Moments of Distances Between Centres of Ford Spheres

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Abstract

Given any positive integer k, we establish asymptotic formulas for the k-moments of the distances between the centres of 'consecutive' Ford spheres with radius less than $\frac{1}{2S^2}$ for any positive integer S. This extends to higher dimensions the work on Ford circles by Chaubey, Malik and Zaharescu in their 2014 paper *k-Moments of Distances Between Centres of Ford Circles*.

To achieve these estimates we bring the current theory of Ford spheres in line with the existing more developed theory for Ford circles and Farey fractions. In particular, we see (i) that a variant of the mediant operation can be used to generate Gaussian rationals analogously to the Stern-Brocot tree construction for Farey fractions and (ii) that two Ford spheres may be considered 'consecutive' for some order S if they are tangent and there is some Ford sphere with radius greater than $\frac{1}{2S^2}$ that is tangent to both of them. We also establish an asymptotic estimate for a version of the Gauss Circle Problem in which we count Gaussian integers in a subregion of a circle in the complex plane that are coprime to a given Gaussian integer.

Contents

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Author's Declaration

I declare that this thesis is a presentation of original work and I am the sole author. This work has not previously been presented for an award at this, or any other, University. All sources are acknowledged as References.

Chapters 2, 3 and 4 are essentially the contents of

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Chapter 1

Introduction

We will begin by introducing the relevant background of Farey fractions, Ford spheres and arithmetical functions required within the thesis. While some of the results in this chapter will be applied directly in our calculation of moments for Ford spheres, many of them will instead inform the necessary higher dimensional analogues. A summary of the rest of the thesis can also be found at the end of this chapter.

1.1 Farey Fractions and Ford Circles

In this section we review various fundamental notions and facts concerning Farey fractions and Ford circles. In particular we focus on the property of two Farey fractions being consecutive, as this notion will later be used to give an analogous definition within the context of Ford spheres. Throughout this section we follow the theory presented in Ford's original paper on the matter [5] and Chapter 3 of Hardy and Wright's book [11]. The definitions and results in this section can be found in these texts.

Given a positive integer Q, the Farey sequence of order Q consists of those reduced fractions in the interval $[0,1]$ with denominator less than or equal to Q, taken in increasing order of size. This set is denoted $\mathcal{F}_Q$; in other words,

$$\mathcal{F}_Q := \left\{ \frac{p}{q} \in [0,1] : p, q \in \mathbb{Z}, (p,q) = 1, q \leq Q \right\}.$$

Here and in the following (p, q) denotes the greatest common divisor of p and q. We denote the number of fractions in $\mathcal{F}_Q$ by $N(Q)$. While these so-called 'vulgar fractions' first appear to be entirely elementary, closer examination reveals many interesting properties and relationships to other areas.

The Farey fractions have a somewhat unusual history, being named not after their original investigator nor even a mathematician, but a geologist, John Farey Sr. In 1816 Farey noticed an interesting property of these fractions which he then wrote about in a letter published by *Philosophical Magazine*. The property he observed was that each term in the Farey sequence of order Q is the mediant of its two neighbours. Given two rationals $\frac{p}{q} < \frac{p'}{q'}$, their mediant is given by $\frac{p+p'}{q+q'}$. This is also sometimes called their 'freshman sum' as it is commonly mistaken for the sum of two fractions when first learning to do such things.

Farey did not provide a proof of his observation but this was later supplied by Cauchy after seeing Farey's letter. The result had actually already been stated and proved by Haros in 1802, but mathematicians have followed Cauchy in attributing the discovery to Farey and the fractions continue to bear his name.

It is worth noting that a mediant will always lie between the two original fractions, that is $\frac{p}{q} < \frac{p+p'}{q+q'} < \frac{p'}{q'}$, but that it does not necessarily lie exactly halfway between them. For the Farey fractions, Farey's conjecture turned out to be true for all orders Q. For example, the Farey fractions of order 5 are

$$\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}.$$

Indeed, $\frac{1}{4}$ is the mediant of $\frac{1}{5}$ and $\frac{1}{3}$, $\frac{3}{5}$ is the mediant of $\frac{1}{2}$ and $\frac{2}{3}$, etc.

A further (and, it transpires, equivalent) defining characteristic of the Farey sequence is that for any pair of successive fractions $\frac{p}{q} < \frac{p'}{q'}$,

$$p'q - pq' = 1.$$

This fact motivates the following definition.

Definition 1.1. *A pair of rationals* $\frac{p}{q} < \frac{p'}{q'}$ *in* $\mathcal{F}_Q$ *will be called adjacent if*

$$p'q - pq' = 1. \tag{1.1}$$

If (1.1) *is satisfied and* $q + q' > Q$*, the rationals are consecutive in* $\mathcal{F}_Q$*.*

Note that this definition coincides with the usual meaning of consecutive, i.e. if $\frac{p}{q}$ and $\frac{p'}{q'}$ satisfy these conditions then $\frac{p'}{q'}$ will immediately follow $\frac{p}{q}$ in $\mathcal{F}_Q$. Moreover, if two fractions in $\mathcal{F}_Q$ are adjacent, they must be consecutive in $\mathcal{F}_{Q'}$ for some $Q' \leq Q$.

Notedly, when we take mediants of adjacent Farey fractions, the resulting fraction is always automatically in its reduced form. Further, if we take the mediant of two fractions which are consecutive in $\mathcal{F}_Q$ we find a new Farey fraction which is not contained in $\mathcal{F}_Q$. This fact provides us with a strategy for finding new Farey fractions from old ones. In fact, if we repeatedly apply this strategy beginning with $\mathcal{F}_1$, i.e. the fractions $\frac{0}{1}$ and $\frac{1}{1}$, we will encounter every rational in the interval $[0, 1]$ at some point.

Lemma 1.1. *Given any two coprime integers* $0 \leq p < q$*, we have*

$$\frac{p}{q} = \frac{a+c}{b+d}$$

for some pair of consecutive fractions $\frac{a}{b}$ *and* $\frac{c}{d}$ *in* $\mathcal{F}_{q-1}$.

Proof. We will argue by induction on q. The fractions $\frac{0}{1}$ and $\frac{1}{1}$ are given so we start with $\mathcal{F}_2$. In this case the only new fraction to check is $\frac{1}{2}$. Indeed, $\frac{0+1}{1+1} = \frac{1}{2}$ and clearly $\frac{0}{1}$ and $\frac{1}{1}$ are consecutive in $\mathcal{F}_1$. Now, as p and q are coprime, we can write

$$bp - aq = 1 \tag{1.2}$$

for some positive integers a and b with $a < p$ and $b < q$. Further, (1.2) implies that a and b are also coprime, thus $\frac{a}{b}$ is a Farey fraction in $\mathcal{F}_{q-1}$. Additionally we have $0 < p - a < p$ and $0 < q - b < q$. Now,

$$\begin{aligned} b(p-a) - a(q-b) &= bp - ab - aq + ab \\ &= bp - aq \\ &= 1 \end{aligned}$$

by (1.2), so $p - a$ and $q - b$ are coprime and $\frac{p-a}{q-b}$ is a Farey fraction in $\mathcal{F}_{q-1}$. This also shows that $\frac{a}{b}$ and $\frac{p-a}{q-b}$ are adjacent fractions. Moreover, the sum of their denominators is $b + (q - b) = q > q - 1$, so $\frac{a}{b}$ and $\frac{p-a}{q-b}$ are consecutive fractions in $\mathcal{F}_{q-1}$ with mediant $\frac{p}{q}$. □

This construction strategy can be visualised in the left hand side of the Stern-

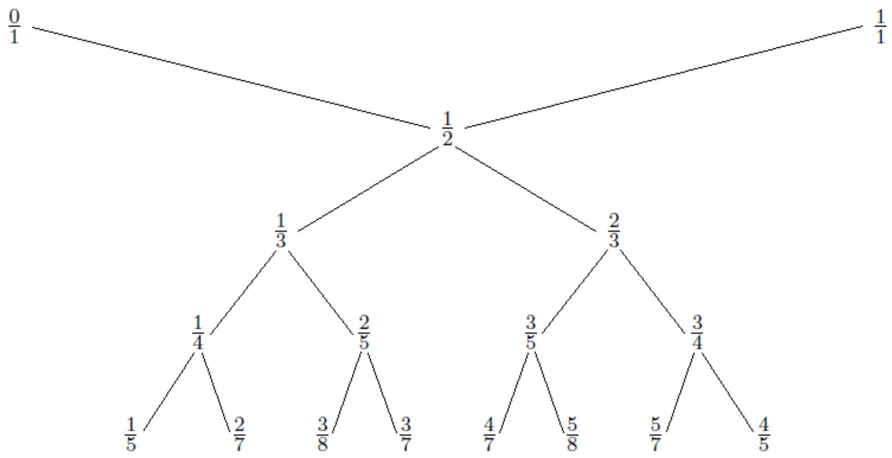

Figure 1.1: The Farey fractions as the left hand side of the Stern-Brocot tree.

Brocot tree, illustrated in figure 1.1. In keeping with the 'story' of the Farey fractions, the Stern-Brocot tree was independently discovered by both a mathematician, Moritz Stern, and a non-mathematician, Achille Brocot. Brocot was a French clockmaker whose interest in this tree laid in its usefulness in establishing sensible gear ratios for the gear systems that drive the hands of a clock. The Stern-Brocot tree is also very intriguing mathematically. As mentioned, it is intimately linked with the Farey fractions, but beyond this we find relations to the Euclidean algorithm, continued fractions and more. These relationships become apparent when we start to navigate the tree.

A natural way to move through the Stern-Brocot tree is to start at the top and slide down through the tree from a fraction to one of its 'children' linked by the branches below it. For example, if we start as convention dictates at $\frac{1}{1}$ we could move left to $\frac{1}{2}$, then move right to $\frac{2}{3}$, then left to $\frac{3}{5}$ and finally left again where we reach $\frac{4}{7}$. Following the usual notation, we may write such a movement down as $LRLL$ or LRL^2. Clearly we can write down something similar for any rational lying between 0 and 1. However, what if we were to consider infinite strings of L's and R's? Doing this allows us to find irrational numbers in the interval $(0, 1)$. Of course, as the tree only contains rationals, we will never actually reach the irrational, but we will find increasingly accurate approximations to it. For an irrational number $0 < \alpha < 1$ we create its string starting from $\frac{1}{1}$ by adding an L and moving down the tree to the left if α is less than the current position or by adding an R and moving down the tree to the right if α is greater than the current position. For example, the infinite string $LRLRLRLR$... corresponds to the fractional part of the golden ratio ϕ.

Our movement through the tree is similarly dictated by a number's continued fraction representation. Accordingly, for positive integers $a_1, a_2, a_3, ...$, the continued fraction

$$\cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots}}}} \tag{1.3}$$

represents the number found in the Stern-Brocot tree by following the string $L^{a_1} R^{a_2} L^{a_3} R^{a_4} ...$ Indeed the continued fraction for the fractional part of ϕ is

$$\cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}$$

as expected from its Stern-Brocot tree string.

We now turn our attention to the Ford Circles, introduced by Lester Ford in [5]. These provide a geometric representation of the Farey fractions according to the following definition.

Definition 1.2. *The Ford circle corresponding to a Farey fraction $\frac{p}{q}$ is the circle of radius $\frac{1}{2q^2}$ touching the x-axis at $\frac{p}{q}$ which lies in the upper half-plane.*

The Ford circles corresponding to the Farey fractions of order 5 can be seen in figure 1.2. From this perspective, the Farey fractions of order Q can be viewed as those rationals whose corresponding Ford circles have centres lying on or above the line $y = \frac{1}{2Q^2}$. As we may speculate from figure 1.2, the Ford circles corresponding to distinct Farey fractions are either tangent or disjoint from one another. We can verify this by considering the distance between their centres, which we denote here by D. According to Pythagoras, for Farey fractions $\frac{p}{q} < \frac{p'}{q'}$, we have

$$D^2 = \left(\frac{p}{q} - \frac{p'}{q'}\right)^2 + \left(\frac{1}{2q^2} - \frac{1}{2q'^2}\right)^2$$

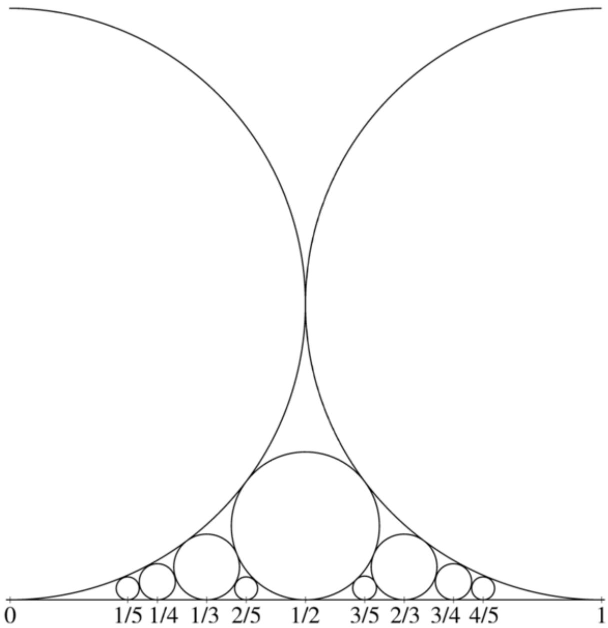

Figure 1.2: The Ford Circles in the interval [0,1].

$$= \left(\frac{p'q - pq'}{qq'}\right)^2 + \frac{1}{4q^4} + \frac{1}{4q'^4} - \frac{1}{2q^2q'^2}$$

$$= \frac{(p'q - pq')^2 - 1}{q^2q'^2} + \left(\frac{1}{2q^2} + \frac{1}{2q'^2}\right)^2$$

$$= \frac{(p'q - pq')^2 - 1}{q^2q'^2} + (r_p + r_{p'})^2 ,$$

where r_p and $r_{p'}$ are the radii of the Ford circles corresponding to $\frac{p}{q}$ and $\frac{p'}{q'}$ respectively. We must now examine three cases,

1. If $|p'q - pq'| > 1$, we must have $D > r_p + r_{p'}$ and the circles are disjoint.
2. If $|p'q - pq'| = 1$ then $D = r_p + r_{p'}$ and the circles are tangent.
3. If $|p'q - pq'| < 1$ then, as $p'q - pq'$ is an integer, we must have $p'q - pq' = 0$ and so $\frac{p}{q} = \frac{p'}{q'}$.

The third case contradicts our assumption that the Farey fractions are distinct and so cannot occur.

The interesting case here is case 2, where the two circles are tangent and we have $p'q - pq' = 1$. For such Ford circles, (1.1) is satisfied and the two corresponding fractions are adjacent. In particular, two Farey fractions $\frac{p}{q}$ and $\frac{p'}{q'}$ are consecutive in $\mathcal{F}_Q$ if they are adjacent and $q + q' > Q$, that is if their mediant is not also in $\mathcal{F}_Q$. Accordingly, we can view two Ford circles as being consecutive at order Q if they are tangent and there is no smaller circle between them at that order. We will recall this perspective in Section 3 when defining consecutivity in higher dimensions.

Thinking back to our scheme for moving through the Stern-Brocot tree according to continued fractions, Ford describes a similar procedure for navigating the Ford circles to the same end in [5]. This time we imagine the Ford circles as clocks which we will move down in steps as before. The numbers are positioned on the 'clock' at the points of tangency with adjacent Ford circles, starting with the zero position at the tangency with the last circle visited. We will start each journey at the circle corresponding to $\frac{0}{1}$, whose zero position is taken to be the top of the circle. For this circle the positions will be labelled clockwise, on the next circle visited they will be labelled anti-clockwise, at the next clockwise again, and so on, alternating each time we move to a new circle. The continued fraction (1.3) will then correspond to moving from the circle at $\frac{0}{1}$ to the circle tangent to it at its ${a_1}^{\text{th}}$ position, then from this circle to the one at its ${a_2}^{\text{th}}$ position, and so on. If the continued fraction is finite, the final circle visited will correspond to the Farey fraction which is equal to the continued fraction.

Returning to our L and R notation, the continued fraction (1.3) corresponded to the string $L^{a_1}R^{a_2}L^{a_3}R^{a_4}$... We can now follow this system again, where the L's denote clockwise movement and the R's anticlockwise movement. For example, the fractional part of e has the string $LR^2LRL^4RLR^6$... and so its sequence of circles will be

$$\frac{0}{1}, \frac{1}{1}, \frac{2}{3}, \frac{3}{4}, \frac{5}{7}, \frac{23}{32}, \frac{28}{39}, \ldots$$

the first four of which are shown in figure 1.3. These rationals coincide with the n^{th} convergents of the fractional part of e, which are obtained by keeping the first n terms of the continued fraction. This is true not only for e, but for any real number.

This connection between Ford circles and continued fractions will be explored further in Section 1.2.2.

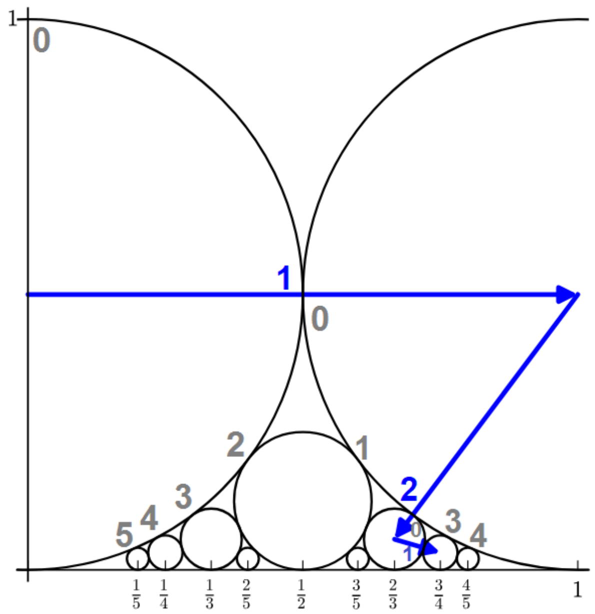

Figure 1.3: The sequence of Ford circles corresponding to e.

1.2 Farey Fractions and Ford Circles – Connections

In this section we examine various interesting uses of Farey fractions and Ford circles in other contexts.

1.2.1 Hurwitz's Theorem

Diophantine approximation is concerned with approximating real numbers by rationals. The first important result in the area is due to Dirichlet and states the following.

Theorem 1.1. (Dirichlet's Approximation Theorem) *For any real number α and positive integer N there exists integers p and q with $1 \leq q \leq N$ such that*

$$|q\alpha - p| < \frac{1}{N}.$$

This theorem immediately implies that for any irrational number α the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$$

has infinitely many solutions $\frac{p}{q}$ with p in $\mathbb{Z}$ and q in $\mathbb{N}$. This inequality has since been improved and we now have the following theorem.

Theorem 1.2. (Hurwitz's Theorem) *For any irrational number α, the inequality*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}, \tag{1.4}$$

has infinitely many solutions $\frac{p}{q}$ with p in $\mathbb{Z}$ and q in $\mathbb{N}$.

Importantly, the constant $\sqrt{5}$ cannot be improved upon. Replacing it by any other number greater than $\sqrt{5}$ results in only finitely many solutions when we take the irrational to be the golden ratio, i.e. $\alpha = \frac{1+\sqrt{5}}{2}$.

The Farey fractions can be used to provide a simple proof of Hurwitz's Theorem, which can be achieved as described below, following the proof laid out by Niven in Chapter 1 of [18].

Proof. We may assume $\alpha \in (0,1)$, as otherwise we can write $\alpha = x + \alpha'$ for some integer x and $\alpha' \in (0,1)$ and proceed as follows with α' then simply add x back at the end. For two consecutive Farey fractions $\frac{p}{q}$ and $\frac{p'}{q'}$ with $\frac{p}{q} < \alpha < \frac{p'}{q'}$ we will show that one of the fractions $\frac{p}{q}$, $\frac{p'}{q'}$ or their mediant $\frac{p+p'}{q+q'} = \frac{a}{b}$ satisfies (1.4).

Now, suppose that (1.4) is false for all three of these rationals and that $\alpha < \frac{a}{b}$. Then we must have

$$\alpha - \frac{p}{q} \geq \frac{1}{\sqrt{5}q^2}, \tag{1.5}$$

$$\frac{p'}{q'} - \alpha \geq \frac{1}{\sqrt{5}q'^2}, \text{ and} \tag{1.6}$$

$$\frac{a}{b} - \alpha \geq \frac{1}{\sqrt{5}b^2}. \tag{1.7}$$

Adding (1.5) to (1.6) and (1.5) to (1.7) gives us

$$\frac{1}{qq'} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{q^2} + \frac{1}{q'^2}\right), \text{ and} \tag{1.8}$$

$$\frac{1}{qb} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{q^2} + \frac{1}{b^2}\right). \tag{1.9}$$

Multiplying (1.8) through by $\sqrt{5}q^2q'^2$ and (1.9) through by $\sqrt{5}q^2b^2$ then adding the results together, we have

$$\sqrt{5}q(q'+b) = \sqrt{5}q(q+2q') \geq 2q^2 + q'^2 + b^2 = 3q^2 + 2q'^2 + 2qq'.$$

Thus,

$$\begin{aligned} 0 &\geq (3-\sqrt{5})q^2 - 2(\sqrt{5}-1)qq' + 2q'^2 \\ &= \frac{1}{2}\left((\sqrt{5}-1)q - 2q'\right)^2. \end{aligned}$$

As the right hand side is a positive multiple of a square and so must be non-negative, this implies that

$$(\sqrt{5}-1)q - 2q' = 0 \Rightarrow \sqrt{5} = \frac{2q'}{q} + 1 \in \mathbb{Q},$$

which is a contradiction and so one of the fractions must satisfy (1.4).

On the other hand, if $\alpha > \frac{a}{b}$, in place of (1.7) we have

$$\alpha - \frac{a}{b} \geq \frac{1}{\sqrt{5}b^2}.$$

This can then be added to (1.6) and the proof proceeds similarly, eventually leading to a contradiction.

Finally, note that we can do this with consecutive fractions in $\mathcal{F}_Q$ for any positive integer Q and so obtain infinitely many solutions to (1.4). □

1.2.2 Lagrange's Theorem

Similarly to Hurwitz's Theorem, we are again concerned with how well real numbers can be approximated by rationals. We say that a rational number $\frac{a}{b}$ is a *best approximation* (of the second kind) of a real number α if, for any rational $\frac{c}{d}$ with $d \leq b$,

$$|b\alpha - a| \leq |d\alpha - c|. \tag{1.10}$$

Note that we have equality in (1.10) if and only if $\frac{a}{b} = \frac{c}{d}$.

