotropic, i.e. the form vanishes when restricted to $\mathfrak{a}$ or restricted to $\mathfrak{a}^\vee$. We refer to this form as the Manin form

There is an obvious notion of a morphism of Manin triples and one has the following relationship between Lie bialgebras and Manin triples.

Proposition 3.2.1. If $\mathfrak{a}$ is a finite dimensional complex Lie algebra then there is a bijection between isomorphism classes of Manin triples of the form $(\mathfrak{k}, \mathfrak{a}, \mathfrak{a}^\vee)$ and isomorphism classes of Lie bialgebra structures on $\mathfrak{a}$.

Proof. See Section 4.1 of [ES01] for all of the details. In particular the Manin form identifies the vector space dual $\mathfrak{a}^*$ with $\mathfrak{a}^\vee$. This gives $\mathfrak{a}^*$ a Lie bracket $[-,-]_{\mathfrak{a}^*}$ coming from the one on $\mathfrak{a}^\vee$. One can then check that $(\mathfrak{a}, [-,-]_{\mathfrak{a}}, [-,-]_{\mathfrak{a}^*})$ is a Lie bialgebra. □

It is easy to see how duality works for Manin triples. If $(\mathfrak{k}, \mathfrak{a}, \mathfrak{a}^\vee)$ is a Manin triple then so too is $(\mathfrak{k}, \mathfrak{a}^\vee, \mathfrak{a})$. Proposition 3.2.1 then says that both $\mathfrak{a}$ and $\mathfrak{a}^\vee$ have the structures of Lie bialgebras and moreover one can check that $\mathfrak{a}^\vee$ is the dual Lie bialgebra of $\mathfrak{a}$.

We now explain the notion of algebraic Manin triples.

Definition 3.2.3. An algebraic Manin triple is a triple of algebraic groups $(\boldsymbol{K}, \boldsymbol{A}, \boldsymbol{A}^\vee)$ where $\boldsymbol{A}$ and $\boldsymbol{A}^\vee$ are closed subgroups of $\boldsymbol{K}$ such that their corresponding triple of tangent Lie algebras $(\mathfrak{k}, \mathfrak{a}, \mathfrak{a}^\vee)$ is a Manin triple.

We will refer to $(\mathfrak{k}, \mathfrak{a}, \mathfrak{a}^\vee)$ as the tangent triple of $(\boldsymbol{K}, \boldsymbol{A}, \boldsymbol{A}^\vee)$. We then have the following proposition which relates algebraic Manin triples to Poisson-Lie groups.

Proposition 3.2.2. If $(\boldsymbol{K}, \boldsymbol{A}, \boldsymbol{A}^\vee)$ is an algebraic Manin triple with tangent Manin triple $(\mathfrak{k}, \mathfrak{a}, \mathfrak{a}^\vee)$ then there exists unique Poisson-Lie group structures on $\boldsymbol{A}$ and $\boldsymbol{A}^\vee$ whose tangent Lie bialgebras are $\mathfrak{a}$ and $\mathfrak{a}^\vee$ respectively.

Proof. A proof of this fact can be found in Section 11 of [DCP93]. Existence is established on page 81 while uniqueness can be found at the end of page 80. □

If $(\boldsymbol{K}, \boldsymbol{A}, \boldsymbol{A}^\vee)$ is an algebraic Manin triple then since $\mathfrak{a}^\vee$ is the dual Lie bialgebra of $\mathfrak{a}$ we have that by definition $\boldsymbol{A}^\vee$ and $\boldsymbol{A}$ are dual as Poisson-Lie groups.

3.3 Standard Dual Poisson-Lie Groups and Bialgebras

In this section we explain how to construct the main example of Poisson-Lie groups and Lie bialgebras that we will use in this thesis: the standard dual Poisson-Lie group $\boldsymbol{G}^\vee$ and standard dual Lie bialgebra $\mathfrak{g}^\vee$. Our reference is Section 11 of [DCP93].

Let $\boldsymbol{G}$ be the simple algebraic group with tangent Lie algebra $\mathfrak{g}$, as in the discussion at the end of Section 1.1. One can give $\boldsymbol{G}$ a certain Poisson structure called the standard Poisson-Lie structure. This in turn endows $\mathfrak{g}$ with a Lie bialgebra structure called the standard bialgebra structure.

In this section – and in this thesis more generally – our interest is not so much in the Poisson-Lie group $\boldsymbol{G}$ but rather its *dual* $\boldsymbol{G}^\vee$. The tangent Lie bialgebra of $\boldsymbol{G}^\vee$ is the standard dual Lie bialgebra $\mathfrak{g}^\vee$. The objective of this section is to explain how to construct $\boldsymbol{G}^\vee$ and give a characterization of $\mathfrak{g}^\vee$. We also relate various subalgebras and subgroups of $\mathfrak{g}^\vee$ and $\boldsymbol{G}^\vee$ to analogous ones in $\mathfrak{g}$ and $\boldsymbol{G}$.

Let's first define $\boldsymbol{G}^\vee$ as an algebraic group. Recall from Section 1.1 that we have chosen positive and negative Borel subgroups $\boldsymbol{B}_\pm$ and a maximal torus $\boldsymbol{T}$ of $\boldsymbol{G}$. Denote by $\pi : \boldsymbol{B}_- \times \boldsymbol{B}_+ \to \boldsymbol{T}$ the product of the two canonical projections from $\boldsymbol{B}_+$ and $\boldsymbol{B}_-$ to the maximal torus. It is easy to see that π is a homomorphism of algebraic groups. The algebraic group $\boldsymbol{G}^\vee$ is then defined to be the kernel of the map π. We use the notation $i^\vee$ for the inclusion of $\boldsymbol{G}^\vee$ into $\boldsymbol{B}_- \times \boldsymbol{B}_+$.

Using Proposition 3.2.2 we now describe how $\boldsymbol{G}^\vee$ obtains the structure of a Poisson-Lie group by showing that it fits into an algebraic Manin triple of the form $(\boldsymbol{G} \times \boldsymbol{G}, \boldsymbol{G}, \boldsymbol{G}^\vee)$. In this triple the simple Lie group $\boldsymbol{G}$ is viewed as a subgroup of $\boldsymbol{G} \times \boldsymbol{G}$ via the diagonal embedding. The algebraic group $\boldsymbol{G}^\vee$ is identified with the subgroup of $\boldsymbol{G} \times \boldsymbol{G}$ via the inclusion $i^\vee$ along with the fact that $\boldsymbol{B}_- \times \boldsymbol{B}_+ \subset \boldsymbol{G} \times \boldsymbol{G}$.

To show that $(\boldsymbol{G} \times \boldsymbol{G}, \boldsymbol{G}, \boldsymbol{G}^\vee)$ is an algebraic Main triple we need to check that its tangent triple, denoted by $(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g}, \mathfrak{g}^\vee)$, is indeed a Manin triple. Here the simple Lie algebra $\mathfrak{g}$ is identified with the diagonal Lie subalgebra of $\mathfrak{g} \oplus \mathfrak{g}$ while $\mathfrak{g}^\vee$ is given by the following Lie subalgebra of $\mathfrak{g} \oplus \mathfrak{g}$.

$$\mathfrak{g}^\vee := \{(X_-, X_+) \in \mathfrak{b}_- \oplus \mathfrak{b}_+ \mid X_-^{\mathfrak{h}} + X_+^{\mathfrak{h}} = 0\} \tag{3.4}$$

The notation $X^{\mathfrak{h}}$ here denotes the projection of an element $X \in \mathfrak{g}$ to the Cartan subalgebra $\mathfrak{h}$. One can check that as a vector space $\mathfrak{g} \oplus \mathfrak{g}$ is the direct sum of its subspaces $\mathfrak{g}$ and $\mathfrak{g}^\vee$ and so we need only establish the existence of a Manin form. Recall from Section 1.1 that we have a normalized Cartan-Killing form $(-,-)_{\mathfrak{g}}$ on $\mathfrak{g}$. One can check that a Manin form on $\mathfrak{g} \oplus \mathfrak{g}$ is determined by the requirement that it restricts to $(-,-)_{\mathfrak{g}}$ on $\{0\} \oplus \mathfrak{g}$, $-(-,-)_{\mathfrak{g}}$ on $\mathfrak{g} \oplus \{0\}$ and that $\mathfrak{g} \oplus \{0\}$ and $\{0\} \oplus \mathfrak{g}$ are mutually orthogonal.

The hypothesis of Proposition 3.2.2 is then satisfied and so $\boldsymbol{G}$ and $\boldsymbol{G}^\vee$ have the structures of Poisson-Lie groups which are dual to each other. In addition the tangent Lie algebras $\mathfrak{g}$ and $\mathfrak{g}^\vee$ have the structures of Lie bialgebras which are again dual.

Definition 3.3.1. Define $\boldsymbol{G}$ to be the standard Poisson-Lie group and $\boldsymbol{G}^\vee$ to be the standard dual Poisson-Lie group. The tangent Lie bialgebra structure on $\mathfrak{g}$ is called the standard bialgebra structure. The Lie bialgebra dual $\mathfrak{g}^\vee$ is called the standard dual Lie bialgebra.

We will now unravel some of the features of $\mathfrak{g}^\vee$ more explicitly. Using the Manin form on $\mathfrak{g} \oplus \mathfrak{g}$ one can identify the vector space $\mathfrak{g}$ with vector space dual of $\mathfrak{g}^\vee$ *which we do from now on*. Using the root vectors of $\mathfrak{g}$ that we defined in Section 1.1 we have the following elements of $\mathfrak{g}^\vee$.

$$e_\alpha^\vee := (-f_\alpha, 0) \qquad f_\alpha^\vee := (0, e_\alpha) \qquad h_i^\vee := \frac{1}{2}(-h_i, h_i) \tag{3.5}$$

With respect to the Manin form identification of $\mathfrak{g}^\vee$ with the vector space dual of $\mathfrak{g}$ one can check that $e_\alpha^\vee$ is the dual vector of e_α, $f_\alpha^\vee$ is the dual vector of f_α and $h_j^\vee(h_i) = a_{ij}$. The elements in Equation (3.5) thus give a basis for $\mathfrak{g}^\vee$. We note that the root lattice Λ_Φ of $\mathfrak{g}$ sits inside $\mathfrak{h}^\vee$ via $\alpha_i \mapsto h_i^\vee$.

We haven't yet discussed the tangent and cotangent Lie brackets on $\mathfrak{g}^\vee$. The cotangent Lie bracket on the vector space dual of $\mathfrak{g}^\vee$ is the one coming from the bracket $[-,-]$ on $\mathfrak{g}$ via the Manin form identification. The tangent Lie bracket on $\mathfrak{g}^\vee$, which we denote by $[-,-]^\vee$, is given by the following proposition.

Proposition 3.3.1. The tangent Lie bracket $[-,-]^\vee$ on $\mathfrak{g}^\vee$ is determined by the following where $\alpha, \beta \in \Phi^+$ and $1 \leqslant i, j \leqslant r$.

$$\begin{aligned}
[h_i^\vee, h_j^\vee]^\vee &= 0 \\
[h_i^\vee, e_\alpha^\vee]^\vee &= \frac{1}{2}(\alpha_i, \alpha)_{\mathfrak{g}}\ e_\alpha^\vee \\
[h_i^\vee, f_\alpha^\vee]^\vee &= \frac{1}{2}(\alpha_i, \alpha)_{\mathfrak{g}}\ f_\alpha^\vee \\
[e_\alpha^\vee, f_\beta^\vee]^\vee &= 0
\end{aligned}$$

The following Serre relations are also required to be satisfied for $i \neq j$.

$$ad_{e^\vee_{\alpha_i}}^{1-a_{ij}}(e^\vee_{\alpha_j}) = 0 \qquad\qquad ad_{f^\vee_{\alpha_i}}^{1-a_{ij}}(f^\vee_{\alpha_j}) = 0$$

Proof. This follows from how $\mathfrak{g}^\vee$ was defined as a subalgebra of $\mathfrak{g} \oplus \mathfrak{g}$ in Equation (3.4) along with the generators and relations description of $\mathfrak{g}$ given in Definition 1.1.1. □

We now define certain Lie subalgebras of $\mathfrak{g}^\vee$ which are analogues of the Cartan, nilpotent and Borel subalgebras of $\mathfrak{g}$. The analogue of the Cartan subalgebra is given by $\mathfrak{h}^\vee := \langle h_i^\vee \rangle$. The following give the analogues of the nilpotent and Borel subalgebras.

$$\mathfrak{n}_+^\vee := \langle e_\alpha^\vee \rangle \qquad\qquad \mathfrak{b}_+^\vee := \langle e_\alpha^\vee, h_i^\vee \rangle$$
$$\mathfrak{n}_-^\vee := \langle f_\alpha^\vee \rangle \qquad\qquad \mathfrak{b}_-^\vee := \langle f_\alpha^\vee, h_i^\vee \rangle$$

How are the above subalgebras of $\mathfrak{g}^\vee$ related to the Cartan, nilpotent and Borel subalgebras of $\mathfrak{g}$? In the Borel case there is a canonical isomorphism between $\mathfrak{b}_\pm^\vee$ and $\mathfrak{b}_\mp$ given by the following.

$$\mathfrak{b}_\pm^\vee \to \mathfrak{b}_- \oplus \mathfrak{b}_+ \to \mathfrak{b}_\mp \tag{3.6}$$

Here the first map is given by the inclusion and the second is the canonical projection. It is easy to check that the isomorphism $\mathfrak{b}_+^\vee \cong \mathfrak{b}_-$ sends $e_\alpha^\vee \mapsto -f_\alpha$ and $h_i^\vee \mapsto -1/2 \cdot h_i$. Similarly under $\mathfrak{b}_-^\vee \cong \mathfrak{b}_+$ we have $f_\alpha^\vee \mapsto e_\alpha$ and $h_i^\vee \mapsto 1/2 \cdot h_i$. The isomorphism $\mathfrak{b}_\pm^\vee \cong \mathfrak{b}_\mp$ restrict to give a canonical isomorphism $\mathfrak{n}_\pm^\vee \cong \mathfrak{n}_\mp$ and two different canonical isomorphisms $\mathfrak{h}^\vee \cong \mathfrak{h}$.

On the algebraic group side of things the Lie subalgebras $\mathfrak{n}_\pm^\vee, \mathfrak{b}_\pm^\vee$ and $\mathfrak{h}^\vee$ of $\mathfrak{g}^\vee$ integrate to subgroups $\boldsymbol{N}_\pm^\vee$, $\boldsymbol{B}_\pm^\vee$ and $\boldsymbol{T}^\vee$ of $\boldsymbol{G}^\vee$ respectively. Integrating Equation (3.6) there is a canonical isomorphism between $\boldsymbol{B}_\pm^\vee$ and $\boldsymbol{B}_\mp$ given by the following.

$$\boldsymbol{B}_\pm^\vee \to \boldsymbol{B}_- \times \boldsymbol{B}_+ \to \boldsymbol{B}_\mp$$

Here the first map is given by inclusion and the last is the canonical projection. The isomorphism $\boldsymbol{B}_\pm^\vee \cong \boldsymbol{B}_\mp$ restricts to a canonical isomorphism between $\boldsymbol{N}_\pm^\vee$ and $\boldsymbol{N}_\mp$ and two canonical isomorphisms between $\boldsymbol{T}^\vee$ and $\boldsymbol{T}$.

Chapter 4

Integral Forms and Poisson-Lie Groups

In this short chapter we discuss integral forms of $\mathbb{C}(t)$-algebras. Integral forms are special types of subalgebras of $\mathbb{C}(t)$-algebras which allow one to rigorously set t to be certain values. In particular, we will explain the formal requirements and process by which Poisson algebras and coordinate algebras of Poisson-Lie groups may arise from integral forms.

4.1 Integral Forms and Poisson-Lie Groups

Suppose that we have a $\mathbb{C}(t)$-algebra which we will denote by B. Clearly B depends on a parameter t and one would often like to set t to be a particular value in order to study the resulting hopefully simpler algebra. A problem arises when one attempts to do this naïvely however: the existence of elements of B with poles can make specializing t behave badly.

Integral forms are certain $\mathbb{C}[t,t^{-1}]$-subalgebras of $\mathbb{C}(t)$-algebras which get around the above problem and allow one to set the parameter t to be any non-zero complex number. There can be lots of different integral forms sitting inside a $\mathbb{C}(t)$-algebra.

Definition 4.1.1. Let B be a $\mathbb{C}(t)$-algebra. An integral form of B is a $\mathbb{C}[t,t^{-1}]$-subalgebra $Z \subseteq B$ which is free as a $\mathbb{C}[t,t^{-1}]$-module and such that the multiplication map $\mathbb{C}(t) \otimes_{\mathbb{C}[t,t^{-1}]} Z \to B$ is an isomorphism of $\mathbb{C}(t)$-algebras.

Once one has an integral form $Z \subseteq B$ one can then set t to be a non-zero complex number w by taking the quotient algebra of Z by the ideal $(t-w)$. We call the resulting algebra the specialization of Z at $t = w$ or alternatively the $t = w$ limit of Z. In this thesis we will usually specialize integral forms at $t = 1$. We also note that for the notion of an integral, one dos not necessarily have to work over $\mathbb{C}$. One could equally work over $\mathbb{Z}$ or $\mathbb{Q}$ instead.

We now explain how Poisson algebras or even coordinate algebras of Poisson-Lie groups arise from integral forms. This phenomenon occurs in the special case that the $t = 1$ limit of an integral form is a commutative algebra.

When the $t = 1$ limit of a $\mathbb{C}[t,t^{-1}]$-algebra Z is commutative we will call the quotient algebra $Z_{sc} := Z/(t-1)$ the semi-classical limit of Z. We then have the following proposition. Note in general we will use the notation $\overline{x}$ for the image of an element $x \in Z$ in the quotient of Z by an ideal.

Proposition 4.1.1. If Z is a $\mathbb{C}[t, t^{-1}]$-algebra with commutative $t = 1$ limit then Z_{sc} is a commutative Poisson $\mathbb{C}$-algebra under the following Poisson bracket.

$$\{\overline{x}, \overline{y}\} := \overline{\left(\frac{xy - yx}{t - 1}\right)}, \qquad x, y \in Z$$

Proof. The Poisson bracket is well-defined since if the product on Z is commutative modulo $(t-1)$ then $(xy - yx)/(t-1)$ is a well-defined element of Z. Skew-symmetry is immediate while the Jacobi identity can be obtained easily using the above formula. For the Leibniz identity note that we have the following identity in Z for any elements $x, y, z \in Z$.

$$\begin{aligned} xyz - zxy &= xyz + (xzy - xzy) - zxy \\ &= x(yz - zy) + (xz - zx)y \end{aligned}$$

Dividing by $t - 1$ and taking the image in Z_{sc} we obtain that $\{\overline{xy}, \overline{z}\} = \overline{x}\{\overline{y}, \overline{z}\} + \{\overline{x}, \overline{z}\}\overline{y}$. □

Suppose our $\mathbb{C}[t, t^{-1}]$-algebra Z from Proposition 4.1.1 has in addition the structure of a $\mathbb{C}[t, t^{-1}]$-Hopf algebra. One can check that the $\mathbb{C}[t, t^{-1}]$-Hopf algebra structure on Z descends to a commutative $\mathbb{C}$-Hopf algebra structure on Z_{sc}. Even better, the Poisson structure from Proposition 4.1.1 is compatible with this $\mathbb{C}$-Hopf algebra structure in the following sense.

Proposition 4.1.2. If Z is a $\mathbb{C}[t, t^{-1}]$-Hopf algebra with commutative $t = 1$ limit then Z_{sc} is a commutative Poisson-Hopf algebra over $\mathbb{C}$. In particular Z_{sc} is the coordinate algebra of a Poisson-Lie group.

Proof. Let Δ denote the coproduct on Z and $\overline{\Delta}$ the induced coproduct on Z_{sc}. The compatibility between the Hopf algebra and Poisson structure required for a Poisson-Hopf algebra was given in Equation (3.1). The requirement, in Sweedler's notation, is that for any $x, y \in Z$ we have the following identity in $Z_{sc} \otimes Z_{sc}$.

$$\overline{\Delta}\{\overline{x}, \overline{y}\} = \{\overline{x}_1, \overline{y}_1\} \otimes \overline{x}_2\overline{y}_2 + \overline{x}_1\overline{y}_1 \otimes \{\overline{x}_2, \overline{y}_2\} \tag{4.1}$$

To see that Equation (4.1) holds consider first the following identity in $Z \otimes Z$.

$$\begin{aligned} \Delta(xy - yx) &= x_1y_1 \otimes x_2y_2 - y_1x_1 \otimes y_2x_2 \\ &= (x_1y_1 - y_1x_1) \otimes x_2y_2 + y_1x_1 \otimes (x_2y_2 - y_2x_2) \\ &= (x_1y_1 - y_1x_1) \otimes x_2y_2 + x_1y_1 \otimes (x_2y_2 - y_2x_2) + O((t-1)^2) \end{aligned}$$

In establishing the last equality we have used the fact that $y_1x_1 = x_1y_1 + O(t-1)$ and that $y_2x_2 - x_2y_2 = O(t-1)$ where O denotes up to order. Dividing the above expression by $t - 1$ and taking the image in $Z_{sc} \otimes Z_{sc}$ we obtain Equation (4.1). □

We end this section with a comment on a direction that we shall not explore this thesis. When the $t = 1$ limit of a $\mathbb{C}[t, t^{-1}]$-Hopf algebra a *cocommutative* Hopf algebra then one obtains something called a coPoisson-Hopf algebra. A coPoisson-Hopf algebra is the dual notion of a Poisson-Hopf algebra.

Often one can show that the underlying Hopf algebra of a coPoisson-Hopf algebra is the universal enveloping algebra of a Lie algebra. In such a case this Lie algebra is in fact a Lie bialgebra and the coPoisson-Hopf structure is the induced structure that the universal enveloping algebra acquires.

Chapter 5

Quantized Enveloping Algebras

In this chapter we give an overview of quantized enveloping algebras. These are $\mathbb{C}(t)$-algebras which are quantizations of $\mathrm{U}(\mathfrak{g})$ the universal enveloping algebra of the simple Lie algebra $\mathfrak{g}$.

For our purposes the most important feature of quantized enveloping algebras is the existence of two natural integral forms. One, which we call the Poisson integral form, specializes at $t = 1$ to the coordinate algebra of functions on a Poisson-Lie group. The other is called the restricted integral form and specializes at $t = 1$ to $\mathrm{U}(\mathfrak{g})$.

We start off this chapter with Section 5.1 where we define $\mathrm{U}_t(\mathfrak{g})$, the quantized enveloping algebra of $\mathfrak{g}$. This is done by deforming a generators and relations description of $\mathrm{U}(\mathfrak{g})$.

Section 5.2 concerns algebra automorphisms of $\mathrm{U}_t(\mathfrak{g})$ due to Lusztig. These automorphisms are used to define analogues in $\mathrm{U}_t(\mathfrak{g})$ of root vectors of $\mathfrak{g}$ called quantum root vectors. For us the main use of quantum root vectors is that they allow one to define the Poisson integral form of Section 5.5.

The aim of Section 5.3 is to introduce an algebra involution Σ of the quantized enveloping algebra. This involution commutes with the automorphisms from Section 5.2. In Section 13.4 we will see that Σ is the analogue of a certain shift functor induced involution of the Bridgeland-Hall algebra.

In Section 5.4 we give the definition of the restricted integral form of the quantized enveloping algebra due to Lusztig. For us the main feature of this integral form is that its specialization at $t = 1$ is universal enveloping algebra $\mathrm{U}(\mathfrak{g})$.

Finally in Section 5.5 we discuss what we call the Poisson integral form of $\mathrm{U}_t(\mathfrak{g})$. This was originally defined by De Concini and Procesi [DCP93] who showed that its $t = 1$ limit is the coordinate algebra of the dual Poisson-Lie group $\boldsymbol{G}^\vee$ from Section 3.3.

5.1 Definition

This section is concerned with defining quantized enveloping algebras. A reference for the material in this section is Section 9.1 of [CP95].

The quantized enveloping algebra $\mathrm{U}_t(\mathfrak{g})$ is a quantization of $\mathrm{U}(\mathfrak{g})$ the universal enveloping algebra of the simple Lie algebra $\mathfrak{g}$. Recall from Definition 1.1.1 that we defined $\mathfrak{g}$ via Lie algebra generators and relations. The *same* generators and relations – viewed instead as generating an associative $\mathbb{C}$-algebra – also define the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$.

Quantization of $\mathrm{U}(\mathfrak{g})$ is achieved by modifying the generators and relations of $\mathrm{U}(\mathfrak{g})$ to obtain a $\mathbb{C}(t)$-algebra $\mathrm{U}_t(\mathfrak{g})$. A key point is that it will be possible to recover $\mathrm{U}(\mathfrak{g})$ by taking the $t = 1$ limit of a suitable integral form of $\mathrm{U}_t(\mathfrak{g})$. We will make this statement rigorous in Section 5.4. In the meantime, an elementary example of quantization is given by the following t-analogues of various integers which we will need subsequently. Here $n \in \mathbb{N}, s \in \mathbb{Z}_{\geqslant 0}$ with $0 \leqslant s \leqslant n$ and $[0]_t! := 1$ by convention.

$$[n]_t := \frac{t^n - t^{-n}}{t - t^{-1}} \qquad [n]_t! := [n]_t[n-1]_t \cdots [1]_t \qquad \begin{bmatrix} n \\ s \end{bmatrix}_t := \frac{[n]_t!}{[s]_t![n-s]_t!} \tag{5.1}$$

It is easy to see that sending $t \to 1$ in the above definitions recovers the integers n, $n!$ and binomial coefficient $\binom{n}{s}$ respectively.

The quantized enveloping algebra of $\mathfrak{g}$ is then given as follows. We remind the reader that $(a_{ij})_{i,j=1}^r$ denotes the symmetric Cartan matrix associated to a simply-laced Dynkin diagram.

Definition 5.1.1. Define the quantized enveloping algebra $\mathrm{U}_t(\mathfrak{g})$ of $\mathfrak{g}$ to be the $\mathbb{C}(t)$-algebra with generators X_i, Y_i and $K_i^{\pm 1}$ for $1 \leqslant i \leqslant r$ such that K_i and K_i^{-1} are mutually inverse and the following relations are satisfied for $1 \leqslant i, j \leqslant r$.

$$K_iK_j = K_jK_i \tag{5.2}$$
$$K_iX_j = t^{a_{ij}}\ X_jK_i \tag{5.3}$$
$$K_iY_j = t^{-a_{ij}}Y_jK_i \tag{5.4}$$
$$[X_i, Y_j] = \delta_{ij}\frac{K_i - K_i^{-1}}{t - t^{-1}} \tag{5.5}$$

We also require that the following t-analogues of the Serre relations hold for $i \neq j$.

$$\sum_{\mu+\nu=1-a_{ij}} (-1)^{\mu} \begin{bmatrix} 1-a_{ij} \\ \mu \end{bmatrix}_t X_i^{\mu}X_jX_i^{\nu} = 0 \qquad \sum_{\mu+\nu=1-a_{ij}} (-1)^{\mu} \begin{bmatrix} 1-a_{ij} \\ \mu \end{bmatrix}_t Y_i^{\mu}Y_jY_i^{\nu} = 0$$

Example 5.1.2. In the case of $\mathfrak{g} = \mathfrak{sl}_2$ then $r = 1$ and the Cartan matrix is the 1x1 matrix with $a_{11} = 2$. Thus $\mathrm{U}_t(\mathfrak{sl}_2)$ is the $\mathbb{C}(t)$-algebra with generators X, Y and $K^{\pm 1}$ such that K and K^{-1} are mutually inverse and the following relations hold.

$$KX = t^2\ XK$$
$$KY = t^{-2}YK$$
$$[X, Y] = \frac{K - K^{-1}}{t - t^{-1}}$$

We end this section by defining $\mathrm{U}_t(\mathfrak{n}_+)$ to be the $\mathbb{C}(t)$-subalgebra of $\mathrm{U}_t(\mathfrak{g})$ generated by the elements X_i. This subalgebra plays an important role in the story of Hall algebras and is called the positive part of the quantized enveloping algebra.

5.2 Lusztig's Automorphisms and Quantum Root Vectors

This section is concerned with certain automorphisms of quantized enveloping algebras due to Lusztig. We will use these automorphisms to define special elements of $\mathrm{U}_t(\mathfrak{g})$ called quantum root vectors. Quantum root vectors are analogues of root vectors of simple Lie algebras and are used to define integral forms of $\mathrm{U}_t(\mathfrak{g})$. A reference for this section is Section 6.8 of [DDPW08].

We begin with some notation. For any element x of $\mathrm{U}_t(\mathfrak{g})$ define the divided powers of x to be $x^{(k)} := x^k/[k]_t!$ where $k \in \mathbb{Z}_{\geqslant 0}$. We then have the following theorem due to Lusztig.

Theorem 5.2.1 (Lusztig)**.** For $1 \leqslant i \leqslant r$ there are $\mathbb{C}(t)$-algebra automorphisms T_i of $\mathrm{U}_t(\mathfrak{g})$ given by the following where $1 \leqslant i, j \leqslant r$.

$$\begin{aligned} T_i(X_i) &= -Y_i K_i \\ T_i(Y_i) &= -K_i^{-1} X_i \\ T_i(K_j) &= K_{s_i(\alpha_j)} \\ T_i(X_j) &= \sum_{\mu+\nu=-a_{ij}} (-1)^{\mu} t^{-\mu} X_i^{(\nu)} X_j X_i^{(\mu)} && (i \neq j) \\ T_i(Y_j) &= \sum_{\mu+\nu=-a_{ij}} (-1)^{\mu} t^{\mu} Y_i^{(\mu)} Y_j Y_i^{(\nu)} && (i \neq j) \end{aligned}$$

Here s_i denotes the simple reflection at the simple root α_i. The notation $K_{s_i(\alpha_j)}$ means that, on decomposing $s_i(\alpha_j) = \sum n_k \alpha_k$ in Λ_Φ, then $K_{s_i(\alpha_j)} = \prod K_k^{n_k}$

Proof. This is Theorem 6.41 in [DDPW08]. □

Lusztig showed that the automorphisms from Theorem 5.2.1 induce an action of the braid group of $\mathfrak{g}$ on $\mathrm{U}_t(\mathfrak{g})$. We refer the interested reader to Theorem 6.45 of [DDPW08] for a formal statement of this fact, although we will not need it in this thesis.

In the remainder of this section we explain how Lusztig's automorphism generate special elements of $\mathrm{U}_t(\mathfrak{g})$ called quantum root vectors. We will need to recall from Section 1.1 that there exists a unique element w_0 of longest length in the Weyl group of $\mathfrak{g}$. In particular the length of the element w_0 is given by the number N of positive roots of $\mathfrak{g}$.

A fact from the theory of simple Lie algebras is that any reduced decomposition of w_0 into simple reflections $w_0 = s_{i_1} \dots s_{i_N}$ allows one to generate the set of positive roots from the simple roots. This is done by setting $\beta_1 := \alpha_{i_1}$ and then setting $\beta_k := s_{i_1} \dots s_{k-1}(\alpha_{i_k})$ for each $1 < k \leqslant N$. All of the positive roots then appear exactly once in the list $\beta_1, \dots, \beta_N$ which we point out also comes equipped with a total ordering.

Quantum root vectors of $\mathrm{U}_t(\mathfrak{g})$ are generated in a similar way except by applying Lusztig's automorphisms to the generators X_i and Y_i instead of simple reflections s_i to the simple roots α_i. In the following definition we fix a reduced decomposition $w_0 = s_{i_1} \dots s_{i_N}$ and thereby fix a total ordering $\beta_1, \dots, \beta_N$ of the set of positive roots .

Definition 5.2.1. For each $1 \leqslant k \leqslant N$ define the quantum root vector of $\mathrm{U}_t(\mathfrak{g})$ associated to β_k to be the following.

$$X_{\beta_k} := T_{i_1}^{-1} \cdots T_{i_{k-1}}^{-1}(X_{i_k}) \qquad\qquad Y_{\beta_k} := T_{i_1}^{-1} \cdots T_{i_{k-1}}^{-1}(Y_{i_k})$$

Our convention here for $k = 1$ is that $X_{\beta_1} := X_{i_1}$ and $Y_{\beta_1} := Y_{i_1}$.

We emphasize that this procedure is not canonical and very much depends on our choice of reduced decomposition of w_0.

5.3 An Involution

In this section we introduce a certain algebra involution of $\mathrm{U}_t(\mathfrak{g})$ and then show that it preserves Lusztig's automorphisms. This involution will come in useful later on in Section 13.4 and Section 16.1.

We begin by defining $\Sigma : \mathrm{U}_t(\mathfrak{g}) \to \mathrm{U}_t(\mathfrak{g})$ as follows. A glance at the generators and relations description of $\mathrm{U}_t(\mathfrak{g})$ from Definition 5.1.1 shows that Σ does indeed give an algebra involution.

$$\Sigma(X_i) = -tY_i \qquad\qquad \Sigma(Y_i) = -t^{-1}X_i \qquad\qquad \Sigma(K_i^{\pm 1}) = K_i^{-\mp 1} \tag{5.6}$$

I could not find this definition in the literature and it appears to be non-standard. A pleasant feature of Σ, however, is that it commutes with Lusztig's automorphisms.

Proposition 5.3.1. For each $1 \leqslant i \leqslant r$ the following diagram commutes.

$$\begin{array}{ccc} U_t(\mathfrak{g}) & \xrightarrow{T_i} & U_t(\mathfrak{g}) \\ \downarrow{\scriptstyle\Sigma} & & \downarrow{\scriptstyle\Sigma} \\ U_t(\mathfrak{g}) & \xrightarrow{T_i} & U_t(\mathfrak{g}) \end{array} \tag{5.7}$$

Proof. Using the formulas in Theorem 5.2.1 and Equation (5.6) we will check the commutativity of Equation (5.7) on the generators X_i, Y_i and K_i. The following gives the case of K_i.

$$T_i \circ \Sigma(K_j) = T_i(K_j^{-1}) = K_{-s_i(\alpha_j)} = \Sigma(K_{s_i(\alpha_j)}) = \Sigma \circ T_i(K_j)$$

For the generators X_i and Y_i we have the following. We point out that Equation (5.3) and Equation (5.4) are used for the third equalities below.

$$\begin{aligned} T_i \circ \Sigma(X_i) = T_i(-tY_i) &= tK_i^{-1}X_i \\ &= t^{-1}X_iK_i^{-1} \\ &= \Sigma(-Y_iK_i) \\ &= \Sigma \circ T_i(X_i) \end{aligned} \qquad\qquad \begin{aligned} T_i \circ \Sigma(Y_i) = T_i(-t^{-1}X_i) &= t^{-1}Y_iK_i \\ &= tK_iY_i \\ &= \Sigma(-K_i^{-1}X_i) \\ &= \Sigma \circ T_i(Y_i) \end{aligned}$$

The case of the generators X_j when $j \neq i$ can be shown as follows.

$$\begin{aligned} T_i \circ \Sigma(X_j) = T_i(-tY_j) &= \sum_{\mu+\nu=-a_{ij}} (-1)^{\mu+1} t^{\mu+1} Y_i^{(\mu)} Y_j Y_i^{(\nu)} \\ &= \Sigma \circ \sum_{\mu+\nu=-a_{ij}} (-1)^{\mu+1-(1-a_{ij})} t^{\mu+1-(1-a_{ij})} X_i^{(\mu)} X_j X_i^{(\nu)} \\ &= \Sigma \circ \sum_{\mu+\nu=-a_{ij}} (-1)^{\nu} t^{-\nu} \; X_i^{(\mu)} X_j X_i^{(\nu)} \\ &= \Sigma \circ T_i(X_j) \end{aligned}$$

Similarly we have the case of the generators Y_j for $j \neq i$.

$$\begin{aligned} T_i \circ \Sigma(Y_j) = T_i(-t^{-1}X_j) &= \sum_{\mu+\nu=-a_{ij}} (-1)^{\mu+1} t^{-\mu-1} X_i^{(\nu)} X_j X_i^{(\mu)} \\ &= \Sigma \circ \sum_{\mu+\nu=-a_{ij}} (-1)^{\mu+1-(1-a_{ij})} t^{-\mu-1+(1-a_{ij})} Y_i^{(\nu)} Y_j Y_i^{(\mu)} \\ &= \Sigma \circ \sum_{\mu+\nu=-a_{ij}} (-1)^{\nu} t^{\nu} \; Y_i^{(\nu)} Y_j Y_i^{(\mu)} \\ &= \Sigma \circ T_i(Y_j) \end{aligned}$$

□

We end by mentioning how the involution Σ interacts with the quantum root vectors from Definition 5.2.1. Indeed using the fact that Σ commutes with Lusztig's automorphisms one can easily establish the following identities for $1 \leqslant k \leqslant N$.

$$\Sigma(X_{\beta_k}) = -tY_{\beta_k} \qquad\qquad \Sigma(Y_{\beta_k}) = -t^{-1}X_{\beta_k}$$

5.4 Restricted Integral Form

In this section we define an integral form of $\mathrm{U}_t(\mathfrak{g})$ due to Lusztig called the restricted integral form. A key property of the restricted integral form is that (up to a slight caveat) it specializes at $t = 1$ to the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$. A reference for this section is Section 9.3 of [CP95].

Recalling the divided powers notation from Section 5.2 we begin with the following definition.

Definition 5.4.1. Define the restricted integral form of the quantized enveloping algebra to be the $\mathbb{C}[t, t^{-1}]$-subalgebra $\mathrm{U}_t^{Res}(\mathfrak{g})$ of $\mathrm{U}_t(\mathfrak{g})$ generated by the divided powers $X_i^{(k)}$, $Y_i^{(k)}$ along with the elements $K_i^{\pm 1}$ where $k \geqslant 0$ and $1 \leqslant i \leqslant r$.

That $\mathrm{U}_t^{Res}(\mathfrak{g})$ is indeed an integral form of $\mathrm{U}_t(\mathfrak{g})$ is Proposition 9.3.1. of [CP95]. Let's discuss how to recover the universal enveloping algebra of $\mathfrak{g}$ from this integral form. One would hope that simply setting $t = 1$ does the job, but that is not quite true – one must also set each $K_i = 1$. Defining U_{qc} to be the quotient of $\mathrm{U}_t^{Res}(\mathfrak{g})$ by the ideal generated by the elements $t - 1$ and $K_i - 1$ where $1 \leqslant i \leqslant r$ we have the following theorem.

Theorem 5.4.1 (Lusztig)**.** There is an isomorphism of $\mathbb{C}$-algebras $\mathrm{U}(\mathfrak{g}) \to \mathrm{U}_{qc}$ given by the following.

$$e_i \mapsto \overline{X}_i \qquad f_i \mapsto \overline{Y}_i \qquad h_i \mapsto \overline{[X_i, Y_i]}$$

Proof. This is (b) in Section 6.7 of Lusztig's [Lus90a]. Note that Lusztig defined $\mathrm{U}(\mathfrak{g})$ and $\mathrm{U}_t(\mathfrak{g})$ over the ground field of $\mathbb{Q}$ rather than $\mathbb{C}$ as we have done. The result follows on base changing however. □

With an integral form we can also specialize t to be values other than 1. In particular for any q is a prime power we will define $\mathrm{U}_q(\mathfrak{g})$ to be the quotient of $\mathrm{U}_t^{Res}(\mathfrak{g})$ by the ideal generated by the element $t - q^{\frac{1}{2}}$. Note we are specializing at the positive square root of q here.

We end by defining positive parts of the algebras considered in this section. Define the positive part of $\mathrm{U}_q(\mathfrak{g})$ to be the subalgebra $\mathrm{U}_q(\mathfrak{n}_+)$ generated by the elements X_i where $1 \leqslant i \leqslant r$. The positive part of the restricted integral form is defined to be the $\mathbb{C}[t, t^{-1}]$-subalgebra $\mathrm{U}_t^{Res}(\mathfrak{n}_+)$ of $\mathrm{U}_t^{Res}(\mathfrak{g})$ generated by the divided powers $X_i^{(k)}$ where $1 \leqslant i \leqslant r$.

5.5 Poisson Integral Form

This section is concerned with an integral form of $\mathrm{U}_t(\mathfrak{g})$ due to De Concini and Procesi which we call the Poisson integral form. One of the main features of this integral form is that it specializes at $t = 1$ to the coordinate algebra of the dual Poisson-Lie group that we defined in Section 3.3. The name Poisson integral form is not standard and indeed does not seem to have been given a name in the literature. A reference for this section is [DCP93], however we will discuss a caveat below.

The key ingredient in defining the Poisson integral form is to renormalize the quantum root vectors X_α and Y_β. To this end define the rescaled quantum root vectors to be the following elements of $\mathrm{U}_t(\mathfrak{g})$ for $1 \leqslant k \leqslant N$.

$$E_{\beta_k} := (t^2 - 1) X_{\beta_k} \qquad F_{\beta_k} := (t^2 - 1) Y_{\beta_k} \tag{5.8}$$

We point out that, as for the unscaled ones, the rescaled quantum root vectors are related by $\Sigma(E_{\beta_k}) = -tF_{\beta_k}$ and $\Sigma(F_{\beta_k}) = -t^{-1}E_{\beta_k}$ where Σ is the involution from Section 5.3. We can then define the Poisson integral form as follows.

Definition 5.5.1. Define the Poisson integral form of the quantized enveloping algebra to be the $\mathbb{C}[t, t^{-1}]$-subalgebra $\mathrm{U}_t^{Poiss}(\mathfrak{g})$ of $\mathrm{U}_t(\mathfrak{g})$ generated by the elements E_{β_k}, F_{β_k} and $K_i^{\pm 1}$ where $1 \leqslant i \leqslant r$ and $1 \leqslant k \leqslant N$.

