Constructive Methods for Frobenius and Cartier Maps

By

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Abstract

Let R be a commutative Noetherian ring of prime characteristic p, and M be an R-module. We may endow M with a new R-module structure given by $r.m = r^{p^e}m$, and we denote this new module with $F_*^e M$, where e is a positive integer.

An e-th Frobenius map on M is an R-linear map from M to $F_*^e M$. When R is a formal power series ring and M is Artinian, given $\phi \in \operatorname{Hom}_R(M, F_*^e M)$ it is known that there are only finitely many annihilators of R-submodules N of M where $\phi(N) \subseteq F_*^e N$ and ϕ restricts to a non nilpotent map on N. A dual notion of this fact shows that there are only finitely many ideals of R which are annihilators of R^α/W for some submodules W with some non degeneracy conditions, where α is a positive integer. There also is an algorithm to find such ideals. In the first part of the thesis, we study these annihilator ideals of R and generalize the dual notion of aforesaid result to polynomial rings, and we present a new algorithm for finding such prime ideals. Further, we provide an application of the new algorithm to Lyubeznik's F-finite F-modules.

An e-th Cartier map on M is an R-linear map from $F_*^e M$ to M. When M is finitely generated, given a surjective Cartier map on M it is again known that there are only finitely many annihilators of Cartier quotients of M. In the second part of the thesis, we study finitely generated modules equipped with a Cartier map. We consider a computational perspective and present an algorithm for finding prime annihilators of Cartier quotients of a given finitely generated module equipped with a surjective Cartier map. Moreover, we use this algorithm to find a lower bound for F-module length of Lyubeznik's F-finite F-modules.

In the last part of the thesis, when R is a power series ring over a perfect field of prime characteristic, we present an explicit correspondence between Artinian R-modules equipped with a Frobenius map and Noetherian R-modules equipped with a Cartier map.

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Contents

Chapter 1

Introduction

Let R be a commutative Noetherian ring of prime characteristic p, and e be a positive integer. Let $f : R \to R$ be the Frobenius homomorphism defined by $f(r) = r^p$ for all $r \in R$, whose e-th iteration is denoted by f^e. The ring R is called F-finite if f^e is a finite map for some e. Let $R[X; f^e]$ be the skew-polynomial ring. Let M be an R-module, an e-th Frobenius map on M is an additive map $\phi : M \to M$ such that $\phi(rm) = r^{p^e}\phi(m)$ for all $m \in M$ and $r \in R$. Notice that defining an e-th Frobenius map on M is equivalent to endowing M with a left $R[X; f^e]$-module structure extending the rule $Xm = \phi(m)$ for all $m \in M$ (see Subsection 2.2.2).(When $e = 1$, we simply drop it from notations.)

The first part of this thesis studies the notion of special ideals. It was introduced by R. Y. Sharp in [20]. For a left $R[X; f^e]$-module M, when X defines an injective e-th Frobenius map on M, he defines an ideal of R to be M-special R-ideal if it is the annihilator of some $R[X; f^e]$-submodule of M (cf. Section 1 of [20]). Later on, it was generalized by M. Katzman and used to study Frobenius maps on injective hulls in [10] and [11]. For a left $R[X; f^e]$-module M, Katzman defines an ideal of R to be M-special if it is the annihilator of some $R[X; f^e]$-submodule of M (cf. Section 6 of [10]). A special case of special ideals is when R is local, M is Artinian, and X defines an injective e-th Frobenius map on M. In this case, Sharp showed that the set of M-special ideals is a finite set of radicals, consisting of all intersections of the finitely many primes in it (Corollary 3.11 in [20]). It was also proved by F. Enescu and M. Hochster independently (Section 3 in [5]). When R is complete regular local and M is Artinian, the notion of special ideals becomes an important device to study Frobenius maps on injective hulls. In particular, since top local cohomology module of R is isomorphic to the injective hull of the residue field of R, it provides an important insight to top local cohomology modules.

In the case that R is a finite dimensional formal power series ring over a field of prime characteristic p, in [13], M. Katzman and W. Zhang focus on the M-special ideals when M is Artinian. In this case, they define the special ideals depending on the $R[\phi; f^e]$-module structures on E^α, where E is the injective hull of the residue field of R and α is a positive integer. They define an ideal of R to be ϕ-special if it is the annihilator of an $R[\phi; f^e]$-submodule of E^α, where $\phi = U^t T^e$ with T is the natural Frobenius on E^α and U is an $\alpha \times \alpha$ matrix with entries in R (see Section 3.2). Furthermore, they use Katzman's Δ^e and Ψ^e functors, which are extensions of Matlis duality keeping track of Frobenius maps, to define ϕ-special ideals equivalently to be the annihilators of R^α/W for some submodule W satisfying $UW \subseteq W^{[p^e]}$, where $W^{[p^e]}$ is the submodule generated by $\{w^{[p^e]} = (w_1^{p^e}, \dots, w_\alpha^{p^e})^t \mid w = (w_1, \dots, w_\alpha)^t \in W\}$ (see Proposition 3.2.2). Katzman and Zhang show that there are only finitely many ϕ-special ideals P of R with the property that P is the annihilator of an $R[\phi; f^e]$-submodule M of E^α such that the restriction of ϕ^e to M is not zero for all e, and introduce an algorithm for finding special prime ideals with this property in [13]. They first present the case $\alpha = 1$, which was considered by M. Katzman and K. Schwede in [12] with a geometric language. Then they extend this to the case $\alpha > 1$.

In the first part of this thesis, we adapt the equivalent definition of ϕ-special ideals above to the polynomial rings, and for an $\alpha \times \alpha$ matrix U we define U-special ideals to be the annihilators of R^α/W for some submodule W of R^α satisfying $UW \subseteq W^{[p^e]}$. We generalize the results in [13] to the case that R is a finite dimensional polynomial ring over a field of prime characteristic p, and show that there are only finitely many U-special ideals with some non degeneracy conditions (see Theorem 3.2.16). We also present an algorithm for finding U-special prime ideals of polynomial rings. Furthermore, we consider the notion of F-finite F-modules, which is a prime characteristic extension of local cohomology modules introduced by G. Lyubeznik in [16], and we show that our new algorithm gives a method for finding the prime ideals of R such that $\operatorname{crk}(H^i_{IR_P}(R_P)) \neq 0$ (see Definition 3.3.2 and Theorem 3.3.3).

The second part of this thesis studies the notion of Cartier modules. An e-th

Cartier map on M is an additive map $C : M \to M$ such that $rC(m) = C(r^{p^e}m)$ for all $m \in M$ and $r \in R$, which is a dual notion of Frobenius maps. An R-module M is called a Cartier module if it is equipped with a Cartier map. M. Blickle and G. Böckle study the notion of nilpotence for finitely generated Cartier modules and present some finiteness results in [2]. One of the main result of this paper states that if R is F-finite and M is a finitely generated R-module equipped with a surjective Cartier map, then the set of annihilators of Cartier quotients of M is a finite set of radical ideals consisting of all intersections of the finitely many primes in it (see Section 4 in [2]). This generalizes the results in [20] and [5] mentioned above. In the second part of this thesis, we consider the case that R is a finite dimensional polynomial ring or a finite dimensional formal power series ring over an F-finite field of prime characteristic p, we take a computational view of this finiteness result of Blickle and Böckle, and we give an alternative proof to the result (see Theorem 4.3.13). We then present an algorithm for finding prime annihilators of Cartier quotients (see Section 4.4). Moreover, we obtain an explicit correspondence between finitely generated Cartier modules and Lyubeznik's F-finite F-modules, which enables us to show that our algorithm gives a method to find a lower bound for F-module length of F-finite F-modules.

When R is complete regular local and F-finite, it is shown that there exists a bijective correspondence between Artininan R-modules equipped with a Frobenius map and Noetherian R-modules equipped with a Cartier map in [2] and independently in [21]. In the last part of this thesis, we obtain an explicit correspondence between these two sets of R-modules which coincides with the correspondences in [2] and [21], more importantly, extends to a computational level. To do this, we define an explicit isomorphism between two modules which are well-known isomorphic modules but an isomorphism has not been given explicitly before (see Lemma 5.1.2).

1.1 Outline of Thesis

In Chapter 2, we collect the necessary concepts from commutative algebra which we need for this thesis as a background. In Section 2.1, we provide brief summaries of localization and completion of modules and rings, as well as a brief introduction to injective and local cohomology modules. In Section 2.2, we provide a technical background to positive characteristic methods in commutative algebra, which we use throughout this thesis.

In Chapter 3, we investigate the notion of special ideals. In Section 3.1, we state the algorithm described in [12] with a more algebraic language and show that it commutes with localization. In Section 3.2, we generalize the results in [13] to polynomial rings. In particular, we present a new algorithm which is very similar to the one defined in [13], and show that it commutes with localization too. Finally, in Section 3.3, we present a connection between special ideals and Lyubeznik's F-finite F-modules using our algorithm. The main result of this chapter, Theorem 3.3.3, not only reproves Proposition 4.14 in [16] but also gives a method for finding the desired prime ideals.

In Chapter 4, we investigate the notion of Cartier modules. In Section 4.1 and 4.2, we study finitely generated Cartier modules in a more algebraic language. In Section 4.3, we prove our technical lemmas which give us computational methods on finitely generated Cartier modules when R is a polynomial ring or a power series ring over an F-finite field of prime characteristic p. In particular, we prove the main result, Theorem 4.3.13, of this chapter using these computational methods which extends Proposition 4.1 and 4.5 in [2] to a computational level. In Section 4.4, we introduce a new algorithm which finds the finite set of prime annihilators of Cartier quotients of a given finitely generated Cartier module. Finally, in Section 4.5, we obtain an explicit correspondence between finitely generated Cartier modules and Lyubeznik's F-finite F-modules which leads us a method for finding a lower bound for F-module length of F-finite F-modules.

In Chapter 5, when R is a power series ring over a perfect field of prime characteristic, we introduce an explicit correspondence between Artinian R-modules equipped with a Frobenius map and Noetherian R-modules equipped with a Cartier map using our computational techniques. This extends the correspondences introduced in [2] and [21] to a computational level. In particular, Lemma 5.1.2 gives an explicit isomorphism for well-known isomorphic modules but was not given explicitly before, which leads us to our explicit correspondence.

1.2 Notation

Throughout this thesis, all rings in consideration are assumed to be commutative and Noetherian with identity, and all modules are assumed to be unital unless otherwise stated.

We use $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}_0$ and $-\mathbb{N}$ to denote the ring of integers, the set of positive integers, the set of non negative integers and the set of negative integers, respectively. We also use $(-)^t$ to denote the transpose of vectors and matrices. Let R be a ring, I be an ideal of R, and M be an R-module. If I is finitely generated by elements $a_1, \ldots, a_m \in R$, we write $I = \langle a_1, \ldots, a_m \rangle$. Similarly, if M is finitely generated by elements $m_1, \ldots, m_s \in M$, we write $M = \langle m_1, \ldots, m_s \rangle$. R is called Noetherian if it satisfies the ascending chain condition on ideals, i.e. every ascending chain $I_1 \subseteq I_2 \subseteq \cdots$ of ideals in R stabilizes, or equivalently every ideal of R is finitely generated. R is called Artinian if it satisfies the descending chain condition on ideals, i.e. every descending chain $I_1 \supseteq I_2 \supseteq \cdots$ of ideals in R stabilizes. Analogously, M is called Noetherian (Artinian) if it satisfies ascending (descending) chain condition on its submodules.

We say that an ideal P of R is a prime ideal if for any $a, b \in R$, $ab \in P \Rightarrow a \in P$ or $b \in P$. The radical of I is the set $\{r \in R \mid r^n \in I \text{ for some } n \in \mathbb{N}\}$, or equivalently is the intersection of all prime ideals which contains I, and denoted by $\sqrt{I}$. I is said to be primary if for any $a, b \in R$, $ab \in I \Rightarrow a \in I$ or $b^n \in I$ for some $n \in \mathbb{N}$. If $P = \sqrt{I}$, then I is called P-primary. A primary decomposition of I is an expression $I = Q_1 \cap \cdots \cap Q_s$ with each Q_i is a primary ideal, and it is called minimal if no Q_i can be omitted in the expression and if $\sqrt{Q_i} \neq \sqrt{Q_j}$ for all $i \neq j$. In Noetherian rings, there always exist minimal primary decompositions of ideals. In this case, if $I = Q_1 \cap \cdots \cap Q_s$ is a minimal primary decomposition, the prime ideals $P_i = \sqrt{Q_i}$ are called associated primes of I.

The set of all prime ideals of R is denoted by $\operatorname{Spec} R$, and $V(I)$ denotes the set $\{P \in \operatorname{Spec} R \mid I \subseteq P\}$. The collection $\{V(I) \mid I \text{ is an ideal of } R\}$ defines a topology on $\operatorname{Spec} R$ that is called the Zariski topology in which $V(I)$ is a closed set. The Krull dimension or simply the dimension of R is the supremum of the lengths of all chains of prime ideals in R, and denoted by $\dim R$. We also define the dimension of a finitely generated module M over R to be the dimension of the ring $R/\operatorname{Ann}_R M$ where $\operatorname{Ann}_R M = \{r \in R \mid rM = 0\}$, and denote it by $\dim_R M$.

A sequence $\cdots \to M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} \cdots$ of R-modules and R-homomorphisms is called exact if $\operatorname{Ker} f_{i+1} = \operatorname{Im} f_i$ for all i. An exact sequence of the form

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is called a short exact sequence. A sequence

$$\mathcal{C} : 0 \to C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{i-1}} C^i \xrightarrow{d^i} \cdots$$

of R-modules and R-homomorphisms is called a cochain complex if $d^i \circ d^{i-1} = 0$, and the module $H^i(\mathcal{C}) := \operatorname{Ker} d^i / \operatorname{Im} d^{i-1}$ is called the i-th cohomology module of $\mathcal{C}$. A sequence

$$\mathcal{C} : 0 \leftarrow C_0 \xleftarrow{d_0} C_1 \xleftarrow{d_1} \cdots \xleftarrow{d_{i-1}} C_i \xleftarrow{d_i} \cdots$$

of R-modules and R-homomorphisms is called a chain complex if $d_{i-1} \circ d_i = 0$, and the module $H_i(\mathcal{C}) := \operatorname{Ker} d^{i-1} / \operatorname{Im} d^i$ is called the i-th homology module of $\mathcal{C}$.

We say that R is local if it has only one maximal ideal, and we write $(R, \mathfrak{m})$ to mean that R is a local ring with the unique maximal ideal $\mathfrak{m}$. If $(R, \mathfrak{m})$ is a local ring, then we call $R/\mathfrak{m}$ the residue field of R. A local ring is called regular if the minimal number of generators of its maximal ideal is equal to its dimension. In general, R is called regular if its localization at every prime ideal is a regular local ring. The regular locus $\operatorname{Reg} R$ of R is the set of prime ideals P such that the localization of R with respect to P is a regular local ring, and $\operatorname{Spec} R \setminus \operatorname{Reg} R$ is the singular locus of R and denoted by $\operatorname{Sing} R$. Furthermore, the characteristic of R is the smallest integer n such that $\sum_1^n 1 = 0$, and if no such integer n exists, then the characteristic of R is zero.

Chapter 2

Preliminaries

In this chapter, we provide a brief introduction to the background and some technical commutative algebra tools that we need for the latter chapters. Section 2.1 contains general background without proofs, and section 2.2 contains some basic tools of positive characteristic methods in commutative algebra and a technical background that is essential for the main results of this thesis. As a reminder, we will always assume all rings are Noetherian even though some of general concepts provided in this chapter are true for the non-Noetherian case.

2.1 General Background

In this section, we provide brief summaries on the concepts of localization and completion of modules and rings. We also provide a brief introduction to injective and local cohomology modules.

2.1.1 Presentations of Finitely Generated Modules

In this subsection, we give a matrix presentation of finitely generated modules using free modules.

An R-module F is called a **free module** if F is isomorphic to a direct sum of copies of R; that is, there is an index set $\mathcal{B}$ with $F = \bigoplus_{b \in \mathcal{B}} R_b$ where $R_b = \langle b \rangle \cong R$

for all $b \in \mathcal{B}$. We call $\mathcal{B}$ is a basis of F and the cardinality of $\mathcal{B}$ is called the rank of F. For each element $m \in F$, we have a unique expression of the form $m = \sum_{b \in \mathcal{B}'} r_b b$ where $\mathcal{B}'$ is a finite subset of $\mathcal{B}$, and $r_b \in R$ for all $b \in \mathcal{B}'$.

Theorem 2.1.1. *[19, Theorem 2.35] Every R-module M is a quotient of a free R-module F. Moreover, M is finitely generated if and only if F can be chosen to be finitely generated.*

Let F and G be free R-modules. Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be a basis of F and $\mathcal{C} = \{c_1, \ldots, c_m\}$ be a basis of G. Let $\phi \in \operatorname{Hom}_R(F, G)$ such that for each b_j, $\phi(b_j) = \sum_{i=1}^{m} a_{ij} c_i$ for some $a_{ij} \in R$. Let A be the $m \times n$ matrix whose ij-th entry is a_{ij}. Then for each element $f = \sum_{j=1}^{n} r_{b_j} b_j \in F$, we have $\phi(f) = \sum_{i=1}^{m} r_{c_i} c_i$ where

$$\begin{pmatrix} r_{c_1} \\ \vdots \\ r_{c_m} \end{pmatrix} = A \begin{pmatrix} r_{b_1} \\ \vdots \\ r_{b_n} \end{pmatrix}.$$

Therefore, for any map $\phi \in \operatorname{Hom}_R(F, G)$ we can associate a matrix A with entries in R.

In particular, any R-linear map $\varphi : R^n \to R^m$ can be represented with an $m \times n$ matrix A with entries in R. In this case, for any element $(r_1, \ldots, r_n)^t \in R^n$, we have

$$\varphi \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} = A \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}.$$

Then we write $R^n \xrightarrow{A} R^m$ to denote φ, $\operatorname{Im} A$ to denote image of φ, $\operatorname{Ker} A$ to denote kernel of φ, and $\operatorname{Coker} A$ to denote cokernel of φ.

Remark 2.1.2. *Let M be a finitely generated R-module. Then there exist an exact sequence*

$$R^\beta \xrightarrow{A} R^\alpha \to M \to 0$$

where $\operatorname{Coker} A = R^\alpha / \operatorname{Im} A \cong M$.

2.1.2 Localization of Modules and Rings

In this subsection, we give a brief summary of localization of modules and rings and its properties which we use throughout this thesis. For the proofs and more details we refer to chapter 3 of [1].

Let R be a ring and M be an R-module. Let W be a multiplicatively closed subset of R, i.e. $1 \in W$ and $ws \in W$ for all $w, s \in W$. For some $(w, m), (s, n) \in W \times M$, we define the equivalence relation $\sim$ on $W \times M$ by $(w, m) \sim (s, n)$ if and only if there is an element $t \in W$ such that $t(wn - sm) = 0$ in M and we denote the equivalence class of $(w, m) \in W \times M$ by $\frac{m}{w}$. We define the **localization** of M at W to be the set of all such equivalence classes and denote it by $W^{-1}M = \{\frac{m}{w} \mid m \in M, w \in W\}$.

If we apply the definition in the case $M = R$, the resulting localization is a commutative ring with addition and multiplication defined respectively by

$$\frac{a}{w} + \frac{b}{s} = \frac{sa + wb}{ws} \text{ and } \frac{a}{w}.\frac{b}{s} = \frac{ab}{ws}$$

for all $\frac{a}{w}, \frac{b}{s} \in W^{-1}R$.

Furthermore, $W^{-1}M$ is an $W^{-1}R$-module with addition and scalar multiplication defined respectively by

$$\frac{m}{w} + \frac{n}{s} = \frac{sm + wn}{ws} \text{ and } \frac{a}{t}\frac{m}{w} = \frac{am}{tw}$$

for all $\frac{m}{w}, \frac{n}{s} \in W^{-1}M$ and $\frac{a}{t} \in W^{-1}R$.

Remarks 2.1.3. 1. *If P is a prime ideal of R, then $W = R \setminus P$ is a multiplicatively closed set and we write M_P to denote $W^{-1}M$ and R_P to denote $W^{-1}R$.*

2. *If $f \in R \setminus \{0\}$, then $W = \{1, f, f^2, \dots\}$ is a multiplicatively closed set and we write M_f to denote $W^{-1}M$ and R_f to denote $W^{-1}R$.*

3. *If $\phi : M \to N$ is an R-module homomorphism, then we have an $W^{-1}R$-module homomorphism $W^{-1}\phi : W^1M \to W^{-1}N$ given by $(W^{-1}\phi)(\frac{m}{w}) = \frac{\phi(m)}{w}$.*

Next we will collect some important properties of localization whose proofs can be found in chapter 3 of [1].

Remarks 2.1.4. *Let M and N be R-modules and W be a multiplicatively closed subset of R.*

1. *The operation W^{-1} is exact.*
2. *The operation W^{-1} commutes with formation of finite sums, products, intersections and quotients and radicals.*
3. *$W^{-1}M \cong W^{-1}R \otimes_R M$ as $W^{-1}R$-modules.*
4. *$W^{-1}M \otimes_{W^{-1}R} W^{-1}N \cong W^{-1}(M \otimes_R N)$ as $W^{-1}R$-modules.*
5. *The prime ideals of $W^{-1}R$ are in one-to-one correspondence with the prime ideals of R which do not intersect W.*
6. *If M is finitely generated, then $W^{-1}(\operatorname{Ann}_R M) = \operatorname{Ann}_{W^{-1}R} W^{-1}M$.*
7. *If M is finitely generated, then $W^{-1}(N :_R M) = (W^{-1}N :_{W^{-1}R} W^{-1}M)$.*
8. *When M is finitely generated, $W^{-1}M = 0$ if and only if there is an element $w \in W$ such that $wM = 0$.*

Proposition 2.1.5. *[1, Proposition 3.9] Let $\phi : M \to N$ be an R-module homomorphism. Then the following are equivalent:*

1. *ϕ is injective(surjective),*
2. *$\phi_P : M_P \to N_P$ is injective(surjective) for each prime ideal P of R,*
3. *$\phi_m : M_m \to N_m$ is injective(surjective) for each maximal ideal m of R.*

Definition 2.1.6. *The support of M is the set of all prime ideals P of R such that $M_p \neq 0$ and it is denoted by $\operatorname{Supp} M$, i.e. $\operatorname{Supp} M = \{P \in \operatorname{Spec} R \mid M_P \neq 0\}$.*

Remark 2.1.7. *Let M be a finitely generated R-module with $M = \langle m_1, \cdots, m_n \rangle$, and let $P \in \operatorname{Spec} R$. Then*

$$\begin{aligned} P \in \operatorname{Supp} M &\Leftrightarrow m_i \neq 0 \textit{ in } M_P, \textit{i.e. } \operatorname{Ann}_R m_i \subseteq P \textit{ for some } i \\ &\Leftrightarrow I := \operatorname{Ann}_R M = \bigcap_{i=1}^{n} \operatorname{Ann}_R m_i \subseteq P. \end{aligned}$$

This means that $\operatorname{Supp} M = V(\operatorname{Ann}_R M)$, and so $\operatorname{Supp} M$ is a Zariski closed subset of $\operatorname{Spec} R$.

2.1.3 Completion of Modules and Rings

In this subsection, we give a brief summary of completion of modules and rings and its properties which we use throughout this thesis. For the proofs and more details we refer to chapter 8 of [22] and section 7 of [6].

Let R be a ring and M be an R-module. A sequence $\{I_n\}_{n\geq 0}$ of ideals is called a **filtration** if $I_0 = R$, $I_n \supseteq I_{n+1}$ and $I_n I_m \subseteq I_{n+m}$ for all $n, m \in \mathbb{N}$. Let $\{I_n\}_{n\geq 0}$ be a filtration on R, a sequence $\{M_n\}_{n\geq 0}$ of submodules of M is called a filtration on M if $M_0 = M$, $M_n \supseteq M_{n+1}$ and $I_m M_n \subseteq M_{m+n}$ for all $n, m \in \mathbb{N}$. In this case, for the condition $I_m M_n \subseteq M_{m+n}$ we say that $\{M_n\}_{n\geq 0}$ is compatible with $\{I_n\}_{n\geq 0}$. The most important case is when the filtration is given by $I_n = I^n$ for all $n \geq 1$ and $I_0 = R$. This is called I-**adic filtration** on R. Analogously, the filtration given by $M_n = I^n M$ for all $n \geq 1$ and $M_0 = M$ is called I-adic filtration on M.

Definition 2.1.8. *Let $A = \{I_n\}_{n\geq 0}$ be a filtration, and let $F = \{M_n\}_{n\geq 0}$ be a filtration on M compatible with A.*

1. *We define the completion of R with respect to the filtration A as*
$$\widehat{R}_A = \varprojlim R/I_n = \{(a_1, a_2, \dots) \in \prod_{n\geq 1} R/I_n \mid a_{n+1} - a_n \in I_n, \forall n \geq 1\}$$
and denote it by $\widehat{R}_A$.

2. *We define the completion of M with respect to the filtration F as*
$$\widehat{M}_F = \varprojlim M/M_n = \{(m_1, m_2, \dots) \in \prod_{n\geq 1} M/M_n \mid m_{n+1} - m_n \in M_n, \forall n \geq 1\}$$
and denote it by $\widehat{M}_F$.

If A and F are I-adic filtrations, then the I-adic completion of R is denoted by $\widehat{R}_I$ and the I-adic completion of M is denoted by $\widehat{M}_I$. If there is no ambiguity, we just drop I from notations, and denote the I-adic completions as $\widehat{R}$ and $\widehat{M}$.

Remark 2.1.9. *Since each R/I_n is a ring, it is easy to see that $\widehat{R}$ is a ring with addition*
$$(a_1, a_2, \dots) + (b_1, b_2, \dots) = (a_1 + b_1, a_2 + b_2, \dots)$$
and multiplication
$$(a_1, a_2, \dots)(b_1, b_2, \dots) = (a_1 b_1, a_2 b_2, \dots)$$

for all $(a_1, a_2, \dots), (b_1, b_2, \dots) \in \widehat{R}$.

Similarly, $\widehat{M}$ *is an* $\widehat{R}$*-module with addition*

$$(m_1, m_2, \dots) + (m'_1, m'_2, \dots) = (m_1 + m'_1, m_2 + m'_2, \dots)$$

and scalar multiplication

$$(a_1, a_2, \dots)(m_1, m_2, \dots) = (a_1 m_1, a_2 m_2, \dots)$$

for all $(m_1, m_2, \dots), (m'_1, m'_2, \dots) \in \widehat{M}$ *and* $(a_1, a_2, \dots) \in \widehat{R}$.

Example 2.1.10. *[6, Section 7.1] If* $R = A[x_1, \dots, x_n]$ *is a polynomial ring over a ring* A*. If* $\mathfrak{m} = \langle x_1, \dots, x_n \rangle$*, then the completion of* R *with respect to* $\mathfrak{m}$ *is the formal power series ring* $A[\![x_1, \dots, x_n]\!]$*, i.e.* $\widehat{R}_{\mathfrak{m}} \cong A[\![x_1, \dots, x_n]\!]$.

Theorem 2.1.11. *[6, Theorem 7.2] Let* M *be a finitely generated* R*-module and* I *be an ideal of* R*. Then:*

1. $\widehat{M_I} \cong M \otimes_R \widehat{R}_I$ *as* $\widehat{R}_I$*-modules.*

2. $\widehat{R}_I$ *is flat as an* R*-module.*

Lemma 2.1.12. *[6, Lemma 7.14] Let* $A = \{I_n\}_{n \geq 0}$ *be a filtration, and* $F = \{M_n\}_{n \geq 0}$ *and* $H = \{N_n\}_{n \geq 0}$ *be two compatible filtrations on an* R*-module* M*, which are cofinal, i.e. for each* M_i *there is an* N_j *such that* $M_i \subset N_j$ *and, for each* N_i *there is an* M_j *such that* $N_i \subset M_j$*. Then* $\widehat{M_F} \cong \widehat{M_H}$ *as* $\widehat{R}_I$*-modules.*

When the natural map $R \to \widehat{R_I}$ is an isomorphism, we call R to be complete with respect to I, and if I is a maximal ideal, R is said to be a **complete local ring**. Next we recall the Cohen structure theorem which states that any complete local ring containing a field is a homomorphic image of a power series ring in finitely many variables over a field.

Theorem 2.1.13. *[6, Theorem 7.7] Let* $(R, \mathfrak{m})$ *be a complete local ring with residue field* K*. If* R *contains a field, then* $R \cong K[\![x_1, \dots, x_n]\!]/I$ *for some* $n \in \mathbb{N}$ *and ideal* I *of* R*.*

2.1.4 Injective Modules and Matlis Duality

In this subsection, we give a brief summary of Injective modules and their important properties which we use throughout this thesis. For the proofs and more details we refer to [6], [4] and Appendix of [9].

Definition 2.1.14. *An R-mdoule E is called injective if it satisfies following equivalent conditions*

1. $\operatorname{Hom}_R(-, E)$ *is an exact functor,*
2. *for any injection of R-modules $N \hookrightarrow M$ the R-linear map* $\operatorname{Hom}_R(M, E) \to \operatorname{Hom}_R(N, E)$ *is surjective.*

Theorem 2.1.15. *[6, Corollary A3.9] Any R-module M can be embedded in an injective R-module E.*

Definition 2.1.16. *The injective hull of an R-module M is the smallest injective R-module containing M which will be denoted by $E_R(M)$.*

Following Appendix of [9] we alternatively define injective hulls using essential extensions. Let M be an R-module and $N \subseteq M$ an R-submodule. M is called an **essential extension** of N if every non zero R-submodule L of M has non zero intersection with N. If also M has no proper essential extension, we say that M is a **maximal essential extension** of N. By Zorn's Lemma there always exist a maximal essential extension of N, and it is unique up to non-canonical isomorphism. The injective hull of N is also defined to be the maximal essential extension of it.

Definition 2.1.17. *Let M be an R-module. An injective resolution of M is a complex of injective R- modules*

$$\mathbf{E} : 0 \to E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \xrightarrow{d^2} \cdots$$

with the cohomology modules $H^0(\mathbf{E}) = M$ and $H^i(\mathbf{E}) = 0$ for all $i \geq 1$. It is called minimal injective resolution if $E^0 = E_R(M)$ and $E^i = E_R(\operatorname{Coker} d^{i-1})$ for each $i \geq 1$ and the length of minimal injective resolution of M is called injective dimension of M denoted by $\operatorname{inj.dim}_R M$.

An R-module M is called a **Gorenstein module** if and only if M has finite injective dimension. If R is a Gorenstein R-module, then we call R a **Gorenstein ring**.

Proposition 2.1.18. *[9, Theorem A.21, Proposition A.22] Let E be an injective R-module. Then*

1. $E \cong \bigoplus_{P \in \operatorname{Spec} R} E_R(R/P)^{\mu_P}$ *where the numbers μ_P are independent of the decomposition,*

2. $E_R(R/P) \cong E_{R_P}(R_P/PR_P)$.

