Models of Dependent Type Theory from Algebraic Weak Factorisation Systems

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Abstract

The main purpose of this dissertation is to analyse the extent to which algebraic weak factorisation systems provide models of Martin-Löf dependent type theory. To this end, we develop the notion of a *type-theoretic awfs*; this is a category equipped with an algebraic weak factorisation system and some additional structure, such that after performing a splitting procedure, a model of Martin-Löf dependent type theory is obtained. We proceed to construct examples of such type-theoretic awfs's; first in the category of small groupoids, which produces the Hofmann and Streicher groupoid model. Later we make use of the machinery of uniform fibrations of Gambino and Sattler to produce type-theoretic awfs's in Grothendieck toposes equipped with an interval object satisfying some additional properties; from this we obtain concrete examples in the categories of simplicial sets and cubical sets. We also study the notion of a *normal* uniform fibration, a strengthening of the notion of a uniform fibration, which allows us to address a question regarding the constructive nature of type-theoretic awfs's. In addition, we show that the procedure of constructing type-theoretic awfs's from uniform fibrations is functorial, thus providing a method for comparing models of dependent type theory.

Contents

Introduction

This primary goal of this dissertation is to investigate the extent to which algebraic weak factorisations systems can be used to give models of Martin-Löf's dependent type theory.

Context and Motivation

A type theory is a formal system consisting of primitive objects called *types* and *terms*, plus some rules for manipulating judgements involving these objects. The basic judgement of any type theory is written as $a : A$ and read as a *is a term of type* A. Informally speaking, there are various possible interpretations of this judgement; for example we could think of a as an element of a set A, or that A is a 'space' and a is a point of this space. We could also think of A as a proposition and a as a code for a proof; and computationally we may think of A as some data type (for example the natural numbers, or the type of finite lists over some type), and a corresponds to an instance of such data.

One of the most important examples of a type theory is the *simply-typed lambda calculus* [Chu40] introduced by Church, which is a variant of the (untyped) lambda calculus where types are introduced in order to provide good computational behaviour to the system. In addition to the basic typing judgement, a new *type constructor* is introduced for the type of *functions*: if A and B are types we can form the type $A \to B$ whose terms correspond to functions mapping terms of type A to terms of type B. Alongside this *formation* rule, there are other rules to manipulate terms of $A \to B$. We have an *introduction* rule that tell us how to build functions, if $t : B$ is a term with (possibly) a free variable of type A, then we can form the term $\lambda x.t : A \to B$. We also have an *elimination* rule, this allow us to use functions as we would expect to: if $f : A \to B$ and $a : A$ then we can apply the function to the term $fa : B$. And finally (but crucially for the computational interpretation) we need a *computation* rule (also known as β*-reduction*), this rule is about equality thus allowing us to reduce new terms constructed from the introduction and elimination rules, to previously known ones: if $t : B$ is a term with a free variable of type A and $a : A$ then $(\lambda x.t)a = t[x/a] : B$. Some explanation is in order. First note that we have introduced a new judgement, one of the kind $b = b' : B$ this is known as a *judgemental equality* meaning that we identify two (possibly syntactically different) terms as equal. Secondly, we have introduced a purely syntactical operation called *substitution*, the term $t[x/a]$ is inductively defined by substituting in t every free occurrence of the variable x by the term a. By construction, this operation is strictly associative, meaning that it does not make a difference the order in which a double substitution is realised.

A *simply-typed lambda theory* augments the simply-typed lambda calculus with a collection of ground types and terms, and some axioms between them. Simply-typed lambda theories have natural semantics in Cartesian closed categories (ccc's for short) [Sco82, LS88]. Types are interpreted as objects and terms as arrows, the type of functions is then interpreted as the exponential or hom-object. Crucially, the substitution operation is interpreted just as the composition of arrows and this soundly models the type theory because composition of arrows is strictly (on-the-nose) associative, just as the syntactic operation. There is, in fact, an equivalence between the category of simply-typed lambda theories and the category of Cartesian closed categories.

One way to increase the expressivity of the simply-typed lambda calculus, is to allow types to depend on free variables from other types. The class of systems that arise from this strengthening are called *dependent type theories*. Along with dependent types, we can introduce new type constructors such as the dependent products, dependent sums, and identity types, a type theory with these constructors is called a *Martin-Löf type theory* [ML75], see Appendix A. The syntactic substitution operation is extended to dependent types: if B is a type depending on a free variable x of type A and $a : A$ we can substitute in B the free occurrences of x by a and have $B[x/a]$.

A first attempt by Seely [See84] to extend the interpretation of the simply-typed lambda calculus in ccc's, leads to an interpretation of extensional Martin-Löf type theories (i.e. equipped with extensional identity types) in locally cartesian closed categories (or lccc's), here a dependent type B on A is interpreted as a arrow $B \to A$ (i.e. an object in the slice over A). This naturally leads to the interpretation of substitution of terms into types as the pullback of the interpretation of the dependent type along the arrow representing the term. We immediately observe that there is a problem: pullbacks are (generally) only associative up to isomorphism, which implies that in general this interpretation does not soundly model the operation of substitution. Moreover, dependent products and dependent sums are interpreted using the right and left adjoints to the substitution functor, which are part of the defining structure of an lccc. These adjoints satisfy a coherence condition call the *Beck-Chevalley condition* which warranties that the interpretations of dependent sums and products commutes with substitution, but only up to isomorphism. We thus encounter a second problem: we cannot soundly model these type constructors, since they commute strictly with substitution in the syntax. These problems were first recognised and dealt with by Hofmann [Hof94]. The crucial point was making use of a *coherence theorem* for lccc's; that is, a method for replacing, or splitting, an lccc with equivalent structured category where the operation of substitution could be modelled soundly, and then carefully making sure that all the structure and type constructors could be transported to this new setting.

Things become more difficult when we try to reproduce the work of Seely and Hofmann to produce models of intensional Martin-Löf type theories. It turns out that the naive interpretation in lccc works only for the extensional identity types, indeed any interpretation of identity types in lccc's will necessarily be extensional. Non-theless, it is possible to define models where the interpretation of the identity types is not trivial; one of the first models of intentional identity types was described by Hofmann and Streicher in the category of groupoids [HS98]. Types in this model correspond to small groupoids G, and dependent types correspond to (split) isofibrations $\mathsf{H} \to \mathsf{G}$. Crucially, the identity type of G corresponds to the discrete groupoid of isomorphisms in G.

Inspired by the Hofmann-Streicher groupoid model, Awodey and Warren [AW09] brought forward earlier ideas from [Car86] and [Tay86] in order to address the question of extending the naive interpretation of extensional identity types in lccc's to the intensional ones. Instead of interpreting dependent types as arbitrary morphisms, they restricted the class of morphisms that could model dependent types. Moreover, in order to be able to model the additional structure of a Martin-Löf type theory (especially intensional identity types), the class of morphisms that are allowed to model dependent types, turned out to have most of the formal properties that classes of *fibrations* (in groupoids, topological spaces, simplicial sets, and more generally in Quillen model categories) enjoy. The Hofmann-Streicher groupoid model is an example of this approach where types are interpreted as isofibrations; more generally, any model that used this naive types-as-fibrations interpretation became known as an *homotopy-theoretic model* of Martin-Löf type theory. These models embodied the intuitive idea that types ought to correspond to some kind of 'spaces' viewed from an homotopical perspective, i.e. as homotopy types, in such a way that identity types correspond to path-spaces [KL12].

However, the problem of substitution being soundly modelled was still an issue; in fact, in the presence of intensional identity types it became even more difficult. Intuitively, the issue is the following; the syntactic rules for dependent sums, products and extensional identities are mirrored in the semantical world by the use of universal properties of some kind, for example for dependent sums and products, these universal properties are the ones coming from the left and right adjoints to the pullback functor. The proof that Hofmann used for adapting Seely's interpretation, heavily relied on these universal properties. However, the rules for intensional identity types are not incarnated via any universal property, thus the methods developed by Hofmann cannot be applied for intensional identities.

In his dissertation, Warren [War06] addressed the problem of adapting Hofmann's method for soundly modelling intensional identities. He identified the precise structure that was needed for this, we refer to this structure by a *pseudo-stable choice of identity types*. Unfortunately, a new problem arises; the common constructions of path-spaces in the context of topological spaces or simplicial sets (or more generally, the construction of path-objects in model categories) will not in general produce pseudo-stable choices of identity types. The crux of the problem is the following: the path-space PX of a space X comes equipped with a map $\mathsf{r} : \mathsf{X} \to \mathsf{PX}$ that maps a point of X to the constant path on it. This map has the left lifting property against fibrations; that is, for every commutative square with r on the left and any fibration on the right, there *exists* a filler. However, the definition of a pseudo-stable choice of identity types requires a *choice* of fillers for such squares, which must be in addition suitably coherent with each other. This need for additional structure, instead of just an existence property, is a reflection of the syntactic construction of the terms in the identity type using the elimination rule.

Various solutions to this problem were proposed, some of them involved using a different coherence method from that used by Hofmann. For example, Voevodsky [Voe15a] proposed the use of a universe for which a 'generic' identity type could be constructed and which would represent all the specific instances of identity types; he used this method for the construction of a model of intensional Martin-Löf's type theory with one univalent universe. It was van den Berg and Garner [dBG12] who realised that it was still possible to adapt Hofmann's method if the types-as-fibrations interpretation was strengthened. They used a more structured, *algebraic*, notion of fibrations where lifting problems come equipped with a specified choice of fillers, and this choice is suitably coherent. However, there is a drawback in their approach, it is still hard to find examples; for instance they introduced a, combinatorially complex, notion of Moore path-space for simplicial sets which was needed to obtain pseudo-stable choices of identity types. They also did not explain how their approach would interact with the additional logical structure of dependent sums and products.

At the same time, Coquand et al. [BCH14, CCHM16], constructed a model of Martin-Löf type theory (including a univalent universe) in the category of cubical sets (with connections). Just as van den Berg and Garner, they also used a structured notion of fibration to model type dependency; and moreover, as opposed to Voevodsky's, their model used only constructive arguments. One drawback of their approach is that the identity types do not coincide precisely with the canonical path-objects obtained by exponentiating by the interval object in the category of cubical sets.

Inspired by these ideas; Gambino and Sattler [GS17] developed a machinery for constructing complex objects called *algebraic weak factorisation systems* out of some general assumptions on a category. Using this data, they showed that the category of right algebras, also called the category of uniform fibrations, could be used to model a dependent type theory equipped with dependent sums and products. As an example of their method, they constructed models of uniform fibrations on simplicial sets and cubical sets, and such that the particular model of the Coquand group could also be obtained as a specific example. They also showed that under some extra assumptions, all of their arguments could be carried out constructively. Their approach however lacked the description of identity types and (univalent) universes. A similar approach was developed simultaneously by Orton and Pitts [OP16], who instead of working with the (external) algebraic structure, they used the internal language of a topos to express a list of axioms that would entail the existence of a model on the given topos.

Summary

This dissertation is an effort to extend the understanding of the (functorial) categorical semantics of Martin-Löf's type theory, using the machinery of algebraic weak factorisation systems.

In Chapter 1, we give an overview of the categorical framework that is needed to model type dependency and additional logical structure. Among the many categorical doctrines available for this purpose, we choose to work with the structure of a *split comprehension category*; the reason being that it fits perfectly with the notion of algebraic weak factorisation system. After going through the basic theory of comprehension categories, we dive into the description of the structure needed to model additional logical structure that the type theory might have. This process can be seen as a straightforward, ad-hoc, translation of the syntactic rules into the categorical framework.

Chapter 2 is dedicated to the subject of a well known procedure for *splitting* comprehension categories. This is a classical construction by Bénabou and Giraud [Gir66] that we have decided to call the *right adjoint splitting* since it forms a 2-coreflection of the inclusion of split comprehension categories into non-split ones. This method will be the main tool for the purpose of splitting algebraic weak factorisation systems. We will provide a detailed example of how to apply the coherence theorem for dependent sums and products in the contexts of Joyal's *tribes* [Joy17]. This example serves also as motivation for moving to the algebraic setting. This is because, in the setting of tribes the problem with the construction of pseudo-stable identity types becomes evident.

After dealing with the right adjoint splitting we provide, in Chapter 3, a brief description of the splitting method developed by Voevodsky using a 'universe' (or 'generic') type. Our approach is slightly more general than the original one allowing to perform the splitting procedure for arbitrary comprehension categories. The main purpose for doing this is to establish a result comparing the two splitting methods.

In Chapter 4 we develop a general theory of *type-theoretic algebraic weak factorisation systems*, or *type-theoretic awfs's* for short, these are algebraic weak factorisation systems equipped with extra structure that makes them suitable to interpret Martin-Löf dependent type theory, after performing the right adjoint splitting procedure. We begin by describing the procedure used to obtain a comprehension category out of an awfs, and identifying the additional structure that an awfs must have in order to produce pseudo-stable choices of dependent sums, products and intentional identity types; we refer by a type-theoretic awfs to the data of an awfs equipped with this additional structure. We move on to provide an example of a type-theoretic awfs in the category of groupoids; we show that it is possible to construct such a type-theoretic awfs by elementary methods, where the underlying awfs has the split isofibrations as right algebras; and thus giving an alternative description of the Hofmann-Streicher original model. The crucial ingredient needed to construct the type-theoretic awfs on groupoids is the existence of a path-object which has a strictly associative and unital composition operation of paths; this is consistent with the results of van den Berg and Garner where

this was an indispensable requirement for their constructions. Unfortunately, it is often difficult to encounter such well behaved path-objects in the intended examples.

For this reason, in Chapter 5, we decide to work with the theory of uniform fibrations as developed by Gambino and Sattler [GS17] as a major source of examples of type-theoretic awfs. In their work, Gambino and Sattler showed that the awfs of uniform fibrations comes equipped with the necessary structure needed to obtain pseudo-stable dependent sums and products. In this chapter, we show that under some additional hypothesis, the requirements for obtaining pseudo-stable identity types are also met. Additionally, we are able to adapt some of their results to obtain type-theoretic awfs in Grothendieck toposes equipped with the structure of an interval object; for example in this way we obtain type-theoretic awfs on simplicial and cubical sets.

We dedicate Chapter 6 to the study of the functorial aspects of the theory of uniform fibrations. In detail, we show that the process of producing type-theoretic awfs of uniform fibrations is just the object part of a functor between suitably defined categories with additional structure. As an application of this result, we show that type-theoretic awfs of uniform fibrations can be transported along the left adjoint of a geometric embedding of toposes with the resulting model being connected to the original one via a morphism of type-theoretic awfs's.

It turns out that, from a constructive perspective, our results regarding the construction of a type-theoretic awfs from the theory of uniform fibrations are not entirely satisfactory. The reason is that, even-though the proof of our main theorem (Theorem 5.1.1) uses only constructive arguments, it is not always possible to show constructively that one of our additional hypothesis hold. For this reason, in Chapter 7, we propose a method for addressing this issue by introducing a notion of *normal uniform fibrations*, which is formally similar to that of normal isofibrations in groupoids.

For the convenience of the reader, we have included a chapter with background material (Chapter 1) as well as three appendices.

Main Contributions

The main contributions of this thesis are aimed at providing a better of understanding of the use of algebraic methods, especially of algebraic weak factorisation systems, in the construction of models of dependent type theory. We proceed to list them for the convenience of the reader.

1. Our first contribution is found in Chapter 2. Here we provide a detailed proof of the coherence theorem for the right adjoint splitting; we do this by carefully analysing the cases of Σ, Π, $\mathtt{Id}$ and universe types. We emphasise that the case of universes has some technical difficulties and, to the best of our knowledge, had not been dealt with before.

2. In Chapter 4 we introduce the notion of a type-theoretic awfs (Definition 4.4.1). This definition is a synthesis of various concepts from the work of [dBG12] and

[GS17]. We show that from a type-theoretic awfs it is possible to obtain a strict model of dependent type theory with Σ, Π and intensional `Id`-types. We do this by first extracting the data of a comprehension category equipped with pseudo-stable choices of the relevant kinds of logical structure, and then using the coherence theorem for the right adjoint splitting of comprehension categories (Theorem 2.6.1).

3. We show, in Chapter 5, that a vast source of examples of type-theoretic awfs's can be obtained by applying and extending the theory of uniform fibrations developed in [GS17]. In detail, our main contribution is Theorem 5.1.1 where we show that the awfs of uniform fibrations can be equipped with the structure of a stable functorial choice of path objects (Definition 4.3.1); and thus, in conjunction with the results of [GS17], provides the necessary structure for a type-theoretic awfs's.

4. Our next contribution can be found in Chapter 6. Here we show that the method of uniform fibrations for constructing a type-theoretic awfs, is functorial. The main motivation for doing this is to have a method for comparing different models of dependent type theory obtained by applying the theory of uniform fibration. The proofs in this chapter are rather technical and are achieved by developing one-by-one the functorial part of the arguments found in [GS17]. The end result is summarise in Theorem 6.6.4 and Theorem 6.6.5. Additionally, we show in Theorem 6.7.1 that the process of obtaining a comprehension category from a type-theoretic awfs is also functorial.

5. Our final main contribution is found in Chapter 7. Here we develop a strengthening of the notion of a uniform fibration, which we call normal uniform fibration (Section 7.2). We show that a type-theoretic awfs of normal uniform fibrations can be constructed by adapting the arguments of [GS17] and of Chapter 5. The main results of this chapter are Theorem 7.5.5 and Theorem 7.6.6. Our motivation for developing this notion is to overcome an issue regarding the constructive nature in the hypothesis of Theorem 5.1.1, as mentioned in Note 5.2.5.

Chapter 1

Background

This dissertation is ultimately about categorical models of dependent type theory; as such, it is important to establish with precision the setting that will be used. The purpose of his first part is precisely that: to give a brief but thorough overview of the categorical structures necessary to produce such models.

The subject of categorical semantics of dependent type theory is vast. There exists a plethora of distinct flavours of structured categories (i.e. doctrines) that enables the interpretation of the basic components of a type theory. Just to mention a few, there are categories with families (CwF) [Dyb96], categories with attributes (CwA) [Car86, Mog91], split comprehension categories [Jac93], contextual categories [Car86, Str91], natural models [Awo16], etc. All of these are suitably equivalent to each other, some of them are more closely related to the syntax in nature (for example contextual categories) and some of them are closer to the semantics (for example comprehension categories).

We choose to work with comprehension categories, the reason being that they fit more naturally with the applications that we have in mind; that is, they streamline the process of creating models of dependent type theory from (algebraic) weak factorisation system.

We will begin with a review of the main aspects of the theory of comprehension categories and of Grothendieck fibrations. Following this, we will explain how additional type-theoretic logical structure is interpreted in a comprehension category. We make no claim of originality for the content of this chapter, however some new notational devices are introduced in the hope of easing the understanding of the more technically difficult definitions and results. Our account on the subject follows [Jac99] and [Str18].

1.1 Grothendieck Fibrations

Let us fix a functor $\rho : \mathbb{E} \to \mathbb{C}$. We say that an object $A \in \mathbb{E}$ (or an arrow $f : B \to A$ in $\mathbb{E}$) is **over** an object $\Gamma \in \mathbb{C}$ (or respectively **over** an arrow $u : \Delta \to \Gamma$) if $\rho(A) = \Gamma$

(respectively $\rho(f) = u$). An arrow in $\mathbb{E}$ is **vertical** if it is over an identity. The category $\mathbb{C}$ will be called the **base** category and $\mathbb{E}$ the **total** category of ρ.

Definition 1.1.1. An arrow $f : B \to A$ in $\mathbb{E}$ is said to be **Cartesian** with respect to ρ (or **ρ-Cartesian**) over $u : \Delta \to \Gamma$ in $\mathbb{C}$ if it is over u and the following universal property holds: for any $g : C \to A$ and $v : \Theta \to \Gamma$ such that $u \circ v = \rho(g)$, there exists a unique arrow $h : C \to B$ over v such that $f \circ h = g$.

This can be represented in a diagram as follows:

$$\begin{tikzcd}
C \arrow[drrr, bend left, "g"] \arrow[dr, dotted, "\exists! h"'] \arrow[dd, dotted, no head] & & & & & \\
& B \arrow[rr, "f"] \arrow[dd, dotted, no head] & & A \arrow[dd, dotted, no head] & & \mathbb{E} \arrow[dd, "\rho"] \\
\Theta \arrow[dr, "v"] \arrow[drrr, bend left, "\rho(g)"] & & & & & \\
& \Delta \arrow[rr, "u"] & & \Gamma & & \mathbb{C}
\end{tikzcd}$$

(1.1)

Diagrammatically, we will denote a ρ-Cartesian arrow f over an arrow u as in the following diagram:

$$\begin{tikzcd}
B \arrow[r, "f"] \arrow[d, dotted, no head] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, dotted, no head] \\
\Delta \arrow[r, "u"'] & \Gamma
\end{tikzcd}$$

This notation suggests that Cartesian arrows resemble in some way pullback squares, the following proposition should evoke some similarities between the two notions. Moreover, as we will see in Example 1.1.5 bellow, there are cases where the two notions coincide.

Proposition 1.1.2. *Suppose that f, g, h are arrows in the total category of a functor ρ such that $f \circ g = h$. Then the following conditions hold:*

1. *If f and g are Cartesian then so is h.*
2. *If h and f are Cartesian then so is g.*

Proof. Straightforward from the definition. If we suppose f, g and h are morphisms in an arrow category $\mathbb{C}^{\to}$ (i.e. squares) and replace Cartesian arrows for pullback squares, then the above lemma takes the form of a well known result; the proofs of both results are very similar. ☐

Definition 1.1.3. The functor ρ is a **fibration** if for any arrow $u : \Delta \to \Gamma$ in $\mathbb{C}$ and for any object $A \in \mathbb{E}$ over Γ, there exists a Cartesian arrow $f : B \to A$ over u. Such an arrow will be called a **Cartesian lift** of u at A.

The following proposition follows easily from the definition of Cartesian lift. The analogous statement with pullbacks is the well known result (which follows from the universal property) that pullback squares are unique up to unique isomorphism.

Proposition 1.1.4. *Any two Cartesian lifts of the same arrow at the same object are isomorphic via a unique vertical isomorphism.* □

Example 1.1.5. There are two canonical functors of signature $\mathbb{C}^{\to} \to \mathbb{C}$; the domain and the codomain functors. For the domain functor Cartesian arrows are squares where the bottom arrow is an isomorphism. For the codomain functor, Cartesian arrows are pullback squares. The domain functor is always a fibration, however the codomain functor is a fibration if and only if the base category has pullbacks.

Definition 1.1.6. A **cleavage** for a fibration ρ is a choice of Cartesian lifts, i.e. for every $u : \Delta \to \Gamma$ and A over Γ a Cartesian lift, which we will denote by:

$$\begin{tikzcd}
A[u] \arrow[r, "u_A"] \arrow[d, dotted, no head] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, dotted, no head] \\
\Delta \arrow[r, "u"] & \Gamma
\end{tikzcd}$$

A **cloven fibration** is a fibration equipped with a cleavage. A cloven fibration is **normal** if the cleavage preserves identities, i.e. the lift of the identity of Γ at A is the identity on A. A normal fibration is **split** if it the cleavage preserves composition; that is, $(u \circ v)_A = u_A \circ v_{A[u]}$ for composable arrows u and v in $\mathbb{C}$.

Assuming the axiom of choice we can always choose a cleavage for any given fibration, moreover the choice can be made in such a way that the resulting cleavage is normal. However it is not always possible to choose a cleavage that makes the fibration split.

Given $\Gamma \in \mathbb{C}$, the **fibre** of ρ over Γ denoted by $\mathbb{E}(\Gamma)$ is the subcategory of $\mathbb{E}$ whose objects are over Γ and whose arrows are over id_Γ (i.e. vertical arrows). Notice that the Cartesian arrows in the fibres (i.e. the arrows which are both vertical and Cartesian) are precisely the vertical isomorphism. The following lemma follows immediately from this observation.

Lemma 1.1.7. *Let $\rho : \mathbb{E} \to \mathbb{C}$ be a fibration and consider $\mathbb{E}_{cart}$ the wide (i.e. containing all objects) subcategory of $\mathbb{E}$ spanned by the Cartesian arrows. Then the functor ρ_{cart} : $\mathbb{E}_{cart} \to \mathbb{C}$ given by composing ρ with the inclusion, is a fibration. Moreover the fibres of ρ_{cart} are groupoids.* □

Lemma 1.1.8. *If ρ is a cloven fibration then every arrow $u : \Delta \to \Gamma$ in $\mathbb{C}$ induces a functor*

$$(-)[u] : \mathbb{E}(\Gamma) \to \mathbb{E}(\Delta).$$

*This is usually called the **reindexing** or **substitution** functor along u.*

Proof. On objects it is defined by the cleavage. Given an arrow $f : A' \to A$ define $f[u] : A'[u] \to A[u]$ as the unique arrow over id_Δ with $u_A \circ f[u] = f \circ u_{A'}$. Functoriality follows because the arrows are defined canonically. In the following diagram, we illustrate how the arrow $f[u]$ is defined:

$$\begin{tikzcd}
\mathbb{E} \arrow[dd, "\rho"] & A'[u] \arrow[rr, "u_{A'}"] \arrow[dr, dotted, "\exists! f{[u]}"'] \arrow[dd, dotted, no head] & & A' \arrow[dr, "f"] \arrow[dd, dotted, no head] & \\
& & A[u] \arrow[rr, "u_A"] \arrow[dd, dotted, no head] \arrow[dr, phantom, very near start, "\lrcorner"] & & A \arrow[dd, dotted, no head] \\
\mathbb{C} & \Delta \arrow[rr, "u"] \arrow[dr, "\mathrm{id}"] & & \Gamma \arrow[dr, "\mathrm{id}"] & \\
& & \Delta \arrow[rr, "u"] & & \Gamma
\end{tikzcd}$$

□

Definition 1.1.9. Let $\rho : \mathbb{E} \to \mathbb{C}$ and $q : \mathbb{D} \to \mathbb{C}$ be Grothendieck fibrations. A functor $H : \mathbb{E} \to \mathbb{D}$ is called **fibred** if the following diagram commutes

$$\begin{tikzcd}
\mathbb{E} \arrow[rr, "H"] \arrow[dr, "\rho"'] & & \mathbb{D} \arrow[dl, "q"] \\
& \mathbb{C} &
\end{tikzcd}$$

and H preserves Cartesian arrows. In the case that ρ and q are cloven fibrations, a fibred functor H is said to **preserve the cleavage** if for any arrow u in $\mathbb{C}$ and A over the codomain of u we have that $H(u_A) = u_{H(A)}$.

For any two fibred functors $H, K : \rho \to q$; a natural transformation $\eta : H \to K$ is said to be **fibred** if its components are vertical, i.e. $\eta_A : H(A) \to K(A)$ is over $\mathrm{id}_{\rho(A)}$ for each A in the total category of ρ.

A fibred functor $H : \rho \to q$ is said to be a **fibred equivalence** if there exist a fibred functor $G : q \to \rho$ and fibred natural isomorphisms $HG \cong \mathrm{id}_q$ and $GH \cong \mathrm{id}_\rho$.

Notice that fibred functors and fibred natural transformations restrict to the fibres, indeed, if H is a fibred functor as in the above definition, then for each Γ in the base category, there is an induced functor

$$H_\Gamma : \mathbb{E}(\Gamma) \to \mathbb{D}(\Gamma)$$

and if $\eta : H \to K$ is a fibred natural transformation then it induces

$$\eta_\Gamma : H_\Gamma \to K_\Gamma$$

whose components are the same as the components of η (i.e. $\eta_{\Gamma,A} = \eta_A$).

The following results leading to Lemma 1.1.12 are elementary and well known. We include the proofs in the spirit of completeness since we could not find detailed proofs in the literature for some of them.

Lemma 1.1.10. *A fibred functor H (as above) is full (respectively faithful) if and only if for each Γ in the base, the restriction H_Γ is full (respectively faithfull).*

Proof. If H is full (faithful) it is clear that each H_Γ is full (faithful). To prove the other direction, let A and B be objects of $\mathbb{E}$ over Γ and Δ respectively. Now suppose that all restriction functors are full, and consider an arrow $f : H(B) \to H(A)$ lying over say $u : \Delta \to \Gamma$. Take $u^* : A^* \to A$ any ρ-Cartesian lift of A along u, since H is fibred we know that $H(u^*)$ is a q-Cartesian lift of $H(A)$ over u and by the defining property of Cartesian arrows we find a unique vertical h that factors f through $H(u^*)$:

$$\begin{tikzcd}
H(B) \arrow[d, "h", "\exists!"'] \arrow[dr, "f"] & \\
H(A^*) \arrow[r, "H(u^*)"] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[d, dotted, no head] & H(A) \arrow[d, dotted, no head] \\
\Delta \arrow[r, "u"] & \Gamma
\end{tikzcd}$$

Since H_Δ is full, there is a vertical $g : B \to A^*$ over Δ such that $H(g) = h$ and thus

$$\begin{aligned} H(u^* \circ g) &= H(u^*) \circ H(g) \\ &= H(u^*) \circ h \\ &= f \end{aligned}$$

A similar argument go through for faithfulness. ☐

Corollary 1.1.11. *A fibred functor H is a fibred equivalence if and only if for each Γ in the base, H_Γ is an equivalence.*

Proof. It is clear that if H is a fibred equivalence, then each H_Γ will be an equivalence. For the converse, we use the previous lemma to show that since all H_Γ are full and faithful functors, then H is also full and faithful. Now consider B in $\mathbb{D}$ over Γ, since H_Γ is surjective on objects there exists A in $\mathbb{D}$ over Γ and a vertical isomorphism

$$H(A) \cong B$$

thus H is surjective on objects and therefore an equivalence. It can be proven from the fact that the above isomorphism is vertical that H is moreover a fibred equivalence. ☐

The fibrations over a common base $\mathbb{C}$ together with the fibred functors and fibred natural transformations assemble into a 2-category $\underline{\mathrm{Fib}}(\mathbb{C})$. The subcategory of $\underline{\mathrm{Fib}}(\mathbb{C})$ of split fibrations and cleavage preserving fibred functors will be denoted $\underline{\mathrm{Sp}}(\mathbb{C})$. Let us denote by $\underline{\mathrm{Cat}}$ the category of locally small categories and by $[\mathbb{C}^{\mathrm{op}}, \underline{\mathrm{Cat}}]$ the functor category.

Lemma 1.1.12. *Given a locally small category $\mathbb{C}$, there is an equivalence:*

$$\underline{\mathrm{Sp}}(\mathbb{C}) \simeq [\mathbb{C}^{\mathrm{op}}, \underline{\mathrm{Cat}}]$$

Proof. Let us start by defining the following functor $\Theta : \underline{\mathrm{Sp}}(\mathbb{C}) \to [\mathbb{C}^{\mathrm{op}}, \underline{\mathrm{Cat}}]$ as follows. Given a split fibration $\rho : \mathbb{E} \to \mathbb{C}$, define $\Theta(\rho) : \mathbb{C}^{\mathrm{op}} \to \underline{\mathrm{Cat}}$ on objects by taking $\Theta(\rho)(\Gamma) := \mathbb{E}(\Gamma)$ the fibre over Γ. The action of $\Theta(\rho)$ on arrows is given by reindexing while functoriality will follow from the fact that ρ is split.

On arrows Θ is defined in the obvious way and naturality follows from the fact that morphisms of split fibrations preserve cleavages. It is straightforward to observe that Θ is full and faithful, we will only show that it is full. Let $\rho : \mathbb{E} \to \mathbb{C}$ and $q : \mathbb{D} \to \mathbb{C}$ be split fibrations and let $\eta : \Theta(\rho) \to \Theta(q)$ be a natural transformation. We define a functor $H : \mathbb{E} \to \mathbb{D}$ such that for each Γ it agrees with η_Γ, i.e. for A over Γ we have $H(A) := \eta_\Gamma(A)$ We now need to define $H(f)$ for an arrow $f : B \to A$ over $u : \Delta \to \Gamma$; for this consider the following factorisation of f as a vertical arrow followed by a Cartesian one:

$$\begin{tikzcd}
B \arrow[d, "h"'] \arrow[dr, "f"] & \\
A[u] \arrow[r, "u_A"] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[d, dotted, no head] & A \arrow[d, dotted, no head] \\
\Delta \arrow[r, "u"] & \Gamma
\end{tikzcd}$$

notice that this factorisation is uniquely determined by the cleavage of ρ. We thus define

$$H(f) := u_{\eta_\Gamma(A)} \circ \eta_\Delta(h)$$

it is clear that H is a well-defined, cleavage preserving, fibred functor. It is also clear that $\Theta(H) = \eta$.

Now we are left to show that Θ is surjective on objects, for this we make use of the Grothendieck construction which we will briefly describe. From a functor $F : \mathbb{C}^{\mathrm{op}} \to \underline{\mathrm{Cat}}$ we can construct a split fibration $\pi : \int F \to \mathbb{C}$ where the category $\int F$ is defined as follows:

Objects: Pairs (Γ, A) where Γ is in $\mathbb{C}$ and A is an object of $F(\Gamma)$.

Arrows: An arrow $(\Delta, B) \to (\Gamma, A)$ is given by a pair (u, α) where $u : \Delta \to \Gamma$ is an arrow in $\mathbb{C}$ and $\alpha : B \to F(u)(A)$ is an arrow in $F(\Delta)$.

The functor π is the evident first projection. To show that π is a cloven fibration it is sufficient to exhibit a cleavage, given $u : \Delta \to \Gamma$ and (Γ, A) over Γ we see that

$$(u, id) : (\Delta, F(u)(A)) \to (\Gamma, A)$$

is a Cartesian arrow over u, thus π is a cloven fibration. The fact that π is split follows from the functoriality of F which we will omit.

Finally we can see that $\Theta(\pi) \cong F$ where the isomorphism is given by a natural transformation λ with components $\lambda_\Gamma : \Theta(\pi)(\Gamma) \to F(\Gamma)$ given by $\lambda_\Gamma(\Gamma, A) = A$. □

We can ask ourselves what happens if we restrict the equivalence given by the previous lemma to the category of presheaves $[\mathbb{C}^{op}, \underline{\mathrm{Set}}]$ instead of the functor category $[\mathbb{C}^{op}, \underline{\mathrm{Cat}}]$.

Definition 1.1.13. A functor $\rho : \mathbb{E} \to \mathbb{C}$ is a **discrete fibration** if it has small fibres (i.e. the fibres are small categories) and for any arrow $u : \Delta \to \Gamma$ in $\mathbb{C}$ and A over Γ, there is a unique arrow $f : A^* \to A$ in $\mathbb{E}$ over u.

Remark 1.1.14. The requirement of small fibres is a technical one since we can always consider a bigger set theoretic universe with respect to which the fibration has small fibres.

Notice that every discrete fibration is in particular a split fibration such that every arrow in the total category is Cartesian. Let $\underline{\mathrm{dFib}}(\mathbb{C})$ denote the full subcategory of $\underline{\mathrm{Sp}}(\mathbb{C})$ consisting of discrete fibrations.

Corollary 1.1.15. *Given a locally small category* $\mathbb{C}$ *there is an equivalence*

$$\underline{\mathrm{dFib}}(\mathbb{C}) \simeq [\mathbb{C}^{op}, \underline{\mathrm{Set}}]$$

Proof. The proof is the same as the one from Lemma 1.1.12 but instead of the Grothendieck construction we consider the category of elements construction and make the appropriate modifications. □

1.2 Comprehension Categories

Definition 1.2.1. Let $\mathbb{C}$ be a category equipped with a terminal object $* \in \mathbb{C}$. A **comprehension category** on $\mathbb{C}$ consists of a strictly commutative diagram of the form:

$$\begin{tikzcd} \mathbb{E} \arrow[rr, "\chi"] \arrow[dr, "\rho"'] & & \mathbb{C}^{\to} \arrow[dl, "\mathrm{cod}"] \\ & \mathbb{C} & \end{tikzcd}$$

where:

- ρ is a Grothendieck fibration.

- χ maps Cartesian arrows in $\mathbb{E}$ to pullback squares in $\mathbb{C}^{\to}$.

The functor χ is called the **comprehension** functor and we will usually refer to a comprehension category by $(\mathbb{C}, \rho, \chi)$ consisting of the base category, the fibration and the comprehension; alternatively when the category in question can be easily inferred from the context we may denote the comprehension category by the pair (ρ, χ).

Note that the base category $\mathbb{C}$ is not required to have pullbacks for all diagrams; in particular, the codomain functor may not be a Grothendieck fibration and in this case χ is not precisely a fibred functor. However, it behaves exactly as one.

Given a comprehension category $(\mathbb{C}, \rho, \chi)$ we say that it is:

- **cloven** if ρ is cloven.
- **split** if ρ is split.
- **full** if χ is full and faithful.
- **discrete** if ρ is a discrete fibration

For an object $A \in \mathbb{E}$ over $\Gamma \in \mathbb{C}$ we denote by

$$\chi_A : \Gamma.A \to \Gamma$$

the arrow in $\mathbb{C}$ resulting from applying the comprehension functor χ to A. Similarly for an arrow $f : B \to A$ over $\sigma : \Delta \to \Gamma$, applying comprehension to f yields the following square:

$$\begin{tikzcd}
\Delta.B \arrow[r, "f"] \arrow[d, "\chi_B"'] & \Gamma.A \arrow[d, "\chi_A"] \\
B \arrow[r, "\sigma"'] & A
\end{tikzcd}$$

here we allow ourselves a mild abuse of notation by giving the same name to the arrows in $\mathbb{E}$ and to the upper horizontal arrows in the resulting square after applying the comprehension functor.

Example 1.2.2. For any category $\mathbb{C}$ with pullbacks, the pair $(\mathsf{cod}, \mathsf{id}_{\mathbb{C}})$ is a comprehension category. More generally, any full subcategory $\mathbb{D}$ of $\mathbb{C}^{\to}$ closed under pullbacks along arbitrary maps induces a comprehension category (cod, ι) as shown

$$\begin{tikzcd}
\mathbb{D} \arrow[rr, hook, "\iota"] \arrow[dr, "\mathsf{cod}"'] & & \mathbb{C}^{\to} \arrow[dl, "\mathsf{cod}"] \\
& \mathbb{C} &
\end{tikzcd}$$

this is an important class of examples of comprehension categories which are closely related to *display map* categories.

The notation established for Grothendieck fibrations and for comprehension categories can become quickly overloaded and difficult to read; for this reason, we will highlight some special cases where we will adopt a slightly different, more compact, notation. Let us suppose that $(\mathbb{C}, \rho, \chi)$ is a cloven comprehension category and let $\sigma : \Delta \to \Gamma$ be an arrow in the base category. For any A over Γ we will alternatively denote by $\sigma^* : A[\sigma] \to A$ the Cartesian lift of σ at A given by the cleavage (which we had previously denoted with σ_A) as shown in the following diagram:

$$\begin{tikzcd}
\Delta.A[\sigma] \arrow[r, "\sigma^*"] \arrow[d, "\chi_{A[u]}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \Gamma.A \arrow[d, "\chi_A"] \\
\Delta \arrow[r, "\sigma"'] & \Gamma
\end{tikzcd}$$

This allows us to ease the notation when we wish to find a further lift of σ^*. For example, let us suppose we have another element B of $\mathbb{E}$ but this time over $\Gamma.A$, following the above convention, the Cartesian lift of B along σ^* will be denoted $\sigma^{**} : A[\sigma] \to B$ and by applying comprehension to it we will get the pullback square:

$$\begin{tikzcd}
\Delta.A[\sigma].B[\sigma] \arrow[r, "\sigma^{**}"] \arrow[d, "\chi_{B[\sigma]}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \Gamma.A.B \arrow[d, "\chi_B"] \\
\Delta.A[\sigma] \arrow[r, "\sigma^*"'] & \Gamma.A
\end{tikzcd}$$

There will be occasions where we will also consider Cartesian lifts of the comprehension morphism of some object (this is the semantic counterpart of context weakening); for example if A and B are above Γ we can consider the Cartesian lift of B along χ_A, we will adopt a further abuse of notation and denote by $\chi_{A,B} : B \to B$ such Cartesian lift (instead of the more cumbersome $(\chi_A)_{B[\chi_A]} : B[\chi_A] \to B$) and the resulting comprehension by:

$$\begin{tikzcd}
\Gamma.A.B \arrow[r, "\chi_{A,B}"] \arrow[d, "\chi_B"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \Gamma.B \arrow[d, "\chi_B"] \\
\Gamma.A \arrow[r, "\chi_A"'] & \Gamma
\end{tikzcd}$$

Definition 1.2.3. Given comprehension categories (ρ, χ) and (q, χ') over a category $\mathbb{C}$. A **morphism of comprehension categories** consist of a pair (H, η) where H is a fibred functor between the underlying fibrations:

$$\begin{tikzcd}
\mathbb{E} \arrow[rr, "H"] \arrow[dr, "\rho"'] & & \mathbb{D} \arrow[dl, "q"] \\
& \mathbb{C} &
\end{tikzcd}$$

and $\eta : \chi \to \chi' \circ H$ is a natural isomorphism such that the whiskering with the codomain functor is the identity. If η is the identity, we will call the morphism **strict.**

For instance, given (H, η) as in Definition 1.2.3 and an object A in $\rho(\Gamma)$, the component of η at A is a commutative diagram of the form

$$\begin{tikzcd}
\Gamma.A \arrow[r, "\eta_A", "\cong"'] \arrow[dr, "\chi_A"'] & \Gamma.H(A) \arrow[d, "\chi'_{H(A)}"] \\
& \Gamma
\end{tikzcd}$$

Comprehension categories over a fixed base $\mathbb{C}$ and strict morphisms between them assemble into a category which we will denote with $\underline{\mathrm{CCat}}(\mathbb{C})$. The subcategory of split comprehension categories and strict cleavage preserving morphisms will be called $\underline{\mathrm{SpCCat}}(\mathbb{C})$.

1.3 Models of Dependent Type Theories

Let us start by considering a dependent type theory $\mathbb{T}_0$ with no additional logical structure, that is we are only interested in the structural rules of substitution and context extension (see Appendix A). We will show that it is possible to build out of the syntax of $\mathbb{T}_0$ a split comprehension category $(\mathbb{C}_{\mathbb{T}_0}, \rho_0, \chi_0)$ over the category $\mathbb{C}_{\mathbb{T}_0}$ of contexts and substitutions. Moreover it is possible to show (by induction on the structure of $\mathbb{T}_0$) that $(\mathbb{C}_{\mathbb{T}_0}, \rho_0, \chi_0)$ is initial (in an appropriate way) in the 2-category the split comprehension categories. Therefore it is justified to consider the category of split comprehension categories as the category of models of $\mathbb{T}_0$, in the same way as Cartesian closed categories are models of simply-typed lambda theories.

Non-split comprehension categories arise more naturally in mathematics (for example the comprehension categories of Example 1.2.2 are non-split in general) however, these will not constitute sound models of the structural rules of dependent type theory. The main reason is that we cannot soundly interpret the operation of substitution if the underlying fibration is not split; for instance, syntactically substitution is strictly associative in the sense that $A[u][v] = A[u[v]]$ but we cannot hope to obtain this strict equality in a general non-split comprehension category.

Moreover, the issue gets more complicated when we start to consider type theories with additional logical structure; for example let $\mathbb{T}_\Pi$ be the extension of $\mathbb{T}_0$ with dependent products. Suppose we have a comprehension category $(\mathbb{C}, \rho, \chi)$ which models $\mathbb{T}_\Pi$. Since $\mathbb{T}_\Pi$ extends $\mathbb{T}_0$ it must be the case that $(\mathbb{C}, \rho, \chi)$ is also a model of $\mathbb{T}_0$ and thus it must be split. Moreover, because of its admissibility, substitution must commute strictly with the formation of Π-types.

For the reasons explain above, we will need to work inside the category of split comprehension categories in order to produce sound models of dependent type theory with additional logical structure. In the following sections we will define the categorical structure needed to model the most important type-formation operations, we will do this for each type structure independently. There is (to the best of our knowledge) no general result that can apply to all type formation operations at once, nevertheless we can give a general heuristic of how to interpret additional logical structure in a split comprehension category. The pattern is the following:

1. We first translate the usual syntactic rules (formation, introduction, elimination and computation) to specify a structured object representing the given logical structure. This can be done in a general comprehension category because there is no substitution operation involved. Notice that this is a *choice of structure* for any given input data.

2. Second we add the appropriate *coherence property* or *Beck-Chevalley condition*. This requires the choice of structure to cohere strictly with the cleavage. Intuitively, this means that 'pulling back' must preserve on-the-nose the choice of structure. Note that this is merely an additional property of the choice made.

We will end this section by constructing a discrete comprehension category associated to the dependent type theory $\mathbb{T}_0$ consisting only of the structural rules for substitution and context extension. In order to do this, let us consider $\mathbb{C}_{\mathbb{T}_0}$ the category of contexts and substitutions of $\mathbb{T}_0$ [Pit01] and let $\rho : \underline{\mathrm{Ty}} \to \mathbb{C}_{\mathbb{T}_0}$ be the **discrete fibration of types**, i.e. $\underline{\mathrm{Ty}}$ is the category that has:

Objects: Pairs (Γ, A) where Γ is a context (i.e. an object of $\mathbb{C}_{\mathbb{T}_0}$) and A is a type in context Γ as shown:

$$\Gamma \vdash A : \mathsf{type}$$

Arrows: An arrow $(\Delta, B) \to (\Gamma, A)$ of $\underline{\mathrm{Ty}}$ consists of a context substitution $u : \Delta \to \Gamma$ (i.e. a morphism of $\mathbb{C}_{\mathbb{T}_0}$) such that:

$$\Delta \vdash B = A[u]$$

The identity morphism on (Γ, A) is given by the identity substitution and composition is given by composition in $\mathbb{C}_{\mathbb{T}_0}$. One can check that $\underline{\mathrm{Ty}}$ is a well defined category.

The functor ρ is given by projecting into the first element (i.e. $\rho(\Gamma, A) = \Gamma$). To see that ρ is a discrete Grothendieck fibration, notice that for $u : \Delta \to \Gamma$ and (Γ, A) over Γ there is a unique lift arising from type substitution:

$$u : (\Delta, A[u]) \to (\Gamma, A)$$

The comprehension functor is given by context extension: if $(\Gamma, A) \in \underline{\mathrm{Ty}}$ the corresponding arrow $\chi_{(\Gamma,A)}$ in $\mathbb{C}_{\mathbb{T}_0}$ is the dependent projection

$$(\Gamma, x : A) \to \Gamma$$

It is straightforward to see that this extends to a functor $\chi : \underline{\mathrm{Ty}} \to \mathbb{C}_{\mathbb{T}_0}^{\to}$ and that χ maps Cartesian arrows to pullback squares.

The resulting discrete comprehension category:

$$\begin{tikzcd}
\underline{\mathsf{Ty}} \arrow[rr, "\chi"] \arrow[dr, "\rho"'] & & \mathbb{C}_{\mathbb{T}_0}^{\rightarrow} \arrow[dl, "\mathsf{cod}"] \\
& \mathbb{C}_{\mathbb{T}_0} &
\end{tikzcd}$$

is called the **syntactic comprehension category** of $\mathbb{T}_0$.

For the sake of readability, we will identify an elements and arrows of $\underline{\mathsf{Ty}}$ only by their second entry (i.e. the object (Γ, A) will be represented by A). Note that for substitutions $\nu : \Theta \to \Delta$, $u : \Delta \to \Gamma$ and for A in $\mathbb{E}$ over Γ the following lifts are equal:

$$\begin{tikzcd}
A[u][\nu] \arrow[r, "\nu"] \arrow[d, dash, dotted] & A[u] \arrow[r, "u"] \arrow[d, dash, dotted] & A \arrow[d, dash, dotted] & A[u \circ \nu] \arrow[r, "u \circ \nu"] \arrow[d, dash, dotted] & A \arrow[d, dash, dotted] \\
\Theta \arrow[r, "\nu"'] & \Delta \arrow[r, "u"'] & \Gamma & \Theta \arrow[r, "u \circ \nu"'] & \Gamma
\end{tikzcd}$$

Consider a type A over some context Γ and a section of the canonical projection (which is also the comprehension of A). Notice that such section is necessarily of the following form

$$(\mathrm{id}_\Gamma, t) : \Gamma \to (\Gamma, x : A)$$

where $\Gamma \vdash t : A$. This gives a canonical bijection between the terms of the syntax and sections of dependent projections.

1.4 Dependent Tuples

In this brief section we introduces a technical machinery that will be useful when defining the structure needed to model additional logical structure in comprehension categories.

Definition 1.4.1. Let (ρ, χ) be a comprehension category. For each $n \in \mathbb{N}$ we define the category $\mathsf{DT}_n(\rho, \chi)$ of **dependent tuples** over (ρ, χ) as follows:

Objects: Tuples $(\Gamma, A_1, \dots, A_n)$ where Γ is an element in the base category, A_1 is over Γ and in general for $i > 1$

$$A_i \in \mathbb{E}(\Gamma.A_1.\dots.A_{i-1})$$

Arrows: An arrow $(\Delta, B_1, \dots, B_n) \to (\Gamma, A_1, \dots, A_n)$ is a tuple of the form $(u, f_1, \dots, f_n)$ where $f_1 : B_1 \to A_1$ is over $u : \Delta \to \Gamma$ and for $i > 1$ we have

$$\begin{tikzcd}
B_i \arrow[r, "f_i"] \arrow[d, dash, dotted] & A_i \arrow[d, dash, dotted] \\
\Delta.B_1.\dots.B_{i-1} \arrow[r, "f_{i-1}"] & \Gamma.A_1.\dots.A_{i-1}
\end{tikzcd}$$

Composition and identities are given component-wise by the structure of the fibration ρ. When no confusion arises from we will denote this category of dependent tuples just by DT_n.

Note that $\mathrm{DT}_0 = \mathbb{C}$ the base category and $\mathrm{DT}_1 = \mathbb{E}$ the total category of the fibration ρ. Also note that for each $n > 0$ there is a canonical projection

$$\rho_n : \mathrm{DT}_n \to \mathrm{DT}_{n-1}$$

given by $\rho_n(\Gamma, A_1, \ldots, A_n) = (\Gamma, A_1, \ldots, A_{n-1})$ and similarly on arrows. It is easy to see that each ρ_n is a fibration (in particular notice that $\rho_1 = \rho$), where a Cartesian arrow with respect to ρ_n is precisely a tuple of ρ-Cartesian arrows. Moreover ρ_n is cloven (respectively normal or split) whenever ρ is cloven (respectively normal or split). It follows that every finite composition of these fibrations is again a fibration, in particular we have that

$$\widehat{\rho_n} := \rho_1 \circ \rho_2 \circ \cdots \circ \rho_n : \mathrm{DT}_n \to \mathbb{C}$$

is a fibration, which is cloven (normal or split) in accordance to ρ.

1.5 Modelling Σ-types

In this section we will describe the structure necessary to model Σ-types, or dependent sums, in a split comprehension category. See Appendix A for the syntactic rules for Σ-types.

The following definition makes sense in a general comprehension category and it will be important to state it in its full generality.

Definition 1.5.1. Let $(\mathbb{C}, \rho, \chi)$ be a comprehension category. A **choice of Σ-types** consists of an operation that assigns to each dependent tuple $(\Gamma, A, B) \in \mathrm{DT}_2(\rho, \chi)$ a tuple $(\Sigma_A B, \mathrm{pair}_{A,B}, \mathrm{sp}_{A,B})$ consisting of the following data:

1. $\Sigma_A B$ is an object of the total category $\mathbb{E}$ lying over Γ.
2. $\mathrm{pair}_{A,B}$ is an arrow over χ_A as shown:

$$\begin{tikzcd}
\Gamma.A.B \arrow[r, "\mathrm{pair}_{A,B}"] \arrow[d, "\chi_B"'] & \Gamma.\Sigma_A B \arrow[d, "\chi_{\Sigma_A B}"] \\
\Gamma.A \arrow[r, "\chi_A"'] & \Gamma
\end{tikzcd}$$

3. $\mathrm{sp}_{A,B}$ is an operation that takes a dependent tuple $(\Gamma, \Sigma_A B, C) \in \mathrm{DT}_2(\chi, \rho)$ and a section t of C over $\mathrm{pair}_{A,B}$, as in the following solid arrowed diagram:

$$\begin{tikzcd}
 & \Gamma.\Sigma_A B.C \arrow[d] \\
\Gamma.A.B \arrow[ru, "t"] \arrow[r, "\mathrm{pair}_{A,B}"'] & \Gamma.\Sigma_A B \arrow[u, bend right, dotted, "\mathrm{sp}_{A,B}(C,t)"']
\end{tikzcd}$$

to a section $\mathsf{sp}_{A,B}(C,t)$ of C, shown in the above diagram as the dotted arrow.

4. The above data must be subject to the following condition: for any section t of C over $\mathsf{pair}_{A,B}$ the following equality holds:

$$\mathsf{sp}_{A,B}(C,t) \circ \mathsf{pair}_{A,B} = t$$

that is, the triangle in the diagram of item (3) involving the dotted arrow, commute.

We will denote a the choice of Σ-types by $(\Sigma, \mathsf{pair}, \mathsf{sp})$.

We will give a brief explanation of why the previous definition reflects the syntactic definition of dependent sums. Consider terms a of A and b of B over a, then the corresponding introduction term (a,b) of $\Sigma_A B$ is given by the composition $(a,b) := \mathsf{pair}_{A,B} \circ b$.

Now given a type C, as in the elimination clause, and a term t of C over $\mathsf{pair}_{A,B}$, we explain the computation rule with the help the following diagram:

$$\begin{tikzcd}
 & & \Gamma.\Sigma_A B.C \arrow[d] \\
 & \Gamma.A.B \arrow[ur, "t"] \arrow[r, "\mathsf{pair}_{A,B}"'] \arrow[d] & \Gamma.\Sigma_A B \arrow[u, bend right, "\mathsf{sp}_{A,B}(C,t)"'] \arrow[d] \\
\Gamma \arrow[ur, "b"] \arrow[r, "a"'] & \Gamma.A \arrow[r, "\chi_A"'] & \Gamma
\end{tikzcd}$$

what we observe is that the computation rule for Σ-types is expressed semantically by the requirement that $\mathsf{sp}_{A,B}(C,t) \circ (a,b) = t \circ b$.

Definition 1.5.2. Let $(\mathbb{C}, \rho, \chi)$ be a split comprehension category. A choice of Σ-types $(\Sigma, \mathsf{pair}, \mathsf{sp})$ is said to be **strictly stable** if for every morphism $\sigma : \Delta \to \Gamma$ in the base category and for any dependent tuple $(\Gamma, A, B) \in \mathrm{DT}_2(\rho, \chi)$, the following conditions are satisfied:

1. $\Sigma_{A[\sigma]} B[\sigma] = (\Sigma_A B)[\sigma]$
2. The following diagram commutes:

$$\begin{tikzcd}
\Delta.A[\sigma].B[\sigma] \arrow[r, "\sigma^{**}"] \arrow[d, "\mathsf{pair}_{A[\sigma],B[\sigma]}"'] & \Gamma.A.B \arrow[d, "\mathsf{pair}_{A,B}"] \\
\Delta.\Sigma_{A[\sigma]} B[\sigma] \arrow[r, "\sigma^{*}"'] & \Gamma.\Sigma_A B
\end{tikzcd}$$

where the horizontal arrows are obtained by the split reindexing along σ.

3. For any dependent tuple $(\Gamma, \Sigma_A B, C)$ in $\mathsf{DT}_2(\rho, \chi)$, and any section t of C over $\mathsf{pair}_{A,B}$ there is a corresponding dependent tuple $(\Delta, \Sigma_{A[\sigma]} B[\sigma], C[\sigma])$ and a section $t[\sigma]$ of C over $\mathsf{pair}_{A[\sigma],B[\sigma]}$ obtained by reindexing. The following diagram is required to commute:

$$\begin{tikzcd}
\Delta.\Sigma_{A[\sigma]} B[\sigma] \arrow[r, "\sigma^*"] \arrow[d, "\mathsf{sp}(C[\sigma], t[\sigma])"'] & \Gamma.\Sigma_A B \arrow[d, "\mathsf{sp}(C, t)"] \\
\Delta.\Sigma_{A[\sigma]} B[\sigma].C[\sigma] \arrow[r, "\sigma^{**}"'] & \Gamma.\Sigma_A B.C
\end{tikzcd}$$

where the horizontal arrows are obtained by the split reindexing along σ.

Definition 1.5.3. A **model of dependent type theory with Σ-types** is given by a split comprehension category (χ, ρ) equipped with a strictly stable choice of Σ-types $(\Sigma, \mathsf{pair}, \mathsf{sp})$.

1.6 Modelling Π-types

In this section we describe the categorical counterpart in split comprehension categories of the syntactic rules of Π-types (or dependent products) in an analogous manner as we did for Σ-types. See Appendix A for the syntactic rules for Π-types.

Definition 1.6.1. Let $(\mathbb{C}, \rho, \chi)$ be a comprehension category. A **choice of Π-types** consists of an operation that assigns to each dependent tuple $(\Gamma, A, B) \in \mathsf{DT}_2(\rho, \chi)$ a tuple $(\Pi_A B, \lambda_{A,B}, \mathsf{app}_{A,B})$ consisting of the following data:

1. $\Pi_A B$ is on object of the total category $\mathbb{E}$ lying over Γ.

2. $\lambda_{A,B}$ is an operation that takes a section $t : \Gamma.A \to \Gamma.A.B$ of χ_B to a section $\lambda_{A,B}(t) : \Gamma \to \Pi_A B$ of $\chi_{\Pi_A B}$, as shown in the following diagram:

$$\begin{tikzcd}
& \Gamma.A.B \arrow[d] & & & \Gamma.\Pi_A B \arrow[d] \\
\Gamma.A \arrow[ur, "t"] \arrow[r, equal] & \Gamma.A & \mapsto & \Gamma \arrow[ur, "\lambda_{A,B}(t)"] \arrow[r, equal] & \Gamma
\end{tikzcd}$$

3. $\mathsf{app}_{A,B}$ is an arrow in the slice over $\Gamma.A$, as shown:

$$\begin{tikzcd}
\Gamma.A.\Pi_A B \arrow[r, "\mathsf{app}_{A,B}"] \arrow[d, "\chi_{\Pi_A B}"'] & \Gamma.A.B \arrow[d, "\chi_B"] \\
\Gamma.A \arrow[r, equal] & \Gamma.A
\end{tikzcd}$$

where $\Gamma.A.\Pi_A B$ is (the comprehension of) *any* reindexing of $\Pi_A B$ along χ_A. Notice that the choice of $\mathsf{app}_{A,B}$ determines uniquely any other choice with respect to a different Cartesian reindexing of $\Pi_A B$, this follow by the universal property of Cartesian arrows.

4. This data must be subject to the following property: for any section $t : \Gamma.A \to \Gamma.A.B$ of χ_B the following equality holds:

   $$\mathsf{app}_{A,B} \circ (\lambda(t)[\chi_A]) = t$$

   where $\lambda(t)[\chi_A]$ is the result of reindexing $\lambda(t)$ along χ_A.

We will denote such a the choice of Π-types by $(\Pi, \lambda, \mathsf{app})$.

Some explanation is in order. Suppose we have a term f of $\Pi_A B$ and a term a of A. According to the syntactic elimination rule, we would like to obtain a term $f[a]$ of $B[a]$ obtained by applying f to a. Let us denote by $\mathsf{app}_{A,B}(f, a)$ the following composite:

$$\begin{tikzcd}
\Gamma \arrow[r, "a"] & \Gamma.A \arrow[r, "f[\chi_A]"] & \Gamma.A.\Pi_A B \arrow[r, "\mathsf{app}_{A,B}"] & \Gamma.A.B
\end{tikzcd}$$

and notice that:

$$\begin{tikzcd}
 & \Gamma.A.B \arrow[d, "\chi_B"] \\
\Gamma \arrow[ur, "\mathsf{app}_{A,B}(f,a)"] \arrow[r, "a"'] & \Gamma.A
\end{tikzcd}$$

and thus we get that $\mathsf{app}_{A,B}(f, a)$ is the semantical counterpart of $f[a]$.

Definition 1.6.2. Let $(\mathbb{C}, \rho, \chi)$ be a split comprehension category. A choice of Π-types $(\Pi, \lambda, \mathsf{app})$ is said to be **strictly stable** if for every morphism $\sigma : \Delta \to \Gamma$ in the base category and for any dependent tuple $(\Gamma, A, B) \in DT_2(\rho, \chi)$, the following conditions are satisfied:

1. $\Pi_{A[\sigma]} B[\sigma] = (\Pi_A B)[\sigma]$
2. For any section t of χ_B there is a corresponding section $t[\sigma]$ of $\chi_{B[\sigma]}$ obtained by reindexing. This two sections must be related by the following commutative diagram:

   $$\begin{tikzcd}
   \Delta \arrow[r, "\sigma"] \arrow[d, "\lambda_{A[\sigma],B[\sigma]}(t[\sigma])"'] & \Gamma \arrow[d, "\lambda_{A,B}(t)"] \\
   \Delta.\Pi_{A[\sigma]} B[\sigma] \arrow[r, "\sigma^*"'] & \Gamma.\Pi_A B
   \end{tikzcd}$$

   where the lower horizontal arrow is obtained by the split reindexing along σ.
3. The following diagram commutes:

   $$\begin{tikzcd}
   \Delta.A[\sigma].\Pi_{A[\sigma]} B[\sigma] \arrow[r, "\sigma^{**}"] \arrow[d, "\mathsf{app}_{A[\sigma],B[\sigma]}"'] & \Gamma.A.\Pi_A B \arrow[d, "\mathsf{app}_{A,B}"] \\
   \Delta.A[\sigma].B[\sigma] \arrow[r, "\sigma^{**}"'] & \Gamma.A.B
   \end{tikzcd}$$

   where the horizontal arrows are obtained by split reindexing along σ.

Definition 1.6.3. A **model of dependent type theory with Π-types** consists of a split comprehension category $(\mathbb{C}, \rho, \chi)$ equipped with a strictly stable choice of Π-types $(\Pi, \lambda, \mathrm{app})$.

1.7 Modelling Id-types

In this section we turn our attention to identity types. We are interested only in the intensional version of these. See Appendix A.

Definition 1.7.1. Let $(\mathbb{C}, \rho, \chi)$ be a comprehension category. A **choice of Id-types** consists of an operation that assigns to each dependent tuple $(\Gamma, A) \in \mathsf{DT}_2(\rho, \chi)$, a tuple $(\mathrm{Id}_A, r_A, j_A)$ where:

1. Id_A is an object of $\mathbb{E}$ over $\Gamma.A.A$.
2. r_A is a section of Id_A over the diagonal morphism δ_A, i.e. a factorisation of δ_A:

$$\begin{tikzcd}
 & \Gamma.A.A.\mathrm{Id}_A \arrow[d, "\chi_{\mathrm{Id}_A}"] \\
\Gamma.A \arrow[ru, "r_A"] \arrow[r, "\delta_A"'] & \Gamma.A.A
\end{tikzcd}$$

3. j_A is an operation that takes a dependent tuple $(\Gamma, A, A, \mathrm{Id}_A, C) \in \mathsf{DT}_4(\rho, \chi)$ and a section t of C over r_A, as in the following solid arrowed diagram:

$$\begin{tikzcd}
\Gamma.A \arrow[r, "t"] \arrow[d, "r_A"'] & \Gamma.A.A.\mathrm{Id}_A.C \arrow[d, "\chi_C"] \\
\Gamma.A.A.\mathrm{Id}_A \arrow[r, equal] \arrow[ru, dotted, "j_A(C,t)"] & \Gamma.A.A.\mathrm{Id}_A
\end{tikzcd}$$

   to a section $j_A(C, t)$ of C, and such that both triangles commute.

We will refer to a choice of Id-types by (Id, r, j).

Both the elimination and the computation rules for Id-types are packed together in the third bullet of the previous definition: the elimination rule is modelled by the section $j_A(C, t)$ of C (i.e. that the lower triangle commutes) and the computation rule is given by the equation $j_A(C, t) \circ r_A = t$.

Definition 1.7.2. Let $(\mathbb{C}, \rho, \chi)$ be a split comprehension category. A choice of Id-types (Id, r, j) is said to be **strictly stable** if for every morphism $\sigma : \Delta \to \Gamma$ in the base category, and for every dependent tuple (Γ, A), the following conditions are satisfied:

1. $\mathrm{Id}_{A[\sigma]} = \mathrm{Id}_A[\sigma]$

2. The following diagram commutes:

$$\begin{tikzcd}
\Delta.A[\sigma] \arrow[r, "\sigma^*"] \arrow[d, "r_{A[\sigma]}"'] & \Gamma.A \arrow[d, "r_A"] \\
\Delta.A[\sigma].A[\sigma].\mathrm{Id}_{A[\sigma]} \arrow[r, "\sigma^{***}"'] & \Gamma.A.A.\mathrm{Id}_A
\end{tikzcd}$$

   where the horizontal arrows are given by split reindexing along σ.

3. For any (C, t) as in (3) of Definition 1.7.1, there is an object $C[\sigma]$ and a section $c[\sigma]$ of $C[\sigma]$ obtained by reindexing along σ. We require following diagram to commute:

$$\begin{tikzcd}
\Delta.A[\sigma].A[\sigma].\mathrm{Id}_{A[\sigma]} \arrow[r, "\sigma^{***}"] \arrow[d, "j_{A[\sigma]}(c[\sigma])"'] & \Gamma.A.A.\mathrm{Id}_A \arrow[d, "j_A(c)"] \\
\Delta.A[\sigma].A[\sigma].\mathrm{Id}_{A[\sigma]}.C[\sigma] \arrow[r, "\sigma^{****}"'] & \Gamma.A.A.\mathrm{Id}_A.C
\end{tikzcd}$$

   where the horizontal arrows are given by split reindexing along σ.

Definition 1.7.3. A **model of dependent type theory with Id-types** consists of a split comprehension category equipped with a strictly stable choice of Id-types (Id, r, j).

1.8 Modelling Universe Types

In this section we will explain what a universe in a comprehension category is, and what does it mean for a universe to be closed under some relevant choice of logical structure.

Definition 1.8.1. Let $(\mathbb{C}, \rho, \chi)$ be a split comprehension category. A **universe** is a dependent tuple $(*, U, \tilde{U}) \in DT_2(\rho, \chi)$, where $*$ denotes the terminal object of $\mathbb{C}$. We will denote a universe by $(U, \tilde{U})$.

We can verify that this is the right notion of universe for a split comprehension category: the syntactic comprehension category admits a universe in the above sense, if and only if, the underlying type theory has a type-theoretic universe à la Tarski (see Appendix A). Reindexing of $\tilde{U}$ correspond the interpretation operation taking a term of the universe to its corresponding type.

Next, we will explain what it means for a universe to be closed under Id-types. We will leave to the reader the task of translating this definition for other choices of logical structure.

Definition 1.8.2. Assume $(\mathbb{C}, \rho, \chi)$ is a split comprehension category with a strictly stable choice of Id-types (Id, r, j) and with a universe $(U, \tilde{U})$. We say that U is **closed under Id-types** if there is an operation that takes a map $a : \Gamma \to U$ to a map $i_a : \Gamma.\tilde{U}[a].\tilde{U}[a] \to U$ such that:

1. The choice is coherent; i.e. for every $\sigma : \Delta \to \Gamma$ we have that $i_{a \circ \sigma} = i_a \circ \sigma^{**}$:

$$\begin{tikzcd}
\Delta.\tilde{U}[a \circ \sigma].\tilde{U}[a \circ \sigma] \arrow[rr, "\sigma^{**}"] \arrow[dr, "i_{a \circ \sigma}"'] & & \Gamma.\tilde{U}[a].\tilde{U}[a] \arrow[dl, "i_a"] \\
& U &
\end{tikzcd}$$

   where σ^{**} is obtained by split reindexing along σ.

2. Reindexing preserves the choices on-the-nose, i.e. we have $\tilde{U}[i_a] = \mathrm{Id}_{\tilde{U}[a]}$ for all $a : \Gamma \to U$.

We see that if a split comprehension category $(\mathbb{C}, \rho, \chi)$ has a strictly stable choice Id-types and universe $(U, \tilde{U})$ closed under Id-types, then we will be able to model a dependent type theory with a Id-types and a universe.

Definition 1.8.3. A **model of dependent type theory with Id-types and a universe** consists of a split comprehension category equipped with a strictly stable choice of Id-types (Id, r, j) and a universe $(U, \tilde{U})$ closed under Id-types.

With this we conclude this chapter dedicated to examining categorical models of dependent type theory with additional logical structure making use of the notion of split comprehension categories as a foundation.

Chapter 2

Models via the Right Adjoint Splitting

The main objective of this chapter is to review a splitting or strictification procedure that acts by replacing an ordinary comprehension category with an equivalent but split one. This procedure will be of main interest for the remainder of this work; it is based on a classical and well known construction, originally introduced by Bénabou and Giraud [Gir66] (but see also [Str18]), which is characterised by the following 2-categorical universal property: it is the right 2-adjoint to the inclusion of split fibrations into fibrations

$$\underline{\mathrm{Sp}}(\mathbb{C}) \hookrightarrow \underline{\mathrm{Fib}}(\mathbb{C}).$$

We will show that this construction can be extended in order to apply also to comprehension categories, and such that the universal property will still hold in this setting. We will call this construction the *right adjoint splitting* of a comprehension category.

After this, we will investigate the appropriate notion of coherence that a choice of logical structure, in a given comprehension category, needs to have in order to become a split choice after applying the right adjoint splitting. These result is a *coherence theorem* for comprehension categories, which we will prove in detail.

As an example of how to apply the right adjoint splitting and the respective coherence theorem, we will use the notion of *tribe* developed by Joyal [Joy17]. A tribe is a full subcategory of arrows with some closure properties. We will see that a tribe has an associated comprehension category with enough structure, such that after applying the right adjoint splitting it produces a model of dependent type theory with Π and Σ types.

However, we will not be able to obtain a model of identity types using the right adjoint splitting in the setting of tribes; the reason is that tribes lack the necessary structure to coherently model the elimination terms. This is one of the main motivation for working with algebraic versions of these notions.

The coherence theorem for Σ and Π types is folklore knowledge, never the less we will develop all the proofs with detail. The coherence theorem for $\mathtt{Id}$-types is due to Warren in his dissertation [War06], however, his proof is very technical and difficult to follow. We include a refined version in this chapter which is hopefully easier to digest. The case of universes has not been dealt with before (to the best of our knowledge); it is more complicated than the other cases and requires manipulating the choices of logical structure in appropriate ways.

2.1 Overview of the Right Adjoint Splitting Construction

We will begin by reviewing the classical Bénabou-Giraud construction following Streicher's account [Str18]. We will adopt a different notation which will hopefully ease the statements and proofs of the coherence theorem for each kind of logical structure.

Definition 2.1.1. Let $(\mathbb{C}, \rho, \chi)$ be a comprehension category and $A \in \mathbb{E}$ over $\Gamma \in \mathbb{C}$. A **local cleavage** for A consists of an operation $A[-]$ that assigns to each map $\sigma : \Delta \to \Gamma$ a Cartesian arrow:

$$\begin{tikzcd}
A[\sigma] \arrow[r, "\sigma^*"] \arrow[d, dotted, phantom, "\lrcorner" very near start] \arrow[d, dotted, no head] & A \arrow[d, dotted, no head] \\
\Delta \arrow[r, "\sigma"'] & \Gamma
\end{tikzcd}$$

We say that a local cleavage $A[-]$ is **normal** if when applied to the identity $\mathtt{id} : \Gamma \to \Gamma$ it outputs the identity Cartesian arrow, i.e. $A[\mathtt{id}] = A$ and $\mathtt{id}_\Gamma^* = \mathtt{id}_A$.

For a fibration $\rho : \mathbb{E} \to \mathbb{C}$ let us define the category $\mathbb{E}^R$ as follows; its objects are pairs $(A, A[-])$ where A is an object of $\mathbb{E}$, and $A[-]$ is a normal local cleavage for A. An arrow $f : (B, B[-]) \to (A, A[-])$ is just an arrow $f : B \to A$ in $\mathbb{E}$. It is clear that identities and the composition operator are just those of $\mathbb{E}$. Notice that there is a functor $\rho^R : \mathbb{E}^R \to \mathbb{C}$ given on objects by $\rho^R(A, A[-]) = \rho(A)$.

Lemma 2.1.2. *The functor $\rho^R : \mathbb{E}^R \to \mathbb{C}$ is a split fibration.*

Proof. Consider $\sigma : \Delta \to \Gamma$ and an object $(A, A[-])$ over Γ. Consider the Cartesian lift given by $\sigma^* : (A[\sigma], A[\sigma][-]) \to (A, A[-])$ where $A[\sigma]$ is obtained by applying the local cleavage of A and $A[\sigma][-]$ is the normal local cleavage of $A[\sigma]$ defined in the following way; it assigns to each $\tau : \Theta \to \Delta$ the Cartesian arrow $\tau^* : A[\sigma \circ \tau] \to A[\sigma]$ given as the unique arrow making the upper triangle in the following diagram commute:

$$\begin{tikzcd}
A[\sigma \circ \tau] \arrow[r, dotted, "\tau^*", "\exists!"'] \arrow[rr, bend left, "(\sigma \circ \tau)^*"] \arrow[d, dotted, no head] & A[\sigma] \arrow[r, "\sigma^*"] \arrow[d, dotted, no head] & A \arrow[d, dotted, no head] \\
\Theta \arrow[r, "\tau"'] & \Delta \arrow[r, "\sigma"'] & \Gamma
\end{tikzcd}$$

It is straightforward to verify that this choice of cleavage is split. ☐

Consider a fibred functor:

$$\begin{tikzcd}
\mathbb{E} \arrow[rr, "H"] \arrow[dr, "\rho"'] & & \mathbb{D} \arrow[dl, "\eta"] \\
& \mathbb{C} &
\end{tikzcd}$$

this induces a split fibred functor $H^R : \rho^R \to \eta^R$ defined on objects as follows; we let $H^R(A, A[-]) := (HA, HA[-])$ where $HA[\sigma] := H(A[\sigma])$, since H preserves Cartesian arrows then $H^R(A, A[-])$ is a well-defined element of $\mathbb{D}^R$.

This induces a 2-functor $(-)^R : \mathsf{Fib}(\mathbb{C}) \to \mathsf{Sp}(\mathbb{C})$. Notice moreover that there is a fibred functor that forgets the local cleavage:

$$\epsilon_\rho : \rho^R \to \rho$$

given by $\epsilon_\rho(A, A[-]) := A$ which is clearly (fibrewise) full and faithful. If we assume the axiom of choice, then it is also surjective on objects an thus a fibred equivalence. The following is a well known result establishing the universal property of the foregoing construction.

