Cohomology of Burnside Rings

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Abstract

We study the groups $\mathrm{Ext}^l_{A(G)}(\mathbb{Z}_H, \mathbb{Z}_J)$ where $A(G)$ is the Burnside ring of a finite group G and for a subgroup $H \subset G$, the $A(G)$-module $\mathbb{Z}_H$ is defined by the mark homomorphism corresponding to H. If $|G|$ is square-free we give a complete description of these groups. If $|G|$ is not square-free we show that for certain $H, J \subset G$ the groups $\mathrm{Ext}^l_{A(G)}(\mathbb{Z}_H, \mathbb{Z}_J)$ have unbounded rank.

We also extend some of these results to the rational and complex representation rings of a finite group, and describe a new generalisation of the Burnside ring for infinite groups.

Contents

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Declaration

I declare that this thesis is a presentation of original work and I am the sole author. This work has not previously been presented for an award at this, or any other, University. All sources are acknowledged as References.

Introduction

Let G be a finite group. The isomorphism classes of finite G-sets form a commutative semi-ring $\widehat{A}(G)$, where addition is given by disjoint union and multiplication is given by cartesian product. For a finite G-set X, write $[X]$ for the isomorphism class of X in $\widehat{A}(G)$. For a subgroup $H \subset G$, we have a homomorphism of semi-rings $\hat{\pi}_H : \widehat{A}(G) \to \mathbb{Z}$ given by putting $\hat{\pi}_H([X]) = |X^H|$. The Burnside ring $A(G)$ is the Grothendieck ring associated to this semi-ring. The homomorphisms $\hat{\pi}_H$ extend to homomorphisms of rings $\pi_H : A(G) \to \mathbb{Z}$ known as the mark homomorphisms of G, and we write $\mathbb{Z}_H$ for the corresponding module structure on $\mathbb{Z}$.

For distinct subgroups $H, J \subset G$, and p a prime, let $\sim_p$ be the equivalence relation on the set of subgroups of G generated by putting $H \sim_p J$ whenever $\pi_H(a) \equiv \pi_J(a) \mod p$ for each $a \in A(G)$. By studying the equivalence classes of $\sim_p$ for each prime p, Dress [10] gave a criterion for the solvability of a finite group in terms of the indecomposability of its Burnside ring. In this thesis we study further the question of when $\pi_H \equiv \pi_J \mod p$, and relate the equivalence classes of $\sim_p$ with the structure of the cohomology groups $\mathrm{Ext}^l_{A(G)}(\mathbb{Z}_H, \mathbb{Z}_J)$.

For I a finite set, define $\mathrm{Gh}(I) = \prod_{i\in I} \mathbb{Z}$. For a subring $R \subset \mathrm{Gh}(I)$, define homomorphisms $\pi_i : R \to \mathbb{Z}$ for each $i \in I$ by projection onto the corresponding factor, and write $\mathbb{Z}_i$ for the corresponding R-module. Say that $R \subset \mathrm{Gh}(I)$ is a B-ring if for each distinct $i, j \in I$, there exists $r \in R$ such that $\pi_i(r) \neq 0$ and $\pi_j(r) = 0$, i.e. if R separates the elements of I. In Chapter 1 we show that the Burnside ring is a natural example of a B-ring, and study the groups $\mathrm{Ext}^l_R(\mathbb{Z}_i, \mathbb{Z}_j)$ for $R \subset \mathrm{Gh}(I)$ an arbitrary B-ring and $i, j \in I$. For each prime p, we establish a link between the groups $\mathrm{Ext}^l_R(\mathbb{Z}_i, \mathbb{Z}_j)$ and the cohomology of the $\mathbb{F}_p$-algebra $\overline{R} = R \otimes_{\mathbb{Z}} \mathbb{F}_p$. Following the case of the Burnside ring, we define an equivalence relation $\sim_p$ on I by putting $i \sim_p j$ if $\pi_i(r) \equiv \pi_j(r) \mod p$ for each $r \in R$. We show that the equivalence classes of of $\sim_p$ are in 1-1 correspondence with the indecomposable summands of the $\mathbb{F}_p$-algebra $\overline{R}$. By examining the corresponding summands, we show that if the equivalence classes of $\sim_p$ on I have cardinality ≤ 2 for each prime p, then $\mathrm{Ext}^l_R(\mathbb{Z}_i, \mathbb{Z}_j) \simeq \mathrm{Ext}^{l+2}_R(\mathbb{Z}_i, \mathbb{Z}_j)$ for all $i, j \in I$ and $l > 0$. It follows that if the order of G is square-free, then $\mathrm{Ext}^l_{A(G)}(\mathbb{Z}_H, \mathbb{Z}_J) \simeq \mathrm{Ext}^{l+2}_{A(G)}(\mathbb{Z}_H, \mathbb{Z}_J)$ for all $l > 0$ and $H, J \subset G$.

In Chapter 2 we consider the converse of this result. Gustafson [13] has shown that if k is a field of characteristic p and p^2 divides the order of G, then the k-algebra $A(G) \otimes_{\mathbb{Z}} k$ is not symmetric. We use this to show that if $p^2 \mid |G|$ then $A(G) \otimes_{\mathbb{Z}} k$ has an indecomposable summand S such that the k-vector spaces $\mathrm{Ext}^l_S(k, k)$ have unbounded dimension. By making use of our link with the integral cohomology, we show that if $|G|$ is not square-free then there exist subgroups $H, J \subset G$ such that the groups $\mathrm{Ext}^l_{A(G)}(\mathbb{Z}_H, \mathbb{Z}_J)$ have unbounded rank. In the remainder of the chapter we consider more generally the question of computing the cohomology of commutative local k-algebras S with maximal ideal $\mathcal{M}$ and $S/\mathcal{M} \simeq k$. We give a formula for the dimension of $\mathrm{Ext}^l_S(k, k)$ when S belongs to the family of commutative noetherian local k-algebras satisfying $\dim \mathcal{M}^2 = 1$ and $\mathcal{M}^3 = 0$. We also give a simple proof of a weak version of Theorem 2.7.

In Chapter 3 we consider a new generalisation of the Burnside ring to infinite groups, using the idea of a Mackey system of subgroups. We establish which properties of the finite group version carry over to this more general setting, such as the mark homomorphisms and Mackey functor structure. Let S be the group of permutations of $\mathbb{N}$ fixing all but finitely many elements. We construct a Mackey system $\mathfrak{M}_S$ for S and study in detail the resulting Burnside ring $A(S, \mathfrak{M}_S)$. We give a combinatorial description of the multiplication operation in $A(S, \mathfrak{M}_S)$ in terms of partial injections between finite sets.

Further examples of B-rings are given by the ring of rational characters $RQ(G)$ of a finite group G. For a cyclic group $H \subset G$, we have a homomorphism $RQ(G) \to \mathbb{Z}$ defined by sending a rational representation to its trace at a generator of H, and we write $\mathbb{Z}_H$ for the corresponding $RQ(G)$-module. In Chapter 4 we apply the results of Chapter 1 to $RQ(G)$, and show that if the order of G is square-free then $\mathrm{Ext}^l_{RQ(G)}(\mathbb{Z}_H, \mathbb{Z}_J) \simeq \mathrm{Ext}^{l+2}_{RQ(G)}(\mathbb{Z}_H, \mathbb{Z}_J)$ for all cyclic groups $H, J \subset G$ and $l > 0$. This leads to the question of the appropriate setting for studying more general rings, such as the ring of complex characters. We introduce the notion of a B'-ring to deal with more general families, and construct a B'-ring embedding for the ring of complex characters of a finite group.

Chapter 1

B-rings

For I a finite set, define $\mathrm{Gh}(I) = \prod_{i \in I} \mathbb{Z}$. Let R be a subring of $\mathrm{Gh}(I)$ and let π_i be the projection $R \to \mathbb{Z}$ corresponding to $i \in I$. For $r \in R$, write $r(i)$ for $\pi_i(r)$.

Definition 1.1. We say that $R \subset \mathrm{Gh}(I)$ as above is a B-ring if R satisfies the following 'separability condition': for each $i, j \in I$ with $i \neq j$ we can find an $r \in R$ with $r(i) \neq 0$ and $r(j) = 0$.

Lemma 1.2. Let I be a finite set and R be a subring of $\mathrm{Gh}(I)$. Then there's a $J \subset I$ and a B-ring $S \subset \mathrm{Gh}(J)$ with R isomorphic to S.

Proof. If $R \subset \mathrm{Gh}(I)$ is a B-ring then there's nothing to show. Otherwise, suppose $R \subset \mathrm{Gh}(I)$ is not a B-ring, i.e. there exists $i, j \in I$ such that the separability condition fails. We claim that $r(i) = r(j)$ for all $r \in R$. To see this, suppose instead that we have some $r \in R$ with $r(i) \neq r(j)$. Then the element $r' = r - r(j)\,\mathrm{id}_R$ has $r'(i) = r(i) - r(j) \neq 0$ and $r'(j) = 0$, which contradicts the choice of i and j. It then follows that R is isomorphic to the ring $S \subset \mathrm{Gh}(I - \{j\})$ obtained by omitting the jth factor. Continuing in this manner gives the result, since I is finite. □

Our rings of interest are then arbitrary subrings of some product of finitely many copies of $\mathbb{Z}$, though we wish to consider them as embedded in some minimal such ring. We give an intrinsic description of these rings as follows.

Proposition 1.3. A ring S is isomorphic to a B-ring $R \subset \mathrm{Gh}(I)$ for some finite set I if and only if S is a commutative ring which is of finite rank and torsion-free as a $\mathbb{Z}$-module, with $\mathbb{Q} \otimes_{\mathbb{Z}} R$ a product of $|I|$ 1-dimensional $\mathbb{Q}$-algebras.

Proof. If $R \subset \mathrm{Gh}(I)$ is a B-ring then it is certainly commutative and torsion-free, since $\mathrm{Gh}(I)$ is. As a $\mathbb{Z}$-module $\mathrm{Gh}(I)$ is finitely generated, so R is of finite rank. For each pair i, j of distinct elements of I, let $r_{i,j}$ be an element of R satisfying $r(i) \neq 0$ and $r(j) = 0$. Then putting $s_i = \prod_{j \neq i} r_{i,j}$ for each $i \in I$, we have $s_i(j) \neq 0$ if and only if $i = j$. Let $N = \prod_{i \in I} s_i(i)$ and $N_i = N/s_i(i)$. For $i \in I$ write e_i for the corresponding primitive idempotent of $\mathrm{Gh}(I)$. Then

$$N \cdot e_i = N_i s_i \in R,$$

and so $N \cdot \mathrm{Gh}(I) \subset R \subset \mathrm{Gh}(I)$. Hence

$$\mathbb{Q} \otimes_{\mathbb{Z}} R \simeq \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Gh}(I) \simeq \prod_{i \in I} \mathbb{Q},$$

i.e. $\mathbb{Q} \otimes R$ is isomorphic to a product of $|I|$ 1-dimensional $\mathbb{Q}$-algebras.

Suppose S is a commutative ring which is of finite rank and torsion-free as a $\mathbb{Z}$-module, with $\mathbb{Q} \otimes_{\mathbb{Z}} S$ a product of 1-dimensional $\mathbb{Q}$-algebras. Then we have an isomorphism

$$\theta : \mathbb{Q} \otimes S \to \prod_{i \in I'} \mathbb{Q}$$

for some finite indexing set I'.

Since S is torsion-free, $\mathbb{Q} \otimes S$ contains a copy of S as the subring $1 \otimes S \subset \mathbb{Q} \otimes S$. Denote the image $\theta(1 \otimes S)$ by $S' \subset \prod_{i \in I'} \mathbb{Q}$, and for each $i \in I'$ let $\hat{\pi}_i$ denote the projection map $\prod_{i \in I'} \mathbb{Q} \to \mathbb{Q}$ onto the ith factor. Write π_i for the restriction of $\hat{\pi}_i$ to S'. Since S' is of finite rank as a $\mathbb{Z}$-module we must have that $\pi_i(S') \subset \mathbb{Z} \subset \mathbb{Q}$ for each $i \in I'$. We can then regard S' as sitting inside $\mathrm{Gh}(I') = \prod_{i \in I'} \mathbb{Z} \subset \prod_{i \in I'} \mathbb{Q}$. We claim that this embedding defines a B-ring. For $s \in S$, write s' for the element $\theta(1 \otimes s)$ of S'. It remains to show that for each distinct pair $i, j \in I'$ we can find an element $s \in S$ such that $\pi_i(s') \neq 0$ and $\pi_j(s') = 0$.

Let $\{f_i\}_{i \in I'}$ be the primitive idempotents of $\mathrm{Gh}(I')$, and note that $\hat{\pi}_j(f_i) = 1$ if $j = i$ and $\hat{\pi}_j(f_i) = 0$ otherwise. For each $i \in I'$, we have $f_i = \theta(q_i \otimes t_i)$ for some $q_i \in \mathbb{Q}$ and $t_i \in S$. Then $t'_i = \theta(1 \otimes t_i) = (1/q_i) \cdot \theta(q_i \otimes t_i) = (1/q_i) f_i$, and so $\pi_i(t'_i) = 1/q_i \neq 0$ and $\pi_j(t'_i) = 0$ for each $j \neq i$. Thus t'_i satisfies the separability condition for any $j \neq i$, and the embedding $S' \subset \mathrm{Gh}(I')$ defines a B-ring. □

For the remainder of this chapter, let I be some fixed finite set. For a B-ring $R \subset \mathrm{Gh}(I)$ and $i \in I$, define a left[1] R-module $\mathbb{Z}_i$ by letting R act on the set $\mathbb{Z}$ via π_i, i.e. for $r \in R, n \in \mathbb{Z}$, put

$$r \cdot n = r(i)n.$$

We will later show that all R-modules which are of rank 1 as a $\mathbb{Z}$-module are of the form $\mathbb{Z}_i$ for some $i \in I$. One advantage of working with an explicit embedding is that these modules are immediately obvious. We will occasionally make use of the intrinsic definition however, as in the following corollary.

We first recall that for a commutative ring R and an R-module N, the functor $\mathrm{Hom}_R(-, N) : R\text{-Mod} \to R\text{-Mod}$ is left exact. For each non-negative integer l, let $\mathrm{Ext}^l_R(-, N)$ denote the lth right derived functor of $\mathrm{Hom}_R(-, N)$. For another R-module M, write $\mathrm{Ext}^l_R(M, N)$ for $\mathrm{Ext}^l_R(-, N)(M)$. For further details see [25] (Chapter 2).

Corollary 1.4. Let $R \subset \mathrm{Gh}(I)$ be a B-ring. Then the R-modules $\mathrm{Ext}^l_R(\mathbb{Z}_i, \mathbb{Z}_j)$ are finite for any $l \geq 1$ and $i, j \in I$.

Proof. Since $R_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} R$ is a product of some finite number of copies of $\mathbb{Q}$, it is

[1] All further modules will be left modules unless otherwise specified.

semisimple. Then $\mathbb{Q} \otimes \mathbb{Z}_i$ is a projective $R_{\mathbb{Q}}$-module for any $i \in I$, and so

$$\mathrm{Ext}^l_{R_{\mathbb{Q}}}(\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}_i, \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}_j) = 0$$

for any $l \geq 1$ and $i, j \in I$. By [25] (Proposition 3.3.10) this is isomorphic to

$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Ext}^l_R(\mathbb{Z}_i, \mathbb{Z}_j),$$

and so $\mathrm{Ext}^l_R(\mathbb{Z}_i, \mathbb{Z}_j)$ is torsion. But it is also finitely generated, so it is finite. □

For a finite R-module M, define the rank of M be its rank considered as a finite group, i.e. the cardinality of a minimal generating set for M.

By the Krull-Schmidt theorem for modules ([16], Section 3.4), for each finite R-module M we have a well-defined integer recording the number of summands appearing in a decomposition of M as a direct sum of indecomposable R-modules. Call this the summand rank of M.

Lemma 1.5. Let $R \subset \mathrm{Gh}(I)$ be a B-ring, and let $i, j \in I$. Then

$$\mathrm{Hom}_R(\mathbb{Z}_i, \mathbb{Z}_j) \simeq \begin{cases} 0 & \text{if } i \neq j \\ \mathbb{Z}_i & \text{if } i = j. \end{cases}$$

Proof. Since $\mathbb{Z}_i$ is generated as an R-module by $1 \in \mathbb{Z}_i$, any $\phi \in \mathrm{Hom}_R(\mathbb{Z}_i, \mathbb{Z}_j)$ is determined by $\phi(1)$.

If $i \neq j$, we can by the definition of a B-ring choose an $r \in R$ such that $r(i) = 0$ and $r(j) \neq 0$. Then $r(j)\phi(1) = \phi(r \cdot 1) = \phi(0) = 0$, and so $\phi = 0$.

Suppose $i = j$. Each $m \in \mathbb{Z}$ determines a unique map $\phi_m \in \mathrm{Hom}_R(\mathbb{Z}_i, \mathbb{Z}_i)$ given by $\phi_m(1) = m$, and $(r \cdot \phi_m)(1) = \phi_m(r \cdot 1) = r(i)\phi_m(1) = \phi_{r(i)m}(1)$. Since any $\phi : \mathbb{Z}_i \to \mathbb{Z}_i$ is determined by $\phi(1)$, all maps $\mathbb{Z}_i \to \mathbb{Z}_i$ are obtained in this way, and so $\mathrm{Hom}_R(\mathbb{Z}_i, \mathbb{Z}_i) \simeq \mathbb{Z}_i$. □

Definition 1.6. Let $R \subset \mathrm{Gh}(I)$ be a B-ring. For distinct $i, j \in I$ define

$$d_R(i, j) = \min\{r(i) \mid r \in R, r(j) = 0, r(i) > 0\}.$$

When the ring R is clear, we will simply write $d(i, j)$.

If $s \in R$ is such that $s(i) = d(i, j)$ and $s(j) = 0$, then $s' = d(i, j)\,\mathrm{id}_R - s$ satisfies $s'(i) = 0$ and $s'(j) = d(i, j)$. It follows that $d(i, j) = d(j, i)$ for all $i, j \in I$. Moreover, if $r \in R$ is such that $r(i) - r(j) = t$ for some $t \in \mathbb{Z}$, then $r' = r - r(j)\,\mathrm{id}_R$ has $r'(j) = 0$ and $r'(i) = t$, hence $d(i, j) \mid t$. It follows then that for any $m \mid d(i, j)$, we have

$$r(i) \equiv r(j) \mod m$$

for each $r \in R$, and so the R-modules $\mathbb{Z}_i/m\mathbb{Z}_i$ and $\mathbb{Z}_j/m\mathbb{Z}_j$ are isomorphic.

Proposition 1.7. Let $R \subset \mathrm{Gh}(I)$ be a B-ring, $i, j \in I$. Then

$$\mathrm{Ext}^1_R(\mathbb{Z}_i, \mathbb{Z}_j) \simeq \begin{cases} \mathbb{Z}_j/d(i,j)\mathbb{Z}_j & \text{if } i \neq j \\ 0 & \text{if } i = j. \end{cases}$$

Proof. Writing K_l for the kernel of $\pi_l : R \to \mathbb{Z}_l$ for each $l \in I$, we have a short exact sequence of R-modules

$$0 \to K_i \xrightarrow{\iota} R \xrightarrow{\pi_i} \mathbb{Z}_i \to 0,$$

where ι denotes the inclusion $K_i \hookrightarrow R$. Applying $\mathrm{Hom}_R(-, \mathbb{Z}_j)$ for $j \neq i$, we obtain a long exact sequence beginning with

$$\begin{array}{l} 0 \longrightarrow \mathrm{Hom}_R(\mathbb{Z}_i, \mathbb{Z}_j) \longrightarrow \mathrm{Hom}_R(R, \mathbb{Z}_j) \longrightarrow \mathrm{Hom}_R(K_i, \mathbb{Z}_j) \\ \longrightarrow \mathrm{Ext}^1_R(\mathbb{Z}_i, \mathbb{Z}_j) \longrightarrow \mathrm{Ext}^1_R(R, \mathbb{Z}_j) \longrightarrow \dots \end{array}$$

Now $\mathrm{Ext}^1_R(R, \mathbb{Z}_j) = 0$ since R is projective, and $\mathrm{Hom}_R(\mathbb{Z}_i, \mathbb{Z}_j) = 0$ by Lemma 1.5. So we obtain a short exact sequence

$$0 \to \mathrm{Hom}_R(R, \mathbb{Z}_j) \xrightarrow{\bar{\iota}} \mathrm{Hom}_R(K_i, \mathbb{Z}_j) \xrightarrow{\partial} \mathrm{Ext}^1_R(\mathbb{Z}_i, \mathbb{Z}_j) \to 0,$$

where $\bar{\iota} = \mathrm{Hom}_R(\iota, \mathbb{Z}_j)$ and ∂ denotes the first connecting homomorphism in the above long exact sequence.