Proposition-Definition 2.1.19. *[9, Theorem A.24] Let M be an R-module and $\mathbf{E}$ be its minimal injective resolution. Then for each i,*

$$E^i \cong \bigoplus_{P \in \operatorname{Spec} R} E_R(R/P)^{\mu_i(P,M)}$$

where $\mu_i(P, M) = \operatorname{rank}_\kappa \operatorname{Ext}^i_{R_P}(\kappa, M_P)$ and $\kappa = R_P/PR_P$. The number $\mu_i(P, M)$ is called the i-th Bass number of M with respect to P.

Lemma 2.1.20. *[9, Theorem A.25] Let $R \to S$ be a local homomorphism and suppose that S is module finite over R. Let E_R and E_S be the injective hulls of residue fields of R and S, respectively. Then, $\operatorname{Hom}_R(S, E_R) \cong E_S$ as S-modules.*

Remark 2.1.21. *A consequence of Lemma 2.1.20 is that if $S = R/I$, where I is an ideal of R, then $E_S \cong \operatorname{Hom}_R(R/I, E_R)$. On the other hand, the elements of $\operatorname{Hom}_R(R/I, E_R)$ are the elements of $\operatorname{Hom}_R(R, E_R)$ which sends I to zero. Since a map from $\operatorname{Hom}_R(R, E_R)$ is completely determined by where it sends $1 \in R$, we get $\operatorname{Hom}_R(R/I, E_R) \cong \operatorname{Ann}_{E_R} I$, and so $E_S \cong \operatorname{Ann}_{E_R} I$.*

Definition 2.1.22. *Let R be local and E_R(or just E if there is no ambiguity) be the injective hull of its residue field. The functor $\operatorname{Hom}_R(-, E)$ is called the Matlis duality functor and denoted by $(-)^\vee$.*

Theorem 2.1.23. *[9, Theorem A.21] Let $(R, \mathfrak{m})$ be a local ring and $\widehat{R}$ its $\mathfrak{m}$-adic completion of R. Let E_R and $E_{\widehat{R}}$ be the injective hulls of residue fields of R and $\widehat{R}$, respectively. Then*

1. $E_R \cong E_{\widehat{R}}$,

2. *the map $\widehat{R} \to \operatorname{Hom}_R(E_R, E_R)$ defined by $r \mapsto (e \mapsto re)$ for any $r \in \widehat{R}$ and $e \in E_R$, is an isomorphism of $\widehat{R}$-modules. In particular, if R is complete, then $R \cong \operatorname{Hom}_R(E_R, E_R)$.*

Theorem 2.1.24. *(Matlis Duality Theorem) Let R be a complete local ring and E be the injective hull of its reside field. Then*

1. *if M is a Noetherian R-module, then $M^\vee$ is Artinian and $(M^\vee)^\vee \cong M$,*
2. *if M is Artinian R-module, then $M^\vee$ is Noetherian and $(M^\vee)^\vee \cong M$.*

Remark 2.1.25. *Let the situation and notation be as in the Matlis Duality Theorem. Let $M \subseteq E$ be an R-submodule. If we apply the Matlis dual functor to the natural injection $M \hookrightarrow E$, we get the surjection $\operatorname{Hom}_R(E, E) \twoheadrightarrow M^\vee$. The kernel of this map is just the set of elements of $\operatorname{Hom}_R(E, E)$ that restrict to 0 on M. On the other hand, by Theorem 2.1.23 2, any map from $\operatorname{Hom}_R(E, E)$ is just a multiplication by an element of R. Therefore, $M^\vee \cong R/J$ where $J = \operatorname{Ann}_R M$. Moreover, by Remark 2.1.21, $(M^\vee)^\vee \cong \operatorname{Ann}_E J$, and so $M \cong \operatorname{Ann}_E J$. Hence, the set of R-submodules of E is $\{\operatorname{Ann}_E J \mid J$ is an ideal of $R\}$.*

Corollary 2.1.26. *Let R be a local ring and E be the injective hull of its residue field. Then E is Artinian.*

Corollary 2.1.27. *[4, 10.2.8 Corollary] Let R be a local ring, E be the injective hull of its residue field and M be an R-module. Then M is Artinian if and only if M is isomorphic to a submodule of E^α for some $\alpha \in \mathbb{N}$.*

2.1.5 Local Cohomology Modules

In this subsection, we summarize topics of local cohomology which are used throughout this thesis and provide some important properties. For proofs and more details, we refer to [4] and [9].

Definition 2.1.28. *For an ideal I of R and an R-module M, we define $\Gamma_I(M)$ to be*

$$\Gamma_I(M) = \bigcup_{n\in\mathbb{N}} (0 :_M I^n) = \{m \in M \mid mI^n = 0 \text{ for some } n \in \mathbb{N}\}.$$

If $\phi : M \to N$ is an R-module homomorphism, then $\Gamma_I(\phi)$ is the restriction map $\Gamma_I(M) \to \Gamma_I(N)$. That is to say that $\Gamma_I(-)$ is a functor on the category of R-modules which is called I-torsion functor.

Definition 2.1.29. *Let M be an R-module. Take an injective resolution*

$$\mathbf{E} : 0 \to E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \xrightarrow{d^2} \cdots$$

of M, so that there is an R-module homomorphism $M \to E^0$ such that the sequence

$$0 \to M \to E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \xrightarrow{d^2} \cdots$$

is exact. Then apply the functor Γ_I to the complex $\mathbf{E}$ to obtain

$$0 \to \Gamma_I(E^0) \xrightarrow{\Gamma_I(d^0)} \Gamma_I(E^1) \xrightarrow{\Gamma_I(d^1)} \Gamma_I(E^2) \xrightarrow{\Gamma_I(d^2)} \cdots$$

The i-th cohomology of this complex is called the i-th local cohomology module of M with respect to I and denoted by $H^i_I(M)$, which is independent of the choice of injective resolution $\mathbf{E}$ up to isomorphism.

Remark 2.1.30. *The I-torsion functor is left exact, and so $H^0_I(M) \cong \Gamma_I(M)$.*

Another characterization of local cohomology modules is the following:

Theorem 2.1.31. *[4, 1.2.11 Theorem, 1.3.8 Theorem] Let M be an R-module and I be an ideal of R. Then*

$$\Gamma_I(M) \cong \varinjlim_{n\in\mathbb{N}} \operatorname{Hom}_R(R/I^n, M) \text{ and } H^i_I(M) \cong \varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^i_R(R/I^n, M).$$

Theorem 2.1.32. *[4, 4.3.2 Flat Base Change Theorem] Let M be an R-module, I be an ideal of R and $\nu : R \to S$ be a flat ring homomorphism. Then*

$$H^i_I(M) \otimes_R S \cong H^i_{IS}(M \otimes_R S)$$

for each $i \in \mathbb{N}_0$.

Corollary 2.1.33. *Let W be a multiplicatively closed subset of R and let J be an ideal of R. Then for each $i \in \mathbb{N}$*

$$W^{-1}H^i_I(M) \cong H^i_{IW^{-1}R}(W^{-1}M),$$

and

$$\widehat{H^i_I(M)}_J \cong H^i_{I\widehat{R}_J}(\widehat{M}_J).$$

Lemma 2.1.34. *[4, 11.2.3 Lemma] Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension n and E be the injective hull of its residue field. Then $E \cong H^n_{\mathfrak{m}}(R)$.*

An alternative definition of local cohomology modules is via use of Čech complex.

Notation 2.1.35. *For postive integers $k \leq n$, $I(k,n)$ will denote the set of k-tuples $\{i = (i(1), \ldots, i(k)) \mid 1 \leq i(1) < i(2) < \cdots < i(k) \leq n\}$. For an element $j \in I(k+1,n)$, $j^{\hat{s}}$ will denote the element $(j(1), \ldots, j(s-1), j(s+1), \ldots, j(k+1)) \in I(k,n)$, and by $a_1, \ldots, \hat{a}_i, \ldots, a_n$ we mean $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$.*

Proposition-Definition 2.1.36. *[4, 5.1.5 Proposition and Definition] The Čech complex of an R-module M with respect to an ideal $I = \langle a_1, \ldots, a_n \rangle \subseteq R$ is the following*

$$\boldsymbol{C} : 0 \to C(M)^0 \xrightarrow{d^0} C(M)^1 \xrightarrow{d^1} C(M)^2 \xrightarrow{d^2} \cdots \to C(M)^{n-1} \xrightarrow{d^{n-1}} C(M)^n \to 0$$

where

1. $C(M)^0 := M$
2. *for each* $k \in \{1, \ldots, n\}$, $C(M)^k := \bigoplus_{i \in I(k,n)} M_{a_{i(1)} \cdots a_{i(k)}}$
3. $d^0 : C(M)^0 \to C(M)^1$ *is to be such that the composition* $C(M)^0 \xrightarrow{d^0} C(M)^1 \xrightarrow{\rho_j} M_{a_j}$ *is just the natural map from* M *to* M_{a_j}*, where* ρ_j *is the canonical projection.*
4. *for* $1 \leq k < n$, $i \in I(k,n)$ *and* $j \in I(k+1,n)$ *the composition*

$$M_{a_{i(1)} \cdots a_{i(k)}} \to C(M)^k \xrightarrow{d^k} C(M)^{k+1} \to M_{a_{j(1)} \cdots a_{j(k+1)}}$$

(in which the first map is the canonical injection and the third map is the canonical projection) is the natural map from $M_{a_{i(1)} \cdots a_{i(k)}}$ *to* $M_{a_{j(1)} \cdots a_{j(k+1)}}$ *multiplied by* $(-1)^{s-1}$ *if* $i = j^{\hat{s}}$ *for an* $s \in \{1, \ldots, k+1\}$*, and it is zero map otherwise.*

Theorem 2.1.37. *[4, 5.1.20 Theorem] Let M be an R-module and $\boldsymbol{C}$ be its Čech complex with respect to ideal $I = \langle a_1, \ldots, a_n \rangle \subseteq R$. Then $H^i(\boldsymbol{C}) \cong H^i_I(M)$.*

Example 2.1.38. *[4, 13.5.3 Example] Let $R = \mathbb{k}[x_1, \dots, x_n]$ or $R = \mathbb{k}[\![x_1, \dots, x_n]\!]$ (the ring of polynomials or the ring of formal power series over a field $\mathbb{k}$) with maximal ideal $\mathfrak{m} = \langle x_1, \dots, x_n \rangle$. Then the top local cohomology module $H^n_{\mathfrak{m}}(R)$ can be computed using the Čech complex of R with respect to $\mathfrak{m}$. Therefore,*

$$H^n_{\mathfrak{m}}(R) \cong \operatorname{Coker}\Big(\bigoplus_{i=1}^{n} R_{x_1 \dots \hat{x}_i \dots x_n} \xrightarrow{d^{n-1}} R_{x_1 \dots x_n} \Big)$$

In addition, $R_{x_1 \dots x_n}$ is a k-vector space with base $\{x_1^{\alpha_1} \dots x_n^{\alpha_n} \mid \alpha_1, \dots, \alpha_n \in \mathbb{Z}\}$ and $R_{x_1 \dots \hat{x}_i \dots x_n}$ is a k-vector space with base $\{x_1^{\alpha_1} \dots x_n^{\alpha_n} \mid \alpha_1, \dots, \alpha_n \in \mathbb{Z}, \alpha_i \geq 0\}$. Thus, $\operatorname{Coker} d^{n-1}$ is the k-vector space with base $\{x_1^{\alpha_1} \dots x_n^{\alpha_n} \mid \alpha_1, \dots, \alpha_n \in -\mathbb{N}\}$, which is the module of inverse polynomials $k[x_1^-, \dots, x_n^-]$ whose R-module structure is extended from the following rule

$$(\lambda x_1^{\alpha_1} \dots x_n^{\alpha_n})(\mu x_1^{-\nu_1} \dots x_n^{-\nu_n}) = \begin{cases} \lambda\mu x_1^{-\nu_1+\alpha_1} \dots x_n^{-\nu_n+\alpha_n} & \text{if } \alpha_i < \nu_i \text{ for all } i \\ 0 & \text{if } \alpha_i \geq \nu_i \text{ for any } i \end{cases}$$

for all $\lambda, \mu \in \mathbb{k}$ and $\alpha_i \geq 0, \nu_i > 0$. Consequently, if $R = \mathbb{k}[\![x_1, \dots, x_n]\!]$, by Lemma 2.1.34, $E \cong k[x_1^-, \dots, x_n^-]$.

2.2 Modules over Rings of Prime Characteristic

Throughout this section all rings are of prime characteristic p. If R is a ring of prime characteristic p, then $(r+s)^{p^e} = r^{p^e} + s^{p^e}$ for all $r, s \in R$ and $e \in \mathbb{N}$. Consequently, the **Frobenius map** $f : R \to R$ defined by $f(r) = r^p$ becomes a ring homomorphism, and so does its e-th iteration $f^e : R \to R$ defined by $f^e(r) = r^{p^e}$ for any $e \in \mathbb{N}$.

2.2.1 General Prime Characteristic Tools

In this subsection, we provide some basics of positive characteristics techniques in commutative algebra which we use throughout this thesis. We also provide some well-known properties with proofs.

Definition 2.2.1. *Let M be an R-module and $e \in \mathbb{N}$. $F^e_* M = \{F^e_* m \mid m \in M\}$ denotes the Abelian group M with the induced R-module structure via the e-th iterated Frobenius map and it is given by*

$$rF_*^e m = F_*^e r^{p^e} m \text{ for all } m \in M \text{ and } r \in R$$

In particular, $F_*^e R$ is the Abelian group R with the induced R-module structure

$$rF_*^e s = F_*^e r^{p^e} s \text{ for all } r, s \in R.$$

Definition 2.2.2. *Let $I \subseteq R$ be an ideal and $e \in \mathbb{N}$, $I^{[p^e]}$ denotes the ideal generated by the set $\{r^{p^e} \mid r \in I\}$. Consequently, if $I = \langle r_1, \ldots, r_n \rangle$, then $I^{[p^e]} = \langle r_1^{p^e}, \ldots, r_n^{p^e} \rangle$.*

One can easily observe the following properties.

Remarks 2.2.3. *Let M and N be R-modules, I be an ideal of R and $e \in \mathbb{N}$.*

1. *$F_*^e(F_*^d M) = F_*^{e+d} M$ for all $d \in \mathbb{N}$.*

2. *$F_*^e R$ is a ring itself with an addition given by $F_*^e r + F_*^e s = F_*^e(r+s)$, and a multiplication given by $F_*^e r \cdot F_*^e s = F_*^e rs$ for all $r, s \in R$, i.e. $F_*^e R \cong R$ as rings.*

3. *$F_*^e M$ is an $F_*^e R$-module and the $F_*^e R$-module structure on $F_*^e M$ is given by $F_*^e r \cdot F_*^e m = F_*^e rm$ for all $m \in M$ and $r \in R$.*

4. *$IF_*^e M = F_*^e(I^{[p^e]} M)$.*

5. *If N is a submodule of M, then $F_*^e N$ is a submodule of $F_*^e M$ and $F_*^e M / F_*^e N$ and $F_*^e(M/N)$ are isomorphic as R-modules.*

6. *If $\phi : M \to N$ is an R-module homomorphism, then the map $F_*^e \phi : F_*^e M \to F_*^e N$ given by $(F_*^e \phi)(F_*^e m) = F_*^e(\phi(m))$ for all $m \in M$ is an $F_*^e R$-module homomorphism.*

7. *$F_*^e(-)$ is an exact functor on the category of R-modules.*

8. *If $\{M_i\}_{i \in I}$ is a family of R-modules, then we have $F_*^e(\prod_{i \in I} M_i) \cong \prod_{i \in I} F_*^e M_i$ and $F_*^e(\bigoplus_{i \in I} M_i) \cong \bigoplus_{i \in I} F_*^e M_i$ as R-modules.*

Proposition 2.2.4. *Let M be an R-module, W be a multiplicatively closed subset of R and I be a finitely generated ideal of R.*

1. *$F_*^e(W^{-1} M) \cong W^{-1}(F_*^e M)$ as $W^{-1} R$-modules.*

2. *If $\widehat{M_I}$ is the I-adic completion of M, then $F_*^e(\widehat{M_I}) \cong \widehat{F_*^e M}_I$ as $\widehat{R}_I$-modules.*

Proof. For *1.* we define $\phi : F_*^e W^{-1}M \to W^{-1}F_*^e M$ by $\phi\Big(F_*^e(\frac{m}{w})\Big) = \dfrac{F_*^e w^{p^e-1}m}{w}$ for all $\dfrac{m}{w} \in W^{-1}M$, and we claim that it is an isomorphism of $W^{-1}R$-modules. Assume that $F_*^e(\frac{m}{w}) = F_*^e(\frac{n}{s})$ for some $\frac{n}{s} \in W^{-1}M$. Then $\frac{m}{w} = \frac{n}{s}$, which implies that $tms = tnw$ for some $t \in W$. Therefore, $(tws)^{p^e-1}(tms) = (tws)^{p^e-1}(tnw)$, and so $tsF_*^e w^{p^e-1}m = twF_*^e s^{p^e-1}n$. This means that $\dfrac{F_*^e w^{p^e-1}m}{w} = \dfrac{F_*^e s^{p^e-1}n}{s}$, i.e. $\phi\Big(F_*^e(\frac{m}{w})\Big) = \phi\Big(F_*^e(\frac{n}{s})\Big)$. This shows that ϕ is well defined. Now for any element $\frac{r}{s} \in W^{-1}R$, we have

$$\begin{aligned}
\phi\Big(\frac{r}{s}F_*^e(\frac{m}{w})\Big) &= \phi\Big(F_*^e(\frac{r^{p^e}m}{s^{p^e}w})\Big) = \frac{F_*^e(s^{p^e-1})^{p^e}w^{p^e-1}r^{p^e}m}{s^{p^e}w} \\
&= \frac{s^{p^e-1}rF_*^e w^{p^e-1}m}{s^{p^e}w} = \frac{rF_*^e w^{p^e-1}m}{sw} \\
&= \frac{r}{s}\frac{F_*^e w^{p^e-1}m}{w} = \frac{r}{s}\phi\Big(F_*^e(\frac{m}{w})\Big),
\end{aligned}$$

i.e. ϕ is $W^{-1}R$-linear. Notice also that for any element $\dfrac{F_*^e m}{w} \in W^{-1}F_*^e M$, we have

$$\frac{F_*^e m}{w} = \frac{w^{p^e-1}F_*^e m}{w^{p^e}} = \frac{F_*^e(w^{p^e})^{p^e-1}m}{w^{p^e}} = \phi\Big(F_*^e(\frac{m}{w^{p^e}})\Big).$$

If also $\phi\Big(F_*^e(\frac{m}{w})\Big) = \dfrac{F_*^e w^{p^e-1}m}{w} = 0$, then there exist an element $s \in W$ such that $sF_*^e w^{p^e-1}m = 0$. Thus, $F_*^e s^{p^e} w^{p^e-1}m = 0$, and so $s^{p^e}w^{p^e-1}m = 0$. This means that $\frac{m}{w} = 0$, i.e. $F_*^e(\frac{m}{w}) = 0$. Hence, ϕ is surjective and injective.

For *2.* since the filtrations $\{F_*^e I^n M\}_{n\geq 0}$ and $\{F_*^e(I^n)^{[p^e]}M\}_{n\geq 0}$ are cofinal by Lemma 2.1.12,

$$\begin{aligned}
\widehat{F_*^e M}_I &= \varprojlim \frac{F_*^e M}{I^n F_*^e M} = \varprojlim \frac{F_*^e M}{F_*^e(I^n)^{[p^e]}M} \\
&\cong \varprojlim \frac{F_*^e M}{F_*^e I^n M} = F_*^e \varprojlim \frac{M}{I^n M} = F_*^e \widehat{M}_I.
\end{aligned}$$

□

Proposition 2.2.5. *Let $\mathcal{C}$ be a subset of R. Then*

1. *R is a free $R^{p^e} := \{r^{p^e} \mid r \in R\}$-module with basis $\mathcal{C}$ if and only if F^e_*R is a free R-module with basis $F^e_*\mathcal{C} = \{F^e_*\lambda \mid \lambda \in \mathcal{C}\}$.*

2. *If R is a free R^{p^e}-module with basis $\mathcal{C}$ and S is the polynomial ring $R[x_1, \dots, x_n]$, then F^e_*S is a free S-module with basis*
$$\mathcal{B} = \{F^e_*\lambda x_1^{\alpha_1} \dots x_n^{\alpha_n} \mid \lambda \in \mathcal{C} \text{ and } 0 \leq \alpha_1, \dots, \alpha_n \leq p^e - 1\}.$$

3. *If R is a free R^{p^e}-module with a finite basis $\mathcal{C}$, and S is the power series ring $R[\![x_1, \dots, x_n]\!]$, then F^e_*S is a free S-module with basis*
$$\mathcal{B} = \{F^e_*\lambda x_1^{\alpha_1} \dots x_n^{\alpha_n} \mid \lambda \in \mathcal{C} \text{ and } 0 \leq \alpha_1, \dots, \alpha_n \leq p^e - 1\}.$$

Proof. For any finite subset Λ of $\mathcal{C}$, and for any $r \in R$, we have $r = \sum_{\lambda\in\Lambda} r_\lambda^{p^e}\lambda \Leftrightarrow F^e_*r = \sum_{\lambda\in\Lambda} r_\lambda F^e_*\lambda$ where $r_\lambda \in R$ for all $\lambda \in \Lambda$. Then the proof of *1.* follows.

For the proof of *2.* we shall show that it holds for $S = R[x]$, then the result follows by induction. Assume that R is free as an R^{p^e}-module with basis $\mathcal{C}$. Since every $n \in \mathbb{N}$ can be written as $n = up^e + \alpha$ where $u, \alpha \in \mathbb{N}$ and $0 \leq \alpha < p^e$, any term rx^n has a unique expression $\sum_{\lambda\in\Lambda} r_\lambda^{p^e}\lambda(x^u)^{p^e}x^\alpha$ for some finite subset Λ of $\mathcal{C}$ and for some $r_\lambda \in R$. Then $F^e_*rx^n$ can be written uniquely as $\sum_{\lambda\in\Lambda} r_\lambda x^u F^e_*\lambda x^\alpha$. Therefore, since every polynomial in S is a finite linear combination of monomials with coefficients in R, any element in F^e_*S can be written uniquely as S-linear combination of elements from $\{F^e_*\lambda x^\alpha \mid \lambda \in \mathcal{C}$ and $0 \leq \alpha < p^e\}$, i.e. this set generates F^e_*S as an S-module freely.

For the proof of *3.* we will similarly show that it holds for $S = R[\![x]\!]$, and the result follows by induction. Assume that R is free as an R^{p^e}-module with basis $\mathcal{C}$. Let $\mathcal{C} = \{\lambda_1, \cdots, \lambda_m\}$ and $g = \sum_{i=0}^\infty r_i x^i \in R[\![x]\!]$. Since every $n \in \mathbb{N}$ can be written as $n = up^e + \alpha$ where $u, \alpha \in \mathbb{N}$ and $0 \leq \alpha < p^e$, every term $r_n x^n$ of g has a unique expression $\sum_{j=1}^m r_{(n,\lambda_j)}^{p^e}\lambda_j(x^u)^{p^e}x^\alpha$, and so $F^e_*r_nx^n$ is uniquely expressed as $\sum_{j=1}^m r_{(n,\lambda_j)}x^uF^e_*\lambda_jx^\alpha$ for some $r_{(n,\lambda_j)} \in R$. Then F^e_*g can be written uniquely as $\sum_{k=0}^{p^e-1}\sum_{j=1}^m g_{k\lambda_j}F^e_*\lambda_jx^k$ where $g_{k\lambda_j} = \sum_{i=0}^\infty r_{(ip^e+k,\lambda_j)}x^i \in S$. This shows that any element in F^e_*S can be written uniquely as S-linear combination of elements from $\{F^e_*\lambda_jx^\alpha \mid 1 \leq j \leq m$ and $0 \leq \alpha < p^e\}$, i.e. this set generates F^e_*S as an S-module freely. □

In general, if S is the power series ring $R[\![x_1, \ldots, x_n]\!]$, the set $\mathcal{B}$ in Proposition 2.2.5 *3.* does not have to generate $F_*^e S$ freely as an S-module. The following example shows that why we need the finiteness condition of the basis set $\mathcal{C}$.

Example 2.2.6. *Let $S = \Bbbk[\![x]\!]$ be the power series ring over a field $\Bbbk$ of prime characteristic p and $\mathcal{C}$ be an infinite free basis of $\Bbbk$ as $\Bbbk^p$ vector space. We claim that the set $\mathcal{B} = \{F_*\lambda x^\alpha \mid \lambda \in \mathcal{C} \text{ and } 0 \le j \le p-1\}$ is not a free basis for F_*S as an S-module. Let $g = \sum_{n=0}^{\infty} a_n x^n \in S$ such that $a_i \neq a_j$ whenever $i \neq j$. Then for every a_n, there exist a finite subset Λ_n of $\mathcal{C}$ such that a_n can be written uniquely as $\sum_{\lambda\in\Lambda_n} r_\lambda^p \lambda$. Since every $n \in \mathbb{N}$ can be written as $n = up + \alpha$ where $u, \alpha \in \mathbb{N}$ and $0 \le \alpha \le p-1$, $F_*a_n x^n$ can be written uniquely as $\sum_{\lambda\in\Lambda_n} r_\lambda x^u F_*\lambda x^\alpha$. This means that $F_*g = \sum_{n=0}^{\infty}\sum_{\lambda\in\Lambda_n} r_\lambda x^{u_n} F_*\lambda x^{\alpha_n}$ where $n = u_n p + \alpha_n$ for some $u_n, \alpha_n \in \mathbb{N}$ and $0 \le \alpha_n < p$. On the other hand, since $\mathcal{C}$ is infinite we have $\Lambda_i \neq \Lambda_j$ almost for all $a_i \neq a_j$. Therefore, F_*g is an infinite S-linear combination of elements from $\mathcal{B}$. Hence, $\mathcal{B}$ is not a free basis of F_*S even though it generates F_*S as an S-module.*

Definition 2.2.7. *R is said to be F-finite if the e-th Frobenius homomorphism makes R into a finitely generated module over the subring $R^{p^e} := \{r^{p^e} \mid r \in R\}$ (or equivalently that $F_*^e R$ is a finitely generated R-module) for any $e \in \mathbb{N}$.*

Proposition 2.2.8. *If R is an F-finite ring, then*

1. *R/I is F-finite for any ideal I of R,*
2. *any localization of R is F-finite,*
3. *$R[x_1, \ldots, x_n]$ and $R[\![x_1, \ldots, x_n]\!]$ are F-finite.*

Proof. Assume that R is F-finite. Let F_*R is generated by $\{F_*\lambda_1, \ldots, F_*\lambda_m\}$ as an R-module. Notice that $F_*(R/I)$ is generated by $\{F_*(\lambda_1 + I), \ldots, F_*(\lambda_m + I)\}$ as an R/I-module, and so R/I is F-finite. For *2.* let W be a multiplicative subset of R. Then $W^{-1}F_*R$ is generated by $\{F_*\lambda_1, \ldots, F_*\lambda_m\}$ as an $W^{-1}R$-module. However, $W^{-1}F_*R \cong F_*W^{-1}R$ by Proposition 2.2.4, and so any localization of R is F-finite. And *3.* follows from Proposition 2.2.5. □

One of the most important flavour of rings of prime characteristic p is the regularity criterion of E. Kunz.

Theorem 2.2.9. *[15, Corollary 2.7] R is regular if and only if R is reduced and F_*R is a flat R-module.*

Theorem 2.2.10. *[15, Theorem 3.3] R is a regular local ring if and only if*

$$l_R(R/\mathfrak{m}^{[p^e]}) = p^{e \dim R} \text{ for some } e \in \mathbb{N}.$$

2.2.2 Modules over the Frobenius Skew Polynomial Ring

In this subsection, we will provide a brief description of the Frobenius skew polynomial rings and modules over such rings. For further details we refer to [**?**] and [21].

Definition 2.2.11. *The Frobenius skew polynomial ring over R is the skew polynomial ring $R[X; f]$ associated to R and the Frobenius map f in the indeterminate X over R, whose multiplication is subject to the rule $Xr = f(r)X = r^pX$ for all $r \in R$.*

Remark 2.2.12. *The Frobenius skew polynomial ring $R[X; f]$ is the free left R-module $\bigoplus_{i=0}^{\infty} RX^i$, and so consist of all polynomials $\sum_{i=0}^{n} r_iX^i$, where $n \in \mathbb{N}_0$ and $r_0, \cdots, r_n \in R$.*

Definition 2.2.13. *Let M be an R-module. An e-th Frobenius map on M is an R-linear map $\varphi : M \to F_*^eM$, or equivalently an additive map $\phi : M \to M$ such that $\phi(rm) = r^{p^e}\phi(m)$ for all $r \in R$ and $m \in M$, where φ and ϕ are related by the formula $\varphi(m) = F_*^e\phi(m)$ for all $m \in M$.*

Remark 2.2.14. *[21, c.f. Discussion 1.6] For given an e-th Frobenius map ϕ on an R-module M, we can turn M into a left $R[X; f^e]$-module by extending the rule $Xm = \phi(m)$ for all $m \in M$, where $X(rm) = \phi(rm) = r^{p^e}\phi(m) = r^{p^e}Xm = f^e(r)Xm = (Xr)m$ for all $r \in R$ and $m \in M$. Conversely, if an R-module M has a left $R[X; f^e]$-module structure, then $X : M \to M$ is an e-th Frobenius map.*

One of the most important examples of modules with Frobenius map is the following.

Example 2.2.15. *Let the situation and notation be as in Example 2.1.38. Then the map $T : E \to E$ defined by $T(\lambda x_1^{-\nu_1} \dots x_n^{-\nu_n}) = \lambda^p x_1^{-p\nu_1} \dots x_n^{-p\nu_n}$ for all $\lambda \in \Bbbk$ and $\nu_1, \dots \nu_n \in \mathbb{N}$ is a Frobenius map on E, which we call it the natural Frobenius map, and so E is a left $R[T; f]$-module. We can further extend this to a natural*

$R[T;f]$-module structure on E^α which is given by

$$T\begin{pmatrix} a_1 \\ \vdots \\ a_\alpha \end{pmatrix} = \begin{pmatrix} Ta_1 \\ \vdots \\ Ta_\alpha \end{pmatrix}$$

Remark 2.2.16. *Note that $F_*^e R$-module structure of $\operatorname{Hom}_R(M, F_*^e M)$ is defined as follow*

$$(F_*^e r.\Theta)(-) = F_*^e r\Theta(-)$$

for all $r \in R$ and $\Theta \in \operatorname{Hom}_R(M, F_*^e M)$.