Using Bridgeland-Hall algebras we will show that $\mathrm{U}_t(\mathfrak{g})$ is indeed an integral form. This will follow from Theorem 16.2.4 along with the commentary following Definition 15.1.1.

In [DCP93], De Concini and Procesi showed that $\mathrm{U}_t^{Poiss}(\mathfrak{g})$ specializes at $t = 1$ to the coordinate algebra of the dual Poisson-Lie group that we defined in Section 3.3. In Chapter 19 we will explain how to use Bridgeland-Hall algebras to give a proof of this fact. We should point out that there are some differences between how we have defined $\mathrm{U}_t^{Poiss}(\mathfrak{g})$ and how De Concini and Procesi go about defining $\mathrm{U}_t^{Poiss}(\mathfrak{g})$ in Section 12.1 of [DCP93].

The main difference is that Lusztig introduced several different automorphisms of $\mathrm{U}_t(\mathfrak{g})$ other than the ones we gave in Theorem 5.2.1 and unfortunately De Concini and Procesi chose different ones to the ones we use. Since $\mathrm{U}_t^{Poiss}(\mathfrak{g})$ is defined in terms of quantum root vectors which in turn are defined in terms of these automorphisms I was not sure that the two definitions coincide.

Another albeit minor difference is that De Concini and Procesi consider a different variant of the quantized enveloping algebra than we have used called the simply connected rational form of $\mathrm{U}_t(\mathfrak{g})$. The one that we have used in Definition 5.1.1 is called the adjoint rational form of $\mathrm{U}_t(\mathfrak{g})$.

We end by defining the positive part of the Poisson integral form $\mathrm{U}_t^{Poiss}(\mathfrak{n}_+)$ to be the $\mathbb{C}[t, t^{-1}]$-subalgebra of $\mathrm{U}_t^{Poiss}(\mathfrak{g})$ generated by the elements E_{β_k} where $1 \leqslant k \leqslant r$.

Chapter 6

Non-Generic (Bridgeland-)Hall Algebras

This chapter is concerned with giving an overview of (non-generic) Hall algebras and Bridgeland-Hall algebras of categories of simply-laced quiver representations.

In Section 6.1 we discuss the background material that will be needed to define the structure constants of Hall algebras and their integral forms. The two integral forms of Hall algebras treated in this thesis are distinguished by whether their product formulas count sets of extensions of quiver representations or flags of subobjects. In particular we carefully explain the difference between these two sets.

Section 6.2 concerns the precise definition of non-generic Hall algebras. We state Ringel's foundational result concerning the relationship between Hall algebras of categories of simply-laced quiver representations and positive parts of quantized enveloping algebras.

Finally in Section 6.3 we define non-generic Bridgeland-Hall algebras. We also explain how Bridgeland used these algebras to extend Ringel's theorem to recover the whole quantized enveloping algebra rather than just its positive part.

6.1 Extensions vs Flags of Subobjects

This section is concerned with background material which will be used to define the structure constants of Hall algebras. We will first discuss the difference between certain sets of extensions of quiver representations and flags of subobjects of quiver representations. We then end with a well-known formula due to Riedtmann which relates the cardinalities of these two types of sets.

Let L_q, M_q and N_q be quiver representations and consider the following set of short exact sequence.

$$\mathrm{SES}(M_q, N_q)_{L_q} := \{0 \to N_q \xrightarrow{f} L_q \xrightarrow{g} M_q \to 0\}$$

The automorphism groups of the quiver representation L_q, M_q and N_q give rise to two different (left) actions on the set $\mathrm{SES}(M_q, N_q)_{L_q}$ each with a different quotient set (i.e. different set of orbits under the group actions). One quotient set is a set of flags of subobjects of quiver representations, the other is a of extensions of quiver representations.

The first action is given by the following. We use the shorthand notation $(f, g) \in \mathrm{SES}(M_q, N_q)_{L_q}$ to denote a short exact sequence of the form $0 \to N_q \xrightarrow{f} L_q \xrightarrow{g} M_q \to 0$.

$$\mathrm{Aut}(N_q) \times \mathrm{Aut}(M_q) \circlearrowright \mathrm{SES}(M_q, N_q)_{L_q}, \qquad (\Psi_N, \Psi_M) \cdot (f, g) = (f \circ \Psi_N^{-1}, \Psi_M \circ g) \tag{6.1}$$

Here injectivity of f and surjectivity of g imply that the group $\mathrm{Aut}(M_q) \times \mathrm{Aut}(N_q)$ acts freely on $\mathrm{SES}(M_q, N_q)_{L_q}$. The quotient set, which we denote by $\mathrm{F}^{L_q}_{M_q,N_q}$, is the set of subobjects $N_q \subseteq L_q$ with quotient object M_q.

Here we are using the categorical definition of subobject and quotient object. In particular a subobject $N_q \subseteq L_q$ is an equivalence class of monomorpisms $N_q \xhookrightarrow{f} L_q$ under the equivalence relation $f \sim f'$ if $f' = f \circ \Psi_N^{-1}$ for some automorphism Ψ_N of N_q. A quotient object is an equivalence class of epimorphisms $L_q \overset{g}{\twoheadrightarrow} M_q$ under the equivalence relation $g \sim g'$ if $g' = \Psi_M \circ g$ for some automorphism Ψ_M of M_q.

The second action is given by the following.

$$\mathrm{Aut}(L_q) \circlearrowright \mathrm{SES}(M_q, N_q)_{L_q}, \qquad \Psi_L \cdot (f, g) = (\Psi_L \circ f, g \circ \Psi_L^{-1}) \tag{6.2}$$

The quotient of the set $\mathrm{SES}(M_q, N_q)_{L_q}$ by $\mathrm{Aut}(L_q)$ is the subset $\mathrm{Ext}^1(M_q, N_q)_{L_q} \subseteq \mathrm{Ext}^1(M_q, N_q)$ consisting of extensions of M_q by N_q with middle term isomorphic to L_q. This interpretation comes from the usual Yoneda description of Ext^1 as equivalence classes of short exact sequences.

Unlike the action in Equation (6.1), the action given by Equation (6.2) is not free. Let $\mathrm{Stab}_{(f,g)} \subseteq \mathrm{Aut}(L_q)$ denote the stabilizer of a short exact sequence (f, g) under the group action in Equation (6.2).

Lemma 6.1.1. The following is an isomorphism between the additive group $\mathrm{Hom}(M_q, N_q)$ and $\mathrm{Stab}_{(f,g)}$.

$$\mathrm{Hom}(M_q, N_q) \to \mathrm{Stab}_{(f,g)}, \qquad h \mapsto id_{L_q} + f \circ h \circ g$$

Proof. We first note that $id_{L_q} + f \circ h \circ g$ is indeed an automorphism of L_q. as its inverse is simply given by $id_{L_q} - f \circ h \circ g$. For injectivity suppose that $id_{L_q} + f \circ h \circ g = id_{L_q} + f \circ h' \circ g$. This is equivalent to $f \circ h \circ g = f \circ h' \circ g$ which implies that $h = h'$ using the fact that f is a monomorphism and g is an epimorphism.

For surjectivity suppose that $\psi_L \in \mathrm{Stab}_{(f,g)}$, that is, $\Psi_L \cdot (f, g) = (f, g)$. Using the formula in Equation (6.1) one can check that this condition is equivalent to $(\Psi_L - id_{L_q}) \circ f = 0$ and $g \circ (\Psi_L - id_{L_q}) = 0$. We may rewrite these two conditions in terms of pullback and pushforwards as $f^*(\Psi_L - id_{L_q}) = 0$ and $g_*(\Psi_L - id_{L_q}) = 0$.

Now applying the functors $\mathrm{Hom}(M_q, -)$ and $\mathrm{Hom}(-, L_q)$ to the short exact sequence $0 \to N_q \xrightarrow{f} L_q \xrightarrow{g} M_q \to 0$ we have the following two usual exact sequences.

$$0 \to \mathrm{Hom}(M_q, N_q) \xrightarrow{f_*} \mathrm{Hom}(M_q, L_q) \xrightarrow{g_*} \mathrm{Hom}(M_q, M_q) \tag{6.3}$$

$$0 \to \mathrm{Hom}(M_q, L_q) \xrightarrow{g^*} \mathrm{Hom}(L_q, L_q) \xrightarrow{f^*} \mathrm{Hom}(N_q, L_q) \tag{6.4}$$

Since $f^*(\Psi_L - id_{L_q}) = 0$, then by exactness of Equation (6.4) there exists an homomorphism $a \in \mathrm{Hom}(M_q, L_q)$ with $g^*(a) = \Psi_L - id_{L_q}$. We then have $0 = g_*(\Psi_L - id_{L_q}) = g_* g^*(a) = g^* g_*(a)$. Injectivity of g^* implies that $g_*(a) = 0$ and so exactness of Equation (6.3) implies that $a = f_*(h)$ for some $h \in \mathrm{Hom}(M_q, N_q)$. We thus have that $\Psi_L = id_{L_q} + f \circ h \circ g$, which establishes surjectivity. □

The distinction between the sets $\mathrm{Ext}^1(M_q, N_q)_{L_q}$ and $\mathrm{F}^{L_q}_{M_q,N_q}$ is of central importance to this thesis and can be confusing when first encountered. The easiest way to get a feel for the difference is via a simple example.

Example 6.1.2. Let $\vec{Q} = \bullet$ be the A_1 quiver, that is, the quiver with one vertex and no arrows. In this case the category $\mathcal{A}_q$ is equivalent to the category of finite dimensional $\mathbb{F}_q$-vector spaces. Up to isomorphism every object in $\mathcal{A}_q$ is of the form $V_n = \mathbb{F}_q^n$ where $n \geqslant 0$.

The set $F^{V_l}_{V_m,V_n}$ is the set of ways that V_n sits inside V_l as a vector subspace such that the quotient vector space V_l/V_n is of dimension m. This set is non-empty only when $l = m + n$ in which case $F^{V_l}_{V_m,V_n}$ is simply the Grassmannian $\mathrm{Gr}_{\mathbb{F}_q}(n, V_{n+m})$ of n dimensional subspaces of V_{m+n} over $\mathbb{F}_q$. It is well known that the cardinality of $\mathrm{Gr}_{\mathbb{F}_q}(n, V_{n+m})$ is given by the quantum binomial $\begin{bmatrix} m+n \\ n \end{bmatrix}_q$ which was defined in Equation (5.1). For example $F^{V_2}_{V_1,V_1}$ has cardinality $\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q = 1 + q$.

On the other hand, the only extension of any two vector spaces is the split extension. Thus $\mathrm{Ext}^1(V_m, V_n)_{V_l}$ is empty unless $l = m + n$ in which case $\mathrm{Ext}^1(V_m, V_n)_{V_{m+n}}$ is a set with cardinality 1.

We point out that the Hom sets of $\mathcal{A}_q$ have finite cardinality as they are finite dimensional vector spaces over a finite field. This implies that the automorphism groups of quiver representations have finitely many elements. Moreover since $\mathrm{SES}(M_q, N_q)_{L_q} \subseteq \mathrm{Hom}(N_q, L_q) \times \mathrm{Hom}(L_q, M_q)$ then the sets $\mathrm{F}^{L_q}_{M_q,N_q}$ and $\mathrm{Ext}^1(M_q, N_q)_{L_q}$ must be of finite cardinality. With these comments in mind we end this section with the following formula due to Riedtmann [Rie94].

Proposition 6.1.1 (Riedtmann)**.** For any quiver representations L_q, M_q and N_q we have the following identity.

$$|\mathrm{F}^{L_q}_{M_q,N_q}| = \frac{|\mathrm{Ext}^1(M_q, N_q)_{L_q}|}{|\mathrm{Hom}(M_q, N_q)|} \frac{|\mathrm{Aut}(L_q)|}{|\mathrm{Aut}(M_q)||\mathrm{Aut}(N_q)|} \tag{6.5}$$

Proof. Since the group $\mathrm{Aut}(L_q)$ acts on $\mathrm{SES}(M_q, N_q)_{L_q}$ with stabilizer $\mathrm{Hom}(M_q, N_q)$ then we have the following via the orbit-stabilizer theorem.

$$\frac{|\mathrm{SES}(M_q, N_q)_{L_q}|}{|\mathrm{Aut}(L_q)|} = \frac{|\mathrm{Ext}^1(M_q, N_q)_{L_q}|}{|\mathrm{Hom}(M_q, N_q)|} \tag{6.6}$$

Similarly since $\mathrm{Aut}(N_q) \times \mathrm{Aut}(M_q)$ acts freely on the set $\mathrm{SES}(M_q, N_q)_{L_q}$ we have the following identity.

$$\frac{|\mathrm{SES}(M_q, N_q)_{L_q}|}{|\mathrm{Aut}(N_q)||\mathrm{Aut}(M_q)|} = |\mathrm{F}^{L_q}_{M_q,N_q}| \tag{6.7}$$

Combining Equation (6.7) and Equation (6.6) we obtain Equation (6.5). □

6.2 Non-Generic Hall Algebras

In this section we give a brief overview of (non-generic) Hall algebras as introduced by Ringel [Rin90a]. For simplicity we will restrict ourselves to Hall algebras of the categories $\mathcal{A}_q$ of finite dimensional $\mathbb{F}_q$-representations of simply-laced quivers.

We will begin with a discussion on Hall algebras in general and the definition of the Hall algebra of $\mathcal{A}_q$ in particular. We then state Ringel's theorem before ending with a commentary on what non-generic means. A superb reference for this section is [Sch09].

Hall algebras are associative algebras which one can assign to any finitary Abelian category. A finitary Abelian category is a small Abelian category such that all Hom and Ext^1 sets have only finitely many elements. Our categories of quiver representations $\mathcal{A}_q$ are finitary but there are of course other examples such as categories of coherent sheaves on projective schemes over $\mathbb{F}_q$.

The basic idea for Hall algebras is to form a vector space whose elements are linear combinations of (isomorphism classes of) objects in one's chosen category. The Hall product then roughly speaking counts numbers of extensions or, equivalently, flags of subobjects as described in Section 6.1. More explicitly we have the following where we use the notation $\mathrm{Iso}(\mathcal{A}_q)$ to denote the set of isomorphism classes of objects in $\mathcal{A}_q$.

Definition 6.2.1. For any q a prime power define the Hall algebra of $\mathcal{A}_q$ to be the $\mathbb{C}$-vector space H_q generated by the set $\mathrm{Iso}(\mathcal{A}_q)$. The product of two basis elements E_{M_q} and E_{N_q} corresponding to two quiver representations M_q and N_q is given by the following.

$$E_{M_q}E_{N_q} = q^{1/2\langle \hat{M}_q,\hat{N}_q\rangle} \sum_{L_q\in \mathrm{Iso}(\mathcal{A}_q)} \frac{|\mathrm{Ext}^1(M_q,N_q)_{L_q}|}{|\mathrm{Hom}(M_q,N_q)|} E_{L_q} \tag{6.8}$$

If we define rescaled basis elements $X_{L_q} := E_{L_q}/|\mathrm{Aut}(L_q)|$ then by Riedtmann's formula from Equation (6.5) the product on H_q admits the following equivalent definition in this alternative basis.

$$X_{M_q}X_{N_q} = q^{1/2\langle \hat{M}_q,\hat{N}_q\rangle} \sum_{L_q\in \mathrm{Iso}(\mathcal{A}_q)} |F^{L_q}_{M_q,N_q}| X_{L_q} \tag{6.9}$$

Of course, one must check that Equation (6.8) or equivalently Equation (6.9) really does define an associative product. Using the isomorphism theorems for Abelian categories one can readily establish associativity of the product for the form it takes in Equation (6.9).

We note that Equation (6.8) and Equation (6.9) remain associative with or without the factor $q^{1/2\langle \hat{M}_q,\hat{N}_q\rangle}$. This factor is referred to as a twist by the Euler form and ensures that one obtains the correct relations making H_q isomorphic to the positive part of the quantized enveloping algebra.

We now state Ringel's theorem on the relationship between quantized enveloping algebras and the Hall algebra of $\mathcal{A}_q$. In the following recall from Section 5.4 that we denoted the specialization at $t = q^{1/2}$ of the positive part of the (restricted integral form of the) quantized enveloping algebra by $\mathrm{U}_q(\mathfrak{n}_+)$.

Theorem 6.2.1 (Ringel)**.** The following describes an isomorphism of $\mathbb{C}$-algebras.

$$\mathrm{U}_q(\mathfrak{n}_+) \to \mathrm{H}_q, \qquad X_i \mapsto X_{S_{i,q}}$$

Here $S_{i,q}$ denotes the simple quiver representation corresponding to the vertex i of $\vec{Q}$.

Proof. See the proof of Theorem 3.16 in [Sch09]. □

We end with the observation that we have defined in fact a family of Hall algebras H_q as q varies over the prime powers. These are non-generic Hall algebras. One might hope that there exists a single 'generic' Hall algebra which specializes to each of the H_q.

In Part II we will show that there does exist such a $\mathbb{C}(t)$-algebra H called the generic Hall algebra. One will be able to recover each of the algebras H_q in our family from H by setting[1] the parameter t to be the positive square root of q. The advantage of having generic Hall algebras is that we can set t to be values other than square roots of prime powers and then study the resulting algebras.

We point out that although the two different Hall products in Equation (6.8) and Equation (6.9) gave rise to the same algebra H_q, we will see in Chapter 9 and Chapter 10 that the analogous formulas for the generic Hall algebra H give rise to two *non-isomorphic* integral forms of H.

6.3 Non-Generic Bridgeland-Hall Algebras

In this section we give a brief overview of non-generic Bridgeland-Hall algebras associated to categories of representations of simply-laced quivers. Bridgeland-Hall algebras were introduced by Bridgeland in [Bri13] to solve the problem of finding a suitable category whose Hall algebra would extend Ringel's Theorem 6.2.1 by recovering the *whole* quantized enveloping algebra.

Bridgeland's key insight was to replace the category $\mathcal{A}_q$ in the definition of H_q with the category $\mathcal{C}_q$ of $\mathbb{Z}_2$-graded complexes in projective quiver representations. The resulting Hall algebra isn't quite the correct object and must be modified in two ways. The first difference is that one must employ a non-standard twist by the Euler form. The other is that certain relations concerning acyclic complexes must be imposed by hand.

We begin with the following naïve definition where we replace $\mathcal{A}_q$ in Definition 6.2.1 with the category $\mathcal{C}_q$. The twist by the Euler form in Equation (6.10) is different to that of Equation (6.8) but we note that this does not affect associativity.

Definition 6.3.1. For any q a prime power define the Hall algebra of $\mathcal{C}_q$ to be the $\mathbb{C}$-vector space $\mathrm{H}(\mathcal{C}_q)$ generated by the set $\mathrm{Iso}(\mathcal{C}_q)$. The product of two basis elements $[M_{\bullet q}]$ and $[N_{\bullet q}]$ corresponding to two complexes $M_{\bullet q}$ and $N_{\bullet q}$ is given by the following.

$$[M_{\bullet q}][N_{\bullet q}] = q^{1/2\langle \hat{M}_0, \hat{N}_0\rangle + 1/2\langle \hat{M}_1, \hat{N}_1\rangle} \sum_{L_{\bullet q} \in \mathrm{Iso}(\mathcal{C}_q)} \frac{|\mathrm{Ext}^1(M_{\bullet q}, N_{\bullet q})_{L_{\bullet q}}|}{|\mathrm{Hom}(M_{\bullet q}, N_{\bullet q})|}[L_{\bullet q}] \tag{6.10}$$

One objection to Definition 6.3.1 is that the category $\mathcal{C}_q$ is not Abelian and so one may wonder if it makes sense to take its Hall algebra. The category $\mathcal{C}_q$ is however a full subcategory of the larger Abelian category of $\mathbb{Z}_2$-graded complexes in arbitrary quiver representations. Moreover the subcategory $\mathcal{C}_q$ is closed under taking extensions of complexes since any extension of two projective objects in $\mathcal{A}_q$ is necessarily projective.

[1] More precisely there will be integral forms of the generic Hall algebra H which recover H_q on setting $t = q^{1/2}$.

The larger category of $\mathbb{Z}_2$-graded complexes in arbitrary quiver representations is finitary Abelian and so has a well-defined Hall algebra. That $\mathcal{C}_q$ is closed under extensions implies that $\mathrm{H}(\mathcal{C}_q)$ is then a subalgebra of this larger Hall algebra and in particular is well-defined.

As observed by Bridgeland in [Bri13] the Hall algebra $\mathrm{H}(\mathcal{C}_q)$ is almost, but not quite the correct one to recover the whole quantized enveloping algebra. If one tries to prove an analogue of Ringel's theorem it quickly becomes apparent that the subalgebra generated by the elements $[K_{P_q}]$ and $[K^*_{P_q}]$ in $\mathrm{H}(\mathcal{C}_q)$ should correspond to the one generated by the elements K_i and K_i^{-1} in $\mathrm{U}_q(\mathfrak{g})$. Here the complexes K_{P_q} and $K^*_{P_q}$ are the acyclic complexes defined in Section 2.3.

In particular one is led to expect that the relations $[K_{P_q}][K^*_{P_q}] = 1$ in $\mathrm{H}(\mathcal{C}_q)$ should be the analogues of the relations $K_iK_i^{-1} = 1$ in $\mathrm{U}_q(\mathfrak{g})$. It is not true however that $[K_{P_q}][K^*_{P_q}] = 1$ in $\mathrm{H}(\mathcal{C}_q)$ and so one must impose this requirement by hand.

Definition 6.3.2. For any q a prime power define the Bridgeland-Hall algebra DH_q to be the quotient algebra of $\mathrm{H}(\mathcal{C}_q)$ by the following ideal.

$$([K_{P_q}][K^*_{P_q}] - 1 \mid P_q \text{ is projective in } \mathcal{A}_q)$$

Armed with this modified Hall algebra, Bridgeland then succeeded in proving the following extension of Ringel's theorem.

Theorem 6.3.1 (Bridgeland)**.** There is an isomorphism of $\mathbb{C}$-algebras $\mathrm{U}_q(\mathfrak{g}) \to \mathrm{DH}_q$.

Proof. This is Theorem 4.9 in [Bri13]. □

As was the case of Section 6.2 later on in Chapter 13 we will discuss the existence of a generic Bridgeland-Hall algebra from which one can recover the non-generic Bridgeland-Hall algebras DH_q.

Part II

Generic Hall Algebras

Definition, Integral Forms, $t = 1$ Limits and Hopf Algebra Structure

Overview

In Part II we will be concerned with generic Hall algebras which were originally introduced by Ringel [Rin90a] in the case of simply-laced quivers.

The basic idea is to take the non-generic Hall algebra from Chapter 6, observe that their structure constants are in fact Laurent polynomial in $q^{\frac{1}{2}}$ and then formally take these polynomials to be the structure constants of a $\mathbb{C}(t)$-algebra. We will define the generic Hall algebra in Chapter 8. There are some minor technical issues in doing so, however, and we deal with these first in Chapter 7.

The upshot of having a generic as opposed to non-generic Hall algebra is that one can talk of integral forms of a generic Hall algebra. In particular the generic Hall algebra H has two natural integral forms each with an interesting $t = 1$ limit.

One integral form H_{ex} which we define in Chapter 9 has a product formula whose structure constants count extensions of quiver representations. The other H_{fl}, which is dealt with in Chapter 10, has structure constants counting flags of quiver representations. The main feature of these integral forms is that the specialization at $t = 1$ of H_{ex} is a Poisson algebra while the $t = 1$ limit of H_{fl} is the universal enveloping algebra of a Lie algebra.

We end Part II with Chapter 11 where we discuss Hopf algebra structures on generic Hall algebras. We will use these Hopf algebra structures to form a doubled version of the generic Hall algebra called its the Drinfeld double.

Almost all of the results and proofs in Part II can be found in the literature in some guise or another, though perhaps not presented in the narrative of integral forms that we will give.

Chapter 7

Background Material

In this chapter we recall various bits of background theory that we will need define and work with generic Hall algebras.

We begin in Section 7.1 by collecting some elementary results that will allow us to pass from statements regarding non-generic Hall algebras to analogous statements about generic Hall algebras. An example of this might be how associativity of the generic Hall algebra will follow from that of the non-generic Hall algebra.

Then in Section 7.2 we explain how various features of the categories $\mathcal{A}_q$ are independent of q. Such a feature might be $\mathrm{Iso}(\mathcal{A}_q)$ the set of isomorphism classes of objects in $\mathcal{A}_q$. The rationale here is that the underlying vector space of a Hall algebra is defined in terms of the set $\mathrm{Iso}(\mathcal{A}_q)$ and we would rather not be tied down to any particular choice of prime power q when defining the generic Hall algebra.

In Section 7.3 we discuss the existence of Hall polynomials. These are polynomial versions of the structure constants of the non-generic Hall algebra which will be used as the structure constants of the generic Hall algebra.

Finally Section 7.4 contains a generic (i.e. polynomial) version of Riedtmann's formula from Equation (6.5) along with an elementary corollary. As we shall see later on in Section 9.1 and Section 10.1, generic Hall algebras have two natural integral forms. The importance of the generic version of Riedtmann's formula is that it gives the relationship between these two forms.

7.1 Passing From Non-Generic to Generic Algebras

In this section we recall some elementary facts that will allow us to pass from statements about non-generic Hall algebras to ones about generic Hall algebras.

Lemma 7.1.1. If $p_1, p_2 \in \mathbb{C}[t, t^{-1}]$ are two Laurent polynomials such that $p_1(w) = p_2(w)$ for infinitely many distinct $w \in \mathbb{C}$ then $p_1 = p_2$.

Proof. This follows from the fundamental theorem of algebra applied to $t^n(p_1 - p_2)$ where the exponent n is chosen so that $t^n(p_1 - p_2)$ is an element of $\mathbb{C}[t]$. □

Lemma 7.1.2. A rational function $f \in \mathbb{Q}(t)$ with $f(k) \in \mathbb{Z}$ for infinitely many distinct $k \in \mathbb{Z}$ is polynomial i.e. we have $f \in \mathbb{Q}[t]$.

Proof. Suppose $f = a/b$ is a non-zero rational function with $a, b \in \mathbb{Q}[t]$ and moreover f has the property that $f(k) \in \mathbb{Z}$ for infinitely many distinct $k \in \mathbb{Z}$. The remainder theorem says that there are polynomials $p, q \in \mathbb{Q}[t]$ such that $a = pb + q$ and either $q = 0$ or $q \neq 0$ and $\deg(q) < \deg(b)$.

For the sake of contradiction suppose that q is non-zero and write $a/b = p + q/b$. Multiplying across by a suitable non-zero integer, without loss of generality may assume that $p \in \mathbb{Z}[t]$. We then have that q/b has the property that its evaluation at infinitely many distinct $k \in \mathbb{Z}$ is an integer since a/b has this property and $p \in \mathbb{Z}[t]$. However $q/b \to 0$ as $t \to \infty$ since $\deg(q) < \deg(b)$. This implies q vanishes at infinitely many points and so $q = 0$ by Lemma 7.1.1. □

7.2 Simply-Laced Quiver Representations: Independence of $\mathbb{F}_q$

In this section we discuss how the categories $\mathcal{A}_q$ are in some respects independent of q. Recall that for each q a prime power $\mathcal{A}_q$ is the category of quiver representations over the finite field $\mathbb{F}_q$.

We first explain how $\mathrm{Iso}(\mathcal{A}_q)$ the set of isomorphisms classes of objects in $\mathcal{A}_q$ is independent of q. We then discuss how the Grothendieck groups $K(\mathcal{A}_q)$ and various properties of quiver representations are also independent of q.

Note that Gabriel's Theorem 2.2.2 and the Krull-Schmidt Theorem 2.2.1 imply that any representation in $\mathcal{A}_q$ is determined uniquely up to isomorphism by a map $\Phi^+ \to \mathbb{Z}_{\geqslant 0}$ prescribing the multiplicities of its indecomposable direct summands. If we abuse notation by writing $\mathrm{Iso}(\mathcal{A})$ for the set of maps $\Phi^+ \to \mathbb{Z}_{\geqslant 0}$ then we have canonical bijections between the following.

$$\mathrm{Iso}(\mathcal{A}_q) \cong \mathrm{Iso}(\mathcal{A})$$

These bijections induce canonical bijections of the following form for any prime powers q and q'.

$$\mathrm{Iso}(\mathcal{A}_q) \cong \mathrm{Iso}(\mathcal{A}_{q'})$$

We will write $L \in \mathrm{Iso}(\mathcal{A})$ for maps $L : \Phi^+ \to \mathbb{Z}_{\geqslant 0}$. Abusing notation again we will frequently refer to the elements $L \in \mathrm{Iso}(\mathcal{A})$ as if were they genuine quiver representations. Should we need to, we will write L_q for a particular choice of actual quiver representation in $\mathcal{A}_q$ determined by an element $L \in \mathrm{Iso}(\mathcal{A})$.

The Grothendieck groups of the categories $\mathcal{A}_q$ are also independent of q. Indeed by Equation (2.2) we have the following canonical isomorphisms.

$$K(\mathcal{A}_q) \cong \Lambda_\Phi$$

We thus have canonical isomorphisms of the following form for any prime powers q and q'.

$$K(\mathcal{A}_q) \cong K(\mathcal{A}_{q'})$$

Motivated by this fact we will *define* $K(\mathcal{A})$ to be the root lattice Λ_Φ of the simple Lie algebra $\mathfrak{g}$. Moreover for any $L \in \mathrm{Iso}(\mathcal{A})$ we will define the class $\hat{L}$ in $K(\mathcal{A})$ to be the image of $\hat{L}_q$ under $K(\mathcal{A}_q) \cong \Lambda_\Phi$.

We will abuse notation and denote by $\langle -, - \rangle$, $(-, -)$ and $(-, -)_{skew}$ respectively the induced usual, symmetrized and skew-symmetrized Euler forms on $K(\mathcal{A})$. By induced here, we mean that we consider the usual, symmetrized and skew-symmetrized Euler forms defined on $K(\mathcal{A}_q)$ in Equation (2.3) and Equation (2.4) as forms on $K(\mathcal{A}) = \Lambda_\Phi$ via the identification $K(\mathcal{A}_q) \cong \Lambda_\Phi$ given in Equation (2.2).

Something we will need to know is that various properties of quiver representations are independent of q. By this we mean that if $L \in \mathrm{Iso}(\mathcal{A})$ such that L_q has a certain property in $\mathcal{A}_q$ for a particular q then L_q has that property in $\mathcal{A}_q$ for all prime powers q.

This will be true for any reasonable property such as being indecomposable, projective or simple. It will then make sense to say things like $I \in \mathrm{Iso}(\mathcal{A})$ is indecomposable which simply means that I_q is indecomposable in any (therefore all) of the categories $\mathcal{A}_q$. We will denote by S_i and I_α the elements of $\mathrm{Iso}(\mathcal{A})$ determining the simple and indecomposable objects $S_{i,q}$ and $I_{\alpha,q}$ of $\mathcal{A}_q$ respectively.

Being the direct sum of two representations is also independent of q. Indeed letting $M, N \in \mathrm{Iso}(\mathcal{A})$ if we define $M \oplus N$ to be the map $M + N : \Phi^+ \to \mathbb{Z}_{\geqslant 0}$ then for any q a prime power by definition we have $M_q \oplus N_q \cong (M \oplus N)_q$ in the category $\mathcal{A}_q$.

We end by observing that for any $M, N \in \mathrm{Iso}(\mathcal{A})$ the following are also independent of q.

$$\dim_{\mathbb{F}_q} \mathrm{End}(M_q) \qquad \dim_{\mathbb{F}_q} \mathrm{Hom}(M_q, N_q) \qquad \dim_{\mathbb{F}_q} \mathrm{Ext}^1(M_q, N_q)$$

That the two on the left are independent of q can be found in the proof of Lemma 3 from [Hub10]. That the Ext^1 dimension is independent of q then follows from the fact that the Euler form is. We will simply write $\dim \mathrm{End}(M)$, $\dim \mathrm{Hom}(M, N)$ and $\dim \mathrm{Ext}^1(M, N)$ for these dimensions respectively.

7.3 Hall Polynomials

In this section we discuss the existence of Hall polynomials. These are polynomials in t which, when evaluated at[1] $t = q^{\frac{1}{2}}$, give the structure constants of the non-generic Hall algebra from Chapter 6 (recall that every choice of finite field $\mathbb{F}_q$ with q elements gave rise to a non-generic Hall algebra H_q). Hall polynomials are used to define generic Hall algebras which were originally introduced by Ringel [Rin90a] in the case of simply-laced quivers.

We briefly recall the notation used for various structure constants of the non-generic Hall algebra from Section 6.2. We used the following notation for the subset of extensions of M_q by N_q with middle term isomorphic to L_q.

$$\mathrm{Ext}^1(M_q, N_q)_{L_q} \subseteq \mathrm{Ext}^1(M_q, N_q)$$

We also defined $\mathrm{F}^{L_q}_{M_q,N_q}$ to be the set of subobjects $N_q \subseteq L_q$ such that the corresponding quotient object is isomorphic to M_q. Finally we wrote $\mathrm{Aut}(L_q)$ for the automorphism group of L_q. The following proposition then establishes the existence of Hall polynomials.

Proposition 7.3.1 (Ringel)**.** For any $L, M, N \in \mathrm{Iso}(\mathcal{A})$ there exists polynomials $e^L_{M,N}$, $h_{M,N}$, $f^L_{M,N}$ and a_L in $\mathbb{Q}[t]$ such that for any q a prime power we have the following. Moreover each of $e^L_{M,N}$, $h_{M,N}$, $f^L_{M,N}$ and a_L in fact lie in the subalgebra $\mathbb{Q}[t^2] \subset \mathbb{Q}[t]$, that is are polynomial in t^2.

[1] One might wonder why we evaluate t at the square root of q, rather than at q. This is purely to accommodate the $q^{1/2\langle \hat{M}_q, \hat{N}_q \rangle}$ factor in Equation (6.8).

$$e^L_{M,N}(q^{\frac{1}{2}}) = |\mathrm{Ext}^1(M_q, N_q)_{L_q}| \qquad h_{M,N}(q^{\frac{1}{2}}) = |\mathrm{Hom}(M_q, N_q)|$$
$$f^L_{M,N}(q^{\frac{1}{2}}) = |\mathrm{F}^{L_q}_{M_q,N_q}| \qquad a_L(q^{\frac{1}{2}}) = |\mathrm{Aut}(L_q)|$$

Proof. The existence of a_L and $f^L_{M,N}$ is well know, see for example Lemma 3 and Theorem 4 of [Hub10]. We also noted at the end of Section 7.2 that $\dim_{\mathbb{F}_q}\mathrm{Hom}(M_q, N_q)$ is independent of q. It follows that $h_{M,N} := t^{2\dim\mathrm{Hom}(M,N)}$ has the correct specialization at $t = q^{\frac{1}{2}}$ and is polynomial in t^2. We will show that the existence of $e^L_{M,N}$ follows from the others. To this end consider the following definition.

$$e^L_{M,N} := f^L_{M,N} h_{M,N} \frac{a_M a_N}{a_L} \tag{7.1}$$

Setting $t = q^{\frac{1}{2}}$ in Equation (7.1) and using the non-generic version of Riedtmann's formula given in Equation (6.5) we have that $e^L_{M,N}(q^{\frac{1}{2}}) = |\mathrm{Ext}^1(M_q, N_q)_{L_q}|$ for any q a prime power. Moreover since $e^L_{M,N}$ is then an element of $\mathbb{Q}(t)$ which satisfies the hypothesis of Lemma 7.1.2 we must have that $e^L_{M,N} \in \mathbb{Q}[t]$. Finally since each of $f^L_{M,N}$, $h_{M,N}$, a_M, a_N and a_L are polynomial in t^2 then so too is $e^L_{M,N}$. □

When $L \neq M \oplus N$ another useful polynomial is the one counting the number of elements of the following projectivization.

$$\mathbb{P}(\mathrm{Ext}^1(M_q, N_q)_{L_q}) := \mathrm{Ext}^1(M_q, N_q)_{L_q}/\mathbb{F}_q^\times \tag{7.2}$$

The $\mathbb{F}_q^\times$-action here is the one coming from the $\mathbb{F}_q$-vector space structure on $\mathrm{Ext}^1(M_q, N_q)$ which induces a free $\mathbb{F}_q^\times$ action on the set $\mathrm{Ext}^1(M_q, N_q)_{L_q}$ when $L_q \neq M_q \oplus N_q$. This motivates the following definition.

$$\mathbb{P}(e)^L_{M,N} := e^L_{M,N}/(t^2 - 1) \tag{7.3}$$

It is easy to see that for any q a prime power the evaluation of $\mathbb{P}(e)^L_{M,N}$ at $t = q^{\frac{1}{2}}$ gives the number of elements in the projectivized set from Equation (7.2). Moreover it follows from Lemma 7.1.2 that $\mathbb{P}(e)^L_{M,N}$ is indeed polynomial i.e. is an element of $\mathbb{Q}[t]$.

Remark 7.3.2. An important example of the automorphism polynomials from Proposition 7.3.1 is the case when we have an indecomposable $I \in \mathrm{Iso}(\mathcal{A})$. From Lemma 3.19 of [Sch09] any indecomposable quiver representation of a simply-laced quiver has a one dimensional endomorphism algebra. It follows that $a_I = t^2 - 1$.

7.4 Generic Riedtmann's Formula

In this section we discuss the generic version of Riedtmann's formula from Equation (6.5) and a useful corollary. For us the formula is important for understanding the relationship between two integral forms of the Hall algebra that we shall introduce in Section 9.1 and Section 10.1.

Proposition 7.4.1 (Riedtmann)**.** For any $L, M, N \in \mathrm{Iso}(\mathcal{A})$ we have the following identity in $\mathbb{Q}[t]$.

$$f^L_{M,N} = \frac{e^L_{M,N}}{h_{M,N}} \frac{a_L}{a_M a_N} \tag{7.4}$$

Proof. By the non-generic Riedtmann's formula from Equation (6.5) we have that Equation (7.4) holds when evaluated at $t = q^{\frac{1}{2}}$ for any q a prime power. The only thing preventing us from using Lemma 7.1.1 to establish equality in Equation (7.4) is that a priori we don't know that both sides of Equation (7.4) live in $\mathbb{Q}[t]$. However that $f^L_{M,N} \in \mathbb{Q}[t]$ follows from Proposition 7.3.1. That the right-hand side of Equation (7.4) is an element of $\mathbb{Q}[t]$ follows from combining Lemma 7.1.2 and Equation (6.5). □

Restricting Riedtmann's formula to indecomposables we have the following useful corollary relating the evaluation at $t = 1$ of the flag counting polynomials $f^L_{M,N}$ with the evaluation at $t = 1$ of the polynomials $\mathbb{P}(e)^L_{M,N}$ defined in Equation (7.3). We recall that we use overline notation to denote the evaluation of a polynomial at $t = 1$.

Corollary 7.4.1. Let $I_1, I_2, J \in \mathrm{Iso}(\mathcal{A})$ be indecomposable. Then we have the following identity.

$$\overline{f}^J_{I_1,I_2} = \overline{\mathbb{P}(e)}^J_{I_1,I_2}$$

Proof. By Remark 7.3.2 any indecomposable I has automorphism polynomial $a_I = t^2 - 1$. Multiplying across by h_{I_1,I_2} in Riedtmann's formula we have the following identity in $\mathbb{Q}[t]$.

$$h_{I_1,I_2} f^J_{I_1,I_2} = e^J_{I_1,I_2} \frac{a_J}{a_{I_1} a_{I_2}} = \frac{e^J_{I_1,I_2}}{(t^2-1)} = \mathbb{P}(e)^J_{I_1,I_2}$$

Setting $t = 1$ we have $\overline{f}^J_{I_1,I_2} = \overline{\mathbb{P}(e)}^J_{I_1,I_2}$ since $h_{I_1,I_2} = t^{2\dim\mathrm{Hom}(I_1,I_2)}$. □

Chapter 8

Generic Hall algebras

In this chapter we recall the definition and properties of generic Hall algebras which are due to Ringel [Rin90b].

The idea behind generic Hall algebras is to replace the structure constants of the non-generic Hall algebra from Chapter 6 with the Hall polynomials we gave in Chapter 7 to obtain a Hall algebra over $\mathbb{C}(t)$. The advantage of doing so is that our Hall algebras now depend on a formal parameter t. We can then talk of integral forms of the generic Hall algebra and moreover set $t = 1$ to obtain degenerate Hall algebras. We will postpone the discussion on integral forms to Chapter 9 and Chapter 10.

8.1 Definition and Properties

In this section we recall the definition of the generic Hall algebra H due to Ringel in [Rin90b]. We also give some elementary properties of H.

Definition 8.1.1. Define the generic Hall algebra be the $\mathbb{C}(t)$-vector space H generated by the set $\mathrm{Iso}(\mathcal{A})$. The product of two basis elements E_M and E_N corresponding to two quiver representations is given by the following.

$$E_M E_N = t^{\langle \hat{M}, \hat{N} \rangle} \sum_{L \in \mathrm{Iso}(\mathcal{A})} \frac{e^L_{M,N}}{h_{M,N}} E_L \tag{8.1}$$

Proposition 8.1.1. H is a unital associative $\mathbb{C}(t)$-algebra with unit element E_0.