We will first describe the R-module $\mathrm{Hom}_R(K_i, \mathbb{Z}_j)$. Consider $\tau \in \mathrm{Hom}_R(K_i, \mathbb{Z}_j)$, and suppose $r_1, r_2 \in K_i$ are such that $\pi_j(r_1) = \pi_j(r_2)$. Then $r_1 - r_2$ belongs to $K_j \cap K_i$. We claim that there exists $N \in \mathbb{Z}$ such that $N(r_1 - r_2) \in K_j K_i$. If this is the case, then we can write

$$N(r_1 - r_2) = \sum_{l=1}^{p} z_l^j z_l^i,$$

where $z_l^j \in K_j$, $z_l^i \in K_i$, for $1 \leq l \leq p$. Then

$$N(\tau(r_1) - \tau(r_2)) = \tau(N(r_1 - r_2)) = \sum_{l=1}^{p} z_l^j \cdot \tau(z_l^i) = 0$$

since each z_l^j acts by zero in $\mathbb{Z}_j$. Hence

$$N\tau(r_1) = N\tau(r_2)$$

and so $\tau(r_1) = \tau(r_2)$.

To show that such an N exists, recall that we can choose for each $l \in I$ an $s_l \in R$

with $s_l(l') \neq 0$ if and only if $l = l'$. For $l \neq i, j$, then $s_l^2 \in K_j K_i$. Putting

$$N = \prod_{\substack{l \in I \\ l \neq i,j}} s_l(l)^2,$$

we note that

$$N(r_1 - r_2) = \sum_{\substack{l \in I \\ l \neq i,j}} \frac{N(r_1 - r_2)(l)}{s_l(l)^2} s_l^2,$$

and so $N(r_1 - r_2) \in K_j K_i$.

Since $\tau(r)$ depends only on $\pi_j(r)$ for each $r \in R$, the homomorphism τ factors through $\pi_j|_{K_i} : K_i \to \mathbb{Z}_j$. Now $\pi_j(K_i)$ is the submodule $D = (d(i,j))$ of $\mathbb{Z}_j$, and for each $m \in \mathbb{Z}$ we have a homomorphism $\sigma_m : D \to \mathbb{Z}_j$ given by

$$\sigma_m(d) = \frac{d}{d(i,j)} m$$

for $d \in D$, and all homomorphisms $D \to \mathbb{Z}_j$ are of this form. It follows that every τ of $\mathrm{Hom}_R(K_i, \mathbb{Z}_j)$ must then be of the form $\tau_m := \sigma_m \circ \pi_j$ for some $m \in \mathbb{Z}$. For $z \in K_i$ we have

$$\tau_m(z) = \sigma_m(\pi_j(z)) = \frac{z(j)}{d(i,j)} m,$$

and so for $r \in R$ we have

$$(r \cdot \tau_m)(z) = (r(j)\tau_m)(z) = \frac{r(j)z(j)}{d(i,j)} m = \tau_{r(j)m}(z),$$

from which we conclude that $\mathrm{Hom}_R(K_i, \mathbb{Z}_j) \simeq \mathbb{Z}_j$.

It remains to determine the image of $\mathrm{Hom}_R(R, \mathbb{Z}_j)$ in $\mathrm{Hom}_R(K_i, \mathbb{Z}_j)$. Now $\mathrm{Hom}_R(R, \mathbb{Z}_j)$ is canonically isomorphic to $\mathbb{Z}_j$, and can be regarded as the set of maps $\phi_m, m \in \mathbb{Z}$, where $\phi_m(r) = mr(j)$. Moreover, for $z \in K_i$ we have

$$(\bar{\iota}(\phi_m))(z) = \phi_m(\iota(z)) = \phi_m(z) = mz(j) = \frac{z(j)}{d(i,j)} d(i,j)m = \tau_{d(i,j)m}(z),$$

and so $\bar{\iota}$ maps $\mathrm{Hom}_R(R, \mathbb{Z}_j)$ onto the submodule generated by $\tau_{d(i,j)}$. Then

$$\mathrm{Ext}^1_R(\mathbb{Z}_i, \mathbb{Z}_j) \simeq \mathbb{Z}_j / d(i,j)\mathbb{Z}_j$$

as claimed.

Next suppose $i = j$. The long exact sequence arising from applying $\mathrm{Hom}_R(-, \mathbb{Z}_i)$ to

$$0 \to K_i \to R \to \mathbb{Z}_i \to 0$$

begins as

$$0 \longrightarrow \mathrm{Hom}_R(\mathbb{Z}_i, \mathbb{Z}_i) \longrightarrow \mathrm{Hom}_R(R, \mathbb{Z}_i) \longrightarrow \mathrm{Hom}_R(K_i, \mathbb{Z}_i) \longrightarrow \mathrm{Ext}^1_R(\mathbb{Z}_i, \mathbb{Z}_i) \longrightarrow 0 \longrightarrow \ldots$$

and so $\mathrm{Hom}_R(K_i, \mathbb{Z}_i)$ surjects onto $\mathrm{Ext}^1_R(\mathbb{Z}_i, \mathbb{Z}_i)$. But by the same argument as before, this time putting

$$N = \prod_{\substack{l \in I \\ l \neq i}} s_l(l)^2,$$

we have that for any $\tau \in \mathrm{Hom}_R(K_i, \mathbb{Z}_i)$, the homomorphism τ factors through π_i. But $\pi_i(z) = 0$ for all $z \in K_i$, so $\mathrm{Hom}_R(K_i, \mathbb{Z}_i) = 0$, and hence $\mathrm{Ext}^1_R(\mathbb{Z}_i, \mathbb{Z}_i) = 0$. $\square$

Our argument that for any $\tau \in \mathrm{Hom}_R(K_i, \mathbb{Z}_j)$, the homomorphism τ factors through π_j applies more generally. We will state it now in a form that will be useful later.

Lemma 1.8. Let M be a submodule of R, and N a torsion-free R-module where r acts by $r \cdot n = r(j)n$ for each $n \in N$. Then any homomorphism of R-modules $g : M \to N$ factors through $\pi_j|_M : M \to \mathbb{Z}_j$.

1.1 The spectrum of a B-ring

Let I be a finite set. The spectrum of $\mathrm{Gh}(I)$ is given by

$$\mathrm{Spec}\,\mathrm{Gh}(I) = \bigsqcup_{i \in I} \mathrm{Spec}\,\mathbb{Z},$$

and for $i \in I$ and $P \in \mathrm{Spec}\,\mathbb{Z}$ we write $Q(i, P)$ for the corresponding prime. For $\hat{\pi}_i : \mathrm{Gh}(I) \to \mathbb{Z}$ the projection onto the ith factor, and s_P the natural map $\mathbb{Z} \to \mathbb{Z}/P\mathbb{Z}$, we have

$$Q(i, P) = \ker(s_P \circ \hat{\pi}_i).$$

Let S be an arbitrary subring of $\mathrm{Gh}(I)$. Since $\mathrm{Gh}(I)$ is generated by a finite set of idempotents, $\mathrm{Gh}(I)$ is integral over S, and it follows that the embedding $S \subset \mathrm{Gh}(I)$ induces a surjection

$$\mathrm{Spec}\,\mathrm{Gh}(I) \to \mathrm{Spec}\,S$$

given by $Q(i, P) \mapsto Q(i, P) \cap S =: q_S(i, P)$ (see [2] Theorem 5.10). When the subring in question is clear we will simply write $q(i, P)$.

Proposition 1.9. Let $R \subset \mathrm{Gh}(I)$ be a B-ring, let i, j be distinct elements of I, and let P, P' be prime ideals in $\mathbb{Z}$. Then

i. $q(i, P) = q(i, P')$ if and only if $P = P'$;

ii. $q(i, P) = q(j, P')$ if and only if $P = P' = (p)$ for some rational prime p with $p \mid d(i, j)$.

Proof. It is immediate that for p a rational prime, $p\,\mathrm{id}_R \in q(i,P)$ if and only if $P=(p)$, and this establishes (i).

By the same reasoning, if $q(i,P)=q(j,P')$, then we must have $P=P'$. By the separability condition for i,j, we moreover must have $P\neq(0)$ and so $P=(p)$ for some rational prime p. It remains to show that $q(i,(p))=q(j,(p))$ if and only if $p\mid d(i,j)$.

Now if $q(i,(p))=q(j,(p))$ then $p\mid r(i)$ whenever $p\mid r(j)$ for each $r\in R$. In particular, $p\mid r(i)$ whenever $r(j)=0$, and so $p\mid d(i,j)$. Conversely, if $p\mid d(i,j)$ and $r\in q(j,(p))$, i.e. $r(j)=lp$ for some $l\in\mathbb{Z}$, then $r'=r-lp\,\mathrm{id}_R$ satisfies $r'(j)=0$, and so $p\mid r'(i)$. Hence $p\mid r(i)$ and $r\in q(i,(p))$. Since $p\mid d(i,j)$ implies $p\mid d(j,i)$, we also have $r\in q(j,(p))$ whenever $r\in q(i,(p))$, and so $q(i,(p))=q(j,(p))$. □

Since any surjective homomorphism of rings $\theta: R\to\mathbb{Z}$ determines a prime ideal $\ker\theta\subset R$, we obtain the following.

Corollary 1.10. For a B-ring $R\subset \mathrm{Gh}(I)$, the modules $\mathbb{Z}_i, i\in I$, give a complete irredundant collection of R-modules (up to isomorphism) which are of rank 1 as a $\mathbb{Z}$-module.

1.2 The Burnside ring

1.2.1 The Grothendieck group associated to a commutative monoid

As we will make use of it repeatedly, we briefly outline the construction of the Grothendieck group associated to a commutative monoid, see [1] (Chapter 2) for more details.

Given a commutative monoid $\widehat{M}$, we wish to associate to $\widehat{M}$ an abelian group M and a homomorphism of monoids $\alpha:\widehat{M}\to M$, such that for any group H and any homomorphism of monoids $\gamma:\widehat{M}\to H$, there exists a unique homomorphism of groups $\beta: M\to H$ such that $\gamma=\beta\circ\alpha$. By the usual universal property argument, if M exists then it must be unique up to isomorphism.

In order to construct M and α, first form the free abelian group $F(\widehat{M})$. Write $+$ for addition in the monoid $\widehat{M}$ and $+'$ and $-'$ for addition and subtraction in $F(\widehat{M})$. Let I be the subgroup generated by all $m+'n-'(m+n)$, $m,n\in\widehat{M}$. Putting $M=F(\widehat{M})/I$, and letting $\alpha:\widehat{M}\to M$ send $m\in\widehat{M}$ to the equivalence class of m in M, it is clear that M satisfies the universal property above.

Furthermore, if $\widehat{M}$ is a semi-ring with unit, then it is immediate that the construction above produces a ring M. We say that M is the Grothendieck ring associated to the semi-ring $\widehat{M}$.

1.2.2 The Burnside ring as a B-ring

Let G be a finite group. The isomorphism classes of finite G-sets form a commutative semi-ring with unit, with addition given by disjoint union and multiplication given by cartesian product. The Burnside ring $A(G)$ is the Grothendieck ring associated to this semi-ring. For a finite G-set X, we write $[X]$ for the isomorphism class of X in $A(G)$. We will first recall some basic facts about Burnside rings; for proofs and further details see [24] (Chapter 1).

Let $\mathrm{ccs}(G)$ denote the set of conjugacy classes of subgroups of G, and for a subgroup $H \subset G$ write (H) for the corresponding conjugacy class. Each finite transitive G-set is isomorphic to one of the form G/H for some unique $(H) \in \mathrm{ccs}(G)$, and the set

$$\{[G/H] \mid (H) \in \mathrm{ccs}(G)\}$$

is a basis for $A(G)$ as an abelian group.

For subgroups $H, K \subset G$, the orbits of G on $G/H \times G/K$ can be put into 1-1 correspondence with the set of double cosets $H\backslash G/K$ as follows. For any orbit O we can choose a representative of the form (H, gK) in O, for some $g \in G$. Two representatives (H, g_1K) and (H, g_2K) belong to the same orbit if and only if there exists a $h \in H$ with $hg_1K = g_2K$, i.e. if and only if $Hg_1K = Hg_2K$. Moreover, for an orbit O containing an element of the form (H, gK) for $g \in G$, note that $H \cap gKg^{-1}$ is the stabilizer of (H, gK), and so O is isomorphic as a G-set to $G/(H \cap gKg^{-1})$.

The multiplication operation of $A(G)$ is then given on the transitive G-sets by

$$[G/H] \cdot [G/K] = \sum_{HgK \in H\backslash G/K} [G/(H \cap gKg^{-1})].$$

Rather than working with double cosets, we will instead embed $A(G)$ in a ring where the multiplication operation is easier to compute.

For a G-set X and subgroup $H \subset G$, write X^H for the set of $x \in X$ fixed by each $h \in H$. For subgroups $H, J \subset G$, say that H is subconjugate to J if H is conjugate to a subgroup of J by some element of G. For each $(H) \in \mathrm{ccs}(G)$ we have a well-defined homomorphism of rings

$$\pi_{(H)} : A(G) \to \mathbb{Z}$$

known as the mark homomorphism associated to (H), given by putting

$$\pi_{(H)}([G/J]) = |(G/J)^H|$$

for $J \subset G$ and extending linearly. Note that $|(G/J)^H| \neq 0$ if and only if H is subconjugate J, and that $|(G/H)^H| = [N_G H : H]$, where $N_G H$ is the normalizer of H in G.

Combining the maps $\pi_{(H)}$ gives an injective homomorphism

$$\pi : A(G) \to \prod_{(H)\in \mathrm{ccs}(G)} \mathbb{Z},$$

which allows us to regard $A(G)$ as a subring of

$$\mathrm{Gh}(\mathrm{ccs}(G)) = \prod_{(H)\in \mathrm{ccs}(G)} \mathbb{Z},$$

the 'ghost ring' of G.

For $H \subset G$, write π_H for $\pi_{(H)}$; for $a \in A(G)$, write $a(H)$ for $\pi_H(a)$.

Lemma 1.11. The embedding $A(G) \subset \mathrm{Gh}(\mathrm{ccs}(G))$ defines a B-ring.

Proof. We need to show that for non-conjugate subgroups $H, J \subset G$ we can find an $a \in A(G)$ with $a(H) \neq 0$ and $a(J) = 0$. Now if J is not subconjugate to H, then $a = [G/H]$ suffices. Otherwise, if J is subconjugate to H then H is not subconjugate to J, and putting $a = [N_G J : J][G/G] - [G/J]$ we have $a(J) = 0$ and $a(H) = [N_G J : J] \neq 0$. □

It follows that we have $A(G)$-modules $\mathbb{Z}_{(H)}$ for each $(H) \in \mathrm{ccs}(G)$, and all rank one modules of the Burnside ring are of this form. Similarly we have prime ideals $q_{A(G)}((H), P)$ for $P \in \mathrm{Spec}\,\mathbb{Z}$, and all prime ideals are of this form. For $H \subset G$ write $\mathbb{Z}_H$ for $\mathbb{Z}_{(H)}$ and $q(H, P)$ for $q_{A(G)}((H), P)$. For $J \subset G$ not conjugate to H, write $d(H, J)$ for $d_{A(G)}((H), (J))$.

For p a rational prime and $H \subset G$, write $O^p(H)$ for the smallest normal subgroup N of H such that H/N is a p-group. We recall the following result due to Dress.

Lemma 1.12 (Proposition 1 of [10])**.** Let H, J be subgroups of G and p a rational prime. Then $q(H, (p)) = q(J, (p))$ if and only if $O^p(H)$ is conjugate to $O^p(J)$.

Lemma 1.12 together with Proposition 1.9 and Proposition 1.7 then gives the following description of the degree 1 cohomology of the Burnside ring.

Theorem 1.13. Let H, J be subgroups of G. Then $\mathrm{Ext}^1_{A(G)}(\mathbb{Z}_H, \mathbb{Z}_J)$ is non-zero if and only if $(H) \neq (J)$ and $(O^p(H)) = (O^p(J))$ for some p, in which case $\mathrm{Ext}^1_{A(G)}(\mathbb{Z}_H, \mathbb{Z}_J)$ has a unique p-power summand.

Example 1.14. Let p be a rational prime and let $G = C_p$, the cyclic group of order p. Write e for the trivial subgroup. As an abelian group, we have

$$A(G) = \mathbb{Z}[G/G] + \mathbb{Z}[G/e].$$

Now $\pi_G([G/G]) = \pi_e([G/G]) = 1$, and $\pi_G([G/e]) = 0, \pi_e([G/e]) = p$. The embedding $\pi : A(G) \to \mathbb{Z} \times \mathbb{Z}$ is then given by

$$\pi([G/G]) = (1,1),$$
$$\pi([G/e]) = (0,p),$$

and for the remainder of this example we will regard $A(G)$ as the subring of $\mathbb{Z} \times \mathbb{Z}$ spanned by these elements. It is clear that for $(a,b) \in \mathbb{Z} \times \mathbb{Z}$ we have $(a,b) \in A(G)$ if and only if $a \equiv b \mod p$. Then $d(G,e) = p$, and so $\mathrm{Ext}^1_{A(G)}(\mathbb{Z}_G, \mathbb{Z}_e) \simeq \mathbb{Z}_e/p\mathbb{Z}_e$.

Let K_G be the kernel of the surjective $A(G)$-module map $A(G) \to \mathbb{Z}_G$ defined by putting $1_{A(G)} = (1,1) \mapsto 1$. K_G is cyclic, generated by $(0,p)$, and so we have a surjection $\alpha : A(G) \to K_G$ defined by putting $(1,1) \mapsto (0,p)$. The kernel of α is then generated by $(p,0)$, and we let β be the surjection $A(G) \to ((p,0))$ defined by putting $(1,1) \mapsto (p,0)$. Then $\ker\beta = K_G$, and so continuing on in this manner we obtain a free $A(G)$-module resolution of $\mathbb{Z}_G$ given by

$$\ldots \xrightarrow{\beta} A(G) \xrightarrow{\alpha} A(G) \xrightarrow{\beta} A(G) \xrightarrow{\alpha} A(G) \to \mathbb{Z}_G \to 0.$$

Applying $\mathrm{Hom}_{A(G)}(-, \mathbb{Z}_e)$ to the above resolution, and writing $\overline{\alpha}$ and $\overline{\beta}$ for $\mathrm{Hom}_{A(G)}(\alpha, \mathbb{Z}_e) : \mathrm{Hom}_{A(G)}(A(G), \mathbb{Z}_e) \to \mathrm{Hom}_{A(G)}(A(G), \mathbb{Z}_e)$ and $\mathrm{Hom}_{A(G)}(\beta, \mathbb{Z}_e) : \mathrm{Hom}_{A(G)}(A(G), \mathbb{Z}_e) \to \mathrm{Hom}_{A(G)}(A(G), \mathbb{Z}_e)$, we obtain a chain complex

$$0 \to \mathrm{Hom}_{A(G)}(A(G), \mathbb{Z}_e) \xrightarrow{\overline{\alpha}} \mathrm{Hom}_{A(G)}(A(G), \mathbb{Z}_e) \xrightarrow{\overline{\beta}} \mathrm{Hom}_{A(G)}(A(G), \mathbb{Z}_e) \xrightarrow{\overline{\alpha}} \ldots$$

Now $\mathrm{Hom}_{A(G)}(A(G), \mathbb{Z}_e)$ is the set of maps $\phi_m : A(G) \to \mathbb{Z}_e$, $m \in \mathbb{Z}$, where $\phi_m(1_{A(G)}) = m$. Then

$$(\overline{\alpha}(\phi_m))(1_{A(G)}) = \phi_m(0,p) = pm,$$

and so $\mathrm{im}\,\overline{\alpha} = (\phi_{pm})_{m \in \mathbb{Z}}$. Moreover,

$$(\overline{\beta}(\phi_m))(1_{A(G)}) = \phi_m(p,0) = 0,$$

and so $\ker\overline{\beta} = \mathrm{Hom}_{A(G)}(A(G), \mathbb{Z}_e)$. Since $\mathrm{Hom}_{A(G)}(A(G), \mathbb{Z}_e)$ is isomorphic to $\mathbb{Z}_e$, it then follows that

$$\mathrm{Ext}^l_{A(G)}(\mathbb{Z}_G, \mathbb{Z}_e) \simeq \begin{cases} \mathbb{Z}_e/p\mathbb{Z}_e & \text{if } l \text{ odd} \\ 0 & \text{if } l \text{ even.} \end{cases}$$

Similarly, applying $\mathrm{Hom}_{A(G)}(-,\mathbb{Z}_G)$ to the above resolution we obtain

$$\mathrm{Ext}^l_{A(G)}(\mathbb{Z}_G,\mathbb{Z}_G) \simeq \begin{cases} 0 & \text{if } l \text{ odd} \\ \mathbb{Z}_G/p\mathbb{Z}_G & \text{if } l > 0 \text{ even.} \end{cases}$$

We also obtain a resolution of $\mathbb{Z}_e$ by swapping the positions of the maps α and β in the resolution for $\mathbb{Z}_G$. Then applying $\mathrm{Hom}_{A(G)}(-,\mathbb{Z}_e)$ and $\mathrm{Hom}_{A(G)}(-,\mathbb{Z}_G)$ we obtain

$$\mathrm{Ext}^l_{A(G)}(\mathbb{Z}_e,\mathbb{Z}_e) \simeq \begin{cases} 0 & \text{if } l \text{ odd} \\ \mathbb{Z}_e/p\mathbb{Z}_e & \text{if } l > 0 \text{ even,} \end{cases}$$

and

$$\mathrm{Ext}^l_{A(G)}(\mathbb{Z}_e,\mathbb{Z}_G) \simeq \begin{cases} \mathbb{Z}_G/p\mathbb{Z}_G & \text{if } l \text{ odd} \\ 0 & \text{if } l \text{ even.} \end{cases}$$

1.3 Higher Ext groups and Tor

Recall that for a commutative ring R and R-module N, the functor $-\otimes_R N : R\text{-Mod} \to R\text{-Mod}$ is right exact. For each non-negative integer l, let $\mathrm{Tor}^R_l(-,N)$ denote the lth left derived functor of $\otimes_R N$. For another R-module M, write $\mathrm{Tor}^R_l(M,N)$ for $\mathrm{Tor}^R_l(-,N)(M)$. For further details see [25] (Chapter 2).