Definition 2.2.17. *Let M be an R-module. An e-th Cartier map on M is an R-linear map $\psi : F_*^e M \to M$, or equivalently an additive map $C : M \to M$ such that $C(mr^{p^e}) = C(m)r$ for all $r \in R$ and $m \in M$, where ψ and C are related by the formula $C(m) = \psi(F_*^e m)$ for all $m \in M$.*

Remark 2.2.18. *[21, c.f. Discussion 1.7] For given an e-th Cartier map C on an R-module M, we can turn M into a right $R[X;f^e]$-module by extending the rule $mX = C(m)$ for all $m \in M$, where $(mX)r = C(m)r = C(mr^{p^e}) = mr^{p^e}X = mf^e(r)X = m(Xr)$ for all $r \in R$ and $m \in M$. Conversely, if an R-module M has a right $R[X;f^e]$-module structure, then $X : M \to M$ is an e-th Cartier map.*

Remark 2.2.19. *Note also that the $F_*^e R$-module structure of $\operatorname{Hom}_R(F_*^e M, M)$ is defined as follow*

$$F_*^e r.\phi(-) = \phi(F_*^e r.-)$$

for all $r \in R$ and $\phi \in \operatorname{Hom}_R(F_*^e M, M)$.

2.2.3 The Frobenius Functor

In this subsection, we give definition and some properties of the Frobenius functor of Peskine and Szpiro introduced in [18].

Definition 2.2.20. *Let M be an R-module. The Frobenius functor F_R from the category of R-modules to itself is defined by $F_R(M) := F_*R \otimes_R M$ where $F_R(M)$*

acquires its R-module structure via the identification of F_*R with R. The resulting R-module structure on $F_R(M)$ satisfies

$$s(F_*r \otimes m) = F_*sr \otimes m \text{ and } F_*s^p r \otimes m = F_*r \otimes sm$$

for all $r, s \in R$ and $m \in M$. The e-th iteration of F_R is denoted by F_R^e, and it is clearly given by $F_R^e(M) = F_*^e R \otimes_R M$.

Remarks 2.2.21. *[16, Remarks 1.0]*

1. *F_R^e commutes with arbitrary direct sums because the tensor product does.*
2. *It is easy to see that the map $\phi : R \to F_R^e(R)$ given by $r \mapsto F_*^e r \otimes 1$ is an R-module isomorphism. If $\Phi : R^\beta \to R^\alpha$ is an R-module homomorphism represented by an $\alpha \times \beta$ matrix A, then by the isomorphism ϕ, $F_R^e(\Phi) : R^\beta \to R^\alpha$ is an R-module homomorphism represented by the matrix $A^{[p^e]}$ which is obtained from A by raising its entries to the p^e-th power.*
3. *F_R^e commutes with limits because the tensor product does.*
4. *If I is an ideal of R, then F_R^e commutes with the torsion functor $\Gamma_I(-)$.*
5. *F_R^e commutes with localization.*

Remarks 2.2.22. *[16, Remarks 1.0] When R is regular the Frobenius functor becomes a useful tool because of the fact that it is exact by Theorem 2.2.9. In this case, we have the following.*

1. *By Remarks 2.2.21 1. and exactness of F_R^e, it commutes with arbitrary sums of submodules and finite intersection of submodules.*
2. *Using the isomorphism in Remarks 2.2.21 2. and exactness of F_R^e, we obtain $F_R^e(I) \cong I^{[p^e]}$ and $R/I^{[p^e]} \cong F_R^e(R/I)$ for any ideal I of R.*
3. *Because of the fact that F_R^e is exact, it commutes with the cohomology of complexes.*
4. *If M and N R-modules with M being finitely generated, then $F_R^e(\mathrm{Ext}_R^i(M, N)) \cong \mathrm{Ext}_R^i(F_R^e(M), F_R^e(N))$ which is induced by the R-module isomorphism*

$$F_R^e(\mathrm{Hom}_R(P, N)) \xrightarrow{1\otimes f \mapsto id \otimes f} \mathrm{Hom}_R(F_R^e(P), F_R^e(N))$$

where P is a finitely generated free R-module.

2.2.4 The $\mathrm{I}_e(-)$ Operation and The $\star$-closure

In this subsection, we will give definitions of $\mathrm{I}_e(-)$ operation and $\star$-closure, and some properties of them. To do this we need the property that $F^e_* R$ are intersection flat R-modules for all $e \in \mathbb{N}$.

Definition 2.2.23. *An R-module M is intersection flat if it is flat and for all sets of R-submodules $\{N_\lambda\}_{\lambda\in\Lambda}$ of a finitely generated R-module N,*

$$M \otimes_R \bigcap_{\lambda\in\Lambda} N_\lambda = \bigcap_{\lambda\in\Lambda} (M \otimes_R N_\lambda)$$

Henceforth in this section R will denote a regular ring with the property that $F^e_* R$ are intersection flat R-modules for all $e \in \mathbb{N}$.

Remark 2.2.24. *Since intersection flat R-modules include R and closed under arbitrary direct sum, free R-modules are intersection flat. For instance, $F^e_* R$ are intersection flat for polynomial rings over a field of prime characteristic p. In addition, for all complete regular local rings of prime characteristic p, $F^e_* R$ are intersection flat [10, cf. Proposition 5.3]. Because of regularity, these rings have the property that for any collection of ideals $\{A_\lambda\}_{\lambda\in\Lambda}$ of R,*

$$(\cap_{\lambda\in\Lambda} A_\lambda)^{[p^e]} \cong F^e_R(\cap_{\lambda\in\Lambda} A_\lambda) \cong \cap_{\lambda\in\Lambda} F^e_R(A_\lambda) \cong \cap_{\lambda\in\Lambda} A_\lambda^{[p^e]},$$

and this is enough to define the minimal ideal $J \subseteq R$ with the property $A \subseteq J^{[p^e]}$.

Proposition-Definition 2.2.25. *Let $e \in \mathbb{N}$.*

1. *For an ideal $A \subseteq R$ there exists a minimal ideal $J \subseteq R$ with the property $A \subseteq J^{[p^e]}$. We denote this minimal ideal by $\mathrm{I}_e(A)$.*
2. *Let $u \in R$ be a non zero element and $A \subseteq R$ an ideal. The set of all ideals $B \subseteq R$ which contain A and satisfy $uB \subseteq B^{[p^e]}$ has a unique minimal element. We call this ideal the star closure of A with respect to u and denote it by $A^{\star^e u}$.*

Proof. We refer to section 5 in [10]. □

Definition 2.2.26. *Let $e \in \mathbb{N}$.*

1. *Given any matrix (or vector) V with entries in R, we define $V^{[p^e]}$ to be the matrix obtained from V by raising its entries to the p^e-th power.*

2. *Given any submodule $K \subseteq R^\alpha$, we define $K^{[p^e]}$ to be the R-submodule of R^α generated by $\{v^{p^e} \mid v \in K\}$.*

The Proposition-Definition below extends the $\mathrm{I}_e(-)$ operation and $\star$-closure defined on ideals to submodules of free R-modules.

Proposition-Definition 2.2.27. *Let $e \in \mathbb{N}$.*

1. *Given a submodule $K \subseteq R^\alpha$ there exists a minimal submodule $L \subseteq R^\alpha$ for which $K \subseteq L^{[p^e]}$. We denote this minimal submodule $\mathrm{I}_e(K)$.*

2. *Let U be an $\alpha \times \alpha$ matrix with entries in R and $V \subseteq R^\alpha$. The set of all submodules $K \subseteq R^\alpha$ which contain V and satisfy $UK \subseteq K^{[p^e]}$ has a unique minimal element. We call this submodule the star closure of V with respect to U and denote it $V^{\star^e U}$.*

Proof. For the proof of *1.* we refer to section 3 of [13]. For the proof of *2.* we shall construct a similar method to that in section 3 of [13]. Let $V_0 = V$ and $V_{i+1} = I_e(UV_i) + V_i$. Then $\{V_i\}_{i\geq 0}$ is an ascending chain and it stabilizes, since R is Noetherian, i.e. $V_j = V_{j+k}$ fo all $k > 0$ for some $j \geq 0$. Therefore, $V_j = I_e(UV_j) + V_j$ implies $I_e(UV_j) \subseteq V_j$, and so $UV_j \subseteq V_j^{[p^e]}$. We show the minimality of V_j by induction on i. Let Z be any submodule of R^α containing V with the property that $UZ \subseteq Z^{[p^e]}$. Then we clearly have $V_0 = V \subseteq Z$, and suppose that $V_i \subseteq Z$ for some i. Thus, $UV_i \subseteq UZ \subseteq Z^{[p^e]}$, which implies $I_e(UV_i) \subseteq Z$ and so $V_{i+1} \subseteq Z$. Hence, $V_j \subseteq Z$. □

For the calculation of $\mathrm{I}_e(-)$ operation, we first fix a free basis $\mathcal{B}$ for R as an R^{p^e}-module, then every element $v \in R^\alpha$ can be expressed uniquely in the form $v = \sum_{b\in\mathcal{B}} u_b^{[p^e]} b$ where $u_b \in R^\alpha$ for all $b \in \mathcal{B}$.

Proposition 2.2.28. *[14, Proposition 2.3] Let $e > 0$.*

1. *For any submodules $V_1, \ldots, V_n$ of R^α, $\mathrm{I}_e(V_1 + \cdots + V_n) = \mathrm{I}_e(V_1) + \cdots + \mathrm{I}_e(V_n)$.*

2. *Let $\mathcal{B}$ be a free basis for R as R^{p^e}-module. Let $v \in R^\alpha$ and $v = \sum_{b\in\mathcal{B}} u_b^{[p^e]} b$ be the unique expression for v where $u_b \in R^\alpha$ for all $b \in \mathcal{B}$. Then $\mathrm{I}_e(\langle v \rangle)$ is the submodule of R^α generated by $\{u_b \mid b \in \mathcal{B}\}$.*

The behaviour of the $\mathrm{I}_e(-)$ operation under localization is very crucial for our results. The following lemma shows that it commutes with localization.

Lemma 2.2.29. *[14, Lemma 2.5] Let $\mathcal{R}$ be a localization of R or a completion at a prime ideal. For all $e \in \mathbb{N}$, and all submodules $K \subseteq R^\alpha$, $\mathrm{I}_e(K \otimes_R \mathcal{R})$ exists and equals to $\mathrm{I}_e(K) \otimes_R \mathcal{R}$.*

Lemma 2.2.30. *Let $e \in \mathbb{N}$, U be a non-zero $\alpha \times \alpha$ matrix with entries in R and $K \subseteq R^\alpha$ a submodule. For any prime ideal $P \subseteq R$,*

$$(\widehat{K_P})^{\star^e U} = (\widehat{K^{\star^e U}})_P.$$

Proof. Define inductively $K_0 = K$ and $K_{i+1} = I_e(UK_i) + K_i$, and also $L_0 = \widehat{K_P}$ and $L_{i+1} = I_e(UL_i) + L_i$ for all $i \geq 0$. Since $\mathrm{I}_e(-)$ operation commutes with localization and completion, an easy induction shows that $L_i = \widehat{(K_i)_P}$, and the result follows. □

2.2.5 Lyubeznik's F-modules

Let R be a regular ring. In this subsection, we will give a brief summary of Lyubeznik's F-modules and their properties which we use in upcoming chapters. For the proofs and details we refer to [16].

Definitions 2.2.31. *An F-module is an R-module $\mathcal{M}$ equipped with an R-module isomorphism $\theta : \mathcal{M} \to F_R(\mathcal{M})$ which we call the structure isomorphism of $\mathcal{M}$.*

An F-module homomorphism is an R-module homomorphism $\phi : \mathcal{M} \to \mathcal{M}'$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\phi} & \mathcal{M}' \\ \theta\downarrow & & \downarrow\theta' \\ F_R(\mathcal{M}) & \xrightarrow[F_R(\phi)]{} & F_R(\mathcal{M}') \end{array}$$

where θ and θ' are the structure isomorphisms of $\mathcal{M}$ and $\mathcal{M}'$, respectively.

A generating morphism of an F-module $\mathcal{M}$ is an R-module homomorphism $\beta : M \to F_R(M)$, where M is an R-module, such that $\mathcal{M}$ is the limit of the inductive system in top row of commutative diagram

$$\begin{array}{ccccccc} M & \xrightarrow{\beta} & F_R(M) & \xrightarrow{F_R(\beta)} & F_R^2(M) & \xrightarrow{F_R^2(\beta)} & \cdots \\ \beta\downarrow & & F_R(\beta)\downarrow & & F_R^2(\beta)\downarrow & & \\ F_R(M) & \xrightarrow[F_R(\beta)]{} & F_R^2(M) & \xrightarrow[F_R^2(\beta)]{} & F_R^3(M) & \xrightarrow[F_R^3(\beta)]{} & \cdots \end{array}$$

and the structure isomorphism of $\mathcal{M}$ is induced by the vertical arrows in this diagram.

The structure isomorphism of an F-module $\mathcal{M}$ is automatically its generating morphism, and so every F-module has at least one generating morphism.

Definition 2.2.32. *An F-module $\mathcal{M}$ is called F-finite if it has a generating morphism $\beta : M \to F_R(M)$ with M a finitely generated R-module. In addition, if β is injective, M is called a root of $\mathcal{M}$ and β is called a root morphism of $\mathcal{M}$.*

Proposition 2.2.33. *[16, Proposition 2.3] Let $\beta : M \to F_R(M)$ be a generating morphism of an F-finite F-module $\mathcal{M}$ and let β_i be the following composition*

$$M \xrightarrow{\beta} F_R(M) \xrightarrow{F_R(\beta)} F_R^2(M) \xrightarrow{F_R^2(\beta)} \cdots \xrightarrow{F_R^{i-1}(\beta)} F_R^i(M).$$

Then:

1. *The ascending chain $\ker \beta_1 \subseteq \ker \beta_2 \subseteq \cdots$ stabilizes at the first integer i where we get $\ker \beta_i = \ker \beta_{i+1}$.*

2. *$\operatorname{Im} \beta_i \cong M/\ker \beta_i$ is a root of $\mathcal{M}$ where $\ker \beta_i$ is the stable kernel of the ascending chain in 1. and $\mathcal{M} = 0$ if it has a zero root.*

Next we gather some important properties of F-finite F-modules which are proved in [16].

Theorem 2.2.34. *[16, Theorem 1.4] Let $\mathcal{M}$ be an F-module. Then*

$$\operatorname{inj.dim}_R \mathcal{M} \leq \dim_R \operatorname{Supp} \mathcal{M}.$$

In particular, if $\dim_R \operatorname{Supp} \mathcal{M} = 0$, then $\mathcal{M}$ is injective as an R-module.

Remarks 2.2.35. *[16, Section 2] Let $\mathcal{M}$ be a F-finite F-module. Then we have the following.*

1. *Every F-finite module $\mathcal{M}$ has a root.*

2. *If $\mathcal{N}$ is an F-submodule of $\mathcal{M}$ and M is a root of $\mathcal{M}$ then $\mathcal{N}$ is F-finite and $N = \mathcal{N} \cap M$ is a root of $\mathcal{N}$. Also, $\mathcal{M}/\mathcal{N}$ is F-finite and M/N is a root of $\mathcal{M}/\mathcal{N}$.*

3. *If M is a root of $\mathcal{M}$, then there is a one-to-one correspondence between the F-submodules $\mathcal{N}$ of $\mathcal{M}$ and the R-submodules N of M such that $\mathcal{N}$ corresponds to $M \cap \mathcal{N}$.*

4. *If $I \subseteq R$ is an ideal, then the local cohomology module $H^i_I(\mathcal{M})$ is F-finite for any i.*

5. *All the Bass numbers of $\mathcal{M}$ are finite.*

6. *If R is a finitely generated algebra over a Noetherian local ring of characteristic p, then $\mathcal{M}$ has finite length in the category of F-modules.*

Example 2.2.36. *Any R-module isomorphism $\phi : R \to F_R(R)$ makes R into an F-module. In particular, the canonical isomorphism*

$$\phi : R \to F_*R \otimes_R R = F_R(R) \text{ defined by } r \mapsto F_*r \otimes 1.$$

Furthermore, R is clearly F-finite F-module. This makes local cohomology modules $H^i_I(R)$ with support on an ideal $I \subseteq R$ into F-finite F-modules. Therefore, there exist a finitely generated module M and an injective map $\beta : M \to F_R(M)$ such that

$$H^i_I(R) = \varinjlim (M \xrightarrow{\beta} F_R(M) \xrightarrow{F_R(\beta)} F^2_R(M) \xrightarrow{F^2_R(\beta)} \cdots)$$

where $\beta : M \to F_R(M)$ is a root morphism.

2.2.6 The Δ^e and Ψ^e Functors

In this subsection, we recall the notions of Δ^e and Ψ^e functors which was defined in Section 3 of [10]. Let R denote a complete local ring and E the injective hull of its residue field. Let $\mathfrak{C}^e$ be the category of Artinian $R[\theta; f^e]$-modules and $\mathfrak{D}^e$ be the category of R-linear maps $M \to F^e_R(M)$ where M is Noetherian R-module and a morphism between $M \to F^e_R(M)$ and $N \to F^e_R(N)$ is a commutative diagram of R-linear maps

$$\begin{array}{ccc} M & \xrightarrow{\phi} & N \\ \downarrow & & \downarrow \\ F^e_R(M) & \xrightarrow{F^e_R(\phi)} & F^e_R(N) \end{array}$$

We define the functor $\Delta^e : \mathfrak{C}^e \to \mathfrak{D}^e$ as follows: given an e-th Frobenius map $\theta : M \to M$, we can obtain an R-linear map $\phi : F_*^e R \otimes M \to M$ such that $\phi(F_*^e r \otimes m) = r\theta(m)$ for all $r \in R$, $m \in M$. Applying Matlis duality to this map gives the R-linear map $M^\vee \to (F_*^e R \otimes M)^\vee \cong F_*^e R \otimes M^\vee$ where the last isomorphism is described in Lemma 4.1 in [16].

Conversely, we define the functor $\Psi^e : \mathfrak{D}^e \to \mathfrak{C}^e$ as follows: given a Noetherian R-module N with an R-linear map $N \to F_R^e(N)$. Applying Matlis duality to this map gives the R-linear map $\varphi : F_R^e(N^\vee) \cong F_R^e(N)^\vee \to N^\vee$ where the first isomorphism is the composition $F_R^e(N^\vee) \cong F_R^e(N^\vee)^{\vee\vee} \cong F_R^e(N^{\vee\vee})^\vee \cong F_R^e(N)^\vee$. Then we define the action of θ on $N^\vee$ by defining $\theta(n) = \varphi(1 \otimes n)$ for all $n \in N^\vee$.

The mutually inverse exact functors Δ^e and Ψ^e are extensions of Matlis duality which also keep track of Frobenius actions. For the details we refer to [10].

Chapter 3

Annihilators of Modules with a Frobenius Map

Throughout this chapter R will denote a polynomial ring in finitely many variables over a field $\mathbb{k}$ of prime characteristic p, i.e. $R = \mathbb{k}[x_1, \ldots, x_n]$. In this chapter, we investigate the algorithms described in [12] and [13]. We present our results on these algorithms, and generalize the algorithm described in [13] to polynomial rings. We finish the chapter with an application to Lyubeznik's F-finite F-modules.

3.1 The Katzman-Schwede Algorithm

The purpose of this section is to redefine the algorithm described in [12] with a more algebraic language and show that it commutes with localization. Let $e \in \mathbb{N}$.

Definition 3.1.1. *For any R-linear map $\phi : F^e_* R \to R$, we say that an ideal $J \subseteq R$ is ϕ-compatible if $\phi(F^e_* J) \subseteq J$.*

Given ϕ which is compatible with J as above definition, there is always a commutative diagram

$$\begin{array}{ccc} F^e_* R & \xrightarrow{\phi} & R \\ \downarrow & & \downarrow \\ F^e_*(R/J) & \xrightarrow{\phi'} & R/J \end{array}$$

where the vertical arrows are the canonical surjections.

Lemma 3.1.2. *[12, Lemma 2.4] Assuming a commutative diagram as above, the ϕ-compatible ideals containing J are in the bijective correspondence with the ϕ'-compatible ideals of R/J, where ϕ' is the induced map $F^e_*(R/J) \xrightarrow{\phi'} R/J$ as in above diagram.*

Next we will explain the $F^e_* R$-module structure of $\mathrm{Hom}_R(F^e_* R, R)$, which is crucial for our computational techniques in this thesis.

Remark 3.1.3. *Let $\mathcal{C}$ be a base for $\Bbbk$ as a $\Bbbk^{p^e}$-vector space which includes the identity element of $\Bbbk$. By Proposition 2.2.5, $F^e_* R$ is a free R-module with the basis set*

$$\mathcal{B} = \{F^e_* \lambda x_1^{\alpha_1} \dots x_n^{\alpha_n} \mid 0 \le \alpha_1, \dots, \alpha_n < p^e, \lambda \in \mathcal{C}\}.$$

Lemma 3.1.4. *[3, cf. Example 3.0.5] Let $\pi_e : F^e_* R \to R$ be the projection map onto the free summand $RF^e_* x_1^{p^e-1} \dots x_n^{p^e-1}$. Then $\mathrm{Hom}_R(F^e_* R, R)$ is generated by π_e as an $F^e_* R$-module.*

Proof. For each basis element $F^e_* \lambda x_1^{\alpha_1} \dots x_n^{\alpha_n} \in \mathcal{B}$, the projection map onto the free summand $RF^e_* \lambda x_1^{\alpha_1} \dots x_n^{\alpha_n}$ is defined by the rule $F^e_* z.\pi_e(-) = \pi_e(F^e_* z.-)$, where $z = \lambda^{-1} x_1^{p^e-1-\alpha_1} \dots x_n^{p^e-1-\alpha_n}$. Since we can obtain all of the projections in this way, the map

$$\Phi : F^e_* R \to \mathrm{Hom}_R(F^e_* R, R) \text{ defined by } \Phi(F^e_* u) = \phi_u,$$

where $\phi_u : F^e_* R \to R$ is the R-linear map $\phi_u(-) = \pi_e(F^e_* u-)$, is surjective. On the other hand, if $\Phi(F^e_* u) = 0$ for some $u \in R$, then we have

$$\phi_u(F^e_* r) = \pi_e(F^e_* ur) = F^e_* u.\pi_e(F^e_* r) = 0 \text{ for all } r \in R.$$

This means that $F^e_* u$ must be zero, and so Φ is injective. Hence, Φ is an $F^e_* R$ isomorphism. In other words, π_e generates $\mathrm{Hom}_R(F^e_* R, R)$ as an $F^e_* R$-module. $\square$

Definition 3.1.5. *Let the notation and situation be as in Lemma 3.1.4. We call the map π_e in Lemma 3.1.4 the trace map on F_*^eR, or just the trace map when the content is clear.*

Next lemma provides an important property of the trace map π_e which gives the relation between elements of $\mathrm{Hom}_R(F_*^eR, R)$ and $\mathrm{I}_e(-)$ operation (cf. Claim 6.2.2 in [3]).

Lemma 3.1.6. *Let A and B be ideals of R. Then $\pi_e(F_*^eA) \subseteq B$ if and only if $A \subseteq B^{[p^e]}$.*

Proof. ($\Rightarrow$) Since R is Noetherian, A is finitely generated, and since π_e is R-linear we may assume that A is a principal ideal, i.e. $A = aR$ for some $a \in R$. Now since F_*^eR is a free R-module with basis $\mathcal{B}$ as in Remark 3.1.3, $F_*^ea = \sum_i r_iF_*^eg_i$ for some $r_i \in R$ and $F_*^eg_i \in \mathcal{B}$. On the other hand, by Lemma 3.1.4, $\pi_e(F_*^ez_ia) = r_i$ for some $z_i \in R$. This implies that $\pi_e(F_*^eRa) = \langle r_i\rangle$. Then by the assumption $\pi_e(F_*^eA) = \langle r_i\rangle \subseteq B$, and since $F_*^ea = F_*^e\sum_i r_i^{p^e}g_i$ we have $a = \sum_i r_i^{p^e}g_i \in B^{[p^e]}$. Hence, $A \subseteq B^{[p^e]}$.

($\Leftarrow$) Assume first that $A \subseteq B^{[p^e]}$ which implies that $F_*^eA \subseteq F_*^eB^{[p^e]}$. Therefore,

$$\pi_e(F_*^eA) \subseteq \pi_e(F_*^eB^{[p^e]}) = \pi_e(BF_*^eR) = B\pi_e(F_*^eR) \subseteq B.$$

□

Corollary 3.1.7. *Let A be an ideal of R, and let $\phi \in \mathrm{Hom}_R(F_*^eR, R)$ be such that $\phi(-) = \pi_e(F_*^eu-)$ for some $u \in R$. Then $\phi(F_*^eA) = \pi_e(F_*^euA) = I_e(uA)$ and $\star$-closure of A gives the smallest ϕ-compatible ideal containing A.*

Proof. Since $uA \subseteq I_e(uA)^{[p^e]}$, the first claim follows from Lemma 3.1.6. The second claim follow from the fact that

$$A \text{ is } \phi-\text{compatible} \Leftrightarrow \phi(F_*^eA) = \pi_e(F_*^euA) = I_e(uA) \subseteq A$$

$$\Leftrightarrow uA \subseteq A^{[p^e]}$$

□

Next we recall Fedder's Lemma which translates the problem of finding compatible ideals of R/I for an ideal I to finding compatible ideals on R. In the case that R is a Gorenstein local ring, this lemma was proved by R. Fedder in [7].

Lemma 3.1.8. *[7, Lemma 1.6][3, Lemma 6.2.1] Let $S = R/I$ for some ideal I and π_e be the trace map, then for any $\phi \in \operatorname{Hom}_R(F_*^e R, R)$ satisfies $\phi(F_*^e I) \subseteq I$ if and only if there exists an element $u \in (I^{[p^e]} : I)$ such that $\phi(-) = \pi_e(F_* u-)$. More generally, there exists an isomorphism of $F_*^e S$-modules*

$$\operatorname{Hom}_S(F_*^e S, S) \cong \frac{\left(F_*^e(I^{[p^e]} : I)\right)}{\left(F_*^e I^{[p^e]}\right)}.$$

Proof. By Lemma 3.1.4, for any $\phi \in \operatorname{Hom}_R(F_*^e R, R)$ there exists an element $u \in R$ such that $\phi(-) = \pi_e(F_* u-)$. Then by Lemma 3.1.6,

$$\phi(F_*^e I) = \pi_e(F_*^e uI) \subseteq I \Leftrightarrow uI \subseteq I^{[p^e]} \Leftrightarrow u \in (I^{[p^e]} : I).$$

For the second claim, we shall show that the map $\Phi : F_*^e(I^{[p^e]} : I) \to \operatorname{Hom}_S(F_*^e S, S)$ which sends $F_*^e z$ to the map $\pi_e(F_*^e z-)$ is surjective. It is easy to verify that this map is well-defined and $F_*^e R$-linear. Since $\operatorname{Hom}_R(F_*^e S, S) = \operatorname{Hom}_S(F_*^e S, S)$, by freeness of $F_*^e R$, for any map $\varphi \in \operatorname{Hom}_S(F_*^e S, S)$ there always exists a map $\psi \in \operatorname{Hom}_R(F_*^e R, R)$ such that I is ψ-compatible. Namely, Φ is surjective. On the other hand, by Lemma 3.1.6 again, $\operatorname{Ker} \Phi = (F_*^e I^{[p^e]})$, and the result follows by the first isomorphism theorem. □

Lemma 3.1.9. *[12, Proposition 2.6.c] If ϕ is surjective, then the set of ϕ-compatible ideals is a finite set of radicals closed under sum and primary decomposition.*

For ϕ-compatible prime ideals $P \subsetneq Q$, we say that Q minimally contains P if there is no ϕ-compatible prime ideal strictly between P and Q. For a given ϕ-compatible prime ideal P, next proposition shows that how to compute ϕ-compatible prime ideals which minimally contain P, and we turn it into an algorithm (cf. Theorem 4.1 in [13] and Section 4 of [12]).

Proposition 3.1.10. *Let $\phi : F_*^e R \to R$ be an R-linear map where $\phi(-) = \pi_e(F_*^e u-)$ for some $u \in R$. Let P and Q be ϕ-compatible prime ideals such that Q minimally contains P, and let J be the ideal whose image in R/P defines the singular locus of R/P. Then:*

1. *If $(P^{[p^e]} : P) \subseteq (Q^{[p^e]} : Q)$ then $J \subseteq Q$,*
2. *If $(P^{[p^e]} : P) \nsubseteq (Q^{[p^e]} : Q)$ then $(uR + P^{[p^e]}) : (P^{[p^e]} : P) \subseteq Q$.*

Proof. For *1.* let $R_{\mathfrak{p}}$ be a localization of R at a prime ideal $\mathfrak{p}$ which contains Q, and let $S = \widehat{R_{\mathfrak{p}}}$ be the completion of $R_{\mathfrak{p}}$ with respect to the maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$. Since colon ideals, Frobenius powers and singular locus commute with localization and completion, $(P^{[p^e]} : P) \subseteq (Q^{[p^e]} : Q) \Rightarrow (PS^{[p^e]} : PS) \subseteq (QS^{[p^e]} : QS)$. Let $Q_1, \dots, Q_s$ be the minimal prime ideals of QS in S, and $Q_i = (QS : s_i)$ for some suitable elements $s_i \in S$. Then $(PS^{[p^e]} : PS) \subseteq (Q_i^{[p^e]} : Q_i)$ for each Q_i since for any element $a \in (QS^{[p^e]} : QS)$,

$$b \in Q_i \Leftrightarrow bs_i \in QS \Rightarrow abs_i \in QS^{[p^e]} \Rightarrow abs_i^{p^e} \in QS^{[p^e]}$$
$$\Leftrightarrow ab \in (QS^{[p^e]} : s_i^{p^e}) = (QS : s_i)^{[p^e]} = Q_i^{[p^e]}$$

Thus, by Theorem 4.1 in [13], $JS \subseteq Q_i$ for each i, which implies that $JS \subseteq QS$, and so $J \subseteq Q$. For *2.* we refer to Theorem 4.1 in [13]. □

The following algorithm is the same algorithm described in [12], which we call it here the Katzman-Schwede algorithm, finds all ϕ-compatible prime ideals of R which do not contain $\mathrm{I}_e(uR)$. We describe it here in a more algebraic language.

Input:

An R linear map $\phi : F_*^e R \to R$ where $\phi(-) = \pi_e(F_*^e u-)$ and $u \in R$.

Output:

Set of all ϕ-compatible prime ideals which do not contain $\mathrm{I}_e(uR)$.

Initialize:

$\mathcal{A}_R = \{0\}$ and $\mathcal{B} = \emptyset$

Execute the following:

While $\mathcal{A}_R \neq \mathcal{B}$ pick any $P \in \mathcal{A}_R - \mathcal{B}$, set $S = R/P$;

1. Find the ideal $J \subseteq R$ whose image in S defines the singular locus of S, and compute $J^{\star^e u}$,
2. Find the minimal prime ideals of $J^{\star^e u}$, add them to $\mathcal{A}_R$,
3. Compute the ideal $B := ((uR + P^{[p^e]}) : (P^{[p^e]} : P))$, and compute $B^{\star^e u}$,
4. Find the minimal prime ideals of $B^{\star^e u}$, add them to $\mathcal{A}_R$,
5. Add P to $\mathcal{B}$.

Output $\mathcal{A}_R$ and stop.