Lemma 2.1.3. *The functor* $(-)^R$ *is a right 2-adjoint to the inclusion* $\underline{\text{Sp}}(\mathbb{C}) \hookrightarrow \underline{\text{Fib}}(\mathbb{C})$ *with* ϵ_ρ *as the counit.*

Proof. Given $\rho : \mathbb{E} \to \mathbb{C}$ a fibration and $\eta : \mathbb{D} \to \mathbb{C}$ a split fibration, we can describe the equivalence of categories:

$$\mathsf{Hom}_{\mathsf{Fib}(\mathbb{C})}(\eta, \rho) \cong \mathsf{Hom}_{\mathsf{Sp}(\mathbb{C})}(\eta, \rho^R)$$

For a split functor $K : \eta \to \rho^R$, we just compose with $\epsilon_\rho : \rho^R \to \rho$ to obtain the corresponding functor in $\underline{\text{Fib}}(\mathbb{C})$. In the other direction, consider a fibred functor $H : \eta \to \rho$, we then define $\hat{H} : \eta \to \rho^R$ to be the functor given on an object $A \in \mathbb{E}$ by:

$$\hat{H}(A) := (HA, HA[-])$$

where $HA[-]$ is the the result of applying H to the split cleavage of η. ☐

The same construction can be used to give a right 2-adjoint of the inclusion of split comprehension categories into comprehension categories:

$$\underline{\text{SpCCat}}(\mathbb{C}) \hookrightarrow \underline{\text{CCat}}(\mathbb{C})$$

Given a comprehension category (ρ, χ) over $\mathbb{C}$ we define the split comprehension category (ρ^R, χ^R) by letting ρ^R be the right adjoint splitting of the fibration ρ and we define χ^R to be the functor given by the following composite:

$$\begin{tikzcd}
\mathbb{E}^R \arrow[r, "\epsilon_\rho"] \arrow[dr, "\rho^R"'] & \mathbb{E} \arrow[r, "\chi"] \arrow[d, "\rho"] & \mathbb{C}^\to \arrow[dl, "\mathsf{cod}"] \\
& \mathbb{C} &
\end{tikzcd}$$

Now, suppose we have a morphism of comprehension categories $(F, \eta) : (\rho, \chi) \to (\rho', \chi')$. Then this will functorially induce a morphism the corresponding split comprehension categories:

$$(\rho^R, \chi^R) \xrightarrow{(F^R, \eta^R)} (\rho'^R, \chi'^R)$$

where F^R is obtain by applying the right adjoint splitting of fibrations to F, and the component of η^R associated to $(A, A[-])$ is given by $\eta^R_{(A,A[-])} = \eta_A$.

This action on morphism gives a functor:

$$\underline{\text{CCat}}(\mathbb{C}) \xrightarrow{R} \underline{\text{SpCCat}}(\mathbb{C})$$

which is a right adjoint to the inclusion of split comprehension categories into comprehension categories. The counit of R is given by $\epsilon_{(\rho,\chi)} := (\epsilon_\rho, id_{\chi^R})$ which is moreover, an equivalences of comprehension categories (assuming axiom of choice).

2.2 Coherence for Σ-types

We describe in this section the structure that a comprehension category that guarantees its right adjoint splitting to be equipped with a strictly-stable choice of Σ-types.

Definition 2.2.1. Let $(\mathbb{C}, \rho, \chi)$ be a comprehension category. A choice of Σ-types $(\Sigma, \mathsf{pair}, \mathsf{sp})$ (see Definition 1.5.1) is said to be **pseudo-stable** if for every Cartesian arrow $(\sigma, f, g) : (\Delta, A', B') \to (\Gamma, A, B)$ of dependent tuples, the following conditions are satisfied:

1. There is a Cartesian arrow $\Sigma_f g : \Sigma_{A'} B' \to \Sigma_A B$ over σ and the assignment:

$$(\sigma, f, g) \mapsto (\sigma, \Sigma_f g)$$

   is functorial, i.e. $\Sigma_{id_A} id_B = id_{\Sigma_A B}$ and $\Sigma_{(f' \circ f)}(g' \circ g) = \Sigma_{f'} g' \circ \Sigma_f g$.

2. The following diagram commutes:

$$\begin{tikzcd}
\Delta.A'.B' \arrow[r, "g"] \arrow[d, "\mathsf{pair}_{A',B'}"'] & \Gamma.A.B \arrow[d, "\mathsf{pair}_{A,B}"] \\
\Delta.\Sigma_{A'} B' \arrow[r, "\Sigma_f g"'] & \Gamma.\Sigma_A B
\end{tikzcd}$$

3. For any Cartesian arrow $h : C' \to C$ above $\Sigma_f g : \Sigma_{A'} B' \to \Sigma_A B$ and for any section t of C over $\mathsf{pair}_{A,B}$ there is a corresponding section t' of C' over $\mathsf{pair}_{A',B'}$ obtained by reindexing. The following diagram is required to commute:

$$\begin{tikzcd}
\Delta.\Sigma_{A'}.B' \arrow[r, "\Sigma_f g"] \arrow[d, "\mathsf{sp}(C',t')"'] & \Gamma.\Sigma_A B \arrow[d, "\mathsf{sp}(C,t)"] \\
\Delta.\Sigma_{A'} B'.C' \arrow[r, "h"'] & \Gamma.\Sigma_A B.C
\end{tikzcd}$$

Theorem 2.2.2 (Coherence for Σ-types)**.** *Let* $(\mathbb{C}, \rho, \chi)$ *be a comprehension category equipped with a pseudo-stable choice of* Σ*-types. Then* (ρ^R, χ^R) *has a strictly stable choice of* Σ*-types. Moreover, the counit* $\epsilon_\rho : (\rho^R, \chi^R) \to (\rho, \chi)$ *preserves strictly the choices of* Σ*-types.*

Proof. We construct a choice of Σ-types for (ρ^R, χ^R) as follows: consider a dependent tuple $(\Gamma, (A, A[-]), (B, B[-]))$ of (ρ^R, χ^R); the Σ-type associated to this tuple has the following form:

$$(\Sigma_A B, \Sigma_A B[-])$$

where $\Sigma_A B$ is the Σ-type given by the psuedo-stable choice of (ρ, χ) applied to (Γ, A, B). Now, the component at $\sigma : \Delta \to \Gamma$ of the local cleavage $\Sigma_A B[-]$ is given using the action on morphisms of the pseudo-stable choice of Σ-types to define:

$$\begin{tikzcd}
\Sigma_A B[\sigma] := \Sigma_{A[\sigma]} B[\sigma] \arrow[r, "\sigma^* := \Sigma_{f^*} g^*"] \arrow[d, dotted, no head] \arrow[dr, phantom, "\lrcorner", very near start] & \Pi_A B \arrow[d, dotted, no head] \\
\Delta \arrow[r, "\sigma"'] & \Gamma
\end{tikzcd}$$

where the Cartesian morphism $(\sigma, f^*, g^*) : (\Delta, A[\sigma], B[\sigma]) \to (\Gamma, A, B)$ is constructed using the local cleavages $A[-]$ and $B[-]$. □

2.3 Coherence for Π-types

We now proceed to state and proof the coherence result for dependent sums, or Π-types. That the definition of pseudo-stability is similar to that of dependent sums.

Definition 2.3.1. Let $(\mathbb{C}, \rho, \chi)$ be a comprehension category. A choice of Π-types $(\Pi, \lambda, \mathsf{app})$ (see Definition 1.6.1) is said to be **pseudo-stable** if for every Cartesian arrow $(\sigma, f, g) : (\Delta, A', B') \to (\Gamma, A, B)$ of dependent tuples, the following conditions are satisfied:

1. There is a Cartesian arrow $\Pi_f g : \Pi_{A'} B' \to \Pi_A B$ over σ and the assignment:

$$(\sigma, f, g) \mapsto (\sigma, \Pi_f g)$$

   is functorial, i.e. $\Pi_{\mathsf{id}_A} \mathsf{id}_B = \mathsf{id}_{\Pi_A B}$ and $\Pi_{(f' \circ f)}(g' \circ g) = \Pi_{f'} g' \circ \Pi_f g$.

2. For any section $t : \Gamma.A \to \Gamma.A.B$ of B there is a corresponding section $t' : \Delta.A' \to \Delta.A'.B'$ of B' obtained by reindexing along $f : \Delta.A' \to \Gamma.A$. Then, the following diagram commutes:

$$\begin{tikzcd}
\Delta \arrow[r, "\sigma"] \arrow[d, "\lambda_{A',B'}(t')"'] & \Gamma \arrow[d, "\lambda_{A,B}(t)"] \\
\Delta.\Pi_{A'} B' \arrow[r, "\Pi_f g"'] & \Gamma.\Pi_A B
\end{tikzcd}$$

3. The following diagram commutes:

$$\begin{tikzcd}
\Delta.A'.\Pi_{A'}B' \arrow[r, "\Pi_f g"] \arrow[d, "\mathsf{app}_{A',B'}"'] & \Gamma.A.\Pi_A B \arrow[d, "\mathsf{app}_{A,B}"] \\
\Delta.A'.B' \arrow[r, "g"'] & \Gamma.A.B
\end{tikzcd}$$

Theorem 2.3.2 (Coherence for Π-types)**.** *Let* $(\mathbb{C}, \rho, \chi)$ *be a comprehension category equipped with a pseudo-stable choice of* Π*-types. Then* (ρ^R, χ^R) *has a strictly stable choice of* Π*-types. Moreover, the counit* $\epsilon_\rho : (\rho^R, \chi^R) \to (\rho, \chi)$ *preserves strictly the choices of* Π*-types.*

Proof. We start by constructing a choice of Π-types for (ρ^R, χ^R). So consider a dependent tuple $(\Gamma, (A, A[-]), (B, B[-]))$ of (ρ^R, χ^R); the Π-type associated to this tuple has the following form:

$$(\Pi_A B, \Pi_A B[-])$$

where $\Pi_A B$ is the Π-type given by the psuedo-stable choice of (ρ, χ) applied to (Γ, A, B). Now, the component at $\sigma : \Delta \to \Gamma$ of the local cleavage $\Pi_A B[-]$ is given as follows: we first use the local cleavages $A[-]$ and $B[-]$ to construct a Cartesian arrow of dependent tuples $(\sigma, f^*, g^*) : (\Delta, A[\sigma], B[\sigma]) \to (\Gamma, A, B)$ and then we use the action on morphisms of the pseudo-stable choice of Π-types to define:

$$\begin{tikzcd}
\Pi_A B[\sigma] := \Pi_{A[\sigma]}B[\sigma] \arrow[r, "\sigma^* := \Pi_{f*} g^*"] \arrow[d, dotted, no head] \arrow[dr, phantom, "\lrcorner", very near start] & \Pi_A B \arrow[d, dotted, no head] \\
\Delta \arrow[r, "\sigma"'] & \Gamma
\end{tikzcd}$$

notice that the local cleavage defined this way is normal because the pseudo-stable is functorial, and in particular maps identities to identities.

We must now show that this choice is strictly stable. By definition we have for each $\sigma : \Delta \to \Gamma$ the following:

$$\begin{aligned}
(\Pi_A B, \Pi_A B[-])[\sigma] &= ((\Pi_A B)[\sigma], (\Pi_A B)[\sigma][-]) && \text{(defn of the cleavage of } (\rho^R, \chi^R)) \\
&= (\Pi_{A[\sigma]} B[\sigma], (\Pi_A B)[\sigma][-]) && \text{(defn of } \Pi_A B[-])
\end{aligned}$$

and thus we must only show that the local cleavages $(\Pi_A B)[\sigma][-]$ and $(\Pi_{A[\sigma]} B[\sigma])[-]$ coincide. But this follows immediately from the functoriality of the pseudo-stable choice of Π-types in (ρ, χ). □

2.4 Coherence for Id-types

We turn our attention now to Id-types. The definition of pseudo-stability will be very similar to that of dependent sums and products; however, the proof of coherence is not as straightforward as before. The coherence theorem for intentional Id-types was first proven by Warren in his dissertation [War06, Theorem 2.48]; unfortunately the proof he gave is technically very involved. Here we aim to present a revised and simpler version of the theorem; the simplification is made possible due to the use of more compact notation and to the alternative description of the right adjoint splitting construction in terms of normal local cleavages. We make use of a classical meta-theory in the proof of the coherence theorem, the reason for this is explained on Remark 2.4.3.

Definition 2.4.1. Let $(\mathbb{C}, \rho, \chi)$ be a comprehension category. A choice of Id-types (Id, r, j) is said to be **pseudo-stable** if for any Cartesian arrow $f : B \to A$ over $\sigma : \Delta \to \Gamma$, the following conditions are satisfied:

1. There is a Cartesian arrow $\mathrm{Id}_f : \mathrm{Id}_B \to \mathrm{Id}_A$ over δ_f (induced by the universal property of pullbacks), and the assignment:

$$\begin{tikzcd}
B \arrow[r, "f"] \arrow[d, dotted, no head] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, dotted, no head] & & \mathrm{Id}_B \arrow[r, "\mathrm{Id}_f"] \arrow[d, dotted, no head] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Id}_A \arrow[d, dotted, no head] \\
\Delta \arrow[r, "\sigma"'] & \Gamma & \mapsto & \Delta.B.B \arrow[r, "\delta_f"'] & \Gamma.A.A
\end{tikzcd}$$

   is functorial, i.e. $\mathrm{Id}_{\mathrm{id}_A} = \mathrm{id}_{\mathrm{Id}_A}$ and $\mathrm{Id}_{f \circ g} = \mathrm{Id}_f \circ \mathrm{Id}_g$.

2. The following diagram commutes:

$$\begin{tikzcd}
\Delta.B \arrow[r, "f"] \arrow[d, "r_B"'] & \Gamma.A \arrow[d, "r_A"] \\
\Delta.B.B.\mathrm{Id}_B \arrow[r, "\mathrm{Id}_f"'] & \Gamma.A.A.\mathrm{Id}_A
\end{tikzcd}$$

3. For any (C, t) as in (3) of Definition 1.7.1 and for any Cartesian $h : C' \to C$ over Id_f, there is an arrow t' obtained by reindexing along the appropriate Cartesian arrow. The following diagram commutes:

$$\begin{tikzcd}
\Delta.B.B.\mathrm{Id}_B \arrow[r, "\mathrm{Id}_f"] \arrow[d, "j_B(C',t')"'] & \Gamma.A.A.\mathrm{Id}_A \arrow[d, "j_A(C,t)"] \\
\Delta.B.B.\mathrm{Id}_B.C' \arrow[r, "h"'] & \Gamma.A.A.\mathrm{Id}_A.C
\end{tikzcd}$$

   where the lower horizontal arrow is the (comprehension of the) Cartesian arrow $h : C' \to C$.

Theorem 2.4.2 (Coherence of Id-types)**.** *Let (ρ, χ) be a comprehension category equipped with a pseudo stable choice of* Id*-types. Then (ρ^R, χ^R) has a strictly stable choice of* Id*-types. Moreover, the counit $\epsilon_\rho : (\rho^R, \chi^R) \to (\rho, \chi)$ preserves the choices of logical structure strictly.*

Proof. We will describe how the choice of $\mathtt{Id}$-types in (ρ^R, χ^R) is made. Let $(A, A[-])$ be an object of $\mathbb{E}^R$ over some Γ. We need to construct an object $(\mathtt{Id}_{(A,A[-])}, \mathtt{Id}_{(A,A[-])}[-])$ in $\mathbb{E}^R$ over $\Gamma.A.A$.

We will abuse notation and let $\mathtt{Id}_{(A,A[-])} := \mathtt{Id}_A$ and $\mathtt{Id}_{(A,A[-])}[-]) = \mathtt{Id}_A[-]$. With this, we define $\mathtt{Id}_A$ using the pseudo-stable choice of $\mathtt{Id}$-types in (ρ, χ) applied to A. For the normal local cleavage $\mathtt{Id}_A[-]$, consider an arrow $(\sigma, a, b) : \Delta \to \Gamma.A.A$ (where $\sigma : \Delta \to \Gamma$, $a : \Delta \to \Gamma.A$ and $b : \Delta \to \Gamma.A.A$, each being a section over of the previous one). First we use the normal local cleavage $A[-]$ as shown:

$$\begin{tikzcd}
A[\sigma] \arrow[r, "\sigma^*"] \arrow[d, dotted, no head] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, dotted, no head] \\
\Delta \arrow[r, "\sigma"'] & \Gamma
\end{tikzcd}$$

we then apply the stable functorial choice of $\mathtt{Id}$-types of (ρ, χ) in order to obtain the following Cartesian arrow:

$$\begin{tikzcd}
\mathtt{Id}_{A[\sigma]} \arrow[r, "\mathtt{Id}_{\sigma^*}"] \arrow[d, dotted, no head] \arrow[dr, phantom, "\lrcorner", very near start] & \mathtt{Id}_A \arrow[d, dotted, no head] \\
\Delta.A[\sigma].A[\sigma] \arrow[r, "\sigma.\sigma^*.\sigma^*"'] & \Gamma.A.A
\end{tikzcd}$$

Notice that the morphism (σ, a, b) factors uniquely through $\sigma.\sigma^*.\sigma^*$ as one of the form $(\mathtt{id}_\Delta, a[\sigma], b[\sigma]) : \Delta \to \Delta.A[\sigma].A[\sigma]$. We let $\mathtt{Id}_A[(\sigma, a, b)]$ be *an arbitrary* reindexing of $\mathtt{Id}_{A[\sigma]}$ along $(\mathtt{id}_\Delta, a[\sigma], b[\sigma])$; and crucially, making sure that whenever $(\mathtt{id}_\Delta, a[\sigma], b[\sigma])$ is the identity morphism, then the reindexing is also the identity. In detail, this means that if $(\sigma, a, b) = \sigma.\sigma^*.\sigma^*$ then $\mathtt{Id}_A[(\sigma, a, b)] = \mathtt{Id}_{A[\sigma]}$.

Let us verify that this choice is strictly stable. Consider $\sigma : \Delta \to \Gamma$, we must verify that

$$(\mathtt{Id}_A[\sigma], \mathtt{Id}_A[\sigma][-]) = (\mathtt{Id}_{A[\sigma]}, \mathtt{Id}_{A[\sigma]}[-])$$

notice that $\mathtt{Id}_A[\sigma]$ is by definition given by the normal local cleavage $\mathtt{Id}_A[-]$ just defined, applied to the arrow $\sigma.\sigma^*.\sigma^*$ which by definition is just $\mathtt{Id}_{A[\sigma]}$. That the local cleavages coincide follows from the functoriality of the pseudo-stable choice of $\mathtt{Id}$-types. ☐

Remark 2.4.3. Notice that in the proof above, one can choose *an arbitrary* identity preserving reindexing of $\mathtt{Id}_{A[\sigma]}$ along $(\mathtt{id}_\Delta, a[\sigma], b[\sigma])$, In other words, there is one *degree of freedom* for the choice of normal local cleavage $\mathtt{Id}_A[-]$. This will be important when modelling universes in the following section, especially in the proof of Proposition 2.5.4. Moreover, notice that in order for the choice of reindexing to preserve identities, we requre a classical meta-theory in order to decide whether a given morphism is the identity or not.

2.5 Coherence for Universe Types

We now turn our attention to the case of splitting type-theoretic universes. Here we encounter an extra difficulty; in order for universes to be of interest, they need to be *closed* under the logical structure that the type theory might have. Semantically, in the non-split setting, one needs to be careful to define the correct notion of a universe closed under the appropriate kind of logical structure. Some of the proofs of these section use a case distinction on whether a given input object is equal, or not, to another one. For this reason, as in the previous section, we will require a classical meta-theory.

Definition 2.5.1. Let $(\mathbb{C}, \rho, \chi)$ be a comprehension category. An **pseudo-stable universe** is a tuple $(\mathsf{U}, \tilde{\mathsf{U}}, \mathsf{El}[-])$ where $(*, \mathsf{U}, \tilde{\mathsf{U}}) \in \mathsf{DT}_2(\rho, \chi)$ is a dependent tuple over the terminal object, and $\mathsf{El}[-]$ is a normal local cleavage of $\tilde{\mathsf{U}}$ (see Definition 2.1.1).

Proposition 2.5.2. *A pseudo-stable universe* $(\mathsf{U}, \tilde{\mathsf{U}}, \mathsf{El}[-])$ *in* (ρ, χ) *determines a universe* $(\mathsf{U}^R, \mathsf{El}^R[-])$ *in the right adjoint splitting* (ρ^R, χ^R) *of* (ρ, χ)*.*

Proof. We construct a dependent tuple $(*, \mathsf{U}^R, \tilde{\mathsf{U}}^R)$ in $\mathsf{DT}_2(\rho^R, \chi^R)$. We let $\mathsf{U}^R = (\mathsf{U}, \mathsf{U}[-])$ where $\mathsf{U}[-]$ is an arbitrary choice of reindexing for U. We now define $\mathsf{U}^R = (\tilde{\mathsf{U}}, \mathsf{El}[-])$ which, by definition of pseudo-stable universe, turns out to be a type in $\mathbb{E}^R$ over U. □

Now we establish the property of a pseudo-stable universe to be closed under a specified type constructor. We will use Id-types as our running example.

Definition 2.5.3. Let (ρ, χ) be a comprehension category with a pseudo-stable choice of Id-types $(\mathsf{Id}, \mathsf{r}, \mathsf{j})$ and with a pseudo-stable universe $(\mathsf{U}, \tilde{\mathsf{U}}, \mathsf{El}[-])$. We say that $(\mathsf{U}, \tilde{\mathsf{U}}, \mathsf{El}[-])$ is **strictly-closed under** Id**-types** if there is an operation that takes a map $a : \Gamma \to \mathsf{U}$ to a map $i_a : \Gamma.\mathsf{El}[a].\mathsf{El}[a] \to \mathsf{U}$ such that:

1. The choice is coherent; i.e. for every $\sigma : \Delta \to \Gamma$ we have that $i_{a \circ \sigma} = i_a \circ (\sigma^{**})$.
2. The choice of reindexing $\mathsf{El}[-]$ preserves the choices of Id-types on-the-nose, i.e. we have $\mathsf{El}[i_a] = \mathsf{Id}_{\mathsf{El}[a]}$ for all $a : \Gamma \to \mathsf{U}$.

Proposition 2.5.4. *Let* (ρ, χ) *be a comprehension category with an pseudo-stable choice of* Id*-types* $(\mathsf{Id}, \mathsf{r}, \mathsf{j})$ *and with an pseudo-stable universe* $(\mathsf{U}, \tilde{\mathsf{U}}, \mathsf{El}[-])$ *strictly closed under* Id*-types. Then the universe* $(\mathsf{U}^R, \tilde{\mathsf{U}}^R)$ *is closed under the strictly-stable choice of* Id*-types induced by* $(\mathsf{Id}, \mathsf{r}, \mathsf{j})$ *in* (ρ^R, χ^R)*.*

Proof. Mostly the proof is a routine verification except for a small detail, we need to take advantage of the 'degree of freedom' (as mentioned in Note 2.4.3) in the construction of the strictly stable choice of Id-type in (ρ^R, χ^R).

Let us describe here the appropriate modifications. In the proof of Theorem 2.4.2 we constructed a strictly stable choice of Id-types:

$$(A, A[-]) \mapsto (\mathsf{Id}_A, \mathsf{Id}_A[-])$$

in the presence of a pseudo-stable universe, we need to modify the choice slightly. We will need to do a case distinction:

Case 1: If the type $(A, A[-])$ is *not* of the form $(\mathsf{El}[a], \mathsf{El}[a][-])$ for some $a : \Gamma \to \mathsf{U}$ then we keep the original choice.

Case 2: If $(A, A[-])$ is equal to some $(\mathsf{El}[a], \mathsf{El}[a][-])$ for a given $a : \Gamma \to \mathsf{U}$ then we modify the choice of reindexing $\mathsf{Id}_{\mathsf{El}[a]}[-]$ as follows: given $\sigma : \Delta \to \Gamma.\mathsf{El}[a].\mathsf{El}[a]$ we define $\mathsf{Id}_{\mathsf{El}[a]}[\sigma] := \mathsf{El}[i_a\sigma]$ as shown in the following diagram:

$$\begin{tikzcd}
\mathsf{Id}_{\mathsf{El}[a]}[\sigma] := \mathsf{El}[i_a\sigma] \arrow[r, dotted, "\exists!"'] \arrow[d, dotted, no head] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Id}_{\mathsf{El}[a\sigma]} \arrow[r, "\mathsf{Id}_{\sigma^*}"] \arrow[d, dotted, no head] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Id}_{\mathsf{El}[a]} \arrow[r, "i_a^*"] \arrow[d, dotted, no head] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{\mathsf{U}} \arrow[d, dotted, no head] \\
\Delta \arrow[r] \arrow[rr, bend right, "\sigma"'] & \Delta.\mathsf{El}[a\sigma].\mathsf{El}[a\sigma] \arrow[r, "\sigma^{**}"'] & \Gamma.\mathsf{El}[a].\mathsf{El}[a] \arrow[r, "i_a"'] & \mathsf{U}
\end{tikzcd}$$

where the Cartesian square in the middle of the diagram is given by the pseudo-stable choice of Id-types.

With this modification in place, we can now verify the rest of the proposition. We need to check that the reindexing of $\tilde{\mathsf{U}}^R = (\tilde{\mathsf{U}}, \mathsf{El}[-])$ preserves the choices of Id-types on-the-nose. So consider $a : \Gamma \to \mathsf{U}$, since $(\mathsf{U}, \tilde{\mathsf{U}}, \mathsf{El}[-])$ is strictly closed under Id-types we have that $\mathsf{El}[i_a] = \mathsf{Id}_{\mathsf{El}[a]}$, thus we only need to make sure that the choice of reindexing $\mathsf{Id}_{\mathsf{El}[a]}[-]$ described above coincide with the choice $\mathsf{El}[i_a][-]$ canonically given by the right adjoint splitting. This is now immediate: consider $\sigma : \Delta \to \Gamma.\mathsf{El}[a].\mathsf{El}[a]$ then:

$$\begin{aligned}
\mathsf{El}[i_a][\sigma] &= \mathsf{El}[i_a\sigma] && \text{(by definition of } \mathsf{El}[i_a][-]\text{)} \\
&= \mathsf{Id}_{\mathsf{El}[a]}[\sigma] && \text{(by the modification made above).}
\end{aligned}$$

□

Remark 2.5.5. The modification made to the choice of the type constructor in the proof of the previous proposition is only needed for Id-types. When dealing with Π or Σ one can stick with the original choice.

In practice, it is hard to encounter a pseudo-stable universe strictly-closed under Id-types. The difficulty lies in the requirement of $\mathsf{El}[-]$ to preserve the choices on-the-nose. A more natural notion is the following.

Definition 2.5.6. Let (ρ, χ) be a comprehension category with an pseudo-stable choice of Id-types (Id, r, j) and with an pseudo-stable universe $(\mathsf{U}, \tilde{\mathsf{U}}, \mathsf{El}[-])$. We say that $(\mathsf{U}, \tilde{\mathsf{U}}, \mathsf{El}[-])$ is **pseudo-closed under Id-types** if there is an operation that takes a map $a : \Gamma \to \mathsf{U}$ to a map $i_a : \Gamma.\mathsf{El}[a].\mathsf{El}[a] \to \mathsf{U}$ such that:

1. The choice is coherent; i.e. for every $\sigma : \Delta \to \Gamma$ we have that $i_{a\circ\sigma} = i_a \circ (\sigma^{**})$.
2. The choice of reindexing $\mathsf{El}[-]$ preserves the choices of Id-types up to natural vertical isomorphism, i.e. we have vertical isomorphisms $\theta_a : \mathsf{El}[i_a] \cong \mathsf{Id}_{\mathsf{El}[a]}$ over $\Gamma.\mathsf{El}[a].\mathsf{El}[a]$ for all $a : \Gamma \to \mathsf{U}$, natural in the slice over U.

Remark 2.5.7. In the presence of item (1), an alternative requirement for (2) in Definition 2.5.6 is the following:

(2') For each $a : \Gamma \to U$, exhibit a Cartesian arrow $i_a^* : \mathrm{Id}_{\mathrm{El}[a]} \to \tilde{U}$ over $i_a : \Gamma.\mathrm{El}[a].\mathrm{El}[a] \to U$.

It is clear that in the presence of (1), both (2) and (2′) are equivalent, the vertical natural isomorphisms being the comparison isomorphisms between two Cartesian lifts of the same arrow.

Luckily we can do a trick to force an pseudo-stable universe pseudo-closed under Id-types to be strictly closed. The cost we have to pay is that we have to modify the pseudo-stable choice of Id-types to a new but isomorphic one.

Proposition 2.5.8. *Let* (ρ, χ) *be a comprehension category with an pseudo-stable choice of* Id*-types* (Id, r, j) *and with a pseudo-stable universe* $(U, \tilde{U}, \mathrm{El}[-])$ *pseudo-closed under* Id*-types. Then there exists a new pseudo-stable choice of* Id*-types* (Id', r', j')*, which is pointwise isomorphic to* (Id, r, j) *(i.e.* $\mathrm{Id}_A \cong \mathrm{Id}'_A$*) and with respect to which* $(U, \tilde{U}, \mathrm{El}[-])$ *is strictly closed.*

Proof. We will do a case distinction to define (Id', r', j'), for this we will need to decide for each $A \in \mathbb{E}$ whether $A = \mathrm{El}[a]$ for some $a : \Gamma \to U$ or not. For clarity of exposition we will avoid writing all explicit contexts, for example instead of writing $\Gamma.\mathrm{El}[a].\mathrm{El}[a].\mathrm{Id}_{\mathrm{El}[a]}$ we will just write $\mathrm{Id}_{\mathrm{El}[a]}$ trusting the reader to distinguish between the different meanings.

Given $A \in \mathbb{E}$ we define Id'_A, r'_A and j' as follows:

Case 1: If A is *not* of the form $\mathrm{El}[a]$ for some $a : \Gamma \to U$. Then we keep the original choice, i.e. $\mathrm{Id}'_A = \mathrm{Id}_A$, $r'_A = r_A$ and $j' = j$.

Case 2: If $A = \mathrm{El}[a]$ for some $a : \Gamma \to U$. Then we define

$$\mathrm{Id}'_{\mathrm{El}[a]} := \mathrm{El}[i_a]$$

since the original Id choice was functorial the new choice Id will still also be functorial. The action on morphisms will be given by factoring in the vertical isomorphism θ (or its inverse) where needed.

We define $r'_{\mathrm{El}[a]}$ using θ_a as shown:

$$\begin{tikzcd}
\Gamma.\mathrm{El}[a] \arrow[d, "r_{\mathrm{El}[a]}"'] \arrow[dr, "r'_{\mathrm{El}[a]}"] & \\
\mathrm{Id}_{\mathrm{El}[a]} \arrow[r, "\cong", "\theta_a"'] & \mathrm{El}[i_a]
\end{tikzcd}$$

Finally, we define j' as follows. Given C' over $\mathrm{El}[i_a]$ and $t' : \mathrm{El}[a] \to C'$ in the slice over $\mathrm{El}[i_a]$ we need to define $j'_{\mathrm{El}[a]}(t', C')$. To do this, we construct C over

$\mathrm{Id}_{\mathrm{El}[a]}$ and $t : \mathrm{El}[a] \to C$ as follows: We let $\theta_a^* : C' \cong C$ be any Cartesian lift of θ_a and we define $t' := \theta_a^* \circ t$. With this in place, we define $j'_{\mathrm{El}[a]}(t', C')$ from $j_{\mathrm{El}[a]}(t, C)$ as indicated in the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Id}_{\mathrm{El}[a]} \arrow[r, "j_{\mathrm{El}[a]}(t{,}C)"] \arrow[d, "\theta_a"', "\cong"] & C \\
\mathrm{El}[i_a] \arrow[r, dotted, "j'_{\mathrm{El}[a]}(t'{,}C')"'] & C' \arrow[u, "\theta_a^*"', "\cong"]
\end{tikzcd}$$

It is straightforward to verify that j' is a coherent choice, this follows essentially from coherence of j and naturality of θ.

It is now evident that $(U, \tilde{U}, \mathrm{El}[-])$ is strictly closed under the new choice of Id-types.

□

If the base category $\mathbb{C}$ of a comprehension category (ρ, χ) has enough structure to represent the categorical premises of the formation rule of some type of logical structure, then it is possible to give a simpler criterion for verifying if an pseudo-stable universe $(U, \tilde{U}, \mathrm{El}[-])$ is pseudo-closed under the logical structure. The extra requirements on (ρ, χ) is called the *(LF)* condition in [LW15] (standing for Logical Framework). This is a slightly weaker requirement than that of being locally cartesian closed.

Definition 2.5.9. A comprehension category $(\mathbb{C}, \rho, \chi)$ satisfies the **(LF) condition** if $\mathbb{C}$ has finite limits and every comprehension map $\chi_A : \Gamma.A \to \Gamma$ is exponentiable, i.e. the categorical exponential $(\chi_A)_* g$ exists for every map $g : X \to \Gamma.A$.

For example, using the *(LF)* condition, and a pseudo-stable universe $(U, \tilde{U}, \mathrm{El}[-])$, we can construct an object representing the premises for the formation rule of Π-types, we will call this object U_Π and it is defined (using the internal language) as:

$$U_\Pi := \Pi_{X:U} U^X$$

This object has the following universal property: maps into U_Π of the form $(a, b) : \Gamma \to U_\Pi$ are in bijection with pairs of maps of the form $a : \Gamma \to U$ and $b : \Gamma.\mathrm{El}[a] \to U$. Notice that this last pair of maps represent the premises of the formation rule for Π-types.

In particular, the identity $\mathrm{id} : U_\Pi \to U_\Pi$ corresponds to two 'generic' types $a_g : U_\Pi \to U$ and $b_g : U_\Pi.\mathrm{El}[a_g] \to U$, making it possible to construct the Π-type $\Pi_{\mathrm{El}[a_g]}\mathrm{El}[b_g]$ (assuming there is a choice of Π-types). We will call the type $\Pi_{\mathrm{El}[a_g]}\mathrm{El}[b_g]$ the **U-generic Π-type**. Analogous definitions are made for the other types logical structure.

The case of Id-types is particularly simple, we have that $U_{\mathrm{Id}} = U$. This is because the formation rule of Id-types only requires one type $a : \Gamma \to U$. Thus the U-generic Id-type is $\mathrm{Id}_{\tilde{U}}$; that is, the Id-type of $\tilde{U}$ in context U.

Definition 2.5.10. Let $(\mathbb{C}, \rho, \chi)$ be a comprehension category that satisfies the (LF) condition. Suppose (ρ, χ) has a pseudo-stable universe $(\mathsf{U}, \tilde{\mathsf{U}}, \mathsf{El}[-])$ and a pseudo-stable choice of Id-types. We say that $(\mathsf{U}, \tilde{\mathsf{U}}, \mathsf{El}[-])$ **reflects Id-types** if we can exhibit the U-generic Id-type as a reindexing of $\tilde{\mathsf{U}}$; that is if we can find a Cartesian diagram:

$$\begin{tikzcd}
\mathsf{Id}_{\tilde{\mathsf{U}}} \arrow[r, "\mathsf{Id}^*"] \arrow[d, dotted, no head] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{\mathsf{U}} \arrow[d, dotted, no head] \\
\tilde{\mathsf{U}} \times_{\mathsf{U}} \tilde{\mathsf{U}} \arrow[r, "\mathsf{Id}"'] & \mathsf{U}
\end{tikzcd}$$

Proposition 2.5.11. *Let* (ρ, χ) *be a comprehension category that satisfies the (LF) condition. Suppose* (ρ, χ) *has a pseudo-stable universe* $(\mathsf{U}, \tilde{\mathsf{U}}, \mathsf{El}[-])$ *and a pseudo-stable choice of* Id*-types. If* $(\mathsf{U}, \tilde{\mathsf{U}}, \mathsf{El}[-])$ *reflects* Id*-types, then* U *is pseudo-closed under* Id*-types.*

Proof. We first need to provide a coherent choice of arrows $i_a : \Gamma.\mathsf{El}[a].\mathsf{El}[a] \to \mathsf{U}$ for each $a : \Gamma \to \mathsf{U}$, we do this as follows. Each arrow $a : \Gamma \to \mathsf{U}$ induces a unique $a^{**} : \Gamma.\mathsf{El}[a].\mathsf{El}[a] \to \tilde{\mathsf{U}} \times_{\mathsf{U}} \tilde{\mathsf{U}}$ by universal properties. We define i_a as the composite:

$$\Gamma.\mathsf{El}[a].\mathsf{El}[a] \xrightarrow{a^{**}} \tilde{\mathsf{U}} \times_{\mathsf{U}} \tilde{\mathsf{U}} \xrightarrow{\mathsf{Id}} \mathsf{U}$$

this choice is coherent because it is defined by universal property and composition. We will use Note 2.5.7 and exhibit an arrow $i_a^* : \mathsf{Id}_{\mathsf{El}[a]} \to \tilde{\mathsf{U}}$ Cartesian over i_a. The arrow i_a^* is defined as follows:

$$\begin{tikzcd}
\mathsf{Id}_{\mathsf{El}[a]} \arrow[r, "\mathsf{Id}_a"] \arrow[d, dotted, no head] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Id}_{\tilde{\mathsf{U}}} \arrow[r, "\mathsf{Id}^*"] \arrow[d, dotted, no head] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{\mathsf{U}} \arrow[d, dotted, no head] \\
\Gamma.\mathsf{El}[a].\mathsf{El}[a] \arrow[r, "a^{**}"'] & \tilde{\mathsf{U}} \times_{\mathsf{U}} \tilde{\mathsf{U}} \arrow[r, "\mathsf{Id}"'] & \mathsf{U}
\end{tikzcd}$$

where $\mathsf{Id}_a : \mathsf{Id}_{\mathsf{El}[a]} \to \mathsf{Id}_{\tilde{\mathsf{U}}}$ is the Cartesian arrow supplied by the pseudo-stable choice of Id-types. ☐

2.6 Main Coherence Theorem

Throughout this chapter we have seen that given a comprehension category $(\mathbb{C}, \rho, \chi)$, if a choice of dependent products, dependent sums, or identity types satisfies the condition for being *pseudo-stable* then in the right adjoint splitting $(\mathbb{C}, \rho^R, \chi^R)$ we obtain a strictly-stable choice of the logical structure. We have also explored the concept of a universe in the non-split setting and under which conditions this can produce a model of a type-theoretic universe in the resulting split comprehension category. In this section we will collect the individual instances of the coherence theorem for each kind of logical structure into a main one.

Theorem 2.6.1 (Coherence Theorem)**.** *Let $(\mathbb{C}, \rho, \chi)$ be a comprehension category equipped with pseudo-stable choices of Σ, Π and $\mathtt{Id}$ types. Then the right adjoint splitting $(\mathbb{C}, \rho^R, \chi^R)$ is equipped with strictly-stable choices of Σ, Π and $\mathtt{Id}$; and the counit ϵ_ρ : $(\rho^R, \chi^R) \to (\rho, \chi)$ preserves each choice of logical structure strictly. Moreover, if $(\mathbb{C}, \rho, \chi)$ has a pseudo-stable universe, strictly-closed under each kind of logical structure, then $(\mathbb{C}, \rho^R, \chi^R)$ is equipped with a universe, closed under each kind of logical structure.*

Proof. The theorem follows immediately from Theorem 2.2.2, Theorem 2.3.2, Theorem 2.4.2 and Proposition 2.5.4. □

To conclude this section, we will provide a strong indication that the pseudo-stability condition on the logical structure is actually a necessary condition to produce strictly stable models after applying the right adjoint splitting. The main result in this direction is that a strictly-stable choice of some type of logical structure in a split comprehension category is necessarily also pseudo-stable.

For this, we first prove a small lemma. The proof of this will be type-theoretic in nature given that we have already seen that split comprehension categories are sound models of for type theory, we can use the internal language.

Lemma 2.6.2. *Let (ρ, χ) be a split comprehension category over $\mathbb{C}$. Suppose that (ρ, χ) has choices of Σ, Π-types and $\mathtt{Id}$-types. Then vertical isomorphisms of dependent tuples induce vertical isomorphisms of the corresponding Σ, Π-types and $\mathtt{Id}$-types, moreover the resulting isomorphisms are coherent with the defining structure of the respective choice of type.*

Let us explain what the above lemma says, we will use Π-types as our running example, the corresponding result for Σ-types and $\mathtt{Id}$-types is analogous. Suppose we have a split comprehension category (ρ, χ) equipped with a strictly-stable choice of Π-types. The first assertion is that a vertical isomorphism of dependent tuples:

$$(\mathrm{id}_\Gamma, f, g) : (\Gamma, A, B) \xrightarrow{\cong} (\Gamma, A', B')$$

induces a vertical isomorphism between the resulting Π-types:

$$(\mathrm{id}_\Gamma, \Pi_f g) : (\Gamma, \Pi_A B) \xrightarrow{\cong} (\Gamma, \Pi_{A'} B')$$

The second assertion is that this resulting isomorphism $\Pi_f g$ commutes with the Π-type structure, what we mean by this is the following:

1. First $\Pi_f g$ is coherent with the introduction terms. This means that for every section t of B the following diagram commutes:

$$\begin{tikzcd} \Gamma.\Pi_A B \arrow[r, "\Pi_f g"] & \Gamma.\Pi_{A'} B' \\ \Gamma \arrow[u, "\lambda(t[g])"] \arrow[r, equal] & \Gamma \arrow[u, "\lambda(t)"'] \end{tikzcd} \tag{2.1}$$

   where $t[g]$ is the section of B obtained by reindexing.

2. Second $\Pi_f g$ must be coherent with respect to the elimination terms, this means that the following diagram commutes:

$$\begin{tikzcd}
\Gamma.A.\Pi_A B \arrow[r, "\Pi_f g"] \arrow[d, "\mathrm{app}"'] & \Gamma.A'.\Pi_{A'}B' \arrow[d, "\mathrm{app}"] \\
\Gamma.A.B \arrow[r, "g"'] & \Gamma.A'.B'
\end{tikzcd} \tag{2.2}$$

Proof of Lemma 2.6.2. We will produce a proof only for Π-types, for the other kind of logical structure, the proof follows the same heuristic. Given that we are working in a split comprehension category, we can reason using the internal dependent type theory with dependent products. First we will translate the hypothesis of the lemma, i.e. what it means in the internal language to have an isomorphism of tuples:

$$(\mathrm{id}_\Gamma, f, g) : (\Gamma, A, B) \overset{\cong}{\longrightarrow} (\Gamma, A', B')$$

this means that we have judgements of the form:

$$\begin{array}{cc} \Gamma, x : A \vdash fx : A' & \Gamma, x : A \vdash f^{-1}x : A \\ \Gamma, x : A, y : Bx \vdash g(x, y) : B'(fx) & \Gamma, x : A', y : B'x \vdash g^{-1}(x, y) : B(f^{-1}x) \end{array}$$

subject to the following equations:

$$\begin{array}{cc} \Gamma, x : A \vdash f^{-1}fx = x : A' & \Gamma, x : A \vdash ff^{-1}x = x : A \\ \Gamma, x : A, y : Bx \vdash g(f^{-1}x, g^{-1}(x, y)) = y : B'x & \Gamma, x : A', y : B'x \vdash g^{-1}(fx, g(x, y)) = y : Bx \end{array}$$

Given this data, we can produce an isomorphism $\Pi_f g : \Pi_A B \to \Pi_{A'}B'$ over the identity of Γ represented by the term:

$$\Gamma, q : \Pi_A B \vdash \Pi_f g(q) = \lambda(x : A').g(f^{-1}x, q(f^{-1}x)) : \Pi_{A'}B'$$

which is well-typed because for each term $a : A'$ we have

$$\Pi_f g(q)(a) = g(f^{-1}a, q(f^{-1}a)) : B'(f^{-1}fa) = B'(a)$$

The inverse of this arrow can be defined in a completely symmetric matter, indeed we have that:

$$(\Pi_f g)^{-1}(q) = \lambda(x : A).g^{-1}(fx, q(fx)) : \Pi_A B$$

we quickly verify that as defined, these two terms are inverses of each other, thus for $q : \Pi_{A'}B'$ and $x : A'$ we have the following (we use η-reduction here):

$$\begin{aligned} \Pi_f g(\Pi_f g^{-1}(q))(x) =& g(f^{-1}x, (\Pi_f g)^{-1}(q)(f^{-1}x)) \\ =& g(f^{-1}x, g^{-1}(ff^{-1}x, q(ff^{-1}x))) \\ =& g(f^{-1}x, g(x, q(x))) \\ =& q(x) \end{aligned}$$

the last equation follows because g is an isomorphism (and by what this means in the internal language). The other direction is completely dual.

We must now check the coherence conditions. Let us do first the coherence with respect of the elimination, i.e. we must prove that the diagram in Eq. (2.1) commutes. Take t a section of B, that is a term:

$$\Gamma, x : A' \vdash tx : B'x$$

Using the fact that g is an isomorphism, we see that the section $t[g]$ of B can be equivalently described as $t[g] = g^{-1} \circ t \circ f$. Thus in the internal language we have

$$\Gamma, x : A \vdash t[g](x) = g^{-1}(fx, t(fx)) : Bx$$

and thus with this we can verify that the diagram in Eq. (2.1) commutes, that is, we verify the following equality:

$$\Gamma \vdash \Pi_f g(\lambda(x : A).t[g](x)) = \lambda(x : A').tx : \Pi_{A'} B'$$

We compute:

$$\begin{aligned}
\Pi_f g(\lambda(x : A).t[g](x)) =&\lambda(x : A').g(f^{-1}x, t[g](f^{-1}))\\
=&\lambda(x : A').g(f^{-1}x, g^{-1}(ff^{-1}x, t(ff^{-1}x)))\\
=&\lambda(x : A').g(f^{-1}x, g(x, t(x)))\\
=&\lambda(x : A').tx
\end{aligned}$$

Finally we verify that $\Pi_f g$ is coherent with respect to the elimination terms, i.e. we verify that the diagram in Eq. (2.2) commutes. The commutation of the diagram is equivalent to the following equation:

$$\Gamma, x : A, q : \Pi_A B \vdash \mathsf{app}(fx, \Pi_f g(q)) = g(x, \mathsf{app}(x, q)) : B'(fx)$$

We compute:

$$\begin{aligned}
\mathsf{app}(fx, \Pi_f g(q)) =&\Pi_f g(fx)\\
=&g(f^{-1}fx, q(f^{-1}fx))\\
=&g(x, q(x))\\
=&g(x, \mathsf{app}(x, q))
\end{aligned}$$

□

With the help of this lemma, we can now easily prove the following result.

Proposition 2.6.3. *Let $(\mathbb{C}, \rho, \chi)$ be a split comprehension category with a strictly-stable choice of Σ, Id and Π-types. Then these choices are also pseudo-stable.*

Proof. We will produce a proof only for Π-types, for the other kind of logical structure, the proof is analogous. Consider a Cartesian morphism of dependent tuples:

$$(u, f, g) : (\Delta, A', B') \to (\Gamma, A, B)$$

using the cleavage we can factor f and g via an isomorphism in the fibres and a morphism in the cleavage as follows:

$$\begin{tikzcd}
A' \arrow[d, dotted, "\exists!\cong"', "\tilde{f}"] \arrow[dr, "f"] & \\
A[u] \arrow[r, "u_A"] \arrow[d, dotted, no head] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, dotted, no head] \\
\Delta \arrow[r, "u"'] & \Gamma
\end{tikzcd}$$

similarly for g:

$$\begin{tikzcd}
B' \arrow[d, dotted, "\exists!\cong"', "\tilde{g}"] \arrow[dr, "g"] & \\
B[u_A] \arrow[r, "u_{A.B}"] \arrow[d, dotted, no head] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, dotted, no head] \\
\Delta.A[u] \arrow[r, "u_A"'] & \Gamma.A
\end{tikzcd}$$

This means that in terms of dependent tuples we have a factorisation of (u, f, g) as a vertical isomorphism followed by a canonical Cartesian morphism in the cleavage:

$$\begin{tikzcd}
(\Delta, A', B') \arrow[rr, "{(u,f,g)}"] \arrow[dr, "\cong", "{(id_\Delta,\tilde{f},\tilde{g})}"'] & & (\Gamma, A, B) \\
& (\Delta, A[u], B[u_A]) \arrow[ur, "{(u,u_A,u_{A.B})}"'] &
\end{tikzcd}$$

Using Lemma 2.6.2 we obtain a vertical isomorphism over Δ commuting with the structure of dependent products:

$$\Pi_{\tilde{f}}\tilde{g} : \Pi_{A'}B' \to \Pi_{A[u]}B[u_A]$$

moreover, the canoncial cleavage will give us a Cartesian arrow over u:

$$u_{\Pi AB} : \Pi_A B[u] \to \Pi_A B.$$

By strict stability we have that $\Pi_A B[u] = \Pi_{A[u]}B[u_A]$, and we also have that $u_{\Pi_A B}$ cohere with the structure of dependent products (this is because reindexing the additional structure associated of $\Pi_A B$ along u matches strictly that of $\Pi_{A[u]}B[u_A]$). Thus we can compose this morphisms to obtain:

$$\Pi_f g = u_{\Pi_A B} \circ \Pi_{\tilde{f}}\tilde{g} : \Pi_{A'}B' \to \Pi_A B$$

over u. Notice that $\Pi_f g$ is Cartesian since it is the composite of an isomorphism and a Cartesian arrow, it also cohere with the structure of dependent products since both of it's composites cohere with it. ☐

We strongly believe that the condition of pseudo-stability is indeed necessary to produce strictly-stable models after splitting. A proof of this conjecture would go in the following direction. We would first need to prove that pseudo-stability is invariant under equivalences; that is, if $(\mathbb{C}, \chi, \rho) \cong (\mathbb{C}, \chi', \rho)$ is an equivalence of comprehension categories and $(\mathbb{C}, \chi, \rho)$ is equipped with a pseudo-stable choice of a given kind of logical structure, then $(\mathbb{C}, \chi', \rho)$ is also equipped with a pseudo-stable choice of the same kind, obtained by transporting the original one along the equivalence.

With this result, we could argue in the following way. Suppose that the right adjoint splitting $(\mathbb{C}, \chi^R, \rho^R)$ of a given comprehension category $(\mathbb{C}, \chi, \rho)$ is equipped with a strictly-stable choice of a given kind of logical structure; by Proposition 2.6.3 this choice would also be pseudo-stable. Moreover given that the counit of the right adjoint splitting adjunction induces an equivalence $(\mathbb{C}, \chi^R, \rho^R) \cong (\mathbb{C}, \chi, \rho)$, we could transport the pseudo-stable choice along this equivalence to obtain a pseudo-stable choice in the original comprhension category $(\mathbb{C}, \chi, \rho)$.

2.7 Example: tribes

In this section we will see how we can obtain pseudo-stable choices of Π and Σ types in a special setting, that of Joyal's tribes [Joy17]. We begin by briefly going through some definitions. Let us fix for the remainder of this section a category $\mathbb{C}$ equipped with terminal object $*$.

Definition 2.7.1. A **tribe structure** consist of a class of maps $\mathcal{R} \subseteq \mathbb{C}^{\rightarrow}$ such that:

- $\mathcal{R}$ is closed under composition and contains all isomorphisms.
- For every map $f : X \to B$ in $\mathcal{R}$ and for any $\sigma : A \to B$, the pullback of f along σ exists and belongs to $\mathcal{R}$.

A **tribe** is a pair $(\mathbb{C}, \mathcal{R})$ consisting of a category and a tribe structure on it. The maps in $\mathcal{R}$ are called **fibrations**.

Given a tribe $(\mathbb{C}, \mathcal{R})$ and an object $A \in \mathbb{C}$, we denote by $\mathcal{R}/A$ the full subcategory of the slice $\mathbb{C}/A$ whose objects are fibrations. We call $\mathcal{R}/A$ the **local tribe at** A, notice that $(\mathbb{C}/A, \mathcal{R}/A)$ is canonically a tribe whose properties are inherited from $(\mathbb{C}, \mathcal{R})$.

Proposition 2.7.2. *A tribe* $(\mathbb{C}, \mathcal{R})$ *has a canonical full comprehension category structure associated to it.*

Proof. Let us consider $\mathcal{R}$ as the full subcategory of $\mathbb{C}^{\to}$ of fibrations. The following diagram

$$\begin{tikzcd}
\mathcal{R} \arrow[rr, hook] \arrow[dr, "\mathsf{cod}"'] & & \mathbb{C}^{\to} \arrow[dl, "\mathsf{cod}"] \\
& \mathbb{C} &
\end{tikzcd}$$

is a comprehension category. The Cartesian morphisms (and their comprehension) are pullback squares between fibrations and the fibres are given by the local tribes. ☐

By definition, any map $\mathsf{f} : \mathsf{B} \to \mathsf{A}$ induces a functor

$$\mathsf{f}^* : \mathcal{R}/\mathsf{A} \to \mathcal{R}/\mathsf{B}$$

defined by pullback along f. Note that when f is a fibration, the pullback functor has a left adjoint

$$\Sigma_\mathsf{f} : \mathcal{R}/\mathsf{B} \to \mathcal{R}/\mathsf{A}$$

defined by composition with f. Consider a square:

$$\begin{tikzcd}
\mathsf{C} \arrow[r, "\mathsf{h}"] \arrow[d, "\mathsf{g}"'] & \mathsf{A} \arrow[d, "\mathsf{f}"] \\
\mathsf{D} \arrow[r, "\mathsf{k}"'] & \mathsf{B}
\end{tikzcd}$$

where g and f are fibrations, then there is a canonical natural transformation

$$\mathsf{BC} : \Sigma_\mathsf{g}\mathsf{h}^* \to \mathsf{k}^*\Sigma_\mathsf{f}$$

given by the following pasting diagram

$$\begin{tikzcd}
& \mathcal{R}/\mathsf{A} \arrow[r, "\mathsf{h}^*"] & \mathcal{R}/\mathsf{C} \arrow[r, "\Sigma_\mathsf{g}"] & \mathcal{R}/\mathsf{D} \\
\mathcal{R}/\mathsf{A} \arrow[ur, "\mathsf{id}"] \arrow[r, "\Sigma_\mathsf{f}"'] \arrow[r, phantom, shift left=4, "\eta \Downarrow"] & \mathcal{R}/\mathsf{B} \arrow[u, "\mathsf{f}^*"'] \arrow[r, "\mathsf{k}^*"'] \arrow[r, phantom, shift left=4, "\cong\Downarrow"] & \mathcal{R}/\mathsf{D} \arrow[u, "\mathsf{g}^*"] \arrow[ur, "\mathsf{id}"'] \arrow[r, phantom, shift left=4, "\epsilon \Downarrow"] &
\end{tikzcd}$$

where η and ϵ are the unit and counit of the adjunction, and the middle isomorphism λ is canonically given. This natural transformation satisfies an important coherence property:

Proposition 2.7.3 (Beck-Chevalley)**.** *For any pullback diagram*

$$\begin{tikzcd}
\mathsf{C} \arrow[r, "\mathsf{h}"] \arrow[d, "\mathsf{g}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathsf{A} \arrow[d, "\mathsf{f}"] \\
\mathsf{D} \arrow[r, "\mathsf{k}"'] & \mathsf{B}
\end{tikzcd}$$

the corresponding arrow $\mathsf{BC} : \Sigma_\mathsf{g}\mathsf{h}^* \to \mathsf{k}^*\Sigma_\mathsf{f}$ *is an isomorphism.* ☐

In general though, the pullback functor will not have a right adjoint, such a right adjoint would have the properties of an exponential of fibrations. The following definition addresses this problem.

Definition 2.7.4. A **π-tribe** is given by a tribe $(\mathbb{C}, \mathcal{R})$ where for each fibration $f : B \to A$ the pullback functor f^* has a right adjoint

$$\Pi_f : \mathcal{R}/B \to \mathcal{R}/A$$

Just as with the left adjoint Σ_f, given fibrations g, f and square $(h, k) : g \to f$, there is an induced natural transformation

$$BC : k^*\Pi_f \to \Pi_g h^*$$

the definition of this natural transformation is dual of that for Σ.

It is not so obvious that when the square (h, k) is a pullback square, then this transformation is an isomorphism. Nevertheless, this follows from an abstract, but well known argument, involving mates pairs of adjoint functors; and by the fact that the dual Beck-Chevalley condition holds for the left adjoint Σ to pullback (see for example [KS74]).

Proposition 2.7.5 (Beck-Chevalley)**.** *For a pullback diagram with* $f, k \in \mathcal{R}$*:*

$$\begin{tikzcd}
B' \arrow[r, "h"] \arrow[d, "g"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, "f"] \\
A' \arrow[r, "k"'] & A
\end{tikzcd}$$

the corresponding arrow $BC : k^*\Pi_f \to \Pi_g h^*$ *is an isomorphism.* ☐

In the next proposition we see that the Beck-Chevalley natural transformation for the right adjoint Π satisfies a coherence condition with respect to the composition of pullback squares in the arrow category.

Proposition 2.7.6. *Given pullback squares* $(l, m) : g' \to g$ *and* $(h, k) : g \to f$ *the following diagram commutes:*

$$\begin{tikzcd}
(km)^*\Pi_f \arrow[rr, "BC"] \arrow[d, "\cong(\Pi_f)"'] & & \Pi_g(hl)^* \arrow[d, "\Pi_g(\cong)"] \\
m^*k^*\Pi_f \arrow[dr, "m^*BC"'] & & \Pi_g l^* h^* \\
& m^*\Pi_g h^* \arrow[ur, "BCh^*"'] &
\end{tikzcd}$$

☐

Every tribe when regarded as a comprehension category admits a pseudo-stable choice of Σ-types, in fact, the explicit definition of the left adjoint Σ_A of A^*, as given by composition, allows the interpretation of dependent sums to be straightforward.

Lemma 2.7.7. *Suppose* $(\mathbb{C}, \mathcal{R})$ *is a tribe, then the associated comprehension category admits a pseudo-stable choice of* Σ*-types.*

Proof. Let (Γ, A, B) be a dependent tuple. Since A is a fibration, the functor $\Sigma_A : \mathcal{R}/\Gamma.A \to \mathcal{R}/\Gamma$ (given by composing any fibration over $\Gamma.A$ (the domain of A) with A) is left adjoint to the chosen pullback A^*.

The formation rule is given by applying Σ_A to B, using the explicit definition of Σ_A we have that:

$$\Sigma_A B := \Sigma_A(B) = A \circ B$$

For the introduction rule, we are required to provide an arrow $\mathsf{pair}_{A,B} : B \to \Sigma_A B$ over A, this means a square:

$$\begin{tikzcd} \cdot \arrow[r, "\mathsf{pair}_{A,B}"] \arrow[d, "B"'] & \cdot \arrow[d, "\Sigma_A B"] \\ \cdot \arrow[r, "A"'] & \cdot \end{tikzcd}$$

we take $\mathsf{pair} = \mathsf{id}_{\Gamma.A.B} = \mathsf{id}_{\Gamma.\Sigma_A B}$ which is well typed by definition of Σ_A. Similarly for the elimination rule, we define for each C over $\Sigma_A B$ and for each section t of C over $\mathsf{pair}_{A,B}$,

$$\mathsf{sp}_{A,B}(C, t) = t$$

The computation (and the corresponding η-rule) are trivially validated by the data defined above, thus giving rise to a choice of Σ-types for each tuple (Γ, A, B). Stable functoriality and coherence of elimination terms are also trivially satisfied by definition. □

The situation with Π-types is formally similar to the previous one for Σ-types, but somewhat more involved given the fact that we do not have an explicit description of the right adjoint functor Π.

Lemma 2.7.8. *Let* $(\mathbb{C}, \mathcal{R})$ *be a* π*-tribe, then the associated comprehension category admits a pseudo-stable choice of* Π*-types.*

Proof. We will first build a Π-type over a dependent tuple (Γ, A, B). In the context of tribes, types are fibrations and thus, a tuple as the one just mentioned is given by a pair of fibrations one on top of the other, which we will denote as follows:

$$\begin{tikzcd} \Gamma.A.B \arrow[d, "B"] \\ \Gamma.A \arrow[d, "A"] \\ \Gamma \end{tikzcd}$$

Since A is a fibration, there is a functor $\Pi_A : \mathcal{R}/\Gamma.A \to \mathcal{R}/\Gamma$ which is right adjoint to pulling-back along A. We have that $B \in \mathcal{R}/\Gamma.A$ and thus we can apply Π_A to obtain a fibration over Γ:

$$\Pi_A B : \Gamma.\Pi_A B \to \Gamma$$

thus applying Π_A correspond to the formation rule for Π-types.

For the introduction rule we need to define the operation λ. Notice that a section t of B is given by an arrow $t : id_{\Gamma.A} \to B$ in the local tribe $\mathcal{R}/\Gamma.A$, we can identify naturally $A^*(id_\Gamma) \cong id_{\Gamma.A}$. Thus we have that a section t of B is the same thing as an arrow

$$t : A^*(id_\Gamma) \to B$$

and taking the transpose yields an arrow:

$$\lambda(t) : id_\Gamma \to \Pi_A B$$

Similarly, for the elimination rule we are required to provide an arrow $app_{A,B} : A^*(\Pi_A B) \to B$, we can take $app_{A,B}$ to be the counit of the adjunction

$$app_{A,B} := \epsilon_B : A^*(\Pi_A B) \to B$$

The computation rule as well as the corresponding uniqueness rule follow from the bijection between hom-sets:

$$\mathcal{R}/\Gamma.A[id_{\Gamma.A}, B] \cong \mathcal{R}/\Gamma[id_\Gamma, \Pi_A B]$$

given in one direction by λ.

We now proceed to show the assignment

$$(\Gamma, A, B) \mapsto (\Pi_A B, \lambda, app)$$

is pseudo-stable (see Definition 2.3.1). Let us first define the action on arrows carefully, given a Cartesian arrow $(u, f, g) : (\Delta, A', B') \to (\Gamma, A, B)$, i.e. a diagram of pullback squares

$$\begin{tikzcd}
\Delta.A'.B' \arrow[r, "g"] \arrow[d, "B'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A.B \arrow[d, "B"] \\
\Delta.A' \arrow[r, "f"] \arrow[d, "A'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[d, "A"] \\
\Delta \arrow[r, "u"'] & \Gamma
\end{tikzcd}$$

we must define a Cartesian arrow $(u, F_\Pi(u, f, g)) : (\Delta, \Pi_{A'} B') \to (\Gamma, \Pi_A B)$, i.e. a pullback square

$$\begin{tikzcd}
\Delta.\Pi_{A'} B' \arrow[r, "F_\Pi(u{,}f{,}g)"] \arrow[d, "\Pi_{A'} B'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.\Pi_A B \arrow[d, "\Pi_A B"] \\
\Delta \arrow[r, "u"'] & \Gamma
\end{tikzcd}$$

The definition is given by composing the following squares.

$$\begin{tikzcd}
\Delta.\Pi_{A'}B' \arrow[r, "\Pi_{A'}(\tilde{g})", "\cong"'] \arrow[d, "\Pi_{A'}B'"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \Delta.\Pi_{A'}(f^*B) \arrow[r, "BC^{-1}", "\cong"'] \arrow[d, "\Pi_{A'}(f^*B)"'] & \Delta.u^*(\Pi_A B) \arrow[r, "u_{\Pi_A B}"] \arrow[d, "u^*(\Pi_A B)"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \Gamma.\Pi_A B \arrow[d, "\Pi_A B"'] \\
\Delta \arrow[r, "id"'] & \Delta \arrow[r, "id"'] & \Delta \arrow[r, "u"'] & \Gamma
\end{tikzcd}$$

the rightmost square is the pullback square given by the functor u^*, the middle one is the component at B of the inverse of the Beck-Chevalley arrow (we need this to be an isomorphism, it follows from Proposition 2.7.5 since the square defined by (f, u) is a pullback square) and the leftmost square is the result of applying the functor $\Pi_{A'}$ to the unique isomorphism $\tilde{g} : B' \cong f^*B$ in $\mathcal{R}/\Delta.A'$ that exists since both B' and f^*B are pullbacks of B along f.

Note that $F_{\Pi}(u, f, g) = u_{\Pi_A B} \circ BC^{-1} \circ \Pi_{A'}(\tilde{g})$ is Cartesian since it is the composite of a Cartesian and two isomorphisms. The proof that F_{Π} is functorial reduces (after factoring out a number of natrurality and functorial diagrams) to Proposition 2.7.6.

We now have to verify that the following diagram commutes:

$$\begin{tikzcd}
\Delta.A'.(\Pi_{A'}B') \arrow[r, "F_{\Pi}(u,f,g)"] \arrow[d, "\epsilon'_{B'}"'] & \Gamma.A.(\Pi_A B) \arrow[d, "\epsilon_B"] \\
\Delta.A'.B' \arrow[r, "g"'] & \Gamma.A.B
\end{tikzcd}$$

By definition of $F_{\Pi}(u, f, g)$ we can factor out a naturality diagram of ϵ', this means that it is sufficient to show that:

$$\begin{tikzcd}
\Delta.A'.(\Pi_{A'}f^*B) \arrow[r, "(u_{\Pi_A B} \circ BC^{-1})"] \arrow[d, "\epsilon'_{f^*B}"'] & \Gamma.A.(\Pi_A B) \arrow[d, "\epsilon_B"] \\
\Delta.A'.f^*B \arrow[r, "f_B"'] & \Gamma.A.B
\end{tikzcd}$$

commutes. The above diagram unwinds to the following one (by reversing the direction of the Beck-Chevalley arrow):

$$\begin{tikzcd}
A'^*u^*(\Pi_A B) \arrow[rr, "A'^*(BC)"] \arrow[d, "u^+_{\Pi_A B}"'] & & A'^*(\Pi_{A'}f^*B) \arrow[d, "\epsilon'_{f^*B}"] \\
A^*(\Pi_A B) \arrow[r, "\epsilon_B"'] & B & f^*B \arrow[l, "f_B"']
\end{tikzcd}$$

we can fill the interior of this diagram with an intermediate one which is easily seen to commute:

$$\begin{tikzcd}
A'^*u^*(\Pi_A B) \arrow[r, "\cong"] \arrow[dr, "u^+_{\Pi_A B}"'] & f^*A^*(\Pi_A B) \arrow[r, "f^*\epsilon_B"] \arrow[d, "f_{A^*(\Pi_A B)}" description] & f^*B \arrow[d, "f_B"] \\
 & A^*(\Pi_A B) \arrow[r, "\epsilon_B"'] & B
\end{tikzcd}$$

and thus it is sufficient to verify that

$$\begin{tikzcd}
A'^{*}u^{*}(\Pi_A B) \arrow[r, "A'^{*}(BC)"] \arrow[d, "\cong"'] & A'^{*}(\Pi_{A'} f^{*}B) \arrow[d, "\epsilon'_{f^{*}B}"] \\
f^{*}A^{*}(\Pi_A B) \arrow[r, "f^{*}\epsilon_B"'] & f^{*}B
\end{tikzcd}$$

commutes. This diagram always commutes [Koc09, Proposition 8.4.1].

Finally, we must show that the choice of introduction terms (i.e. the lambda terms) is coherent, this means that we must verify that for every section t of B the following diagram commutes:

$$\begin{tikzcd}
\Pi_{A'}B' \arrow[r, "F_{\Pi}(u,f,g)"] & \Pi_A B \\
\mathrm{id}_{\Delta} \arrow[u, "\lambda'(t')"] \arrow[r, "u"'] & \mathrm{id}_{\Gamma} \arrow[u, "\lambda(t)"']
\end{tikzcd}$$

Recall that the operation λ was given by transposing the term t, in other words, it was given as the composition $\lambda(t) = \Pi_A(t) \circ \eta_{\mathrm{id}_{\Gamma}}$ and similarly for λ'. Using this, notice that functoriality of $\Pi_{A'}$ and naturality of η' allows us to factor out a square on the leftmost side, leaving us with the task of verifying the commutativity of the square:

$$\begin{tikzcd}
\Pi_{A'}f^{*}B \arrow[r, "u_{\Pi_A B} \circ BC^{-1}"] & \Pi_A B \\
\mathrm{id}_{\Delta} \arrow[u, "\lambda'(f^{*}t)"] \arrow[r, "u"'] & \mathrm{id}_{\Gamma} \arrow[u, "\lambda(t)"']
\end{tikzcd}$$

And to verify this we will split the problem into two parts. We will assume that identities are pullback stable (as mentioned before), otherwise we will have to insert extra naturality squares at some places.

1. For the first part we will show the appropriate commutation with the units η and η', for this consider:

$$\begin{tikzcd}
\Pi_{A'}(A'^{*}(\mathrm{id}_{\Delta})) \arrow[r, "\Pi_{A'}(\cong)"] & \Pi_{A'}f^{*}A^{*}(\mathrm{id}_{\Gamma}) \arrow[r, "BC^{-1}"] & u^{*}\Pi_A A^{*}(\mathrm{id}_{\Gamma}) \arrow[r, "u_{(-)}"] \arrow[dr, phantom, "\lrcorner", very near start] & \Pi_A A^{*}(\mathrm{id}_{\Gamma}) \\
 & \mathrm{id}_{\Delta} \arrow[ul, "\eta'_{\mathrm{id}_{\Delta}}"] \arrow[r, "\mathrm{id}"'] & \mathrm{id}_{\Delta} \arrow[u, "u^{*}\eta_{\mathrm{id}_{\Gamma}}"'] \arrow[r, "u"'] & \mathrm{id}_{\Gamma} \arrow[u, "\eta_{\mathrm{id}_{\Gamma}}"']
\end{tikzcd}$$

   the left diagram always commutes [Koc09, Proposition 8.4.1], the right diagram is just the pullback square defined by applying u^{*} to $\eta_{\mathrm{id}_{\Gamma}}$, thus in particular it is commutative.

2. The second part correspond to the appropriate commutation with the arrows $\Pi_{A'}f^*t$ and $\Pi_A t$. Consider the diagram:

$$\begin{tikzcd}
\Pi_{A'}f^*B \arrow[r, "BC^{-1}"] & u^*\Pi_A B \arrow[r, "u_{\Pi_A B}"] \arrow[dr, phantom, "\lrcorner", very near start] & \Pi_A B \\
\Pi_{A'}f^*A^*(\mathrm{id}_\Gamma) \arrow[u, "\Pi_{A'}f^*t"'] \arrow[r, "BC^{-1}"] & u^*\Pi_A A^*(\mathrm{id}_\Gamma) \arrow[u, "u^*\Pi_A t"'] \arrow[r, "u_{(-)}"] & \Pi_A A^*(\mathrm{id}_\Gamma) \arrow[u, "\Pi_A t"']
\end{tikzcd}$$

   the left diagram commutes by naturality of BC and the right one is once again just the pullback diagram defined by the functor u^* applied to $\Pi_A t$.

Pasting the diagram from item (2) on top of the diagram from item (1) gives the desired property, i.e. the commutation with the introduction terms. □

Remark 2.7.9. We can observe in the proof of the previous proposition, that pseudo-stability of Π-types follows from the following three properties:

- A choice of pullback of fibrations, in the case of tribes, this is implicitly assumed when working with pullback functors.
- Comparison isomorphisms, in this case given by Beck-Chevalley.
- Isomorphism in the fibres, in this case given by functoriality of Π_A for each type A.

Example 2.7.10. The following are examples of Π-tribes:

- Every locally cartesian closed category is a π-tribe where every arrow is a fibration.
- The category of small groupoids is a π-tribe, where a fibration is an isofibration.
- The category of Kan complexes is a π-tribe, where a fibration is a Kan fibration.

Remark 2.7.11. We strongly emphasise that the case of Id-types for tribes is much more delicate. Although in some cases it will be possible to give a functorial and stable factorisation of the diagonal (giving the formation and introduction rules for Id-types), it will be very difficult in general to give a *coherent choice of elimination terms*, and therefore to be able to interpret intentional identity types in the resulting split comprehension category obtained by applying the right adjoint splitting. The main reason for this, is that in the vast majority of examples, the elimination terms are obtained by a non-unique *lifting property* and in this case, coherence is impossible to obtain. This is one of the main motivations for us to work in an algebraic setting.

Chapter 3

The Voevodsky Splitting for Comprehension Categories

In this chapter we will discuss an alternative to the right adjoint splitting of a comprehension category. This is a slight generalisation of Voevodsky's work [Voe15b, Voe15a]. He considers conditions on a universe in a category that ensure that, after applying this new splitting, it produces a contextual category supporting the various type constructors. Here, we consider a comprehension category equipped with a type classifier and instead of a contextual category we consider a split comprehension category.

The key motivation for doing this is to have Voevodsky's ideas in the same setting as ours in order to compare the two approaches.

3.1 Overview of the Construction

Definition 3.1.1. Let $\rho : \mathbb{E} \to \mathbb{C}$ be fibration. A **type classifier** on ρ consists of a tuple $(\mathsf{U}, \tilde{\mathsf{U}}, \mathsf{El}[-]))$ where $\mathsf{U} \in \mathbb{C}$, $\tilde{\mathsf{U}} \in \mathbb{E}$ is over U; and $\mathsf{El}[-]$ is a normal local cleavage for $\tilde{\mathsf{U}}$ in the sense of Definition 2.1.1.

Remark 3.1.2. The definition of a type classifier is essentially the same as that of a pseudo-stable universe (Definition 2.5.1). There is one subtle difference, in a type classifier the object U need not be (the comprehension of) an object in $\mathbb{E}$, i.e. U need not be 'fibrant'. The main reason we introduce this new concept is the following: a type classifier is used for splitting, and a pseudo-stable universe is used to model a type theoretic universe.

The data of a type classifier $(\mathsf{U}, \tilde{\mathsf{U}}, \mathsf{El}[-])$ determines a new split fibration which will be denote by $\rho^{\mathsf{U}} : \mathbb{E}^{\mathsf{U}} \to \mathbb{C}$.

The objects of $\mathbb{E}^{\mathsf{U}}$ are arrows of the form $a : \Gamma \to \mathsf{U}$, and a morphisms $f : a \to b$ is a map $f : \mathsf{El}[a] \to \mathsf{El}[b]$ in $\mathbb{E}$. The functor ρ^{U} is given on objects by taking the domain of the arrow into U and on morphisms by applying ρ. The split choice of reindexing for ρ^{U} is given by composition.