Theorem 1.3. (Lagrange's Theorem) *Let α be a real number and $\frac{a}{b}$ be a rational that is not an integer. Then $\frac{a}{b}$ is a best approximation for α if and only if it is a convergent of α.*

At the end of Section 1.1 we saw how Ford circles are linked to continued fractions by thinking of the circles like clocks. This idea can be used to give a nice proof of Lagrange's Theorem. Indeed, the proof given by Ian Short in [20], involves first showing that a statement about Ford circles is equivalent to $\frac{a}{b}$ being a best approximation, and then showing that this statement is true if and only if $\frac{a}{b}$ is a convergent of α. Before we can describe this further it will be helpful to introduce some notation, we follow that laid out by Short in [20]. We denote the Ford circle corresponding to a rational $x = \frac{a}{b}$ by C_x and its radius by $\mathrm{rad}[C_x]$. For a real number α we also define

$$R_x(\alpha) := \frac{1}{2}|b\alpha - a|^2.$$

When $\alpha = x$ this is zero, otherwise this is the radius of the unique circle that is tangent to both C_x and the x-axis at α. Finally, we denote the n^{th} convergent of the continued fraction of α by $\frac{A_n}{B_n}$ and define the *continued fraction chain* of α to be the sequence of Ford circles $C_{A_0/B_0}, C_{A_1/B_1}, C_{A_2,B_2}, \ldots$. We can now state the main theorem of [20].

Theorem 1.4. *Let α be a real number and x be a rational that is not an integer. The following are equivalent.*

(i) *x is a convergent of α.*

(ii) *C_x is a member of the continued fraction chain of α.*

(iii) *x is a best approximation of α.*

(iv) *For any rational z with $\mathrm{rad}[C_x] \leq \mathrm{rad}[C_z]$ we have $R_x(\alpha) \leq R_z(\alpha)$, with equality if and only if $x = z$.*

The equivalence of statements (i) and (ii) follows immediately from the definition of the continued fraction chain. The equivalence of statements (iii) and (iv) is clear from (1.10) and the definition of $R_x(\alpha)$ using the fact that $\mathrm{rad}[C_x] = \frac{1}{2b^2}$. The equivalence of statements (i) and (iii) is Lagrange's Theorem. The proof of this theorem requires two additional results, the proofs of which can be found in [20].

Lemma 1.2. [20, Lemma 2.2] *Let x and y be rationals such that their Ford circles C_x and C_y are tangent. Any rational z lying strictly between x and y must have Ford circle C_z of radius less than both C_x and C_y.*

Lemma 1.3. [20, Lemma 2.3] *Let x and y be as in the previous lemma with $\text{rad}[C_x] > \text{rad}[C_y]$ and let α be a real number lying strictly between them. For any rational z lying strictly outside the interval bounded by x and y we must have $R_x(\alpha) < R_z(\alpha)$.*

We now outline the proof of Theorem 1.4. Our strategy is to show that statements (i) and (iv) are equivalent. First, assume that $x = \frac{A_n}{B_n}$ and $y = \frac{A_{n+1}}{B_{n+1}}$ are consecutive convergents of α (so C_x and C_y are tangent with $\text{rad}[C_x] < \text{rad}[C_y]$) and z is a rational with $\text{rad}[C_x] \leq \text{rad}[C_z]$. Then by Lemma 1.2 z must lie outside of the region bounded by x and y and so by Lemma 1.3 we must have $R_x(\alpha) < R_z(\alpha)$ as required. Now, assume that x is not a convergent of α. We may also assume that $\text{rad}[C_x] > \text{rad}[C_\alpha$. Now, there must exist a unique $n \geq 1$ such that $\text{rad}[C_{A_n/B_n}] \geq \text{rad}[C_x] > \text{rad}[C_{A_{n+1}/B_{n+1}}]$. Further, α must lie strictly in the interval between $\frac{A_n}{B_n}$ and $\frac{A_{n+1}}{B_{n+1}}$ and by Lemma 1.2 x must lie outside that interval. Thus, by Lemma 1.3, $R_{A_n/B_n}(\alpha) < R_x(\alpha)$ and statement (iv) fails with $z = \frac{A_n}{B_n}$.

1.2.3 The Riemann Hypothesis

The Riemann Hypothesis is arguably the most important unsolved problem in mathematics today. If proven true it will have many meaningful consequences, notably for the distribution of the primes. The Farey fractions can be used to form a statement that is equivalent to the Riemann Hypothesis, providing a new avenue for its potential proof. This equivalence was first presented by Franel [7] and Landau [14]. Further, their theorem has since been generalised by Huxley [12], showing that the Riemann Hypothesis for a Dirichlet L-function is also equivalent to a statement about the distribution of Farey fractions, weighted by the values that the corresponding Dirichlet character takes at their denominators.

The Riemann Hypothesis concerns the zeros of the Riemann zeta function $\zeta(s)$, which is defined for complex numbers $s = \sigma + i\tau$ with $Re(s) > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

and for s with $0 < Re(s) < 1$ by

$$\zeta(s) = \lim_{x\to\infty}\left(\sum_{n\leq x}\frac{1}{n^s} - \frac{x^{1-x}}{1-s}\right).$$

It can also be analytically continued to be defined at all complex numbers except $s = 1$. By doing this we can find the trivial zeros of ζ, which are at all the negative even integers. The non-trivial zeros, however, are much more intriguing. Bernhard Riemann conjectured in [19] that they all lie on the so-called critical line $s = \frac{1}{2} + i\tau$.

The Riemann Hypothsis. *Every non-trivial zero of the Riemann zeta function has real part* $\frac{1}{2}$.

The Farey fractions are related to the Riemann Hypothesis via a further equivalent statement regarding the growth of Mertens' function. To define it we must first define the Möbius function $\mu(n)$, which takes the value of the sum of the primitive n^{th} roots of unity.

Throughout, given an integer $n \geq 2$

$$n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}.$$

will denote its canonical representation; thus $p_1 < p_2 < \cdots < p_k$ are distinct primes and $\alpha_i \in \mathbb{N}$.

Definition 1.3. *The Möbius function is defined by*

$$\mu(n) = \begin{cases} 1 & \textit{if } n = 1, \\ (-1)^k & \textit{if } \alpha_1 = \cdots = \alpha_k = 1, \\ 0 & \textit{otherwise.} \end{cases}$$

Mertens' function $M(x)$ is then defined as a finite sum of the Möbius function, so

$$M(x) = \sum_{n\leq x} \mu(n)$$

for any positive real number x.

In 1912 Littlewood [16] proved that the following conjecture is equivalent to the Riemann Hypothesis.

Conjecture 1.1. *For every $\epsilon > 0$,*

$$\lim_{x\to\infty} \frac{M(x)}{x^{\frac{1}{2}+\epsilon}} = 0.$$

We now outline the connection with Conjecture 1.1 and Farey fractions as observed by Franel and Landau – full details can be found in [4].

Recall that for any positive integer Q, $N(Q)$ denotes the number of Farey fractions in $\mathcal{F}_Q$. For this section only, it will be practical to exclude 0 from $\mathcal{F}_Q$ so, for example,

$$\mathcal{F}_5 = \left\{ \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, 1 \right\}$$

and $N(5) = 10$. Now, consider the $N(Q)$ evenly spaced points $\frac{1}{N(Q)}, \frac{2}{N(Q)}, \ldots,$ $\frac{N(Q)}{N(Q)} = 1$. The Farey fractions are not evenly spaced in the interval $[0,1]$, so for $\nu = 1, 2, \ldots, N(Q)$, the ν^{th} Farey fraction r_ν differs from $\frac{\nu}{N(Q)}$ by some amount that we denote by δ_ν. We then define a function $D(Q)$ to be the sum of these differences, that is,

$$D(Q) = \sum_{\nu=1}^{N(Q)} |\delta_\nu|. \tag{1.11}$$

Theorem 1.5. (Franel-Landau Theorem) *The Riemann Hypothesis is equivalent to the following statement. For every $\epsilon > 0$,*

$$\lim_{Q\to\infty} \frac{D(Q)}{Q^{\frac{1}{2}+\epsilon}} = 0, \tag{1.12}$$

where $D(Q)$ is as defined in (1.11).

We can prove that the statement associated with (1.11) is equivalent to Conjecture 1.1 using the formula

$$\sum_{\nu=1}^{N(Q)} f(r_\nu) = \sum_{k=1}^{\infty} \sum_{j=1}^{k} f\left(\frac{j}{k}\right) M\left(\frac{Q}{k}\right), \tag{1.13}$$

where f is a real-valued function defined on $[0,1]$ and r_ν denotes the ν^{th} term of $\mathcal{F}_Q$. To see why this equality holds we start by defining the function

$$L(x) = \begin{cases} 1 & \text{if } x \geq 1, \\ 0 & \text{if } x < 1 \end{cases}$$

for positive real numbers x. We will show that

$$\sum_{k\geq 1} M\left(\frac{x}{k}\right) = L(x).$$

First, note that if $x < 1$ then $\frac{x}{k} < 1$ and so $M\left(\frac{x}{k}\right) = 0$ for all integers $k \geq 1$. Thus, when $x < 1$, $\sum_{k\geq 1} M\left(\frac{x}{k}\right) = 0$. Now, assume $x \geq 1$. We have $M\left(\frac{x}{k}\right) = 0$ for all integers $k > x$, and so

$$\begin{aligned}
\sum_{k\geq 1} M\left(\frac{x}{k}\right) &= \sum_{k=1}^{\lfloor x\rfloor} M\left(\frac{x}{k}\right) \\
&= \sum_{k=1}^{\lfloor x\rfloor} \sum_{l=1}^{\lfloor \frac{x}{k}\rfloor} \mu(l) \\
&= \sum_{1\leq n\leq \lfloor x\rfloor} \sum_{k|n} \mu(k),
\end{aligned}$$

where $n = kl$. Now, later in Section 1.3.2 we will see that the inner sum is non-zero only when $n = 1$, in which case the sum is equal to 1. Thus, when $x \geq 1$, $\sum_{k\geq 1} M\left(\frac{x}{k}\right) = 1$. Finally, let $0 < \frac{p}{q} \leq 1$ be a fraction written in lowest terms and consider the coefficient of $f(\frac{p}{q})$ in (1.13). On the left-hand side this is 1 if $\frac{p}{q}$ is in $\mathcal{F}_Q$, i.e. if $q \leq Q$, and is 0 otherwise. On the right-hand side note that $f(\frac{p}{q}) = f(\frac{2p}{2q}) = f(\frac{3p}{3q}) = \ldots$ and so the coefficient of $f(\frac{p}{q})$ is

$$M\left(\frac{Q}{q}\right) + M\left(\frac{Q}{2q}\right) + M\left(\frac{Q}{3q}\right) + \cdots = L\left(\frac{Q}{q}\right),$$

which is also 1 when $q \leq Q$ and 0 when $q > Q$.

To show that (1.12) implies the Riemann Hypothesis we start by substituting $f(u) = \mathrm{e}^{2\pi i u}$ into (1.13), so that

$$\sum_{\nu=1}^{N(Q)} e^{2\pi i r_\nu} = \sum_{k=1}^{\infty} \sum_{j=1}^{k} e^{2\pi i \frac{j}{k}} M\left(\frac{Q}{k}\right).$$

Now, $e^{2\pi i \frac{j}{k}}$ for $1 \leq j \leq k$ are the k^{th} roots of unity and so their sum is zero except when $k = 1$. In this case the sum is equal to 1 and so the right-hand side of the equality above is just $M(Q)$. Thus, writing $N = N(Q)$, we have

$$\begin{aligned} M(Q) &= \sum_{\nu=1}^{N} e^{2\pi i r_\nu} \\ &= \sum_{\nu=1}^{N} e^{2\pi i(\frac{\nu}{N}+\delta_\nu)} \\ &= \sum_{\nu=1}^{N} e^{2\pi i \frac{\nu}{N}} e^{2\pi i \delta_\nu} - \sum_{\nu=1}^{N} e^{2\pi i \frac{\nu}{N}} + \sum_{\nu=1}^{N} e^{2\pi i \frac{\nu}{N}} \\ &= \sum_{\nu=1}^{N} e^{2\pi i \frac{\nu}{N}} \left(e^{2\pi i \delta_\nu} - 1\right) + \sum_{\nu=1}^{N} e^{2\pi i \frac{\nu}{N}}. \end{aligned}$$

The case when $Q = 1$ is trivial as $\mathcal{F}_Q$ contains only the element 1, so we can assume $Q \geq 2$, in which case $N \geq 2$ and so $\sum_{\nu=1}^{N} e^{2\pi i \frac{\nu}{N}} = 0$. Hence,

$$\begin{aligned} |M(Q)| &\leq \sum_{\nu=1}^{N} |e^{2\pi i \frac{\nu}{N}}||e^{2\pi i \delta_\nu} - 1| \\ &= \sum_{\nu=1}^{N} |e^{2\pi i \delta_\nu} - 1| \\ &= \sum_{\nu=1}^{N} |e^{\pi i \delta_\nu} - e^{-\pi i \delta_\nu}| \\ &= 2\sum_{\nu=1}^{N} |\sin \pi \delta_\nu| \\ &\leq 2\pi \sum_{\nu=1}^{N} |\delta_\nu|. \end{aligned}$$

Thus, (1.12) implies the Riemann Hypothesis.

The proof of the converse is significantly longer and beyond the scope of this thesis. The key in this direction is to take $f(u)$ in (1.13) to be the periodic Bernoulli polynomial $\bar{B}_1(u) = u - \lfloor u \rfloor + \frac{1}{2}$, where $\lfloor u \rfloor$ denotes the integer part of u, i.e. the greatest integer that is less than or equal to u. The full proof can be found in Section 12.2 of [4].

1.3 Arithmetical Functions

This section contains notions and results on arithmetical functions required for the thesis. Some of these results will be used directly where others are noted here as they will be the base for results on the Gaussian integers detailed in Chapter 2. The results in the next four sections can be found in many texts on the theory of arithmetical functions, see Chapters 2 and 3 of [1], and Chapters 16 and 17 of [11] in particular. Section 1.3.5 follows Chapter 2 of [9].

1.3.1 Multiplicative Functions

An arithmetical function is a function defined on the natural numbers taking values in the complex numbers, i.e. $f : \mathbb{N} \to \mathbb{C}$. We will be concerned mainly with a particular type of these functions called *multiplicative functions*. These are those functions f that are not identically zero and satisfy

$$f(mn) = f(m)f(n) \tag{1.14}$$

whenever m and n in $\mathbb{N}$ are coprime. Further, if (1.14) holds for all $m, n \in \mathbb{N}$, then f is called *completely multiplicative*.

For example, the Möbius function $\mu(n)$, which we encountered in the previous section, is an arithmetical function that is multiplicative but not completely multiplicative.

Lemma 1.4. *The Möbius function is multiplicative.*

Proof. Let m and n be coprime natural numbers. If either m or n has a square factor then $\mu(m)\mu(n) = 0$ and, since mn must then also have a square factor, $\mu(mn) = 0$ also. So suppose neither m nor n has a square factor and write $m = p_1...p_k$ and $n = q_1...q_j$ for distinct primes p_i and q_i. Then we have

$$\begin{aligned}\mu(mn) &= \mu(p_1...p_k q_1...q_j)\\ &= (-1)^{k+j}\\ &= (-1)^k(-1)^j\\ &= \mu(m)\mu(n).\end{aligned}$$

Thus, μ is multiplicative. □

However, μ is not completely multiplicative since, for example, $\mu(2)\mu(6) = -1 \neq 0 = \mu(12)$.

Another important example of a multiplicative function is Euler's totient function ϕ.

Definition 1.4. *Euler's totient function $\phi(n)$ counts positive integers less than or equal to n that are coprime to n, so that*

$$\phi(n) = \sum_{\substack{a=1 \\ (a,n)=1}}^{n} 1.$$

Two more simple but useful arithmetical functions are the identity and unit functions. The identity function $I(n)$ is defined by

$$I(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The unit function u is defined by

$$u(n) = 1, \text{ for all } n.$$

These two functions are both clearly completely multiplicative.

1.3.2 Dirichlet Convolution

We now introduce a type of multiplication for arithmetical functions called the Dirichlet convolution and take note of some of its properties.

Definition 1.5. *For two arithmetical functions f and g, the Dirichlet convolution (or Dirichlet product) of f and g is a new arithmetical function defined by*

$$(f * g)(n) = \sum_{d|n} f(d)g\left(\frac{n}{d}\right) = \sum_{ab=n} f(a)g(b), \tag{1.15}$$

where the sum is taken over all positive divisors d of n.

In particular, the function $I(n)$ is an identity function for $*$. To verify this, consider $(f * I)(n)$ for any arithmetical function f and natural number n. We have,

$$\begin{aligned}(f * I)(n) &= \sum_{d|n} f(d) I\left(\frac{n}{d}\right)\\ &= f(n)\end{aligned}$$

since $I\left(\frac{n}{d}\right) = 0$ for all divisors d of n except for $d = n$. The same idea shows that $(I * f)(n) = f(n)$ also, thus $f * I = f = I * f$.

Lemma 1.5. *The Dirichlet convolution $*$ is associative, i.e. for arithmetical functions f, g and h we have*

$$(f * g) * h = f * (g * h).$$

Proof. Denote $k = f * g$ and $l = g * h$. Using the latter form of (1.15) we have

$$\begin{aligned}((f * g) * h)(n) &= (k * h)(n)\\ &= \sum_{ab=n} k(a)h(b)\\ &= \sum_{ab=n} \sum_{cd=a} f(c)g(d)h(b)\\ &= \sum_{bcd=n} f(c)g(d)h(b)\\ &= \sum_{cz=n} f(c) \sum_{db=z} g(d)h(b)\\ &= \sum_{cz=n} f(c)l(z)\\ &= (f * l)(n)\\ &= (f * (g * h))(n).\end{aligned}$$

□

For an arithmetical function f with $f(1) \neq 0$, we can define its inverse with respect to Dirichlet convolution as the unique arithmetical function f^{-1} such that

$$f * f^{-1} = I = f^{-1} * f.$$

Lemma 1.6. *The Dirichlet inverse of f exists and can be found using the recursion formulae*

$$\begin{aligned} f^{-1}(1) &= \frac{1}{f(1)}, \\ f^{-1}(n) &= \frac{-1}{f(1)} \sum_{\substack{d \mid n \\ d<n}} f\left(\frac{n}{d}\right) f^{-1}(d), \ \textit{for } n > 1. \end{aligned}$$

Proof. We will show by induction on n that given f as above there exists a solution to $(f * f^{-1})(n) = I(n)$ and that this solution is unique for the values $f^{-1}(n)$. First, let $n = 1$. Then

$$\begin{aligned} (f * f^{-1})(1) &= 1 \\ \Rightarrow f(1)f^{-1}(1) &= 1 \\ \Rightarrow f^{-1}(1) &= \frac{1}{f(1)}, \end{aligned}$$

which exists and is unique as $f(1) \neq 0$.

Now, assume for $k < n$ that the value of $f^{-1}(k)$ exists and is unique. For $n > 1$ we have $I(n) = 0$ and so

$$\begin{aligned} 0 &= (f * f^{-1})(n) \\ &= \sum_{d \mid n} f\left(\frac{n}{d}\right) f^{-1}(d) \\ &= f(1)f^{-1}(n) + \sum_{\substack{d \mid n \\ d<n}} f\left(\frac{n}{d}\right) f^{-1}(d). \end{aligned}$$

Thus,

$$f^{-1}(n) = \frac{-1}{f(1)} \sum_{\substack{d \mid n \\ d<n}} f\left(\frac{n}{d}\right) f^{-1}(d)$$

which exists and is uniquely determined as the values of $f^{-1}(d)$ are already known for d less than n. Thus, by induction, $f^{-1}(n)$ exists and is unique for all values of n, and hence so is f^{-1}. □

The previous two lemmas combined with the fact that $I(n)$ is an identity function for $*$ shows that the set of arithmetical functions with $f(1) \neq 0$ forms a group with the operation $*$, which we denote $\mathcal{A}$. In fact, as $*$ is easily shown to be commutative, $\mathcal{A}$ is an abelian group. The multiplicative functions then form a subgroup of $\mathcal{A}$, as the following two lemmas show.

Lemma 1.7. *If f and g are multiplicative arithmetical functions so is their Dirichlet convolution $f * g$.*

Proof. Let $h = f * g$ and choose $m, n \in \mathbb{N}$ such that $(m, n) = 1$. Note that if d divides mn then $d = ab$ where a divides m and b divides n. Further, since $(m, n) = 1$, we must have $(a, b) = 1$ and $\left(\frac{m}{a}, \frac{n}{b}\right) = 1$ also. Thus the products ab are exactly the divisors d of mn and we have

$$\begin{aligned} h(mn) &= \sum_{d|mn} f(d)g\left(\frac{mn}{d}\right) \\ &= \sum_{ab|mn} f(ab)g\left(\frac{mn}{ab}\right) \\ &= \sum_{\substack{a|m \\ b|n}} f(a)f(b)g\left(\frac{m}{a}\right)g\left(\frac{n}{b}\right) \\ &= \sum_{a|m} f(a)g\left(\frac{m}{a}\right)\sum_{b|n} f(b)g\left(\frac{n}{b}\right) \\ &= h(m)h(n). \end{aligned}$$

Thus, h is multiplicative. □

Further, a similar method shows that if g and $f * g$ are both multiplicative functions then so is f.

Lemma 1.8. *If g is multiplicative so is its Dirichlet inverse.*

Proof. We have $g * g^{-1} = I$, which is clearly multiplicative. As g is also multiplicative, by the result above g^{-1} is multiplicative. □

Another useful property of multiplicative functions emerges when we take their sum over the divisors of a positive integer n.

Lemma 1.9. *If f is a multiplicative function then*

$$\sum_{d|n} f(d) = \prod_{p^\alpha || n} \left(1 + f(p) + f(p^2) + ... + f(p^\alpha)\right)$$

where the product is over prime powers p^α which exactly divide n.

Further, if f is completely multiplicative then

$$\sum_{d|n} f(d) = \prod_{p^\alpha || n} \frac{f(p)^{\alpha+1} - 1}{f(p) - 1}.$$

In particular, we can apply this lemma with f equal to the Möbius function to obtain the following result.