The Katzman-Schwede algorithm produces a list of all ϕ-compatible prime ideals which do not contain $L := I_e(uR)$. Because for any prime ideal Q, whenever $L \subseteq Q$ we have the property that Q is ϕ-compatible if and only if Q/L is ϕ'-compatible where ϕ' is the induced map from ϕ. But Q/L is clearly compatible since ϕ' is zero. Thus, we do not need to assume that ϕ is surjective.

Discussion 3.1.11. *Let $R_\mathfrak{p}$ be a localization of R at a prime ideal $\mathfrak{p}$, and let $S = \widehat{R_\mathfrak{p}}$ be the completion of $R_\mathfrak{p}$ with respect to the maximal ideal $\mathfrak{p}R_\mathfrak{p}$. We know that $\mathfrak{p}\widehat{R_\mathfrak{p}}$ is the maximal ideal of $\widehat{R_\mathfrak{p}}$. Now let $X_1, \ldots, X_s$ be minimal generators of $\mathfrak{p}\widehat{R_\mathfrak{p}}$, and let $\mathbb{K}[\![X_1, \ldots, X_s]\!]$ be the formal power series ring over the residue field $\mathbb{K}$ of $R_\mathfrak{p}$. By the Cohen's structure theorem $S \cong \mathbb{K}[\![X_1, \ldots, X_s]\!]$. Let $E = E_S(S/\mathfrak{m})$ be the injective hull of the residue field. Then by Example 2.1.38, E is isomorphic to the module of inverse polynomials $\mathbb{K}[X_1^-, \ldots, X_s^-]$. Let $T : E \to E$ be the natural Frobenius map as in the Example 2.2.15.*

We can also view the Katzman-Schwede algorithm from the point of Frobenius maps on injective hull of residue fields (cf. section 4 of [13]). By Remark 2.1.25, the set of S-submodules of E is $\{\operatorname{Ann}_E J \mid J$ is an ideal of $R\}$. Also Theorem 4.3 in [10] shows that an S-submodule $\operatorname{Ann}_E J \subseteq E$ is an $S[\Theta; f^e]$-submodule if and only if $uJ \subseteq J^{[p^e]}$ where $\Theta = uT^e$ and $u \in S$. Thus, the Katzman-Schwede algorithm finds all submodules $\operatorname{Ann}_E P$ of E which are preserved by the Frobenius map Θ, under the assumptions that P is a prime ideal of S and the restriction of Θ to $\operatorname{Ann}_E P$ is not the zero map (i.e. it finds all the Θ-special prime ideals of S, see Definition 3.2.6).

All of the operations used in the Katzman-Schwede algorithm are defined for localizations of R. Therefore, we can apply the algorithm to any localization of R at a prime ideal. In the rest of this section, we investigate behaviour of the Katzman-Schwede algorithm under localization. Let $R_\mathfrak{p}$ be a localization of R at a prime ideal $\mathfrak{p}$. Our next theorem gives the exact relation between the output sets $\mathcal{A}_R$ and $\mathcal{A}_{R_\mathfrak{p}}$ of the Katzman-Schwede algorithm for R and $R_\mathfrak{p}$, respectively.

Theorem 3.1.12. *The Katzman-Schwede algorithm commutes with localization: for a given $u \in R$, if $\mathcal{A}_R$ and $\mathcal{A}_{R_\mathfrak{p}}$ are the output sets of the Katzman-Schwede algorithm for R and $R_\mathfrak{p}$, respectively, then*

$$\mathcal{A}_{R_\mathfrak{p}} = \{PR_\mathfrak{p} \mid P \in \mathcal{A}_R \text{ and } P \subseteq \mathfrak{p}\}$$

Proof. We shall show that the Katzman-Schwede algorithm commutes with localization step by step. Since the ideal defining singular locus commutes with localization, so is step *1.* Since Frobenius powers and colon ideals commute with localization under Noetherian hypothesis, so is step *3.* Then by Lemma 2.2.30, $\star$-closure commutes with localization. Therefore, step *2.* and *4.* follow from the fact that primary decomposition commutes with localization.

Let P be a ϕ-compatible prime ideal of R. Then since $uP \subseteq P^{[p^e]} \Leftrightarrow uPR_\mathfrak{p} \subseteq P^{[p^e]}R_\mathfrak{p}$, $PR_\mathfrak{p}$ is a ϕ-compatible prime ideal of $R_\mathfrak{p}$. Since the Katzman-Schwede algorithm commutes with localization, Q is a ϕ-compatible prime ideal of R minimally containing P if and only if $QR_\mathfrak{p}$ is a ϕ-compatible prime ideal of $R_\mathfrak{p}$ minimally containing $PR_\mathfrak{p}$. Hence, $\mathcal{A}_{R_\mathfrak{p}} = \{PR_\mathfrak{p} \mid P \in \mathcal{A}_R$ and $P \subseteq \mathfrak{p}\}$. □

3.2 A Generalization of the Katzman-Zhang Algorithm

Let $R_\mathfrak{p}$ be a localization of R at a prime ideal $\mathfrak{p}$, and let $S = \widehat{R_\mathfrak{p}}$ be the completion of $R_\mathfrak{p}$ with respect to the maximal ideal $\mathfrak{m} = \mathfrak{p}R_\mathfrak{p}$. Let $E = E_S(S/\mathfrak{m})$ be the injective hull of residue field of S. The purpose of this section is to generalize the algorithm defined in Section 6 of [13] to R, and show that it commutes with localization.

Remark 3.2.1. *Given an Artininan S-module M, by Corollary 2.1.26, we can embed M in E^α for some positive integer α, we can then embed* $\operatorname{Coker}(M \hookrightarrow E^\alpha)$ *in E^β for some positive integer β. Continuing in this way, we get an injective resolution*

$$0 \to M \to E^\alpha \xrightarrow{A^t} E^\beta \to \cdots$$

of M, where A is an $\alpha \times \beta$ matrix with entries in S since $\operatorname{Hom}_S(E^\alpha, E^\beta) \cong \operatorname{Hom}_S(S^\beta, S^\alpha)$, and so $M \cong \operatorname{Ker} A^t$.

Proposition 3.2.2. *[13, Proposition 2.1] Let $M \cong \operatorname{Ker} A^t$ be an Artininan S-module where A is an $\alpha \times \beta$ matrix with entries in S. For a given e-th Frobenius map on M, $\Delta^e(M) \in \operatorname{Hom}_S(\operatorname{Coker} A, \operatorname{Coker} A^{[p^e]})$ and is given by an $\alpha \times \alpha$ matrix U such that $U \operatorname{Im} A \subseteq \operatorname{Im} A^{[p^e]}$, conversely any such U defines an $S[\Theta; f^e]$-module structure on M which is given by the restriction to M of the Frobenius map $\Theta : E^\alpha \to E^\alpha$ defined by $\Theta(a) = U^t T^e(a)$ for all $a \in E^\alpha$, where Δ^e as in subsection 2.2.6 and T is the natural Frobenius map on E^α.*

Remark 3.2.3. *By Proposition 3.2.2, for any Artinian submodule $M \cong \operatorname{Ker} A^t$ of E^α with a given $S[\Theta; f^e]$-module structure, where $\Theta = U^t T^e$, there is a submodule V of S^α such that $M = \operatorname{Ann}_{E^\alpha} V^t := \{a \in E^\alpha \mid V^t a = 0\}$ and $UV \subseteq V^{[p^e]}$, (in fact $V = \operatorname{Im} A$). For simplicity, for $V \subseteq S^\alpha$ we denote $E(V) = \operatorname{Ann}_{E^\alpha} V^t$.*

Lemma 3.2.4. *[13, Lemma 3.6, Lemma 3.7] Let $\Theta = U^t T : E^\alpha \to E^\alpha$ be a Frobenius map where U is an $\alpha \times \alpha$ matrix with entries in S and let $K \subset S^\alpha$. Then*

1. $E(I_e(\operatorname{Im} U^{[p^e-1]} U^{[p^e-2]} \cdots U)) = \{a \in E^\alpha \mid \Theta^e(a) = 0\}$,
2. $E(I_1(UK)) = \{a \in E^\alpha \mid \Theta(a) \in E(K)\}$.

Remark 3.2.5. *Let $M = \operatorname{Ann}_{E^\alpha} V^t$ be as in Remark 3.2.3. Then $\operatorname{Ann}_S M = \operatorname{Ann}_S S^\alpha/V$ because $\operatorname{Ann}_S M \subseteq \operatorname{Ann}_S M^\vee \subseteq \operatorname{Ann}_S M^{\vee\vee} \cong \operatorname{Ann}_S M$.*

Definition 3.2.6. *Let $\Theta = U^t T^e : E^\alpha \to E^\alpha$ be a Frobenius map, where U is an $\alpha \times \alpha$ matrix with entries in S. We call an ideal of S a Θ-special ideal if it is an annihilator of an $S[\Theta; f^e]$-submodule of E^α, equivalently if it is the annihilator of S^α/W for some $W \subset S^\alpha$ with $UW \subseteq W^{[p^e]}$.*

Notice that the concept of injective hull of the residue field is not available for polynomial rings. Therefore, we adapt above definition for a more general setting and define special ideals depending on a given square matrix as follows.

Definition 3.2.7. *Let $\mathcal{R}$ be R or $R_\mathfrak{p}$ or S. For a given $\alpha \times \alpha$ matrix U with entries in $\mathcal{R}$, we call an ideal of $\mathcal{R}$ a U-special ideal if it is the annihilator of $\mathcal{R}^\alpha/V$ for some submodule $V \subseteq \mathcal{R}^\alpha$ satisfying $UV \subseteq V^{[p^e]}$.*

Next we will provide some properties of special ideals. The following lemma gives the most important properties which are actually generalization of Lemma 3.8 and 3.10 in [13] to R with similar proofs.

Lemma 3.2.8. *Let $\mathcal{R}$ be R or $R_{\mathfrak{p}}$ or S. Let U be an $\alpha \times \alpha$ matrix with entries in $\mathcal{R}$ and J be a U-special ideal of $\mathcal{R}$. Then*

1. *Associated primes of J are U-special,*
2. *$V = (JR^{\alpha})^{\star^{e}U}$ is the smallest submodule of $\mathcal{R}^{\alpha}$ such that $J = \operatorname{Ann}_{\mathcal{R}} \mathcal{R}^{\alpha}/V$ and $UV \subseteq V^{[p^e]}$.*

Proof. For *1.* let P be an associated prime of J and $J = \operatorname{Ann}_{\mathcal{R}} \mathcal{R}^{\alpha}/V$ for some $V \subseteq \mathcal{R}^{\alpha}$ such that $UV \subseteq V^{[p^e]}$. Then for a suitable element $r \in \mathcal{R}$ we have $P = (J : r)$. If $W = (V :_{\mathcal{R}^{\alpha}} r) = \{w \in \mathcal{R}^{\alpha} \mid rw \in V\}$ then $P = \operatorname{Ann}_{\mathcal{R}} \mathcal{R}^{\alpha}/W$ since $s \in P \Leftrightarrow rs \in J \Leftrightarrow rs\mathcal{R}^{\alpha} \subseteq V \Leftrightarrow s\mathcal{R}^{\alpha} \subseteq W$. On the other hand, since $UV \subseteq V^{[p^e]}$ and $rW \subseteq V$ we have $rUW \subseteq UV$ and so $r^{p^e}UW \subseteq r^{p^e-1}UV \subseteq r^{p^e-1}V^{[p^e]} \subseteq V^{[p^e]}$. This means that $UW \subseteq (V^{[p^e]} :_{\mathcal{R}^{\alpha}} r^{p^e}) = (V :_{\mathcal{R}^{\alpha}} r)^{[p^e]} = W^{[p^e]}$.

For *2.* let $J = \operatorname{Ann}_{\mathcal{R}} \mathcal{R}^{\alpha}/V$ for some $V \subseteq \mathcal{R}^{\alpha}$ such that $UV \subseteq V^{[p^e]}$. It is clear that $J\mathcal{R}^{\alpha} \subseteq (J\mathcal{R}^{\alpha})^{\star^{e}U}$ and $J\mathcal{R}^{\alpha} \subseteq V \Rightarrow (J\mathcal{R}^{\alpha})^{\star^{e}U} \subseteq V^{\star^{e}U} = V$. Therefore, $J \subseteq \operatorname{Ann}_{\mathcal{R}} \mathcal{R}^{\alpha}/(J\mathcal{R}^{\alpha})^{\star^{e}U} \subseteq \operatorname{Ann}_{\mathcal{R}} \mathcal{R}^{\alpha}/V = J$, and so $J = \operatorname{Ann}_{\mathcal{R}} \mathcal{R}/(J\mathcal{R}^{\alpha})^{\star^{e}U}$. □

Theorem 3.2.9. *[13, Theorem 5.1] There are only finitely many Θ-special prime ideals P of S with the property that for some $S[\Theta; f]$-submodule $M \subseteq E^{\alpha}$ with $\operatorname{Ann}_S M = P$ and the restriction of Θ to M is not zero.*

Theorem 3.2.9 was proved by induction on α using the aid of injective hull of the residue field of S, and turned into an algorithm in [13], which we call it here Katzman-Zhang Algorithm. Since injective hulls of residue fields are not available for polynomial rings, we only use techniques of $\mathrm{I}_e(-)$ operation and $\star$-closure to generalize the Katzman-Zhang Algorithm to R. Next theorem allows us to prove polynomial version of Theorem 3.2.9.

Theorem 3.2.10. *[14, Theorem 3.2] Let U be an $\alpha \times \alpha$ matrix with entries in R and $\alpha \in \mathbb{N}$.*

1. *If $\mathrm{I}_e(U^{[p^{e-1}]}U^{[p^{e-2}]} \cdots UR^{\alpha}) = \mathrm{I}_{e+1}(U^{[p^e]}U^{[p^{e-1}]} \cdots UR^{\alpha})$ then*

$$\mathrm{I}_e(U^{[p^{e-1}]}U^{[p^{e-2}]} \cdots UR^{\alpha}) = \mathrm{I}_{e+j}(U^{[p^{e+j-1}]}U^{[p^{e+j-2}]} \cdots UR^{\alpha})$$

for all $j \geq 0$.

2. *There exists an integer e such that (1) holds.*

For the rest of this section, we will fix an $\alpha \times \alpha$ matrix U with entries in R, and $\mathcal{K}$ will denote the stable value of $\{\mathrm{I}_e(U^{[p^{e-1}]}U^{[p^{e-2}]}\cdots UR^\alpha)\}_{e\geq 1}$ as in Theorem 3.2.10.

Proposition 3.2.11. *If P is a prime ideal of R with the property that $\mathcal{K} \subseteq PR^\alpha$ where $\mathcal{K} = \mathrm{I}_e(U_eR^\alpha)$ and $U_e = U^{[p^{e-1}]}U^{[p^{e-2}]}\cdots U$, then P is U_e-special.*

Proof. Let P be a prime ideal of R such that $\mathcal{K} \subseteq PR^\alpha$. Then

$$\mathcal{K} \subseteq PR^\alpha \Rightarrow U_eR^\alpha \subseteq P^{[p^e]}R^\alpha \Rightarrow U_ePR^\alpha \subseteq P^{[p^e]}R^\alpha \Rightarrow PR^\alpha = (PR^\alpha)^{\star U_e}.$$

Therefore, P is U_e-special. □

By Proposition 3.2.11, any prime ideal containing $\mathcal{K}$ is U_e-special. This is equivalent to saying that the action of U_e on submodules PR^α containing $\mathcal{K}$ with P being a prime is the same as the action of zero matrix. Henceforth, we will assume that $\mathcal{K} \neq 0$.

Our next theorem is the generalization of Theorem 3.2.10 to R, and we will prove it using a very similar method to that in [13, Section 5].

Theorem 3.2.12. *The set of all U-special prime ideals P of R with the property that $\mathcal{K} \nsubseteq PR^\alpha$ is finite.*

We will prove Theorem 3.2.12 by induction on α. Assume that $\alpha = 1$. For a prime ideal P being a u-special prime, i.e. $P = \mathrm{Ann}_R R/P^{\star u}$, is equivalent to the property that $uP \subseteq P^{[p]}$. This means, by Corollary 3.1.7, that P is a ϕ-compatible ideal where $\phi(-) = \pi(F_* u-)$. Then the set of all u-special prime ideals are finite and the Katzman-Schwede algorithm finds such primes. Henceforth in this section, we will assume that Theorem 3.2.12 holds for $\alpha - 1$.

For a U-special prime ideal P, we will present an effective method for finding all U-special prime ideals $Q \supsetneq P$ for which there is no U-special prime ideal strictly between P and Q, and we will call such U-special prime ideals Q as minimally containing P. The following lemma is a generalization of Lemma 5.2 in [13] to R, which is our starting point of finding U-special prime ideals minimally containing P.

Lemma 3.2.13. *Let $P \subsetneq Q$ be U-special prime ideals of R such that Q contains P minimally. If $a \in Q \setminus P$, then Q is among the minimal prime ideals of $\mathrm{Ann}_R R^\alpha/W$ where $W = ((P + aR)R^\alpha)^{\star U}$.*

Proof. Since $PR^\alpha \subseteq (P+aR)R^\alpha \subseteq QR^\alpha$ we have

$$(PR^\alpha)^{\star U} \subseteq ((P+aR)R^\alpha)^{\star U} \subseteq (QR^\alpha)^{\star U}.$$

Then by Lemma 3.2.8,

$$P = \operatorname{Ann}_R \frac{R^\alpha}{(PR^\alpha)^{\star U}} \subseteq \operatorname{Ann}_R \frac{R^\alpha}{W} \subseteq \operatorname{Ann}_R \frac{R^\alpha}{(QR^\alpha)^{\star U}} = Q$$

which implies that Q contains a minimal prime ideal of $\operatorname{Ann}_R R^\alpha/W$. Therefore, by Lemma 3.2.8 again, this minimal prime is U-special. Since Q contains P minimally, it has to be Q itself. □

Next, we will prove a generalization of Lemma 5.3 in [13] to R, which is a crucial step for proving Theorem 3.2.12.

Lemma 3.2.14. *Let Q be a U-special prime ideal of R, where $Q = \operatorname{Ann}_R R^\alpha/W$ for some submodule $W \subseteq R^\alpha$ satisfying $UW \subseteq W^{[p]}$. Let $a \notin Q$ and X be an invertible $\alpha \times \alpha$ matrix with entries in the localization R_a. Let $\nu \gg 0$ be such that $U_1 = a^\nu X^{[p]} U X^{-1}$ has entries in R and $W_1 = XW_a \cap R^\alpha$. Then*

1. *Q is a minimal prime of $\operatorname{Ann}_R R^\alpha/W_1$ and $U_1 W_1 \subseteq W_1^{[p]}$, i.e. Q is U_1-special.*
2. *If $\mathrm{I}_e(U^{[p^{e-1}]}U^{[p^{e-2}]}\cdots UR^\alpha) \nsubseteq W$, then $\mathrm{I}_e(U_1^{[p^{e-1}]}U_1^{[p^{e-2}]}\cdots U_1R^\alpha) \nsubseteq W_1$.*

Proof. Let $J = \operatorname{Ann}_R R^\alpha/W_1$. Then

$$\begin{aligned} J_a &= (\operatorname{Ann}_R R^\alpha/W_1)_a = \operatorname{Ann}_{R_a} R_a^\alpha/(W_1)_a = \operatorname{Ann}_{R_a} R_a^\alpha/XW_a \\ &\cong \operatorname{Ann}_{R_a} R_a^\alpha/W_a = (\operatorname{Ann}_R R^\alpha/W)_a = Q_a. \end{aligned}$$

Therefore, Q is a minimal prime ideal of J. We also have

$$\begin{aligned} U_1W_1 &= a^\nu X^{[p]}UX^{-1}(XW_a \cap R^\alpha) \subseteq (a^\nu X^{[p]}UX^{-1}XW_a) \cap R^\alpha \\ &\subseteq X^{[p]}W_a^{[p]} \cap R^\alpha = (XW_a)^{[p]} \cap R^\alpha = (XW_a \cap R^\alpha)^{[p]} = W_1^{[p]}. \end{aligned}$$

This means that J is U_1-special. Therefore, by Lemma 3.2.8, Q is U_1-special.

Assume that

$$\mathrm{I}_e(U^{[p^{e-1}]}U^{[p^{e-2}]}\cdots UR^\alpha) \nsubseteq W, \text{ i.e. } U^{[p^{e-1}]}U^{[p^{e-2}]}\cdots UR^\alpha \nsubseteq W^{[p^e]}.$$

Now suppose the contrary that

$$\mathrm{I}_e(U_1^{[p^{e-1}]}U_1^{[p^{e-2}]}\cdots U_1R^\alpha)\subseteq W_1,\text{ i.e. } U_1^{[p^{e-1}]}U_1^{[p^{e-2}]}\cdots U_1R^\alpha\subseteq W_1^{[p^e]}.$$

Since

$$\begin{aligned}U_1^{[p^{e-1}]}U_1^{[p^{e-2}]}\cdots U_1 &= (a^\nu X^{[p]}UX^{-1})^{[p^{e-1}]}(a^\nu X^{[p]}UX^{-1})^{[p^{e-2}]}\cdots a^\nu X^{[p]}UX^{-1}\\ &= a^{\nu(p^{e-1})}X^{[p^e]}U^{[p^{e-1}]}(X^{-1})^{[p^{e-1}]}a^{\nu(p^{e-2})}X^{[p^{e-1}]}U^{[p^{e-2}]}(X^{-1})^{[p^{e-2}]}\cdots a^\nu X^{[p]}UX^{-1}\\ &= a^{\nu(p^{e-1}+p^{e-2}+\cdots+1)}X^{[p^e]}U^{[p^{e-1}]}U^{[p^{e-2}]}\cdots UX^{-1},\end{aligned}$$

we have $bX^{[p^e]}U^{[p^{e-1}]}U^{[p^{e-2}]}\cdots UX^{-1}R^\alpha\subseteq W_1^{[p^e]}=(XW_a\cap R^\alpha)^{[p^e]}=X^{[p^e]}W_a^{[p^e]}\cap R^\alpha$, where $b=a^{\nu(p^{e-1}+p^{e-2}+\cdots+1)}$. Therefore, $X^{[p^e]}U^{[p^{e-1}]}U^{[p^{e-2}]}\cdots UX^{-1}R_a^\alpha\subseteq X^{[p^e]}W_a^{[p^e]}$, and so $U^{[p^{e-1}]}U^{[p^{e-2}]}\cdots UR_a^\alpha\subseteq W_a^{[p^e]}$. Then $U^{[p^{e-1}]}U^{[p^{e-2}]}\cdots UR^\alpha\subseteq W^{[p^e]}$ since a is not a zero divisor on $R^\alpha/W^{[p^e]}$, which contradicts with our assumption. □

Next, we will give a generalization of Proposition 5.4 in [13] to R, which will give us an effective method for finding the U-special prime ideals containing a U-special prime P minimally in an important case.

Proposition 3.2.15. *Let P be a U-special prime ideal of R such that $\mathcal{K}\nsubseteq PR^\alpha$. Assume that the α-th column of U is zero and $PR^\alpha=(PR^\alpha)^{\star U}$. Then the set of U-special prime ideals minimally containing P is finite.*

Proof. Let Q be a U-special prime ideal minimally containing P and $W=(QR^\alpha)^{\star U}$. Let U_0 be the top left $(\alpha-1)\times(\alpha-1)$ submatrix of U. Since $PR^\alpha=(PR^\alpha)^{\star U}\Leftrightarrow UPR^\alpha\subseteq P^{[p]}R^\alpha$, all entries of U are in $(P:P^{[p]})$. Therefore, $U_0PR^{\alpha-1}\subseteq P^{[p]}R^{\alpha-1}$, and so P is U_0-special. Let $\mathcal{K}_0$ be the stable value of $\{\mathrm{I}_e(U_0^{[p^{e-1}]}U_0^{[p^{e-2}]}\cdots U_0R^{\alpha-1})\}_{e>0}$ as in Theorem 3.2.10. We now split our proof into two parts. Assume first that $\mathcal{K}_0\subseteq PR^{\alpha-1}$, i.e. $\mathrm{I}_e(U_0^{[p^{e-1}]}U_0^{[p^{e-2}]}\cdots U_0R^{\alpha-1})\subseteq PR^{\alpha-1}$ for some $e>0$.

1) Let $(g_1,\ldots,g_{\alpha-1},0)$ be the last row of the matrix $U^{[p^{e-1}]}U^{[p^{e-2}]}\cdots U$. Note that its top left $(\alpha-1)\times(\alpha-1)$ submatrix is $U_0^{[p^{e-1}]}U_0^{[p^{e-2}]}\cdots U_0$. By our assumption, all entries of $U_0^{[p^{e-1}]}U_0^{[p^{e-2}]}\cdots U_0$ are in $P^{[p^e]}\subseteq Q^{[p^e]}$. Therefore, $\mathrm{I}_e(U_0^{[p^{e-1}]}U_0^{[p^{e-2}]}\cdots U_0R^{\alpha-1})\subseteq QR^{\alpha-1}$. Then by Proposition 3.2.11, P and Q are $U_0^{[p^{e-1}]}U_0^{[p^{e-2}]}\cdots U_0$-special, and so the action of $U^{[p^{e-1}]}U^{[p^{e-2}]}\cdots U$ is the same action of a matrix U_e whose first $\alpha-1$ rows are zero and last row is $(g_1,\ldots,g_{\alpha-1},0)$, and so we replace $U^{[p^{e-1}]}U^{[p^{e-2}]}\cdots U$ with U_e without effecting

any issues. We now define inductively $V_0 = QR^\alpha$ and $V_{i+1} = \mathrm{I}_e(U_e V_i) + V_i$ for all $i \geq 0$. Since

$$U_e Q R^\alpha = \{(0, \dots, 0, \sum_{i=1}^{\alpha-1} g_i q_i)^t \mid \forall i, q_i \in Q\},$$

$$\mathrm{I}_e(U_e Q R^\alpha) = \{(0, \dots, 0, v) \mid v \in \mathrm{I}_e(\sum_{i=1}^{\alpha-1} g_i Q)\}.$$

Therefore, the sequence $\{V_i\}_{i\geq 0}$ stabilizes at $V_1 = \mathrm{I}_e(U_e Q R^\alpha) + QR^\alpha$. By definition of $\star$-closure, we have $QR^\alpha \subsetneq V_1 \subseteq W$, and so $\mathrm{Ann}_R\, R^\alpha / V_1 = Q$. Furthermore, we have

$$\mathrm{Ann}_R \frac{R}{\mathrm{I}_e(\sum_{i=1}^{\alpha-1} g_i Q)} = \mathrm{Ann}_R \frac{R^\alpha}{\mathrm{I}_e(U_e Q R^\alpha)} \subseteq Q \text{ since } \mathrm{I}_e(U_e Q R^\alpha) \subseteq V_1,$$

which implies that

$$\mathrm{I}_e(\sum_{i=1}^{\alpha-1} g_i Q) = \sum_{i=1}^{\alpha-1} \mathrm{I}_e(g_i Q) \subseteq Q,$$

i.e. $\mathrm{I}_e(g_i Q) \subseteq Q \Leftrightarrow g_i Q \subseteq Q^{[p^e]}$ for all $1 \leq i < \alpha$. Hence, Q is g_i-special for all $1 \leq i < \alpha$. On the other hand, at least for one g_i we must have $g_i \notin P^{[p^e]}$ so that we do not get a contradiction with our assumption $\mathcal{K} \nsubseteq PR^\alpha$. We can now produce all such Q using the Katzman-Schwede algorithm.

Let $\tau \subset R$ be intersection of the finite set of U_0-special prime ideals of R minimally containing P. Let $\rho : R^\alpha \to R^{\alpha-1}$ be the projection onto first $\alpha-1$ coordinates, and let $J = \mathrm{Ann}_R\, R^{\alpha-1}/\rho(W)$. Then since $U_0 \rho(W) = \rho(UW) \subseteq \rho(W^{[p]}) = \rho(W)^{[p]}$, J is U_0-special. Note that $Q \subseteq J$, and so $P \subsetneq J$. Assume now that $\mathcal{K}_0 \nsubseteq PR^{\alpha-1}$.

2) We now compute $(\tau^{[p^e]}\mathcal{K}_0)^{\star U_0}$ as the stable value of

$$L_0 = \tau^{[p^e]}\mathcal{K}_0$$

$$L_1 = \mathrm{I}_1(U_0 L_0) + L_0 = \tau^{[p^{e-1}]}\,\mathrm{I}_1(U_0\mathcal{K}_0) + \tau^{[p^e]}\mathcal{K}_0 = \tau^{[p^{e-1}]}\mathcal{K}_0 + \tau^{[p^e]}\mathcal{K}_0$$

$$\vdots$$

$$L_e = \tau\mathcal{K}_0 + L_{e-1}$$

$$\vdots$$

and we deduce that $\tau\mathcal{K}_0 \subseteq L_e \subseteq (\tau^{[p^e]}\mathcal{K}_0)^{\star U_0}$. On the other hand, since J is a U_0-special ideal strictly containing P, $\tau \subseteq \sqrt{J}$. Thus, for all large $e \geq 0$, we have $\tau^{[p^e]} \subseteq J$. Therefore,

$$\tau\mathcal{K}_0 \subset (\tau^{[p^e]}\mathcal{K}_0)^{\star U_0} \subseteq (JR^{\alpha-1})^{\star U} \subseteq \rho(W)^{\star U_0} = \rho(W).$$

where the last equality follows from the fact that $UW \subseteq W^{[p]}$. Moreover, since $\tau \nsubseteq P$, we have $\tau\mathcal{K}_0 \nsubseteq PR^{\alpha-1}$.

3) Now we define $\bar{v} = (v_1, \ldots, v_{\alpha-1}, 0)^t$ for any element $v = (v_1, \ldots, v_{\alpha-1}, v_\alpha)^t$, and $\bar{V} = \{\bar{v} \mid v \in V\}$ for any submodule V. Let $l : R^{\alpha-1} \to R^{\alpha-1} \oplus R$ be the natural inclusion $l(v) = v \oplus 0$. Note that $\bar{V} = l(\rho(V))$. Then we also define $W_0 = \{w \in W \mid \rho(w) \in \tau\mathcal{K}_0\}$ and note that (2) implies that $\rho(W_0) = \tau\mathcal{K}_0$. We have $W_0^{\star U} \subseteq W^{\star U} = W$ and $W_0^{\star U} = \mathrm{I}_1(UW_0)^{\star U} + W_0$. Since $UW_0 = U\overline{W_0} = Ul(\tau\mathcal{K}_0)$, $\mathrm{I}_1(Ul(\tau\mathcal{K}_0))^{\star U} \subseteq W_0^{\star U} \subseteq W$. On the other hand, if $\mathrm{I}_1(Ul(\tau\mathcal{K}_0))^{\star U} \subseteq PR^\alpha$, then

$$\begin{aligned}\mathrm{I}_1(Ul(\tau\mathcal{K}_0)) \subseteq PR^\alpha &\Rightarrow Ul(\tau\mathcal{K}_0) \subseteq P^{[p]}R^\alpha \Rightarrow \rho(Ul(\tau\mathcal{K}_0)) \subseteq \rho(P^{[p]}R^\alpha)\\ &\Rightarrow U_0\tau\mathcal{K}_0 \subseteq P^{[p]}R^{\alpha-1} \Rightarrow \tau^{[p]}U_0\mathcal{K}_0 \subseteq P^{[p]}R^{\alpha-1}\\ &\Rightarrow \mathrm{I}_1(\tau^{[p]}U_0\mathcal{K}_0) \subseteq PR^{\alpha-1} \Rightarrow \tau\,\mathrm{I}_1(U_0\mathcal{K}_0) \subseteq PR^{\alpha-1}\\ &\Rightarrow \tau\mathcal{K}_0 \subseteq PR^{\alpha-1}\end{aligned}$$

which contradicts with (2). Hence, we also have $\mathrm{I}_1(Ul(\tau\mathcal{K}_0))^{\star U} \nsubseteq PR^\alpha$.