Notice that there is a fibred functor $\mathsf{El} : \rho^{\mathsf{U}} \to \rho$ which maps an object $a \in \mathbb{E}^{\mathsf{U}}$ to $\mathsf{El}[a]$:

$$\begin{tikzcd}
\mathbb{E}^{\mathsf{U}} \arrow[rr, "\mathsf{El}"] \arrow[dr, "\rho^{\mathsf{U}}"'] & & \mathbb{E} \arrow[dl, "\rho"] \\
& \mathbb{C} &
\end{tikzcd}$$

If we start with a comprehension category (ρ, χ) we obtain a split comprehension category $(\rho^{\mathsf{U}}, \chi^{\mathsf{U}})$ where the comprehension functor χ^{U} is given by the composition:

$$\begin{tikzcd}
\mathbb{E}^{\mathsf{U}} \arrow[r, "\mathsf{El}"] \arrow[dr, "\rho^{\mathsf{U}}"'] & \mathbb{E} \arrow[r, "\chi"] \arrow[d, "\rho"] & \mathbb{C}^{\to} \arrow[dl, "\mathsf{cod}"] \\
& \mathbb{C} &
\end{tikzcd}$$

We will call the resulting comprehension category $(\rho^{\mathsf{U}}, \chi^{\mathsf{U}})$ the **Voevodsky splitting** of (ρ, χ).

Remark 3.1.3. The fibred functor $\mathsf{El} : \rho^{\mathsf{U}} \to \rho$ is not an equivalence in general.

3.2 Condition for Π and Σ types

We will review here the necessary structure in $(\mathbb{C}, \rho, \chi)$ to produce a strictly-stable choice of Π-types and Σ-types in $(\mathbb{C}, \rho^{\mathsf{U}}, \chi^{\mathsf{U}})$. For ease of presentation, we will assume that the base categories of the comprehension categories used in this section are locally cartesian closed (although it is sufficient for the base category to satisfy the (LF) condition of Definition 2.5.9). We will make liberal use the internal language of locally cartesian closed categories.

Definition 3.2.1. Let (ρ, χ) be a comprehension category with a type classifier $(\mathsf{U}, \tilde{\mathsf{U}}, \mathsf{El}[-])$. Let $\mathsf{U}_{\Pi} = \Pi_{X:\mathsf{U}} \mathsf{U}^{X}$ (in the internal language of $\mathbb{C}$) the object representing the premises for the formation rules for Π-types. A **Π-structure** for (ρ, χ) consists of a pair of arrows $(\Pi, \tilde{\Pi})$ making the following diagram a pullback square:

$$\begin{tikzcd}
\Sigma_{(a,b):\mathsf{U}_{\Pi}} \Pi_{\mathsf{El}[a]} \mathsf{El}[b] \arrow[r, "\tilde{\Pi}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{U}.\tilde{\mathsf{U}} \arrow[d, "\chi_{\tilde{\mathsf{U}}}"] \\
\mathsf{U}_{\Pi} \arrow[r, "\Pi"'] & \mathsf{U}
\end{tikzcd}$$

Remark 3.2.2. The definition of a Π-structure is the same in spirit as that of reflecting Π-types (Definition 2.5.10). We are essentially reflecting inside of U the dependent products of the locally cartesian closed category $\mathbb{C}$. One main difference is that we will work in the level of $\mathbb{C}^{\to}$ instead of working in $\mathbb{E}$ (i.e. the diagram in the previous definition is a pullback diagram instead of a Cartesian lift).

Proposition 3.2.3. *Let* (ρ, χ) *be a comprehension category with a type classifier* $(U, \tilde{U}, El[-])$ *and a* Π*-structure* $(\Pi, \tilde{\Pi})$. *Then the Voevodsky splitting* (ρ^{U}, χ^{U}) *has a strictly-stable choice of* Π*-types.*

Proof. Let us first construct a choice of Π-types (Π, λ, app) in (ρ^{U}, χ^{U}). Consider a dependent tuple $(\Gamma, a, b) \in DT_2(\rho^{U}, \chi^{U})$ (recall Definition 1.4.1), by the universal property of U_Π this is precisely the same as an arrow $(a, b) : \Gamma \to U_\Pi$. We let $\Pi_a b := \Pi \circ (a, b)$, it is immediate that this choice is strictly stable.

We turn our attention to the λ operation. Let $t : \Gamma.El[a] \to \Gamma.El[a].El[b]$ be a section of $\chi^{U}(b)$. In the slice over $\Gamma.El[a]$ this is an arrow $t : id_{\chi^{U}(a)} \to \chi^{U}(b)$ which transpose to an arrow $\lambda(t) : id_\Gamma \to \Pi_{\chi^{U}(a)}\chi^{U}(b)$. This arrow fits in the following diagram:

$$\begin{tikzcd}
 & \Gamma.El[\Pi_a b] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Sigma_{(a,b):U_\Pi}\Pi_{El[a]}El[b] \arrow[r, "\tilde{\Pi}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U.\tilde{U} \arrow[d, "\chi_{\tilde{U}}"] \\
\Gamma \arrow[ur, "\lambda(t)"] \arrow[r, equal] & \Gamma \arrow[r, "{(a,b)}"'] & U_\Pi \arrow[r, "\Pi"'] & U
\end{tikzcd}$$

The arrow $app_{a,b} : \Gamma.El[a].El[\Pi_a b] \to \Gamma.El[a].El[b]$ correspond to the counit of the dependent product adjunction in $\mathbb{C}$. The computation rule follows from the universal properties of this adjunction.

Moreover it is clear that these choices are strictly stable, this follows because of the uniqueness of the universal properties involved. ☐

The case of Σ-types is entirely analogous; that is, we 'reflect' inside the type classifier the dependent sums of the locally cartesian closed category $\mathbb{C}$. We will only state without a proof the corresponding coherence condition.

Definition 3.2.4. Let (ρ, χ) be a comprehension category with a type classifier $(U, \tilde{U}, El[-])$. Let $U_\Sigma = \Pi_{X:U}U^X$ (in the internal language of $\mathbb{C}$) the object representing the premises for the formation rules for Σ-types. A **Σ-structure** for (ρ, χ) consists of a pair of arrows $(\Sigma, \tilde{\Sigma})$ making the following diagram a pullback square:

$$\begin{tikzcd}
\Sigma_{(a,b):U_\Sigma}\Sigma_{El[a]}El[b] \arrow[r, "\tilde{\Sigma}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U.\tilde{U} \arrow[d, "\chi_{\tilde{U}}"] \\
U_\Sigma \arrow[r, "\Sigma"'] & U
\end{tikzcd}$$

Proposition 3.2.5. *Let* (ρ, χ) *be a comprehension category with a type classifier* $(U, \tilde{U}, El[-])$ *and a* Σ*-structure* $(\Sigma, \tilde{\Sigma})$. *Then the Voevodsky splitting* (ρ^{U}, χ^{U}) *has a strictly-stable choice of* Σ*-types.*

3.3 Condition on Id-types

The condition on Id-types are more involved. Intuitively, the reason is that there is a priory no pseudo-stable choice of Id-types in the base category $\mathbb{C}$ that we can reflect inside the type classifier.

We will need some preliminary notions, first of all for simplicity we will refer to the comprehension $\chi_{\tilde{U}} : U.\tilde{U} \to U$ just as $p : \tilde{U} \to U$.

For any map $q : E \to B$ in $\mathbb{C}$, define the functor $I_q : \mathbb{C} \to \mathbb{C}/B$ given on objects by:

$$I_q : X \mapsto (\sum_{x:B} X^{E(x)} \to B)$$

by the universal properties of the constructions involved, maps $(A, q) : \Gamma \to I_q(V)$ correspond bijectively with pairs consisting of $A : \Gamma \to B$ and $q : \Gamma \times_B E \to V$. Note that $I_q(X)$ is not only functorial on X but also (contravariantly) on q, that is given a diagram:

$$\begin{tikzcd} E' \arrow[rr, "\alpha"] \arrow[dr, "q'"'] & & E \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

there is a natural transformation $I_\alpha : I_q \to I_{q'}$. For details, we refer the reader to [Voe15b].

Definition 3.3.1. Let (ρ, χ) be a comprehension category with a type classifier $(U, \tilde{U}, El[-])$. A **partial Id-structure** on U consists of a pair (Id, r) of maps making the following diagram commute:

$$\begin{tikzcd} \tilde{U} \arrow[r, "r"] \arrow[d, "\delta_{\tilde{U}}"'] & \tilde{U} \arrow[d, "p"] \\ \tilde{U} \times_U \tilde{U} \arrow[r, "\mathsf{Id}"'] & U \end{tikzcd}$$

where $\delta_{\tilde{U}} : \tilde{U} \to \tilde{U} \times_U \tilde{U}$ is the diagonal map.

Given a partial Id-structure, consider Eq to be the pullback of p along Id and denote by p_{Eq} the composite

$$\mathsf{Eq} \to \tilde{U} \times_U \tilde{U} \to \tilde{U} \to U$$

the canonical induced map $\tilde{U} \to \mathsf{Eq}$ will be denoted by w. Thus we have a situation:

$$\begin{tikzcd} \tilde{U} \arrow[rr, "w"] \arrow[dr, "p"'] & & \mathsf{Eq} \arrow[dl, "p_{\mathsf{Eq}}"] \\ & U & \end{tikzcd}$$

which induces a natural transformation $I_w : I_{p_{Eq}} \to I_p$, and applying naturality to $p : \tilde{U} \to U$ we obtain the following diagram:

$$\begin{tikzcd}
I_{p_{Eq}}(\tilde{U}) \arrow[r, "I_w(\tilde{U})"] \arrow[d, "I_{p_{Eq}}(p)"'] & I_p(\tilde{U}) \arrow[d, "I_p(p)"] \\
I_{p_{Eq}}(U) \arrow[r, "I_w(U)"'] & I_p(U)
\end{tikzcd}$$

This in turn induces a canonical map to the pullback:

$$I_{elim} : I_{p_{Eq}}(\tilde{U}) \to I_{p_{Eq}}(U) \times_{I_p(U)} I_p(\tilde{U})$$

We can explain the action of I_{elim} representably as follows. Consider a generalised element of the domain of I_{elim}, that is a map of the form $(A, C, c) : \Gamma \to I_{p_{Eq}}(\tilde{U})$, this corresponds to an arrow $A : \Gamma \to U$ and a diagram

$$\begin{tikzcd}
 & \tilde{U} \arrow[d, "p"] \\
\Gamma.A.A.Id_A \arrow[ru, "c"] \arrow[r, "C"'] & U
\end{tikzcd}$$

where $\Gamma.A.A.Id_A$ denotes the pullback of Eq along $A : \Gamma \to U$. The arrow I_{elim} maps (A, C, c) to the pair $[(A, C), (A, C \circ r_A, c \circ r_A)]$, where r_A is the pullback of w along $A : \Gamma \to U$.

Definition 3.3.2. Let $(\mathbb{C}, \rho, \chi)$ be a comprehension category with a type classifier $(U, \tilde{U}, El[-])$ and a partial Id-structure (Id, r) on it. An **Id-structure** on the previous data consists of a section

$$j : I_{p_{Eq}}(U) \times_{I_p(U)} I_p(\tilde{U}) \to I_{p_{Eq}}(\tilde{U})$$

of I_{elim}.

The following theorem is a rephrasing in the language of comprehension categories and type classifiers of constructions in section 2 of [Voe15b].

Theorem 3.3.3. *Let* $(\mathbb{C}, \rho, \chi)$ *be a comprehension category with a type classifier* $(U, \tilde{U}, El[-])$ *and an Id-structure* (Id, r, j) *on it. Then* (ρ^U, χ^U) *has a choice of strictly-stable* Id*-types.*

Proof. Consider an object $A : \Gamma \to U$ of $\mathbb{E}^U$, we need an Id-type structure on it. For this we let $Id_A : \Gamma.A.A \to U$ be the composite

$$\Gamma.A.A \xrightarrow{A^*} \tilde{U} \times_U \tilde{U} \xrightarrow{Id} U.$$

The reflexivity map $r_A : \Gamma.A \to \Gamma.A.A.\mathrm{Id}_A$ is the result of pulling back w along A. Now, for the elimination consider a pair (C, d) of maps making the diagram

$$\begin{tikzcd}
\Gamma.A \arrow[r, "d"] \arrow[d, "r_A"'] & \tilde{U} \arrow[d, "p"] \\
\Gamma.A.A.\mathrm{Id}_A \arrow[r, "C"'] & U
\end{tikzcd}$$

commute. Transposing, this is the same as an arrow

$$[(A, C), (A, C \circ r_A, d)] : \Gamma \to I_{p_{\mathrm{Eq}}}(U) \times_{I_p(U)} I_p(\tilde{U})$$

thus composing with the section j, gives an arrow $(A, C, j(d)) : \Gamma \to I_{p_{\mathrm{Eq}}}(\tilde{U})$, which computes the elimination term required. Strict stability follows by the strict naturality of the constructions involved. ☐

3.4 Condition for Universe Types

Definition 3.4.1. Let $(\mathbb{C}, \rho, \chi)$ be a comprehension category with a type classifier $(U, \tilde{U}, \mathrm{El}[-])$. A **$U$-universe** corresponds to the following structure:

1. A map $v : * \to U$ in $\mathbb{C}$.
2. A map $i : \mathrm{El}(v) \to U$ in $\mathbb{C}$.

We will denote $V := \mathrm{El}(v)$ and $\tilde{V} := \mathrm{El}(i)$, and refer to the structure by $(V, \tilde{V})$.

Given a U-universe $(V, \tilde{V})$, we obtain a Cartesian arrow over i as shown in the following diagram:

$$\begin{tikzcd}
\tilde{V} \arrow[r, "i^*"] \arrow[d, dotted, no head] \arrow[dr, phantom, very near start, "\lrcorner"] & \tilde{U} \arrow[d, dotted, no head] \\
V \arrow[r, "i"'] & U
\end{tikzcd}$$

Moreover, $(V, \tilde{V})$ inherits the choice of reindexing El of $(U, \tilde{U})$, thus giving rise to a pseudo-stable universe $(V, \tilde{V}, \mathrm{El}[-])$ in (ρ, χ).

Definition 3.4.2. Let $(\mathbb{C}, \rho, \chi)$ be a comprehension category with a type classifier $(U, \tilde{U}, \mathrm{El}[-])$, a U-universe $(V, \tilde{V})$ and an Id-structure (Id, r, j) on U. We say that V is **U-closed under Id-types** if there is an arrow $\mathrm{Id}^V : V.\tilde{V}.\tilde{V} \to V$ making the following diagram commute:

$$\begin{tikzcd}
V.\tilde{V}.\tilde{V} \arrow[r, "i^{**}"] \arrow[d, "\mathrm{Id}^V"'] & \tilde{U} \times_U \tilde{U} \arrow[d, "\mathrm{Id}"] \\
V \arrow[r, "i"'] & U
\end{tikzcd}$$

Theorem 3.4.3. *Let* $(\mathbb{C}, \rho, \chi)$ *be a comprehension category with a type classifier* $(\mathsf{U}, \tilde{\mathsf{U}}, \mathsf{El}[-])$ *and a* U*-universe* $(\mathsf{V}, \tilde{\mathsf{V}})$ U*-closed under* $\mathtt{Id}$*-types. The* $(\rho^{\mathsf{U}}, \chi^{\mathsf{U}})$ *has a universe closed under* $\mathtt{Id}$*-types.*

Proof. Notice that the data for a U-universe $(\mathsf{V}, \tilde{\mathsf{V}})$ in (ρ, χ) corresponds precisely to that of a universe in $(\rho^{\mathsf{U}}, \chi^{\mathsf{U}})$.

We are required to provide a coherent choice of maps into V representing the identity types, this is given by composing with $\mathtt{Id}^{\mathsf{V}}$. The preservation of choices on-the-nose, is precisely the commutativity of the diagram in Definition 3.4.2. ☐

3.5 Comparison between the two Splittings

We will now investigate how the right adjoint splitting and the Voevodsky splitting are related. Let us fix for the remainder of this section a comprehension category $(\mathbb{C}, \rho, \chi)$ where $\mathbb{C}$ is locally cartesian closed, and equipped with a type classifier $(\mathsf{U}, \tilde{\mathsf{U}}, \mathsf{El}[-])$.

We observe that, by the universal property of the right adjoint splitting, we obtain a unique dotted functor as in the following diagram, factoring the morphism of comprehension categories $\mathsf{El} : (\rho^{\mathsf{U}}, \chi^{\mathsf{U}}) \to (\rho, \chi)$:

$$\begin{tikzcd}
\mathbb{E}^{\mathsf{U}} \arrow[r, dotted, "\tilde{\mathsf{El}}"] \arrow[rr, bend left, "\mathsf{El}"] \arrow[dr, "\rho^{\mathsf{R}}"'] & \mathbb{E}^{\mathsf{R}} \arrow[r, "\epsilon"] \arrow[d, "\rho^{\mathsf{R}}"] & \mathbb{E} \arrow[dl, "\rho"] \\
 & \mathbb{C} &
\end{tikzcd}$$

Explicitly this morphism $\tilde{\mathsf{El}} : (\rho^{\mathsf{U}}, \chi^{\mathsf{U}}) \to (\rho^{\mathsf{R}}, \chi^{\mathsf{R}})$ is given on objects by:

$$\tilde{\mathsf{El}}(a : \Gamma \to \mathsf{U}) := (\mathsf{El}[a], \mathsf{El}[a][-])$$

where $\mathsf{El}[a][-]$ is the canonical normal local cleavage on $\mathsf{El}[a]$ induced by $\mathsf{El}[-]$ as in the proof of Lemma 2.1.2.

Proposition 3.5.1. *The functor* $\tilde{\mathsf{El}}$ *is fully faithful and thus constitutes an embedding of split comprehension categories.* ☐

We now want to see how this embedding $\tilde{\mathsf{El}}$ interacts with the additional logical structure. We will investigate only the case of $\mathtt{Id}$-types and leave the rest to the reader.

Let us suppose that $(\mathbb{C}, \rho, \chi)$ is equipped with a pseudo-stable choice of Id-types. Just as we did for pseudo-stable universe in Definition 2.5.10, we will require that the type classifier 'reflects' the choice of $\mathtt{Id}$-types.

Definition 3.5.2. We say that $(\mathsf{U}, \tilde{\mathsf{U}}, \mathrm{El}[-])$ **reflects** Id**-types** if we can exhibit the U-generic Id-type as a reindexing of $\tilde{\mathsf{U}}$; that is if we can find a Cartesian diagram:

$$\begin{tikzcd}
\mathrm{Id}_{\tilde{\mathsf{U}}} \arrow[r, "\mathrm{Id}^*"] \arrow[d, dotted] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{\mathsf{U}} \arrow[d, dotted] \\
\tilde{\mathsf{U}} \times_{\mathsf{U}} \tilde{\mathsf{U}} \arrow[r, "\mathrm{Id}"'] & \mathsf{U}
\end{tikzcd}$$

Proposition 3.5.3. *Let* $(\mathbb{C}, \rho, \chi)$ *be a comprehension category with locally cartesian closed base and equipped with a pseudo-stable choice of* Id*-types. If* $(\mathsf{U}, \tilde{\mathsf{U}}, \mathrm{El}[-])$ *reflects* Id*-types, then there is a canonical* Id*-structure* $(\mathrm{Id}', \mathsf{r}', \mathsf{j}')$ *on* U.

Proof. The partial Id-structure $(\mathrm{Id}', \mathsf{r}')$ is given by applying comprehension to the diagram in Definition 3.5.2 and using the factorisation of the diagonal given by r:

$$\begin{tikzcd}
\tilde{\mathsf{U}} \arrow[r, "\mathsf{r}"] \arrow[rr, bend left, "\mathsf{r}'"] \arrow[dr, "\delta_{\tilde{\mathsf{U}}}"'] & \mathrm{Id}_{\tilde{\mathsf{U}}} \arrow[r, "\mathrm{Id}^*"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{\mathsf{U}} \arrow[d] \\
 & \tilde{\mathsf{U}} \times_{\mathsf{U}} \tilde{\mathsf{U}} \arrow[r, "\mathrm{Id}' := \mathrm{Id}"'] & \mathsf{U}
\end{tikzcd}$$

We now are required to define a section:

$$\mathsf{j}' : \mathrm{I}_{\mathsf{p}_{\mathrm{Eq}}}(\mathsf{U}) \times_{\mathrm{I}_{\mathsf{p}}(\mathsf{U})} \mathrm{I}_{\mathsf{p}}(\tilde{\mathsf{U}}) \to \mathrm{I}_{\mathsf{p}_{\mathrm{Eq}}}(\tilde{\mathsf{U}})$$

for this, we consider an arrow $[(A, C), (A, C \circ \mathsf{r}_A, \mathsf{d})] : \Gamma \to \mathrm{I}_{\mathsf{p}_{\mathrm{Eq}}}(\mathsf{U}) \times_{\mathrm{I}_{\mathsf{p}}(\mathsf{U})} \mathrm{I}_{\mathsf{p}}(\tilde{\mathsf{U}})$, representing a diagram:

$$\begin{tikzcd}
\Gamma.A \arrow[r, "\mathsf{d}"] \arrow[d, "\mathsf{r}_A"'] & \tilde{\mathsf{U}} \arrow[d, "\mathsf{p}"] \\
\Gamma.A.A.\mathrm{Id}_A \arrow[r, "C"'] \arrow[ur, dotted, "\mathsf{j}"] & \mathsf{U}
\end{tikzcd}$$

this has a filler j (as shown) induced in a canonical way by the pseudo-stable choice of Id-types, which in turn induces an arrow $\mathsf{j}'[(A, C), (A, C \circ \mathsf{r}_A, \mathsf{d})] : \Gamma \to \mathrm{I}_{\mathsf{p}_{\mathrm{Eq}}}(\tilde{\mathsf{U}})$. This whole process in natural in Γ precisely because the choice of fillers is coherent, thus giving the arrow j' by Yoneda. $\square$

Following what we did in Section 2.5, if $(\mathsf{U}, \tilde{\mathsf{U}}, \mathrm{El}[-])$ reflects Id-types, then U will be pseudo-closed under Id-types. Furthermore, we can modify the pseudo-stable choice of Id-types in order to make U strictly-closed under the new choice, i.e. for each $a : \Gamma \to \mathsf{U}$ we have:

$$\mathrm{El}[\mathrm{Id}' \circ a^{**}] = \mathrm{Id}_{\mathrm{El}[a]}$$

Proposition 3.5.4. *Let us consider* $(\mathtt{Id}', \mathtt{r}', \mathtt{j}')$ *the* $\mathtt{Id}$*-structure on* U *induced by the pseudo-stable choice of* $\mathtt{Id}$*-types* $(\mathtt{Id}, \mathtt{r}, \mathtt{j})$ *as in Proposition 3.5.3, for which we can assume that* U *is strictly closed. Then the embedding* $\tilde{\mathsf{El}} : (\rho^{\mathsf{U}}, \chi^{\mathsf{U}}) \to (\rho^{\mathsf{R}}, \chi^{\mathsf{R}})$ *preserves the corresponding strictly-stable choice of* $\mathtt{Id}$*-types on-the-nose.*

Proof. Lets consider an object $a : \Gamma \to \mathsf{U}$ in $\mathbb{E}^{\mathsf{U}}$, the $\mathtt{Id}$-type that corresponds to a is $\mathtt{Id}_a := \mathtt{Id}' \circ a^{**} : \Gamma.\mathsf{El}[a].\mathsf{El}[a] \to \mathsf{U}$. Now, $\tilde{\mathsf{El}}$ maps $\mathtt{Id}_a$ to $(\mathsf{El}[\mathtt{Id}_a], \mathsf{El}[\mathtt{Id}_a][-])$ which by the previous paragraph is just $(\mathtt{Id}_{\mathsf{El}[a]}, \mathtt{Id}_{\mathsf{El}[a]}[-])$. ☐

We summarise the results of this section as follows. Let us suppose are given a comprehension category equipped with a pseudo-stable choice of $\mathtt{Id}$-types and with a type classifier which is 'big enough' that it can 'reflect' the choice of $\mathtt{Id}$-types as in Definition 3.5.2. Then, after manipulating the pseudo-stable choice of $\mathtt{Id}$-types as we did in Section 2.5 to obtain an isomorphic strictly-stable choice of $\mathtt{Id}$-types; the embedding of split comprehension categories from the Voevodsky splitting to the right adjoint splitting, will preserve the choices of $\mathtt{Id}$-types on-the-nose by Proposition 3.5.4.

Applying this same heuristic ideas to the other types of logical structure, we obtain the following theorem which we state without a proof.

Theorem 3.5.5. *Let* $(\mathbb{C}, \rho, \chi)$ *be a comprehension category over a locally Cartesian closed base, equipped with pseudo-stable choices of* Σ*,* Π *and* $\mathtt{Id}$*-types; and a type classifier* $(\mathsf{U}, \tilde{\mathsf{U}}, \mathsf{El}[-])$ *which reflects the appropriate choice in the sense of Definition 3.5.2. Then* U *has canonical choices of* Σ*,* Π *and* $\mathtt{Id}$ *structures. Moreover, after modifying the relevant pseudo-stable choices of logical structure (as done in Section 2.5), the embedding of comprehension categories* $\tilde{\mathsf{El}} : (\rho^{\mathsf{U}}, \chi^{\mathsf{U}}) \to (\rho^{\mathsf{R}}, \chi^{\mathsf{R}})$ *preserves the corresponding strictly-stable choice of logical structure on-the-nose.* ☐

This theorem makes precise the idea that the Voevodsky splitting gives an embedding of models of Martin-Löf dependent type theory preserving strictly all the logical structure.

Chapter 4

Type-Theoretic Algebraic Weak Factorisation Systems

In section Section 2.7, we saw that by working with the notion of tribe (or, for that matter, with any similar non-algebraic framework) we were able to obtain pseudo-stable choices of Π and Σ types. In summary, we do this by using the right and left adjoints, respectively, to the pullback functor between the corresponding local tribes. Coherence follows essentially from the universal properties of the adjoint functors.

In some settings we could also obtain a choice of Id-types by looking at path-object factorisations of the diagonal morphism, satisfying some lifting conditions. For example, we can work in a model category and take our tribe to be the class of fibrations. With some extra work we could even get the choice of be suitably functorial and stable. However, we will hardly obtain a coherent choice of elimination terms. Briefly, the situation is the following. In the setting of weak factorisation systems and model categories, the classes of (acyclic) fibrations and (acyclic) cofibrations are defined as classes of maps satisfying some lifting *properties*, therefore when applying this to model Id-types, the elimination terms will be only required to exist, and there is no reason to expect that a given choice will satisfy the extra coherence properties needed.

A very neat way to fix this problem was proposed in [dBG12]. The solutions is to categorify the notion of lifting properties to that of lifting *structure*, and in this way, instead of a class of fibrations, we obtain a category of structured fibrations. We may apply this to the problem of finding pseudo-stable choices of Id-types, and in this case, the coherence problem of the elimination terms will be an instance of functoriality in this category.

In this chapter of the dissertation we will develop a framework for doing exactly this, making use the theory of algebraic weak factorisation systems (a review of which can be found in Appendix B, but see [BG16a, BG16b]) in order to construct models of Martin-Löf type theory.

In order to obtain a wide range of examples, we will work with the theory of Uniform Fibrations developed in [GS17] based on ideas from [CCHM16], a brief review is available in Appendix C.

4.1 From AWFS to Comprehension Categories

Recall from Section 2.7 that tribes naturally give rise to comprehension categories. Analogously any algebraic weak factorisation systems (or awfs for short) also has an associated comprehension category with the difference is that the comprehension functor will no longer be full. We will examine in this section how this is done.

We will begin this section with an observation. Consider an awfs (L, R) on a category $\mathbb{C}$; there are two categories of arrows that we might want to consider as generalising the class of fibrations in a Tribe. We may consider R-**Map** (i.e. the category of algebras for the pointed endofunctor R) or alternatively R-**Alg** (i.e. the category of algebras of R). We choose to work with R-**Map**; the main reason being that this category is better behaved with respect to lifting structures (see Proposition B.4.3). However, let us point out that, in the case of algebraically-free awfs, it is possible to use both categories of arrows, this follows because we have functors back and forth between them (see Proposition B.4.6).

Lemma 4.1.1. *Let* (L, R) *be an awfs over* $\mathbb{C}$*. The functor* R-***Map*** $\to \mathbb{C}$ *mapping an* R*-map* (f, s) *to* $\mathtt{cod}(f)$ *is a Grothendieck fibration. Moreover, the Cartesian arrows are the morphisms of* R*-maps whose underlying square is a pullback square.*

Proof. Let us first see that a morphism $(h, q) : (f', s') \to (f, s)$ of R-maps such that $\mathsf{U}(h, q)$ is a pullback square (i.e. forgetting the algebraic structure), is Cartesian. For this consider $(u, v) : (g, t) \to (f, s)$ any R-map morphism and an arrow $b : \mathtt{cod}(g) \to \mathtt{cod}(f')$ making the obvious diagram commute. Since $\mathsf{U}(h, q)$ is a pullback square, we obtain a unique arrow $a : \mathtt{dom}(g) \to \mathtt{dom}(f')$ as shown:

$$\begin{tikzcd}
\cdot \arrow[dr, dotted, "a"] \arrow[drr, "u"] \arrow[dd, "g"'] & & \\
& \cdot \arrow[r, "h"] \arrow[dd, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[dd, "f"] \\
\cdot \arrow[dr, "b"'] \arrow[drr, "v"] & & \\
& \cdot \arrow[r, "q"'] & \cdot
\end{tikzcd}$$

we need to show that the square $(a, b) : (g, t) \to (f', s')$ is a morhpism of R-maps. More precisely, we need to show that:

$$a \circ t = s' \circ P(a, b)$$

where P is the underlying functorial factorisation. But this follows by the universal property of the pullback square (h, q) using that the other two squares are morphisms of R-maps.

We now proceed to show that R-**Map** $\to \mathbb{C}$ is a Grothendieck fibration. For this let $q : A \to B$ be an arrow in $\mathbb{C}$ and $(f, s) : X \to B$ an R-map. Let $f' : \cdot \to A$ be the pullback of f along q, by Lemma B.2.4 there is a unique R-map structure on f' making the pullback square into a morphisms of R-maps, i.e. into a Cartesian arrow. ☐

With this lemma in place, it is now easy to verify the following proposition.

Proposition 4.1.2. *For a given awfs* (L, R) *on a category* $\mathbb{C}$ *the following commutative diagram is a comprehension category:*

$$\begin{tikzcd}
\mathsf{R}\text{-}\boldsymbol{Map} \arrow[rr, "\mathsf{U}"] \arrow[dr] & & \mathbb{C}^{\to} \arrow[dl, "\mathsf{cod}"] \\
& \mathbb{C} &
\end{tikzcd}$$

where the horizontal functor is the forgetful one.

Proof. By the previous lemma, the functor R-**Map** $\to \mathbb{C}$ is a Grothendieck fibration. Moreover, by the characterisation of the Cartesian arrows, we see that U maps Cartesian arrows to pullback squares. ☐

Remark 4.1.3. The comprehension category induced by an awfs is not in general split and also not in general full. This is a crucial difference with the comprehension category associated to a tribe.

We will now proceed to investigate additional structure on an awfs (L, R) such that the comprehension category given by Proposition 4.1.2 has pseudo-stable choices of the required logical structure.

The first thing we notice is that the category of R-**Map** has a canonical vertical composition (Proposition B.2.2). This implies that for each map $(f, s) : B \to A$ in R-**Map** there is a lift of the functor Σ_f as can be seen in the diagram:

$$\begin{tikzcd}
\mathsf{R}\text{-}\mathbf{Map}/B \arrow[r, "\Sigma_f"] \arrow[d] & \mathsf{R}\text{-}\mathbf{Map}/A \arrow[d] \\
\mathbb{C}/B \arrow[r, "\Sigma_f"'] & \mathbb{C}/A
\end{tikzcd}$$

here the slice category R-**Map**$/A$ (or analogously R-**Map**$/B$) is defined as the pullback of $\mathsf{U} : \mathsf{R}$-**Map** $\to \mathbb{C}^{\to}$ along the inclusion $\mathbb{C}/A \to \mathbb{C}^{\to}$; that is, objects are R-maps of the form $f : X \to A$ and arrows are morphisms of R-maps over the identity on A.

We can prove directly by inspection that the adjunction $\Sigma_f \vdash f^*$ also lifts to the categories of R-**Map**, i.e. as in the following diagram:

$$\begin{tikzcd}
\mathsf{R}\text{-}\mathbf{Map}/A \arrow[r, "f^*"'] \arrow[r, phantom, shift left=2, "\perp"] & \mathsf{R}\text{-}\mathbf{Map}/B \arrow[l, bend right, "\Sigma_f"']
\end{tikzcd}$$

Using this fact, we can follow the same ideas as we did in Section 2.7 (specifically in Lemma 2.7.7), in order to prove the following result:

Proposition 4.1.4. *Let* (L, R) *be an awfs. The comprehension category associated to* (L, R) *has a canonical pseudo-stable choice of* Σ*-types.*

4.2 Functorial Frobenius Structure

The case of Π-types is more complicated. First of all, throughout this section we will assume that the base category of an awfs (L, R) satisfies the (LF) condition (see Definition 2.5.9) with respect to the comprehension category induced by (L, R); this is needed in order for the pullback functor relative to a right map f, to posses a right adjoint, which we denote by Π_{f}. We have the following:

Proposition 4.2.1. *Let* (L, R) *an awfs satisfying the (LF) condition and equipped with a functorial Frobenius structure (Definition B.6.1). Then the comprehension category associated to* (L, R) *has a pseudo-stable choice of* Π*-types.*

Proof. First we construct a choice of Π-types $(\Pi, \lambda, \mathsf{app})$. Using Proposition B.6.4, we obtain a generalised Frobenius structure on $(\mathsf{L}\text{-}\mathbf{Map}, \mathsf{R}\text{-}\mathbf{Map},^{\boxslash} (\mathsf{L}\text{-}\mathbf{Map}^{\boxslash}))$ and by Proposition B.6.7 (using that $(-)^{\boxslash}$ commutes with slicing [GS17, Proposition 5.3] and that there are functors back-and-forth $\mathsf{R}\text{-}\mathbf{Map} \leftrightarrow \mathsf{L}\text{-}\mathbf{Map}^{\boxslash}$ by Proposition B.4.3), we have that for each $(\mathsf{f}, \mathsf{s}) : A \to \Gamma$, there is a lift of the pushforward functor as shown:

$$\mathsf{R}\text{-}\mathbf{Map}/A \xrightarrow{\Pi_{\mathsf{f}}} \mathsf{R}\text{-}\mathbf{Map}/\Gamma$$

we can use this to construct the choice $\Pi_{\mathsf{f}}\mathsf{g}$. To see that this choice is functorial, we proceed just as in the case of tribes (Lemma 2.7.8), and for this we also need the Beck-Chevalley isomorphism to lift to the category of R-**Map** which is also guaranteed by the functorial Frobenius structure (Proposition B.6.7). Finally, for the choices of λ and app, we proceed exactly as we did in Lemma 2.7.8. ☐

Remark 4.2.2. Even-though there are lifts of the pushforward functor Π_{f} to the category of R-**Map**, it is not in general the case that the adjunction $\mathsf{f}^* \vdash \Pi_{\mathsf{f}}$ lifts to R-**Map**. For this we need a further strengthening of the functorial Frobenius structure, as we show in the following proposition. We emphasis that this extra assumption on the functorial Frobenius is not necessary for the purpose of modelling dependent products.

Proposition 4.2.3. *Let* (L, R) *be an awfs satisfying the (LF) condition and equipped with a* strong *functorial Frobenius structure (Definition B.6.2). Then for every* $(\mathsf{f}, \mathsf{s}) : B \to A$ *in* R-***Map*** *there is a lift of the adjunction* $\mathsf{f}^* \vdash \Pi_{\mathsf{f}}$ *as shown:*

$$\begin{tikzcd}
\mathsf{R}\text{-}\boldsymbol{Map}/A \arrow[r, "\mathsf{f}^*"] \arrow[r, phantom, shift right=2, "\perp"] & \mathsf{R}\text{-}\boldsymbol{Map}/B \arrow[l, bend left, "\Pi_{\mathsf{f}}"]
\end{tikzcd}$$

Proof. The idea is to use Proposition B.4.9 to lift the unit and counit of the corresponding adjunction to the category of R-**Map**, to do this it will be necessary to slightly generalise the statement. Let's fix for the reminder of the proof an R-map $(f, s) : B \to A$.

Consider the category of arrows $u : \mathsf{L}\text{-}\mathbf{Map}/B \to (\mathbb{C}/B)^{\to}$ given on objects as follows:

$$(X \xrightarrow{g} B) \mapsto (X \xrightarrow{g} B \xrightarrow{id_B} B)$$

the action on morphisms is defined similarly. Notice that the arrow category $(\mathbb{C}/B)^{\to}$ has as objects composable pairs of arrows such that the second arrow has codomain B. In a completely similar manner, we define $\nu : \mathsf{R}\text{-}\mathbf{Map}/B \to (\mathbb{C}/B)^{\to}$.

We now fix some notation. We define the adjunctions $F_1 \vdash U_1$ and $F_2 \vdash U_2$ as given by the following functors:

$$F_1 = U_1 := id_{\mathbb{C}/B} : \mathbb{C}/B \to \mathbb{C}/B$$

and

$$F_2 := f^*\Sigma_f : \mathbb{C}/B \to \mathbb{C}/B \qquad U_2 := f^*\Pi_f : \mathbb{C}/B \to \mathbb{C}/B.$$

Next, we define the following natural transformations forming mates, respectively as the unit and the counit of the previous adjunctions, that is:

$$n := \epsilon : f^*\Pi_f \to id \qquad m := \eta : id \to f^*\Sigma_f$$

With this definitions in place, we may instantiating Proposition B.4.9. The conclusion tells us that we can find a lift the unit of $\Sigma_f \vdash f^*$ if and only if we can find a lift the counit of $f^* \vdash \Pi_f$ as shown:

$$\begin{tikzcd}
\mathsf{L}\text{-}\mathbf{Map}/B \arrow[r, bend left, "id"{name=U}] \arrow[r, bend right, "f^*\Sigma_f"'{name=D}] \arrow[from=U, to=D, phantom, "\Downarrow \eta"] & {}^{\boxslash}(\mathsf{R}\text{-}\mathbf{Map}/B)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{R}\text{-}\mathbf{Map}/B \arrow[r, bend left, "f^*\Pi_f"{name=U}] \arrow[r, bend right, "id"'{name=D}] \arrow[from=U, to=D, phantom, "\Downarrow \epsilon"] & (\mathsf{L}\text{-}\mathbf{Map}/B)^{\boxslash}
\end{tikzcd}$$

Given that the functor Σ_f is defined explicitly by composition with f, it will be easier to show that $\eta : id \to f^*\Sigma_f$ lifts, and this can be seen directly for each component.

For this consider $(g, \lambda) : X \to B$ a L-map and $(h, t) : Z \to B$ an R-map. Let us first notice that the component $\eta_g : u(g, \lambda) \to f^*\Sigma g$ is given as the left rectangle in the following diagram:

$$\begin{tikzcd}
X \arrow[r, dotted, "\eta_g"'] \arrow[d, "g"'] \arrow[rr, bend left, "id"] & X' \arrow[r, "f''"'] \arrow[d, "g'"'] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d, "g"] \\
B \arrow[r, "\delta_f"] \arrow[d, equal] & B' \arrow[r, "f'"] \arrow[d, "f'"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, "f"] \\
B \arrow[r, equal] & B \arrow[r, "f"'] & A
\end{tikzcd}$$

We will now describe what it mean for $f^*\Sigma_f g = \Sigma_{f'}g'$ to be an object of the category ${}^{\boxslash}(\mathsf{R}\text{-}\mathbf{Map}/B)$. Following our definitions, let's consider a morphism $(\alpha, f') : \Sigma_{f'}g' \to \nu(h,t)$ in $(\mathbb{C}/B)^{\to}$ with $(h,t) \in \mathsf{R}\text{-}\mathbf{Map}/B$, i.e. a diagram as the following one with solid arrows:

$$\begin{tikzcd}
X' \arrow[r, "\alpha"] \arrow[d, "g'"'] & Z \arrow[d, "h"] \\
B' \arrow[r, "f'"] \arrow[ur, dotted, "\theta"] \arrow[d, "f'"'] & B \arrow[d, equal] \\
B \arrow[r, equal] & B
\end{tikzcd}$$

then $\Sigma_{f'}g'$ is in ${}^{\boxslash}(\mathsf{R}\text{-}\mathbf{Map}/B)$ if there is a coherent choice of lifts θ as shown above. This coherent choice of functors clearly exists by the functorial Frobenius structure.

Now let us consider the diagram that results from pasting $(\eta_g, \mathsf{id}) : u(g,\lambda) \to \Sigma_{f'}g'$ to the left of the previous diagram while also inserting the defining pullback squares of f' and g':

$$\begin{tikzcd}
X \arrow[r, dotted, "\eta_g"'] \arrow[d, "g"'] & X' \arrow[r, "f''"'] \arrow[rr, bend left, "\alpha"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[r, "\alpha\eta_g"'] \arrow[d, "g"] & Z \arrow[d, "h"] \\
B \arrow[r, "\delta_f"] \arrow[d, equal] & B' \arrow[r, "f'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r, equal] \arrow[d, "f"] & B \arrow[d, equal] \\
B \arrow[r, equal] & B \arrow[r, "f"] \arrow[rr, bend right, "\mathsf{id}"'] & A & B
\end{tikzcd}$$

What we must show now is that the given lift $l : B \to Z$ of g against h is equal to the composite of the diagonal δ_f with the given lift $m : B' \to Z$ of g' against h.

Using the strong functorial Frobenius condition, we have that $g' \to g$ is a morphism of L-maps, this means that $l \circ f' = m$ and composing both sides with the diagonal we obtain that $l = m \circ \delta_f$, which is what we wanted.

By Proposition B.4.9 we obtain that $\epsilon : f^*\Pi_f \to \mathsf{id} : \mathsf{R}\text{-}\mathbf{Map}/B \to (\mathsf{L}\text{-}\mathbf{Map}/B)^{\boxslash}$ lifts. Finally using that $(-)^{\boxslash}$ commutes with slicing [GS17, Proposition 5.3] and that there are back-and-forth functors $\mathsf{R}\text{-}\mathbf{Map} \leftrightarrow \mathsf{L}\text{-}\mathbf{Map}^{\boxslash}$ by Proposition B.4.3, we obtain the following back-and-forth maps over the slice $\mathbb{C}/B$:

$$(\mathsf{L}\text{-}\mathbf{Map}/B)^{\boxslash} \cong \mathsf{L}\text{-}\mathbf{Map}^{\boxslash}/B \leftrightarrow \mathsf{R}\text{-}\mathbf{Map}/B$$

which implies that the counit of $f^* \vdash \Pi$ lifts to $\mathsf{R}\text{-}\mathbf{Map}/B$.

It remains to show that the unit of $f^* \vdash \Pi_f$ lifts, for this we instantiate Proposition B.4.9 in the dual manner, in order to obtain a lift of $\eta : \mathsf{id} \to \Pi_f f^*$ from a lift of $\epsilon : \Sigma_f f^* \to \mathsf{id}$ (and viceversa) as described in the following diagram:

$$\begin{tikzcd}
\mathsf{L}\text{-}\mathbf{Map}/A \arrow[r, bend left, "\Sigma_f f^*"{name=U}] \arrow[r, bend right, "\mathsf{id}"'{name=D}] \arrow[from=U, to=D, phantom, "\Downarrow \epsilon"] & {}^{\boxslash}(\mathsf{R}\text{-}\mathbf{Map}/A) & \mathsf{R}\text{-}\mathbf{Map}/A \arrow[r, bend left, "\mathsf{id}"{name=U2}] \arrow[r, bend right, "\Pi_f f^*"'{name=D2}] \arrow[from=U2, to=D2, phantom, "\Downarrow \eta"] & (\mathsf{L}\text{-}\mathbf{Map}/A)^{\boxslash}
\end{tikzcd}$$

As before, we prove that $\epsilon : \Sigma_f f^* \to \mathrm{id}$ lifts using the explicit description of Σ_f. First let's describe first what the component $\epsilon_g : \Sigma_f f^* g \to g$ looks like for some L-map $(g, \lambda) : X \to A$ as a morphism in $(\mathbb{C}/A)^{\to}$:

$$\begin{tikzcd}
X' \arrow[r, "\epsilon_g"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "g"] \\
B \arrow[r, "f"'] \arrow[d, "f"'] & A \arrow[d, equal] \\
A \arrow[r, equal] & A
\end{tikzcd}$$

That this is a map in ${}^{\boxslash}(\mathsf{R}\text{-}\mathbf{Map}/A)$ will again follow from the strong functorial Frobenius condition. We thus obtain a lift of $\eta : \mathrm{id} \to \Pi_f f^* : \mathsf{R}\text{-}\mathbf{Map}/A \to \mathsf{R}\text{-}\mathbf{Map}/A$ as desired. ☐

4.3 Stable Functorial Choice of Path Objects

In this section we will explore sufficient conditions on an awfs, in order, for the comprehension category associated to it, to posses a pseudo-stable choice of Id-types. The idea of using algebraic structure to construct models of Id-types was first introduced in [dBG12].

Definition 4.3.1. Let (L, R) be an awfs. A **stable functorial choice of path objects** (or **sfpo** for conciseness), consists of a functor

$$\mathcal{P} : \mathsf{R}\text{-}\mathbf{Map} \to \mathsf{L}\text{-}\mathbf{Map} \times_{\mathbb{C}} \mathsf{R}\text{-}\mathbf{Map}$$

that lifts a functorial and stable choice of factorisation of the diagonal morphism.

Let us explain the previous definition in detail. We require first a choice of path objects, that is, for every R-map $f : X \to Y$, a factorisation of the diagonal morphism as given in the following diagram:

$$X \xrightarrow{r_f} PX \xrightarrow{\rho_f} X \times_Y X$$

together with an L-map structure on r_f and an R-map structure on ρ_f. This choice is functorial if for any morphism of R-maps $(h, k) : (f', s') \to (f, s)$, there a diagram as shown:

$$\begin{tikzcd}
X' \arrow[r, "h"] \arrow[d, "r_{f'}"'] & X \arrow[d, "h"] \\
PX' \arrow[r, "P(h{,}k)"] \arrow[d, "\rho_{f'}"'] & PX \arrow[d, "\rho_f"] \\
X' \times_{Y'} X' \arrow[r, "h \times_k h"'] & X \times_Y X
\end{tikzcd}$$

such that the upper square is a morphism of L-maps and the lower square is a morphism of R-maps. Stability means that if (h, k) underlies a pullback square (i.e. if it is a Cartesian morphism of R-maps), then the bottom square of the previous diagram must be a pullback too.

We will denote an sfpo either with the notation $\mathcal{P}$ as in the definition or by its components $\langle \mathsf{r}, \rho \rangle$ where r is the first leg and ρ is the second leg of the factorisation.

Proposition 4.3.2. *Let* (L, R) *be an awfs equipped with a sfpo of the form* $\mathcal{P} = \langle \mathsf{r}, \rho \rangle$*. Then* (L, R) *is equipped with the structure of a pseudo-stable choice of* $\mathtt{Id}$*-types*

Proof. We need to construct a choice $(\mathtt{Id}, \mathsf{r}, \mathsf{j})$ of $\mathtt{Id}$-types. The choices for $\mathtt{Id}$ and r are canonically given by the stable functorial choice of path objects. It is straightforward to verify that these choices are pseudo-stable.

Since the maps r_f are equipped with an L-map structure, there are given lifts against R-maps and thus we get a choice of canonical elimination terms (i.e. j-terms). We are left to verify that this choice is coherent. For this, it is sufficient to show the following: given a Cartesian morphism of R-maps $(\mathsf{h}, \mathsf{k}) : \mathsf{f}' \to \mathsf{f}$, an R-map $\mathsf{q} : \mathsf{C} \to \mathsf{PX}$, and a commutative diagram:

$$\begin{tikzcd}
\mathsf{X} \arrow[r, "\mathsf{d}"] \arrow[d, "\mathsf{r}_\mathsf{f}"'] & \mathsf{C} \arrow[d, "\mathsf{q}"] \\
\mathsf{PX} \arrow[r, equal] & \mathsf{PX}
\end{tikzcd}$$

then, the following diagram commutes:

$$\begin{tikzcd}
\mathsf{C}^* \arrow[r, "\mathsf{P(h,k)}^*"] & \mathsf{C} \\
\mathsf{PX}' \arrow[u, "\mathsf{j(d^*)}"] \arrow[r, "\mathsf{P(h,k)}"'] & \mathsf{PX} \arrow[u, "\mathsf{j(d)}"']
\end{tikzcd}$$

where C^* is defined as a pullback of q along $\mathsf{P}(\mathsf{h}, \mathsf{k})$, the arrows denoted by j are the canonical choices of lifts. We notice that the arrow d^* is the pullback of the map d along $\mathsf{P}(\mathsf{h}, \mathsf{k})$, i.e. it is defined to be the unique arrow $\mathsf{d}^* : \mathsf{X}' \to \mathsf{C}^*$ such that:

$$\mathsf{q}^* \circ \mathsf{d}^* = \mathsf{r}_{\mathsf{f}'} \quad \text{and} \quad \mathsf{P}(\mathsf{h}, \mathsf{k})^* \circ \mathsf{d}^* = \mathsf{d} \circ \mathsf{h}. \tag{4.1}$$

We will split the problem into two parts. First consider the following canoncial lifts:

$$\begin{tikzcd}
\mathsf{X}' \arrow[r, "\mathsf{d}^*"] \arrow[d, "\mathsf{r}_{\mathsf{f}'}"'] & \mathsf{C}^* \arrow[r, "\mathsf{P(h,k)}^*"] \arrow[d, "\mathsf{q}^*"] & \mathsf{C} \arrow[d, "\mathsf{q}"] \\
\mathsf{PX}' \arrow[r, equal] \arrow[ur, dotted, "\mathsf{j(d^*)}"] \arrow[urr, dotted, "\mathsf{j}"] & \mathsf{PX} \arrow[r, "\mathsf{P(h,k)}"'] & \mathsf{PX}
\end{tikzcd}$$

note that by Proposition B.1.2 we obtain that $\mathsf{j} = \mathsf{P}(\mathsf{h}, \mathsf{k})^* \circ \mathsf{j}(\mathsf{d}^*)$.

Now for the second part, consider the following lifting problem

$$\begin{tikzcd}
X' \arrow[r, "h"] \arrow[d, "r_{f'}"'] & X \arrow[r, "d"] \arrow[d, "r_f"] & C \arrow[d, "q"] \\
PX' \arrow[r, "P(h{,}k)"'] \arrow[urr, dotted, "j'"] & PX \arrow[r, equal] \arrow[ur, dotted, "j(d)"] & PX
\end{tikzcd}$$

once more, by Proposition B.1.2 we obtain that $j' = j(d) \circ P(h, k)$. Finally we notice that Eq. (4.1) tells us that the outer squares of the two previous diagrams are the same, implying that they have the same lift $j = j'$; thus $P(h, k)^* \circ j(d^*) = j(d) \circ P(h, k)$ as needed. ☐

4.4 Type-Theoretic Algebraic Weak Factorisation Systems

In this section we summarise the results we had obtain on this chapter so far. We will do this by stating a general definition accompanied by a general theorem about awfs.

Definition 4.4.1. A **type-theoretic algebraic weak factorisation system** consists of the following data:

1. A category $\mathbb{C}$ equipped with an awfs (L, R) and satisfying the (LF) condition with respect to the comprehension category associated to (L, R) (see Definition 2.5.9).

2. A functorial Frobenius structure on (L, R).

3. A stable functorial choice of path objects for (L, R).

We will use the abbreviation of type-theoretic awfs for conciseness.

The proof of the following theorem follows immediately from the results of this chapter.

Theorem 4.4.2. *Let (L, R) be an awfs on $\mathbb{C}$ with the structure of an type-theoretic awfs. Then the comprehension category associated to (L, R) has pseudo-stable choices of Σ, Π and Id-types.*

We can apply the techniques from Chapter 2 as follows. Given an type-theoretic awfs, we can perform the right adjoint splitting to the comprehension category associated to it, and by Theorem 2.6.1, we obtain a split model of of Martin Löf type theory equipped with dependent sums and products, and intensional identity types. Yet the problem arrises as to how one obtains a type-theoretic awfs. It is to this problem that we turn our attention next.

4.5 Example: Groupoids

One of the first models of dependent type theory with intensional identity types was constructed by Hofmann and Streicher using groupoids as closed types [HS98]. In this paper, they constructed, from the category of groupoids, a category with families (CwF) which is a closely related structure to that of a split comprehension category; the advantage of doing things this way is that there is no need to develop a general framework for splitting as we did; i.e. they bypassed the need to apply any splitting procedure. However, the downside of their construction is that it is much more complex from the beginning and it obscures the important intuition of the interpretation of dependent type theory in groupoids where dependent types are modelled as isofibrations.

Here we will revisit their model using the theory we have exposed so far. We will construct an algebraic weak factorisation system $(\mathsf{C}_f, \mathsf{F})$ on the category $\underline{\mathrm{Grd}}$ of groupoids and functors. We will show explicitly how to construct a functorial Frobenious structure and a stable functorial choice of path objects.

We will denote by $\underline{\mathrm{Grd}}$ the category of groupoids and functors. Given a groupoid G, we refer to its objects and morphisms by *points* and *path* respectively, this is justified by thinking of a groupoid as an homotopy 1-type.

As first shown in [And78] the category $\underline{\mathrm{Grd}}$ has a Quillen model structure, known as the *canonical model structure*. Here we list some basic facts about this model structure for future reference:

- The fibrations are the isofibrations; these are the maps that have the right lifting property against the endpoint inclusion $\delta^0 : \mathbb{0} \to \mathbb{1}$ where $\mathbb{0}$ is the groupoid that has a single point and $\mathbb{1}$ has two distinct objects and a single path between them.
- The cofibration are the functors that are injective on objects.
- The weak equivalences are the categorical equivalences; these are fully-faithful and essentially surjective functors.

There are (at least) three different notions of structured isofibrations which arise as a natural categorification of the property of being an isofibration. To describe these, let us fix a functor of groupoids $f : G \to H$.

- A **cloven isofibration structure** on f consists of an operation θ that given any commutative square as the one below:

$$\begin{tikzcd} \mathbb{0} \arrow[r, "a"] \arrow[d, "\delta^0"'] & G \arrow[d, "f"] \\ \mathbb{1} \arrow[r, "p"'] \arrow[ur, dotted, "\theta(a,p)"] & H \end{tikzcd}$$

  produces a lift, shown as the dotted arrow. In other words, given as input an object $a \in G$ and a path $p : b \rightsquigarrow f(a)$ in H, then θ outputs a path $\theta(a, p) : b^* \rightsquigarrow a$ in H which lies over p. We refer to the operation θ as a **cleavage**.

- A **normal isofibration structure** of f consists of a cleavage θ which has the additional property that identities lift to identities. Precisely, this means that given any $a \in G$, then $\theta(a, \mathrm{id}_{f(a)}) = \mathrm{id}_a$. This is a compatibility condition between the constant paths and the lifting structure.

- A **split isofibration structure** of f consists of a normal cleavage θ which has the additional property of being compatible with composition. This means that given $a \in G$ and two paths $p : b \rightsquigarrow f(a)$ and $q : c \rightsquigarrow b$, then $\theta(p \cdot q, a) = \theta(p, a) \cdot \theta(q, b^*)$. Here, we demand the further compatibility of the lifting structure with the composition of paths operation.

Remark 4.5.1. Isofibrations are precisely Grothendieck fibrations between groupoids. The notions of cloven and split isofibrations also coincide with the analogous definitions for Grothendieck fibrations.

Notice that if a functor $f : G \to H$ has the structure θ of either a cloven, normal or split isofibration; then by forgetting the algebraic structure, the functor f retains the property of being a classical isofibration.

There is a natural notion of morphism between structured isofibrations. In order to describe this, let us consider (f, θ) and (f', θ') a pair of cloven isofibrations, and a morphism in the arrow category $(l, m) : f \to f'$ between the underlying maps, that is a commutative square as shown:

$$\begin{tikzcd}
G \arrow[r, "l"] \arrow[d, "f"'] & G' \arrow[d, "f'"] \\
H \arrow[r, "m"'] & H'
\end{tikzcd}$$

We say that (l, m) is **cleavage preserving** if for every $a \in G$ and $p : b \rightsquigarrow f(a)$ in H we have that $l \cdot \theta(a, p) = \theta'(l(a), m(p))$. Diagrammatically, this means that the triangle created by the respective lifts commute:

$$\begin{tikzcd}
\mathbb{0} \arrow[r, "a"] \arrow[d, "\delta^0"'] & G \arrow[r, "l"] \arrow[d, "f"] & G' \arrow[d, "f'"] \\
\mathbb{1} \arrow[r, "p"'] \arrow[ru, dotted, "\theta"] \arrow[rru, dotted, "\theta'"'] & H \arrow[r, "m"'] & H'
\end{tikzcd}$$

We will denote the categories of arrows (Definition B.4.1) of cloven, normal, and split isofibrations respectively as shown in the following diagram:

$$\begin{tikzcd}
\underline{\mathrm{ClFib}} \arrow[rd] & \underline{\mathrm{NrmFib}} \arrow[l, hook'] \arrow[d] & \underline{\mathrm{SpFib}} \arrow[l, hook'] \arrow[ld] \\
& \underline{\mathrm{Grd}}^{\to} &
\end{tikzcd}$$

We will now proceed to construct an awfs on the $\underline{\mathrm{Grd}}$ and an type-theoretic awfs structure on top of it. We first recall the following well-known construction. Let

$f : X \to Y$ be a functor between groupoids, the **comma category of** f denoted by $\downarrow \mathbf{f}$ has:

Objects: Triples (a, b, p) with $a \in X$, $b \in Y$ and $p : b \to fa$.

Arrows: $(\alpha, \beta) : (a, b, p) \to (a', b', p')$ where:

$$\begin{tikzcd}
b \arrow[r, "\beta"] \arrow[d, "p"'] & b' \arrow[d, "p'"] \\
fa \arrow[r, "F\alpha"'] & fa'
\end{tikzcd}$$

Identities and composition are component-wise those of G and H respectively.

Note that the the comma category $\downarrow \mathbf{f}$ is again a groupoid, and moreover the construction is functorial, thus giving rise to a functor $\downarrow (-) : \underline{\mathrm{Grd}}^{\to} \to \underline{\mathrm{Grd}}$. This will be the middle functor of a functorial factorisation assigning to $f : X \to Y$:

$$\begin{tikzcd}
X \arrow[r, "C_t f"] & \downarrow \mathbf{f} \arrow[r, "Ff"] & Y
\end{tikzcd}$$

where $C_t f(a) = (a, fa, id_{fa})$ on points and $C_t f(p) = (p, p)$ on paths. And Ff is the projection on the second coordinate, i.e. $Ff(a, b, p) = b$ and similarly on paths.

Let us examine the categories of C_t-maps and F-maps. Let's start with F-**Map**, we know that an F-map structure on a map $f : X \to Y$ is a lift s as shown:

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "C_t f"'] & X \arrow[d, "f"] \\
\downarrow \mathbf{f} \arrow[r, "Ff"'] \arrow[ur, dotted, "s"] & Y
\end{tikzcd}$$

a closer analysis will show that s gives $f : X \to Y$ precisely the structure of a normal isofibration, and that morphisms of F-maps correspond to cleavage preserving maps.

Let us now examine the category L-**Map**. An L-map structure on a map $g : A \to B$, is given by a lift λ as shown:

$$\begin{tikzcd}
A \arrow[r, "C_t g"] \arrow[d, "g"'] & \downarrow \mathbf{g} \arrow[d, "Fg"] \\
B \arrow[r, equal] \arrow[ur, dotted, "\lambda"] & B
\end{tikzcd}$$

If we examine the structure obtained from the lift λ, we observe that it corresponds to the data of a retraction $\lambda_1 : B \to A$ of g, and a natural transformation (homotopy) $\lambda_2 : id_B \to g \circ \lambda_1$ constant on the image of f. This data is given by $\lambda(b) = (\lambda_1(b), b, \lambda_2(b))$. In other words, a C_t-map $(g, \lambda_1, \lambda_2)$ is the same thing as a **strong deformation retraction**.

Proposition 4.5.2. *The functorial factorisation* $(\downarrow(-), \mathsf{C_t}, \mathsf{F})$ *is an algebraic weak factorisation system on* Grd.

Proof. We have to give the corresponding structures of a comonad and a monad to $\mathsf{C_t}$ and F respectively, we will only provide a brief description and leave the details to the reader.

We first define a comultiplication $\delta_f : \downarrow \mathbf{f} \to \downarrow \mathbf{C_t f}$ for $\mathsf{C_t}$ as follows:

$$(a, b, p) \mapsto (a, (a, b, p), (1_a, p) : (a, b, p) \to (a, Fa, 1_{fa}))$$

Similarly we have that the endofunctor F has a multiplication $\mu_f : \downarrow \mathbf{Ff} \to \downarrow \mathbf{f}$ given by:

$$((a, b, p), \tilde{b}, \tilde{p} : \tilde{b} \to b) \mapsto (a, \tilde{b}, p \circ \tilde{p})$$

□

Remark 4.5.3. Notice that in the definition of the multiplication $\mu_f : \downarrow \mathbf{Ff} \to \downarrow \mathbf{f}$ for F, the fact that paths can be composed is used. Moreover, the fact that the composition is strictly associativity and unital is crucial in proving the monad axioms.

A close analysis of the category of F-algebras, reveals that these are precisely the split isofibrations. In summary we have the following correspondence; the algebras for the pointed endofunctor (F, η) correspond to normal isofibrations and the algebras for the monad (F, η, μ) correspond to the split isofibrations.

We now proceed to show that the awfs $(\mathsf{C_t}, \mathsf{F})$ in Grd has a functorial Frobenius structure. This is done by elementary methods whose details we will omit.

Proposition 4.5.4. *The awfs* $(\mathsf{C_t}, \mathsf{F})$ *satisfies the strong functorial Frobenius condition.*

Proof. We need to show that pulling back a $\mathsf{C_t}$-map along an F-map is uniformly a $\mathsf{C_t}$-map. Consider $(g, \lambda) : A \to Y$ a $\mathsf{C_t}$-map and $(f, s) : X \to Y$ an F-map. Consider the following pullback square:

$$\begin{tikzcd}
A \times_Y X \arrow[r, "f'"] \arrow[d, "g'"'] & A \arrow[d, "g"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

we will define a $\mathsf{C_t}$-map structure λ' on g' which by the discussion above corresponds to a strong deformation retraction $(g', \lambda'_1, \lambda'_2)$. Using that the F-map structure s on f correspond to a normal isofibration, we can find for each point $x \in X$ a point $x' \in X$ and a lift $\lambda'_2(x)$ as shown:

$$\begin{tikzcd}
x \arrow[r, "\lambda'_2(x)"] \arrow[d, dash, dotted] & x' \arrow[d, dash, dotted] \\
fx \arrow[r, "\lambda_2(fx)"'] & g\lambda_1 fx
\end{tikzcd}$$

and we define $\lambda'_1(x) = (\lambda_1 fx, x')$. The homotopy $\lambda' : 1 \to g' \circ \lambda'_1$ is defined using the top arrow in the previous diagram.

Finally, we need to verify that $(f', f) : g' \to f$ is a morphism of C_t-maps (i.e. the strong Frobenius condition). By spelling out the details, we must show that the following diagram commutes:

$$\begin{tikzcd}
\downarrow \mathbf{g'} \arrow[r, "\downarrow(\mathbf{f'},\mathbf{f})"] & \downarrow \mathbf{g} \\
X \arrow[u, "\lambda'"] \arrow[r, "f"'] & Y \arrow[u, "\lambda"']
\end{tikzcd}$$

so let $x \in X$, and notice that:

$$\begin{aligned}
\downarrow (\mathbf{f'}, \mathbf{f})(\lambda'(x)) &= \downarrow (\mathbf{f'}, \mathbf{f})(\lambda'_1(x), x, \lambda'_2(x)) \\
&= \downarrow (\mathbf{f'}, \mathbf{f})((\lambda_1 fx, x'), x, \lambda'_2(x)) \\
&= (\lambda_1 fx, fx, f\lambda'_2(x)) \\
&= (\lambda_1 fx, fx, \lambda_2(fx)) \\
&= \lambda(fx)
\end{aligned}$$

□

We turn our attention to identity types. We start by noticing that the category Grd has an interval object $\mathbb{1}$ given by the groupoid with two points 0 and 1 and only one non-trivial path $0 \to 1$. The endpoint inclusions are the only two possible maps from the terminal groupoid into $\mathbb{1}$. It is straightforward to verify that interval path-object factorisation (Appendix C.3) of a map $f : X \to Y$ is given by:

$$\begin{tikzcd}
X \arrow[r, "r_f"] & P_w f \arrow[r, "\rho_f"] & X \times_Y X
\end{tikzcd}$$

where the points of $P_w f$ are:

$$P_w f = \{(a, a', p) | a, a' \in X, p : a \to a' \text{ such that } fa = fa' \text{ and } fp = \mathrm{id}_{fa}\}$$

an map in $P_w f$ from (a, a', p) to (b, b', q) is given by a pair (α, β) such that $\alpha : a \to b$, $\beta : a' \to b'$ and the following diagram commutes:

$$\begin{tikzcd}
a \arrow[r, "\alpha"] \arrow[d, "p"'] & b \arrow[d, "q"] \\
a' \arrow[r, "\beta"'] & b'
\end{tikzcd}$$

The map r_f is given by $a \mapsto (a, a, \mathrm{id}_a)$ and the map ρ_f is given by $(a, b, p) \mapsto (a, b)$.

Proposition 4.5.5. *In the category* Grd, *the interval path-object factorisation lifts to a stable functorial choice of path objects with respect to the awfs* (C_t, F).

Proof. Suppose we have an F-map $(f, s) : X \to Y$, we need to uniformly provide a $\mathsf{C_t}$-map structure to r_f and an F-map structure to ρ_f.

A $\mathsf{C_t}$-map structure on r_f is the same thing as the data for a strong deformation retract, take $\lambda_1 := t_f : P_w f \to X$ the canonical target map. We define the natural transformation $\lambda_2 : \mathrm{id}_{P_w f} \to r_f \circ t_f$ as:

$$\lambda_2(a, a', p) := (p, \mathrm{id}_{a'}) : (a, a', p) \to (a', a', \mathrm{id}'_a)$$

Now, a F-map structure on ρ_f corresponds to the data of a split isofibration. Consider a lifting situation:

$$\begin{tikzcd}
 & (a, a', p) \arrow[d, dotted, no head] & & P_w f \arrow[d, "\rho_f"] \\
(b, b') \arrow[r, "{(\alpha, \beta)}"'] & (a, a') & & X \times_Y X
\end{tikzcd}$$

we let $q := \beta \circ p \circ \alpha^{-1} : b \to b'$ and thus $(\alpha, \beta) : (b, b', q) \to (a, a', p)$ is the desired lift. □

Using the two previous propositions we see that the awfs $(\mathsf{C_t}, \mathsf{F})$ on the category Grd is equipped with a functorial Frobenius structure and with a stable functorial choice of path objects. From this, the following theorem follows immediately.

Theorem 4.5.6. *The category Grd is equipped with the structure of a type-theoretic awfs (Definition 4.4.1).* □

Applying Theorem 4.4.2 we obtain a model of dependent type theory with Π, Σ and Id-types. This is essentially the same model as the Hofmann-Streicher one.

Chapter 5

Type-Theoretic AWFS from Uniform Fibrations

In this chapter we will investigate how to obtain type-theoretic awfs in the setting of uniform fibrations of [GS17].

We will work with an awfs $(\mathsf{C}_t, \mathsf{F})$ of uniform fibrations (Appendix C.4) where the base category $\mathbb{C}$ is closed symmetric monoidal and is equipped with an interval object $(\mathrm{I}, \delta^0, \delta^1)$ with contractions and connections.

One of the main theorem of [GS17] is that the awfs of uniform fibrations $(\mathsf{C}_t, \mathsf{F})$ has a functorial Frobenius structure (Theorem C.4.3). In this chapter we will show how to obtain an type-theoretic awfs by constructing a stable functorial choice of path objects.

Given an interval object, there is a natural choice of path objects, the interval path-object factorisation $\mathcal{P}_\mathrm{I}$ as explained in Appendix C.3. Let us briefly recall the construction here for the convenience of the reader. For a morphism $f : B \to A$, consider the following factorisation of the diagonal morphism $\delta_f : X \to X \times_Y X$:

$$B \xrightarrow{r_f} P_w f \xrightarrow{\rho_f} B \times_A B$$

where the morphism $r_f : B \to P_w f$ is given by the universal property of pullback squares as shown in the following diagram:

$$\begin{tikzcd}
B \arrow[drr, bend left, "B^{\epsilon}"] \arrow[dr, dotted, "r_p"] \arrow[ddr, bend right, "f"'] & & \\
& P_w f \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B^{\mathrm{I}} \arrow[d, "f^{\mathrm{I}}"] \\
& A \arrow[r, "A^{\epsilon}"'] & A^{\mathrm{I}}
\end{tikzcd} \tag{5.1}$$

and $\rho_f : P_w f \to B \times_A B$ is given again by the universal property of the pullback of f along itself applied to the canonical source and target maps $s_f, t_f : P_w f \to B$.

We will give an alternative construction of this factorisation making evident some intermediate steps and using the adjunction $-\hat{\otimes} i \vdash \widehat{\mathrm{hom}}(i, -)$ given by the Leibniz construction (or pushout-product) explicitly (see Appendix C.1). First of all, we define an arrow $i : \partial I \to I$ corresponding to the boundary inclusion of the interval object, this is given by taking the coproduct $\partial I := \bot + \bot$ and using the universal property:

$$\begin{tikzcd}
\bot \arrow[r, "\iota_0"] \arrow[dr, "\delta^0"'] & \partial I \arrow[d, "i"'] & \bot \arrow[l, "\iota_1"'] \arrow[dl, "\delta^1"] \\
& I &
\end{tikzcd}$$

Consider the following diagram that expands the above diagram Eq. (5.1), i.e. the exterior part of the following diagram is exactly the foregoing one.

$$\begin{tikzcd}
B \arrow[dr, "r_f"] \arrow[rr, bend left, "B^{\epsilon}"] \arrow[ddr, "\Delta_f"'] \arrow[dddr, "\mathrm{id}_B"'] \arrow[ddddr, bend right, "f"'] & & \\
& P_w f \arrow[r] \arrow[d, "\rho_f"] \arrow[dr, phantom, very near start, "\lrcorner"] & B^I \arrow[d, "\widehat{\mathrm{hom}}(i,f)"'] \arrow[dd, bend left, "\widehat{\mathrm{hom}}(\delta^1,f)"] \arrow[ddd, bend left=80, "f^I"] \\
& B \times_A B \arrow[r, "\langle \alpha_f, \lambda_f \rangle"] \arrow[d, "\pi_2"] \arrow[dr, phantom, very near start, "\lrcorner"] & A^I \times_{A^{\partial I}} B^{\partial I} \arrow[d, "\mathrm{id} \times_{A^{\iota_1}} B^{\iota_1}"'] \\
& B \arrow[r, "\langle \beta_f, \mathrm{id}_B \rangle"] \arrow[d, "f"] \arrow[dr, phantom, very near start, "\lrcorner"] & A^I \times_A B \arrow[d] \\
& A \arrow[r, "A^{\epsilon}"'] & A^I
\end{tikzcd} \tag{5.2}$$

This diagram will be used a couple of times in the next section.

The two middle horizontal arrows are defined as follows. First, the arrow $\langle \alpha_f, \lambda_f \rangle$ is given by the universal property of pullbacks using two intermediate arrows λ_f and α_f defined as follows:

$$\begin{tikzcd}
B \times_A B \arrow[r, "\langle \pi_1, \pi_2 \rangle"] \arrow[dr, "\lambda_f"'] & B \times B \arrow[d, "\cong"] \\
& B^{\partial I}
\end{tikzcd}
\qquad
\begin{tikzcd}
B \times_A B \arrow[r, "\pi_1"] \arrow[drr, "\alpha_f"'] & B \arrow[r, "f"] & A \arrow[d, "A^{\epsilon}"] \\
& & A^I
\end{tikzcd}$$

then we let $\langle \alpha_f, \lambda_f \rangle : B \times_A B \to A^I \times_{A^{\partial I}} B^{\partial I}$. For the second horizontal arrow $\langle \beta_f, \mathrm{id}_B \rangle$, we define β_f as follows:

$$\begin{tikzcd}
B \arrow[r, "f"] \arrow[dr, "\beta_f"'] & A \arrow[d, "A^{\epsilon}"] \\
& A^I
\end{tikzcd}$$

and we let $\langle \beta_f, \mathrm{id}_B \rangle : B \to A^I \times_A B$.

5.1 Id-types in Uniform Fibrations

With the help of the machinery of uniform fibrations, we are now able to state and prove the following theorem.

Theorem 5.1.1. *Let* $(\mathsf{C}, \mathsf{F}_t)$ *be a suitable awfs (Definition C.4.1) on* $\mathbb{C}$. *Suppose that the following additional hypothesis hold:*

1. *Taking the Leibniz product to the boundary inclusion of the interval* $i : \partial I \to I$ *uniformly preserves* C*-maps as shown:*

$$\begin{tikzcd}
\mathsf{C}\text{-}\boldsymbol{Map} \arrow[r, "i\hat{\otimes}-"] \arrow[d] & \mathsf{C}\text{-}\boldsymbol{Map} \arrow[d] \\
\mathbb{C}^{\to} \arrow[r, "i\hat{\otimes}-"'] & \mathbb{C}^{\to}
\end{tikzcd}$$

2. *The reflexivity* $r : \mathbb{C}^{\to} \to \mathbb{C}^{\to}$ *functor, of the interval path-object factorisation* $\mathcal{P}_I$*, uniformly lifts to the category of* C*-maps:*

$$\begin{tikzcd}
 & \mathsf{C}\text{-}\boldsymbol{Map} \arrow[d] \\
\mathbb{C}^{\to} \arrow[ur, "r"] \arrow[r, "r"'] & \mathbb{C}^{\to}
\end{tikzcd}$$

Then the factorisation $\mathcal{P}_I$ *from Appendix C.3 lifts to a stable functorial choice of path objects, as shown:*

$$\begin{tikzcd}
\mathsf{F}\text{-}\boldsymbol{Map} \arrow[r, dotted, "\tilde{\mathcal{P}}_I"] \arrow[d] & \mathsf{C}_t\text{-}\boldsymbol{Map} \times_{\mathbb{C}} \mathsf{F}\text{-}\boldsymbol{Map} \arrow[d] \\
\mathbb{C}^{\to} \arrow[r, "\mathcal{P}_I"'] & \mathbb{C}^{\to} \times_{\mathbb{C}} \mathbb{C}^{\to}
\end{tikzcd}$$

Proof. We will divide the proof into two parts.