Lemma 1.15. Let $R \subset \mathrm{Gh}(I)$ be a B-ring, $i \in I$. Then $\mathbb{Z}_i \otimes_R \mathbb{Z}_i \simeq \mathbb{Z}_i$. If $j \in I$ is distinct from i, then $\mathbb{Z}_i \otimes_R \mathbb{Z}_j \simeq \mathbb{Z}_i/d(i,j)\mathbb{Z}_i \simeq \mathbb{Z}_j/d(i,j)\mathbb{Z}_j$.

Proof. Suppose $i = j$, and let K_i be the kernel of the map $\pi_i : R \to \mathbb{Z}_i$. Note that $R/K_i \otimes_R R/K_i \simeq R/K_i$ ([5] Section 4.1, Corollary 2). Then

$$\mathbb{Z}_i \otimes_R \mathbb{Z}_i \simeq R/K_i \otimes R/K_i \simeq R/K_i \simeq \mathbb{Z}_i.$$

If $i \neq j$, then for $m, m' \in \mathbb{Z}$, we have $m \otimes m' = mm' \otimes 1$ in $\mathbb{Z}_i \otimes \mathbb{Z}_j$, and so we have a surjective homomorphism of R-modules $\theta : \mathbb{Z}_i \to \mathbb{Z}_i \otimes \mathbb{Z}_j$ given by $m \mapsto m \otimes 1$. Put $d = d(i,j)$ and let $r \in R$ be such that $r(i) = d$ and $r(j) = 0$. Then

$$\theta(d) = d \otimes 1 = r \cdot 1 \otimes 1 = 1 \otimes r \cdot 1 = 0.$$

Moreover if m is a positive integer with $m < d$ and $\theta(m) = 0$, then there exists $r \in R$ with $r(i) \mid m$ and $r(j) = 0$. But then $d \mid m$, a contradiction, and so $\ker\theta = (d(i,j))$ and $\mathbb{Z}_i \otimes_R \mathbb{Z}_j \simeq \mathbb{Z}_i/d(i,j)\mathbb{Z}_i$. □

Recalling Proposition 1.7, we note that for $i \neq j$ we have an isomorphism

$$\mathrm{Tor}^R_0(\mathbb{Z}_i,\mathbb{Z}_j) \simeq \mathrm{Ext}^1_R(\mathbb{Z}_i,\mathbb{Z}_j).$$

We will show that this also holds in the next degree up. In order to do this we first need to identify a family of injective R-modules. For $i \in I$, write $\mathbb{Q}_i$ for the R-module structure on the field $\mathbb{Q}$ given by $r \cdot q = r(i)q$.

Lemma 1.16. The R-modules $\mathbb{Q}_i$ are injective.

Proof. By Baer's criterion ([7] Theorem 57.14), it is sufficient to show that for any ideal $M \subset R$ and homomorphism of R-modules $g : M \to \mathbb{Q}_i$, we can extend g to a homomorphism $R \to \mathbb{Q}_i$.

Let

$$d = \min\{m(i) \mid m \in M, m(i) > 0\},$$

and let $m' \in M$ be an element with $m'(i) = d$. Note that $d \mid m(i)$ for each $m \in M$. Define $\hat{g} : R \to \mathbb{Q}_i$ by

$$\hat{g}(r) = \frac{\pi_i(r)g(m')}{d}.$$

Certainly $\hat{g}$ is a homomorphism of R-modules, and so it remains to show that $\hat{g}|_M = g$. By Lemma 1.8, g factors through $\pi_i|_M : M \to \mathbb{Z}_i$. It follows that if $m \in M$ is such that $m(i) = tm'(i)$ for some $t \in \mathbb{Z}$, then $g(m) = tg(m')$. Then for such an $m \in M$,

$$\hat{g}(m) = \frac{\pi_i(m)g(m')}{d} = \frac{tdg(m')}{d} = tg(m') = g(m),$$

and so $\hat{g}$ is an extension of g. □

We also introduce the notion of an i-special R-module as follows.

Definition 1.17. Let M be an R-module and $i \in I$ such that for each $r \in R$ and $m \in M$ we have

$$r \cdot m = r(i)m.$$

Then M is said to be i-special.

Example 1.18. Let M be an R-module and $(\mathcal{F}_\bullet, \partial_\bullet)$ a free R-module resolution of M, where $\mathcal{F}_l = R^{\oplus n_l}$. Applying $\mathrm{Hom}_R(-, \mathbb{Z}_i)$ to $\mathcal{F}_\bullet$ then gives a chain complex where each term is of the form $\mathrm{Hom}_R(R^{\oplus n_l}, \mathbb{Z}_i) \simeq \mathbb{Z}_i^{\oplus n_l}$. Since $\mathrm{Ext}^l_R(M, \mathbb{Z}_i)$ is a subquotient of such a module, it follows that each $\mathrm{Ext}^l_R(M, \mathbb{Z}_i)$ is i-special.

Similarly, each $\mathrm{Ext}^l_R(\mathbb{Z}_i, M)$ is i-special. It follows that the modules $\mathrm{Ext}^l_R(\mathbb{Z}_i, \mathbb{Z}_j)$ are both i-special and j-special. Since we can choose an $r \in R$ with $r(i) = d(i,j)$ and $r(j) = 0$, it follows that $d(i,j)$ annihilates the modules $\mathrm{Ext}^l_R(\mathbb{Z}_i, \mathbb{Z}_j)$.

Lemma 1.19. Let M, N be i-special R-modules for some $i \in I$. Then

i. $\mathrm{Hom}_R(M, N) = \mathrm{Hom}_{\mathbb{Z}}(M, N)$;

ii. $M \otimes_R N = M \otimes_{\mathbb{Z}} N$;

iii. $M \otimes_R \mathbb{Z}_i \simeq M$;

where the R-module structure on $\mathrm{Hom}_{\mathbb{Z}}(M, N)$ and $M \otimes_{\mathbb{Z}} N$ is given by $r \in R$ acting by $r(i)$.

Proof. If $\phi : M \to N$ is a homomorphism of R-modules, then it is certainly a homomorphism of $\mathbb{Z}$-modules. And if $\phi' : M \to N$ is a homomorphism of $\mathbb{Z}$-modules, then for any $r \in R$ and $m \in M$,

$$\phi'(r \cdot m) = \phi'(r(i)m) = r(i)\phi'(m) = r \cdot \phi'(m)$$

and so ϕ is a homomorphism of R-modules. This proves (i), and the proof for (ii) is similar. For (iii), since $\mathbb{Z}_i$ is i-special, we have

$$M \otimes_R \mathbb{Z}_i = M \otimes_{\mathbb{Z}} \mathbb{Z}_i \simeq M.$$

□

Proposition 1.20. Let $R \subset \mathrm{Gh}(I)$ be a B-ring, and let $i, j \in I$. Then

$$\mathrm{Ext}^2_R(\mathbb{Z}_i, \mathbb{Z}_j) \simeq \mathrm{Tor}^R_1(\mathbb{Z}_i, \mathbb{Z}_j).$$

Proof. For each $l \in I$ write K_l for the kernel of $\pi_l : R \to \mathbb{Z}_l$. Dimension shifting (see [4], Proposition 2.5.5) with the short exact sequence

$$0 \to K_i \to R \to \mathbb{Z}_i \to 0 \qquad (\dagger)$$

gives isomorphisms

$$\mathrm{Ext}^l_R(\mathbb{Z}_i, \mathbb{Z}_j) \simeq \mathrm{Ext}^{l-1}_R(K_i, \mathbb{Z}_j)$$

for $l \geq 2$ and any i, j. Let

$$0 \to \mathbb{Z}_i \to \mathbb{Q}_i \to \mathbb{Q}_i/\mathbb{Z}_i \to 0$$

be the short exact sequence associated with the inclusion $\mathbb{Z}_i \hookrightarrow \mathbb{Q}_i$. Since $\mathbb{Q}_i$ is injective, $\mathrm{Ext}^1_R(K_i, \mathbb{Q}_i) = 0$, and so the long exact sequence arising from applying $\mathrm{Hom}_R(K_i, -)$ to the above short exact sequence begins as

$$0 \longrightarrow \mathrm{Hom}_R(K_i, \mathbb{Z}_i) \longrightarrow \mathrm{Hom}_R(K_i, \mathbb{Q}_i) \longrightarrow \mathrm{Hom}_R(K_i, \mathbb{Q}_i/\mathbb{Z}_i) \longrightarrow \mathrm{Ext}^1_R(K_i, \mathbb{Z}_i) \longrightarrow 0.$$

By Lemma 1.8, $\mathrm{Hom}_R(K_i, \mathbb{Q}_i) = 0$, and we obtain

$$\mathrm{Ext}^2_R(\mathbb{Z}_i, \mathbb{Z}_i) \simeq \mathrm{Ext}^1_R(K_i, \mathbb{Z}_i) \simeq \mathrm{Hom}_R(K_i, \mathbb{Q}_i/\mathbb{Z}_i).$$

Moreover, it is clear that any $\phi \in \mathrm{Hom}_R(K_i, \mathbb{Q}_i/\mathbb{Z}_i)$ must vanish on K_i^2, so we can instead consider the R-module $\mathrm{Hom}_R(K_i/K_i^2, \mathbb{Q}_i/\mathbb{Z}_i)$.

For $z \in K_i$, write $[z]$ for the element $z + K_i^2$ in K_i/K_i^2. Note that for $r \in R$, we have $r - r(i) \in K_i$, and so

$$[r - r(i)\,\mathrm{id}_R][z] = 0$$

in K_i/K_i^2 for any $z \in K_i$. Hence

$$r \cdot [z] = [r(i)\,\mathrm{id}_R\, z] = r(i)[z],$$

and K_i/K_i^2 is i-special. Since the R-module $\mathbb{Q}_i/\mathbb{Z}_i$ is also i-special, we then have

$$\mathrm{Hom}_R(K_i/K_i^2, \mathbb{Q}_i/\mathbb{Z}_i) = \mathrm{Hom}_{\mathbb{Z}}(K_i/K_i^2, \mathbb{Q}_i/\mathbb{Z}_i).$$

Now for a prime p and $\alpha \in \mathbb{N}$ we have $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^\alpha\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \simeq \mathbb{Z}/p^\alpha\mathbb{Z}$. Since K_i/K_i^2 is finite, and since the Hom functor respects direct sums, it follows that $\mathrm{Hom}_{\mathbb{Z}}(K_i/K_i^2, \mathbb{Q}_i/\mathbb{Z}_i) \simeq K_i/K_i^2$.

On the other hand, applying $- \otimes_R \mathbb{Z}_i$ to (†) and noting that $\mathrm{Tor}_1^R(R, \mathbb{Z}_i) = 0$ since R is projective, we obtain an exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^R(\mathbb{Z}_i, \mathbb{Z}_i) \longrightarrow K_i \otimes \mathbb{Z}_i \longrightarrow R \otimes \mathbb{Z}_i \longrightarrow \mathbb{Z}_i \otimes \mathbb{Z}_i \longrightarrow 0.$$

It is immediate that the map $R \otimes \mathbb{Z}_i \to \mathbb{Z}_i \otimes \mathbb{Z}_i$ is an isomorphism, and so

$$\mathrm{Tor}_1^R(\mathbb{Z}_i, \mathbb{Z}_i) \simeq K_i \otimes \mathbb{Z}_i.$$

Note that if $z_1 z_2 \in K_i^2$, then

$$z_1 z_2 \otimes 1 = z_1 \otimes z_2(i) = 0$$

in $K_i \otimes \mathbb{Z}_i$, and so $K_i \otimes \mathbb{Z}_i \simeq K_i/K_i^2 \otimes \mathbb{Z}_i$. Moreover K_i/K_i^2 is i-special, so by the previous lemma we have $K_i/K_i^2 \otimes \mathbb{Z}_i \simeq K_i/K_i^2$. Then

$$\mathrm{Ext}_R^2(\mathbb{Z}_i, \mathbb{Z}_i) \simeq K_i/K_i^2 \simeq \mathrm{Tor}_1^R(\mathbb{Z}_i, \mathbb{Z}_i)$$

as claimed.

Next consider $\mathrm{Ext}_R^2(\mathbb{Z}_i, \mathbb{Z}_j)$ with $i \neq j$. Returning again to our sequence

$$0 \to \mathbb{Z}_j \to \mathbb{Q}_j \to \mathbb{Q}_j/\mathbb{Z}_j \to 0,$$

and now applying $\mathrm{Hom}_R(K_i, -)$, we identify $\mathrm{Ext}^2(\mathbb{Z}_i, \mathbb{Z}_j)$ as the cokernel of the map

$$\overline{\sigma} : \mathrm{Hom}_R(K_i, \mathbb{Q}_j) \to \mathrm{Hom}_R(K_i, \mathbb{Q}_j/\mathbb{Z}_j)$$

induced by the quotient map $\sigma : \mathbb{Q}_j \to \mathbb{Q}_j/\mathbb{Z}_j$. Again we note that any R-module homomorphism $K_i \to \mathbb{Q}_j$ must vanish on K_iK_j, and similarly for any homomorphism $K_i \to \mathbb{Q}_j/\mathbb{Z}_j$, so we can instead consider the cokernel of the map

$$\mathrm{Hom}_R(K_i/K_iK_j, \mathbb{Q}_j) \to \mathrm{Hom}_R(K_i/K_iK_j, \mathbb{Q}_j/\mathbb{Z}_j).$$

Since the modules $K_i/K_iK_j, \mathbb{Q}_j$, and $\mathbb{Q}_j/\mathbb{Z}_j$ are all j-special, this is the same as considering the cokernel of the map

$$\overline{\sigma}' : \mathrm{Hom}_{\mathbb{Z}}(K_i/K_iK_j, \mathbb{Q}_j) \to \mathrm{Hom}_{\mathbb{Z}}(K_i/K_iK_j, \mathbb{Q}_j/\mathbb{Z}_j).$$

Let K_i/K_iK_j have a free part of rank s as an abelian group and write T for the torsion submodule. Now any $\mathbb{Z}$-module homomorphism $K_i/K_iK_j \to \mathbb{Q}_j$ must map the torsion submodule of K_i/K_iK_j to zero, and so $\mathrm{Hom}_{\mathbb{Z}}(K_i/K_iK_j, \mathbb{Q}_j)$ is isomorphic to $\mathbb{Q}_j^{\oplus s}$. Similarly, $\mathrm{Hom}_{\mathbb{Z}}(K_i/K_iK_j, \mathbb{Q}_j/\mathbb{Z}_j)$ is isomorphic to $(\mathbb{Q}_j/\mathbb{Z}_j)^{\oplus s} \oplus T$. By inspection, $\overline{\sigma}'$ maps $\mathbb{Q}_j^{\oplus s}$ onto $(\mathbb{Q}_j/\mathbb{Z}_j)^{\oplus s}$, and so

$$\mathrm{Ext}^2(\mathbb{Z}_i, \mathbb{Z}_j) \simeq T.$$

Moving on to $\mathrm{Tor}_1^R(\mathbb{Z}_i, \mathbb{Z}_j)$, from the long exact sequence obtained from applying $- \otimes_R \mathbb{Z}_j$ to (†) we have an exact sequence

$$0 \to \mathrm{Tor}_1^R(\mathbb{Z}_i, \mathbb{Z}_j) \to K_i \otimes \mathbb{Z}_j \xrightarrow{\alpha} R \otimes \mathbb{Z}_j \to \mathbb{Z}_i \otimes \mathbb{Z}_j \to 0,$$

and so

$$\mathrm{Tor}_1^R(\mathbb{Z}_i, \mathbb{Z}_j) \simeq \ker(\alpha).$$

Suppose $z \otimes n$ a non-zero element of $\ker \alpha$. Then $z(j) = 0$ and so $z \in K_i \cap K_j$. By the proof of Proposition 1.7, there exists some $N \in \mathbb{N}$ with $Nz \in K_iK_j$. Then $N(z \otimes n) = 0$ in $K_i \otimes \mathbb{Z}_j$, and so $z \otimes n$ is torsion. Conversely, if some non-zero $z \otimes n \in K_i \otimes \mathbb{Z}_j$ is torsion then so is its image in $R \otimes \mathbb{Z}_j$. But $R \otimes \mathbb{Z}_j \simeq \mathbb{Z}_j$ is torsion-free, so $z \otimes n \in \ker(\alpha)$. So $\ker \alpha$ is the torsion submodule of $K_i \otimes \mathbb{Z}_j$. Identifying $K_i \otimes \mathbb{Z}_j$ with $K_i/K_iK_j \otimes \mathbb{Z}_j \simeq K_i/K_iK_j$ then gives

$$\mathrm{Tor}_1^R(\mathbb{Z}_i, \mathbb{Z}_j) \simeq T \simeq \mathrm{Ext}_R^2(\mathbb{Z}_i, \mathbb{Z}_j),$$

for all i, $j \in I$.

□

1.4 Reducing to the modular case

Fix some rational prime p and put $k = \mathbb{F}_p$. For a B-ring $R \subset \mathrm{Gh}(I)$, we have an associated commutative k-algebra $\overline{R} = R \otimes_{\mathbb{Z}} k \simeq R/pR$. For $i \in I$, we have an R-module k_i where r acts on the field k by $r(i)$. For an R-module N, write $\overline{N}$ for the associated $\overline{R}$-module N/pN. If N is annihilated by p, we will also denote the associated $\overline{R}$-module by N.

Lemma 1.21. Let X be a torsion-free R-module and M an R-module annihilated by pR. Then

$$\mathrm{Ext}^l_R(X, M) \simeq \mathrm{Ext}^l_{\overline{R}}(\overline{X}, M)$$

and

$$\mathrm{Tor}^R_l(X, M) \simeq \mathrm{Tor}^{\overline{R}}_l(\overline{X}, M)$$

for each $l \geq 0$.

Proof. First note that for any R-module N, any homomorphism of R-modules $\phi : N \to M$ must vanish on pN, and so we have an induced homomorphism $\overline{\phi} : \overline{N} \to M$. Similarly, any homomorphism of $\overline{R}$-modules $\psi : \overline{N} \to M$ lifts to a homomorphism of R-modules $N \to M$. It follows that

$$\mathrm{Hom}_R(N, M) \simeq \mathrm{Hom}_{\overline{R}}(\overline{N}, M).$$

Let $(\mathcal{F}_\bullet, \partial_\bullet)$ be a free R-module resolution of X. In particular $\mathcal{F}_\bullet$ is a free $\mathbb{Z}$-module resolution of the $\mathbb{Z}$-module X, and so applying $- \otimes_{\mathbb{Z}} k$ gives a chain complex over $\overline{X}$ with homology groups $\mathrm{Tor}^{\mathbb{Z}}_l(X, k)$. But X is torsion-free so the homology groups vanish and the chain complex is exact. Since $\mathcal{F}_\bullet$ is a free R-module resolution, $\mathcal{F}_\bullet \otimes_{\mathbb{Z}} k$ is a free $\overline{R}$-module resolution.