Proof. First of all note that the product formula is a well-defined element of H since by Proposition 7.3.1 the structure constants are Laurent polynomials and in particular elements of $\mathbb{C}(t)$. That E_0 is the unit is obvious. For associativity let A, B and C be quiver representations. Expanding the product $E_A E_B E_C$ in the two different ways gives two expressions $\sum_{L \in \mathrm{Iso}(\mathcal{A})} \Psi^i_L E_L$ for some Laurent polynomials Ψ^1_L and Ψ^2_L.

By Proposition 7.3.1 the structure constants of Equation (8.1) evaluated at $t = +q^{\frac{1}{2}}$ give the structure constants of the non-generic Hall algebra H_q for each q a prime power. Thus $\Psi^1_L(q^{\frac{1}{2}}) = \Psi^2_L(q^{\frac{1}{2}})$ for each q a prime power by associativity of H_q. However by Lemma 7.1.1 if two Laurent polynomials coincide at infinitely many points they must coincide. ☐

We will apply this proof strategy again and again to pass from results about non-generic Hall algebras to generic ones.

We note that the generic Hall algebra is graded as an algebra by the Grothendieck group. In particular let $\mathrm{H}_\alpha \subset \mathrm{H}$ be the subspace spanned by the elements E_L of class $\hat{L} = \alpha$ in $K(\mathcal{A})$. Since the quiver representations in Equation (8.1) have the property that $\hat{L} = \hat{M} + \hat{N}$ then we have the following grading of H.

$$\mathrm{H} = \bigoplus_{\alpha \in K(\mathcal{A})} \mathrm{H}_\alpha, \qquad \mathrm{H}_\alpha \cdot \mathrm{H}_\beta \subseteq \mathrm{H}_{\alpha+\beta} \tag{8.2}$$

In the literature the product on the generic Hall algebra usually takes a different form which we now explain. Using the automorphism polynomials from Proposition 7.3.1 we first rescale the basis vectors of H by letting $X_L := E_L / a_L$. Employing Riedtmann's formula from Proposition 7.4.1 the product in this basis becomes the following.

$$X_M X_N = t^{\langle \hat{M}, \hat{N} \rangle} \sum_{L \in \mathrm{Iso}(\mathcal{A})} \frac{e^L_{M,N}}{h_{M,N}} \frac{a_L}{a_M a_N} X_L = t^{\langle \hat{M}, \hat{N} \rangle} \sum_{L \in \mathrm{Iso}(\mathcal{A})} f^L_{M,N} X_L \tag{8.3}$$

The dichotomy between these two products should be understood as giving rise to two different integral forms of the generic Hall algebra. That natural integral forms of Hall algebras arise in this way is one of the core messages of this thesis and appears to be only partially appreciated in the literature. We define and explore these integral forms in the following two chapters and in particular show that their $t = 1$ limits have very different flavours.

Chapter 9

Extension Counting Integral Form

This chapter is concerned with what we call the extension counting integral form H_{ex} of the generic Hall algebra H. The name comes from the fact that the structure constants of H_{ex} in some sense count extensions of quiver representations. The key property of this integral form is that its $t = 1$ limit is a commutative Poisson $\mathbb{C}$-algebra which we call the semi-classical Hall algebra. We will define H_{ex} in Section 9.1 and then explain in Section 9.2 how its specialization at $t = 1$ is commutative and Poisson.

9.1 Definition

In this section we define the extension counting integral form H_{ex} of H. This is the obvious integral form in light of Definition 8.1.1.

Definition 9.1.1. Define the extension counting integral form of H to be the $\mathbb{C}[t, t^{-1}]$-subalgebra H_{ex} spanned by the elements E_L. The product is given by the following.

$$E_M E_N = t^{\langle \hat{M}, \hat{N} \rangle} \sum_{L \in \mathrm{Iso}(\mathcal{A})} \frac{e^L_{M,N}}{h_{M,N}} E_L$$

Recalling Definition 4.1.1 of an integral form we need to verify that H_{ex} really is closed under the product, free as a $\mathbb{C}[t, t^{-1}]$-module and has the property that the multiplication map induces an isomorphism of $\mathbb{C}(t)$-algebras $\mathbb{C}(t) \otimes_{\mathbb{C}[t,t^{-1}]} \mathrm{H}_{ex} \to \mathrm{H}$.

The subalgebra property follows from the fact that the structure constants $t^{\langle \hat{M}, \hat{N} \rangle} e^L_{M,N} / h_{M,N}$ are all Laurent polynomials by Proposition 7.3.1. That H_{ex} is free as a $\mathbb{C}[t, t^{-1}]$-module follows from the fact that the elements E_L form a $\mathbb{C}(t)$-basis for H. Finally it is easy to see that multiplication induces an isomorphism of $\mathbb{C}(t)$-algebras $\mathrm{H}_{ex} \otimes_{\mathbb{C}[t,t^{-1}]} \mathbb{C}(t) \to \mathrm{H}$.

Since we are dealing with the Hall algebras of an Abelian category we shall sometimes refer to H_{ex} as the Abelian extension counting integral form to distinguish it from the extension counting integral form of the Bridgeland-Hall algebra that we shall introduce in Section 15.1.

9.2 The Semi-Classical Hall Algebra

In this section we discuss the semi-classical Hall algebra H_{sc}. This is the $t = 1$ limit of the extension counting integral form of H. In particular H_{sc} is shown to be a finitely generated commutative Poisson algebra. This section is based on Bridgeland's work in [Bri12] where he showed that analogous results hold for a type of Hall algebra called the motivic Hall algebra.

Definition 9.2.1. Define the semi-classical Hall algebra H_{sc} to be the quotient algebra of H_{ex} by the ideal $(t-1)$.

The following proposition gives the key property of the semi-classical Hall algebra. We will use the overline notation to denote the image of an element in H_{sc}.

Proposition 9.2.1. The semi-classical limit H_{sc} is a commutative Poisson $\mathbb{C}$-algebra with product and Poisson bracket given by the following.

$$\overline{E}_M\overline{E}_N = \overline{E}_{M\oplus N} \qquad \{\overline{E}_M, \overline{E}_N\}_{sc} := \overline{\left(\frac{E_M E_N - E_N E_M}{2(t-1)}\right)} \tag{9.1}$$

Proof. We establish the product formula first. The basic idea is to use the $\mathbb{F}_q$-vector space structure on $\mathrm{Ext}^1(M_q, N_q)$. Recall that the zero vector is given by the split exact sequence, addition by the Baer sum and scalar multiplication of an extension class by the following formula.

$$\lambda \cdot [N_q \xrightarrow{\varphi} L_q \xrightarrow{\psi} M_q] = \begin{cases} [N_q \xrightarrow{\lambda^{-1}\varphi} L_q \xrightarrow{\psi} M_q] & \lambda \in \mathbb{F}_q^\times \\ [N_q \to M_q \oplus N_q \to M_q] & \lambda = 0 \end{cases} \tag{9.2}$$

If $L_q \ncong M_q \oplus N_q$ by Equation (9.2) we have that $\mathbb{F}_q^\times$ acts freely on $\mathrm{Ext}^1(M_q, N_q)_{L_q}$ and in particular $(q-1)$ divides $|\mathrm{Ext}^1(M_q, N_q)_{L_q}|$. When $L \neq M \oplus N$ the rational function $e^L_{M,N}/(t^2-1)$ then has the property that its value at $t = q^{\frac{1}{2}}$ is an integer and so $t^2 - 1$ divides $e^L_{M,N}$ by Lemma 7.1.2.

Now the structure constants of the product $E_M E_N$ are $t^{\langle \hat{M}, \hat{N}\rangle} e^L_{M,N}/h_{M,N}$. By Proposition 2.2.3 we have $e^{M\oplus N}_{M,N} = 1$ and so evaluating these structure constants of $E_M E_N$ at $t = 1$ gives the value 1 if $L = M \oplus N$ and 0 otherwise. Thus the product on H_{sc} takes the following form which is clearly commutative.

$$\overline{E}_M\overline{E}_N = \overline{E}_{M\oplus N}$$

Finally by Proposition 4.1.1 the formula in Equation (9.1) endows H_{sc} with the structure of a Poisson algebra. □

Note that the reason for the $1/2$ in the Poisson bracket formula is that morally we are dividing by $t^2 - 1$ instead of $t - 1$ and $\overline{t+1} = 2$. However due to the term $(t^{\langle \hat{M}, \hat{N}\rangle} - t^{\langle \hat{N}, \hat{M}\rangle})E_{M\oplus N}$ in its expansion, $E_M E_N - E_N E_M$ is not always divisible by $t^2 - 1$ and so dividing out by $t^2 - 1$ might not land in H_{ex}.

A nice property of the interplay between homological algebra and Hall algebras is that H_{sc} has natural algebra generators given by the indecomposables. Indeed decomposing any representation L

as a direct sum of indecomposable representations $I_1^{\oplus n_1} \oplus \cdots \oplus I_k^{\oplus n_k}$ we have the following identity in H_{sc}.

$$\overline{E}_L = \overline{E}_{I_1}^{n_1} \cdots \overline{E}_{I_k}^{n_k}$$

By Theorem 2.2.2 a simply-laced quiver has up to isomorphism only finitely many indecomposable representations and so we have the following corollary to Proposition 9.2.1.

Corollary 9.2.2. H_{sc} is a finitely generated polynomial $\mathbb{C}$-algebra in the $|\Phi^+|$-many indecomposable representations.

Note that one could also have defined a $t = -1$ limit of H_{ex}. One would then get a skew-commutative algebra with product given as follows.

$$\overline{E}_M \overline{E}_N = (-1)^{\langle \hat{M}, \hat{N} \rangle} \overline{E}_{M \oplus N} \qquad \overline{E}_M \overline{E}_N = (-1)^{(\hat{M}, \hat{N})} \overline{E}_N \overline{E}_M$$

Chapter 10

Flag Counting Integral Form

In this chapter we consider what we call the flag counting integral form H_{fl} of the generic Hall algebra H. The name here comes from the fact that the structure constants of H_{fl} in some sense count flags of quiver representations. The key property of this integral form is that its $t = 1$ limit, which we refer to as the quasi-classical Hall algebra, is the universal enveloping algebra of a Lie algebra. We will define H_{fl} in Section 10.1 and then explain in Section 10.2 how its specialization at $t = 1$ is an enveloping algebra.

10.1 Definition

In this section we define the flag counting integral form H_{fl} of H. The form that the product on H_{fl} takes is the more usual Hall product that one finds in the literature.

Definition 10.1.1. Define the flag counting integral form of H to be the $\mathbb{C}[t, t^{-1}]$-subalgebra H_{fl} spanned by the elements X_L. The product is given by the following.

$$X_M X_N = t^{\langle \hat{M}, \hat{N} \rangle} \sum_{L \in \mathrm{Iso}(\mathcal{A})} f^L_{M,N} X_L$$

That H_{fl} is an integral form of H follows from the exact same reasoning as for the extension counting integral form H_{ex}. As with the case of H_{ex} we shall sometimes refer to H_{fl} as the Abelian flag counting integral form to distinguish it from the flag counting form of the Bridgeland-Hall algebra that we shall introduce in Section 15.2.

It is worth comparing the two integral forms H_{ex} and H_{fl}. We first note that there is an embedding of $\mathbb{C}[t, t^{-1}]$-algebras given by the following.

$$\mathrm{H}_{ex} \hookrightarrow \mathrm{H}_{fl}, \qquad E_L \mapsto a_L X_L \tag{10.1}$$

However this is *not* an isomorphism as surjectivity fails. If we were to work over $\mathbb{C}(t)$ instead, Equation (10.1) would become an isomorphism. This contrasts with the non-generic case that the reader may be more familiar with where the analogue of the above map *does* induce an isomorphism of $\mathbb{C}$-algebras.

10.2 The Quasi-Classical Hall Algebra

In this section we discuss the $t = 1$ limit of the flag counting integral form of H which we will call the quasi-classical Hall algebra H_{qc}. We will explain how the subspace of H_{qc} spanned by indecomposable quiver representations is a Lie algebra under the commutator bracket. Moreover we will show that H_{qc} is in fact isomorphic to the universal enveloping algebra of this Lie algebra. The following material is due to Ringel [Rin90b].

Definition 10.2.1. Define the quasi-classical Hall algebra H_{qc} to be the quotient algebra of H_{fl} by the ideal $(t-1)$.

To see how H_{qc} is an enveloping algebra we will first place an elementary cocommutative bialgebra structure on H_{qc}. That H_{qc} contains a Lie algebra of indecomposables and can by identified with its enveloping algebra will then follow from standard results from the theory of bialgebras. That H_{qc} is a bialgebra hinges on the following lemma which tells us how the Hall polynomials $f^L_{M,N}$ behave on setting $t = 1$.

Lemma 10.2.2. We have the following identity for any representations M and N.

$$\overline{f}^{L_1 \oplus L_2}_{M,N} = \sum_{\substack{M = M_1 \oplus M_2 \\ N = N_1 \oplus N_2}} \overline{f}^{L_1}_{M_1,N_1} \overline{f}^{L_2}_{M_2,N_2} \tag{10.2}$$

Here the summation ranges over all the ways that the representations M and N decompose as direct sums $M = M_1 \oplus M_2$ and $N = N_1 \oplus N_2$.

Proof. This is Proposition 3 in [Rin90b]. □

Using Lemma 10.2.2 we have the following proposition.

Proposition 10.2.1. The quasi-classical Hall algebra H_{qc} is a cocommutative $\mathbb{C}$-bialgebra with the following coproduct Δ_{qc} and counit ε_{qc}.

$$\Delta_{qc}(\overline{X}_L) = \sum_{L = L_1 \oplus L_2} \overline{X}_{L_1} \otimes \overline{X}_{L_2} \qquad\qquad \varepsilon_{qc}(\overline{X}_L) = \delta_{L,0}$$

Here the summation ranges over all the ways that the representation L decomposes as a direct sum $L = L_1 \oplus L_2$.

Proof. Cocommutativity follows by definition of Δ_{qc} and the counit axioms are trivial to verify. To see that Δ_{qc} is an algebra homomorphism consider the following where we use Lemma 10.2.2 for the third equality.

$$\begin{aligned}\Delta_{qc}(\overline{X}_M)\Delta_{qc}(\overline{X}_N) &= \sum_{\substack{M=M_1\oplus M_2\\ N=N_1\oplus N_2}} \overline{X}_{M_1}\overline{X}_{N_1}\otimes \overline{X}_{M_2}\overline{X}_{N_2}\\ &= \sum_{\substack{M=M_1\oplus M_2\\ N=N_1\oplus N_2}} \sum_{L_1,L_2} \overline{f}^{L_1}_{M_1,N_1}\overline{f}^{L_2}_{M_2,N_2}\overline{X}_{L_1}\otimes \overline{X}_{L_2}\\ &= \sum_{L_1,L_2} \overline{f}^{L_1\oplus L_2}_{M,N}\overline{X}_{L_1}\otimes \overline{X}_{L_2}\\ &= \sum_{L} \overline{f}^{L}_{M,N}\Delta_{qc}(\overline{X}_L)\\ &= \Delta_{qc}(\overline{X}_M\overline{X}_N)\end{aligned}$$

□

Let's explain how to use Proposition 10.2.1 to extract a Lie algebra from H_{qc}. Recall that an element x of a bialgebra is called primitive if its image under the coproduct is given by the following.

$$x \mapsto x\otimes 1 + 1\otimes x$$

It is well known that the subspace of primitive elements of a bialgebra form a Lie algebra under the commutator bracket. Using Proposition 10.2.1 it is easy to check that the primitive elements of H_{qc} consists of the span of the elements $\overline{X}_I$ where I is indecomposable. This motivates the following definition.

Definition 10.2.3. Define the quasi-classical Lie algebra $\mathfrak{n}_{qc}$ to be the subspace of H_{qc} spanned by the elements $\overline{X}_I$ where I is indecomposable. The Lie bracket is the commutator bracket.

The quasi-classical Hall algebra is essentially determined completely by $\mathfrak{n}_{qc}$.

Theorem 10.2.2. The map $\mathrm{U}(\mathfrak{n}_{qc}) \to \mathrm{H}_{qc}$ induced by the inclusion of the quasi-classical Lie algebra is an isomorphism of $\mathbb{C}$-bialgebras.

Proof. This follows via an application of the Milnor-Moore theorem. Recall that Milnor-Moore says that any connected graded cocommutative bialgebra over $\mathbb{C}$ is isomorphic to the universal enveloping algebra of its Lie algebra of primitive elements via the induced map as above. Connected here means that the degree 0 part of the graded bialgebra is isomorphic to $\mathbb{C}$.

In our case note that the grading on the generic Hall algebra H given in Equation (8.2) descends to a grading of the quasi-classical Hall algebra H_{qc}. For connectedness by definition the degree 0 part is spanned by elements of the form $\overline{E}_L$ where $\hat{L} = 0$. However for dimensional reasons the zero quiver representation is the only representation with such a class $K(\mathcal{A})$ and so the result follows by Milnor-Moore.

The interested reader may wish to glance at the Theorem in Section 3 of [Rin90b] for further details. □

It is worth describing the structure constants of the quasi-classical Lie algebra $\mathfrak{n}_{qc}$. It is easy to see that the Lie bracket is determined by the following where I_1 and I_2 are indecomposable and $\Gamma^J_{I_1,I_2} := \overline{f}^J_{I_1,I_2} - \overline{f}^J_{I_2,I_1}$.

$$[\overline{X}_{I_1}, \overline{X}_{I_2}]_{qc} = \sum_{J \text{ indecomposable}} \Gamma^J_{I_1,I_2} \overline{X}_J \tag{10.3}$$

It will follow from Proposition 18.4.3 and Theorem 18.4.1 that $\mathfrak{n}_{qc}$ is isomorphic to the nilpotent subalgebra $\mathfrak{n}_+$ of the simple Lie algebra $\mathfrak{g}$ as given in Section 1.1.

Chapter 11

Hopf Algebras and Drinfeld Doubles

In this chapter we describe how to upgrade a slightly extended version of the generic Hall algebra to a $\mathbb{C}(t)$-Hopf algebra. Using this we will construct another natural $\mathbb{C}(t)$-Hopf algebra called the Drinfeld double. In Chapter 14 of Part III we will use the material in this chapter to give the generic Bridgeland-Hall algebra the structure of a Hopf algebra.

11.1 Hopf Algebras and Drinfeld Doubles

We begin this section by defining a certain extended version of the generic Hall algebra. Using results of Green [Gre95] and Xiao [Xia97] we will place two Hopf algebra structures on these algebras. We then describe a Hopf algebra called the Drinfeld double which, roughly speaking, pieces these two Hopf algebras together to form a new one.

It is not quite true that the generic Hall algebra H as defined in Definition 8.1.1 admits the structure of a genuine Hopf algebra. One must first extend the algebra H by adding a copy of the group algebra of the Grothendieck group $\mathbb{C}(t)[K(\mathcal{A})]$. In the following definition we will denote by K_α the element of the group algebra corresponding to a class $\alpha \in K(\mathcal{A})$.

Definition 11.1.1. Define the extended Hall algebra to be $\mathrm{H}^{\geqslant 0} := \mathrm{H} \otimes_{\mathbb{C}(t)} \mathbb{C}(t)[K(\mathcal{A})]$. The $\mathbb{C}(t)$-algebra structure on $\mathrm{H}^{\geqslant 0}$ is determined by the fact that H and $\mathbb{C}(t)[K(\mathcal{A})]$ are subalgebras along with the following relation for any $\alpha \in K(\mathcal{A})$ and $L \in \mathrm{Iso}(\mathcal{A})$. We recall that $(-,-)$ here denotes the symmetrized Euler form on $K(\mathcal{A})$ and was defined in Section 7.2.

$$K_\alpha E_L = t^{(\alpha, \hat{L})} E_L K_\alpha \tag{11.1}$$

The extended Hall algebra admits two slightly different Hopf algebra structures which we will wish to differentiate between when defining the Drinfeld double. To this end we will denote by $\mathrm{H}^{\leqslant 0}$ another copy of $\mathrm{H}^{\geqslant 0}$. To avoid confusion we will write F_L and K^*_α in place of E_L and K_α respectively in $\mathrm{H}^{\leqslant 0}$. Recall also that we introduced the elements $X_L := E_L / a_L$ of H in Section 10.1. In particular $X_L \in \mathrm{H}^{\geqslant 0}$ and we will denote by Y_L the corresponding elements in $\mathrm{H}^{\leqslant 0}$.

The following theorem due to Green endows $\mathrm{H}^{\geqslant 0}$ and $\mathrm{H}^{\leqslant 0}$ with the structure of bialgebras. In Theorem 11.1.2 we will extend these bialgebras to Hopf algebras by giving the antipode due to Xiao
.

Theorem 11.1.1 (Green)**.** The extended Hall algebras $\mathrm{H}^{\geqslant 0}$ and $\mathrm{H}^{\leqslant 0}$ have the structure of $\mathbb{C}(t)$-bialgebras with coproducts Δ given as follows.

$$\begin{aligned}
&\Delta : \mathrm{H}^{\geqslant 0} \to \mathrm{H}^{\geqslant 0} \otimes_{\mathbb{C}(t)} \mathrm{H}^{\geqslant 0} \\
&\quad E_L \mapsto \sum_{M,N \in \mathrm{Iso}(\mathcal{A})} t^{\langle \hat{M},\hat{N}\rangle} f^L_{M,N} E_M K_{\hat{N}} \otimes E_N \\
&\quad K_\alpha \mapsto K_\alpha \otimes K_\alpha
\end{aligned}
\qquad
\begin{aligned}
&\Delta : \mathrm{H}^{\leqslant 0} \to \mathrm{H}^{\leqslant 0} \otimes_{\mathbb{C}(t)} \mathrm{H}^{\leqslant 0} \\
&\quad F_L \mapsto \sum_{M,N \in \mathrm{Iso}(\mathcal{A})} t^{\langle \hat{M},\hat{N}\rangle} f^L_{M,N} F_N \otimes F_M K^*_{\hat{N}} \\
&\quad K^*_\alpha \mapsto K^*_\alpha \otimes K^*_\alpha
\end{aligned}$$

The counits ε are given by $\varepsilon(E_L) = \delta_{L,0}$, $\varepsilon(K_\alpha) = 1$ and $\varepsilon(F_L) = \delta_{L,0}$, $\varepsilon(K^*_\alpha) = 1$.

Proof. We must check that the non-generic version proved in the literature implies the generic version. Note that the claimed bialgebra structure on $\mathrm{H}^{\leqslant 0}$ coincides with the *opposite* bialgebra structure on $\mathrm{H}^{\geqslant 0}$. We thus need only establish the result for $\mathrm{H}^{\geqslant 0}$.

The unit and counit axioms are trivial to verify. We are left check that the coproduct on $\mathrm{H}^{\geqslant 0}$ is an algebra homomorphism. It is immediate that $\Delta(K_\alpha K_\beta) = \Delta(K_\alpha)\Delta(K_\beta)$. Expanding the expressions one can check that the equality of $\Delta(E_M E_N)$ and $\Delta(E_M)\Delta(E_N)$ is equivalent to the following identity holding for each representations $M, N, A, B \in \mathrm{Iso}(\mathcal{A})$.

$$\sum_L \frac{e^L_{M,N}}{h_{M,N}} f^L_{A,B} = \sum_{\substack{A_1,A_2 \\ B_1,B_2}} t^{-2\langle A_1,B_2\rangle} \frac{e^A_{A_1,A_2}}{h_{A_1,A_2}} \frac{e^B_{B_1,B_2}}{h_{B_1,B_2}} f^M_{A_1,B_1} f^N_{A_2,B_2} \tag{11.2}$$

Using Reidtmann's formula from Proposition 7.4.1 and multiplying across by $a_A a_B$ one can see that Equation (11.2) is equivalent to the following identity.

$$\sum_L f^L_{M,N} f^L_{A,B} \frac{a_M a_N a_A a_B}{a_L} = \sum_{\substack{A_1,A_2 \\ B_1,B_2}} t^{-2\langle A_1,B_2\rangle} f^A_{A_1,A_2} f^B_{B_1,B_2} f^M_{A_1,B_1} f^N_{A_2,B_2} a_{A_1} a_{A_2} a_{B_1} a_{B_2} \tag{11.3}$$

This is an identity in $\mathbb{C}[t, t^{-1}]$ and so holds by Lemma 7.1.1 if it holds on setting $t = +q^{\frac{1}{2}}$ for all q a prime power. However by Proposition 7.3.1 setting $t = +q^{\frac{1}{2}}$ in Equation (11.3) gives precisely the identity that one can find proved in part (b) of the proof of Theorem 4.5. on page 124 of [Xia97]. $\square$

To treat the case of the antipode we will need some notation for the structure constants of k-fold products. For each $k \geqslant 1$ denote by $[f]^L_{L_1,\dots,L_k}$, $[e]^L_{L_1,\dots,L_k}$ the Laurent polynomials such that one has the following.

$$X_{L_1} \cdots X_{L_k} = t^{\sum\limits_{i<j} \langle \hat{L}_i, \hat{L}_j\rangle} \sum_L [f]^L_{L_1,\dots,L_k} X_L$$

$$E_{L_1} \cdots E_{L_k} = t^{\sum\limits_{i<j} \langle \hat{L}_i, \hat{L}_j\rangle} \sum_L [e]^L_{L_1,\dots,L_k} E_L$$

Note that $\hat{L} = \hat{L}_1 + \ldots + \hat{L}_k$ in the Grothendieck group and that $[f]^L_{L_1} = [e]^L_{L_1} = \delta_{L,L_1}$. Moreover we have the following identities.

$$[e]^A_{L_0,\dots,L_k} = \sum_T [e]^A_{L_0,T} [e]^T_{L_1 \dots L_k} \tag{11.4}$$

$$[f]^A_{L_0,\dots,L_k} = \sum_N [f]^A_{L_0,N} [f]^N_{L_1 \dots L_k} \tag{11.5}$$

Theorem 11.1.2 (Xiao)**.** The bialgebras $\mathrm{H}^{\geqslant 0}$ and $\mathrm{H}^{\leqslant 0}$ have the structure of $\mathbb{C}(t)$-Hopf algebras with antipodes S given as follows.

$$\begin{aligned} S : \mathrm{H}^{\geqslant 0} &\to \mathrm{H}^{\geqslant 0} \\ E_N &\mapsto \delta_{N,0} + \sum_{k \geqslant 1} (-1)^k \sum_{L} \sum_{\substack{L_1,\ldots,L_k \\ \neq 0}} t^{2\sum_{i<j}\langle \hat{L}_i, \hat{L}_j \rangle} [e]^L_{L_1,\ldots,L_k} [f]^N_{L_1,\ldots,L_k} K^{-1}_{\hat{N}} E_L \qquad (11.6) \\ K_\alpha &\mapsto K_\alpha^{-1} \end{aligned}$$

$$\begin{aligned} S : \mathrm{H}^{\leqslant 0} &\to \mathrm{H}^{\leqslant 0} \\ F_N &\mapsto \delta_{N,0} + \sum_{k \geqslant 1} (-1)^k \sum_{L} \sum_{\substack{L_1,\ldots,L_k \\ \neq 0}} [e]^L_{L_k,\ldots,L_1} [f]^N_{L_1,\ldots,L_k} F_L (K^*_{\hat{N}})^{-1} \qquad (11.7) \\ K^*_\alpha &\mapsto (K^*_\alpha)^{-1} \end{aligned}$$

Proof. We will prove the antipode axioms for $\mathrm{H}^{\geqslant 0}$ with the case of $\mathrm{H}^{\leqslant 0}$ being entirely similar. Denoting the multiplication by m and the unit morphism by i. Recall that the antipode is required to satisfy the following identities.

$$m \circ (id \otimes S) \circ \Delta = i \circ \varepsilon \qquad\qquad m \circ (S \otimes id) \circ \Delta = i \circ \varepsilon$$

We will prove the identity on the left as the one on the right is similar. The identity is easy to verify when applied to K_α so we need to show that $m \circ (id \otimes S) \circ \Delta(E_A) = \delta_{A,0}$. Using the formula for the coproduct from Theorem 11.1.1 with $L_0 = M$ we have the following.

$$m \circ (id \otimes S) \circ \Delta(E_A) = \sum_{L_0, N} t^{\langle \hat{L}_0, \hat{N} \rangle} [f]^A_{L_0,N} E_{L_0} K_{\hat{N}} S(E_N) \qquad (11.8)$$

The large summation term over $k \geqslant 1$ appearing in the expansion of $S(E_N)$ from Equation (11.6) is unwieldy to work with. Let us write $\sigma_1(N)$ and $\sigma_2(N)$ for the summations over $k \geqslant 1$ and $k \geqslant 2$ respectively. In particular the following identities hold for any $N, A \in \mathrm{Iso}(\mathcal{A})$.

$$S(E_N) = \delta_{N,0} + \sigma_1(N) \qquad (11.9)$$
$$S(E_A) = 2\delta_{A,0} - K^{-1}_{\hat{A}} E_A + \sigma_2(A) \qquad (11.10)$$

We can reduce the RHS of Equation (11.8) by first splitting the summation into its $L_0 = 0$ and $L_0 \neq 0$ terms and then substituting Equation (11.9) for $S(E_N)$ and Equation (11.10) for $S(E_A)$.

$$\begin{aligned} m \circ (id \otimes S) \circ \Delta(E_A) &= K_{\hat{A}} S(E_A) + \sum_N \sum_{L_0 \neq 0} t^{\langle \hat{L}_0, \hat{N} \rangle} [f]^A_{L_0,N} E_{L_0} K_{\hat{N}} S(E_N) \\ &= K_{\hat{A}} S(E_A) + (E_A - \delta_{A,0}) + \sum_N \sum_{L_0 \neq 0} t^{\langle \hat{L}_0, \hat{N} \rangle} [f]^A_{L_0,N} E_{L_0} K_{\hat{N}} \sigma_1(N) \\ &= \delta_{A,0} + K_{\hat{A}} \sigma_2(A) + \sum_N \sum_{L_0 \neq 0} t^{\langle \hat{L}_0, \hat{N} \rangle} [f]^A_{L_0,N} E_{L_0} K_{\hat{N}} \sigma_1(N) \qquad (11.11) \end{aligned}$$

Thus if the sum of the last two terms in Equation (11.11) vanish then we are done. On the one hand consider the last term of Equation (11.11). On substituting the $k \geqslant 1$ summation term of

Equation (11.6) for $\sigma_1(N)$ we obtain at the following equalities. Note we have replaced L by T in Equation (11.6) and instead used L in the sum coming from the product of E_{L_0} and E_T to ease notation later on.

$$\begin{aligned}
&\sum_N \sum_{L_0\neq 0} t^{\langle \hat{L}_0,\hat{N}\rangle}[f]^A_{L_0,N} E_{L_0} K_{\hat{N}} \sigma_1(N) \\
&= \sum_N \sum_{L_0\neq 0} t^{\langle \hat{L}_0,\hat{N}\rangle}[f]^A_{L_0,N} E_{L_0} K_{\hat{N}} \sum_{k\geqslant 1}(-1)^k \sum_T \sum_{\substack{L_1,\dots,L_k\\ \neq 0}} t^{2\sum_{i<j}\langle \hat{L}_i,\hat{L}_j\rangle}[e]^T_{L_1,\dots,L_k}[f]^N_{L_1,\dots,L_k} K^{-1}_{\hat{N}} E_T \\
&= \sum_{k\geqslant 1}(-1)^k \sum_{N,T} \sum_{\substack{L_0,\dots,L_k\\ \neq 0}} t^{\langle \hat{L}_0,\hat{N}\rangle + 2\sum_{i<j}\langle \hat{L}_i,\hat{L}_j\rangle}[e]^T_{L_1,\dots,L_k}[f]^A_{L_0,N}[f]^N_{L_1,\dots,L_k} E_{L_0} E_T \\
&= \sum_{k\geqslant 1}(-1)^k \sum_{L,N,T} \sum_{\substack{L_0,\dots,L_k\\ \neq 0}} t^{\langle \hat{L}_0,\hat{N}+\hat{T}\rangle + 2\sum_{i<j}\langle \hat{L}_i,\hat{L}_j\rangle}[e]^L_{L_0,T}[e]^T_{L_1,\dots,L_k}[f]^A_{L_0,N}[f]^N_{L_1,\dots,L_k} E_L \qquad (11.12)
\end{aligned}$$

On the other hand substituting the $k \geqslant 2$ summation term from Equation (11.6) for $\sigma_2(A)$ in $K_{\hat{A}}\sigma_2(A)$ and reordering the summation over k we get the following.

$$\begin{aligned}
K_{\hat{A}}\sigma_2(A) &= \sum_{k\geqslant 2}(-1)^k \sum_L \sum_{\substack{L_1,\dots,L_k\\ \neq 0}} t^{2\sum_{i<j}\langle \hat{L}_i,\hat{L}_j\rangle}[e]^L_{L_1,\dots,L_k}[f]^A_{L_1,\dots,L_k} E_L \\
&= -\sum_{k\geqslant 1}(-1)^k \sum_L \sum_{\substack{L_0,\dots,L_k\\ \neq 0}} t^{2\sum_{i<j}\langle \hat{L}_i,\hat{L}_j\rangle}[e]^L_{L_0,\dots,L_k}[f]^A_{L_0,\dots,L_k} E_L \qquad (11.13)
\end{aligned}$$

Using Equation (11.4) and Equation (11.5) along with the fact that $\hat{N} = \hat{T} = \hat{L}_1 + \ldots + \hat{L}_k$ it follows that the sum of Equation (11.12) and Equation (11.13) vanishes. □

We will now construct the Drinfeld double of $\mathrm{H}^{\geqslant 0}$ and $\mathrm{H}^{\leqslant 0}$. A reference for the details on the construction of the Drinfeld double is Section 3.2.1, Section 3.2.2 and Section 3.2.3 of [Jos95]. The definition of the Drinfeld double depends on a certain pairing. In the case of Hall algebras Green [Gre95] defined such a pairing P between $\mathrm{H}^{\geqslant 0}$ and $\mathrm{H}^{\leqslant 0}$ as follows.

$$P : \mathrm{H}^{\geqslant 0} \otimes \mathrm{H}^{\leqslant 0} \to \mathbb{C}(t), \qquad P(E_A K_\alpha, F_B K^*_\beta) = t^{(\alpha,\beta)} a_A \cdot \delta_{A,B} \qquad (11.14)$$

To form the Drinfeld double one needs to ensure that the pairing is skew-Hopf pairing. A skew-Hopf pairing is one which satisfies the following requirements for all $x, y \in \mathrm{H}^{\geqslant 0}$ and $a, b \in \mathrm{H}^{\leqslant 0}$.

$$\begin{aligned}
P(1,a) &= \varepsilon(a) & \qquad P(x,ab) &= P(\Delta(x), a\otimes b) \\
P(x,1) &= \varepsilon(x) & \qquad P(xy,a) &= P(x\otimes y, \Delta^{op}(a))
\end{aligned}$$

Here Δ^{op} denotes the opposite coproduct on $\mathrm{H}^{\leqslant 0}$ and we have extended the pairing to tensor products via $P(x\otimes y, a\otimes b) = P(x,a)B(y,b)$. One can check that Green's pairing is indeed skew-Hopf.

Theorem 11.1.3 (Drinfeld)**.** The vector space $\mathrm{H}^{\geqslant 0} \otimes_{\mathbb{C}(t)} \mathrm{H}^{\leqslant 0}$ has the structure of a $\mathbb{C}(t)$-Hopf algebra determined by the following requirements.

(i) The antipode is an anti-homomorphism and $\mathrm{H}^{\geqslant 0}$ and $\mathrm{H}^{\leqslant 0}$ are Hopf subalgebras via their inclusions.

$$\mathrm{H}^{\geqslant 0} \to \mathrm{H}^{\geqslant 0} \otimes_{\mathbb{C}(t)} \mathrm{H}^{\leqslant 0}, \quad x \mapsto x \otimes 1$$
$$\mathrm{H}^{\leqslant 0} \to \mathrm{H}^{\geqslant 0} \otimes_{\mathbb{C}(t)} \mathrm{H}^{\leqslant 0}, \quad a \mapsto 1 \otimes a$$

(ii) For any two elements $x \in \mathrm{H}^{\geqslant 0}$ and $a \in \mathrm{H}^{\leqslant 0}$ we have $(x \otimes 1)(1 \otimes a) = x \otimes a$ along with the following.

$$P(x_1, a_1) \cdot (1 \otimes a_2)(x_2 \otimes 1) = P(x_2, a_2) \cdot x_1 \otimes a_1 \tag{11.15}$$

Here the subscripts $1, 2$ denote Sweedler's notation.

Proof. The details of this are standard and rely on P being a skew-Hopf pairing. See for example Section 3.2.1, Section 3.2.2 and Section 3.2.3 of [Jos95]. □

There is a slightly more useful variant of the Drinfeld double where one formally sets each $K_\alpha \otimes 1$ to be the inverse of $1 \otimes K_\alpha^*$. Recall that a Hopf ideal is an algebra ideal in the kernel of the counit which is also a coalgebra coideal and is preserved by the antipode.

Definition 11.1.2. Define the reduced Drinfeld double to be the quotient Hopf algebra of the Drinfeld double $\mathrm{H}^{\geqslant 0} \otimes_{\mathbb{C}(t)} \mathrm{H}^{\leqslant 0}$ by the Hopf ideal $(K_\alpha \otimes K_\alpha^* - 1)$.

Part III

Generic Bridgeland-Hall Algebras

Definition, Hopf Algebra Structure, Integral Forms and Quasi-Classical Limit

Overview

Part III is concerned with generic Bridgeland-Hall algebras and their integral forms. Generic Bridgeland-Hall algebras are the generic versions of the algebras we discussed in Section 6.3. The existence of these algebras was established by Deng and Chen [CD15] for the case of categories of simply-laced quiver representations.

We will begin in Chapter 12 with the background material required to define Bridgeland-Hall algebras. In particular we will recall from [CD15] the existence of Hall polynomials for Bridgeland-Hall algebras. We also discuss how various properties of the categories of complexes $\mathcal{C}_q$ are independent of q.

In Chapter 13 we define the generic Bridgeland-Hall algebra DH. Following [CD15] we will develop some properties of these algebras and in particular explain how DH is isomorphic to the quantized enveloping algebra $\mathrm{U}_t(\mathfrak{g})$. We also introduce new basis elements of DH and establish a product expansion in terms of these. This expansion will be used in calculating the Poisson structure on the semi-classical Bridgeland-Hall algebra in Part IV.

Chapter 14 deals with a natural Hopf algebra structure which one can place on DH. This is achieved by establishing a generic version of results due to Yanagida [Yan16] which says that DH is isomorphic to the Drinfeld double of the (extended) Abelian Hall algebra that we had in Chapter 11. The Hopf algebra structure on the Drinfeld double then induces a Hopf algebra structure on DH.

The material in Chapter 15 concerns integral forms of DH. As in the Abelian case there will be two natural integral forms of DH. One integral form DH_{ex} has product counting extensions while the other DH_{fl} roughly speaking has product counting flags. Deng and Chen introduced DH_{fl} in [CD15] and to our knowledge we are the first to consider DH_{ex} from the Hall algebra perspective.

In Chapter 15 we will also define the quasi-classical Bridgeland-Hall algebra DH_{qc} which is the $t = 1$ limit of DH_{fl}. Deng and Chen showed that DH_{qc} is isomorphic to $\mathrm{U}(\mathfrak{g})$ and used this fact to recover the whole simple Lie algebra $\mathfrak{g}$ from Hall algebras.

Both DH_{fl} and DH_{qc} seem tricky to work with and we will not develop their properties in great detail. This contrasts with the semi-classical Bridgeland-Hall algebra DH_{sc} which is the $t = 1$ limit of DH_{ex}. The main feature of DH_{sc} is that it is the algebra of functions on a Poisson-Lie group. We will fully explore DH_{sc} from a Hall algebraic point of view in Part IV.

Finally we end this part with Chapter 16 where we prove the first main result of this thesis: the extension counting integral form of the Bridgeland-Hall algebra is isomorphic to the Poisson integral form of the quantized enveloping algebra.

Chapter 12

Background Material

In Chapter 12 we recall various bits of background theory that is needed to define and work with generic Bridgeland-Hall algebras. The results contained here are the $\mathbb{Z}_2$-graded complexes analogues of those in Chapter 7.

In Section 12.1 we start off this chapter by explaining how various features of the categories of complexes $\mathcal{C}_q$ are independent of q. We then discuss the existence of Hall polynomials for Bridgeland-Hall algebras in Section 7.3. The existence of these polynomials in the case of simply-laced quivers was established by Deng and Chen in [CD15].

12.1 $\mathbb{Z}_2$-Graded Complexes: Independence of $\mathbb{F}_q$

In this section we discuss how the categories $\mathcal{C}_q$ are independent of q, just as we did for the categories $\mathcal{A}_q$ in Section 7.2. Recall that for each q a prime power $\mathcal{C}_q$ is the category of $\mathbb{Z}_2$-graded complexes in projective objects in $\mathcal{A}_q$. The material here is from [CD15].

We begin by explaining how $\mathrm{Iso}(\mathcal{C}_q)$ the set of isomorphisms classes of objects in $\mathcal{C}_q$ is independent of q. We then describe how various properties of $\mathbb{Z}_2$-graded complexes are also independent of q.