Claim 5.1.1.1. *The functor* $\rho : \mathbb{C}^{\to} \to \mathbb{C}^{\to}$ *lifts to a functor* $\rho : \mathsf{F}\text{-}\boldsymbol{Map} \to \mathsf{F}\text{-}\boldsymbol{Map}$.

Proof of Claim 5.1.1.1. Since $(\mathsf{C}, \mathsf{F}_t)$ is suitable, we have that $\delta^k \hat{\otimes} -$ lifts to **C-Map** and by Lemma C.5.4 we have that $\delta^k \hat{\otimes} -$ also factors though the category S_k of k-oriented homotopy equivalences. Combining this two facts, we obtain a lift of $\delta^k \hat{\otimes} -$ as shown:

$$\begin{tikzcd}
\mathbf{C\text{-}Map} \arrow[r, "\delta^k\hat{\otimes}-"] \arrow[d] & \mathbf{C\text{-}Map} \times_{\mathbb{C}^{\to}} \mathsf{S}_k \arrow[d] \\
\mathbb{C}^{\to} \arrow[r, "\delta^k\hat{\otimes}-"'] & \mathbb{C}^{\to}
\end{tikzcd}$$

By Lemma C.5.4 again, we have a functor $C\text{-}\mathbf{Map} \times_{\mathbb{C}^{\to}} S_k \to C_t\text{-}\mathbf{Map}$ over $\mathbb{C}^{\to}$, composing with the one above, we get a lift of $\delta^k \hat{\otimes} -$ as shown:

$$\begin{tikzcd}
C\text{-}\mathbf{Map} \arrow[r, "\delta^k \hat{\otimes} -"] \arrow[d] & C_t\text{-}\mathbf{Map} \arrow[d] \\
\mathbb{C}^{\to} \arrow[r, "\delta^k \hat{\otimes} -"'] & \mathbb{C}^{\to}
\end{tikzcd}$$

Using the hypothesis that $i \hat{\otimes} -$ lifts to $C\text{-}\mathbf{Map}$ and that (C, F_t) is algebraically-free on a category of arrows $u : \mathcal{J} \to \mathbb{C}^{\to}$ we obtain:

$$\begin{tikzcd}
\mathcal{J} \arrow[r, "\eta"] \arrow[dr, "u"'] & C\text{-}\mathbf{Map} \arrow[r, "i \hat{\otimes} -"] \arrow[d] & C\text{-}\mathbf{Map} \arrow[r, "\delta^k \hat{\otimes} -"] \arrow[d] & C_t\text{-}\mathbf{Map} \arrow[d] \\
& \mathbb{C}^{\to} \arrow[r, "i \hat{\otimes} -"'] & \mathbb{C}^{\to} \arrow[r, "\delta^k \hat{\otimes} -"'] & \mathbb{C}^{\to}
\end{tikzcd}$$

Since the monoidal structure on $\mathbb{C}$ is symmetric, the lifted monoidal structure on the category of arrows by the Leibniz construction is also symmetric, this means in particular that following diagram commutes up-to-iso:

$$\begin{tikzcd}
\mathbb{C}^{\to} \arrow[r, "i \hat{\otimes} -"] \arrow[d, "\delta^k \hat{\otimes} -"'] \arrow[dr, phantom, "\cong"] & \mathbb{C}^{\to} \arrow[d, "\delta^k \hat{\otimes} -"] \\
\mathbb{C}^{\to} \arrow[r, "i \hat{\otimes} -"'] & \mathbb{C}^{\to}
\end{tikzcd}$$

It is easy to see that we can transfer the algebraic structure along this natural isomorphism, this means that we obtain from the last two diagrams, the following lift:

$$\begin{tikzcd}
\mathcal{J} \arrow[rr] \arrow[d, "u"'] \arrow[dr, "\delta^k \hat{\otimes} u"] & & C_t\text{-}\mathbf{Map} \arrow[d] \\
\mathbb{C}^{\to} \arrow[r, "\delta_k \hat{\otimes} -"'] & \mathbb{C}^{\to} \arrow[r, "i \hat{\otimes} -"'] & \mathbb{C}^{\to}
\end{tikzcd}$$

now combining the above lifts for $k = 0, 1$, and using the definition of $u_{\otimes} : \mathcal{J}_{\otimes} \to \mathbb{C}^{\to}$ we obtain a lift of $i \hat{\otimes} -$ as shown:

$$\begin{tikzcd}
\mathcal{J}_{\otimes} \arrow[r, "i \hat{\otimes} -"] \arrow[d, "u_{\otimes}"'] & C_t\text{-}\mathbf{Map} \arrow[d] \\
\mathbb{C}^{\to} \arrow[r, "i \hat{\otimes} -"'] & \mathbb{C}^{\to}
\end{tikzcd}$$

For the following we will require some results of the previous chapter. First recall that by the Leibniz construction, there is an adjunction adjunction between $i \hat{\otimes} -$ and

$\widehat{\hom}(i,-)$ in the category of arrows. Now, using the nice interplay between adjunctions and lifting structure (Proposition C.1.6) and noticing that C_t-**Map** $\cong^{\boxslash}$ F-**Alg** by Proposition B.4.3, we obtain the following lift of $\widehat{\hom}(i,-)$:

$$\begin{tikzcd}
F\text{-}\mathbf{Alg} \arrow[r, "\widehat{\hom}(i,-)"] \arrow[d] & (\mathcal{J}_{\otimes})^{\boxslash} \arrow[d, "(u_{\otimes})^{\boxslash}"] \\
\mathbb{C}^{\to} \arrow[r, "\widehat{\hom}(i,-)"'] & \mathbb{C}^{\to}
\end{tikzcd}$$

Finally we use that (C_t, F) is algebraically-free on the category of arrows $u_{\otimes} : \mathcal{J}_{\otimes} \to \mathbb{C}^{\to}$ we obtain an equivalence over $\mathbb{C}^{\to}$ between $(\mathcal{J}_{\otimes})^{\boxslash}$ and F-**Alg** thus composing with what we had before, we obtain a lift of $\widehat{\hom}(i,-)$ to the category of F-algebras:

$$\begin{tikzcd}
F\text{-}\mathbf{Alg} \arrow[r, "\widehat{\hom}(i,-)"] \arrow[d] & (I_{\otimes})^{\boxslash} \arrow[r, "\cong"] \arrow[d, "(u_{\otimes})^{\boxslash}"'] & F\text{-}\mathbf{Alg} \arrow[dl] \\
\mathbb{C}^{\to} \arrow[r, "\widehat{\hom}(i,-)"'] & \mathbb{C}^{\to} &
\end{tikzcd}$$

If we look at the top pullback square of Eq. (5.2) we see that the morphism $\rho_f : P_w B \to B \times_A B$ is obtained by the following two steps:

$$f \mapsto \widehat{\hom}(i,f) \mapsto \langle \alpha_f, \lambda_f \rangle^* \widehat{\hom}(i,f) = \rho_f$$

i.e. by first applying $\widehat{\hom}(i,-)$ and then pulling back along $\langle \alpha_f, \lambda_f \rangle$. Thus since we have lifts of $\widehat{\hom}$ and of the pullback functor to the category of F-algebras, we obtain a lift of ρ as shown:

$$\begin{tikzcd}
F\text{-}\mathbf{Alg} \arrow[r, "\widehat{\hom}(i,-)"'] \arrow[rr, bend left, dotted, "\rho"] \arrow[d] & F\text{-}\mathbf{Alg} \arrow[r, "\mathsf{PB}(-,\langle\alpha,\lambda\rangle)"'] & F\text{-}\mathbf{Alg} \arrow[d] \\
\mathbb{C}^{\to} \arrow[rr, "\rho"'] & & \mathbb{C}^{\to}
\end{tikzcd}$$

Now, since we are working in an algebraically-free awfs, we have lifts back-and-forth between R-**Alg** and R-**Map** over $\mathbb{C}^{\to}$ (Proposition B.4.6). Composing with this lifts we obtain the desired lift of ρ. ☐

Claim 5.1.1.2. *The functor* $r : \mathbb{C}^{\to} \to \mathbb{C}^{\to}$ *lifts to a functor* $r : F$-***Map*** $\to C_t$-***Map***.

Proof of Claim 5.1.1.2. We will first show that $r : \mathbb{C}^{\to} \to \mathbb{C}^{\to}$ lifts to a functor r : F-**Map** $\to S_0$ where S_0 is the category of 0-oriented strong left homotopy equivalence (see Definition C.5.1).

For this we will make use of the fact that the functor that maps f to the target map t_f lifts to a functor from F-**Map** to F_t-**Map**, the proof of this will require us to recreate some arguments from Claim 5.1.1.1. Using that we have a lift $\delta^1\hat{\otimes}- : \mathsf{C}\text{-}\mathbf{Map} \to \mathsf{C}_t\text{-}\mathbf{Map}$ as shown in the previous claim, we can transpose using Proposition C.1.6 to obtain a lift:

$$\begin{tikzcd}
\mathsf{F}\text{-}\mathbf{Alg} \arrow[r, "\widehat{\hom}(\delta^1{,}-)"] \arrow[d] & \mathsf{F}_t\text{-}\mathbf{Alg} \arrow[d] \\
\mathbb{C}^{\to} \arrow[r, "\widehat{\hom}(\delta^1{,}-)"'] & \mathbb{C}^{\to}
\end{tikzcd}$$

Looking at Eq. (5.2) we see that $t_f : P_w f \to B$ is obtained by applying $\widehat{\hom}(\delta^1, -)$ to f and then pulling back along $\langle \beta_f, id_B \rangle$, thus the functor mapping $f \mapsto t_f$ lifts as shown:

$$\begin{tikzcd}
\mathsf{F}\text{-}\mathbf{Alg} \arrow[r, "\widehat{\hom}(\delta^1{,}-)"'] \arrow[d] \arrow[rr, bend left, dotted, "t_{(-)}"] & \mathsf{F}_t\text{-}\mathbf{Alg} \arrow[r, "\mathsf{PB}(-{,}\langle\beta{,}id\rangle)"'] & \mathsf{F}_t\text{-}\mathbf{Alg} \arrow[d] \\
\mathbb{C}^{\to} \arrow[rr, "t_{(-)}"'] & & \mathbb{C}^{\to}
\end{tikzcd} \tag{5.3}$$

since the awfs are algebraically-free we can apply Proposition B.4.6 to obtain the desired lift.

Now let's return to our task of finding a lift of the functor r as shown in the following diagram:

$$\begin{tikzcd}
\mathsf{F}\text{-}\mathbf{Map} \arrow[r, "r"] \arrow[d] & S_0 \arrow[d] \\
\mathbb{C}^{\to} \arrow[r, "r"'] & \mathbb{C}^{\to}
\end{tikzcd}$$

for this, we will first show that for each uniform fibration $f : B \to A$ the target map $t_f : P_w f \to B$ is an strong homotopy retraction of $r_f : B \to P_w f$.

Looking again at Eq. (5.2) it is clear that $t_f \circ r_f = id_B$. Thus we are left with the task of constructing an homotopy $H : r_f \circ t_f \sim id_{P_w f}$, for this consider the following diagram:

$$\begin{tikzcd}
P_w f \arrow[r, "\langle r_f \circ t_f{,} id_{P_w f}\rangle"] \arrow[d, "B^{\epsilon} \circ t_f"'] & P_w f^{\partial I} \arrow[d, "t_f^{\partial I}"] \\
B^I \arrow[r, "B^i"'] & B^{\partial I}
\end{tikzcd}$$

where the top horizontal arrow is given by the universal product of the product $P_w f^{\partial I} \cong P_w f \times P_w f$. This diagram commutes, as shown in the following calculation:

$$\begin{aligned}
t_f^{\partial I} \circ \langle r_f \circ t_f, id \rangle = t \times t \circ \langle r_f \circ t_f, id \rangle &= \langle t_f \circ r_f \circ t_f, t \rangle \\
= \langle t, t \rangle = \Delta_f \circ t &= B^i \circ B^{\epsilon} \circ t
\end{aligned}$$

This gives us (by universal property) an arrow into the pullback:

$$\tilde{H} : P_w f \to B^I \times_{B^{\partial I}} P_w f^{\partial I}.$$

Now, we already have a lift of the target map $t_{(-)} : F\textbf{-Map} \to F_t\textbf{-Map}$. And notice that by hypothesis, $i \hat{\otimes} -$ lifts to $C\textbf{-Map}$, then using Proposition C.1.6 we find a lift of $\widehat{\hom}(i, -)$ to $F_t\textbf{-Map}$. Composing this two lifts, we find that $\widehat{\hom}(i, t_{(-)})$ lifts to a functor:

$$\begin{tikzcd}
F\textbf{-Map} \arrow[r, dotted, "\widehat{\hom}(i{,}t_{(-)})"] \arrow[d] & F_t\textbf{-Map} \arrow[d] \\
\mathbb{C}^{\to} \arrow[r, "\widehat{\hom}(i{,}t_{(-)})"'] & \mathbb{C}^{\to}
\end{tikzcd}$$

let's apply this to f to obtain a uniform trivial fibration $\widehat{\hom}(i, t_f)$.

We now have that, since (C, F_t) is suitable, every object is uniformly cofibrant, thus we obtain the desired morphism H as a lift in the following diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & P_w f^I \arrow[d, "\widehat{\hom}(i{,}t_f)"] \\
P_w f \arrow[r, "\tilde{H}"'] \arrow[ur, dotted, "H"] & B^I \times_{B^{\partial I}} P_w f^{\partial I}
\end{tikzcd}$$

In order to verify that this H is actually an homotopy from $r_f \circ t_f$ to $id_{P_w B}$, consider the following diagram:

$$\begin{tikzcd}
 & & P_w f \arrow[dll, bend right, "r_f \circ t_f"'] \arrow[dl, "\langle r_f \circ t_f{,}id \rangle"'] \arrow[d, "H"] \arrow[dr, "\langle r_f \circ t_f{,}id \rangle"] \arrow[drr, bend left, "id_{P_w f}"] & & \\
P_w f & P_w f^{\partial I} \arrow[l, "P_w f^{\iota_0}"'] & P_w f^I \arrow[l, "P_w f^{i}"'] \arrow[r, "P_w f^{i}"] \arrow[ll, bend left, "P_w f^{\delta^0}"] \arrow[rr, bend right, "P_w f^{\delta^1}"'] & P_w f^{\partial I} \arrow[r, "P_w f^{\iota_1}"] & P_w f
\end{tikzcd}$$

This shows that t_f is a deformation retract of r_f, but every deformation retraction is in particular an homotopy equivalence, in this case the object of S_0 that gives the strong homotopy equivalence is the tuple $(r_f, t_f, B^\epsilon, H)$ (it is straightforward to verify that this homotopy equivalence is strong).

So far we have given the action on objects of the desired lift $r : F\textbf{-Map} \to S_0$. Now we have to show that this construction is functorial on f. For this, consider a square $(h, k) : f' \to f$ in $F\textbf{-Map}$, using the fact that the interval path-object factorisation is functorial, we obtain the following:

$$\begin{tikzcd}
B' \arrow[r, "h"] \arrow[d, "r_{f'}"'] & B \arrow[d, "r_f"] \\
P_w f' \arrow[r, "P_w(h,k)"] \arrow[d, "t_{f'}"'] & P_w f \arrow[d, "t_f"] \\
B' \arrow[r, "h"'] & B
\end{tikzcd}$$

Note that the bottom square is a morphism of F_t-**Map** since it is the result of applying the lift of $t_{(-)}$ of Eq. (5.3) to the square (h, k).

Let us prove that $(h, P_w(h, k)) : (r_{f'}, t_{f'}, B'^{\epsilon}, \tilde{H}') \to (r_f, t_f, B^{\epsilon}, \tilde{H})$ is a morphism of strong 0-oriented homotopy equivalences. Looking at the Definition C.5.1, we observe that the only thing we need to prove is that the following diagram commutes:

$$\begin{tikzcd}
P_w f' \arrow[r, "P_w(h,k)"] \arrow[d, "H'"'] & P_w f \arrow[d, "H"] \\
P_w f'^{I} \arrow[r, "P_w(h,k)^{I}"'] & P_w f^{I}
\end{tikzcd}$$

We make use of the naturality of the filling operations, consider the following two diagrams:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & P_w f^{I} \arrow[d, "\widehat{\hom}(i,t_f)"] \\
P_w f' \arrow[r, "P_w(h,k)"'] \arrow[urr, dotted, "L" description] & P_w f \arrow[r, "\tilde{H}"'] \arrow[ur, dotted, "H" description] & f^{I} \times_{B^{\partial I}} P_w f^{I}
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & P_w f'^{I} \arrow[r, "P_w(h,k)^{I}"] \arrow[d, "\widehat{\hom}(i,t_{f'})" description] & P_w f^{I} \arrow[d, "\widehat{\hom}(i,t_f)"] \\
P_w f' \arrow[r, "\tilde{H}'"'] \arrow[ur, dotted, "H'" description] \arrow[urr, dotted, "L" description] & f'^{I} \times_{B'^{\partial I}} P_w f'^{I} \arrow[r, "h^{I} \times_{h^{\partial I}} P_w(h,k)^{\partial I}"'] & f^{I} \times_{B^{\partial I}} P_w f^{I}
\end{tikzcd}$$

The left square of the top diagram is a morphism of C-maps by the requirement that every object is uniformly cofibrant. The right square of the bottom diagram is a morphism of F_t-maps since it is the result of applying the lift $\widehat{\hom}(i, t_{(-)}) : F$-**Map** $\to$ F_t-**Map** to (h, k) that is a morphism of F-maps. Thus the corresponding lifts cohere.

Finally, since the construction of the maps $\tilde{H}$ and $\tilde{H}'$ is functorial (given by a universal property), we have that the following diagram commutes:

$$\begin{tikzcd}
P_w f' \arrow[r, "P_w(h,k)"] \arrow[d, "\tilde{H}'"'] & P_w f \arrow[d, "\tilde{H}"] \\
B'^I \times_{B'^{\partial I}} P_w f'^I \arrow[r, "h^I \times_{h^{\partial I}} P_w(h,k)^{\partial I}"'] & B^I \times_{B^{\partial I}} P_w f^I
\end{tikzcd}$$

this means that that the composition of the bottom horizontal arrows in the previous two lifting diagrams coincide, this makes the lift L in both diagrams the same morphism, and thus we have:

$$H \circ P_w(h,k) = L = P_w(h,k)^I \circ H'$$

as required.

Given that we have a lift $r : F\text{-}\mathbf{Map} \to S_0$ and that by hypothesis we also have a lift $r : \mathbb{C}^{\to} \to C\text{-}\mathbf{Map}$; we can combining these two lifts and applying Lemma C.5.4 in order to obtain a lift of r as shown:

$$\begin{tikzcd}
F\text{-}\mathbf{Map} \arrow[r] \arrow[d] & C\text{-}\mathbf{Map} \times_{\mathbb{C}^{\to}} S_0 \arrow[r] \arrow[d] & C_t\text{-}\mathbf{Map} \arrow[dl] \\
\mathbb{C}^{\to} \arrow[r, "r"'] & \mathbb{C}^{\to} &
\end{tikzcd}$$

□

Putting together Claim 5.1.1.1 and Claim 5.1.1.2 we obtain a lift of the interval path-object factorisation $\mathcal{P}_I$ to a stable functorial choice of path objects. □

5.2 Type-Theoretic AWFS in Toposes

In this section, we will show that there are a large number of examples where the hypothesis of Theorem 5.1.1 hold. We will fix a category $\mathbb{E}$, and we will make the following two assumptions on it:

1. $\mathbb{E}$ is a Grothendieck topos where we identify the monoidal structure $\otimes$ with the canonical Cartesian one.

2. $\mathbb{E}$ is equipped with an interval object with connections (Appendix C.2). Notice that because the unit object is the terminal one, the interval will trivially have contractions.

we will show that, under these assumptions, it is possible to equip $\mathbb{E}$ with a type-theoretic awfs of uniform fibrations. For this, we will follow [GS17, Section 9].

First of all, let us denote by $\mathcal{M}_{\mathfrak{all}}$ the subcategory of $\mathbb{E}^{\to}$ whose objects are monomorphisms and whose arrows are Cartesian squares. We can apply Theorem C.6.3 in order to obtain a suitable awfs (C, F_t) algebraically-free on $\mathcal{M}_{\mathfrak{all}}$.

We will proceed to show that the hypothesis of Theorem 5.1.1 are satisfied for $(\mathsf{C}, \mathsf{F}_t)$. As a first step, we prove the following statement.

Proposition 5.2.1. *The objects of* $\mathcal{M}_{\mathrm{all}}$ *are closed under taking Leibniz product with the boundary inclusion* $i : \partial I \to I$.

Proof. It follows since $i : \partial I \to I$ is a monomorphism and pushout-product with i preserve monomorphisms and Cartesian squares by Lemma C.1.5. ☐

In other words, we obtain a lift of $i\hat{\otimes}(-) : \mathbb{E}^{\to} \to \mathbb{E}^{\to}$ to the category $\mathcal{M}_{\mathrm{all}}$ as shown in the following diagram:

$$\begin{tikzcd}
\mathcal{M}_{\mathrm{all}} \arrow[r, "i\hat{\otimes}(-)"] \arrow[d] & \mathcal{M}_{\mathrm{all}} \arrow[d] \\
\mathbb{E}^{\to} \arrow[r, "i\hat{\otimes}(-)"'] & \mathbb{E}^{\to}
\end{tikzcd}$$

This is, in fact, enough to find a lift of $i\hat{\otimes}(-)$ to the category of C-maps as we will show using some orthogonality arguments. The unit of the adjunction of the orthogonality functors is an arrow over $\mathbb{E}^{\to}$:

$$\eta_{\mathcal{M}_{\mathrm{all}}} : \mathcal{M}_{\mathrm{all}} \to {}^{\boxslash}(\mathcal{M}_{\mathrm{all}}^{\boxslash})$$

Thus by composing with the lift of $i\hat{\otimes}(-)$ to $\mathcal{M}_{\mathrm{all}}$ from the proposition, we obtain the following lift:

$$\begin{tikzcd}
\mathcal{M}_{\mathrm{all}} \arrow[r, "i\hat{\otimes}(-)"] \arrow[d] & {}^{\boxslash}(\mathcal{M}_{\mathrm{all}}^{\boxslash}) \arrow[d] \\
\mathbb{E}^{\to} \arrow[r, "i\hat{\otimes}(-)"'] & \mathbb{E}^{\to}
\end{tikzcd}$$

Now, using the adjunction $i\hat{\otimes}(-) \vdash \widehat{\mathrm{hom}}(i, -)$, and Proposition C.1.6 we are able to transpose the previous lift to the following one:

$$\begin{tikzcd}
\mathcal{M}_{\mathrm{all}}^{\boxslash} \arrow[r, "\widehat{\mathrm{hom}}(i{,}-)"] \arrow[d] & \mathcal{M}_{\mathrm{all}}^{\boxslash} \arrow[d] \\
\mathbb{E}^{\to} \arrow[r, "\widehat{\mathrm{hom}}(i{,}-)"'] & \mathbb{E}^{\to}
\end{tikzcd}$$

we know that $(\mathsf{C}, \mathsf{F}_t)$ is algebraically-free on $\mathcal{M}_{\mathrm{all}}$ (i.e. we have that $\mathsf{F}_t\textbf{-Alg} \cong \mathcal{M}_{\mathrm{all}}^{\boxslash}$ over $\mathbb{E}^{\to}$). Using this and composing with the counit of the adjunction between the ortogonality functors $\epsilon_{\mathsf{F}_t\textbf{-Alg}} : \mathsf{F}_t\textbf{-Alg} \to ({}^{\boxslash}\mathsf{F}_t\textbf{-Alg})^{\boxslash}$ we obtain a lift:

$$\begin{tikzcd}
{}^{\boxslash}\mathsf{F}_t\textbf{-Alg} \arrow[r, "\widehat{\mathrm{hom}}(i{,}-)"] \arrow[d] & ({}^{\boxslash}\mathsf{F}_t\textbf{-Alg})^{\boxslash} \arrow[d] \\
\mathbb{E}^{\to} \arrow[r, "\widehat{\mathrm{hom}}(i{,}-)"'] & \mathbb{E}^{\to}
\end{tikzcd}$$

finally, using that C**-Map** $\cong^{\boxslash}$ $\mathsf{F_t}$**-Alg** and transposing the lift again with respect to the same adjunction, we obtain the lift:

$$\begin{tikzcd}
\mathsf{C}\textbf{-Map} \arrow[r, "i\hat{\otimes}(-)"] \arrow[d] & \mathsf{C}\textbf{-Map} \arrow[d] \\
\mathbb{E}^{\to} \arrow[r, "i\hat{\otimes}(-)"'] & \mathbb{E}^{\to}
\end{tikzcd}$$

which shows the first hypothesis of Theorem 5.1.1.

For the second hypothesis, we require the reflexivity functor $r_{(-)} : \mathbb{E}^{\to} \to \mathbb{E}^{\to}$ to lift to the category of C-maps. We can easily see that for each $f : X \to Y$, r_f is a monomorphism (since it has a rectract $t_f \circ r_f = id_X$). Moreover, $r_{(-)}$ preserves pullbacks, because the interval path-object factorisation is stable. Thus we obtain a lift as follows:

$$\begin{tikzcd}
 & \mathcal{M}_{all} \arrow[d] \\
\mathbb{E}^{\to} \arrow[ur, "r_{(-)}"] \arrow[r, "r_{(-)}"'] & \mathbb{E}^{\to}
\end{tikzcd}$$

again, composing with the unit $\eta_{\mathcal{M}_{all}} : \mathcal{M}_{all} \to^{\boxslash} (\mathcal{M}_{all}^{\boxslash})$ and using that $(\mathsf{C}, \mathsf{F_t})$ is algebraically-free on $\mathcal{M}_{all}$ we obtain a lift:

$$\begin{tikzcd}
 & \mathsf{C}\textbf{-Map} \arrow[d] \\
\mathbb{E}^{\to} \arrow[ur, "r_{(-)}"] \arrow[r, "r_{(-)}"'] & \mathbb{E}^{\to}
\end{tikzcd}$$

and by composing with the forgetful functor from F**-Map**, we obtain the required lift, showing that the second hypothesis of Theorem 5.1.1 holds. We summarise this in the following theorem.

Theorem 5.2.2. *Let* $(\mathsf{C}, \mathsf{F_t})$ *be the suitable awfs on* $\mathbb{E}$ *obtained by Theorem C.6.3, and let* $(\mathsf{C_t}, \mathsf{F})$ *be the resulting awfs of Uniform Fibrations. Then* $(\mathsf{C_t}, \mathsf{F})$ *has a stable functorial choice of path objects that lifts the interval path-object factorisation from Appendix C.3.*

We can combine this theorem with [GS17, Theorem 8.8] which says that the awfs of Uniform Fibrations has a functorial Frobenius structure, in order to obtain the following one.

Theorem 5.2.3. *The awfs of uniform fibrations* $(\mathsf{C_t}, \mathsf{F})$ *on* $\mathbb{E}$ *has the structure of an type-theoretic awfs.*

Let us describe some specific applications of this theorem.

Example 5.2.4. We can instantiate the result on the presheaf toposes of simplicial sets $\underline{\mathrm{sSet}}$ and of cubical sets $\underline{\mathrm{cSet}}$ equipped with the obvious choices of interval objects given by the representable 1-simplex and 1-cube respectively. We thus obtain type-theoretic awfs on $\underline{\mathrm{sSet}}$ and $\underline{\mathrm{cSet}}$; moreover, using [GS17, Theorem 9.9] we observe that the underlying morphisms of Uniform Fibrations, on either example, corresponds exactly to Kan fibrations.

We conclude this section with the following observation. Although the proof of Theorem 5.2.3 uses only constructive arguments; it has been pointed out, for example in [OP16], that in order to construct a univalent universe á la Hofmann-Streicher in a constructive setting, it is necessary to restrict the category $\mathcal{M}_{\mathsf{all}}$ of generating monomorphisms to that of decidable ones. A mononomorphism $i : A \to B$ in $\underline{\mathrm{sSet}}$ or $\underline{\mathrm{cSet}}$ (or more generally in any presheaf category) is **decidable** if each level-wise inclusion of sets has decidable image.

Remark 5.2.5. It turns out that the arguments in this section will not apply if we take $\mathcal{M}_{\mathsf{dec}}$ as the category of generating monomorphisms, where $\mathcal{M}_{\mathsf{dec}}$ is the subcategory of $\mathcal{M}_{\mathsf{all}}$ on decidable monomorphisms (for either $\underline{\mathrm{sSet}}$ or $\underline{\mathrm{cSet}}$). The issue lies on verifying that the first leg of the interval path-object factorisation $r_{(-)} : \mathbb{C}^{\to} \to \mathbb{C}^{\to}$ lifts to the category $\mathcal{M}_{\mathsf{dec}}$; intuitively, for a given $f : X \to Y$, the morphism $r_f : X \to P_w f$ maps an object of x of X to the degenerate path on x, this morphism is not decidable because, in general, it is not posible to decide degeneracies [BC15].

Chapter 6

Functoriality of Uniform Fibrations

In this chapter we observe that the results of [GS17] admit a functorial description. In detail, we show that there are functors, as follows:

$$\begin{tikzcd} \begin{array}{c}\text{Toposes with}\\ \text{stable class of monos}\end{array} \arrow[r, "(\text{Theorem C.6.3})"] & \begin{array}{c}\text{Suitable}\\ \text{AWFS}\end{array} \arrow[r, "(\text{Theorem C.4.3})"] & \begin{array}{c}\text{Type-Theoretic}\\ \text{AWFS}\end{array} \end{tikzcd}$$

whose action on objects is given by the theorem referenced in the label of the arrows in the diagram.

The motivation for doing this is to produce morphisms of type-theoretic awfs, thus ultimately giving a natural way for comparing different models of dependent type theories that arise in this way. As an example, we are able to produce a morphism of type-theoretic awfss from the category of homotopy n-types (modelled as $n+1$ coskeletal simplicial sets) to the category of simplicial sets.

Because we need some minimum level of generality in order to find meaningful examples, we will require the base of the awfs to vary, and we will relate the bases of different awfs by a special kind of adjunctions which we will describe in the first section.

6.1 GF and GFI Adjunctions

Let us first establish some notation. Consider a functor $G : \mathbb{C} \to \mathbb{D}$, we will denote by $\mathbb{D} /\!\!/ G\mathbb{C}$ the category whose objects are commutative triangles in $\mathbb{D}$ of the following form:

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr] & & Y \arrow[dl] \\ & GC & \end{tikzcd}$$

an whose morphism are triangular prisms with an obvious edge of the form $Gf : GC \to GC'$ with $f : C \to C'$ in $\mathbb{C}$. We omit G from the notation if it is the identity functor. It is clear that any given adjunction $\mathbb{C}\underset{G}{\overset{T}{\rightleftarrows}}\mathbb{D}$ (with $\perp$), induces a second adjunction:

$$\begin{tikzcd} \mathbb{C} \mathbin{/\!\!/} \mathbb{C} \arrow[r, shift right, "G/\!\!/\mathbb{C}"'] \arrow[r, phantom, "\perp"] & \mathbb{D} \mathbin{/\!\!/} G\mathbb{C} \arrow[l, shift right, "T/\!\!/\mathbb{C}"'] \end{tikzcd}$$

where $G \mathbin{/\!\!/} \mathbb{C}$ is given by applying G to a diagram and $T \mathbin{/\!\!/} \mathbb{C}$ is given by transposition, that is:

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr] & & Y \arrow[dl] & \mapsto & TX \arrow[rr, "Tg"] \arrow[dr] & & TY \arrow[dl] \\ & GC & & & & C & \end{tikzcd}$$

If we fix an object $C \in \mathbb{C}$, we denote $\mathbb{C} \mathbin{/\!\!/} C$ for the arrow category of the slice over C. As before, an adjunction $\begin{tikzcd} \mathbb{C} \arrow[r, shift right, "G"'] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, shift right, "T"'] \end{tikzcd}$ also induces an adjunction of the form:

$$\begin{tikzcd} \mathbb{C} \mathbin{/\!\!/} C \arrow[r, shift right, "G/\!\!/C"'] \arrow[r, phantom, "\perp"] & \mathbb{D} \mathbin{/\!\!/} GC \arrow[l, shift right, "T/\!\!/C"'] \end{tikzcd}$$

which is defined analogously as the foregoing one.

Given a category of arrows $\nu : \mathcal{J} \to \mathbb{C}^{\to}$, we denote by $\mathcal{J} \mathbin{/\!\!/} \mathbb{C}$ the category whose objects are tuples (C, i, a, b) where $i \in \mathcal{J}$, $C \in \mathbb{C}$ and a, b are arrows in $\mathbb{C}$ making the following diagram commute.

$$\begin{tikzcd} \cdot \arrow[rr, "u_i"] \arrow[dr, "a"'] & & \cdot \arrow[dl, "b"] \\ & C & \end{tikzcd}$$

and whose morphisms are commutative triangular prisms. If we fix $C \in \mathbb{C}$ we denote $\mathcal{J} \mathbin{/\!\!/} C$ the category whose objects are triangles as in the previous diagram but with the base object fixed. There are obvious forgetful functors:

$$\mathcal{J} \mathbin{/\!\!/} \mathbb{C} \to \mathbb{C} \mathbin{/\!\!/} \mathbb{C} \quad \text{and} \quad \mathcal{J} \mathbin{/\!\!/} C \to \mathbb{C} \mathbin{/\!\!/} C$$

We refer the reader to Appendix B.6 for more information.

Remark 6.1.1. We adopt the double-slash notation (i.e. $\mathbb{C} \mathbin{/\!\!/} C$) to distinguish these categories from the slice categories. For example in Section 4.1, we used the notation R-**Map**/C to refer to the category whose objects are R-maps over C; in contrast, the objects of R-**Map** $\mathbin{/\!\!/} C$ are commutative triangles between R-**Map** with codomain C.

Definition 6.1.2. A **Generalised Frobenius Adjunction**, or **GF-adjunction** for short, is an adjunction between locally cartesian closed categories:

$$\begin{tikzcd} \mathbb{C} \arrow[r, shift right, "G"'] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, shift right, "T"'] \end{tikzcd}$$

such that:

1. T preserves Cartesian squares

2. The counit $\epsilon : \mathsf{TG} \to 1$ is equifibred, i.e. all naturality squares are Cartesian.

Remark 6.1.3. If $\mathsf{G} \vdash \mathsf{T}$ is a GF-adjuction and if T moreover preserves the terminal object (and hence all finite limits), then the counit ϵ is in fact a natural isomorphism, as can be seen by observing the naturality square associated to each of the unique arrows into the terminal object. In particular, any geometric embedding between toposes is a GF-adjunction.

Proposition 6.1.4. *Let* $\mathbb{C} \underset{\mathsf{G}}{\overset{\mathsf{T}}{\underset{\perp}{\leftrightarrows}}} \mathbb{D}$ *be a GF-adjunction. Then for any arrow* $\mathsf{f} : \mathsf{X} \to \mathsf{Y}$ *in* $\mathbb{C}$*, the following diagram commutes up to isomorphism:*

$$\begin{tikzcd}
\mathbb{D} /\!\!/ \mathsf{GY} \arrow[r, "(\mathsf{Gf})^*"] \arrow[d, "\mathsf{T}/\!\!/\mathsf{Y}"'] & \mathbb{D} /\!\!/ \mathsf{GX} \arrow[d, "\mathsf{T}/\!\!/\mathsf{X}"] \\
\mathbb{C} /\!\!/ \mathsf{Y} \arrow[r, "\mathsf{f}^*"'] & \mathbb{C} /\!\!/ \mathsf{X}
\end{tikzcd}$$

Proof. Consider an arrow $\mathsf{g} : \mathsf{A} \to \mathsf{B}$ over GY, that is an object of $\mathbb{D} /\!\!/ \mathsf{GY}$. The result follows by observing the following diagram:

$$\begin{tikzcd}
\mathsf{T(Gf)^*A} \arrow[rr] \arrow[dr, "\mathsf{T(Gf)^*g}"] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathsf{TA} \arrow[dr, "\mathsf{Tg}"] \arrow[dd] & \\
& \mathsf{T(Gf)^*B} \arrow[rr] \arrow[dl] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathsf{TB} \arrow[dl] \\
\mathsf{TGX} \arrow[rr, "\mathsf{TGf}"] \arrow[d, "\epsilon_{\mathsf{X}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathsf{TGY} \arrow[d, "\epsilon_{\mathsf{Y}}"] & \\
\mathsf{X} \arrow[rr, "\mathsf{f}"'] & & \mathsf{Y} &
\end{tikzcd}$$

□

Corollary 6.1.5. *Let* $\mathbb{C} \underset{\mathsf{G}}{\overset{\mathsf{T}}{\underset{\perp}{\leftrightarrows}}} \mathbb{D}$ *be a GF-adjunction. Then for any arrow* $\mathsf{f} : \mathsf{X} \to \mathsf{Y}$ *in* $\mathbb{C}$*, the following diagram commutes up to isomorphism:*

$$\begin{tikzcd}
\mathbb{D} /\!\!/ \mathsf{GY} & \mathbb{D} /\!\!/ \mathsf{GX} \arrow[l, "\Pi_{\mathsf{Gf}}"'] \\
\mathbb{C} /\!\!/ \mathsf{Y} \arrow[u, "\mathsf{G}/\!\!/\mathsf{Y}"] & \mathbb{C} /\!\!/ \mathsf{X} \arrow[l, "\Pi_{\mathsf{f}}"] \arrow[u, "\mathsf{G}/\!\!/\mathsf{X}"']
\end{tikzcd}$$

Proof. This is an easy consequence of the previous theorem using the facts that adjoints are unique up-to-isomorphism. ☐

Remark 6.1.6. The definition of a GF-adjunction is precisely what is needed for a right adjoint functor to preserve dependent products.

We can also prove that the right adjoint of a GF-adjunction preserves the Beck-Chevalley natural transformation as we shall see. For this we are going to need some intermediate results. To avoid confusion, we will denote with $\hat{\eta}$ and $\hat{\epsilon}$ the unit and the counit of the adjunction $f^* \vdash \Pi_f$, respectively.

In the following lemmas we will make use of the following notation. We will say that a cylinder diagram of categories of the form:

$$\begin{tikzcd}
\cdot \arrow[r, bend left, ""{name=U, below}] \arrow[r, bend right, ""{name=D, above}] \arrow[d] \arrow[phantom, from=U, to=D, "\Downarrow"] & \cdot \arrow[d] \\
\cdot \arrow[r, bend left, ""{name=U2, below}] \arrow[r, bend right, ""{name=D2, above}] \arrow[phantom, from=U2, to=D2, "\Downarrow"] & \cdot
\end{tikzcd}$$

is **commutative** if the following equality of pasting diagrams holds:

$$\begin{tikzcd}
\cdot \arrow[r, bend left, ""{name=U, below}] \arrow[r, bend right, ""{name=D, above}] \arrow[d] \arrow[phantom, from=U, to=D, "\Downarrow"] & \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\cdot \arrow[r] \arrow[d] & \cdot \arrow[d] \\
\cdot \arrow[r, bend left, ""{name=U2, below}] \arrow[r, bend right, ""{name=D2, above}] \arrow[phantom, from=U2, to=D2, "\Downarrow"] & \cdot
\end{tikzcd}$$

Lemma 6.1.7. *Let* $\begin{tikzcd} \mathbb{C} \arrow[r, shift right, "G"'] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, shift right, "T"'] \end{tikzcd}$ *be a GF-adjunction. Then for every arrow* $f : X \to Y$ *in* $\mathbb{C}$*, the following diagram commutes:*

$$\begin{tikzcd}
\mathbb{C} /\!\!/ Y \arrow[r, bend left, "\mathrm{id}", ""{name=U, below}] \arrow[r, bend right, "\Pi_f f^*"', ""{name=D, above}] \arrow[d, "G/\!\!/Y"'] \arrow[phantom, from=U, to=D, "\Downarrow \hat{\eta}"] & \mathbb{C} /\!\!/ Y \arrow[d, "G/\!\!/Y"] \\
\mathbb{D} /\!\!/ GY \arrow[r, bend left, "\mathrm{id}", ""{name=U2, below}] \arrow[r, bend right, "\Pi_{(Gf)}(Gf)^*"', ""{name=D2, above}] \arrow[phantom, from=U2, to=D2, "\Downarrow \hat{\eta}"] & \mathbb{D} /\!\!/ GY
\end{tikzcd}$$

Proof. For a given arrow $h : A \to B$ in $\mathbb{C}$ over Y, the arrow $\hat{\eta}_{Gh} : Gh \to \Pi_{Gf}(Gf)^*h$ (over GY) has the universal property of transposing to the identity. We will verify that the arrow $G\hat{\eta}_h : Gh \to G(\Pi_f f^* h)$ has the same universal property (modulo the isomorphism in Corollary 6.1.5).

Transposing $G\hat{\eta}_h$ twice (using a triangular identity) we obtain $f^*\epsilon_h : f^*TGh \to f^*h$ and using that both G and T commute with pullbacks and that ϵ is equifibred, we obtain $\epsilon_{f^*h} : TGf^*h \to f^*h$. The result follows since ϵ_{f^*h} transposes to the identity. ☐

Lemma 6.1.8. *Let* $\mathbb{C}\underset{G}{\overset{T}{\underset{\perp}{\leftrightarrows}}}\mathbb{D}$ *be a GF-adjunction. Then for every arrow* $f : X \to Y$ *in* $\mathbb{C}$*, the following diagram commutes:*

$$\begin{tikzcd}
\mathbb{C} /\!\!/ X \arrow[r, bend left, "f^*\Pi_f"] \arrow[r, bend right, "\mathrm{id}"'] \arrow[r, phantom, "\Downarrow \hat{\varepsilon}"] \arrow[d, "G/\!\!/X"'] & \mathbb{C} /\!\!/ X \arrow[d, "G/\!\!/X"] \\
\mathbb{D} /\!\!/ GX \arrow[r, bend left, "(Gf)^*\Pi_{Gf}"] \arrow[r, bend right, "\mathrm{id}"'] \arrow[r, phantom, "\Downarrow \hat{\varepsilon}"] & \mathbb{D} /\!\!/ GX
\end{tikzcd}$$

Proof. The proof is similar to the one of the previous lemma. We need to verify that $G\hat{\varepsilon}_h : G(f^*\Pi_f h) \to Gh$ has the same universal property to $\hat{\varepsilon}_{Gh} : (Gf)^*\Pi_{Gf}Gh \to Gh$. We leave the details to the reader. □

Given a square $\tau = (u, v) : g \to f$ in $\mathbb{C}^{\to}$, we will denote by $BC_\tau : v^*\Pi_f \to \Pi_g u^*$ the Beck-Chevalley natural transformation and by BC_τ^h the component at a given arrow h.

Proposition 6.1.9. *Let* $\mathbb{C}\underset{G}{\overset{T}{\underset{\perp}{\leftrightarrows}}}\mathbb{D}$ *be a GF-adjunction. And let* $\tau = (u, v) : g \to f$ *be a square in* $\mathbb{C}^{\to}$ *as shown:*

$$\begin{tikzcd}
X' \arrow[r, "u"] \arrow[d, "g"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "v"'] & Y
\end{tikzcd}$$

Then the following diagram commutes:

$$\begin{tikzcd}
\mathbb{C} /\!\!/ X \arrow[r, bend left, "v^*\Pi_f"] \arrow[r, bend right, "\Pi_g u^*"'] \arrow[r, phantom, "\Downarrow BC_\tau"] \arrow[d, "G/\!\!/X"'] & \mathbb{C} /\!\!/ Y' \arrow[d, "G/\!\!/Y'"] \\
\mathbb{D} /\!\!/ GX \arrow[r, bend left, "(Gv)^*(\Pi_f)"] \arrow[r, bend right, "\Pi_{(Gg)}(Gu)^*"'] \arrow[r, phantom, "\Downarrow BC_{G\tau}"] & \mathbb{D} /\!\!/ GY'
\end{tikzcd}$$

Proof. Let $h : A \to B$ be a morphism over X. We must show that $G(BC_\tau^h) = BC_{G\tau}^{Gh}$. In order to do this, we recall the definition of the Beck-Chevalley natural transformation, BC_τ^h is given by the following composite:

$$v^*\Pi_f h \xrightarrow{\hat{\eta}_{v^*\Pi_f h}} \Pi_g g^* v^* \Pi_f h \xrightarrow[\cong]{\Pi_g \lambda_{\Pi_f h}} \Pi_g u^* f^* \Pi_f h \xrightarrow{\Pi_g u^* \hat{\varepsilon}_h} \Pi_g u^* h$$

where $\lambda : g^*v^* \to u^*f^*$ is the canonical isomorphism. Applying G to the above composite and using the previous lemmas we reach the desired conclusion. □

Let us suppose that $\mathbb{C}$ and $\mathbb{D}$ are equipped with an interval object with contraction and connections. We will need that the interval objects are suitably related by adjunctions. For this we introduce the following definitions.

Definition 6.1.10. Let $\mathbb{C}$ and $\mathbb{D}$ be symmetric closed monoidal categories equipped with interval objects. We say that $\begin{tikzcd}\mathbb{C} \arrow[r, shift right, "G"'] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, shift right, "T"']\end{tikzcd}$ is:

- an **I-adjunction** if T is strong monoidal and it preserves the interval objects, including endpoints, contractions and connections.
- an **GFI-adjunction** if it is both a GF-adjunction and a I-adjunction.

6.2 Morphisms of suitable AWFS

In this section we will assume that our categories are locally cartesian closed and symmetric closed monoidal equipped with an interval object with contraction and connections. Recall the definitions of adjunction of awfs and the change of base theorem from Appendix B.5.

Definition 6.2.1. Let (C, F_t) and (C', F_t') be suitable awfs's (see Definition C.4.1) on $\mathbb{C}$ and $\mathbb{D}$ algebraically-free on $\mathcal{J}$ and $\mathcal{J}'$ respectively. A **morphism of suitable awfs** denoted by:

$$\begin{tikzcd}
(C, F_t) \arrow[r, "{(T,G,\theta)}"] \arrow[d, dotted, no head] & (C', F_t') \arrow[d, dotted, no head] \\
\mathbb{C} \arrow[r, shift right, "G"'] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, shift right, "T"']
\end{tikzcd}$$

consists of the following data:

1. A GFI-adjunction $T \vdash G$
2. A lift of T the the generating categories of arrows:

$$\begin{tikzcd}
\mathcal{J} \arrow[d] & \mathcal{J}' \arrow[l, "\theta"'] \arrow[d] \\
\mathbb{C}^{\to} & \mathbb{D}^{\to} \arrow[l, "T"]
\end{tikzcd}$$

subject to the condition that the lift of T to the category of coalgebras (Theorem B.5.3) induced by θ cohere with the structure of suitable awfs; that is, that the following diagrams commute:

$$\begin{tikzcd}
\mathbf{C}\text{-}\mathbf{Map} & \mathbf{C'}\text{-}\mathbf{Map} \arrow[l, "T"'] \\
\mathbb{C} \arrow[u, "\perp"] & \mathbb{D} \arrow[l, "T"] \arrow[u, "\perp"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{C}\text{-}\mathbf{Map} \arrow[d, "\delta^k \otimes (-)"'] & \mathbf{C'}\text{-}\mathbf{Map} \arrow[l, "T"'] \arrow[d, "\delta^k \otimes (-)"] \\
\mathbf{C}\text{-}\mathbf{Map} & \mathbf{C'}\text{-}\mathbf{Map} \arrow[l, "T"]
\end{tikzcd}$$

and that for each arrow $h : D \to C$ in $\mathbb{C}$, the following diagram commutes:

$$\begin{tikzcd}
\mathbf{C\text{-}Map} /\!\!/ C \arrow[r, "h^*"] & \mathbf{C\text{-}Map} /\!\!/ D \\
\mathbf{C'\text{-}Map} /\!\!/ GC \arrow[u, "T/\!\!/C"] \arrow[r, "(Gh)^*"'] & \mathbf{C'\text{-}Map} /\!\!/ GD \arrow[u, "T/\!\!/D"']
\end{tikzcd}$$

We will denote by $\mathsf{sAWFS}/(-)_{\mathrm{radj}}$ the category whose objects are suitable awfs with a chosen category of arrows on which it is algebraically free, i.e. the objects of $\mathsf{sAWFS}/(-)_{\mathrm{radj}}$ are tuples $(\mathbb{C}, (C, F_t), \mathcal{I})$ where (C, F_t) is suitable on $\mathbb{C}$ and algebraically free on $\mathcal{I}$. The morphisms are the ones given by the previous definition.

6.3 Morphisms of GF-Structures

Recall from Definition B.6.3 the definition of a Generalized Frobenius structure or GF-structure for short. For categories of arrows $u, v, w : \mathcal{I}, \mathcal{J}, \mathcal{K} \to \mathbb{C}^{\to}$ we will denote by $(\mathcal{I}, \mathcal{J}, \mathcal{K}, \widetilde{PB})$ a GF-structure on $\mathbb{C}$ where $\widetilde{PB}$ is a lift of the pullback functor; we will also denote the structure just by the tuple $(\mathcal{I}, \mathcal{J}, \mathcal{K})$ whenever the GF-structure $\widetilde{PB}$ is implicit.

Definition 6.3.1. A **morphism of GF-structures** denoted by:

$$\begin{tikzcd}
(\mathcal{I}, \mathcal{J}, \mathcal{K}, \widetilde{PB}) \arrow[r, "(a,b,c)"] \arrow[d, dotted, no head] & (\mathcal{I}', \mathcal{J}', \mathcal{K}', \widetilde{PB}') \arrow[d, dotted, no head] \\
\mathbb{C} \arrow[r, bend right, "G"'] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, bend right, "T"']
\end{tikzcd}$$

consists of:

1. A GF-adjunction (or GFI-adjunction depending on the situation) $T \vdash G$.
2. Lifts a, b, c of T and G as shown:

$$\begin{tikzcd}
\mathcal{I} \arrow[d] & \mathcal{I}' \arrow[l, "a"'] \arrow[d] \\
\mathbb{C}^{\to} & \mathbb{D}^{\to} \arrow[l, "T"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{J} \arrow[r, "b"] \arrow[d] & \mathcal{J}' \arrow[d] \\
\mathbb{C}^{\to} \arrow[r, "G"'] & \mathbb{D}^{\to}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{K} \arrow[d] & \mathcal{K}' \arrow[l, "c"'] \arrow[d] \\
\mathbb{C}^{\to} & \mathbb{D}^{\to} \arrow[l, "T"]
\end{tikzcd}$$

such that for each $j \in \mathcal{J}$ over an arrow $v_j : D_j \to C_j$, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{I} /\!\!/ C_j \arrow[r, "v_j^*"] & \mathcal{K} /\!\!/ D_j \\
\mathcal{I}' /\!\!/ GC_j \arrow[u, "a/\!\!/j"] \arrow[r, "(v_{bj})^*"'] & \mathcal{K}' /\!\!/ GD_j \arrow[u, "c/\!\!/j"']
\end{tikzcd}$$

where $a \mathbin{/\!\!/} j$ and $c \mathbin{/\!\!/} j$ are the obvious lifts of $T \mathbin{/\!\!/} C_j$ and $T \mathbin{/\!\!/} D_j$ induced by a and c respectively.

Definition 6.3.2. Categories equipped with GF-structures and morphisms of GF-structures define in a natural way a category which we will denote by $\mathsf{GF}/(-)_{\mathrm{radj}}$.

We will proceed to prove an alternative characterisation of morphisms of GF-structures using the lifts of the pushforward functors instead of the pullback ones (Proposition B.6.7). This will be a straightforward consequence of the following lemma extending Proposition B.4.8.

Lemma 6.3.3. *Let* $u : \mathcal{I} \to \mathbb{C}^{\to}$ *and* $v : \mathcal{J} \to \mathbb{D}^{\to}$ *be category of arrows and* $\mathbb{C} \underset{\mathsf{G}}{\overset{\mathsf{T}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbb{D}$ *be an adjunction. Then the bijection between lifts of* G *and* T *from Proposition B.4.8 is natural in the following sense:*

Given $u' : \mathcal{I}' \to \mathbb{C}'^{\to}$ *and* $v' : \mathcal{J}' \to \mathbb{D}'^{\to}$*, an adjunction* $\mathbb{C}' \underset{\mathsf{G}'}{\overset{\mathsf{T}'}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbb{D}'$*, and lifts as shown:*

$$\begin{tikzcd}
\mathcal{I} \arrow[r, "y"] \arrow[d] & \mathcal{I}' \arrow[d] \\
\mathbb{C}^{\to} \arrow[r, shift right, "\mathsf{Y}"'] \arrow[r, phantom, "\perp"] & \mathbb{C}'^{\to} \arrow[l, shift right, "\mathsf{X}"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{J} \arrow[d] & \mathcal{J}' \arrow[l, "w"'] \arrow[d] \\
\mathbb{D}^{\to} \arrow[r, shift right, "\mathsf{Z}"'] \arrow[r, phantom, "\perp"] & \mathbb{D}'^{\to} \arrow[l, shift right, "\mathsf{W}"']
\end{tikzcd}$$

such that $\mathsf{TW} \cong \mathsf{XT}'$ *(or equivalently* $\mathsf{ZG} \cong \mathsf{G}'\mathsf{Y}$*). Then one of the following diagrams commutes if, and only if, the other one also commutes.*

$$\begin{tikzcd}
\mathcal{J} \arrow[r, "\mathsf{T}"] & {}^{\boxslash}\mathcal{I} \\
\mathcal{J}' \arrow[u, "w"] \arrow[r, "\mathsf{T}'"'] & {}^{\boxslash}\mathcal{I}' \arrow[u, "{}^{\boxslash}y"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{I} \arrow[r, "\mathsf{G}"] \arrow[d, "y"'] & \mathcal{J}^{\boxslash} \arrow[d, "w^{\boxslash}"] \\
\mathcal{I}' \arrow[r, "\mathsf{G}'"'] & \mathcal{J}'^{\boxslash}
\end{tikzcd}$$

Proof. This is a straightforward diagram chase making use of the definitions involved. For example, supposing that the diagram in the left commutes let us show that the one on the right commutes.

Let $i \in \mathcal{I}$, the object $\mathsf{G}(i) = (\mathsf{G}u_i, \theta)$ where theta is the right $\mathcal{J}$-lifting structure given object-wise by transposing the lifting structures given by the lift $\mathsf{T} : \mathcal{J} \to^{\boxslash} \mathcal{I}$, now $w^{\boxslash}(\mathsf{G}u_i, \theta) = (\mathsf{ZG}u_i, w\theta)$ where $w\theta$ is the lifting structure given object-wise by transposing and using the lift $w : \mathcal{J}' \to \mathcal{J}$.

On the other hand $\mathsf{G}'yi = (\mathsf{G}'\mathsf{Y}u_i, \theta')$ where θ' is given by transposing and using the lift of T'. We must know verify that modulo the isomorphism $\mathsf{ZG} \cong \mathsf{G}'\mathsf{Y}$, the lifts $w\theta$ and θ' coincide. But by plugging in a specific lifting problem and transposing twice we reduce the problem to the commutativity of the square on the left. □

Proposition 6.3.4. *Let* $\begin{tikzcd} \mathbb{C} \arrow[r, shift right, "\mathsf{G}"'] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, shift right, "\mathsf{T}"'] \end{tikzcd}$ *be a GF-adjunction and let*

$$(\mathcal{I}, \mathcal{J},^{\boxslash} (\mathcal{K}^{\boxslash}), \widetilde{\mathsf{PB}}) \quad \textit{and} \quad (\mathcal{I}', \mathcal{J}',^{\boxslash} (\mathcal{K}'^{\boxslash}), \widetilde{\mathsf{PB}}')$$

be GF-structures on $\mathbb{C}$ *and* $\mathbb{D}$ *respectively. Consider lifts of* G *and* T *as follows:*

$$\begin{tikzcd} \mathcal{I} \arrow[d] & \mathcal{I}' \arrow[l, "a"'] \arrow[d] \\ \mathbb{C}^{\to} & \mathbb{D}^{\to} \arrow[l, "\mathsf{T}"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{J} \arrow[r, "b"] \arrow[d] & \mathcal{J}' \arrow[d] \\ \mathbb{C}^{\to} \arrow[r, "\mathsf{G}"'] & \mathbb{D}^{\to} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{K} \arrow[d] & \mathcal{K}' \arrow[l, "c"'] \arrow[d] \\ \mathbb{C}^{\to} & \mathbb{D}^{\to} \arrow[l, "\mathsf{T}"] \end{tikzcd}$$

Then the following are equivalent:

1. $(a, b,^{\boxslash} (c^{\boxslash}))$ *is a morphism of GF-structures.*
2. *For each* $j \in \mathcal{J}$ *over* $v_j : \mathsf{D}_j \to \mathsf{C}_j$, *the following diagram commutes:*

$$\begin{tikzcd} \mathcal{K}^{\boxslash} \sslash \mathsf{D}_j \arrow[r, "\Pi_{v_j}"] \arrow[d, "c^{\boxslash} \sslash j"'] & \mathcal{I}^{\boxslash} \sslash \mathsf{C}_j \arrow[d, "a^{\boxslash} \sslash j"] \\ \mathcal{K}'^{\boxslash} \sslash \mathsf{G}\mathsf{D}_j \arrow[r, "\Pi_{v_{bj}}"'] & \mathcal{I}'^{\boxslash} \sslash \mathsf{G}\mathsf{C}_j \end{tikzcd}$$

Proof. We instantiate Lemma 6.3.3 as follows:

- For the adjuctions $\mathsf{T} \vdash \mathsf{G}$ and $\mathsf{T}' \vdash \mathsf{G}'$ we plug in: $(v_j)^* \vdash \Pi_{v_j}$ and $(v_{bj})^* \vdash \Pi_{v_{bj}}$.
- For the adjunctions $\mathsf{X} \vdash \mathsf{Y}$ and $W \vdash \mathsf{Z}$ we plug in: $\mathsf{T} \sslash \mathsf{D}_j \vdash \mathsf{G} \sslash \mathsf{D}_j$ and $\mathsf{T} \sslash \mathsf{C}_j \vdash \mathsf{G} \sslash \mathsf{C}_j$.
- For the category of arrows $u : \mathcal{I} \to \mathbb{C}^{\to}$ and $v : \mathcal{J} \to \mathbb{D}^{\to}$ we plug in: $\mathcal{K}^{\boxslash} \sslash \mathsf{D}_j \to \mathbb{C} \sslash \mathsf{D}_j$ and $\mathcal{I} \sslash \mathsf{C}_j \to \mathbb{C} \sslash \mathsf{C}_j$.
- For the category of arrows $u' : \mathcal{I}' \to \mathbb{C}'^{\to}$ and $v' : \mathcal{J}' \to \mathbb{D}'^{\to}$ we plug in: $\mathcal{K}'^{\boxslash} \sslash \mathsf{G}\mathsf{D}_j \to \mathbb{C} \sslash \mathsf{G}\mathsf{D}_j$ and $\mathcal{I}' \sslash \mathsf{G}\mathsf{C}_j \to \mathbb{C} \sslash \mathsf{G}\mathsf{C}_j$.
- For the lifts $y : \mathcal{I} \to \mathcal{I}'$ and $w : \mathcal{J}' \to \mathcal{J}$ we plug in: $c^{\boxslash} \sslash j : \mathcal{K}^{\boxslash} \sslash \mathsf{D}_j \to \mathcal{K}'^{\boxslash} \sslash \mathsf{G}\mathsf{D}_j$ and $a \sslash j : \mathcal{I}' \sslash \mathsf{G}\mathsf{C}_j \to \mathcal{I} \sslash \mathsf{C}_j$.

With this in place, the diagram of the left of the conclusion of Lemma 6.3.3 commutes if, and only if, $(a, b,^{\boxslash} (c^{\boxslash}))$ is a morphism of GF-structure while the diagram on the right correspond to the second part of this proposition. $\square$

Proposition 6.3.5. *Let* $(\mathcal{I}, \mathcal{J},^{\boxslash} (\mathcal{K}^{\boxslash}), \widetilde{\mathsf{PB}}) \xrightarrow{(a,b,^{\boxslash}(c^{\boxslash}))} (\mathcal{I}', \mathcal{J}',^{\boxslash} (\mathcal{K}'^{\boxslash}), \widetilde{\mathsf{PB}}')$ *be a morphism of GF-structures over* $\mathsf{T} \vdash \mathsf{G}$. *Then there is a lift of the Beck-Chevalley natural* *transformation as follows: for each* $\tau = (l, m) : j \to k$ *in* $\mathcal{J}$ *the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{K}^{\boxslash} /\!\!/ D_k \arrow[r, bend left, "m^*\Pi_{\nu_k}"] \arrow[r, bend right, "\Pi_{\nu_j} l^*"'] \arrow[r, phantom, "\Downarrow BC_\tau"] \arrow[d, "c^{\boxslash} /\!\!/ k"'] & \mathcal{K}^{\boxslash} /\!\!/ C_j \arrow[d, "c^{\boxslash} /\!\!/ j"] \\
\mathcal{K}'^{\boxslash} /\!\!/ GD_k \arrow[r, bend left, "(Gm)^*(\Pi_{\nu_k})"] \arrow[r, bend right, "\Pi_{(G\nu_j)}(Gl)^*"'] \arrow[r, phantom, "\Downarrow BC_{G\tau}"] & \mathcal{K}'^{\boxslash} /\!\!/ GC_j
\end{tikzcd}$$

Proof. This follows by faithfulness of the functor $\mathcal{K}'^{\boxslash} /\!\!/ GC_j \to \mathbb{D} /\!\!/ GC_j$, and by the fact that the underlying diagram commutes by Proposition 6.1.9. □

6.4 From Suitable AWFS to GF-structure: Functoriality

In the main result of this section we will show that the assignment of a GF-structure $(C_t\textbf{-Map}, F\textbf{-Alg}, C_t\textbf{-Map})$ from a suitable awfs (C, F_t) of Gambino and Sattler can be extended to a functor:

$$\mathsf{sAWFS}/(-)_{radj} \longrightarrow \mathsf{GF}/(-)_{radj}$$

In order to prove this, we will need to go step-by-step through the proof of [GS17, Theorem 8.8]. To aid the reader through this section, we will list here the main steps in the proof. Let (C, F_t) be a suitable awfs on $\mathbb{C}$:

1. Start from the fact that $(C\textbf{-Map}, \mathbb{C}^{\to}, C\textbf{-Map})$ has a GF-structure.
2. Construct a GF-structure on $(C\textbf{-Map}, F\textbf{-Alg}, C\textbf{-Map})$ using [GS17, Proposition 6.3].
3. Construct a GF-structure on $(S, F\textbf{-Alg}, S)$ using [GS17, Lemma 8.7]. Here S is the category of strong homotopy equivalences (Appendix C.5).
4. Construct a GF-structure on $(C\textbf{-Map} \times_{\mathbb{C}} S, F\textbf{-Alg}, C\textbf{-Map} \times_{\mathbb{C}} S)$ from items 2 and 3 using [GS17, Proposition 6.10].
5. Construct a GF-structure on $(\mathcal{I}_{\otimes}, F\textbf{-Alg}, C_t\textbf{-Map})$ by [GS17, Lemma 8.5] using [GS17, Proposition 6.3].
6. Construct a GF-structure on $(C_t\textbf{-Map}, F\textbf{-Alg}, C_t\textbf{-Map})$ by [GS17, Proposition 6.8].

We will now state and prove a series of results which will give the functorial action to each of the intermediate steps. Each lemma will refer to the result from [GS17] of which it is an extension. These extension lemas turn out to be rather technically complicated, the most difficult lemma is Lemma 6.4.9 because of the delicate issues regarding the coherence of some of the lifting structure.

Lemma 6.4.1 (Proposition 6.3)**.** *Consider category of arrows* $u_t, v_t, z_t : \mathcal{I}_t, \mathcal{J}_t, \mathcal{K}_t \to \mathbb{C}^{\to}$ *for* $t \in \{1, 2\}$ *and functors over* $\mathbb{C}^{\to}$*:*

$$\begin{tikzcd} \mathcal{I}_1 & \mathcal{I}_2 \arrow[l, "l"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{J}_1 & \mathcal{J}_2 \arrow[l, "m"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{K}_1 \arrow[r, "n"] & \mathcal{K}_2 \end{tikzcd}$$

then, a GF-structure $(\mathcal{I}_1, \mathcal{J}_1, \mathcal{K}_1, \mathsf{PB}_1)$ *induces a GF-structure* $(\mathcal{I}_2, \mathcal{J}_2, \mathcal{K}_2, \mathsf{PB}_2)$*.*

Moreover, if in addition we have $u'_t, v'_t, z'_t : \mathcal{I}'_t, \mathcal{J}'_t, \mathcal{K}'_t \to \mathbb{D}^{\to}$ *for* $t \in \{1, 2\}$*, functors over* $\mathbb{D}^{\to}$*:*

$$\begin{tikzcd} \mathcal{I}'_1 & \mathcal{I}'_2 \arrow[l, "l'"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{J}'_1 & \mathcal{J}'_2 \arrow[l, "m'"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{K}'_1 \arrow[r, "n'"] & \mathcal{K}'_2 \end{tikzcd}$$

a morphism of GF-structures $(\mathcal{I}_1, \mathcal{J}_1, \mathcal{K}_1, \mathsf{PB}_1) \xrightarrow{(a_1, b_1, c_1)} (\mathcal{I}'_1, \mathcal{J}'_1, \mathcal{K}'_1, \mathsf{PB}'_1)$ *and functors* a_2, b_2, c_2 *as shown, making the following diagram commute:*

$$\begin{tikzcd} \mathcal{I}_1 & \mathcal{I}_2 \arrow[l, "l"'] \\ \mathcal{I}'_1 \arrow[u, "a_1"] & \mathcal{I}'_2 \arrow[l, "l'"] \arrow[u, "a_2"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{J}_1 \arrow[d, "b_1"'] & \mathcal{J}_2 \arrow[l, "m"'] \arrow[d, "b_2"] \\ \mathcal{J}'_1 & \mathcal{J}'_2 \arrow[l, "m'"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{K}_1 \arrow[r, "n"] & \mathcal{K}_2 \\ \mathcal{K}'_1 \arrow[u, "c_1"] \arrow[r, "n'"'] & \mathcal{K}'_2 \arrow[u, "c_2"'] \end{tikzcd}$$

Then (a_2, b_2, c_2) *is a morphism of the induced GF-structures:*

$$(\mathcal{I}_2, \mathcal{J}_2, \mathcal{K}_2, \mathsf{PB}_2) \xrightarrow{(a_2, b_2, c_2)} (\mathcal{I}'_2, \mathcal{J}'_2, \mathcal{K}'_2, \mathsf{PB}'_2)$$

Proof. Given category of arrows u_t, v_t, z_t for $t \in \{1, 2\}$, functors l, m, n and a GF-structure $(\mathcal{I}_1, \mathcal{J}_1, \mathcal{K}_1, \mathsf{PB}_1)$, we define the GF-structure PB_2 for $(\mathcal{I}_2, \mathcal{J}_2, \mathcal{K}_2)$ as the dotted arrow in the following diagram:

$$\begin{tikzcd} \mathcal{I}_1 /\!\!/ \mathbb{C} \times_{\mathbb{C}} \mathcal{J}_1 \arrow[r, "\mathsf{PB}_1"] & \mathcal{K}_1 /\!\!/ \mathbb{C} \arrow[d, "n /\!\!/ \mathbb{C}"] \\ \mathcal{I}_2 /\!\!/ \mathbb{C} \times_{\mathbb{C}} \mathcal{J}_2 \arrow[u, "\langle l /\!\!/ \mathbb{C}, m \rangle"] \arrow[r, dotted, "\mathsf{PB}_2"'] & \mathcal{K}_2 /\!\!/ \mathbb{C} \end{tikzcd}$$

Now, given a morphism of GF-structures $(\mathcal{I}_1, \mathcal{J}_1, \mathcal{K}_1, \mathsf{PB}_1) \xrightarrow{(a_1, b_1, c_1)} (\mathcal{I}'_1, \mathcal{J}'_1, \mathcal{K}'_1, \mathsf{PB}'_1)$ over a GF-adjunction $\mathsf{T} \vdash \mathsf{G}$, and functors (a_2, b_2, c_2) as in the hypothesis, we must show that (a_2, b_2, c_2) is a morphism of GF-structures. By definition, this means showing that for each $j \in \mathcal{J}_2$ over an arrow $v_{2,j} : \mathsf{D}_j \to \mathsf{C}_j$, the following diagram commutes:

$$\begin{tikzcd} \mathcal{I}_2 /\!\!/ \mathsf{C}_j \arrow[r, "v^*_{2,j}"] & \mathcal{K}_2 /\!\!/ \mathsf{D}_j \\ \mathcal{I}'_2 /\!\!/ \mathsf{G}\mathsf{C}_j \arrow[u, "a_2 /\!\!/ j"] \arrow[r, "(v'_{2,b_2 j})^*"'] & \mathcal{K}'_2 /\!\!/ \mathsf{G}\mathsf{D}_j \arrow[u, "c_2 /\!\!/ j"'] \end{tikzcd}$$

this is an easy diagram chase: let $h \in \mathcal{I}'_2 /\!\!/ \mathsf{G}C_j$, by definition of PB_2 and PB'_2 we have the following:

$$\begin{aligned} c_2(v'_{2,b_2 j})^*(h) &= c_2(n'(v'_{1,m'b_2 j})^* l')(h) \\ &= n(c_1(v'_{1,b_1 mj})^* l')(h) \\ &= n((v_{1,mj})^* a_1 l')(h) \\ &= n((v_{1,mj})^* l a_2)(h) \\ &= ((v_{2,j})^* a_2)(h) \end{aligned}$$

□

Lemma 6.4.2 (Proposition 6.8)**.** *Let* $u, v : \mathcal{I}, \mathcal{J} \to \mathbb{C}^{\to}$ *be categories of arrows. Then* $(\mathcal{I}, \mathcal{J}, {}^{\boxslash}(\mathcal{I}^{\boxslash}))$ *has a GF-structure if, and only if,* $({}^{\boxslash}(\mathcal{I}^{\boxslash}), \mathcal{J}, {}^{\boxslash}(\mathcal{I}^{\boxslash}))$ *has a GF-structure.*

Moreover, given a GF-adjunction $\mathsf{T} \vdash \mathsf{G}$, *lifts* $u', v' : \mathcal{I}', \mathcal{J}' \to \mathbb{D}^{\to}$ *and lifts of* T *and* G *as follows:*

$$\begin{tikzcd} \mathcal{I} \arrow[d, "u"'] & \mathcal{I}' \arrow[l, "a"'] \arrow[d, "u'"] \\ \mathbb{C}^{\to} & \mathbb{D}^{\to} \arrow[l, "\mathsf{T}"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{J} \arrow[r, "b"] \arrow[d, "v"'] & \mathcal{J}' \arrow[d, "v'"] \\ \mathbb{C}^{\to} \arrow[r, "\mathsf{G}"'] & \mathbb{D}^{\to} \end{tikzcd}$$

Then:

$$(\mathcal{I}, \mathcal{J}, {}^{\boxslash}(\mathcal{I}^{\boxslash})) \xrightarrow{(a, b, {}^{\boxslash}(a^{\boxslash}))} (\mathcal{I}', \mathcal{J}', {}^{\boxslash}(\mathcal{I}'^{\boxslash}))$$

is a morphism of GF-structures if, and only if:

$$({}^{\boxslash}(\mathcal{I}^{\boxslash}), \mathcal{J}, {}^{\boxslash}(\mathcal{I}^{\boxslash})) \xrightarrow{({}^{\boxslash}(a^{\boxslash}), b, {}^{\boxslash}(a^{\boxslash}))} ({}^{\boxslash}(\mathcal{I}'^{\boxslash}), \mathcal{J}', {}^{\boxslash}(\mathcal{I}'^{\boxslash}))$$

is a morphism of GF-structures.

Proof. For objects the result follows from the characterisation of GF-structures given by Proposition B.6.5 and Proposition B.6.7 by composing respectively with the unit and counit of the adjunction of the orthogonality functors.

To prove the result with respect to morphisms we will split the problem into the following two claims.

Claim 6.4.2.1. *Let* $b : \mathcal{J} \to \mathcal{J}'$ *be a lift of* G. *Then the counit of* $(-)^{\boxslash} \vdash^{\boxslash} (-)$ *commutes with* b *as shown in the following diagram:*

$$\begin{tikzcd} \mathcal{J}' \arrow[r, "\epsilon_{\mathcal{J}'}"] & ({}^{\boxslash}\mathcal{J}')^{\boxslash} \\ \mathcal{J} \arrow[u, "b"] \arrow[r, "\epsilon_{\mathcal{J}}"'] & ({}^{\boxslash}\mathcal{J})^{\boxslash} \arrow[u, "({}^{\boxslash}b)^{\boxslash}"'] \end{tikzcd}$$

Proof. Let $j \in \mathcal{J}$ over $\nu_j : D_j \to C_j$ in $\mathbb{C}$. By definition we have that $\epsilon_{\mathcal{J}}(j) = (\nu_j, \epsilon_j)$ where ϵ_j is the left-${}^{\boxslash}\mathcal{J}$ lifting structure given by $\epsilon_j(l, m, (g, \psi)) = \psi(l, m, j)$; here $(g, \psi) \in^{\boxslash} \mathcal{J}$ and $(l, m) : g \to \nu_j$ is a square in $\mathbb{C}$.

Let us also recall the action of ${}^{\boxslash}b :^{\boxslash} \mathcal{J}' \to^{\boxslash} \mathcal{J}$. This is given by ${}^{\boxslash}b(f, \theta) = (Tf, b\theta)$ where $b\theta(l, m, j) = \hat{\theta}(\hat{l}, \hat{m}, b_j)$ and $(\hat{-})$ denotes the transpose with respect to the adjunction $T \vdash G$. The definition of $a^{\boxslash}$ is dual.