Applying $\mathrm{Hom}_{\overline{R}}(-, M)$ to $\mathcal{F}_\bullet \otimes_{\mathbb{Z}} k$ and computing cohomology then computes the groups $\mathrm{Ext}^l_{\overline{R}}(\overline{X}, M)$. But $\mathrm{Hom}_{\overline{R}}(\overline{R}, M) \simeq \mathrm{Hom}_R(R, M)$, so this is the same as applying $\mathrm{Hom}_R(-, M)$ to $\mathcal{F}_\bullet$ and taking cohomology, i.e. computing the groups $\mathrm{Ext}^l_R(X, M)$. The proof for Tor is analogous. $\square$

For each $i \in I$, we have a short exact sequence of R-modules

$$0 \to \mathbb{Z}_i \xrightarrow{p} \mathbb{Z}_i \to k_i \to 0. \tag{‡}$$

Applying $\mathrm{Hom}_R(\mathbb{Z}_i, -)$ we obtain

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Hom}_R(\mathbb{Z}_i, \mathbb{Z}_i) & \xrightarrow{p} & \mathrm{Hom}_R(\mathbb{Z}_i, \mathbb{Z}_i) & \longrightarrow & \mathrm{Hom}_{\overline{R}}(k_i, k_i) & \\
\longrightarrow & \mathrm{Ext}^1_R(\mathbb{Z}_i, \mathbb{Z}_i) & \xrightarrow{p} & \mathrm{Ext}^1_R(\mathbb{Z}_i, \mathbb{Z}_i) & \longrightarrow & \mathrm{Ext}^1_{\overline{R}}(k_i, k_i) & \\
\longrightarrow & \mathrm{Ext}^2_R(\mathbb{Z}_i, \mathbb{Z}_i) & \xrightarrow{p} & \mathrm{Ext}^2_R(\mathbb{Z}_i, \mathbb{Z}_i) & \longrightarrow & \dots &
\end{array}$$

where we make use of the additivity of $\mathrm{Ext}^l_R(\mathbb{Z}_i, -)$ for each $l \geq 1$ to identify each map $\mathrm{Ext}^l_R(\mathbb{Z}_i, \mathbb{Z}_i) \to \mathrm{Ext}^l_R(\mathbb{Z}_i, \mathbb{Z}_i)$ as multiplication by p, and make use of Lemma 1.21 above to replace each $\mathrm{Ext}^l_R(\mathbb{Z}_i, k_i)$ with $\mathrm{Ext}^l_{\overline{R}}(k_i, k_i)$.

Recall that by Example 1.18, each $\mathrm{Ext}^l_R(\mathbb{Z}_i, \mathbb{Z}_i)$ is i-special. Then any decomposition of $\mathrm{Ext}^l_R(\mathbb{Z}_i, \mathbb{Z}_i)$ as an abelian group is also a decomposition of R-modules. For each $l \geq 1$ and each rational prime q, let $M_{l,q}$ be the submodule of $\mathrm{Ext}^l_R(\mathbb{Z}_i, \mathbb{Z}_i)$ annihilated by some power of q. Since $\mathrm{Ext}^l_R(\mathbb{Z}_i, \mathbb{Z}_i)$ is finite, it follows that we have a decomposition of R-modules

$$\mathrm{Ext}^l_R(\mathbb{Z}_i, \mathbb{Z}_i) \simeq \bigoplus_q M_{l,q}$$

where the sum is over all rational primes q.

For $l \geq 1$, let a_l be the summand rank of $M_{l,p}$ as defined in the remark following Corollary 1.4, and let b_l be the k-dimension of $\mathrm{Ext}^l_{\overline{R}}(k_i, k_i)$. Note that for $l \geq 1$ the kernel of the map $\left(\mathrm{Ext}^l_R(\mathbb{Z}_i, \mathbb{Z}_i) \xrightarrow{p} \mathrm{Ext}^l_R(\mathbb{Z}_i, \mathbb{Z}_i)\right)$ is a k-vector space with dimension a_l. Moreover, the cokernel of this map is also a k-vector space of dimension a_l, and so the image of the connecting homomorphism $\mathrm{Ext}^l_{\overline{R}}(k_i, k_i) \to \mathrm{Ext}^{l+1}_R(\mathbb{Z}_i, \mathbb{Z}_i)$ has dimension $b_l - a_l$. Hence

$$\begin{aligned} a_{l+1} &= \dim_k \ker \left(\mathrm{Ext}^{l+1}_R(\mathbb{Z}_i, \mathbb{Z}_i) \xrightarrow{p} \mathrm{Ext}^{l+1}_R(\mathbb{Z}_i, \mathbb{Z}_i)\right) \\ &= \dim_k \mathrm{im} \left(\mathrm{Ext}^l_{\overline{R}}(k_i, k_i) \to \mathrm{Ext}^{l+1}_R(\mathbb{Z}_i, \mathbb{Z}_i)\right) \\ &= b_l - a_l \end{aligned}$$

for $l \geq 1$. In order to determine the sequence a_l, it is then sufficient to compute the sequence b_l. Note that by Proposition 1.7, $a_1 = 0$.

Similarly, applying $\mathrm{Hom}_R(\mathbb{Z}_i, -)$ to the corresponding short exact sequence for $j \in I$ with $j \neq i$, writing c_l for the summand rank of the p-part of $\mathrm{Ext}^l_R(\mathbb{Z}_i, \mathbb{Z}_j)$, and writing d_l for the dimension of $\mathrm{Ext}^l_{\overline{R}}(k_i, k_j)$, we obtain

$$c_{l+1} = d_l - c_l$$

for $l \geq 1$. By Proposition 1.7, $c_1 = 1$ if $p \mid d(i, j)$ and $c_1 = 0$ otherwise.

If we instead apply the (covariant) functor $- \otimes_R \mathbb{Z}_j$ to (‡), we obtain a long exact sequence

$$\begin{aligned} \ldots &\longrightarrow \mathrm{Tor}^R_2(\mathbb{Z}_i, \mathbb{Z}_i) \longrightarrow \mathrm{Tor}^{\overline{R}}_2(k_i, k_i) \\ &\longrightarrow \mathrm{Tor}^R_1(\mathbb{Z}_i, \mathbb{Z}_i) \xrightarrow{p} \mathrm{Tor}^R_1(\mathbb{Z}_i, \mathbb{Z}_i) \longrightarrow \mathrm{Tor}^{\overline{R}}_1(k_i, k_i) \\ &\longrightarrow \mathbb{Z}_i \otimes \mathbb{Z}_i \xrightarrow{p} \mathbb{Z}_i \otimes \mathbb{Z}_i \longrightarrow k_i \otimes \mathbb{Z}_i \longrightarrow 0. \end{aligned}$$

Putting

$$\mathrm{Tor}^R_l(\mathbb{Z}_i, \mathbb{Z}_i) = \bigoplus_q N_{l,q},$$

and writing z_l for the summand rank of $N_{l,p}$, and y_l for the dimension of $\mathrm{Tor}_l^{\overline{R}}(k_i, k_i)$, this time we obtain

$$z_l = y_{l+1} - z_{l+1}$$

for $l \geq 1$.

Repeating with $-\otimes \mathbb{Z}_j$ for $j \in I$ with $j \neq i$, and writing x_l for the summand rank of the p-part of $\mathrm{Tor}_l^R(\mathbb{Z}_i, \mathbb{Z}_j)$, and w_l for the dimension of $\mathrm{Tor}_l^{\overline{R}}(k_i, k_j)$, we obtain

$$x_l = w_{l+1} - x_{l+1}$$

for $l \geq 1$.

Example 1.22. Let $G = C_{p^2}$, the cyclic group of order p^2. Let e denote the trivial subgroup and H the subgroup of index p. The Burnside ring $A(G)$ can be regarded as the subring of $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ generated by $t_1 = (1,1,1)$, $t_2 = (0,p,p)$, $t_3 = (0,0,p^2)$. As a k-vector space, $\overline{A(G)} = A(G) \otimes k$ then has a basis consisting of $\overline{t_1}, \overline{t_2}, \overline{t_3}$, where we write $\overline{t_i}$ for the image of t_i in $\overline{A(G)}$.

Now $t_2^2 = (0, p^2, p^2) = pt_2$, $t_2t_3 = pt_3$, and $t_3^2 = p^2t_3$; hence $\overline{t_2}^2 = \overline{t_2t_3} = \overline{t_3}^2 = 0$. It follows that $\overline{A(G)}$ is isomorphic to the k-algebra $A = k[x,y]/(x,y)^2$. Moreover since $p \mid d(G,H)$ and $p \mid d(H,e)$ we have $k_G \simeq k_H \simeq k_e$, and we can denote the module simply by k. In order to determine the summand rank of the p-part of the groups $\mathrm{Ext}^l_{A(G)}(\mathbb{Z}_U, \mathbb{Z}_V)$ for $U, V \in \{e, H, G\}$ and $l \geq 1$, it is then sufficient to compute $\mathrm{Ext}^l_A(k,k)$ for $l \geq 1$.

Let K be the kernel of the quotient map $A \to A/(x,y) \simeq k$. Then $K \simeq (x,y) \simeq k^{\oplus 2}$ as an A-module, and we have a surjection $A^{\oplus 2} \to K$ with kernel $K^{\oplus 2}$. Continuing in this manner we obtain a resolution

$$\ldots \to A^{\oplus 2^l} \to \ldots \to A^{\oplus 2} \to A \to k \to 0.$$

Applying $\mathrm{Hom}_A(-,k)$ gives a chain complex where all maps are identically zero, and so $\mathrm{Ext}^l_A(k,k) = k^{\oplus 2^l}$ for each $l \geq 0$.

Now suppose $U \in \{e, H, G\}$, let a_l be the summand rank of the p-part of $\mathrm{Ext}^l_{A(G)}(\mathbb{Z}_U, \mathbb{Z}_U)$, and let $b_l = \dim \mathrm{Ext}^l_{\overline{A(G)}}(k_U, k_U)$. Computing a_l corresponds to solving the recurrence $a_{l+1} = b_l - a_l$ with $a_1 = 0$. Since we already know that $b_l = 2^l$, we can immediately solve the recurrence to obtain

$$a_l = \frac{2^l + 2(-1)^l}{3}$$

for $l \geq 1$. Similarly, for distinct $U, V \in \{e, H, G\}$, let c_l be the summand rank of the p-part of $\mathrm{Ext}^l_{A(G)}(\mathbb{Z}_U, \mathbb{Z}_V)$, and let $d_l = \dim \mathrm{Ext}^l_{\overline{A(G)}}(k_U, k_V)$. Computing c_l corresponds to solving the recurrence $c_{l+1} = d_l - c_l$ where now $c_1 = 1$. Again $d_l = 2^l$, and solving the recurrence gives

$$c_l = \frac{2^l - (-1)^l}{3}$$

for $l \geq 0$.

1.5 B-rings modulo a prime

Let k be any field. For a commutative, finite-dimensional k-algebra A we have a decomposition of A as a left A-module

$$A = \bigoplus_{i=1}^{n} A_i$$

into a finite direct sum of n indecomposable projective A-modules A_i, for some $n \in \mathbb{N}$. Since A is commutative, this is a decomposition of A into indecomposable local k-algebras. By the usual block theory considerations (see [3] II.5), studying the cohomology of A is reduced to studying the cohomology of the individual algebras A_i.

Let I be a finite set and let $R \subset \mathrm{Gh}(I)$ be a B-ring. Define a relation $\sim_p'$ on I by putting $i \sim_p' j$ if and only if $p \mid d(i,j)$ for $i \neq j$. Note that by the definition of $d(i,j)$ this relation is symmetric and transitive, and we write $\sim_p$ for the equivalence relation defined by taking its reflexive closure. Let $\mathcal{E}$ denote the set of equivalence classes of I with respect to $\sim_p$. Let $k = \mathbb{F}_p$ and recall that we write $\overline{R}$ for the k-algebra $R \otimes_{\mathbb{Z}} k$. For an equivalence class $E \in \mathcal{E}$, write k_E for the (well-defined) R-module which is k as an abelian group and where $r \cdot m = r(i)m$ for i any element of E. Since pR annihilates k_E for each $E \in \mathcal{E}$, we can also regard k_E as a module for $\overline{R}$.

Lemma 1.23. For each $E \in \mathcal{E}$ there exists an $r \in R$ such that $r(i) \equiv 1 \mod p$ for each $i \in E$ and $r(j) \equiv 0 \mod p$ for each $j \notin E$.

Proof. For each equivalence class E' distinct from E, we have an $r_{E'} \in R$ with $r_{E'}(i) \not\equiv r_{E'}(j) \mod p$ for $i \in E$ and $j \in E'$. Subtracting $r_{E'}(j)$ if required, we can assume $r_{E'}(j) = 0$, and hence $p \nmid r_{E'}(i)$. Replacing $r_{E'}$ by $r_{E'}^{p-1}$ if required, we can assume $r_{E'}(i) \equiv 1 \mod p$. Then putting $r = \prod_{\substack{E' \in \mathcal{E} \\ E' \neq E}} r_{E'}$ it is clear that r has the claimed properties. □

Proposition 1.24. We have a surjective homomorphism of $\overline{R}$-modules

$$\theta : \overline{R} \to \prod_{E \in \mathcal{E}} k_E$$

with kernel the radical of $\overline{R}$.

Proof. We have a homomorphism of R-modules

$$R \to \prod_{E \in \mathcal{E}} k_E$$

given by

$$r \mapsto (r(i) \mod p)_{E \in \mathcal{E}}$$

where $i \in E$. By Lemma 1.23 this homomorphism is surjective. The kernel of this homomorphism contains pR, and we write θ for the induced surjective homomorphism of $\overline{R}$-modules. Since $\prod_{E \in \mathcal{E}} k_E \simeq \bigoplus_{E \in \mathcal{E}} k_E$ is a semisimple $\overline{R}$-module, the kernel of θ certainly contains the radical of $\overline{R}$. It remains to show the reverse inclusion.

As in the proof of Proposition 1.3, choose for each $i \in I$ an element $s_i \in R$ such that $s_i(j) \neq 0$ if and only if $j = i$. Let $s_i(i) = p^{t_i} n_i$ where n_i is coprime to p; put $t = \max_i t_i$ and put $N = \prod_{i \in I} n_i$. Let $r \in R$ be such that $r(i) \equiv 0 \mod p$ for each $i \in I$. Putting $q = Nr^{t+1}$, we have that $ps_i(i) \mid q(i)$ for each $i \in I$, and so we can define integers $m_i \in \mathbb{Z}$ by requiring $q(i) = pm_i s_i(i)$. It follows that

$$q = p \cdot \sum_{i \in I} m_i s_i$$

and so $q \in pR$. Then the image of q in $\overline{R}$ is zero, and hence the image of r^{t+1} in $\overline{R}$ is zero, since N is coprime with p.

It follows that the kernel of θ is nilpotent, and hence equal to the radical of $\overline{R}$. □

Corollary 1.25. Let R, I and $\mathcal{E}$ be as above.

i. $\overline{R}$ is the direct sum of $|\mathcal{E}|$ indecomposable k-algebras;

ii. Each indecomposable k-algebra summand of $\overline{R}$ is a local k-algebra with maximal ideal of codimension 1.

iii. the set $\{k_E\}_{E \in \mathcal{E}}$ is a complete irredundant set of simple modules for $\overline{R}$;

iv. for $i, j \in I$, the modules k_i and k_j belong to the same block of $\overline{R}$ if and only if $i \sim_p j$ and hence $k_i \simeq k_j$;

v. the dimension of the block of $\overline{R}$ corresponding to k_i is $|E|$, the size of the $\sim_p$-equivalence class of $i \in I$.

Proof. The only part that does not follow immediately is v. For an equivalence class E, let R' be the B-ring $R' \subset \prod_{i \in E} \mathbb{Z}$ induced by R, and $\pi_E : R \to R'$ the corresponding homomorphism. Then π_E descends to a map $\overline{R} \to \overline{R}'$ and this is clearly a surjection of k-algebras. Since $\overline{R}'$ has a single block of dimension $|E|$, the block of $\overline{R}$ corresponding to E has dimension $\geq |E|$. Since we can do this for each equivalence class in $\mathcal{E}$, the block must have dimension $|E|$. □

Corollary 1.26. Let $R \subset \mathrm{Gh}(I)$ be a B-ring with $p \nmid d(i,j)$ for all distinct $i, j \in I$. Then $\overline{R}$ is semisimple.

Proof. Since $p \nmid d(i,j)$ for all distinct $i, j \in I$, we have $|\mathcal{E}| = |I|$ and the homomorphism θ of Proposition 1.24 is an isomorphism. □

Corollary 1.27. Let $R \subset \mathrm{Gh}(I)$ be a B-ring and let $i, j \in I$ be distinct with $d(i,j) = 1$. Then

$$\mathrm{Ext}^l_R(\mathbb{Z}_i, \mathbb{Z}_j) = 0$$

for all $l \geq 0$.

Proof. We already have the result for $l = 0$ by Lemma 1.5.

Since $p \nmid d(i,j)$, the modules k_i and k_j belong to different blocks of $\overline{R}$. Then $\mathrm{Ext}^l_{\overline{R}}(k_i, k_j) = 0$ for each $l \geq 0$, and so the p-part of $\mathrm{Ext}^l_R(\mathbb{Z}_i, \mathbb{Z}_j)$ is zero for each $l \geq 1$. Since this is true of any prime p, it follows that $\mathrm{Ext}^l_R(\mathbb{Z}_i, \mathbb{Z}_j) = 0$ for all $l \geq 1$. □

Corollary 1.28. Let $R \subset \mathrm{Gh}(I)$ be a B-ring and suppose that distinct elements i, j of I are such that $\{i, j\}$ is an equivalence class for the relation $\sim_p$ on I. Write a_l and c_l for the summand ranks of the p-parts of $\mathrm{Ext}^l_R(\mathbb{Z}_i, \mathbb{Z}_i)$ and $\mathrm{Ext}^l_R(\mathbb{Z}_i, \mathbb{Z}_j)$ respectively. Then

$$a_l = \begin{cases} 0 & \text{if } i \text{ odd} \\ 1 & \text{if } i \text{ even} \end{cases}$$

$$c_l = \begin{cases} 1 & \text{if } i \text{ odd} \\ 0 & \text{if } i \text{ even} \end{cases}$$

for all $l \geq 1$.

Proof. The algebra $\overline{R}$ has an indecomposable 2-dimensional k-algebra summand corresponding to the pair $\{i, j\}$, and this summand has a maximal ideal of codimension 1. Any such k-algebra is isomorphic to $k[x]/(x^2)$. Note that for $n > 1$ and $A = k[x]/(x^n)$, we have a free A-module resolution of k given by

$$\ldots \xrightarrow{\beta} A \xrightarrow{\alpha} A \xrightarrow{\beta} A \xrightarrow{\alpha} A \to k \to 0$$

where $\alpha(1_A) = x$ and $\beta(1_A) = x^{n-1}$. It follows immediately that $\mathrm{Ext}^l_{k[x]/(x^2)}(k, k) = k$ for all $l \geq 0$, i.e. in the notation of Section 1.4 we have $b_l = d_l = 1$ for all $l \geq 1$. The corollary follows by the recurrence relations $a_{l+1} = b_l - a_l$ and $c_{l+1} = d_l - c_l$, together with the conditions $a_1 = 0$, $c_1 = 1$. □

We rephrase the above corollaries in terms of the Burnside ring of a finite group. The first was first shown by Solomon in [22].

Corollary 1.29. Suppose $p \nmid |G|$. Then $\overline{A(G)}$ is semisimple.

Proof. If $p \nmid |G|$ then certainly $p \nmid [N_G H : H] = \pi_H([G/H])$ for each $H \subset G$. Now for any distinct pair $(H), (J)$, we must have that H is not subconjugate to J or J is not subconjugate to H. Suppose H is not subconjugate to J. Then $\pi_H([G/J]) = 0$. But $p \nmid \pi_J([G/J])$, and so $p \nmid d(J, H)$. By Corollary 1.26, $\overline{A(G)}$ is semisimple. □

Recall that for a prime p and $H \subset G$, we write $O^p(H)$ for the smallest normal subgroup $N \lhd H$ such that H/N is a p-group.

Corollary 1.30. Let $H, J \subset G$ be subgroups such that $O^p(H)$ is not conjugate to $O^p(J)$ for any rational prime p. Then

$$\mathrm{Ext}^l_{A(G)}(\mathbb{Z}_H, \mathbb{Z}_J) = 0$$

for all $l \geq 0$.