Lemma 1.10. *For $n \in \mathbb{N}$,*

$$\sum_{d|n} \mu(d) = I(n).$$

Proof. If $n = 1$ we have $\sum_{d|1} \mu(d) = \mu(1) = 1 = I(1)$. Now, using Lemma 1.9, for $n > 1$ we have

$$\begin{aligned}
\sum_{d|n} \mu(d) &= \prod_{p^\alpha || n} \left(1 + \mu(p) + \mu(p^2) + ... + \mu(p^\alpha)\right) \\
&= \prod_{p^\alpha || n} (1 + \mu(p)) \\
&= \prod_{p^\alpha || n} (1 + (-1)) \\
&= 0 \\
&= I(n)
\end{aligned}$$

since $\mu(p^a) = 0$ for $a > 1$ and $\mu(p) = -1$ for any prime p. □

We can write this result using the Dirichlet convolution notation as

$$\mu * u = I. \tag{1.16}$$

This implies that the Dirichlet inverse of the Möbius function is u (and vice versa). We can use this property to prove the following important theorem.

Theorem 1.6. (Möbius Inversion Formula) *For two arithmetical functions f and g,*

$$f(n) = \sum_{d|n} g(d)$$

if and only if

$$g(n) = \sum_{d|n} \mu(d) f\left(\frac{n}{d}\right).$$

Proof. The first equation is the same as writing $f = g * u$. Taking the Dirichlet convolution with μ and recalling that $*$ is commutative, this implies that $\mu * f = (g * u) * \mu$. Now, using (1.16) and the associativity of $*$, we have $\mu * f = g * I = g$, which is the same as the second equation of the theorem. The inverse is proved similarly, by taking the Dirichlet convolution with u on both sides of $g = \mu * f$. □

Möbius inversion can be used to prove the following fact about the Euler totient function.

Lemma 1.11. *For $n \geq 1$ we have*

$$\sum_{d|n} \phi(d) = n. \tag{1.17}$$

Proof. First, note that

$$\phi(n) = \sum_{\substack{a=1 \\ (a,n)=1}}^{n} 1 = \sum_{a=1}^{n} \sum_{d|(a,n)} \mu(d)$$

since the inner sum is equal to 0 unless $(a, n) = 1$ by Lemma 1.10. Now, note that $d|(a, n)$ if and only if $d|a$ and $d|n$. We then change the order of summation using $a = bd$, so that

$$\begin{aligned} \phi(n) &= \sum_{d|n} \mu(d) \sum_{b=1}^{n/d} 1 \\ &= \sum_{d|n} \mu(d) \left(\frac{n}{d}\right) \\ &= (\mu * N)(n), \end{aligned} \tag{1.18}$$

where N is the function $N(n) = n$ for all $n \in \mathbb{N}$. We can then take the Dirichlet convolution on both sides with u to find that

$$\phi * u = N$$

which is the same as (1.17). □

Note that since both μ and N are multiplicative, (1.18) with Lemma 1.7 shows that ϕ is multiplicative.

1.3.3 Dirichlet Series

We have already seen the most famous Dirichlet series in Section 1.2.3, the Riemann zeta function $\zeta(s)$. In general, they are defined as follows.

Definition 1.6. *For an arithmetical function f, its Dirichlet series is defined by*

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

for some complex variable s.

Recalling that for $s > 1$ the Riemann zeta function is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

it is thus clearly the Dirichlet series of the unit function $u(n)$.

Ignoring issues of convergence, we can add and multiply Dirichlet series together using the rules stated in the lemma below.

Lemma 1.12. *For arithmetical functions f and g with Dirichlet series $F(s)$ and $G(s)$ respectively, we have*

1. $F(s) + G(s) = \sum_{n=1}^{\infty} \frac{f(n) + g(n)}{n^s}$,
2. $F(s)G(s) = \sum_{n=1}^{\infty} \frac{(f * g)(n)}{n^s}$.

For example, we can use this lemma to find the inverse of the Riemann zeta function. We have

$$\zeta(s) \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \sum_{n=1}^{\infty} \frac{(u * \mu)(n)}{n^s} = \sum_{n=1}^{\infty} \frac{I(n)}{n^s} = 1.$$

Then dividing through by $\zeta(s)$ gives the required result

$$\zeta^{-1}(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$

We can also use this along with the fact that $\phi = \mu * N$ to rewrite the Dirichlet series of Euler's totient function as follows.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} &= \sum_{n=1}^{\infty} \frac{(\mu * N)(n)}{n^s} \\
&= \left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\right) \left(\sum_{n=1}^{\infty} \frac{N(n)}{n^s}\right) \\
&= \zeta^{-1}(s) \left(\sum_{n=1}^{\infty} \frac{1}{n^{s-1}}\right) \\
&= \frac{\zeta(s-1)}{\zeta(s)}.
\end{aligned}$$

Dirichlet series can be quickly differentiated according to the following result, obtained simply by differentiating the sum term by term.

Lemma 1.13. *For an arithmetical function f with Dirichlet series $F(s)$,*

$$\frac{d}{ds} F(s) = -\sum_{n=1}^{\infty} \frac{(\ln n) f(n)}{n^s}.$$

Thus, for example, the derivative of the Riemann zeta function is

$$\zeta'(s) = -\sum_{n=1}^{\infty} \frac{\ln n}{n^s}.$$

1.3.4 Summing Arithmetical Functions

Many arithmetical functions oscillate substantially as n increases and this can make it difficult to study their behaviour. However, we can manage these oscillations somewhat by taking averages.

We will make use of the following notation.

Definition 1.7. *We write*

$$f(x) = \mathcal{O}\left(g(x)\right)$$

to mean that there exists a constant $M > 0$ *such that*

$$|f(x)| \leq M|g(x)|$$

for all sufficiently large x*. We write*

$$f(x) = o(g(x))$$

to mean that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

These are known as the Landau symbols 'big-oh' and 'little-oh' respectively.

When it is more convenient we will also use the alternative Vinogradov notation, writing $f(x) \ll g(x)$ when $f(x) = \mathcal{O}\left(g(x)\right)$.

We can now study a useful method for estimating the sums of arithmetical functions known as Abel's summation formula or partial summation.

Theorem 1.7. (Abel's Summation Formula) *Suppose we have functions* $a : \mathbb{N} \to \mathbb{R}$ *and* $f : \mathbb{R} \to \mathbb{R}$*, and that* $f'(x)$ *exists and is continuous. Let* $A(x) = \sum\limits_{n\leq x} a(n)$*. Then*

$$\sum_{n\leq x} a(n)f(n) = A(x)f(x) - \int_1^x A(t)f'(t)\,\mathrm{d}t.$$

Proof. Let $k = \lfloor x \rfloor$. Then $A(x) = A(k)$ and we have

$$\begin{aligned}
\sum_{n\leq x} a(n)f(n) &= \sum_{n=1}^{k} a(n)f(n) \\
&= \sum_{n=1}^{k} (A(n) - A(n-1)f(n)) \\
&= \sum_{n=1}^{k} A(n)f(n) - \sum_{n=0}^{k-1} A(n)f(n+1) \\
&= \sum_{n=1}^{k-1} A(n)(f(n) - f(n+1)) + A(k)f(k) - A(0)f(1)
\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{n=1}^{k-1} A(n) \int_n^{n+1} f'(t)\,\mathrm{d}t + A(k)f(k) \\
&= -\sum_{n=1}^{k-1} \int_n^{n+1} A(t)f'(t)\,\mathrm{d}t + A(k)f(k) \\
&= -\int_1^k A(t)f'(t)\,\mathrm{d}t + A(x)f(x) - \int_k^x A(t)f'(t)\,\mathrm{d}t \\
&= A(x)f(x) - \int_1^x A(t)f'(t)\,\mathrm{d}t.
\end{aligned}
$$

□

Alternatively this result can also be quickly proved using Riemann-Stieltjes integration.

Abel's Summation Formula can, for example, be used to find $\sum_{n\leq x} n^\alpha$ for any $\alpha \geq -1$. If $\alpha > -1$, we use Theorem 1.7 with $a(n) = 1$ and $f(n) = n^\alpha$. We then have $A(x) = \lfloor x \rfloor$ and so, denoting $\{t\} = t - \lfloor t \rfloor$,

$$
\begin{aligned}
\sum_{n\leq x} n^\alpha &= A(x)f(x) - \int_1^x A(t)f'(t)\,\mathrm{d}t \\
&= \lfloor x \rfloor x^\alpha - \int_1^x \alpha \lfloor t \rfloor t^{\alpha-1}\,\mathrm{d}t \\
&= x^{\alpha+1} - \{x\}x^\alpha - \alpha \left(\int_1^x t^\alpha\,\mathrm{d}t - \int_1^x \{t\}t^{\alpha-1}\,\mathrm{d}t \right) \\
&= \frac{1}{\alpha+1} x^{\alpha+1} + \mathcal{O}\left(x^\alpha\right).
\end{aligned}
$$

Otherwise if $\alpha = -1$, we use Theorem 1.7 with $a(n) = 1$ and $f(n) = \frac{1}{n}$. We then have $A(x) = \lfloor x \rfloor$ as before and so

$$
\begin{aligned}
\sum_{n\leq x} \frac{1}{n} &= A(x)f(x) - \int_1^x A(t)f'(t)\,\mathrm{d}t \\
&= \frac{\lfloor x \rfloor}{x} + \int_1^x \frac{\lfloor t \rfloor}{t^2}\,\mathrm{d}t \\
&= \frac{\lfloor x \rfloor}{x} + \int_1^x \frac{1}{t}\,\mathrm{d}t - \int_1^x \frac{\{t\}}{t^2}\,\mathrm{d}t
\end{aligned}
$$

$$= \ln x + \mathcal{O}(1) - \mathcal{O}\left(\int_1^x \frac{1}{t^2}\,\mathrm{d}t\right)$$

$$= \ln x + \mathcal{O}(1).$$

We could also use another technique for estimating sums of arithmetical functions called Euler-Maclaurin summation to improve the last estimate to

$$\sum_{n\leq x}\frac{1}{n} = \ln x + \gamma + \frac{1}{2N} + \frac{1}{12N^2} + \mathcal{O}\left(\frac{1}{N^3}\right),$$

where γ is Euler's constant. However, the extra main terms will not be needed in what follows and so we omit the details of this method.

1.3.5 Gauss Circle Problem

In Chapter 4 we will encounter a variant of the Gauss Circle Problem. This famous problem concerns an arithmetical function $r_2(n)$ called the sum of squares function. In short, $r_2(n)$ counts the number of integer solutions to the equation

$$a^2 + b^2 = n;$$

i.e. it counts integer points that lie on the circle or radius $\sqrt{n}$ centred at the origin. The problem we are interested in is called the Gauss Circle Problem, named after Carl Friedrich Gauss as he was the first person to study it [8].

Gauss Circle Problem. *For a given real number m, determine the number of pairs of integers a and b that satisfy the inequality*

$$a^2 + b^2 \leq m.$$

In terms of the sum of squares function, the problem asks for the value of

$$A(m) = \sum_{n\leq m} r_2(n).$$

It is clear that $A(m)$ is equal to the number of integer points inside a circle

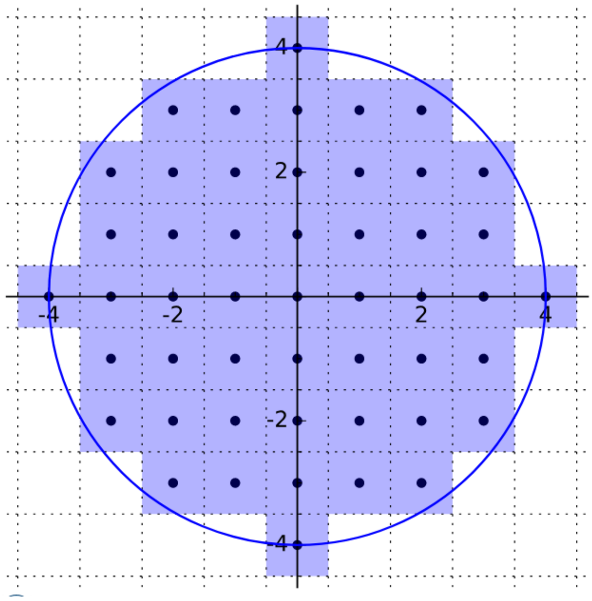

Figure 1.4: Unit squares around points of the integer lattice inside the circle of radius 4. The shaded area is equal to the number of points.

of radius $\sqrt{m}$ centred at the origin. From this perspective, we observe that

$$A(m) = \pi m + \mathcal{O}\left(m^{\kappa}\right) \tag{1.19}$$

for some $0 < \kappa < 1$. To see this, draw a unit square with sides parallel to the co-ordinate axes around each point on the integer lattice that is inside the circle, as shown in figure 1.4. Clearly the combined area of these squares is equal to the number of lattice points inside the circle. This area is also obviously close to the area of the circle, which is πm, but how close is it?

Gauss proved that this holds when $\kappa = \frac{1}{2}$, which can be seen as follows. The diagonal of a unit square has length $\sqrt{2}$. So for any point in the circle its square must be fully contained within the circle of radius $\sqrt{m}+\frac{\sqrt{2}}{2}$ centred at the origin. On the other hand, this also means that the squares must completely cover the circle of radius $\sqrt{m}-\frac{\sqrt{2}}{2}$ centred at the origin. Thus we have

$$\pi\left(\sqrt{m}-\frac{\sqrt{2}}{2}\right)^2 \leq A(m) \leq \pi\left(\sqrt{m}+\frac{\sqrt{2}}{2}\right)^2.$$

This simplifies to

$$\pi m - \pi\sqrt{2}\sqrt{m} + \frac{\pi}{2} \leq A(m) \leq \pi m + \pi\sqrt{2}\sqrt{m} + \frac{\pi}{2},$$

leading us to Gauss's result that $\kappa \leq \frac{1}{2}$.

This bound has since been improved upon multiple times and it has been conjectured that the true bound is $\mathcal{O}(m^{\frac{1}{4}+\epsilon})$ for any $\epsilon > 0$. However, the best we can do for now is $\mathcal{O}(m^{\kappa})$ where $\frac{1}{4} < \kappa < \frac{131}{416}$. The lower bound for κ was provided independently by both by Hardy [10] and Landau [15] in 1915 and the upper bound follows from Huxley's work in 2003 [13].

As already mentioned, in Chapter 4 we will be concerned with a variant of the Gauss Circle Problem also dealt with by the work of Huxley. In particular, the points of interest are limited to a subregion of the circle.

1.4 k-Moments for Ford Circles

In this section we review the work of Chaubey et. al. in [3] on k-moments of distances between centres of Ford circles. This will motivate the work carried out in this thesis which attempts to generalise their work to higher dimensions. While their calculations are valid for any subinterval I of $[0,1]$, we focus on the case when $I = [0,1]$ as that is what will be most relevant to us in higher dimensions. Following their notation, we denote the Ford circle corresponding to the j^{th} Farey fraction in $\mathcal{F}_Q$ by $C_{Q,j}$ and its centre by $O_{Q,j}$. We write $D(a,b)$ for the Euclidean distance between two points a and b. We can now define the k^{th} moment of the distance between the centres of consecutive Ford circles in $\mathcal{F}_Q$ as

$$\mathcal{M}_k(Q) = \sum_{j=1}^{N(Q)-1} \left(D(O_{Q,j}, O_{Q,j+1})\right)^k.$$

Chaubey et. al. study the averages of these moments for all large X,

$$\mathcal{A}_k(X) = \frac{1}{X}\int_X^{2X} \mathcal{M}_k(Q)\,\mathrm{d}Y, \tag{1.20}$$

where $Q = \lfloor Y \rfloor$.

1.4.1 First Moment for Ford Circles

For the first moment Chaubey et al. proved the following result.

Theorem 1.8. *[3, Theorem 1.1] When $k = 1$ in (1.20), we have*

$$\mathcal{A}_1(X) = \frac{6}{\pi^2}\ln(4X) + B_1 + \mathcal{O}\left(\frac{1}{Xe^{c_0(\ln X)^{3/5}(\ln\ln X)^{-1/5}}}\right),$$

where

$$B_1 = \frac{\gamma - 1}{\zeta(2)} - \frac{\zeta'(2)}{\zeta^2(2)}.$$

We now briefly review their proof strategy, as we will later use similar techniques in proving the equivalent theorems for Ford spheres. Note that this is adjusted here to account for the fact that we are only considering moments on the full interval $[0, 1]$.

First, notice that because the circles $C_{Q,j}$ and $C_{Q,j+1}$ are Ford circles of consecutive Farey fractions, they must be tangent, and so the distance between their centres is $D(O_{Q,j}, O_{Q,j+1}) = \frac{1}{2q_j^2} + \frac{1}{2q_{j+1}^2}$ where q_i is the denominator of the i^{th} fraction in $\mathcal{F}_Q$. Thus we have

$$\begin{aligned}\mathcal{M}_1(Q) &= \sum_{j=1}^{N(Q)-1}\left(\frac{1}{2q_j^2} + \frac{1}{2q_{j+1}^2}\right)\\ &= \sum_{j=2}^{N(Q)}\frac{1}{q_j^2} + \frac{1}{2q_1^2} - \frac{1}{2q_{N(Q)}^2}\\ &= \sum_{1\le q\le Q}\frac{1}{q^2}\sum_{\substack{0<a\le q\\(a,q)=1}} 1\\ &= \sum_{1\le q\le Q}\frac{\phi(q)}{q^2}\end{aligned}$$

since $q_1 = q_{N(Q)} = 1$ when working over the full interval $[0, 1]$. We then have

$$\mathcal{A}_1(X) = \frac{1}{X}\int_X^{2X}\sum_{1\le q\le Q}\frac{\phi(q)}{q^2}\,\mathrm{d}Y = \frac{1}{X}\int_1^{2X}\sum_{1\le q\le Q}\frac{\phi(q)}{q^2}\,\mathrm{d}Y - \frac{1}{X}\int_1^{X}\sum_{1\le q\le Q}\frac{\phi(q)}{q^2}\,\mathrm{d}Y.$$

To evaluate this integral we write $f(q) = \frac{\phi(q)}{q^2}$ and split into sections where $\lfloor Y \rfloor$

is constant.

$$\begin{aligned}\int_1^X \sum_{q\le\lfloor Y\rfloor} f(q)\,\mathrm{d}Y &= \int_1^2 \sum_{q\le 1} f(q)\,\mathrm{d}Y + \int_2^3 \sum_{q\le 2} f(q)\,\mathrm{d}Y + \cdots + \int_{X-1}^X \sum_{q\le X-1} f(q)\,\mathrm{d}Y \\ &= \sum_{q\le 1} f(q) + \sum_{q\le 2} f(q) + \cdots + \sum_{q\le X-1} f(q) \\ &= \sum_{q\le X-1} f(q)(X-q) \\ &= \sum_{q\le X} f(q)(X-q).\end{aligned}$$

Now, $f(q)$ is a multiplicative arithmetical function with Dirichlet series

$$\sum_{q=1}^{\infty} \frac{\phi(q)q^{-2}}{q^s} = \frac{\zeta(s+1)}{\zeta(s+2)}$$

for complex $s = \sigma + i\tau$, which converges when $\sigma > 0$. Applying Perron's formula as stated in [22], for example, for $c > 0$ gives us

$$\begin{aligned}\frac{1}{X}\sum_{q\le X} f(q)(X-q) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{X^s\zeta(s+1)}{s(s+1)\zeta(s+2)}\,\mathrm{d}s \\ &=: \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} g(s)\,\mathrm{d}s.\end{aligned}$$

Chaubey et. al. then modify a section of the path of integration and use the residue theorem to show that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} g(s)\,\mathrm{d}s = \sum \mathrm{Res}(g(s)) + \sum_{m=1}^{9} J_m,$$

where the J_m are integrals along the new path and the sum of $\mathrm{Res}(g(s))$ is taken over all poles of $g(s)$ inside the region bounded by the new path and the unmodified path section. The only such pole in this case is at $s = 0$ which is of order 2 and has residue

$$\mathrm{Res}(g(s)) = \frac{\ln X}{\zeta(2)} + \frac{\gamma - 1}{\zeta(2)} - \frac{\zeta'(2)}{\zeta^2(2)}.$$

Using standard bounds for $\zeta(s)$ and $\zeta^{-1}(s)$ they find that

$$\sum_{m=1}^{9} J_m = \mathcal{O}\left(\frac{1}{Xe^{c_0(\ln X)^{3/5}(\ln\ln X)^{-1/5}}}\right),$$

for some suitable positive absolute constant c_0.

Putting all of this together they obtain

$$\mathcal{A}_1(X) = \frac{6}{\pi^2}\ln(4X) + \frac{\gamma-1}{\zeta(2)} - \frac{\zeta'(2)}{\zeta^2(2)} + \mathcal{O}\left(\frac{1}{Xe^{c_0(\ln X)^{3/5}(\ln\ln X)^{-1/5}}}\right)$$

as required.

1.4.2 Second and Higher Moments for Ford Circles

For $k \geq 2$ Chaubey et al. prove the next two theorems.

Theorem 1.9. *[3, Theorem 1.2] When $k = 2$ in (1.20), we have*

$$\mathcal{A}_2(X) = \frac{\zeta(3)}{2\zeta(4)} + \frac{3}{\pi^2}\frac{\log X}{X^2} + \frac{3}{\pi^2}\left(\gamma - \frac{\zeta'(2)}{\zeta(2)} + \frac{5}{4} - \log 2\right)\frac{1}{X^2} + \mathcal{O}_\epsilon\left(\frac{\log^{5/3} X(\log\log X)^{1+\epsilon}}{X^3}\right).$$

Theorem 1.10. *[3, Theorem 1.3] When $k \geq 3$ in (1.20), we have*

$$\mathcal{A}_k(X) = \frac{\zeta(2k-1)}{2^{k-1}\zeta(2k)} + \frac{k\zeta(2k-3)}{2^k\zeta(2k-2)}\frac{1}{X^2} + \mathcal{O}\left(\frac{1}{X^3}\right).$$

Chaubey et al.'s proofs of these theorems begin similarly to that of Theorem 1.8, again adjusted here to only include the case when I is the full interval $[0,1]$. We have,

$$\begin{aligned}\mathcal{M}_k(Q) &= \sum_{j=1}^{N(Q)-1}\left(\frac{1}{2q_j^2} + \frac{1}{2q_{j+1}^2}\right)^k \\ &= \frac{1}{2^{k-1}}\sum_{j=2}^{N(Q)}\frac{1}{q_j^{2k}} + \frac{1}{2^k}\sum_{j=1}^{N(Q)-1}\sum_{i=1}^{k-1}\binom{k}{i}\frac{1}{q_j^i q_{j+1}^{k-i}}^2\end{aligned}$$

$$= \frac{1}{2^{k-1}} \sum_{1 \leq q \leq Q} \frac{\phi(q)}{q^{2k}} + \frac{1}{2^k} \sum_{i=1}^{k-1} \binom{k}{i} \sum_{j=1}^{N(Q)-1} \frac{1}{q_j^{2i} q_{j+1}^{2k-2i}}$$
$$=: S_k + S_k'.$$

Thus,

$$\begin{aligned} \mathcal{A}_k(X) &= \frac{1}{X} \int_X^{2X} S_k + S_k' \,\mathrm{d}Y \\ &= \frac{1}{X} \int_1^{2X} S_k \,\mathrm{d}Y - \frac{1}{X} \int_1^{X} S_k \,\mathrm{d}Y + \frac{1}{X} \int_X^{2X} S_k' \,\mathrm{d}Y. \end{aligned} \tag{1.21}$$

Using the same method as for the first moments yields

$$\frac{1}{X} \int_X^{2X} S_k \,\mathrm{d}Y = \frac{\zeta(2k-1)}{2^{k-1}\zeta(2k)} + \frac{1 - 2^{2k-3}}{2^{3k-4}(2k-3)(2k-2)\zeta(2)X^{2k-2}} + \mathcal{O}\left(\frac{\log X}{X^{2k-1}}\right).$$

Now, evaluating S_k' for $k \geq 2$ is a key part of their proof and will be split in cases for $k = 2$ and $k \geq 3$. To proceed they have a geometric criterion for determining when two integers appear as consecutive denominators, which allows them to transform the problem into a lattice point counting problem. This is the basis for the strategy we will use in the proof for the higher dimensional analogue with Ford spheres.