Note that we observed in the commentary following Theorem 2.3.1 that any complex in $\mathcal{C}_q$ is determined uniquely up to isomorphism by a map $\Phi \coprod(\mathbb{Z}_2 \times Q_0) \to \mathbb{Z}_{\geqslant 0}$. If we abuse notation by writing $\mathrm{Iso}(\mathcal{C})$ for the set of maps $\Phi \coprod(\mathbb{Z}_2 \times Q_0) \to \mathbb{Z}_{\geqslant 0}$ then we have canonical bijections between the following sets.

$$\mathrm{Iso}(\mathcal{C}_q) \cong \mathrm{Iso}(\mathcal{C}) \tag{12.1}$$

These bijections induce canonical bijections of the following form for any prime powers q and q'.

$$\mathrm{Iso}(\mathcal{C}_q) \cong \mathrm{Iso}(\mathcal{C}_{q'})$$

We will write $L_\bullet \in \mathrm{Iso}(\mathcal{C})$ for maps $L_\bullet : \Phi \coprod(\mathbb{Z}_2 \times Q_0) \to \mathbb{Z}_{\geqslant 0}$. Abusing notation again we will frequently refer to the elements $L_\bullet \in \mathrm{Iso}(\mathcal{C})$ as if were they genuine $\mathbb{Z}_2$-graded complexes. Should we need to, we will write $L_{\bullet q}$ for a particular choice of an actual complex in the category $\mathcal{C}_q$ determined by the element $L_\bullet \in \mathrm{Iso}(\mathcal{C})$.

For any representations $L, P \in \mathrm{Iso}(\mathcal{A})$ with P projective we will define elements C_L, C_L^*, K_P and K_P^* to be the elements of $\mathrm{Iso}(\mathcal{C})$ which for any q a prime power determine the objects C_{L_q}, $C_{L_q}^*$, K_{P_q} and $K_{P_q}^*$ of $\mathcal{C}_q$ that we defined in Section 2.3.

As for quiver representations, we will need to know that various properties of $\mathbb{Z}_2$-graded complexes are independent of q. By this we mean that if $L_\bullet \in \mathrm{Iso}(\mathcal{C})$ such that $L_{\bullet q}$ has a certain property in $\mathcal{C}_q$ for a particular q then $L_{\bullet q}$ has that property in $\mathcal{C}_q$ for all prime powers q. This will be true for any reasonable property.

Given two complexes $M_\bullet, N_\bullet \in \mathrm{Iso}(\mathcal{C})$ we define their direct sum $M_\bullet \oplus N_\bullet$ to be the following map.

$$M_\bullet + N_\bullet : \Phi \coprod (\mathbb{Z}_2 \times Q_0) \to \mathbb{Z}_{\geqslant 0}$$

We then have that the property of being a direct sum of complexes is independent of q since $(M_\bullet \oplus N_\bullet)_q \cong M_{\bullet q} \oplus N_{\bullet q}$ for any q a prime power. By definition then, any complex $L_\bullet$ decomposes uniquely as follows.

$$L_\bullet = C_A \oplus C_B^* \oplus K_P \oplus K_Q^*$$

We define the degree zero and degree one homology objects of $L_\bullet$ to be $\mathrm{H}_0(L_\bullet) := A$ and $\mathrm{H}_1(L_\bullet) := B$ respectively. We have $\mathrm{H}_i(L_\bullet)_q \cong \mathrm{H}_i(L_{\bullet q})$ for $i = 0, 1$ and all q a prime power. In particular it follows that property of being an acyclic complex is independent of q.

For a complex $L_\bullet \in \mathrm{Iso}(\mathcal{C})$ we define its class in the Grothendieck group $K(\mathcal{A})$ to be given by $\hat{L}_\bullet := \hat{A} - \hat{B}$. For example we have $\hat{C}_L = \hat{L}$ and $\hat{C}_L^* = -\hat{L}$. We also note that the shift functor involution $* : \mathcal{C}_q \to \mathcal{C}_q$ induces an involution $* : \mathrm{Iso}(\mathcal{C}) \to \mathrm{Iso}(\mathcal{C})$. This sends the complex $L_\bullet$ to the shifted complex $L_\bullet^* = C_B \oplus C_A^* \oplus K_Q \oplus K_P^*$. We have $(L_{\bullet q})^* \cong (L_\bullet^*)_q$ for any q a prime power.

The projectives appearing in minimal projective resolutions, as defined in Equation (2.5), are independent of q. If $L \in \mathrm{Iso}(\mathcal{A})$ is a representation then there exists projectives $P_L, Q_L \in \mathrm{Iso}(\mathcal{A})$ such that for each q a prime power there is a minimal projective resolution of L_q of the form $P_{L_q} \to Q_{L_q} \to L_q$. The statement of this fact can be found at the end of page 22 in [CD15].

We end by observing that Lemma 3.5. (1) of [CD15] says that for any $M_\bullet, N_\bullet \in \mathrm{Iso}(\mathcal{C})$ the following dimensions are independent of q.

$$\dim_{\mathbb{F}_q} \mathrm{Hom}(M_{\bullet q}, N_{\bullet q})$$

Consequently we will simply write $\dim \mathrm{Hom}(M_\bullet, N_\bullet)$ for these dimensions.

12.2 Hall Polynomials for Bridgeland-Hall Algebras

This section is concerned with Hall polynomials for Bridgeland-Hall algebras. We begin by establishing the existence of such polynomials due to Deng and Chen [CD15]. We then discuss some new interpretations of these polynomials and prove some simple properties which to our knowledge have not been considered in the literature.

We briefly recall the notation used for various structure constants of the non-generic Bridgeland-Hall algebra from Chapter 6. We used the following notation for the subset of extensions of the complex $M_{\bullet q}$ by $N_{\bullet q}$ whose middle term isomorphic to $L_{\bullet q}$.

$$\mathrm{Ext}^1(M_{\bullet q}, N_{\bullet q})_{L_{\bullet q}} \subseteq \mathrm{Ext}^1(M_{\bullet q}, N_{\bullet q})$$

We also defined $\mathrm{F}^{L_{\bullet q}}_{M_{\bullet q},N_{\bullet q}}$ to be the set of subobjects $N_{\bullet q} \subseteq L_{\bullet q}$ such that the corresponding quotient object is isomorphic to $M_{\bullet q}$. Finally we wrote $\mathrm{Aut}(L_{\bullet q})$ for the set of automorphisms of $L_{\bullet q}$. The following proposition then establishes the existence of Hall polynomials for Bridgeland-Hall algebras.

Proposition 12.2.1 (Deng, Chen)**.** For any complexes $L_\bullet, M_\bullet, N_\bullet \in \mathrm{Iso}(\mathcal{C})$ there exists polynomials $e^{L_\bullet}_{M_\bullet,N_\bullet}$, $h_{M_\bullet,N_\bullet}$, $f^{L_\bullet}_{M_\bullet,N_\bullet}$ and $a_{L_\bullet}$ in $\mathbb{Q}[t]$ such that for any q a prime power we have the following.

$$e^{L_\bullet}_{M_\bullet,N_\bullet}(q^{\frac{1}{2}}) = |\mathrm{Ext}^1(M_{\bullet q}, N_{\bullet q})_{L_{\bullet q}}| \qquad h_{M_\bullet,N_\bullet}(q^{\frac{1}{2}}) = |\mathrm{Hom}(M_{\bullet q}, N_{\bullet q})|$$
$$f^{L_\bullet}_{M_\bullet,N_\bullet}(q^{\frac{1}{2}}) = |\mathrm{F}^{L_{\bullet q}}_{M_{\bullet q},N_{\bullet q}}| \qquad a_{L_\bullet}(q^{\frac{1}{2}}) = |\mathrm{Aut}(L_{\bullet q})|$$

Proof. This follows from Theorem 3.11, Corollary 3.12 and Lemma 3.5. in [CD15]. □

Using Proposition 12.2.1 we can derive the existence of some other polynomials. In Part IV these polynomials will be integral to describing the Poisson and associated Lie algebra structures of semi-classical Bridgeland-Hall algebras.

Recall from the end of Section 2.3 that for any $A, B, M, N \in \mathrm{Iso}(\mathcal{A})$ and any q a prime power we have the set $\mathrm{Hom}(M_q, N_q)_{A_q,B_q}$ of morphisms $M_q \to N_q$ with kernel A_q and cokernel B_q. By Equation (2.11) for any q a prime power we have an isomorphism of sets of the following form.

$$\mathrm{Ext}^1(C_{M_q}, C^*_{N_q})_{L_{\bullet q}} \cong \mathrm{Hom}(M_q, N_q)_{A_q,B_q} \tag{12.2}$$

Here $L_\bullet \in \mathrm{Iso}(\mathcal{C})$ is the complex $C_A \oplus C^*_B \oplus K_{P_{MA}} \oplus K^*_{Q_{NB}}$ where the acyclic direct summands are uniquely determined by the requirement that $P_M = P_{MA} \oplus P_A$ and $Q_N = Q_{NB} \oplus Q_B$. Note that the notation P_{MA} signifies that P_{MA} is determined uniquely by P_M and P_A. Similarly Q_{NB} is determined uniquely by Q_N and Q_B.

It follows then from Proposition 12.2.1 that the following defines a well-defined polynomial which, when evaluated at $t = q^{\frac{1}{2}}$, counts the number of morphisms $M_q \to N_q$ with kernel A_q and cokernel B_q.

$$h^{A,B}_{M,N} := e^{L_\bullet}_{C_M,C^*_N} \tag{12.3}$$

When $(A, B) \neq (M, N)$ we also have the following polynomial which counts the number of elements in the projectivization of the set $\mathrm{Hom}(M_q, N_q)_{A_q,B_q}$.

$$\mathbb{P}(h)^{A,B}_{M,N} := h^{A,B}_{M,N}/(t^2 - 1) \tag{12.4}$$

The $\mathbb{F}_q^\times$-action on $\mathrm{Hom}(M_q, N_q)_{A_q,B_q}$ is given by the scaling of morphisms $M_q \to N_q$. This action is free away from the zero morphism (which is the only morphism with kernel M_q and cokernel N_q). That Equation (12.4) really does give a polynomial when $(A, B) \neq (M, N)$ then follows from Lemma 7.1.2.

The polynomials $\mathbb{P}(h)^{A,B}_{M,N}$ in certain special cases have the following useful interpretation in terms of the Hall polynomials which count flags of quiver representations.

Lemma 12.2.1. For any $A, B, M, N \in \mathrm{Iso}(\mathcal{A})$ with M and N indecomposable we have the following two identities.

$$\mathbb{P}(h)^{A,0}_{M,N} = f^{M}_{N,A} \qquad\qquad \mathbb{P}(h)^{0,B}_{M,N} = f^{N}_{B,M}$$

Proof. We will establish the right-hand identity with the proof of the other being analogous. Recall from Remark 7.3.2 that the automorphism polynomial of an indecomposable quiver representation is $t^2 - 1$. We thus have the following identity.

$$\mathbb{P}(h)^{0,B}_{M,N} = h^{0,B}_{M,N}/(t^2 - 1) = h^{0,B}_{M,N}/a_M \tag{12.5}$$

Let q be a prime power and set $t = q^{\frac{1}{2}}$ in Equation (12.5). By definition the evaluation at $t = q^{\frac{1}{2}}$ of the polynomial $h^{0,B}_{M,N}$ counts *injective homomorphisms* $M_q \hookrightarrow N_q$ with corresponding quotient object B_q. Dividing out by the number of automorphisms of M_q we get the number of *subobjects* $M_q \subseteq N_q$ with quotient B_q. However this is exactly the evaluation of $f^{N}_{B,M}$ at $t = q^{\frac{1}{2}}$. The result then follows by Lemma 7.1.1. □

Chapter 13

Generic Bridgeland-Hall Algebras

This chapter concerns the generic Bridgeland-Hall algebra DH due to Deng and Chen [CD15]. Generic Bridgeland-Hall algebras are the generic versions of the algebras we discussed in Section 6.3.

We start off this chapter in Section 13.1 by defining the generic Bridgeland-Hall algebra DH using the Hall polynomials from Chapter 12. We then develop some properties of generic Bridgeland-Hall algebras in Section 13.2 regarding subalgebras and tensor product descriptions.

In Section 13.3 we introduce new basis elements of DH and establish a product expansion in terms of these elements. This expansion will be used in calculating the Poisson structure on the semi-classical Bridgeland-Hall algebra in Part IV.

Finally we end with Section 13.4 where we explain how DH is isomorphic to the quantized enveloping algebra $\mathrm{U}_t(\mathfrak{g})$. This is the generic version of Bridgeland's Theorem 6.3.1 and is due to Deng and Chen. We also give a slight modification of this isomorphism and show that it intertwines a certain shift functor induced involution of DH and the involution Σ on $\mathrm{U}_t(\mathfrak{g})$ from Equation (5.6).

13.1 Definitions

This section is concerned with the definition of generic Bridgeland-Hall algebras. These are the generic counterparts of the Bridgeland-Hall algebras that we defined in Section 6.3. The existence of generic Bridgeland-Hall algebras for categories of simply-laced quiver representations is due to Deng and Chen [CD15].

We begin with the following generic version of Definition 6.3.1.

Definition 13.1.1. Define $\mathrm{H}(\mathcal{C})$ to be the $\mathbb{C}(t)$-vector space generated by the set $\mathrm{Iso}(\mathcal{C})$. The product of two basis elements $[M_\bullet]$ and $[N_\bullet]$ corresponding to two complexes is given by the following.

$$[M_\bullet][N_\bullet] = t^{\langle \hat{M}_0, \hat{N}_0\rangle + \langle \hat{M}_1, \hat{N}_1\rangle} \sum_{L_\bullet \in \mathrm{Iso}(\mathcal{C})} \frac{e^{L_\bullet}_{M_\bullet, N_\bullet}}{h_{M_\bullet, N_\bullet}} [L_\bullet] \tag{13.1}$$

The unit element $[0_\bullet]$ is given by the zero complex $0_\bullet$.

Note that there is not a genuine category $\mathcal{C}$ whose Hall algebra we are taking here. However, in light of Equation (12.1), the set of isomorphism classes of each category of $\mathbb{Z}_2$-graded complexes $\mathcal{C}_q$ is canonically isomorphic to the set $\mathrm{Iso}(\mathcal{C})$ that was defined in Section 12.1. Thus it is as if there is an actual category $\mathcal{C}$ which gives the Hall algebra in Definition 13.1.1.

Proposition 13.1.1. $\mathrm{H}(\mathcal{C})$ is a unital associative $\mathbb{C}(t)$-algebra.

Proof. The product is a well-defined element of $\mathrm{H}(\mathcal{C})$ since by Proposition 12.2.1 the structure constants are Laurent polynomial and in particular elements of $\mathbb{C}(t)$. Associativity then follows by the same kind of argument that we had Proposition 8.1.1 in the Abelian case. That the unit is $[0_\bullet]$ is obvious. □

As we observed in Section 6.3 the algebra $\mathrm{H}(\mathcal{C})$ is not quite the correct algebra to take. A problem arises for example when one attempts to recover the whole quantized enveloping algebra $\mathrm{U}_t(\mathfrak{g})$. In particular the subalgebra of $\mathrm{H}(\mathcal{C})$ generated by the elements $[K_P \oplus K_Q^*]$ given by acyclic complexes should correspond to the subalgebra of $\mathrm{U}_t(\mathfrak{g})$ generated by the elements K_i and K_i^{-1}. However $[K_P]$ and $[K_Q^*]$ are neither invertible nor is $[K_P]$ the inverse of $[K_P^*]$. To remedy this one must formally invert the elements $[K_P \oplus K_Q^*]$ and then set $[K_P]$ to be the inverse of $[K_P^*]$.

One needs to take care in order to invert the elements $[K_P \oplus K_Q^*]$ as $\mathrm{H}(\mathcal{C})$ is a non-commutative algebra. It is easy to check that $[M_\bullet][N_\bullet] = [M_\bullet \oplus N_\bullet]$ in $\mathrm{H}(\mathcal{C})$ for any two acyclic complexes $M_\bullet$ and $N_\bullet$. The following lemma will then allow one to formally invert the elements $[K_P \oplus K_Q^*]$ and indeed is very useful in its own right.

Lemma 13.1.2. If $L_\bullet \in \mathrm{Iso}(\mathcal{C})$ and $P \in \mathrm{Iso}(\mathcal{A})$ is projective then we have the following identities in $\mathrm{H}(\mathcal{C})$.

$$[K_P][L_\bullet] = t^{\langle \hat{P}, \hat{L}_\bullet \rangle}[L_\bullet \oplus K_P] \qquad [L_\bullet][K_P] = t^{-\langle \hat{L}_\bullet, \hat{P} \rangle}[L_\bullet \oplus K_P]$$
$$[K_P^*][L_\bullet] = t^{-\langle \hat{P}, \hat{L}_\bullet \rangle}[L_\bullet \oplus K_P^*] \qquad [L_\bullet][K_P^*] = t^{\langle \hat{L}_\bullet, \hat{P} \rangle}[L_\bullet \oplus K_P^*]$$

Proof. This is an easy calculation using the acyclicity of the complexes K_P and K_P^*, which were given in Equation (2.6). □

Lemma 13.1.2 ensures that the Ore conditions are satisfied for localizing $\mathrm{H}(\mathcal{C})$ at the set of elements corresponding to acyclic complexes $\{[M_\bullet] \mid M_\bullet \text{ acyclic}\}$. Accordingly we make the following definition.

Definition 13.1.3. Define the generic localized Bridgeland-Hall algebra to be the following localized algebra.

$$\mathrm{DH}_{loc} := \mathrm{H}(\mathcal{C}) \left[[M_\bullet]^{-1} \mid M_\bullet \text{ acyclic} \right]$$

Unfortunately we would still not have that $[K_P]$ is the inverse of $[K_P^*]$. We will introduce and discuss some elements of DH_{loc} and then formally set $[K_P]$ to be the inverse of $[K_P^*]$. Let α be a class in $K(\mathcal{A})$ which we decompose into $\hat{P} - \hat{Q}$ where $P, Q \in \mathrm{Iso}(\mathcal{A})$ are projective. We then define the following elements of DH_{loc} which we note do not depend on the choice of decomposition $\alpha = \hat{P} - \hat{Q}$.

$$K_\alpha := [K_P][K_Q]^{-1} \qquad\qquad K_\alpha^* := [K_P^*][K_Q^*]^{-1}$$

We observe that the elements K_α and K_α^* were used in a different context to define generic extended Hall algebras in Definition 11.1.1. We will see in Proposition 13.2.2 that no confusion should arise from this an abuse of notation. Note that it follows from Lemma 13.1.2 that the following identities hold in DH_{loc} for any $L_\bullet \in \mathrm{Iso}(\mathcal{C})$ and class α in $K(\mathcal{A})$.

$$K_\alpha[L_\bullet] = t^{(\alpha,\hat{L}_\bullet)}[L_\bullet]K_\alpha \qquad\qquad K_\alpha^*[L_\bullet] = t^{-(\alpha,\hat{L}_\bullet)}[L_\bullet]K_\alpha^* \tag{13.2}$$

By definition the elements $[L_\bullet]$ form a basis for $\mathrm{H}(\mathcal{C})$. Using the fact that a complex $L_\bullet$ decomposes as a direct sum of the form $C_A \oplus C_B^* \oplus K_P \oplus K_Q^*$ it is easy to see from Lemma 13.1.2 that the elements $K_\alpha K_\beta^*[C_A \oplus C_B^*]$ form a basis for DH_{loc}. Moreover using Equation (13.1) one can see that the product in this basis has structure constants which are Laurent polynomials.

We may now define the following algebra which is the correct one for recovering the whole quantized enveloping algebra.

Definition 13.1.4. Define the generic (reduced) Bridgeland-Hall algebra DH to be the quotient of DH_{loc} by the ideal $(K_\alpha K_\alpha^* - 1)$. We will almost always refer to DH simply as the generic Bridgeland-Hall algebra.

It is instructive to compare Definition 13.1.4 with that of Definition 11.1.2 for the reduced Drinfeld double. We will show in Chapter 14 that these two algebras in fact coincide.

Similar to the case of DH_{loc} it is easy to see that the elements $K_\alpha[C_A \oplus C_B^*]$ form a basis for DH. Note also that the shift involution $* : \mathcal{C} \to \mathcal{C}$ induces well-defined $\mathbb{C}(t)$-algebra involutions of both DH_{loc} and DH given by $[L_\bullet] \mapsto [L_\bullet^*]$. We will abuse notation and also denote both of these involutions by $*$. For example under this involution we have $K_\alpha \mapsto K_\alpha^*$.

We end this section by pointing out that we could also have defined DH directly to be the quotient of $\mathrm{H}(\mathcal{C})$ by the following ideal.

$$\mathrm{DH} := \mathrm{H}(\mathcal{C})/([K_P][K_P^*] - 1 \mid P \text{ projective}) \tag{13.3}$$

The reason we have opted not to take this short-cut, which bypasses the definition of DH_{loc}, is that we will need to use the algebra DH_{loc} in Section 14.1.

13.2 Properties of Generic Bridgeland-Hall Algebras

In this section we explain some of the structure of DH_{loc} and DH in terms of their subalgebras. The results in this section are generic versions of ones proved by Bridgeland in [Bri13].

Our definition of the subalgebras of DH_{loc} and DH require the following special elements of DH_{loc} and DH associated to any $L \in \mathrm{Iso}(\mathcal{A})$.

$$E_L := t^{\langle \hat{P}_L, \hat{L}\rangle} K_{-\hat{P}_L}[\mathcal{C}_L] \qquad X_L := E_L / a_L \tag{13.4}$$

$$F_L := t^{\langle \hat{P}_L, \hat{L}\rangle} K^*_{-\hat{P}_L}[\mathcal{C}^*_L] \qquad Y_L := F_L / a_L \tag{13.5}$$

Here recall from the end of Section 12.1 that $P_L \in \mathrm{Iso}(\mathcal{A})$ is one of the projectives determining a minimal projective resolution $P_{L_q} \to Q_{L_q}$ of L_q for any q a prime power. Note that under the shift involution we have $E_L \mapsto F_L$.

By abuse of notation we have already used E_L and F_L to denote basis vectors of two copies of the generic Hall algebra H (sitting inside the extended Hall algebras $\mathrm{H}^{\geqslant 0}$ and $\mathrm{H}^{\leqslant 0}$ as defined in Definition 11.1.1). The following proposition shows that no confusion may arise.

Proposition 13.2.1 (Bridgeland)**.**

(i) DH_{loc} contains two copies of the group algebra $\mathbb{C}(t)[K(\mathcal{A})]$ as subalgebras. One is given by the span of elements K_α while the other coincides with the span of the elements K^*_α.

(ii) DH_{loc} contains two copies of the generic Hall algebra H as subalgebras. One is given by the span of elements E_L while the other coincides with the span of the elements F_L.

Proof. Using the shift involution it suffices to prove the assertions for the span of the elements K_α and E_L only. Using acyclicity of the complexes involved one can check that the elements K_α all commute with each other. Moreover a straightforward calculation, exactly as in Lemma 4.3 of [Bri13], shows that the product of the elements E_L as defined in Equation (13.4) coincides with the product on H as defined in Equation (8.1).

The only thing left to check is that the elements K_α and E_L are linearly independent in DH_{loc}. Working through the definitions of Ore localization one can check that linear independence of the elements K_α in DH_{loc} follows from the fact that the elements $[K_P]$ in $\mathrm{H}(\mathcal{C})$ are linearly independent by definition.

To see that the elements E_L in DH_{loc} are linearly independent note that taking homology of a complex $M_\bullet$ determines a linear map $\mathrm{DH}_{loc} \to \mathrm{H}$ as follows.

$$[M_\bullet] \mapsto t^{-\langle \hat{M}_1, \widehat{H_0(M_\bullet)}\rangle} E_{H_0(M_\bullet)}$$

Linear independence then follows from the fact that under this map the elements E_L in DH_{loc} are sent to the basis elements E_L in H. □

Recall that in Definition 11.1.1 we defined the extended Hall algebra which was denoted by $\mathrm{H}^{\geqslant 0}$ and alternatively $\mathrm{H}^{\leqslant 0}$. These were isomorphic as algebras (though differed as Hopf algebra). The following proposition realises these two copies as subalgebras of the generic localized Bridgeland-Hall algebra.

Proposition 13.2.2 (Bridgeland)**.** DH_{loc} contains two copies of the extended Hall algebra as subalgebras. One copy $\mathrm{H}^{\geqslant 0}$ coincides with the $\mathbb{C}(t)$-subspace spanned by elements of the form by $E_L K_\alpha$ the other copy $\mathrm{H}^{\leqslant 0}$ is the span of the elements $F_L K^*_\alpha$.

Proof. This is the generic version of Lemma 4.6 in [Bri13]. □

We will see subsequently in Chapter 14 that the generic localized Bridgeland-Hall algebra in fact coincides with the Drinfeld double that we defined in Chapter 11. The following proposition should be viewed as the first step towards proving this statement.

Proposition 13.2.3 (Bridgeland)**.** Multiplication induces the following isomorphism of $\mathbb{C}(t)$-vector spaces.

$$\mathrm{H}^{\geqslant 0} \otimes_{\mathbb{C}(t)} \mathrm{H}^{\leqslant 0} \to \mathrm{DH}_{loc}, \qquad E_A K_\alpha \otimes F_B K_\beta^* \mapsto E_A K_\alpha F_B K_\beta^*$$

Proof. This stated in the commentary preceding Equation 5.3 in [CD15]. □

Since the set of elements of the form $E_A K_\alpha$ and $F_B K_\beta^*$ give a basis for $\mathrm{H}^{\geqslant 0}$ and $\mathrm{H}^{\leqslant 0}$ respectively then Proposition 13.2.3 implies that the set of elements of the form $E_A K_\alpha F_B K_\beta^*$ give a basis for DH_{loc}.

In the following proposition we use Proposition 13.2.1 to identify the generic Hall algebra H with the two subalgebras of DH spanned by the elements E_L and F_L respectively.

Proposition 13.2.4 (Bridgeland)**.** Multiplication induces the following isomorphism of $\mathbb{C}(t)$-vector spaces.

$$\mathrm{H} \otimes_{\mathbb{C}(t)} \mathbb{C}(t)[K(\mathcal{A})] \otimes_{\mathbb{C}(t)} \mathrm{H} \to \mathrm{DH}, \qquad E_A \otimes K_\alpha \otimes F_B \mapsto E_A K_\alpha F_B$$

Proof. This is Lemma 5.4 in [CD15]. □

Similar to the case of DH_{loc}, the above proposition implies that the Bridgeland-Hall algebra has a $\mathbb{C}(t)$-basis given by the elements of the form $E_A K_\alpha F_B$.

13.3 An Identity

In this section we introduced certain new basis elements of DH. We then prove a product expansion formula in terms of these elements. This formula will be used in Section 17.2 to calculate the Poisson algebra structure on the semi-classical Bridgeland-Hall algebra that will be introduced in Part IV. The contents of this section are to our knowledge new and do not appear in Bridgeland's original paper [Bri13].

Recall from Equation (13.4) and Equation (13.5) that we defined the following elements DH which were slight modifications of the elements $[C_L]$ and $[C_L^*]$ respectively.

$$E_L := t^{\langle \hat{P}_L, \hat{L} \rangle} K_{-\hat{P}_L} [\mathcal{C}_L] \qquad\qquad F_L := t^{\langle \hat{P}_L, \hat{L} \rangle} K_{\hat{P}_L} [\mathcal{C}_L^*] \tag{13.6}$$

Here C_L is the complex given by a minimal projective resolution $P_L \to Q_L$ of the representation L. Now for any two representations $A, B \in \mathrm{Iso}(\mathcal{A})$ we introduce the following modification of the element $[C_A \oplus C_B^*]$ which generalizes E_A and F_B.

$$D_{A,B} := t^{\langle \hat{P}_A - \hat{P}_B, \hat{A} - \hat{B} \rangle} K_{\hat{P}_B - \hat{P}_A} [C_A \oplus C_B^*] \tag{13.7}$$

Indeed we observe that $D_{A,0} = E_A$ and $D_{0,B} = F_B$. A useful fact about these elements is that the shift involution $*$ of DH sends $D_{A,B}$ to $D_{B,A}$.

We will now show that products of the form $E_M F_N$ and $F_N E_M$ have natural expansions in terms of the elements $D_{A,B}$. The structure constants of these expansions involve the polynomials $h^{A,B}_{M,N}$ that we defined in Section 12.2. Recall that these polynomials count morphisms $M \to N$ with kernel A and cokernel B. We observe that the Grothendieck group classes of the representations involved are related via $\hat{M} - \hat{A} = \hat{N} - \hat{B}$.

Proposition 13.3.1. For any $M, N \in \mathrm{Iso}(\mathcal{A})$ we have the following identities[1] in DH.

$$E_M F_N = \sum_{A,B \in \mathrm{Iso}(\mathcal{A})} t^{\langle \hat{N}-\hat{B}, \hat{M}-\hat{N}\rangle} h^{A,B}_{M,N} K_{\hat{B}-\hat{N}} D_{A,B} \tag{13.8}$$

$$F_N E_M = \sum_{A,B \in \mathrm{Iso}(\mathcal{A})} t^{\langle \hat{B}-\hat{N}, \hat{M}-\hat{N}\rangle} h^{B,A}_{N,M} K_{\hat{N}-\hat{B}} D_{A,B} \tag{13.9}$$

Proof. We begin by showing that Equation (13.9) follows from Equation (13.8). To see this we first apply the shift functor involution to Equation (13.8) and use the fact that $D^*_{A,B} = D_{B,A}$. Switching M with N, A with B and using $\hat{M} - \hat{A} = \hat{N} - \hat{B}$ one arrives at Equation (13.9).

It remains to establish Equation (13.8) which we do via a lengthy expansion of the product $E_M F_N$. We begin with the following identity in DH where $n_1 = \langle \hat{P}_M, \hat{M}\rangle + \langle \hat{P}_N, \hat{N}\rangle - (\hat{P}_N, \hat{M})$.

$$\begin{aligned} E_M F_N &= t^{\langle \hat{P}_M, \hat{M}\rangle + \langle \hat{P}_N, \hat{N}\rangle} K_{-\hat{P}_M}[C_M] K_{\hat{P}_N}[C^*_N] \\ &= t^{n_1} K_{\hat{P}_N - \hat{P}_M}[C_M][C^*_N] \end{aligned}$$

Here we have used Equation (13.6) to substitute for E_M and F_N and Equation (13.2) to skew commute the element $K_{\hat{P}_N}$ past $[C_M]$. Expanding the product $[C_M][C^*_N]$ using Equation (13.1) we obtain the following where $n_2 = n_1 + \langle \hat{P}_M, \hat{Q}_N\rangle + \langle \hat{Q}_M, \hat{P}_N\rangle - 2\langle \hat{P}_M, \hat{Q}_N\rangle$.

$$E_M F_N = t^{n_2} K_{\hat{P}_N - \hat{P}_M} \sum_{L_\bullet \in \mathrm{Iso}(\mathcal{C})} e^{L_\bullet}_{C_M, C^*_N} [L_\bullet] \tag{13.10}$$

To obtain Equation (13.10) we have used the fact that $\dim \mathrm{Hom}(C_M, C^*_N)$ is the same as $\dim \mathrm{Hom}(P_M, Q_N)$ and moreover $\dim \mathrm{Hom}(P_M, Q_N) = \langle \hat{P}_M, \hat{Q}_N\rangle$ since P_M and Q_N are projective.

We will now rearrange Equation (13.10) by summing over the homology objects of the complexes $L_\bullet$ rather than the complexes themselves. Recall from Equation (12.3) that by definition $e^{L_\bullet}_{C_M, C^*_N} = h^{A,B}_{M,N}$ where A and B are the two homology objects of the complex $L_\bullet$. Moreover from the discussion at the end of Section 2.3 we have that $L_\bullet = C_A \oplus C^*_B \oplus K_{P_{MA}} \oplus K^*_{Q_{NB}}$ where P_{MA} and Q_{NB} are the unique projectives satisfying $P_{MA} \oplus P_A = P_M$ and $Q_{NB} \oplus Q_B = Q_N$. We may thus sum over $A, B \in \mathrm{Iso}(\mathcal{A})$ in Equation (13.10) to obtain the following identity.

$$E_M F_N = t^{n_2} K_{\hat{P}_N - \hat{P}_M} \sum_{A,B \in \mathrm{Iso}(\mathcal{A})} h^{A,B}_{M,N} [C_A \oplus C^*_B \oplus K_{P_{MA}} \oplus K^*_{Q_{NB}}]$$

[1] An interesting feature of these identities is that the coefficient of the $A = B = 0$ term is given by Green's pairing that was defined in Equation (11.14).

Using Lemma 13.1.2 to pull out the term $K_{P_{MA}} \oplus K^*_{Q_{NB}}$ we obtain the following equation where $n_3 = n_2 - \langle \hat{P}_{MA} - \hat{Q}_{NB}, \hat{A} - \hat{B}\rangle$ and $\gamma = \hat{P}_N - \hat{P}_M + \hat{P}_{MA} - \hat{Q}_{NB}$.

$$E_M F_N = \sum_{A,B \in \mathrm{Iso}(\mathcal{A})} t^{n_3} h^{A,B}_{M,N} K_\gamma [C_A \oplus C^*_B] \tag{13.11}$$

Now we have the following simplifications of γ and n_3. The proofs of these are messy and so we will postpone them to Lemma 13.3.1 for ease of reading.

$$n_3 = \langle \hat{N} - \hat{B} + \hat{P}_A - \hat{P}_B, \hat{M} - \hat{N}\rangle \qquad \gamma = \hat{B} - \hat{N} + \hat{P}_B - \hat{P}_A$$

Substituting these identities into Equation (13.11) and using the definition of the elements $D_{A,B}$ from Equation (13.7) we obtain Equation (13.8) as desired. □

It remains to establish the following lemma.

Lemma 13.3.1. In the notation from the proof of Proposition 13.3.1 the following identities hold.

$$n_3 = \langle \hat{N} - \hat{B} + \hat{P}_A - \hat{P}_B, \hat{M} - \hat{N}\rangle \qquad \gamma = \hat{B} - \hat{N} + \hat{P}_B - \hat{P}_A$$

Proof. We first observe that we have the following relationships in $K(\mathcal{A})$.

$$\begin{aligned} \hat{P}_{MA} &= \hat{P}_M - \hat{P}_A \\ \hat{Q}_{NB} &= \hat{Q}_N - \hat{Q}_B \end{aligned} \tag{13.12}$$

$$\begin{aligned} \hat{B} &= \hat{Q}_B - \hat{P}_B \\ \hat{N} &= \hat{Q}_N - \hat{P}_N \end{aligned} \tag{13.13}$$

Now using Equation (13.12) and Equation (13.13) we can establish the case of γ as follows.

$$\begin{aligned} \gamma &= \hat{P}_N - \hat{P}_M + \hat{P}_{MA} - \hat{Q}_{NB} \\ &= \hat{P}_N - \hat{P}_M + (\hat{P}_M - \hat{P}_A) - (\hat{Q}_N - \hat{Q}_B) \\ &= (\hat{Q}_B - \hat{P}_B) - (\hat{Q}_N - \hat{P}_N) + \hat{P}_B - \hat{P}_A \\ &= \hat{B} - \hat{N} + \hat{P}_B - \hat{P}_A \end{aligned}$$

For n_3 we will first simplify n_2. Using the facts that $\hat{M} = \hat{Q}_M - \hat{P}_M$ and $\hat{N} = \hat{Q}_N - \hat{P}_N$ one can check that we have the following.

$$\begin{aligned} n_2 &= \langle \hat{P}_M, \hat{M}\rangle + \langle \hat{P}_N, \hat{N}\rangle - (\hat{P}_N, \hat{M}) - \langle \hat{P}_M, \hat{Q}_N\rangle + \langle \hat{Q}_M, \hat{P}_N\rangle \\ &= \langle \hat{P}_M - \hat{P}_N, \hat{M} - \hat{N}\rangle \end{aligned} \tag{13.14}$$

Substituting Equation (13.14) into n_3 and using Equation (13.12), Equation (13.13) and the fact that $\hat{A} - \hat{B} = \hat{M} - \hat{N}$ we establish the case of n_3 as follows.

$$\begin{aligned} n_3 &= \langle \hat{P}_M - \hat{P}_N, \hat{M} - \hat{N}\rangle - \langle \hat{P}_{MA} - \hat{Q}_{NB}, \hat{A} - \hat{B}\rangle \\ &= \langle \hat{P}_M - \hat{P}_N, \hat{M} - \hat{N}\rangle - \langle \hat{P}_M - \hat{P}_A - \hat{Q}_N + \hat{Q}_B, \hat{M} - \hat{N}\rangle \\ &= \langle \hat{Q}_N - \hat{P}_N - \hat{Q}_B + P_B + \hat{P}_A - P_B, \hat{M} - \hat{N}\rangle \\ &= \langle \hat{N} - \hat{B} + \hat{P}_A - \hat{P}_B, \hat{M} - \hat{N}\rangle \end{aligned}$$

□

13.4 Relationship with Quantized Enveloping Algebras

In this section we give the generic version of Bridgeland's realization of the whole quantized enveloping algebra from Theorem 6.3.1. We also give a slight modification of this isomorphism and show that it intertwines the shift functor involution of DH and the involution Σ on $\mathrm{U}_t(\mathfrak{g})$ that we introduced in Equation (5.6).

The following theorem is due to Deng and Chen [CD15] in the generic case and Bridgeland [Bri13] in the non-generic case.

Theorem 13.4.1 (Bridgeland, Deng, Chen)**.** There is a $\mathbb{C}(t)$-algebra isomorphism $R : \mathrm{U}_t(\mathfrak{g}) \to \mathrm{DH}$ given by the following.

$$X_i \mapsto X_{S_i} \qquad\qquad Y_i \mapsto -tY_{S_i} \qquad\qquad K_i^{\pm 1} \mapsto K_{\hat{S}_i}^{\pm 1}$$

Proof. This is Theorem 5.5 in [CD15] coupled with the facts that $X_{S_i} = E_{S_i}/a_{S_i}$ and $Y_{S_i} = F_{S_i}/a_{S_i}$. □

We now introduce a modified version of the isomorphism R from Theorem 13.4.1 and show that it intertwines the shift functor involution $*$ of DH and the involution Σ on $\mathrm{U}_t(\mathfrak{g})$ that we introduced in Equation (5.6). The following does not appear in the literature to my knowledge.

We begin with the following provisional algebra involution ω of the quantized enveloping algebra which we stress is *not* the same as the involution Σ given in Equation (5.6).

$$\omega(X_i) = -Y_i \qquad\qquad \omega(Y_i) = -X_i \qquad\qquad \omega(K_i) = K_i^{-1}$$

A glance at the generators and relations description of $\mathrm{U}_t(\mathfrak{g})$ from Definition 5.1.1 shows that ω does indeed give an algebra involution. Using ω we then modify Bridgeland's isomorphism to get a new isomorphism $\bar{R} : \mathrm{U}_t(\mathfrak{g}) \to \mathrm{DH}$ given by $\overline{R} := * \circ R \circ \omega$. The map $\bar{R}$ is indeed an isomorphism since each of the algebra homomorphisms in its definition are. One can check that $\overline{R}$ is determined by the following.

$$\bar{R}(X_i) = tX_{S_i} \qquad\qquad \bar{R}(Y_i) = -Y_{S_i} \qquad\qquad \bar{R}(K_i) = K_{\hat{S}_i} \tag{13.15}$$

A remark we will use in later chapters is that it is not hard to see that the isomorphism $\bar{R}$ descends to an isomorphism between the positive part of the quantized enveloping algebra $\mathrm{U}_t(\mathfrak{n}_+)$ and the copy of the generic Hall algebra H in DH spanned by the elements of the form X_L. Recall from Equation (5.6) that the algebra involution $\Sigma : \mathrm{U}_t(\mathfrak{g}) \to \mathrm{U}_t(\mathfrak{g})$ was defined via the following.

$$\Sigma(X_i) = -tY_i \qquad\qquad \Sigma(Y_i) = -t^{-1}X_i \qquad\qquad \Sigma(K_i) = K_i^{-1} \tag{13.16}$$

We end this section by showing that $\overline{R}$ intertwines the two involutions Σ and $*$.

Proposition 13.4.2. The following is a commutative diagram of $\mathbb{C}(t)$-algebra isomorphisms.

$$\begin{array}{ccc} \mathrm{U}_t(\mathfrak{g}) & \xrightarrow{\Sigma} & \mathrm{U}_t(\mathfrak{g}) \\ \big\downarrow{\scriptstyle \bar{R}} & & \big\downarrow{\scriptstyle \bar{R}} \\ \mathrm{DH} & \xrightarrow{*} & \mathrm{DH} \end{array}$$

Proof. This is immediate on using Equation (13.15), Equation (13.16) and the fact that $X^*_{S_i} = Y_{S_i}$, $Y^*_{S_i} = X_{S_i}$ and $K^*_{\hat{S}_i} = K_{-\hat{S}_i}$. $\square$

Chapter 14

Hopf Algebra Structure

In this chapter we endow the Bridgeland-Hall algebra with the structure of a $\mathbb{C}(t)$-Hopf algebra. We do so by giving an isomorphism between DH and the (reduced) Drinfeld double of the extended Abelian Hall algebra from Chapter 11. The Drinfeld double Hopf algebra structure then induces such a structure on DH. The results in this section follow from work due to Yanagida [Yan16] who identified Bridgeland-Hall algebras with Drinfeld doubles in the non-generic case.

The reason why this chapter is important for us is that in Part IV we will see that the Hopf algebra structure on DH descends to a Hopf algebra structure on the semi-classical Bridgeland-Hall algebra DH_{sc}. This in particular ensures that the spectrum of DH_{sc} is an algebraic group.

14.1 Drinfeld Double and Hopf Algebra Structure

We start off this section by showing that the generic localized Bridgeland-Hall DH_{loc} coincides with the Drinfeld double defined in Theorem 11.1.3. We then show that this identification descends to an isomorphism between the (reduced) Bridgeland-Hall algebra DH and the reduced Drinfeld double. We will end with an explicit description of the resulting Hopf algebra structure on DH induced from that of the reduced Drinfeld double. The reference for this section is [Yan16].