With this in mind we have:

$$\begin{aligned} ({}^{\boxslash}b)^{\boxslash}(\epsilon_{\mathcal{J}}(j)) &= ({}^{\boxslash}b)^{\boxslash}(\nu_j, \epsilon_j) \\ &= (G\nu_j, {}^{\boxslash} b\epsilon_j) \end{aligned} \qquad\qquad \epsilon_{\mathcal{J}'}(bj) = (\nu'_{bj}, \epsilon'_{bj})$$

we have that $G\nu_j = \nu'_{bj}$ thus we only need to check that ${}^{\boxslash}b\epsilon_j = \epsilon'_{bj}$ for this, consider $(f, \theta) \in^{\boxslash} \mathcal{J}'$ and notice:

$$\begin{aligned} {}^{\boxslash}b\epsilon_j(l, m, (f, \theta)) &= \hat{\epsilon}_j(\hat{l}, \hat{m}, {}^{\boxslash} b(f, \theta)) \\ &= \hat{\epsilon}_j(\hat{l}, \hat{m}, (Tf, b\theta)) \\ &= \widehat{b\theta}(\hat{l}, \hat{m}, j) \\ &= \hat{\hat{\theta}}(\hat{\hat{l}}, \hat{\hat{m}}, bj) \\ &= \theta(l, m, bj) = \epsilon'_{bj}(l, m, (f, \theta)) \end{aligned}$$

$\square$

Claim 6.4.2.2. *Let* $a : \mathcal{I}' \to \mathcal{I}$ *be a lift of* T. *Then the unit of* $(-)^{\boxslash} \vdash^{\boxslash} (-)$ *commutes with* a *as shown in the following diagram:*

$$\begin{tikzcd} \mathcal{I}' \arrow[r, "\eta_{\mathcal{I}'}"] \arrow[d, "a"'] & {}^{\boxslash}(\mathcal{I}'^{\boxslash}) \arrow[d, "{}^{\boxslash}(a^{\boxslash})"] \\ \mathcal{I} \arrow[r, "\eta_{\mathcal{I}}"'] & {}^{\boxslash}(\mathcal{I}^{\boxslash}) \end{tikzcd}$$

Proof. Dual to the prove of the previous claim. $\square$

We proceed to prove the proposition. Let us first suppose that $({}^{\boxslash}(a^{\boxslash}), b, {}^{\boxslash}(a^{\boxslash}))$ is a morphism of GF-structures, and let $j \in \mathcal{J}$ over $\nu_j : D_i \to C_i$. Consider the following diagram:

$$\begin{tikzcd} \mathcal{I} /\!\!/ C_j \arrow[r, "\eta_{\mathcal{I}} /\!\!/ C_j"] \arrow[d, "a /\!\!/ j"'] & {}^{\boxslash}(\mathcal{I}^{\boxslash}) /\!\!/ C_j \arrow[r, "\nu_j^*"] \arrow[d, "{}^{\boxslash}(a^{\boxslash}) /\!\!/ j"] & {}^{\boxslash}(\mathcal{I}^{\boxslash}) /\!\!/ D_j \arrow[d, "{}^{\boxslash}(a^{\boxslash}) /\!\!/ j"] \\ \mathcal{I}' /\!\!/ GC_j \arrow[r, "\eta'_{\mathcal{I}} /\!\!/ GC_j"'] & {}^{\boxslash}(\mathcal{I}'^{\boxslash}) /\!\!/ GC_j \arrow[r, "(\nu'_{bj})^*"'] & {}^{\boxslash}(\mathcal{I}'^{\boxslash}) /\!\!/ GD_j \end{tikzcd}$$

the left square commutes by Claim 6.4.2.2 and the fact that orthogonality commutes with slicing [GS17, Proposition 5.3]. The right square commutes by definition of morphism of GF-structures. Thus $(a, b, {}^{\boxslash}(a^{\boxslash}))$ is a morphism of GF-structures.

Let us now suppose that $(a, b, {}^{\boxslash}(a^{\boxslash}))$ is a morphism of GF-structures, let $j \in \mathcal{J}$ and consider the following diagram:

$$\begin{tikzcd}
\mathcal{I}^{\boxslash} /\!\!/ D_j \arrow[r, "\Pi_{v_j}"] \arrow[d, "a^{\boxslash} /\!\!/ j"'] & \mathcal{I}^{\boxslash} /\!\!/ C_j \arrow[r, "\epsilon_{\mathcal{I}^{\boxslash}} /\!\!/ C_j"] \arrow[d, "a^{\boxslash} /\!\!/ j"'] & ({}^{\boxslash}(\mathcal{I}^{\boxslash}))^{\boxslash} /\!\!/ C_j \arrow[d, "({}^{\boxslash}(a^{\boxslash}))^{\boxslash} /\!\!/ j"] \\
\mathcal{I}'^{\boxslash} /\!\!/ GD_j \arrow[r, "\Pi_{v'_{bj}}"'] & \mathcal{I}'^{\boxslash} /\!\!/ GC_j \arrow[r, "\epsilon_{\mathcal{I}'^{\boxslash}} /\!\!/ GC_j"'] & ({}^{\boxslash}(\mathcal{I}'^{\boxslash}))^{\boxslash} /\!\!/ GC_j
\end{tikzcd}$$

the left square commutes by Proposition 6.3.4 because $(a, b, {}^{\boxslash}(a^{\boxslash}))$ is a morphism of GF-strucutes and the right square commutes by Claim 6.4.2.1. Using Proposition 6.3.4 again, we see that $({}^{\boxslash}(a^{\boxslash}), b, {}^{\boxslash}(a^{\boxslash}))$ is a morphism of GF-structures. ☐

Lemma 6.4.3 (Proposition 6.10). *Let* $u_t, z_t : \mathcal{I}_t, \mathcal{K}_t \to \mathbb{C}^{\to}$ *for* $t \in \{1, 2\}$ *and* $v : \mathcal{J} \to \mathbb{C}^{\to}$ *be category of arrows. Suppose* $(\mathcal{I}_1, \mathcal{J}, \mathcal{K}_1, PB_1)$ *and* $(\mathcal{I}_2, \mathcal{J}, \mathcal{K}_2, PB_2)$ *are GF-structures. These induce GF-structures:*

$$(\mathcal{I}_1 \times_{\mathbb{C}^{\to}} \mathcal{I}_2, \mathcal{J}, \mathcal{K}_1 \times_{\mathbb{C}^{\to}} \mathcal{K}_2, PB_{\times}) \qquad (\mathcal{I}_1 +_{\mathbb{C}^{\to}} \mathcal{I}_2, \mathcal{J}, \mathcal{K}_1 +_{\mathbb{C}^{\to}} \mathcal{K}_2, PB_{+})$$

Moreover, given $u'_t, z'_t : \mathcal{I}'_t, \mathcal{K}'_t \to \mathbb{D}^{\to}$ *for* $t \in \{1, 2\}$ *and* $v' : \mathcal{J}' \to \mathbb{D}^{\to}$*, GF-structures* $(\mathcal{I}'_1, \mathcal{J}', \mathcal{K}'_1, PB'_1)$ *and* $(\mathcal{I}'_2, \mathcal{J}', \mathcal{K}'_2, PB'_2)$ *and morphisms of GF-structures over a GF-adjunction* $T \vdash G$*:*

$$(\mathcal{I}_1, \mathcal{J}, \mathcal{K}_1, PB_1) \xrightarrow{(a_1, b, c_1)} (\mathcal{I}'_1, \mathcal{J}', \mathcal{K}'_1, PB'_1)$$

$$(\mathcal{I}_2, \mathcal{J}, \mathcal{K}_2, PB_2) \xrightarrow{(a_2, b, c_2)} (\mathcal{I}'_2, \mathcal{J}', \mathcal{K}'_2, PB'_2)$$

they induce morphisms of GF-structures over $T \vdash G$*:*

$$(\mathcal{I}_1 \times_{\mathbb{C}^{\to}} \mathcal{I}_2, \mathcal{J}, \mathcal{K}_1 \times_{\mathbb{C}^{\to}} \mathcal{K}_2, PB_{\times}) \xrightarrow{(a_1 \times a_2, b, c_1 \times c_2)} (\mathcal{I}'_1 \times_{\mathbb{D}^{\to}} \mathcal{I}'_2, \mathcal{J}', \mathcal{K}'_1 \times_{\mathbb{D}^{\to}} \mathcal{K}'_2, PB'_{\times})$$

$$(\mathcal{I}_1 +_{\mathbb{C}^{\to}} \mathcal{I}_2, \mathcal{J}, \mathcal{K}_1 +_{\mathbb{C}^{\to}} \mathcal{K}_2, PB_{+}) \xrightarrow{(a_1 + a_2, b, c_1 + c_2)} (\mathcal{I}'_1 +_{\mathbb{D}^{\to}} \mathcal{I}'_2, \mathcal{J}', \mathcal{K}'_1 +_{\mathbb{D}^{\to}} \mathcal{K}'_2, PB'_{+})$$

Proof. We focus first on the pullback case. For this, let us consider an object $(X, i_1, i_2, a, b) \in (\mathcal{I}_1 \times_{\mathbb{C}^{\to}} \mathcal{I}_2) /\!\!/ \mathbb{C}$, that is $i_t \in \mathcal{I}_t$ for $t \in \{1, 2\}$ and the morphism $u_1(i_1) = u_2(i_2) = l : A \to B$ is in the slice over X via the maps $a : A \to X$ and $b : B \to X$. Now given $j \in \mathcal{J}$ over $v_j : Y \to X$, we can use both the GF-structures PB_1 and PB_2 to pullback (X, i_1, i_2, a, b) along v_j to an object $(Y, PB_1(i_1), PB_2(i_2), v_j^*(a), v_j^*(b)) \in (\mathcal{K}_1 \times_{\mathbb{C}^{\to}} \mathcal{K}_2) /\!\!/ \mathbb{C}$. It is clear that this operation can be done functorially thus giving the desired GF-structure $PB_{\times}$.

The coproduct case is similar. An object of $\mathcal{I}_1 +_{\mathbb{C}^{\to}} \mathcal{I}_2$ is a pair (t, i) where $t \in \{1, 2\}$ and $i \in \mathcal{I}_t$, this is a category of arrows via the map $(t, i) \mapsto u_t(i)$. Now consider an object $(X, (t, i), a, b) \in (\mathcal{I}_1 +_{\mathbb{C}^{\to}} \mathcal{I}_2) /\!\!/ \mathbb{C}$, that is $u_t(i)$ is in the slice over X via a and

b. Given $j \in \mathcal{J}$ over $v_j : Y \to X$, we can pullback $(X, (t, i), a, b)$ along v_j by using either PB_1 if $t = 1$ or PB_2 if $t = 2$, either way we obtain $(Y, (t, PB_t(i)), v_j^*(a), v_j^*(b)) \in (\mathcal{K}_1 +_{\mathbb{C}^{\to}} \mathcal{K}_2) \mathbin{/\!\!/} \mathbb{C}$. This can be done functorially, giving PB_+.

Let us focus now on the functorial part for the pullback case. Following the definition of morphism of GF-structures, we need to show that for $j \in \mathcal{J}$ over $v_j : D_j \to C_j$ the following diagram commutes:

$$\begin{tikzcd}
(\mathcal{J}_1 \times_{\mathbb{C}^{\to}} \mathcal{J}_2) \mathbin{/\!\!/} C_j \arrow[r, "v_j^*"] \arrow[d, "(a_1 \times a_2)/\!\!/j"'] & (\mathcal{K}_1 \times_{\mathbb{C}^{\to}} \mathcal{K}_2) \mathbin{/\!\!/} D_j \arrow[d, "(c_1 \times c_2)/\!\!/j"] \\
(\mathcal{J}'_1 \times_{\mathbb{D}^{\to}} \mathcal{J}'_2) \mathbin{/\!\!/} GC_j \arrow[r, "(v_{bj})^*"'] & (\mathcal{K}'_1 \times_{\mathbb{D}^{\to}} \mathcal{K}'_2) \mathbin{/\!\!/} GD_j
\end{tikzcd}$$

it is clear that by construction this follows from the fact that (a_t, b, c_t) are morphisms of GF-structures for $t \in \{1, 2\}$. A dual argument shows that $(a_1 + a_2, b, c_1 + c_2)$ is a morphism of GF-structures. □

We will now proceed to generalise some results from section 8 of [GS17]. Recall from Lemma C.5.3 that the structure of a k-strong homotopy equivalence for a map $f : X \to Y$ can be equivalently described as that of a retract of the square $\theta\hat{\otimes}f$; that is a square $\rho_f : \delta_k\hat{\otimes}f \to f$ such that $\rho_f \circ \theta\hat{\otimes}f = id_f$.

Proposition 6.4.4. *Let* $\mathbb{C}\underset{G}{\overset{T}{\underset{\perp}{\leftrightarrows}}}\mathbb{D}$ *be an I-adjunction. Then for* $k \in \{1, 2\}$ *there is a lift of* T *to the categories of k-oriented strong homotopy equivalences as shown:*

$$\begin{tikzcd}
S_k \arrow[d] & S_k \arrow[l, "T"'] \arrow[d] \\
\mathbb{C}^{\to} & \mathbb{D}^{\to} \arrow[l, "T"]
\end{tikzcd}$$

Proof. We know that T preserves colimits, the monoidal strucuture and the interval object. Thus we have that $T(\delta_k\hat{\otimes}(-)) = \delta_k\hat{\otimes}T(-)$ and $T(\theta_k\hat{\otimes}(-)) = \theta_k\hat{\otimes}T(-)$. The result follows since functors preserve retractions. □

Lemma 6.4.5 (Lemma 8.4)**.** *There is a lift of* $\delta_k\hat{\otimes}(-) : \mathbb{C}^{\to} \to \mathbb{C}^{\to}$ *to the category of k-oriented strong homotopy equivalences* S_k *as shown:*

$$\begin{tikzcd}
 & S_k \arrow[d] \\
\mathbb{C}^{\to} \arrow[ru, "\delta_k\hat{\otimes}(-)"] \arrow[r, "\delta_k\hat{\otimes}(-)"'] & S_k
\end{tikzcd}$$

Moreover, if $\begin{tikzcd}\mathbb{C} \arrow[r, shift right, "G"'] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, shift right, "T"']\end{tikzcd}$ *be an I-adjunction, then this lift cohere with the lift of* T *as shown:*

$$\begin{tikzcd}
\mathbb{C}^{\to} \arrow[d, "\delta_k\hat{\otimes}(-)"'] & \mathbb{D}^{\to} \arrow[l, "T"'] \arrow[d, "\delta_k\hat{\otimes}(-)"] \\
S_k & S_k \arrow[l, "T"]
\end{tikzcd}$$

Proof. We will need to prove a claim first.

Claim 6.4.5.1 (Remark 8.3)**.** *There is a lift of* $\delta_k \otimes (-) : \mathbb{C} \to \mathbb{C}^{\to}$ *to* S_k *and these lifts cohere with* T *as shown:*

$$\begin{tikzcd}
\mathbb{C} \arrow[d, "\delta_k\otimes(-)"'] & \mathbb{D} \arrow[l, "T"'] \arrow[d, "\delta_k\otimes(-)"] \\
S_k & S_k \arrow[l, "T"]
\end{tikzcd}$$

Proof. This follows since for each $X \in \mathbb{C}$, the arrow $\delta_k \otimes (-) : X \to I \otimes X$ is a k-oriented strong deformation retract (in particular an homotopy equivalence) with the retraction given by the contraction $\epsilon \otimes X : I \otimes X \to X$ and the homotopy between $(\delta_k \otimes X) \circ (\epsilon \otimes X)$ and $\mathrm{id}_{I\otimes X}$ is given by the connection $c_k \otimes X : I \otimes I \otimes X \to I \otimes X$. It is clear that the previous diagram commutes, since T preserves all the structure of the interval I. □

To prove the lemma we will use some formal manipulations of the Leibniz construction. Consider the category $\mathrm{End}(\mathbb{C})$ of endofunctors of $\mathbb{C}$, notice that the functor $\delta_k \otimes (-) : \mathbb{C} \to \mathbb{C}^{\to}$ can be seen as an object of $\mathrm{End}(\mathbb{C})^{\to}$ i.e. as a natural transformation from $\mathrm{id}_{\mathbb{C}}$ to $I \otimes (-)$. Similarly, the transformation $\theta_k \otimes (-) : \perp \to \delta_k \otimes (-)$ may be regarded as a morphism in $\mathrm{End}(\mathbb{C})^{\to}$ (i.e. as a square of natural transformation); where $\perp : 0 \to \mathrm{id}_{\mathbb{C}}$ is the natural transformation whose component at $X \in \mathbb{C}$ is the unique arrow from the initial object to X. To avoid overloading the notation, we will denote $\delta_k \otimes (-)$ and $\theta_k \otimes (-)$ by δ_k and θ_k respectively; trusting the reader to distinguish between the two different meanings.

Let's denote $\mathrm{app} : \mathrm{End}(\mathbb{C}) \times \mathbb{C} \to \mathbb{C}$ the application functor (i.e. $\mathrm{app}(F, X) = FX$), and consider it's Leibniz construction $\mathrm{a\hat{p}p} : \mathrm{End}(\mathbb{C})^{\to} \times \mathbb{C}^{\to} \to \mathbb{C}^{\to}$, notice that:

$$\mathrm{a\hat{p}p}(\delta_k, f) = \delta_k\hat{\otimes}f \quad \text{and} \quad \mathrm{a\hat{p}p}(\theta_k, f) = \theta_k\hat{\otimes}f$$

Now let's denote by $- \circ - : \mathrm{End}(\mathbb{C}) \times \mathrm{End}(\mathbb{C}) \to \mathrm{End}(\mathbb{C})$ the composition functor. For any fixed $X \in \mathbb{C}$ we have that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{End}(\mathbb{C}) \times \mathbb{C} \arrow[r, "\mathrm{app}"] & \mathbb{C} \\
\mathrm{End}(\mathbb{C}) \times \mathrm{End}(\mathbb{C}) \arrow[u, "{\langle \mathrm{id}, \mathrm{app}(-,X) \rangle}"] \arrow[r, "-\circ-"'] & \mathrm{End}(\mathbb{C}) \arrow[u, "{\mathrm{app}(-,X)}"']
\end{tikzcd}$$

Applying the Leibniz construction (and using that $\mathsf{app}(-, X)$ preserves pullbacks) we obtain the following commutative square:

$$\begin{tikzcd}
\mathsf{End}(\mathbb{C})^{\to} \times \mathbb{C}^{\to} \arrow[r, "\widehat{\mathsf{app}}"] & \mathbb{C}^{\to} \\
\mathsf{End}(\mathbb{C})^{\to} \times \mathsf{End}(\mathbb{C})^{\to} \arrow[r, "-\hat{\circ}-"'] \arrow[u, "{\langle \mathsf{id}, \mathsf{app}(-,X)\rangle}"] & \mathsf{End}(\mathbb{C})^{\to} \arrow[u, "{\mathsf{app}(-,X)}"']
\end{tikzcd}$$

which implies that:

$$\begin{aligned}
\delta_k \widehat{\otimes}(\delta_k \otimes X) &= \delta_k \widehat{\otimes}(\mathsf{app}(\delta_k, X)) \\
&= \widehat{\mathsf{app}}(\delta_k, \mathsf{app}(\delta_k, X)) \\
&= (\delta_k \hat{\circ} \delta_k)(X)
\end{aligned}$$

We thus have that $\delta_k \widehat{\otimes}(\delta_k \otimes (-)) = \delta_k \hat{\circ} \delta_k$ since both are natural transformation with the same components. Similarly, we can show that,

$$\theta_k \widehat{\otimes}(\delta_k \otimes (-)) = \theta_k \hat{\circ} \delta_k : \delta_k \to \delta_k \hat{\circ} \delta_k$$

.

By Claim 6.4.5.1 and the alternative characterisation of homotopy equivalences as retracts of $\theta_k \widehat{\otimes}(-)$, we have that $\theta_k \widehat{\otimes}(\delta_k \otimes X)$ has a retract ρ_X for each X, it can be seen by the description of the retractions that ρ_X is natural in X, this gives, by Yoneda, a retract of ρ of $\theta_k \hat{\circ} \delta_k : \delta_k \to \delta_k \hat{\circ} \delta_k$.

Denote by $\mathsf{c} : \mathbb{C} \to \mathsf{End}(\mathbb{C})$ the functor that sends an object X to the constant endofunctor $\mathsf{c}(X) : \mathbb{C} \to \mathbb{C}$. Notice that:

$$\begin{tikzcd}
\mathsf{End}(\mathbb{C}) \times \mathbb{C} \arrow[r, "\mathsf{app}"] \arrow[d, "{\langle \mathsf{id}, \mathsf{c}\rangle}"'] & \mathbb{C} \arrow[d, "\mathsf{c}"] \\
\mathsf{End}(\mathbb{C}) \times \mathsf{End}(\mathbb{C}) \arrow[r, "-\circ-"'] & \mathsf{End}(\mathbb{C})
\end{tikzcd}$$

applying the Leibniz construction, using that c preserves pullbacks, we get:

$$\begin{tikzcd}
\mathsf{End}(\mathbb{C})^{\to} \times \mathbb{C}^{\to} \arrow[r, "\widehat{\mathsf{app}}"] \arrow[d, "{\langle \mathsf{id}, \mathsf{c}\rangle}"'] & \mathbb{C}^{\to} \arrow[d, "\mathsf{c}"] \\
\mathsf{End}(\mathbb{C})^{\to} \times \mathsf{End}(\mathbb{C})^{\to} \arrow[r, "-\hat{\circ}-"'] & \mathsf{End}(\mathbb{C})^{\to}
\end{tikzcd}$$

chasing the diagram we get for some $f \in \mathbb{C}^{\to}$:

$$\begin{aligned} c(\delta_k \hat{\otimes} (\delta_k \hat{\otimes} f)) &= c(a\hat{p}p(\delta_k, \delta_k \hat{\otimes} f)) \\ &= \delta_k \hat{\circ} c(\delta_k \hat{\otimes} f) \\ &= \delta_k \hat{\circ} (\delta_k \hat{\circ} c(f)) \\ &= (\delta_k \hat{\circ} \delta_k) \hat{\circ} c(f) \\ &= c(a\hat{p}p(\delta_k \hat{\circ} \delta_k, f)) = c((\delta_k \hat{\circ} \delta_k) \hat{\otimes} f) \end{aligned}$$

and thus in particular, we have that $\delta_k \hat{\otimes} (\delta_k \hat{\otimes} f) = (\delta_k \hat{\circ} \delta_k) \hat{\otimes} f)$. Similarly we obtain that $\theta_k \hat{\otimes} (\delta_k \hat{\otimes} f) = (\theta_k \hat{\circ} \delta_k) \hat{\otimes} f$.

Our first goal is to show that $\delta_k \hat{\otimes} f$ is an k-oriented strong homotopy equivalence, i.e. that there is a retract of $\theta_k \hat{\otimes} (\delta_k \hat{\otimes} f) = (\theta_k \hat{\circ} \delta_k) \hat{\otimes} f$ functorially on f, but since the functor (any functor) $(-) \hat{\otimes} f$ preserves section-retraction pairs, and as we saw, ρ is a retract of $\theta_k \hat{\circ} \delta_k$, we have that $\rho \hat{\otimes} f$ is a retract of the desired map. This is clearly functorial in f.

We now proceed to show the functorial action, that is, the coherence with the lift of T. For this let $g \in \mathbb{D}^{\to}$ and apply the lifted functor $\delta_k \hat{\otimes} (-)$, as we saw, this maps g to the pair $(g, \rho \hat{\otimes} g)$. Now applying the lift of T to this object, we obtain that $g \mapsto Tg$ and $\rho \hat{\otimes} g \mapsto T\rho \hat{\otimes} Tg$. Observe that the retraction ρ of $\theta_k \hat{\circ} \delta_k$ in $\mathbb{D}$ is mapped by T to the retraction ρ of $\theta_k \hat{\circ} \delta_k$ in $\mathbb{C}$; this follows easily from the construction of ρ and the functorial part of Claim 6.4.5.1. Thus we obtain that T maps $(g, \rho \hat{\otimes} g) \mapsto (Tg, \rho \hat{\otimes} Tg)$. □

Before proving the generalised version of [GS17, Lemma 8.5], we need some observations and some results. Consider $(T, G, \theta) : (C, F_t) \to (C', F'_t)$ a morphism of suitable awfs over a GFI-adjunction $\mathbb{C} \underset{G}{\overset{T}{\leftrightarrows}}_{\perp} \mathbb{D}$ where $\theta : \mathcal{I}' \to \mathcal{I}$ is a lift of T to the categories of arrows generating the suitable awfs's. Notice that by Theorem B.5.3, the lift $\theta : \mathcal{I}' \to \mathcal{I}$ of T induces lifts of T and G respectively to both the categories of (co)algebras for the (co)monads and for the (co)pointed endofunctors making the following diagrams commute:

$$\begin{tikzcd} C\textbf{-Coalg} \arrow[d] & C'\textbf{-Coalg} \arrow[l, "T"'] \arrow[d] \\ C\textbf{-Map} \arrow[d] & C'\textbf{-Map} \arrow[l, "T"'] \arrow[d] \\ \mathbb{C}^{\to} & \mathbb{D}^{\to} \arrow[l, "T"] \end{tikzcd} \qquad \begin{tikzcd} F_t\textbf{-Alg} \arrow[r, "G"] \arrow[d] & F'_t\textbf{-Alg} \arrow[d] \\ F_t\textbf{-Map} \arrow[r, "G"] \arrow[d] & F'_t\textbf{-Map} \arrow[d] \\ \mathbb{C}^{\to} \arrow[r, "G"'] & \mathbb{D}^{\to} \end{tikzcd}$$

Furthermore, since T preserves the monoidal structure and the interval objects we obtain a canonically induced lift as shown:

$$\begin{tikzcd}
\mathcal{J}_{\otimes} \arrow[d] & \mathcal{J}'_{\otimes} \arrow[l, "\theta_{\otimes}"'] \arrow[d] \\
\mathbb{C}^{\to} & \mathbb{D}^{\to} \arrow[l, "\mathsf{T}"]
\end{tikzcd}$$

defined in the obvious way as $\theta_{\otimes} = \theta + \theta$ using the universal property of $\mathcal{J}_{\theta} = \mathcal{J} + \mathcal{J}$ (and similarly for $\mathcal{J}'$). The diagram commutes since $\mathsf{T}(\delta_k \hat{\otimes} u'_i) = \delta_k \hat{\otimes} \mathsf{T}(u'_i) = \delta_k \hat{\otimes} u_{\theta i}$.

The lift $\theta_{\otimes} : \mathcal{J}'_{\otimes} \to \mathcal{J}_{\otimes}$ in turn induces (again by Theorem B.5.3) lifts of T and G respectively:

$$\begin{tikzcd}
\mathsf{C}_t\text{-}\mathbf{Coalg} \arrow[d] & \mathsf{C}'_t\text{-}\mathbf{Coalg} \arrow[l, "\mathsf{T}"'] \arrow[d] \\
\mathsf{C}_t\text{-}\mathbf{Map} \arrow[d] & \mathsf{C}'_t\text{-}\mathbf{Map} \arrow[l, "\mathsf{T}"'] \arrow[d] \\
\mathbb{C}^{\to} & \mathbb{D}^{\to} \arrow[l, "\mathsf{T}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{F}\text{-}\mathbf{Alg} \arrow[r, "\mathsf{G}"] \arrow[d] & \mathsf{F}'\text{-}\mathbf{Alg} \arrow[d] \\
\mathsf{F}\text{-}\mathbf{Map} \arrow[r, "\mathsf{G}"] \arrow[d] & \mathsf{F}'\text{-}\mathbf{Map} \arrow[d] \\
\mathbb{C}^{\to} \arrow[r, "\mathsf{G}"'] & \mathbb{D}^{\to}
\end{tikzcd}$$

Apart from the above observation, we will also make use of the algebraic counterpart of the classical fact that classes of left (or right) maps in a weak factorisation system are closed under retracts. For a category of arrows $u : \mathcal{J} \to \mathbb{C}^{\to}$ we denote by $\bar{u} : \bar{\mathcal{J}} \to \mathbb{C}^{\to}$ the category of arrows where the objects of $\bar{\mathcal{J}}$ are tuples (i, e, σ, ρ) where $i \in \mathcal{J}$, $e \in \mathbb{C}^{\to}$ and $\sigma : e \to u_i$ and $\rho : u_i \to e$ are arrows such that $\rho \circ \sigma = id_e$. A morphism in $\bar{\mathcal{J}}$ is of the form $(\theta, \kappa) : (i, e, \sigma, \rho) \to (i', e', \sigma', \rho')$ where $\theta : i \to i'$ in $\mathcal{J}$ and $\kappa : e \to e$ in $\mathbb{C}^{\to}$ making the obvious two squares commute. The map $\bar{u}$ is given by $(i, e, \sigma, \rho) \mapsto e$ (see the discussion before [GS17, Proposition 5.2]).

If we have an adjunction $\mathbb{C} \underset{\mathsf{G}}{\overset{\mathsf{T}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbb{D}$, categories of arrows $u : \mathcal{J} \to \mathbb{C}$ and $u' : \mathcal{J}' \to \mathbb{D}$ and a lift $a : \mathcal{J} \to \mathcal{J}'$ of G we obtain $\bar{a} : \bar{\mathcal{J}} \to \bar{\mathcal{J}}'$ given by $(i, e, \sigma, \rho) \mapsto (a(i), \mathsf{G}e, \mathsf{G}\sigma, \mathsf{G}\rho)$. Similarly for lift of T.

Lemma 6.4.6 (Proposition 5.2). *For every $u : \mathcal{J} \to \mathbb{C}^{\to}$, we have back and forth functors over $\mathbb{C}^{\to}$ as shown:*

$$\mathcal{J}^{\bar{\boxslash}} \longleftrightarrow \mathcal{J}^{\boxslash} \qquad\qquad {}^{\bar{\boxslash}}\mathcal{J} \longleftrightarrow {}^{\boxslash}\mathcal{J}$$

Moreover, if $\mathbb{C} \underset{\mathsf{G}}{\overset{\mathsf{T}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbb{D}$ is an adjunction and $u : \mathcal{J} \to \mathbb{C}$, $u' : \mathcal{J}' \to \mathbb{D}$ are categories of arrows and we have a lift $b : \mathcal{J} \to \mathcal{J}'$ of G and a lift $a : \mathcal{J}' \to \mathcal{J}$ of T, then the following *diagrams commute.*

$$\begin{tikzcd}
\bar{\mathcal{J}}^{\boxslash} \arrow[r, "\bar{a}^{\boxslash}"] \arrow[d, leftrightarrow] & \bar{\mathcal{J}}'^{\boxslash} \arrow[d, leftrightarrow] & & \bar{\mathcal{J}}^{\boxslash} \arrow[d, leftrightarrow] & \bar{\mathcal{J}}'^{\boxslash} \arrow[l, "\bar{b}^{\boxslash}"'] \arrow[d, leftrightarrow] \\
\mathcal{J}^{\boxslash} \arrow[r, "a^{\boxslash}"'] & \mathcal{J}'^{\boxslash} & & \mathcal{J}^{\boxslash} & \mathcal{J}'^{\boxslash} \arrow[l, "b^{\boxslash}"]
\end{tikzcd}$$

Proof. We will give a sketch of the proof leaving the details for the reader. For any category of arrows, we always obtain a functor $\mathcal{J} \to \bar{\mathcal{J}}$ by mapping $i \mapsto (i, u_i, id, id)$. This obviously commutes with adjunctions. Thus we only need to construct the function $\bar{\mathcal{J}}^{\boxslash} \to \mathcal{J}^{\boxslash}$.

We define the functor on objects as follows $((f,\theta), e, \sigma, \rho) \mapsto (e, \bar{\theta})$ where $\bar{\theta}$ is the right-$\mathcal{J}$ lifting structure defined as follows: consider a square $(l, m) : u_i \to e$, then:

$$\begin{tikzcd}
\cdot \arrow[r, "l"] \arrow[d, "u_i"'] & \cdot \arrow[r, "\sigma_1"] \arrow[d, "e"] & \cdot \arrow[r, "\rho_1"] \arrow[d, "f"] & \cdot \arrow[d, "e"] \\
\cdot \arrow[r, "m"'] \arrow[urr, dotted, "\theta"] & \cdot \arrow[r, "\sigma_2"'] & \cdot \arrow[r, "\rho_2"'] & \cdot
\end{tikzcd}$$

so we define $\bar{\theta}(l, m, i) = \rho_1\theta(\sigma_1 l, \sigma_2 m, i)$. There is an obvious way to extend this operation to a functor.

Now we proceed to observe that this definition commutes with adjunctions on the base category. Let us consider a situation as in the statement of the lemma; we will prove that the left diagram with the arrows pointing downwards commute. For this let $((f,\theta), e, \sigma, \rho) \in \bar{\mathcal{J}}^{\boxslash}$ and let us denote by $R : \bar{\mathcal{J}}^{\boxslash} \to \mathcal{J}^{\boxslash}$ the functor which was just defined. We obtain the following:

$$R\bar{a}^{\boxslash}((f,\theta), e, \sigma, \rho) = R((Gf, a\theta), Ge, G\sigma, G\rho) = (Ge, (\bar{a\theta}))$$

and

$$a^{\boxslash}R((f,\theta), e, \sigma, \rho) = a^{\boxslash}(e, \bar{\theta}) = (Ge, a\bar{\theta})$$

thus we have to verify that the two lifting structures $a\bar{\theta}$ and $(\bar{a\theta})$ are equal. So let $i \in \mathcal{J}'$ and $(l, m) : u'_i \to Ge$. We have that:

$$\begin{aligned}
(\bar{a\theta})(l, m, i) &= G\rho_1 \circ (a\theta(G(\sigma_1)l, G(\sigma_2)m, i)) \\
&= G\rho_1 \circ \hat{\theta}((G(\widehat{\sigma_1})l, G(\widehat{\sigma_2})m, a_i)) \\
&= G\rho_1 \circ \hat{\theta}(\sigma_1\hat{l}, \sigma_2\hat{m}, a_i) \\
&= \mathbf{trps}((\rho_1 \circ \theta(\sigma_1\hat{l}, \sigma_2\hat{m}, a_i))) \\
&= \mathbf{trps}(\bar{\theta}(\hat{l}, \hat{m}, a_i)) \\
&= a\bar{\theta}(l, m, i)
\end{aligned}$$

where **trps** denotes the operation of transposing along $G \vdash T$. $\square$

Lemma 6.4.7 (Corollary 7.7)**.** *Let* $(\mathsf{C}, \mathsf{F}_t)$ *be a suitable awfs, then there is a lift of* $\delta_k \hat{\otimes}(-)$ *as shown:*

$$\begin{tikzcd}
\mathsf{C}\text{-}\boldsymbol{Map} \arrow[r, dotted, "\delta_k \hat{\otimes}(-)"] \arrow[d] & \mathsf{C}_t\text{-}\boldsymbol{Map} \arrow[d] \\
\mathbb{C}^{\to} \arrow[r, "\delta_k \hat{\otimes}(-)"'] & \mathbb{C}^{\to}
\end{tikzcd}$$

Moreover, if $(\mathsf{T}, \mathsf{G}, \theta) : (\mathsf{C}, \mathsf{F}_t) \to (\mathsf{C}', \mathsf{F}_t')$ *is a morphism of suitable awfs, then the following diagram commutes:*

$$\begin{tikzcd}
\mathsf{C}\text{-}\boldsymbol{Map} \arrow[d, "\delta_k \hat{\otimes}(-)"'] & \mathsf{C}'\text{-}\boldsymbol{Map} \arrow[l, "\mathsf{T}"'] \arrow[d, "\delta_k \hat{\otimes}(-)"] \\
\mathsf{C}_t\text{-}\boldsymbol{Map} & \mathsf{C}_t'\text{-}\boldsymbol{Map} \arrow[l, "\mathsf{T}"]
\end{tikzcd}$$

Proof. We will make repeated use of Lemma 6.3.3. First note that by definition of $\mathcal{J}_{\otimes}$, there is a lift $\mathcal{J} \to \mathcal{J}_{\otimes}$ (given by the k-th inclusion) of $\delta_k \hat{\otimes}(-)$, and composing with the unit of the orthogonality adjunction, we get a lift $\mathcal{J} \to^{\boxslash} (\mathcal{J}_{\otimes}^{\boxslash})$ of the same functor. By orthogonality (i.e. Lemma 6.3.3) we get a lift $\mathcal{J}_{\otimes}^{\boxslash} \to \mathcal{J}^{\boxslash}$ of $\exp(\hat{\delta}_k, -)$ or equivalently a lift $\mathsf{F}\textbf{-Alg} \to \mathsf{F}_t\textbf{-Alg}$ by algebraic freeness. Composing with the counit this time, we obtain a lift $\mathsf{F}\textbf{-Alg} \to ({}^{\boxslash}\mathsf{F}_t\textbf{-Alg})^{\boxslash} = \mathsf{C}\textbf{-Map}^{\boxslash}$ and by orthogonality again we get a lift $\mathsf{C}\textbf{-Map} \to^{\boxslash} \mathsf{F}\textbf{-Alg} = \mathsf{C}_t\textbf{-Map}$ of $\delta_k \hat{\otimes}(-)$ as desired.

To verify the functoriality, let $(\mathsf{T}, \mathsf{G}, \theta) : (\mathsf{C}, \mathsf{F}_t) \to (\mathsf{C}', \mathsf{F}_t')$ be a morphism of suitable awfs, following the appropriate definitions, we see that there is a commutative square:

$$\begin{tikzcd}
\mathcal{J} \arrow[d] & \mathcal{J}' \arrow[l, "\theta"'] \arrow[d] \\
\mathcal{J}_{\otimes} & \mathcal{J}'_{\otimes} \arrow[l, "\theta_{\otimes}"]
\end{tikzcd}$$

where the vertical arrows are the functors over $\delta_k \hat{\otimes}(-)$. The rest follows from the functoriality of Lemma 6.3.3 and the coherence of the unit and counit of the orthogonality adjunction (Claim 6.4.2.2 and Claim 6.4.2.1). ☐

With these results in place, we can now prove the desired extension of [GS17, Lemma 8.5].

Lemma 6.4.8 (Lemma 8.5)**.** *Let* $(\mathsf{C}, \mathsf{F}_t)$ *be a suitable awfs on* $\mathbb{C}$ *algebraically free on* $\mathcal{J}$ *and denote by* S *the category of strong homotopy equivalences. There are functors in the slice over* $\mathbb{C}^{\to}$ *as shown:*

$$\begin{tikzcd}
\mathcal{J}_{\otimes} \arrow[r, "\mathsf{L}_1"] & \mathsf{C}\text{-}\boldsymbol{Map} \times_{\mathbb{C}^{\to}} \mathsf{S} \arrow[r, "\mathsf{L}_2"] & \mathsf{C}_t\text{-}\boldsymbol{Map}
\end{tikzcd}$$

Moreover, if $(T, G, \theta) : (C, F_t) \to (C', F'_t)$ *is a morphism of suitable awfs, then the following diagrams commute where the horizontal arrows are lifts of* T *to the respective categories of arrows:*

$$\begin{tikzcd}
\mathcal{J}_{\otimes} \arrow[d, "L_1"'] & & \mathcal{J}'_{\otimes} \arrow[ll, "\theta_{\otimes}"'] \arrow[d, "L'_1"] \\
C\text{-}\boldsymbol{Map} \times_{\mathbb{C}^{\to}} S \arrow[d, "L_2"'] & & C'\text{-}\boldsymbol{Map} \times_{\mathbb{D}^{\to}} S' \arrow[ll, "T \times_T T"'] \arrow[d, "L'_2"] \\
C_t\text{-}\boldsymbol{Map} & & C'_t\text{-}\boldsymbol{Map} \arrow[ll, "T"]
\end{tikzcd}$$

Proof. We define L_1 as follows. First we build functors as shown:

$$\begin{tikzcd}
\mathcal{J} \arrow[rr, "M^1_k"] \arrow[dr, "\delta_k \hat{\otimes} u"'] & & \textbf{C-Map} \arrow[dl] \\
& \mathbb{C}^{\to} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{J} \arrow[rr, "M^2_k"] \arrow[dr, "\delta_k \hat{\otimes} u"'] & & S_k \arrow[dl] \\
& \mathbb{C}^{\to} &
\end{tikzcd}$$

We construct M^1_k as the follow composite:

$$\begin{tikzcd}
\mathcal{J} \arrow[r, "\eta"] \arrow[d, "u"'] & \textbf{C-Map} \arrow[r, "\delta_k \hat{\otimes} (-)"] \arrow[d] & \textbf{C-Map} \arrow[d] \\
\mathbb{C}^{\to} \arrow[r, equal] & \mathbb{C}^{\to} \arrow[r, "\delta_k \otimes (-)"'] & \mathbb{C}^{\to}
\end{tikzcd}$$

the first square is the unit of the orthogonality adjunction and the second square is the lift given by one of the hypothesis of suitable awfs. The second functor M^2_k is given by Lemma 6.4.5 by first composing with the forgetful functor. Using the universal properties of coproducts, we obtain functors over $\mathbb{C}^{\to}$:

$$\mathcal{J}_{\otimes} \xrightarrow{M^1} \textbf{C-Map} \qquad\qquad \mathcal{J}_{\otimes} \xrightarrow{M^2} S$$

and using the universal property of pullbacks we obtain L_1.

To show the functoriality of L_1, it is enough to show the functoriality of M^t_k for $k, t \in \{1, 2\}$. So let's consider a morphism of suitable awfs $(T, G, \theta) : (C, F_t) \to (C', F'_t)$, by definition and by Claim 6.4.2.2 we obtain that M^1_k commutes with the lift of T. By Lemma 6.4.5 we have that M^2_k commutes with the lift of T.

We now proceed to construct $L_2 : \textbf{C-Map} \times_{\mathbb{C}^{\to}} S \to C_t\textbf{-Map}$. We will construct this functor using the following composite of functors over $\mathbb{C}$ for $k \in \{1, 2\}$:

$$\textbf{C-Map} \times_{\mathbb{C}^{\to}} S_k \xrightarrow{N_k} C_t\textbf{-}\bar{\textbf{M}}\textbf{ap} \xrightarrow{H} C_t\textbf{-Map}$$

the map N_k is given as follows. Consider an object $((g, \lambda), \rho) \in \mathbb{C}\text{-}\mathbf{Map} \times_{\mathbb{C}^{\to}} S_k$ where $(g, \lambda) \in \mathbb{C}\text{-}\mathbf{Map}$ and $\rho : \delta_k \hat{\otimes} g \to g$ is a retract of $\theta_k \hat{\otimes} g$ exhibiting g as a k-oriented strong homotopy equivalence. Then we have that $\delta_k \hat{\otimes} g \in \mathbb{C}_t\text{-}\mathbf{Map}$ by Lemma 6.4.7 and thus we define N_k as:

$$((g, \lambda), \rho) \mapsto (\delta_k \hat{\otimes} g, g, \theta_k \hat{\otimes} g, \rho)$$

The functor H is given by Lemma 6.4.6 since $\mathbb{C}_t\text{-}\mathbf{Map} =^{\boxslash} \mathsf{F}\text{-}\mathbf{Alg}$. The functorial part follows from the functorial parts of Lemma 6.4.7 and Lemma 6.4.6. ☐

The last result for which for which we need to develop the functorial extension is [GS17, Lemma 8.7]. This lemma contains one of the crucial arguments needed in order to obtain the functorial Frobenius structure in [GS17].

Lemma 6.4.9 (Lemma 8.7)**.** *Let* $(\mathbb{C}, \mathsf{F}_t)$ *be a suitable awfs on a category* $\mathbb{C}$*. Then the tuple* $(S, \mathsf{F}\text{-}\boldsymbol{Map}, S)$ *has a GF-structure. Moreover, if* $(T, G, \theta) : (\mathbb{C}, \mathsf{F}_t) \to (\mathbb{C}'\mathsf{F}'_t)$ *is a morphism of suitable awfs over a GFI-adjunction* $\mathbb{C}\underset{G}{\overset{T}{\underset{\perp}{\leftrightarrows}}}\mathbb{D}$*, then the following is a morphism of GF-structures:*

$$(S, \mathsf{F}\text{-}\boldsymbol{Alg}, S) \xrightarrow{(T,G,T)} (S', \mathsf{F}'\text{-}\boldsymbol{Alg}, S')$$

Proof. We will briefly recall the proof that $(S, \mathsf{F}\text{-}\mathbf{Alg}, S)$ has a GF-structure. The main point of the argument is to construct a lift of the pullback functor as in the following diagram:

$$\begin{tikzcd}
S_k \times_{\mathbb{C}} \mathsf{F}\text{-}\mathbf{Alg} \arrow[r, dotted, "PB"] \arrow[d] & S_k \arrow[d] \\
\mathbb{C}^{\to} \times_{\mathbb{C}} \mathbb{C}^{\to} \arrow[r, "PB"'] & \mathbb{C}^{\to}
\end{tikzcd}$$

On objects, the lift is given as follows. Let $(g, f) \in S_k \times_{\mathbb{C}} \mathsf{F}\text{-}\mathbf{Map}$ and consider the pullback square $\sigma = (h, f) : \bar{g} \to g$:

$$\begin{tikzcd}
A' \arrow[r, "h"] \arrow[d, "\bar{g}"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "g"] \\
X' \arrow[r, "f"'] & X
\end{tikzcd}$$

we need to show that $\bar{g} \in S_k$. Since $g \in S_k$, there is a retract $\rho : \delta_k \hat{\otimes} g \to g$ of $\theta_k \hat{\otimes} g : g \to \delta_k \hat{\otimes} g$. We construct a retract $\bar{\rho}$ of $\theta_k \hat{\otimes} \bar{g}$ fitting in the following diagram:

$$\begin{tikzcd}
\bar{g} \arrow[r, "\theta_k \hat{\otimes} \bar{g}"] \arrow[d, "\sigma"'] & \delta_k \hat{\otimes} \bar{g} \arrow[r, dotted, "\bar{\rho}"] \arrow[d, "\delta_k \hat{\otimes} \sigma"] & \bar{g} \arrow[d, "\sigma"] \\
g \arrow[r, "\theta_k \hat{\otimes} g"'] & \delta_k \hat{\otimes} g \arrow[r, "\rho"'] & g
\end{tikzcd}$$

and since σ is a Cartesian square, it is enough to find a retract in the above diagram when restricted to the codomains:

$$\begin{tikzcd}
X' \arrow[r, "\delta_{1-k}\hat{\otimes}X'"] \arrow[d, "f"'] & I \otimes X' \arrow[r, dotted, "\mathrm{cod}(\bar{\rho})"] \arrow[d, "I\otimes f"] & X' \arrow[d, "\sigma"] \\
X \arrow[r, "\delta_{1-k}\hat{\otimes}X"'] & I \otimes X \arrow[r, "\mathrm{cod}(\rho)"'] & X
\end{tikzcd}$$

which is equivalent to a lift in the following diagram:

$$\begin{tikzcd}
X' \arrow[rr, equal] \arrow[d, "\delta_{1-k}\otimes X'"'] & & X' \arrow[d, "f"] \\
I \otimes X' \arrow[urr, dotted, "\mathrm{cod}(\bar{\rho})"] \arrow[r, "I\otimes f"'] & I \otimes X \arrow[r, "\mathrm{cod}(\rho)"'] & X
\end{tikzcd}$$

and this is immediate since $f \in F$-**Alg** and $\delta_{1-k} \otimes X = \delta_{1-k}\hat{\otimes}\perp_X$ and $\perp_X \in C$-**Map** by hypothesis of suitable awfs. The action on morphisms is a consequence of the coherence of lifts in an awfs. We can combine the cases $k = 1, 2$ to obtain a lift $\mathsf{PB} : S \times_{\mathbb{C}} F$-**Alg** $\to S$.

Now, in order to obtain the GF-structure we argue as follows; we need a lift of the pullback functor to:

$$S \mathbin{/\!\!/} \mathbb{C} \times_{\mathbb{C}} F\text{-}\mathbf{Alg} \xrightarrow{\bar{\mathsf{PB}}} S \mathbin{/\!\!/} \mathbb{C}$$

to define this functor consider an object $((g, X), f) \in S \mathbin{/\!\!/} \mathbb{C} \times_{\mathbb{C}} F$-**Alg**, this is represented by the following diagram:

$$\begin{tikzcd}
 & A \arrow[dd] \arrow[dr, "g"] & \\
 & & B \arrow[dl, "l"] \\
X \arrow[r, "f"'] & Y &
\end{tikzcd}$$

first we pullback f along l to obtain a Cartesian square $\tau : f' \to f$ and by Lemma B.2.4 there is a unique F-**Alg** structure on f'. We then obtain the object $\bar{\mathsf{PB}}((g, X), f))$ by applying the lift $\mathsf{PB} : S \times_{\mathbb{C}} F$-**Alg** $\to S$ to f' and g. A similar argument applies for the morphism part.

Now we examine the functorial part. Let $(T, G, \theta) : (C, F_t) \to (C', F'_t)$ be a morphism of suitable awfs over a GFI adjunction $\begin{tikzcd} \mathbb{C} \arrow[r, shift right, "G"'] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, shift right, "T"'] \end{tikzcd}$ and notice that in order to obtain a morphism of GF-structures $(S, F\text{-}\mathbf{Alg}, S) \xrightarrow{(T,G,T)} (S', F'\text{-}\mathbf{Alg}, S')$ we need to verify that for each $f : X \to Y$ in F-**Alg**, the following diagram commutes:

$$\begin{tikzcd}
S /\!\!/ Y \arrow[r, "\bar{PB}(-{,}f)"] & S /\!\!/ X \\
S' /\!\!/ GY \arrow[u, "T/\!\!/Y"] \arrow[r, "\bar{PB}(-{,}Gf)"'] & S' /\!\!/ GX \arrow[u, "T/\!\!/X"']
\end{tikzcd}$$

For this, let's consider $g' : A' \to B'$ an object of S' in the slice over GY. We will first chase the diagram from the lower-right part, thus we first apply $\bar{PB}(-, Gf)$ which by the construction explained above, produces the following diagram:

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr, "\bar{g}'"] \arrow[dr, phantom, very near start, "\lrcorner"] \arrow[dd] & & D_i \arrow[dr, "u'_i"] \arrow[dd] & \\
& B \arrow[rr, crossing over, "f'"] \arrow[dr, phantom, very near start, "\lrcorner"] \arrow[dl] & & B' \arrow[dl] \\
GX \arrow[rr, "Gf"'] & & GY &
\end{tikzcd}$$

where $\bar{g}'$ has the structure of a strong homotopy equivalence given by a retract $\bar{\rho}' : \delta_k \hat{\otimes} \bar{g}' \to \bar{g}'$ of $\theta_k \hat{\otimes} \bar{g}'$ whose codomain is given by the lift $\mathbf{cod}(\bar{\rho}')$ as shown:

$$\begin{tikzcd}
B \arrow[rr, equal] \arrow[d, "\delta_{1-k} \otimes B"'] & & B \arrow[r] \arrow[d, "f'"] \arrow[dr, phantom, very near start, "\lrcorner"] & GX \arrow[d, "Gf"] \\
I \otimes B \arrow[r, "I \otimes f"'] \arrow[urr, dotted, "\mathbf{cod}(\bar{\rho}')"] \arrow[urrr, dotted, "l"'] & I \otimes B' \arrow[r, "\mathbf{cod}(\rho)"'] & B' \arrow[r] & GY
\end{tikzcd}$$

Notice that there is a further lift l as shown, such that both lifts in the diagram cohere since $f' \to Gf$ is a morphism of F-**Alg**. The lift of T to the categories of strong homotopy equivalences is given by applying T to the retract $\bar{\rho}'$, which restricted to the codomain looks as follows (applying T to the whole diagram):

$$\begin{tikzcd}
TB \arrow[rr, equal] \arrow[d, "\delta_{1-k} \otimes TB"'] & & TB \arrow[r] \arrow[d, "Tf'"] \arrow[dr, phantom, very near start, "\lrcorner"] & TGX \arrow[r, "\epsilon_X"] \arrow[d, "TGf"] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d, "f"] \\
I \otimes TB \arrow[r, "I \otimes Tf"'] \arrow[urr, dotted, "\mathbf{cod}(T\bar{\rho}')"] \arrow[urrr, dotted, "Tl"'] & I \otimes TB' \arrow[r, "\mathbf{cod}(T\rho)"'] & TB' \arrow[r] & TGY \arrow[r, "\epsilon_Y"'] & Y
\end{tikzcd}$$

we furthermore attach the counit diagram to the right of the diagram for future reference. The resulting strong homotopy equivalence structure of $T\bar{g}'$ over X is given by $T\bar{\rho}'$ which is uniquely induced by the lift $\mathbf{cod}(T\bar{\rho}')$ shown above.

We now chase the diagram through the left-upper part. So we first apply T and then pullback using $\mathsf{PB}(\bar{-}, f)$. We thus obtain the following diagram:

$$\begin{tikzcd}
TA \arrow[rr] \arrow[dr, "T\bar{g}'"] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dd] & & TD_i \arrow[dr, "Tu'_i"] \arrow[dd] & \\
& TB \arrow[rr, "Tf'"] \arrow[dl] \arrow[dr, phantom, "\lrcorner", very near start] & & TB' \arrow[dl] \\
TGX \arrow[rr, "TGf"'] \arrow[d, "\epsilon_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & & TGY \arrow[d, "\epsilon_Y"] & \\
X \arrow[rr, "f"'] & & Y &
\end{tikzcd}$$

Now, the strong homotopy equivalence structure on $T\bar{g}'$ is given by some retract ν of $\theta_k \hat{\otimes} T\bar{g}'$ which is induced by the homotopy equivalence structure $T\rho$ of Tu'_i and by the unique F-**Alg** structure on Tf' induced by the Cartesian square $Tf' \to f$ shown. This is captured in the following diagram:

$$\begin{tikzcd}
TB \arrow[rr, equal] \arrow[d, "\delta_{1-k}\otimes TB"'] & & TB \arrow[r] \arrow[d, "Tf'"] \arrow[dr, phantom, "\lrcorner", very near start] & TGX \arrow[r, "\epsilon_X"] \arrow[d, "TGf"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
I \otimes TB \arrow[r, "I\otimes Tf"'] \arrow[urr, dotted, "\mathsf{cod}(\nu)"] \arrow[urrrr, dotted, "m"'] & I \otimes TB' \arrow[r, "\mathsf{cod}(T\rho)"'] & TB' \arrow[r] & TGY \arrow[r, "\epsilon_Y"'] & Y
\end{tikzcd}$$

here m is the canonical lift given by the structure of F-algebra of f against the C_t-coalgebra $\delta_{1-k} \otimes TB$. Thus the proof is reduced to showing that the lifts $\mathsf{cod}(\nu)$ and $\mathsf{cod}(T\bar{\rho}')$ are the same. And in turn this reduces to proving that the lifts m and $\mathsf{cod}(T\bar{\rho}')$ cohere (since both squares on the right are Cartesian). As we already show, $\mathsf{cod}(T\bar{\rho}')$ and n cohere, thus we need only show that n and m cohere.

First notice that the C_t-map structure on $T(\delta_{1-k} \otimes B)$ and $\delta_{1-k} \otimes TB$ is the same, this follows by one defining property of a morphism of suitable awfs (mainly that T preserves the C-map structure on the maps $\perp_X : 0 \to X$) and by the functorial part of Lemma 6.4.7.

With this in place, the fact that the lifts n and m cohere follows from the way that the lifting structure of Gf is defined from that of f: given a lifting problem with Gf on the right, we first transpose the square (i.e. apply T and compose with the counit), then we solve the problem (in this case the solution is m) and then we transpose the solution, which means that $m = \epsilon_X \circ n$ which is exactly what we need. □

We now have all the pieces that we need in order to prove the main result of this section.

Theorem 6.4.10. *There is a functor:*

$$\mathsf{sAWFS}/(-)_{\mathsf{radj}} \longrightarrow \mathsf{GF}/(-)_{\mathsf{radj}}$$

which on objects is given by [GS17, Theorem 8.8]:

$$(\mathbb{C}, (C, F_t), \mathcal{I}) \mapsto (\mathbb{C}, (C_t\textbf{-}\boldsymbol{Map}, F\textbf{-}\boldsymbol{Alg}, C_t\textbf{-}\boldsymbol{Map}, \mathsf{PB}))$$

Proof. Consider a morphism of suitable awfs $(\mathsf{T}, \mathsf{G}, \theta) : (\mathsf{C}, \mathsf{F}_t) \to (\mathsf{C}', \mathsf{F}'_t)$ over a GFI-adjunction $\begin{tikzcd}\mathbb{C} \arrow[r, bend right, "\mathsf{G}"'] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, bend right, "\mathsf{T}"']\end{tikzcd}$.

The definition of morphism of suitable awfs implies that that the following is a morphism of GF-structures:

$$(\mathsf{C}\text{-}\mathbf{Map}, \mathbb{C}^{\to}, \mathsf{C}\text{-}\mathbf{Map}) \xrightarrow{(\mathsf{T},\mathsf{G},\mathsf{T})} (\mathsf{C}'\text{-}\mathbf{Map}, \mathbb{D}^{\to}, \mathsf{C}'\text{-}\mathbf{Map})$$

we use the functorial part of Lemma 6.4.1 to verify that the following is also a morphism of GF-strucutures

$$(\mathsf{C}\text{-}\mathbf{Map}, \mathsf{F}\text{-}\mathbf{Alg}, \mathsf{C}\text{-}\mathbf{Map}) \xrightarrow{(\mathsf{T},\mathsf{G},\mathsf{T})} (\mathsf{C}'\text{-}\mathbf{Map}, \mathsf{F}'\text{-}\mathbf{Alg}, \mathsf{C}'\text{-}\mathbf{Map})$$

Using Lemma 6.4.9 we also have a morphism of GF-structures

$$(\mathsf{S}, \mathsf{F}\text{-}\mathbf{Alg}, \mathsf{S}) \xrightarrow{(\mathsf{T},\mathsf{G},\mathsf{T})} (\mathsf{S}', \mathsf{F}'\text{-}\mathbf{Alg}, \mathsf{S}')$$

By Lemma 6.4.3 we can combine this two previous morphisms of GF-structures, in order to obtain a morphism of GF-structures:

$$(\mathsf{C}\text{-}\mathbf{Map} \times_{\mathbb{C}} \mathsf{S}, \mathsf{F}\text{-}\mathbf{Alg}, \mathsf{C}\text{-}\mathbf{Map} \times_{\mathbb{C}} \mathsf{S}) \xrightarrow{(\mathsf{T},\mathsf{G},\mathsf{T})} (\mathsf{C}'\text{-}\mathbf{Map} \times_{\mathbb{D}} \mathsf{S}', \mathsf{F}'\text{-}\mathbf{Alg}, \mathsf{C}'\text{-}\mathbf{Map} \times_{\mathbb{D}} \mathsf{S}')$$

Now using this last morphism, by Lemma 6.4.1 and by the functorial part of Lemma 6.4.8 we obtain that the following is a morphism of GF-structures:

$$(\mathcal{I}_{\otimes}, \mathsf{F}\text{-}\mathbf{Alg}, \mathsf{C}_t\text{-}\mathbf{Map}) \xrightarrow{(\theta_{\otimes},\mathsf{G},\mathsf{T})} (\mathcal{I}'_{\otimes}, \mathsf{F}'\text{-}\mathbf{Alg}, \mathsf{C}'_t\text{-}\mathbf{Map})$$

And finally, we apply the functorial part of Lemma 6.4.2 to obtain the desired morphism of GF-structures:

$$(\mathsf{C}_t\text{-}\mathbf{Map}, \mathsf{F}\text{-}\mathbf{Alg}, \mathsf{C}_t\text{-}\mathbf{Map}) \xrightarrow{(\mathsf{T},\mathsf{G},\mathsf{T})} (\mathsf{C}'_t\text{-}\mathbf{Map}, \mathsf{F}\text{-}\mathbf{Alg}, \mathsf{C}'_t\text{-}\mathbf{Map})$$

Finally we just mention that it this correspondence does satisfy the functor laws; i.e. it preserves composition and identities. This can be verified at each step of the proof. □

6.5 Suitable AWFS in Toposes: Functoriality

Let us examine the second main contribution from [GS17]. They described a way for obtaining suitable awfs in presheaf categories, equipped with a distinguished class of monomorphisms, closed under some operations. In the appendix we gave a slight generalisation of this result to include more generally Grothendieck toposes (see Theorem C.6.3).

In this section we will show that this construction admits a functorial description as well. We will use this to build examples of morphisms of type-theoretic awfs. We start by defining the category which will be the domain of this functor.

Definition 6.5.1. Denote by $\mathsf{sTopos}/(-)_{\mathsf{radj}}$ the category of **suitable toposes** consisting of:

- **Objects:** These are tuples $(\mathbb{C}, \mathbb{A}, \mathcal{M})$ where $\mathbb{C}$ is a Grothendieck topos equipped with a closed symmetric monoidal structure and with an interval object with contractions and connections. $\mathbb{A}$ is a dense subcategory of $\mathbb{C}$ and $\mathcal{M}$ is a subcategory of $\mathbb{C}^{\rightarrow}$ that satisfies the axioms of Theorem C.6.3.
- **Morphisms:** A morphism of suitable toposes, denoted by:

$$(\mathbb{C}, \mathbb{A}, \mathcal{M}) \xrightarrow{(\mathsf{T},\mathsf{G})} (\mathbb{D}, \mathbb{B}, \mathcal{M}')$$

  consists of a GFI-adjunction $\begin{tikzcd} \mathbb{C} \arrow[r, shift right, "\mathsf{G}"'] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, shift right, "\mathsf{T}"'] \end{tikzcd}$ such that:

  1. T restricts to the dense subcategories, i.e. $\mathbb{B} \xrightarrow{\mathsf{T}} \mathbb{A}$.
  2. T preserves the categories of monomorphisms, i.e. $\mathsf{Ti} \in \mathcal{M}$ for every $\mathsf{i} \in \mathcal{M}'$.

With this definition in place, we can now state and prove the main theorem of this section.

Theorem 6.5.2. *There is a functor:*

$$\mathsf{sTopos}/(-)_{\mathsf{radj}} \longrightarrow \mathsf{sAWFS}/(-)_{\mathsf{radj}}$$

whose action on objects is given by Theorem C.6.3.

Proof. We briefly review the action on objects since we will need it later. We start with $(\mathbb{C}, \mathbb{A}, \mathcal{M})$ an object of $\mathsf{sTopos}/(-)_{\mathsf{radj}}$. Define $\mathcal{I} = \{\mathsf{i} \in \mathcal{M} | \mathsf{cod}(\mathsf{i}) \in \mathbb{A}\}$ and let us consider it as a full subcategory of $\mathcal{M}$. By Garner's small object argument, we obtain an algebraically-free awfs $(\mathsf{C}, \mathsf{F}_{\mathsf{t}})$ on $\mathcal{I}$, which is also algebraically free on $\mathcal{M}$ as shown in the proof of Theorem C.6.3. We have to verify that $(\mathsf{C}, \mathsf{F}_{\mathsf{t}})$ is suitable in the sense of Definition C.4.1, by construction we have that $(\mathsf{C}, \mathsf{F}_{\mathsf{t}})$ is algebraically-free on $\mathcal{I}$, thus we need to show that every object is uniformly cofibrant and that **C-Map** is uniformly closed under pullback and Leibniz product with the endpoint inclusions of the interval.

We first show that every object is uniformly cofibrant. For this, we notice that we can lift the functor $\perp : \mathbb{C} \to \mathbb{C}^{\rightarrow}$ mapping $\mathsf{X} \mapsto \perp_{\mathsf{X}} : 0 \to \mathsf{X}$ to the subcategory $\mathcal{M}$ as shown:

$$\begin{tikzcd} & \mathcal{M} \arrow[d] \\ \mathbb{C} \arrow[ur, dotted, "\perp"] \arrow[r, "\perp"'] & \mathbb{C}^{\rightarrow} \end{tikzcd}$$

this follows easily from the requirement of $\mathcal{M}$ to contain all arrows $\perp_{\mathsf{X}} : 0 \to \mathsf{X}$ and from the fact that any square $\perp_{\mathsf{f}} : \perp_{\mathsf{X}} \to \perp_{\mathsf{Y}}$ induced by an arrow $\mathsf{f} : \mathsf{X} \to \mathsf{Y}$ is Cartesian.

Now, composing with the counit of the orthogonality adjunction, we obtain:

$$\begin{tikzcd}
 & \mathcal{M} \arrow[r, "\epsilon"] \arrow[d] & {}^{\boxslash}(\mathcal{M}^{\boxslash}) = \mathbf{C\text{-}Map} \arrow[dl] \\
\mathbb{C} \arrow[ur, "\perp"] \arrow[r, "\perp"'] & \mathbb{C}^{\to} &
\end{tikzcd}$$

Now, to show that $\mathbf{C\text{-}Map}$ is uniformly closed under pullbacks, we first notice that by definition $(\mathcal{M}, \mathbb{C}^{\to}, \mathcal{M})$ has a GF-structure. By Lemma 6.4.1, using the counit $\epsilon : \mathcal{M} \to \mathbf{C\text{-}Map}$, we obtain that $(\mathcal{M}, \mathbb{C}^{\to}, \mathbf{C\text{-}Map})$ has a GF-structure. But them by Lemma 6.4.2 we obtain that $(\mathbf{C\text{-}Map}, \mathbb{C}^{\to}, \mathbf{C\text{-}Map})$ has a GF-structure.

Lastly, we show that $\mathbf{C\text{-}Map}$ is uniformly closed under Leibniz product with endpoint inclusion. By the hypothesis on $\mathcal{M}$ and since $\delta_k \hat{\otimes}(-) : \mathbb{C}^{\to} \to \mathbb{C}^{\to}$ preserves Cartesian squares, we get a lift of $\delta_k \hat{\otimes}(-)$ to the category $\mathcal{M}$:

$$\mathcal{M} \xrightarrow{\delta_k \hat{\otimes}(-)} \mathcal{M}$$

composing to the right with $\epsilon : \mathcal{M} \to \mathbf{C\text{-}Map}$ and transposing we obtain:

$$F_t\text{-}\mathbf{Alg} \xrightarrow{\hat{\exp}(\delta_k, -)} F_t\text{-}\mathbf{Alg}$$

transposing again we get:

$$\mathbf{C\text{-}Map} \xrightarrow{\delta_k \hat{\otimes}(-)} \mathbf{C\text{-}Map}$$

We now verify the functoriality of the construction. Let us consider a morphism of suitable toposes $(\mathbb{C}, \mathbb{A}, \mathcal{M}) \xrightarrow{(T,G)} (\mathbb{D}, \mathbb{B}, \mathcal{M}')$. Let $\mathcal{I} = \{i \in \mathcal{M} | \mathtt{cod}(i) \in \mathbb{A}\}$ and $\mathcal{I}' = \{i \in \mathcal{M}' | \mathtt{cod}(i) \in \mathbb{B}\}$. Notice that we have a lift of T as shown:

$$\begin{tikzcd}
\mathcal{I} \arrow[d] & \mathcal{I}' \arrow[l, "\theta"'] \arrow[d] \\
\mathbb{C}^{\to} & \mathbb{D}^{\to} \arrow[l, "T"]
\end{tikzcd}$$

this follows since if $i \in \mathcal{I}'$ in particular $i \in \mathcal{M}'$ and by hypothesis, $Ti \in \mathcal{M}$ but also $\mathtt{cod}(Ti) = T\mathtt{cod}(i) \in \mathbb{A}$ since T restricts to the corresponding dense subcategories; thus $\theta(i) := Ti \in \mathcal{I}$.

We now have to show that $(C, F_t) \xrightarrow{(T,G,\theta)} (C', F'_t)$ is a morphism of suitable awfs. This follows essentially from the fact that on objects all the operations are functorial by the lemmas of the previous section, let's see how this works.

from the construction of the lift $\perp : \mathbb{C} \to \mathbf{C\text{-}Map}$, by making use of Claim 6.4.2.1, we obtain that T cohere with the C-map structure on the cofibrant objects.

Now, the map of suitable toposes $(\mathbb{C}, \mathbb{A}, \mathcal{M}) \xrightarrow{(T,G)} (\mathbb{D}, \mathbb{B}, \mathcal{M}')$ gives a morphism of GF-structures $(\mathcal{M}, \mathbb{C}^{\rightarrow}, \mathcal{M}) \xrightarrow{(T,G,T)} (\mathcal{M}', \mathbb{D}^{\rightarrow}, \mathcal{M}')$, and by using the functorial part of Lemma 6.4.1 and Lemma 6.4.2 we obtain a morphism of GF-structures

$$(\mathsf{C}\textbf{-Map}, \mathbb{C}^{\rightarrow}, \mathsf{C}\textbf{-Map}) \xrightarrow{(T,G,T)} (\mathsf{C}'\textbf{-Map}, \mathbb{D}^{\rightarrow}, \mathsf{C}'\textbf{-Map})$$

which immediately implies one of the requirements of a morphism of GF-structures.

Finally, the coherence of T and the lifts $\delta_k \hat{\otimes} (-)$ follows from the construction of the lift as explained above by evoking the functorial part of Lemma 6.3.3 twice. ☐

6.6 Compatibility with Id-types

Putting together the results from this chapter, we have shown that the main results for [GS17] admit the following functorial description:

$$\mathsf{sTopos}/(-)_{\mathsf{radj}} \longrightarrow \mathsf{sAWFS}/(-)_{\mathsf{radj}} \longrightarrow \mathsf{GF}/(-)_{\mathsf{radj}}$$

In this section we will show that our results on Id-types from Chapter 5 can also be lifted to the functorial setting. In order to do this it will be necessary to adapt the categories of suitable toposes, suitable awfs and GF structures to include the additional structure needed to accommodate Id-types.

Let us start by describing the additional structure we need to impose to $\mathsf{sTopos}/(-)_{\mathsf{radj}}$. Recall that an object of this category is a tuple $(\mathbb{C}, \mathbb{A}, \mathcal{M})$ as explained in Definition 6.5.1.

Definition 6.6.1. The category $\mathsf{sTopos{+}Id}/(-)_{\mathsf{radj}}$ of **suitable toposes compatible with Id-types** is defined as follows:

- **Objects:** The objects are objects $(\mathbb{C}, \mathbb{A}, \mathcal{M})$ of $\mathsf{sTopos}/(-)_{\mathsf{radj}}$ such that the subcategory of arrows $\mathcal{M}$ is closed under the following operations:
  1. Taking Leibniz product with the inclusion of the interval boundary $i : \partial I \rightarrow I$. That is if $m \in \mathcal{M}$, then $i \hat{\otimes} m \in \mathcal{M}$.
  2. For each $f : B \rightarrow A$ in $\mathbb{C}^{\rightarrow}$, the 'reflexivity' arrow $r_f : B \rightarrow P_w f$ constructed in Appendix C.3 is an object of $\mathcal{M}$.
- **Morphisms:** The morphisms are exactly those of $\mathsf{sTopos}/(-)_{\mathsf{radj}}$.

We now adapt the category of suitable awfs to be compatible with Id-types. But first, we will need to make an observation. Recall the construction of the interval path-object factorisation from Appendix C.3, and suppose that $\mathbb{C} \underset{G}{\overset{T}{\underset{\rightarrow}{\overset{\leftarrow}{\perp}}}} \mathbb{D}$ is a GFI-adjunction. It is easy to see that $G : \mathbb{C} \rightarrow \mathbb{D}$ preserves the corresponding path objects, i.e. if $f : B \rightarrow A$ in $\mathbb{C}$, then:

$$B \xrightarrow{r_f} P_w f \xrightarrow{\rho_f} B \times_A B \quad \overset{G}{\mapsto} \quad GB \xrightarrow{r_{Gf}} PGf \xrightarrow{\rho_{Gf}} GB \times_{GA} GB$$

this follows easily from the fact that G preserves limits, the interval object, and the exponentiation operation.

Definition 6.6.2. The category $sAWFS + \mathrm{Id}/(-)_{\mathrm{radj}}$ of **suitable awfs compatible with Id-types** is defined as follows:

- **Objects:** These are suitable awfs (C, F_t) over a category $\mathbb{C}$, that satisfies the additional hypothesis of Theorem 5.1.1.

- **Morphisms:** These are morphisms of suitable awfs $(T, G, \theta) : (C, F_t) \to (C', F'_t)$ over a GFI-adjunction $\mathbb{C} \underset{G}{\overset{T}{\leftrightarrows}} \mathbb{D}$ such that the following diagram commute:

$$\begin{tikzcd} C\text{-}\mathbf{Map} & C'\text{-}\mathbf{Map} \arrow[l, "T"'] \arrow[d, "i\hat{\otimes}(-)"] \\ C\text{-}\mathbf{Map} \arrow[from=u, "i\hat{\otimes}(-)"'] & C'\text{-}\mathbf{Map} \arrow[l, "T"] \end{tikzcd}$$

where $i : \partial I \to I$ is the boundary inclusion on the interval (notice that $Ti = i$). And for each $f : B \to A$ in $\mathbb{C}$ the following naturality square

$$\begin{tikzcd} TGB \arrow[r, "\epsilon_B"] \arrow[d, "Tr_{Gf}"'] & B \arrow[d, "r_f"] \\ TGP_w f \arrow[r, "\epsilon_{P_w f}"'] & P_w f \end{tikzcd}$$

is a morphism of C-maps.

We now proceed to adapt the category of GF-structures, whose objects we will more generally refer to as type-theoretic awfs in virtu of Definition 4.4.1.

Definition 6.6.3. The category $AM/(-)_{\mathrm{radj}}$ of **type-theoretic awfs** consists of:

- **Objects:** These are awfs (C_t, F) over a category $\mathbb{C}$ equipped with the following structure:

  1. A GF-structure (Definition B.6.3) on the tuple of arrow categories:

  $$(C_t\text{-}\mathbf{Map}, F\text{-}\mathbf{Map}, C_t\text{-}\mathbf{Map})$$

  2. A lift of the stable functorial choice of path objects constructed from the interval of $\mathbb{C}$ (as described on Appendix C.3):

  $$F\text{-}\mathbf{Map} \xrightarrow{\langle r,\rho \rangle} C_t\text{-}\mathbf{Map} \times_{\mathbb{C}} F\text{-}\mathbf{Map}$$

- **Morphisms:** A morphism of type-theoretic awfs $(T, G) : (C_t, F) \to (C'_t, F')$ consists of a morphism between the GF-structures of the objects:

$$\begin{tikzcd} (C_t\textbf{-Map}, F\textbf{-Alg}, C_t\textbf{-Map}) \arrow[r, "{(T,G,T)}"] & (C'_t\textbf{-Map}, F'\textbf{-Alg}, C'_t\textbf{-Map}) \end{tikzcd}$$

over a GFI-adjunction $\begin{tikzcd} \mathbb{C} \arrow[r, bend right, "G"'] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, bend right, "T"'] \end{tikzcd}$. Such that $G \vdash T$ is coherent with the lifts of the corresponding choices of path objects in the following sense: the following diagram commutes:

$$\begin{tikzcd} F\textbf{-Map} \arrow[r, "\rho"] \arrow[d, "G"'] & F\textbf{-Map} \arrow[d, "G"] \\ F'\textbf{-Map} \arrow[r, "\rho"'] & F'\textbf{-Map} \end{tikzcd}$$

and for each $f : B \to A$ in R**-Map**, the following naturality square of the counit:

$$\begin{tikzcd} TGB \arrow[r, "\epsilon_B"] \arrow[d, "Tr_{Gf}"'] & B \arrow[d, "r_f"] \\ TGP_w f \arrow[r, "\epsilon_{P_w f}"'] & P_w f \end{tikzcd}$$

is a morphism of C_t-maps.