When $k = 2$ we have,

$$S_2' = \frac{1}{2} \sum_{j=1}^{N(Q)-1} \left(\frac{1}{q_j q_{j+1}}\right)^2.$$

Noting that we are dealing with the full interval $[0, 1]$, Chaubey et al. then use Theorem 2 of [2] to obtain

$$S_2' = \frac{6}{\pi^2 Q^2} \left(\log Q + \gamma - \frac{\zeta'(2)}{\zeta(2)} + \frac{1}{2}\right) + \mathcal{O}_\epsilon\left(\frac{\log^{5/3} X (\log\log X)^{1+\epsilon}}{X^3}\right).$$

Putting this and S_k with $k = 2$ into (1.21) then gives

$$\mathcal{A}_2(X) = \frac{\zeta(3)}{2\zeta(4)} + \frac{3}{\pi^2}\frac{\log X}{X^2} + \frac{3}{\pi^2}\left(\gamma - \frac{\zeta'(2)}{\zeta(2)} + \frac{5}{4} - \log 2\right)\frac{1}{X^2} + \mathcal{O}_\epsilon\left(\frac{\log^{5/3} X(\log\log X)^{1+\epsilon}}{X^3}\right),$$

proving Theorem 1.9.

Now, when $k \geq 3$ we proceed with S'_k by first considering the sum

$$S_{k,i} := \sum_{j=1}^{N(Q)-1} \frac{1}{q_j^{2i} q_{j+1}^{2k-2i}}.$$

This sum can then be rewritten using the fact that two positive integers r and q are the denominators of consecutive Farey fractions in $\mathcal{F}_Q$ if and only if they satisfy

1. $q, r \leq Q$,
2. $(q, r) = 1$, and
3. $q + r > Q$.

Thus,

$$S_{k,i} = \sum_{\substack{1 \leq q,r \leq Q \\ (q,r)=1 \\ q+r>Q}} \frac{1}{q^{2i} r^{2k-2i}}.$$

Chaubey et al. then split this into three further sums, depending on whether we have $r < \frac{Q}{2}$, $q < \frac{Q}{2}$, or $q, r > \frac{Q}{2}$. Finally, they use the fact that for $x \geq 2$ we have,

$$\sum_{n \leq x} \frac{\phi(n)}{n} = \mathcal{O}(x), \quad \sum_{n \leq x} \frac{\phi(n)}{n^2} = \frac{\log x}{\zeta(2)} + \mathcal{O}(1), \text{ and}$$
$$\sum_{n \leq x} \frac{\phi(n)}{n^a} = \frac{\zeta(a-1)}{\zeta(a)} + \mathcal{O}\left(x^{2-a}\right) \text{ for } a \geq 3,$$

to evaluate each of these three sums. This leaves us with

$$S'_k = \frac{k\zeta(2k-3)}{2^{k-1}\zeta(2k-2)}\frac{1}{Q^2} + \mathcal{O}\left(\frac{1}{Q^3}\right).$$

As in the previous case, substituting this along with S_k into (1.21) proves Theorem 1.10.

1.5 Higher Dimensions – Ford Spheres

Ford spheres are the three dimensional analogue of Ford circles and were likewise first defined by Ford in [5]. These spheres arise when in place of ratios of rational integers we consider ratios of Gaussian integers. The definitions and results concerning Ford spheres in this section can be found in Section 8 of Ford's paper [5] and those concerning Gaussian integers and Gaussian rationals can be found, for example, in [6] or [21].

The Gaussian integers are defined as those complex numbers $a+bi$ for which a and b are integers. The set of all such numbers forms an integral domain with the usual addition and multiplication of complex numbers, and it is denoted by $\mathbb{Z}[i]$, i.e. i adjoined to the rational integers. We will denote the set of Gaussian integers $a+bi$ with $a>0$ and $b\geq 0$ by $\mathbb{Z}[i]^+$. The units of $\mathbb{Z}[i]$ are $1,-1,i$ and $-i$.

The Gaussian rationals form the field of fractions of $\mathbb{Z}[i]$. They consist of those complex numbers $a+bi$ where a and b are real rationals and are denoted by $\mathbb{Q}[i]$. We can also write the Gaussian rationals as fractions of Gaussian integers and this is the form we will generally use in this thesis. Note that for Gaussian integers $r=r_1+r_2i$ and $s=s_1+s_2i$ we have

$$\begin{aligned}\frac{r}{s} &= \frac{r_1+r_2i}{s_1+s_2i}\\ &= \frac{(r_1+r_2i)(s_1-s_2i)}{(s_1+s_2i)(s_1-s_2i)}\\ &= \frac{r_1s_1+r_2s_2}{s_1^2+s_2^2}+\frac{r_2s_1-r_1s_2}{s_1^2+s_2^2}i.\end{aligned}$$

As with the usual integers, two fractions of distinct Gaussian integers may be equal to the same Gaussian rational. However, like $\mathbb{Z}$, $\mathbb{Z}[i]$ is a unique factorisation domain and so we can define a greatest common divisor for any pair r and s of Gaussian integers. We denote this by (r,s) and it is defined as a Gaussian integer d such that

1. d divides both r and s, and

2. if another Gaussian integer c satisfies condition 1 then c divides d.

The greatest common divisor is unique up to multiplication by a unit. We will follow the convention of choosing the divisor that lies in $\mathbb{Z}[i]^+$. We can then write a Gaussian fraction $\frac{r}{s}$ in its reduced form by dividing r and s by (r, s).

It will also be pertinent to mention the prime elements of $\mathbb{Z}[i]$, called the Gaussian primes. These turn out to be those Gaussian integers $a + bi$ such that either

1. a or b is zero and the other has absolute value equal to a prime that is congruent to 3 modulo 4, or

2. a and b are both non-zero and $a^2 + b^2$ is prime.

In particular, this means that not all prime numbers are Gaussian primes. For example, $2 = (1+i)(1-i)$. As $\mathbb{Z}[i]$ is a unique factorisation domain, any Gaussian integer r can be written uniquely up to some choice of units, as a product of Gaussian primes and a unit. That is, we can write

$$r = up_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$$

for a unit u, real $\alpha_j \geq 1$ and distinct Gaussian primes p_j in $\mathbb{Z}[i]^+$.

The Ford spheres give a geometric representation of these Gaussian fractions according to the following definition.

Definition 1.8. *The Ford sphere corresponding to a reduced Gaussian fraction* $\frac{r}{s}$ *is the sphere of radius* $\frac{1}{2|s|^2}$ *in the upper half-space* $\mathbb{C} \times \mathbb{R}^+$*, tangent to the complex plane at* $\frac{r}{s}$.

We will be concerned with the Ford spheres of those Gaussian rationals lying in the unit square in the upper-right quadrant of the complex plane, which we denote I_2. If we take a vertical slice of these spheres along the x-axis we obtain an image of the Ford circles in the interval $[0, 1]$. Two Ford spheres corresponding to distinct Gaussian fractions are either tangent or disjoint. This can be shown using a similar calculation as we used for Ford circles in Section 1.1. As before, the interesting case is when the two spheres are tangent and, for the corresponding Gaussian rationals $\frac{r}{s}$ and $\frac{r'}{s'}$, we have

$$|r's - rs'| = 1.$$

When this is true we call the fractions adjacent. Unlike Ford circles, there is no obvious way to define 'consecutive' for Ford spheres as the Gaussian rationals do not have a natural ordering like the real integers do. This issue will be addressed in Chapter 3. There we will also establish a way to generate new Gaussian fractions from old ones, which builds on Ford's observation that for adjacent fractions $\frac{r}{s}$ and $\frac{r'}{s'}$, any fraction of the form

$$\frac{r'_n}{s'_n} = \frac{r' + nr}{s' + ns}$$

is also adjacent to $\frac{r}{s}$, where n is any Gaussian integer.

1.5.1 Statements of Main Theorems

In the following chapters of this thesis we will work to prove Theorems 1.11 and 1.12, stated below. These results are the higher dimensional analogues of Theorems 1.8, 1.9 and 1.10 for Ford circles, in which we examine the k-moments of the distances between centres of consecutive Ford spheres. As our definition of consecutive will require the spheres to be adjacent and so tangent, the distance between their centres is given by the sum of their radii. Thus, the k^{th} moment is defined as

$$\mathcal{M}_{k,I_2}(S) = \sum_{\substack{\frac{r}{s}, \frac{r'}{s'} \in \mathcal{G}_S \\ \textit{consecutive}}} \left(\frac{1}{2|s|^2} + \frac{1}{2|s'|^2} \right)^k, \tag{1.22}$$

for positive integers k and S.

For the first moment we have the following theorem, which is proved in Chapter 4. Here ζ_i denotes the Dedekind zeta function for $\mathbb{Q}(i)$ defined for a complex number s with $Re(s) > 1$ by

$$\zeta_i(s) = \sum_{q \in \mathbb{Z}[i]^+} \frac{1}{|q|^{2s}}.$$

Theorem 1.11. *For $S \in \mathbb{N}$ and any $\epsilon > 0$,*

$$\mathcal{M}_{1,I_2}(S) = \frac{\pi}{4}\zeta_i^{-1}(2)(8z_1'' - 1)S^2 + \mathcal{O}_\epsilon(S^{1+\epsilon}).$$

where $z_1'' = -\int_0^{\frac{1}{\sqrt{2}}} \ln(\sqrt{2}u)(1-u^2)^{\frac{1}{2}}du$.

For all higher moments we prove the following in Chapter 5.

Theorem 1.12. *For each integer $k \geq 2$, there exists a constant $\xi_k > 0$ with the property that, for any $\epsilon > 0$ and for any $S \in \mathbb{N}$,*

$$\mathcal{M}_{k,I_2}(S) = \xi_k S + \mathcal{O}_\epsilon(S^{2\kappa+\epsilon}),$$

with $\frac{1}{4} < \kappa \leq \frac{131}{416}$.

1.6 Outlook

In the next two chapters we assemble the foundations required to work with the moments described in Section 1.5.1. Chapter 2 concerns Gaussian integers and the analogues of the arithmetical functions we saw in Section 1.3. In particular, we will define Möbius- and Euler-phi-type functions for $\mathbb{Z}[i]^+$, then go on to explore a number of their valuable properties.

Chapter 3 explores the Ford spheres and the higher dimensional analogue of the Farey fractions that underpins them in the same way that the Farey fractions underpin the theory of Ford circles. Specifically, we investigate the best way to define 'consecutivity' for Ford spheres and use this to demonstrate how Gaussian fractions in I_2 can be constructed from just $0, 1, i$ and $1+i$ with an approach comparable to that used in creating the Stern-Brocot tree. Furthermore, we classify consecutivity for pairs of denominators of Gaussian fractions and use this to count the Gaussian integers that are denominators of Gaussian fractions which are consecutive to another Gaussian fraction with a given denominator.

With the groundwork laid out, in Chapter 4 we prove Theorem 1.11. This will involve calculating the area of Ω_s, a subregion of a circle, then applying this along with Abel's Summation Formula and the results of the preceding chapters in order to achieve our asymptotic estimate of the first moment.

Finally, Chapter 5 contains the proof of Theorem 1.12, which will be split into cases for $k = 2$ and $k \geq 3$. This proof will again make use of Abel's Summation Formula and many of the prerequisites of Chapters 2 and 3, but requires more care as the error terms become more delicate in the higher moment calculations.

Chapter 2

Preliminaries – Gaussian Integers

In this chapter we lay out various definitions and results involving Gaussian integers and associated functions, which will be crucial to the rest of the thesis. Although many of these are analogues of well known facts for arithmetical functions, for completeness their proofs are also included.

Specifically, in Section 2.1 we define Möbius and Euler-phi-type functions with domain $\mathbb{Z}[i]^+$. We also describe when functions such as these may be considered multiplicative and detail how their sum may be rewritten as a product in this case.

In Section 2.2 we state and prove numerous useful lemmas involving the Möbius and Euler-phi functions from the previous section. In particular, Lemma 2.4 illustrates Möbius inversion for functions defined on $\mathbb{Z}[i]^+$.

Lastly, in Section 2.3 we define the Dedekind zeta function for $\mathbb{Q}(i)$, which gives the constant in our asymptotic formula for the partial sums of the Euler-phi-type function. In proving our estimate of this sum we revisit the sum of squares function from the Gauss Circle Problem and utilise the lemmas from the preceding sections.

2.1 Multiplicative Functions on $\mathbb{Z}[i]$

Arithmetical functions were integral to the study of moments of distances between centres of Ford circles, as their underlying fractions are ratios of positive integers. Now that we wish to study Ford spheres we will need analogous functions which are defined not on $\mathbb{N}$, but instead on some corresponding subset of the Gaussian integers. Thus, in place of functions defined on $\mathbb{N}$ we work with functions defined on $\mathbb{Z}[i]^+$, which we defined in Section 1.5 as those Gaussian integers s with $Re(s) > 0$ and $Im(s) \geq 0$. In fact, the functions introduced in this section will actually be considered to be defined on all the non-zero elements of $\mathbb{Z}[i]$, simply by taking them such that their value is not affected when the input is multiplied by a unit, that is,

$$f(z) = f(-z) = f(iz) = f(-iz),$$

for any z in $\mathbb{Z}[i]^+$.

Analogously to the usual arithmetical functions, a function $f : \mathbb{Z}[i]^+ \to \mathbb{C}$ will be called multiplicative if

$$f(qr) = f(q)f(r) \tag{2.1}$$

whenever $(q, r) = 1$. The function is called completely multiplicative if (2.1) is true for all q and r in $\mathbb{Z}[i]^+$. For example, we can define an identity function $I_i(q)$ on $\mathbb{Z}[i]^+$ comparable to the function $I(n)$ on $\mathbb{N}$ to be

$$I_i(q) = \begin{cases} 1 & \text{if } q = 1, \\ 0 & \text{otherwise.} \end{cases}$$

for any q in $\mathbb{Z}[i]^+$. The function is clearly completely multiplicative.

We can also define functions analogous to the Möbius and Euler-phi funtions we saw in Section 1.3 as follows, recalling that any Gaussian integer q can be written uniquely in the form

$$q = u p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k} \tag{2.2}$$

for a unit u, real $\alpha_j \geq 1$ and distinct Gaussian primes p_j in $\mathbb{Z}[i]^+$.

Definition 2.1. *For a Gaussian integer q as in* (2.2)*, define $\mu_i : \mathbb{Z}[i]^+ \to \mathbb{Z}$ by*

$$\mu_i(q) := \begin{cases} 1 & \text{if } q = 1, \\ (-1)^k & \text{if } \alpha_1 = \cdots = \alpha_k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.2. *For a Gaussian integer q as in (2.2), define $\phi_i : \mathbb{Z}[i]^+ \to \mathbb{Z}$ by*

$$\phi_i(q) := |(\mathbb{Z}[i]/q\mathbb{Z}[i])^*| .$$

The function $\mu_i(q)$ can be shown to be multiplicative using the same argument as for the usual Möbius function.

In Section 1.3 we saw that sums of multiplicative arithmetical functions can be rewritten in terms of a product. The following lemma shows that we can do the same thing here with multiplicative functions defined on $\mathbb{Z}[i]^+$. Here and in the rest of this chapter, for all $q \in \mathbb{Z}[i]$, $\sum_{d|q}$ denotes a sum over $d \in \mathbb{Z}[i]^+$ which divide q.

Lemma 2.1. *For a multiplicative function $f : \mathbb{Z}[i]^+ \to \mathbb{C}$ and $q = up_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$, we have*

$$\sum_{d|q} f(d) = \prod_{j=1}^{k} (f(1) + f(p_j) + \ldots + f(p_j^{\alpha_j})).$$

If f is completely multiplicative then

$$\sum_{d|q} f(d) = \prod_{j=1}^{k} \left(\frac{f(p_j)^{\alpha_j+1} - 1}{f(p_j) - 1} \right).$$

The proof follows from multiplicativity together with the unique factorisation property mentioned above.

2.2 Elementary Lemmas

The functions μ_i and ϕ_i have various interesting properties that will be of use to us in calculating the moments for Ford spheres in Chapters 4 and 5. We begin by seeing what happens when we take the sum of $\mu_i(q)$ and $\phi_i(q)$ over the divisors of q. Our first result is an analogue of Lemma 1.10.

Lemma 2.2. *For q as in (2.2) we have*

$$\sum_{d|q} \mu_i(d) = \begin{cases} 1 & \text{if } q = u, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Clearly this is true when $q = u$ so suppose $q \neq u$. Then we have

$$\begin{aligned}
\sum_{d|q} \mu_i(d) &= \sum_{d|p_1 \ldots p_k} \mu_i(d) \\
&= 1 - \sum_{p_i} 1 + \sum_{\substack{p_i, p_j \\ i \neq j}} 1 - \cdots \\
&= (1-1)^k \\
&= 0.
\end{aligned}$$

□

The next lemma is analogous to Lemma 1.11.

Lemma 2.3. *For every $q \in \mathbb{Z}[i]^+$ we have*

$$\sum_{d|q} \phi_i(d) = |q|^2$$

Proof. Let $U = \{x + iy | 0 \leq x, y < 1\}$. Then $U_q := q \cdot U$ is a fundamental domain for ${}^{\mathbb{C}}/_{q\mathbb{Z}[i]}$ and we have

$$\begin{aligned}
|{}^{\mathbb{Z}[i]}/_{q\mathbb{Z}[i]}| &= |U_q \cap \mathbb{Z}[i]| \\
&= \sum_{d|q} \#\{r \in U_q \cap \mathbb{Z}[i] \mid (r, q) = d\} \\
&= \sum_{d|q} \#\{r \in U_{q/d} \cap \mathbb{Z}[i] \mid (r, \frac{q}{d}) = 1\} \\
&= \sum_{d|q} \phi_i\left(\frac{q}{d}\right) = \sum_{d|q} \phi_i(d).
\end{aligned}$$

Also, we have $|{}^{\mathbb{Z}[i]}/_{q\mathbb{Z}[i]}| = vol(U_q) = |q|^2$, so we are done. □

The function μ_i also provides us with a result analogous to Theorem 1.6 for Möbius inversion.

Lemma 2.4. **(Möbius Inversion on $\mathbb{Z}[i]^+$)** *Two functions $f, g : \mathbb{Z}[i]^+ \to \mathbb{C}$ satisfy*

$$f(q) = \sum_{d|q} g(d), \tag{2.3}$$

for all $q \in \mathbb{Z}[i]^+$ if and only if they satisfy

$$\begin{aligned} g(q) &= \sum_{d|q} \mu_i(d) f\left(\frac{q}{d}\right) \\ &= \sum_{d|q} \mu_i\left(\frac{q}{d}\right) f(d), \end{aligned} \tag{2.4}$$

for all $q \in \mathbb{Z}[i]^+$.

Proof. Suppose (2.3) holds. Then

$$\begin{aligned} \sum_{d|q} \mu_i\left(\frac{q}{d}\right) f(d) &= \sum_{d|q} \mu_i\left(\frac{q}{d}\right) \sum_{e|d} g(e) \\ &= \sum_{e|q} g(e) \sum_{d'|\frac{q}{e}} \mu_i\left(\frac{q}{d'e}\right) \\ &= \sum_{e|q} g(e) \sum_{d'|\frac{q}{e}} \mu_i(d') \\ &= g(q) \end{aligned}$$

since $\sum_{d'|\frac{q}{e}} \mu_i(d') = 0$ unless $\frac{q}{e} = 1$. The other direction is proved similarly. □

We can use this technique with $f(q) = |q|^2$ and $g(q) = \phi_i(q)$ along with Lemma 2.3 to relate these Möbius and Euler-phi functions to one another according to our next lemma.

Lemma 2.5. *For every q in $\mathbb{Z}[i]^+$ we have*

$$\phi_i(q) = |q|^2 \sum_{d|q} \frac{\mu_i(d)}{|d|^2}.$$

2.3 The Dedekind Zeta Function for $\mathbb{Q}(i)$ and Summing ϕ_i

In addition to the higher dimensional analogues of the arithmetical functions, we will also make use of the Dedekind zeta function for $\mathbb{Q}(i)$, which we denote ζ_i. We have, for a complex number s with $Re(s) > 1$ and Gaussian primes p,

$$\begin{aligned}\zeta_i(s) &:= \sum_{q \in \mathbb{Z}[i]^+} \frac{1}{|q|^{2s}} \\ &= \prod_{\substack{p \in \mathbb{Z}[i]^+ \\ (p \text{ prime})}} (1 - |p|^{-2s})^{-1}.\end{aligned}$$

Further, the inverse of ζ_i is

$$\zeta_i^{-1}(s) = \sum_{q \in \mathbb{Z}[i]^+} \frac{\mu_i(q)}{|q|^{2s}}.$$

With this in mind we now aim to prove the following lemma, which will be integral to our proofs of Theorem 1.11 and Theorem 1.12.

Lemma 2.6. *For $Q \geq 1$, we have*

$$\sum_{\substack{q \in \mathbb{Z}[i]^+ \\ |q| \leq Q}} \phi_i(q) = \frac{\pi}{8} \zeta_i^{-1}(2) Q^4 + \mathcal{O}\left(Q^{2+2\kappa}\right).$$

To begin, recall the sum of squares function,

$$r_2(n) = \#\{(a, b) \in \mathbb{Z}^2 \mid a^2 + b^2 = n\},$$

for any positive integer n. In Section 1.3.5 we saw how a finite sum of this function behaves. Now, however, we will be interested in its sum when multiplied by some power of the integer n.