In Proposition 13.2.3 we identified DH_{loc} with $\mathrm{H}^{\geqslant 0}\otimes_{\mathbb{C}(t)}\mathrm{H}^{\leqslant 0}$ as vector spaces. Moreover the latter is precisely the vector space underlying the Drinfeld double (Hopf) algebra we had in Theorem 11.1.3. Via this identification we thus a priori have two different algebra structures on DH_{loc}. The following theorem says that these two algebra structures in fact coincide.

Theorem 14.1.1 (Yanagida)**.** The Drinfeld double algebra structure on $\mathrm{H}^{\geqslant 0} \otimes_{\mathbb{C}(t)} \mathrm{H}^{\leqslant 0}$ coincides with the Hall algebra structure on DH_{loc} under the following vector space identification from Proposition 13.2.3.

$$\mathrm{H}^{\geqslant 0} \otimes_{\mathbb{C}(t)} \mathrm{H}^{\leqslant 0} \to \mathrm{DH}_{loc}, \qquad E_A K_\alpha \otimes F_B K_\beta^* \mapsto E_A K_\alpha F_B K_\beta^* \tag{14.1}$$

Proof. Recall from Theorem 11.1.3 that there are two conditions determining the Drinfeld double algebra structure. We must verify that under Equation (14.1) these conditions are also satisfied by the algebra structure on DH_{loc}.

The first says that $\mathrm{H}^{\geqslant 0}$ and $\mathrm{H}^{\leqslant 0}$ are subalgebras of DH_{loc}. This was shown in Proposition 13.2.2. The only non-trivial requirement from Theorem 11.1.3 to check then is that under Equation (14.1) we have that Equation (11.15) holds in DH_{loc}. Recall that Equation (11.15) says that for all $x \in \mathrm{H}^{\geqslant 0}$ and $a \in \mathrm{H}^{\leqslant 0}$ the following identity holds in $\mathrm{H}^{\geqslant 0} \otimes_{\mathbb{C}(t)} \mathrm{H}^{\leqslant 0}$.

$$P(x_1, a_1) \cdot (1 \otimes a_2)(x_2 \otimes 1) = P(x_2, a_2) \cdot x_1 \otimes a_1 \tag{14.2}$$

Set $x = E_A K_\alpha$ and $a = F_B K^*_\beta$ in Equation (14.2) and substitute the formula for Green's pairing from Equation (11.14) and the coproduct Δ in Theorem 11.1.1. Applying Equation (14.1) to Equation (14.2) we are then required to check that the following holds in DH_{loc}.

$$\sum_{\substack{A_1,A_2\\B_1,B_1}} t^{\langle A_1,A_2\rangle+\langle B_1,B_2\rangle} f^A_{A_1,A_2} f^B_{B_1,B_2} P(E_{A_1}K_{\hat{A}_2+\alpha}, F_{B_2}K^*_\beta) F_{B_1} K^*_{\hat{B}_2+\beta} E_{A_2} K_\alpha \tag{14.3}$$

$$= \sum_{\substack{A_1,A_2\\B_1,B_1}} t^{\langle A_1,A_2\rangle+\langle B_1,B_2\rangle} f^A_{A_1,A_2} f^B_{B_1,B_2} P(E_{A_2}K_\alpha, F_{B_1}K^*_{\hat{B}_2+\beta}) E_{A_1} K_{\hat{A}_2+\alpha} F_{B_2} K^*_\beta \tag{14.4}$$

Using Equation (13.2) and Equation (11.14) we have the following identities.

$$K^*_{\hat{B}_2+\beta} E_{A_2} = t^{-(\hat{A}_2,\beta)} K^*_{\hat{B}_2} E_{A_2} K^*_\beta \tag{14.5}$$

$$K_{\hat{A}_2+\alpha} F_{B_2} = t^{-(\hat{B}_2,\alpha)} K_{\hat{A}_2} F_{B_2} K_\alpha \tag{14.6}$$

$$P(E_{A_1}K_{\hat{A}_2+\alpha}, F_{B_2}K^*_\beta) = t^{(\hat{A}_2+\alpha,\beta)} P(E_{A_1}, F_{B_2}) \tag{14.7}$$

$$P(E_{A_2}K_\alpha, F_{B_1}K^*_{\hat{B}_2+\beta}) = t^{(\alpha,\hat{B}_2+\beta)} P(E_{A_2}, F_{B_1}) \tag{14.8}$$

Substituting Equation (14.5) and Equation (14.7) into Equation (14.3) and Equation (14.6) and Equation (14.8) into Equation (14.4) and then cancelling the factor $t^{(\alpha,\beta)} K_\alpha K^*_\beta$ the equality of Equation (14.3) and Equation (14.4) is equivalent to the following.

$$\sum_{\substack{A_1,A_2\\B_1,B_1}} t^{\langle A_1,A_2\rangle+\langle B_1,B_2\rangle} f^A_{A_1,A_2} f^B_{B_1,B_2} P(E_{A_1}, F_{B_2}) F_{B_1} K^*_{\hat{B}_2} E_{A_2} \tag{14.9}$$

$$= \sum_{\substack{A_1,A_2\\B_1,B_1}} t^{\langle A_1,A_2\rangle+\langle B_1,B_2\rangle} f^A_{A_1,A_2} f^B_{B_1,B_2} P(E_{A_2}, F_{B_1}) E_{A_1} K_{\hat{A}_2} F_{B_2} \tag{14.10}$$

The non-generic version of *exactly* this equation can be found stated in Equation (2.1) of [Yan16] and is also proved there. To see why the non-generic case implies the generic case we will use the usual trick.

Note first that by definition $P(E_M, F_N) = a_M \delta_{M,N}$. We also observed in Section 13.1 that DH_{loc} has a set of basis vectors of the form $K_\gamma K^*_\delta [C_M \oplus C_N]$ and moreover the structure constants of the product on DH_{loc} in this basis were Laurent polynomials. Using Equation (13.4) and Equation (13.5) it follows that any product of elements of the form E, F, K and K^* can be expressed as $\mathbb{C}[t, t^{-1}]$-linear combination of the basis vectors of DH_{loc}.

The previous paragraph implies that Equation (14.9) and Equation (14.10) can be both be expressed as a $\mathbb{C}[t, t^{-1}]$-linear combination of the above basis vectors of DH_{loc}. Moreover the Laurent polynomial coefficients in these expansions are expressions in Hall polynomials.

Since Yanagida checked the equality of Equation (14.9) and Equation (14.10) in the non-generic case, Proposition 7.3.1 and Proposition 12.2.1 then imply that the corresponding Laurent polynomial coefficients in the expansions of Equation (14.9) and Equation (14.10) above coincide on setting $t = q^{1/2}$ for all q a prime power. Lemma 7.1.1 says then that such Laurent polynomials must coincide and this establishes the result. □

We also have a reduced version of the above theorem. Recall from Definition 11.1.2 that reduced Drinfeld double was given by the quotient of the Drinfeld double by the (Hopf) ideal $(K_\alpha \otimes K_\alpha^* - 1)$. Under the identification with DH_{loc} this ideal is sent to $(K_\alpha K_\alpha^* - 1)$ which precisely the algebra ideal defining the (reduced) Bridgeland-Hall algebra in Definition 13.1.4.

Corollary 14.1.1 (Yanagida)**.** The (reduced) Bridgeland-Hall algebra DH coincides with the reduced Drinfeld double as algebras.

The Bridgeland-Hall algebra DH was defined in Definition 13.1.3 only as an algebra. The upshot of combining Corollary 14.1.1, Theorem 11.1.3 and Definition 11.1.2 is that DH has the structure of a $\mathbb{C}(t)$-Hopf algebra.

Corollary 14.1.2. The Bridgeland-Hall algebra DH is a $\mathbb{C}(t)$-Hopf algebra with coproduct Δ, antipode S and counit ε given by the following formulas.

$$\begin{aligned}
\Delta : \mathrm{DH} &\to \mathrm{DH} \otimes_{\mathbb{C}(t)} \mathrm{DH}\\
E_L &\mapsto \sum_{M,N \in \mathrm{Iso}(\mathcal{A})} t^{\langle \hat{M}, \hat{N}\rangle} f^L_{M,N} \cdot E_M K_{\hat{N}} \otimes E_N\\
F_L &\mapsto \sum_{M,N \in \mathrm{Iso}(\mathcal{A})} t^{\langle \hat{M}, \hat{N}\rangle} f^L_{M,N} \cdot F_N \otimes F_M K_{-\hat{N}}\\
K_\alpha &\mapsto K_\alpha \otimes K_\alpha
\end{aligned}$$

$$\begin{aligned}
S : \mathrm{DH} &\to \mathrm{DH}\\
E_M &\mapsto \delta_{M,0} + \sum_{k \geqslant 1} (-1)^k \sum_L \sum_{\substack{L_1,\dots,L_k\\ \neq 0}} t^{2\sum_{i<j} \langle \hat{L}_i, \hat{L}_j\rangle} [e]^L_{L_1,\dots,L_k} [f]^M_{L_1,\dots,L_k} \cdot K_{-\hat{M}} E_L\\
F_M &\mapsto \delta_{M,0} + \sum_{k \geqslant 1} (-1)^k \sum_L \sum_{\substack{L_1,\dots,L_k\\ \neq 0}} [e]^L_{L_k,\dots,L_1} [f]^M_{L_1,\dots,L_k} \cdot F_L K_{\hat{M}}\\
K_\alpha &\mapsto K_{-\alpha}
\end{aligned}$$

The counit is determined by $E_L \mapsto \delta_{L,0}$, $F_L \mapsto \delta_{L,0}$ and $K_\alpha \mapsto 1$.

Chapter 15

Integral Forms and Quasi-Classical Limit

This chapter deals with two natural integral forms of DH along with the quasi-classical Bridgeland-Hall algebra. These two integral forms will roughly speaking be the Bridgeland-Hall versions of the extension and flag counting integral forms of H that we had in Part II. The quasi-classical Bridgeland-Hall algebra is the analogue of the quasi-classical Hall algebra from Section 10.2.

In Section 15.1 we define DH_{ex} the extension counting integral form of DH. This is defined in a completely analogous way to how H_{ex} was defined in Chapter 9. In spite of its simple definition this integral form does not seem to have been considered before in the Bridgeland-Hall algebra setting.

Later on in Chapter 16 we will give the first main result of this thesis by proving that DH_{ex} is isomorphic to the Poisson integral form of the quantized enveloping algebra. The $t = 1$ limit of DH_{ex} is what we will call the semi-classical Bridgeland-Hall algebra and will be explored in-depth from a Hall algebraic point of view in Part IV.

Section 15.2 concerns DH_{fl} the flag counting integral form of DH. Unlike DH_{ex} this integral form is not defined in an entirely analogous way to how H_{fl} was in Chapter 10. We will explain what goes wrong and then give the correct definition of DH_{fl} due to Deng and Chen [CD15]. In this chapter we will also explain how the flag counting integral form is isomorphic to $\mathrm{U}_t^{Res}(\mathfrak{g})$ the restricted integral form of the quantized enveloping algebra.

Finally Section 15.3 is a short section on the quasi-classical Bridgeland-Hall algebra. This is the $t = 1$ limit of DH_{fl} and is isomorphic to the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$.

15.1 Extension Counting Integral Form

In this section we define and discuss the extension counting integral form DH_{ex} of DH. This is the $\mathbb{Z}_2$-graded version of the extension counting integral H_{ex} of the generic Hall algebra H that we had in Section 9.1. The main feature of DH_{ex}, as we shall see in Section 17.1, is that its semi-classical limit is the algebra of functions on a Poisson-Lie group.

Definition 15.1.1. Define the extension counting integral form of DH to be the $\mathbb{C}[t, t^{-1}]$-submodule DH_{ex} spanned by the elements of the form $[M_\bullet]$.

We should make sure that DH_{ex} really is an integral form, as defined in Definition 4.1.1. Since by Definition 13.1.1 the structure constants of the product on $\mathrm{H}(\mathcal{C})$ (in the basis given by the elements $[M_\bullet]$) are Laurent polynomials, then by Equation (13.3) we must have that DH_{ex} is closed under the product.

To see that DH_{ex} is free as a $\mathbb{C}[t,t^{-1}]$-module note that by Lemma 13.1.2 the $\mathbb{C}[t,t^{-1}]$-span of the elements $[M_\bullet]$ coincides with the $\mathbb{C}[t,t^{-1}]$-span of the elements $K_\alpha[C_M \oplus C_N^*]$. Freeness follows from the fact which we observed in Section 13.1 that the latter elements form a $\mathbb{C}(t)$-basis of DH. Finally multiplication clearly an induces and isomorphism $\mathbb{C}(t) \otimes_{\mathbb{C}[t,t^{-1}]} \mathrm{DH}_{ex} \to \mathrm{DH}$.

Remark 15.1.1. Setting $t = q^{\frac{1}{2}}$ in DH_{ex} recovers the non-generic Bridgeland-Hall algebra that we had in Chapter 6. In particular for any q a prime power we have the following isomorphism of $\mathbb{C}$-algebras.

$$\mathrm{DH}_{ex}/(t - q^{\frac{1}{2}}) \to \mathrm{DH}_q, \qquad [M_\bullet] \mapsto [M_{\bullet q}] \text{ and } t \mapsto q^{\frac{1}{2}} \tag{15.1}$$

We make some remarks as to why this is true. Comparing Definition 6.3.1 and Definition 6.3.2 with Definition 15.1.1, Definition 13.1.1 and Equation (13.3) shows that DH_{ex} and DH_q have completely analogous definitions. That Equation (15.1) is an isomorphism of algebras follows from the following two facts. First the Hall polynomials from Proposition 12.2.1 determining the product on DH_{ex} specialize at $t = q^{\frac{1}{2}}$ to give the structure constants determining the product on the non-generic Bridgeland-Hall algebra as given in Equation (6.10). The second is that since DH_{ex} is free as a $\mathbb{C}[t,t^{-1}]$-algebra then $\mathrm{DH}_{ex}/(t - q^{\frac{1}{2}})$ is isomorphic to DH_q as a $\mathbb{C}$-vector space.

In Chapter 16 we will show that DH_{ex} is isomorphic to the Poisson integral form of the quantized enveloping algebra $\mathrm{U}_t^{Poiss}(\mathfrak{g})$. A nice feature of the Hall algebra approach to integral forms is that the definition of DH_{ex} is almost tautological. This contrasts with the definition of $\mathrm{U}_t^{Poiss}(\mathfrak{g})$ in Section 5.5 which relied on the non-trivial machinery of Lusztig's braid group action which we had to introduce in Section 5.2.

How does H_{ex} relate to DH_{ex}? Recall from Definition 9.1.1 that the extension counting integral form H_{ex} of the generic Hall algebra H was defined as the $\mathbb{C}[t,t^{-1}]$-span of elements of the form E_M. In Section 13.2 we observed that the Bridgeland-Hall algebra DH contains two copies of the generic Hall algebra H as $\mathbb{C}(t)$-subalgebras. One was given by the $\mathbb{C}(t)$-spans of elements of the form E_M. Applying the shift involution gives the other which is spanned by the elements F_N. Since the elements E_M and F_N both lie in DH_{ex} we thus have that DH_{ex} contains two copies of H_{ex} as $\mathbb{C}[t,t^{-1}]$-subalgebras.

The triangular decomposition of DH from Proposition 13.2.4 descends to one of DH_{ex}. In the following we use the previous paragraph to consider H_{ex} as a subspace of DH_{ex} in two ways.

Proposition 15.1.2. Multiplication induces the following isomorphism of $\mathbb{C}[t,t^{-1}]$-modules.

$$\mathrm{H}_{ex} \otimes \mathbb{C}[t,t^{-1}][K(\mathcal{A})] \otimes \mathrm{H}_{ex} \to \mathrm{DH}_{ex}, \qquad E_A \otimes K_\alpha \otimes F_B \mapsto E_A K_\alpha F_B$$

Proof. This follows by identical reasoning to the proof of Lemma 4.8. in [Bri13]. □

We end by observing that the $\mathbb{C}(t)$-Hopf algebra structure on DH descends to a $\mathbb{C}[t, t^{-1}]$-Hopf algebra structure on DH_{ex}. To see this first note that elements of the form E_M, F_N and K_α all lie in DH_{ex}. By Corollary 14.1.2 the Hopf algebra structure on DH was determined by the values of the coproduct, antipode and counit on these elements. Since the formulas in Corollary 14.1.2 for these operations all have Laurent polynomials as coefficients then the Hopf algebra structure must descend to DH_{ex}.

15.2 Flag Counting Integral Form

In this section we define DH_{fl} the flag counting integral form of the Bridgeland-Hall algebra. The correct definition was given by Deng and Chen in [CD15] who used it to recover the whole simple Lie algebra $\mathfrak{g}$ from DH. The main feature of DH_{fl} is that it is isomorphic to the restricted integral form of the quantized enveloping algebra and that its $t = 1$ limit is isomorphic to $\mathrm{U}(\mathfrak{g})$.

Unfortunately the flag counting integral form of DH does not seem particularly easy to work with in contrast with the extension counting integral forms of H and DH or indeed the flag counting integral form of H. Another trait of DH_{fl} is that it is not defined in the obvious way that one would expect.

We begin by explaining what goes wrong with the obvious attempt to define a flag counting integral form of DH. We then discuss the correct definition and explain how it is isomorphic to the restricted integral form of the quantized enveloping algebra. The reference for this section is [CD15].

Recall that in the Abelian case that there were two sets of basis vectors of the generic Hall algebra H given by the elements E_L and $X_L := E_L/a_L$ respectively. The $\mathbb{C}[t, t^{-1}]$-span of the former gave rise to the extension counting integral form of H while the span of the latter gave the flag counting one.

Similarly for Bridgeland-Hall algebras it was the $\mathbb{C}[t, t^{-1}]$-span of the elements $[L_\bullet]$ which defined the extension counting integral form of DH_{ex}. The obvious way to try and define DH_{fl} then is to define it to be the $\mathbb{C}[t, t^{-1}]$-span of the elements $[L_\bullet]/a_{L_\bullet}$.

This isn't quite the correct definition however. To see why, let's provisionally define DH'_{fl} to be the $\mathbb{C}[t, t^{-1}]$-submodule of DH spanned by the elements $[L_\bullet]/a_{L_\bullet}$ where $L_\bullet \in \mathrm{Iso}(\mathcal{C})$. It turns out that this subspace is too large and includes certain elements corresponding to acyclic complexes which prevent it from having a good $t = 1$ limit.

In particular to see what goes awry first note that if $P \in \mathrm{Iso}(\mathcal{A})$ is projective indecomposable then a simple calculation shows that $a_{K_P} = a_{K_P^*} = t^2 - 1$. Now if DH'_{fl} was an integral form of DH then in particular the following element would lie in DH'_{fl}.

$$\frac{[K_P]}{a_{K_P}}\frac{[K_P^*]}{a_{K_P^*}} = \frac{[K_P]}{(t^2-1)}\frac{[K_P^*]}{(t^2-1)} = \frac{1}{(t^2-1)^2}$$

Obviously one is going to run into problems on setting $t = 1$ and so DH'_{fl} would not have a good $t = 1$ limit.

The correct way to define the flag counting integral form of DH is to take the smaller $\mathbb{C}[t, t^{-1}]$-subalgebra generated by the two copies of H_{fl} along with the copy of $\mathbb{C}[t, t^{-1}][K(\mathcal{A})]$.

Definition 15.2.1. Define the flag counting integral form of the Bridgeland-Hall algebra to be the $\mathbb{C}[t,t^{-1}]$-subalgebra DH_{fl} generated by the elements X_L, Y_L and $K^{\pm 1}_{\hat{S}_i}$ where $L \in \mathrm{Iso}(\mathcal{A})$ and $1 \leqslant i \leqslant r$.

We will now discuss a smaller set of generators of DH_{fl} before establishing that it really is an integral form of DH. In particular let's show that DH_{fl} is also generated by the elements $X_{S_i^k}$, $Y_{S_i^k}$ and $K^{\pm 1}_{\hat{S}_i}$ where $k \geqslant 0$ and $1 \leqslant i \leqslant r$. Note this claim is equivalent to showing that H_{fl} is generated by the elements $X_{S_i^k}$, which we establish using the following argument due to Reineke which can be found in Lemma 4.4 of [Rei03].

First choose an ordering of vertices $1,\ldots,r$ of $\vec{Q}$ in such a way that $i > j$ if there is an arrow $i \to j$. Such an ordering is always possible for a simply-laced quiver. One can then check that any $L \in \mathrm{Iso}(\mathcal{A})$ then has a unique filtration of subobjects $0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_r = L$ such that $L_k/L_{k-1} = S_k^{l_k}$ for any $1 \leqslant k \leqslant r$ and where $l_k \in \mathbb{Z}_{\geqslant 0}$. It follows that X_L is equal to $X_{S_1^{l_1}} \cdots X_{S_r^{l_r}}$ up to some power of t and this establishes the claim.

Now the fact that DH_{fl} really is an integral form of DH follows from the following proposition along with the fact that $\mathrm{U}_t^{Res}(\mathfrak{g})$ is an integral form of $\mathrm{U}_t(\mathfrak{g})$.

Proposition 15.2.1 (Deng, Chen)**.** The isomorphism $R : \mathrm{U}_t(\mathfrak{g}) \to \mathrm{DH}$ restricts to an isomorphism of $\mathbb{C}[t,t^{-1}]$-algebras between $\mathrm{U}_t^{Res}(\mathfrak{g})$ and DH_{fl}.

Proof. Recall from Definition 5.4.1 that the restricted integral form of the quantized enveloping algebra is defined as the $\mathbb{C}[t,t^{-1}]$-subalgebra $\mathrm{U}_t^{Res}(\mathfrak{g})$ of $\mathrm{U}_t(\mathfrak{g})$ generated by the divided powers $X_i^{(k)}$, $Y_i^{(k)}$ along with the elements $K_i^{\pm 1}$. Moreover recall from Theorem 13.4.1 that the isomorphism $R : \mathrm{U}_t(\mathfrak{g}) \to \mathrm{DH}$ was given by the following.

$$X_i \mapsto X_{S_i} \qquad\qquad Y_i \mapsto -tY_{S_i} \qquad\qquad K_i^{\pm 1} \mapsto K^{\pm 1}_{\hat{S}_i}$$

The result then follows from the fact that, up to a factor of t, we have that $X_{S_i}^{(k)}$ is equal to $X_{S_i^k}$ and $Y_{S_i}^{(k)}$ is equal to $Y_{S_i^k}$. □

We remark that the above proof also holds for the modified isomorphism $\bar{R} : \mathrm{U}_t(\mathfrak{g}) \to \mathrm{DH}$ that we introduced in Section 13.4. It is also easy to see that $\bar{R} : \mathrm{U}_t(\mathfrak{g}) \to \mathrm{DH}$ restricts to an isomorphism $\bar{R} : \mathrm{U}_t(\mathfrak{n}_+) \to \mathrm{H}$ and that this isomorphism further restricts to a $\mathbb{C}[t,t^{-1}]$-algebra isomorphism between $\mathrm{U}_t^{Res}(\mathfrak{n}_+)$ and H_{fl}.

Note that we have used the fact that $\mathrm{U}_t^{Res}(\mathfrak{g})$ is an integral form of $\mathrm{U}_t(\mathfrak{g})$ to establish that DH_{fl} is an integral form of DH. By contrast in Section 15.1 we saw that the extension counting integral form of DH was tautologically an integral form.

It would be nice to find an elementary proof of the fact that DH_{fl} is an integral form using the language of Bridgeland-Hall algebras and without invoking the isomorphism between DH and $\mathrm{U}_t(\mathfrak{g})$. However after some thought we could not find such a simple proof.

On the other hand, a point worth making is that Proposition 15.2.1 was easy to prove. That the analogous statement to Proposition 15.2.1 holds between $\mathrm{U}_t^{Poiss}(\mathfrak{g})$ and DH_{ex} is more involved and is the subject of Chapter 16.

15.3 The Quasi-Classical Algebra

Following Deng and Chen, in this short section we define the quasi-classical Bridgeland-Hall algebra and establish that it is isomorphic to $\mathrm{U}(\mathfrak{g})$. Unlike the semi-classical Bridgeland-Hall algebra which is the subject of Part IV we will not explore the structure of the quasi-classical Bridgeland-Hall algebra in great detail and include this section only for completeness. The reference for this section is [CD15].

The quasi-classical Bridgeland-Hall algebra is defined as follows. It may be instructive to compare Definition 15.3.1 to the definition of U_{qc} in Section 5.4.

Definition 15.3.1. Define the quasi-classical Bridgeland-Hall algebra DH_{qc} to be the quotient algebra of DH_{fl} by the ideal generated by $t-1$ and the elements $K_{\hat{S}_i} - 1$ where $1 \leqslant i \leqslant r$.

We then have the following theorem which allowed Deng and Chen to recover the whole simple Lie algebra $\mathfrak{g}$ from the generic Bridgeland-Hall algebra. In Chapter 18 we will give an alternative way to recover this Lie algebra using the semi-classical Bridgeland-Hall algebra.

Theorem 15.3.1 (Deng, Chen, Lusztig)**.** There is an isomorphism of $\mathbb{C}$-algebras $\mathrm{U}(\mathfrak{g}) \to \mathrm{DH}_{qc}$ given by the following where $1 \leqslant i \leqslant r$.

$$e_i \mapsto \overline{X}_{S_i} \qquad\qquad f_i \mapsto -\overline{Y}_{S_i} \qquad\qquad h_i \mapsto \overline{[Y_{S_i}, X_{S_i}]}$$

Proof. This follows from Proposition 15.2.1 due to Deng and Chen along with Lusztig's Theorem 5.4.1. □

Chapter 16

Identification of Integral Forms

In this chapter we establish the first main result of this thesis: the isomorphism $\overline{R} : \mathrm{U}_t(\mathfrak{g}) \to \mathrm{DH}$ from Section 13.4 descends to an isomorphism between $\mathrm{U}_t^{Poiss}(\mathfrak{g})$ the Poisson integral form of $\mathrm{U}_t(\mathfrak{g})$ and DH_{ex} the extension counting integral form of DH.

One of the merits of this result is that DH_{ex} admits an almost tautological definition whereas $\mathrm{U}_t^{Poiss}(\mathfrak{g})$ is non-trivial to define, depending as it does on the construction of quantum root vectors via the machinery of Lusztig's braid group action. Moreover it seems to us that $\mathrm{U}_t^{Poiss}(\mathfrak{g})$ is slightly messy to work with whereas proofs involving DH_{ex} and its $t = 1$ limit seem more straightforward.

Let us give an outline of the proof method and then explain which steps correspond to which sections. Recall that the Poisson integral form $\mathrm{U}_t^{Poiss}(\mathfrak{g})$ was defined in Definition 5.5.1 as the $\mathbb{C}[t, t^{-1}]$-subalgebra of $\mathrm{U}_t(\mathfrak{g})$ generated by the rescaled quantum root vectors E_β, F_β along with $K_i^{\pm 1}$. The rescaled quantum root vectors were defined via Lusztig's braid group action and moreover depended on a choice of reduced decomposition of the longest element of the Weyl group w_0.

On the Bridgeland-Hall algebra side of things we have similar elements which are naturally assigned to each positive root β. Indeed Gabriel's Theorem 2.2.2 says there is a unique indecomposable I_β associated to each β and so we have corresponding elements E_{I_β} and F_{I_β} in DH. Moreover one can show that the elements E_{I_β}, F_{I_β} and $K_{\hat{S}_i}^{\pm 1}$ generate DH_{ex} as a $\mathbb{C}[t, t^{-1}]$-subalgebra of DH.

The idea for the proof then is to identify the elements E_β and F_β and $\mathrm{U}_t(\mathfrak{g})$ with (a scalar multiple of) E_{I_β} and F_{I_β} in DH under the isomorphism $\overline{R} : \mathrm{U}_t(\mathfrak{g}) \to \mathrm{DH}$. This will involve making a suitable choice of reduced decomposition of w_0. It will then follow that $\mathrm{U}_t^{Poiss}(\mathfrak{g})$ and DH_{ex} are isomorphic.

Since the quantum root vectors E_β and F_β are defined via Lusztig's braid group action on $\mathrm{U}_t(\mathfrak{g})$, the way to understand where they map under $\overline{R}$ involves understanding how this braid group action behaves on the Bridgeland-Hall algebra side of things. It is known from [SVdB99], [XY01] and [Gor] that in the non-generic case the braid group action corresponds to certain isomorphisms of Bridgeland-Hall algebras induced by BGP reflection functors.

Thus in Section 16.1 we will give an overview of these reflection functor induced isomorphisms, extending the details to the generic case in the process. Using this technology, in Section 16.2 we then describe the proof that $\mathrm{U}_t^{Poiss}(\mathfrak{g})$ and DH_{ex} are isomorphic as $\mathbb{C}[t, t^{-1}]$-algebras.

16.1 Reflection Functor Induced Isomorphisms of DH

In this section we discuss how BGP reflection functors induce isomorphisms of generic Bridgeland-Hall algebras. In addition we will describe how these isomorphisms relate to Lusztig's automorphisms T_i that were defined in Section 5.2. The results in this section are due to Gorsky [Gor] and Ringel [Rin96].

Let $\mathrm{Iso}(\mathcal{A}^{\vec{Q}}\langle i\rangle) \subset \mathrm{Iso}(\mathcal{A})$ be the subset consisting of representations of $\vec{Q}$ which do not have S_i as a direct summand. It follows from Theorem 2.4.1 that for each sink i of $\vec{Q}$ the BGP reflection functors induce mutually inverse bijections of the following form.

$$\sigma_i^+ : \mathrm{Iso}(\mathcal{A}^{\vec{Q}}\langle i\rangle) \to \mathrm{Iso}(\mathcal{A}^{\sigma_i\vec{Q}}\langle i\rangle) \tag{16.1}$$

$$\sigma_i^- : \mathrm{Iso}(\mathcal{A}^{\sigma_i\vec{Q}}\langle i\rangle) \to \mathrm{Iso}(\mathcal{A}^{\vec{Q}}\langle i\rangle) \tag{16.2}$$

Here $\sigma_i\vec{Q}$ is the quiver obtained from $\vec{Q}$ by reversing all arrows incident at i. In particular the above bijections take indecomposables to indecomposables and for any $L \in \mathrm{Iso}(\mathcal{A})$ the class of $\sigma_i^{\pm}(L)$ in $K(\mathcal{A})$ is $s_i(\hat{L})$ where s_i denotes a simple reflection.

We then have the following theorem, due to Gorsky in the non-generic case, relating BGP reflection functors to Bridgeland-Hall algebras.

Theorem 16.1.1 (Gorsky). Let i be a sink for $\vec{Q}$ or equivalently a source for $\sigma_i\vec{Q}$. The following determine mutually inverse $\mathbb{C}(t)$-algebra isomorphisms with the property that $\mathcal{T}_i^{\pm 1} \circ * = * \circ \mathcal{T}_i^{\pm 1}$ where $*$ is the shift involution on DH.

$$\begin{aligned} \mathcal{T}_i : \mathrm{DH}(\vec{Q}) &\to \mathrm{DH}(\sigma_i\vec{Q}) \\ E_{S_i} &\mapsto t^{-1}F_{S_i}K_{\hat{S}_i} \\ E_M &\mapsto E_{\sigma_i^+ M} \\ K_\alpha &\mapsto K_{s_i(\alpha)} \end{aligned} \qquad \begin{aligned} \mathcal{T}_i^{-1} : \mathrm{DH}(\sigma_i\vec{Q}) &\to \mathrm{DH}(\vec{Q}) \\ E_{S_i} &\mapsto tF_{S_i}K_{-\hat{S}_i} \\ E_N &\mapsto E_{\sigma_i^- N} \\ K_\alpha &\mapsto K_{s_i(\alpha)} \end{aligned}$$

Here M and N are required to have no S_i as a direct summand and s_i are simple reflections.

Proof. We first note that $\mathcal{T}_i$ and $\mathcal{T}_i^{-1}$ as defined above induce the following well-defined $\mathbb{C}[t,t^{-1}]$-linear maps on restricting to the extension counting integral form of the Bridgeland-Hall algebra.

$$\mathcal{T}_i : \mathrm{DH}_{ex}(\vec{Q}) \to \mathrm{DH}_{ex}(\sigma_i\vec{Q}) \qquad \mathcal{T}_i^{-1} : \mathrm{DH}_{ex}(\sigma_i\vec{Q}) \to \mathrm{DH}_{ex}(\vec{Q})$$

We will prove that these give mutually inverse isomorphisms of $\mathbb{C}[t,t^{-1}]$-algebras with the result following by tensoring along $\mathbb{C}(t) \otimes_{\mathbb{C}[t,t^{-1}]} (-)$.

It is easy to check using the properties of the bijections in Equation (16.1) and Equation (16.2) that the formulas defining $\mathcal{T}_i$ and $\mathcal{T}_i^{-1}$ give mutually inverse linear maps. Moreover if one of the maps is an algebra homomorphism then the other one is necessarily is. The only thing we need to check then is that $\mathcal{T}_i : \mathrm{DH}_{ex}(\vec{Q}) \to \mathrm{DH}_{ex}(\sigma_i\vec{Q})$ is a $\mathbb{C}[t,t^{-1}]$-algebra homomorphism.

Theorem 9.27 and Proposition 9.23 of [Gor] says that the same formulas for $\mathcal{T}_i$ above except with $t = q^{\frac{1}{2}}$ give a $\mathbb{C}$-algebra isomorphism of non-generic Bridgeland-Hall algebras $\mathrm{DH}_q(\vec{Q}) \to \mathrm{DH}_q(\sigma_i\vec{Q})$.

Recall Remark 15.1.1 says that DH_{ex} specializes to DH_q on setting $t = q^{\frac{1}{2}}$. We thus have that for any $x, y \in \mathrm{DH}_{ex}(\vec{Q})$ the image of $\mathcal{T}_i(xy) - \mathcal{T}_i(x)\mathcal{T}_i(y)$ in $\mathrm{DH}_{ex}(\sigma_i\vec{Q})/(t - q^{\frac{1}{2}})$ vanishes for all q a prime power. Combining this with Lemma 7.1.1 on the vanishing of Laurent polynomials along with the fact that $\mathrm{DH}_{ex}(\sigma_i\vec{Q})$ is free as a $\mathbb{C}[t, t^{-1}]$-module gives the result. □

We would like to show how the above isomorphisms relate to Lusztig's automorphisms T_i of the quantized enveloping algebra which were defined in Section 5.2. To do so we will need to know the value of $\mathcal{T}_i$ on certain elements. For i a sink and any vertex $j \neq i$ we have the following.

$$\mathcal{T}_i(X_{S_i}) = t^{-1}Y_{S_i}K_{\hat{S}_i} \qquad\qquad \mathcal{T}_i(X_{S_j}) = X_{\sigma_i^+ S_j} \tag{16.3}$$

This follows by rescaling the formulas defining $\mathcal{T}_i$ in Theorem 16.1.1 by dividing across by $t^2 - 1$. Indeed by definition $X_I := E_I/(t^2 - 1)$ and $Y_I := F_I/(t^2 - 1)$ since for any I indecomposable we have $a_I = t^2 - 1$. The only thing we need to be sure of is that if $j \neq i$ then $\sigma_i^+ S_j$ is in fact indecomposable. However this follows from the fact that the bijection Equation (16.1) preserves indecomposables. To prove Theorem 16.1.2 below we will need the following key Lemma due to Ringel.

Lemma 16.1.1 (Ringel)**.** Let i be a source for $\sigma_i\vec{Q}$. Then for $i \neq j$ the following identity holds in $\mathrm{DH}(\sigma_i\vec{Q})$.

$$X_{\sigma_i^+ S_j} = \sum_{\mu+\nu=-a_{ij}} (-1)^\mu t^\nu X_{S_i}^{(\nu)} X_{S_j} X_{S_i}^{(\mu)} \tag{16.4}$$

Proof. A proof of this can be found in Proposition 3 of [Rin96]. Ringel uses slightly different notation. Define $\langle X_L \rangle := t^{-dim L + \epsilon(L)} X_L$ where $\epsilon(L) = \mathrm{dimEnd}(L)$. The identity proved in [Rin96] is then the following.

$$\langle X_{\sigma_i^+ S_j} \rangle = \sum_{\mu+\nu=-a_{ij}} (-1)^\mu t^{-\mu} X_{S_i}^{(\nu)} X_{S_j} X_{S_i}^{(\mu)} \tag{16.5}$$

Since $\sigma_i^+ S_j$ is indecomposable and our quiver is assumed to be simply-laced then it has a one dimensional endomorphism algebra. Moreover by the properties of the bijection in Equation (16.1) the class of $\sigma_i^+ S_j$ in the Grothendieck group is $\hat{S}_j - a_{ij}\hat{S}_i$ and so its dimension is $1 - a_{ij}$. It is easy to see from these observations that Equation (16.4) is equivalent to Equation (16.5). □

Armed with the above identities we can prove that, under the modified isomorphism $\bar{R}$ between $\mathrm{U}_t(\mathfrak{g})$ and DH introduced in Equation (13.15), the isomorphisms $\mathcal{T}_i^{\pm 1}$ commute with Lusztig's automorphisms $T_i^{\pm 1}$.

Theorem 16.1.2. Let i be a sink for $\vec{Q}$. Then the following diagrams commute.

$$\begin{array}{ccc} U_t(\mathfrak{g}) & \xrightarrow{T_i} & U_t(\mathfrak{g}) \\ \downarrow \bar{R} & & \downarrow \bar{R} \\ \mathrm{DH}(\vec{Q}) & \xrightarrow{\mathcal{T}_i} & \mathrm{DH}(\sigma_i\vec{Q}) \end{array} \qquad\qquad \begin{array}{ccc} U_t(\mathfrak{g}) & \xrightarrow{T_i^{-1}} & U_t(\mathfrak{g}) \\ \downarrow \bar{R} & & \downarrow \bar{R} \\ \mathrm{DH}(\sigma_i\vec{Q}) & \xrightarrow{\mathcal{T}_i^{-1}} & \mathrm{DH}(\vec{Q}) \end{array}$$

Proof. Commutativity of the right-hand diagram follows from that of the left-hand one. We need to check then that this diagram commutes when applied to the generators X_k, Y_k and K_k of the quantized enveloping algebra where $1 \leqslant k \leqslant r$.

We can cut down on the number of calculations required as follows. First recall that by Proposition 13.4.2 we have $\bar{R} \circ \Sigma = * \circ \bar{R}$ where Σ is the involution of $\mathrm{U}_t(\mathfrak{g})$ defined in Equation (5.6). Moreover Σ commutes with the T_i by Proposition 5.3.1 and $*$ commutes with the $\mathcal{T}_i$ by Theorem 16.1.1. It is easy to see that these facts imply that we need only check the statement for the generators X_j and K_j.

We will use the formulas in Theorem 5.2.1 for Lusztig's braid group action T_i. The definition of $\bar{R}$ can be found in Equation (13.15). Theorem 16.1.1 and Equation (16.3) tell us the action of $\mathcal{T}_i$. For the convenience of the reader we reproduce the relevant formulas here. For any vertices k, j with $j \neq i$ we have the following.

$$\begin{aligned} \bar{R}(K_k) &= K_{\hat{S}_k} & \mathcal{T}_i(K_{\hat{S}_k}) &= K_{s_i(\hat{S}_k)} & T_i(K_k) &= K_{s_i(\alpha_k)} \\ \bar{R}(X_k) &= tX_{S_k} & \mathcal{T}_i(X_{S_i}) &= t^{-1} Y_{S_i} K_{\hat{S}_i} & T_i(X_i) &= -Y_i K_i \\ \bar{R}(Y_k) &= -Y_{S_k} & \mathcal{T}_i(X_{S_j}) &= X_{\sigma_i^+ S_j} & T_i(X_j) &= \sum_{\mu+\nu=-a_{ij}} (-1)^\mu t^{-\mu} X_i^{(\nu)} X_j X_i^{(\mu)} \end{aligned}$$

For the generators K_j and X_i we then have the following identities.

$$\begin{aligned} \mathcal{T}_i \circ \bar{R}(K_k) = \mathcal{T}_i(K_{\hat{S}_k}) &= K_{s_i(\hat{S}_k)} \\ &= \bar{R}(K_{s_i(\alpha_k)}) \\ &= \bar{R} \circ T_i(K_k) \end{aligned} \qquad \begin{aligned} \mathcal{T}_i \circ \bar{R}(X_i) = \mathcal{T}_i(tX_{S_i}) &= Y_{S_i} K_{\hat{S}_i} \\ &= \bar{R}(-Y_i K_i) \\ &= \bar{R} \circ T_i(X_i) \end{aligned}$$

For the generators X_j where $j \neq i$ the following equalities hold.

$$\begin{aligned} \mathcal{T}_i \circ \bar{R}(X_j) = \mathcal{T}_i(tX_{S_j}) &= tX_{\sigma_i^+ S_j} \\ &= \sum_{\mu+\nu=-a_{ij}} (-1)^\mu t^{\nu+1} X_i^{(\nu)} X_j X_i^{(\mu)} && [\text{Lemma 16.1.1}] \\ &= \sum_{\mu+\nu=-a_{ij}} (-1)^\mu t^{-\mu+(1-a_{ij})} X_i^{(\nu)} X_j X_i^{(\mu)} && [\nu = -\mu - a_{ij}] \\ &= \sum_{\mu+\nu=-a_{ij}} (-1)^\mu t^{-\mu} \bar{R}(X_i^{(\nu)} X_j X_i^{(\mu)}) && [1 - a_{ij} = \nu + 1 + \mu] \\ &= \bar{R} \circ T_i(X_j) \end{aligned}$$

□

16.2 The Identification

In this section we prove that $\mathrm{U}_t^{Poiss}(\mathfrak{g})$ the Poisson integral form of the quantized enveloping algebra is isomorphic to DH_{ex} the extension counting integral form of the generic Bridgeland-Hall algebra.