Proof. By Lemma 1.12 and Proposition 1.9, we have $p \nmid d(H, J)$ for each p, and the result then follows by Corollary 1.27. □

Corollary 1.31. Suppose $p \mid |G|$ and $p^2 \nmid |G|$. Let $\mathcal{E}$ be the set of equivalence classes of the relation $\sim_p$ on $\mathrm{ccs}(G)$, and let $E_1, \ldots, E_q$ be the equivalence classes of cardinality 2. For $H \subset G$ and $l \geq 1$, write $M_{H,l}$ for the p-part of $\mathrm{Ext}^l_{A(G)}(\mathbb{Z}_H, \mathbb{Z}_H)$; for $J \subset G$ not conjugate to H and $l \geq 1$, write $N_{H,J,l}$ for the p-part of $\mathrm{Ext}^l_{A(G)}(\mathbb{Z}_H, \mathbb{Z}_J)$. Then

i. each $E \in \mathcal{E}$ has cardinality ≤ 2,

ii. $M_{H,l}$ is non-zero for some $l \geq 1$ if and only if $H \in E_i$ for some $1 \leq i \leq q$,

iii. $N_{H,J,l}$ is non-zero for some $l \geq 1$ if and only if $\{H, J\} = E_i$ for some $1 \leq i \leq q$,

iv. if $H \in E_i$ for some $1 \leq i \leq q$ then

$$M_{H,l} \simeq \begin{cases} 0 & \text{if } i \text{ odd} \\ \mathbb{Z}/p\mathbb{Z} & \text{if } i \text{ even,} \end{cases}$$

v. if $\{H, J\} = E_i$ for some $1 \leq i \leq q$, then

$$N_{H,J,l} \simeq \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{if } i \text{ odd} \\ 0 & \text{if } i \text{ even.} \end{cases}$$

Proof. Suppose E is an equivalence class of cardinality ≥ 3, and consider distinct conjugacy classes of subgroups $(H), (J), (J')$ in E. Without loss of generality we can assume $O^p(J) = O^p(J') = H$. Then J/H and J'/H are p-groups in $N_G H/H$. Since $p^2 \nmid |G|$, we certainly have $p^2 \nmid |N_G H/H|$, so J/H and J'/H are Sylow p-groups in $N_G H/H$. Then J/H and J'/H are conjugate in $N_G H/H$, and so J and J' are conjugate in G, a contradiction. So each class $E \in \mathcal{E}$ has cardinality at most 2.

For parts ii and iii, we know by Corollary 1.28 that if $H \in E_i$ for some i then $M_{H,l}$ is non-zero whenever l is even, and if $\{H, J\} = E_i$ for some i then $N_{H,J,l}$ is non-zero whenever l is odd. If $H \notin E_i$ for each i, then by part i the block corresponding to k_H is 1-dimensional, and $M_{H,l} = 0$ for all $l \geq 1$. Similarly, if $\{H, J\} \neq E_i$ for each i, then H and J belong to distinct equivalence classes and $N_{H,J,l} = 0$ for all $l \geq 0$.

It remains to show parts iv and v. Again by Corollary 1.28 we know that $M_{H,l}$ has a p-power summand only when l is even, and $N_{H,J,l}$ has a p-power summand only when l is odd. We still need to show that the p-power summand is in fact $\mathbb{Z}/p\mathbb{Z}$. For $N_{H,J,l}$ this follows from Example 1.18, since $d(H, J)$ annihilates $N_{H,J,l}$ and $p^2 \nmid d(H, J)$ since $p^2 \nmid |G|$. For $M_{H,l}$ note that by dimension shifting this is the p-part of $\mathrm{Ext}^{l-1}_{A(G)}(K_H, \mathbb{Z}_H)$ where $K_H = \ker \pi_H$. Since $p^2 \nmid d(H, J')$ for any $J' \subset G$, and since $p \mid d(H, J')$ if and only if $(J') = (J)$, we can construct $s_H \in A(G)$ such that $s_H(J') \neq 0$ if and only if $(H) = (J')$, and such that $p^2 \nmid s_H(H)$. Now s_H annihilates K_H, so it follows that s_H annihilates $\mathrm{Ext}^l_{A(G)}(\mathbb{Z}_H, \mathbb{Z}_H)$, and so a fortiori s_H annihilates $M_{H,l}$. So any p-power summand of $M_{H,l}$ must be of the form $\mathbb{Z}/p\mathbb{Z}$. □

The following is then immediate.

Theorem 1.32. Suppose $|G|$ is square-free. Then for all $H, J \subset G$ and $l > 0$, we have an isomorphism of $A(G)$-modules $\mathrm{Ext}^l_{A(G)}(\mathbb{Z}_H, \mathbb{Z}_J) \simeq \mathrm{Ext}^{l+2}_{A(G)}(\mathbb{Z}_H, \mathbb{Z}_J)$.

In order to consider the case where G does not have square-free order, we need to consider $\mathrm{Ext}^l_{\overline{R}}(k, k)$ for more general k-algebras $\overline{R}$. In the following chapter we consider the problem of computing $\dim \mathrm{Ext}^l_{\overline{R}}(k, k)$ for $\overline{R}$ a commutative local k-algebra.

Chapter 2

Cohomology of commutative local k-algebras

Let k be a field and S a commutative local k-algebra with maximal ideal $\mathcal{M}$ and residue field $S/\mathcal{M} \simeq k$. For $l \geq 0$, put $a_l = \dim \operatorname{Ext}_S^l(k,k)$. In this chapter we consider the problem of studying the sequence $(a_l)_{l \in \mathbb{N}}$.

In the first section we observe that we can just as well consider $\dim \operatorname{Tor}_l^S(k,k)$. Next we consider the case of the Burnside ring $A(G)$ of a finite group G, and show that if $p^2 \mid |G|$ then $\overline{A(G)} = A(G) \otimes_{\mathbb{Z}} \mathbb{F}_p$ has an indecomposable summand with corresponding sequence (a_l) unbounded, and hence there exist subgroups $H, J \subset G$ such that the groups $\operatorname{Ext}_R^l(\mathbb{Z}_H, \mathbb{Z}_J)$ have unbounded rank.

Turning then to the problem of actually computing the sequence (a_l) for a given k-algebra, we consider the family of k-algebras of the form $\dim \mathcal{M}^2 = 1$ and $\mathcal{M}^3 = 0$, and obtain an explicit description of the numbers a_l for this family.

In obtaining our results on the Burnside ring, we rely on work of Tate [23] and Gulliksen [12] on differential graded algebras. In the final part of this chapter, we give shorter arguments using spectral sequences for some of these results.

2.1 Relating $\operatorname{Ext}_S^l(k,k)$ and $\operatorname{Tor}_l^S(k,k)$

Lemma 2.1. Let $(\mathcal{F}_\bullet, \partial_\bullet)$ be a minimal projective S-module resolution of an S-module N. Then the maps $\operatorname{Hom}_S(\partial_l, k) : \operatorname{Hom}_S(\mathcal{F}_l, k) \to \operatorname{Hom}_S(\mathcal{F}_{l+1}, k)$ and $\partial_l \otimes_S k : \mathcal{F}_l \otimes_S k \to \mathcal{F}_{l-1} \otimes_S k$ are all zero.

Proof. Let X be an S-module with a minimal generating set of cardinality n. Then $\dim X/\mathcal{M}X = n$, and we have a surjective homomorphism of S-modules $S^{\oplus n} \to X/\mathcal{M}X$. Since S is free, this map lifts to a surjection $S^{\oplus n} \to X$ with kernel contained in $\mathcal{M} \cdot S^{\oplus n}$.

It follows that we can choose a minimal resolution $(\mathcal{F}_\bullet, \partial_\bullet)$ of N so that $\operatorname{im} \partial_l \subset \mathcal{M} \cdot \mathcal{F}_{l-1}$ for each l. Since any map $S \to k$ vanishes on $\mathcal{M}$, it follows that $\operatorname{Hom}_S(\partial_l, k) = 0$ for each l. Similarly each $\partial_l \otimes k$ has image contained in $\mathcal{M} \cdot (\mathcal{F}_{l-1} \otimes k) = 0$. □

Corollary 2.2. With S as above, $\dim \operatorname{Ext}_S^l(k,k) = \dim \operatorname{Tor}_l^S(k,k)$ for each $l \geq 0$.

Corollary 2.3. Let I be a finite set and $R \subset \operatorname{Gh}(I)$ a B-ring. Then for all $i, j \in I$

and $l \geq 1$, the summand rank of the p-part of $\mathrm{Tor}_l^R(\mathbb{Z}_i, \mathbb{Z}_j)$ is equal to the summand rank of the p-part of $\mathrm{Ext}_R^{l+1}(\mathbb{Z}_i, \mathbb{Z}_j)$.

Proof. Suppose $i = j$. Recall that in Section 1.4 we derived recurrence relations

$$a_{l+1} = b_l - a_l$$

and

$$z_l = y_{l+1} - z_{l+1}$$

for $l \geq 1$, where a_l is the summand rank of the p-part of $\mathrm{Ext}_R^l(\mathbb{Z}_i, \mathbb{Z}_i)$, z_l is the summand rank of the p-part of $\mathrm{Tor}_l^R(\mathbb{Z}_i, \mathbb{Z}_i)$, and b_l and y_l are the dimensions of the k-vector spaces $\mathrm{Ext}_{\overline{R}}^l(k_i, k_i)$ and $\mathrm{Tor}_l^{\overline{R}}(k_i, k_i)$ respectively. By Corollary 2.2 we have $b_l = y_l$. Since our recurrence relations begin with $z_1 = b_1$ and $a_1 = 0$, we then have $z_l = a_{l+1}$ for all $l \geq 1$.

Similarly, for $i \neq j$, the relations were

$$c_{l+1} = d_l - c_l$$

and

$$x_l = w_{l+1} - x_{l+1}$$

for $l \geq 1$, where c_l is the summand rank of the p-part of $\mathrm{Ext}_R^l(\mathbb{Z}_i, \mathbb{Z}_j)$, x_l is the summand rank of the p-part of $\mathrm{Tor}_l^R(\mathbb{Z}_i, \mathbb{Z}_j)$, and d_l and w_l are the dimensions of the k-vector spaces $\mathrm{Ext}_{\overline{R}}^l(k_i, k_j)$ and $\mathrm{Tor}_l^{\overline{R}}(k_i, k_j)$ respectively. If k_i and k_j belong to different blocks of $\overline{R}$ then the result is trivially true. Otherwise we have $k_i \simeq k_j$ and can again apply Corollary 2.2 to get $d_l = w_l$. Now the relations begin with $x_1 = d_1 - 1$ and $c_1 = 1$, and once more we get $x_l = c_{l+1}$ for all $l \geq 1$. □

Let G be a finite group. Corollary 1.30 is then equivalent to the following:

Corollary 2.4. Let $H, J \subset G$ be subgroups such that $O^p(H)$ is not conjugate to $O^p(J)$ for any rational prime p. Then

$$\mathrm{Tor}_l^{A(G)}(\mathbb{Z}_H, \mathbb{Z}_J) = 0$$

for all $l \geq 0$.

Corollary 2.5. For $H, J \subset G$, the groups $\mathrm{Ext}_{A(G)}^l(\mathbb{Z}_H, \mathbb{Z}_J)$ are of unbounded rank if and only if the groups $\mathrm{Tor}_l^{A(G)}(\mathbb{Z}_H, \mathbb{Z}_J)$ are of unbounded rank.

Proof. Suppose $H, J \subset G$ are such that the groups $\mathrm{Ext}_{A(G)}^l(\mathbb{Z}_H, \mathbb{Z}_J)$ are of unbounded rank. We know by Corollary 1.29 that if p is a rational prime with $p \nmid |G|$ then $\mathrm{Ext}_{A(G)}^l(\mathbb{Z}_H, \mathbb{Z}_J)$ has no p-power summand for any l. Since there are only finitely many primes dividing $|G|$, there must be some rational prime q such that the q-parts of the groups $\mathrm{Ext}^l_{A(G)}(\mathbb{Z}_H, \mathbb{Z}_J)$ are also of unbounded rank. Since the rank of the q-part is the same as the summand rank of the q-part, it follows that the q-parts of the groups $\mathrm{Tor}^R_l(\mathbb{Z}_H, \mathbb{Z}_J)$ have unbounded rank, and so a fortiori the groups $\mathrm{Tor}^R_l(\mathbb{Z}_H, \mathbb{Z}_J)$ have unbounded rank. The reverse argument is identical. □

2.2 Indecomposable summands of the modular Burnside ring

We have seen already in Section 1.5 that if p is a rational prime with $p^2 \nmid |G|$ then $A(G) \otimes_{\mathbb{Z}} \mathbb{F}_p$ is a sum of indecomposable uniserial $\mathbb{F}_p$-algebras. Moreover if $|G|$ is square-free then for arbitrary subgroups $H, J \subset G$, the groups $\mathrm{Ext}^l_{A(G)}(\mathbb{Z}_H, \mathbb{Z}_J)$ are periodic. In this section we show that if $p^2 \mid |G|$ for some rational prime p, then $A(G)$ has Ext groups of unbounded rank. This is an easy consequence of a result of Gustafson [13] on the modular Burnside ring together with a result of Gulliksen [12] on the homology of commutative local noetherian rings.

Let k be a field and recall that a k-algebra S is said to be symmetric if there exists a k-linear map $\lambda : S \to k$ such that $\lambda(st) = \lambda(ts)$ for each $s, t \in S$, and $\ker \lambda$ contains no non-zero ideals of S. Recall that a chain $P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_q$ of prime ideals of S is said to have length q. Recall that for a commutative ring S the Krull dimension $d(S)$ is the supremum of the lengths of all chains of prime ideals of S. The two theorems we need are then as follows.

Theorem 2.6 (Gustafson)**.** If $p^2 \mid |G|$ and k is a field of characteristic p then the k-algebra $A(G) \otimes_{\mathbb{Z}} k$ is not symmetric.

Theorem 2.7 (Gulliksen)**.** Let S be a commutative noetherian local k-algebra with maximal ideal $\mathcal{M}$ and with $k = S/\mathcal{M}$. Then the sequence $(\dim \mathrm{Tor}^S_l(k,k))_{l \in \mathbb{N}}$ is bounded if and only if

$$d(S) \geq \dim \mathcal{M}/\mathcal{M}^2 - 1.$$

Lemma 2.8. Let S be a finite-dimensional commutative local k-algebra. Then $d(S) = 0$.

Proof. Since S is finite dimensional the maximal ideal $\mathcal{M}$ is nilpotent. So it is the only prime of S, and so $d(S) = 0$. □

Lemma 2.9. Let S be a finite-dimensional commutative local k-algebra with maximal ideal $\mathcal{M}$ and $\dim \mathcal{M}/\mathcal{M}^2 = 1$. Then S is symmetric.

Proof. Let $t \in S$ generate $\mathcal{M}$ and note that $\{1_S, t, \ldots, t^q\}$ is a vector space basis for S for some $q \geq 0$. Define $\lambda : S \to k$ by putting

$$\lambda\left(\sum_{i=0}^{q} a_i t^i\right) = a_q.$$

Now if $J \subset S$ is a non-zero ideal then we can choose some non-zero element $s = \sum_{i=0}^{q} b_i t^i$ in J, and choose m to be minimal such that $b_m \neq 0$. Then $t^{q-m}s = b_m t^q \in J$ is not in $\ker \lambda$, so $\ker \lambda$ contains no non-zero ideals and S is symmetric. □

We can then provide our converse to Theorem 1.32 as follows.

Theorem 2.10. If $|G|$ is not square-free then there exist subgroups $H, J \subset G$ such that the groups $\mathrm{Ext}^l_{A(G)}(\mathbb{Z}_H, \mathbb{Z}_J)$ have unbounded rank.

Proof. Let p be a rational prime with $p^2 \mid |G|$. The algebra $\overline{A(G)} = A(G) \otimes_{\mathbb{Z}} \mathbb{F}_p$ is not symmetric by Theorem 2.6, and so it has an indecomposable k-algebra summand S which is not symmetric. By Lemma 2.9, the maximal ideal $\mathcal{M}$ of S satisfies $\dim \mathcal{M}/\mathcal{M}^2 > 1$. By Lemma 2.8 and Theorem 2.7 the sequence $(\dim \mathrm{Tor}^S_l(k,k))_{l \in \mathbb{N}}$ is unbounded, and so by Lemma 2.2 the sequence $(\dim \mathrm{Ext}^l_S(k,k))_{l \in \mathbb{N}}$ is unbounded.

Let $\mathcal{E}$ be the set of equivalence classes on $\mathrm{ccs}(G)$ with respect to the equivalence relation $\sim_p$ as defined in Section 1.5. By Corollary 1.25, the summand S corresponds to some equivalence class $E \in \mathcal{E}$, and we have

$$\dim \mathrm{Ext}^l_S(k,k) = \dim \mathrm{Ext}^l_{\overline{A(G)}}(k_E, k_E).$$

Let H, J be subgroups of G with $(H), (J) \in E$, let α_l be the summand rank of the p-part of $\mathrm{Ext}^l_{A(G)}(\mathbb{Z}_H, \mathbb{Z}_J)$, and let β_l be the dimension of $\mathrm{Ext}^l_{\overline{A(G)}}(k_E, k_E)$. By Section 1.4, we have a recurrence

$$\alpha_{l+1} = \beta_l - \alpha_l$$

for $l \geq 0$. Since β_l is unbounded, it follows immediately that α_l is unbounded, and hence the groups $\mathrm{Ext}^l_{A(G)}(\mathbb{Z}_H, \mathbb{Z}_J)$ have unbounded rank. □

2.3 The case $\dim \mathcal{M}^2 = 1$

Recall that for k a field and S a commutative local k-algebra with maximal ideal $\mathcal{M}$ and with $S/\mathcal{M} \simeq k$, we are interested in determining the sequence $(a_l) = (\dim \mathrm{Ext}^l_S(k,k))$. In this section we consider the family of finite-dimensional commutative local k-algebras satisfying $\dim \mathcal{M}^2 = 1$ and $\mathcal{M}^3 = 0$, and give an explicit description of the sequence (a_l). We assume that k is algebraically closed and that $\mathrm{char}\, k \neq 2$.

Put $n = \dim \mathcal{M}/\mathcal{M}^2$, let $v_1, \ldots, v_n$ generate $\mathcal{M}$, and let w span $W = \mathcal{M}^2$. Let V be the vector space spanned by $v_1, \ldots, v_n$. The multiplication operation of S then defines a symmetric k-bilinear form $\hat{B}$ on V by $v_i v_j = \hat{B}(v_i, v_j) w$. Let m be the rank of $\hat{B}$ and note that $m > 0$ since $v_1, \ldots, v_n$ generate the ideal $\mathcal{M}$. Write B for the corresponding linear map $V \otimes V \to k$.

2.3.1 Degenerate cases

Before proceeding with a general analysis we deal with two exceptional cases.

First, if $n = 1$, then clearly S is isomorphic to the algebra $k[x]/(x^3)$. We have already computed the cohomology groups for this algebra in the proof of Corollary 1.28, and we recall that $a_l = 1$ for all $l \geq 0$.

Next, if $n > 1$ and $m = 1$, then since k is algebraically closed we can suppose $\hat{B}(v_i, v_j) \neq 0$ if and only if $i = j = 1$. Writing U for the submodule generated by v_1, note that we have an S-module decomposition

$$\mathcal{M} \simeq U \oplus k^{\oplus n-1}.$$

Moreover, $w, v_2, \ldots, v_n$ span a a sub-module $T \subset S$ isomorphic to $k^{\oplus n}$, and it is clear that $S/T \simeq U$. Then dimension shifting using the short exact sequences

$$0 \to \mathcal{M} \to S \to k \to 0$$

and

$$0 \to T \to S \to U \to 0,$$

and using the fact that $\mathrm{Ext}^l_S(-, k)$ respects direct summands, we obtain

$$\begin{aligned} a_{l+1} &= \dim \mathrm{Ext}^l_S(\mathcal{M}, k) \\ &= \dim \mathrm{Ext}^l_S(k^{\oplus n-1}, k) + \dim \mathrm{Ext}^l_S(U, k) \\ &= (n-1) \dim \mathrm{Ext}^l_S(k, k) + \dim \mathrm{Ext}^{l-1}_S(k^{\oplus n}, k) \\ &= (n-1)a_l + n a_{l-1}, \end{aligned}$$

for $l \geq 1$. Now $a_0 = 1$ and $a_1 = n$, so an easy induction yields $a_l = n^l$ for $l \geq 0$.

2.3.2 Constructing a resolution

For the remainder we will assume that $n > 1$ and $m > 1$. By diagonalising our quadratic form and making use of the fact that k is algebraically closed with characteristic not equal to 2, we can assume that our basis of V is chosen so that $\hat{B}(v_i, v_i) = 1$ for $1 \leq i \leq m$ and $\hat{B}(v_i, v_j) = 0$ for all other i, j (see e.g. [17] Chapter 1.2). We proceed to describe a minimal resolution of the S-module k.