With this in place, we will show that the functors described in the previous sections cohere also with the identity type structure, thus lifting to the categories defined above:

$$\begin{tikzcd} \mathsf{sTopos} + \mathsf{Id}/(-)_{\mathsf{radj}} \arrow[r] & \mathsf{sAWFS} + \mathsf{Id}/(-)_{\mathsf{radj}} \arrow[r] & \mathsf{AM}/(-)_{\mathsf{radj}} \end{tikzcd}$$

Let us first focus on the leftmost functor, i.e. from suitable topos to suitable awfs compatible with Id-types.

Theorem 6.6.4. *There is a functor:*

$$\begin{tikzcd} \mathsf{sTopos} + \mathsf{Id}/(-)_{\mathsf{radj}} \arrow[r] & \mathsf{sAWFS} + \mathsf{Id}/(-)_{\mathsf{radj}} \end{tikzcd}$$

which lifts the one from Theorem 6.5.2.

Proof. Consider $(\mathbb{C}, \mathbb{A}, \mathcal{M})$ a suitable topos compatible with Id-types. We apply the construction of Theorem 6.5.2 to obtain a suitable awfs (C, F_t) which is algebraically free on $\mathcal{I} = \{k \in \mathcal{M} | \mathsf{cod}(k) \in \mathbb{A}\}$.

We have to verify that it is compatible with Id-types, First, we compose with the counit of the orthogonality adjunction and apply orthogonality arguments to construct a lift:

$$\begin{tikzcd} C\textbf{-Map} \arrow[r, "i\hat{\otimes}(-)"] & C\textbf{-Map} \end{tikzcd}$$

from the lift:

$$\mathcal{M} \xrightarrow{i\hat{\otimes}(-)} \mathcal{M}$$

given by hypothesis (notice that $i\hat{\otimes}(-)$ preserves pullback squares). Now we have to verify that the functor $r : \mathbb{C}^{\to} \to \mathbb{C}^{\to}$ given on objects by $f \mapsto r_f$ lifts to $r : \mathbb{C}^{\to} \to$ C**-Map**. To see this, we first notice that by hypothesis, we already have a functor $r : \mathbb{C}^{\to} \to \mathcal{M}$, since r preserves pullback squares, and we obtain the desired lift by composing with the counit of the orthogonality adjunction $\epsilon : \mathcal{M} \to$ C**-Map**, using that $(\mathsf{C}, \mathsf{F}_t)$ is algebraically free on $\mathcal{M}$.

We now focus on the action of morphisms. Let $(\mathsf{T}, \mathsf{G}) : (\mathbb{C}, \mathbb{A}, \mathcal{M}) \to (\mathbb{D}, \mathbb{B}, \mathcal{M}')$ be a morphism of suitable toposes compatible with Id-types. By Theorem 6.5.2 we already have a morphism of suitable awfs $(\mathsf{T}, \mathsf{G}) : (\mathsf{C}, \mathsf{F}_t) \to (\mathsf{C}', \mathsf{F}'_t)$ we need to verify that is is compatible with Id-types. We see that the lifts of $i\hat{\otimes}(-)$ are compatible with the left adjoint T by applying the functorial part of the Claim 6.4.2.1 and of Lemma 6.3.3.

We now have to verify that for a given map $f : B \to A$, the naturality square of the counit of $\mathsf{T} \vdash \mathsf{G}$ applied to the reflexivity map $r_f : B \to P_w f$ is a morphism of C-maps. First, we notice that r_f and $\mathsf{T}r_{\mathsf{G}f}$ are both objects of $\mathcal{M}$ and since $\mathsf{T} \vdash \mathsf{G}$ is a GF-adjunction, the naturality square $\epsilon : \mathsf{T}r_{\mathsf{G}f} \to r_f$ is Cartesian, and thus it is a morphism in $\mathcal{M}$. But then applying the counit of the orthogonality adjunction $\mathcal{M} \to$ C**-Map** will produce a morphism of C-maps as required. ☐

Now we proceed to show that the rightmost functor, i.e. from suitable awfs compatible with Id-types to type-theoretic awfs, also lifts.

Theorem 6.6.5. *There is a functor:*

$$\mathsf{sAWFS} + \mathrm{Id}/(-)_{\mathsf{radj}} \longrightarrow \mathsf{AM}/(-)_{\mathsf{radj}}$$

which lifts the one from Theorem 6.4.10.

Proof. The action on object is given by Theorem 5.1.1: given $(\mathsf{C}, \mathsf{F}_t)$ a suitable awfs compatible with Id-types, there is an type-theoretic awfs on the awfs $(\mathsf{C}_t, \mathsf{F})$ of uniform fibrations.

Now, let's consider a morphism of suitable awfs compatible with Id-types $(\mathsf{T}, \mathsf{G}, \theta) :$ $(\mathsf{C}, \mathsf{F}_t) \to (\mathsf{C}', \mathsf{F}'_t)$ over a GFI-adjunction $\begin{tikzcd}\mathbb{C} \arrow[r, bend right, "\mathsf{G}"'] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, bend right, "\mathsf{T}"']\end{tikzcd}$. We know, from Theorem 6.4.10, that there is a morphism of GF-structures:

$$(\mathsf{C}_t\textbf{-Map}, \mathsf{F}\textbf{-Alg}, \mathsf{C}_t\textbf{-Map}) \xrightarrow{(\mathsf{T},\mathsf{G},\mathsf{T})} (\mathsf{C}'_t\textbf{-Map}, \mathsf{F}'\textbf{-Alg}, \mathsf{C}'_t\textbf{-Map})$$

as required. We need to show that the lifts of the adjunction $\mathsf{T} \vdash \mathsf{G}$ cohere with the lifts of the path objects.

First we show that $\mathsf{G} : \mathsf{F}\textbf{-Map} \to \mathsf{F}'\textbf{-Map}$ is compatible with the lifts of ρ to the categories of F and F' maps. Making use of the definition of morphism of suitable awfs and of the Lemma 6.4.5, Lemma 6.4.8 and Claim 6.4.2.2, we obtain that the following diagrams commute:

$$\begin{tikzcd}
\mathcal{J} \arrow[r, "\eta"] & \text{C-}\mathbf{Map} \arrow[r, "i\widehat{\otimes}(-)"] & \text{C-}\mathbf{Map} \arrow[r, "\delta_k\widehat{\otimes}(-)"] & \text{C-}\mathbf{Map} \times_{\mathbb{C}^{\to}} \mathcal{S}_k \arrow[r, "L_2"] & \text{C}_t\text{-}\mathbf{Map} \\
\mathcal{J}' \arrow[u, "\theta"] \arrow[r, "\eta"'] & \text{C}'\text{-}\mathbf{Map} \arrow[u, "T"'] \arrow[r, "i\widehat{\otimes}(-)"'] & \text{C}'\text{-}\mathbf{Map} \arrow[u, "T"] \arrow[r, "\delta_k\widehat{\otimes}(-)"'] & \text{C}'\text{-}\mathbf{Map} \times_{\mathbb{C}^{\to}} \mathcal{S}_k \arrow[u, "T\times T"] \arrow[r, "L_2"'] & \text{C}'_t\text{-}\mathbf{Map} \arrow[u, "T"']
\end{tikzcd} \tag{6.1}$$

and this implies using the symmetry of the monoidal structure and combining the cases $k = 0$ and $k = 1$ that:

$$\begin{tikzcd}
\mathcal{J}_{\otimes} \arrow[r, "i\widehat{\otimes}(-)"] & \text{C}_t\text{-}\mathbf{Map} \\
\mathcal{J}'_{\otimes} \arrow[u, "\theta_{\otimes}"] \arrow[r, "i\widehat{\otimes}(-)"'] & \text{C}'_t\text{-}\mathbf{Map} \arrow[u, "T"']
\end{tikzcd}$$

using the orthogonality Lemma 6.3.3 we obtain that the following diagram commutes:

$$\begin{tikzcd}
\text{F-}\mathbf{Alg} \arrow[r, "\widehat{\hom}(i{,}-)"] \arrow[d, "G"'] & \text{F-}\mathbf{Alg} \arrow[d, "G"] \\
\text{F}'\text{-}\mathbf{Alg} \arrow[r, "\widehat{\hom}(i{,}-)"'] & \text{F}'\text{-}\mathbf{Alg}
\end{tikzcd}$$

We recall from Eq. (5.2) that for $f : B \to A$ in F-**Alg**, the map $\rho_f : P_w f \to B \times_A B$ arrises as the pullback of $\widehat{\hom}(i, f) : B^I \to A_I \times_{A^{\partial I}} B^{\partial I}$ along the map $\langle \alpha_f, \lambda_f \rangle : B \times_A B \to A_I \times_{A^{\partial I}} B^{\partial I}$. Morally, the map $\langle \alpha_f, \lambda_f \rangle$ maps a pair of objects (b_1, b_2) of B (in the same fibre over A) to the pair $(\mathtt{refl}_{fb_1}, (b_1, b_2))$ that consists of the constant loop $\mathtt{refl}_{fb_1} \in A^I$ and the pair $(b_1, b_2) \in B^{\partial I}$. It is easy to verify that these morphisms are preserved by the application of the functor G, that is $G\langle \alpha_f, \lambda_f \rangle = \langle \alpha_{Gf}, \lambda_{Gf} \rangle$; this follows by the fact that G preserves pullbacks, products, the interval object and exponentiation.

Now, there are two ways to assign an F-**Alg** structure to the map ρ_{Gf}. Either by first pulling back $\widehat{\hom}(i, f)$ along $\langle \alpha_f, \lambda_f \rangle$ and then applying G or by first applying G to obtain $G\widehat{\hom}(i, f) = \widehat{\hom}(i, Gf)$ and then pullback along $G\langle \alpha_f, \lambda_f \rangle = \langle \alpha_{Gf}, \lambda_{Gf} \rangle$. It follows easily from the previous diagram and from Lemma B.2.4 that these two F-algebra structures are in fact the same one, this completes the proof that the desired diagram commutes for objects. The result for morphisms of F-algebras follows from faith-fullness of the forgetful functor F′-**Alg** $\to \mathbb{D}^{\to}$.

We proceed to show that for each $f : B \to A$ in F-**Map** the naturality $\epsilon_{r_f} : Tr_{Gf} \to r_f$ square of the counit of $T \vdash G$ is a morphism of C_t-maps. We know by hypotesis that it is a morphism of C-maps, so it will be sufficient to show that it is a morphism of strong homotopy equivalences (by Lemma 6.4.8).

Let's recall from the proof of Theorem 5.1.1, that for an F-map $f : B \to A$, the target map $t_f : P_w f \to B$ is a strong deformation retract of $r_f : B \to P_w f$ where the homotopy $\mathsf{H}^{\mathsf{f}} : \mathsf{P}_w\mathsf{f} \to \mathsf{P}_w\mathsf{f}^{\mathrm{I}}$ from $\mathsf{r}_{\mathsf{f}} \circ \mathsf{t}_{\mathsf{f}}$ to $\mathsf{id}_{\mathsf{P}_w\mathsf{f}}$ is given as the following lift:

$$\begin{tikzcd}
 & \mathsf{P}_w\mathsf{f}^{\mathrm{I}} \arrow[d, "\widehat{\mathrm{hom}}(\mathrm{i},\mathrm{t}_{\mathrm{f}})"] \\
\mathsf{P}_w\mathsf{f} \arrow[ur, dotted, "\mathsf{H}^{\mathsf{f}}"] \arrow[r, "\bar{\mathsf{H}}^{\mathsf{f}}"'] & \mathsf{B}^{\mathrm{I}} \times_{\mathsf{B}^{\partial\mathrm{I}}} \mathsf{P}_w\mathsf{f}^{\partial\mathrm{I}}
\end{tikzcd}$$

and the map $\bar{\mathsf{H}}^{\mathsf{f}}$ is morally given by taking a path $\mathsf{p} : \mathsf{b}_1 \rightsquigarrow \mathsf{b}_2 \in \mathsf{P}_w\mathsf{f}$ to the pair $(\mathsf{refl}_{\mathsf{b}_2}, (\mathsf{r}_{\mathsf{f}}\mathsf{t}_{\mathsf{f}}(\mathsf{p}), \mathsf{p})) \in \mathsf{B}^{\mathrm{I}} \times_{\mathsf{B}^{\partial\mathrm{I}}} \mathsf{P}_w\mathsf{f}^{\partial\mathrm{I}}$ where $\mathsf{relf}_{\mathsf{b}_2} \in \mathsf{B}^{\mathrm{I}}$ is the constant path and $(\mathsf{refl}_{\mathsf{b}_2}, \mathsf{p}) \in \mathsf{P}_w\mathsf{f}^{\partial\mathsf{f}}$ is a pair of paths in $\mathsf{P}_w\mathsf{f}$.

Now to show that $\epsilon_{\mathsf{r}_{\mathsf{f}}} : \mathsf{Tr}_{\mathsf{Gf}} \to \mathsf{r}_{\mathsf{f}}$ is a morphism if strong homotopy equivalences Definition C.5.1, it is enough to show that the following diagram commutes:

$$\begin{tikzcd}
\mathsf{T}(\mathsf{PGf}) \arrow[r, "\epsilon_{\mathsf{P}_w\mathsf{f}}"] \arrow[d, "\mathsf{TH}^{\mathsf{Gf}}"'] & \mathsf{P}_w\mathsf{f} \arrow[d, "\mathsf{H}^{\mathsf{f}}"] \\
\mathsf{T}(\mathsf{PGf})^{\mathrm{I}} \arrow[r, "\epsilon^{\mathrm{I}}_{\mathsf{P}_w\mathsf{f}}"'] & \mathsf{P}_w\mathsf{f}^{\mathrm{I}}
\end{tikzcd}$$

it is equally well to show that the transpose under $\mathsf{T} \vdash \mathsf{G}$ commutes, but transposing the counit gives the identity. Thus it will be sufficient to show that $\mathsf{H}^{\mathsf{Gf}} \cong \mathsf{GH}^{\mathsf{f}}$.

Notice first that GH^{f} is the canonical lift that results from the lifting situation of the (cofibrant) object $\mathsf{G}(\mathsf{P}_w\mathsf{f})$ against $\mathsf{G}(\widehat{\mathrm{hom}}(\mathsf{i}, \mathsf{t}_{\mathsf{f}}))$, let's explain why this follows; the canonical lift we need is given by first transposing the lifting situation, then lifting, and then transposing the lift back, and recall that in a suitable awfs, every object is uniformly cofibrant, in particular any square between arrows from the terminal object is a morphism of C-maps, thus the lifts in the following diagram cohere:

$$\begin{tikzcd}
0 \arrow[r, equal] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdot \arrow[d, "\widehat{\mathrm{hom}}(\mathrm{i},\mathrm{t}_{\mathrm{f}})"] \\
\mathsf{TG}(\mathsf{P}_w\mathsf{f}) \arrow[r, "\epsilon_{\mathsf{P}_w\mathsf{f}}"'] \arrow[urr, dotted] & \mathsf{P}_w\mathsf{f} \arrow[r, "\bar{\mathsf{H}}^{\mathsf{f}}"'] \arrow[ur, dotted, "\mathsf{H}^{\mathsf{f}}"] & \cdot
\end{tikzcd}$$

so the canonical lift we need is the transpose of $\mathsf{H}^{\mathsf{f}} \circ \epsilon_{\mathsf{P}_w\mathsf{f}}$ which is just GH^{f}.

Now, it is clear from the construction that $\mathsf{G}(\bar{\mathsf{H}}^{\mathsf{f}}) \cong (\bar{\mathsf{H}})^{\mathsf{Gf}}$. Thus to show that the lifts H^{Gf} and GH^{f} are the same, it is sufficient to show that the two F_{t}-map structures on $\mathsf{G}\widehat{\mathrm{hom}}(\mathsf{i}, \mathsf{t}_{\mathsf{f}}) \cong \widehat{\mathrm{hom}}(\mathsf{i}, \mathsf{t}_{\mathsf{Gf}})$ are the same.

We first notice that by virtue of the orthogonality Lemma 6.3.3 and the hypothesis that $(\mathsf{T}, \mathsf{G}, \theta)$ is a morphism of suitable awfs structures, the following square commutes:

$$\begin{tikzcd}
\mathsf{F}_{\mathsf{t}}\textbf{-Map} \arrow[r, "\mathsf{G}"] \arrow[d, "\widehat{\mathrm{hom}}(\mathrm{i},-)"'] & \mathsf{F}'_{\mathsf{t}}\textbf{-Map} \arrow[d, "\widehat{\mathrm{hom}}(\mathrm{i},-)"] \\
\mathsf{F}_{\mathsf{t}}\textbf{-Map} \arrow[r, "\mathsf{G}"'] & \mathsf{F}'_{\mathsf{t}}\textbf{-Map}
\end{tikzcd}$$

So we have reduced the problem to showing that the two possible F_t-map structures on $\mathsf{G}t_f \cong t_{\mathsf{G}f}$ are equal. By looking at Eq. (5.2) we see that t_f is the pullback of $\widehat{\hom}(\delta_1, f)$ along the map $\langle \beta_f, \mathtt{id}_B \rangle$ where $\beta_f : B \to A^I$ is given by $b \mapsto \mathtt{refl}_{fb}$. Thus we see that $\mathsf{G}\langle \beta_f, \mathtt{id}_B \rangle = \langle \beta_{\mathsf{G}f}, \mathtt{id}_{\mathsf{G}B} \rangle$; using Lemma B.2.4 this means that two show that the two F_t-map structures on $\mathsf{G}t_f \cong t_{\mathsf{G}f}$ coincide, it is sufficient to show that the following diagram commutes:

$$\begin{tikzcd}
\mathsf{F}\text{-}\mathbf{Map} \arrow[r, "\mathsf{G}"] \arrow[d, "\widehat{\hom}(\delta_k{,}-)"'] & \mathsf{F}'\text{-}\mathbf{Map} \arrow[d, "\widehat{\hom}(\delta_k{,}-)"] \\
\mathsf{F}_t\text{-}\mathbf{Map} \arrow[r, "\mathsf{G}"'] & \mathsf{F}'_t\text{-}\mathbf{Map}
\end{tikzcd}$$

By the orthogonality Lemma 6.3.3 it is sufficient to show that the following diagram commutes:

$$\begin{tikzcd}
\mathsf{C}\text{-}\mathbf{Map} \arrow[d, "\delta_k \hat{\otimes} (-)"'] & \mathsf{C}'\text{-}\mathbf{Map} \arrow[l, "\mathsf{T}"'] \arrow[d, "\delta_k \hat{\otimes} (-)"] \\
\mathsf{C}_t\text{-}\mathbf{Map} & \mathsf{C}'_t\text{-}\mathbf{Map} \arrow[l, "\mathsf{T}"]
\end{tikzcd}$$

but this commutes, as we can see by pasting the rightmost two squares in Eq. (6.1). □

6.7 From Type-Theoretic AWFS to Comprehension Categories: Functoriality

The last piece we need to complete the functorial construction of models of Martin-Löf type theory, is to explain how the process of converting an type-theoretic awfs to a comprehension category equipped with pseudo-stable choices of Π, Σ and Id types; is also functorial. That is, we need to construct a functor of the following form:

$$\mathsf{AM}/(-)_{\mathtt{radj}} \longrightarrow \underline{\mathrm{CCat}}^{\Sigma}_{\Pi,\mathrm{Id}}$$

which on object is the construction explained on Theorem 4.4.2. In order to achieve this, we will first need to explain the category $\underline{\mathrm{CCat}}^{\Sigma}_{\Pi,\mathrm{Id}}$, paying special attention to the morphisms.

The objects of $\underline{\mathrm{CCat}}^{\Sigma}_{\Pi,\mathrm{Id}}$ are comprehension categories equipped with pseudo-stable choices of Π, Σ and Id-types, as seen in Chapter 2. Given to objects of $\underline{\mathrm{CCat}}^{\Sigma}_{\Pi,\mathrm{Id}}$ which we will denote $(\mathbb{C}, \rho, \chi)$ and $(\mathbb{D}, \rho', \chi')$ where $\mathbb{C}$ and $\mathbb{D}$ are the base categories. A morphism between these two objects, consists of a strict morphism of comprehension categories (Definition 1.2.3):

$$\begin{tikzcd}
\mathbb{E} \arrow[rr, "\bar{G}"] \arrow[dr, "\chi"] \arrow[ddr, "\rho"'] & & \mathbb{E}' \arrow[dr, "\chi'"] \arrow[ddr, "\rho'"'] & \\
& \mathbb{C}^{\to} \arrow[rr] \arrow[d] & & \mathbb{D}^{\to} \arrow[d] \\
& \mathbb{C} \arrow[rr, "G"'] & & \mathbb{D}
\end{tikzcd}$$

that preserves the psuedo-stable choices of Π, Σ and Id types up-to-isomorphism. We will make precise what we mean by this only for Id-type, leaving the reader to state the equivalent definitions for the other logical structures.

Let (Id, r, j) and (Id', r', j') be the pseudo-stable choices of Id-types associated to $(\mathbb{C}, \rho, \chi)$ and $(\mathbb{D}, \rho', \chi')$ respectively. We say that the morphism $(G, \bar{G}) : (\mathbb{C}, \rho, \chi) \to (\mathbb{D}, \rho', \chi')$ preserve the choices of Id-types up-to-isomprphism if for each dependent tuple (A, Γ) in $(\mathbb{C}, \rho, \chi)$ there is a vertical isomorphism over $G\Gamma$:

$$\eta_A : \bar{G}\mathrm{Id}_A \cong \mathrm{Id}'_{\bar{G}A}$$

which is natural in the obvious way. It must cohere with the reflexivity term, in the sense that $\eta_A \circ Gr_A = r'_{\bar{G}A}$; and with the elimination term in the following way: for each pair (C, c) where $C \in \mathbb{E}$ is over Id_A and c a section of C over r_A, the diagonal square (of pointed arrows) in the following diagram commutes:

$$\begin{tikzcd}
GA \arrow[rr, "c'"] \arrow[dr, equal] \arrow[dd, "r_{\bar{G}A}"'] & & C' \arrow[dr, "\cong"] \arrow[dd] & \\
& GA \arrow[rr, "Gc"] \arrow[dd, "Gr_A"] & & GC \arrow[dd] \\
\mathrm{Id}'_{\bar{G}A} \arrow[rr, equal] \arrow[dr, "\eta_A"', "\cong"] \arrow[uurr, dotted, "j'(C',c')"] & & \mathrm{Id}'_{\bar{G}A} \arrow[dr, "\eta_A"', "\cong"] & \\
& G\mathrm{Id}_A \arrow[rr, equal] \arrow[uurr, dotted, "Gj(C,c)"] & & G\mathrm{Id}_A
\end{tikzcd}$$

here C' is any reindexing of GC along η_A and c' is the only arrow making the upper square commute.

Theorem 6.7.1. *There is a functor:*

$$\mathsf{AM}/(-)_{radj} \longrightarrow \underline{\mathrm{CCat}}^{\Sigma}_{\Pi,\mathrm{Id}}$$

which on objects acts as described in Theorem 4.4.2.

Proof. The action on objects has been already described. We focus on the action on morphisms, for this, consider $(T, G) : (C_t, F) \to (C'_t, F')$ a morphism of type-theoretic awfs over a GFI-adjunction $\mathbb{C} \underset{G}{\overset{T}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbb{D}$. In particular we have a lift of the right adjoint $G : \mathbb{C} \to \mathbb{D}$ to the categories of F-maps, giving a strict morphism of the induced comprehension categories:

$$\begin{tikzcd}
F\text{-}\mathbf{Map} \arrow[rr, "G"] \arrow[dr, "u"] \arrow[ddr] & & F'\text{-}\mathbf{Map} \arrow[dr, "u"] \arrow[ddr] & \\
& \mathbb{C}^{\to} \arrow[rr] \arrow[d] & & \mathbb{D}^{\to} \arrow[d] \\
& \mathbb{C} \arrow[rr, "G"'] & & \mathbb{D}
\end{tikzcd}$$

Note that G preserves Cartesian morphisms because they are just pullback squares and G is right adjoint.

The first thing to notice is that the lift of G to the categories of F-maps is part of an adjunction of awfs, in particular it is a functor of double categories, meaning that it preserves the vertical composition structure of F-maps. This implies that G preserves the pseudo-stable choices of Σ-types, this is immediate by the description of such choices as seen in Proposition 4.1.4.

It is also easy to verify that G preserves the choices of Π-types. This follows by the description of such choices as seen in Proposition 4.2.1, using Proposition 6.3.4 and Proposition 6.3.5.

We will focus on the preservation of Id-types. Let us consider (Id, r, j) and (Id', r', j') the pseudo-stable choices of Id-types induced on $\mathbb{C}$ and $\mathbb{D}$ by the respective type-theoretic awfs as explained on the proof of Theorem 6.6.5. By construction, we have a canonical comparison isomorphism (of F'-maps) $\eta : G\mathrm{Id} \cong \mathrm{Id}'_G$, this follows by the construction of interval path-object factorisation (Appendix C.3), using the relevant property from the definition of morphisms of type-theoretic awfs (Definition 6.6.3). It is also clear by construction that η cohere with the reflexivity maps r and r'.

We now proceed to the difficult part which consists on showing that the isomorphism η cohere with the choice of elimination terms j and j' in the precise sense that was explained before. For this consider a pair (C, c) where $C \to \mathrm{Id}_A$ is in F-**Map** over Id_A and c a section of C over r_A. Consider the following diagram:

$$\begin{tikzcd}
TGA \arrow[r, "\epsilon_A"] \arrow[d, "Tr_{GA}"'] & A \arrow[r, "c"] \arrow[d, "r_A"] & C \arrow[d] \\
T\mathrm{Id}_{GA} \arrow[r, "\epsilon_{\mathrm{Id}_A}"'] \arrow[urr, dotted, "x"] & \mathrm{Id}_A \arrow[r, equal] \arrow[ur, dotted, "j(C,c)"] & \mathrm{Id}_A
\end{tikzcd}$$

the lifts x and $j(C, c)$ cohere because by hypothesis the square on the left is a morphism of C_t-maps. Applying G to the whole diagram and composing with the unit of $T \vdash G$

on the left we obtain the following:

$$\begin{tikzcd}
\mathsf{G}A \arrow[r, equal] \arrow[d, "r_{\mathsf{G}A}"'] & \mathsf{G}A \arrow[r, "\mathsf{G}c"] \arrow[d, "\mathsf{G}r_A"] & \mathsf{G}C \arrow[d] \\
\mathrm{Id}_{\mathsf{G}A} \arrow[r, "\cong", "\eta_A"'] \arrow[urr, dotted, "j'(\mathsf{G}C,\mathsf{G}c)"] & \mathsf{G}\mathrm{Id}_A \arrow[r, equal] \arrow[ur, dotted, "\mathsf{G}j(C,c)"'] & \mathsf{G}\mathrm{Id}_A
\end{tikzcd}$$

this follows because the lift x in the previous diagram is by definition the transpose of the lift $j'(\mathsf{G}C, \mathsf{G}c)$. Since $C' \to C$ is a morphism of F-**Map**, it follows immediately that the lifts $j'(C', c')$ and $\mathsf{G}j(C, c)$ cohere, as required. □

6.8 Examples

In this section we will look at some applications of the functoriality aspects of the theory of uniform fibrations, developed in this chapter.

Let us describe a general scenario. Consider a suitable topos compatible with Id-types $(\mathbb{D}, \mathbb{B}, \mathcal{M}')$(see Definition 6.6.1) where the symmetric monoidal structure is taken as the Cartesian one. Consider $\mathbb{C}$ a second topos, and let

$$\begin{tikzcd}
\mathbb{C} \arrow[r, bend right, "\mathsf{G}"'] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, bend right, "\mathsf{T}"']
\end{tikzcd}$$

be a geometric embedding; that is, G is full and faithful and T preserves finite limits. Notice that $\mathsf{G} \vdash \mathsf{T}$ is a GF-adjunction, since the counit of $\mathsf{G} \vdash \mathsf{T}$ is an isomorphism.

We will explain how to transfer the suitable topos strucutre on $\mathbb{D}$ to a suitable topos structure on $\mathbb{C}$. For this, let us define $\mathbb{A} := \mathsf{T}(\mathbb{B})$, which is easily seen to be a dense subcategory of $\mathbb{C}$. It is also easy to verify that if

$$(I, \delta^0, \delta^1, \epsilon, c^0, c^1)$$

is the interval object with contraction and connections in $\mathbb{D}$, then by applying T to all the structure, we obtain

$$(\mathsf{T}I, \mathsf{T}\delta^0, \mathsf{T}\delta^1, \mathsf{T}\epsilon, \mathsf{T}c^0, \mathsf{T}c^1)$$

which is a new interval object with contraction and connections in $\mathbb{C}$; moreover T will trivially preserves such interval objects. Finally, we define $\mathcal{M} := \mathsf{T}(\mathcal{M}')$.

It is straightforward to verify that $(\mathbb{C}, \mathbb{A}, \mathcal{M})$ is again a suitable topos compatible with Id-types. And moreover, the following result follows easily from the foregoing construction.

Theorem 6.8.1. *Let* $\begin{tikzcd} \mathbb{C} \arrow[r, bend right, "\mathsf{G}"'] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, bend right, "\mathsf{T}"'] \end{tikzcd}$ *be a geometric embedding of Grothendieck topos, such that* $\mathbb{D}$ *is equipped with an interval object with contraction and connections. Suppose that* $(\mathbb{D}, \mathbb{B}, \mathcal{M}')$ *is a structure of suitable topos compatible with Id-types and that* $(\mathbb{C}, \mathbb{A}, \mathcal{M})$ *is the induced structure on* $\mathbb{C}$ *described before. Then* $\mathsf{G} \vdash \mathsf{T}$ *is a morphism of suitable topos compatible with Id-types.*

Proof. By the above discussion, we have that $G \vdash T$ is a GFI-adjunction. We have that T preserves the dense subcategories by definition of $\mathbb{A}$ and finally we have that T preserves the subcategories of monomorphism since it preserves finite limits. $\square$

This theorem provides a uniform way of transporting type-theoretic awfs from a topos to a subtopos; and moreover connecting the resulting models via a morphism of type-theoretic awfs.

Let us see a concrete example of this construction. Consider the presheaf topos $\underline{\mathrm{sSet}}$ of simplicial sets equipped with the suitable topos structure $(\underline{\mathrm{sSet}}, \mathsf{y}\Delta, \mathcal{M}_{\mathrm{all}})$ where $\mathsf{y}\Delta$ is the dense subcategory of representables and $\mathcal{M}_{\mathrm{all}}$ consists of all monomophisms. Now, Theorem 6.8.1 allows us to transfer the structure along any embedding, in particular we can consider, for each $n \geq 0$, the following one:

$$\begin{tikzcd} \underline{\mathrm{Cosk}}_{n+1} \arrow[r, phantom, "\perp"] & \underline{\mathrm{sSet}} \arrow[l, bend right, "\mathrm{tr}_{(n+1)}"'] \arrow[l, leftarrow, bend left] \end{tikzcd}$$

where $\underline{\mathrm{Cosk}}_{n+1}$ is the full subcategory of $\underline{\mathrm{sSet}}$ on $(n+1)$-coskeletal objects (see for example [May92]); we thus obtain an type-theoretic awfs on type theory on $\underline{\mathrm{Cosk}}_{n+1}$ and a comparison morphism to the model on $\underline{\mathrm{sSet}}$.

It is well known that Kan $(n+1)$-coskeletal complexes are models of homotopy n-types, so intuitively what Theorem 6.8.1 is saying, is that we can coherently restrict the model of dependent type theory from $\underline{\mathrm{sSet}}$, whose uniform fibrant objects model homotopy types, to a model on $\underline{\mathrm{Cosk}}_{n+1}$, whose uniform fibrant objects model homotopy n-types.

Chapter 7

Type-Theoretic AWFS from Normal Uniform Fibrations

Let us recall the statement of Theorem 5.1.1. We showed that for a given a suitable awfs $(\mathsf{C}, \mathsf{F}_t)$ on a category $\mathbb{C}$, under some assumptions, the interval path-object factorisation from Appendix C.3 lifted to a stable functorial choice of path objects. One of the main requirements was that the first leg of the interval path-object factorisation $\mathsf{r} : \mathbb{C}^{\to} \to \mathbb{C}^{\to}$ lifted to the the category of C-maps on its codomain. As pointed out in Note 5.2.5, it is not possible to construct such a lift of r to the category of C-maps unless the category of generating monomorphisms consists only of the decidable ones.

In this chapter we will address the question of whether there exists a natural strengthening of the notion of a uniform fibration, such that we can prove a version of Theorem 5.1.1 without the requirement that first leg of the interval path-object factorisation lifts to the category of C-maps; while still conserving the functorial Frobenius structure. We achieve this by introducing the notion of *normality* to the theory of uniform fibrations.

7.1 Cloven Isofibrations and Uniform Fibrations

We begin the chapter by comparing the categories of uniform fibrations in simplicial sets and that of cloven isofibrations in the category of groupoids. This will serve as motivation for introducing a notion of normal uniform fibrations, which will appropriately generalise the structure of a normal isofibration on groupoids.

For this section, we will work with the suitable topos (Definition 6.5.1) of simplicial sets $(\underline{\mathrm{sSet}}, \Delta, \mathcal{M}_{\mathsf{all}})$ where the category of generating monomorphisms $\mathcal{M}_{\mathsf{all}}$ consists of all monos, and the dense subcategory is that of representables. We will also use a slightly differnet notation, we will denote by $\underline{\mathrm{UniFib}}$ the category of Uniform fibrations, while the category of cloven isofibrations will be denoted by $\underline{\mathrm{ClFib}}$.

In what follows, we will show that uniform fibrations are a generalisation of cloven isofibrations. Specifically, this means that there is a pullback diagram of categories as the following one:

$$\begin{tikzcd}
\underline{\mathrm{ClFib}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{UniFib}} \arrow[d] \\
\underline{\mathrm{Grd}}^{\to} \arrow[r, "\mathsf{N}"'] & \underline{\mathrm{sSet}}^{\to}
\end{tikzcd}$$

where the vertical arrows are the forgetful functors and the lower horizontal arrow is the nerve functor lifted to the arrow categories. At a first glance, this might be slightly surprising because the algebraic structure on uniform fibrations satisfy some coherence conditions (i.e. that the lifting structures are coherent with respect to morphisms of generating trivial cofibrations) while cloven isofibrations do not have any coherence. Before proving this, we will first need to review some notions.

First of all, recall that there is a nerve-realisation adjunction between groupoids and simplicial sets:

$$\begin{tikzcd}
\underline{\mathrm{Grd}} \arrow[r, shift right, "\mathsf{N}"'] \arrow[r, phantom, "\perp"] & \underline{\mathrm{sSet}}. \arrow[l, shift right, "|{-}|"']
\end{tikzcd}$$

It is induced by the cosimplicial object in Grd whose value at $[n] \in \Delta$ is the groupoid with n composable paths which we will denote by $\mathbb{n}$. We describe with some detail the action on objects of the realisation functor $|-| : \underline{\mathrm{sSet}} \to \underline{\mathrm{Grd}}$; consider a simplicial set X, then the groupoid $|X|$ has:

Objects: The points (0-simplices) of X.

Arrows: These are freely generated from the paths (1-simplices) of X by taking composites and inverses; subject to the relation naturally imposed by the 2-simplices of X. In other words, an arrow in $|X|$ can be represented by an equivalence class of zig-zags of 1-simplices of X.

It follows easily from this construction that $|-| : \underline{\mathrm{sSet}} \to \underline{\mathrm{Grd}}$ preserves cofibrations, that is, it maps monomorphism to functors injective on objects. Moreover, it is well known that the nerve $\mathsf{N} : \underline{\mathrm{Grd}} \to \underline{\mathrm{sSet}}$ preserves fibrations. In other words, we have that $\mathsf{N} \vdash |-|$ is a Quillen pair.

Another way to construct $|X|$ would be to first consider the realisation of X as a category (i.e. in Cat) and then localise at all arrows. It is well known that both these functors preserve finite products, thus we obtain the following folklore result.

Proposition 7.1.1. *The realisation functor* $|-| : \underline{\mathrm{sSet}} \to \underline{\mathrm{Grd}}$ *preserves finite products.* □

If we apply the realisation functor to the interval object Δ^1 of sSet, we obtain the canonical interval object in Grd:

$$\begin{tikzcd}
\mathbb{0} \arrow[r, shift left, "\delta^0"] \arrow[r, shift right, "\delta^1"'] & \mathbb{1}
\end{tikzcd}$$

Since the realisation functor is left adjoint, it also preserves colimits, in particular it preserves pushouts. Using this together with the previous proposition, we can prove the following:

Proposition 7.1.2. *The realisation functor* $|-| : \underline{\mathrm{sSet}} \to \underline{\mathrm{Grd}}$ *preserves the pushout-product (Appendix C.1) against the endpoint inclusions of the interval. That is, for any monomorphism* $i : A \rightarrowtail B$ *we have:*

$$|\delta^k \hat{\times} i| = \delta^k \hat{\times} |i| : \mathbb{n} +_{|A|} (\mathbb{1} \times |A|) \to \mathbb{1} \times \mathbb{n}$$

Let us now establish some notational conventions for the rest of the section. For each $n \in \mathbb{N}$, the groupoid $\mathbb{n}$ consist of n composable arrows, we will denote the objects and arrows of this groupoid as follows:

$$\mathbb{n} = \qquad \begin{tikzcd} 0 \arrow[r, squiggly, "p_1"] & 1 \arrow[r, squiggly, "p_2"] & 2 \arrow[r, squiggly, "p_3"] & \cdots \arrow[r, squiggly, "p_{n-1}"] & n-1 \arrow[r, squiggly, "p_n"] & n \end{tikzcd}$$

the groupoid $\mathbb{1} \times \mathbb{n}$ can also be easily pictured in a similar way. It consists of n composable squares, which we denote as follows:

$$\mathbb{1}\times\mathbb{n} = \begin{tikzcd} 01 \arrow[r, squiggly, "p_{11}"] & 11 \arrow[r, squiggly, "p_{21}"] & 21 \arrow[r, squiggly, "p_{31}"] & \cdots \arrow[r, squiggly, "p_{n1}"] & n1 \\ 00 \arrow[u, squiggly, "q_0"] \arrow[r, squiggly, "p_{10}"'] & 10 \arrow[u, squiggly, "q_1"] \arrow[r, squiggly, "p_{20}"'] & 20 \arrow[u, squiggly, "q_2"] \arrow[r, squiggly, "p_{30}"'] & \cdots \arrow[r, squiggly, "p_{n0}"'] & n0 \arrow[u, squiggly, "q_n"'] \end{tikzcd}$$

We have the tools we need to start proving the main result of this section. We start with the following lemma.

Lemma 7.1.3. *Consider a cloven isofibration* $(f, \theta) : G \to H$ *in* $\underline{\mathrm{Grd}}$*. Then there is a uniform fibration structure on its nerve* $\mathrm{N}f : \mathrm{N}G \to \mathrm{N}H$*, which will be denoted by* $\mathrm{N}\theta$*.*

Proof. For simplicity, we will use the same names for an arrow $a : X \to \mathrm{N}G$ and for its transpose $a : |X| \to G$ under the nerve-realisation adjunction $\mathrm{N} \vdash |-|$.

Let $(f, \theta) : G \to H$ be a cloven isofibration. In order to define the uniform fibration structure on the nerve of f, we fix $i : A \rightarrowtail \Delta^n$ a generating monomorphism of simplicial sets. Consider a lifting problem as on the left of the following diagram:

$$\begin{tikzcd} \Delta^n +_A (\Delta^1 \times A) \arrow[r, "{[b_1,u]}"] \arrow[d, "\delta^0 \hat{\times} i"'] & \mathrm{N}G \arrow[d, "\mathrm{N}f"] \\ \Delta^1 \times \Delta^n \arrow[r, "b"'] & \mathrm{N}H \end{tikzcd} \qquad \begin{tikzcd} \mathbb{n} +_{|A|} (\mathbb{1} \times |A|) \arrow[r, "{[b_1,u]}"] \arrow[d, "\delta^0 \hat{\times} |i|"'] & G \arrow[d, "f"] \\ \mathbb{1} \times \mathbb{n} \arrow[r, "b"'] & H \end{tikzcd}$$

transposing along the adjunction $\mathrm{N} \vdash |-|$ and using Proposition 7.1.2 we obtain a lifting diagram as on the right. We proceed to define the lift $\mathrm{N}\theta_i([b_0, u], b) : \mathbb{1} \times \mathbb{n} \to G$ (which we will refer to just as $\mathrm{N}\theta_i$) by doing case analysis on the objects and arrows of $\mathbb{1} \times \mathbb{n}$.

$\mathrm{N}\theta_i$ **on objects:** Recall that the objects of $\mathbb{1} \times \mathbb{n}$ are denoted by (kt) where $k \in \{0, 1\}$ and $0 \leq t \leq n$. So let us consider an object $(kt) \in \mathbb{1} \times \mathbb{n}$ and define $\mathrm{N}\theta_i(kt) \in G$ as follows:

Case $k = 1$**:** We define $N\theta_i(kt) := b_1(t)$.

Case $k = 0$ **and** $t \in |A|$**:** We define $N\theta_i(kt) := u(0, t)$.

Case $k = 1$ **and** $t \notin |A|$**:** Consider the path H given by $b(\mathbb{1} \times \{t\}) : b(0t) \rightsquigarrow b(1t)$ and notice that $b_1(t)$ is over $b(1t)$. Thus we use the cleavage θ to obtain a path

$$\theta(b_1(t), b(\mathbb{1} \times \{t\})) : \theta^*(1t) \rightsquigarrow b_1(t)$$

in G over $b(\mathbb{1} \times \{t\})$. We define $N\theta_i(kt) := \theta^*(1t)$.

It is straightforward to verify that this definition of $N\theta_i$ makes the relevant triangles commute, on objects.

$N\theta_i$ **on arrows:** We will use the notation for paths in $\mathbb{1} \times \mathbb{n}$ explained previously in this section. Let g be a path in $\mathbb{1} \times \mathbb{n}$ and define the path $N\theta_i(g)$ in G as follows:

Case $g = q_t$ **for** $0 \leq t \leq n$ **and** $t \in |A|$**:** We define $N\theta_i(g) := u(\mathbb{1} \times \{t\})$.

Case $g = q_t$ **for** $0 \leq t \leq n$ **and** $t \notin |A|$**:** We define $N\theta_i(g) = \theta(b_1(t), b(\mathbb{1} \times \{t\}))$ the lift of $b(\mathbb{1} \times \{t\})$ at $b_1(t)$ given by the cleavage of f.

Case $g = p_{t1}$ **for** $0 < t \leq n$**:** We define $N\theta_i(g) := b_1(p_t)$.

Case $g = p_{t0}$ **for** $0 < t \leq n$**:** We define $N\theta_i(g) = N\theta_i(q_t)^{-1} \cdot N\theta_i(p_{t_1}) \cdot N\theta_i(q_{t-1})$.

It is easy to verify that this data gives rise to a well defined collection of functors $N\theta_i : \mathbb{1} \times \mathbb{n} \to G$ which are lifts of Nf against $\delta^0 \hat{\times} i$, for each generating monomorphism $i : A \rightarrowtail \Delta^n$. Dually, we can make the same definitions to obtain lifting structures of Nf against $\delta^1 \hat{\times} i$.

We now have to verify that this data defines a uniform fibration structure on Nf, for this we have to check that the lifting structures are coherent with respect to the morphisms of the arrow category $u_\otimes : \mathcal{I}_\otimes \to \underline{\mathrm{sSet}}^{\to}$. In order to do this, we consider a morphism in the arrow category $\mathcal{I} \to \underline{\mathrm{sSet}}^{\to}$, that is a Cartesian square:

$$\begin{tikzcd}
B \arrow[r, "\sigma"] \arrow[d, "j"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "i"] \\
\Delta^m \arrow[r, "\tau"'] & \Delta^n
\end{tikzcd}$$

we need to verify that in the following diagram, the triangle created by the lifting structures commute (we will only deal with $k = 0$):

$$\begin{tikzcd}
\mathbb{m} +_{|B|} (\mathbb{1} \times |B|) \arrow[r, "|\tau| +_{|\sigma|} (\mathbb{1} \times |\sigma|)"] \arrow[d, "\delta^0 \hat{\times} |j|"'] & \mathbb{n} +_{|A|} (\mathbb{1} \times |A|) \arrow[r, "{[b_1, u]}"] \arrow[d, "\delta^0 \hat{\times} |i|"'] & G \arrow[d, "f"] \\
\mathbb{1} \times \mathbb{m} \arrow[r, "\mathbb{1} \times |\tau|"'] \arrow[urr, dotted, "N\theta_j"] & \mathbb{1} \times \mathbb{n} \arrow[r, "b"'] \arrow[ur, dotted, "N\theta_i"] & G
\end{tikzcd}$$

We will show that this triangle commutes by doing the same case analysis as before. Let g be a path in $\mathbb{1} \times \mathbb{m}$ and observe that:

Case $g = q_t$ for $0 \leq t \leq m$ and $t \in |B|$: Then:

$$\begin{aligned} N\theta_j(g) &= u \cdot (\mathbb{1} \times |\sigma|)(\mathbb{1} \times \{t\}) = u(\mathbb{1} \times \{|\sigma|(t)\}) \\ &= N\theta_i(q_{|\sigma|(t)}) = N\theta_i(q_{|\tau|(t)}) \\ &= N\theta_i \cdot (\mathbb{1} \times |\tau|)(g) \end{aligned}$$

Case $g = q_t$ for $0 \leq t \leq m$ and $t \notin |B|$: Notice that since the square (σ, τ) is a pullback then the points of $|B|$ are precisely those of $|A|$ in the image of $|\tau|$; in particular, since $t \notin |B|$, then $|\tau|(t) \notin |A|$. Then we have:

$$\begin{aligned} N\theta_j(g) &= \theta(b_1(|\tau|(t)), b(\mathbb{1} \times \{|\tau|(t)\})) \\ &= N\theta_i(q_{|\tau|(t)}) \\ &= N\theta_i \cdot (\mathbb{1} \times |\tau|)(g) \end{aligned}$$

Case $g = p_{t1}$ for $0 < t \leq m$: Then:

$$\begin{aligned} N\theta_j(g) &= |\tau| \cdot b_1(p_t) \\ &= N\theta_i \cdot (\mathbb{1} \times |\tau|)(g) \end{aligned}$$

Case $g = p_{t0}$ for $0 < t \leq m$: Then

$$\begin{aligned} N\theta_j(g) &= N\theta_j(q_t)^{-1} \cdot N\theta_j(p_{t1}) \cdot N\theta_j(q_{t-1}) \\ &= N\theta_i(\mathbb{1} \times |\tau|)(q_t)^{-1} \cdot N\theta_i(\mathbb{1} \times |\tau|)(p_{t_1}) \cdot N\theta_i(\mathbb{1} \times |\tau|)(q_{t-1}) \\ &= N\theta_i(q_{|\tau|(t)})^{-1} \cdot N\theta_i(|\tau|(p_{t1})) \cdot N\theta_i(q_{|\tau|(t-1)}) \\ &= N\theta_i \cdot (\mathbb{1} \times |\tau|)(g) \end{aligned}$$

We obtain that for all $g \in \mathbb{1} \times \mathbb{m}$, the lifts cohere, $N\theta_j(g) = N\theta_i \cdot (\mathbb{1} \times |\tau|)(g)$. In this way, the collection of lifting structures $N\theta_i$, for $i : A \rightarrowtail \Delta^n$, define a uniform fibration structure on Nf. ☐

Lemma 7.1.4. *The construction of Lemma 7.1.3 is the action on objects of a lift of the nerve functor* $N : \underline{\text{Grd}} \longrightarrow \underline{\text{sSet}}$ *as shown in the following diagram:*

$$\begin{tikzcd} \underline{\text{ClFib}} \arrow[r, dotted, "\tilde{N}"] \arrow[d] & \underline{\text{UniFib}} \arrow[d] \\ \underline{\text{Grd}}^{\to} \arrow[r, "N"'] & \underline{\text{sSet}}^{\to} \end{tikzcd}$$

Proof. The action on objects of the lift will be given by Lemma 7.1.3, that is, we define $\tilde{N}(f, \theta) := (Nf, N\theta)$. We have to verify that this operation is functorial, for this consider a mono $i : A \rightarrowtail \Delta^n$, a morphism of cloven fibrations $(l, m) : (f', \theta') \to (f, \theta)$ and a lifting problem of $\delta^0 \hat{\times} |i|$ against f' as in the following diagram:

$$\begin{tikzcd}
\mathbb{n} +_{|A|} (\mathbb{1} \times |A|) \arrow[r, "{[b_1,u]}"] \arrow[d, "\delta^0 \hat{\times} |i|"'] & G' \arrow[r, "l"] \arrow[d, "f'"'] & G \arrow[d, "f"] \\
\mathbb{1} \times \mathbb{n} \arrow[r, "b"'] \arrow[ur, dotted, "N\theta'_i"] \arrow[urr, dotted, "N\theta_i"'] & H' \arrow[r, "m"'] & H
\end{tikzcd}$$

we need to show that the triangle created by the lifts cohere. We do a case analysis as before. Let g be a path in $\mathbb{1} \times \mathbb{m}$ and observe that:

Case $g = q_t$ for $0 \leq t \leq n$ and $t \in |A|$: Then:

$$\begin{aligned} N\theta_i(g) &= (l \cdot u)(\mathbb{1} \times \{t\}) \\ &= l \cdot N\theta'_i(g) \end{aligned}$$

Case $g = q_t$ for $0 \leq t \leq n$ and $t \notin |A|$: Here is the only clause where we make use of the fact that (l, m) preserves cleavages. We have:

$$\begin{aligned} N\theta_i(g) &= \theta(l \cdot b_1(t), m \cdot b(\mathbb{1} \times \{t\})) \\ &= l \cdot \theta'(b_1(t), b(\mathbb{1} \times \{t\})) \\ &= l \cdot N\theta'_i(g) \end{aligned}$$

Case $g = p_{t1}$ for $0 < t \leq n$: Then:

$$\begin{aligned} N\theta_i(g) &= (l \cdot b_1)(p_t) \\ &= l \cdot N\theta'_i(g) \end{aligned}$$

Case $g = p_{t0}$ for $0 < t \leq m$: Then

$$\begin{aligned} N\theta_i(g) &= N\theta_i(q_t)^{-1} \cdot N\theta_i(p_{t1}) \cdot N\theta_i(q_{t-1}) \\ &= lN\theta'_i(q_t)^{-1} \cdot lN\theta'_i(p_{t1}) \cdot lN\theta'_i(q_{t-1}) \\ &= l \cdot (N\theta'_i(q_t)^{-1} \cdot N\theta'_i(p_{t1}) \cdot N\theta'_i(q_{t-1})) \\ &= l \cdot N\theta'_i(g) \end{aligned}$$

Thus, we see that the construction of Lemma 7.1.3 is functorial and produces a lift $\tilde{N} : \underline{\text{ClFib}} \to \underline{\text{UniFib}}$ of the nerve functor as desired. ☐

Lemma 7.1.5. *The functor given by the universal property of pullbacks applied to the square of Lemma 7.1.4:*

$$\underline{\text{ClFib}} \xrightarrow{P} \underline{\text{Grp}}^{\to} \times_{\underline{\text{sSet}}^{\to}} \underline{\text{UniFib}}$$

is an isomorphism.

Proof. We can define an inverse explicitly:

$$\underline{\mathrm{Grp}}^{\rightarrow} \times_{\underline{\mathrm{sSet}}^{\rightarrow}} \underline{\mathrm{UniFib}} \xrightarrow{Q} \underline{\mathrm{ClFib}}$$

given by $(f, (Nf, \phi)) \mapsto (f, C\phi)$ where $C\phi$ is the cleavage on the functor $f : G \to H$ given as follows: a path $p : b \rightsquigarrow f(a)$ in H, induces the following diagram:

$$\begin{tikzcd}
* \arrow[r, "a"] \arrow[d, "\delta^0"'] & G \arrow[d, "f"] \\
\mathbb{1} \arrow[r, "p"'] \arrow[ur, dotted, "\phi(a{,}p)"] & H
\end{tikzcd}$$

recall that $\delta^0 = \delta^0 \hat{\times} |\bot_{\Delta^0}|$ where $\bot_{\Delta^0} : \emptyset \to \Delta^0$ is the unique arrow from the initial object. Thus, there is a lift given by the uniform fibration strucutre ϕ. We define

$$C\phi(a, p) := \phi(a, p)$$

It is straightforward to see that this operation is functorial: a morphism between uniform fibrations will in particular preserve these lifts.

We only have to show that P and Q are inverses. One direction is easy; let us show that $Q \cdot P = \mathrm{id}$. For this consider $(f, \theta) \in \underline{\mathrm{ClFib}}$, then $Q(P(f, \theta)) = (f, CN\theta)$, we must show that the lifting structures θ and $CN\theta$ coincide. So let $p : a \to f(a)$ be a path in H, and consider a square $(a, p) : \delta^0 \to f$ as in the beginning of the proof. It is clear that the unique arrow in the codomain of δ^0 is of the form q_0 (using our previous notation), and clearly 0 is not in the domain of δ^0, so using the definition of $N\theta$ by case analysis we see that:

$$CN\theta(a, p) = N\theta(a, p) = \theta(a, p)$$

The other direction is a bit more involved. In order show that $P \circ Q = \mathrm{id}$, let's consider $(f, (Nf, \phi))$ in $\underline{\mathrm{Grp}}^{\rightarrow} \times_{\underline{\mathrm{sSet}}^{\rightarrow}} \underline{\mathrm{UniFib}}$, we see that $P(Q(f, (Nf, \phi))) = P(f, C\phi) = (f, (Nf, NC\phi))$, so we have to show that the uniform fibration structures ϕ and $NC\phi$ coincide. For this consider a lifting problem as on the right square of the following diagram:

$$\begin{tikzcd}
* \arrow[r, "in_0(t)"] \arrow[d, "\delta^0"'] & \mathbb{n} +_{|A|} (\mathbb{1} \times |A|) \arrow[r, "{[b_1,u]}"] \arrow[d, "\delta^0 \hat{\times} |i|"'] & G \arrow[d, "f"] \\
\mathbb{1} \arrow[r, "\mathbb{1}\times\{t\}"'] \arrow[urr, dotted, "\phi"] & \mathbb{1} \times \mathbb{n} \arrow[r, "b"'] \arrow[ur, dotted, "\phi"] & H
\end{tikzcd}$$

Now let g be a path in $\mathbb{1} \times \mathbb{m}$, and let's check that $\phi(g) = NC\phi(g)$. As before, we prove this by case analysis; there is only one non obvious case:

> **Case $g = q_t$ for $0 \leq t \leq n$ and $t \notin |A|$:** By definition of the uniform fibration structure C we have that:
>
> $$NC\phi(g) = C\phi(b_1(t), b(\mathbb{1} \times \{t\})) = \phi(b_1(t), b(\mathbb{1} \times \{t\})).$$

Now consider the square on the left of the above diagram, by uniformity, the triangle created by the lifts must commute, using this together with the fact that $g = g_t = \mathbb{1} \times \{t\}$ we have that:

$$NC\phi(g) = \phi(b_1(t), b(\mathbb{1} \times \{t\})) = \phi(g).$$

It is not necessary to check that P and Q are inverses on arrows. This is because P and Q are functors between categories of arrows in the slice over $\underline{\mathrm{Grd}}^{\to}$ whose forgetful functors are fully-faithful. □

Remark 7.1.6. The previous lemmas can be proven constructively, if we restrict to decidable monomorphisms. This is needed when doing case analysis, where one of the cases depends on whether an object is in the image of a generating cofibration or not.

The following theorem is a summary of the results from this section and follows immediately from Lemma 7.1.4 and Lemma 7.1.5.

Theorem 7.1.7. *The following is a pullback square:*

$$\begin{tikzcd}
\underline{\mathrm{ClFib}} \arrow[r, "\tilde{N}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{UniFib}} \arrow[d] \\
\underline{\mathrm{Grd}}^{\to} \arrow[r, "N"'] & \underline{\mathrm{sSet}}^{\to}.
\end{tikzcd}$$

□

7.2 Normal Uniform Fibrations

In this section, we will develop the notion of normal uniform fibration in the general context $(\mathbb{C}, \mathbb{A}, \mathcal{M})$ of a suitable topos (Definition 6.5.1). We will denote the objects of $\mathbb{A}$ with cursive letters to differentiate them from the objects of $\mathbb{C}$; for example $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{A}$ and $A, B, C \in \mathbb{C}$.

Recall from Appendix C.4 that the category of arrows of uniform fibrations was constructed from the categories of arrows of generating cofibrations $\mathcal{I}$ and of generating trivial cofibrations $\mathcal{I}_\otimes$ over $\mathbb{C}$. Recall that $\mathcal{I}$ was obtained as the full subcategory of $\mathcal{M}$ whose object has codomain in $\mathbb{A}$; that is

$$\mathcal{I} := \{i \in \mathcal{M} | \mathrm{cod}(i) \in \mathbb{A}\}$$

We will define a new category of generating trivial cofibrations:

$$u^{\mathbf{n}}_\otimes : \mathcal{I}^{\mathbf{n}}_\otimes \to \mathbb{C}^{\to}$$

in such a way that a right $\mathcal{I}^{\mathbf{n}}_\otimes$-map will be uniform fibration with an extra ‘normality’ property.

Intuitively the idea is that $\mathfrak{u}^{\mathbf{n}}_{\otimes} : \mathcal{I}^{\mathbf{n}}_{\otimes} \to \mathbb{C}^{\to}$ will be obtained from $\mathfrak{u}_{\otimes} : \mathcal{I}_{\otimes} \to \mathbb{C}^{\to}$ by adding for each generating monomorphism $i : A \rightarrowtail B$ and for $k \in \{0, 1\}$ the coherence square on the left of the following diagram:

$$\begin{tikzcd}
B +_A (I \times A) \arrow[r, "\mathbf{sq}_k(i)"] \arrow[d, "\delta^k \hat{\times} i"'] & B \arrow[d, equal] \\
I \times B \arrow[r, "\epsilon \times B"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "\delta^k \times A"'] \arrow[dr, phantom, "\ulcorner", very near end] & B \arrow[d, "\delta^k \times B"] \arrow[ddr, bend left, "id_B"] & \\
I \times A \arrow[r] \arrow[dr, bend right, "\epsilon \times A"'] & B +_A (I \times A) \arrow[dr, dotted, "\mathbf{sq}_k(i)"] & \\
& A \arrow[r, "i"'] & B
\end{tikzcd}$$

where the map $\mathbf{sq}_k(i) : B +_A (I \times A) \to B$ is the universal map out of the pushout as described on the right of the previous diagram. The arrows $\epsilon \times B$ and $\epsilon \times A$ are the product of the identity map (on B or A respectively) with the terminal map $\epsilon : I \to *$, in other words, they are the projections from the second component of the product.

We will refer to the square on the left of the previous diagram as the **k-squash square** of $i : A \rightarrowtail B$ and we will denote the whole square by

$$\mathbf{squash}_k(i) : \delta^k \hat{\times} i \to id_B.$$

The name follows the intuition of squashing the mapping cylinder in the direction of the interval (i.e. the filling direction). The following technical result about squash squares will be used in the following section.

Lemma 7.2.1. *Let* $k \in \{0, 1\}$ *and consider monomophisms* $i : A \rightarrowtail B$ *and* $j : C \rightarrowtail D$. *Then applying the Leibniz pushout-product functor* $(j \hat{\times} -) : \mathbb{C}^{\to} \to \mathbb{C}^{\to}$ *to the k-squash square of* $i : A \rightarrowtail B$*, produces the k-squash square of* $j \hat{\times} i$*; that is:*

$$j \hat{\times} (\boldsymbol{squash}_k(i)) \cong \boldsymbol{squash}_k(j \hat{\times} i) : \delta^k \hat{\times} (j \hat{\times} i) \to id_{D \times B}$$

Proof. If we apply $(j \hat{\times} -) : \mathbb{C}^{\to} \to \mathbb{C}^{\to}$ to the k-squash square of $i : A \rightarrowtail B$, using that the pushout-product is symmetric and associative, we will get the following square:

$$\begin{tikzcd}
\mathrm{dom}(\delta^k \hat{\times} (j \hat{\times} i)) \arrow[r, "\Theta"] \arrow[d, "\delta^k \hat{\times} (j \hat{\times} i)"'] & D \times B \arrow[d, equal] \\
I \times (D \times B) \arrow[r, "\epsilon \times (D \times B)"'] & D \times B
\end{tikzcd}$$

where we only need to verify that the top horizontal arrow Θ is the squash morphism, that is, we need to verify that $\Theta = \mathbf{sq}_k(j \hat{\times} i) : \mathrm{dom}(\delta^k \hat{\times} (j \hat{\times} i)) \to D \times B$, but this follows since the diagram above commutes. □

We now proceed to construct the arrow category $u^{\mathbf{n}}_{\otimes} : \mathcal{J}^{\mathbf{n}}_{\otimes} \to \mathbb{C}^{\to}$ that will generate the category of normal uniform fibrations. We do this a follows. First let us denote by $\mathbb{I}$ the 'walking arrow', that is the poset with two objects $0 < 1$ considered as a category, this has the structure of an interval object in Cat and we denote the inclusions by:

$$\begin{tikzcd} * \arrow[r, shift left, "\iota^0"] \arrow[r, shift right, "\iota^1"'] & \mathbb{I} \end{tikzcd}$$

Using this we construct for $k \in \{0, 1\}$ the category of arrows $u^{\mathbf{n}}_{\otimes} : \mathcal{J}^{\mathbf{n}}_{\otimes} \to \mathbb{C}^{\to}$ where we define $\mathcal{J}^{\mathbf{n}}_{\otimes} := \mathbb{I} \times \mathcal{J}_{\otimes}$, and where $\mathcal{J}_{\otimes}$ is the generating category of uniform fibrations. The functor $u^{\mathbf{n}}_{\otimes}$ is determined by the following two properties.

1. The following diagram commutes:

$$\begin{tikzcd} \mathcal{J}_{\otimes} \arrow[dr, "u_{\otimes}"'] & \mathcal{J}^{\mathbf{n}}_{\otimes} \arrow[l, "\rho_0"'] \arrow[r, "\rho_1"] \arrow[d, "u^{\mathbf{n}}_{\otimes}"] & \mathcal{J}_{\otimes} \arrow[dl, "\epsilon_{cod}"] \\ & \mathbb{C}^{\to} & \end{tikzcd}$$

   where the map $\epsilon_{cod} : \mathcal{J}_{\otimes} \to \mathbb{C}^{\to}$ sends an object $i \in \mathcal{J}_{\otimes}$ to the identity arrow on the codomain of i (where we recall that $\mathcal{J}_{\otimes} = \mathcal{J} + \mathcal{J}$).

2. For $k \in \{0, 1\}$ and for each $i : A \rightarrowtail \Delta^n$ in $\mathcal{J}$, the functor $u^{\mathbf{n}}_{\otimes}$ takes the arrow in $I \times \mathcal{J}_{\otimes}$ of the form $I \times i : \{0\} \times i \to \{1\} \times i$, to the k-squash square of i, that is $u^{\mathbf{n}}_{\otimes}(I \times i) := \mathbf{squash}_k(i) : \delta^k \hat{\times} i \to \mathrm{id}_{\Delta^n}$.

in other words, $u^{\mathbf{n}}_{\otimes}$ is a natural transformation: $u^{\mathbf{n}}_{\otimes} : u_{\otimes} \to \epsilon_{cod} : \mathcal{J}_{\otimes} \to \mathbb{C}^{\to}$ whose components are the k-squash squares.

Just as we did for uniform fibrations before, we define NrmUniFib $\to \mathbb{C}^{\to}$ to be the category of arrows of right $\mathcal{J}^{\mathbf{n}}_{\otimes}$-maps in $\mathbb{C}$, and we call its objects **normal uniform fibrations**. Using Garner's small object argument [Gar09], we can easily obtain the following result.

Theorem 7.2.2. *There is an algebraically-free awfs on the category of arrows* $u^{\mathbf{n}}_{\otimes} : u_{\otimes} \to \mathbb{C}^{\to}$*, denoted by* $(\mathsf{NC}_t, \mathsf{NF})$*, whose category of* NF*-algebras is that of normal uniform fibrations.*

Let us observe that the forgetful functor into $\mathbb{C}^{\to}$ factors through the category of uniform fibrations, i.e. we have a commutative diagram:

$$\begin{tikzcd} \underline{\mathrm{NrmUniFib}} \arrow[rr, "u"] \arrow[dr] & & \underline{\mathrm{UniFib}} \arrow[dl] \\ & \mathbb{C}^{\to} & \end{tikzcd}$$

moreover, we can prove the following lemma.

Lemma 7.2.3. *The forgetful functor* $U : \underline{\text{NrmUniFib}} \to \underline{\text{UniFib}}$ *is fully-faithful.*

Proof. This follows intuitively by noticing that the structure of a normal uniform fibration does not add any new lifting problems to that of a uniform fibrations; this is because the only new vertical arrows we are adding are identities and every morphism has a unique lift against them. Concretely, if $(f, \phi) \in \underline{\text{NrmUniFib}}$ and if $(f, \theta) \in \underline{\text{UniFib}}$, then both lifting structures ϕ and θ produce lifts against the exactly the same squares, the difference is that ϕ may have additional coherence properties. ☐

Indeed in the following proposition we characterise those uniform fibration structures that are normal. We will use the following notation: we say that a morphism $\theta : I \times B \to X$ is **degenerate in the lifting direction** if it factors through the projection $\rho_1 : I \times B \to B$ via some arrow $\theta^* : B \to Y$; we call θ^* the **lifting degeneracy** of b.

Proposition 7.2.4. *Let* $(f, \theta) \in \underline{\text{UniFib}}$ *then the following are equivalent:*

1. (f, θ) *is a normal uniform fibration.*
2. *For any generating monomorphism* $i : A \rightarrowtail \mathcal{A}$ *in* $\mathcal{I}$ *(i.e. with* $\mathcal{A} \in \mathbb{A}$*) and for any square:*

$$\begin{tikzcd}
\mathcal{A} +_A (I \times A) \arrow[r, "a"] \arrow[d, "\delta^k \hat{\times} i"'] & X \arrow[d, "f"] \\
I \times \mathcal{A} \arrow[r, "b"'] \arrow[ur, dotted, "\theta_i(a,b)"] & Y
\end{tikzcd}$$

   if the square factors through the squash square of i *as* $\delta^k \hat{\times} i \xrightarrow{\textbf{\textit{squash}}_k(i)} id_{\mathcal{A}} \xrightarrow{(a^*,b^*)} f$ *, then the lift* $\theta_i(a, b)$ *is degenerate in the lifting direction with* a^* *as lifting degeneracy.*
3. *For any generating monomorphism* $i : A \rightarrowtail B$ *in* $\mathcal{M}$ *and for any square:*

$$\begin{tikzcd}
B +_A (I \times A) \arrow[r, "a"] \arrow[d, "\delta^k \hat{\times} i"'] & X \arrow[d, "f"] \\
I \times B \arrow[r, "b"'] \arrow[ur, dotted, "\theta_i(a,b)"] & Y
\end{tikzcd}$$

   if the square factors through the squash square of i *as* $\delta^k \hat{\times} i \xrightarrow{\textbf{\textit{squash}}_k(i)} id_B \xrightarrow{(a^*,b^*)} f$ *, then the lift* $\theta_i(a, b)$ *is degenerate in the lifting direction with* a *as lifting degeneracy.*

Proof. Let us first assume that (f, θ) is a normal uniform fibration. It is easy to see that item (2) holds, for this consider the diagram:

$$\begin{tikzcd}
\mathcal{A} +_A (I \times A) \arrow[r, "\mathbf{sq}_k(i)"] \arrow[d, "\delta^k \hat{\times} i"'] & \mathcal{A} \arrow[r, "a^*"] \arrow[d, equal] & X \arrow[d, "f"] \\
I \times \mathcal{A} \arrow[r, "\rho_1"'] \arrow[urr, dotted, "\theta"] & \mathcal{A} \arrow[r, "b*"'] \arrow[ur, dotted, "a^*"] & Y
\end{tikzcd}$$

it is clear that the lifts cohere because the left square is by definition a morphism in (the image of) $u^{\mathbf{n}}_{\otimes} : \mathcal{J}^{\mathbf{n}}_{\otimes} \to \mathbb{C}^{\to}$.

It is also easy to see that (2) implies (1), this follows since the uniform fibration structure θ already provides lifts against all lifting problems coming from $\mathcal{J}^{\mathbf{n}}_{\otimes}$, moreover, the lifts will also cohere with all the squares coming from $u_{\otimes} : \mathcal{J}_{\otimes} \to \mathbb{C}^{\to}$. So we only need to verify that it coheres with the squash squares, but these squares are precisely those as in the hypothesis of item (2).

It is clear that (3) implies (2). For the converse let us first observe, using that colimits in $\mathbb{C}$ are universal, that any monomorphism $i : A \rightarrowtail B$, is the colimit over the generalised elements from the dense subcategory $\mathbb{A}$; that is

$$i \cong \operatorname{colim}_{\substack{x : \mathcal{A} \to B \\ \mathcal{A} \in \mathbb{A}}} x^*(i)$$

where for each $x : \mathcal{A} \to B$ we denote by $x^*(i)$ the pullback of i along x. Now, since $\delta^k \hat{\times} - : \mathbb{C}^{\to} \to \mathbb{C}^{\to}$ is cocontinuous, we have that:

$$\operatorname{colim}_{\substack{x : \mathcal{A} \to B \\ \mathcal{A} \in \mathbb{A}}} (\delta^k \hat{\times} (x^*(i))) \cong \delta^k \hat{\times} \operatorname{colim}_{\substack{x : \mathcal{A} \to B \\ \mathcal{A} \in \mathbb{A}}} x^*(i) \cong \delta^k \hat{\times} i$$

Now let us suppose that (2) holds, and we have a diagram as in item (3). Then for each generalised element $x : \mathcal{A} \to B$ with $\mathcal{A} \in \mathbb{A}$, we have a square:

$$\begin{tikzcd}
\mathcal{A} +_{x^*(A)} (I \times x^*(A)) \arrow[r, "\iota_x"] \arrow[d, "\delta^k \hat{\times} x^*(i)"'] & B +_A (I \times A) \arrow[r, tail, "a"] \arrow[d, "\delta^k \hat{\times} i"] & X \arrow[d, "f"] \\
I \times \mathcal{A} \arrow[r, "I \times x"'] \arrow[urr, dotted, "\theta_{x*(i)}"] & I \times B \arrow[r, "b"'] \arrow[ur, dotted, "\theta_i"] & Y
\end{tikzcd}$$

where the left square is the colimit inclusion corresponding to $x : \mathcal{A} \to B$. The commutation of the respective triangle is obtained by the universal property of the colimit.

Finally, if the right square (in the previous diagram) factors through a squash square as

$$\delta^k \hat{\times} i \xrightarrow{\mathbf{squash}_k(i)} id_B \xrightarrow{(a^*, b^*)} f$$

then (by naturality) the outer square also factor through a squash square and thus the lift $\theta_{x^*(i)}$ is degenerate with $a^* \iota_x$ as lifting degeneracy. This implies by the uniqueness of the universal property, that also θ_i is degenerate with a^* as lifting degeneracy. □

7.3 Normal Isofibrations and Normal Uniform Fibrations

In Section 7.1 we compared the categories of arrows of uniform fibrations on simplicial sets and cloven isofibrations on groupoids, using the nerve functor. In this section we will extend this analysis to the categories of normal uniform fibrations and normal isofibrations. Thus, in this section we will again work on the suitable topos of simplicial sets ($\underline{\text{sSet}}, \Delta, \mathcal{M}_{all}$). We will denote by $\underline{\text{NrmFib}}$ the category of normal isofibrations in groupoids.

Lemma 7.3.1. *The functor* $\tilde{N} : \underline{\mathrm{ClFib}} \to \underline{\mathrm{UniFib}}$ *of Lemma 7.1.4 lifts to a functor:*

$$\begin{tikzcd}
\underline{\mathrm{NrmFib}} \arrow[r, dotted, "\tilde{N}"] \arrow[d] & \underline{\mathrm{NrmUniFib}} \arrow[d] \\
\underline{\mathrm{ClFib}} \arrow[r, "\tilde{N}"'] & \underline{\mathrm{UniFib}}
\end{tikzcd}$$

Proof. Because the vertical forgetful functors are fully-faithful, we only need to show the following: given a normal isofibration (f, θ), then $(Nf, N\theta)$ is a normal uniform fibration. This follows easily from the case analysis in the proof of Lemma 7.1.3 as we proceed to show.

We will assume without loss of generality that $k = 0$. Consider a monomorphism $i : A \rightarrowtail \Delta^n$ and let us suppose that the following lifting problem $(a, b) : \delta^0 \hat{\times} i \to Nf$ factors through a squash square. Next, we transposes the square to the category of groupoids, to obtain the one shown as the outer square below:

$$\begin{tikzcd}
\mathbb{n} +_{|A|} (\mathbb{1} \times A) \arrow[r, "sq_0(|i|)"] \arrow[d, "\delta^0 \hat{\times} |i|"'] \arrow[rr, bend left, "a"] & \mathbb{n} \arrow[r, "a^*"] \arrow[d, equal] & G \arrow[d, "f"] \\
\mathbb{1} \times \mathbb{n} \arrow[r, "\epsilon \times \mathbb{n}"'] \arrow[urr, dotted, "N\theta_i"] \arrow[rr, bend right, "b"'] & \mathbb{n} \arrow[r, "b^*"'] \arrow[ur, dotted, "a^*"] & H
\end{tikzcd}$$

which will factor through a squash square as shown. The property of factoring through a squash square translates to the statement that both maps a and b are the identity on all 'vertical' arrows in $\mathbb{1} \times \mathbb{n}$ (i.e. the ones called q_k).