Lemma 2.7. *For $a \geq 0$,*

$$\sum_{n \leq N} n^a r_2(n) = \frac{\pi N^{a+1}}{a+1} + \mathcal{O}(N^{a+\kappa})$$

where κ is the exponent in the error term from the Gauss circle problem and satisfies $\frac{1}{4} < \kappa < \frac{1}{2}$.

Proof. We apply Abel's Summation Formula with $x = N$, $f(n) = n^b$ for $b \geq 0$ and $a(n) = r_2(n)$ so that

$$A(n) = \sum_{n \leq N} r_2(n) = \pi N + \mathcal{O}\left(n^{\kappa}\right)$$

from (1.19). Thus,

$$\begin{aligned}
\sum_{n \leq N} n^b r_2(n) &= A(N)f(N) - \int_1^N A(t)f'(t)\,\mathrm{d}t \\
&= \pi N^{b+1} + \mathcal{O}\left(N^{\kappa}\right) - b\int_1^N (\pi t - (A(t) - \pi t))t^{b-1}\,\mathrm{d}t \\
&= \pi N^{b+1} + \mathcal{O}\left(N^{\kappa}\right) - b\pi \int_1^N t^b\,\mathrm{d}t + b\int_1^N (A(t) - \pi t)t^{b-1}\,\mathrm{d}t \\
&= \pi N^{b+1} - \frac{b\pi}{b+1}N^{b+1} + \frac{b\pi}{b+1} + \mathcal{O}\left(N^{b+\kappa}\right) \\
&= \frac{\pi}{b+1}N^{b+1} + \mathcal{O}\left(N^{b+\kappa}\right).
\end{aligned}$$

□

The following lemma gives a bound for the tail of $\zeta_i(s)$.

Lemma 2.8. *For $s > 1$, we have*

$$\sum_{\substack{q \in \mathbb{Z}[i]^+ \\ |q| \geq Q}} \frac{1}{|q|^{2s}} \ll_s \frac{1}{Q^{2(s-1)}}.$$

Proof. Rewriting the left hand side as a sum over annuli and using (1.19), we have

$$\begin{aligned}
\sum_{|q| \geq Q} \frac{1}{|q|^{2s}} &= \sum_{k=0}^{\infty} \sum_{2^k Q \leq |q| < 2^{k+1} Q} \frac{1}{|q|^{2s}} \\
&\leq \sum_{k=0}^{\infty} \frac{1}{2^{2ks} Q^{2s}} \left(\sum_{n \leq 2^{2|k+1|} Q^2} r_2(n) - \sum_{n \leq 2^{2k} Q^2} r_2(n) \right)
\end{aligned}$$

$$\ll \sum_{k=0}^{\infty} \frac{2^{2k}Q^2}{2^{2ks}Q^{2s}}$$

$$= \frac{1}{Q^{2(s-1)}} \sum_{k=0}^{\infty} \frac{1}{2^{2k(s-1)}}$$

$$\ll_s \frac{1}{Q^{2(s-1)}}.$$

□

We can now use these two lemmas along with Lemma 2.3 to prove Lemma 2.6 as follows.

Proof. We have

$$\begin{aligned}
\sum_{\substack{q\in\mathbb{Z}[i]^+ \\ |q|\le Q}} \phi_i(q) &= \sum_{\substack{q\in\mathbb{Z}[i]^+ \\ |q|\le Q}} |q|^2 \sum_{d|q} \frac{\mu_i(d)}{|d|} \\
&= \sum_{\substack{d\in\mathbb{Z}[i]^+ \\ |d|\le Q}} \mu_i(d) \sum_{\substack{q'\in\mathbb{Z}[i]^+ \\ |q'|\le \frac{Q}{|d|}}} |q'|^2 \\
&= \sum_{\substack{d\in\mathbb{Z}[i]^+ \\ |d|\le Q}} \mu_i(d) \sum_{k\le \frac{Q^2}{|d|^2}} \frac{r_2(k)}{4} k \\
&= \frac{1}{4} \sum_{\substack{d\in\mathbb{Z}[i]^+ \\ |d|\le Q}} \mu_i(d) \left(\frac{\pi Q^4}{2|d|^4} + \mathcal{O}\left(\frac{Q^{2+2\kappa}}{|d|^{2+2\kappa}} \right) \right) \quad \text{(by Lemma 2.7)} \\
&= \frac{\pi}{8} \zeta_i^{-1}(2)\, Q^4 + \mathcal{O}\left(Q^{2+2\kappa} \right).
\end{aligned}$$

The last equality uses Lemma 2.8 and the definition of $\zeta_i^{-1}(s)$ with $s = 2$ and $s = 1 + \kappa$. □

Chapter 3

Ford Spheres

This chapter comprises of four results concerning the structure of Ford spheres that will be essential in calculating the k-moments $\mathcal{M}_{k,I_2}(S)$ as described in Section 1.5.1.

First, in Section 3.1 we determine the best way to define consecutivity for the Gaussian fractions related to Ford spheres, in light of their lack of a natural ordering by size as we had in the two-dimensional case. With this in mind, in Section 3.2 we see how every Gaussian rational in I_2 can be generated starting from just $0, 1, i$ and $1+i$. In doing so we also introduce a variant of the mediant that is more appropriate now that we are working in higher dimensions.

In Section 3.3 we establish three criteria for when two Gaussian integers will appear as the denominators of consecutive fractions. Thereby we also determine how many distinct pairs of consecutive Gaussian fractions will have these denominators.

Finally, in Section 3.4 we consider, for a given Gaussian integers s, how to count the Gaussian integers s' which are denominators of a fraction that is consecutive to a fraction of the form $\frac{r}{s}$. To do this we will use the three previous results to restate our task in terms of counting lattice points in a region of the complex plane. From here we will apply our lemmas from Chapter 2 along with the Prime Number Theorem for Gaussian primes to achieve our goal.

3.1 Defining Consecutivity for Ford Spheres

When we studied $\mathcal{M}_k(Q)$ in Section 1.4 we were interested in the distances between the centres of consecutive Ford circles. In this thesis we are working with a higher dimensional analogue of $\mathcal{M}_k(Q)$ and so we are now concerned with the distances between the centres of consecutive Ford spheres instead. But what does it mean to say that two Ford spheres are consecutive?

Two Ford circles are consecutive if their corresponding fractions are consecutive in $\mathcal{F}_Q$, the elements of which are taken to be in increasing order of size. On the other hand, the fractions corresponding to Ford spheres are Gaussian fractions and so in place of $\mathcal{F}_Q$ we define $\mathcal{G}_S$ as follows.

Definition 3.1. *For a positive integer S,*

$$\mathcal{G}_S := \left\{ \tfrac{r}{s} \in I_2 : r, s \in \mathbb{Z}[i], (r, s) = 1, |s| \leq S \right\}$$

where I_2 is the unit square in the upper right quadrant of the complex plane.

However, unlike $\mathcal{F}_Q$, as $\mathcal{G}_S$ consists of elements of $\mathbb{Q}[i]$ we cannot simply order the elements by size and use this order to define consecutivity as we did before.

Alternatively, consecutivity for fractions in $\mathcal{F}_Q$ is equivalent to the fractions being adjacent and the sum of their denominators being greater than Q. If we try this approach for $\mathcal{G}_S$, the question becomes, how should we "add" the denominators and then compare the result to S? To compare their size to S it makes sense to take an absolute value, but should we do this before or after we add the denominators? It turns out that both of these options create problems later when we look at the wider picture of the Ford spheres, particularly when we look for a way to construct $\mathcal{G}_S$ using consecutive fractions in an analogous way to what we have for $\mathcal{F}_Q$.

So, instead of defining consecutivity for Farey fractions directly, we can reverse our thinking and consider what two fractions being consecutive means for their corresponding Ford circles. It is easy to see that two Farey fractions are consecutive in $\mathcal{F}_Q$ if and only if their Ford circles are tangent and there is no smaller circle between them for that order Q. Using this idea we can give an equivalent definition for consecutivity in $\mathcal{G}_S$ in terms of Ford spheres.

Definition 3.2. *Let $\frac{r}{s}$ and $\frac{r'}{s'}$ be fractions in $\mathcal{G}_S$ with spheres R and R' respectively. The fractions are consecutive in $\mathcal{G}_S$ if they are adjacent and there is at least one other fraction in $\mathbb{Q}[i]$ with sphere of radius less than $\frac{1}{2S^2}$ which is tangent to both R and R'.*

This definition is consistent with the analogues of other properties of Farey fractions and Ford circles in higher dimensions, which will be explored in the rest of this chapter.

Note that having a sphere of radius less than $\frac{1}{2S^2}$ means that the fraction will not be in $\mathcal{G}_S$ itself. The structure of the spheres means that for any two given tangent spheres there will be multiple smaller spheres which are tangent to both (unlike Ford circles where there is only one such circle). The definition above ensures that two spheres are considered consecutive until all of those smaller spheres are in $\mathcal{G}_S$; this will be necessary in the next section, when we study a method for generating $\mathcal{G}_S$ comparable to that used in the Stern-Brocot tree for $\mathcal{F}_Q$.

3.2 Generating $\mathcal{G}_S$

The Farey fractions can be generated from 0 and 1 by taking mediants, and we have seen that doing this will eventually generate every rational number in $[0, 1]$. In the complex case, in place of the interval $[0, 1]$ we are working in I_2, the unit square in the upper-right quadrant of the complex plane. Thus, rather than beginning with just 0 and 1 it makes sense for us to start with i and $1 + i$ as well, so that we have all four corners of I_2 to begin with.

If we take mediants of consecutive fractions from here in the same way as before, we do still find some fractions, namely those on the boundary of I_2. However, none of the fractions in the interior of I_2 will ever appear. Even if we extend our starting set to include $\frac{i}{1+i}$ for example, or indeed any finite set of Gaussian fractions within I_2, we still miss infinitely many of the rationals in I_2. The issue then, must come from taking mediants.

Ford remarks in [5] that given any adjacent Gaussian fractions $\frac{r}{s}$ and $\frac{r'}{s'}$, any fraction of the form

$$\frac{r'_n}{s'_n} = \frac{r' + nr}{s' + ns}$$

is also adjacent to $\frac{r}{s}$ for any Gaussian integer n. If we choose n to be 1 then we have our usual mediant and the resulting fraction is adjacent to $\frac{r'}{s'}$ as well as $\frac{r}{s}$. In fact, $\frac{r'_n}{s'_n}$ is adjacent to both $\frac{r}{s}$ and $\frac{r'}{s'}$ if we choose n to be any unit of $\mathbb{Z}[i]$. This is because two Gaussian rationals $\frac{a}{b}$ and $\frac{a'}{b'}$ are adjacent if $|a'b - ab'| = 1$, and for $\frac{r'}{s'}$ and $\frac{r'+ur}{s'+us}$ where $u \in \mathbb{Z}[i]^*$ we have,

$$\begin{aligned}
|(r' + ur)s' - r'(s' + us)| &= |r's' + urs' - r's' - ur's| \\
&= |u(rs' - r's)| \\
&= |rs' - r's| \\
&= 1,
\end{aligned}$$

using the fact that the original fractions $\frac{r}{s}$ and $\frac{r'}{s'}$ are adjacent also. Thus, given two adjacent fractions $\frac{r}{s}$ and $\frac{r'}{s'}$, the four fractions which are adjacent to both are given by

$$\frac{r + ur'}{s + us'} \quad \text{for} \quad u \in \{\pm 1, \pm i\}. \tag{3.1}$$

We will use this complex version of mediants for Gaussian fractions in place of the usual mediant, noting that unlike taking mediants of the usual Farey fractions, which always results in a denominator larger than that of either initial fraction, this type of mediant can result in a smaller denominator. Analogously, when taking mediants of real Farey fractions $\frac{p}{q}$ and $\frac{p'}{q'}$ we could instead consider

$$\frac{p + u'p'}{q + u'q'} \text{ for } u' \in \{\pm 1\}.$$

This would sometimes give us a denominator smaller than either q or q', specifically when we take $u' = -1$. Taking mediants in this way only gives us new Farey fractions when $u' = 1$, so we can discard $u' = -1$ and still generate every fraction. However, when taking complex mediants it is not clear which choices of unit u will lead to larger denominators and which will lead to smaller, so we need to consider all four units and ignore any repeated resulting fractions.

We now prove that beginning with 0, 1, i, and $1 + i$, taking mediants as in (3.1) will generate every Gaussian rational in the unit square of $\mathbb{C}$.

Lemma 3.1. *Given Gaussian integers $r = r_1 + ir_2$ and $s = s_1 + is_2$ such that $\frac{r}{s} \in I_2$ and $(r, s) = 1$, $\frac{r}{s}$ occurs as a complex mediant of two consecutive fractions in I_2 with denominators of modulus less than $|s|$.*

Proof. We argue by induction on $|s|$. Assume that all Gaussian rationals in I_2 with denominator of modulus less than $|s|$ have already been found. Now, since $(r, s) = 1$ we can find $x, y \in \mathbb{Z}[i]$ such that

$$rx - sy = 1$$

with $|x| < |s|$. Further, we can always choose $\frac{y}{x}$ to be in I_2. To see this, first note that each edge of I_2 looks like the Ford circles, with a sphere sat on every rational point, creating a 'wall' of spheres. This means that a sphere in $\mathring{I}_2$ cannot be tangent to any sphere outside of I_2. So if $\frac{r}{s} \in \mathring{I}_2$ then we must have $\frac{y}{x} \in I_2$. On the other hand, if $\frac{r}{s} \in I_2 \backslash \mathring{I}_2$, it is possible to choose $\frac{y'}{x'} \notin I_2$ which is adjacent to $\frac{r}{s}$, but in this case we could always instead choose $\frac{y}{x} \in I_2$, the mirror image of $\frac{y'}{x'}$ over the boundary of I_2. Thus, $\frac{y}{x} \in I_2$ and by the assumption, $\frac{y}{x}$ has already been found. Let

$$\frac{a}{b} = \frac{r - vy}{s - vx}$$

where v is a unit to be decided later. We claim that $\frac{a}{b}$ is adjacent to $\frac{r}{s}$ and $\frac{y}{x}$, and $|b| < |s|$. First,

$$\begin{aligned} ax - by &= (r - vy)x - (s - vx)y \\ &= rx - sy \\ &= 1 \end{aligned}$$

so $\frac{a}{b}$ is adjacent to $\frac{y}{x}$. Similarly,

$$\begin{aligned} as - br &= (r - vy)s - (s - vx)r \\ &= v(rx - sy) \\ &= v \in \mathbb{Z}[i]^* \end{aligned}$$

so $\frac{a}{b}$ is adjacent to $\frac{r}{s}$.

Now, to show that $|b| < |s|$, consider Figure 3.1. The circle UC contains all points which are within $|s|$ of s, this is where we want vx to be. The circle LC contains all the possible locations of x, since $|x| < |s|$. If we split LC into quarters we can force vx to lie in any one of those quarters by choosing the unit v accordingly. In particular, we can choose v so that vx lies in QC, and so in UC. Thus $|b| = |s - vx| < |s|$ as required.

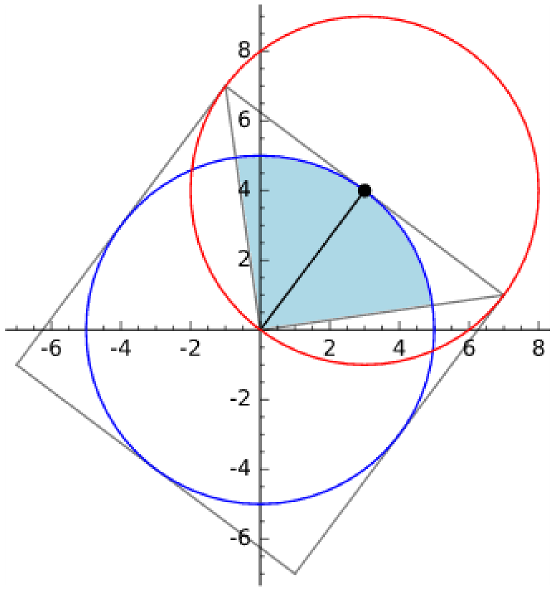

Figure 3.1: Diagram for the proof of Lemma 3.1. The point s is shown in black. UC is the red circle, LC is the blue circle and QC is the shaded quarter circle.

Now all that is left to check is that $\frac{a}{b} \in I_2$. We know that $\frac{r}{s}, \frac{y}{x} \in I_2$ so there are three possibilities:

1. $\frac{r}{s}, \frac{y}{x} \in \mathring{I}_2$,
2. $\frac{r}{s}, \frac{y}{x} \in I_2 \backslash \mathring{I}_2$, or
3. $\frac{r}{s} \in \mathring{I}_2$ and $\frac{y}{x} \in I_2 \backslash \mathring{I}_2$ or vice versa.

In cases 1 and 3 at least one of the two fractions is in $\mathring{I}_2$ and $\frac{a}{b}$ is adjacent to that fraction so, as their spheres are tangent, we must have $\frac{a}{b} \in I_2$. In case 2 however, as when choosing $\frac{y}{x}$, it is possible to choose $\frac{a'}{b'} \notin I_2$ with $|b'| < |s|$ which is adjacent to both $\frac{r}{s}$ and $\frac{y}{x}$. But in this case we could again always instead choose $\frac{a}{b} \in I_2$, the mirror image of $\frac{a'}{b'}$ over the boundary of I_2. Thus, $\frac{a}{b} \in I_2$. Note also that the fractions $\frac{a}{b}$ and $\frac{y}{x}$ are consecutive in $\mathcal{G}_S$ for S such that $S < |s|$ as their spheres are tangent (since they are adjacent) and $\frac{r}{s}$ is a fraction with a sphere which is tangent to both and has radius less than $\frac{1}{2S^2}$. $\square$

3.3 Classifying Consecutive Ford Spheres

In Section 3.1 we gave a geometric definition of consecutivity for Ford spheres. However, for our moment calculation we will need a set of criteria that tell us exactly when a given pair of denominators are consecutive in $\mathcal{G}_S$.

In $\mathcal{F}_Q$, q and q' are called consecutive if they are denominators of two fractions which are consecutive. For $\mathcal{F}_Q$ we have the following classification of consecutivity for denominators.

Lemma 3.2. *Denominators q and q' will be consecutive in $\mathcal{F}_Q$ if and only if all of the following are satisfied:*

1. $1 \leq q, q' \leq Q$,
2. $(q, q') = 1$, *and*
3. $q + q' > Q$.

Furthermore, for each pair of denominators q, q' satisfying these conditions there will be exactly two pairs of consecutive fractions with denominators q and q'. In one case $\frac{p_1}{q} < \frac{p_1'}{q'}$, and in the other $\frac{p_2}{q} > \frac{p_2'}{q'}$

In $\mathcal{G}_S$, s and s' are called consecutive if they are denominators of two fractions which are consecutive. Now that we know how to generate Gaussian fractions, we can also classify what it means for two denominators to be consecutive in $\mathcal{G}_S$.

In the case of the usual Farey fractions, the three requirements for $q, q' \in \mathbb{Z}$ to be consecutive can be thought of as

1. The fractions are in $\mathcal{F}_Q$. $(q, q' \leq Q)$
2. The fractions are adjacent. $((q, q') = 1)$
3. There is no fraction between the original two which is also in $\mathcal{F}_Q$. $(q + q' > Q)$

The classification for $\mathcal{G}_S$ should have conditions analogous to these statements, but take into account that we are now taking complex mediants. The next lemma details such a classification.

Lemma 3.3. *Two Gaussian integers s and s' appear as consecutive denominators in $\mathcal{G}_S$ if and only if all of the following are satisfied:*

1. $|s|, |s'| \leq S$*,*

2. $(s, s') = 1$*, and*

3. $|s' + u's| > S$ *for some unit* u'*.*

Furthermore, there are exactly four distinct pairs $r, r' \in \mathbb{Z}[i]$ *which give consecutive fractions* $\frac{r}{s}, \frac{r'}{s'}$*, when these three conditions are satisfied.*

The three conditions follow directly from the geometric definition of consecutive in $\mathcal{G}_S$, which we previously described. The final statement is proved as follows.

Proof. Suppose $s, s' \in \mathbb{Z}[i]$ satisfy the three conditions above. Then there are $r, r' \in \mathbb{Z}[i]$ for which $\frac{r}{s}$ and $\frac{r'}{s'}$ are consecutive in $\mathcal{G}_S$. So for r, r' we have

$$rs' - r's = u$$

where u is a unit in $\mathbb{Z}[i]$. There are four choices for the unit u, each of which corresponds to a pair r, r'. We claim that each of these four pairs is distinct. We have

$$r = us'^{-1} \mod s, \text{ and}$$
$$r' = \frac{rs' - u}{s},$$

so r' is determined by r. When the unit u is changed either r is changed, or r remains the same and so r' is changed. Either way a new pair is found for each choice of u, and so there are four distinct possibilities for the pair r, r'. □

3.4 Counting Consecutive Denominators

To estimate $\mathcal{M}_{k,I_2}(S)$ it will be necessary, given $s \in \mathbb{Z}[i]^+$ with $|s| \leq S$, to count how many different $s' \in \mathbb{Z}[i]^+$ have at least one fraction $\frac{r'}{s'} \in \mathcal{G}_S$ which is consecutive to a fraction with denominator s. In other words, given s, how many $s' \in \mathbb{Z}[i]^+$ satisfy the three conditions of Lemma 3.3?

Ignoring the coprimality condition for now, we need to know how many s' satisfy

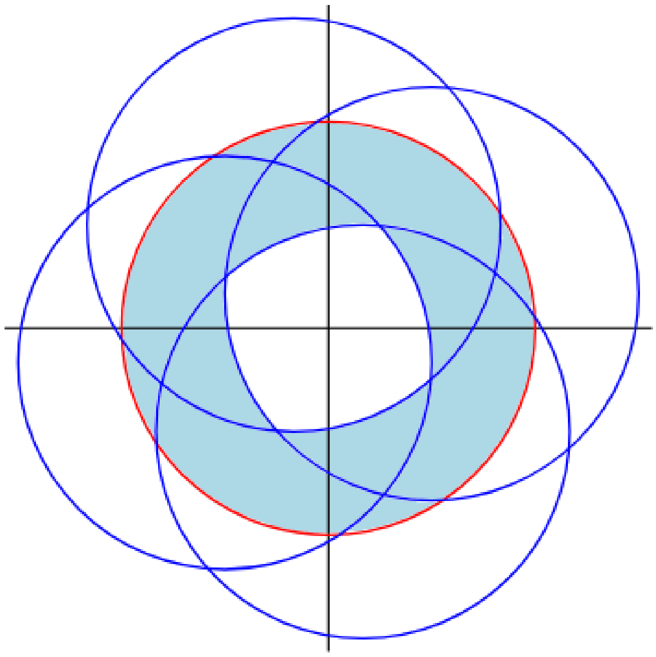

Figure 3.2: Circles of radius S with centres the origin, $\pm s$ and $\pm is$. To be consecutive to s, denominators s' must lie in the shaded region.