Let K_1 be the kernel of the natural module map $S \to k$. Since $\hat{B}$ is non-zero, the map $S \otimes_k V \to S$ given by $\sum_i s_i \otimes v_i \mapsto \sum_i s_i v_i$ is a surjection onto the kernel K_1. We denote the kernel of this map by K_2, and begin the resolution by

$$0 \to K_2 \to S \otimes V \to S \to k \to 0.$$

Regarding $\ker B \subset V \otimes V$ as a subspace of $S \otimes V$, note that $K_2 = \ker B \oplus (W \otimes V)$.

We then have a map $S \otimes \ker B \to K_2$ given by putting

$$s \otimes \left(\sum_{i,j} \alpha_{i,j} v_i \otimes v_j \right) \mapsto \sum_{i,j} \alpha_{i,j} s v_i \otimes v_j$$

and extending linearly. We claim that this is a surjection. Certainly $\ker B$ is in the image, so it remains to check that $w \otimes v_i$ is in the image for each $1 \leq i \leq n$. But since $m \geq 2$, for any v_i we can choose $j \neq i$ with $\hat{B}(v_j, v_j) = 1$, and then $v_j \otimes (v_j \otimes v_i)$ is an element of $S \otimes \ker B$ mapping onto $w \otimes v_i$.

We'd like to continue this process, identifying subspaces A_l of the k-vector space $\bigoplus_{r=1}^{\infty} V^{\otimes r}$ such that we have a resolution given by

$$\ldots \to S \otimes A_l \to S \otimes A_{l-1} \to \ldots \to S \otimes A_1 \to S \to k$$

where the maps are given by multiplication of the first two terms as above.

We claim that putting $A_1 = V$, and

$$A_l = \bigcap_{j=0}^{l-2} \left(V^{\otimes j} \otimes \ker B \otimes V^{\otimes l-2-j} \right)$$

for $l \geq 2$ gives such a resolution.

Proof. It is clear that we have an S-module map $\delta_l : S \otimes A_l \to S \otimes V^{\otimes l-1}$ given by multiplication in the first two positions. In order to show that δ_l maps into $S \otimes A_{l-1}$, it is then sufficient to check elements of the form $1 \otimes q$ for $q \in A_l$.

Now $\delta_l(1 \otimes q) = q$. Since

$$q \in \bigcap_{j=0}^{l-2} V^{\otimes j} \otimes \ker B \otimes V^{\otimes l-2-j}$$

we certainly have that

$$q \in \bigcap_{j=1}^{l-2} V^{\otimes j} \otimes \ker B \otimes V^{\otimes l-2-j} = V \otimes \bigcap_{j=0}^{(l-1)-2} V^{\otimes j} \otimes \ker B \otimes V^{\otimes (l-1)-2-j}.$$

So $q \in S \otimes A_{l-1}$ as required.

Since $S \otimes A_l \subset S \otimes \ker B \otimes V^{\otimes l-2}$ for each l, it is clear that $\delta_{l-1} \circ \delta_l = 0$, so we do indeed have a chain complex. It remains to show that the complex is exact, i.e. that each δ_{l+1} is a surjection onto $\ker \delta_l$. Note that $\ker \delta_l \subset \mathcal{M} \cdot (S \otimes A_l) = (V \oplus W) \otimes A_l$.

Now if $t \in \ker \delta_l \cap (V \otimes A_l)$ then

$$t \in \left(\ker B \otimes V^{\otimes l-1} \right) \cap (V \otimes A_l)$$

and hence $t \in A_{l+1}$. So the element $1 \otimes t$ of $S \otimes A_{l+1}$ maps onto t.

Suppose $t \in \ker \delta_l \cap (W \otimes A_l)$, and write t as

$$t = w \otimes \left(\sum_{i_1,\ldots,i_l} \alpha_{i_1,\ldots,i_l} v_{i_1} \otimes \ldots \otimes v_{i_l} \right).$$

For each $1 \leq i \leq n$, choose some $1 \leq j^{(i)} \leq m$ with $j^{(i)} \neq i$. Then $v_{j^{(i)}} \otimes v_i \in \ker B$ for each $1 \leq i \leq n$, and so

$$\sum_{i_1,\ldots,i_l} \alpha_{i_1,\ldots,i_s} v_{j^{(i_1)}} \otimes v_{i_1} \otimes \ldots \otimes v_{i_l} \in A_{l+1}$$

and

$$\sum_{i_1,\ldots,i_l} \alpha_{i_1,\ldots,i_l} v_{j^{(i_1)}} \otimes v_{j^{(i_1)}} \otimes v_{i_1} \otimes \ldots \otimes v_{i_l}$$

is an element of $S \otimes A_{l+1}$ mapping onto t.

Finally, since $\ker \delta_l \subset \mathcal{M} \cdot (S \otimes A_{l-1})$ for each l, the map

$$\mathrm{Hom}_S(\delta_l, k) : \mathrm{Hom}_S(S \otimes A_l, k) \to \mathrm{Hom}_S(S \otimes A_{l+1}, k)$$

is the zero map for each l, and so our resolution is minimal. Then $\dim \mathrm{Ext}^l_S(k,k) = \dim A_l$ for each $l \geq 1$. □

2.3.3 Determining A_l

Define a non-singular bilinear form Q on V by putting $Q(v_i, v_j) = \delta_{i,j}$. For each $l \geq 2$ we have a non-singular bilinear form $Q^{(l)}$ on $V^{\otimes l}$ by putting

$$Q^{(l)}(v_{i_1} \otimes \ldots \otimes v_{i_l}, v'_{i_1} \otimes \ldots \otimes v'_{i_l}) = Q(v_{i_1}, v'_{i_1}) \cdot \ldots \cdot Q(v_{i_l}, v'_{i_l})$$

and extending linearly.

Now

$$\begin{aligned} A_l^{\perp} &= \left(\bigcap_{j=0}^{l-2} V^{\otimes j} \otimes \ker B \otimes V^{\otimes l-2-j} \right)^{\perp} \\ &= \sum_{j=0}^{l-2} \left(V^{\otimes j} \otimes \ker B \otimes V^{\otimes l-2-j} \right)^{\perp}. \end{aligned}$$

Note that $\ker B$ is spanned by vectors of the form $v_i \otimes v_j$ for $i \neq j$, $v_1 \otimes v_1 - v_i \otimes v_i$ for $1 \leq i \leq m$ and $v_j \otimes v_j$ for $j \geq m+1$. Putting $u = \sum_{i=1}^m v_i \otimes v_i$, we have

$$Q^{(2)}(v_i \otimes v_j, u) = 0,$$

for all $i \neq j$, and

$$Q^{(2)}(v_1 \otimes v_1 - v_i \otimes v_i, u) = 0,$$

for $1 \leq i \leq m$, and

$$Q^{(2)}(v_j \otimes v_j, u) = 0$$

for $j \geq m+1$, so $k \cdot u \subset (\ker B)^\perp$. Since $\ker B$ has dimension $n^2 - 1$ and $Q^{(2)}$ is non-singular, it follows that $(\ker B)^\perp = k \cdot u$, and that $\bigoplus_{l \geq 2} A_l^\perp$ is spanned by tensors of the form $v \otimes u \otimes w$ for $v, w \in \bigoplus_{l \geq 0} V^{\otimes s}$.

Let F be the free non-commutative k-algebra on $v_1, \ldots, v_n$ and write f for the element $\sum_{i=1}^m v_i^2$ of F. The vector space spanned by elements of the form bfc for $b, c \in F$ is then the two-sided ideal I generated by f. Note that F comes with a natural grading by degree $F = \bigoplus_{i \geq 0} F_i$, and that since the ideal I is homogeneous, this grading descends to the quotient $F/I = \bigoplus_{i \geq 0} F_i/(I \cap F_i)$. Since $\dim A_l^\perp = \dim I \cap F_l$, and $\dim V^{\otimes l} = \dim F_l$, the problem of computing the dimension $a_l = \dim A_l$ is then equivalent to the problem of computing $\dim F_l/(I \cap F_l)$. In order to do this we will construct a Gröbner basis for I.

We first recall some results from the theory of non-commutative Gröbner bases (see [8] Chapter 6 for proofs and further details).

2.3.4 Gröbner bases

Let X denote the set of monomials of F. Introduce a total order on X by first ordering by degree, and breaking ties with the lexicographic ordering $v_n > v_{n-1} > \ldots > v_1$. Explicitly, if $x = v_{i_1} \ldots v_{i_s}$ and $y = v_{j_1} \ldots v_{j_s}$ are monomials of the same degree, we have $x > y$ if and only if there is some $1 \leq r \leq s$ such that $i_r > j_r$ and $i_q = j_q$ for all $1 \leq q < r$. For an element $w = \sum_{x \in X} \lambda_x x$ of F, say that x is occurring in w if $\lambda_x \neq 0$. To each element w of F, we have a unique element of X occurring in w which is maximal with respect to the total order, called the leading monomial of w and denoted $\mathrm{LM}(w)$.

For J an ideal of F, let $N(J)$ be the set of monomials which are not leading monomials of elements of J, and $C(J)$ the subspace of F spanned by the elements of $N(J)$.

Lemma 2.11 (Proposition 6.1.1 of [8])**.** *We have a vector space decomposition $F = J \oplus C(J)$.*

For monomials $v, w \in X$, say that w is a subword of v if there exist monomials $w_1, w_2 \in X$ such that $w_1 w w_2 = v$. Let $\mathcal{G}$ be a set of generators for an ideal J. We say that $\mathcal{G}$ is a Gröbner basis for J if for each $j \in J$ there is an element g of $\mathcal{G}$ such that $\mathrm{LM}(g)$ is a subword of $\mathrm{LM}(j)$.

Lemma 2.12 (Theorem 6.1.4 of [8])**.** *If $\mathcal{G}$ is a Gröbner basis for J then $N(J)$ is the set of monomials which do not contain $\mathrm{LM}(g)$ as a subword for any $g \in \mathcal{G}$.*

Call a (not necessarily distinct) pair (g, f) of elements of F an overlap if we have $\mathrm{LM}(g)v = w\,\mathrm{LM}(h)$ for some monomials v, w in X. We then have the following recognition condition for Gröbner bases.

Lemma 2.13 (Theorem 6.1.6 of [8])**.** If $\mathcal{G}$ is a generating set for J, then $\mathcal{G}$ is a Gröbner basis for J if and only if for each overlap $g, h \in \mathcal{G}$ with $\mathrm{LM}(g)v = w\,\mathrm{LM}(h)$, we have that $gv - wh$ is a linear combination of elements of the form $v'g'w'$, where $v', w' \in X$, $g' \in \mathcal{G}$, and $v'\,\mathrm{LM}(g')w' < \mathrm{LM}(g)v$ with respect to our total ordering of monomials.

2.3.5 A Gröbner basis for I

Recall that $f = v_1^2 + \ldots + v_m^2$ and $I = (f)$. Certainly $f_2 = v_m f - f v_m \in I$, so the set $\mathcal{G} = \{f, f_2\}$ is a generating set for I. We claim that it is also a Gröbner basis.

The set of leading monomials of $\mathcal{G}$ is $\{v_m^2, v_m v_{m-1}^2\}$. The pair (f, f) is an overlap since $(v_m^2)v_m = v_m(v_m^2)$, and the pair (f, f_2) is an overlap since $v_m^2(v_{m-1}^2) = (v_m)(v_m v_{m-1}^2)$, and these give all possible overlaps. Now $fv_m - v_m f = f_2$, and

$$f v_{m-1}^2 - v_m f_2 = f\left(\sum_{i=1}^{m-2} v_i^2\right) + f_2 v_m - \left(\sum_{i=1}^{m-1} v_i^2\right) f,$$

where all leading monomials appearing are $< v_m v_{m-1}^2$, so both overlaps satisfy the conditions of Lemma 2.13. Thus $\mathcal{G}$ is a Gröbner basis for I.

Call a word in $v_1, \ldots, v_n$ special if it does not contain v_m^2 or $v_m v_{m-1}^2$ as a subword. By Lemma 2.12 the special words then span the vector space $C(I)$, and by Lemma 2.11 we have that a_l is the number of special words of length l.

2.3.6 Counting special words

Let b_l be the number of special words of length l which do not begin with v_m, and let c_l be the number of special words of length l that begin with v_m.

Given any special word of length $l-1$, we have $n-1$ special words of length l not beginning with v_m obtained by appending one of $v_1, \ldots, v_{m-1}, v_{m+1}, \ldots, v_n$. So

$$b_l = (n-1)(b_{l-1} + c_{l-1}).$$

Then $\frac{1}{n-1} b_{l-1}$ is the number of special words of length l beginning with v_{m-1}, and also the number of special words of length l beginning with v_{m-1}^2. So

$$c_l = b_{l-1} - \frac{1}{n-1} b_{l-2}.$$

We then have a system of recurrences for $l \geq 3$ with $b_1 = n-1, c_1 = 1, b_2 = n^2 - n$, and $c_2 = n - 1$.

Adding the two together and rewriting in terms of the quantity of interest a_l, we get

$$a_l = (n-1)a_{l-1} + b_{l-1} - \frac{1}{n-1} b_{l-2},$$

which we can rewrite as

$$a_l = na_{l-1} - a_{l-2} - (a_{l-1} - na_{l-2} + a_{l-3}).$$

Noting that $a_2 = na_1 - a_0$, it then follows by induction that

$$a_l = na_{l-1} - a_{l-2}$$

for all $l \geq 2$. If $n = 2$, it is clear by a further induction argument that we then have $a_l = l + 1$ for all $l \geq 0$. Suppose $n > 2$.

We can express the recurrence as the matrix equation

$$\begin{bmatrix} a_{l+1} \\ a_l \end{bmatrix} = \begin{bmatrix} n & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_l \\ a_{l-1} \end{bmatrix},$$

and repeatedly making use of the recurrence we can rewrite this as

$$\begin{bmatrix} a_{l+1} \\ a_l \end{bmatrix} = \begin{bmatrix} n & -1 \\ 1 & 0 \end{bmatrix}^l \begin{bmatrix} a_1 \\ a_0 \end{bmatrix}.$$

Putting the above matrix into Jordan normal form allows us to rewrite this further as

$$\begin{bmatrix} a_{l+1} \\ a_l \end{bmatrix} = \frac{1}{\theta} \begin{bmatrix} 1 & 1 \\ \frac{n-\theta}{2} & \frac{n+\theta}{2} \end{bmatrix} \begin{bmatrix} \left(\frac{n+\theta}{2}\right)^l & 0 \\ 0 & \left(\frac{n-\theta}{2}\right)^l \end{bmatrix} \begin{bmatrix} \frac{n+\theta}{2} & -1 \\ \frac{-n+\theta}{2} & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_0 \end{bmatrix},$$

where $\theta = \sqrt{n^2 - 4}$. We then get

$$a_l = \frac{1}{\theta 2^{l+1}} \left((n+\theta)^{l+1} - (n-\theta)^{l+1} \right)$$

for $l \geq 0$.

2.4 Studying (a_l) with spectral sequences

Let S be a commutative noetherian local ring with maximal ideal $\mathcal{M}$ and with $S/\mathcal{M} \simeq k$. In [23] Tate gives a method for constructing a resolution of the S-module k which has the structure of a differential graded S-algebra. In [12] Gulliksen uses this construction to prove Theorem 2.7 together with the following:

Theorem 2.14. With S, $\mathcal{M}$, k as above, let $x_1, \ldots, x_n$ be a minimal generating set for $\mathcal{M}$ and let N be the ideal of S generated by $x_1, \ldots, x_{n-1}$. If N is prime then

$$\dim \operatorname{Tor}_l^S(k,k) = \dim \operatorname{Tor}_l^S(S/N,k) + \dim \operatorname{Tor}_{l-1}^S(S/N,k)$$

for each $l \geq 1$.

In this section we use spectral sequences to provide significantly shorter arguments for Theorem 2.14 and a weaker version of Theorem 2.7.

2.4.1 The Grothendieck spectral sequence and Theorem 2.14

We first recall some facts about the Grothendieck spectral sequence, see [25] (Chapter 5.8) for further details. Let B, C, and D be abelian categories with B and C having enough projectives. Let $\mathcal{G} : \mathsf{B} \to \mathsf{C}$ and $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ be right exact functors. Then $\mathcal{F} \circ \mathcal{G}$ is right exact, and we write $L_*\mathcal{G}, L_*\mathcal{F}$ and $L_*(\mathcal{F} \circ \mathcal{G})$ for the corresponding left derived functors. Suppose that $\mathcal{G}$ sends projective objects of B to acyclic objects for $\mathcal{F}$. Then for each object B in B, we have a first quadrant homological spectral sequence E with E^2 page given by

$$E^2_{pq} = (L_p\mathcal{F}) \circ (L_q\mathcal{G})(B),$$

and with E converging to $L_*(\mathcal{F} \circ \mathcal{G})(B)$.

Proof of Theorem 2.14. Consider the right-exact functors

$$\begin{aligned} - \otimes_S k &: S\text{-Mod} \to k\text{-Mod}, \\ - \otimes_S S/N &: S\text{-Mod} \to S/N\text{-Mod}, \\ - \otimes_{S/N} k &: S/N\text{-Mod} \to k\text{-Mod}, \end{aligned}$$

and note that we have a factorisation

$$- \otimes_S k = (- \otimes_{S/N} k) \circ (- \otimes_S S/N).$$

Note moreover that $- \otimes_S S/N$ sends projective (i.e. free) S-modules to free S/N-modules, which are acyclic for $- \otimes_{S/N} k$. We are then in a situation to apply the Grothendieck spectral sequence.

Letting k play the rôle of the object B above, the E^2 page of our spectral sequence is then given by

$$E^2_{pq} = \mathrm{Tor}^{S/N}_p(\mathrm{Tor}^S_q(k, S/N), k),$$

and the sequence converges to $\mathrm{Tor}^S_*(k, k)$.

Given a projective S-module resolution of any S-module X, it is clear that tensoring by k produces a chain complex where each term is some number of copies of the S-module k. It follows that as an S-module, $\mathrm{Tor}^S_q(k, S/N)$ is isomorphic to some number of copies of k for any $q \geq 0$, and hence as an S/N-module $\mathrm{Tor}^S_q(k, S/N)$ is isomorphic to some number of copies of the S/N-module k.

Since the ideal N is prime, we have that $x_n^l \notin N$ for each $l \in \mathbb{N}$, and hence $S/N \simeq k[t]$, the polynomial ring over k in one variable. We have a projective resolution of the $k[t]$-module k given by

$$0 \to k[t] \to k[t] \to k \to 0,$$

where the map $k[t] \to k[t]$ is given by multiplication by t. It follows immediately that

$$\mathrm{Tor}_1^{k[t]}(k,k) \simeq \mathrm{Tor}_0^{k[t]}(k,k) \simeq k$$

and that $\mathrm{Tor}_l^{k[t]}(k,k)$ vanishes for all $l \geq 2$.

The E^2 page of our spectral sequence is then given by

$$E^2_{pq} = \begin{cases} \mathrm{Tor}_q^R(k, S/N) & \text{if } p \in \{0,1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Since the maps on the E^2 page are of the form $\partial^2_{pq} : E^2_{pq} \to E^2_{p-2,q+1}$, all such maps are zero, and similarly all maps on subsequent pages are zero. So the sequence converges immediately. Recall that in general the terms on the lth diagonal of E^∞ are a set of successive quotients in a filtration of $L_l(F \circ G)$. Since in this case our object is in k-Mod, it follows that summing up the terms on the lth diagonal gives $\mathrm{Tor}_l^A(k,k)$. Then

$$\mathrm{Tor}_l^A(k,k) \simeq \mathrm{Tor}_l^S(k,k) \oplus \mathrm{Tor}_{l-1}^S(k,k)$$

as required. □

2.4.2 A weak form of Theorem 2.7

We state our weaker form of Theorem 2.7 as follows.

Theorem 2.15. Let S be a commutative local finite-dimensional k-algebra with maximal ideal $\mathcal{M}$ satisfying $S/\mathcal{M} \simeq k$ and $\dim \mathcal{M}/\mathcal{M}^2 = 2$. For $l \geq 0$ put $a_l = \dim \mathrm{Tor}_l^S(k,k)$. Then the sequence $(a_l)_{l \in \mathbb{N}}$ is unbounded.

Proof. Let $A = k[x,y]$, and note that without loss of generality we can suppose S is a quotient A/I for some ideal $I \subset (x,y)$.

Consider the right-exact functors

$$\begin{aligned} - \otimes_A k &: A\text{-Mod} \to k\text{-Mod}, \\ - \otimes_A A/I &: A\text{-Mod} \to S\text{-Mod}, \\ - \otimes_S k &: S\text{-Mod} \to k\text{-Mod}, \end{aligned}$$

and note that we have a factorisation

$$- \otimes_A k = (- \otimes_S k) \circ (- \otimes_A A/I).$$

As before, $-\otimes_A A/I$ sends projective A-modules to S-modules which are acyclic for $-\otimes_S k$.