4) Let M' be a matrix whose columns generate $\mathrm{I}_1(Ul(\tau\mathcal{K}_0))^{\star U} \subseteq W$. Choose an entry a of M' which is not in P. Then

(a) If $a \in Q$, Lemma 3.2.13 shows that Q is among the minimal prime ideals of $\mathrm{Ann}_R\, R^\alpha/((P + aR)R^\alpha)^{\star U}$.

(b) If $a \notin Q$, we shall apply Lemma 3.2.14 with the matrix X with entries in R_a such that the α-th elementary vector $e_\alpha \in W_1 = XW_a \cap R^\alpha$ and U_1 as in Lemma 3.2.14. Then $R^\alpha/W_1 \cong R^{\alpha-1}/\rho(W_1)$, and so Q is a minimal prime $\mathrm{Ann}_R\, R^{\alpha-1}/\rho(W_1)$. Let U_2 be the top left $(\alpha-1)\times(\alpha-1)$ submatrix of U_1. Then since $U_2\rho(W_1) \subseteq \rho(U_1W_1) \subseteq \rho(W_1^{[p]}) = \rho(W_1)^{[p]}$, $\mathrm{Ann}_R\, R^{\alpha-1}/\rho(W_1)$ is U_2-special, and so is Q.

This shows that in any case Q is an element of a finite set of prime ideals. Hence, there are only finitely many U-special prime ideals of R which contain P minimally. □

Next Theorem is a generalization of Theorem 5.5 in [13] to R, and it provides an effective algorithm for finding all U-special prime ideals P of R with the property that $\mathcal{K} \nsubseteq PR^\alpha$.

Theorem 3.2.16. *Let P a U-special prime ideal of R such that $\mathcal{K} \nsubseteq PR^\alpha$, and Q be a U-special prime ideal minimally containing P. Let M be a matrix whose columns generate $(PR^\alpha)^{\star U}$.*

1. *If $PR^\alpha \subsetneq \operatorname{Im} M$, then either*
   - (a) *all entries of M are in Q, and so there exist an element $a \in Q \setminus P$ and Q is among the minimal prime ideals of $\operatorname{Ann}_R R^\alpha/((P+aR)R^\alpha)^{\star U}$, or*
   - (b) *there exists an entry of M which is not in Q, and Q is a special prime over an $(\alpha-1)\times(\alpha-1)$ matrix.*
2. *If$PR^\alpha = \operatorname{Im} M$, then there exist an element $a_1 \in R \setminus P$, an element $g \in (P^{[p]} : P)$, and an $\alpha \times \alpha$ matrix V such that for some $\mu \gg 0$, we have $a_1^\mu U \equiv gV$ modulo $P^{[p]}$. If $d = \det V$, then either*
   - (a) *$d \in P$, and Q is a special prime ideal over an $(\alpha-1)\times(\alpha-1)$ matrix, or*
   - (b) *$d \in Q \setminus P$, and Q is among the minimal prime ideals of $\operatorname{Ann}_R R^\alpha/((P+dR)R^\alpha)^{\star U}$, or*
   - (c) *$d \notin Q$, and Q is a g-special ideal of R.*

Proof. Let $W \subseteq R^\alpha$ be such that $UW \subseteq W^{[p]}$ and $Q = \operatorname{Ann}_R R^\alpha/W$. When all entries of M are in P, $\operatorname{Im} M \subseteq PR^\alpha$, i.e., $\operatorname{Im} M = (PR^\alpha)^{\star U} = PR^\alpha$. Thus, if we are in case 1., we have at least one entry a of M which is not in P. If $a \in Q$, by Lemma 3.2.13, Q is among the minimal primes of $\operatorname{Ann}_R R^\alpha/((P+aR)R^\alpha)^{\star U}$. If $a \notin Q$, by Lemma 3.2.14, Q is a minimal prime of $\operatorname{Ann}_R R^\alpha/W_1$ such that $U_1 W_1 \subseteq W_1^{[p]}$, where U_1 and W_1 as in Lemma 3.2.14. On the other hand, since a becomes a unit in R_a, we can choose the invertible matrix X with entries in R_a such that $W_1 = XW_a \cap R^\alpha$ contains the α-th elementary vector e_α. Then we have $R^\alpha/W_1 \cong R^{\alpha-1}/\rho(W_1)$, where $\rho : R^\alpha \to R^{\alpha-1}$ is the projection onto first $\alpha-1$ coordinates. Let U_2 be the top left

$(\alpha-1)\times(\alpha-1)$ submatrix of U_1. Then $\operatorname{Ann}_R R^\alpha/W_1 = \operatorname{Ann}_R R^{\alpha-1}/\rho(W_1)$ and $U_2\rho(W_1) \subseteq \rho(U_1W_1) \subseteq \rho(W_1^{[p]}) = \rho(W_1)^{[p]}$. Therefore, $\operatorname{Ann}_R R^\alpha/W_1$ is U_2-special, and so is Q.

Assume now that we are in case 2., by definition of $\star$-closure $UPR^\alpha \subseteq P^{[p]}R^\alpha$, i.e., the entries of U are in $(P^{[p]} : P)$. On the other hand, by Lemma 3.1.8, if $A = R/P$, $F_*((P^{[p]} : P)/P^{[p]}) \cong \operatorname{Hom}_A(F_*A, A)$ is rank one F_*A-module. This means that $(P^{[p]} : P)/P^{[p]}$ is rank one A-module, and so we can find an element $g \in (P^{[p]} : P)\setminus P^{[p]}$ such that $(P^{[p]} : P)/P^{[p]}$ is generated by $g+P^{[p]}$ as an A-module. Also we can find an element $a_1 \in R\setminus P$ such that the localization of $(P^{[p]} : P)/P^{[p]}$ at a_1 is generated by $g/1 + P^{[p]}_{a_1}$ as an A_{a_1}-module and hence as an R_{a_1}-module. If $a_1 \in Q$, we can find Q as in the case 1.(a), thus, we assume that $a_1 \notin Q$. Then for any entry u of U, working in the localization, we have an expression

$$\frac{u}{1} + P^{[p]}_{a_1} = \frac{r}{a_1^{w_1}}\frac{g}{1} + P^{[p]}_{a_1}$$

which implies that $\dfrac{u-rg}{a_1^{w_1}} \in P^{[p]}_{a_1}$, i.e., $\dfrac{u-rg}{a_1^{w_1}} = \dfrac{r'}{a_1^{w_2}}$, where $r \in R$, $r' \in P^{[p]}$ and $w_1, w_2 \in \mathbb{N}$. Thus,

$$a_1^{w_1+w_2}u = a_1^{w_2}rg + a_1^{w_1}r'$$

Therefore, we can write $a_1^{\mu}U = gV + V'$ for some $\mu \gg 0$ and $\alpha\times\alpha$ matrices V and V' with entries in R and $P^{[p]}$, respectively. Then by Proposition 3.2.11, we may replace V' with the zero matrix, since $\mathrm{I}(V'R^\alpha) \subseteq PR^\alpha$. Let $d = \det V$. We now consider three cases:

1. If $d \in P$, then the determinant of V in the fraction field $\mathbb{F}$ of A, say $\bar{d}$, will be zero. So we can find an invertible matrix X with entries in $\mathbb{F}$ such that the last column of VX^{-1} is zero, and so is UX^{-1}. Let a_2 is the product of all denominators of entries of X and X^{-1}, i.e. the entries of X and X^{-1} are in R_{a_2}. If $a_2 \in Q$, we can find Q as in the case 1.(a) again, thus, we also assume that $a_2 \notin Q$. Let $a = a_1a_2$. By Lemma 3.2.14, P and Q are U_1-special prime ideals where $U_1 = a^\nu X^{[p]}UX^{-1}$ whose last column is zero. Then since $PR^\alpha = (PR^\alpha)^{\star U} \Leftrightarrow UPR^\alpha \subseteq P^{[p]}R^\alpha$, we also have

$$U_1PR^\alpha = a^\nu X^{[p]}UX^{-1}PR^\alpha \subseteq a^\nu X^{[p]}UPR^\alpha \subseteq UPR^\alpha \subseteq P^{[p]}R^\alpha$$

which implies $PR^\alpha = (PR^\alpha)^{\star U_1}$. Hence, we can produce Q as in Proposition 3.2.15.

2. If $d \in Q \setminus P$, then by Lemma 3.2.13, Q is among minimal prime ideals of $\operatorname{Ann}_R R^\alpha/((P+dR)R^\alpha)^{\star U}$.

3. If $d \notin Q$, let $a = da_1$, $W = (QR^\alpha)^{\star U}$ and $X = I_\alpha$ be the $\alpha \times \alpha$ identity matrix. Then by Lemma 3.2.14, Q is a minimal prime ideal of $\operatorname{Ann}_R R^\alpha/W_1$ where $W_1 = (QR^\alpha)^{\star U}_a \cap R^\alpha$. By definition of $\star$-closure $(QR^\alpha)^{\star U}_a = (QR^\alpha_a)^{\star U}$ is the stable value of the sequence

$$L_0 = QR^\alpha_a$$

$$L_1 = \mathrm{I}_1(UQR^\alpha_a) + QR^\alpha = \mathrm{I}_1(gVQR^\alpha_a) + QR^\alpha_a = \mathrm{I}_1(gQR^\alpha)_a + QR^\alpha_a$$

$$L_2 =$$

$$\vdots$$

which also equals to $(QR^\alpha)^{\star gI_\alpha}$. The third equality for L_1 is because of the fact that $\mathrm{I}_e(-)$-operation commutes with localization and V is invertible. This implies that $\operatorname{Ann}_R R^\alpha/W_1$ is gI_α-special, and so is Q. Therefore, Q is g-special and can be computed using the Katzman-Schwede algorithm, since $g \notin P^{[p]}$.

This method also shows that for a given U-special ideal P, there are only finitely many U-special prime ideals minimally containing P. □

For the sake of integrity, we shall give the proof of Theorem 3.2.12. The main difference between our methods and the methods in [13, Section 5] is that we do not use the aid of injective hulls of residue fields although our results are identical with the results in [13, Section 5] over power series rings.

Proof.[Proof of Theorem 3.2.12] The proof is by induction on α. The case $\alpha = 1$ is established in section 3.1. Assume that $\alpha > 0$ and the claim is true for $\alpha - 1$. Since zero ideal is always a U-special prime ideal of R, we start with 0 and use Theorem 3.2.16 to find U-special prime ideals minimally containing 0. Continuing this process recursively gives us bigger U-special prime ideals at each steps. Therefore, since R is of finite dimension, the number of steps in this process is bounded by the dimension of R. Hence, there are only finitely many U-special prime ideals with the desired property. □

Next we turn Theorem 3.2.16 into an algorithm which gives us a generalization of the Katzman-Zhang algorithm to R. Note also that over power series rings the following is identical with the Katzman-Zhang algorithm.

Intput:

An $\alpha \times \alpha$ matrix U with entries in R such that $\mathcal{K} \neq 0$.

Output:

Set of all U-special prime ideals P of R with the property that $\mathcal{K} \nsubseteq PR^\alpha$.

Initialize:

$\mathcal{A}_{R^\alpha} = \{0\}, \mathcal{B} = \emptyset$.

Execute the following:

If $\alpha = 1$, use the Katzman-Schwede Algorithm to find desired primes, put these in $\mathcal{A}_{R^\alpha}$, output $\mathcal{A}_{R^\alpha}$ and stop.
If $\alpha > 1$, then while $\mathcal{A}_{R^\alpha} \neq \mathcal{B}$, pick any $P \in \mathcal{A}_{R^\alpha} \setminus \mathcal{B}$. If $\mathcal{K} \subseteq PR^\alpha$, add P to $\mathcal{B}$, if not, write $W = (PR^\alpha)^{\star U}$ as the image of a matrix M and do the following:

1. If there is an entry a of M which is not in P, then;
   (a) Find the minimal primes of $\operatorname{Ann}_R \dfrac{R^\alpha}{((P+aR)R^\alpha)^{\star U}}$, and add them to $\mathcal{A}_{R^\alpha}$,
   (b) Find an invertible $\alpha \times \alpha$ matrix X with entries in R_a such that the α-th elementary vector $e_\alpha \in XW_a \cap R^\alpha$, and choose $\nu \gg 0$ such that $U_1 = a^\nu X^{[p]} U X^{-1}$ has entries in R. Let U_0 be the top left $(\alpha-1)\times(\alpha-1)$ submatrix of U_1. Then apply the algorithm recursively to U_0 and add resulting primes to $\mathcal{A}_{R^\alpha}$.
2. If $\operatorname{Im} M = PR^\alpha$, then find elements $a_1 \in R \setminus P$, $g \in (P^{[p]} : P)$, and an $\alpha \times \alpha$ matrix V, and $\mu \gg 0$ such that $a_1^\mu U \equiv gV$ modulo $P^{[p]}$. Compute $d = \det V$ and do the following:
   (a) If $d \in P$, find an element $a_2 \in R \setminus P$ and an invertible matrix X with entries in R_{a_2} such that the last column of UX^{-1} is zero. Find $\nu \gg 0$ such that the entries of $U_1 = (a_1 a_2)^\nu X^{[p]} U X^{-1}$ are in R. Let U_0 be the

top left $(\alpha-1)\times(\alpha-1)$ submatrix of U_1, and $\mathcal{K}_0$ be the stable value of $\{\mathrm{I}_e(\operatorname{Im} U_0^{[p^e-1]}U_0^{[p^e-2]}\cdots U_0)\}_{e>0}$ as in Theorem 3.2.10. Then;

i. If $\mathcal{K}_0 \subseteq PR^{\alpha-1}$, write the last row of the matrix $U_1^{[p^{e-1}]}U_1^{[p^{e-2}]}\cdots U_1$ as $(g_1,\ldots,g_{\alpha-1},0)$ and apply the Katzman-Schwede Algorithm to the case $u=g_i$ for each i, and add resulting primes to $\mathcal{A}_{R^\alpha}$,

ii. If $\mathcal{K}_0 \nsubseteq PR^{\alpha-1}$, find recursively all prime ideals for U_0 which contain P minimally and denote their intersection with τ. Compute $\mathrm{I}_1(U_1 l(\tau\mathcal{K}_0))^{\star U_1}$, and write this as the image of a matrix M'. Find an entry a' of M' not in P. Now;

A. Add the minimal primes of $\operatorname{Ann}_R \dfrac{R^\alpha}{((P+a'R)R^\alpha)^{\star U_1}}$ to $\mathcal{A}_{R^\alpha}$,

B. Find an invertible matrix X with entries in $R_{a'}$ such that the α^{th} elementary vector $e_\alpha \in X(\operatorname{Im} M')_{a'} \cap R^\alpha$. Find $\nu \gg 0$ such that $U_2=(a')^v X^{[p]}U_1X^{-1}$ has entries in R. Let U_3 be the top left $(\alpha-1)\times(\alpha-1)$ submatrix of U_2. Apply the algorithm recursively to U_3, and add resulting primes to $\mathcal{A}_{R^\alpha}$.

(b) If $d \notin P$, then;

i. add the minimal primes of $\operatorname{Ann}_R \dfrac{R^\alpha}{((P+dR)R^\alpha)^{\star U}}$ to $\mathcal{A}_{R^\alpha}$,

ii. apply the Katzman-Schwede algorithm to the case $u=g$, and add resulting primes to $\mathcal{A}_{R^\alpha}$.

3. Add P to $\mathcal{B}$

Output $\mathcal{A}_{R^\alpha}$ and stop.

Since all the operations used in the above algorithm are defined for localizations of R, we can apply our algorithm to any localization of R at a prime ideal $\mathfrak{p}$. In the rest of this section, we investigate the relations between output sets of our algorithm applied to R and $R_\mathfrak{p}$.

Lemma 3.2.17. *Let $\mathcal{R}$ be R or $R_\mathfrak{p}$ or $\widehat{R_\mathfrak{p}}$. P is a U-special ideal of R not contained in $\mathfrak{p}$ if and only if $P\mathcal{R}$ is a U-special ideal of $\mathcal{R}$.*

Proof. Let P be a prime ideal of R. Then

$$\begin{aligned} P \text{ is } U\text{-special } &\Leftrightarrow P=\operatorname{Ann}_R R^\alpha/(PR^\alpha)^{\star U}\\ &\Leftrightarrow P\mathcal{R}=\operatorname{Ann}_\mathcal{R} \mathcal{R}^\alpha/(P\mathcal{R}^\alpha)^{\star U} \Leftrightarrow P\mathcal{R} \text{ is } U\text{-special}\end{aligned}$$

□

Our next theorem gives the exact relation between the output sets $\mathcal{A}_{R^\alpha}$ and $\mathcal{A}_{R_\mathfrak{p}^\alpha}$ of our algorithm for R and $R_\mathfrak{p}$, respectively.

Theorem 3.2.18. *Let U be an $\alpha \times \alpha$ matrix with entries in R. Our algorithm commutes with localization: if $\mathcal{A}_{R^\alpha}$ and $\mathcal{A}_{R_\mathfrak{p}^\alpha}$ are the output sets of our algorithm for R and $R_\mathfrak{p}$, respectively, then*

$$\mathcal{A}_{R_\mathfrak{p}^\alpha} = \{PR_\mathfrak{p} \mid P \in \mathcal{A}_{R^\alpha} \text{ and } P \subseteq \mathfrak{p}\}.$$

Before proving our claim we need a remark which we will use it in step 2. of the proof.

Remark 3.2.19. *Keeping the notations of above theorem, for any prime ideal P of R, and any submodule K of R^α we have the property that $K \subseteq PR^\alpha \Leftrightarrow K_\mathfrak{p} \subseteq PR_\mathfrak{p}^\alpha$. We already know that $K \subseteq PR^\alpha$ implies $K_\mathfrak{p} \subseteq PR_\mathfrak{p}^\alpha$. For the converse, suppose the contrary that there is an element $k = (k_1, \dots, k_\alpha)^t \in K \setminus PR^\alpha$ where $k_i \in R \setminus P$ for some i. Then there exists an element $s \in R \setminus \mathfrak{p}$ such that $sk \in PR^\alpha$, i.e. $sk_i \in P$. Since P is prime, $k_i \in P$ or $s \in P$, which is impossible. Therefore, $K_\mathfrak{p} \subseteq PR_\mathfrak{p}^\alpha$ implies that $K \subseteq PR^\alpha$.*

Proof. By Theorem 3.1.12, the Katzman-Schwede Algorithm commutes with localization. Therefore, we can, and do, assume $\alpha > 1$. Let P be the prime ideal of R in the initial step of our algorithm, and $R_\mathfrak{p}$ be a localization of R at a prime ideal $\mathfrak{p}$ containing P. Since $\star$-closure commutes with localization, whenever we write $(PR^\alpha)^{\star U}$ as the image of a matrix M with entries in R, we can write $(PR_\mathfrak{p}^\alpha)^{\star U} = (PR^\alpha)^{\star U} R_\mathfrak{p}$ as the image of same matrix but working in $R_\mathfrak{p}$.

1. Since $a \notin P \Leftrightarrow a \notin PR_\mathfrak{p}$, a is an entry of M not in $(PR_\mathfrak{p}^\alpha)^{\star U}$. Then, by Lemma 2.2.30, step 1.(a) commutes with localization. However, for step 1.(b), we can take the same matrix X with entries in R_a but working in $R_\mathfrak{p}$. Then while we do operations in $R_\mathfrak{p}$, we see that $e_\alpha \in X(\operatorname{Im} M)_a \cap R^\alpha$ implies that $e_\alpha \in (X(\operatorname{Im} M)_a \cap R^\alpha)R_\mathfrak{p} \cong X(\operatorname{Im} M)_a \cap R_\mathfrak{p}^\alpha$. Also $U_1 = a^\nu X^{[p]} U X^{-1}$ has entries in R (and in $R_\mathfrak{p}$) for the same $\nu \gg 0$. Therefore, we end up with the same matrix U_0.

2. We first note that $(PR^\alpha)^{\star U} = PR^\alpha \Leftrightarrow (PR_\mathfrak{p}^\alpha)^{\star U} = PR_\mathfrak{p}^\alpha$. Therefore, if $(PR_\mathfrak{p}^\alpha)^{\star U} = PR_\mathfrak{p}^\alpha$, we can have the same construction working in $R_\mathfrak{p}$, i.e., we can take $a_1 \in R_\mathfrak{p} \setminus PR_\mathfrak{p}$, $g \in ((PR_\mathfrak{p})^{[p]} : PR_\mathfrak{p})$, $\alpha \times \alpha$ matrix V for the same $\mu \gg 0$ such that $a_1^\mu U = gV$ modulo $(PR_\mathfrak{p})^{[p]}$ and compute $d = \det V$.

(a) For any $r \in R$, we have the property that $r \in P \Leftrightarrow r \in PR_{\mathfrak{p}}$. Thus, if $d \in PR_{\mathfrak{p}}$, then we can have the same construction again, and so we can take $a_2 \in R_{\mathfrak{p}} \setminus PR_{\mathfrak{p}}$ and the same invertible matrix X with entries in R_{a_2} (and in $(R_{\mathfrak{p}})_{a_2} \cong (R_{a_2})_{\mathfrak{p}}$) such that the last column of UX^{-1} is zero, working in $R_{\mathfrak{p}}$. We also can take the same $\nu \gg 0$ such that the entries of $U_1 = (a_1a_2)^{\nu} X^{[p]} U X^{-1}$ are in R (and in $R_{\mathfrak{p}}$), and U_0 to be the same matrix. In addition, since $\mathrm{I}_e(-)$ operation commutes with localization, if we do calculations in $R_{\mathfrak{p}}$, then the stable value of

$$\{\mathrm{I}_e(U_0^{[p^{e-1}]} U_0^{[p^{e-2}]} \cdots U_0 R_{\mathfrak{p}})\}_{e>0}$$

is going to equal to the stable value of $\{\mathrm{I}_e(U_0^{[p^{e-1}]} U_0^{[p^{e-2}]} \cdots U_0 R) R_{\mathfrak{p}}\}_{e>0}$ which is $\mathcal{K}_0 R_{\mathfrak{p}}$. Now, since $\mathcal{K}_0 \subseteq PR^{\alpha-1} \Leftrightarrow \mathcal{K}_0 R_{\mathfrak{p}} \subseteq PR_{\mathfrak{p}}^{\alpha-1}$, we can do next:

   i. Working in $R_{\mathfrak{p}}$, if $\mathcal{K}_0 R_{\mathfrak{p}} \subseteq PR_{\mathfrak{p}}^{\alpha-1}$ we can write the last row of the matrix $U_1^{[p^{e-1}]} U_1^{[p^{e-2}]} \cdots U_1$ as $(g_1, \ldots, g_{\alpha-1}, 0)$.

   ii. Working in $R_{\mathfrak{p}}$, if $\mathcal{K}_0 R_{\mathfrak{p}} \nsubseteq PR_{\mathfrak{p}}^{\alpha-1}$, we can apply our algorithm recursively to U_0 and find all prime ideals which contain $PR_{\mathfrak{p}}$ minimally and denote their intersection with $\bar{\tau}$, which is $\tau R_{\mathfrak{p}}$, as we have showed all steps of algorithm commute with localization. Then we have

$$\mathrm{I}_1(U_1 \bar{l}(\bar{\tau} \mathcal{K}_0 R_{\mathfrak{p}}))^{\star U_1} = (\mathrm{I}_1(U_1 l(\tau K))^{\star U_1}) R_{\mathfrak{p}},$$

   where $\bar{l} : R_{\mathfrak{p}}^{\alpha-1} \to R_{\mathfrak{p}}^{\alpha-1} \oplus R_{\mathfrak{p}}$ is the extension map induced by l.

All other steps are similar to previous steps, and so all steps of our algorithm commute with localization.

Since our algorithm commutes with localization, by Lemma 3.2.17, the output set $\mathcal{A}_{R_{\mathfrak{p}}^{\alpha}}$ is the set of all U-special prime ideals of $R_{\mathfrak{p}}$, and hence,

$$\mathcal{A}_{R_{\mathfrak{p}}^{\alpha}} = \{PR_{\mathfrak{p}} \mid P \in \mathcal{A}_{R^{\alpha}} \text{ and } P \subseteq \mathfrak{p}\}.$$

□

Let U be an $\alpha \times \alpha$ matrix with entries in R, and let $\mathcal{A}_{R^{\alpha}}$ and $\mathcal{A}_{S^{\alpha}}$ be the output sets of our algorithm for R and S, respectively. Let P be a U-special prime ideal of R, i.e. $P \in \mathcal{A}_{R^{\alpha}}$. Since PS is not always a prime ideal of S, we do not have a relation between $\mathcal{A}_{R^{\alpha}}$ and $\mathcal{A}_{S^{\alpha}}$ like in Theorem 3.2.18. However, by Lemma 3.2.17, we can say that the minimal prime ideals of PS are in $\mathcal{A}_{S^{\alpha}}$. Therefore, the set of minimal prime ideals of elements from $\{PS \mid P \in \mathcal{A}_{R^{\alpha}}\}$ is contained in $\mathcal{A}_{S^{\alpha}}$.

3.3 An Application to Lyubeznik's F-modules

In this section, we investigate the connections between special ideals and local cohomology modules using Lyubeznik's theory of F-finite F-modules.

By Example 2.2.36, the i-th local cohomology module of R with respect to an ideal I is an F-finite F-module and there exist a finitely generated module M with an injective map $\beta : M \to F_R(M)$ such that

$$H^i_I(R) = \varinjlim(M \xrightarrow{\beta} F_R(M) \xrightarrow{F_R(\beta)} F_R^2(M) \xrightarrow{F_R^2(\beta)} \cdots)$$

where $\beta : M \to F_R(M)$ is a root morphism. Since M is finitely generated, we also have $M \cong \operatorname{Coker} A = R^\alpha / \operatorname{Im} A$ for some matrix A with entries in R as in subsection 2.1.1. Hence,

$$H^i_I(R) \cong \varinjlim(\operatorname{Coker} A \xrightarrow{U} \operatorname{Coker} A^{[p]} \to \cdots)$$

for some $\alpha \times \alpha$ matrix U with entries in R such that $U \operatorname{Im} A \subseteq \operatorname{Im} A^{[p]}$. Furthermore, U defines an injective map on $\operatorname{Coker} A$, since β is a root morphism.

Remark 3.3.1. *[16, Section 4] If $(R, \mathfrak{m})$ is a local ring, $\mathcal{M}$ is an F-finite module and $\mathcal{M}'$,$\mathcal{M}'' \subset \mathcal{M}$ are two F-submodules with the property that*

$$\dim_R \operatorname{Supp}(\mathcal{M}/\mathcal{M}') = \dim_R \operatorname{Supp}(\mathcal{M}/\mathcal{M}'') = 0,$$

then their intersection also has this property, and there exists a smallest F-submodule $\mathcal{N}$ of $\mathcal{M}$ with this property, since $\mathcal{M}$ is Artinian as an F-module. Since $\mathcal{L} = \mathcal{M}/\mathcal{N}$ is an F-module, Theorem 2.2.34 implies that it is injective. Since it is also F-finite, the Bass numbers of it are finite. Hence, $\mathcal{L} \cong E^k$ as R-modules, where $k = \mu_1(\mathfrak{m}, \mathcal{L})$ and E is the injective hull of the residue field of R.

Definition 3.3.2. *If R is local, we define the corank of an F-finite F-module $\mathcal{M}$ the number k in Remark 3.3.1, and denote it by $\operatorname{crk} \mathcal{M} = k$.*

In Section 4 of [16], Lyubeznik uses the theory of corank to shed more light on the notion of F-depth of a scheme in characteristic p, which is analogous to the notion of DeRham depth of a scheme in characteristic 0. Following [16, Section 4], in equicharacteristic 0 one can interpret the DeRham depth in terms of closed points only. Proposition 4.14 in [16] shows that in characteristic p we can not interpret the F-depth of a scheme Y in terms of closed points only. To show this Lyubeznik proves that there are only finitely many prime ideals P of A such that $\operatorname{crk}(H^i_{IA_P}(A_P)) \neq 0$.

Here $Y = \operatorname{Spec} B$, where B is a finitely generated algebra over a regular local ring S, $A = S[x_1, \cdots, x_n]$ and I is the kernel of the surjection $A \to B$. Our next theorem not only reproves this result but also gives us an effective way to compute desired prime ideals.

Theorem 3.3.3. *Let I be an ideal of R and $P \subset R$ a prime ideal. If $H^i_{IR_P}(R_P)$ has non zero corank then P is in the output of our algorithm introduced in section 3.2, i.e.*

$$\operatorname{crk}(H^i_{IR_P}(R_P)) \neq 0 \Rightarrow P \in \mathcal{A}_{R^\alpha}.$$

for some $\alpha \times \alpha$ matrix U with entries in R.

Proof. Since $H^i_{IR_P}(R_P) \cong R_p \otimes_R H^i_I(R)$, we have

$$H^i_{IR_P}(R_P) \cong \varinjlim(\operatorname{Coker} A_P \xrightarrow{U_P} \operatorname{Coker} A_P^{[p]} \to \cdots)$$

where A_P and U_P are localizations of A and U, respectively. We also have that U_P defines an injective map on $\operatorname{Coker} A_P$ since U defines a root morphism for $H^i_I(R)$.