We will first give a relationship between the quantum root vectors E_β and F_β of $\mathrm{U}_t(\mathfrak{g})$ and the elements E_{I_β} and F_{I_β} of DH where I_β are indecomposables corresponding to positive roots β. We then use this relationship to establish our result. In this section Proposition 16.2.2 and Theorem 16.2.3 are Bridgeland-Hall algebra versions of results due to Ringel [Rin96] in the Abelian case.

Recall from Section 5.2 that the definition of the quantum root vectors E_β and F_β depended on a choice of reduced decomposition of w_0 the longest element of the Weyl group. This contrasts with the fact that the elements E_{I_β} and F_{I_β} are canonical to $\mathrm{DH}(\vec{Q})$. This apparent discrepancy is hidden in the fact that one has fixed a choice of orientation for $\vec{Q}$. In [Lus90b] Lusztig gives the following way to choose a reduced decomposition of w_0 depending on a quiver $\vec{Q}$.

Proposition 16.2.1 (Lusztig)**.** Let $\vec{Q}$ be a simply-laced quiver. Then there exists a sequence of vertices $i_1, \ldots, i_N$ of $\vec{Q}$, where N is the number of positive roots, with the following properties.

1. $i_1, \ldots, i_N$ is a source sequence, that is, for $1 \leqslant k \leqslant N$ then i_k is a source for $\sigma_{i_k} \ldots \sigma_{i_1}\vec{Q}$.
2. $w_0 = s_{i_1} \cdots s_{i_N}$ is a reduced decomposition of the longest element of the Weyl group into simple reflections.

Proof. This is part (b) of Proposition 4.12. in [Lus90b]. □

For the remainder of this section we fix a sequence of vertices $i_1, \ldots, i_N$ as in Proposition 16.2.1. For $1 < k \leqslant N$ let $\beta_1 := \alpha_{i_1}$ and $\beta_k := s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})$ be the total ordering of positive roots induced by $w_0 = s_{i_1} \cdots s_{i_N}$ as in Section 5.2. By Gabriel's Theorem 2.2.2 this determines a total ordering of the indecomposables $I_{\beta_1}, \ldots, I_{\beta_N}$. In particular we have elements $E_{I_{\beta_k}}$ and $F_{I_{\beta_k}}$ in $\mathrm{DH}(\vec{Q})$. On the other hand by Definition 5.2.1 the reduced decomposition $w_0 = s_{i_1} \cdots s_{i_N}$ determines quantum root vectors E_{β_k} and F_{β_k} in $\mathrm{U}_t(\mathfrak{g})$.

The following Lemma says that the elements $E_{I_{\beta_k}}$ and $F_{I_{\beta_k}}$ of DH are generated by the isomorphism $\mathcal{T}_i$ from Theorem 16.1.1 in an analogous way to how the quantum root vectors E_{β_k} and F_{β_k} are generated by Lusztig's automorphisms T_i.

Proposition 16.2.2. For any $1 < k \leqslant N$ the following identities hold in $\mathrm{DH}(\vec{Q})$.

$$\mathcal{T}_{i_1}^{-1} \cdots \mathcal{T}_{i_{k-1}}^{-1}(E_{S_{i_k}}) = E_{I_{\beta_k}} \qquad \mathcal{T}_{i_1}^{-1} \cdots \mathcal{T}_{i_{k-1}}^{-1}(F_{S_{i_k}}) = F_{I_{\beta_k}}$$

Proof. We prove the first identity as the second follows from the fact that the $\mathcal{T}_i^{-1}$ commute with the shift involution $*$. Since $i_1, \ldots, i_N$ is a source sequence and $\sigma_j\sigma_j\vec{Q} = \vec{Q}$ then by Theorem 16.1.1 the following is a well-defined isomorphism.

$$\mathcal{T}_{i_1}^{-1} \cdots \mathcal{T}_{i_{k-1}}^{-1} : \mathrm{DH}(\sigma_{i_{k-1}} \cdots \sigma_{i_1}\vec{Q}) \to \mathrm{DH}(\vec{Q})$$

Consider $E_{S_{i_k}}$ as an element in $\mathrm{DH}(\sigma_{i_{k-1}} \cdots \sigma_{i_1}\vec{Q})$. We claim that a repeated application of $\mathcal{T}_j^{-1}(E_N) = E_{\sigma_j^- N}$, where N does not have S_j as a direct summand, yields the following.

$$\mathcal{T}_{i_1}^{-1} \cdots \mathcal{T}_{i_{k-1}}^{-1}(E_{S_{i_k}}) = E_{\sigma_{i_1}^- \cdots \sigma_{i_{k-1}}^-(S_{i_k})} \tag{16.6}$$

Since σ_j^- takes an indecomposable different from S_j to an indecomposable different from S_j the only thing we need to check is that $S_{i_k} \neq S_{i_{k-1}}$ and for any $1 < j \leqslant k-1$ we *do not* have the following equality of indecomposables.

$$\sigma_{i_j}^- \cdots \sigma_{i_{k-1}}^-(S_{i_k}) = S_{i_{j-1}} \tag{16.7}$$

If $S_{i_k} = S_{i_{k-1}}$ then $i_k = i_{k-1}$ and $w_0 = s_{i_1} \cdots s_{i_N}$ would not be a reduced decomposition of w_0. For the other case, suppose Equation (16.7) held. Taking the image of Equation (16.7) in $K(\mathcal{A})$ then we would have the following implications (which we view as taking place in the root lattice Λ_ϕ).

$$\begin{aligned} & s_{i_j} \cdots s_{i_{k-1}}(\alpha_{i_k}) = \alpha_{i_{j-1}} \\ \Longrightarrow\ & s_{i_1} \ldots s_{i_{k-1}}(\alpha_{i_k}) = s_{i_1} \cdots s_{i_{j-2}}(-\alpha_{i_{j-1}}) \\ \Longrightarrow\ & \beta_k = -\beta_{j-1} \end{aligned}$$

However this is a contradiction since no positive root is the negative of another. We have established Equation (16.6) and it remains to show that $\sigma_{i_1}^- \cdots \sigma_{i_{k-1}}^-(S_{i_k}) = I_{\beta_k}$. Since σ_j^- takes an indecomposable different from S_j to an indecomposable then we know $\sigma_{i_1}^- \cdots \sigma_{i_{k-1}}^-(S_{i_k})$ is indecomposable. Moreover the class of $\sigma_{i_1}^- \cdots \sigma_{i_{k-1}}^-(S_{i_k})$ in the Grothendieck group is $s_{i_1} \cdots s_{i_{k-1}}(\hat{S}_{i_k})$ which is equal to $\hat{I}_{\beta_k}$ by definition of β_k. The result follows by Gabriel's Theorem 2.2.2. □

The following theorem gives the relationship between the quantum root vectors E_β and F_β of $\mathrm{U}_t(\mathfrak{g})$ and the elements E_{I_β} and F_{I_β} of DH. We recall from Equation (13.15) that we introduced a modified version $\bar{R}$ of Bridgeland's isomorphism between $\mathrm{U}_t(\mathfrak{g})$ and DH. In particular in the following we will need to use the fact that $\bar{R}(E_{S_j}) = tE_{S_j}$.

Theorem 16.2.3. For $1 \leqslant k \leqslant N$ we have the following identities in $\mathrm{DH}(\vec{Q})$.

$$\bar{R}(E_{\beta_k}) = tE_{I_{\beta_k}} \qquad\qquad \bar{R}(F_{\beta_k}) = -F_{I_{\beta_k}}$$

Proof. We can reduce to the case of showing that $\bar{R}(E_{\beta_k}) = tE_{I_{\beta_k}}$. Indeed by Proposition 13.4.2 we have $\bar{R} \circ \Sigma = \Sigma \circ *$ and from Section 5.5 that $F_{\beta_k} = \Sigma(-t^{-1}E_{\beta_k})$. Thus we would have the following.

$$\bar{R}(F_{\beta_k}) = \bar{R} \circ \Sigma(-t^{-1}E_{\beta_k}) = * \circ \bar{R}(-t^{-1}E_{\beta_k}) = -F_{I_{\beta_k}}$$

Using Equation (13.15) the result is obvious for $k = 1$ since $\beta_1 = \alpha_{i_1}$ and $I_{\beta_1} = S_{i_1}$. Suppose that $1 < k \leqslant N$. Using the fact that $i_1, \ldots, i_N$ is a source sequence, we can repeatedly apply Theorem 16.1.2 to get the following commutative diagram of algebra isomorphisms.

$$\begin{array}{ccc} U_t(\mathfrak{g}) & \xrightarrow{T_{i_1}^{-1} \cdots T_{i_{k-1}}^{-1}} & U_t(\mathfrak{g}) \\ \downarrow{\scriptstyle \bar{R}} & & \downarrow{\scriptstyle \bar{R}} \\ \mathrm{DH}(\sigma_{i_{k-1}} \cdots \sigma_{i_1}\vec{Q}) & \xrightarrow{\mathcal{T}_{i_1}^{-1} \cdots \mathcal{T}_{i_{k-1}}^{-1}} & \mathrm{DH}(\vec{Q}) \end{array}$$

By combining Equation (5.8) and Definition 5.2.1 we have that $E_{\beta_k} = T_{i_1}^{-1} \cdots T_{i_{k-1}}^{-1}(E_{i_k})$ in $\mathrm{U}_t(\mathfrak{g})$. Moreover by Proposition 16.2.2 we have that $E_{I_{\beta_k}} = \mathcal{T}_{i_1}^{-1} \cdots \mathcal{T}_{i_{k-1}}^{-1}(E_{S_{i_k}})$ in $\mathrm{DH}(\vec{Q})$. It follows then that in $\mathrm{DH}(\vec{Q})$ we have the following.

$$\begin{aligned}\bar{R}(E_{\beta_k}) &= \bar{R} \circ T_{i_1}^{-1} \cdots T_{i_{k-1}}^{-1}(E_{i_k}) \\ &= \mathcal{T}_{i_1}^{-1} \cdots \mathcal{T}_{i_{k-1}}^{-1} \circ \bar{R}(E_{i_k}) \\ &= \mathcal{T}_{i_1}^{-1} \cdots \mathcal{T}_{i_{k-1}}^{-1}(tE_{S_{i_k}}) \\ &= tE_{I_{\beta_k}}\end{aligned}$$

□

We can now show that the Poisson integral form $\mathrm{U}_t^{Poiss}(\mathfrak{g})$ of $\mathrm{U}_t(\mathfrak{g})$ is identified with the extension counting integral form DH_{ex} of DH under $\bar{R}$, the modified version of Bridgeland's isomorphism.

Theorem 16.2.4. The $\mathbb{C}(t)$-algebra isomorphism $\bar{R} : \mathrm{U}_t(\mathfrak{g}) \to \mathrm{DH}$ restricts to a $\mathbb{C}[t, t^{-1}]$-algebra isomorphism of integral forms between $\mathrm{U}_t^{Poiss}(\mathfrak{g})$ and DH_{ex}.

Proof. Recall that in Definition 5.5.1 the Poisson integral form $U_t^{Poiss}(\mathfrak{g})$ of $\mathrm{U}_t(\mathfrak{g})$ is defined as the $\mathbb{C}[t, t^{-1}]$-subalgebra of $\mathrm{U}_t(\mathfrak{g})$ generated by the elements E_{β_k}, F_{β_k} and $K_k^{\pm 1}$. Moreover by Theorem 16.2.3 we have that $\bar{R}(E_{\beta_k}) = tE_{I_{\beta_k}}$, $\bar{R}(F_{\beta_k}) = -F_{I_{\beta_k}}$ and $\bar{R}(K_k^{\pm 1}) = K_{\hat{S}_k}^{\pm 1}$. Thus we need only show that $E_{I_{\beta_k}}, F_{I_{\beta_k}}$ and $K_{\hat{S}_k}^{\pm 1}$ generate DH_{ex} as a $\mathbb{C}[t, t^{-1}]$-subalgebra of DH.

Lemma 3.19 from [Sch09] says that there is a total ordering[1] $\gamma_1, \ldots, \gamma_N$ of the positive roots such that if $j < k$ then $\dim \mathrm{Ext}^1(I_{\gamma_j}, I_{\gamma_k}) = 0$. Using this fact, a simple calculation shows that for any representation $L = I_{\gamma_1}^{n_1} \oplus \cdots \oplus I_{\gamma_N}^{n_N}$ then $E_L = t^d E_{I_{\gamma_1}}^{n_1} \cdots E_{I_{\gamma_N}}^{n_N}$ for some $d \in \mathbb{Z}$. Applying the shift involution establishes the analogous result for F_L. The triangular decomposition of DH_{ex} from Proposition 15.1.2 coupled with the above show that the elements $E_{I_{\beta_k}}$, $F_{I_{\beta_k}}$, and $K_{\hat{S}_k}^{\pm 1}$ generate DH_{ex} as a $\mathbb{C}[t, t^{-1}]$-algebra. □

We mentioned in Section 13.4 that the isomorphism $\bar{R} : \mathrm{U}_t(\mathfrak{g}) \to \mathrm{DH}$ restricts to an isomorphism between the positive part of the quantized enveloping algebra $\mathrm{U}_t(\mathfrak{n}_+)$ and the copy of the generic Hall algebra H in DH spanned by the elements of the form X_L or equivalently E_L.

Recall also that by definition the positive part of the Poisson integral form $\mathrm{U}_t^{Poiss}(\mathfrak{n}_+)$ is the $\mathbb{C}[t, t^{-1}]$-subalgebra of $\mathrm{U}_t(\mathfrak{g})$ generated by the rescaled quantum root vectors E_{β_k} where $1 \leqslant k \leqslant r$. The proof of Theorem 16.2.4 thus also establishes the following theorem.

Theorem 16.2.5. The $\mathbb{C}(t)$-algebra isomorphism $\bar{R} : \mathrm{U}_t(\mathfrak{n}_+) \to \mathrm{H}$ restricts to a $\mathbb{C}[t, t^{-1}]$-algebra isomorphism of integral forms between $\mathrm{U}_t^{Poiss}(\mathfrak{n}_+)$ and H_{ex}.

[1] $\gamma_1, \ldots, \gamma_N$ is a priori a different ordering to $\beta_1, \ldots \beta_N$. Probably one can take them to coincide, however we will not need to establish this fact.

Part IV

Semi-Classical Bridgeland-Hall Algebras

Poisson Structure, Lie Bialgebras and Poisson Lie Groups

Overview

Part IV is concerned with the semi-classical Bridgeland-Hall algebra DH_{sc}. This is the $t = 1$ limit of the extension counting integral form of DH that was defined in Section 15.1. The main attribute of DH_{sc} is that it is a commutative Poisson-Hopf algebra or equivalently the algebra of functions on a Poisson-Lie group. To our knowledge the properties of semi-classical Bridgeland-Hall algebras have not been studied from the Hall algebraic viewpoint before.

We begin with Chapter 17 where we define the semi-classical Bridgeland-Hall algebra. We explain how DH_{sc} is a commutative Poisson-Hopf algebra and give an explicit calculation of its Poisson structure. In this chapter we emphasise how various homological properties of categories of quiver representations and complexes translate into algebraic properties of DH_{sc}. By definition, DH_{sc} is the coordinate algebra of a Poisson-Lie group $\boldsymbol{G}_{sc}^{\vee}$ which we call the semi-classical Poisson-Lie group.

In Chapter 18 we discuss the tangent Lie bialgebra $\mathfrak{g}_{sc}^{\vee}$ of the semi-classical Poisson-Lie group $\boldsymbol{G}_{sc}^{\vee}$. We give an explicit calculation of the structure constants of $\mathfrak{g}_{sc}^{\vee}$ and then show that it is isomorphic to the standard dual Lie bialgebra $\mathfrak{g}^{\vee}$ from Section 3.3. This chapter should be viewed as giving a way to associate Lie bialgebras to Hall algebras and in particular a new way to recover the whole simple Lie algebra $\mathfrak{g}$ from Bridgeland-Hall algebras.

Finally in Chapter 19 we prove that the semi-classical Poisson-Lie group $\boldsymbol{G}_{sc}^{\vee}$ is isomorphic to the standard dual Poisson-Lie group $\boldsymbol{G}^{\vee}$ that was constructed in Section 3.3. This is a new proof using Hall algebraic methods of the classical result of De Concini and Procesi [DCP93] that the $t = 1$ limit of the Poisson integral form of $\mathrm{U}_t(\mathfrak{g})$ is the algebra of functions on $\boldsymbol{G}_{sc}^{\vee}$. We feel our Hall algebraic proof is more direct and conceptual than De Concini and Procesi's and in particular avoids the case by case analysis of the proof in [DCP93].

We expect that the methods in this part should apply more generally to generic Bridgeland-Hall algebras of categories other than representations of simply-laced quivers. In particular it should be possible to extract Poisson-Lie groups from Hall algebras of hereditary categories other than the categories considered in this thesis.

Chapter 17

Semi-Classical Algebra and Poisson-Lie Group

In this chapter we define and discuss the semi-classical Bridgeland-Hall algebra DH_{sc} which is the $t = 1$ limit of DH_{ex}. We also explain how Poisson-Lie groups arise from such algebras and give an explicit computation of the Poisson structure.

We start off with Section 17.1 where we give the definition of DH_{sc} and show that it is a commutative Poisson-Hopf algebra over $\mathbb{C}$. In particular DH_{sc} is the algebra of functions on a Poisson-Lie group $\boldsymbol{G}_{sc}^{\vee}$ which we call the semi-classical Poisson-Lie group. Later on in Chapter 19 we will give a proof of the fact that $\boldsymbol{G}_{sc}^{\vee}$ coincides with the standard dual Poisson-Lie group $\boldsymbol{G}^{\vee}$ that was defined in Section 3.3.

Section 17.2 is concerned with explicitly calculating the structure constants of the Poisson structure on DH_{sc}. We will linearize this Poisson structure in Chapter 18 to get the cotangent Lie algebra of $\boldsymbol{G}_{sc}^{\vee}$ which is isomorphic to the simple Lie algebra $\mathfrak{g}$. This will yield an alternative way to realize the full simple Lie algebra in terms of Hall algebras than that of Section 15.3.

17.1 Definitions and Properties

In this section we define the semi-classical Bridgeland-Hall algebra DH_{sc} and its associated semi-classical Poisson-Lie group. In particular, DH_{sc} will be the $t = 1$ limit of the extension counting integral form DH_{ex} of DH. We will show that this $t = 1$ limit is commutative and so, as outlined in Chapter 4, will inherit the structure of a Poisson-Hopf algebra. This structure is equivalent to the data of a Poisson-Lie group which we shall call the semi-classical Poisson-Lie group.

Definition 17.1.1. Define the semi-classical Bridgeland-Hall algebra DH_{sc} to be the quotient algebra of DH_{ex} by the ideal $(t - 1)$.

Let us describe how the semi-classical Bridgeland-Hall algebra is a Poisson-Hopf algebra, as defined in Definition 3.1.1. In Corollary 14.1.2 we gave the generic Bridgeland-Hall algebra DH the structure of a $\mathbb{C}(t)$-Hopf algebra. We observed at the end of Section 15.1 that this Hopf algebra structure descended to a $\mathbb{C}[t, t^{-1}]$-Hopf algebra structure on the extension counting integral form DH_{ex} of

DH. As outlined in Chapter 4 then, this $\mathbb{C}[t,t^{-1}]$-Hopf algebra structure descends to a $\mathbb{C}$-Hopf algebra structure on DH_{sc}.

Proposition 17.1.1. The semi-classical Bridgeland-Hall algebra DH_{sc} is a finitely generated commutative Poisson-Hopf algebra over $\mathbb{C}$. The product and Poisson bracket are determined by the requirement that $\overline{[K_P]}\ \overline{[K_P^*]} = 1$ along with the following.

$$\overline{[M_\bullet]}\ \overline{[N_\bullet]} = \overline{[M_\bullet \oplus N_\bullet]} \qquad\qquad \{\overline{[M_\bullet]}, \overline{[N_\bullet]}\}_{sc} := \overline{\left(\frac{[M_\bullet][N_\bullet] - [N_\bullet][M_\bullet]}{2(t-1)}\right)}$$

Proof. That $\overline{[K_P]}\ \overline{[K_P^*]} = 1$ holds is obvious from Definition 13.1.4 and Definition 15.1.1. To show that we have $\overline{[M_\bullet]}\ \overline{[N_\bullet]} = \overline{[M_\bullet \oplus N_\bullet]}$ the exact same reasoning holds as in the proof of Proposition 9.2.1, once one replaces $\mathrm{Ext}^1(M_q, N_q)$ everywhere with $\mathrm{Ext}^1(M_{\bullet q}, N_{\bullet q})$. The commutativity of DH_{sc} follows from that of the operation $\oplus$, while the Poisson-Hopf structure then follows from Proposition 4.1.1 and Proposition 4.1.2.

To see that DH_{sc} is finitely generated note that by definition any complex $M_\bullet \in \mathrm{Iso}(\mathcal{C})$ decomposes as a direct sum of the form $\mathcal{C}_A \oplus \mathcal{C}_B \oplus K_P \oplus K_Q^*$. Thus DH_{sc} is spanned by the following elements.

$$\overline{[\mathcal{C}_A \oplus \mathcal{C}_B^* \oplus K_P \oplus K_Q^*]} = \overline{[\mathcal{C}_A]}\ \overline{[\mathcal{C}_B^*]}\ \overline{[K_P]}\ \overline{[K_Q^*]}$$

Moreover each of the complexes $\mathcal{C}_A$, $\mathcal{C}_B^*$, K_P and K_Q^* further decompose as a direct sum of their indecomposable direct summands. One can then see that, as an algebra, DH_{sc} is generated by the elements $\overline{[\mathcal{C}_{I_1}]}$, $\overline{[\mathcal{C}_{I_2}^*]}$, $\overline{[K_{P_1}]}$ and $\overline{[K_{P_2}^*]}$ where I_1, I_2 are indecomposable and P_1, P_2 are projective indecomposable. By Theorem 2.2.2 this is a finite collection of elements. □

We now describe the semi-classical Bridgeland-Hall algebra completely as a $\mathbb{C}$-algebra. Recall from Section 13.2 that we had that DH_{ex} contains two copies of the Abelian extension counting integral form H_{ex} of H, given by the $\mathbb{C}[t,t^{-1}]$-span of the elements E_L and F_L respectively. The algebra DH_{ex} also contains a copy of the group algebra $\mathbb{C}[t,t^{-1}][K(\mathcal{A})]$ given by the span of the K_α.

Using Corollary 9.2.2 we can take the $t = 1$ semi-classical limit of these subalgebras of DH_{ex}. The algebra H_{sc} is the polynomial algebra in the elements $\overline{E}_I$ where I is indecomposable while the quotient of the group algebra $\mathbb{C}[t,t^{-1}][K(\mathcal{A})]$ by the ideal $(t-1)$ yields the algebra $\mathbb{C}[K(\mathcal{A})]$. We have the following algebra triangular decomposition of DH_{sc} in terms of these semi-classical subalgebras.

Proposition 17.1.2. Multiplication induces the following isomorphism of $\mathbb{C}$-algebras.

$$\mathrm{H}_{sc} \otimes_{\mathbb{C}} \mathbb{C}[K(\mathcal{A})] \otimes_{\mathbb{C}} \mathrm{H}_{sc} \to \mathrm{DH}_{sc}, \qquad \overline{E}_A \otimes \overline{K}_\alpha \otimes \overline{F}_B \mapsto \overline{E}_A \overline{K}_\alpha \overline{F}_B$$

Moreover the two copies of H_{sc} are polynomial algebras in the elements $\overline{E}_I$ and $\overline{F}_I$ respectively with I indecomposable.

Proof. Taking the $t = 1$ limit of Proposition 15.1.2 gives us an isomorphism of $\mathbb{C}$-vector spaces. Using commutativity of the product of the algebras involved it is easy to see that multiplication is also preserved. □

We end this section by discussing the semi-classical Poisson-Lie group. As outlined in Section 3.1, the geometric counterpart of a commutative Poisson-Hopf algebra is a Poisson-Lie group. Proposition 17.1.1 then allows us to make the following definition.

Definition 17.1.2. Define the semi-classical Poisson-Lie group to be $\boldsymbol{G}_{sc}^{\vee} := \operatorname{Spec} \mathrm{DH}_{sc}$.

By Proposition 17.1.2 the semi-classical Poisson-Lie group $\boldsymbol{G}_{sc}^{\vee}$ is not particularly complicated as a variety: it is isomorphic to $\mathbb{A}_{\mathbb{C}}^{N} \times \mathbb{G}_{m}^{r} \times \mathbb{A}_{\mathbb{C}}^{N}$ where N is the number of indecomposables or equivalently the number of positive roots.

We have chosen notation $\boldsymbol{G}_{sc}^{\vee}$ to emphasize that, as we shall show later in Section 16.2, the semi-classical Poisson-Lie group is isomorphic to the dual Poisson-Lie group $\boldsymbol{G}^{\vee}$ that we defined in Section 3.3. More generally we shall append the subscript sc to various objects such as Lie algebras and subgroups related to $\boldsymbol{G}_{sc}^{\vee}$ to indicate that they come from semi-classical Bridgeland-Hall algebra as opposed to analogous objects related to $\boldsymbol{G}^{\vee}$ that one finds in Section 3.3.

17.2 Calculation of Poisson Structure

In this section we give an explicit calculation of the Poisson bracket on the semi-classical Bridgeland-Hall algebra DH_{sc}. Before doing so, we shall find it convenient to introduce some new notation for structure constants of the Poisson bracket.

For representations $L, M, N \in \operatorname{Iso}(\mathcal{A})$ recall from Equation (7.3) that if $L \neq M \oplus N$ then we have polynomials $\mathbb{P}(e)_{M,N}^{L} := e_{M,N}^{L}/(t^2-1)$ counting the projectivization of the set $\operatorname{Ext}^1(M,N)_L$. This allows us to define the following structure constants.

$$\Gamma_{M,N}^{L} := \begin{cases} \overline{\mathbb{P}(e)}_{M,N}^{L} - \overline{\mathbb{P}(e)}_{N,M}^{L} & \text{if } L \neq M \oplus N \\ 1/2(\hat{M},\hat{N})_{\text{skew}} + \dim\operatorname{Hom}(N,M) - \dim\operatorname{Hom}(M,N) & \text{if } L = M \oplus N \end{cases}$$

Recall that the overline bar notation here denotes the evaluation at $t=1$ of a polynomial. Note also that, for L, M and N indecomposable, the above notation appears to clash with that of the structure constants $\Gamma_{M,N}^{L} := \overline{f}_{M,N}^{L} - \overline{f}_{N,M}^{L}$ defined in Definition 10.2.3. However by Corollary 7.4.1 these two definitions coincide.

Recall from Equation (12.3) that for representations $A, B, M, N \in \operatorname{Iso}(\mathcal{A})$ we have polynomials $h_{M,N}^{A,B}$ counting morphisms $M \to N$ with kernel A and cokernel B. When $(A,B) \neq (M,N)$, by Equation (12.4), we also have polynomials $\mathbb{P}(h)_{M,N}^{A,B} := h_{M,N}^{A,B}/(t^2-1)$ counting the projectivization of these sets. We then define the following structure constants, which in general are elements of $\mathbb{C}[K(\mathcal{A})]$.

$$\Gamma_{M,N}^{A,B} := \begin{cases} \overline{\mathbb{P}(h)}_{M,N}^{A,B}\overline{K}_{\hat{B}-\hat{N}} - \overline{\mathbb{P}(h)}_{N,M}^{B,A}\overline{K}_{\hat{N}-\hat{B}} & \text{if } (A,B) \neq (M,N) \\ 0 & \text{if } (A,B) = (M,N) \end{cases}$$

Using the above structure constants we can now give an explicit calculation of the Poisson bracket. The Leibniz rule implies that the Poisson bracket is determined by its value on the algebra generators $\overline{E}_I$, $\overline{F}_J$ and $\overline{K}_\alpha$ of DH_{sc} where I and J are indecomposable and $\alpha \in K(\mathcal{A})$.

Proposition 17.2.1. The Poisson structure on the semi-classical Bridgeland-Hall algebra DH_{sc} is determined by the following identities, where I and J are indecomposable and $\alpha \in K(\mathcal{A})$.

$$\{\overline{K}_\alpha, \overline{K}_\beta\}_{sc} = 0 \tag{17.1}$$

$$\{\overline{E}_I, \overline{E}_J\}_{sc} = \sum_{L \in \mathrm{Iso}(\mathcal{A})} \Gamma^L_{I,J} \overline{E}_L \tag{17.2}$$

$$\{\overline{F}_I, \overline{F}_J\}_{sc} = \sum_{L \in \mathrm{Iso}(\mathcal{A})} \Gamma^L_{I,J} \overline{F}_L \tag{17.3}$$

$$\{\overline{K}_\alpha, \overline{E}_I\}_{sc} = \frac{1}{2}(\alpha, \hat{I})\overline{E}_I \overline{K}_\alpha \tag{17.4}$$

$$\{\overline{K}_\alpha, \overline{F}_I\}_{sc} = -\frac{1}{2}(\alpha, \hat{I})\overline{F}_I \overline{K}_\alpha \tag{17.5}$$

$$\{\overline{E}_I, \overline{F}_J\}_{sc} = \sum_{A,B \in \mathrm{Iso}(\mathcal{A})} \Gamma^{A,B}_{I,J} \overline{E}_A \overline{F}_B \tag{17.6}$$

Proof. Equation (17.1) is trivial as the elements K_α all commute with each other. For Equation (17.2) consider the following identity in DH_{ex} which holds by Equation (8.1) and Proposition 13.2.1.

$$E_I E_J - E_J E_I = \sum_{L \in \mathrm{Iso}(\mathcal{A})} \left(t^{\langle \hat{I}, \hat{J} \rangle} \frac{e^L_{I,J}}{h_{I,J}} - t^{\langle \hat{J}, \hat{I} \rangle} \frac{e^L_{J,I}}{h_{J,I}} \right) E_L$$

Split the summation into $L = I \oplus J$ and $L \neq I \oplus J$ components and note that by Proposition 2.2.3 we have $e^{I \oplus J}_{I,J} = e^{I \oplus J}_{J,I} = 1$. We get the required formula on dividing across by $2(t-1)$ taking the $t = 1$ image in DH_{sc}. Moreover by applying the shift involution we similarly obtain Equation (17.3).

For Equation (17.4) and Equation (17.5), by Equation (13.2) we have the following in DH_{ex}.

$$K_\alpha E_I - E_I K_\alpha = (t^{(\alpha, \hat{I})} - 1) E_I K_\alpha \qquad K_\alpha F_I - F_I K_\alpha = (t^{-(\alpha, \hat{I})} - 1) F_I K_\alpha$$

Dividing across by $2(t-1)$ and taking the image in DH_{sc} gives the result. Finally, for the case of Equation (17.6), by Proposition 13.3.1 the following holds in DH_{ex}.

$$E_M F_N - F_N E_M = \sum_{A,B \in \mathrm{Iso}(\mathcal{A})} \left(t^{\langle \hat{N} - \hat{B}, \hat{M} - \hat{N} \rangle} h^{A,B}_{M,N} K_{\hat{B} - \hat{N}} - t^{\langle \hat{B} - \hat{N}, \hat{M} - \hat{N} \rangle} h^{B,A}_{N,M} K_{\hat{N} - \hat{B}} \right) D_{A,B}$$

Using the following two facts, on dividing across by $2(t-1)$ and taking the $t = 1$ image in DH_{sc} we obtain Equation (17.6). The first is that the $(A, B) = (M, N)$ component of the summation above vanishes. This is true as $h^{M,N}_{M,N} = 1$ since the zero morphism is the only morphism $M \to N$ with kernel M and cokernel N. The second fact is that in DH_{sc} we have $\overline{D}_{A,B} = \overline{E}_A \overline{F}_B$. This is established by using that the product on DH_{sc} is given by taking direct sums of complexes and then comparing Equation (13.7) with Equation (13.4) and Equation (13.5). □

Chapter 18

Semi-Classical Lie Bialgebras

This chapter is concerned with Lie bialgebras arising of Bridgeland-Hall algebras. In particular we will see that these Lie bialgebras give a new way to recover the whole simple Lie algebra from Bridgeland-Hall algebras. We will also use the results contained here in Chapter 19 to show that the semi-classical Poisson-Lie group $\boldsymbol{G}_{sc}^{\vee}$ coincides with the dual Poisson-Lie group $\boldsymbol{G}^{\vee}$.

In Section 3.2 we explained how tangent Lie bialgebra arise from Poisson-Lie groups on linearizing the Poisson-Lie structure at the group identity. Furthermore in Section 17.1 we obtained a Poisson-Lie group from the Bridgeland-Hall algebra called the semi-classical Poisson-Lie group $\boldsymbol{G}_{sc}^{\vee}$. Taking the tangent Lie bialgebra of $\boldsymbol{G}_{sc}^{\vee}$ we can thus extract a Lie bialgebra from the Bridgeland-Hall algebra which we call the semi-classical Lie bialgebra $\mathfrak{g}_{sc}^{\vee}$.

We will begin with Section 18.1 with the definition of $\mathfrak{g}_{sc}^{\vee}$ as the tangent Lie bialgebra of $\boldsymbol{G}_{sc}^{\vee}$. In this section we also explain how $\mathfrak{g}_{sc}^{\vee}$ comes equipped with a natural basis in terms of indecomposable quiver representations.

In Section 18.2 we then give an explicit computation of the structure constants of $\mathfrak{g}_{sc}^{\vee}$. As for all objects arising from Hall algebras, these will depend on the homological properties of the underlying categories of quiver representations and their complexes.

The material in Section 18.3 concerns the definition of certain Lie subalgebras of $\mathfrak{g}_{sc}^{\vee}$ and its dual $\mathfrak{g}_{sc}$ along with a root space decomposition of $\mathfrak{g}_{sc}$.

Finally in Section 18.4 we construct an isomorphism between $\mathfrak{g}_{sc}^{\vee}$ and the standard dual bialgebra $\mathfrak{g}^{\vee}$ introduced in Definition 3.3.1. Since the dual bialgebra of $\mathfrak{g}^{\vee}$ is $\mathfrak{g}$ then this gives a new way to recover the whole simple Lie algebra from Bridgeland-Hall algebras. The other way due to Deng and Chen was discussed in Section 15.3.

18.1 Definitions and Basis

In this section we will define the semi-classical Lie bialgebra $\mathfrak{g}_{sc}^{\vee}$ and its bialgebra dual $\mathfrak{g}_{sc}$. We also give a natural basis for these bialgebras.

Recall that in Section 3.2 we explained how a Lie bialgebra arises from linearizing the structure of a Poisson-Lie group at the group identity. Applying this to the semi-classical Poisson-Lie group $\boldsymbol{G}_{sc}^{\vee}$

we get a semi-classical Lie bialgebra.

Definition 18.1.1. Define the semi-classical Lie bialgebra $\mathfrak{g}_{sc}^{\vee}$ to be the tangent Lie bialgebra to the semi-classical Poisson-Lie group $\boldsymbol{G}_{sc}^{\vee}$. The bialgebra dual of $\mathfrak{g}_{sc}^{\vee}$ will be denoted by $\mathfrak{g}_{sc}$.

We will write $[-,-]_{sc}^{\vee}$ for the tangent Lie bracket on $\mathfrak{g}_{sc}^{\vee}$ and $[-,-]_{sc}$ for the cotangent Lie bracket on $\mathfrak{g}_{sc}$. A comment on why we have chosen the notation above: in Section 18.4 we will show that the semi-classical Lie bialgebra $\mathfrak{g}_{sc}^{\vee}$ coincides with the standard dual Lie bialgebra $\mathfrak{g}^{\vee}$ that was given in Definition 3.3.1. The dual of this statement is that the bialgebra dual of the semi-classical Lie bialgebra $\mathfrak{g}_{sc}$ coincides with the standard Lie bialgebra structure on the simple Lie algebra $\mathfrak{g}$.

To work with the semi-classical Lie bialgebra we will need a basis for $\mathfrak{g}_{sc}^{\vee}$ and its dual $\mathfrak{g}_{sc}$. By definition the underlying vector space of the semi-classical Lie bialgebra is $T_e\boldsymbol{G}_{sc}^{\vee}$. Recall that the Cartan matrix $(a_{ij})_{i,j=1}^{r}$ is non-degenerate. Since DH_{sc} is the coordinate algebra of the semi-classical Poisson-Lie group $\boldsymbol{G}_{sc}^{\vee}$, by Proposition 17.1.2 then we have the following basis for $T_eG_{sc}^{\vee}$ of partial derivatives at e.

$$e_I^{\vee} := \left.\frac{\partial}{\partial \overline{E}_I}\right|_e \qquad f_I^{\vee} := \left.\frac{\partial}{\partial \overline{F}_I}\right|_e \qquad h_{\hat{S}_i}^{\vee} := \frac{1}{2}\sum_{j=1}^{r} a_{ij}\left.\frac{\partial}{\partial \overline{K}_{\hat{S}_i}}\right|_e \tag{18.1}$$

Here I ranges over the indecomposables and $1 \leqslant i \leqslant r$. On the dual side of things the underlying vector space of $\mathfrak{g}_{sc}$ is $T_e^*\boldsymbol{G}_{sc}^{\vee}$ and so has a basis of differentials at e of the coordinate functions on $\boldsymbol{G}_{sc}^{\vee}$. Rescaling the differentials $d_e\overline{K}_{\hat{S}_i}$ by a factor of 2 gives us the following basis.

$$e_I := d_e\overline{E}_I \qquad f_I := d_e\overline{F}_I \qquad h_{\hat{S}_i} := 2d_e\overline{K}_{\hat{S}_i} \tag{18.2}$$

For any class $\hat{M}$ in the Grothendieck group we can extend the assignment of the class $\hat{S}_i$ to the vectors $h_{\hat{S}_i}^{\vee}$ and $h_{\hat{S}_i}$ to obtain elements $h_{\hat{M}}^{\vee}$ and $h_{\hat{M}}$

We end by remarking that the two bases of $\mathfrak{g}_{sc}$ and $\mathfrak{g}_{sc}^{\vee}$ above are chosen to match up with basis vectors introduced in Definition 1.1.1 and Section 1.1 for $\mathfrak{g}$ and in Equation (3.5) in the case of $\mathfrak{g}^{\vee}$.

18.2 Calculation of Lie Bialgebra Structure

In this section we compute the Lie bialgebra structure on the semi-classical Lie bialgebra $\mathfrak{g}_{sc}^{\vee}$. We will split this up into computing the tangent Lie bracket $[-,-]_{sc}^{\vee}$ in Theorem 18.2.1 before considering the cotangent Lie bracket $[-,-]_{sc}$ in Theorem 18.2.2.

We begin by making a few remarks on how to compute tangent Lie algebras in terms of the algebraic Hopf algebra data. The underlying vector space of the semi-classical Lie bialgebra is $T_e\boldsymbol{G}_{sc}^{\vee}$. We view $T_eG_{sc}^{\vee}$ as the set of $\mathbb{C}$-derivations on DH_{sc} evaluated at the group identity $e \in \boldsymbol{G}_{sc}^{\vee}$, that is $\mathbb{C}$-linear maps $D : \mathrm{DH}_{sc} \to \mathbb{C}$ such that for any two functions $u, v \in \mathrm{DH}_{sc}$ the following holds.

$$D(uv) = \overline{\varepsilon}(u)D(v) + \overline{\varepsilon}(v)D(u) \tag{18.3}$$

Here recall that $\overline{\varepsilon}$ is the counit of DH_{sc} which picks out the group identity e. Remark 3.9.1 of [Car07] says that, for the coordinate Hopf algebra of an algebraic group, the Lie bracket can be computed directly from the coproduct. Applied to our case, given any two derivations D_1 and D_2 in $\mathfrak{g}_{sc}^\vee$ then their Lie bracket is the derivation $[D_1, D_2] : \mathrm{DH}_{sc} \to \mathbb{C}$ given as follows.

$$[D_1, D_2] = (D_1 \otimes D_2 - D_2 \otimes D_1) \circ \overline{\Delta} \tag{18.4}$$

Here $D_1 \otimes D_2(u \otimes v) = D_1(u)D_2(v)$ for any u, v in DH_{sc}.

We now compute the tangent Lie bracket of $\mathfrak{g}_{sc}^\vee$. For convenience, given three positive roots α, β and γ we will write $\Gamma_{\alpha,\beta}^{\gamma}$ in place of $\Gamma_{I_\alpha,I_\beta}^{I_\gamma} := \overline{f}_{I_\alpha,I_\beta}^{I_\gamma} - \overline{f}_{I_\beta,I_\alpha}^{I_\gamma}$.