1. $|s'| \leq S$, and
2. $|s' + us| > S$ for some unit u,

for a given s. The s' satisfying condition 1 are those points on the $\mathbb{Z}[i]$ lattice that lie inside R, the circle of radius S centred on the origin. In condition 2 we consider mediants of s' with s, taking $s' + us$ for each unit $u \in \{\pm 1, \pm i\}$. For s' to satisfy condition 2, one of these four points must lie outside of R. We can look at this condition in another way by translating R. For example, consider s' for which $|s + s'| > S$, so $s + s'$ lies outside of R. Then if we translate R by $-s$, the point s' will lie outside of the translated circle. Similarly, if s' has $|s' + us| > S$, $s' + us$ lies outside of R and so s' lies outside the circle of radius S centred at $-us$.

Translating the circle R in each of the four directions s, $-s$, is and $-is$, we have the picture in Figure 3.2. Points s' satisfy the two conditions above if and only if they lie inside the red circle R and outside at least one of the blue circles, i.e. in the shaded area. Our aim then is to count the points on the $\mathbb{Z}[i]$ lattice in this region that are coprime to s.

We will denote the shaded region in this diagram by Ω_s and its boundary by

$\partial\Omega_s$. The following theorem concerns any convex region Ω in the complex plane, but still holds for our region Ω_s from Figure 3.2 in particular as it is the set difference of two convex regions.

Theorem 3.1. *For a convex region Ω in the complex plane with boundary $\partial\Omega$, we have*

$$\sum_{\substack{z\in\Omega \\ (z,s)=1}} 1 = \frac{\phi_i(s)}{|s|^2}|\Omega| + \mathcal{O}_\epsilon\left(|\partial\Omega||s|^\epsilon\right)$$

for all $\epsilon > 0$.

Proof. By Lemma 2.5, we have

$$\begin{aligned}
\sum_{\substack{z\in\Omega \\ (z,s)=1}} 1 &= \sum_{z\in\Omega}\sum_{d|z,s} \mu_i(d) \\
&= \sum_{d|s} \mu_i(d) \sum_{\substack{z\in\Omega \\ d|z}} 1 \\
&= \sum_{d|s} \mu_i(d) \left(\frac{|\Omega|}{|d|^2} + \mathcal{O}\left(\frac{|\partial\Omega|}{|d|}\right)\right) \\
&= \frac{\phi_i(s)}{|s|^2}|\Omega| + \mathcal{O}\left(|\partial\Omega| \sum_{d|s} \frac{|\mu_i(d)|}{|d|}\right).
\end{aligned}$$

Now, using Lemma 2.1, we have that

$$\sum_{d|s} \frac{|\mu_i(d)|}{|d|} = \prod_{p|s}\left(1 + \frac{1}{|p|}\right),$$

and we also observe that

$$\log\left(\prod_{p|s}\left(1 + \frac{1}{|p|}\right)\right) \ll \sum_{p|s}\frac{1}{|p|},$$

where p is a Gaussian prime. Therefore, the worst case scenario is when s is the product of the smallest possible distinct primes (replacing any such prime with a larger prime will reduce the value of the sum). So we have

$$\sum_{p|s}\frac{1}{|p|} \le \sum_{|p|^2\le X}\frac{1}{|p|}$$

for some X depending on s.

To estimate X, first note that

$$|s| = \prod_{|p|^2 \le X} |p|, \text{ and}$$
$$\log \prod_{|p|^2 \le X} |p| = \sum_{|p|^2 \le X} \log |p|.$$

Now, using Stieltjes integration and the Prime Number Theorem for Gaussian primes (Proposition 7.17 in [17]),

$$\begin{aligned}
\sum_{|p|^2 \le X} \log |p| &= \int_{1^-}^{X^+} (\log t^{\frac{1}{2}})\, \mathrm{d}\pi_i(t) \\
&= \frac{(\log X)\pi_i(X)}{2} - \int_1^X \frac{\pi_i(t)}{2t}\, \mathrm{d}t \\
&= \frac{X}{2} + o(X),
\end{aligned}$$

where $\pi_i(t)$ is the prime counting function for Gaussian integers, which counts Gaussian primes p with $|p|^2 \le t$. So

$$\prod_{|p|^2 \le X} |p| = e^{\frac{X}{2} + o(X)}.$$

If $X \gg \log |s|$, this says that $|s| = \prod_{|p|^2 \le X} |p| \gg |s|$, so we must have $X \le c \log |s|$, for some $c > 0$. So using Stieltjes integration and the Prime Number Theorem for Gaussian primes again, we have

$$\begin{aligned}
\sum_{p|s} \frac{1}{|p|} &\le \sum_{|p|^2 \le c \log |s|} \frac{1}{|p|} \\
&= \int_{1^-}^{(c \log |s|)^+} t^{-\frac{1}{2}}\, \mathrm{d}\pi_i(t) \\
&= \frac{\pi_i(c \log |s|)}{c^{\frac{1}{2}} (\log |s|)^{\frac{1}{2}}} + \frac{1}{2} \int_1^{c \log |s|} \frac{\pi_i(t)}{t^{\frac{3}{2}}}\, \mathrm{d}t \\
&\ll \frac{(\log |s|)^{\frac{1}{2}}}{\log(\log |s|)}.
\end{aligned}$$

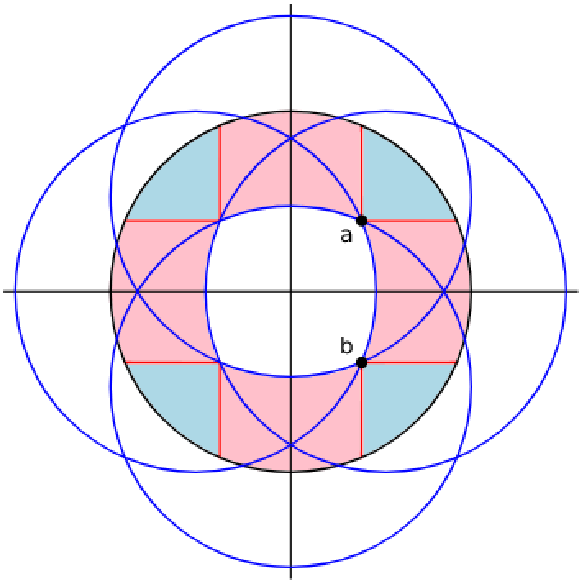

Figure 3.3: The rotated view of Figure 3.2. The shaded area is Ω_s.

Thus, we have

$$\sum_{d|s} \frac{|\mu_i(d)|}{|d|} = \exp\left(\mathcal{O}\left(\sum_{p|s} \frac{1}{|p|}\right)\right)$$

$$= \exp\left(\mathcal{O}\left(\frac{(\log|s|)^{\frac{1}{2}}}{\log(\log|s|)}\right)\right)$$

$$\ll_\epsilon |s|^\epsilon$$

for all $\epsilon > 0$. □

We now show that in this estimation the main term will always be asymptotically larger than the error term when Ω is Ω_s, the shaded region region in Figure 3.2. In this case the sum in Theorem 3.1 is counting points s' which satisfy all three conditions for being consecutive to s and so is equal to the inner sum in (4.1). The following argument uses only the area of the region and the length of its boundary, not its position. So to simplify the calculations we rotate our view of the diagram so that the circles' centres lie on the axes, as shown in Figure 3.3.

We call the top right blue corner region C and the right hand red region A. Clearly, $Area(\Omega_s) = 4(Area(A) + Area(C))$ and $|\partial\Omega_s| \ll S$. Let A_h be the height of A. This will be given by the difference in the imaginary parts of the points a and b. The symmetry of the diagram means that $Im(b) = -Im(a)$ and so their difference is $2Im(a)$.

Now, the circles with intersection point a have equations

$$\begin{aligned} y^2 + (x + |s|)^2 &= S^2 \text{ , and} \\ (y + |s|)^2 + x^2 &= S^2. \end{aligned}$$

The points of intersection of these two circles lie on the line $y = x$. Substituting this into one of the equations gives us

$$2y^2 + 2|s|y + |s|^2 - S^2 = 0.$$

The point a has positive imaginary part and so $Im(a)$ is the positive solution to this equation, i.e.,

$$Im(a) = \frac{1}{2}\left(-|s| + \sqrt{2S^2 - |s|^2}\right)$$

Thus, the height of A is

$$\begin{aligned} A_h &= 2Im(a) \\ &= \sqrt{2S^2 - |s|^2} - |s| \\ &= \sqrt{2}S\left(\sqrt{1 - \frac{|s|^2}{2S^2}} - \frac{|s|}{\sqrt{2}S}\right), \end{aligned}$$

and so

$$\begin{aligned} Area(A) &= |s|A_h \\ &= \sqrt{2}|s|S\left(\sqrt{1 - \frac{|s|^2}{2S^2}} - \frac{|s|}{\sqrt{2}S}\right). \end{aligned}$$

For the area of C note that the two red lines at the edges of C are two sides of a square of side length $|s|$ which completely contains C. Further, C will always make up more than half of this square and so,

$$Area(C) \geq \frac{1}{2}|s|^2.$$

Now, if $S \leq 2|s|$, note that $Area(C) \geq \frac{|s|^2}{2}$ so $Area(\Omega_s) \geq 2|s|^2$, and $|\partial\Omega_s||s|^\epsilon \ll |s|^{1+\epsilon}$.

On the other hand, if $S > 2|s|$, then $Area(\Omega_s) \geq 4Area(A) \geq 2(\sqrt{7} - 1)S|s|$, and $|\partial\Omega_s||s|^\epsilon \ll S|s|^\epsilon$. So for any choice of S, the error term is (on average over s) asymptotically smaller than the main term in Theorem 3.1.

Chapter 4

Ford Spheres – First Moment

In this chapter we aim to prove Theorem 1.11, restated below for convenience.

Theorem. 1.11 *For $S \in \mathbb{N}$ and any $\epsilon > 0$,*

$$\mathcal{M}_{1,I_2}(S) = \frac{\pi}{4}\zeta_i^{-1}(2)(2z_1'' - 1)S^2 + \mathcal{O}_\epsilon(S^{1+\epsilon}).$$

where $z_1'' = -\int_0^{\frac{1}{\sqrt{2}}} \ln(\sqrt{2}u)(1-u^2)^{\frac{1}{2}}du.$

To begin, in Section 4.1 we will use the classification of consecutivity for denominators of Gaussian fractions to rewrite $M_{1,I_2}(S)$ in terms of only the denominators s and s'. The resulting sum will involve counting lattice points in the region Ω_s from Section 3.4, to which we apply Theorem 3.1.

In order to continue we will require the area of Ω_s, which is calculated in Section 4.2. We are left with a main term and an error term for $M_{1,I_2}(S)$, which are estimated in Sections 4.3 and 4.4 respectively. The main term calculation in particular will require many of the lemmas from Chapter 2 and two applications of Abel's Summation Formula.

4.1 Rewriting $\mathcal{M}_{1,I_2}(S)$

In accordance with (1.22) with $k = 1$, the sum we are aiming to evaluate in this chapter is,

$$\mathcal{M}_{1,I_2}(S) = \sum_{\substack{\frac{r}{s}, \frac{r'}{s'} \in \mathcal{G}_S \\ \text{consec}}} \left(\frac{1}{2|s|^2} + \frac{1}{2|s'|^2} \right).$$

First of all, we use Lemma 3.3, our classification of consecutivity for denominators of Gaussian fractions, to rewrite this as

$$\mathcal{M}_{1,I_2}(S) = \frac{1}{2} \sum_{\substack{s \in \mathbb{Z}[i]^+ \\ |s| \leq S}} \sum_{\substack{s' \in \mathbb{Z}[i]^+ \\ s' \text{ consec to } s}} 4 \left(\frac{1}{2|s|^2} + \frac{1}{2|s'|^2} \right)$$

where the factor of 4 comes from the final statement of the lemma. Note that here and in the rest of the thesis we are choosing to write the Gaussian fractions so that their denominators are in $\mathbb{Z}[i]^+$. Doing this means fractions $\frac{ur}{us}$ for a unit u will only be counted one time, rather than once for each choice of the unit u. We have also multiplied the right hand side by $\frac{1}{2}$ to account for the fact that the double sum will count each distance twice, once for "s is consecutive to s' " and again when the roles are reversed and we have "s' is consecutive to s". Now, as s and s' run through the same numbers, this also means that for every $a \in \mathbb{Z}[i]^+$, for every time $s = a$ produces a $\frac{1}{2|a|^2}$ term, $s' = a$ will also at some point produce another $\frac{1}{2|a|^2}$ term. Thus we have,

$$\mathcal{M}_{1,I_2}(S) = 2 \sum_{\substack{s \in \mathbb{Z}[i]^+ \\ |s| \leq S}} \sum_{\substack{s' \in \mathbb{Z}[i]^+ \\ s' \text{ consec to } s}} \frac{1}{|s|^2}.$$

Now, we can use Theorem 3.1, which tells us how to count consecutive denominators, to rewrite $\mathcal{M}_{1,I_2}(S)$ further as follows.

$$\begin{aligned} \mathcal{M}_{1,I_2}(S) &= 2 \sum_{\substack{s \in \mathbb{Z}[i]^+ \\ |s| \leq S}} \frac{1}{|s|^2} \sum_{\substack{s' \in \mathbb{Z}[i]^+ \\ s' \text{ consec to } s}} 1 \qquad (4.1) \\ &= 2 \sum_{\substack{s \in \mathbb{Z}[i]^+ \\ |s| \leq S}} \frac{1}{|s|^2} \sum_{\substack{z \in \Omega_s \\ (z,s)=1}} \frac{1}{4} \end{aligned}$$

$$= \frac{1}{2} \sum_{\substack{s\in\mathbb{Z}[i]^+ \\ |s|\le S}} \frac{1}{|s|^2} \left(\frac{\phi_i(s)}{|s|^2} |\Omega_s| + \mathcal{O}_\epsilon \left(|\partial\Omega_s| |s|^\epsilon \right) \right)$$

$$= \frac{1}{2} \sum_{\substack{s\in\mathbb{Z}[i]^+ \\ |s|\le S}} \frac{\phi_i(s)}{|s|^4} |\Omega_s| + \mathcal{O}_\epsilon \left(\sum_{\substack{s\in\mathbb{Z}[i]^+ \\ |s|\le S}} \frac{1}{|s|^{2-\epsilon}} |\partial\Omega_s| \right)$$

$$=: \frac{1}{2} A + \mathcal{O}_\epsilon(B), \tag{4.2}$$

for all $\epsilon > 0$, where Ω_s is the shaded region in figure 3.2 with boundary $\partial\Omega_s$ as before. We now aim to estimate A and B.

4.2 The Area of Ω_s

The first step we need to take in estimating the main term of $\mathcal{M}_{1,I_2}(S)$ is to determine the area of Ω_s. We will calculate this as follows.

Proposition 4.1. *The area of the region Ω_s is given by*

$$|\Omega_s| = -2|s|^2 + I_1(|s|)$$

where $I_1(|s|) = 8S^2 \int_0^{\sin^{-1}\left(\frac{|s|}{\sqrt{2}S}\right)} \cos^2 u \, \mathrm{d}u$.

Proof. To find the area of Ω_s, consider the circles with equations $x^2 + y^2 = S^2$ and $(x + |s|)^2 + y^2 = S^2$, and call them C_1 and C_2 respectively. Then the region between these two circles, the line $y = x$ and the x-axis (as shown in Figure 4.1) will be equal to $\frac{1}{8}|\Omega_s|$. Now, working in polar coordinates, C_1 and C_2 have equations $r = S$ and $r = -|s| \cos\theta + (S^2 - |s|^2 \sin^2\theta)^{\frac{1}{2}} =: r_\theta$ respectively, so

$$|\Omega_s| = 8 \int_0^{\frac{\pi}{4}} \int_{r_\theta}^{S} r \, \mathrm{d}r \, \mathrm{d}\theta.$$

We have

$$\int_{r_\theta}^{S} r \, \mathrm{d}r = \left[\frac{1}{2} r^2 \right]_{r_\theta}^{S}$$
$$= \frac{1}{2} S^2 - \frac{1}{2} \left(-|s| \cos\theta + \left(S^2 - |s|^2 \sin^2\theta \right)^{\frac{1}{2}} \right)^2$$

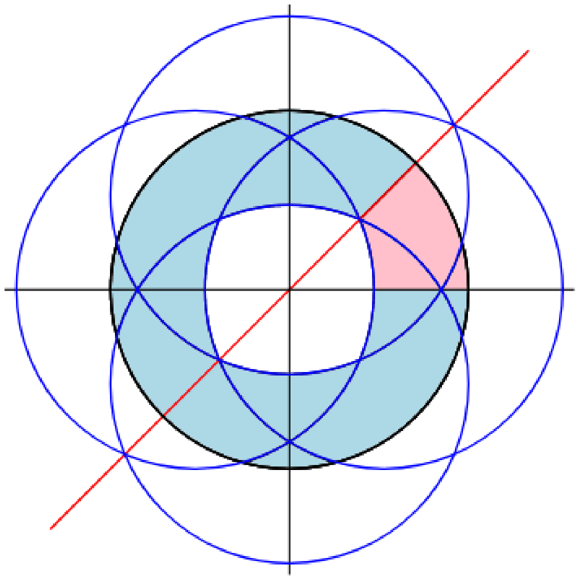

Figure 4.1: The area Ω_s. The pink area is one eighth of the whole shaded region.

$$= -\frac{1}{2}|s|^2\left(\cos^2\theta - \sin^2\theta\right) + |s|\cos\theta\left(S^2 - |s|^2\sin^2\theta\right)^{\frac{1}{2}}$$
$$= |s|\cos\theta\left(S^2 - |s|^2\sin^2\theta\right)^{\frac{1}{2}} - \frac{1}{2}|s|^2\cos 2\theta.$$

So, substituting this back into the integral for $|\Omega_s|$, we have

$$|\Omega_s| = 4\int_0^{\frac{\pi}{4}} 2\left(|s|\cos\theta\left(S^2 - |s|^2\sin^2\theta\right)^{\frac{1}{2}} - \frac{1}{2}|s|^2\cos 2\theta\,\mathrm{d}\theta\right)$$
$$= -4\int_0^{\frac{\pi}{4}} |s|^2\cos 2\theta\,\mathrm{d}\theta + I_1(|s|)$$
$$= -4\left[\frac{1}{2}|s|^2\sin 2\theta\right]_0^{\frac{\pi}{4}} + I_1(|s|)$$
$$= -2|s|^2 + I_1(|s|),$$

where $I_1(|s|) = 8\int_0^{\frac{\pi}{4}}|s|\cos\theta\left(S^2 - |s|^2\sin^2\theta\right)^{\frac{1}{2}}\,\mathrm{d}\theta$. Now, into $I_1(|s|)$ we substitute $\sin u = \frac{|s|}{S}\sin\theta$, giving us,

$$I_1(|s|) = 8S|s|\int_0^{\sin^{-1}\left(\frac{|s|}{\sqrt{2}S}\right)} \frac{S}{|s|}\cos u\left(\cos^2 u\right)^{\frac{1}{2}}\,\mathrm{d}u$$
$$= 8S^2\int_0^{\sin^{-1}\left(\frac{|s|}{\sqrt{2}S}\right)} \cos^2 u\,\mathrm{d}u. \qquad \square$$

4.3 First Moment - Main Term Calculation

In this section we prove the proposition below, which details an asymptotic formula for the main term of the first moment. We start by applying Proposition 4.1 and then complete the proof using Abel's Summation Formula (Theorem 1.7) and the lemmas from Chapter 2.

Proposition 4.2. *For A as defined in* (4.2)*,*

$$A = \frac{\pi}{2}\zeta_i^{-1}(2)\left(8z_1'' - 1\right)S^2 + \mathcal{O}\left(S\ln S\right),$$

where $z_1'' = -\int_0^{\frac{1}{\sqrt{2}}} \ln(\sqrt{2}u)(1-u^2)^{\frac{1}{2}}du.$

Proof. We begin by substituting our value for the area of Ω_s from Proposition 4.1 into A, which gives us

$$A = \sum_{\substack{s\in\mathbb{Z}[i]^+\\|s|\le S}} \frac{\phi_i(s)}{|s|^4} I_1(|s|) - 2\sum_{\substack{s\in\mathbb{Z}[i]^+\\|s|\le S}} \frac{\phi_i(s)}{|s|^2}.$$

We focus first on the second sum, applying Abel's Summation Formula with $x = S^2$, $f(t) = \frac{1}{t}$ and

$$a(n) = \sum_{\substack{s\in\mathbb{Z}[i]^+\\|s|=n^{\frac{1}{2}}}} \phi_i(s),$$

so that, by Lemma 2.6,

$$A(t) = \sum_{\substack{s\in\mathbb{Z}[i]^+\\|s|\le t^{\frac{1}{2}}}} \phi_i(s) = \frac{\pi}{8}\zeta_i^{-1}(2)t^2 + \mathcal{O}\left(t^{1+\kappa}\right).$$

Thus, we have

$$\sum_{\substack{s\in\mathbb{Z}[i]^+\\|s|\le S}} \frac{\phi_i(s)}{|s|^2} = \sum_{n\le S^2} \frac{a(n)}{n}$$

$$= A(S^2)f(S^2) - \int_1^{S^2} A(t)t^{-2}\,\mathrm{d}t$$

$$= \left(\frac{z_1}{4}S^4 + \mathcal{O}\left(S^{2+2\kappa}\right)\right)S^{-2} + \int_1^{S^2}\left(\frac{z_1}{4}t^2 + \left(A(t) - \frac{z_1}{4}t^2\right)\right)t^{-2}\,\mathrm{d}t$$

$$= \frac{z_1}{2}S^2 + \mathcal{O}\left(S^{2\kappa}\right) + \int_1^{S^4} \left(A(t) - \frac{z_1}{4}t^2\right) t^{-2}\,\mathrm{d}t$$
$$= \frac{z_1}{2}S^2 + \mathcal{O}\left(S^{2\kappa}\right), \tag{4.3}$$

where $z_1 = \frac{\pi}{2}\zeta_i^{-1}(2)$ and κ is the exponent from the Gauss Circle Problem, which satisfies $\frac{1}{4} < \kappa < \frac{1}{2}$.