Now, if we look at the procedure by case analysis for the construction of the lift $N\theta$, we see that all vertical arrows lift to the identity because θ is normal, and this in turns implies that $N\theta_i$ is degenerate in the lifting direction with a^* as lifting degeneracy. ☐

Lemma 7.3.2. *The following square produced by the lift of Lemma 7.3.1:*

$$\begin{tikzcd}
\underline{\mathrm{NrmFib}} \arrow[r, "\tilde{N}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{NrmUniFib}} \arrow[d] \\
\underline{\mathrm{ClFib}} \arrow[r, "\tilde{N}"'] & \underline{\mathrm{UniFib}}
\end{tikzcd}$$

is a pullback square.

Proof. By fully-faithfulness of the vertical forgetful functors (Lemma 7.2.3), we only need to show that the universal arrow from $\underline{\mathrm{NrmFib}}$ to the pullback $\underline{\mathrm{ClFib}} \times_{\underline{\mathrm{UniFib}}} \underline{\mathrm{NrmUniFib}}$ is surjective on objects. Let us consider an object of the pullback, that is a cloven isofibration (f, θ) such that $(Nf, N\theta)$ is a normal uniform fibration, and consider the square:

$$\begin{tikzcd}
* \arrow[r, "a"] \arrow[d, "\delta^0"'] & G \arrow[d, "f"] \\
\mathbb{1} \arrow[r, "\mathrm{id}_{f(a)}"'] \arrow[ur, dotted, "\theta"] & H
\end{tikzcd}$$

we must show that (f, θ) is normal, i.e. that the lift θ is the identity on a. But this follows directly from the normality property of $N\theta$ using the squash square of the arrow $\perp_* : \emptyset \rightarrow *$. ☐

The next result is the counterpart of Theorem 7.1.7 but in the context of normal uniform fibrations in simplicial sets and normal isofibrations in groupoids.

Theorem 7.3.3. *The following is a pullback square:*

$$\begin{tikzcd}
\underline{\mathrm{NrmFib}} \arrow[r, "\tilde{N}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{NrmUniFib}} \arrow[d] \\
\underline{\mathrm{Grd}}^{\rightarrow} \arrow[r, "N"'] & \underline{\mathrm{sSet}}^{\rightarrow}
\end{tikzcd}$$

Proof. This follows from Theorem 7.1.7 and Lemma 7.3.2 by vertically pasting together the relevant pullback squares. ☐

7.4 Normal Trivial Cofibrations and Strong Deformation Retracts

We would like to have a way of characterising normal trivial cofibrations (i.e. the maps uniformly equipped with a left lifting structure against normal uniform fibrations); however a complete characterisation remains elusive. The next best thing we can have is a general method for constructing normal trivial cofibrations, and this we can do. We will work in the context of an arbitrary suitable topos $(\mathbb{C}, \mathbb{A}, \mathcal{M})$.

Definition 7.4.1. The category of arrows of **normal trivial cofibrations** is defined to be the category of NC_t-maps with respect to the awfs of normal uniform fibrations (Theorem 7.2.2). Alternatively, it is the left orthogonal category of arrows of $\underline{\mathrm{NrmUniFib}}$. We will denote it by $\underline{\mathrm{NrmTrivCof}}$.

In this section, we will observe that every strong deformation retract has the structure of a normal trivial cofibration. That is, we will construct a functor over $\mathbb{C}^{\rightarrow}$ from a category of arrows of strong deformation retracts, which we define in Definition C.5.5, to that of normal trivial cofibrations. Normality is an essential ingredient in the proof, in particular, a similar result would not hold for uniform fibrations.

Theorem 7.4.2. *There is a functor from the category strong deformation retracts* <u>SDR</u> *to that of normal trivial cofibrations* <u>NrmTrivCof</u> *as shown in the following diagram:*

$$\begin{tikzcd}
\underline{\mathrm{SDR}} \arrow[rr, "\Psi"] \arrow[dr] & & \underline{\mathrm{NrmTrivCof}} \arrow[dl] \\
& \mathbb{C}^{\to} &
\end{tikzcd}$$

Proof. First let us define the action of Ψ on objects. Let $(g, r, h) \in \mathsf{SDR}$ which, we assume to be 0-oriented (the other case being analogous). We have to define $\Psi(g, r, h) := (g, \Psi h)$ with Ψh a left <u>NrmUniFib</u>-map; to do this, let's consider a normal uniform fibration (f, ϕ) and a square $(a, b) : g \to f$ for which we will construct a lift $\Psi h_f : B \to X$ as shown:

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "g"'] & X \arrow[d, "f"] \\
B \arrow[r, "b"'] \arrow[ur, dotted, "\Psi h_f"] & Y
\end{tikzcd}$$

For this, we first consider the lift $H : I \times B \to X$, in the following square (which commutes because the deformation retraction is 0-oriented), produced by the normal uniform fibration structure of f:

$$\begin{tikzcd}
B \arrow[r, "r"] \arrow[d, "\delta^0 \times B"'] & A \arrow[r, "a"] & X \arrow[d, "f"] \\
I \times B \arrow[r, "h"'] \arrow[urr, dotted, "H"] & B \arrow[r, "b"'] & Y
\end{tikzcd}$$

and we define $\Psi h_f := H \cdot (\delta^1 \times B)$, that is, the lift Ψh_f is defined to be H on restricted to the top of the cylinder $I \times B$.

We need to verify that the triangles created by the lift Ψh_f commute. We first do the bottom one, that is, we need to check that $f \cdot \Phi h_f = b$:

$$\begin{aligned}
f \cdot \Phi h_f &= f \cdot H \cdot (\delta^1 \times B) && \text{(by defn of } \Phi h_f\text{)} \\
&= b \cdot h \cdot (\delta^1 \times B) && \text{(by defn of } H\text{)} \\
&= b && \text{(by defn of } h\text{)}
\end{aligned}$$

Notice that until now we have not used the normality assumption on (f, ψ). We now need to verify that the top triangle commutes, i.e. we check that $\Psi h_f \cdot g = a$, for this we first observe the following diagram:

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "\delta^0 \times A"'] & B \arrow[r, "r"] \arrow[d, "\delta^0 \times B"] & A \arrow[r, "a"] & X \arrow[d, "f"] \\
I \times A \arrow[r, "I \times g"'] \arrow[urrr, dotted, "H_0"] & I \times B \arrow[r, "h"'] \arrow[urr, dotted, "H"] & B \arrow[r, "b"'] & Y
\end{tikzcd}$$

here, the lift H_0 is also defined by the uniform fibration structure of f, and moreover the triangle created by the lifts commute, since the square on the left is a morphism of left UniFib-maps.

We now use that $rg = \mathrm{id}_a$ and the strength of the homotopy retraction tuple (g, r, h), to replace the horizontal arrows in the previous diagram in order to obtain the following:

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "\delta^0 \times A"'] & A \arrow[rr, "a"] \arrow[d, equal] & & X \arrow[d, "f"] \\
I \times A \arrow[r, "\rho_1"'] \arrow[urrr, dotted, "H_0"] & A \arrow[r, "g"'] \arrow[urr, dotted, "a"] & B \arrow[r, "b"'] & Y
\end{tikzcd}$$

where now the lifts cohere by Proposition 7.2.4 using the squash square of the arrow $\perp_A : \emptyset \to A$. With this in place, we can compute the desired equation:

$$\begin{aligned}
\Psi h_f \cdot g &= H \cdot (\delta^1 \times B) \cdot g && \text{(by defn of } \Phi h_f\text{)} \\
&= H \cdot (I \times g) \cdot (\delta^1 \times A) && \text{(by naturality of } \delta^1 \times -\text{)} \\
&= H_0 \cdot (\delta^1 \times A) && \text{(by construction of } H_0\text{)} \\
&= a \cdot \rho_1 \cdot (\delta^1 \times A) && \text{(by normality of } (f, \phi)\text{)} \\
&= a
\end{aligned}$$

Moreover, if $(l, m) : (f, \phi) \to (f', \phi')$ is a morphism of normal uniform fibrations, then we first define $\Phi h_{f'}$ relative to the square $(la, mb) : g \to f'$ by creating the intermediate homotopy $H' : I \times B \to X'$. But since (l, m) is a morphism of normal uniform fibrations, we have that $m \cdot H = H'$, and thus we have:

$$\begin{aligned}
m \cdot \Psi h_f &= m \cdot H \cdot (\delta^1 \times B) && \text{(by defn of } \Phi h_f\text{)} \\
&= H' \cdot (\delta^1 \times B) && \text{(since } (l, m) \text{ is structure preserving)} \\
&= \Psi h_{f'} && \text{(by defn of } \Phi h_{f'}\text{)}
\end{aligned}$$

and this shows that $(g, \Psi h)$ is a normal trivial cofibration.

We now have to show that the assignment $\Psi : (g, r, h) \mapsto (g, \Psi h)$ is functorial. For this let's consider a morphism of (0-oriented) strong deformation retracts $(s, t) : (g', r', h') \to (g, r, h)$ for which we need to verify that the underlying square $(s, t) : g' \to g$ is a morphism of NrmTrivCof. So, let us consider a normal uniform fibration (f, ψ) and a square $(a, b) : g \to f$, we have lifts as shown in the following diagram:

$$\begin{tikzcd}
A' \arrow[r, "s"] \arrow[d, "g'"'] & A \arrow[r, "a"] \arrow[d, "g"] & X \arrow[d, "f"] \\
B' \arrow[r, "t"'] \arrow[urr, dotted, "\Psi h_f'"] & B \arrow[r, "b"'] \arrow[ur, dotted, "\Psi h_f"] & Y
\end{tikzcd}$$

where $\Psi h_f'$ is the lifting structure defined relative to the square $(as, bt) : g' \to f$ via the intermediate homotopy $H' : I \times B' \to X$. We need to verify that the triangle created by the lifts commute. For this, let's first consider the diagram:

$$\begin{tikzcd}
B' \arrow[r, "t"] \arrow[d, "\delta^0 \times A"'] & B \arrow[r, "r"] \arrow[d, "\delta^0 \times B"] & A \arrow[r, "a"] & X \arrow[d, "f"] \\
I \times B' \arrow[r, "I \times t"'] \arrow[urrr, dotted, "H'"] & I \times B \arrow[r, "h"'] \arrow[urr, dotted, "H"] & B \arrow[r, "b"'] & Y
\end{tikzcd}$$

where we notice that the lifts cohere by uniformity, and also we observe that the leftmost lift H' coincide with the intermediate homotopy used to define $\Psi h'_f$. To see this, we use the hypothesis that (s, t) is a morphism of strong deformation retracts, and thus we have that $r \cdot t = s \cdot r'$ and $h \cdot (I \times t) = t \cdot h'$. We now can compute the desired equality:

$$\begin{aligned}
\Psi h_f \cdot t &= H \cdot (\delta^1 \times B) \cdot t && \text{(by defn of } \Psi h_f) \\
&= H \cdot (I \times t) \cdot (\delta^1 \times B') && \text{(by naturality of } \delta^1 \times -) \\
&= H' \cdot (\delta^1 \times B') && \text{(by uniformity of } f) \\
&= \Psi h'_f && \text{(by defn of } \Psi h'_f)
\end{aligned}$$

□

7.5 Compatibility with Path Objects

We now proceed to show that stable functorial choice of path objects $\mathcal{P}_I$ on Uniform Fibrations from Theorem 5.1.1 is compatible with the category of arrows of normal uniform fibrations. That is, we need to exhibit a lift of the interval path-object factorisation $\mathcal{P}_I$ (Appendix C.3) as shown in the following diagram:

$$\begin{tikzcd}
\underline{\mathrm{NrmUniFib}} \arrow[r, dotted, "\tilde{\mathcal{P}}_I"] \arrow[d] & \underline{\mathrm{NrmTrivCof}} \times_{\mathbb{C}} \underline{\mathrm{NrmUniFib}} \arrow[d] \\
\mathbb{C}^{\to} \arrow[r, "\mathcal{P}_I"'] & \mathbb{C}^{\to} \times_{\mathbb{C}} \mathbb{C}^{\to}
\end{tikzcd}$$

The first thing to notice that we can split the problem in two. If we denote by $r, \rho : \mathbb{C}^{\to} \to \mathbb{C}^{\to}$ the two legs of the sfpo (i.e. by composing $\mathcal{P}_I$ with the two projections from the pullback). Then it is sufficient to show that there are lifts of these functors as shown below.

$$\begin{tikzcd}
\underline{\mathrm{NrmUniFib}} \arrow[r, dotted, "\tilde{r}"] \arrow[d] & \underline{\mathrm{NrmTrivCof}} \arrow[d] \\
\mathbb{C}^{\to} \arrow[r, "r"'] & \mathbb{C}^{\to}
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{\mathrm{NrmUniFib}} \arrow[r, dotted, "\tilde{\rho}"] \arrow[d] & \underline{\mathrm{NrmUniFib}} \arrow[d] \\
\mathbb{C}^{\to} \arrow[r, "\rho"'] & \mathbb{C}^{\to}
\end{tikzcd}$$

To do this we are going to use some constructions and results from Section 5.1. In fact we will only use the following lemmas whose proofs are found inlined in the proof of Theorem 5.1.1.

Lemma 7.5.1. *There is a lift of the functor* $r : \mathbb{C}^{\to} \to \mathbb{C}^{\to}$ *to the category of strong deformation retracts as shown:*

$$\begin{tikzcd}
 & \underline{\mathrm{SDR}} \arrow[d] \\
\mathbb{C}^{\to} \arrow[ur, dotted, "\tilde{r}"] \arrow[r, "r"'] & \mathbb{C}^{\to}
\end{tikzcd}$$

Lemma 7.5.2. *There is a lift of the functor* $\rho : \mathbb{C}^{\to} \to \mathbb{C}^{\to}$ *to the category of uniform fibrations as shown:*

$$\begin{tikzcd}
\underline{\mathrm{UniFib}} \arrow[r, dotted, "\tilde{\rho}"] \arrow[d] & \underline{\mathrm{UniFib}} \arrow[d] \\
\mathbb{C}^{\to} \arrow[r, "\rho"'] & \mathbb{C}^{\to}
\end{tikzcd}$$

Using Lemma 7.5.1 and the results from the previous section, it is easy to see that we obtain a lift of $r : \mathbb{C}^{\to} \to \mathbb{C}^{\to}$ as desired.

Lemma 7.5.3. *There is a lift the functor* $r : \mathbb{C}^{\to} \to \mathbb{C}^{\to}$ *to the category of uniform trivial cofibrations as shown:*

$$\begin{tikzcd}
\underline{\mathrm{NrmUniFib}} \arrow[r, dotted, "\tilde{r}"] \arrow[d] & \underline{\mathrm{NrmTrivCof}} \arrow[d] \\
\mathbb{C}^{\to} \arrow[r, "r"'] & \mathbb{C}^{\to}
\end{tikzcd}$$

Proof. We construct the desired lift as the following composite:

$$\begin{tikzcd}
\underline{\mathrm{NrmUniFib}} \arrow[r] \arrow[d] & \mathbb{C}^{\to} \arrow[r, "\tilde{r}"] \arrow[d, equal] & \underline{\mathrm{SDR}} \arrow[r, "\Psi"] \arrow[d] & \underline{\mathrm{NrmTrivCof}} \arrow[d] \\
\mathbb{C}^{\to} \arrow[r, equal] & \mathbb{C}^{\to} \arrow[r, "r"'] & \mathbb{C}^{\to} \arrow[r, equal] & \mathbb{C}^{\to}
\end{tikzcd}$$

where the lift in the leftmost square is the forgetful functor, that on the middle square is that from Lemma 7.5.1 and the lift in the rightmost square is the one from Theorem 7.4.2. ☐

Unfortunately, the construction of the lift for the other functor $\rho : \mathbb{C}^{\to} \to \mathbb{C}^{\to}$ is not quite as direct; we will have to recall the construction of the uniform fibration structure produced by Lemma 7.5.2. For this let us consider a map $f : X \to Y$ in $\mathbb{C}$; recall (from Eq. (5.2)) that the map $\rho_f : P_w f \to X \times_Y X$ is alternatively obtained as in the following pullback:

$$\begin{tikzcd}
P_w f \arrow[r] \arrow[d, "\rho_f"'] \arrow[dr, phantom, "\lrcorner", very near start] & X^I \arrow[d, "\hat{\hom}(j_1, f)"] \\
X \times_Y X \arrow[r, "\langle \alpha_f, \lambda_f \rangle"'] & Y^I \times_{Y^{\partial I}} X^{\partial I}
\end{tikzcd}$$

where the map $j_1 : \partial I \to I$ stands for the inclusion of the boundary of the interval.

Let us assume from now that (f, θ) is a uniform fibration. We know that right orthogonal categories of arrows are closed under pulbacks, thus to give a uniform fibration structure to ρ_f it is sufficient to give one to $\widehat{\hom}(j_1, f)$. Now, in order to construct a uniform fibration structure for $\widehat{\hom}(j_1, f)$, let us consider a lifting problem with respect to the generating category of arrows $\mathcal{I}_\otimes$ for uniform fibrations; i.e. a square of the form $(u, b) : \delta^k \hat{\times} i \to \widehat{\hom}(j_1, f)$ where $i : A \rightarrowtail \mathcal{A}$ is in $\mathcal{I}$ for which we show how to construct a lift. This is shown in the left side of the following diagram.

$$\begin{tikzcd}
\mathcal{A} +_A (I \times A) \arrow[r, "u"] \arrow[d, "\delta^k \hat{\times} i"'] & X^I \arrow[d, "\widehat{\hom}(j_1,f)"] \\
I \times \mathcal{A} \arrow[r, "b"'] \arrow[ur, dotted, "\rho\theta_i"] & Y^I \times_{Y^{\partial I}} X^{\partial I}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{dom}(j_1 \hat{\times} (\delta^k \hat{\times} i)) \arrow[r, "\hat{u}"] \arrow[d, "j_1 \hat{\times} (\delta^k \hat{\times} i)"'] & X \arrow[d, "f"] \\
I \times (I \times \mathcal{A}) \arrow[r, "\hat{b}"'] \arrow[ur, dotted, "\rho\theta_i"] & X
\end{tikzcd}$$

Now, transposing along the adjunction $(j_1 \hat{\times} -) \vdash \widehat{\hom}(j_1, -)$ we obtain a square as on the right of the previous diagram. Next, we use that the pushout-product construction is symmetric and associative, and in particular we obtain that $j_1 \hat{\times} (\delta^k \hat{\times} i) \cong \delta^k \hat{\times} (j_1 \hat{\times} i)$; by the properties of the category of generating cofibrations $\mathcal{M}$ we see that $j_1 \times i$ is a generating monomorphism, thus we can find a lift denoted by $\rho\theta_i$, and by transposing everything back we obtain the desired lift for the original square. This construction produces a uniform fibration structure for $\widehat{\hom}(j_1, f)$ which we denote by $\rho\theta$. This finishes the description of the action on objects of the functor from Lemma 7.5.2.

With this in place we can now state and prove the following lemma. We will make use of the explicit construction of the uniform fibration structure $\rho\theta$ described above.

Lemma 7.5.4. *There is a lift of the functor from Lemma 7.5.2 as shown:*

$$\begin{tikzcd}
\underline{\mathrm{NrmUniFib}} \arrow[r, dotted, "\tilde{\rho}"] \arrow[d] & \underline{\mathrm{NrmUniFib}} \arrow[d] \\
\underline{\mathrm{UniFib}} \arrow[r, "\tilde{\rho}"'] & \underline{\mathrm{UniFib}}
\end{tikzcd}$$

Proof. Fortunately, since the forgetful functor $\underline{\mathrm{NrmUniFib}} \to \underline{\mathrm{UniFib}}$ is fully faithful, and using that right ortogonal categories are closed under pullbacks; it is sufficient to prove that given (f, ψ) a normal uniform fibration, the uniform fibration structure $\rho\psi$ of $\widehat{\hom}(j_1, f)$ described above, is also normal. Using the characterisation of normal uniform fibrations from Proposition 7.2.4, we need to show that for any generating monomorphism $i : A \rightarrowtail B$ the lifts in the following diagram on the left cohere:

$$\begin{tikzcd}
B +_A (I \times A) \arrow[r, "\mathbf{sq}_k(i)"] \arrow[d, "\delta^k \hat{\times} i"'] & B \arrow[r, "u^*"] \arrow[d, equal] & X^I \arrow[d, "\widehat{\hom}(j_1,f)"] \\
I \times B \arrow[r, "\epsilon \times B"'] \arrow[urr, dotted, "\rho\theta_i"] & B \arrow[r, "b^*"'] \arrow[ur, dotted, "u^*"] & Y^I \times_{Y^{\partial I}} X^{\partial I}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{dom}(\delta^k \hat{\times} (j_1 \hat{\times} i)) \arrow[r, "\mathbf{sq}_k(j_1 \hat{\times} i)"] \arrow[d, "\delta^k \hat{\times} (j_1 \hat{\times} i)"'] & I \times B \arrow[r, "u^*"] \arrow[d, equal] & X \arrow[d, "f"] \\
I \times (I \times B) \arrow[r, "\epsilon \times (I \times B)"'] \arrow[urr, dotted, "\rho\theta_i"] & I \times B \arrow[r, "b^*"'] \arrow[ur, dotted, "u^*"] & Y
\end{tikzcd}$$

by transposing the whole diagram along $(j_1 \hat{\times} -) \vdash \widehat{\hom}(j_1, -)$, and using the symmetry and associativity of the pushout-product, we obtain the lifting problem as on the right of the previous diagram, for which we need to show that the lifts cohere. Observe that the lift $\rho\theta_i$ on the left (on either diagram) is, by construction, the lift obtained from the uniform fibration structure $\rho\theta$ on $\widehat{\hom}(j_1, f)$. The result follows by applying Lemma 7.2.1. □

In the following theorem is a synthesis of the results from the foregoing section.

Theorem 7.5.5. *The stable functorial choice of path objects* $\mathcal{P}_I$ *of Theorem 5.1.1 is compatible with the category of arrows of normal uniform fibrations as shown in the following diagram:*

$$\begin{tikzcd}
\underline{\mathrm{NrmUniFib}} \arrow[r, dotted, "\tilde{\mathcal{P}}_I"] \arrow[d] & \underline{\mathrm{NrmTrivCof}} \times_{\mathbb{C}} \underline{\mathrm{NrmUniFib}} \arrow[d] \\
\mathbb{C}^{\to} \arrow[r, "\mathcal{P}_I = \langle r, \rho \rangle"'] & \mathbb{C}^{\to} \times_{\mathbb{C}} \mathbb{C}^{\to}
\end{tikzcd}$$

Proof. This follows by applying Lemma 7.5.3 to lift the functor $r : \mathbb{C}^{\to} \to \mathbb{C}^{\to}$ and by applying Lemma 7.5.2 and Lemma 7.5.4 to lift the functor $\rho : \mathbb{C}^{\to} \to \mathbb{C}^{\to}$. □

7.6 Compatibility with Functorial Frobenius

In what follows, we will provide a proof that the category of arrows of normal uniform fibrations has a functorial Frobenius structure. This will be given by adapting the functorial Frobenius structure on uniform fibrations. Here we will work on an arbitrary suitable topos $(\mathbb{C}, \mathbb{A}, \mathcal{M})$.

In order to prove the main result of this section, we will need to use the explicit construction of the functorial Frobenius structure on uniform fibrations. For the convenience of the reader, we will state the theorem here providing a sketch of the proof. In what follows we will denote by $\underline{\mathrm{TrivCof}}$ the category of arrows of trivial cofibrations, the objects of which are arrows having a left lifting structure against uniform fibrations.

Theorem 7.6.1. *[GS17, Theorem 8.8] The category of arrows of uniform fibrations has a functorial Frobenius structure. That is, we have a lift of the pullback functor as in the following diagram:*

$$\begin{tikzcd}
\underline{\mathrm{TrivCof}} \times_{\mathbb{C}} \underline{\mathrm{UniFib}} \arrow[r, "\tilde{\mathsf{PB}}"] \arrow[d] & \underline{\mathrm{TrivCof}} \arrow[d] \\
\mathbb{C}^{\to} \times_{\mathbb{C}} \mathbb{C}^{\to} \arrow[r, "\mathsf{PB}"'] & \mathbb{C}^{\to}
\end{tikzcd}$$

Sketch of proof: We will give a brief overview the object-wise construction of the lift $\widetilde{\mathsf{PB}} : \underline{\mathrm{TrivCof}} \times_{\mathbb{C}} \underline{\mathrm{UniFib}} \to \underline{\mathrm{TrivCof}}$. We will start with the special case where the trivial normal cofibration is of the form $\delta^k \hat{\times} i$ for some generating monomorphism $i : A \rightarrowtail B$; that is, we start by taking a Cartesian diagram:

$$\begin{tikzcd}
A' \arrow[r] \arrow[d, "g'"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B +_A (\Delta^1 \times A) \arrow[d, "\delta^k \hat{\times} i"] \\
B' \arrow[r, "f"'] & \Delta^1 \times B
\end{tikzcd}$$

where (f, θ) is a uniform fibration, and we show that $g' : A' \to B'$ is a trivial cofibration. By hypothesis the generating category of monomophism is closed under pushout-product by the endpoint inclusion, it is also closed under pullback along any arrow; this implies that $g' : A' \to B'$ is a generating monomorphism. We also know that $g' : A' \to B'$ is a k-oriented strong homotopy equivalence, this follows from lemma [GS17, Lemma 8.4] which shows that $\delta^k \hat{\times} i$ is a k-oriented strong homotopy equivalence and by lemma [GS17, Lemma 8.7] which says that k-oriented strong homotopy equivalences are closed under pullbacks along uniform fibrations.

Using the characterisation of k-oriented strong homotopy equivalences described in Lemma C.5.3, we have that g' is a retract of $\delta^k \hat{\times} g'$, and this later morphism is a normal trivial cofibration since g' is a generating monomorphism. Thus, g' is also a normal trivial cofibration since these are closed under retracts. The general case follows by [GS17, Proposition 6.8]. □

To show that the proof of this theorem can be adapted to the case of normal uniform fibrations, we will first need a couple of lemmas.

Lemma 7.6.2. *Let $i : A \rightarrowtail B$ be a monomorphism, then the following holds:*

1. *For any map $f : X \to B$, there is an isomorphism*

$$\delta^k \hat{\times} (f^* i) \cong (I \times f)^* (\delta^k \hat{\times} i)$$

2. *Pulling back the k-squash square of i along the square $(I \times f, f)$ produces the k-squash square of $f^* i$; concretely, for $k \in \{0, 1\}$, there is an isomorphism:*

$$\boldsymbol{squash}_k(f^* i) \cong (I \times f, f)^* (\boldsymbol{squash}_k(i))$$

Proof. To show item (1), let us first consider the following cube:

$$\begin{tikzcd}
f^*A \arrow[rr, "\pi"] \arrow[dr, "f^*i"] \arrow[dd, "\delta^k\times f^*A"'] & & A \arrow[dr, "i"] \arrow[dd, dotted, "\delta^k\times A" near start] & \\
& X \arrow[rr, "f"] \arrow[dd, "\delta^k\times X" near start] & & B \arrow[dd, "\delta^k\times B"] \\
I\times(f^*A) \arrow[rr, dotted, "I\times\pi"' near end] \arrow[dr, "I\times f^*i"'] & & I\times A \arrow[dr, dotted, "I\times i"] & \\
& I\times X \arrow[rr, "I\times f"'] & & I\times B
\end{tikzcd}$$

here, the square on the top is the pullback of i along f. It is straightforward to verify that all squares pointing from left to right are Cartesian, and notice that the squares on the left and right are the outer squares used for defining the pushout-products $\delta^k\hat{\times}(f^*i)$ and $\delta^k\hat{\times}i$ respectively. All of this implies that there is a comparison map $\delta^k\hat{\times}(f^*i) \to (I\times f)^*(\delta^k\hat{\times}i)$, which is an isomorphism because colimits in $\mathbb{C}$ are universal. Item (2) follows directly from item (1). $\square$

The following lemma is a technical result about the squares $\theta^k\hat{\times}i : i \to \delta^k\hat{\times}i$ (see Appendix C.5).

Lemma 7.6.3. *For any morphism* $i : A \to B$ *the square* $\theta^k\hat{\times}i$ *depicted bellow:*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & B +_A (I\times A) \arrow[d, "\delta^k\hat{\times}i"] \\
B \arrow[r, "\delta^{1-k}\times B"'] & I\times B
\end{tikzcd}$$

is Cartesian.

Proof. The proof uses once again the fact that colimits in $\mathbb{C}$ are universal. Let us compute the pullback of $\delta^k\hat{\times}i$ along $\delta^{1-k}\times B$; by universality of colimits, this is the same as pulling back the diagram (defining $B +_A (I\times A)$) and then calculating the colimit.

We can observe in the following picture, the result of first pulling back the defining diagram of $B +_A (I\times A)$ which appears as the upper span of the right-most square on the following cube:

$$\begin{tikzcd}
\emptyset \arrow[rr] \arrow[dr] \arrow[dd] & & A \arrow[dr, "i"] \arrow[dd, dotted, "\delta^k\times A"] & \\
& \emptyset \arrow[rr] \arrow[dd] & & B \arrow[dd, "\delta^k\times B"] \\
A \arrow[rr, dotted, "\delta^{1-k}\times A"] \arrow[dr, "i"'] & & I\times A \arrow[dr, dotted, "I\times i"] & \\
& B \arrow[rr, "\delta^{1-k}\times B"'] & & I\times B
\end{tikzcd}$$

Let us notice that the pullback of $\delta^k\times B$ (respectively $\delta^k\times A$) along $\delta^{1-k}\times B$ (respectively $\delta^{1-k}\times A$) is empty since the interval has disjoint endpoints. We conclude that the colimit of the upper span of the left-most square on the cube must be equal to A and moreover, the universal arrow down to B has to be $i : A \to B$. ☐

Here is an important thing to notice. Consider a generating monomorphism $i : A \rightarrowtail B$ and a uniform fibration $f : X \to B$, then there are two possible trivial uniform cofibration structures on the map $\delta^k \hat{\times} (f^*i)$: the first one is the canonical one, i.e. the one given by the fact that f^*i is also a generating monomorphism. The second one is the one provided by the functorial Frobenius structure on uniform fibrations using the isomorphism $\delta^k \hat{\times} (f^*i) \cong (I\times f)^*(\delta^k \hat{\times} i)$ of Lemma 7.6.2. Luckily, it turns out that these two are actually the same structure as we proceed to show in the following lemma.

Lemma 7.6.4. *Consider* $i : A \rightarrowtail B$ *be a monomorphism and* $f : X \to B$ *a uniform fibration. Then the two possible trivial uniform cofibration structures on* $\delta^k \hat{\times} (f^*i)$ *coincide.*

Proof. Let us denote by λ^1 and λ^2, respectively, the canonical trivial uniform cofibration structure on $\delta^k \hat{\times} (f^*i)$ and the one obtained by applying the functorial Frobenius structure.

In order to prove they are the same, let us consider $g : Z \to Y$ a uniform fibration and a square $(a, b) : \delta^k \hat{\times} (f^*i) \to g$. Without loss of generality, let's denote by λ^1 and λ^2 the two fillers of this square given by the uniform trivial cofibration structure with the same name.

We have to show that $\lambda^1 = \lambda^2$. If we go over the proof of Theorem 7.6.1 (applied to this situation), before concluding, we see that there is a retract diagram as in the two left-most squares shown bellow:

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "\delta^k \hat{\times} (f^*i)"'] & \cdot \arrow[r] \arrow[d, "\delta^k \hat{\times} \delta^k \hat{\times} (f^*i)"] & \cdot \arrow[r, "a"] \arrow[d, "\delta^k \hat{\times} (f^*i)"] & Z \arrow[d, "f"] \\
\cdot \arrow[r, "t"'] & \cdot \arrow[r] & \cdot \arrow[r, "b"'] & Y
\end{tikzcd}$$

where the left-most square is $\theta^k \hat{\times} \delta^k \hat{\times} (f^* i)$. Now, the square $\delta^k \hat{\times} \delta^k \hat{\times} (f^* i) \to f$ has a lift which we denote by λ, notice that by definition, the lift λ^2 is equal to $\lambda \cdot t$ where t is the horizontal arrow on the lower left part of the diagram. Moreover, we have that the lift of the outer square is λ^1. Thus if we want to show that $\lambda^1 = \lambda^2$ it is sufficient to show that the square $\theta^k \hat{\times} \delta^k \hat{\times} (f^* i)$ is a morphism of trivial uniform cofibrations.

To show this, we use that the pushout-product is symmetric and associative, and thus $\theta^k \hat{\times} \delta^k \hat{\times} (f^* i) \cong \delta^k \hat{\times} \theta^k \hat{\times} (f^* i)$. From this, we see that the square is a morphism of trivial uniform cofibrations if the square $\theta^k \hat{\times} (f^* i)$ is a morphism of generating cofibrations, i.e. if it is Cartesian, but this is precisely the statement of Lemma 7.6.3. ☐

We now have enough tools to show that the functorial Frobenius structure on uniform fibrartion given by Theorem 7.6.1 can be extended to a functorial Frobenius strucutre on normal uniform fibrations. We start with the following lemma.

Lemma 7.6.5. *There is a lift of the pullback functor as shown:*

$$\begin{tikzcd}
\mathcal{I}^{\mathbf{n}}_{\otimes} \times_{\mathbb{C}} \underline{\mathrm{UniFib}} \arrow[r, "\widetilde{\mathsf{PB}}"] \arrow[d] & \underline{\mathrm{NrmTrivCof}} \arrow[d] \\
\mathbb{C}^{\to} \times_{\mathbb{C}} \mathbb{C}^{\to} \arrow[r, "\mathsf{PB}"'] & \mathbb{C}^{\to}
\end{tikzcd}$$

Proof. Object-wise, this follows directly from Theorem 7.6.1. To see this, we notice that there are no more objects in $\mathcal{I}^{\mathbf{n}}_{\otimes}$ that in $\mathcal{I}_{\otimes}$ thus we can apply the functorial Frobenius structure for uniform fibrations. Then we use the functor $\underline{\mathrm{TrivCof}} \to \underline{\mathrm{NrmTrivCof}}$, obtain by functoriality of the left orthogonal functor ${}^{\boxslash}(-)$ applied to the forgetful functor $\underline{\mathrm{NrmUniFib}} \to \underline{\mathrm{UniFib}}$.

For the morphism case, we first notice that the only morphisms in $\mathcal{I}^{\mathbf{n}}_{\otimes}$ that we need to consider are the squash squares. Thus let us consider a cospan of squares as in the following diagram:

$$\begin{tikzcd}
 & & \cdot \arrow[dr, "\mathbf{sq}_k(i)"] \arrow[dd, "\delta^k \hat{\times} i"'] & \\
 & & & B \arrow[dd, equal] \\
X' \arrow[rr, "f'"] \arrow[dr, "m"'] & & I \times B \arrow[dr, "\epsilon \times B"'] & \\
 & X \arrow[rr, "f"'] & & B
\end{tikzcd}$$

such that the vertical square is the squash square of a generating monomorphism $i : A \rightarrowtail B$ and the horizontal square is a morphism of uniform fibrations $(m, \epsilon \times B) : f' \to f$. We need to verify that pulling back the squash square along the morphism of uniform fibrations is a morphism of normal trivial cofibrations.

The first thing we do is to split this cospan of squares into two, by factoring through the pullback square of f along $\epsilon \times B$. That is we obtain the following:

$$\begin{tikzcd}
 & & \cdot \arrow[dd, "\delta^k \hat{\times} i"'] \arrow[dr, "\mathbf{sq}_k(i)"] & \\
 & & & B \arrow[dd, equal] \\
I \times X \arrow[rr, "I\times f"] \arrow[dr, "\epsilon\times X"'] \arrow[dr, phantom, very near start, "\lrcorner"] & & I \times B \arrow[dr, "\epsilon\times B"'] & \\
 & X \arrow[rr, "f"'] & & B
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & \cdot \arrow[dd, "\delta^k \hat{\times} i"'] \arrow[dr, equal] & \\
 & & & \cdot \arrow[dd, "\delta^k \hat{\times} i"] \\
X' \arrow[rr, "f'"] \arrow[dr, dotted, "m^*"'] & & I \times B \arrow[dr, equal] & \\
 & I \times X \arrow[rr, "I\times f"'] & & I \times B
\end{tikzcd}$$

where the dotted arrow $m^* : X' \to (I \times X)$ is obtain by the universal property of the pullback. Notice that composing the two cospans of squares along their common face, produces the original one. Notice also that the two horizontal squares are morphisms of uniform fibrations.

Let us focus first on the cospan of the right. The identity morphism $\mathrm{id} : (\delta^k \hat{\times} i) \to (\delta^k \hat{\times} i)$ is a morphism of trivial uniform cofibrations, thus if we pull-back this along the morphism of uniform fibrations $(f', I \times f) : m^* \to \mathrm{id}_{\delta^k \hat{\times} i}$ we obtain a morphism of trivial uniform cofibrations by Theorem 7.6.1 to which we can apply the functor $\underline{\mathrm{TrivCof}} \to \underline{\mathrm{NrmTrivCof}}$ to obtain a morphism of normal trivial cofibrations.

With this we have reduced the situation to that of the cospan of squares on the left of the previous diagram. But now, using item (2) of Lemma 7.6.2 we see that the pullback of the squash square of $i : A \rightarrowtail B$ along the square $(I \times f, f) : \epsilon \times X \to \epsilon \times B$ is the squash square of $f^*i : f^*A \rightarrowtail X$. This square is a morphism in $\mathcal{J}^{\mathbf{n}}_{\otimes}$ provided that the canonical trivial normal cofibration structure of $\delta^k \hat{\times} (f^* i)$ is the same as that obtained from the functorial Frobenius structure; and this follows from Lemma 7.6.4. ☐

We these lemmas in place, we are now ready to state and proof the main theorem of this section.

Theorem 7.6.6. *The category* $\underline{\mathrm{NrmUniFib}}$ *has a functorial Frobenius structure which is an extension of that of uniform fibrations* $\underline{\mathrm{UniFib}}$.

Proof. Using the lift of Lemma 7.6.5 and the forgetful functor $\underline{\mathrm{NrmUniFib}} \to \underline{\mathrm{UniFib}}$, we find a lift of the pullback functor as the one shown below.

$$\begin{tikzcd}
\mathcal{J}^{\mathbf{n}}_{\otimes} \times_{\mathbb{C}} \underline{\mathrm{NrmUniFib}} \arrow[r, "\tilde{\mathsf{PB}}"] \arrow[d] & \underline{\mathrm{NrmTrivCof}} \arrow[d] \\
\mathbb{C}^{\to} \times_{\mathbb{C}} \mathbb{C}^{\to} \arrow[r, "\mathsf{PB}"'] & \mathbb{C}^{\to}
\end{tikzcd}$$

The fact that we can extend this structure from $\mathcal{J}^{\mathbf{n}}_{\otimes}$ to the whole category $\underline{\mathrm{NrmTrivCofi}}$ follows from [GS17, Proposition 6.8]. ☐

We conclude this chapter by pointing out that we can combine Theorem 7.6.6 and Theorem 7.5.5 in order to obtain a type-theoretic awfs of normal uniform fibrations.

Chapter 8

Conclusions

In this dissertation we have shown that by making use of algebraic techniques it is possible to obtain sufficient structure to model a version of Martin-Löf's dependent type theory, equipped with dependent sums, products and intensional identity types. In order to do so, we introduced the notion of a type-theoretic algebraic weak factorisation system. There are two main reasons for the interest in this notion, as opposed to its non-algebraic counterpart. First, the condition of pseudo-stability for intensional identity types is extremely hard to come by in nature (for example in simplicial sets and cubical sets), but in a type-theoretic awfs the extra algebraic structure makes it possible to construct pseudo-stable identity types from the more natural notion of a path-objects. Secondly, making use of the theory of uniform fibrations of Gambino and Sattler, we have shown that type-theoretic awfs are abundant; in particular any Grothendieck topos with an interval object (with connections) can be equipped with a type-theoretic awfs. In addition, the original Hofmann-Streicher groupoid model is also shown to be an example of a type-theoretic awfs whose right algebras correspond to split isofibrations.

Moreover, we have shown that by adapting slightly the methodology of Gambino and Sattler, we are able to produce a type-theoretic awfs of normal uniform fibrations. This allows us to circumvent one of the requirements that the interval path-object factorisation need to satisfy in order to produce a stable functorial choice of path objects. With this we are able to carry out some arguments in a constructive metatheory instead of a classical one.

In a nutshell, we have shown that most of the type-theoretic properties that are present in the non-algebraic approaches to the categorical semantics of type theory, have a direct categorification in the language of awfs. We expect that this approach can be extended to accommodate additional kinds of logical structure such as W-types and universes. The payoff of working with the additional algebraic structure is that we are able to apply the right adjoint splitting to obtain models, which in some contemporary approaches to the semantics of dependent type theory has been abandoned in favour of other methods (such as the left adjoint splitting) due to the difficulty of satisfying the pseudo-stability conditions.

Future work includes adapting the definition of type-theoretic awfs in order to include the relevant structure needed to produce models with additional logical structure. Of particular interest is the case of universes. As the results from Section 2.5 show, it is possible to apply the right adjoint splitting to model universes closed under the relevant kinds of logical structure in the resulting split comprehension category. The next step is to identify sufficient additional structure that a type-theoretic awfs should posses in order to model these universes. Afterwards, we could ask if the methodology of uniform fibrations can be adapted to produce such structure, the models based on uniform fibrations in cubical sets would provide useful guidance to develop this theory. Additionally we could investigate under which circumstances the universes produced in this manner are univalent.

Appendix A

Type Theory

In this section we will give a short but self-contained introduction to the type theory we are interested with. We refer the reader to [NPS00] and to [Uni13] for additional information.

A type theory is a formal theory consisting of syntactic *judgements* and of *inference rules* that specify when a judgement is valid or well formed. Every type theory has at least two common judgements, those specifying valid *types* and valid *terms* of an already given valid type. Type theories also come equipped with an equality judgement, called *judgemental equality* which specifies when two types and when two terms of the same type must be considered equal.

A.1 Structural Rules

In this document we are interested in a class of type theories called *dependent type theories*, these are characterise by the fact that both types and terms are allowed to depend on variables of some specified types. The variables used in a judgement must be declared beforehand in a *context*, thus a context is just a collection of typed variables of the form:

$$\Gamma = (x_1 : A_1, x_2, : A_2, \ldots, x_n : A_n)$$

Thus for the class of type theories we will deal with have five kinds of judgements which are displayed in Table A.1, the column to the right of each judgement specify how to read them.

There is also a collection of *structural* inference rules which are present in any theory under consideration. These rules are presented on Table A.2.

It is necessary to introduce rules specifying that the judgemental equality is a reflexive, symmetric and transitive relation; and that each of the structural rules preserve equality. For example, we have the following rule for substitution

$$\frac{\Gamma \vdash a : A \quad \Gamma, x : A, \Delta \vdash b = c : B}{\Gamma, \Delta[a/x] \vdash b[a/x] = c[a/x] : B[a/x]}$$

Γ ctx	Γ is a valid context
$\Gamma \vdash A$ type	A is a type on context Γ
$\Gamma \vdash a : A$	a is a term of type A
$\Gamma \vdash A = B$ type	A and B are equal types
$\Gamma \vdash a = b : A$	a and b are equal terms of type A

Table A.1: Judgements of dependent type theory

In order to bootstrap such a theory we may also need some *ground types* and *ground terms*. We will call such a theory a *structural dependent type theory*.

A.2 Logical Structure

On top of our structural dependent type theory we can add various kinds of *logical structure*; we do these by introducing new rules that specify how to create new types. In this document we are considering a dependent type theory with four kinds of logical structure: dependent products (or Π-types), dependent sums (or Σ-types), intensional identities (or Id-types) and universe types. A dependent type theory together with these rules is usually called a *Martin-Löf intensional type theory.*

There is a general pattern for introducing new types. This pattern consists of four types of rules: formation, introduction, elimination and computation, there is also an optional uniqueness principle that can be further assumed.

We will start by introducing Π-types (or dependent product types). In Table A.3 we lay out the various rules.

Here we are able to see the general pattern for introducing new types in action. The *formation* rule tells us how to form the new type and the *introduction* rule specifies how to form the 'canonical' terms of this new type, these canonical terms are also

$\dfrac{}{\cdot \text{ ctx}}$	Empty context
$\dfrac{x_1 : A, \dots, x_{n-1} : A_{n-1} \vdash A_n \text{ type}}{(x_1 : A, \dots, x_n : A_n) \text{ ctx}}$	Context Extension
$\dfrac{(x_1 : A, \dots, x_n : A_n) \text{ ctx}}{x_1 : A, \dots, x_n : A_n \vdash x_i : A_i}$	Variable Declaration
$\dfrac{\Gamma \vdash a : A \quad \Gamma, x : A, \Delta \vdash b : B}{\Gamma, \Delta[a/x] \vdash b[a/x] : B[a/x]}$	Substitution
$\dfrac{\Gamma \vdash A \text{ type} \quad \Gamma, \Delta \vdash b : B}{\Gamma, x : A, \Delta \vdash b : B}$	Weakening

Table A.2: Structural Rules of dependent type theory

called *constructors*. The *elimination* rule tells us how to use terms of the new type and the *computation* rule specify how to reduce constructors terms to already known ones. Finally the optional *uniqueness* rule tells us that any term of the new type is actually of the form of some constructor term.

Together with the rules in the previous table there must also be also rules that guarantee that the new type in question is compatible with judgemental equality and with the structural rule of substitution, for example we must have that:

$$\frac{x : C \vdash A \text{ type} \quad x_1 : C, x_2 : A \vdash B \text{ type} \quad \vdash t : C}{\vdash (\prod_{x:A} B)[t/x] = \prod_{x:A[t/x]} B[t/x] \text{ type}}$$

Next we will introduce Σ-types (or dependent sum types). As with dependent products we will use the general pattern of rules, these are depicted in Table A.4.

Using the elimination rule for dependent sum types, it is possible to construct the following two projection terms.

$$\Gamma, z : \sum_{x:A} B \vdash \pi_1 : A \qquad \Gamma, z : \sum_{x:A} B \vdash \pi_2 : B[\pi_1/x]$$

The uniqueness rule for Σ-types is stated using the two previous projections, as can be seen in Table A.5.

$\dfrac{\Gamma \vdash A \text{ type} \quad \Gamma, x : A \vdash B \text{ type}}{\Gamma \vdash \prod_{x:A} B \text{ type}}$	Formation
$\dfrac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : \prod_{x:A} B}$	Introduction
$\dfrac{\Gamma \vdash a : A \quad \Gamma \vdash f : \prod_{x:A} B}{\Gamma \vdash f(a) : B[a/x]}$	Elimination
$\dfrac{\Gamma \vdash a : A \quad \Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t(a) = t[a/x] : B[a/x]}$	Computation
$\dfrac{\Gamma \vdash f : \prod_{x:A} B}{\Gamma \vdash f = \lambda x.f(x) : \prod_{x:A} B}$	Uniqueness

Table A.3: Rules for Dependent Product Types

As with dependent products, there is a further set of rules that specify the coherence of the dependent sum types with judgemental equality and substitution.

The models of dependent type theory we are considering in this thesis will validate the uniqueness rules for both dependent products and sums, this is a consequence of this types being modelled using adjoint functors.

We turn our attention now to $\mathtt{Id}$-types or identity types. Table A.6 contains the corresponding rules.

As before, these rules must cohere with judgemental equality and substitution. We will not assume the uniqueness principle for identity types; doing this will have the effect of reducing all proofs of equality to the trivial reflexivity one. This is precisely the difference between *intensional* identity types (without the uniqueness principle) and *extensional* identity types (with the uniqueness principle).

Finally we lay out the rules for universes à la Tarski. Intuitively, a universe is a type of 'codes' for types in the theory equipped with an interpretation operation that takes a code into an actual type. Moreover, for it to be an interesting notion, a universe most be suitably closed under all the previous types of logical structure. The rules for universes don't follow the usual pattern instead, in Table A.7, we describe the axioms establishing the existence of the universe and the interpretation operation.

$\dfrac{\Gamma \vdash A \text{ type} \quad \Gamma, x : A \vdash B \text{ type}}{\Gamma \vdash \sum_{x:A} B \text{ type}}$	Formation
$\dfrac{\Gamma \vdash a : A \quad \Gamma \vdash b : B[a/x]}{\Gamma \vdash (a, b) : \sum_{x:A} B}$	Introduction
$\dfrac{\begin{array}{c}\Gamma, z : \sum_{x:A} B \vdash C \text{ type} \quad \Gamma, x : A, y : B \vdash d : C[(x, y)/z] \\ \Gamma \vdash p : \sum_{x:A} B\end{array}}{\Gamma \vdash \mathbf{sp}(C, d, p) : C[p/z]}$	Elimination
$\dfrac{\begin{array}{c}\Gamma, z : \sum_{x:A} B \vdash C \text{ type} \quad \Gamma, x : A, y : B \vdash d : C[(x, y)/z] \\ \Gamma \vdash a : A \qquad \Gamma \vdash b : B[a/x]\end{array}}{\Gamma \vdash \mathbf{sp}(C, d, (a, b)) = d[a, b/x, y] : C[(a, b)/z]}$	Computation

Table A.4: Rules for Dependent Sum Types

$\dfrac{\Gamma \vdash p : \sum_{x:A} B}{\Gamma \vdash p = (\pi_1[p/z], \pi_2[p/z]) : \sum_{x:A} B}$	Uniqueness

Table A.5: Uniqueness Rule for Dependent Sum Types

We must also require additional rules to express that the universe is closed under additional logical structure. We will only describe these rules for Id-types in Table A.8.

$\dfrac{\Gamma \vdash A \text{ type}}{\Gamma, x, y : A \vdash \mathsf{Id}_A(x, y) \text{ type}}$	Formation
$\dfrac{\Gamma \vdash a : A}{\Gamma \vdash \mathsf{refl}_a : \mathsf{Id}_A[a, a/x, y]}$	Introduction
$\dfrac{\Gamma, x, y : A, z : \mathsf{Id}_A(x, y) \vdash C \text{ type} \quad \Gamma, x : A, \vdash d : C[x, x, \mathsf{refl}_x/x, y, z] \quad \Gamma, x, y : A \vdash p : \mathsf{Id}_A(x, y)}{\Gamma, x, y : A \vdash \mathsf{j}(C, d, p) : C[x, y, p/x, y, z]}$	Elimination
$\dfrac{\Gamma, x, y : A, z : \mathsf{Id}_A(x, y) \vdash C(x, y, z) \text{ type} \quad \Gamma, x : A, \vdash d : C[x, x, \mathsf{refl}_x/x, y, z]}{\Gamma, x : A \vdash \mathsf{j}(C, d, \mathsf{refl}_x) = d : C[x, x, \mathsf{refl}_x/x, y, z]}$	Computation

Table A.6: Rules for Identity Types

$\dfrac{}{\cdot \vdash U \text{ type}}$
$\dfrac{}{x : U \vdash \mathsf{El}(x) \text{ type}}$

Table A.7: Rules for Universes

$\dfrac{\Gamma \vdash x : U \quad \Gamma \vdash a, b : \mathsf{El}(x)}{\Gamma \vdash \mathsf{id}_x(a, b) : U}$
$\dfrac{\Gamma \vdash x : U \quad \Gamma \vdash a, b : \mathsf{El}(x)}{\Gamma \vdash \mathsf{El}(\mathsf{id}_x(a, b)) = \mathsf{Id}_{\mathsf{El}(x)}(a, b)}$

Table A.8: Closure of U under Id-types

Appendix B

Algebraic Weak Factorisation Systems

In this appendix section we will recall the necessary notions and tools to categorify the work done for tribes in Section 2.7. The idea is to replace the *class* of fibrations $\mathcal{R}$ of a tribe, by a *category* of fibrations R**-Map**. The material in this section is built on top of the machinery of algebraic weak factorisation systems developed by Bourke, Garner and van den Berg [BG16a, BG16b, dBG12] and that of uniform fibrations by Gambino and Sattler [GS17], also borrowing ideas from Riehl and Swan [Rie11, Swa15].

B.1 Functorial Factorisations

As a first step towards the definition of algebraic weak factorisation systems, we will explore the concept of functorial factorisation together with the rich amount of structure attached to it.

Definition B.1.1. A **functorial factorisation** $(\mathsf{Q}, \mathsf{L}, \mathsf{R})$ on a category $\mathbb{C}$ consists of an operation that assigns to each arrow $\mathsf{f} : \mathsf{X} \to \mathsf{Y}$ a factorisation of f:

$$\mathsf{X} \xrightarrow{\mathsf{Lf}} \mathsf{Qf} \xrightarrow{\mathsf{Rf}} \mathsf{Y}$$

and to each square $(\mathsf{h}, \mathsf{k}) : \mathsf{g} \to \mathsf{f}$ (with f as before and $\mathsf{g} : \mathsf{A} \to \mathsf{B}$) a diagram:

$$\begin{tikzcd}
\mathsf{A} \arrow[r, "\mathsf{h}"] \arrow[d, "\mathsf{Lg}"'] & \mathsf{X} \arrow[d, "\mathsf{Lf}"] \\
\mathsf{Qg} \arrow[r, "\mathsf{Q(h,k)}"] \arrow[d, "\mathsf{Rg}"'] & \mathsf{Qf} \arrow[d, "\mathsf{Rf}"] \\
\mathsf{B} \arrow[r, "\mathsf{k}"'] & \mathsf{Y}
\end{tikzcd}$$

functorial in the obvious way.

Abstractly, a functorial factorisation is a functor $\hat{Q} : \mathbb{C}^{\mathbb{2}} \to \mathbb{C}^{\mathbb{3}}$ between functor categories where $\mathbb{2}$ and $\mathbb{3}$ are the shape categories

$$0 \to 1 \qquad 0 \to 1 \to 2$$

respectively, moreover $\hat{Q}$ should satisfy that $d_1 \circ Q = id$ where $d_1 : \mathbb{C}^{\mathbb{3}} \to \mathbb{C}^{\mathbb{2}}$ is the functor induced by composition.

Let (Q, L, R) be a functorial factorisation on a category $\mathbb{C}$, we see from the definition that L and R are in fact endofunctors on the arrow category $\mathbb{C}^{\to}$, they are moreover copointed and pointed respectively as we proceed to show:

The unit associated to R is given by the natural transformation $\hat{\eta} : id_{\mathbb{C}^{\to}} \to R$, whose component at f is:

$$\begin{tikzcd}
\cdot \arrow[r, "Lf"] \arrow[d, "f"'] & \cdot \arrow[d, "Rf"] \\
\cdot \arrow[r, equal] & \cdot
\end{tikzcd}$$

that is $\hat{\eta}_f = (Lf, id)$. Dually, the counit for L is given by $\hat{\varepsilon} : L \to id_{\mathbb{C}^{\to}}$, with component at f is:

$$\begin{tikzcd}
\cdot \arrow[r, equal] \arrow[d, "Lf"'] & \cdot \arrow[d, "f"] \\
\cdot \arrow[r, "Rf"'] & \cdot
\end{tikzcd}$$

that is $\hat{\varepsilon}_f = (id, Rf)$.

We will be mainly interested in arrows carrying an algebra, respectively coalgebra, structure for the pointed endofunctor $(R, \hat{\eta})$, respectively for the copointed endofunctor $(L, \hat{\varepsilon})$.

Let us examine carefully what it means for an arrow f to have an $(R, \hat{\eta})$-algebra structure. By definition, such a structure corresponds to a morphism $\hat{s} : Rf \to f$ in $\mathbb{C}^{\to}$, such that $\hat{s} \circ \hat{\eta}_f = id_f$, this means that the arrow $\hat{s}$ is necessarily of the form $\hat{s} = (s, id)$ where s satisfies $s \circ Lf = id$ (i.e. a section of Lf) and $f \circ s = Rf$. Dually, an $(L, \hat{\varepsilon})$-coalgebra structure $\hat{\lambda} : g \to Lg$ on an arrow g is a morphism of the form $\hat{\lambda} = (id, \lambda)$ such that $Rg \circ \lambda = id$ and $Lg = \lambda \circ g$.

We can easily and completely characterise such (co)algebra structures as diagonal fillers in the squares corresponding to the unit and counit of the respective endofunctors:

$$\begin{tikzcd}
A \arrow[r, "Lg"] \arrow[d, "g"'] & Qg \arrow[d, "Rg"] \\
B \arrow[r, equal] \arrow[ur, dotted, "\lambda"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "Lf"'] & X \arrow[d, "f"] \\
Qf \arrow[r, "Rf"'] \arrow[ur, dotted, "s"] & Y
\end{tikzcd}$$

A morphism $(h, k) : (f', s') \to (f, s)$ of $(R, \hat{\eta})$-algebras consists of a morphism on the arrow category $(h, k) : f' \to f$ that preserves the structure in the sense that $h \circ s' = s \circ Q(h, k)$. Dually, a morphism $(u, v) : (g', \lambda') \to (g, \lambda)$ of $(L, \hat{\varepsilon})$-coalgebras is a morphism of the underlying arrows, such that $Q(u,v) \circ \lambda' = \lambda \circ v$. These definitions are illustrated in the following diagrams:

$$\begin{tikzcd}
Qg' \arrow[r, "Q(u,v)"] & Qg \\
B' \arrow[u, "\lambda'"] \arrow[r, "v"'] & B \arrow[u, "\lambda"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X' \arrow[r, "h"] & X \\
Qf' \arrow[u, "s'"] \arrow[r, "Q(h,k)"'] & Qf \arrow[u, "s"']
\end{tikzcd}$$

We will denote the category of $(R, \hat{\eta})$-algebras and morphisms as R-**Map**; similarly we'll denote by L-**Map** the category of $(L, \hat{\varepsilon})$-coalgebras and morphisms. We will refer to the objects R-**Map** by (f, s) and call them simply R-maps, dually the objects of L-**Map** will be denote by (g, λ) and called L-maps. Notice that there is a pair of faithful (but not full) forgetful functors down to the arrow category:

$$U : L\text{-}\mathbf{Map} \to \mathbb{C}^{\to} \quad \text{and} \quad V : R\text{-}\mathbf{Map} \to \mathbb{C}^{\to}$$

As we mentioned before, L-maps and R-maps canonically lift against each other. Let us record this fact in the following proposition.

Proposition B.1.2. *Let* $(g, \lambda) : A \to B$ *be an* L*-map,* $(f, s) : X \to Y$ *be an* R*-map and* $(h, k) : g \to f$ *be a square in the underlying category. Then there is a canonical filler* $j : B \to X$ *for the square* (h, k) *defined as:*

$$j := s \circ Q(h,k) \circ \lambda : B \to X$$

Proof. By functoriality of the factorisation Q and by the structure on the maps g and f, we have the following diagram:

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "Lg"] & X \arrow[d, "Lf"'] \\
Qg \arrow[r, "Q(h,k)"] \arrow[d, "Rg"] & Qf \arrow[d, "Rf"'] \arrow[u, bend right, "s"'] \\
B \arrow[r, "k"] \arrow[u, bend left, "\lambda"] & Y
\end{tikzcd}$$

so letting $j := p \circ Q(h,k) \circ s$. It follows from the characterisation of R-map and L-map structures that j is indeed a filler for the square (h, k) □

Another property that we will make use of is that the canonical lifts are natural with respect to morphisms of L and R maps, this is made precise in the following proposition.

Proposition B.1.3. *Let* $(g, \lambda) : A \to B$ *be an* L*-map,* $(f, s) : X \to Y$ *an* R*-map, and* $(h, k) : g \to f$ *a square between the underlying arrows. Suppose* $(l, m) : (f, s) \to (f', s')$

is a morphism of R*-maps; then the canonical lift associated to the square* $(l \circ h, m \circ k)$ *is equal to the canonical lift of* (h, k) *composed with* l*. This is illustrated by the diagram:*

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "g"] & X \arrow[r, "l"] \arrow[d, "f"] & X' \arrow[d, "f'"] \\
B \arrow[r, "k"'] \arrow[ru, dotted] \arrow[rru, dotted] & Y \arrow[r, "m"'] & Y'
\end{tikzcd}$$

where the dotted arrows are the canoncial lifts.

Proof. The proof follows immediately from the definitions of the canonical lifts and from that of morphism of R-maps. ☐

Of course, an analogous proposition holds if we take a morphism of L-maps pasted to the left of the square (h, k). In other words, the canonical choice of lifts correspond to a natural transformation:

$$j : \mathbb{C}^{\to}(U-, V-) \to \mathbb{C}(\mathrm{cod}U-, \mathrm{dom}V-) : (L\text{-}\mathbf{Map})^{op} \times R\text{-}\mathbf{Map} \to \mathbf{Set}.$$

There are a couple of properties that we would want the category of R-**Map** to satisfy. For example, it is not true in general that the (vertical) composition of two R-maps is again an R-map. We also don't have in general R-map structures on maps of the form Rf and L-map structures on maps Lf. In the following section we will address these problems.

B.2 Algebraic Weak Factorisation Systems

The notion of algebraic weak factorisation systems was introduced by Grandis and Tholen [GT06], and studied extensively by Bourke, Garner and Riehl [BG16a, BG16b, Rie11]. It is a very succinct algebraic enhancement to a functorial factorisation, although at first glance the connection with classic weak factorisation system is not entirely obvious. We will study this notions here.

Definition B.2.1. An **algebraic weak factorisation system** or **AWFS** on a category $\mathbb{C}$ consists of the following data:

- A functorial factorisation (Q, L, R).
- A monad $(R, \hat{\eta}, \hat{\mu})$ over the pointed endofunctor $(R, \hat{\eta})$.
- A comonad $(L, \hat{\epsilon}, \hat{\delta})$ over the copointed endofuctor $(L, \hat{\epsilon})$.

We will refer to the AWFS just as (L, R).

Let us unwind this definition and examine some properties. First of all, let's examine the multiplication $\hat{\mu} : R^2 \to R$ associated to R; given any arrow f, by the monad axioms we have (in particular) that $\hat{\mu}_f \circ \hat{\eta}_{Rf} = id_{Rf}$, this means that the top and bottom arrows of the following diagram must be equal to the identities

$$\begin{tikzcd}
\cdot \arrow[r, "LRf"] \arrow[d, "Rf"'] & \cdot \arrow[r, "\mu_f"] \arrow[d, "R^2f"] & \cdot \arrow[d, "Rf"] \\
\cdot \arrow[r, equal] & \cdot \arrow[r] & \cdot
\end{tikzcd}$$

thus we have a description for the component of the multiplication at f, mainly $\hat{\mu}_f = (\mu_f, id)$ where $\mu_f \circ LRf = id$. We thus see that $\hat{\mu}$ further induces a natural transformation $\mu : QR \to Q : \mathbb{C}^{\to} \to \mathbb{C}$, such that $\mu \circ LR = id$.

In a completely dual manner we find that the comultiplication $\hat{\delta} : L \to L^2$ has component at f given by $\hat{\delta}_f = (id, \delta_f)$ such that $RLf \circ \delta_f = id$. Thus $\hat{\delta}$ induces a natural transformation $\delta : Q \to QL$ such that $LR \circ \delta = id$.

Note that we can paste together the properties of μ and δ and give rise to a commutative square

$$\begin{tikzcd}
\cdot \arrow[r, "\delta_f"] \arrow[d, "LRf"'] & \cdot \arrow[d, "RLf"] \\
\cdot \arrow[r, "\mu_f"'] & \cdot
\end{tikzcd}$$

for each f (indeed the diagonal arrow is just the identity). This is the component of a natural transformation $\Delta : LR \to RL$. In Garner's definition of AWFS [Gar09, Section 2.18] this natural transformation is required to be a distributive law, this becomes essential in the proof of the algebraic version of the small object argument.

We can now talk about $(R, \hat{\eta}, \hat{\mu})$-algebras; and observe that such algebras are objects of R-**Map** with the additional property that they cohere with the multiplication $\hat{\mu}$ (but have no additional structure). Thus, if we denote R-**Alg** the category of such algebras, we find that we have a full and faithful functor:

$$R\textbf{-Alg} \hookrightarrow R\textbf{-Map}$$

Dually, we denote by L-**Coalg** the category of coalgebras of $(L, \hat{\epsilon}, \hat{\delta})$ and we obtain another full and faithful functor

$$L\textbf{-Coalg} \hookrightarrow L\textbf{-Map}$$

We will refer to the objects of R-**Alg** and L-**Coalg** respectively as R-algebras and L-coalgebras.

As with every (co)monad, we can consider the *free* (co)algebras; in this special case, this tells us that for every arrow f of $\mathbb{C}$ we have a L-coalgebra and a R-algebra respectively:

$$(Lf, \delta_f) \in L\textbf{-Coalg} \quad \text{and} \quad (Rf, \mu_f) \in R\textbf{-Alg}$$

these free (co)algebras have a special universal property. For details, we refer the reader to [BG16a] .

As we mentioned before, it is not obvious at first which (if any) is the underlying weak factorisation system associated to an algebraic one. Let's denote by $\mathcal{R}$ the class of maps that admit an R-algebra structure (i.e. $\mathcal{R}$ is the image of the forgetful functor R-**Alg** $\to \mathbb{C}^{\to}$). Similarly, we denote by $\mathcal{L}$ the class of maps that admit an L-coalgebra structure.

It makes sense to think that pair of classes $(\mathcal{L}, \mathcal{R})$ form a weak factorisation system, indeed, we have that every arrow factors as a map in $\mathcal{L}$ followed by a map on $\mathcal{R}$ and that $\mathcal{L} \boxslash \mathcal{R}$. However, it might not be true in general that these classes of maps are closed under retracts. Thus we must instead consider $(\overline{\mathcal{L}}, \overline{\mathcal{R}})$ where the operation $\overline{(-)}$ is that of retract closure. We indeed have that $(\overline{\mathcal{L}}, \overline{\mathcal{R}})$ forms a weak factorisation system on $\mathbb{C}$ which we will refer to as the **underlying wfs** of (L, R). It can also be easily checked that the classes $\overline{\mathcal{L}}$ and $\overline{\mathcal{R}}$ consist of exactly those arrows that admit and L-map and respectively and R-map structure (as opposed to a L-coalgebra and R-algebra structure).

With the additional structure of an AWFS, we have that the class R-**Map** is closed under vertical composition, we will see this in the following proposition.

Proposition B.2.2. *Composition lifts to a functor in the category* R-***Map*** *as shown in the following diagram:*

$$\begin{tikzcd}
\mathsf{R}\text{-}\boldsymbol{Map} \times_{\mathbb{C}} \mathsf{R}\text{-}\boldsymbol{Map} \arrow[r, "- \circ -"] \arrow[d] & \mathsf{R}\text{-}\boldsymbol{Map} \arrow[d] \\
\mathbb{C}^{\to} \times_{\mathbb{C}} \mathbb{C}^{\to} \arrow[r, "- \circ -"'] & \mathbb{C}^{\to}
\end{tikzcd}$$

Proof. For this we will make use of the natural transformation $\delta : \mathsf{Q} \to \mathsf{QL}$ obtained from the comultiplication of L. Given R-maps (f, s) and (g, t) such that $\mathsf{cod}(\mathsf{f}) = \mathsf{dom}(\mathsf{g})$, the composite gf has an R-map structure $\mathsf{t} \star \mathsf{s}$ given by the composite

$$\mathsf{Q}(\mathsf{gf}) \xrightarrow{\delta_{\mathsf{gf}}} \mathsf{QL}(\mathsf{gf}) \xrightarrow{\mathsf{Q}(\mathsf{id}, \mathsf{t} \circ \mathsf{Q}(\mathsf{f}, \mathsf{id}))} \mathsf{Qf} \xrightarrow{\mathsf{s}} \mathsf{dom}(\mathsf{f})$$

it can be verified that this is indeed an R-map structure on gf and that it is moreover compatible with the morphisms of R-maps. ☐

It can be seen that the composition functor lifts further to the category R-**Alg**. In fact, finding such a vertical composition operation gives an alternative definition of an algebraic weak factorisation systems as stated in the following result.

Theorem B.2.3. *[BR13, Theorem 4.15] Suppose* R *is a monad on* $\mathbb{C}^{\to}$ *over* $\mathsf{cod} : \mathbb{C}^{\to} \to \mathbb{C}$. *Specifying a vertical composition operation on* R-***Alg*** *is equivalent to specifying an AWFS* (L, R) *on* $\mathbb{C}$.

A monad $(R, \hat{\eta}, \hat{\mu})$ on $\mathbb{C}^{\to}$ is said to be *over* the codomain functor $\mathsf{cod} : \mathbb{C}^{\to} \to \mathbb{C}$ if we have that $\mathsf{cod} \circ R = \mathsf{cod}$, $\mathsf{cod} \circ \hat{\eta} = \mathsf{id}_{\mathsf{cod}}$ and $\mathsf{cod} \circ \hat{\mu} = \mathsf{id}_{\mathsf{cod}}$.

It is easy to show that isomorphisms admit a canonical R-algebra structures; indeed if f is an isomorphism then Lf is too, thus (f, Lf^{-1}) is an R-algebra structure on f. For the property of closure under pullbacks, we have the following.

Lemma B.2.4. *Given an* R*-algebra* (f, s) *and a pullback square* $(h, k) : f' \to f$*, then there is a unique* R*-algebra structure* s' *on* f' *making* (h, k) *a morphism of* R*-algebras.*

Proof. Use the universal property of pullback squares to define an arrow s' as shown in the following diagram:

$$\begin{tikzcd}
Qf' \arrow[rr, "Q(h{,}k)"] \arrow[dr, dotted, "s'"] \arrow[ddr, "Qf'"'] & & Qf \arrow[dr, "s"] & \\
& X' \arrow[rr, "h"] \arrow[d, "f'"] \arrow[dr, phantom, "\lrcorner", very near start] & & X \arrow[d, "f"] \\
& Y' \arrow[rr, "k"'] & & Y
\end{tikzcd}$$

it is straightforward to verify that $s' : Qf' \to X'$ equipes f' with the structure of an R-algebra. ☐

In other words, the pullback functor lifts to the to the category of R-algebras as can be seen from the following diagram:

$$\begin{tikzcd}
R\text{-}\mathbf{Alg} \times_{\mathbb{C}} \mathbb{C}^{\to} \arrow[r, "PB"] \arrow[d] & R\text{-}\mathbf{Alg} \arrow[d] \\
\mathbb{C}^{\to} \times_{\mathbb{C}} \mathbb{C}^{\to} \arrow[r, "PB"'] & \mathbb{C}^{\to}
\end{tikzcd}$$

Remark B.2.5. The previous result also applies if we restrict to the case of R-**Map**, that is of R-algebras for the pointed endofunctor (R, η) coming from the functorial factorisation Q.

B.3 Morphisms of AWFS

In this section, we will describe a category $\mathbf{AWFS}(\mathbb{C})$ whose objects are algebraic weak factorisation systems over some base category $\mathbb{C}$. There will be a forgetful functor to the slice category $\mathbf{CAT}/\mathbb{C}^{\to}$ of small categories over the arrow category $\mathbb{C}^{\to}$:

$$\mathbf{AWFS}(\mathbb{C}) \to \mathbf{CAT}/\mathbb{C}^{\to}$$

sending (L, R) to the forgetful functor from the category of L-coalgebras, L-**Coalg** $\to \mathbb{C}^{\to}$.

Definition B.3.1. Let (Q, L, R) and (Q', L', R') be two functorial factorisations over $\mathbb{C}$. A **morphism of functorial factorisations** from (Q, L, R) to (Q', L', R') consists of a natural transformation $\phi : Q \to Q'$ making the following diagram commute:

$$\begin{tikzcd}
& \mathsf{dom} \arrow[dl, "L"'] \arrow[dr, "L'"] & \\
Q \arrow[rr, "\phi"] \arrow[dr, "R"'] & & Q' \arrow[dl, "R'"] \\
& \mathsf{cod} &
\end{tikzcd}$$

Given such a morphism ϕ we can define a pair of natural transformations between the corresponding endofunctors

$$(1, \phi) : L \to L' \quad \text{and} \quad (\phi, 1) : R \to R'$$

and the fact that ϕ makes both triangles of the above diagram commute, implies that these two natural transformations preserve respectively the counit and the unit of the endofunctors. This in turn implies that ϕ induces a pair of functors over $\mathbb{C}^{\to}$

$$L\textbf{-Map} \to L'\textbf{-Map} \quad \text{and} \quad R'\textbf{-Map} \to R\textbf{-Map}$$

these are defined by respectively post and precomposing with the appropriate components of ϕ.

Definition B.3.2. Let (L, R) and (L', R') be two AWFS over $\mathbb{C}$. A **morphism of AWFS** from (L, R) to (L', R') consists of a morphism of the underlying functorial factorisations $\phi : (Q, L, R) \to (Q', L', R')$ such that the following diagrams commute.

$$\begin{tikzcd}
Q \arrow[rr, "\phi"] \arrow[d, "\delta"'] & & Q' \arrow[d, "\delta'"] \\
QL \arrow[r, "\phi_L"'] & Q'L \arrow[r, "Q'(1,\phi)"'] & Q'L'
\end{tikzcd}
\qquad
\begin{tikzcd}
QR \arrow[r, "\phi_R"] \arrow[d, "\mu"'] & Q'R \arrow[r, "Q'(\phi,1)"] & Q'R' \arrow[d, "\mu'"] \\
Q \arrow[rr, "\phi"'] & & Q'
\end{tikzcd}$$

We can characterise morphisms of AWFS in different ways. This is made precise in the following proposition.

Proposition B.3.3. *[Rie11, Lemma 6.9] Let (L, R) and (L', R') be two AWFS over $\mathbb{C}$ and $\phi : Q \to Q'$ a natural transformation. The following are equivalent:*

- $\phi : (L, R) \to (L', R')$ *is a morphism of AWFS.*
- ϕ *induces functors over $\mathbb{C}^{\to}$:*

$$L\text{-}\textbf{\textit{Coalg}} \to L'\text{-}\textbf{\textit{Coalg}} \quad \textit{and} \quad R'\text{-}\textbf{\textit{Alg}} \to R\text{-}\textbf{\textit{Alg}}.$$

that preserve the vertical composition of (co)algebras.

- $(1, \phi) : L \to L'$ *is a comonad morphism and* $(\phi, 1) : R \to R'$ *is a monad morphism.*

With these definitions in place, we can construct the category **AWFS**$(\mathbb{C})$ and the corresponding forgetful functor to **CAT**$/\mathbb{C}^{\to}$ as was mentioned in the beginning of this section.