Theorem 18.2.1. The tangent Lie bracket on the semi-classical Lie bialgebra $\mathfrak{g}_{sc}^\vee$ is given by the following identities where $1 \leqslant i, j \leqslant r$ and α, β are positive roots.

$$[h_{\hat{S}_i}^\vee, h_{\hat{S}_j}^\vee]_{sc}^\vee = 0 \tag{18.5}$$

$$[e_{I_\alpha}^\vee, e_{I_\beta}^\vee]_{sc}^\vee = \begin{cases} \Gamma_{\alpha,\beta}^{\alpha+\beta} e_{I_{\alpha+\beta}}^\vee & \alpha+\beta \text{ a positive root} \\ 0 & \text{otherwise} \end{cases} \tag{18.6}$$

$$[f_{I_\alpha}^\vee, f_{I_\beta}^\vee]_{sc}^\vee = \begin{cases} \Gamma_{\beta,\alpha}^{\alpha+\beta} f_{I_{\alpha+\beta}}^\vee & \alpha+\beta \text{ a positive root} \\ 0 & \text{otherwise} \end{cases} \tag{18.7}$$

$$[h_{\hat{S}_i}^\vee, e_{I_\alpha}^\vee]_{sc}^\vee = \frac{1}{2}(\hat{S}_i, \hat{I}_\alpha) e_{I_\alpha}^\vee \tag{18.8}$$

$$[h_{\hat{S}_i}^\vee, f_{I_\alpha}^\vee]_{sc}^\vee = \frac{1}{2}(\hat{S}_i, \hat{I}_\alpha) f_{I_\alpha}^\vee \tag{18.9}$$

$$[e_{I_\alpha}^\vee, f_{I_\beta}^\vee]_{sc}^\vee = 0 \tag{18.10}$$

Proof. To prove the above identities we will need the formulas for the coproduct $\overline{\Delta}$ from Corollary 14.1.2 along with the formula for the Lie bracket in Equation (18.4). One then checks the equality of derivations above on the algebra generators $\overline{E}_J$, $\overline{F}_J$ and $\overline{K}_\gamma$ of DH_{sc} where J is indecomposable. In doing so we will need to use the formulas for the counit when employing Equation (18.3) and so we recall that $\overline{\varepsilon}(\overline{E}_L)$ and $\overline{\varepsilon}(\overline{F}_L)$ are zero unless $L = 0$ and that $\overline{\varepsilon}(\overline{K}_\alpha) = 1$. We begin by establishing Equation (18.5).

$$\begin{aligned} [h_{\hat{S}_i}^\vee, h_{\hat{S}_j}^\vee]_{sc}^\vee &= \left(h_{\hat{S}_i}^\vee \otimes h_{\hat{S}_j}^\vee - h_{\hat{S}_j}^\vee \otimes h_{\hat{S}_i}^\vee\right) \circ \overline{\Delta}(\overline{K}_\gamma) \\ &= \left(h_{\hat{S}_i}^\vee \otimes h_{\hat{S}_j}^\vee - h_{\hat{S}_j}^\vee \otimes h_{\hat{S}_i}^\vee\right) \overline{K}_\gamma \otimes \overline{K}_\gamma \\ &= h_{\hat{S}_i}^\vee(\overline{K}_\gamma) h_{\hat{S}_j}^\vee(\overline{K}_\gamma) - h_{\hat{S}_j}^\vee(\overline{K}_\gamma) h_{\hat{S}_i}^\vee(\overline{K}_\gamma) \\ &= 0 \end{aligned}$$

Since the derivations $h_{\hat{S}_i}^\vee$ vanish on elements of the form $\overline{E}_M$ and $\overline{F}_N$ of DH_{sc} it is easy to check that one also has $[h_{\hat{S}_i}^\vee, h_{\hat{S}_j}^\vee]_{sc}^\vee(\overline{E}_J) = 0$ and $[h_{\hat{S}_i}^\vee, h_{\hat{S}_j}^\vee]_{sc}^\vee(\overline{F}_J) = 0$. For Equation (18.6) note first that for any γ a positive root we have the following.

$$e_{I_\gamma}^\vee(E_M K_{\hat{N}}) = \overline{\varepsilon}(\overline{K}_{\hat{N}}) e_{I_\gamma}^\vee(\overline{E}_M) + \overline{\varepsilon}(\overline{E}_M) e_{I_\gamma}^\vee(\overline{K}_{\hat{N}}) = e_{I_\gamma}^\vee(\overline{E}_M)$$

Using this identity we have the following computation.

$$\begin{aligned}
[e^\vee_{I_\alpha}, e^\vee_{I_\beta}]^\vee_{sc}(\overline{E}_J) &= \left(e^\vee_{I_\alpha} \otimes e^\vee_{I_\beta} - e^\vee_{I_\beta} \otimes e^\vee_{I_\alpha}\right) \circ \overline{\Delta}(\overline{E}_J) \\
&= \left(e^\vee_{I_\alpha} \otimes e^\vee_{I_\beta} - e^\vee_{I_\beta} \otimes e^\vee_{I_\alpha}\right) \sum_{M,N \in \mathrm{Iso}(\mathcal{A})} \overline{f}^J_{M,N} \overline{E}_M \overline{K}_{\hat{N}} \otimes \overline{E}_N \\
&= \sum_{M,N \in \mathrm{Iso}(\mathcal{A})} \overline{f}^J_{M,N} e^\vee_{I_\alpha}(\overline{E}_M \overline{K}_{\hat{N}}) e^\vee_{I_\beta}(\overline{E}_N) - \sum_{M,N \in \mathrm{Iso}(\mathcal{A})} \overline{f}^J_{M,N} e^\vee_{I_\beta}(\overline{E}_M \overline{K}_{\hat{N}}) e^\vee_{I_\alpha}(\overline{E}_N) \\
&= \overline{f}^J_{I_\alpha, I_\beta} - \overline{f}^J_{I_\beta, I_\alpha} \\
&= \begin{cases} \Gamma^{\alpha+\beta}_{\alpha,\beta} & \alpha + \beta \text{ a positive root} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Since the derivations $e^\vee_J$ vanish on elements of the form $\overline{F}_M$ and $\overline{K}_\gamma$ of DH_{sc} it is easy to see check that the derivation $[e^\vee_{I_\alpha}, e^\vee_{I_\beta}]^\vee_{sc}$ vanishes when applied to $\overline{F}_J$ and $\overline{K}_\gamma$. One can establish Equation (18.7) in an entirely analogous manner.

Note that from Equation (18.2) one can check that $h^\vee_{\hat{S}_i}(\overline{K}_{\hat{I}_\alpha}) = \frac{1}{2}(\hat{S}_i, \hat{I}_\alpha)$. We establish Equation (18.8) then as follows.

$$\begin{aligned}
[h^\vee_{\hat{S}_i}, e^\vee_{I_\alpha}]^\vee_{sc}(\overline{E}_J) &= \left(h^\vee_{\hat{S}_i} \otimes e^\vee_{I_\alpha} - e^\vee_{I_\alpha} \otimes h^\vee_{\hat{S}_i}\right) \sum_{M,N \in \mathrm{Iso}(\mathcal{A})} \overline{f}^J_{M,N} \overline{E}_M \overline{K}_{\hat{N}} \otimes \overline{E}_N \\
&= \sum_{M,N \in \mathrm{Iso}(\mathcal{A})} \overline{f}^J_{M,N} h^\vee_{\hat{S}_i}(\overline{E}_M \overline{K}_{\hat{N}}) e^\vee_{I_\alpha}(\overline{E}_N) \\
&= \sum_{M \in \mathrm{Iso}(\mathcal{A})} \overline{f}^J_{M,I_\alpha} \left(\overline{\varepsilon}(\overline{E}_M) h^\vee_{\hat{S}_i}(\overline{K}_{\hat{I}_\alpha}) + \overline{\varepsilon}(\overline{K}_{\hat{I}_\alpha}) h^\vee_{\hat{S}_i}(\overline{E}_M)\right) \\
&= \sum_{M \in \mathrm{Iso}(\mathcal{A})} \overline{f}^J_{M,I_\alpha} \overline{\varepsilon}(\overline{E}_M) h^\vee_{\hat{S}_i}(\overline{K}_{\hat{I}_\alpha}) \\
&= \overline{f}^J_{0,I_\alpha} h^\vee_{\hat{S}_i}(\overline{K}_{\hat{I}_\alpha}) \\
&= \begin{cases} \frac{1}{2}(\hat{S}_i, \hat{I}_\alpha) & \text{if } J = I_\alpha \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Since the derivations $e^\vee_{I_\alpha}$ vanish on elements of the form $\overline{F}_M$ and $\overline{K}_\gamma$ in DH_{sc} one can check that the derivation $[h^\vee_{\hat{S}_i}, e^\vee_{I_\alpha}]^\vee_{sc}$ vanishes when applied to $\overline{F}_J$ and $\overline{K}_\gamma$. Equation (18.9) is proved entirely analogously to Equation (18.8).

For Equation (18.10) we use the fact that $f^\vee_{I_\beta}(\overline{E}_N)$ and $f^\vee_{I_\beta}(\overline{E}_M \overline{K}_{\hat{N}})$ are both zero to make the following computation.

$$\begin{aligned}
[e^\vee_{I_\alpha}, f^\vee_{I_\beta}]^\vee_{sc}(\overline{E}_J) &= \left(e^\vee_{I_\alpha} \otimes f^\vee_{I_\beta} - f^\vee_{I_\beta} \otimes e^\vee_{I_\alpha}\right) \circ \overline{\Delta}(\overline{E}_J) \\
&= \left(e^\vee_{I_\alpha} \otimes f^\vee_{I_\beta} - f^\vee_{I_\beta} \otimes e^\vee_{I_\alpha}\right) \sum_{M,N \in \mathrm{Iso}(\mathcal{A})} \overline{f}^J_{M,N} \overline{E}_M \overline{K}_{\hat{N}} \otimes \overline{E}_N \\
&= \sum_{M,N \in \mathrm{Iso}(\mathcal{A})} \overline{f}^J_{M,N} \left(e^\vee_{I_\alpha}(\overline{E}_M \overline{K}_{\hat{N}}) f^\vee_{I_\beta}(\overline{E}_N) - f^\vee_{I_\beta}(\overline{E}_M \overline{K}_{\hat{N}}) e^\vee_{I_\alpha}(\overline{E}_N)\right) \\
&= 0
\end{aligned}$$

A similar calculation shows that $[e^\vee_{I_\alpha}, f^\vee_{I_\beta}]^\vee_{sc}$ vanishes when applied to $\overline{F}_J$. Finally applying $[e^\vee_{I_\alpha}, f^\vee_{I_\beta}]^\vee_{sc}$ to $\overline{K}_\gamma$ gives zero as $e^\vee_{I_\alpha}(\overline{K}_\gamma) = 0$ and $f^\vee_{I_\beta}(\overline{K}_\gamma) = 0$. □

It remains to compute the Lie bracket $[-,-]_{sc}$. As described in Section 3.2 the cotangent bracket is given by the following formula where $u, v \in \mathrm{DH}_{sc}$.

$$[d_e u, d_e v]_{sc} := d_e\{u, v\}_{sc} \tag{18.11}$$

Since we have described the Poisson bracket explicitly in Proposition 17.2.1 we can compute the cotangent Lie bracket as follows.

Theorem 18.2.2. The cotangent Lie bracket on the semi-classical Lie bialgebra $\mathfrak{g}^\vee_{sc}$ is given by the following identities where $1 \leqslant i, j \leqslant r$ and α, β are positive roots.

$$[h_{\hat{S}_i}, h_{\hat{S}_j}]_{sc} = 0 \tag{18.12}$$

$$[e_{I_\alpha}, e_{I_\beta}]_{sc} = \begin{cases} \Gamma^{\alpha+\beta}_{\alpha,\beta} e_{I_{\alpha+\beta}} & \alpha+\beta \text{ a positive root} \\ 0 & \text{otherwise} \end{cases} \tag{18.13}$$

$$[f_{I_\alpha}, f_{I_\beta}]_{sc} = \begin{cases} \Gamma^{\alpha+\beta}_{\alpha,\beta} f_{I_{\alpha+\beta}} & \alpha+\beta \text{ a positive root} \\ 0 & \text{otherwise} \end{cases} \tag{18.14}$$

$$[h_{\hat{S}_i}, e_{I_\alpha}]_{sc} = \ \ (\hat{S}_i, \hat{I}_\alpha) e_{I_\alpha} \tag{18.15}$$

$$[h_{\hat{S}_i}, f_{I_\alpha}]_{sc} = -(\hat{S}_i, \hat{I}_\alpha) f_{I_\alpha} \tag{18.16}$$

$$[e_{I_\alpha}, f_{I_\beta}]_{sc} = \begin{cases} \Gamma^{\alpha}_{\beta,\alpha-\beta} e_{I_{\alpha-\beta}} & \alpha-\beta \text{ a positive root} \\ \Gamma^{\beta}_{\beta-\alpha,\alpha} f_{I_{\beta-\alpha}} & \beta-\alpha \text{ a positive root} \\ -h_{\hat{I}_\beta} & \alpha = \beta \\ 0 & \text{otherwise} \end{cases} \tag{18.17}$$

Proof. We will compute the above identities by using Equation (18.11) to linearize the formulas for the Poisson bracket that we derived in Proposition 17.2.1.

Throughout we will frequently need to make computations of the form $d_e(uv) = \overline{\varepsilon}(u) d_e v + \overline{\varepsilon}(v) d_e v$ for various u and v in DH_{sc}. To this end we will find it useful to recall that both $\overline{\varepsilon}(\overline{E}_L)$ and $\overline{\varepsilon}(\overline{F}_L)$ are zero unless $L = 0$ and that $\overline{\varepsilon}(\overline{K}_\alpha) = 1$. A particular consequence of this is that $d_e\overline{E}_L$ and $d_e\overline{F}_L$ vanish if L is not indecomposable.

We begin by noting that Equation (18.12) follows from the fact that $\{\overline{K}_\alpha, \overline{K}_\beta\}_{sc} = 0$. For Equation (18.13) consider the following.

$$[e_{I_\alpha}, e_{I_\beta}]_{sc} = d_e\{\overline{E}_{I_\alpha}, \overline{E}_{I_\beta}\}_{sc} = \sum_{L \in \mathrm{Iso}(\mathcal{A})} \Gamma^L_{I_\alpha, I_\beta} d_e \overline{E}_L$$

The identity in Equation (18.13) follows from the fact that if $d_e\overline{E}_L \neq 0$ then L must be indecomposable of class $\alpha + \beta$ in the Grothendieck group. An identical calculation establishes Equation (18.14).

For Equation (18.15), recalling that $h_{\hat{S}_i} := 2d_e\overline{K}_{\hat{S}_i}$, we have the following.

$$\begin{aligned}[h_{\hat{S}_i}, e_{I_\alpha}]_{sc} = 2d_e\{\overline{K}_{\hat{S}_i}, \overline{E}_{I_\alpha}\}_{sc} &= (\hat{S}_i, \hat{I}_\alpha)d_e(\overline{E}_{I_\alpha}\overline{K}_{\hat{S}_i}) \\ &= (\hat{S}_i, \hat{I}_\alpha)\left(\overline{\varepsilon}(\overline{K}_{\hat{S}_i})d_e(\overline{E}_{I_\alpha}) + \overline{\varepsilon}(\overline{E}_{I_\alpha})d_e(\overline{K}_{\hat{S}_i})\right) \\ &= (\hat{S}_i, \hat{I}_\alpha)d_e(\overline{E}_{I_\alpha}) \\ &= (\hat{S}_i, \hat{I}_\alpha)e_{I_\alpha}\end{aligned}$$

A similar calculation proves Equation (18.16). Of all the identities, Equation (18.17) will take the most work to establish. We provisionally make the following computation.

$$\begin{aligned}[e_{I_\alpha}, f_{I_\beta}]_{sc} &= \{\overline{E}_{I_\alpha}, \overline{F}_{I_\beta}\}_{sc} \\ &= \sum_{A,B\in \mathrm{Iso}(\mathcal{A})} d_e\left(\Gamma^{A,B}_{I_\alpha,I_\beta}\overline{E}_A\overline{F}_B\right) \\ &= \sum_{A,B\in \mathrm{Iso}(\mathcal{A})} \overline{\varepsilon}(\Gamma^{A,B}_{I_\alpha,I_\beta}\overline{F}_B)d_e\overline{E}_A + \overline{\varepsilon}(\Gamma^{A,B}_{I_\alpha,I_\beta}\overline{E}_A)d_e\overline{F}_B + \overline{\varepsilon}(\overline{E}_A\overline{F}_B)d_e\Gamma^{A,B}_{I_\alpha,I_\beta} \\ &= \sum_{A\in \mathrm{Iso}(\mathcal{A})} \overline{\varepsilon}(\Gamma^{A,0}_{I_\alpha,I_\beta})d_e\overline{E}_A + \sum_{B\in \mathrm{Iso}(\mathcal{A})} \overline{\varepsilon}(\Gamma^{0,B}_{I_\alpha,I_\beta})d_e\overline{F}_B + d_e\Gamma^{0,0}_{I_\alpha,I_\beta} \end{aligned} \tag{18.18}$$

Let us make some comments on how to simplify Equation (18.18). Recall from Section 17.2 that by definition $\Gamma^{A,B}_{M,N}$ necessarily vanishes unless $(A,B) \neq (M,N)$ in which case we have the following.

$$\Gamma^{A,B}_{M,N} := \overline{\mathbb{P}(h)}^{A,B}_{M,N}\overline{K}_{\hat{B}-\hat{N}} - \overline{\mathbb{P}(h)}^{B,A}_{N,M}\overline{K}_{\hat{N}-\hat{B}}$$

Now in Lemma 12.2.1 we showed that, in the case that M and N are indecomposable, we have $\mathbb{P}(h)^{A,0}_{M,N} = f^M_{N,A}$ and $\mathbb{P}(h)^{0,B}_{M,N} = f^N_{B,M}$. Thus evaluating $\overline{\varepsilon}$ on $\Gamma^{A,B}_{M,N}$ and respectively setting B and A to be 0 we obtain the following.

$$\begin{aligned}\overline{\varepsilon}(\Gamma^{A,0}_{I_\alpha,I_\beta}) &= \overline{\mathbb{P}(h)}^{A,0}_{I_\alpha,I_\beta} - \overline{\mathbb{P}(h)}^{0,A}_{I_\beta,I_\alpha} \\ &= \overline{f}^{I_\alpha}_{I_\beta,A} - \overline{f}^{I_\alpha}_{A,I_\beta} \\ &= \Gamma^{I_\alpha}_{I_\beta,A}\end{aligned} \tag{18.19}$$

$$\begin{aligned}\overline{\varepsilon}(\Gamma^{0,B}_{I_\alpha,I_\beta}) &= \overline{\mathbb{P}(h)}^{0,B}_{I_\alpha,I_\beta} - \overline{\mathbb{P}(h)}^{B,0}_{I_\beta,I_\alpha} \\ &= \overline{f}^{I_\beta}_{B,I_\alpha} - \overline{f}^{I_\beta}_{I_\alpha,B} \\ &= \Gamma^{I_\beta}_{B,I_\alpha}\end{aligned} \tag{18.20}$$

Let's turn our attention to $d_e\Gamma^{0,0}_{I_\alpha,I_\beta}$. First note that $h^{0,0}_{M,N}$ counts *isomorphisms* $M \to N$ since both kernel and cokernel must vanish. Thus $h^{0,0}_{M,N}$ vanishes unless $M = N$ in which case we have $h^{0,0}_{M,N} = a_M$. If $M = N$ is indecomposable then, since $a_M = t^2 - 1$, we have $\mathbb{P}(h)^{0,0}_{M,N} = h^{0,0}_{M,N}/(t^2-1) = 1$. Thus $d_e\Gamma^{0,0}_{I_\alpha,I_\beta} = 0$ unless $\alpha = \beta$ in which case we have the following.

$$\begin{aligned}d_e\Gamma^{0,0}_{I_\alpha,I_\beta} &= d_e\left(\overline{\mathbb{P}(h)}^{0,0}_{I_\alpha,I_\beta}\overline{K}_{-\hat{I}_\beta} - \overline{\mathbb{P}(h)}^{0,0}_{I_\beta,I_\alpha}\overline{K}_{\hat{I}_\beta}\right) \\ &= d_e\overline{K}_{-\hat{I}_\beta} - d_e\overline{K}_{\hat{I}_\beta} \\ &= -h_{\hat{I}_\beta}\end{aligned} \tag{18.21}$$

We observe from Equation (18.19) and Equation (18.20) that if $A \neq 0$ then its class in the Grothendieck group is $\alpha - \beta$ and similarly if $B \neq 0$ then its class is $\beta - \alpha$. We obtain Equation (18.17) then on substituting Equation (18.19), Equation (18.20) and Equation (18.21) into Equation (18.18) and using that $d_e\overline{E}_L$ and $d_e\overline{F}_L$ vanish if L is not indecomposable. □

18.3 Subalgebras and Properties

In this section we discuss various properties of $\mathfrak{g}_{sc}$ and $\mathfrak{g}_{sc}^\vee$. We will introduce various Lie subalgebras of $\mathfrak{g}_{sc}$ and $\mathfrak{g}_{sc}^\vee$ which will be used in Section 16.2 when proving that the semi-classical Poisson-Lie group $\boldsymbol{G}_{sc}^\vee$ is isomorphic to the dual Poisson-Lie group $\boldsymbol{G}^\vee$. We also discuss the root and coroot lattices of $\mathfrak{g}_{sc}$ along with its decomposition into root spaces.

We begin by defining some Lie subalgebras. Recall from Equation (1.2) that we introduced various Lie subalgebras of the simple Lie algebra $\mathfrak{g}$. The semi-classical analogue in $\mathfrak{g}_{sc}$ of the Cartan subalgebra of $\mathfrak{g}$ is given by $\mathfrak{h}_{sc} := \mathrm{Span}_{\mathbb{C}}\{h_{\hat{S}_i}\}$. The analogues of the nilpotent and Borel subalgebras are give by the following.

$$\mathfrak{n}_{sc,+} := \mathrm{Span}_{\mathbb{C}}\{e_I\} \qquad \mathfrak{b}_{sc,+} := \mathrm{Span}_{\mathbb{C}}\{e_I, h_{\hat{S}_i}\}$$
$$\mathfrak{n}_{sc,-} := \mathrm{Span}_{\mathbb{C}}\{f_I\} \qquad \mathfrak{b}_{sc,-} := \mathrm{Span}_{\mathbb{C}}\{f_I, h_{\hat{S}_i}\}$$

Similarly in Section 3.3 we defined certain Lie subalgebras of the standard dual $\mathfrak{g}^\vee$. The semi-classical analogues of these in $\mathfrak{g}_{sc}^\vee$ are given by $\mathfrak{h}_{sc}^\vee := \mathrm{Span}_{\mathbb{C}}\{h_{\hat{S}_i}^\vee\}$ along with the following.

$$\mathfrak{n}_{sc,+}^\vee := \mathrm{Span}_{\mathbb{C}}\{e_I^\vee\} \qquad \mathfrak{b}_{sc,+}^\vee := \mathrm{Span}_{\mathbb{C}}\{e_I^\vee, h_{\hat{S}_i}^\vee\}$$
$$\mathfrak{n}_{sc,-}^\vee := \mathrm{Span}_{\mathbb{C}}\{f_I^\vee\} \qquad \mathfrak{b}_{sc,-}^\vee := \mathrm{Span}_{\mathbb{C}}\{f_I^\vee, h_{\hat{S}_i}^\vee\}$$

That the above really are Lie subalgebras of $\mathfrak{g}_{sc}$ or respectively $\mathfrak{g}_{sc}^\vee$ can be verified by glancing at the formulas in Theorem 18.2.2 and Theorem 18.2.1.

Let's discuss the relationship between the Grothendieck group and the root and coroot lattices. In Theorem 18.4.1 we will show that the underlying Lie algebra of $\mathfrak{g}_{sc}$ coincides with the simple Lie algebra $\mathfrak{g}$. On the semi-classical side of things note that the elements $h_{\hat{S}_j}^\vee$ and $h_{\hat{S}_i}$ pair to give the Cartan matrix as follows.

$$h_{\hat{S}_j}^\vee(h_{\hat{S}_i}) = a_{ij}$$

Thus the elements $h_{\hat{S}_i} \in \mathfrak{h}_{sc}$ give simple coroots of $\mathfrak{g}_{sc}$ while the elements $h_{\hat{S}_i}^\vee \in \mathfrak{h}_{sc}^\vee$ give simple roots of $\mathfrak{g}_{sc}$. Moreover the Grothendieck group is identified with the root and coroot lattices via the following embeddings.

$$K(\mathcal{A}) \hookrightarrow \mathfrak{h}_{sc}^\vee, \qquad \hat{S}_i \mapsto h_{\hat{S}_i}^\vee$$
$$K(\mathcal{A}) \hookrightarrow \mathfrak{h}_{sc}, \qquad \hat{S}_i \mapsto h_{\hat{S}_i}$$

We end with a remark on the root space decomposition of $\mathfrak{g}_{sc}$. For each positive root α consider the following one dimensional subspaces of $\mathfrak{g}_{sc}$.

$$\mathfrak{g}_{sc}[\alpha] := \mathrm{Span}_{\mathbb{C}}\{e_{I_\alpha}\} \qquad \mathfrak{g}_{sc}[-\alpha] := \mathrm{Span}_{\mathbb{C}}\{f_{I_\alpha}\}$$

These are the root spaces for the adjoint action of $\mathfrak{h}_{sc}$ on $\mathfrak{g}_{sc}$. Indeed $\mathfrak{g}_{sc}$ is graded as a Lie algebra by $K(\mathcal{A})$ and Theorem 18.2.2 implies that we have the following root space decomposition.

$$\mathfrak{g}_{sc} = \mathfrak{h}_{sc} \oplus \bigoplus_{\alpha \text{ a root}} \mathfrak{g}_{sc}[\alpha]$$

A nice feature about the Hall algebra approach is that the indecomposable representations I_α furnish us with natural positive and negative root vectors e_{I_α} and f_{I_α} of $\mathfrak{g}_{sc}$.

18.4 Relationship with Standard Bialgebras

In this section we prove that the semi-classical Lie bialgebra $\mathfrak{g}_{sc}^\vee$ and the standard dual Lie bialgebra $\mathfrak{g}^\vee$ are isomorphic. It will be easier to prove the equivalent dual statement that $\mathfrak{g}_{sc}$ is isomorphic to the standard Lie bialgebra structure on the simple Lie algebra $\mathfrak{g}$. In this section we also relate various nilpotent subalgebras of $\mathfrak{g}_{sc}$ and $\mathfrak{g}_{sc}^\vee$ to the Lie algebra of indecomposables $\mathfrak{n}_{\mathrm{Ind}}$ defined in Section 10.2.

We will split the proof that $\mathfrak{g}_{sc}$ and $\mathfrak{g}$ are isomorphic up into two parts. We will first establish an isomorphism of the underlying Lie algebras before upgrading it to one of Lie bialgebras.

Theorem 18.4.1. The following determines an isomorphism of Lie algebras $\theta : \mathfrak{g} \to \mathfrak{g}_{sc}$ between the simple Lie algebra and the dual semi-classical Lie algebra.

$$\theta(e_i) = e_{S_i} \qquad \theta(f_i) = -f_{S_i} \qquad \theta(h_i) = h_{\hat{S}_i}$$

Proof. We will first show θ induces a homomorphism of Lie algebras. This amounts to showing that the elements e_{S_i}, $-f_{S_i}$ and $h_{\hat{S}_i}$ of $\mathfrak{g}_{sc}$ satisfy the same relations as e_i, f_i and h_i of $\mathfrak{g}$ that were given in Definition 1.1.1. This follows easily from Theorem 18.2.2 with the only relations not completely trivial to verify being $[e_{S_i}, -f_{S_j}] = \delta_{i,j} h_{\hat{S}_i}$ and that the elements e_{S_i} and $-f_{S_i}$ satisfy the Serre relations.

The first of these follows from Equation (18.17) along with the fact that $\alpha_i - \alpha_j$ is not a positive root of $\mathfrak{g}$ and so by Theorem 2.2.2 there are no indecomposables with class $\hat{S}_i - \hat{S}_j$ in the Grothendieck group. The Serre relations follow from Equation (18.13) and Equation (18.14) along with the fact that there is no positive root of the form $(1 - a_{ij})\alpha_i + \alpha_j$.

It remains to show that θ is an isomorphism. Since $\mathfrak{g}$ and $\mathfrak{g}_{sc}$ have the same finite dimension this is equivalent to establishing the injectivity of θ. Recall from the theory of simple Lie algebras that the adjoint action of the Cartan subalgebra $\mathfrak{h}$ on $\mathfrak{g}$ induces the following root space decomposition.

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \text{ a root}} \mathfrak{g}[\alpha]$$

First note that θ restrict to an isomorphism on the Cartan subalgebras $\mathfrak{h} \to \mathfrak{h}_{sc}$ as both have the same dimension. To establish injectivity we will show that θ respects the root space decompositions of $\mathfrak{g}$ and $\mathfrak{g}_{sc}$ and that θ is injective on each root space.

Using the facts that θ is a homomorphism, $\mathfrak{n}_+$ is generated by the e_i and $\mathfrak{g}_{sc}$ is graded by $K(\mathcal{A})$ one can see that for α a positive root the map θ takes $\mathfrak{g}[\alpha]$ to $\mathfrak{g}_{sc}[\alpha]$. Similarly for α a positive root θ takes $\mathfrak{g}[-\alpha]$ to $\mathfrak{g}_{sc}[-\alpha]$.

We claim then that θ is injective on the one dimensional root spaces $\mathfrak{g}[\alpha] \hookrightarrow \mathfrak{g}_{sc}[\alpha]$. Supposing that this isn't the case, let $v \in \mathfrak{g}[\alpha]$ be nonzero with $\theta(v) = 0$. The theory of simple Lie algebras says there is a $u \in \mathfrak{g}[-\alpha]$ such that $[v, u] \in \mathfrak{h}$ is nonzero. However then we would have the following contradiction.

$$0 = [\theta(v), \theta(u)] = \theta([v, u]) \neq 0$$

Identical reasoning shows that $\mathfrak{g}[-\alpha] \hookrightarrow \mathfrak{g}_{sc}[-\alpha]$. □

Before upgrading θ to an isomorphism of Lie bialgebras we'll need a discussion on root vectors and the linear dual of the map θ. Recall that in Section 1.1 we made a choice of root vectors for $\mathfrak{g}$ with the property that $(e_\alpha, f_\beta)_{\mathfrak{g}} = \delta_{\alpha,\beta}$. This choice was arbitrary but we now fix it so that we have the following.

$$\theta(e_\alpha) = e_{I_\alpha} \qquad\qquad \theta(f_\alpha) = -f_{I_\alpha} \tag{18.22}$$

In the proof of Theorem 18.4.2 we will view the linear dual to the map $\theta : \mathfrak{g} \to \mathfrak{g}_{sc}$ as a map $\theta^\vee : \mathfrak{g}_{sc}^\vee \to \mathfrak{g}^\vee$. We need to be careful as to what we mean by $\theta^\vee$ being the vector space dual of θ. For the vector space dual of the domain of θ, recall that $\mathfrak{g}$ was defined by generators and relations in Definition 1.1.1. In Section 3.3 we identified $\mathfrak{g}$ with $T_e^*\boldsymbol{G}^\vee$ via the Manin triple form. By the canonical isomorphism between a vector space and its double dual then, we can then identify the vector space dual of $\mathfrak{g}$ with $T_e\boldsymbol{G}^\vee$ which is by Definition 18.1.1 the underlying vector space of $\mathfrak{g}^\vee$.

For the dual of the codomain of θ, recall from Definition 18.1.1 that the underlying vector space of $\mathfrak{g}_{sc}$ is by definition $T_e^*\boldsymbol{G}_{sc}^\vee$. Using the canonical isomorphism between a vector space and its double dual we can then identify the vector space dual of $\mathfrak{g}_{sc}$ with $T_e\boldsymbol{G}_{sc}^\vee$ which is by Definition 18.1.1 the underlying vector space of $\mathfrak{g}_{sc}^\vee$. A simple calculation using the above identifications shows that $\theta^\vee$ is given by the following.

$$\theta^\vee(e_{I_\alpha}^\vee) = e_\alpha^\vee \qquad \theta^\vee(f_{I_\alpha}^\vee) = -f_\alpha^\vee \qquad \theta^\vee(h_{\hat{S}_i}^\vee) = h_i^\vee \tag{18.23}$$

Theorem 18.4.2. The isomorphism of Lie algebras $\theta : \mathfrak{g} \to \mathfrak{g}_{sc}$ is an isomorphism of Lie bialgebras between the standard Lie bialgebra and the dual semi-classical Lie bialgebra.

Proof. To show θ is a bialgebra homomorphism it is enough to check that the *inverse* of the dual map $(\theta^\vee)^{-1} : \mathfrak{g}^\vee \to \mathfrak{g}_{sc}^\vee$ is a Lie algebra homomorphism. However using Theorem 18.2.1 and Equation (18.23) one can check that the basis vectors $e_{I_\alpha}^\vee$, $-f_{I_\alpha}^\vee$ and $h_{\hat{S}_i}^\vee$ of $\mathfrak{g}_{sc}^\vee$ satisfy the same relations as the basis vectors $e_\alpha^\vee$, $f_\alpha^\vee$ and $h_i^\vee$ of $\mathfrak{g}^\vee$ which were given in Proposition 3.3.1.

The only relations which are not immediate to check is that $e_{I_{\alpha_i}}^\vee$ and $-f_{I_{\alpha_i}}^\vee$ satisfy the Serre relations. This follows from Equation (18.6) and Equation (18.7) coupled with the fact that there is no positive root of the form $(1 - a_{ij})\alpha_i + \alpha_j$. □

Applying bialgebra duality to the statement of Theorem 18.4.2 we obtain the following corollary.

Corollary 18.4.1. The following determines an isomorphism $\theta^\vee : \mathfrak{g}_{sc}^\vee \to \mathfrak{g}^\vee$ between semi-classical Lie bialgebra $\mathfrak{g}_{sc}^\vee$ and standard dual Lie bialgebra $\mathfrak{g}^\vee$.

$$\theta^\vee(e_{I_\alpha}^\vee) = e_\alpha^\vee \qquad \theta^\vee(f_{I_\alpha}^\vee) = -f_\alpha^\vee \qquad \theta^\vee(h_{\hat{S}_i}^\vee) = h_i^\vee$$

We end this section discussing how subalgebras of $\mathfrak{g}_{sc}$ and $\mathfrak{g}_{sc}^{\vee}$ relate to the quasi-classical Lie algebra $\mathfrak{n}_{qc}$ that we gave in Definition 10.2.3.

Proposition 18.4.3. The quasi-classical Lie algebra $\mathfrak{n}_{qc}$ is isomorphic to each of the Lie algebras $\mathfrak{n}_{sc,\pm}$ and $\mathfrak{n}_{sc,\pm}^{\vee}$

Proof. All of these Lie algebras are vector spaces on the set of indecomposable representations. We need only show that for each the structure constants in the basis of indecomposables all coincide. By Equation (10.3) the structure constants of $\mathfrak{n}_{qc}$ are given by the following where I_1, I_2 and J are indecomposable.

$$\Gamma_{I_1,I_2}^{J} := \overline{f}_{I_1,I_2}^{J} - \overline{f}_{I_2,I_1}^{J}$$

By Equation (18.6) and Equation (18.7) these give the structure constants of $\mathfrak{n}_{sc,\pm}^{\vee}$. The same holds for $\mathfrak{n}_{sc,\pm}$ by Equation (18.13) and Equation (18.14). □

Chapter 19

Identification of Poisson-Lie Groups

In this chapter we will show that the semi-classical Poisson-Lie group $\boldsymbol{G}^{\vee}_{sc}$ coincides with the dual Poisson-Lie group $\boldsymbol{G}^{\vee}$ that was defined in Section 3.3. We do so by first characterizing various subgroups of $\boldsymbol{G}^{\vee}_{sc}$ in Section 19.1 and then using these characterizations to build up an explicit isomorphism of Poisson-Lie groups between $\boldsymbol{G}^{\vee}_{sc}$ and $\boldsymbol{G}^{\vee}$ in Section 19.2.

The contents of this chapter should be viewed as a new proof of an old result. Indeed in the case of quantized enveloping algebras the corresponding identification of Poisson-Lie groups was originally proved by De Concini and Procesi in [DCP93]. In particular they showed that the $t = 1$ limit of the Poisson integral form $\mathrm{U}_t^{Poiss}(\mathfrak{g})$ of $\mathrm{U}_t(\mathfrak{g})$ coincides with the coordinate algebra of $\boldsymbol{G}^{\vee}$.

We could of course use [DCP93] to show that $\boldsymbol{G}^{\vee}_{sc}$ and $\boldsymbol{G}^{\vee}$ are isomorphic. Indeed in Theorem 16.2.4 we identified the extension counting integral form DH_{ex} of the Bridgeland-Hall algebra with $\mathrm{U}_t^{Poiss}(\mathfrak{g})$. It should then follow that the semi-classical Bridgeland-Hall algebra DH_{sc} is also isomorphic to $\mathbb{C}[\boldsymbol{G}^{\vee}]$.

We will opt to give a direct proof, however, that $\boldsymbol{G}^{\vee}_{sc}$ and $\boldsymbol{G}^{\vee}$ are isomorphic rather than using De Concini and Procesi's result. There are two reasons why we choose to do so. The first is that De Concini and Procesi's proof involves a lengthy case-by-case computation. Using machinery of Bridgeland-Hall algebras we feel that one can give a direct and in our view more conceptual proof of the fact that $\boldsymbol{G}^{\vee}_{sc}$ is isomorphic to $\boldsymbol{G}^{\vee}$.

The second (minor) obstacle to simply directly using De Concini and Procesi's result is that, as outlined in Section 5.5, the way that $\mathrm{U}_t^{Poiss}(\mathfrak{g})$ is defined in [DCP93] differs slightly from the one we have given in Section 5.5.

19.1 Subgroups of Semi-Classical Poisson-Lie Groups

In this section we characterize various subgroups of the semi-classical Poisson-Lie group $\boldsymbol{G}^{\vee}_{sc}$. We will use these characterizations to construct an explicit isomorphism between $\boldsymbol{G}^{\vee}_{sc}$ and $\boldsymbol{G}^{\vee}$ in Section 19.2.

Let's introduce semi-classical analogues of the subgroups $\boldsymbol{B}^{\vee}_{\pm}$, $\boldsymbol{N}^{\vee}_{\pm}$ and $\boldsymbol{T}^{\vee}$ of $\boldsymbol{G}^{\vee}$ discussed at the end of Section 3.3. Define subgroups $\boldsymbol{B}^{\vee}_{sc,+}$, $\boldsymbol{B}^{\vee}_{sc,-}$ and $\boldsymbol{T}^{\vee}_{sc}$ of $\boldsymbol{G}^{\vee}_{sc}$ to be those given by the Hopf ideals $(\overline{F}_I)$, $(\overline{E}_I)$ and $(\overline{E}_I, \overline{F}_J)$ of DH_{sc} respectively. Also let $\boldsymbol{N}^\vee_{sc,\pm}$ be the subgroup of $\boldsymbol{B}^\vee_{sc,\pm}$ determined by the Hopf ideal $(\overline{K}_{\hat{S}_i} - 1)$ of the coordinate algebra of $\boldsymbol{B}^\vee_{sc,\pm}$.

For convenience we will use notation of the form, for example, $\mathbb{C}[\boldsymbol{B}^\vee_{sc,+}] := \mathrm{DH}_{sc}/(\overline{F}_I)$ for the coordinate algebras of these subgroups. We will abuse notation and simply write, for example, $\overline{E}_I$ for the image of $\overline{E}_I$ in the quotient Hopf algebra $\mathrm{DH}_{sc}/(\overline{F}_I)$.

The reason we are considering these subgroups is that, as we will see, $\boldsymbol{G}^\vee_{sc}$ is almost $\boldsymbol{B}^\vee_{sc,+} \times \boldsymbol{B}^\vee_{sc,-}$ and each $\boldsymbol{B}^\vee_{sc,\pm}$ is a semi-direct product $\boldsymbol{N}^\vee_{sc,\pm} \rtimes \boldsymbol{T}^\vee_{sc}$. Moreover as outlined in Section 3.3 the group $\boldsymbol{G}^\vee$ is almost $\boldsymbol{B}_- \times \boldsymbol{B}_+$ and it is well known that $\boldsymbol{B}_\pm$ is a semi-direct product $\boldsymbol{N}_\pm \rtimes \boldsymbol{T}$. Thus if we want to build an isomorphism $\boldsymbol{G}^\vee_{sc}$ between and $\boldsymbol{G}^\vee$ we should first relate these two collections of groups. Indeed the remainder of this section is devoted to constructing isomorphisms of algebraic groups between $\boldsymbol{B}^\vee_{sc,\pm}$ and $\boldsymbol{B}_\mp$ by constructing isomorphisms $\boldsymbol{N}^\vee_{sc,\pm} \to \boldsymbol{N}_\mp$ and $\boldsymbol{T}^\vee_{sc} \to \boldsymbol{T}$.

Proposition 19.1.1. There are isomorphisms of unipotent algebraic groups $\nu_\pm : \boldsymbol{N}^\vee_{sc,\pm} \to \boldsymbol{N}_\mp$ inducing the following isomorphisms of Lie algebras $d_e\nu_\pm : \mathfrak{n}^\vee_{sc,\pm} \to \mathfrak{n}_\mp$.

$$\begin{aligned} \mathfrak{n}^\vee_{sc,+} &\to \mathfrak{n}_- \\ e^\vee_{I_\alpha} &\mapsto -f_\alpha \end{aligned} \tag{19.1}$$

$$\begin{aligned} \mathfrak{n}^\vee_{sc,-} &\to \mathfrak{n}_+ \\ f^\vee_{I_\alpha} &\mapsto -e_\alpha \end{aligned} \tag{19.2}$$

Proof. We will first construct the isomorphisms of nilpotent Lie algebras in Equation (19.1) and Equation (19.2) before using Baker-Campbell-Hausdorff to obtain the required isomorphisms $\nu_\pm$ of unipotent groups.