Now, before moving on to the first term of A, it will be helpful to consider the sum $\sum_{\substack{s\in\mathbb{Z}[i]\\|s|\le S}} \frac{\phi_i(s)}{|s|^4}$. We will apply Abel's Summation Formula with $x = S^4$, $f(t) = \frac{1}{t}$ and

$$\hat{a}(n) = \sum_{\substack{s\in\mathbb{Z}[i]\\|s|=n^{\frac{1}{4}}}} \phi_i(s),$$

so that

$$\hat{A}(t) = \sum_{\substack{s\in\mathbb{Z}[i]\\|s|\le t^{\frac{1}{4}}}} \phi_i(s).$$

This combined with Lemma 2.6 gives us,

$$\sum_{\substack{s\in\mathbb{Z}[i]\\|s|\le S}} \frac{\phi_i(s)}{|s|^4} = \sum_{n\le S^4} \frac{\hat{a}(n)}{n}$$
$$= \sum_{|s|\le S} \phi_i(s)S^{-4} + \int_1^{S^4} \hat{A}(t)t^{-2}\,\mathrm{d}t$$
$$= \left(z_1S^4 + \mathcal{O}\left(S^{2+2\kappa}\right)\right)S^{-4} + \int_1^{S^4}\left(z_1t + \left(\hat{A}(t) - z_1t\right)\right)t^{-2}\,\mathrm{d}t$$
$$= z_1 + \mathcal{O}\left(S^{2\kappa-2}\right) + z_1\int_1^{S^4} t^{-1}\,\mathrm{d}t + \int_1^{S^4}\left(\hat{A}(t) - z_1t\right)t^{-2}\,\mathrm{d}t$$
$$= 4z_1\ln S + z_1 + \int_1^{\infty}\left(\hat{A}(t) - z_1t\right)t^{-2}\,\mathrm{d}t - \int_{S^4}^{\infty}\left(\hat{A}(t) - z_1t\right)t^{-2}\,\mathrm{d}t + \mathcal{O}\left(S^{2\kappa-2}\right).$$

Define

$$z_1' := \int_1^{\infty}\left(\hat{A}(t) - z_1t\right)t^{-2}\,\mathrm{d}t, \tag{4.4}$$

and note that it is absolutely convergent and so is well-defined. Note also that we have

$$\int_{S^4}^{\infty}\left(\hat{A}(t) - z_1t\right)t^{-2}\,\mathrm{d}t \ll \int_{S^4}^{\infty}\frac{1}{t^{\frac{3}{2}-\frac{\kappa}{2}}}\,\mathrm{d}t$$

$$\ll \left[t^{\frac{\kappa}{2}-\frac{1}{2}}\right]_{S^4}^{\infty}$$
$$\ll S^{2\kappa-2}.$$

Thus,

$$\sum_{\substack{s\in\mathbb{Z}[i]\\|s|\le S}} \frac{\phi_i(s)}{|s|^4} = 4z_1 \ln S + (z_1 + z_1') + \mathcal{O}\left(S^{2\kappa-2}\right). \tag{4.5}$$

Now, returning to A, we need to estimate $\sum_{\substack{s\in\mathbb{Z}[i]\\|s|\le S}} \frac{\phi_i(s)}{|s|^4} I_1$. We apply Abel's Summation Formula with $x = S^4$, $f(t) = I_1(t^{1/4})$, and $a(n) = \sum_{\substack{s\in\mathbb{Z}[i]\\|s|=n^{\frac{1}{4}}}} \phi_i(s)n^{-1}$. Then by (4.5),

$$A(t) = \sum_{\substack{s\in\mathbb{Z}[i]\\|s|\le t^{\frac{1}{4}}}} \frac{\phi_i(s)}{|s|^4} = z_1 \ln t + (z_1 + z_1') + \mathcal{O}\left(t^{\frac{\kappa}{2}-\frac{1}{2}}\right).$$

In addition, by the Fundamental Theorem of Calculus,

$$\begin{aligned}
f'(t) &= 8S^2 \cos^2\left(\sin^{-1}\left(\frac{t^{\frac{1}{4}}}{\sqrt{2}S}\right)\right) \frac{d}{dt}\left(\sin^{-1}\left(\frac{t^{\frac{1}{4}}}{\sqrt{2}S}\right)\right) \\
&= 8S^2 \left(1 - \frac{t^{\frac{1}{2}}}{2S^2}\right) \frac{t^{-\frac{3}{4}}}{4\sqrt{2}S} \left(1 - \frac{t^{\frac{1}{2}}}{2S^2}\right)^{-\frac{1}{2}} \\
&= \sqrt{2}St^{-\frac{3}{4}} \left(1 - \frac{t^{\frac{1}{2}}}{2S^2}\right)^{\frac{1}{2}}.
\end{aligned}$$

Thus, we have,

$$\begin{aligned}
\int_1^{S^4} A(t)f'(t)\,\mathrm{d}t &= \int_1^{S^4} \sqrt{2}St^{-\frac{3}{4}} \left(z_1 \ln t + (z_1 + z_1') + \mathcal{O}\left(t^{\frac{\kappa}{2}-\frac{1}{2}}\right)\right) \left(1 - \frac{t^{\frac{1}{2}}}{2S^2}\right)^{\frac{1}{2}} \mathrm{d}t \\
&= \sqrt{2}z_1 S \int_1^{S^4} t^{-\frac{3}{4}} \ln t \left(1 - \frac{t^{\frac{1}{2}}}{2S^2}\right)^{\frac{1}{2}} \mathrm{d}t \\
&\quad + \sqrt{2}\,(z_1 + z_1')\, S \int_1^{S^4} t^{-\frac{3}{4}} \left(1 - \frac{t^{\frac{1}{2}}}{2S^2}\right)^{\frac{1}{2}} \mathrm{d}t \\
&\quad + \mathcal{O}\left(S \int_1^{S^4} t^{\frac{\kappa}{2}-\frac{5}{4}} \left(1 - \frac{t^{\frac{1}{2}}}{2S^2}\right)^{\frac{1}{2}} \mathrm{d}t\right)
\end{aligned}$$

$$= X_1 + X_2 + \mathcal{O}\left(S\right). \tag{4.6}$$

Now, substituting $\sin\theta = \frac{t^{\frac{1}{4}}}{\sqrt{2}S}$,

$$\int_1^{S^4} t^{-\frac{3}{4}} \left(1 - \frac{t^{\frac{1}{2}}}{2S^2}\right)^{\frac{1}{2}} \mathrm{d}t = 4\sqrt{2}S \int_{\sin^{-1}\left(\frac{1}{\sqrt{2}S}\right)}^{\frac{\pi}{4}} \cos^2\theta \,\mathrm{d}\theta$$
$$= \sqrt{2}S\left(\frac{\pi}{2} + 1\right) + \mathcal{O}\left(1\right)$$

and so

$$X_2 = (z_1 + z_1')(\pi + 2)S^2 + \mathcal{O}\left(S\right). \tag{4.7}$$

Finally, letting $u = \frac{t^{\frac{1}{4}}}{\sqrt{2}S}$,

$$\int_1^{S^4} t^{-\frac{3}{4}} \ln t \left(1 - \frac{t^{\frac{1}{2}}}{2S^2}\right)^{\frac{1}{2}} \mathrm{d}t = 16\sqrt{2}S \int_{\frac{1}{\sqrt{2}S}}^{\frac{1}{\sqrt{2}}} \ln(\sqrt{2}uS)\left(1 - u^2\right)^{\frac{1}{2}} \mathrm{d}u$$
$$= 16\sqrt{2}S\ln S \int_{\frac{1}{\sqrt{2}S}}^{\frac{1}{\sqrt{2}}} \left(1 - u^2\right)^{\frac{1}{2}} \mathrm{d}u$$
$$+ 16\sqrt{2}S \int_{\frac{1}{\sqrt{2}S}}^{\frac{1}{\sqrt{2}}} \ln(\sqrt{2}u)\left(1 - u^2\right)^{\frac{1}{2}} \mathrm{d}u$$
$$= 16\sqrt{2}S\ln S \left(\int_0^{\frac{1}{\sqrt{2}}} \left(1 - u^2\right)^{\frac{1}{2}} \mathrm{d}u - \int_0^{\frac{1}{\sqrt{2}S}} \left(1 - u^2\right)^{\frac{1}{2}} \mathrm{d}u\right)$$
$$+ 16\sqrt{2}S \int_{\frac{1}{\sqrt{2}S}}^{\frac{1}{\sqrt{2}}} \ln(\sqrt{2}u)\left(1 - u^2\right)^{\frac{1}{2}} \mathrm{d}u$$
$$= \sqrt{2}\left(2\pi + 4\right) S\ln S + \mathcal{O}\left(\ln S\right)$$
$$+ 16\sqrt{2}S \int_0^{\frac{1}{\sqrt{2}}} \ln(\sqrt{2}u)\left(1 - u^2\right)^{\frac{1}{2}} \mathrm{d}u$$
$$- 16\sqrt{2}S \int_0^{\frac{1}{\sqrt{2}S}} \ln(\sqrt{2}u)\left(1 - u^2\right)^{\frac{1}{2}} \mathrm{d}u$$
$$= \sqrt{2}\left(2\pi + 4\right) S\ln S - 16\sqrt{2}z_1''S + \mathcal{O}\left(\ln S\right),$$

where $z_1'' = -\int_0^{\frac{1}{\sqrt{2}}} \ln(\sqrt{2}u)\left(1 - u^2\right)^{\frac{1}{2}} \mathrm{d}u > 0$. Thus,

$$X_1 = (4\pi + 8)z_1S^2\ln S - 32z_1z_1''S^2 + \mathcal{O}\left(S\ln S\right). \tag{4.8}$$

Now (4.6), (4.7) and (4.8) give us

$$\begin{aligned}\sum_{\substack{s\in\mathbb{Z}[i]^+\\|s|\le S}}\frac{\phi_i(s)}{|s|^4}I_1(|s|) &= \frac{1}{4}\sum_{\substack{s\in\mathbb{Z}[i]\\|s|\le S}}\frac{\phi_i(s)}{|s|^4}I_1\\&=\frac{1}{4}\left(A(S^4)f(S^4)-\int_1^{S^4}A(t)f'(t)\,\mathrm{d}t\right)\\&=\frac{1}{4}\left[\left(4z_1\ln S+(z_1+z_1')+\mathcal{O}\left(S^{2\kappa-2}\right)\right)(\pi+2)S^2-X_1-X_2\right]+\mathcal{O}\left(S\right)\\&=8z_1z_1''S^2+\mathcal{O}\left(S\ln S\right).\end{aligned}$$

This, together with (4.3), gives our estimate for A,

$$A=\frac{\pi}{2}\zeta_i^{-1}(2)\left(8z_1''-1\right)S^2+\mathcal{O}\left(S\ln S\right).$$

□

4.4 First Moment - Error Term Calculation

The final component we require is an estimate for the sum B associated with (4.2). This will be achieved by splitting the sum over dyadic annuli.

Proposition 4.3. *For B as defined in* (4.2),

$$B\ll S^{1+\epsilon}$$

for all $\epsilon>0$.

Proof. Clearly $|\partial\Omega_s|\ll S$ so, substituting this into B and splitting the resulting sum over dyadic annuli,

$$\begin{aligned}B&=\sum_{|s|\le S}\frac{|\partial\Omega|}{|s|^{2-\epsilon}}\\&\ll S\sum_{|s|\le S}\frac{1}{|s|^{2-\epsilon}}\\&\ll S\sum_{k\le\log_2 S}\sum_{2^{k-1}\le|s|<2^k}\frac{1}{|s|^{2-\epsilon}}\end{aligned}$$

$$
\begin{aligned}
&\ll S \sum_{k \le \log_2 S} \frac{1}{2^{k(2-\epsilon)}} \sum_{2^{k-1} \le |s| < 2^k} 1 \\
&\ll S \sum_{k \le \log_2 S} \left(2^k\right)^{\epsilon} \\
&\ll S^{1+\epsilon},
\end{aligned}
$$

using the fact that

$$\sum_{2^{k-1} \le |s| < 2^k} 1 \asymp 2^{2k}.$$

□

Thus, putting together our estimates for A and B, we have

$$M_{1,I_2}(S) = \frac{\pi}{4} \zeta_i^{-1}(2)\,(8z_1'' - 1)\, S^2 + \mathcal{O}_\epsilon(S^{1+\epsilon}).$$

Note that $z_1'' \approx 0.68644 > \frac{1}{2}$, so $8z_1'' - 1$ is positive.

Chapter 5

Ford Spheres – Higher Moments

In this chapter we prove Theorem 1.12, restated here for convenience.

Theorem. 1.12 *For each integer $k \geq 2$, there exists a constant $\xi_k > 0$ with the property that, for any $\epsilon > 0$ and for any $S \in \mathbb{N}$,*

$$\mathcal{M}_{k,I_2}(S) = \xi_k S + \mathcal{O}_\epsilon(S^{2\kappa+\epsilon}),$$

with $\frac{1}{4} < \kappa \leq \frac{131}{416}$.

The proof will be split into cases for $k = 2$ and $k > 2$. The methods for each will be mostly the same but there is some difference in the details, making it worthwhile to consider them separately.

For both the second and higher moments, the proof will follow the same lines as that of the first moment. However, as we are now dealing with powers of the distances between the centres of the spheres, we will also require a number of further results to deal with this. These are detailed in Section 5.1. Furthermore, if we attempt the same method of proof as we used for $k = 1$, the error terms in the corresponding calculations become of the same order of magnitude as what would otherwise be considered the main terms. However, by more refined lattice point counting techniques we will be able to prove Theorem 1.12.

In Section 5.2 we prove Theorem 1.12 for the second moment, applying again our results from Chapters 2 and 3 alongside the new lemmas of the preceding section and repeated applications of Abel's Summation Formula. These results will then be used again in Section 5.3 to prove Theorem 1.12 for all higher moments.

5.1 Preliminary Results

Before proving Theorem 1.12 it will be constructive to lay out a few results involving sums over a circle in the complex plane, which will then be applied in the subsequent sections.

Lemma 5.1. *For $S \geq 1$ we have that*

$$\sum_{\substack{s\in\mathbb{Z}[i]\\0<|s|\leq S}} \frac{1}{|s|^2} = 2\pi \log S + \mathcal{O}(1).$$

Proof. Using partial summation and (1.19), we have that

$$\begin{aligned}\sum_{0<|s|\leq S} \frac{1}{|s|^2} &= \sum_{\ell\leq S^2} \frac{r_2(\ell)}{\ell} \\ &= \sum_{\ell\leq S^2} \frac{1}{\ell(\ell+1)} \sum_{j=1}^{\ell} r_2(j) + \frac{1}{S^2+1} \sum_{j\leq S^2} r_2(j) \\ &= \sum_{\ell\leq S^2} \frac{\pi\ell + \mathcal{O}(\ell^\kappa)}{\ell(\ell+1)} + \frac{\pi S^2 + \mathcal{O}(S^{2\kappa})}{S^2+1} \\ &= \pi \sum_{\ell\leq S^2} \frac{1}{\ell+1} + \mathcal{O}(1) \\ &= 2\pi \log S + \mathcal{O}(1).\end{aligned}$$

□

Lemma 5.2. *For $S \geq 1$,*

$$\sum_{\substack{s\in\mathbb{Z}[i]^+\\|s|\leq S}} \frac{\phi_i(s)}{|s|^4} = z_1 \log S + \frac{z_1 + z_1'}{4} + \mathcal{O}(S^{2\kappa-2}),$$

where $z_1 = \frac{\pi}{2}\zeta_i^{-1}(2)$ and

$$z_1' = \int_1^\infty \left(\sum_{\substack{s\in\mathbb{Z}[i]^+\\|s|\leq t^{1/4}}} \phi_i(s) - z_1 t \right) t^{-2} \, \mathrm{d}t$$

from (4.4).

The proof of this lemma is included in the proof of Proposition 4.2.

Lemma 5.3. *For each $k > 2$ there is a constant $z_k > 0$ such that, for $S \geq 1$,*

$$\sum_{\substack{s\in\mathbb{Z}[i]^+ \\ |s|\leq S}} \frac{\phi_i(s)}{|s|^{2k}} = z_k + \mathcal{O}_k\left(\frac{1}{S^{2(k-2)}}\right).$$

Proof. Since $0 < \phi_i(s) \leq |s|^2$, it follows from Lemma 2.8 that

$$z_k = \sum_{s\in\mathbb{Z}[i]^+} \frac{\phi_i(s)}{|s|^{2k}}$$

is positive and finite. By the same result, we also have that

$$\begin{aligned}\sum_{\substack{s\in\mathbb{Z}[i]^+ \\ |s|\leq S}} \frac{\phi_i(s)}{|s|^{2k}} &= z_k - \sum_{\substack{s\in\mathbb{Z}[i]^+ \\ |s|>S}} \frac{\phi_i(s)}{|s|^{2k}} \\ &= z_k + \mathcal{O}_k\left(\frac{1}{S^{2(k-2)}}\right).\end{aligned}$$

□

5.2 Proof of Theorem 1.12: $k = 2$ case

We will begin with the $k = 2$ case of Theorem 1.12. First of all, recall that the k^{th} moment is defined for $k, S \in \mathbb{N}$ by

$$\mathcal{M}_{k,I_2}(S) = \sum_{\substack{\frac{r}{s},\frac{r'}{s'}\in\mathcal{G}_S \\ \text{consecutive}}} \left(\frac{1}{2|s|^2} + \frac{1}{2|s'|^2}\right)^k$$

We start by expanding the square in the summand defining $\mathcal{M}_{2,I_2}(S)$

$$\begin{aligned}\mathcal{M}_{2,I_2}(S) &= \sum_{\frac{r}{s}\in\mathcal{G}_S} \frac{\#\{r'/s' \in \mathcal{G}_S \text{ consecutive to } r/s\}}{4|s|^4} + \sum_{\substack{\frac{r}{s},\frac{r'}{s'}\in\mathcal{G}_S \\ \text{consecutive}}} \frac{1}{2|s|^2|s'|^2} \\ &= \Sigma_1 + \Sigma_2. \qquad (5.1)\end{aligned}$$

It is important to keep in mind that the sum defining $\mathcal{M}_{2,I_2}(S)$ is over unordered consecutive pairs r/s and r'/s' in $\mathcal{G}_S$, which is why there is 4 in the denominator of Σ_1 and not a 2.

The sum Σ_2 will end up being asymptotically smaller than our main error term, so we estimate it first. From the characterization of consecutivity provided in Section 3.3 we know that if a pair s, s' occurs as a pair of consecutive denominators in $\mathcal{G}_S$ then there are four possible choices for the corresponding pairs of numerators. By multiplying by appropriate units we may also assume without loss of generality that s and s' lie in $\mathbb{Z}[i]^+$, so we have that

$$\Sigma_2 = \sum_{\substack{s\in\mathbb{Z}[i]^+ \\ |s|\leq S}} \sum_{\substack{s'\in\mathbb{Z}[i]^+ \\ s'\text{cons to } s}} \frac{1}{|s|^2|s'|^2}.$$

Although the constant here is not important, for completeness we mention that we have also divided by an extra factor of 2 to account for the fact that the double sum on the right hand side counts each pair s, s' twice. Using Lemma 5.1, we now have that

$$\begin{aligned} \Sigma_2 &= \sum_{\substack{s\in\mathbb{Z}[i]^+ \\ |s|\leq S}} \sum_{\substack{s'\in\mathbb{Z}[i]^+ \\ s' \text{ cons to } s}} \frac{1}{|s|^2|s'|^2} \\ &\ll \sum_{\substack{s,s'\in\mathbb{Z}[i]^+ \\ |s|,|s'|\leq S}} \frac{1}{|s|^2|s'|^2} \\ &\leq \left(\sum_{|s|\leq S} \frac{1}{|s|^2}\right)^2 \\ &\ll \log^2 S. \end{aligned} \tag{5.2}$$

Now, for Σ_1, we have that

$$\begin{aligned} \Sigma_1 &= \sum_{\substack{s\in\mathbb{Z}[i]^+ \\ |s|\leq S}} \sum_{\substack{s'\in\mathbb{Z}[i]^+ \\ s'\text{cons to} s}} \frac{1}{|s|^4} \\ &= \frac{1}{4} \sum_{\substack{s\in\mathbb{Z}[i]^+ \\ |s|\leq S}} \frac{1}{|s|^4} \sum_{\substack{s'\in\Omega_s\cap\mathbb{Z}[i] \\ (s,s')=1}} 1, \end{aligned} \tag{5.3}$$

where $\Omega_s = \Omega_s(S)$ is the region defined in Section 3.4. Using Möbius inversion on the inner sum gives

$$\begin{aligned}\sum_{\substack{s'\in\Omega_s\cap\mathbb{Z}[i]\\(s,s')=1}} 1 &= \sum_{s'\in\Omega_s\cap\mathbb{Z}[i]}\sum_{d|s,s'}\mu_i(d)\\ &= \sum_{d|s}\mu_i(d)\sum_{z\in d^{-1}\Omega_s} 1.\end{aligned}$$

For each d, the number of lattice points in $d^{-1}\Omega_s$ is equal to the number of lattice points in the circle of radius $S/|d|$ centered at the origin, minus the number of lattice points in the intersection of the four translates of this circle by $\pm s/|d|$ and $\pm is/|d|$. The number of lattice points in each of these two convex regions can be calculated, via the Gauss circle method, using the machinery in [13], and (see the comments immediately following [13, Equation (1.1)]) the implied constants in the resulting error terms can be taken to be the same. This leads to the estimate

$$\sum_{z\in d^{-1}\Omega_s} 1 = \frac{|\Omega_s|}{|d|^2} + \mathcal{O}\left(\frac{S^{2\kappa}}{|d|^{2\kappa}}\right),$$

and using this in the equation above, we have that

$$\begin{aligned}\sum_{\substack{s'\in\Omega_s\cap\mathbb{Z}[i]\\(s,s')=1}} 1 &= |\Omega_s|\sum_{d|s}\frac{\mu_i(d)}{|d|^2} + \mathcal{O}\left(S^{2\kappa}\sum_{d|s}\frac{|\mu_i(d)|}{|d|^{2\kappa}}\right)\\ &= \frac{\phi_i(s)}{|s|^2}|\Omega_s| + \mathcal{O}_\epsilon(S^{2\kappa+\epsilon}). \qquad (5.4)\end{aligned}$$