Moving to the Grothendieck spectral sequence associated to the above factorisation and the A-module k, we have that the E^2 page is given by

$$E^2_{pq} = \mathrm{Tor}^S_p(\mathrm{Tor}^A_q(k,S),k),$$

and this sequence converges to $\mathrm{Tor}^A_*(k,k)$.

Computing the derived functors of $-\otimes_A k$ is straightforward: there's a canonical minimal projective resolution of the A-module k given by the Koszul resolution

$$0 \to A \xrightarrow{\beta} A^{\oplus 2} \xrightarrow{\alpha} A \to k \to 0 \tag{††}$$

where $\alpha(a,b) = ax+by$ and $\beta(a) = a\cdot(y,-x)$. This gives

$$\mathrm{Tor}^A_1(k,k) \simeq k^{\oplus 2},$$
$$\mathrm{Tor}^A_2(k,k) \simeq k,$$

and $\mathrm{Tor}^A_l(k,k) = 0$ for $l \geq 3$.

It remains to compute $\mathrm{Tor}^A_l(k,S)$. For $a \in A$, write $[a]$ for its image in S. Applying $-\otimes_A S$ to (††) we obtain a chain complex

$$0 \to S \xrightarrow{\overline{\beta}} S^{\oplus 2} \xrightarrow{\overline{\alpha}} S \to 0,$$

where $\overline{\alpha}([a],[b]) = [ax+by]$ and $\overline{\beta}([a]) = [a]\cdot([y],[-x])$. Let Z be the socle of S and note that $Z = \ker\overline{\beta}$. Let $z = \dim Z$. Then $\dim\mathrm{im}\,\overline{\beta} = \dim S - z$. But $\dim\ker\overline{\alpha} = 2\dim S - (\dim S - 1) = \dim S + 1$, and hence $\dim \mathrm{Tor}^A_1(k,S) = z+1$.

It is clear that $\mathrm{Tor}^A_2(k,S) \simeq Z \simeq k^{\oplus z}$ as an S-module. By the same reasoning as before, each A-module $\mathrm{Tor}^A_l(k,S)$ is isomorphic to some number of copies of k. Since the S-module structure of $\mathrm{Tor}^A_1(k,S)$ is induced by the A-module structure, it follows that $\mathrm{Tor}^A_1(k,S) \simeq k^{\oplus z+1}$ as an S-module.

Making use of the fact that $\mathrm{Tor}^S_l(-,k)$ respects direct sums, the E^2 page of our spectral sequence is then

$$\begin{array}{ccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & k^{\oplus z} & \mathrm{Tor}^S_1(k,k)^{\oplus z} & \mathrm{Tor}^S_2(k,k)^{\oplus z} & \mathrm{Tor}^S_3(k,k)^{\oplus z} & \cdots \\
\cdots & 0 & k^{\oplus z+1} & \mathrm{Tor}^S_1(k,k)^{\oplus z+1} & \mathrm{Tor}^S_2(k,k)^{\oplus z+1} & \mathrm{Tor}^S_3(k,k)^{\oplus z+1} & \cdots \\
\cdots & 0 & k & \mathrm{Tor}^S_1(k,k) & \mathrm{Tor}^S_2(k,k) & \mathrm{Tor}^S_3(k,k) & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

with maps $\partial^2_{p,q} : E^2_{pq} \to E^2_{p-2,q+1}$.

In order to determine E^3, note that all maps out of E^2_{pq} are zero for $q = 2$ and p arbitrary, and for q arbitrary and $p = 0, 1$. Similarly all maps into E^2_{pq} are zero for $q = 0$. The E^3 page is then given by

$$\begin{array}{lllllll}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\dots & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & \operatorname{coker} \partial^2_{2,1} & \operatorname{coker} \partial^2_{3,1} & \operatorname{coker} \partial^2_{4,1} & \operatorname{coker} \partial^2_{5,1} & \dots \\
\dots & 0 & \operatorname{coker} \partial^2_{2,0} & \operatorname{coker} \partial^2_{3,0} & \ker \partial^2_{2,1} / \operatorname{im} \partial^2_{4,0} & \ker \partial^2_{3,1} / \operatorname{im} \partial^2_{5,0} & \dots \\
\dots & 0 & k & \operatorname{Tor}^S_1(k,k) & \ker \partial^2_{2,0} & \ker \partial^2_{3,0} & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & \dots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

with maps $\partial^3_{pq} : E^3_{pq} \to E^3_{p-3,q+2}$.

On the E^3 page the only potentially non-zero maps are ∂^3_{pq} for $q = 0$ and $p \geq 3$, so it follows that the E^4 is given by

$$\begin{array}{lllllll}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\dots & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & \operatorname{coker} \partial^3_{3,0} & \operatorname{coker} \partial^3_{4,0} & \operatorname{coker} \partial^3_{5,0} & \operatorname{coker} \partial^3_{6,0} & \dots \\
\dots & 0 & \operatorname{coker} \partial^2_{2,0} & \operatorname{coker} \partial^2_{3,0} & \ker \partial^2_{2,1} / \operatorname{im} \partial^2_{4,0} & \ker \partial^2_{3,1} / \operatorname{im} \partial^2_{5,0} & \dots \\
\dots & 0 & k & \operatorname{Tor}^S_1(k,k) & \ker \partial^2_{2,0} & \ker \partial^3_{3,0} & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & \dots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

and since all maps ∂^4 are 'out of bounds' and hence zero, the sequence converges and the above is the page E^∞. As before, summing up the terms on the lth diagonal then gives $\operatorname{Tor}^A_l(k,k)$.

Considering the $l = 1$ diagonal, note $\operatorname{Tor}^S_1(k,k) \simeq \mathcal{M}/\mathcal{M}^2 \simeq k^{\oplus 2} \simeq \operatorname{Tor}^A_1(k,k)$, and so $\operatorname{coker} \partial^2_{2,0} = 0$ and $\operatorname{Tor}^S_2(k,k)$ surjects onto $k^{\oplus z+1}$. This implies $a_2 \geq (z+1)$.

Considering the $l = 2$ diagonal, the sum of the three modules appearing is 1-dimensional, and so in particular $\dim \operatorname{coker} \partial^2_{3,0} \leq 1$. Then the map $\operatorname{Tor}^S_3(k,k) \to \operatorname{Tor}^S_1(k,k)^{\oplus z+1}$ has image of dimension at least $(z+1)(\dim \operatorname{Tor}^S_1(k,k)) - 1 = 2z+1$, and hence $a_3 \geq 2z+1$.

For $l \geq 3$, the lth diagonal is zero, and so in particular

$$\ker \partial^2_{l-1,1} / \operatorname{im} \partial^2_{l+1,0} = 0.$$

Then for any $l \geq 2$, we have

$$a_{l+2} \geq \dim \operatorname{im} \partial^2_{l+2,0} = \dim \ker \partial^2_{l,1} \geq (z+1)a_l - z a_{l-2}.$$

Now $a_2 \geq z + 1 > za_0$, and $a_3 \geq 2z + 1 > za_1$, so it follows that $a_l > a_{l-2}$ for each l, and hence $a_l \to \infty$ as $l \to \infty$. □

A small advantage of this proof is that we can say more about the growth of $\mathrm{Tor}^l_S(k, k)$ when S is not symmetric, as in the case of certain summands appearing in the modular Burnside ring.

Corollary 2.16. Suppose S as above is in addition not symmetric. Then the sequence $(\dim \mathrm{Tor}^l_S(k, k))_{l\in\mathbb{N}}$ is bounded below by a sequence which grows exponentially.

Proof. The k-algebra S is symmetric if and only if it has a 1-dimensional socle ([9] Lemma 9). Put $z = \dim \mathrm{soc}\, S$ and $a_l = \dim \mathrm{Tor}^l_S(k, k)$ for $l \geq 0$. By the proof of Theorem 2.15 we have an inequality

$$a_{l+2} \geq (z + 1)a_l - a_{l-2}$$

for each $l \geq 2$.

Consider the recurrence

$$b_{l+2} = (z + 1)b_l - zb_{l-2}$$

defined for all even $l \geq 2$, with $b_0 = a_0$ and $b_2 = a_2$. We then have

$$\begin{bmatrix} b_{2l+2} \\ b_{2l} \end{bmatrix} = \begin{bmatrix} z+1 & -z \\ 1 & 0 \end{bmatrix}^l \begin{bmatrix} b_2 \\ b_0 \end{bmatrix}$$

for $l \geq 0$. We can rewrite this as

$$\begin{bmatrix} b_{2l+2} \\ b_{2l} \end{bmatrix} = \frac{1}{z-1} \begin{bmatrix} z & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} z^l & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & z \end{bmatrix} \begin{bmatrix} b_2 \\ b_0 \end{bmatrix}$$

and obtain

$$b_{2l+2} = \frac{(z^{l+1} - z)(b_2 - b_0)}{z - 1}.$$

Similarly we can consider the recurrence

$$b_{l+2} = (z + 1)b_l - zb_{l-2}$$

defined for all odd $l \geq 3$, with $b_1 = a_1$ and $b_3 = a_3$, and we obtain

$$b_{2l+3} = \frac{(z^{l+1} - z)(b_3 - b_1)}{z - 1}.$$

Since $z > 1$, $b_2 > b_0$, and $b_3 > b_1$, it follows that the subsequences $(a_l)_{l\in 2\mathbb{N}}$ and $(a_l)_{l\in 2\mathbb{N}+1}$ of even and odd terms are both bounded below by sequences which grow exponentially. We conclude that the sequence $(a_l)_{l\in\mathbb{N}}$ is bounded below by a sequence which grows exponentially. $\square$

Chapter 3

Mackey systems for Burnside rings

A survey of generalisations of the Burnside ring to larger classes of groups is given in [18]. In this chapter we introduce the notion of a Mackey system in order to present a new generalisation of the Burnside ring, and establish which properties of the Burnside ring of a finite group carry forward to this setting.

Let S be the group of permutations of $\mathbb{N}$ which fix all but finitely many elements. In the latter half of the chapter we construct a Burnside ring for S and study this ring in detail.

Definition 3.1. Let G be a group. A collection of subgroups $\mathfrak{M}$ of G is a Mackey system for G if:

1. $G \in \mathfrak{M}$;
2. for $H, K \in \mathfrak{M}$ we have $H \cap K \in \mathfrak{M}$;
3. for $H \in \mathfrak{M}$ and $g \in G$ we have $gHg^{-1} \in \mathfrak{M}$;
4. for each $H, K \in \mathfrak{M}$ there are only finitely many (H, K) double cosets.

Definition 3.2. A G-set X is said to be $\mathfrak{M}$-admissible if it is finitely generated (i.e. has finitely many orbits), and if for any $x \in X$ we have $\mathrm{Stab}_G(x) \in \mathfrak{M}$.

Let X be a transitive $\mathfrak{M}$-admissible G-set. Then X is isomorphic to the G-set G/A for some subgroup $A \in \mathfrak{M}$. Similarly, any $\mathfrak{M}$-admissible G-set is isomorphic to a finite disjoint union of G-sets of the form G/A_i, $A_i \in \mathfrak{M}$.

Proposition 3.3. Let X_1, X_2 be $\mathfrak{M}$-admissible G-sets. Then $X_1 \sqcup X_2$ and $X_1 \times X_2$ are $\mathfrak{M}$-admissible G-sets.

Proof. If $x \in X_1 \sqcup X_2$ then either $x \in X_1$ or $x \in X_2$, and in each case we have $\mathrm{Stab}_G(x) \in \mathfrak{M}$. If $x_1, \ldots, x_m$ generates X_1 as a G-set and $x'_1, \ldots, x'_{m'}$ generates X_2, then $x_1, \ldots, x_m, x'_1, \ldots, x'_{m'}$ generates $X_1 \sqcup X_2$. So $X_1 \sqcup X_2$ is $\mathfrak{M}$-admissible.

Without loss of generality, suppose X_1 and X_2 are transitive. Then there exist $A_1, A_2 \in \mathfrak{M}$ with $X_1 \simeq G/A_1$ and $X_2 \simeq G/A_2$. For $(g_1A_1, g_2A_2) \in G/A_1 \times G/A_2$, we then have $\mathrm{Stab}_G((g_1A_1, g_2A_2)) = g_1A_1g_1^{-1} \cap g_2A_2g_2^{-1} \in \mathfrak{M}$.

In order to show that $G/A_1 \times G/A_2$ is $\mathfrak{M}$-admissible, it remains to show that it is finitely generated. Let O be an orbit of $G/A_1 \times G/A_2$. We can choose a representative $(A_1, gA_2) \in O$ for some $g \in G$. Any other $(A_1, g'A_2) \in G/A_1 \times G/A_2$

belongs to the same orbit if and only if there is an $a \in A_1$ with $agA_2 = g'A_2$, i.e. if and only if g and g' generate the same (A_1, A_2)-double coset of G. Since there are finitely many such double cosets, we have that $G/A_1 \times G/A_2$ is finitely generated, and so $G/A_1 \times G/A_2$ is $\mathfrak{M}$-admissible. □

It follows that the isomorphism classes of $\mathfrak{M}$-admissible G-sets form a semi-ring $\widehat{A}(G, \mathfrak{M})$, with addition given by disjoint union, multiplication by cartesian product, and the identity provided by the isomorphism class of the trivial G-set G/G.

Definition 3.4. The Burnside ring $A(G, \mathfrak{M})$ of a group G together with a Mackey system $\mathfrak{M}$ is the Grothendieck ring of $\widehat{A}(G, \mathfrak{M})$.

3.1 Examples and basic properties

Example 3.5. Let G be a finite group. Then any collection $\mathfrak{M}$ of subgroups containing G and closed under conjugacy and intersection is automatically a Mackey system for G, and the ring $A(G, \mathfrak{M})$ naturally embeds in the usual Burnside ring $A(G)$. When $\mathfrak{M}$ is the collection of all subgroups of G then $A(G, \mathfrak{M})$ is the usual Burnside ring of G.

Example 3.6. For n a positive integer, let S_n be the symmetric group on n letters. For a partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ of n, let S_λ denote the subgroup $S_{\lambda_1} \times \ldots \times S_{\lambda_m}$ of S_n, called the Young subgroup associated to λ. Let $\mathfrak{M}$ be the collection of all S_λ together with their conjugates, as λ ranges over the partitions of n. It is clear that $\mathfrak{M}$ is a Mackey system for S_n. For a partition λ, write s_λ for the element $[S_n/S_\lambda]$ of $A(S_n, \mathfrak{M})$. As an abelian group, $A(S_n, \mathfrak{M})$ is free on the elements $s_\lambda = [S_n/S_\lambda]$, where λ ranges over the partitions of n.

For ν, μ partitions of n and $t \in S_n$, the subgroup $S_\nu \cap tS_\mu t^{-1}$ of S_n is conjugate to a unique Young subgroup of S_n, and we denote the corresponding partition $\nu \cap \mu^t$. By our description of multiplication in the Burnside ring in Section 1.2.2, we have

$$s_\nu s_\mu = \sum_{S_\nu t S_\mu \in S_\nu \backslash S_n / S_\mu} s_{\nu \cap \mu^t}.$$

To each s_λ we can associate a rational character r_λ, where for $t \in S_n$, the integer $r_\lambda(t)$ is the number of elements of S/S_λ fixed by t. Let $R(S_n)$ denote the character ring of S_n. Our formula above then mirrors the Mackey formula for the product of induced characters, and so we have a homomorphism of rings $A(S_n, \mathfrak{M}) \to R(S_n)$ given by mapping $s_\lambda \mapsto r_\lambda$. Since the characters r_λ span the character ring $R(S_n)$ as λ ranges over the partitions of n ([15] Lecture 4), this homomorphism is an isomorphism of rings $A(S_n, \mathfrak{M}) \simeq R(S_n)$.

Example 3.7. Let G be a group and let $\mathfrak{M}$ be the collection of subgroups of finite index. This is the 'finite-G-set-version' given in [18].

Example 3.8. For a set $A \subset \mathbb{N}$ write $\operatorname{Sym}(A)$ for the group of permutations of A fixing all but finitely many elements, and write A' for the complement of A. For $n \in \mathbb{N}$, write $\mathbf{n}$ for the subset $\{1, \ldots, n\} \subset \mathbb{N}$. Let $S = \operatorname{Sym}(\mathbb{N})$, the restricted infinite symmetric group. The representation theory of S is studied in [20], and the stability properties of such are important in the theory of FI-modules (see for instance [6]).

Let $\mathfrak{M}_S$ be the collection of subgroups of the form $H \times \operatorname{Sym}(A)$ where $A \subset \mathbb{N}$ is cofinite and $H \subset \operatorname{Sym}(A')$. We claim that $\mathfrak{M}_S$ is a Mackey system for S.

Proof. Certainly $S \in \mathfrak{M}_S$ and $\mathfrak{M}_S$ is closed under conjugation.

For cofinite subsets A, B and subgroups $H \times \operatorname{Sym}(A)$, $K \times \operatorname{Sym}(B) \in \mathfrak{M}$, let

$$I = (H \times \operatorname{Sym}(A)) \cap (K \times \operatorname{Sym}(B)).$$

Recall Dedekind's modular law, which states that for subgroups J_1, J_2, J_3, with $J_1 \subset J_2$, we have

$$J_2 \cap J_1 J_3 = J_1(J_2 \cap J_3).$$

Putting $J_1 = \operatorname{Sym}(A \cap B), J_2 = I$, and $J_3 = \operatorname{Sym}(A' \cup B')$, and noting that $I \subset \operatorname{Sym}(A' \cup B') \times \operatorname{Sym}(A \cap B)$, we then have

$$\begin{aligned} I &= I \cap (\operatorname{Sym}(A' \cup B') \times \operatorname{Sym}(A \cap B)) \\ &= (I \cap \operatorname{Sym}(A' \cup B')) \times \operatorname{Sym}(A \cap B), \end{aligned}$$

which identifies I as being of the form $L \times \operatorname{Sym}(A \cap B)$ with $A \cap B$ cofinite and $L \subset \operatorname{Sym}((A \cap B)')$.

In order to establish that for any $X, Y \in \mathfrak{M}_S$ there are only finitely many $X \backslash S / Y$ double cosets, it is sufficient to show this for $X = Y = \operatorname{Sym}(A)$, where A is an arbitrary cofinite subset of $\mathbb{N}$. Without loss of generality, put $A = \mathbb{N} - \mathbf{m}$ for some $m \in \mathbb{N}$. Let Ω be the set of distinct m-tuples in $\mathbb{N}$, and note that we have a bijective correspondence between Ω and the set of cosets $S / \operatorname{Sym}(A)$ by assigning to a coset $\sigma \operatorname{Sym}(A)$ the tuple $(\sigma(1), \ldots, \sigma(m))$. Now $\operatorname{Sym}(A)$ acts on Ω, and we have a bijective correspondence between $\operatorname{Sym}(A) \backslash S / \operatorname{Sym}(A)$ and $\operatorname{Sym}(A) \backslash \Omega$. Since we can assign to each orbit in $\operatorname{Sym}(A) \backslash \Omega$ a tuple $(j_1, \ldots, j_m)$ with all $j_k \leq 2m$, it follows that the number of orbits is finite, and hence there are finitely many double cosets. □

Example 3.9. For a set of positive integers $A \subset \mathbb{Z}$ write $A+$ for the set $A \cup -A$, and write $\operatorname{Sym}+(A)$ for the group of permutations σ on $A+$ such that σ fixes all but finitely many elements and such that $\sigma(i) = -\sigma(-i)$ for each $i \in A$. When A is finite, $\operatorname{Sym}+(A)$ is the Weyl group of type BC on $|A|$ letters, and the subgroup of elements which reverse the sign of an even number of elements is the Weyl group of type D on $|A|$ letters.

Let $W_{BC} = \mathrm{Sym}{+}(\mathbb{N})$, and let $\mathfrak{M}_{BC}$ be the collection of subgroups of the form $H \times \mathrm{Sym}{+}(A)$ where $A \subset \mathbb{N}$ is cofinite and $H \subset \mathrm{Sym}{+}(A')$. Similarly, let W_D be the subgroup of W_{BC} of elements which reverse the sign of an even number of elements, and let $\mathfrak{M}_D$ be the subset of $\mathfrak{M}_{BC}$ of subgroups of W_D.

By a proof analogous to the one above, $\mathfrak{M}_{BC}$ is a Mackey system for W_{BC} and $\mathfrak{M}_D$ is a Mackey system for W_D.

Example 3.10. For a rational prime p, let V be the $\mathbb{F}_p$-vector space with basis $E = \{e_1, e_2, \ldots\}$, and for $I \subset E$, write $\mathrm{GL}(I, p)$ for the group of invertible linear transformations on the $\mathbb{F}_p$-span of I which fix some cofinite subset of I. Write I' for the complement of I in E. Put $G = \mathrm{GL}(E, p)$ and let M be the collection of subgroups of the form $H \times \mathrm{GL}(I, p)$, where $I \subset E$ is cofinite and $H \subset \mathrm{GL}(I', p)$. Let $\mathfrak{M}$ be the smallest set containing M which is closed under conjugation and intersection.