It is easy to see that the isomorphism from Corollary 18.4.1 between $\mathfrak{g}^\vee_{sc}$ and $\mathfrak{g}^\vee$ restricts to isomorphisms between the subalgebras $\mathfrak{n}^\vee_{sc,\pm}$ and $\mathfrak{n}^\vee_\pm$. Moreover the isomorphisms $\mathfrak{n}^\vee_\pm \to \mathfrak{n}_\mp$ from Equation (3.6) restrict to give isomorphisms $\mathfrak{n}^\vee_\pm \to \mathfrak{n}_\mp$. Combining these we get the isomorphisms of Lie algebras given in Equation (19.1) and Equation (19.2).

Now since the $\mathfrak{n}_\mp$ are finite dimensional nilpotent Lie algebras then so too are the $\mathfrak{n}^\vee_{sc,\pm}$. Using the Baker-Campbell-Hausdorff formula one can upgrade these Lie algebras to algebraic groups which we will also denote by $\mathfrak{n}^\vee_{sc,\pm}$ and $\mathfrak{n}_\mp$ by abuse of notation. The $\mathbb{C}$-valued points of these algebraic groups are precisely the underlying complex vector spaces of the Lie algebras $\mathfrak{n}^\vee_{sc,\pm}$ and $\mathfrak{n}_\mp$. Moreover the isomorphisms of Lie algebras in Equation (19.1) and Equation (19.2) induce isomorphisms of algebraic groups $u_\pm : \mathfrak{n}^\vee_{sc,\pm} \to \mathfrak{n}_\mp$, that is on $\mathbb{C}$-valued points we have the following.

$$u_+(e^\vee_{I_\alpha}) = -f_\alpha \qquad\qquad u_-(f^\vee_{I_\alpha}) = -e_\alpha \tag{19.3}$$

It is well known that there are canonical exponential and logarithm maps giving algebraic group isomorphisms between finite dimensional unipotent groups and their nilpotent Lie algebras, viewed as algebraic groups. Thus we have a chain of isomorphisms of algebraic groups whose composition we denote by $\nu_\pm : \boldsymbol{N}^\vee_{sc,\pm} \to \boldsymbol{N}_\mp$.

$$\boldsymbol{N}^\vee_{sc,\pm} \xrightarrow{log} \mathfrak{n}^\vee_{sc,\pm} \xrightarrow{u_\pm} \mathfrak{n}_\mp \xrightarrow{exp} \boldsymbol{N}_\mp, \qquad \nu_\pm = exp \circ u_\pm \circ log \tag{19.4}$$

Since the differential of the exponential and logarithm maps at the group identities give the identity Lie algebra homomorphism one can check that $d_e\nu_\pm$ gives the claimed isomorphisms of Lie algebras. □

Let's turn our attention to the tori. We'll show first that there are two canonical isomorphisms $\tau_\pm : \boldsymbol{T}^\vee_{sc} \to \boldsymbol{T}$. Since $\boldsymbol{T}$ is a complex algebraic torus then $\mathbb{C}[\boldsymbol{T}]$ is canonically the group algebra $\mathbb{C}[X^\bullet(\boldsymbol{T})]$ of the character lattice $X^\bullet(\boldsymbol{T})$ of $\boldsymbol{T}$. Since we assumed that $\boldsymbol{G}$ is of adjoint type in Section 1.1 then the theory of simple Lie groups says that then $X^\bullet(\boldsymbol{T})$ is canonically identified with the root lattice Λ_Φ and hence with the Grothendieck group $K(\mathcal{A})$ by definition from Section 7.2.

Denoting by $k_\alpha \in \mathbb{C}[X^\bullet(\boldsymbol{T})]$ the function corresponding to an element α of the root lattice Λ_Φ or equivalently $K(\mathcal{A})$ then the isomorphisms $\tau_\pm : \boldsymbol{T}^\vee_{sc} \to \boldsymbol{T}$ are determined by the following.

$$\begin{aligned} \tau_+^* : \mathbb{C}[\boldsymbol{T}] &\to \mathbb{C}[\boldsymbol{T}^\vee_{sc}] \\ k_\alpha &\mapsto \overline{K}_{-\alpha} \end{aligned} \tag{19.5}$$

$$\begin{aligned} \tau_-^* : \mathbb{C}[\boldsymbol{T}] &\to \mathbb{C}[\boldsymbol{T}^\vee_{sc}] \\ k_\alpha &\mapsto \overline{K}_{\alpha} \end{aligned} \tag{19.6}$$

Proposition 19.1.2. The isomorphisms of algebraic tori $\tau_\pm : \boldsymbol{T}^\vee_{sc} \to \boldsymbol{T}$ induce isomorphisms of Cartan subalgebras $d_e\tau_\pm : \mathfrak{h}^\vee_{sc} \to \mathfrak{h}$ given by $h^\vee_{\hat{S}_i} \mapsto \mp\frac{1}{2}h_i$.

Proof. Let's check that $d_e\tau_+(h^\vee_{\hat{S}_i}) = -1/2\ h_i$ as the other calculation is similar. Recall that by definition we have $\mathfrak{h}_{sc} = T^*_e\boldsymbol{T}^\vee_{sc}$ and $\mathfrak{h}^* = T^*_e\boldsymbol{T}$. The map $\tau_+ : \boldsymbol{T}^\vee_{sc} \to \boldsymbol{T}$ induces the pullback cotangent spaces at the identity $(d_e\tau_+)^* : \mathfrak{h}^* \to \mathfrak{h}_{sc}$. Applying this to a simple root α_j in $\mathfrak{h}^*$ we have the following identity in $\mathfrak{h}_{sc}$.

$$(d_e\tau_+)^*(\alpha_j) = (d_e\tau_+)^*(d_e k_{\alpha_j}) = d_e\overline{K}_{-\hat{S}_j} = -\frac{1}{2}h_{\hat{S}_j}$$

That we have $d_e\tau_+(h^\vee_{\hat{S}_i}) = -\frac{1}{2}h_i$ then follows from the following computation.

$$\alpha_j(d_e\tau_+(h^\vee_{\hat{S}_i})) = (d_e\tau_+)^*(\alpha_j)(h^\vee_{\hat{S}_i}) = -\frac{1}{2}h_{\hat{S}_j}(h^\vee_{\hat{S}_i}) = -\frac{1}{2}a_{ij}$$

□

We now consider the case of the Borel subgroups.

Lemma 19.1.1. As an algebraic group each $\boldsymbol{B}^\vee_{sc,\pm}$ splits as a semi-direct product $\boldsymbol{N}^\vee_{sc,\pm} \rtimes \boldsymbol{T}^\vee_{sc}$.

Proof. We will treat the case of $\boldsymbol{B}^\vee_{sc,+}$ as the case of $\boldsymbol{B}^\vee_{sc,-}$ is similar. We must show that there is a homomorphism of algebraic groups $\boldsymbol{B}^\vee_{sc,+} \to \boldsymbol{T}^\vee_{sc}$ restricting to the identity on the subgroup $\boldsymbol{T}^\vee_{sc} \subset \boldsymbol{B}^\vee_{sc,+}$ and with kernel $\boldsymbol{N}^\vee_{sc,+}$. To this end consider the homomorphism $\boldsymbol{B}^\vee_{sc,+} \to \boldsymbol{T}^\vee_{sc}$ given by the following homomorphism of Hopf algebras.

$$\mathbb{C}[\boldsymbol{T}^\vee_{sc}] \to \mathbb{C}[\boldsymbol{B}^\vee_{sc,+}], \qquad \overline{K}_\alpha \mapsto \overline{K}_\alpha \tag{19.7}$$

That the restriction of $\boldsymbol{B}^\vee_{sc,+} \to \boldsymbol{T}^\vee_{sc}$ to $\boldsymbol{T}^\vee_{sc} \subset \boldsymbol{B}^\vee_{sc,+}$ is the identity comes from the fact that the map in Equation (19.7) composed with the quotient map $\mathbb{C}[\boldsymbol{B}^\vee_{sc,+}] \to \mathbb{C}[\boldsymbol{T}^\vee_{sc}]$ is the identity. It remains to check the kernel condition i.e. that $\boldsymbol{N}^\vee_{sc,+}$ is the fibre over the torus group identity of the homomorphism $\boldsymbol{B}^\vee_{sc,+} \to \boldsymbol{T}^\vee_{sc}$. However one can easily verify the equivalent Hopf algebraic statement that the following is a pushout diagram in the category of Hopf algebras over $\mathbb{C}$.

$$\begin{array}{ccc} \mathbb{C}[\boldsymbol{N}^\vee_{sc,+}] & \longleftarrow & \mathbb{C}[\boldsymbol{B}^\vee_{sc,+}] \\ \uparrow & & \uparrow \\ \mathbb{C} & \longleftarrow & \mathbb{C}[\boldsymbol{T}^\vee_{sc}] \end{array}$$

Here the top map is the quotient map by the ideal $(\overline{K}_\alpha - 1)$, the right-hand map is the homomorphism from Equation (19.7) and the bottom one is given by the Hopf algebra counit sending $\overline{K}_\alpha \mapsto 1$. □

In Proposition 19.1.3 we will show that $\boldsymbol{B}^\vee_{sc,\pm}$ and $\boldsymbol{B}_\mp$ are isomorphic as semi-direct products. In order to do so will need to discuss how actions of tori on nilpotent Lie algebras induce root space decompositions. For the remainder of this section will continue to abuse notation and view nilpotent Lie algebras as schemes rather than as complex vector spaces. For example $\mathfrak{n}_-$ will mean the Lie algebra viewed as an algebraic group via Baker-Campbell-Hausdorff rather than the underlying vector space.

Recall that for a simple Lie group the adjoint action of the maximal torus $\boldsymbol{T}$ on the nilpotent Lie algebra $\mathfrak{n}_-$ induces a decomposition of varieties into one dimensional root spaces.

$$\mathfrak{n}_- = \prod_{\alpha \text{ a positive root}} \mathfrak{n}[-\alpha] \tag{19.8}$$

Letting A denote a $\mathbb{C}$-algebra, the adjoint action $Ad : \boldsymbol{T} \times \mathfrak{n}_- \to \mathfrak{n}_-$ is determined by the following where g and ξ are A-valued points of the torus $\boldsymbol{T}$ and root space $\mathfrak{n}[-\alpha]$ respectively.

$$Ad_g(\xi) = k_{-\alpha}(g)\xi \tag{19.9}$$

An analogous decomposition of $\mathfrak{n}^\vee_{sc,+}$ is also induced from the adjoint action of the torus $\boldsymbol{T}^\vee_{sc}$ on $\mathfrak{n}^\vee_{sc,+}$. First note that the basis of $\mathfrak{n}^\vee_{sc,+}$ given by the vectors $e^\vee_{I_\alpha}$ in Equation (18.1) induces the following decomposition of varieties into one dimensional spaces.

$$\mathfrak{n}^\vee_{sc,+} = \prod_{\alpha \text{ a positive root}} \mathfrak{n}^\vee_{sc}[\alpha] \tag{19.10}$$

Recall that in Section 1.2 we gave a formula for the adjoint action of an algebraic group on its Lie algebra. Applying this to the algebraic group $\boldsymbol{B}^\vee_{sc,+}$ and suitably restricting, one obtains an adjoint action of the torus $\boldsymbol{T}^\vee_{sc} \subset \boldsymbol{B}^\vee_{sc,+}$ on the nilpotent Lie algebra $\mathfrak{n}^\vee_{sc,+} \subset \mathfrak{b}^\vee_{sc,+}$.

A straightforward calculation using Section 1.2, along with the formulas for the coproduct and antipode of the Bridgeland-Hall algebra given in Corollary 14.1.2, shows that the adjoint action $Ad : \boldsymbol{T}^\vee_{sc} \times \mathfrak{n}^\vee_{sc,+} \to \mathfrak{n}^\vee_{sc,+}$ is determined by the following where g and ξ are A-valued points of $\boldsymbol{T}^\vee_{sc}$ and $\mathfrak{n}^\vee_{sc}[\alpha]$ respectively.

$$Ad_g(\xi) = \overline{K}_\alpha(g)\xi \tag{19.11}$$

We now relate the algebraic groups $\boldsymbol{B}^\vee_{sc,\pm}$ and $\boldsymbol{B}_\mp$.

Proposition 19.1.3. There are isomorphisms of algebraic groups $b_\pm : \boldsymbol{B}^\vee_{sc,\pm} \to \boldsymbol{B}_\mp$ inducing the following isomorphisms of Lie algebras $d_e b_\pm : \mathfrak{b}^\vee_{sc,\pm} \to \mathfrak{b}_\mp$.

$$\begin{aligned} \mathfrak{b}^\vee_{sc,+} &\to \mathfrak{b}_- \\ e^\vee_{I_\alpha} &\mapsto -f_\alpha \\ h^\vee_{\hat{S}_i} &\mapsto -\frac{1}{2}h_i \end{aligned} \tag{19.12}$$

$$\begin{aligned} \mathfrak{b}^\vee_{sc,-} &\to \mathfrak{b}_+ \\ f^\vee_{I_\alpha} &\mapsto -e_\alpha \\ h^\vee_{\hat{S}_i} &\mapsto \frac{1}{2}h_i \end{aligned} \tag{19.13}$$

Proof. It is well known that for Borel subgroups of simple algebraic groups one has $\boldsymbol{B}_\mp = \boldsymbol{N}_\mp \rtimes \boldsymbol{T}$. Similarly in Lemma 19.1.1 we showed that $\boldsymbol{B}^\vee_{sc,\pm} = \boldsymbol{N}^\vee_{sc,\pm} \rtimes \boldsymbol{T}^\vee_{sc}$. Now by Proposition 19.1.1 and Proposition 19.1.2 we have isomorphisms of algebraic groups $\nu_\pm : \boldsymbol{N}^\vee_{sc,\pm} \to \boldsymbol{N}_\mp$ and $\tau_\pm : \boldsymbol{T}^\vee_{sc} \to \boldsymbol{T}$. We will show that $b_\pm := \nu_\pm \times \tau_\pm$ gives an isomorphism of semi-direct products $\boldsymbol{N}^\vee_{sc,\pm} \rtimes \boldsymbol{T}^\vee_{sc} \to \boldsymbol{N}_\mp \rtimes \boldsymbol{T}$. In particular we observe that by Proposition 19.1.1 and Proposition 19.1.2 the induced maps of Lie algebras are the ones claimed in Equation (19.12) and Equation (19.13).

We will show that b_+ is an isomorphism with the case of b_- being similar. Since ν_+ and τ_+ are isomorphisms of algebraic groups then $b_+ := \nu_+ \times \tau_+$ will be an isomorphism of semi-direct products if we can show that it preserves the adjoint actions of $\boldsymbol{T}^\vee_{sc}$ on $\boldsymbol{N}^\vee_{sc,+}$ and $\boldsymbol{T}$ on $\boldsymbol{N}_-$. In particular, recalling how ν_+ was defined in Equation (19.4), we need to show that the following diagram commutes.

$$\begin{array}{ccccccc} \boldsymbol{T}^\vee_{sc} \times \boldsymbol{N}^\vee_{sc,+} & \xrightarrow{id\times log} & \boldsymbol{T}^\vee_{sc} \times \mathfrak{n}^\vee_{sc,+} & \xrightarrow{\tau_+ \times u_+} & \boldsymbol{T} \times \mathfrak{n}_- & \xrightarrow{id\times exp} & \boldsymbol{T} \times \boldsymbol{N}_- \\ \downarrow Ad & & \downarrow Ad & & \downarrow Ad & & \downarrow Ad \\ \boldsymbol{N}^\vee_{sc,+} & \xrightarrow{log} & \mathfrak{n}^\vee_{sc,+} & \xrightarrow{u_+} & \mathfrak{n}_- & \xrightarrow{exp} & \boldsymbol{N}_- \end{array}$$

We need only verify that the central square commutes as the cases of the left-hand and right-hand squares follow from the fact that adjoint actions commute with exponentiation. Letting A denote a $\mathbb{C}$-algebra, take g and ξ to be A-valued points of $\boldsymbol{T}^\vee_{sc}$ and $\mathfrak{n}^\vee_{sc}[\alpha]$ respectively. We need to check that we have the following equality of A-valued points of $\mathfrak{n}_-$.

$$Ad_{\tau_+(g)} \circ u_+(\xi) = u_+ \circ Ad_g(\xi) \tag{19.14}$$

To establish Equation (19.14) we will expand the left-hand and right-hand sides while explaining what is going in words as the notation is a little cumbersome. Expanding the left-hand side we obtain the following.

$$Ad_{\tau_+(g)} \circ u_+(\xi) = k_{-\alpha}(\tau_+(g))u_+(\xi) = \tau_+^*(k_{-\alpha})(g)u_+(\xi) \tag{19.15}$$

To explain Equation (19.15) we first note that map $u_+ : \mathfrak{n}^\vee_{sc,+} \to \mathfrak{n}_-$ preserves the root space decompositions from Equation (19.8) and Equation (19.10). This is true since u_+ preserves root vectors $e^\vee_{I_\alpha} \mapsto -f_\alpha$ by Equation (19.3). In particular we have that $u_+(\xi)$ is an A-valued point of the root space $\mathfrak{n}[-\alpha]$. Moreover by definition $\tau_+ : \boldsymbol{T}^\vee_{sc} \to \boldsymbol{T}$ sends the point g of $\boldsymbol{T}^\vee_{sc}(A)$ to $\tau_+(g)$ in $\boldsymbol{T}(A)$.

Now for the first equality in Equation (19.15) we use Equation (19.11) to see that the adjoint action of $\tau_+(g)$ on $u_+(\xi)$ gives $k_{-\alpha}(\tau_+(g))u_+(\xi)$. For the second equality we note that evaluating the function $k_{-\alpha}$ on the point $\tau_+(g)$ is the same as evaluating the pullback function $\tau_+^*(k_{-\alpha})$ on the point g.

Expanding the right-hand side of Equation (19.14) we have the following.

$$u_+ \circ Ad_g(\xi) = u_+(\overline{K}_\alpha(g)\xi) = \overline{K}_\alpha(g)u_+(\xi) = \tau_+^*(k_{-\alpha})(g)u_+(\xi) \tag{19.16}$$

For the first equality in Equation (19.16) we use the fact from Equation (19.11) that the adjoint action of g on ξ gives $\overline{K}_\alpha(g)\xi$. For the second equality we use the A-linearity of u_+ while the last equality comes from Equation (19.5) which says that $\tau_+^*(k_{-\alpha}) = \overline{K}_\alpha$.

Combining Equation (19.15) and Equation (19.16) we obtain Equation (19.14). □

19.2 Statement and Proof

In this section we give an isomorphism of Poisson-Lie groups between the semi-classical Poisson-Lie group $\boldsymbol{G}_{sc}^\vee$ and the dual Poisson-Lie group $\boldsymbol{G}^\vee$ using the results from the previous section. In particular the goal of this section is to prove the following theorem.

Theorem 19.2.1. There is an isomorphism of Poisson-Lie groups $g : \boldsymbol{G}_{sc}^\vee \to \boldsymbol{G}^\vee$.

Let us assume for the moment that there exists an isomorphism $g : \boldsymbol{G}_{sc}^\vee \to \boldsymbol{G}^\vee$, a priori only of *algebraic groups*. We will suppose in addition that the induced isomorphism of tangent Lie algebras is in fact an isomorphism of Lie bialgebras. Under this hypothesis we would have the following proof of Theorem 19.2.1.

Proof. We have two Poisson-Lie groups $\boldsymbol{G}_{sc}^\vee$ and $\boldsymbol{G}^\vee$ which are isomorphic as algebraic groups. The only extra condition we need to check for g to be an isomorphism of Poisson-Lie groups is that it is Poisson. Now g being Poisson is equivalent to the requirement that the pushforward of the semi-classical Poisson bracket $\{-,-\}_{sc}$ under g coincides with the Poisson bracket $\{-,-\}$ on $\boldsymbol{G}^\vee$. Here the pushforward is given by the following formula where $a, b \in \mathbb{C}[\boldsymbol{G}^\vee]$.

$$g_*\{a,b\}_{sc} := \{a \circ g, b \circ g\}_{sc} \circ g^{-1}$$

Note that by virtue of the fact that g is an isomorphism of algebraic groups, $g_*\{-,-\}_{sc}$ endows $\boldsymbol{G}^\vee$ with the structure of a Poisson-Lie group. We then have two (a priori distinct) Poisson structures $g_*\{-,-\}_{sc}$ and $\{-,-\}$ on the algebraic group $\boldsymbol{G}^\vee$ which both turn it into a Poisson-Lie group. Moreover since g induces an isomorphism of tangent Lie bialgebras then both $g_*\{-,-\}_{sc}$ and $\{-,-\}$ give $\boldsymbol{G}^\vee$ the exact same tangent Lie bialgebra structure

Stepping for a moment into the category of complex manifolds (as opposed to complex varieties) Theorem 11.39 (1) of [LGPV13] says the following: given a complex Lie group $\boldsymbol{K}$ and the structure of a complex Lie bialgebra on its tangent Lie algebra then there is at most one Poisson structure on $\boldsymbol{K}$ turning it into a complex Poisson-Lie group inducing the prescribed Lie bialgebra.

It is easy to see that this implies that $g_*\{-,-\}_{sc} = \{-,-\}$ on our algebraic group $\boldsymbol{G}^\vee$.

□

The remainder of this section is dedicated to showing the hypothesized existence of an isomorphism $g : \boldsymbol{G}_{sc}^\vee \to \boldsymbol{G}$ inducing an isomorphism of tangent Lie bialgebras.

Theorem 19.2.2. There is an isomorphism of algebraic groups $g : \boldsymbol{G}^\vee_{sc} \to \boldsymbol{G}^\vee$.

Proof. We will show that the isomorphisms $b_+ \times b_- : \boldsymbol{B}^\vee_{sc,+} \times \boldsymbol{B}^\vee_{sc,-} \to \boldsymbol{B}_- \times \boldsymbol{B}_+$ induced by Proposition 19.1.3 descends to an isomorphism $g : \boldsymbol{G}^\vee_{sc} \to \boldsymbol{G}^\vee$. Recall from Section 3.3 that the algebraic group $\boldsymbol{G}^\vee$ fits into the following short exact sequence of algebraic groups.

$$\boldsymbol{G}^\vee \xrightarrow{i^\vee} \boldsymbol{B}_- \times \boldsymbol{B}_+ \xrightarrow{\pi} \boldsymbol{T} \tag{19.17}$$

Here π was the product of the two canonical projections to the torus while $i^\vee$ was the inclusion of the kernel. We will define maps so that the following is an analogous short exact sequence of semi-classical groups. Showing that Equation (19.17) is isomorphic to Equation (19.18) as short exact sequences will provide us with the required isomorphism g.

$$\boldsymbol{G}^\vee_{sc} \xrightarrow{i^\vee_{sc}} \boldsymbol{B}^\vee_{sc,+} \times \boldsymbol{B}^\vee_{sc,-} \xrightarrow{\pi_{sc}} \boldsymbol{T}^\vee_{sc} \tag{19.18}$$

Let's define the maps π_{sc} and $i^\vee_{sc}$ and show that they do indeed give a short exact sequence. Recall that the coordinate algebras of $\boldsymbol{T}^\vee_{sc}$ and $\boldsymbol{B}^\vee_{sc,\pm}$ were defined as quotient Hopf algebras at the beginning of Section 19.1. Using these definitions along with the Hopf algebra formulas in Corollary 14.1.2 one can check that the following is a homomorphism of Hopf algebras. This gives the map π_{sc}.

$$\mathbb{C}[\boldsymbol{T}^\vee_{sc}] \to \mathbb{C}[\boldsymbol{B}^\vee_{sc,+}] \otimes_{\mathbb{C}} \mathbb{C}[\boldsymbol{B}^\vee_{sc,-}], \qquad \overline{K}_\alpha \mapsto \overline{K}_\alpha \otimes \overline{K}^*_\alpha$$

Similarly the following two Hopf algebra homomorphisms give the two maps $\boldsymbol{G}^\vee_{sc} \to \boldsymbol{B}^\vee_{sc,\pm}$ determining $i^\vee_{sc}$.

$$\begin{aligned} \mathbb{C}[\boldsymbol{B}^\vee_{sc,+}] &\to \mathbb{C}[\boldsymbol{G}^\vee_{sc}] \\ \overline{E}_I &\mapsto \overline{E}_I \\ \overline{K}_\alpha &\mapsto \overline{K}_\alpha \end{aligned} \tag{19.19}$$

$$\begin{aligned} \mathbb{C}[\boldsymbol{B}^\vee_{sc,-}] &\to \mathbb{C}[\boldsymbol{G}^\vee_{sc}] \\ \overline{F}_I &\mapsto \overline{F}_I \\ \overline{K}^*_\alpha &\mapsto \overline{K}^*_\alpha \end{aligned} \tag{19.20}$$

It is easy to see that π_{sc} is surjective. That $i^\vee_{sc}$ is the kernel of π_{sc} is equivalent to verifying that the following is a pushout in the category of $\mathbb{C}$-Hopf algebras which one can readily check.

$$\begin{array}{ccc} \mathbb{C}[\boldsymbol{G}^\vee_{sc}] & \xleftarrow{(i^\vee_{sc})^*} & \mathbb{C}[\boldsymbol{B}^\vee_{sc,+}] \otimes_{\mathbb{C}} \mathbb{C}[\boldsymbol{B}^\vee_{sc,-}] \\ \uparrow & & \uparrow \pi^*_{sc} \\ \mathbb{C} & \xleftarrow{\overline{\varepsilon}} & \mathbb{C}[\boldsymbol{T}^\vee_{sc}] \end{array}$$

Here bottom map is the counit given by $\overline{\varepsilon}(\overline{K}_\alpha) = 1$ which picks out the group identity of the torus.

It remains to give the isomorphism of short exact sequences between Equation (19.18) and Equation (19.17). Recall from Proposition 19.1.3 and Proposition 19.1.2 that we have isomorphisms of algebraic groups $b_\pm : \boldsymbol{B}^\vee_{sc,\pm} \to \boldsymbol{B}_\mp$ and $\tau_+ : \boldsymbol{T}^\vee_{sc} \to \boldsymbol{T}$. Using the semi-direct product decomposition of the Borel subgroups involved one can check the right-hand square in the diagram below commutes. This induces a required isomorphism of algebraic groups $g : \boldsymbol{G}^\vee_{sc} \to \boldsymbol{G}^\vee$ via the universal property of the kernel.

$$\begin{array}{ccccc} \boldsymbol{G}^\vee_{sc} & \xrightarrow{i^\vee_{sc}} & \boldsymbol{B}^\vee_{sc,+} \times \boldsymbol{B}^\vee_{sc,-} & \xrightarrow{\pi_{sc}} & \boldsymbol{T}^\vee_{sc} \\ \downarrow g & & \downarrow b_+ \times b_- & & \downarrow \tau_+ \\ \boldsymbol{G}^\vee & \xrightarrow{i^\vee} & \boldsymbol{B}_- \times \boldsymbol{B}_+ & \xrightarrow{\pi} & \boldsymbol{T} \end{array} \tag{19.21}$$

□

The only thing we haven't yet checked is that the isomorphism of algebraic groups $g : \boldsymbol{G}^\vee_{sc} \to \boldsymbol{G}^\vee$ does what we want on the level of Lie bialgebras.

Proposition 19.2.3. The isomorphism of algebraic groups $g : \boldsymbol{G}^\vee_{sc} \to \boldsymbol{G}^\vee$ induces exactly the isomorphism of tangent Lie bialgebras that we constructed in Corollary 18.4.1.

Proof. Recall that the isomorphism of tangent Lie bialgebras $\theta^\vee : \mathfrak{g}^\vee_{sc} \to \mathfrak{g}^\vee$ from Corollary 18.4.1 was given by the following.

$$\theta^\vee(e^\vee_{I_\alpha}) = e^\vee_\alpha \qquad \theta^\vee(f^\vee_{I_\alpha}) = -f^\vee_\alpha \qquad \theta^\vee(h^\vee_{\hat{S}_i}) = h^\vee_i$$

We will check that $d_e g$ coincides with $\theta^\vee$ using the following commutative diagram of tangent Lie algebras induced from Equation (19.21).

$$\begin{array}{ccc} \mathfrak{g}^\vee_{sc} & \xrightarrow{d_e i^\vee_{sc}} & \mathfrak{b}^\vee_{sc,+} \oplus \mathfrak{b}^\vee_{sc,-} \\ \downarrow{\scriptstyle d_e g} & & \downarrow{\scriptstyle d_e b_+ \oplus d_e b_-} \\ \mathfrak{g}^\vee & \xrightarrow{d_e i^\vee} & \mathfrak{b}_- \oplus \mathfrak{b}_+ \end{array} \tag{19.22}$$

We will compute that $d_e g = \theta^\vee$ using the three other maps in this diagram. For the first map using Equation (19.19) and Equation (19.20) a simple calculation shows that $d_e i^\vee_{sc}$ is given by the following.

$$e^\vee_{I_\alpha} \mapsto (e^\vee_{I_\alpha}, 0) \qquad f^\vee_{I_\alpha} \mapsto (0, f^\vee_{I_\alpha}) \qquad h^\vee_{\hat{S}_i} \mapsto (h^\vee_{\hat{S}_i}, h^\vee_{\hat{S}_i})$$

For the next map it follows from Proposition 19.1.3 that under $d_e b_+ \oplus d_e b_-$ we have the following.

$$(e^\vee_{I_\alpha}, 0) \mapsto (-f_\alpha, 0) \qquad (0, f^\vee_{I_\alpha}) \mapsto (0, -e_\alpha) \qquad (h^\vee_{\hat{S}_i}, h^\vee_{\hat{S}_i}) \mapsto \frac{1}{2}(-h_i, h_i)$$

Finally from Section 3.3 we have that $d_e i^\vee$ is simply the inclusion of the subalgebra $\mathfrak{g}^\vee \subset \mathfrak{b}_- \oplus \mathfrak{b}_+$ since $i^\vee$ is the inclusion of the subgroup $\boldsymbol{G}^\vee \subset \boldsymbol{B}_- \times \boldsymbol{B}_+$. We arrive at our result by combining the above observations with the fact that the basis vectors for $\mathfrak{g}^\vee$ were defined in Equation (3.5) to be the following.

$$e^\vee_\alpha := (-f_\alpha, 0) \qquad f^\vee_\alpha := (0, e_\alpha) \qquad h^\vee_i := \frac{1}{2}(-h_i, h_i)$$

□

Glossary and Bibliography

Glossary

Hall Algebras and Bridgeland-Hall Algebras

$D_{A,B}$	Basis vector of Bridgeland-Hall algebra 70
E_L	Basis vector of (Bridgeland)-Hall algebra; $E_L = a_L X_L$ 45, 69
F_L	Basis vector of (Bridgeland)-Hall algebra; $F_L = a_L Y_L$ 54, 69
R	Isomorphism between quantized enveloping algebra and generic Bridgeland-Hall algebra 73
S	Antipode of various Hopf algebras such as DH; underlying algebra indicated by context 77
X_L	Basis vector of (Bridgeland)-Hall algebra; $X_L = E_L/a_L$ 46, 69
Y_L	Basis vector of (Bridgeland)-Hall algebra; $Y_L = F_L/a_L$ 54, 69
Δ	Coproduct of various Hopf algebras such as DH; underlying algebra indicated by context 77
$*$	$\mathbb{Z}_2$-graded complexes shift functor involution; induced involution on Bridgeland-Hall algebras 12
$\mathcal{T}_i$	BGP reflection functor induced isomorphisms of Bridgeland-Hall algebras 84
$\overline{R}$	Modification of isomorphism R 73
DH_{ex}	Extension counting integral form of generic Bridgeland-Hall algebra DH 78
DH_{fl}	Flag counting integral form of generic Bridgeland-Hall algebra DH 81
DH_{loc}	Generic localized Bridgeland-Hall algebra 67
DH_q	Non-generic Bridgeland-Hall algebra 36
DH_{sc}	Semi-classical Bridgeland-Hall algebra; $t = 1$ limit of DH_{ex} 93
DH	Generic Bridgeland-Hall algebra 68
$\mathrm{H}(\mathcal{C})$	Generic Hall algebra of category of $\mathbb{Z}_2$-graded complexes in projective representations 66
H_{ex}	Extension counting integral form of generic Hall algebra H 47
H_{fl}	Flag counting integral form of generic Hall algebra H 50
H_{qc}	Quasi-classical Hall algebra H_{fl} 51
H_q	Non-generic Hall algebra of category of quiver representations $\mathcal{A}_q$ 34
H_{sc}	Semi-classical Hall algebra; $t = 1$ limit of H_{ex} 48
H	Generic Hall algebra 45
ε	Counit of various Hopf algebras such as DH; underlying algebra indicated by context 77
$\{-,-\}_{sc}$	Semi-classical Poisson bracket on H_{sc} and DH_{sc} 48, 94

Structure Constants for Algebras and Lie Algebras

$\Gamma_{M,N}^{A,B}$	Hall algebra structure constant for Poisson brackets 95
$\Gamma_{M,N}^{L}$	Hall algebra structure constant for Poisson brackets and Lie algebras 53, 95
$\mathbb{P}(e)_{M,N}^{L}$	Polynomial counting projectivization of set $\mathrm{Ext}^1(M,N)_L$ 43
$\mathbb{P}(h)_{M,N}^{A,B}$	Polynomial counting projectivization of set of maps $M \to N$ with kernel A and cokernel B 64
$a_{L_\bullet}$	Polynomial counting automorphisms of complex $L_\bullet$ 64
a_L	Polynomial counting automorphisms of representation L 42
$e_{M_\bullet,N_\bullet}^{L_\bullet}$	Polynomial counting extensions of complexes $M_\bullet$ by $N_\bullet$ with middle term isomorphic to $L_\bullet$ 64
$e_{M,N}^{L}$	Polynomial counting extensions of M by N with middle term isomorphic to L 42
$f_{M_\bullet,N_\bullet}^{L_\bullet}$	Polynomial counting number of subobjects of complexes $N_\bullet \subseteq L_\bullet$ with quotient $M_\bullet$ 64
$f_{M,N}^{L}$	Polynomial counting number of subobjects $N \subseteq L$ with quotient M 42
$h_{M,N}^{A,B}$	Polynomial counting homomorphisms $M \to N$ with kernel A and cokernel B 64
$h_{M,N}$	Polynomial counting homomorphisms from M to N 42
$h_{M_\bullet,N_\bullet}$	Polynomial counting homomorphisms of complexes from $M_\bullet$ to $N_\bullet$ 64

Quantized Enveloping Algebras

E_{β_k}	Rescaled quantum root vector; $E_{\beta_k} = (t^2-1)X_{\beta_k}$ 29
F_{β_k}	Rescaled quantum root vector; $F_{\beta_k} = (t^2-1)Y_{\beta_k}$ 29
$K_i^{\pm 1}$	Generator of quantized enveloping algebra $\mathrm{U}_t(\mathfrak{g})$ 25
T_i	Lusztig's braid group automorphisms 26
X_{β_k}	Quantum root vector; $X_{\beta_k} = E_{\beta_k}/(t^2-1)$ 27
X_i	Generator of quantized enveloping algebra $\mathrm{U}_t(\mathfrak{g})$ 25
Y_{β_k}	Quantum root vector; $Y_{\beta_k} = F_{\beta_k}/(t^2-1)$ 27
Y_i	Generator of quantized enveloping algebra $\mathrm{U}_t(\mathfrak{g})$ 25
Σ	Algebra involution of $\mathrm{U}_t(\mathfrak{g})$; commutes with Lusztig's braid group action 27
$\mathrm{U}_q(\mathfrak{g})$	Specialization at $t = q^{\frac{1}{2}}$ of the restricted integral form of $\mathrm{U}_t(\mathfrak{g})$ 29
$\mathrm{U}_q(\mathfrak{n}_+)$	Positive part of $\mathrm{U}_q(\mathfrak{g})$ 29
$\mathrm{U}_t(\mathfrak{g})$	Quantized enveloping algebra of simple Lie algebra $\mathfrak{g}$ 25
$\mathrm{U}_t(\mathfrak{n}_+)$	Positive part of quantized enveloping algebra 25
$\mathrm{U}_t^{Poiss}(\mathfrak{g})$	Poisson integral form of quantized enveloping algebra 29
$\mathrm{U}_t^{Res}(\mathfrak{g})$	Restricted integral form of quantized enveloping algebra 28

Miscellaneous

$[n]_t!$	t-analogue of $n!$ 25
$[n]_t$	t-analogue of integer n 25
$\mathbb{F}_q$	Finite field with q elements 9
q	Cardinality of finite field $\mathbb{F}_q$; a prime power 9
$\begin{bmatrix} n \\ s \end{bmatrix}_t$	t-analogue of binomial coefficient $\binom{n}{s}$ 25

Quivers, Representations and Complexes

Algebraic Groups and Poisson-Lie Groups

Lie Algebras and Lie Bialgebras

$(-,-)_{\mathfrak{g}}$	Normalized Cartan-Killing form on $\mathfrak{g}$ 5
$(a_{ij})_{i,j=1}^{r}$	Symmetric Cartan matrix of simple Lie algebra assigned to a simply-laced quiver $\vec{Q}$ 4
N	Number of indecomposable representations of $\vec{Q}$; equivalently number of positive roots of $\mathfrak{g}$ 5
$[-,-]_{qc}$	Lie bracket on quasi-classical Lie algebra 53
$[-,-]_{sc}^{\vee}$	Lie bracket on dual semi-classical Lie algebra 98
$[-,-]_{sc}$	Lie bracket on semi-classical Lie algebra 98
$[-,-]$	Lie bracket on simple Lie algebra 4
Λ_{Φ}	Root lattice of simple Lie algebra $\mathfrak{g}$ 5
Φ^{+}	Set of positive roots of simple Lie algebra $\mathfrak{g}$ 5
Φ	Set of roots of simple Lie algebra $\mathfrak{g}$ 5
α_i	Simple root of $\mathfrak{g}$ 5
$\mathfrak{b}_{\pm}^{\vee}$	Borel subalgebras of standard dual Lie bialgebra $\mathfrak{g}^{\vee}$ 20
$\mathfrak{b}_{\pm}$	Borel subalgebras of simple Lie algebra $\mathfrak{g}$ 5
$\mathfrak{b}_{sc,\pm}^{\vee}$	Borel subalgebras of semi-classical Lie bialgebra $\mathfrak{g}_{sc}^{\vee}$ 103
$\mathfrak{b}_{sc,\pm}$	Borel subalgebras of $\mathfrak{g}_{sc}$ 103
$\mathfrak{g}[\alpha]$	Root space of simple Lie algebra $\mathfrak{g}$ 5
$\mathfrak{g}^{\vee}$	Standard dual Lie bialgebra 19
$\mathfrak{g}_{sc}[\alpha]$	Root space of $\mathfrak{g}_{sc}$ 103
$\mathfrak{g}_{sc}^{\vee}$	Semi-classical Lie bialgebra 98
$\mathfrak{g}_{sc}$	Bialgebra dual of semi-classical Lie bialgebra $\mathfrak{g}_{sc}^{\vee}$ 98
$\mathfrak{g}$	Simple Lie algebra; endowed with standard Lie bialgebra structure 4, 19
$\mathfrak{h}^{\vee}$	Cartan subalgebra of standard dual Lie bialgebra $\mathfrak{g}^{\vee}$ 20
$\mathfrak{h}_{sc}^{\vee}$	Cartan subalgebra of semi-classical Lie bialgebra $\mathfrak{g}_{sc}^{\vee}$ 103
$\mathfrak{h}_{sc}$	Cartan subalgebra of $\mathfrak{g}_{sc}$ 103
$\mathfrak{h}$	Cartan subalgebra of simple Lie algebra $\mathfrak{g}$ 5
$\mathfrak{n}_{\pm}^{\vee}$	Nilpotent subalgebras of standard dual Lie bialgebra $\mathfrak{g}^{\vee}$ 20
$\mathfrak{n}_{\pm}$	Nilpotent subalgebras of simple Lie algebra $\mathfrak{g}$ 5
$\mathfrak{n}_{qc}$	Abelian quasi-classical Lie algebra 52
$\mathfrak{n}_{sc,\pm}^{\vee}$	Nilpotent subalgebras of semi-classical Lie bialgebra $\mathfrak{g}_{sc}^{\vee}$ 103
$\mathfrak{n}_{sc,\pm}$	Nilpotent subalgebras of $\mathfrak{g}_{sc}$ 103
r	Number of vertices of $\vec{Q}$; equivalently number of simple roots of $\mathfrak{g}$ 5
s_i	Simple reflection of root lattice or equivalently of Grothendieck group 5, 10
w_0	Longest element of Weyl group 5, 26

Variables for Lie Algebras and Lie Bialgebras

$e_I^{\vee}$	Positive root vector of semi-classical Lie bialgebra $\mathfrak{g}_{sc}^{\vee}$ 98
e_I	Positive root vector of $\mathfrak{g}_{sc}^{\vee}$ 98
$e_{\alpha}^{\vee}$	Positive root vector of standard dual Lie bialgebra $\mathfrak{g}^{\vee}$ 19
e_{α}	Positive root vector of simple Lie algebra $\mathfrak{g}$ 6
e_i	Generator of simple Lie algebra $\mathfrak{g}$ 4
$f_I^{\vee}$	Negative root vector of semi-classical Lie bialgebra $\mathfrak{g}_{sc}^{\vee}$ 98

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