To briefly explain the estimate used in the error term here, first of all notice that

$$\sum_{d|s}\frac{|\mu_i(d)|}{|d|^{2\kappa}} = \prod_{p|s}\left(1+\frac{1}{|p|^{2\kappa}}\right) \le \exp\left(c_1\sum_{p|s}\frac{1}{|p|^{2\kappa}}\right),$$

for some constant $c_1 > 0$. Using the Prime Number Theorem for Gaussian integers (as we did in the proof of Theorem 3.1), there is a constant $c_2 > 0$ with the property that

$$\sum_{p|s}\frac{1}{|p|^{2\kappa}} \le \sum_{p^2\le c_2\log|s|}\frac{1}{|p|^{2\kappa}} \ll \frac{(\log|s|)^{1-2\kappa}}{\log\log|s|},$$

and since

$$\exp\left(\frac{(\log|s|)^{1-2\kappa}}{\log\log|s|}\right) \ll_\epsilon |s|^\epsilon,$$

for any $\epsilon > 0$, this explains the error term in (5.4). Returning to our estimate of Σ_1, we now have that

$$\begin{aligned}\Sigma_1 &= \frac{1}{4}\sum_{\substack{s\in\mathbb{Z}[i]^+\\ |s|\le S}} \frac{\phi_i(s)}{|s|^6}|\Omega_s| + \mathcal{O}_\epsilon\left(S^{2\kappa+\epsilon}\right)\\ &= \frac{1}{4}\Sigma_3 + \mathcal{O}_\epsilon(S^{2\kappa+\epsilon}). \end{aligned} \tag{5.5}$$

Using the formula for $|\Omega_s|$ from Section 4.2 (Proposition 4.1), we have that

$$\Sigma_3 = \sum_{\substack{s\in\mathbb{Z}[i]^+\\ |s|\le S}} \frac{\phi_i(s)}{|s|^6} I_1(|s|) - 2\sum_{\substack{s\in\mathbb{Z}[i]^+\\ |s|\le S}} \frac{\phi_i(s)}{|s|^4},$$

where

$$I_1(t) = 8S^2\int_0^{\sin^{-1}\left(\frac{t}{\sqrt{2}S}\right)} \cos^2 u \; du.$$

From Lemma 5.2 we have that

$$\sum_{\substack{s\in\mathbb{Z}[i]^+\\ |s|\le S}} \frac{\phi_i(s)}{|s|^4} = z_1 \ln S + \frac{z_1 + z_1'}{4} + \mathcal{O}(S^{2\kappa-2}).$$

In order to estimate the other sum that appears above we first apply Abel's Summation Formula (Theorem 1.7) with $x = S^6$, $f(t) = 1/t$, and

$$a_1(n) = \sum_{\substack{s\in\mathbb{Z}[i]^+\\ |s|=n^{1/6}}} \phi_i(s)$$

so that, by Lemma 2.6,

$$A_1(t) = \sum_{\substack{s\in\mathbb{Z}[i]^+\\ |s|\le t^{1/6}}} \phi_i(s) = \frac{z_1}{4}t^4 + \mathcal{O}(t^{2\kappa+2})$$

This gives us

$$\begin{aligned}\sum_{\substack{s\in\mathbb{Z}[i]^+\\|s|\le S}}\frac{\phi_i(s)}{|s|^6} &= \sum_{n\le x} a_1(n)f(n)\\ &= S^{-6}\sum_{\substack{s\in\mathbb{Z}[i]^+\\|s|\le S}}\phi_i(s)+\int_1^{S^6}A_1(t)t^{-2}\,\mathrm{d}t\\ &= \frac{z_1}{4S^2}+\mathcal{O}(S^{2\kappa-4})+\frac{3z_1}{4}-\frac{3z_1}{4S^2}+z_2'+\mathcal{O}(S^{2\kappa-4})\\ &= \frac{3z_1}{4}+z_2'-\frac{z_1}{2S^2}+\mathcal{O}(S^{2\kappa-4}),\end{aligned}$$

with

$$z_2'=\int_1^\infty (A_1(t)-\frac{z_1}{4}t^{2/3})t^{-2}\,\mathrm{d}t.$$

Next we apply Abel's Summation Formula again, this time with $x=S^6$, $f(t)=I_1(t^{1/6})$, and

$$b_2(n)=\frac{1}{n}\sum_{\substack{s\in\mathbb{Z}[i]^+\\|s|=n^{1/6}}}\phi_i(s).$$

We have that

$$B_2(t)=\sum_{\substack{s\in\mathbb{Z}[i]^+\\|s|\le t^{1/6}}}\frac{\phi_i(s)}{|s|^6}=\frac{3z_1}{4}+z_2'-\frac{z_1}{2t^{1/3}}+\mathcal{O}\left(t^{\frac{\kappa-2}{3}}\right), \tag{5.6}$$

and that

$$\begin{aligned}f'(t) &= 8S^2\cos^2\left(\sin^{-1}\left(\frac{t^{1/6}}{\sqrt{2}S}\right)\right)\frac{\mathrm{d}}{\mathrm{d}t}\left(\sin^{-1}\left(\frac{t^{1/6}}{\sqrt{2}S}\right)\right)\\ &= \frac{2\sqrt{2}}{3}St^{-5/6}\left(1-\frac{t^{1/3}}{2S^2}\right)^{1/2}.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_1^{S^6}B_2(t)f'(t)\,\mathrm{d}t &= \frac{(3z_1+z_2')}{3\sqrt{2}}S\int_1^{S^6}t^{-5/6}\left(1-\frac{t^{1/3}}{2S^2}\right)^{1/2}\,\mathrm{d}t\\ &\quad+\frac{2\sqrt{2}}{3}S\int_1^{S^6}t^{-5/6}\left(B_2(t)-\left(\frac{3z_1}{4}+z_2'\right)\right)\left(1-\frac{t^{1/3}}{2S^2}\right)^{1/2}\,\mathrm{d}t\\ &= X_1+X_2. \end{aligned}\tag{5.7}$$

Making the substitution $\sin u = t^{1/6}/(\sqrt{2}S)$, we find that

$$\begin{aligned} X_1 &= 2(3z_1 + 4z_2')S^2 \int_{\sin^{-1}(1/\sqrt{2}S)}^{\pi/4} \cos^2 u \, \mathrm{d}u \\ &= \frac{1}{4}(3z_1 + 4z_2')(\pi + 2)S^2 - \sqrt{2}(3z_1 + 4z_2')S + \frac{3z_1 + 4z_2'}{6\sqrt{2}S} + \mathcal{O}(S^{-3}). \end{aligned}$$

Next let

$$z_2'' = \frac{2\sqrt{2}}{3} \int_1^\infty t^{-5/6} \left(B_2(t) - \left(\frac{3z_1}{4} + z_2' \right) \right) \, \mathrm{d}t, \tag{5.8}$$

which by (5.6) is finite. Using a first order approximation for the function $(1-x)^{1/2}$ in the compact subregion $\{|x| \leq 1/2\}$ of its interval of convergence, together with the estimate in (5.6), we have that

$$\begin{aligned} &\frac{2\sqrt{2}}{3} \int_1^{S^6} t^{-5/6} \left(B_2(t) - \left(\frac{3z_1}{4} + z_2' \right) \right) \left(1 - \frac{t^{1/3}}{2S^2} \right)^{1/2} \mathrm{d}t \\ &\quad = \frac{2\sqrt{2}}{3} \int_1^{S^6} t^{-5/6} \left(B_2(t) - \left(\frac{3z_1}{4} + z_2' \right) \right) \mathrm{d}t + \mathcal{O}\left(\frac{1}{S^2} \int_1^{S^6} t^{-5/6} \, \mathrm{d}t \right) \\ &\quad = z_2'' + \mathcal{O}\left(\frac{1}{S} \right), \end{aligned}$$

which proves that

$$X_2 = z_2'' S + \mathcal{O}(1).$$

Substituting into (5.7) and using Abel's Summation Formula now gives that

$$\begin{aligned} \sum_{\substack{s \in \mathbb{Z}[i]^+ \\ |s| \leq S}} \frac{\phi_i(s)}{|s|^6} I_1(|s|) &= B_2(S^6) f(S^6) - \int_1^{S^6} B_2(t) f'(t) \, \mathrm{d}t \\ &= \left(\frac{3z_1}{4} + z_2' - \frac{z_1}{2S^2} + \mathcal{O}(S^{2\kappa - 4}) \right) (\pi + 2)S^2 - X_1 - X_2 \\ &= \left(\sqrt{2}(3z_1 + 4z_2') - \frac{z_2''}{4} \right) S + \mathcal{O}(1). \end{aligned}$$

Using these formulas in (5.1) and (5.5) then gives the statement of Theorem 1.12, with

$$\xi_2 = \frac{3z_1 + 4z_2'}{2\sqrt{2}} - \frac{z_2''}{4}.$$

Finally, in order to verify that $\xi_2 > 0$, consider the contribution to Σ_1, as written in (5.3), coming from the $s = 1$ term. The coprimeness condition on the inner sum is automatically satisfied and, again using the machinery from [13] (see comments above relating to the uniformness of the errors terms), we find that

$$\#\{s' \in \Omega_s \cap \mathbb{Z}[i]\} \geq \#\{s' \in \mathbb{Z}[i] : |s| \leq S, |s-1| \geq S\} \gg S.$$

Therefore $\mathcal{M}_{2,I_2}(S) \gg S$ and $\xi_2 > 0$.

5.3 Proof of Theorem 1.12: $k > 2$ case

Suppose now that $k > 2$. First we write

$$\begin{aligned}
\mathcal{M}_{k,I_2}(S) &= \frac{1}{2^k} \sum_{\frac{r}{s} \in \mathcal{G}_S} \frac{\#\{r'/s' \in \mathcal{G}_S \text{ consecutive to } r/s\}}{|s|^{2k}} \\
&\quad + \frac{1}{2^k} \sum_{\ell=1}^{k-1} \binom{k}{\ell} \sum_{\substack{\frac{r}{s}, \frac{r'}{s'} \in \mathcal{G}_S \\ \text{consecutive}}} \frac{1}{|s|^{2\ell} |s'|^{2(k-\ell)}} \\
&= \Sigma_1 + \Sigma_2.
\end{aligned} \tag{5.9}$$

Using Lemmas 5.1 and 2.8, we have that

$$\begin{aligned}
\Sigma_2 &\ll_k \sum_{\ell=1}^{k-1} \sum_{\substack{s \in \mathbb{Z}[i]^+ \\ |s| \leq S}} \sum_{\substack{s' \in \mathbb{Z}[i]^+ \\ s' \text{cons to } s}} \frac{1}{|s|^{2\ell} |s'|^{2(k-\ell)}} \\
&\ll \sum_{\ell=1}^{k-1} \left(\sum_{\substack{s \in \mathbb{Z}[i]^+ \\ |s| \leq S}} \frac{1}{|s|^{2\ell}} \right) \left(\sum_{\substack{s' \in \mathbb{Z}[i]^+ \\ |s'| \leq S}} \frac{1}{|s'|^{2(k-\ell)}} \right) \\
&\ll_k \log S.
\end{aligned} \tag{5.10}$$

This is asymptotically smaller than what we obtained in the $k = 2$ case because the inner sum is divergent only if $\ell = 1$ or $k - 1$, and in either of these cases the other exponent appearing in the inner summand, $2(k - \ell)$ or 2ℓ, is at least 4.

Next, using (5.4) we have that

$$\begin{aligned}
\Sigma_1 &= \frac{1}{2^{k-2}} \sum_{\substack{s\in\mathbb{Z}[i]^+ \\ |s|\le S}} \sum_{\substack{s'\in\mathbb{Z}[i]^+ \\ s'\text{cons to } s}} \frac{1}{|s|^{2k}} \\
&= \frac{1}{2^k} \sum_{\substack{s\in\mathbb{Z}[i]^+ \\ |s|\le S}} \frac{1}{|s|^{2k}} \sum_{\substack{s'\in\Omega_s\cap\mathbb{Z}[i] \\ (s,s')=1}} 1 \\
&= \frac{1}{2^k} \sum_{\substack{s\in\mathbb{Z}[i]^+ \\ |s|\le S}} \frac{\phi_i(s)}{|s|^{2k+2}} |\Omega_s| + \mathcal{O}_\epsilon\left(S^{2\kappa+\epsilon}\right) \\
&= \frac{1}{2^k}\Sigma_3 + \mathcal{O}_\epsilon(S^{2\kappa+\epsilon}). \qquad (5.11)
\end{aligned}$$

As before, we write

$$\Sigma_3 = \sum_{\substack{s\in\mathbb{Z}[i]^+ \\ |s|\le S}} \frac{\phi_i(s)}{|s|^{2k+2}} I_1(|s|) - 2 \sum_{\substack{s\in\mathbb{Z}[i]^+ \\ |s|\le S}} \frac{\phi_i(s)}{|s|^{2k}}.$$

From Lemma 5.3 we have that

$$\sum_{\substack{s\in\mathbb{Z}[i]^+ \\ |s|\le S}} \frac{\phi_i(s)}{|s|^{2k}} = z_k + \mathcal{O}\left(\frac{1}{S^{2(k-2)}}\right),$$

and for the other sum we first apply Abel's Summation Formula with $x = S^{2k+2}$, $f(t) = 1/t$, and

$$a_k(n) = \sum_{\substack{s\in\mathbb{Z}[i]^+ \\ |s|=n^{1/(2k+2)}}} \phi_i(s)$$

so that

$$A_k(t) = \sum_{\substack{s\in\mathbb{Z}[i]^+ \\ |s|\le t^{1/(2k+2)}}} \phi_i(s) = \frac{z_1}{4} t^{2/(k+1)} + \mathcal{O}(t^{(\kappa+1)/(k+1)}),$$

to obtain

$$\sum_{\substack{s\in\mathbb{Z}[i]^+ \\ |s|\le S}} \frac{\phi_i(s)}{|s|^{2k+2}} = \sum_{n\le x} a_k(n) f(n)$$

$$
\begin{aligned}
&= S^{-2-2k} \sum_{\substack{s\in\mathbb{Z}[i]^+ \\ |s|\le S}} \phi_i(s) + \int_1^{S^{2k+2}} A_k(t)t^{-2}\ \mathrm{d}t \\
&= \frac{z_1}{4S^{2k-2}} + \mathcal{O}(S^{2(\kappa-k)}) + \left(\frac{k+1}{k-1}\right)\frac{z_1}{4} + \mathcal{O}(S^{2(1-k)}) + z_k' + \mathcal{O}(S^{2(\kappa-k)}) \\
&= \left(\frac{k+1}{k-1}\right)\frac{z_1}{4} + z_k' + \mathcal{O}(S^{2(1-k)}),
\end{aligned}
$$

with

$$
z_k' = \int_1^{\infty} (A_k(t) - \frac{z_1}{4}t^{2/(k+1)})t^{-2}\ \mathrm{d}t.
$$

Next we apply Abel's Summation Formula again, with $x = S^{2k+2}, f(t) = I_1\left(t^{1/(2k+2)}\right)$, and

$$
b_k(n) = \frac{1}{n} \sum_{\substack{s\in\mathbb{Z}[i]^+ \\ |s|=n^{1/(2k+2)}}} \phi_i(s).
$$

We have that

$$
B_k(t) = \sum_{\substack{s\in\mathbb{Z}[i]^+ \\ |s|\le t^{1/(2k+2)}}} \frac{\phi_i(s)}{|s|^{2k+2}} = \left(\frac{k+1}{k-1}\right)\frac{z_1}{4} + z_k' + \mathcal{O}\left(t^{\frac{1-k}{1+k}}\right), \tag{5.12}
$$

and that

$$
\begin{aligned}
f'(t) &= 8S^2\cos^2\left(\sin^{-1}\left(\frac{t^{1/(2k+2)}}{\sqrt{2}S}\right)\right)\frac{\mathrm{d}}{\mathrm{d}t}\left(\sin^{-1}\left(\frac{t^{1/(2k+2)}}{\sqrt{2}S}\right)\right) \\
&= \frac{4\sqrt{2}}{2k+2}St^{-(2k+1)/(2k+2)}\left(1 - \frac{t^{1/(k+1)}}{2S^2}\right)^{1/2}.
\end{aligned}
$$

Therefore,

$$
\int_1^{S^{2k+2}} B_k(t)f'(t)\ \mathrm{d}t = X_1 + X_2,
$$

where

$$
X_1 = \frac{4\sqrt{2}}{2k+2}\left(\left(\frac{k+1}{k-1}\right)\frac{z_1}{4} + z_k'\right)S\int_1^{S^{2k+2}} t^{-(2k+1)/(2k+2)}\left(1 - \frac{t^{1/(k+1)}}{2S^2}\right)^{1/2}\ \mathrm{d}t
$$

and

$$
X_2 = \frac{4\sqrt{2}}{2k+2}S\int_1^{S^{2k+2}} t^{-(2k+1)/(2k+2)}\left(B_k(t) - \left(\frac{k+1}{k-1}\right)\frac{z_1}{4} - z_k'\right)\left(1 - \frac{t^{1/(k+1)}}{2S^2}\right)^{1/2}\ \mathrm{d}t.
$$

Making the substitution $\sin u = t^{1/(2k+2)}/(\sqrt{2}S)$, we find that

$$\begin{aligned} X_1 &= 8\left(\left(\frac{k+1}{k-1}\right)\frac{z_1}{4}+z_k'\right)S^2\int_{\sin^{-1}(1/\sqrt{2}S)}^{\pi/4}\cos^2 u\ \mathrm{d}u \\ &= \left(\left(\frac{k+1}{k-1}\right)\frac{z_1}{4}+z_k'\right)(\pi+2)S^2 - 4\sqrt{2}\left(\left(\frac{k+1}{k-1}\right)\frac{z_1}{4}+z_k'\right)S + \mathcal{O}\left(S^{-1}\right). \end{aligned}$$

Using (5.12), we have that

$$X_2 = z_k''S + \mathcal{O}\left(S^{-2k+3}\right) = z_k''S + \mathcal{O}\left(S^{-2}\right),$$

with

$$z_k'' = \frac{4\sqrt{2}}{2k+2}\int_1^\infty t^{-(2k+1)/(2k+2)}\left(B_k(t) - \left(\frac{k+1}{k-1}\right)\frac{z_1}{4} - z_k'\right)\ \mathrm{d}t.$$

This gives that

$$\begin{aligned} \sum_{\substack{s\in\mathbb{Z}[i]^+\\ |s|\le S}}\frac{\phi_i(s)}{|s|^{2k+2}}I_1(|s|) &= B_k(S^{2k+2})f(S^{2k+2}) - \int_1^{S^{2k+2}} B_k(t)f'(t)\ \mathrm{d}t \\ &= \left(\left(\frac{k+1}{k-1}\right)\frac{z_1}{4}+z_k'+\mathcal{O}\left(S^{2-2k}\right)\right)(\pi+2)S^2 - X_1 - X_2 \\ &= \left(\left(\frac{k+1}{k-1}\right)\sqrt{2}z_1 + 4\sqrt{2}z_k' - z_k''\right)S + \mathcal{O}(S^{-1}). \end{aligned}$$

Using these formulas gives the statement of Theorem 1.12, with

$$\xi_k = \frac{1}{2^k}\left(\left(\frac{k+1}{k-1}\right)\sqrt{2}z_1 + 4\sqrt{2}z_k' - z_k''\right).$$

As in the $k=2$ case, the contribution to Σ_1 from $s=1$ is large enough to guarantee that $\xi_k > 0$.

Bibliography

[1] T. M. Apostol. *Introduction to Analytic Number Theory*. Springer-Verlag New York Inc., 1976.

[2] F. P. Boca, C. Cobeli, and A. Zaharescu. A conjecture of R. R. Hall on Farey points. *J. Reine Angew. Math.*, 535:207–236, 2001.

[3] S. Chaubey, A. Malik, and A. Zaharescu. k-moments of distances between centers of Ford circles. *Journal of Mathematical Analysis and Applications*, 422:906–919, 02 2015.

[4] H. M. Edwards. *Riemann's Zeta Function*. Dover Publications Inc., 2001.

[5] L. R. Ford. Fractions. *The American Mathematical Monthly*, 45(9):586–601, 1938.

[6] J. B Fraleigh. *A First Course in Abstract Algebra*. Pearson, 7 edition, 2002.

[7] J. Franel. Les suites de Farey et le problème des nombres premiers. *Göttinger Nachrichten*, pages 198–201, 1924.

[8] C. F. Gauss. *De nexu inter multitudinem classium, in quas formae binariae secundi gradus distribuuntur, Earumque Determinantem*, volume 2 of *Cambridge Library Collection - Mathematics*, page 269–291. Cambridge University Press, 2011.

[9] E. Grosswald. *Representations of Integers as Sums of Squares*. Springer-Verlag New York Inc., 1985.

[10] G. H. Hardy. On the expression of a number as the sum of two squares. *Quart. J. Math.*, 46:263–283, 1915.

[11] G. H. Hardy and E. M. Wright. *An Introduction to the Theory of Numbers.* Oxford, fourth edition, 1975.

[12] M. N. Huxley. The distribution of Farey points i. *Acta Arithmetica*, 18:281–287, 1971.

[13] M. N. Huxley. Exponential sums and lattice points iii. *Proc. London Math. Soc.*, 87, 2003.

[14] E. Landau. Bemerkungen zu der vorstehenden abhandlung von herrn Franel. *Göttinger Nachrichten*, 8:202–206, 1924.

[15] E. Landau. Neue untersuchungen über die pfeiffer'sche methode zur abschätzung von gitterpunktanzahlen. *Sitzungsber. d. math-naturw. Classe der Kaiserl. Akad. d. Wissenschaften*, 124:469–505, 1926.

[16] J. E. Littlewood. Quelques conséquences de l'hypothèse que la fonction $\zeta(s)$ n'a pas des zéros dans le demiplan $\operatorname{Re}(s) > \frac{1}{2}$. *C.R. Acad. Sci. Paris*, 154:263–266, 1912.

[17] W. Narkiewicz. *Elementary and Analytic Theory of Algebraic Numbers.* Springer-Verlag Berlin Heidelberg, 3 edition, 2004.

[18] I. Niven. *An Introduction to Diophantine Approximations.* Dover Publications Inc., 2008.

[19] B. Riemann. On the number of primes less than a given magnitude. *Monatsberichte der Berliner Akademie*, 1859.

[20] I. Short. Ford circles, continued fractions, and rational approximation. *The American Mathematical Monthly*, 118:130–135, 02 2011.

[21] J. Stillwell. *Elements of Number Theory.* Springer-Verlag New York, 2003.

[22] G. Tenenbaum. *Introduction to Analytic and Probabilistic Number Theory*, page 130. Cambridge University Press, 1995.