$\operatorname{crk}(H^i_{IR_P}(R_P)) \neq 0$ implies that there exists a proper F_{R_P}-submodule $\mathcal{N}$ of $H^i_{IR_P}(R_P)$ such that $\dim_{R_P} \operatorname{Supp}(H^i_{IR_P}(R_P)/\mathcal{N}) = 0$. Since $H^i_{IR_P}(R_P)$ is F_{R_P}-finite, by Remarks 2.2.35 (3), we have

$$\mathcal{N} = \varinjlim(N \to F_{R_P}(N) \to F^2_{R_P}(N) \to \cdots)$$

where $N = \mathcal{N} \cap \operatorname{Coker} A_P$ is an R_P-submodule of $\operatorname{Coker} A_P$. Thus, $N \cong V/\operatorname{Im} A_P$ for some submodule $V \subseteq R_P^\alpha$ such that $U_P V \subseteq V^{[p]}$. Then

$$\begin{aligned} H^i_{IR_P}(R_P)/\mathcal{N} &\cong \varinjlim(\operatorname{Coker} A_P/N \xrightarrow{U_P} F_{R_P}(\operatorname{Coker} A_P/N) \to \cdots) \\ &\cong \varinjlim(R_P^\alpha/V \xrightarrow{U_P} R_P^\alpha/V^{[p]} \to \cdots). \end{aligned}$$

Furthermore,

$$\begin{aligned} \dim_{R_P} \operatorname{Supp}(H^i_{IR_P}(R_P)/\mathcal{N}) = 0 &\Rightarrow \operatorname{Ass}(H^i_{IR_P}(R_P)/\mathcal{N}) = \{PR_P\} \\ &\Rightarrow \operatorname{Ass}(R_P^\alpha/V) = \{PR_P\} \\ &\Rightarrow \operatorname{Ann}_{R_P}(R_P^\alpha/V) \text{ is } PR_P\text{-primary} \end{aligned}$$

Therefore, $\operatorname{Ann}_{R_P}(R_P^\alpha/V)$ is U_P-special and so is PR_P by Lemma 3.2.8, because it is the only minimal prime ideal of $\operatorname{Ann}_{R_P}(R_P^\alpha/V)$, i.e. $PR_P \in \mathcal{A}_{R_P^\alpha}$. Then by Theorem 3.2.18, $P \in \mathcal{A}_{R^\alpha}$ $\square$

Corollary 3.3.4. $\mathcal{C}_R := \{P \in \mathcal{A}_{R^\alpha} \mid (\operatorname{Im} A_P + PR_P^\alpha)^{\star U_P} \neq R_P^\alpha\}$ *is the set of all prime ideals of* R *which satisfy* $\operatorname{crk}(H^i_{IR_P}(R_P)) \neq 0$

Proof. By Theorem 3.3.3, $\operatorname{crk}(H^i_{IR_P}(R_P)) \neq 0$ implies that PR_P is a U_P-special prime ideal of R_P such that $PR_P = \operatorname{Ann}_{R_P}(R_p^\alpha/W)$ for some proper submodule $W \subset R_p^\alpha$ where $\operatorname{Im} A_P \subseteq W$ and A_P as in Theorem 3.3.3. Since $(\operatorname{Im} A_P + PR_P^\alpha)^{\star U_P}$ is the smallest submodule of R_P^α which satisfies $PR_P = \operatorname{Ann}_{R_P}(R_p^\alpha/(\operatorname{Im} A_P + PR_P^\alpha)^{\star U_P})$, if $(\operatorname{Im} A_P + PR_P^\alpha)^{\star U_P} = R_P^\alpha$, then we have a contradiction with the existence of W. Hence, the set of primes ideals of R which satisfy $\operatorname{crk}(H^i_{IR_P}(R_P)) \neq 0$ is the set $\{P \in \mathcal{A}_{R^\alpha} \mid (\operatorname{Im} A_P + PR_P^\alpha)^{\star U_P} \neq R_P^\alpha\}$. □

Corollary 3.3.4 says that if we want to compute the prime ideals of R which satisfy $\operatorname{crk}(H^i_{IR_P}(R_P)) \neq 0$, we pick an element $P \in \mathcal{A}_{R^\alpha}$ and need to check whether $(\operatorname{Im} A_P + PR_P^\alpha)^{\star U_P}$ is equal to R_P^α.

Chapter 4

Annihilators of Cartier Quotients

Let R be a ring of prime characteristic p. In this chapter, we investigate finitely generated Cartier modules over R and present our computational results on these. In particular, we introduce a new algorithm for finding annihilators of Cartier quotients for a given finitely generated Cartier module. We finish the chapter with the connections between Cartier modules and Lyubeznik's F-modules.

4.1 Cartier Modules

In this section, we recall the notion of Cartier modules over R, and we give some properties of finitely generated Cartier modules which are proven in [2]. We also provide some technical lemmas with their proofs.

Definition 4.1.1. *A Cartier module is an R-module M equipped with an additive map $C : M \to M$, which we call the structural map of M, such that $C(r^{p^e}m) = rC(m)$ for all $m \in M$ and $r \in R$, i.e. $C \in \operatorname{Hom}_R(F^e_* M, M)$.*

A map of Cartier modules is a map $\varphi : M \to N$ such that the following diagram commutes

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ {\scriptstyle C_M}\downarrow & & \downarrow{\scriptstyle C_N} \\ M & \xrightarrow[\varphi]{} & N \end{array}$$

where C_M and C_N are the structural maps of M and N, respectively.

We generally fix $e = 1$, when we take a Cartier module M with the structural map C as a pair (M, C).

Remark 4.1.2. *We can define the composition of Cartier module structures on M as additive maps. If $C_1, C_2 : M \to M$ are two structural maps on M which satisfy $C_1(r^{p^{e_1}}m) = rC_1(m)$ and $C_2(r^{p^{e_2}}m) = rC_2(m)$ for all $m \in M$ and $r \in R$, respectively, then the composition maps satisfy*

$$(C_1 \circ C_2)(r^{p^{e_1+e_2}}m) = C_1(C_2((r^{p^{e_1}})^{e_2}m)) = C_1(r^{p^{e_1}}C_2(m))$$

$$= rC_1(C_2(m)) = r(C_1 \circ C_2)(m)$$

and similarly

$$(C_2 \circ C_1)(r^{p^{e_1+e_2}}m) = r(C_2 \circ C_1)(m),$$

i.e. $C_1 \circ C_2, C_2 \circ C_1 \in \mathrm{Hom}_R(F_*^{e_1+e_2}M, M)$. In particular, if $C \in \mathrm{Hom}_R(F_*M, M)$, then the e-th iteration C^e defines a Cartier structure on M and $C^e \in \mathrm{Hom}_R(F_*^eM, M)$.

Definition 4.1.3. *A Cartier module (M, C) is called nilpotent if $C^k(M) = 0$ for some $k \in \mathbb{N}$, and the smallest k such that $C^k(M) = 0$ is called the order of nilpotence of M which is denoted by $\mathrm{on}(M) = k$.*

Remark 4.1.4. *Let (M, C) be a Cartier module, and W be a multiplicative subset of R. By Proposition 2.2.4, we know that $W^{-1}F_*M \cong F_*W^{-1}M$. Therefore, localization of the structural map $C : F_*M \to M$ with respect to W gives $W^{-1}M$ a Cartier module structure over $W^{-1}R$, which is $C_W : W^{-1}M \to W^{-1}M$ defined by $C_W(\frac{m}{r}) = \frac{C(r^{p-1}m)}{r}$ for all $m \in M$ and $r \in W$.*

Remarks 4.1.5. *[2, Section 2.2] Let (M, C) be a finitely generated Cartier module.*

1. *M is nilpotent if and only if the localization M_P is nilpotent for every prime ideal P.*

2. *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of finitely generated Cartier modules. Then M is nilpotent if and only if M' and M'' are nilpotent.*

3. *We define M_{nil} to be the sum of all nilpotent Cartier submodules of M. Then $\overline{M} := M/M_{nil}$ becomes the smallest Cartier quotient of M such that the kernel of $M \twoheadrightarrow \overline{M}$ is nilpotent.*

4. *By Proposition 2.14 in [2], the descending chain*

$$M \supseteq C(M) \supseteq C^2(M) \supseteq \cdots$$

stabilizes. We denote the stable image by $\underline{M}$ which is the smallest Cartier submodule of M such that $M/\underline{M}$ is nilpotent.

Furthermore, if M has no proper nilpotent quotients, then the structural map C is surjective. Because, if C is not surjective, $M/\underline{\mathrm{M}}$ becomes a nonzero nilpotent quotient. It might also be expected that $\ker C^e$ is a nilpotent Cartier submodule of M. However, $\ker C^e$ is not even an R-submodule of M in general, since for any $m \in M$ and $r \in R$, $C^e(m) = 0$ does not imply $C^e(rm) = 0$ unless $r = s^{p^e}$ for some $s \in R$.

Facts 4.1.6. *Let R be F-finite, and (M, C) be a finitely generated Cartier module, and $X = \operatorname{Spec} R$.*

1. *By Proposition 4.1 in [2], if X is irreducible, then there is an open subset $\mathcal{U}$ of X such that for all non-minimal prime ideal $P \in \mathcal{U}$ we have:*

   ($\star$) *All finite length Cartier quotients of M_P are nilpotent.*

2. *By Proposition 4.5 in [2], there is a finite subset $\mathcal{S} \subseteq X$ such that for all $P \in X \backslash \mathcal{S}$, M satisfies ($\star$) at P.*

3. *By Proposition 4.9 in [2], if C is surjective, then the collection of ideals $\mathcal{A} := \{\operatorname{Ann}_R M/N \mid N$ is a Cartier submodule of $M\}$ is a finite set of radical ideals consisting of all intersections of the finitely many primes in it.*

Next we state two important properties of Cartier modules which we use repeatedly in this chapter (cf. footnotes 6 and 7 in [2]).

Lemma 4.1.7. *Let (M, C) be a finitely generated Cartier module. If $\underline{M}$ (or $\overline{M}$) satisfies ($\star$) at any prime ideal P of R, then M satisfies ($\star$) at P.*

Proof. Let P be a prime ideal of R. Fix a finite length Cartier quotient M_P/N. Assume that $\underline{\mathrm{M}}$ satisfies ($\star$) P. Then

$$(\underline{\mathrm{M}}_P + N)/N \cong \underline{\mathrm{M}}_P/(N \cap \underline{\mathrm{M}}_P)$$

has finite length, and so it is nilpotent by our assumption. Since $M_P/\underline{\mathrm{M}}_P$ is nilpotent, we also have

$$M_P/(N+\underline{\mathrm{M}}_P) \cong \frac{M_P/\underline{\mathrm{M}}_P}{(N+\underline{\mathrm{M}}_P)/\underline{\mathrm{M}}_P}$$

is nilpotent. On the other hand,

$$M_P/(N+\underline{\mathrm{M}}_P) \cong \frac{M_P/N}{(\underline{\mathrm{M}}_P+N)/N}.$$

Hence M_P/N is nilpotent by Remarks 4.1.5 *2.*

Similarly, we assume now that $\overline{M}$ satisfies $(\star)$ at P.

$$M_P/((M_{nil})_P+N) \cong \frac{M_P/(M_{nil})_P}{((M_{nil})_P+N)/(M_{nil})_P}$$

has finite length and it is nilpotent by the assumption. In addition, $\dfrac{(M_{nil})_P+N}{N} \cong \dfrac{(M_{nil})_P}{(M_{nil})_P \cap N}$ has finite length, and so it is nilpotent. On the other hand, we have

$$M_P/((M_{nil})_P+N) \cong \frac{M_P/N}{((M_{nil})_P+N)/N}.$$

Therefore, M_P/N is nilpotent by Remarks 4.1.5 *2.* again. □

Lemma 4.1.8. *Let (M, C) be a finitely generated Cartier module with surjective C, and I be an ideal of R, and $M(I) := \dfrac{M}{\sum_{e\geq 0} C^e(IM)}$. If as an R/I-module $M(I)$ satisfies $(\star)$ at any $P \in V(I)$, then M as an R-module satisfies $(\star)$ at P as well.*

Proof. Suppose that $M(I)$ satisfies $(\star)$ at any $P \in V(I)$, i.e. any finite length Cartier quotient of $M(I)_P$ is nilpotent. Let $N \subseteq M_P$ be such that M_P/N has finite length. Thus, $(PR_P)^k(M_P/N) = 0$ for some $k \in \mathbb{N}$. On the other hand, for some $i \gg 0$ we have $(PR_P)^{[p^i]}(M_P/N) \subseteq (PR_P)^k(M_P/N)$ which implies that $(PR_P)^{[p^i]}(M_P/N) = 0$. Then, since C is surjective,

$$\begin{aligned}(PR_P)^{[p^i]}(M_P/N) = 0 &\Rightarrow C^i\big((PR_P)^{[p^i]}(M_P/N)\big) = 0\\ &\Rightarrow (PR_P)C^i(M_P/N) = 0 \Rightarrow (PR_P)(M_P/N) = 0\end{aligned}$$

and so $(PR_P)M_P \subseteq N$. Since $P \in V(I)$, $IR_P \subseteq PR_P$ and we have $(IR_P)M_P \subseteq N$. It follows that $K := \sum_{e\geq 0} C^e((IR_P)M_P) \subseteq N$. Now let N' denote the submodule $N/K \subseteq M(I)_P$. Then $M(I)_P/N' \cong (M_P/K)/(N/K) \cong M_P/N$ has finite length. However, since $M(I)$ satisfies $(\star)$, $M(I)_P/N'$ is nilpotent. Hence, M_P/N is nilpotent. □

Lemma 4.1.9. *Let (M, C) be a finitely generated Cartier module with surjective C. Let $\mathcal{A}$ and $\mathcal{S}$ be as in Facts 4.1.6. Then:*

1. *If an ideal P of R is an element of $\mathcal{A}$, then* $P = \operatorname{Ann}_R \dfrac{M}{\sum_{e\geq 0} C^e(PM)}$,

2. *If also P is a prime ideal in $\mathcal{A}$, then $P \in \mathcal{S}$.*

Proof. For *1.* suppose that $P \in \mathcal{A}$, i.e. $P = \operatorname{Ann}_R M/N$ for some Cartier submodule N of M. It is clear that $P \subseteq \operatorname{Ann}_R M(P)$, where $M(P) := \dfrac{M}{\sum_{e\geq 0} C^e(PM)}$. On the other hand, since $\sum_{e\geq 0} C^e(PM)$ is the smallest Cartier submodule of M which contains PM, we also have that $\operatorname{Ann}_R M(P) \subseteq \operatorname{Ann}_R M/N$. Hence, $P \subseteq \operatorname{Ann}_R M(P) \subseteq \operatorname{Ann}_R M/N = P$, and so $P = \operatorname{Ann}_R M(P)$.

For *2.* suppose $P = \operatorname{Ann}_R M/N$ for some Cartier submodule $N \subseteq M$. Then $(M/N)_P = M_P/N_P$ is a non-zero finite length quotient of M_P as an R_P-module. Since C is surjective, the structural map of M_P is surjective and hence that of $(M/N)_P$ is surjective. Therefore, $(M/N)_P$ can not be nilpotent, and so $P \in \mathcal{S}$. □

4.2 Adjoint map to the structural map

In this section, we use the Hom-Tensor adjunction (cf. Theorem 2.75 in [19]) to define an adjoint map to the structural map of a given Cartier module M, which will help us to compute the nilpotent Cartier submodule M_{nil} of M (cf. Section 2.3 of [2]). Let $e \in \mathbb{N}$, and M be an R-module. If we consider F^e_*R as an (F^e_*R, R)-bimodule, then we have the following isomorphism

$$\operatorname{Hom}_R(F^e_*M \otimes_{F^e_*R} F^e_*R, M) \cong \operatorname{Hom}_{F^e_*R}(F^e_*M, \operatorname{Hom}_R(F^e_*R, M)).$$

Thus, for a given Cartier map

$$C \in \operatorname{Hom}_R(F^e_*M, M) \cong \operatorname{Hom}_R(F^e_*M \otimes_{F^e_*R} F^e_*R, M)$$

we have an adjoint map, which is F^e_*R-linear,

$$\kappa : F^e_*M \to \mathrm{Hom}_R(F^e_*R, M)$$

given by $\kappa(F^e_*m) = \phi_m$ where $\phi_m(-) = C(F^e_*m-)$.

Proposition 4.2.1. *Let (M, C) be a finitely generated Cartier module and*

$$\kappa^i : F^i_*M \to \mathrm{Hom}_R(F^i_*R, M)$$

be the adjoint map to C^i. Let K_i be the R-submodule of M such that $F^i_*K_i = \ker \kappa^i$. Then:

1. *K_i is the largest nilpotent Cartier submodule of M such that $\mathrm{on}(K_i) \leq i$,*
2. *$M_{nil} = \bigcup_i K_i$,*
3. *the sequence of nilpotent Cartier submodules $K_1 \subseteq K_2 \subseteq \cdots \subseteq K_i \subseteq \cdots$ stabilizes at the first integer where we get $K_i = K_{i+1}$.*

Proof. Since R is Noetherian, each K_i is finitely generated, and let $K_i = \langle k_1, \ldots, k_s \rangle$. Then for each generator k_j, $\kappa^i(F^i_*k_j) = \phi_{k_j}$ is a zero map, i.e. $\mathrm{Im}\, \phi_{k_j} = \phi_{k_j}(F^i_*R) = C^i(F^i_*Rk_j) = 0$. On the other hand,

$$C^i(F^i_*K_i) = C^i(F^i_*(Rk_1 + \cdots + Rk_s)) = C^i(F^i_*Rk_1) + \cdots + C^i(F^i_*Rk_s)$$
$$= \mathrm{Im}\, \phi_{k_1} + \cdots + \mathrm{Im}\, \phi_{k_s} = 0$$

Now let N be another nilpotent Cartier submodule of M with $\mathrm{on}(n) \leq i$ and let $N = \langle n_1, \cdots, n_k \rangle$. Then

$$0 = C^i(F^i_*N) = C^i(F^i_*(Rn_1 + \cdots + Rn_k)) = C^i(F_*Rn_1) + \cdots + C^i(F^i_*Rn_k),$$

and so $C^i(F^i_*Rn_j) = 0$ for each generator n_j. However, $\kappa^i(F^i_*n_j) = \phi_{n_j}$ is an F^i_*R-linear map where $\mathrm{Im}\, \phi_{n_j} = \phi_{n_j}(F^i_*R) = C^i(F^i_*Rn_j) = 0$. Thus, $F^i_*n_j \in \mathrm{Ker}\, \kappa^i$, and so $n_j \in K_i$, i.e. $N \subseteq K_i$. This proves *1.*

We clearly have the following ascending sequence $K_1 \subseteq K_2 \subseteq \cdots \subseteq K_i \subseteq \cdots$ of nilpotent Cartier submodules of M. Therefore, $M_{nil} = \bigcup_i K_i$.

By the Noetherian hypothesis the ascending chain above stabilizes. Assume now that i is the first integer such that $K_i = K_{i+1}$. We shall show that $K_{i+1} = K_{i+2}$

and (3) follows by induction. Then $C^{i+2}(K_{i+2}) = C^{i+1}(C(K_{i+2})) = 0$ implies that $C(K_{i+2})$ is a nilpotent Cartier submodule of M with $\mathrm{on}(C(K_{i+2})) \leq i+1$. Then since K_{i+1} is the largest nilpotent Cartier submodule of M with $\mathrm{on}(K_{i+1}) \leq i+1$, $C(K_{i+2}) \subseteq K_{i+1}$. Then by assumption $C(K_{i+2}) \subseteq K_i$, and so $C^i(C(K_{i+2})) = 0$. Therefore, $C^{i+1}(K_{i+2}) = 0$ which implies that $K_{i+2} \subseteq K_{i+1}$. Hence, $K_{i+1} = K_{i+2}$. □

4.3 The Polynomial case and computations

In this section, we prove some technical lemmas which will be used to compute $\underline{\mathrm{M}}$ and $\overline{M}$ for a given Cartier module M. Moreover, we prove the main theorem of this chapter using our computational methods. Henceforth, we will assume that $R = \Bbbk[x_1, \ldots, x_n]$ is a polynomial ring (or $R = \Bbbk[\![x_1, \ldots, x_n]\!]$ a power series ring) over an F-finite field $\Bbbk$ of prime characteristic p.

Lemma 4.3.1. *For any $\Phi \in \mathrm{Hom}_R(F^e_* R^\alpha, R^\alpha)$, there exists an $\alpha \times \alpha$ matrix U with entries in R such that $\Phi(-) = \Pi_e(F^e_* U-)$ where*

$$\Pi_e \begin{pmatrix} F^e_* v_1 \\ \vdots \\ F^e_* v_\alpha \end{pmatrix} = \begin{pmatrix} \pi_e(F^e_* v_1) \\ \vdots \\ \pi_e(F^e_* v_\alpha) \end{pmatrix}$$

for all $(F_* v_1, \ldots, F^e_* v_\alpha)^t \in F^e_* R^\alpha$, and $\pi_e \in \mathrm{Hom}_R(F^e_* R, R)$ is the trace map.

Proof. If $\alpha = 1$, by Lemma 3.1.4, $\mathrm{Hom}_R(F^e_* R, R)$ is generated as an $F^e_* R$-module by the trace map π_e. If $\alpha > 1$, we first need to describe elements of $\mathrm{Hom}_R(F^e_* R, R^\alpha)$. Since $\mathrm{Hom}_R(F^e_* R, R^\alpha) \cong \mathrm{Hom}_R(F^e_* R, R)^\alpha$, any R-linear map $\varphi \in \mathrm{Hom}_R(F^e_* R, R^\alpha)$ can be expressed as a direct sum of elements of $\mathrm{Hom}_R(F^e_* R, R)$. Therefore, we have $\varphi(-) = (\phi_1(-), \ldots, \phi_\alpha(-))^t$ for some $\phi_i \in \mathrm{Hom}_R(F^e_* R, R)$ where $1 \leq i \leq \alpha$, and by Lemma 3.1.4, $\varphi(-) = (\pi_e(F^e_* u_1-), \ldots, \pi_e(F^e_* u_\alpha-))^t$ for some $u_1, \ldots, u_\alpha \in R$.

Since $\mathrm{Hom}_R(F^e_* R^\alpha, R^\alpha) \cong \mathrm{Hom}_R(F^e_* R, R^\alpha)^\alpha$, any $\Phi \in \mathrm{Hom}_R(F^e_* R^\alpha, R^\alpha)$ can be expressed as a direct sum of elements of $\mathrm{Hom}_R(F^e_* R, R^\alpha)$. Therefore, for any element $(v_1, \ldots, v_\alpha)^t \in R^\alpha$, we have

$$\Phi((F^e_* v_1, \ldots, F^e_* v_\alpha)^t) = \sum_{1 \leq j \leq \alpha} \varphi_j(F^e_* v_j)$$

for some $\varphi_j \in \mathrm{Hom}_R(F^e_*R, R^\alpha)$. By the previous observation of $\mathrm{Hom}_R(F^e_*R, R^\alpha)$, for each j, we also have $\varphi_j(F^e_*v_j) = (\pi_e(F^e_*u_{1j}v_j), \ldots, \pi_e(F^e_*u_{\alpha j}v_j))^t$ for some elements $u_{1j}, \ldots, u_{\alpha j} \in R$. Thus,

$$\Phi\begin{pmatrix} F^e_*v_1 \\ \vdots \\ F^e_*v_\alpha \end{pmatrix} = \sum_{1\le j\le\alpha}\begin{pmatrix} \pi_e(F^e_*u_{1j}v_j) \\ \vdots \\ \pi_e(F^e_*u_{\alpha j}v_j) \end{pmatrix}.$$

Hence, for any $\Phi \in \mathrm{Hom}_R(F^e_*R^\alpha, R^\alpha)$, there exist an $\alpha \times \alpha$ matrix U with entries $u_{ij} \in R$ such that $\Phi(-) = \Pi_e(F^e_*U-)$ where Π_e takes the components of elements in $F^e_*R^\alpha$ to their images under the trace map π_e. □

Definition 4.3.2. *Let the notation and situation be as in Lemma 4.3.1. We call the map Π_e in Lemma 4.3.1 the trace map on $F^e_*R^\alpha$, or just the trace map when the content is clear.*

The following lemma extends Lemma 3.1.6 to submodules of free modules, and gives a way to connect $I_e(-)$ operation to the images of elements in $\mathrm{Hom}_R(F^e_*R^\alpha, R^\alpha)$.

Lemma 4.3.3. *Let V and W be submodules of R^α. Then $\Pi_e(F^e_*V) \subseteq W$ if and only if $V \subseteq W^{[p^e]}$.*

Proof. Assume that $\Pi_e(F^e_*V) \subseteq W$. Then by the Noetherian hypothesis V and W are finitely generated, and since Π_e is R-linear, we may assume that $V = \langle v \rangle$ for some element $v = (v_1, \ldots, v_\alpha)^t \in V$. Additionally, since F^e_*R is a free R-module with basis $\mathcal{B}$ as in Remark 3.1.3, for each v_i, we have

$$F^e_*v_i = \sum_{F^e_*g\in\mathcal{B}} r_{ig}F^e_*g \text{ for some } r_{ig} \in R \text{ and } F^e_*g \in \mathcal{B}.$$

Then $F^e_*v = (F^e_*v_1, \ldots, F^e_*v_\alpha)^t$ can be expressed uniquely in the form

$$F^e_*v = \sum_{F^e_*g\in\mathcal{B}} u_gF^e_*g \text{ where } u_g = (r_{1g}, \ldots, r_{\alpha g})^t.$$

Since $\Pi_e(F^e_*v) = (\pi_e(F^e_*v_1), \ldots, \pi_e(F^e_*v_\alpha))^t$, a similar way in the proof of Lemma 3.1.6 implies that $\Pi_e(F^e_*V) = \langle u_g \rangle$ for u_g's from the above expression of F^e_*v. Then by the assumption, we have $\Pi_e(F^e_*V) = \langle u_g \rangle \subseteq W$, and since

$$F^e_*v = \sum_{F^e_*g\in\mathcal{B}} u_gF^e_*g = F^e_*(\sum_{F^e_*g\in\mathcal{B}} u_g^{[p^e]}g)$$

we also have $v = \sum_g u_g^{[p^e]} g \in W^{[p^e]}$. Therefore, $V \subset W^{[p^e]}$.

For the converse, we first note that

$$V \subseteq W^{[p^e]} \Rightarrow F_*^e V \subseteq F_*^e W^{[p^e]} \Rightarrow \Pi_e(F_*^e V) \subseteq \Pi_e(F_*^e W^{[p^e]}).$$

If $W = \langle w_1, \cdots, w_s \rangle$ for some $w_i \in W$, then $W^{[p^e]} = \langle w_1^{[p^e]}, \ldots, w_s^{[p^e]} \rangle$. Now take an element $z = \sum_i r_i w_i^{[p^e]} \in W^{[p^e]}$ for some $r_i \in R$. Then $\Pi_e(F_*^e z) = \sum_i w_i \pi_e(F_*^e r_i) \in W$, and so $\Pi_e(F_*^e V) \subseteq W$. □

Corollary 4.3.4. *Let V be a submodule of R^α and let $C : F_*^e R^\alpha \to R^\alpha$ be a Cartier map such that $C(-) = \Pi_e(F_*^e U -)$ for some $\alpha \times \alpha$ matrix U with entries in R. Then $C(F_*^e V) = \Pi_e(F_*^e UV) = \mathrm{I}_e(UV)$ and the $\star$-closure of V gives the smallest C Cartier submodule of R^α which contains V.*

Proof. Since $UV \subseteq \mathrm{I}_e(UV)^{[p^e]}$, $\Pi_e(F_*^e UV) \subseteq \mathrm{I}_e(UV)$ by Lemma 4.3.3. On the other hand, we have $UV \subseteq \Pi_e(F_*^e UV)^{[p^e]}$ by Lemma 4.3.3 again. Then by minimality of $\mathrm{I}_e(UV)$, we must have $\mathrm{I}_e(UV) = \Pi_e(F_*^e UV)$. The second claim follows from the fact that

$$\begin{aligned} V \text{ is } C \text{ Cartier submodule of } R^\alpha &\Leftrightarrow C(F_*^e V) = \Pi_e(F_*^e UV) \subseteq V \\ &\Leftrightarrow UV \subseteq V^{[p^e]}. \end{aligned}$$

□

Lemma 4.3.5. *Let $C : F_*^e R^\alpha \to R^\alpha$ be a Cartier map with the $\alpha \times \alpha$ matrix U such that $C(-) = \Pi_e(F_*^e U -)$. If C is surjective, then $\det U$ is not zero.*

Proof. We will assume $\det U = 0$ and try to get a contradiction to our assumption. In this case, there exist an invertible matrix V with entries in the fraction field $\mathbb{F}$ of R such that UV has a zero column. If f is the multiplication of denominators of entries of the matrix V, then V is an invertible matrix with entries in R_f. On the other hand, since C is surjective, the localization map $C_f : F_*^e R_f^\alpha \to R_f^\alpha$ is surjective. Then $C_f(F_*^e R_f^\alpha) = \Pi_e(F_*^e U R_f^\alpha) = \Pi_e(F_*^e UV R_f^\alpha) \subseteq R_f^{\alpha-1}$ since UV has a zero column. But this contradicts with the surjectivity, and so $\det U$ must be non zero. □

Next we investigate how Cartier structures behave on finitely generated R-modules.

Proposition 4.3.6. *Let (M, C) be a finitely generated Cartier module and let $M \cong R^\alpha / \operatorname{Im} A$ as in Subsection 2.1.1. Then there exist a Cartier module structure C' on R^α such that the diagram below is commutative*

$$\begin{array}{ccc} F_*R^\alpha & \xrightarrow{C'} & R^\alpha \\ \downarrow & & \downarrow \\ F_*(R^\alpha/\operatorname{Im} A) & \xrightarrow[\tilde{C}]{} & R^\alpha/\operatorname{Im} A \end{array} \tag{4.1}$$

where the vertical arrows are natural surjections and $\tilde{C}$ is isomorphic to C. In particular, if C is surjective then C' is surjective.

Proof. Since F_*R is a free R-module, there exist an R-linear map $C' : F_*R^\alpha \to R^\alpha$ such that the diagram 4.1 is commutative. In the case that C is surjective, let $\{m_1, \cdots, m_\alpha\}$ be a minimal generating set for M and let $\rho : R^\alpha \twoheadrightarrow M$ be the projection which sends each elementary vector e_i to m_i. Since C is surjective, the composition $F_*R^\alpha \twoheadrightarrow F_*M \xrightarrow{C} M$ is surjective, and so is the composition $F_*R^\alpha \xrightarrow{C'} R^\alpha \twoheadrightarrow M$. Therefore, there exists $F_*a_i \in F_*R^\alpha$ such that $\rho(C'(F_*a_i)) = m_i$ for each m_i. Thus, $e_i = C'(F_*a_i) + b_i$ for some $b_i \in \ker \rho$. On the other hand, we claim that the set $\{e_1 - b_1, \ldots, e_\alpha - b_\alpha\}$ generates R^α freely. Otherwise, we would be able to write one of $e_i - b_i$'s as an R-linear combination of others, i.e.

$$e_i - b_i = r_1(e_1 - b_1) + \cdots + r_{i-1}(e_{i-1} - b_{i-1}) + r_{i+1}(e_{i+1} - b_{i+1}) + \cdots + r_\alpha(e_\alpha - b_\alpha)$$

for some i and $r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_\alpha \in R$. This means that

$$\begin{aligned} m_i &= \rho(e_i - b_i) \\ &= \rho\big(r_1(e_1 - b_1) + \cdots + r_{i-1}(e_{i-1} - b_{i-1}) + r_{i+1}(e_{i+1} - b_{i+1}) + \cdots + r_\alpha(e_\alpha - b_\alpha)\big) \\ &= r_1\rho(e_1 - b_1) + \cdots + r_{i-1}\rho(e_{i-1} - b_{i-1}) + r_{i+1}\rho(e_{i+1} - b_{i+1}) + \cdots + r_\alpha\rho(e_\alpha - b_\alpha) \\ &= r_1m_1 + \cdots + r_{i-1}m_{i-1} + r_{i+1}m_{i+1} + \cdots + r_\alpha m_\alpha. \end{aligned}$$

However, this contradicts with the minimality of $\{m_1, \ldots, m_\alpha\}$. Hence, our claim is true, and so C' is surjective. □

Notation 4.3.7. *By Proposition 4.3.6, for a given finitely generated Cartier module (M, C), there exist a Cartier module structure C' on R^α such that $C'(F_* \operatorname{Im} A) \subseteq$*

$\operatorname{Im} A$. Then by Lemma 4.3.1, there exist an $\alpha \times \alpha$ matrix U with entries in R such that $C'(-) = \Pi_e(F_* U -)$, and by Corollary 4.3.4, $U \operatorname{Im} A \subseteq \operatorname{Im} A^{[p]}$. Therefore, by $(\operatorname{Coker} A, U)$ we mean a finitely generated Cartier module with a square matrix U defining the structural map on it.