B.4 Free and Algebraically-Free AWFS

In [Gar09] it is shown that, assuming some set theoretic conditions on $\mathbb{C}$ (it will be enough to have $\mathbb{C}$ locally presentable), it is possible to construct a left adjoint to the forgetful functor from **AWFS**$(\mathbb{C})$ to **CAT**$/\mathbb{C}^{\to}$ as illustrated in the following diagram.

$$\begin{tikzcd}
\mathbf{AWFS}(\mathbb{C}) \arrow[r] \arrow[r, phantom, bend right=40, "\perp"] & \mathbf{CAT}/\mathbb{C}^{\to} \arrow[l, bend right=40]
\end{tikzcd}$$

This tells us that given any functor $\mathfrak{J} \to \mathbb{C}^{\to}$ it is possible to construct a *free* AWFS (L, R) equipped with a functor $\eta : \mathfrak{J} \to L\text{-}\mathbf{Coalg}$ over $\mathbb{C}^{\to}$ with the following universal property: for any other AWFS (L', R') and any other functor $F : \mathfrak{J} \to L'\text{-}\mathbf{Coalg}$ over $\mathbb{C}^{\to}$, there exists a unique morphism of AWFS $\phi : (L, R) \to (L', R')$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{J} \arrow[r, "\eta"'] \arrow[rr, bend left, "F"] & L\text{-}\mathbf{Coalg} \arrow[r, "\tilde{\phi}"'] & L'\text{-}\mathbf{Coalg}
\end{tikzcd}$$

where $\tilde{\phi}$ is the functor induced by ϕ.

In fact, the construction generates an AWFS with a stronger notion of freeness, we call an AWFS generated this way over some category $\mathfrak{J} \to \mathbb{C}^{\to}$ *algebraically-free* on $\mathfrak{J}$. In this section we will review this notion as well; but in order to do this, we will first need to review a categorification of the classical Galois connection between the orthogonality operations ${}^{\boxslash}(-)$ and $(-)^{\boxslash}$ in the poset of subsets of arrows of $\mathbb{C}$.

Definition B.4.1. Let $\mathbb{C}$ be a category. By a **category of arrows** over $\mathbb{C}$ we mean a functor $u : \mathfrak{J} \to \mathbb{C}^{\to}$ where $\mathfrak{J}$ is a (possibly small) category, we denote by u_i, u_j and by u_σ, u_τ the action of u on objects and arrows of $\mathfrak{J}$ respectively. When we can infer from the context the name of the functor u, we will denote the category of arrows just by its domain category $\mathfrak{J}$.

Definition B.4.2. Consider a category of arrows $u : \mathfrak{J} \to \mathbb{C}^{\to}$. A **right $\mathfrak{J}$-map** consists of a pair (f, θ) where $f : X \to Y$ is an arrow or $\mathbb{C}$ and θ is a lifting structure against $\mathfrak{J}$: that is θ assigns to each commutative square of the form $(l, m) : u_i \to f$, with $i \in \mathfrak{J}$, a filler $\theta(i, l, m)$. These fillers, in addition, must be compatible with the arrows in $\mathfrak{J}$ in the following way: if $\sigma : i \to j$ is an arrow in $\mathcal{J}$, then in the following diagram:

$$\begin{tikzcd}
D_i \arrow[r, "D_\sigma"] \arrow[d, "u_i"'] & D_j \arrow[r, "l"] \arrow[d, "u_j"'] & G \arrow[d, "f"] \\
C_i \arrow[r, "C_\sigma"'] \arrow[urr, dotted, "\theta(i)"] & C_j \arrow[r, "m"'] \arrow[ur, dotted, "\theta(j)"] & H
\end{tikzcd}$$

the triangle created by the lifts given by θ must commute.

Given a pair of right $\mathcal{J}$-maps (f, θ) and (f', θ'), a **right $\mathcal{J}$-map morphism** consists of a square $(\alpha, \beta) : f \to f'$ such that for every $i \in \mathcal{J}$ we have that the triangle created by the corresponding choices of diagonal fillers commute:

$$\begin{tikzcd}
D_i \arrow[r] \arrow[d, "u_i"'] & X \arrow[r, "\alpha"] \arrow[d, "f"] & X' \arrow[d, "f'"] \\
C_i \arrow[r] \arrow[ur, dotted] \arrow[urr, dotted] & Y \arrow[r, "\beta"'] & Y'
\end{tikzcd}$$

Let us consider a category of arrow $u : \mathcal{J} \to \mathbb{C}^{\to}$, and from this we can define a new category $\mathcal{J}^{\boxslash}$ consisting of right $\mathcal{J}$-maps (f, θ) together with the corresponding morphisms; moreover there is a functor

$$u^{\boxslash} : \mathcal{J}^{\boxslash} \to \mathbb{C}^{\to}$$

that forgets the lifting structure, which produces a new category of arrows. It can be shown that this operation defines a contravariant functor:

$$(-)^{\boxslash} : (\mathbf{CAT}/\mathbb{C}^{\to})^{\mathrm{op}} \to \mathbf{CAT}/\mathbb{C}^{\to}$$

In a completely analogous manner, we can define the concepts of **left $\mathcal{J}$-map** and **left $\mathcal{J}$-map morphism**, and in this way we construct a dual contraviariant functor:

$$^{\boxslash}(-) : \mathbf{CAT}/\mathbb{C}^{\to} \to (\mathbf{CAT}/\mathbb{C}^{\to})^{\mathrm{op}}$$

It turns out that these two functors form an adjunction, which generalises the classical Galois connection between orthogonal classes of maps:

$$\begin{tikzcd}
\mathbf{CAT}/\mathbb{C}^{\to} \arrow[r, bend left, "^{\boxslash}(-)"] \arrow[r, phantom, "\perp"] & (\mathbf{CAT}/\mathbb{C}^{\to})^{\mathrm{op}} \arrow[l, "(-)^{\boxslash}"]
\end{tikzcd}$$

The relation between these lifting operations and algebraic weak factorisation systems can already be seen in Proposition B.1.2. Specifically, we have the following:

Proposition B.4.3. *Let* (L, R) *be an AWFS on* $\mathbb{C}$*. There are lifting functors over* $\mathbb{C}^{\to}$ *as shown in the following commutative diagram:*

$$\begin{tikzcd}
\mathsf{R}\text{-}\boldsymbol{Alg} \arrow[r, "lift"] \arrow[d] & (\mathsf{L}\text{-}\boldsymbol{Coalg})^{\boxslash} \\
\mathsf{R}\text{-}\boldsymbol{Map} \arrow[ru, "lift", "\simeq"'] \arrow[r, leftrightarrow, "lift"'] & (\mathsf{L}\text{-}\boldsymbol{Map})^{\boxslash} \arrow[u]
\end{tikzcd}$$

All three functors are full and faithful and only the diagonal one is an equivalence. Moreover, there is a functor $(\mathsf{L}\text{-}\boldsymbol{Map})^{\boxslash} \to \mathsf{R}\text{-}\boldsymbol{Map}$*; but it will not, in general, be an inverse of the bottom horizontal functor.*

Proof. The proof that the functors are full and faithful is the same one for each of them, thus it is enough to show it for lift : $\mathsf{R}\text{-}\mathbf{Map} \to (\mathsf{L}\text{-}\mathbf{Coalg})^{\boxslash}$. Let (f, s) be an R-map, we define lift$(\mathsf{f}, \mathsf{s}) = (\mathsf{f}, \theta_{\mathsf{s}})$, where $\theta_{\mathsf{s}}((\mathsf{g}, \lambda), \mathsf{h}, \mathsf{k}) = \mathsf{s} \circ \mathsf{Q}(\mathsf{h}, \mathsf{k}) \circ \lambda$ for a given L-coalgebra (g, λ). It follows from Proposition B.1.2 that this defines a functor.

We will show that all three functors are full and faithful. The first thing to notice is that for any R-map (f, s) we have:

$$\theta_{\mathsf{s}}((\mathsf{Lf}, \delta_{\mathsf{f}}), \mathsf{id}, \mathsf{Rf}) = \mathsf{s}$$

using the comultiplication axioms of the comonad structure on L. Let's show that lift is full; for this consider $(\mathsf{f}, \mathsf{s}), (\mathsf{f}', \mathsf{s}')$ two R-maps and a morphism of right L-**Coalg**-maps $(\mathsf{h}, \mathsf{k}) : (\mathsf{f}, \theta_{\mathsf{s}}) \to (\mathsf{f}', \theta_{\mathsf{s}'})$. Consider the following lifting diagrams:

$$\begin{tikzcd}
\cdot \arrow[r, equal] \arrow[d, "\mathsf{Lf}"'] & \cdot \arrow[r, "\mathsf{h}"] \arrow[d, "\mathsf{f}"] & \cdot \arrow[d, "\mathsf{f}'"] \\
\cdot \arrow[r, "\mathsf{Rf}"'] \arrow[ru, dotted, "s"] \arrow[rru, dotted, "\theta"] & \cdot \arrow[r, "\mathsf{k}"'] & \cdot
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\cdot \arrow[r, "\mathsf{h}"] \arrow[d, "\mathsf{Lf}"'] & \cdot \arrow[r, equal] \arrow[d, "\mathsf{Lf}'"'] & \cdot \arrow[d, "\mathsf{f}'"] \\
\cdot \arrow[r, "\mathsf{Q(h,k)}"'] \arrow[rru, dotted, "\theta"] & \cdot \arrow[r, "\mathsf{Rf}'"'] \arrow[ru, dotted, "s'"] & \cdot
\end{tikzcd}$$

since the bottom composite of both diagrams agree, we have that $\mathsf{h} \circ \mathsf{s} = \theta = \mathsf{s}' \circ \mathsf{Q}(\mathsf{h}, \mathsf{k})$ as required. Faithfulness is immediate.

We proceed show that the diagonal lifting functor is surjective on objects; for this consider (f, θ) a right L-**Coalg**-map. We let

$$\mathsf{s} := \theta((\mathsf{Lf}, \delta_{\mathsf{f}}), \mathsf{id}, \mathsf{Rf})$$

this shows that (f, s) is an R-map. We also need to show that moreover $\theta_{\mathsf{s}} = \theta$, for this consider an L-coalgebra (g, λ) and a square $(\mathsf{h}, \mathsf{k}) : (\mathsf{g}, \lambda) \to (\mathsf{f}, \theta)$, consider the following derived diagram:

$$\begin{tikzcd}
\cdot \arrow[r, equal] \arrow[d, "\mathsf{g}"'] & \cdot \arrow[r, "\mathsf{h}"] \arrow[d, "\mathsf{Lg}"'] & \cdot \arrow[r, equal] \arrow[d, "\mathsf{Lf}"'] & \cdot \arrow[d, "\mathsf{f}"] \\
\cdot \arrow[r, "\lambda"'] \arrow[rrru, dotted, "\theta"] & \cdot \arrow[r, "\mathsf{Q(h,k)}"'] & \cdot \arrow[r, "\mathsf{Rf}"'] \arrow[ru, dotted, "s"] & \cdot
\end{tikzcd}$$

we know $(h, \Theta(h, k))$ is a morphism of L-**Coalg** in general, but $(id, \lambda) : g \to \mathsf{L}g$ is a morphism of L-**Coalg** precisely because (g, λ) is an L-coalgebra (and not just an L-map). Then since the lift cohere, we have desired property. If (g, λ) is only an L-map we still obtain a functor $(\mathsf{L}\textbf{-Map})^{\boxslash} \to \mathsf{R}\textbf{-Map}$ but it will not form an equivalence. □

Remark B.4.4. There is a dual result of the above proposition using ${}^{\boxslash}(-)$, instead of $(-)^{\boxslash}$, thus we obtain lifting functors as shown in the following diagram:

$$\begin{tikzcd}
\mathsf{L}\textbf{-Coalg} \arrow[r, "\text{lift}"] \arrow[d] & {}^{\boxslash}(\mathsf{R}\textbf{-Alg}) \\
\mathsf{L}\textbf{-Map} \arrow[r, leftrightarrow, "\text{lift}"'] \arrow[ur, "\text{lift}", "\simeq"'] & {}^{\boxslash}(\mathsf{R}\textbf{-Map}) \arrow[u]
\end{tikzcd}$$

With these concepts in place, we can now introduce the definition of algebraically free AWFS.

Definition B.4.5. Let $u : \mathcal{J} \to \mathbb{C}^{\to}$ be a functor. We say that an AWFS (L, R) is **algebraically-free** on $\mathcal{J}$ if there is a functor $\eta : \mathcal{J} \to \mathsf{L}\textbf{-Coalg}$ over $\mathbb{C}^{\to}$, such that the composition

$$\mathsf{R}\textbf{-Alg} \xrightarrow{\text{lift}} (\mathsf{L}\textbf{-Coalg})^{\boxslash} \xrightarrow{\eta^{\boxslash}} (\mathcal{J})^{\boxslash}$$

is an isomorphism of categories.

We proceed to state a brief proposition regarding algebraically-free AWFS. This proposition relates F-maps and F-algebras, we'll offer a brief sketch of the proof.

Proposition B.4.6. *If* (L, R) *is algebraically-free on some category of arrows* $\mathcal{J} \to \mathbb{C}^{\to}$*, then there are maps back-and-forth over* $\mathbb{C}^{\to}$*:*

$$\begin{tikzcd}
\mathsf{R}\textbf{-\textit{Map}} \arrow[rr, leftrightarrow] \arrow[dr] & & \mathsf{R}\textbf{-\textit{Alg}} \arrow[dl] \\
& \mathbb{C}^{\to} &
\end{tikzcd}$$

Sketch of proof. There is a 'retract closure' operation on categories of arrows $\overline{(-)}$ whose underlying class of arrows is the usual retract closure . In general for an ordinary AWFS we have that $\overline{\mathsf{R}\textbf{-Alg}} \leftrightarrow \mathsf{R}\textbf{-Map}$ over $\mathbb{C}^{\to}$. Since we are in the algebraically-free case, the category of R-algebras is the category of right-maps of $\mathcal{J}$ and it is automatically closed under retracts, i.e. we have $\mathsf{R}\textbf{-Alg} \leftrightarrow \overline{\mathsf{R}\textbf{-Alg}}$ [GS17, Proposition 5.2]. □

We now proceed to state a version of Garner's small object argument, the actual result is more general.

Theorem B.4.7. *[Gar09, Theorem 4.4] Let* $\mathbb{C}$ *be a locally presentable category and let* $u : \mathfrak{I} \to \mathbb{C}^{\to}$ *be a small category of arrows over* $\mathbb{C}$*. Then the free AWFS on* $\mathfrak{I}$ *exists, and it is algebraically-free on* $\mathfrak{I}$*.*

Garner also proved that algebraically-free AWFS implies free (which is not at all trivial, in fact the converse is unknown). Notice that the notion of algebraically-free AWFS generalises the non-algebraic concept of cofibrantly generated weak factorisation systems (since we have an isomorphism of categories $\mathfrak{J}^{\boxslash} \cong \mathsf{R}$-**Alg**). Moreover, the construction of an algebraically-free AWFS from a category of arrows $\mathfrak{J}$ generalises Quillen's small object argument for normal weak factorisation systems, because of this, the method has been called Garner's small object argument. Just as Quillen's, Garner's small object arguments builds the AWFS (L, R) via a transfinite inductive process. However, even if the category $\mathfrak{J}$ is discrete, the constructions may not be equal. For more on this see [Gar09, Rie11].

Finally we will state, without giving a proof, some propositions which are needed in the thesis. These results are about relating lifting structures and adjoint functors, they are generalisations of widely known results in the non-algebraic setting.

Proposition B.4.8. *[GS17, Proposition 5.7] Consider an adjunction* $\mathsf{F} : \mathbb{C} \rightleftarrows \mathbb{D} : \mathsf{U}$ *and let* $\mathsf{u} : \mathfrak{J} \to \mathbb{C}^{\to}$ *and* $\mathsf{v} : \mathfrak{I} \to \mathbb{D}^{\to}$ *be categories of arrows. Then there is a bijection between lifts of* $\mathsf{F}^{\to}$ *and lifts of* $\mathsf{U}^{\to}$*, as illustrated in the following diagram:*

$$\begin{tikzcd}
\mathfrak{J} \arrow[r, dotted, "\mathsf{F}^{\to}"] \arrow[d, "\mathsf{u}"'] & {}^{\boxslash}\mathfrak{I} \arrow[d, "{}^{\boxslash}\mathsf{v}"] \\
\mathbb{C}^{\to} \arrow[r, "\mathsf{F}^{\to}"'] & \mathbb{D}^{\to}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{I} \arrow[r, dotted, "\mathsf{U}^{\to}"] \arrow[d, "\mathsf{v}"'] & \mathfrak{J}^{\boxslash} \arrow[d, "\mathsf{u}^{\boxslash}"] \\
\mathbb{D}^{\to} \arrow[r, "\mathsf{U}^{\to}"'] & \mathbb{C}^{\to}
\end{tikzcd}$$

Proposition B.4.9. *[GS17, Proposition 5.8] Consider functors* $\mathsf{u} : \mathfrak{J} \to \mathbb{C}^{\to}$ *and* $\mathsf{v} : \mathfrak{I} \to \mathbb{D}^{\to}$ *and two adjunctions* $\mathsf{F}_1 : \mathbb{C} \rightleftarrows \mathbb{D} : \mathsf{U}_1$ *and* $\mathsf{F}_2 : \mathbb{C} \rightleftarrows \mathbb{D} : \mathsf{U}_2$ *. Let*

$$\mathsf{m} : \mathsf{F}_1 \to \mathsf{F}_2 \qquad \mathsf{n} : \mathsf{U}_2 \to \mathsf{U}_1$$

be natural transformation forming mates. Then the transformation m *can be lifted if and only if the transformation* n *can be lifted as illustrated in the following diagram:*

$$\begin{tikzcd}
\mathfrak{J} \arrow[r, bend left, "\mathsf{F}_1^{\to}"] \arrow[r, bend right, "\mathsf{F}_2^{\to}"'] \arrow[r, phantom, "\Downarrow \mathsf{m}"] \arrow[d, "\mathsf{u}"'] & {}^{\boxslash}\mathfrak{I} \arrow[d, "{}^{\boxslash}\mathsf{v}"] \\
\mathbb{C}^{\to} \arrow[r, bend left, "\mathsf{F}_1^{\to}"] \arrow[r, bend right, "\mathsf{F}_2^{\to}"'] \arrow[r, phantom, "\Downarrow \mathsf{m}"] & \mathbb{D}^{\to}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{I} \arrow[r, bend left, "\mathsf{U}_2^{\to}"] \arrow[r, bend right, "\mathsf{U}_1^{\to}"'] \arrow[r, phantom, "\Downarrow \mathsf{n}"] \arrow[d, "\mathsf{v}"'] & \mathfrak{J}^{\boxslash} \arrow[d, "\mathsf{u}^{\boxslash}"] \\
\mathbb{D}^{\to} \arrow[r, bend left, "\mathsf{U}_2^{\to}"] \arrow[r, bend right, "\mathsf{U}_1^{\to}"'] \arrow[r, phantom, "\Downarrow \mathsf{n}"] & \mathbb{C}^{\to}
\end{tikzcd}$$

It is possible to generalise Proposition B.4.8 to the setting of adjunctions between arrow categories induced by the Leibniz construction (instead of being induced by a usual adjunction as in the previous cases).

For this, let us consider bifunctors $\mathsf{F} : \mathbb{C} \times \mathbb{D} \to \mathbb{E}$ and $\mathsf{G} : \mathbb{C}^{\mathsf{op}} \times \mathbb{E} \to \mathbb{D}$, such that for every $\mathsf{C} \in \mathbb{C}$ there is an adjunction

$$\begin{tikzcd}
\mathbb{D} \arrow[r, shift left=2, "\mathsf{F}(\mathsf{C},-)"] \arrow[r, phantom, "\perp"] & \mathbb{E} \arrow[l, shift left=2, "\mathsf{G}(\mathsf{C},-)"]
\end{tikzcd}$$

We can apply the Leibniz construction (Appendix C.1, but see also [RV14]) to these bifunctors in order to obtain $\hat{\mathsf{F}} : \mathbb{C}^{\to} \times \mathbb{D}^{\to} \to \mathbb{E}^{\to}$ and $\hat{\mathsf{G}} : (\mathbb{C}^{\mathsf{op}})^{\to} \times \mathbb{E}^{\to} \to \mathbb{D}^{\to}$. The original family of adjuctions, will induce a new one of the form, for each $\mathsf{h} \in \mathbb{C}^{\to}$:

$$\begin{tikzcd}
\mathbb{D}^{\to} \arrow[r, shift left=2, "\hat{\mathsf{F}}(\mathsf{h},-)"] \arrow[r, phantom, "\perp"] & \mathbb{E}^{\to} \arrow[l, shift left=2, "\hat{\mathsf{G}}(\mathsf{h},-)"]
\end{tikzcd}$$

Proposition B.4.10. *[GS17, Proposition 5.9] Consider the situation described above and let $\mathsf{u} : \mathcal{J} \to \mathbb{C}^{\to}$ and $\mathsf{v} : \mathcal{I} \to \mathbb{D}^{\to}$ be categories of arrows. Then for each $\mathsf{h} \in \mathbb{C}^{\to}$ there is a bijection between lifts of $\hat{\mathsf{F}}(\mathsf{h}, -)$ and lifts of $\hat{\mathsf{G}}(\mathsf{h}, -)$, as illustrated in the following diagram:*

$$\begin{tikzcd}
\mathcal{J} \arrow[r, dotted, "\hat{\mathsf{F}}(\mathsf{h},-)"] \arrow[d, "\mathsf{u}"'] & {}^{\boxslash}\mathcal{I} \arrow[d, "{}^{\boxslash}\mathsf{v}"] \\
\mathbb{D}^{\to} \arrow[r, "\hat{\mathsf{F}}(\mathsf{h},-)"'] & \mathbb{E}^{\to}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{I} \arrow[r, dotted, "\hat{\mathsf{G}}(\mathsf{h},-)"] \arrow[d, "\mathsf{v}"'] & \mathcal{J}^{\boxslash} \arrow[d, "\mathsf{u}^{\boxslash}"] \\
\mathbb{E}^{\to} \arrow[r, "\hat{\mathsf{G}}(\mathsf{h},-)"'] & \mathbb{D}^{\to}
\end{tikzcd}$$

B.5 Adjunction of AWFS and Change of Base

In [Rie11] Riehl pointed out that the universal property of free AWFS can be extended to a more powerful universal property where the base of the AWFS is allowed to change, provided this change happens through an adjunction. We will explain the basic reasoning here.

Definition B.5.1. Let (L, R) and $(\mathsf{L}', \mathsf{R}')$ be AWFS over $\mathbb{C}$ and $\mathbb{D}$ respectively. An **adjunction of AWFS** consists of an adjunction $\begin{tikzcd} \mathbb{C} \arrow[r, shift right=1.5, "\mathsf{G}"'] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, shift right=1.5, "\mathsf{T}"'] \end{tikzcd}$ and a lift:

$$\begin{tikzcd}
\mathsf{R}\textbf{-Alg} \arrow[r, "\tilde{\mathsf{G}}"] \arrow[d] & \mathsf{R}'\textbf{-Alg} \arrow[d] \\
\mathbb{C}^{\to} \arrow[r, "\mathsf{G}"'] & \mathbb{C}^{\to}
\end{tikzcd}$$

such that $\tilde{\mathsf{G}}$ preserves the vertical composition of algebras (i.e. the double categorical structure).

Remark B.5.2. An equivalent definition of an adjunction of AWFS is that T lifts to a functor $\tilde{\mathsf{T}} : \mathsf{L}\textbf{-Coalg} \to \mathsf{L}'\textbf{-Coalg}$ that preserves the vertical composition [Rie11, Lemma 6.12]. Thus an adjunction of AWFS gives automatically lifts of both G and T that preserve the double categorical structure.

With this in place we now explain the generalised universal property of algebraically-free AWFS. This is understood as a change of base of AWFS along adjunctions.

Theorem B.5.3. *[Rie11, Theorem 6.22] Consider an AWFS* $(\mathsf{L}', \mathsf{R}')$ *which is algebraically-free on* $\mathfrak{u}' : \mathcal{J}' \to \mathbb{D}^{\to}$*, and consider an adjunction*

$$\begin{tikzcd} \mathbb{C} \arrow[r, phantom, "\perp"] \arrow[r, bend right, "\mathsf{G}"'] & \mathbb{D} \arrow[l, bend right, "\mathsf{T}"'] \end{tikzcd}.$$

Then, given any AWFS (L, R) *on* $\mathbb{C}$ *and any functor* $\mathsf{f} : \mathcal{J}' \to \mathsf{L}$*-**Coalg** over* T*; there is a unique adjunction of AWFS between* (L, R) *and* $(\mathsf{L}', \mathsf{R}')$ *over* $\mathsf{G} \vdash \mathsf{T}$*, such that the following diagram commutes:*

$$\begin{tikzcd} & \mathcal{J}' \arrow[dl, "\mathsf{f}"'] \arrow[d, "\eta"] \\ \mathsf{L}\text{-}\textbf{\textit{Coalg}} & \mathsf{L}'\text{-}\textbf{\textit{Coalg}} \arrow[l, "\tilde{\mathsf{T}}"] \end{tikzcd}$$

where $\tilde{\mathsf{T}}$ *is the lift of the left adjoint to the categories of coalgebras.*

B.6 Functorial Frobenius and Generalised Frobenius Structure

In this section we define two notions of Frobenius structures for categories of arrows. Let us recall that in the non-algebraic setting, a weak factorisation system satisfies the Frobenius property if the left class is stable under pullback along all arrows on the right class. The material from this section is taken from [dBG12] and [GS17].

Definition B.6.1. Let (L, R) be an algebraic weak factorisation system on a category $\mathbb{C}$. A **functorial Frobenius structure (or FF-structure)** is given by a lift of the pullback functor:

$$\begin{tikzcd} \mathsf{R}\textbf{-Map} \times_{\mathbb{C}} \mathsf{L}\textbf{-Map} \arrow[r, dotted, "\tilde{\mathsf{PB}}"] \arrow[d] & \mathsf{L}\textbf{-Map} \arrow[d] \\ \mathbb{C}^{\to} \times_{\mathbb{C}} \mathbb{C}^{\to} \arrow[r, "\mathsf{PB}"'] & \mathbb{C}^{\to} \end{tikzcd}$$

where $\mathsf{PB}(\mathsf{f}, \mathsf{g})$ denotes the pullback of g along f.

There is a slightly stronger notion than that of a functorial Frobenius structure that we will be interested in.

Definition B.6.2. Let (L, R) be an AWFS on a category $\mathbb{C}$ with a FF-structure $\tilde{\mathsf{PB}}$, we say that the structure is **strong** if for each $(\mathsf{f}, \mathsf{g}) \in \mathsf{R}$**-Map** $\times_{\mathbb{C}}$ L**-Map**, the pullback square:

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "\mathsf{f}^*\mathsf{g}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[d, "\mathsf{g}"] \\ \cdot \arrow[r, "\mathsf{f}"'] & \cdot \end{tikzcd}$$

is a morphism of L-maps.

We now provide an equivalent reformulation of the property of having a functorial Frobenius structure. In order to do this, we first need some notation. Let $u : \mathcal{I} \to \mathbb{C}^{\to}$ be a category of arrows over $\mathbb{C}$, we denote by $\mathcal{I} /\!\!/ \mathbb{C}$ the category whose objects are tuples (X, i, a, b) where $i \in \mathcal{I}$, $X \in \mathbb{C}$ and a, b are arrows in $\mathbb{C}$ making the following diagram commute.

$$\begin{tikzcd}
\cdot \arrow[rr, "u_i"] \arrow[dr, "a"'] & & \cdot \arrow[dl, "b"] \\
& X &
\end{tikzcd}$$

The arrows of $\mathcal{I} /\!\!/ \mathbb{C}$ are commutative triangular prisms where the two triangular faces are objects of $\mathcal{I} /\!\!/ \mathbb{C}$ and the square face is (the image of) a morphism of $\mathcal{I}$; that is, a morphism $(\sigma, f, g) : (X', i', a', b') \to (X, i, a, b)$ in $\mathcal{I} /\!\!/ \mathbb{C}$ can be pictured as in the following diagram:

$$\begin{tikzcd}
\cdot \arrow[rr, "f"] \arrow[dr, "u_{i'}"] \arrow[dd, "a'"'] & & \cdot \arrow[dr, "u_i"] \arrow[dd, "a"' near end] & \\
& \cdot \arrow[rr, "g"] \arrow[dl, "b'"] & & \cdot \arrow[dl, "b"] \\
X' \arrow[rr, "\sigma"'] & & X &
\end{tikzcd}$$

The projection $s : \mathcal{I} /\!\!/ \mathbb{C} \to \mathbb{C}$ maps an object (X, i, a, b) to X and a morphism (σ, f, g) to σ.

Given a second category of arrows $v : \mathcal{J} \to \mathbb{C}$, we can consider the following pullback square:

$$\begin{tikzcd}
\mathcal{I} /\!\!/ \mathbb{C} \times_{\mathbb{C}} \mathcal{J} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \mathcal{J} \arrow[d, "\mathsf{cod} \circ v"] \\
\mathcal{I} /\!\!/ \mathbb{C} \arrow[r, "s"'] & \mathbb{C}
\end{tikzcd}$$

an object of $\mathcal{I} /\!\!/ \mathbb{C} \times_{\mathbb{C}} \mathcal{J}$ is an object $(X, i, a, b) \in \mathcal{I} /\!\!/ \mathbb{C}$ and an object $j \in \mathcal{J}$ such that $\mathsf{cod}(v_j) = X$. This can be illustrated as the following diagram:

$$\begin{tikzcd}
& \cdot \arrow[dd, "a"'] \arrow[dr, "u_i"] & \\
& & \cdot \arrow[dl, "b"] \\
\cdot \arrow[r, "v_j"'] & X &
\end{tikzcd}$$

Notice that it is possible to pullback the arrow u_i along v_j in order to obtain an arrow in the slice over the codomain of v_j.

Definition B.6.3. Consider category of arrows $u : \mathcal{I} \to \mathbb{C}^{\to}$, $v : \mathcal{J} \to \mathbb{C}^{\to}$ and $z : \mathcal{Z} \to \mathbb{C}^{\to}$. A **generalised Frobenius structure (or GF-structure)** for $(\mathcal{I}, \mathcal{J}, \mathcal{Z})$ is given by a lift of the pullback functor as follows:

$$\begin{tikzcd}
\mathcal{I} /\!\!/ \mathbb{C} \times_{\mathbb{C}} \mathcal{J} \arrow[r, dotted, "\tilde{\mathsf{PB}}"] \arrow[d] & \mathcal{Z} /\!\!/ \mathbb{C} \arrow[d] \\
\mathbb{C} /\!\!/ \mathbb{C} \times_{\mathbb{C}} \mathbb{C}^{\to} \arrow[r, "\mathsf{PB}"'] & \mathbb{C}^{\to}
\end{tikzcd}$$

At a first glance, a GF-structure appears to be a more general notion than that of a FF-structure. Nonetheless, we have the following proposition.

Proposition B.6.4. *[GS17, Propositions 6.5 and 6.9] Let (L, R) be an AWFS on $\mathbb{C}$. Then, the following are equivalent:*

1. *A functorial Frobenius structures on (L, R).*
2. *A GF-structures on $(\mathsf{L}\text{-}\boldsymbol{Map}, \mathsf{R}\text{-}\boldsymbol{Map}, \mathsf{L}\text{-}\boldsymbol{Map})$.*
3. *A GF-structures on $(\mathsf{L}\text{-}\boldsymbol{Map}, \mathsf{R}\text{-}\boldsymbol{Map},^{\boxslash} (\mathsf{L}\text{-}\boldsymbol{Map}^{\boxslash}))$*
4. *If (L, R) is algebraically-free on $\mathcal{I} \to \mathbb{C}^{\to}$, a GF-structure on $(\mathcal{I}, \mathcal{I}^{\boxslash},^{\boxslash} (\mathcal{I}^{\boxslash}))$.*

The main reason to introduced generalised Frobenius structures is because these admit an equivalent formulation in terms of pushforward functors. We proceed to describe the reformulation.

Proposition B.6.5. *[GS17, Proposition 6.6] Consider category of arrows $u : \mathcal{I} \to \mathbb{C}^{\to}$, $v : \mathcal{J} \to \mathbb{C}^{\to}$ and $z : \mathcal{Z} \to \mathbb{C}^{\to}$. Then $(\mathcal{I}, \mathcal{J},^{\boxslash} (\mathcal{Z}^{\boxslash}))$ has a GF-structure if and only if we can provide the following data:*

1. *For each $\mathsf{j} \in \mathcal{J}$ with $v_{\mathsf{j}} : \mathsf{D}_{\mathsf{j}} \to \mathsf{C}_{\mathsf{j}}$ a lift of the pullback functor to the slices:*

$$\begin{tikzcd}
\mathcal{I} /\!\!/ \mathsf{C}_{\mathsf{j}} \arrow[r, dotted, "v_{\mathsf{j}}^{*}"] \arrow[d] & {}^{\boxslash}(\mathcal{Z}^{\boxslash}) /\!\!/ \mathsf{D}_{\mathsf{j}} \arrow[d] \\
\mathbb{C} /\!\!/ \mathsf{C}_{\mathsf{j}} \arrow[r, "v_{\mathsf{j}}^{*}"'] & \mathbb{C} /\!\!/ \mathsf{D}_{\mathsf{j}}
\end{tikzcd}$$

2. *For each morphism $\tau : \mathsf{j} \to \mathsf{k}$ in $\mathcal{J}$, the canonical Beck-Chevalley (Proposition 2.7.3) natural transformation:*

$$\mathsf{BC}_{\tau} : \Sigma_{\mathsf{D}_{\tau}} v_{\mathsf{k}}^{*} \to v_{\mathsf{j}}^{*} \Sigma_{\mathsf{C}_{\tau}} : \mathbb{C} /\!\!/ \mathsf{C}_{\mathsf{j}} \to \mathbb{C} /\!\!/ \mathsf{D}_{\mathsf{k}}$$

   induced by the square $v_{\tau} = (\mathsf{D}_{\tau}, \mathsf{C}_{\tau}) : v_{\mathsf{j}} \to v_{\mathsf{k}}$ lifts to a natural tranformation as shown:

$$\mathsf{BC}_{\tau} : \Sigma_{\mathsf{D}_{\tau}} v_{\mathsf{k}}^{*} \to v_{\mathsf{j}}^{*} \Sigma_{\mathsf{C}_{\tau}} : \mathcal{I} /\!\!/ \mathsf{C}_{\mathsf{j}} \to^{\boxslash} (\mathcal{Z}^{\boxslash}) /\!\!/ \mathsf{D}_{\mathsf{k}}$$

Remark B.6.6. Because ${}^{\boxslash}(z^{\boxslash}) :{}^{\boxslash} (\mathcal{Z}^{\boxslash}) \to \mathbb{C}^{\to}$ is faithful, condition (2) is not extra structure but only an extra property. Specifically, it is the property that for each $j \in \mathcal{J}$ (with v_j in the slice over C_j) the square $BC_\tau(v_j)$ is in the image under ${}^{\boxslash}(z^{\boxslash})$ of a morphism in ${}^{\boxslash}(\mathcal{Z}^{\boxslash})$.

Using the relationship between orthogonality and adjoints, we can reformulate the previous proposition in terms of pushforward.

Proposition B.6.7. *[GS17, Proposition 6.7] Consider category of arrows $u : \mathcal{I} \to \mathbb{C}^{\to}$, $v : \mathcal{J} \to \mathbb{C}^{\to}$ and $z : \mathcal{Z} \to \mathbb{C}^{\to}$. Then $(\mathcal{I}, \mathcal{J},{}^{\boxslash} (\mathcal{Z}^{\boxslash}))$ has a GF-structure if and only if we can provide the following data:*

1. *For each $j \in \mathcal{J}$ with $v_j : D_j \to C_j$ a lift of the pushforward functor to the slices:*

$$\begin{tikzcd}
\mathcal{Z}^{\boxslash} /\!\!/ D_j \arrow[r, dotted, "\Pi_{v_j}"] \arrow[d] & \mathcal{I}^{\boxslash} /\!\!/ C_j \arrow[d] \\
\mathbb{C} /\!\!/ D_j \arrow[r, "\Pi_{v_j}"'] & \mathbb{C} /\!\!/ C_j
\end{tikzcd}$$

2. *For each morphism $\tau : j \to k$ in $\mathcal{J}$, the canonical Beck-Chevalley (Proposition 2.7.3) natural transformation:*

$$BC_\tau : C_\tau^* \Pi_{v_k} \to \Pi_{v_j} D_\tau^* : \mathbb{C} /\!\!/ D_j \to \mathbb{C} /\!\!/ Cj_k$$

   induced by the square $v_\tau = (D_\tau, C_\tau) : v_j \to v_k$ lifts to a natural tranformation as shown:

$$BC_\tau : C_\tau^* \Pi_{v_k} \to \Pi_{v_j} D_\tau^* : \mathcal{Z}^{\boxslash} /\!\!/ D_k \to (\mathcal{I}^{\boxslash}) /\!\!/ C_j$$

Remark B.6.8. In the previous two propositions, if the morphism $\tau : j \to k$ has an underlying pullback square, then BC_τ is a natural isomorphism.

Appendix C

Theory of Uniform Fibrations

In this section we will give an overview of the theory of Uniform Fibrations developed by Gambino and Sattler [GS17]. Among other things, they developed a machinery to build, under some general assumptions, an algebraically-free AWFS that satisfies the functorial Frobenious property. First, we need a brief review of the Leibniz construction (also known as pushout-product) which we will provide in the next section.

C.1 Leibniz construction

The Leibniz construction, is a very useful tool for working with lifting problems and orthogonality. This constructions are crucial in the development of the theory of Uniform fibrations and thus we will give a quick review of these.

Definition C.1.1. Let $\mathbb{C}, \mathbb{D}$ and $\mathbb{E}$ be categories such that $\mathbb{E}$ has finite colimits and let $- \otimes - : \mathbb{C} \times \mathbb{D} \to \mathbb{E}$ be a bifunctor. The **Leibniz construction** or **pushout-product** outputs from this data a bifunctor on the arrow categories

$$-\hat{\otimes}- : \mathbb{C}^{\to} \times \mathbb{D}^{\to} \to \mathbb{E}^{\to}$$

whose action on a pair of arrows $f : X \to Y$ in $\mathbb{C}$ and $g : A \to B$ in $\mathbb{D}$ is given as in the following pushout diagram in $\mathbb{E}$:

$$\begin{tikzcd}
X \otimes A \arrow[r, "f\otimes A"] \arrow[d, "X\otimes g"'] & Y \otimes A \arrow[d] \arrow[ddr, bend left, "Y\otimes g"] & \\
X \otimes B \arrow[r] \arrow[drr, bend right, "f\otimes B"'] & (Y \otimes A) +_{X\otimes A} (X \otimes B) \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dotted, "f\hat{\otimes} g"] & \\
 & & Y \otimes B
\end{tikzcd}$$

There are some general results about this construction that we will state without providing a proof.

Lemma C.1.2. *[RV14, Lemma 4.8] Suppose that $\mathbb{C}$ (respectively $\mathbb{D}$) and $\mathbb{E}$ are cocomplete and that $- \otimes - : \mathbb{C} \times \mathbb{D} \to \mathbb{E}$ is cocontinuous in its first (respectively second) variable. Then the resulting $-\hat{\otimes}- : \mathbb{C}^{\to} \times \mathbb{D}^{\to} \to \mathbb{E}^{\to}$ is cocontinuous in its first (respectively second) variable.*

Lemma C.1.3. *[RV14, Lemma 4.10] Suppose that for each $A \in \mathbb{D}$, the functor $- \otimes A$ has a right adjoint $\mathrm{hom}_r(A, -)$, then $-\hat{\otimes} f$ also has a right adjoint $\widehat{\mathrm{hom}}_r(f, -)$ for each $f \in \mathbb{C}^{\to}$. Moreover the bifunctor $\widehat{\mathrm{hom}}_r : \mathbb{D}^{\to} \times (\mathbb{E}^{\to})^{op} \to (\mathbb{C}^{\to})^{op}$ is given as in the following pullback diagram:*

$$\begin{tikzcd}
\mathrm{hom}_r(A, Y) \arrow[drr, bend left, "\mathrm{hom}_r(g,Y)"] \arrow[dr, dotted, "\widehat{\mathrm{hom}}_r(f,g)"] \arrow[ddr, bend right, "\mathrm{hom}_r(A,f)"'] & & \\
& \mathrm{hom}_r(A, X) \times_{\mathrm{hom}_r(A,Y)} \mathrm{hom}_r(B, Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{hom}_r(B, Y) \arrow[d, "\mathrm{hom}_r(B,f)"] \\
& \mathrm{hom}_r(A, X) \arrow[r, "\mathrm{hom}_r(g,X)"'] & \mathrm{hom}_r(B, X)
\end{tikzcd}$$

This construction is also known as **pullback-exponential**.

Lemma C.1.4. *[RV14, Observation 4.12] Suppose $(\otimes, \perp)$ defines a (symmetric) monoidal structure on $\mathbb{C}$, then $(\hat{\otimes}, \mathrm{id}_{\perp})$ defines a (symmetric) monoidal structure on $\mathbb{C}^{\to}$. Moreover, if $(\otimes, \mathrm{hom}_r, \perp)$ is a closed monoidal structure, so is $(\hat{\otimes}, \widehat{\mathrm{hom}}_r, \mathrm{id}_{\perp})$.*

The following result is an easy application of the construction of the join or union of subobjects in the context of elementary toposes.

Lemma C.1.5. *Suppose $\mathbb{C}$ is an elementary topos equipped with a monoidal product $- \otimes -$, such that it preserves monomorphism in both variables. Consider a monomorphism $i : A \to B$ such that $i \otimes - : (A \otimes -) \to (B \otimes -) : \mathbb{C} \to \mathbb{C}$ is an equifibred natural transformation (i.e. all naturality squares are Cartesian). Then for any monomorphism $j \in \mathbb{C}^{\to}$, the morphism $i \hat{\otimes} j$ is again a monomorphism.*

Proof. Let $j : X \to Y$ be a monomorphism in $\mathbb{C}$. Then, the arrow $i \hat{\otimes} j$ coincides with the join (or union) of the subobjects $i \otimes Y$ and $B \otimes j$. ☐

The following proposition is the algebraic counterpart to the fact that the Leibniz construction has a nice behaviour with respect to lifting problems. It is a special case of Proposition B.4.10.

Proposition C.1.6. *Consider $\mathbb{C}$ a category equipped with a closed symmetric monoidal strucutre. Let $u : \mathcal{J} \to \mathbb{C}^{\to}$ and $v : \mathcal{I} \to \mathbb{C}^{\to}$ be categories of arrows and fix an arrow $i : A \to B$ in $\mathbb{C}$.*

Then there is a bijective correspondence between lifts of $i \hat{\otimes} -$ and lifts of $\widehat{\mathrm{hom}}(i, -)$ as illustrated in the following diagram:

$$\begin{tikzcd}
\mathscr{J} \arrow[r, dotted, "i\hat{\otimes}-"] \arrow[d, "u"'] & {}^{\boxslash}\mathcal{I} \arrow[d, "{}^{\boxslash}v"] \\
\mathbb{C}^{\to} \arrow[r, "i\hat{\otimes}-"'] & \mathbb{C}^{\to}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{I} \arrow[r, dotted, "\hat{\mathrm{hom}}(i{,}-)"] \arrow[d, "v"'] & \mathscr{J}^{\boxslash} \arrow[d, "u^{\boxslash}"] \\
\mathbb{C}^{\to} \arrow[r, "\hat{\mathrm{hom}}(i{,}-)"'] & \mathbb{C}^{\to}
\end{tikzcd}$$

C.2 Interval Objects

An interval object in a category $\mathbb{C}$ corresponds to an abstraction of the closed interval $[0, 1]$ in the category of topological spaces (or some other 'nice' category of spaces). The main motivation for introducing this kind of structure is to construct a path object factorisation reminiscent of the classical one for a given topological space X

$$X \to X^{[0,1]} \to X \times X$$

where the left-most map sends a point to the constant path (or loop) on that point, and the right-most map sends a path to it's source and target points.

We will start by assuming that $\mathbb{C}$ has a symmetric monoidal structure which is given by a bifunctor:

$$- \otimes - : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$$

which is associative and symmetric, and by a unit $\perp \in \mathbb{C}$ for $\otimes$. The associativity, symmetry and identity axioms are taken up to a coherent choice of natural isomorphisms. We will also assume that $\mathbb{C}$ has an initial object $\emptyset \in \mathbb{C}$.

Definition C.2.1. An **interval object** in $(\mathbb{C}, \otimes, \perp)$ consists a object I together with two morphisms:

$$\delta^0, \delta^1 : \perp \to I$$

respectively called the left and right **endpoint inclusions** which are disjoint, meaning that the following is a Cartesian square:

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \perp \arrow[d, "\delta^0"] \\
\perp \arrow[r, "\delta^1"'] & I
\end{tikzcd}$$

We will denote the whole structure by (I, δ^0, δ^1).

It might be useful to require the interval object to have additional structure, other than two disjoint points. In this thesis, we will assume that an interval objects comes equipped with two additional structures.

Definition C.2.2. A **contraction** for an interval object $(\mathrm{I}, \delta^0, \delta^1)$ is given by a morphism

$$\epsilon : \mathrm{I} \to \perp$$

which is a common retraction to both endpoint inclusions, i.e.

$$\begin{tikzcd}
\perp \arrow[r, "\delta^0"] \arrow[dr, equal] & \mathrm{I} \arrow[d, "\epsilon"] & \perp \arrow[l, "\delta^1"'] \arrow[dl, equal] \\
& \perp &
\end{tikzcd}$$

Definition C.2.3. Consider an interval with contraction $(\mathrm{I}, \delta^0, \delta^1, \epsilon)$. The **connection** operations on I are given by two arrows

$$c^k : \mathrm{I} \otimes \mathrm{I} \to \mathrm{I} \quad \text{for} \quad k \in \{0, 1\}$$

making the following diagrams commute:

$$\begin{tikzcd}
\mathrm{I} \arrow[r, "\delta^k \otimes \mathrm{I}"] \arrow[d, "\epsilon"'] & \mathrm{I} \otimes \mathrm{I} \arrow[d, "c^k"] \\
\perp \arrow[r, "\delta^k"'] & \mathrm{I}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{I} \arrow[r, "\delta^{1-k} \otimes \mathrm{I}"] \arrow[dr, equal] & \mathrm{I} \otimes \mathrm{I} \arrow[d, "c^k"] \\
& \mathrm{I}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{I} \otimes \mathrm{I} \arrow[r, "c^k"] \arrow[d, "\epsilon \otimes \mathrm{I}"'] & \mathrm{I} \arrow[d, "\epsilon"] \\
\mathrm{I} \arrow[r, "\epsilon"'] & \perp
\end{tikzcd}$$

The contraction operations correspond to the two contraction of $[0, 1]$ fixing each endpoint. Meanwhile the connections correspond to special type of degeneracy maps that can be pictured as two kinds of continuous deformations of the square $[0, 1] \times [0, 1]$ into its diagonal, as illustrated in the following diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow", near start] & 1 \\
0 \arrow[r, equal] \arrow[ur, "c^0"] \arrow[ur, phantom, "\Leftarrow"', near end, shift right=3] & 0 \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, equal] \arrow[dr, phantom, "\Rightarrow", near start] & 1 \arrow[d, equal] \\
0 \arrow[u] \arrow[r] \arrow[ur, "c^1"] \arrow[ur, phantom, "\Uparrow"', near end, shift right=3] & 1
\end{tikzcd}$$

For the topological interval $[0, 1]$, these maps are indeed given by $c^0(i, j) = \min(i, j)$ and $c^1(i, j) = \max(i, j)$ respectively.

C.3 Path Objects from an Interval

In order to construct a path object such as $X^{[0,1]}$ for spaces, we will need to have an appropriate notion of exponentiation. For this reason, we will further assume that the monoidal structure on $\mathbb{C}$ is *closed*, this means that there is another bifunctor:

$$\hom : \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{C}$$

such that for each object $A \in \mathbb{C}$, there is an adjunction between the following endofunctors:

$$(-) \otimes A : \mathbb{C} \rightleftarrows \mathbb{C} : \hom(A, -)$$

the objects $\hom(A, B)$ are usually known as *hom-objects* and we will adopt the following notation:

$$\hom(\#1, \#2) := (\#2)^{\#1} : \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{C}$$

From now on, we will consider a category $\mathbb{C}$ equipped with a closed symmetric monoidal structure and an interval object with contraction and connections. With this structure available, we are able to construct a factorisation of the diagonal map $\delta_f : B \to B \times_A B$ for each $f : B \to A$ in $\mathbb{C}$ of the following form:

$$\begin{tikzcd}
B \arrow[r, "r_f"] \arrow[dr, "\delta_f"'] & P_w f \arrow[d, "\rho_f"] \\
& B \times_A B
\end{tikzcd}$$

The object $P_w f$ in the middle, is intended to be the object of paths in B contained in the fibres of f. Formally, $P_w f$ is constructed as the following pullback:

$$\begin{tikzcd}
P_w f \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B^I \arrow[d, "f^I"] \\
A \arrow[r, "A^\epsilon"'] & A^I
\end{tikzcd}$$

built using the structure of the interval and the closed monoidal structure on $\mathbb{C}$. The morphism $r_f : B \to P_w f$ will take an object of B to the constant path; it is induced by the universal property to the pullback square as in the following diagram:

$$\begin{tikzcd}
B \arrow[drr, bend left, "B^\epsilon"] \arrow[dr, dotted, "r_p"] \arrow[ddr, bend right, "f"'] & & \\
& P_w f \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B^I \arrow[d, "f^I"] \\
& A \arrow[r, "A^\epsilon"'] & A^I
\end{tikzcd}$$

The morphism $\rho_f : P_w f \to B \times_A B$ takes a path and outputs its endpoints. To construct it, first consider the canonical source and target arrows $s_f, t_f : P_w f \to B^I$ defined as follows:

$$\begin{tikzcd}
& P_w f \arrow[dl, "s_f"'] \arrow[d] \arrow[dr, "t_f"] & \\
B & B^I \arrow[l, "B^{\delta^0}"] \arrow[r, "B^{\delta^1}"'] & B
\end{tikzcd}$$

using the properties of the interval, we see that $f \circ s_f = f \circ t_f$ and so we obtain an arrow into the pullback of f along itself:

$$\rho_f : P_w f \to B \times_A B$$

it is immediate, by construction, that $\rho_f \circ r_f = \delta_f$

It is an easy observation that the construction of r_f and ρ_f are functorial; that is, there are the action on objects of functors:

$$r, \rho : \mathbb{C}^{\to} \to \mathbb{C}^{\to}$$

the action on arrows is canonically given by the universal property of pullback squares, in particular, if $(h, k) : f' \to f$ is a morphism in the arrow category, we obtain the following commutative squares:

$$\begin{tikzcd}
B' \arrow[r, "h"] \arrow[d, "r_{f'}"'] & B \arrow[d, "r_f"] \\
P_w f' \arrow[r, "P_w(h,k)"'] & P_w f
\end{tikzcd}
\qquad
\begin{tikzcd}
P_w f' \arrow[r, "P_w(h,k)"] \arrow[d, "\rho_{f'}"'] & P_w f \arrow[d, "\rho_f"] \\
B' \times_{A'} B' \arrow[r, "h \times_k h"'] & B \times_A B
\end{tikzcd}$$

Moreover, using some basic properties of pullback squares and the fact that $(-)^I$ preserves limits (since it is a right adjoint) we obtain that the functor ρ preserves Cartesian squares.

We have constructed a stable and functorial choice of factorisations of the diagonal morphism; that is a functor which we will denote as follows:

$$\mathcal{P}_I := \langle r, \rho \rangle : \mathbb{C}^{\to} \to \mathbb{C}^{\to} \times_{\mathbb{C}} \mathbb{C}^{\to}$$

we will refer to it by the name **interval path-object factorisation**.

Notice that if the monoidal unit coincide with the terminal object, and we apply the above construction to an arrow of the form $X \to \perp$, we obtain that $P_w X = X^I$, and with this we confirm our initial intuition.

C.4 Uniform Fibrations

The following section follows the work done in [GS17], although we will work in a slightly less general setting. The main idea is that using some general properties on a category $\mathbb{C}$ equipped with a interval object and a sufficiently 'nice' AWFS (C, F_t), we will be able to construct another AWFS (C_t, F) that will satisfy the functorial Frobenious property. One is encourage to think of these two AWFS as those corresponding to the ones of (trivial) cofibrations and (trivial) fibrations respectively, in a model category.

For the rest of this section we will work in a locally presentable $\mathbb{C}$ equipped with a closed symmetric monoidal structure and an interval object (I, δ^0, δ^1) with contraction and connections.

Definition C.4.1. An AWFS $(\mathsf{C}, \mathsf{F}_t)$ is said to be **suitable** if the following conditions hold:

1. $(\mathsf{C}, \mathsf{F}_t)$ is algebraically-free on a category of arrows $u : \mathcal{J} \to \mathbb{C}^{\to}$.
2. Every object is uniformly cofibrant: i.e. the functor $\perp : \mathbb{C} \to \mathbb{C}^{\to}$ mapping $X \in \mathbb{C}$ to the unique arrow $\perp_X : 0 \to X$ factors through C-**Map**:

$$\begin{tikzcd}
\mathbb{C} \arrow[r, dotted, "\perp"] \arrow[dr, "\perp"'] & \mathsf{C}\text{-}\mathbf{Map} \arrow[d] \\
 & \mathbb{C}^{\to}
\end{tikzcd}$$

3. C-**Map** is uniformly closed under pullback: i.e. there is a lift of the pullback functor (here $\mathsf{PB}(f, g)$ means the pullback of g along f):

$$\begin{tikzcd}
\mathbb{C}^{\to} \times_{\mathbb{C}} \mathsf{C}\text{-}\mathbf{Map} \arrow[r, dotted, "\mathsf{PB}"] \arrow[d] & \mathsf{C}\text{-}\mathbf{Map} \arrow[d] \\
\mathbb{C}^{\to} \times_{\mathbb{C}} \mathbb{C}^{\to} \arrow[r, "\mathsf{PB}"'] & \mathbb{C}^{\to}
\end{tikzcd}$$

4. C-**Map** is uniformly closed under Leibniz product with endpoint inclusions: i.e. there are lifts for $k \in \{0, 1\}$ as shown:

$$\begin{tikzcd}
\mathsf{C}\text{-}\mathbf{Map} \arrow[r, dotted, "\delta^k \hat{\otimes} -"] \arrow[d] & \mathsf{C}\text{-}\mathbf{Map} \arrow[d] \\
\mathbb{C}^{\to} \arrow[r, "\delta^k \hat{\otimes} -"'] & \mathbb{C}^{\to}
\end{tikzcd}$$

Remark C.4.2. Notice that for part 3 in the previous definition, it is equivalent to have a GF-structure (Definition B.6.3) on the tuple $(\mathsf{C}\text{-}\mathbf{Map}, \mathbb{C}^{\to}, \mathsf{C}\text{-}\mathbf{Map})$.

Let $(\mathsf{C}, \mathsf{F}_t)$ be a suitable AWFS algebraically-free on $u : \mathcal{J} \to \mathbb{C}^{\to}$. We define the category of arrows $u_{\otimes} : \mathcal{J}_{\otimes} \to \mathbb{C}^{\to}$ by letting $\mathcal{J}_{\otimes} := \mathcal{J} + \mathcal{J}$ and $u_{\otimes}$ defined by the following diagram:

$$\begin{tikzcd}
\mathcal{J} \arrow[r] \arrow[dr, "\delta^0 \hat{\otimes} u"'] & \mathcal{J}_{\otimes} \arrow[d, dotted, "u_{\otimes}"] & \mathcal{J} \arrow[l] \arrow[dl, "\delta^1 \hat{\otimes} u"] \\
 & \mathbb{C}^{\to} &
\end{tikzcd}$$

where $(\delta^k \hat{\otimes} u)_i := \delta^k \hat{\otimes} u_i$ for $i \in \mathcal{J}$. Using Garner's small object argument we generate the algebraically-free AWFS on $u_{\otimes} : \mathcal{J}_{\otimes} \to \mathbb{C}^{\to}$ which we denote $(\mathsf{C}_t, \mathsf{F})$. The objects of F-**Map** are called **uniform fibrations**.

Theorem C.4.3. *[GS17, Theorem 8.8] The AWFS* $(\mathsf{C}_t, \mathsf{F})$ *has a functorial Frobenious structure.*

The proof of this theorem uses the crucial fact that homotopy equivalences can be given algebraic structure and can be organised in a category of arrows; as we will see in the following section.

Remark C.4.4. Notice that the uniform fibration construction has a non-algebraic counterpart. Let us give a quick account of this. Let us consider a suitable weak factorisation system (C, F_t); i.e. it is cofibrantly generated from a set $\mathcal{I}$, all arrows $\perp : 0 \to X$ are in C, the class C is pullback stable and closed under Leibniz product with endpoint inclusions. From this we can generate using the small object argument a second weak factorisation system (C_t, F) generated by the set

$$\{\delta^k \hat{\otimes} u | k \in \{0,1\} \text{ and } u \in \mathcal{I}\}$$

We will obtain that (C_t, F) satisfies the Frobeinius condition.

As an example, let sSet be the category of simplicial sets, and take $\mathcal{I}$ to be the set of boundary inclusion of representables. The resulting class F will turn out to coincide with the class of Kan fibrations.

C.5 Homotopy Equivalences and Deformation Retracts

We will recall some basic facts about homotopy and homotopy equivalences in the context of an interval object. We will see how some classical results lift to the algebraic setting.

Definition C.5.1. Consider a category $\mathbb{C}$ be as before.

1. Let $f, g : X \to Y$ be morphisms in $\mathbb{C}$. An **homotopy from f to g** denoted $\theta : f \sim g$ consists of an arrow $\theta : I \otimes X \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\delta^0 \otimes X"] \arrow[dr, "f"'] & I \otimes X \arrow[d, dotted, "\theta"] & X \arrow[l, "\delta^1 \otimes X"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

2. We say that $f : X \to Y$ is an **left (or 0-oriented) homotopy equivalence** if there is a map $h : Y \to X$ and homotopies $\theta : h \circ f \sim id_X$ and $\psi : f \circ h \sim id_Y$. Dually, f is a **right (or 1-oriented) homotopy equivalence** if there is $h : Y \to X$ and homotopies $\theta : id_X \sim h \circ f$ and $\psi : id_Y \sim f \circ h$.

3. A k-oriented homotopy equivalence (f, h, θ, ψ) is said to be **strong** if the following diagram commutes:

$$\begin{tikzcd} I \otimes X \arrow[r, "I \otimes f"] \arrow[d, "\theta"'] & I \otimes Y \arrow[d, "\psi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

4. A **morphism of strong k-oriented homotopy equivalences** $(s,t) : (f,h,\theta,\psi) \to (f',h',\theta',\psi')$ consists of maps $s : X \to X'$ and $t : Y \to Y'$ making the following diagrams commute:

$$\begin{tikzcd}
X \arrow[r, "s"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\
Y \arrow[r, "t"'] & Y'
\end{tikzcd}
\quad
\begin{tikzcd}
Y \arrow[r, "t"] \arrow[d, "h"'] & Y' \arrow[d, "h'"] \\
X \arrow[r, "s"'] & X'
\end{tikzcd}
\quad
\begin{tikzcd}
I \otimes X \arrow[r, "I\otimes s"] \arrow[d, "\theta"'] & I \otimes X' \arrow[d, "\theta'"] \\
X \arrow[r, "s"'] & X'
\end{tikzcd}
\quad
\begin{tikzcd}
I \otimes Y \arrow[r, "I\otimes t"] \arrow[d, "\psi"'] & I \otimes Y' \arrow[d, "\psi'"] \\
Y \arrow[r, "t"'] & Y'
\end{tikzcd}$$

We denote by S_k the category of strong k-oriented homotopy equivalences and morphisms. Notice that projecting to the first component gives us a functor $S_k \to \mathbb{C}^{\to}$ making S_k a category of arrows. We define $S := S_0 + S_1$ with the corresponding functor $S \to \mathbb{C}^{\to}$ given by the universal property of the coproduct, we call S the **category of strong homotopy equivalences**.

Remark C.5.2. Since we are working with closed monoidal category $(\mathbb{C}, \otimes, \text{hom}, \perp)$ using the adjuctions between $- \otimes -$ and $\text{hom}(-,-)$, we can translate the definitions from above using the internal-hom instead of the monoidal product. For example, an homotopy from f to g consists of a morphism $\tilde{\theta} : X \to Y^I$ (the transpose of $\theta : I \otimes X \to Y$) such that the following diagram commutes:

$$\begin{tikzcd}
 & X \arrow[dl, "f"'] \arrow[d, dotted, "\tilde{\theta}"] \arrow[dr, "g"] & \\
Y & Y^I \arrow[l, "(\delta^0)^I"] \arrow[r, "(\delta^1)^I"'] & Y
\end{tikzcd}$$

All the other concepts from can be translated in a similar fashion. The category of strong homotopy equivalences defined using the internal-hom functor will turn out to be equivalent over $\mathbb{C}^{\to}$ to that of the foregoing definition, thus we can choose to work with either one.

There is an alternative characterisation of the category of strong homotopy equivalences which we proceed to describe. For $k \in \{0,1\}$, we use the notation θ^k for the following Cartesian square:

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d, "\perp_*"'] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d, "\delta_k"] \\
* \arrow[r, "\delta_{1-k}"'] & I
\end{tikzcd}$$

For any map $f : X \to Y$, we can take the Leibniz pushout-product of θ^k with f in order to obtain the square $\theta^k \hat{\otimes} f : f \to \delta^k \hat{\otimes} f$ depicted below:

$$\begin{tikzcd}
X \arrow[r, "Lf"] \arrow[d, "f"'] & Y +_X (I \otimes X) \arrow[d, "\delta^k \hat{\otimes} f"] \\
Y \arrow[r, "\delta^{1-k} \otimes Y"'] & I \otimes Y
\end{tikzcd}$$

Lemma C.5.3. *[GS17, Lemma 8.1] For $k \in \{0, 1\}$, the category S_k of k-oriented strong homotopy equivalences can be described as the category of arrows $f \in \mathbb{C}^{\to}$ equipped with a retraction ρ of $\theta^k \hat{\otimes} f$. In detail, we have*

Objects: *Pairs (f, ρ) where $f \in \mathbb{C}^{\to}$ and $\rho : \theta^k \hat{\otimes} f \to f$ such that $\rho \circ (\theta^k \hat{\otimes} f) = id_f$.*

Arrows: *An arrow $\tau : (f, \rho) \to (f', \rho')$ consists of a square $\tau : f \to f'$ such that the following diagram commutes:*

$$\begin{tikzcd}
\delta^k \hat{\otimes} f \arrow[r, "\delta^k \hat{\otimes} \tau"] \arrow[d, "\rho"'] & \delta^k \hat{\otimes} f' \arrow[d, "\rho'"] \\
f \arrow[r, "\tau"'] & f'
\end{tikzcd}$$

We will now state without a proof some useful results that relate the AWFS (C, F_t) and (C_t, F) with the categories of strong homotopy equivalences.

Lemma C.5.4. *[GS17, 8.4, 8.5 and 8.7] Let (C, F_t) be a suitable AWFS. Then we have the following lifts of functors:*

1. *The functor $\delta^k \otimes - : \mathbb{C} \to \mathbb{C}^{\to}$ lifts to S_k as shown:*

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\delta^k \otimes -"] \arrow[d, equal] & S_k \arrow[d] \\
\mathbb{C} \arrow[r, "\delta_k \otimes -"'] & \mathbb{C}^{\to}
\end{tikzcd}$$

2. *The functor $\delta^k \hat{\otimes} - : \mathbb{C}^{\to} \to \mathbb{C}^{\to}$ lifts to S_k as shown:*

$$\begin{tikzcd}
\mathbb{C}^{\to} \arrow[r, "\delta^k \hat{\otimes} -"] \arrow[d, equal] & S_k \arrow[d] \\
\mathbb{C}^{\to} \arrow[r, "\delta_k \hat{\otimes} -"'] & \mathbb{C}^{\to}
\end{tikzcd}$$

3. *There is a functor over $\mathbb{C}^{\to}$:*

$$\begin{tikzcd}
C\text{-}\boldsymbol{Map} \times_{\mathbb{C}^{\to}} S \arrow[rr] \arrow[dr] & & C_t\text{-}\boldsymbol{Map} \arrow[dl] \\
& \mathbb{C}^{\to} &
\end{tikzcd}$$

4. *There is a functor over $\mathbb{C}^{\to}$:*

$$\begin{tikzcd}
F\text{-}\boldsymbol{Map} \times_{\mathbb{C}^{\to}} S \arrow[rr] \arrow[dr] & & F_t\text{-}\boldsymbol{Map} \arrow[dl] \\
& \mathbb{C}^{\to} &
\end{tikzcd}$$

5. *There are a lifts of the pullback functor for* $k \in \{0, 1\}$*:*

$$\begin{tikzcd}
S_k \times_{\mathbb{C}} F\text{-}\boldsymbol{Map} \arrow[r, "PB"] \arrow[d] & S_k \arrow[d] \\
\mathbb{C}^{\to} \times_{\mathbb{C}} \mathbb{C}^{\to} \arrow[r, "PB"'] & \mathbb{C}^{\to}
\end{tikzcd}$$

where $PB(g, h) = h^* g$ *the pullback of* g *along* h*.*

A very similar, but more restrictive, notion to that of strong homotopy equivalence is that of strong deformation retract.

Definition C.5.5. Let $g : A \to B$ be a map in $\mathbb{C}$. For $k \in \{0, 1\}$, a **k-oriented strong deformation retraction structure** for g corresponds to the data of maps $r : B \to A$ and $h : I \times B \to B$ subject to the following conditions:

1. r is a retract of g, that is $r \cdot g = id_A$.
2. h is a k-oriented simplicial homotopy between $g \cdot r$ and id_B. That is depending on whether $k = 0$ or $k = 1$ we have that one of the following diagrams commutes:

$k{=}0$

$$\begin{tikzcd}
B \arrow[r, "\delta^0"] \arrow[dr, "g\cdot r"'] & I \times B \arrow[d, "h"] & B \arrow[l, "\delta^1"'] \arrow[dl, equal] \\
& B &
\end{tikzcd}$$

$k{=}1$

$$\begin{tikzcd}
B \arrow[r, "\delta^0"] \arrow[dr, equal] & I \times B \arrow[d, "h"] & B \arrow[l, "\delta^1"'] \arrow[dl, "g\cdot r"] \\
& B &
\end{tikzcd}$$

3. The retraction has a strength, which we express by requiring the following diagram to commute:

$$\begin{tikzcd}
I \times A \arrow[r, "I\times g"] \arrow[d, "\rho_1"'] & I \times B \arrow[d, "h"] \\
A \arrow[r, "g"'] & B
\end{tikzcd}$$

intuitively, we are requiring the homotopy h to be degenerate on the image of g.

We thus define a k-*strong deformation retraction* to be a tuple (g, r, h) where $g : A \to B$ is an arrow in $\mathbb{C}$ with a k-oriented strong deformation retraction structure given by r and h. A *morphism of k-strong deformation retractions*

$$(s, t) : (g, r, h) \to (g', r', h')$$

consists of maps $s : A \to A'$ and $t : B \to B'$ such that the following diagrams commute:

$$\begin{tikzcd}
A \arrow[r, "s"] \arrow[d, "g"'] & A' \arrow[d, "g'"] \\
B \arrow[r, "t"'] & B'
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "t"] \arrow[d, "r"'] & B' \arrow[d, "r'"] \\
A \arrow[r, "s"'] & A'
\end{tikzcd}
\qquad
\begin{tikzcd}
I \times B \arrow[r, "I\times t"] \arrow[d, "h"'] & I \times B' \arrow[d, "h'"] \\
B \arrow[r, "t"'] & B'
\end{tikzcd}$$

We have that k-strong deformation retractions and morphisms of such form a category of arrows, which we denote: $\underline{\text{SDR}}_k \to \mathbb{C}^{\to}$ by mapping $(g, r, h) \mapsto g$. The category of arrows of *strong deformation retractions* is defined as the coproduct in the slice over $\mathbb{C}^{\to}$ of $\underline{\text{SDR}}_0$ and $\underline{\text{SDR}}_1$, we denote this as:

$$\underline{\text{SDR}} \to \mathbb{C}^{\to}$$

It is easy to verify that there is a functor over $\mathbb{C}^{\to}$ form the category of strong deformation retracts to that of strong homotopy equivalences as shown in the following diagram:

$$\begin{tikzcd}
\underline{\text{SDR}} \arrow[rr] \arrow[dr] & & S \arrow[dl] \\
& \mathbb{C}^{\to} &
\end{tikzcd}$$

the action on objects is given by $(g, r, h) \mapsto (g, r, h, \epsilon)$ where epsilon denotes the constant homotopy.

C.6 Uniform Fibrations in Toposes

In this section we will describe a slightly more general result than [GS17, Theorem 9.1]. We will generalise from the setting of presheaves to that of a Grothendieck topos.

Definition C.6.1. A subcategory $\mathbb{D}$ of a category $\mathbb{C}$ is said to be **dense** if it is full, small and every object of $\mathbb{C}$ is the canonical colimit over $\mathbb{D}$, that is, for every $C \in \mathbb{C}$, we have:

$$\operatorname{colim}_{\substack{D \to C \\ D \in \mathbb{D}}} D \cong C$$

Lemma C.6.2. *[GS17, Lemma 5.15] Let $\mathbb{E}$ be a cocomplete category equipped with universal colimits and a dense subcategory $\mathbb{D} \subset \mathbb{E}$. Let $\mathfrak{J}$ be a full subcategory of $\mathbb{C}^{\to}_{\mathrm{cart}}$ closed under pullback along maps with domain in $\mathbb{D}$. Denote by $\mathfrak{I}$ the restriction of $\mathfrak{J}$ to arrows with codomain in $\mathbb{D}$, that is*

$$\mathfrak{I} := \{i \in \mathfrak{J} | \mathrm{cod}(i) \in \mathbb{D}\}$$

Then the inclusion $\mathfrak{J} \to \mathbb{C}^{\to}$ is the left Kan extension of $\nu : \mathfrak{I} \to \mathbb{C}^{\to}$ along $q : \mathfrak{I} \to \mathfrak{J}$, as shown in the diagram:

$$\begin{tikzcd}
\mathfrak{I} \arrow[rr, "q"] \arrow[dr, "\nu"'] & & \mathfrak{J} \arrow[dl, "\mathrm{Lan}_q \nu"] \\
& \mathbb{C}^{\to} &
\end{tikzcd}$$

Proof. Because $\mathbb{C}$ is cocomplete, the left Kan extension admits the following description as a colimit:

$$\mathrm{Lan}_q \nu(j) = \int^{i \in \mathfrak{I}} \mathfrak{J}(i, j) \cdot i$$

that is, the colimit (in the arrow category) of the $i \in \mathcal{I}$ indexed by the pullback squares of the form:

$$\begin{tikzcd} A_f \arrow[r, "\hat{f}"] \arrow[d, "i"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "j"] \\ D \arrow[r, "f"'] & C \end{tikzcd}$$

with $D \in \mathbb{D}$. Because $\mathcal{J}$ is closed under pullbacks along morphisms with domain in $\mathbb{D}$, we can write the colimit in question as:

$$\mathsf{Lan}_q \nu(j) \cong \mathsf{colim}_{\substack{f:D \to C \\ D \in \mathbb{D}}} f^* j$$

Now, because $\mathbb{D}$ is dense, we have that $\mathsf{colim}_{\substack{f:D \to C \\ D \in \mathbb{D}}} D \cong C$, and because colimits are universal, we have that $\mathsf{colim}_{\substack{f:D \to C \\ D \in \mathbb{D}}} A_f \cong A$. Finally, because colimits in $\mathbb{C}^{\to}$ are computed pointwise, we have:

$$j \cong \mathsf{colim}_{\substack{f:D \to C \\ D \in \mathbb{D}}} (A_f \to D) \cong \mathsf{colim}_{\substack{f:D \to C \\ D \in \mathbb{D}}} f^* j \cong \mathsf{Lan}_q \nu(j)$$

□

Theorem C.6.3. *[GS17, Theorem 9.1] Let $\mathbb{E}$ be a Grothendieck topos with a closed symmetric monoidal structure, a dense subcategory $\mathbb{D}$ and with an interval object with contractions and connections such that:*

(C1) $I \otimes (-) : \mathbb{E} \to \mathbb{E}$ preserves pullbaks

(C2) $\delta^k \otimes (-) : \mathsf{id}_{\mathbb{E}} \to I \otimes (-)$ is a Cartesian natural transformation for $k \in \{0, 1\}$.

Consider $\mathcal{M}$ a full subcategory of $\mathbb{E}^{\to}_{\mathsf{cart}}$ satisfying:

(M1) the objects of $\mathcal{M}$ are monomorphisms

(M2) $\perp : 0 \to X$ is in $\mathcal{M}$ for every $X \in \mathbb{E}$.

(M3) the objects of $\mathcal{M}$ are closed under pullbacks.

(M4) the elements of $\mathcal{M}$ are closed under Leibniz product with the endpoint inclusions.

Then there exists a suitable AWFS (C, F_t) and it is algebraically-free on $\mathcal{M}$.

Proof. Consider the subcategory of $\mathcal{M}$ of arrows with codomain in $\mathbb{D}$, that is $\mathcal{I} := \{j \in \mathcal{M} | \mathsf{cod}(j) \in \mathbb{D}\}$. By the previous lemma we have that:

$$\begin{tikzcd} \mathcal{I} \arrow[rr, "q"] \arrow[dr, "\nu"'] & & \mathcal{M} \arrow[dl, "\mathsf{Lan}_q \nu"] \\ & \mathbb{C}^{\to} & \end{tikzcd}$$

By Garner's small object argument, there is an algebraically-free AWFS (C, F_t) on $\mathcal{I}$ and by [GS17, Proposition 5.14] we have that $\mathcal{M}^{\boxslash} = \mathcal{I}^{\boxslash} = F\textbf{-Alg}$. The verification that (C, F_t) is suitable is straightforward using orthogonality arguments. □

Remark C.6.4. Notice that $\mathbb{E}$, being a topos, is in particular a locally presentable category and as such it can always be equipped with a dense subcategory (the full subcategory of compact objects for a large enough cardinal).

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