We now start explaining how to compute the Cartier modules $\underline{\mathrm{M}}$ and $\overline{M}$ for a finitely generated R-module M. First we need to discuss how to define a composition of trace maps.

Notation 4.3.8. *Let $k, k_1, k_2 \in \mathbb{Z}$. We will write*

1. *$\bar{x}$ to denote $x_1 \dots x_n$,*
2. *$\bar{\alpha}$, $\bar{\alpha} + \bar{\beta}$ and $k\bar{\alpha}$ to denote the n-tuples $(\alpha_1, \dots, \alpha_n)$, $(\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$ and $(k\alpha_1, \dots, k\alpha_n)$, respectively,*
3. *$\bar{x}^{\bar{\alpha}}$ to denote $x_1^{\alpha_1} \dots x_n^{\alpha_n}$, and $\bar{x}^k$ to denote $x_1^k \dots x_n^k$,*
4. *$k_1 < \bar{\alpha} < k_2$ to mean that $k_1 < \alpha_i < k_2$ for each i,*
5. *$r_{\bar{\alpha}}$ and $r_{\bar{k}}$ to denote the elements of R indexed with the n-tuples $\bar{\alpha}$ and $(k, \dots, k)$, respectively.*

Lemma 4.3.9. *Let $\Pi_e \in \operatorname{Hom}_R(F_*^e R^\alpha, R^\alpha)$ be the trace map as in Lemma 4.3.1.*

1. *For any $\alpha \times \alpha$ matrix U with entries in R, $U\Pi_e(-) = \Pi_e(F_*^e U^{[p^e]} -)$.*
2. *The trace map $\Pi_{e_1+e_2}$ is equal to following compositions*

$$F_*^{e_1+e_2} R^\alpha \xrightarrow{F_*^{e_2}\Pi_{e_1}} F_*^{e_2} R^\alpha \xrightarrow{\Pi_{e_2}} R^\alpha \text{ and } F_*^{e_1+e_2} R^\alpha \xrightarrow{F_*^{e_1}\Pi_{e_2}} F_*^{e_1} R^\alpha \xrightarrow{\Pi_{e_1}} R^\alpha.$$

Proof. For *1.* if $(u_{ij})_{1 \le i,j \le \alpha}$ are the entries of U, for any $v = (v_1, \dots, v_\alpha)^t \in R^\alpha$,

$$\begin{aligned} U\Pi_e(F_*^e v) &= U(\pi_e(F_*^e v_1), \dots, \pi_e(F_*^e v_\alpha))^t \\ &= \begin{pmatrix} \sum_{j=1}^{\alpha} u_{1j}\pi_e(F_*^e v_j) \\ \vdots \\ \sum_{j=1}^{\alpha} u_{\alpha j}\pi_e(F_*^e v_j) \end{pmatrix} = \begin{pmatrix} \pi_e(F_*^e \sum_{j=1}^{\alpha} u_{1j}^{p^e} v_j) \\ \vdots \\ \pi_e(F_*^e \sum_{j=1}^{\alpha} u_{\alpha j}^{p^e} v_j) \end{pmatrix} \\ &= \Pi_e\Big(\big(F_*^e \sum_{j=1}^{\alpha} u_{1j}^{p^e} v_j, \cdots, F_*^e \sum_{j=1}^{\alpha} u_{\alpha j}^{p^e} v_j\big)^t\Big) = \Pi_e(F_*^e U^{[p^e]} v) \end{aligned}$$

For (2), since Π_e takes the components of elements in $F_*^e R^\alpha$ to their images under the trace map π_e, it is enough to show that the assumption is satisfied for π_e. To do that we shall show $\pi(F_*\pi(F_*-)) = \pi_2(F_*^2-)$ and the result follows inductively. Since for any $e > 0$, $F_*^e R$ is a free R-module, for any $r \in R$, there is a unique expression $F_*r = \sum_{0\le\bar{\alpha}\le p-1} r_{\bar{\alpha}} F_*\bar{x}^{\bar{\alpha}}$ for some $r_{\bar{\alpha}} \in R$. Also for each $r_{\bar{\alpha}}$ there is a unique expression $r_{\bar{\alpha}} = \sum_{0\le\bar{\beta}\le p-1} (s_{\bar{\alpha}})_{\bar{\beta}} F_*\bar{x}^{\bar{\beta}}$ for some $(s_{\bar{\alpha}})_{\bar{\beta}} \in R$, which implies that we have the following unique expression of $F_*^2 r$

$$\begin{aligned} F_*^2 r &= \sum_{0\le\bar{\alpha}\le p-1} \left(\sum_{0\le\bar{\beta}\le p-1} (s_{\bar{\alpha}})_{\bar{\beta}} F_*^2 \bar{x}^{p\bar{\beta}+\bar{\alpha}} \right) \\ &= (s_{\overline{p-1}})_{\overline{p-1}} F_*^2 \bar{x}^{p^2-1} + \sum_{0\le\bar{\alpha}<p-1} \left(\sum_{0\le\bar{\beta}<p-1} (s_{\bar{\alpha}})_{\bar{\beta}} F_*^2 \bar{x}^{p\bar{\beta}+\bar{\alpha}} \right) \end{aligned}$$

Hence, $\pi(F_*\pi(F_*r)) = \pi(F_*r_{\overline{p-1}}) = (s_{\overline{p-1}})_{\overline{p-1}} = \pi_2(F_*^2 r)$. □

Lemma 4.3.10. *Let (M, C) be a finitely generated Cartier module isomorphic to $(\operatorname{Coker} A, U)$, and let κ^i be the adjoint map to C^i. Then:*

1. $\underline{M} = \dfrac{I_i(U^{[p^{i-1}]}U^{[p^{i-2}]}\cdots UR^\alpha) + \operatorname{Im} A}{\operatorname{Im} A}$*, and i is the first integer where we get* $I_i(U^{[p^{i-1}]}U^{[p^{i-2}]}\cdots UR^\alpha) = I_{i+1}(U^{[p^i]}U^{[p^{i-1}]}\cdots UR^\alpha)$.
2. $\overline{M} = \dfrac{R^\alpha}{\{m \in R^\alpha \mid U^{[p^{i-1}]}U^{[p^{i-2}]}\cdots Um \in \operatorname{Im} A^{[p^i]}\}}$*, and i is the first integer where we get* $\operatorname{Ker}\kappa^i = \operatorname{Ker}\kappa^{i+1}$.

Proof. An easy application of Lemma 4.3.9 shows that if $\Pi(F_*U-)$ defines the Cartier structure on $\operatorname{Coker} A$, then $\Pi_i(F_*^i U^{[p^{i-1}]}U^{[p^{i-2}]}\cdots U-)$ defines the composition map C^i on $\operatorname{Coker} A$.

Hence, by Corollary 4.3.4, the stable image $\underline{M}$ of C is

$$\frac{I_i(U^{[p^{i-1}]}U^{[p^{i-2}]}\cdots UR^\alpha) + \operatorname{Im} A}{\operatorname{Im} A}$$

for some i. Furthermore, by Theorem 3.2.10, i is the first integer where we get the equality $I_i(U^{[p^{i-1}]}U^{[p^{i-2}]}\cdots UR^\alpha) = I_{i+1}(U^{[p^i]}U^{[p^{i-1}]}\cdots UR^\alpha)$.

By Proposition 4.2.1, to compute $\overline{M}$, we need to find kernels of adjoint maps κ^i to $C^i(-) = \Pi_i(F_*^i U^{[p^{i-1}]}U^{[p^{i-2}]}\cdots U-)$, which give us the following sequence of nilpotent submodules $K_i = \{m \in M \mid F_*^i m \in \operatorname{Ker}\kappa^i\}$ of M,

$$K_1 \subseteq K_2 \subseteq \cdots \subseteq K_i \subseteq \cdots$$

By Proposition 4.2.1 again, if i is the first integer where we have $K_i = K_{i+1}$, then the sequence above stabilizes at K_i, and so $M_{nil} = K_i$. Therefore, to compute $\overline{M}$ we need to compute $\operatorname{Ker}\kappa^i$.

$$\begin{aligned} m \in K_i &\Leftrightarrow F^i_* m \in \operatorname{Ker}\kappa^i \\ &\Leftrightarrow \phi_m(F^i_* R) = C^i(F^i_* mR) = \Pi_i(F^i_* U^{[p^{i-1}]}U^{[p^{i-2}]}\cdots UmR) \\ &\qquad = I_i(U^{[p^{i-1}]}U^{[p^{i-2}]}\cdots UmR) \subseteq \operatorname{Im} A \\ &\Leftrightarrow (U^{[p^{i-1}]}U^{[p^{i-2}]}\cdots UmR) \subseteq \operatorname{Im} A^{[p^i]} \\ &\Leftrightarrow m \in \{w \in R^\alpha \mid U^{[p^{i-1}]}U^{[p^{i-2}]}\cdots Uw \in \operatorname{Im} A^{[p^i]}\} \end{aligned}$$

Hence, $M_{nil} = K_i = \{w \in R^\alpha \mid U^{[p^{i-1}]}U^{[p^{i-2}]}\cdots Uw \in \operatorname{Im} A^{[p^i]}\}$, and so

$$\overline{M} = \frac{R^\alpha}{\{w \in R^\alpha \mid U^{[p^{i-1}]}U^{[p^{i-2}]}\cdots Uw \in \operatorname{Im} A^{[p^i]}\}}.$$

Alternatively, see Corollary 4.5.3. If β_i is the map $U^{[p^{i-1}]}U^{[p^{i-2}]}\cdots U : \operatorname{Coker} A \to \operatorname{Coker} A^{[p^i]}$, then $M_{nil} = \ker\beta_i$ where $\ker\beta_i$ is the stable kernel. □

Lemma 4.3.11. *Let (M, C) be a finitely generated Cartier module isomorphic to $(\operatorname{Coker} A, U)$. If U is invertible, then the adjoint map $\kappa : F_*M \to \operatorname{Hom}_R(F_*R, M)$ is surjective.*

Proof. Since F_*R is a free R-module, any map $\Phi \in \operatorname{Hom}_R(F_*R, \operatorname{Coker} A)$ can be written as a composition $F_*R \xrightarrow{\varphi} R^\alpha \twoheadrightarrow \operatorname{Coker} A$ for a map $\varphi \in \operatorname{Hom}_R(F_*R, R^\alpha)$, where the last map is the natural surjection. We know that any $\varphi \in \operatorname{Hom}_R(F_*R, R^\alpha)$ can be written as $\varphi(-) = (\pi(F_*v_1-), \cdots, \pi(F_*v_\alpha-))^t$ for some $v_1, \cdots, v_\alpha \in R$. On the other hand, invertibility of the matrix U implies that $\kappa(F_*U^{-1}\bar{v}) = \phi_{U^{-1}\bar{v}} \in \operatorname{Hom}_R(F_*R, \operatorname{Coker} A)$ where $\bar{v}$ is the image of $v = (v_1, \cdots, v_\alpha)^t$ in $\operatorname{Coker} A$. However, for any $r \in R$, we have $\phi_{U^{-1}\bar{v}}(F_*r) = C(F_*U^{-1}\bar{v}r)$. Therefore,

$$\begin{aligned} C(F_*U^{-1}\bar{v}r) &= \Pi(F_*UU^{-1}vr) + \operatorname{Im} A = \Pi(F_*vr) + \operatorname{Im} A \\ &= (\pi(F_*v_1r), \cdots, \pi(F_*v_\alpha r))^t + \operatorname{Im} A \\ &= \varphi(F_*r) + \operatorname{Im} A = \Phi(F_*r). \end{aligned}$$

This shows that $\kappa(F_*U^{-1}\bar{v}) = \Phi$, and so κ is surjective. $\square$

Before proving the main theorem of this chapter, we need to recall a crucial property of finite length modules (cf. footnote 8 in [2]).

Lemma 4.3.12. *Let S be regular local F-finite ring of prime characteristic p, and let $\mathfrak{m}$ be the maximal ideal of S. Let N be a finite length S-module. Then*

$$l_{F_*S}(\operatorname{Hom}_S(F_*S, N)) = p^{\dim S} l_S(N).$$

Proof. Let $0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_m = N$ be a maximal chain of submodules of N. Since F_*S is finitely generated and flat S-module, $\operatorname{Hom}_S(F_*S, -)$ is exact, and we get the following chain of F_*S-modules

$$\operatorname{Hom}_S(F_*S, N_0) \subsetneq \operatorname{Hom}_S(F_*S, N_1) \subsetneq \cdots \subsetneq \operatorname{Hom}_S(F_*S, N_m).$$

Therefore, for each j, we need to check the length of $\operatorname{Hom}_S(F_*S, N_{j+1}/N_j)$ over F_*S. Furthermore, since N_{j+1}/N_j is a simple S-module,

$$\operatorname{Hom}_S(F_*S, N_{j+1}/N_j) \cong \operatorname{Hom}_S(F_*S, S/\mathfrak{m}).$$

We also have $\operatorname{Hom}_S(F_*S, S/\mathfrak{m}) = \operatorname{Hom}_{S/\mathfrak{m}}(F_*(S/\mathfrak{m}^{[p]}), S/\mathfrak{m})$ since for any $x \in \mathfrak{m}$ we have $0 = x\varphi(F_*s) = \varphi(xF_*s) = \varphi(F_*x^p s)$ for all $\varphi \in \operatorname{Hom}_S(F_*S, S/\mathfrak{m})$. By Theorem 2.2.10, $l_S(S/\mathfrak{m}^{[p]}) = p^{\dim S}$. This means that $S/\mathfrak{m}^{[p]}$ is free of dimension $p^{\dim S}$ as an $S/\mathfrak{m}$ module. Therefore,

$$\operatorname{Hom}_{S/\mathfrak{m}}(F_*(S/\mathfrak{m}^{[p]}), S/\mathfrak{m}) \cong \operatorname{Hom}_{S/\mathfrak{m}}(F_*(S/\mathfrak{m}), S/\mathfrak{m})^{p^{\dim S}}.$$

Then since $\operatorname{Hom}_{S/\mathfrak{m}}(F_*(S/\mathfrak{m}), S/\mathfrak{m}) \cong F_*(S/\mathfrak{m})$ as $F_*(S/\mathfrak{m})$ modules, length of $\operatorname{Hom}_{S/\mathfrak{m}}(F_*(S/\mathfrak{m}^{[p]}), S/\mathfrak{m})$ over $F_*(S/\mathfrak{m})$ is $p^{\dim S}$. Hence the length of $\operatorname{Hom}_S(F_*S, S/\mathfrak{m})$ over F_*S is $p^{\dim S}$, and so

$$l_{F_*S}(\operatorname{Hom}_S(F_*S, N)) = p^{\dim S} l_S(N).$$

$\square$

Next theorem gives a computational proof of Facts 4.1.6 2. in a more algebraic language. We will use the proof to provide an effective algorithm for finding the finite set $\mathcal{S}$.

Theorem 4.3.13. *[2, cf. Proposition 4.1 and Proposition 4.5] Let (M, C) be a finitely generated Cartier module. There is a finite subset $\mathcal{S} \subseteq \operatorname{Supp} M$ such that for all prime ideals $P \in \operatorname{Supp} M \backslash \mathcal{S}$*

($\star$) *All finite length Cartier quotients of M_P are nilpotent.*

Proof. If $\dim M = 0$, then we take $\mathcal{S} = \operatorname{Supp} M$ and we are done. Therefore, we suppose that $\dim M > 0$ and prove the claim by induction on $\dim M$. By Lemma 4.1.7, we can replace M with $\underline{\mathrm{M}}$ and assume that the structural map C is surjective. Let $P_1, \ldots, P_m$ be the minimal primes of $\operatorname{Ann}_R M$. Then, by Lemma 4.1.8, $(\star)$ condition for $M(P_i) := \dfrac{M}{\sum_{e \geq 0} C^e(P_i M)}$ at any prime $P \in V(P_i)$ implies that $(\star)$ condition holds for M at P. For each i, by Lemma 4.1.7 again, we can replace $M(P_i)$ with $M_i := \underline{M(P_i)}$. We now claim to show that for each i, there exists a finite subset $\mathcal{S}_i$ such that $(\star)$ condition holds for M_i at any prime ideal $P \in \operatorname{Supp} M_i \setminus \mathcal{S}_i$, and we choose $\mathcal{S} = \bigcup \mathcal{S}_i$. If $\dim M_i = 0$ we take $\mathcal{S}_i = \operatorname{Supp} M_i$. Otherwise, to find $\mathcal{S}_i$ for which $\dim M_i > 0$, we will then show that there is an open set $\mathcal{U}_i$ such that $(\star)$ condition holds for M_i at every prime in $\mathcal{U}_i \setminus \{P_i\}$.

Let $(\operatorname{Coker} A_i, U_i)$ be the Cartier module (M_i, C_{M_i}) where U_i is the square matrix with entries in R such that $C_{M_i}(-) = \Pi(F_* U_i -)$, and d_i be the determinant of U_i. Then by Lemma 4.3.5 and Proposition 4.3.6, d_i is a non-zero element of R. Now let $\mathcal{U}_i$ be the open set $X_{d_i} \cap \operatorname{Reg} R/P_i$ where $\operatorname{Reg} R/P_i$ denotes the regular locus of R/P_i and X_{d_i} denotes the complement of $V(d_i R)$. On the other hand, by Proposition 4.2.1, the adjoint map $\kappa_i : F_* M_i \to \operatorname{Hom}_R(F_* R, M_i)$ to the structural map C_{M_i} is injective. Next we take any prime ideal $Q \in \mathcal{U}_i$ which properly contains P_i and localize the adjoint map κ_i at Q. Then the map

$$(\kappa_i)_Q : F_*(M_i)_Q \to \operatorname{Hom}_{R_Q}(F_* R_Q, (M_i)_Q)$$

is an isomorphism by Lemma 4.3.11, since d_i is an invertible element of R_Q. Therefore, we have a natural surjective map of F_*S-modules

$$\psi : F_*(M_i)_Q \twoheadrightarrow \operatorname{Hom}_S(F_* S, (M_i)_Q)$$

where $S = R_Q / P_i R_Q$. Now let N is a finite length Cartier quotient of $(M_i)_Q$. Since P_i is contained in Q properly, S is a regular local ring of dimension ≥ 1. Moreover, $\operatorname{Hom}_S(F_*S, -)$ is exact, since F_*S is a finitely generated flat S-module. Therefore, the map $F_*N \to \operatorname{Hom}_S(F_*S, N)$ induced from ψ is surjective, and so $l_{F_*S}(F_*N) \geq l_{F_*S}(\operatorname{Hom}_S(F_*S, N))$. On the other hand, by Lemma 4.3.12, $l_{F_*S}(\operatorname{Hom}_S(F_*S, N)) = p^{\dim S} l_S(N)$. Therefore, since $l_S(N) = l_{F_*S}(F_*N)$, we have $l_S(N) \geq p^{\dim S} l_S(N)$. This only happens when $N = 0$ or $\dim S = 0$. However, since $\dim S > 0$, we must have $N = 0$, in particular, N is nilpotent. If it was in the case that $Q = P_i$, we would have $\dim S = 0$, and $l_S(N) \geq p^{\dim S} l_S(N)$ would hold. Therefore, $\mathcal{S}_i = \mathcal{S}_i' \cup \{P_i\}$

where $\mathcal{S}_i'$ is a finite subset contained in the complement of $\mathcal{U}_i$. Hence, we will find the set $\mathcal{S}_i'$ in the complement of $\mathcal{U}_i$.

Let I_i be the ideal of R whose image in R/P_i defines the complement of $\mathcal{U}_i$. Then, by Lemma 4.1.8 again, $(\star)$ condition for $M(I_i) := \dfrac{M}{\sum_{e\geq 0} C^e(I_i M)}$ at any prime $P \in V(I_i)$ implies that $(\star)$ condition holds for M at P, and since $P_i \subsetneqq I_i$, $\dim M > \dim M(I_i)$ for each i. If $\dim M(I_i) = 0$, then we choose $\mathcal{S}_i' = \operatorname{Supp} M(I_i)$. Otherwise, by induction, we find $\mathcal{S}_i'$ in $\operatorname{Supp} M(I_i)$. Therefore, by the induction, $\mathcal{S}$ consists of such P_i's which are finitely many, and supports of finitely many Cartier quotients of M whose dimension is zero. □

4.4 An Algorithm for Finding Annihilators of Cartier Quotients

In this section, we introduce a new algorithm for finding explicitly determining a finite set of prime ideals $\mathcal{S}$ satisfying the hypothesis of Theorem 4.3.13 by following its proof. Suppose that $R = \Bbbk[x_1, \ldots, x_n]$ is a polynomial ring (or $R = \Bbbk[\![x_1, \ldots, x_n]\!]$ a power series ring) over an F-finite field $\Bbbk$ of prime characteristic p. Given a finitely generated Cartier module (M, C), here are steps of the algorithm.

Input

A finitely generated Cartier R-module (M, C).

Output

Prime annihilators of Cartier quotients of M.

Initialize

$\mathcal{S} = \emptyset$, $\mathcal{M} = \{M\}$ and $\mathcal{M}' = \emptyset$.

Execute the following:

While $\mathcal{M} \setminus \mathcal{M}' \neq \emptyset$, pick an element $M \in \mathcal{M} \setminus \mathcal{M}'$ and do the following:

1. If $\dim M = 0$, then add the ideals from $\operatorname{Supp} M$ to $\mathcal{S}$ and add M to $\mathcal{M}'$.

2. If $\dim M > 0$, do the following:

   (a) Replace M by $\underline{\mathrm{M}}$.

   (b) Find the minimal prime ideals $P_1, \ldots, P_m$ in $X := V(\operatorname{Ann}_R M)$ and add them to $\mathcal{S}$.

   (c) For each i, find the modules $M_i := \overline{M(P_i)}$ where $M(P_i) = \dfrac{M}{\sum_{e \geq 0} C^e(P_i M)}$.

   (d) For each M_i with $\dim M_i = 0$, add the ideals from $\operatorname{Supp} M_i$ to $\mathcal{S}$.

   (e) For each M_i with $\dim M_i > 0$, find the square matrix U_i which gives the Cartier module structure on M_i, and compute its determinant $d_i = \det U_i$, and do the following:

      i. Find the open set $\mathcal{U}_i := X_{d_i} \bigcap \operatorname{Reg} R/P_i$.

      ii. Find the ideal $I_i \subseteq R$ whose image in R/P_i defines the complement of $\mathcal{U}_i$

      iii. Add the modules $\dfrac{M}{\sum_{e \geq 0} C^e(I_i M)}$ to $\mathcal{M}$ and add M to $\mathcal{M}'$.

3. Output $\mathcal{S}$, and stop.

Theorem 4.4.1. *Given a finitely generated Cartier module (M, C), the algorithm described above terminates and the output set $\mathcal{S}$ is a finite set of primes ideals such that M satisfies $(\star)$ condition on the complement of $\mathcal{S}$.*

Proof. To prove the claim, we shall explain how the steps of the algorithm work.At step *1.* we choose $\mathcal{S}$ to be $\operatorname{Supp} M$, since supports of zero dimensional modules are finite.

The main idea of step *2.* is to divide $\operatorname{Supp} M$ into irreducible components and find an open set for each irreducible component on which M satisfies $(\star)$ condition. By Lemma 4.1.7, at step *2.*(a) we can reduce our assumptions to the surjective case. Then by Lemma 4.1.8, we can look for the desired ideals for $M(P_i) = \dfrac{M}{\sum_{e \geq 0} C^e(P_i M)}$ where P_i's are minimal prime ideals in $\operatorname{Supp} M$ computed at step *2.*(b).

By Lemma 4.1.7 again, at step *2.*(c) we can replace $M(P_i)$ with M_i. At step *2.*(d), we eliminate M_i's with zero dimension for the sake of induction. Afterwards, at step *2.*(e) we find the square matrix U_i which represents the Cartier module structure on M_i whose dimension is strictly bigger than zero, and surjectivity guaranties that the determinant of U_i is not zero. Then at step *2.*(e)i. we find the open set $\mathcal{U}_i$ on which M_i satisfies $(\star)$ condition except P_i. Since $S := (R/P_i)_{P_i}$ is a local ring of dimension 0, by Lemma 4.3.12, $l_{F_*S}(\mathrm{Hom}_S(F_*S, N)) = l_S(N)$ for any finite length S-module N. However, this is not enough to say that anything about nilpotency of N. Therefore, we put P_i's in $\mathcal{S}$ at step *2.*(b).

The crucial step of the algorithm is step *2.*(e)ii. We find the ideals I_i, because we want to apply same process to the modules $M(I_i) := \dfrac{M}{\sum_{e\geq 0} C^e(I_i M)}$ inductively and find such open sets on which $M(I_i)$ satisfies $(\star)$ condition. By Lemma 4.1.8 again, we know that if $M(I_i)$ satisfies $(\star)$ condition at a prime ideal $P \in V(I_i)$ then M satisfies $(\star)$ condition at P. Hence, at step *2.*(e)iii., we add $M(I_i)$ to $\mathcal{M}$. The most important point here is that $\dim M > \dim M(I_i)$ since $P_i \subsetneqq I_i$. Therefore, since the dimension drops, the algorithm terminates.

After all the output set $\mathcal{S}$ is a finite set of prime ideals and on the complement of $\mathcal{S}$, by Theorem 4.3.13, M satisfies $(\star)$ condition as desired. □

If the structural map of M is surjective, it is easy to find prime ideals in the collection $\mathcal{A} := \{\mathrm{Ann}_R M/N \mid N \text{ is a Cartier submodule of } M\}$. By Lemma 4.1.9, prime ideals of $\mathcal{A}$ is also in $\mathcal{S}$, and for any prime ideal $P \in \mathcal{S}$ to decide whether $P \in \mathcal{A}$ we just need to check that if the annihilator of the module $\dfrac{M}{\sum_{e\geq 0} C^e(PM)}$ is equal to P.

4.5 An Application to Lyubeznik's F-modules

In this section, we investigate the connections between finitely generated Cartier modules and Lyubeznik's F-finite F-modules. We start with an important observation.

Discussion 4.5.1. *Let $\mathcal{M}$ be an F-finite F-module with a generating morphism $\beta : M \to F_R(M)$ and let M be presented by a matrix A as in subsection 2.1.1, and*

$M \cong \operatorname{Coker} A$. *Then we can write the generating morphism of* $\mathcal{M}$ *as* $\operatorname{Coker} A \xrightarrow{U} \operatorname{Coker} A^{[p]}$ *where* U *is an* $\alpha \times \alpha$ *matrix with entries in* R *such that* $U \operatorname{Im} A \subseteq \operatorname{Im} A^{[p]}$. *By Lemma 4.3.3, we have* $U \operatorname{Im} A \subseteq \operatorname{Im} A^{[p]} \Leftrightarrow I_1(U \operatorname{Im} A) \subseteq \operatorname{Im} A$, *and by Corollary 4.3.4 we have* $I_1(U \operatorname{Im} A) \subseteq \operatorname{Im} A \Leftrightarrow \Pi(F_* U \operatorname{Im} A) \subseteq \operatorname{Im} A$. *Therefore, we can use the matrix* U *to define a Cartier module structure on* M *given by the map* $C : F_*M \to M$ *where* $C(-) = \Pi_R(F_*U-)$.

Conversely, let (N, C') *be a finitely generated Cartier* R*-module represented by a matrix* B *and denoted by* $(\operatorname{Coker} B, V)$ *where* $C'(F_*V \operatorname{Im} B) = \Pi(F_*V \operatorname{Im} B) \subseteq \operatorname{Im} B$, *which implies* $V \operatorname{Im} B \subseteq \operatorname{Im} B^{[p]}$. *Then, we can define a generating morphism* $\operatorname{Coker} B \xrightarrow{V} \operatorname{Coker} B^{[p]}$ *for an* F*-finite* F*-module*

$$\mathcal{N} = \varinjlim(\operatorname{Coker} B \xrightarrow{V} \operatorname{Coker} B^{[p]} \xrightarrow{V^{[p]}} \operatorname{Coker} B^{[p^2]} \xrightarrow{V^{[p^2]}} \cdots).$$

Proposition 4.5.2. *Let* (M, C) *be a finitely generated Cartier module isomorphic to* $(\operatorname{Coker} A, U)$ *and let* $\mathcal{M}$ *be the* F*-finite* F*-module*

$$\mathcal{M} = \varinjlim(\operatorname{Coker} A \xrightarrow{U} \operatorname{Coker} A^{[p]} \xrightarrow{U^{[p]}} \operatorname{Coker} A^{[p^2]} \xrightarrow{U^{[p^2]}} \cdots)$$

Then M *is nilpotent if and only if* $\mathcal{M}$ *is zero.*

Proof. Let β_i denote the composition

$$M \xrightarrow{\beta} F_R(M) \xrightarrow{F_R(\beta)} F_R^2(M) \xrightarrow{F_R^2(\beta)} \cdots \xrightarrow{F_R^{i-1}(\beta)} F_R^i(M).$$

Then we can write β_i as the composition

$$\operatorname{Coker} A \xrightarrow{U} \operatorname{Coker} A^{[p]} \xrightarrow{U^{[p]}} \operatorname{Coker} A^{[p^2]} \xrightarrow{U^{[p^2]}} \cdots \xrightarrow{U^{[p^{i-1}]}} \operatorname{Coker} A^{[p^i]}.$$

Therefore, β_i is the map $U^{[p^{i-1}]}U^{[p^{i-2}]}\cdots U : \operatorname{Coker} A \to \operatorname{Coker} A^{[p^i]}$.

$$\begin{aligned}
M \text{ is nilpotent} &\Leftrightarrow C^e(M) = 0 \text{ for some } e > 0 \\
&\Leftrightarrow \Pi_e(F^e_* U^{[p^{e-1}]}\cdots UR^\alpha) \subseteq \operatorname{Im} A \\
&\Leftrightarrow I_e(U^{[p^{e-1}]}\cdots UR^\alpha) \subseteq \operatorname{Im} A \\
&\Leftrightarrow U^{[p^{e-1}]}\cdots UR^\alpha \subseteq \operatorname{Im} A^{[p^e]} \\
&\Leftrightarrow \operatorname{Im}(U^{[p^{e-1}]}\cdots U) \subseteq \operatorname{Im} A^{[p^e]} \\
&\Leftrightarrow U^{[p^{e-1}]}\cdots U \text{ is a zero map on } M \text{, i.e. } \operatorname{Im}\beta_e = 0 \\
&\Leftrightarrow \mathcal{M} = 0 \text{ by Proposition 2.2.33 .}
\end{aligned}$$

□

Corollary 4.5.3. *Let $\mathcal{M}$ be a non-zero F-finite F-module with a generating morphism $\beta : M \to F_R(M)$ where $M = \operatorname{Coker} A$ and U is the square matrix for which the map $\operatorname{Coker} A \xrightarrow{U} \operatorname{Coker} A^{[p]}$ is isomorphic to the generating morphism. Let $C : F_*M \to M$ be the Cartier structure given by U, and N a Cartier submodule of M. Then*

1. *N is nilpotent if and only if $N \subseteq \ker \beta_i$ for some i, where β_i is the composition map $M \xrightarrow{\beta} F_R(M) \xrightarrow{F_R(\beta)} F^2_R(M) \xrightarrow{F^2_R(\beta)} \cdots \xrightarrow{F^{i-1}_R(\beta)} F^i_R(M)$.*
2. *$M_{nil} = \ker \beta_i$, where $\ker \beta_i$ is the stable kernel of the ascending chain $\ker \beta_1 \subseteq \ker \beta_2 \subseteq \ldots$, and so $\overline{M}$ is a root of $\mathcal{M}$.*
3. *If $\mathcal{M}'$ is the F-finite F-module whose generating morphism is $\beta_{\underline{M}} : \underline{M} \to F_R(\underline{M})$, then $\mathcal{M} \cong \mathcal{M}'$.*

Proof. For (1), since N is nilpotent, there exist an integer i such that $C^i(F^i_*N) = \Pi_i(F^i_*U^{[p^{e-1}]}\cdots UN) \subseteq \operatorname{Im} A$ which implies that $U^{[p^{e-1}]}\cdots UN \subseteq \operatorname{Im} A^{[p^i]}$. Thus, $N \subseteq \ker \beta_i$.

By (1), $M_{nil} \subseteq \ker \beta_i$. Also $C^i(F^i_* \ker \beta_i) = \Pi_i(F^i_*U^{[p^{e-1}]}\cdots U \ker \beta_i) = \Pi_e(0) = 0$ implies that $\ker \beta_i$ is nilpotent for each i, and so $\ker \beta_i \subseteq M_{nil}$. It shows part (2).

Since $M/\underline{\mathrm{M}}$ is nilpotent, the F-finite F-module $\mathcal{M}/\mathcal{M}'$ whose generating morphism $M/\underline{\mathrm{M}} \to F_R(M/\underline{\mathrm{M}})$, which is induced from β, is zero, and so $\mathcal{M} \cong \mathcal{M}'$ □

When $R = \Bbbk[\![x_1, \ldots, x_n]\!]$ is a power series ring over a perfect field $\Bbbk$ of prime characteristic p and $R \to S$ is a surjective ring homomorphism, for any $N \in \mathfrak{C}$, we write $\Delta(N) = (M \xrightarrow{\beta} F_R(M))$, and we define the functor $\mathcal{H}_{R,S}$ to be

$$\mathcal{H}_{R,S}(N) = \varinjlim(M \xrightarrow{\beta} F_R(M) \xrightarrow{F_R(\beta)} F_R^2(M) \xrightarrow{F_R^2(\beta)} \cdots)$$

where $\mathfrak{C}$ is the category of Artininan $R[\theta; f]$-modules and Δ is the functor as in Subsection 2.2.6. The functor $\mathcal{H}_{R,S}$ and some of its useful properties are introduced in [16, Theorem 4.2]. One can use the correspondence in Theorem 5.2.3 and reprove Theorem 4.2 in [16] using Proposition 4.5.2 and Corollary 4.5.3.

Definition 4.5.4. *Let (M, C) be a finitely generated Cartier R-module. M is called minimal if C is surjective and $M_{nil} = 0$.*

Discussion 4.5.1 and Corollary 4.5.3 gives us next Theorem.

Theorem 4.5.5. *There is a bijective correspondence between the category of F-finite F-modules and the category of finitely generated minimal Cartier modules.*

Remark 4.5.6. *Let (M, C) be a finitely generated Cartier R-module denoted by $(\operatorname{Coker} A, U)$. By Lemma 4.3.10, $\underline{(\overline{M})}$ and $(\overline{\underline{M}})$ are clearly equal to*

$$\frac{I_i(U^{[p^{i-1}]}U^{[p^{i-2}]}\cdots UR^\alpha) + \operatorname{Im} A}{\{m \in R^\alpha \mid U^{[p^{i-1}]}U^{[p^{i-2}]}\cdots Um \in \operatorname{Im} A^{[p^i]}\}},$$

which is a minimal Cartier module, and so we denote it by M_{min}.

Theorem 4.5.7. *Let (M, C) be a finitely generated Cartier R-module and let $\mathcal{M}$ be the corresponding F-finite F-module. Then the maximum length of a chain of ideals in the collection $\mathcal{A} := \{\operatorname{Ann}_R M_{min}/N \mid N$ is a Cartier submodule of $M_{min}\}$ is a lower bound for the F-module length of $\mathcal{M}$.*

Proof. Let $J_0 \subsetneqq J_1 \subsetneqq \cdots \subsetneqq J_m$ be a chain of ideals in the collection $\mathcal{A}$ with maximum length. Then by Lemma 4.1.9, we have a chain of Cartier submodules of M_{min}, $N_0 \subsetneqq N_1 \subsetneqq \cdots \subsetneqq N_m$ where N_i is the smallest Cartier submodule of M_{min} containing J_iM, i.e. $N_i = \sum_{e \geq 0} C^e(J_iM)$. Then by Corollary 4.5.3, M_{min} is a root for $\mathcal{M}$, and by Remarks 2.2.35, we have a bijective correspondence between F-submodules of $\mathcal{M}$ and R-submodules of M_{min}. Therefore, each N_i corresponds an F-submodule $\mathcal{N}_i$ of $\mathcal{M}$ where $\mathcal{N}_i = \varinjlim(N_i \xrightarrow{\gamma_i} F_R(N_i) \xrightarrow{F_R(\gamma_i)} F_R^2(N_i) \xrightarrow{F_R^2(\gamma_i)} \cdots)$ and γ_i defines the Cartier structure on N_i induced by C, and so m is a lower bound for F-module length of $\mathcal{M}$. □

Theorem 4.5.7 shows that the algorithm described in section 4.4 gives a method to find a lower bound for F-module length of $\mathcal{M}$.

Chapter 5

An Explicit Correspondence

Let R be a formal power series ring over a perfect field $\Bbbk$ of prime characteristic p, i.e. $R = \Bbbk[\![x_1, \ldots, x_n]\!]$, and let $E = E_R(R/\mathfrak{m})$ be the injective hull of its residue field. In this chapter, we introduce our computational correspondence between finitely generated Cartier modules and Artininan modules equipped with a Frobenius map over R, and we show that it coincides with the correspondences introduced in [2] and [21].

5.1 An Explicit Isomorphism

Let $\Bbbk[x_1^-, \ldots, x_n^-]$ denote the module of inverse polynomials. By Example 2.1.38, we know that $E \cong \Bbbk[x_1^-, \ldots, x_n^-]$. In the rest of this section, we identify E with $\Bbbk[x_1^-, \ldots, x_n^-]$, and we will write $-\bar{\nu}$ to denote n-tuples $(-\nu_1, \ldots, -\nu_n)$ in addition to Notation 4.3.8.

Since F_*E is the injective hull of residue field of F_*R, an application of Lemma 2.1.20 with $S = F_*R$ gives us the following corollary.

Corollary 5.1.1. $\operatorname{Hom}_R(F_*R, E) \cong F_*E$ *as* F_*R*-modules*

By Proposition 2.2.5, we know that F_*R is a free R-module with a basis set $\mathcal{B} = \{F_*\bar{x}^{\bar{\alpha}} \mid 0 \leq \alpha_i \leq p-1 \text{ for all } i = 1, \ldots, n\}$. Therefore, an R-linear map from F_*R to any other R-module is simply a choice of where to send these basis elements.

Next we fix an explicit F_*R isomorphism between $\mathrm{Hom}_R(F_*R, E)$ and F_*E which we use repeatedly in this chapter.

Lemma 5.1.2. *The map* $\Phi : \mathrm{Hom}_R(F_*R, E) \to F_*E$ *given by*

$$\Phi(g) = \sum_{0 \le \bar{\alpha} < p} F_*[\bar{x}^{p-1-\bar{\alpha}} T(g(F_*\bar{x}^{\bar{\alpha}}))] \tag{5.1}$$

for all $g \in \mathrm{Hom}_R(F_*R, E)$*, where* T *is the natural Frobenius map on* E*, is an* F_*R*-isomorphism.*

Proof. By the definitions of T and g, it can easily be seen that Φ is well-defined and additive. For any $r \in R$, we further have the following

$$\begin{aligned} r\Phi(g) &= \sum_{0 \le \bar{\alpha} < p} F_*[\bar{x}^{p-1-\bar{\alpha}} r^p T(g(F_*\bar{x}^{\bar{\alpha}}))] \\ &= \sum_{0 \le \bar{\alpha} < p} F_*[\bar{x}^{p-1-\bar{\alpha}} T(rg(F_*\bar{x}^{\bar{\alpha}}))] = \Phi(rg) \end{aligned}$$

which means that Φ is R-linear. Thus, for F_*R-linearity of Φ, since F_*R is a free R-module, it is enough to show that $F_*\bar{x}^{\bar{\beta}}\Phi(g) = \Phi(F_*\bar{x}^{\bar{\beta}} g)$ for any basis element $F_*\bar{x}^{\bar{\beta}} \in \mathcal{B}$, and so we will show that the right hand sides of following equations are equal.

$$F_*\bar{x}^{\bar{\beta}}\Phi(g) = \sum_{0 \le \bar{\alpha} < p} F_*[\bar{x}^{p-1-\bar{\alpha}+\bar{\beta}} T(g(F_*\bar{x}^{\bar{\alpha}}))], \tag{5.2}$$

$$\Phi(F_*\bar{x}^{\bar{\beta}} g) = \sum_{0 \le \bar{\alpha} < p} F_*[\bar{x}^{p-1-\bar{\alpha}} T(g(F_*\bar{x}^{\bar{\alpha}+\bar{\beta}}))]. \tag{5.3}$$

Moreover, since $F_*\bar{x}^{\bar{\beta}} = F_*x_1^{\beta_1} \cdots F_*x_n^{\beta_n}$, it is enough to show that

$$\begin{aligned} F_*x_i^{\beta_i}\Phi(g) &= \sum_{0 \le \bar{\alpha} < p} F_*[x_1^{p-1-\alpha_1} \ldots x_i^{p-1-\alpha_i+\beta_i} \ldots x_n^{p-1-\alpha_n} T(g(F_*\bar{x}^{\bar{\alpha}}))] \\ &= \sum_{0 \le \bar{\alpha} < p} F_*[\bar{x}^{p-1-\bar{\alpha}} T(g(F_*x_1^{\alpha_1} \ldots x_i^{\alpha_i+\beta_i} \ldots x_n^{\alpha_n}))] = \Phi(F_*x_i^{\beta_i} g) \end{aligned}$$

for each $F_*x_i^{\beta_i}$. To do that we will show the following sets are the same

$$S_1 = \{F_*[\bar{x}^{p-1-\bar{\alpha}}T(g(F_*x_1^{\alpha_1}\dots x_i^{\alpha_i+\beta_i}\dots x_n^{\alpha_n}))] \mid 0 \le \alpha_i < p\},$$
$$S_2 = \{F_*[x_1^{p-1-\alpha_1}\dots x_i^{p-1-\alpha_i+\beta_i}\dots x_n^{p-1-\alpha_n}T(g(F_*\bar{x}^{\bar{\alpha}}))] \mid 0 \le \alpha_i < p\}.$$

In the case that $\alpha_i + \beta_i < p$,

$$\{F_*[\bar{x}^{p-1-\bar{\alpha}}T(g(F_*x_1^{\alpha_1}\dots x_i^{\alpha_i+\beta_i}\dots x_n^{\alpha_n}))] \mid 0 \le \alpha_i < p-\beta_i\} =$$
$$\{F_*[x_1^{p-1-\alpha_1}\dots x_i^{p-1-\alpha_i+\beta_i}\dots x_n^{p-1-\alpha_n}T(g(F_*\bar{x}^{\bar{\alpha}}))] \mid \beta_i \le \alpha_i < p\}$$

since substituting α_i with $\alpha_i + \beta_i$ in the latter set gives us the former set. On the other hand, in the case that $\alpha_i + \beta_i \ge p$,

$$\{F_*[\bar{x}^{p-1-\bar{\alpha}}T(g(F_*x_1^{\alpha_1}\dots x_i^{\alpha_i+\beta_i}\dots x_n^{\alpha_n}))] \mid p-\beta_i \le \alpha_i < p\} =$$
$$\{F_*[x_1^{p-1-\alpha_1}\dots x_i^{p-1-\alpha_i+\beta_i}\dots x_n^{p-1-\alpha_n}T(g(F_*\bar{x}^{\bar{\alpha}}))] \mid 0 \le \alpha_i < \beta_i\}$$

since for each $k \in \{0,\dots,\beta_i-1\}$, where $\alpha_i+\beta_i = p+k$ (i.e. $p-\alpha_i = \beta_i - k$),

$$F_*[\bar{x}^{p-1-\bar{\alpha}}T(g(F_*x_1^{\alpha_1}\dots x_i^{\alpha_i+\beta_i}\dots x_n^{\alpha_n}))] =$$
$$F_*[x_1^{p-1-\alpha_1}\dots x_i^{p-1-\alpha_i+p}\dots x_n^{p-1-\alpha_n}T(g(F_*x_1^{\alpha_1}\dots x_i^{k}\dots x_n^{\alpha_n}))] =$$
$$F_*[x_1^{p-1-\alpha_1}\dots x_i^{p-1-k+\beta_i}\dots x_n^{p-1-\alpha_n}T(g(F_*x_1^{\alpha_1}\dots x_i^{k}\dots x_n^{\alpha_n}))].$$

Therefore, $S_1 = S_2$, and so the right hand sides of 5.2 and 5.3 are equal.

For injectivity of Φ, we first need the following. For any $g \in \mathrm{Hom}_R(F_*R, E)$, we have $g(F_*\bar{x}^{\bar{\alpha}}) \in \mathbb{k}[x_1^-,\dots,x_n^-]$, and so $g(F_*\bar{x}^{\bar{\alpha}})$ is a finite $\mathbb{k}$-linear combination of monomials $x_1^{-\nu_1}\dots x_n^{-\nu_n}$, where ν_i's are positive integers. Therefore, for each $F_*\bar{x}^{\bar{\alpha}} \in \mathcal{B}$, $F_*[\bar{x}^{p-1-\bar{\alpha}}T(g(F_*\bar{x}^{\bar{\alpha}}))]$ is a finite $\mathbb{k}$-linear combination of monomials $F_*x_1^{p-1-\alpha_1-p\nu_1}\dots x_n^{p-1-\alpha_n-p\nu_n}$. This means that

$$\Phi(g) = \sum_{0\le\bar{\alpha}<p} F_*[\bar{x}^{p-1-\bar{\alpha}}T(g(F_*\bar{x}^{\bar{\alpha}}))] = \sum_{0\le\bar{\alpha}<p}\left(\sum_{0<\bar{\nu}<\infty}\lambda_\nu F_*\bar{x}^{p-1-\bar{\alpha}-p\bar{\nu}}\right)$$

Therefore, $\Phi(g) = 0$ if and only if $\lambda_\nu = 0$ for all $\bar{\nu} > 0$ since $p-1-\bar{\alpha} < p$ implies that $p-1-\bar{\alpha}-p\bar{\nu} < 0$ and the terms $F_*\bar{x}^{p-1-\bar{\alpha}-p\bar{\nu}} \ne F_*\bar{x}^{p-1-\bar{\beta}-p\bar{\mu}}$ unless $\bar{\alpha} = \bar{\beta}$ and $\bar{\mu} = \bar{\nu}$ at the same time. Hence,

$$\begin{aligned}\Phi(g) = 0 &\Leftrightarrow T(g(F_*\bar{x}^{\bar{\alpha}})) = 0 \text{ for all } 0 \le \bar{\alpha} < p\\ &\Leftrightarrow g(F_*\bar{x}^{\bar{\alpha}}) = 0 \text{ for all } 0 \le \bar{\alpha} < p\\ &\Leftrightarrow g = 0.\end{aligned}$$

For surjectivity of Φ, we take an element F_*e of $F_*\Bbbk[x_1^-, \dots, x_n^-]$. We know that it can be written as a finite sum of terms $F_*\mu x_1^{-\nu_1} \dots x_n^{-\nu_n}$, where $\mu \in \Bbbk$, and these terms can be written as

$$\begin{aligned} F_*\mu x_1^{-\nu_1} \dots x_n^{-\nu_n} &= F_*x_1^{k_1} \dots x_n^{k_n} F_*\lambda^p x_1^{-p\beta_1} \dots x_n^{-p\beta_n} \\ &= F_*x_1^{k_1} \dots x_n^{k_n} F_*T(\lambda x_1^{-\beta_1} \dots x_n^{-\beta_n}) \end{aligned}$$

where $\mu = \lambda^p$, and for each i, $k_i = p\beta_i - \nu_i$ and $0 \leq k_i < p$. Now we rewrite F_*e as a finite sum

$$\sum_{0 \leq k_1, \dots, k_n < p} F_*x_1^{k_1} \dots x_n^{k_n} F_*T(e_{k_1, \dots, k_n})$$

where $e_{k_1,\dots,k_n} \in \Bbbk[x_1^-, \dots, x_n^-]$, and we choose a map $g \in \mathrm{Hom}_R(F_*R, E)$ which sends $F_*x_1^{p-1-k_1} \dots x_n^{p-1-k_n}$ to $e_{k_1,\dots,k_n}$. This means that $\Phi(g) = F_*e$, i.e. Φ is surjective. Hence, it is an isomorphism of F_*R-modules. □

5.2 The Correspondence

Proposition 5.2.1. *Let α be a non negative integer. There is a bijective correspondence between $\mathrm{Hom}_R(F_*R^\alpha, R^\alpha)$ and $\mathrm{Hom}_R(E^\alpha, F_*E^\alpha)$ such that the trace map Π on F_*R^α corresponds to the natural Frobenius map T on E^α and $\Pi(F_*U-)$ corresponds to U^tT for any $\alpha \times \alpha$ matrix U with entries in R.*

Proof. We start by identifying $\mathrm{Hom}_R(F_*R, E)$ with F_*E using the isomorphism Φ defined in Lemma 5.1.2. Then we first assume that $\alpha = 1$ and let $\phi : F_*R \to R$ be a Cartier map. We know that there is an element $u \in R$ such that $\phi(-) = \pi(F_*u-)$. Applying Matlis duality to this map gives us $\mathrm{Hom}_R(R, E) \xrightarrow{f \mapsto f\circ\phi} \mathrm{Hom}_R(F_*R, E)$. Next we use the isomorphism $E \xrightarrow{e \mapsto f_e} \mathrm{Hom}_R(R, E)$, where $f_e(1) = e$, to get the following composition

$$\begin{array}{cccc} E \to & \mathrm{Hom}_R(R,E) \to & \mathrm{Hom}_R(F_*R,E) \to & F_*E \\ e \mapsto & f_e \mapsto & f_e \circ \phi \mapsto & \Phi(f_e \circ \phi) \end{array}$$

$$\begin{aligned}
\Phi(f_e \circ \phi) &= \Phi(f_e \circ \pi(F_*u-)) \text{ (by } F_*R\text{-linearity of } \Phi) \\
&= F_*u\Phi(f_e \circ \pi) = F_*u \sum_{0 \le \bar{\alpha} < p} F_*[\bar{x}^{p-1-\bar{\alpha}} T(f_e \circ \pi(F_*\bar{x}^{\bar{\alpha}}))] \\
&= F_*u[F_*T(f_e \circ \pi(F_*\bar{x}^{p-1}))] = F_*u[F_*T(f_e(1))] \\
&= F_*uF_*T(e) = F_*uT(e)
\end{aligned}$$

Therefore, the composition above gives us the Frobenius map uT on E. In particular, if $u = 1$ we get the natural Frobenius T on E.

We now assume that $\alpha > 1$ and let $\phi : F_*R^\alpha \to R^\alpha$ be a Cartier map. We know that there is an $\alpha \times \alpha$ matrix U with entries $(u_{ij})_{1 \le i,j \le \alpha}$ in R such that $\phi(-) = \Pi(F_*U-)$. Applying Matlis duality to this map gives $\mathrm{Hom}_R(R^\alpha, E) \xrightarrow{f \mapsto f \circ \phi} \mathrm{Hom}_R(F_*R^\alpha, E)$. Then we get the following composition

$$\begin{array}{cccc}
E^\alpha & \dashrightarrow & \dashrightarrow & F_*E^\alpha \\
a = (a_1, \ldots, a_\alpha)^t & & & (\Phi(f_a \circ \phi \circ \epsilon_1), \ldots, \Phi(f_a \circ \phi \circ \epsilon_\alpha))^t \\
\downarrow & & & \uparrow \\
(f_{a_1}, \ldots, f_{a_\alpha})^t \mapsto & f_a \mapsto & f_a \circ \phi \mapsto & (f_a \circ \phi \circ \epsilon_1, \ldots, f_a \circ \phi \circ \epsilon_\alpha)^t \\
\mathrm{Hom}_R(R, E)^\alpha \to & \mathrm{Hom}_R(R^\alpha, E) \to & \mathrm{Hom}_R(F_*R^\alpha, E) \to & \mathrm{Hom}_R(F_*R, E)^\alpha
\end{array}$$

where $a \in E^\alpha$ and we use the following obvious isomorphisms

$$E^\alpha \to \mathrm{Hom}_R(R, E)^\alpha \text{ given by } (a_1, \ldots, a_\alpha)^t \mapsto (f_{a_1}, \ldots, f_{a_\alpha})^t$$

such that $f_{a_i}(1) = a_i$ for each i,

$$\mathrm{Hom}_R(R, E)^\alpha \to \mathrm{Hom}_R(R^\alpha, E) \text{ given by } (g_1, \ldots, g_\alpha)^t \mapsto g$$

such that $g(e_i) = g_i(1)$ for each elementary vector e_i, and

$$\mathrm{Hom}_R(F_*R^\alpha, E) \to \mathrm{Hom}_R(F_*R, E)^\alpha \text{ given by } h \mapsto (h \circ \epsilon_1, \ldots, h \circ \epsilon_\alpha)^t$$

such that the map $\epsilon_i : F_*R \to F_*R^\alpha$ given by $F_*r \mapsto F_*rF_*e_i$ is the canonical injection for each i. Then for a fixed i where $1 \le i \le \alpha$, we have

$$\Phi(f_a \circ \phi \circ \epsilon_i) = \sum_{0 \le \bar{\alpha} < p} F_*\left[\bar{x}^{p-1-\bar{\alpha}} T\left(f_a\left(\Pi\big(F_*U\epsilon_i(F_*\bar{x}^{\bar{\alpha}})\big)\right)\right)\right]$$

Since

$$
\begin{aligned}
f_a\Big(\Pi\big(F_*U\epsilon_i(F_*\bar{x}^{\bar{\alpha}})\big)\Big) &= f_a\Big(\Pi\big((F_*u_{1i}\bar{x}^{\bar{\alpha}},\dots,F_*u_{\alpha i}\bar{x}^{\bar{\alpha}})^t\big)\Big) \\
&= f_a\Big(\big(\pi(F_*u_{1i}\bar{x}^{\bar{\alpha}}),\dots,\pi(F_*u_{\alpha i}\bar{x}^{\bar{\alpha}})\big)^t\Big) \\
&= \sum_{1\le j\le\alpha} f_{a_j}\big(\pi(F_*u_{ji}\bar{x}^{\bar{\alpha}})\big),
\end{aligned}
$$

we have

$$
\Phi(f_a\circ\phi\circ\epsilon_i) = \sum_{1\le j\le\alpha}\Bigg(\sum_{0\le\bar{\alpha}<p} F_*\Big[\bar{x}^{p-1-\bar{\alpha}}T\Big(f_{a_j}\big(\pi(F_*u_{ji}\bar{x}^{\bar{\alpha}})\big)\Big)\Big]\Bigg).
$$

Then by definition of Φ,

$$
\begin{aligned}
\Phi(f_a\circ\phi\circ\epsilon_i) &= \sum_{1\le j\le\alpha} F_*u_{ji}\Phi(f_{a_j}\circ\pi) \\
&= \sum_{1\le j\le\alpha} F_*u_{ji}F_*T\Big(f_{a_j}\big(\pi(\bar{x}^{p-1})\big)\Big) \\
&= \sum_{1\le j\le\alpha} F_*u_{ji}T(a_j) = F_*\big[\sum_{1\le j\le\alpha} u_{ji}T(a_j)\big].
\end{aligned}
$$

Therefore, for $1\le i\le\alpha$

$$
\begin{pmatrix}\Phi(f_a\circ\phi\circ\epsilon_1)\\ \vdots \\ \Phi(f_a\circ\phi\circ\epsilon_\alpha)\end{pmatrix} = \begin{pmatrix} F_*\big[\sum_{1\le j\le\alpha} u_{j1}T(a_j)\big] \\ \vdots \\ F_*\big[\sum_{1\le j\le\alpha} u_{j\alpha}T(a_j)\big]\end{pmatrix}
$$

which is equal to

$$
F_*[U^t(T((a_1,\dots,a_\alpha)^t)] = F_*[U^tT(a)].
$$

Hence, the composition above gives us the Frobenius map U^tT on E^α. In particular, if U is the identity matrix we get the natural Froebenius T on E^α.

The construction above gives us a map $\Omega : \mathrm{Hom}_R(F_*R^\alpha, R^\alpha) \to \mathrm{Hom}_R(E^\alpha, F_*E^\alpha)$ defined by $\Omega(\phi) = \Theta$ such that $\phi(-) = \Pi(F_*U-)$ and $\Theta(-) = F_*U^tT(-)$. We claim that this map actually is an F_*R-linear isomorphism. Let $\Omega(F_*r.\phi) = \Theta'$ for any $r \in R$. Then since $(F_*r.\phi)(-) = \phi(F_*r-) = \Pi(F_*Ur-)$, we have $\Theta'(-) =$

$F_*(Ur)^tT(-) = F_*rF_*U^tT(-) = F_*r\Theta(-)$, i.e. $\Omega(F_*r.\phi) = \Theta' = F_*r\Theta = F_*r\Omega(\phi)$, and so Ω is F_*r-linear. Surjectivity of Ω is clear since for any Frobenius map $\Theta(-) = F_*U^tT(-)$ we can define a Cartier map $\phi(-) = \Pi(F_*U-)$. We also have $\Omega(\phi) = 0 \Rightarrow U^tT = 0 \Rightarrow U = 0 \Rightarrow \phi = 0$, because if any entry of U was non zero there would be non zero elements in the image of U^tT, i.e. Ω is injective. This means that we get the promised bijective correspondence. □

Next we see that the Matlis duality functor $(-)^\vee = \mathrm{Hom}_R(-, E)$ commutes with $F_*(-)$ (cf. Lemma 5.1 in [2]).

Lemma 5.2.2. *Let M be a finitely generated or an Artinian R-module. Then $F_*M^\vee \cong (F_*M)^\vee$.*

Proof. We first assume that M is finitely generated. Then M has a presentation $\cdots \to R^\beta \xrightarrow{A} R^\alpha \twoheadrightarrow M \to 0$ where A is an $\alpha \times \beta$ matrix with entries in R. If we apply the Matlis dual to this presentation we get $0 \to M^\vee \hookrightarrow E^\alpha \xrightarrow{A^t} E^\beta \to \ldots$. So $M^\vee = \mathrm{Ker}\, A^t = \mathrm{Ann}_{E^\alpha} A^t$. On the other hand, F_*M has the presentation $\cdots \to F_*R^\beta \xrightarrow{F_*A} F_*R^\alpha \twoheadrightarrow F_*M \to 0$. Then if we apply the Matlis dual again and identify $\mathrm{Hom}_R(F_*R, E)$ with F_*E using the isomorphism Φ defined in Lemma 5.1.2, we get $0 \to (F_*M)^\vee \hookrightarrow F_*E^\alpha \xrightarrow{F_*A^t} F_*E^\beta \to \ldots$, and so $(F_*M)^\vee = \mathrm{Ker}\, F_*A^t = \mathrm{Ann}_{F_*E^\alpha} F_*A^t = F_*(\mathrm{Ann}_{E^\alpha} A^t) = F_*M^\vee$.

If now M is Artinian, we know that $M^\vee$ is Noetherian and $M \cong M^{\vee\vee}$. Then it follows from first assumption, $F_*M^\vee \cong (F_*M^\vee)^{\vee\vee} \cong (F_*M^{\vee\vee})^\vee \cong (F_*M)^\vee$ □

Next theorem extends Proposition 5.2 in [2] to a computational level.

Theorem 5.2.3. *Matlis duality induces a bijective correspondence between finitely generated Cartier modules and Artinian modules equipped with Frobenius maps given as follows: if M is a finitely generated Cartier module with a square matrix U defining the Cartier module structure on M, then the corresponding Artinian module is $M^\vee$ with the corresponding Frobenius map U^tT, which preserves the nilpotency.*

Proof. Let (M, C) be a finitely generated Cartier module with a square matrix U defining Cartier module structure on M. Then we have a presentation of M as follows $\cdots \to R^\beta \xrightarrow{A} R^\alpha \twoheadrightarrow M \to 0$ and the following commutative diagram with exact rows

$$\begin{array}{ccccc} F_*R^\alpha & \longrightarrow & F_*M & \longrightarrow & 0 \\ \downarrow{\scriptstyle \Pi(F_*U-)} & & \downarrow{\scriptstyle C} & & \\ R^\alpha & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

where C is induced by $\Pi(F_*U-)$ on M. If we apply the Matlis dual to the diagram above and if we use Lemma 5.1.2, Proposition 5.2.1 and Lemma 5.2.2 we get

$$\begin{array}{ccccc} 0 & \longrightarrow & M^\vee & \longrightarrow & E^\alpha \\ & & \theta\big\downarrow & & \big\downarrow F_*U^tT \\ 0 & \longrightarrow & F_*M^\vee & \longrightarrow & F_*E \end{array}$$

where θ is the restriction of F_*U^tT on $M^\vee$. The same construction follows the converse. We also have

$$\begin{aligned} M \text{ is nilpotent } &\Leftrightarrow C^e(M) = 0 \\ &\Leftrightarrow \mathrm{I}_e(U^{[p^e-1]} \cdots UR^\alpha) \subseteq \operatorname{Im} A \text{ by Lemma 4.3.10} \\ &\Leftrightarrow \theta \text{ is a nilpotent Frobenius map on } M^\vee \text{ by Lemma 3.2.4.} \end{aligned}$$

Hence, this construction preserves nilpotency. □

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