We claim that $\mathfrak{M}$ is a Mackey system for G.

Proof. If $A \in \mathfrak{M}$, then there a cofinite $I \subset E$ with $\mathrm{GL}(I, p) \subset A$. In order to establish the double coset property for pairs (A, B), it is then sufficient to show it for $A = B = \mathrm{GL}(I, p)$ and I an arbitrary cofinite subset. Without loss of generality put $I = \{e_i \mid i \geq m\}$ for some $m \in \mathbb{N}$.

Given any $h, h' \in G$, there exists an $n \in \mathbb{N}$ such that h, h' fix all e_i satisfying $i > n$. Let W be the $\mathbb{F}_p$-span of all e_i with $i > n$ and let W' be the span of all e_i with $i \leq n$, and note that $V = W \oplus W'$. We can then represent h and h' by $n \times n$-dimensional matrices X, X' with entries in $\mathbb{F}_p$, where $h \cdot w = w$ if $w \in W$ and $h \cdot w' = Xw'$ if $w' \in W'$, and similarly for h'. Note that h and h' belong to the same (A, A)-double coset if X can be arrived at from X' using row operations on the rows beyond row m, and column operations on the columns beyond column m. But using row and column reduction we can replace X, X' with matrices Y, Y' such that for $i, j \geq 2m$, we have $Y_{ij} = Y'_{ij} = \delta_{ij}$.

It follows that the number of double cosets is bounded above by the number of $2m \times 2m$ matrices with entries in $\mathbb{F}_p$, and hence there are only finitely many double cosets. □

For a group G and Mackey system $\mathfrak{M}$, write $\mathrm{ccs}(G, \mathfrak{M})$ for the collection of conjugacy classes of subgroups in $\mathfrak{M}$. We record basic properties of the Burnside ring of a finite group which carry over to the Burnside ring of a group with Mackey system.

Proposition 3.11. As an abelian group we have

$$A(G, \mathfrak{M}) = \bigoplus_{(H) \in \mathrm{ccs}(G, \mathfrak{M})} \mathbb{Z}[G/H].$$

Proposition 3.12. For each $(H) \in \mathrm{ccs}(G, \mathfrak{M})$, we have a homomorphism of rings $\pi_{(H)} : A(G, \mathfrak{M}) \to \mathbb{Z}$ given by mapping $[G/K]$ for $K \in \mathfrak{M}$ to $|(G/K)^H|$.

Proof. The only thing we need to check is that $|(G/K)^H|$ is finite. But to each coset gK fixed by H we can assign a unique double coset $HgK = gK$. So $|(G/K)^H| \leq |H \backslash G/K|$ is finite. □

Proposition 3.13. Combining the maps $\pi_{(H)}$ for each $(H) \in \mathrm{ccs}(G, \mathfrak{M})$ forms an injective homomorphism of rings

$$\pi : A(G, \mathfrak{M}) \to \prod_{(H) \in \mathrm{ccs}(G, \mathfrak{M})} \mathbb{Z}.$$

Proof. In order to establish injectivity, we first show that we can define a partial order on $\mathrm{ccs}(G, \mathfrak{M})$ by putting $(H) \leq (K)$ whenever H is subconjugate to K. For a proper inclusion of subgroups $H \subset K$, we can choose a set of double coset representatives for $H \backslash G/H$ such that some $k \in K - H$ is a representative. Then this defines a degenerate set of representatives for $K \backslash G/K$. Since the number of double cosets is stable under conjugation, it follows that K is not subconjugate to H, and so $\leq$ defines a partial order.

Now suppose we have an $x \in A(G; \mathfrak{M})$ with $\pi(x) = 0$. Write $x = \sum a_H[G/H]$, and let J be maximal with respect to the partial order such that $a_J \neq 0$. Now $\pi_{(J)}([G/H]) = 0$ whenever J is not subconjugate to H. Then

$$\pi_{(J)}(x) = \pi_{(J)}(a_J[G/J]) = a_J[N_G J : J] \neq 0,$$

a contradiction, and so π is injective. □

3.2 Mackey functors

Induction and restriction homomorphisms between Burnside rings are well understood, and typically described in the context of Mackey functors. In this section we quickly recall the theory of Mackey functors, and give an analogous description of induction and restriction for the Burnside ring of a group with Mackey system. There are several ways of defining Mackey functors, here we follow [18].

Let FGI be the category of finite groups, with morphisms injective homomorphisms of groups. For a finite group G and $g \in G$, write $c(g)$ for the distinguished morphism $G \to G$ given by conjugation by g.

For R a commutative ring and A a category, a bifunctor $\mathsf{A} \to R$-Mod is a pair $M = (M_*, M^*)$, where M_* is a covariant functor $\mathsf{A} \to R$-Mod and M^* is a contravariant functor $\mathsf{A} \to R$-Mod, with M_* and M^* agreeing on objects. For an object A, we write $M(A)$ for $M_*(A) = M^*(A)$. For a bifunctor $\mathsf{FGI} \to R$-Mod and an injective homomorphism of groups $f : H \to G$, denote by ind_f the map

$M_*(f) : M(H) \to M(G)$ and by res_f the map $M^*(f) : M(G) \to M(H)$. When f is an inclusion of groups, write $\mathrm{ind}_H^G = \mathrm{ind}_f$ and $\mathrm{res}_H^G = \mathrm{res}_f$.

Definition 3.14. A bifunctor $(M_*, M^*) : \mathsf{FGI} \to R\text{-Mod}$ is said to be a Mackey functor if:

1. for each distinguished morphism $c(g)$, $M_*(c(g))$ is the identity map;
2. for f an isomorphism of groups, the composites $\mathrm{res}_f \circ \mathrm{ind}_f$ and $\mathrm{ind}_f \circ \mathrm{res}_f$ are the identity map;
3. for inclusions of subgroups $H, K \subset G$,

$$\mathrm{res}_K^G \circ \mathrm{ind}_H^G = \sum_{KgH \in K\backslash G/H} \mathrm{ind}_{c(g)|_{H\cap g^{-1}Kg}} \circ \mathrm{res}^H_{H\cap g^{-1}Kg},$$

   where $c(g)|_{H\cap g^{-1}Kg}$ is regarded as a map $H \cap g^{-1}Kg \to K$.

Recall that for a homomorphism of groups $f : H \to G$ and a H-set X, we can define an equivalence relation $\sim$ on the set $G\times X$ by putting $(gf(h), x) \sim (g, h\cdot x)$ for $g \in G, h \in H, x \in X$. Write $[g, x]$ for the equivalence class of $(g, x) \in G\times X$. The set of equivalence classes $G\times_f X$ then has a natural G-action given by $g\cdot[g', x] = [gg', x]$.

Put $R = \mathbb{Z}$ and define a bifunctor $M = (M^*, M_*)$ on objects by sending a finite group G to the Burnside ring $A(G)$. For $f : H \to G$ an injective homomorphism and X a finite H-set, let $\mathrm{ind}_f(X)$ be the G-set $G \times_f X$, and for Y a finite G-set, let $\mathrm{res}_f Y$ be the set Y with H-action given by $h \cdot y = f(h)y$. It is easy to check that this then defines a Mackey functor. We will now give the appropriate generalisation for the Burnside ring of an infinite group with Mackey system.

Let GMI be the category whose objects are pairs $(H, \mathfrak{M})$ where H is a group and $\mathfrak{M}$ is a Mackey system for H. A morphism $\phi : (H, \mathfrak{M}) \to (G, \mathfrak{N})$ in GMI is given by an injective homomorphism of groups $H \to G$ (also denoted ϕ) such that $\phi(H)$ is of finite index in G, $\phi(M) \in \mathfrak{N}$ for each $M \in \mathfrak{M}$, and $\phi^{-1}(N) \in \mathfrak{M}$ for each $N \in \mathfrak{N}$.

We can identify FGI as a full subcategory of GMI by associating to each finite group G the pair $(G, \mathrm{sub}(G))$, where $\mathrm{sub}(G)$ is the set of subgroups of G.

Proposition 3.15. Define a bifunctor $M = (M_*, M^*) : \mathsf{GMI} \to \mathbb{Z}\text{-Mod}$ on objects by putting $M((H, \mathfrak{M})) = A(H, \mathfrak{M})$. For a morphism $\phi : (H, \mathfrak{M}) \to (G, \mathfrak{N})$, define $M_*(\phi) = \mathrm{ind}_\phi$ on an $\mathfrak{M}$-admissible H-set X by putting $\mathrm{ind}_\phi X = G \times_\phi X$, and extend to a homomorphism $A(H, \mathfrak{M}) \to A(G, \mathfrak{N})$. Define $M^*(\phi) = \mathrm{res}_\phi$ on an $\mathfrak{N}$-admissible G-set Y by putting $\mathrm{res}_\phi Y = Y$ with H-action $h \cdot y = \phi(h)y$, and extend to a homomorphism $A(G, \mathfrak{N}) \to A(H, \mathfrak{M})$. Then M satisfies properties $1 - 3$ of Definition 3.14.

Proof. The only thing to be checked is that the maps ind_ϕ and res_ϕ are well-defined, and the category GMI has been chosen to make sure that they are.

Indeed, let $\phi : (H, \mathfrak{M}) \to (G, \mathfrak{N})$ be a morphism and X an $\mathfrak{M}$-admissible H-set. Then for $(g, x) \in G \times_\phi X$, we have $\mathrm{Stab}_G(g, x) = g\phi(\mathrm{Stab}_H x)g^{-1}$, which is in $\mathfrak{N}$ since $\mathrm{Stab}_H x \in \mathfrak{M}$. Moreover, since $\phi(H)$ is of finite index in G, we can choose a finite set of (right) coset representatives $g_1, \ldots, g_l$ for $\phi(H)\backslash G$, and since X is $\mathfrak{M}$-admissible we have a finite generating set $x_1, \ldots, x_m$ for X as a H-set. Then $G \times_\phi X$ is finitely generated as a G-set by the elements (g_i, x_j) for $1 \le i \le l$ and $1 \le j \le m$.

Similarly, for an $\mathfrak{N}$-admissible G-set Y and $y \in Y$, note that we have $\mathrm{Stab}_H y = \phi^{-1}(\mathrm{Stab}_G(y))$ and $\mathrm{Stab}_G y \in \mathfrak{N}$, so $\phi^{-1}(\mathrm{Stab}_G y) \in \mathfrak{M}$. Moreover, let $y_1, \ldots, y_s$ be a finite set of generators for Y as a G-set. Then any $y \in Y$ is of the form gy_k for some $1 \le k \le s$, and hence of the form hg_iy_k for some $1 \le i \le l$, and so the elements g_iy_k generate Y as a H-set. □

3.3 The restricted infinite symmetric group

In Example 3.8 we introduced the restricted infinite symmetric group S and Mackey system $\mathfrak{M}_S$. In order to describe the Burnside ring $A(S, \mathfrak{M}_S)$, we first give a description of the set of conjugacy classes of subgroups in $\mathfrak{M}_S$.

Consider the class T of pairs (G, X) where G is a finite group and X is a finite set with faithful G-action. Write $\hat{G}$ for the natural embedding of G in $\mathrm{Sym}(X)$. For pairs (G, X) and (H, Y) in T and a bijection $s : X \to Y$, note that s induces a homomorphism of groups $\hat{s} : \hat{G} \to \mathrm{Sym}(Y)$ by requiring

$$\hat{s}(g)(y) = s(g(s^{-1}(y)))$$

for each $y \in Y$ and $g \in \hat{G}$. Call two pairs (G, X), (H, Y) isomorphic if there exists a bijection $X \to Y$ which induces an isomorphism of groups $\hat{G} \simeq \hat{H}$. Let $\mathcal{T}$ be the set of isomorphism classes of pairs (G, X) and write $[G, X]$ for the isomorphism class of (G, X).

Given a subgroup $G \times \mathrm{Sym}(A) \in \mathfrak{M}_S$, the group G acts faithfully on A', and so we have a pair (G, A') in T. Suppose $G \times \mathrm{Sym}(A)$ and $H \times \mathrm{Sym}(B)$ are elements of $\mathfrak{M}_S$ belonging to the same conjugacy class in $\mathrm{ccs}(S, \mathfrak{M}_S)$, with say $\sigma\,(G \times \mathrm{Sym}(A))\,\sigma^{-1} = H \times \mathrm{Sym}(B)$ for some $\sigma \in S$. Since σ conjugates $\mathrm{Sym}(A)$ to $\mathrm{Sym}(B)$, we must have that σ induces a bijection $s : A' \to B'$. The homomorphism $G \to H$ induced by this bijection is the one induced by the conjugation action of σ, and so the pairs (G, A') and (H, B') are isomorphic. We then have a well-defined map $t : \mathrm{ccs}(S, \mathfrak{M}_S) \to \mathcal{T}$.

Given a pair (G, X) in T and an embedding $\pi : X \hookrightarrow \mathbb{N}$, we have an injective homomorphism $\tilde{\pi} : G \hookrightarrow S$ by requiring $\tilde{\pi}(g)(\pi(x)) = \pi(g \cdot x)$ for each $x \in X$, and $\tilde{\pi}(g)(y) = y$ for each $y \in \pi(X)'$. In this way we can assign to the pair (G, X), and the embedding π, a subgroup $\tilde{\pi}(G) \times \mathrm{Sym}(\pi(X)')$ in $\mathfrak{M}_S$. For $\pi' : X \to \mathbb{N}$ another embedding, any $\sigma \in S$ mapping $\pi(x)$ to $\pi'(x)$ for each $x \in X$ then conjugates

$\tilde{\pi}(G) \times \mathrm{Sym}(\pi(X)')$ to $\tilde{\pi}'(G) \times \mathrm{Sym}(\pi'(X)')$, and so to a pair (G, X) we can assign a well-defined conjugacy class of subgroups in $\mathfrak{M}_S$.

Suppose pairs (G, X) and (H, Y) in T are isomorphic, with $s : X \to Y$ the bijection between X and Y inducing the isomorphism, and suppose we have some fixed embeddings $\pi_1 : X \to \mathbb{N}$, $\pi_2 : Y \to \mathbb{N}$. Our recipe above allows us to construct subgroups $\tilde{\pi}_1(G) \times \mathrm{Sym}(\pi_1(X)')$ and $\tilde{\pi}_2(H) \times \mathrm{Sym}(\pi_2(Y)')$ in $\mathfrak{M}_S$. Then for any $\sigma \in S$ satisfying $\sigma(\pi_2(s(x))) = \pi_1(x)$ for each $x \in X$, we have that σ conjugates $\tilde{\pi}_2(H) \times \mathrm{Sym}(\pi_2(Y)')$ to $\tilde{\pi}_1(G) \times \mathrm{Sym}(\pi_1(X)')$. Our construction then gives a well-defined map $c : \mathcal{T} \to \mathrm{ccs}(S, \mathfrak{M}_S)$.

The following is then immediate:

Lemma 3.16. The maps t and c as above are mutually inverse, and define a 1-1 correspondence between $\mathcal{T}$ and $\mathrm{ccs}(S, \mathfrak{M})$.

By Proposition 3.11, $A(S, \mathfrak{M}_S)$ can be regarded as free on the set $\mathrm{ccs}(S, \mathfrak{M}_S)$, and hence on the set $\mathcal{T}$. For a finite group G and finite set X with faithful G-action, write $\gamma_{[G,X]}$ for the corresponding $\mathbb{Z}$-basis element of $A(S, \mathfrak{M}_S)$.

Recall that a filtration of a ring R is a collection $\{R_i\}_{i \in \mathbb{N}}$ where each R_i is a subgroup of R viewed as an abelian group, where $R_i \subset R_{i+1}$ for each $i \in \mathbb{N}$, and where $R_i \cdot R_j \subset R_{i+j}$ for each $i, j \in \mathbb{N}$. We say that the filtration is exhaustive if $R = \bigcup_{i \in \mathbb{N}} R_i$. The graded ring $\mathrm{gr}\, R$ associated to an exhaustive filtration $\{R_i\}_{i \in \mathbb{N}}$ of a ring R is given by

$$\mathrm{gr}\, R = \bigoplus_{i \in \mathbb{N}} Q_i$$

where $Q_0 = R_0$ and $Q_i = R_i / R_{i-1}$ for $i > 0$. Multiplication is defined by putting

$$(r + Q_{i-1}) \cdot (s + Q_{j-1}) = rs + Q_{i+j-1}$$

for $r \in R_i$ and $s \in R_j$, and extending by distributivity.

Lemma 3.17. Let

$$A_n = \sum_{\substack{[G,X] \in \mathcal{T} \\ |X| \leq n}} \mathbb{Z}\gamma_{[G,X]}.$$

Then $\{A_n\}_{n \in \mathbb{N}}$ is an exhaustive filtration of $A(S, \mathfrak{M}_S)$.

Proof. We have already seen in Proposition 3.11 that $A(S, \mathfrak{M}_S) = \bigcup_n A_n$. Moreover, we have seen in Example 3.8 that computing some product

$$[S/(G \times \mathrm{Sym}(A))] \cdot [S/(H \times \mathrm{Sym}(B))]$$

gives terms of the form $[S/(L \times \mathrm{Sym}(C))]$ where $|C'| \leq |A'| + |B'|$. It follows then that $A_n \cdot A_m \subset A_{n+m}$. □

Given $[G, X]$, $[H, Y] \in \mathcal{T}$ we have a faithful action of $G \times H$ on $X \sqcup Y$ and hence a well-defined element of $\mathcal{T}$ given by $[G \times H, X \sqcup Y]$. Let $\mathcal{A}$ be the free abelian group on the set of $\zeta_{[G,X]}, [G, X] \in \mathcal{T}$. Define a multiplication on $\mathcal{A}$ by

$$\zeta_{[G,X]} \cdot \zeta_{[H,Y]} = \zeta_{[G \times H, X \sqcup Y]}.$$

One can check that this makes $\mathcal{A}$ into a unital ring, with unity given by the pair $[e, \emptyset]$, where e is the trivial group and $\emptyset$ is the empty set.

Theorem 3.18. Let $\operatorname{gr} A(S, \mathfrak{M}_S)$ be the graded ring associated to the filtration $\{A_n\}_{n \in \mathbb{N}}$ of $A(S, \mathfrak{M}_S)$. Then $\operatorname{gr} A(S, \mathfrak{M}_S) \simeq \mathcal{A}$.

We will defer the proof to the next section.

3.3.1 Constructing the multiplication operation

While we have described a basis of $A(S, \mathfrak{M}_S)$ as an abelian group in terms of the set $\mathcal{T}$, in order to compute the multiplication operation we are still required to choose embeddings into $\mathbb{N}$ and work with double cosets. In what follows we give a description of $\gamma_{[G,X]} \cdot \gamma_{[H,Y]}$ only in terms of the pairs $[G, X]$ and $[H, Y]$.

Let $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_m\}$. A partial injection $f : X \to Y$ is a (possibly empty) set of pairs (x_i, y_j), with each x_i, y_j appearing at most once. If (x_i, y_j) is a pair contained in f we write $f(x_i) = y_j$; otherwise we say that $f(x_i)$ is undefined. Let $\operatorname{im} f \subset Y$ be the set of all $y_j \in Y$ such that there exists some $x_i \in X$ with $f(x_i) = y_j$. Note that the set of partial injections $Y \to X$ is just the set of partial injections $X \to Y$ with each pair reversed. Write $\Delta_{X,Y}$ for the set of partial injections $X \to Y$.

For finite groups G and H, with G acting on a finite set X and H acting on a finite set Y, we have an action of $G \times H$ on $\Delta_{X,Y}$, where if

$$f = \{(x_{i_1}, y_{j_1}), \ldots, (x_{i_k}, y_{j_k})\},$$

then we put

$$(g, h) \cdot f = \{(g \cdot x_{i_1}, h \cdot y_{j_1}), \ldots, (g \cdot x_{i_k}, h \cdot y_{j_k})\}.$$

This partitions $\Delta_{X,Y}$ into a set of orbits $\mathcal{O}$ for the action of $G \times H$. Consider an orbit O, and $f \in O$ with stabiliser $K_O \subset G \times H$. Let $Z_O = (X \sqcup Y) - \operatorname{im}(f)$. Since K_O stabilises f, we have a well-defined action of K_O on Z_O. If G has a faithful action on X and H has a faithful action on Y, then K_O acts faithfully on Z_O. So to pairs $(G, X), (H, Y)$ in T, and $\mathcal{O}$ the set of orbits of $G \times H$ on $\Delta_{X,Y}$, we have a well-defined collection $\{(K_O, Z_O)\}_{O \in \mathcal{O}}$ in T. Moreover, it is immediate that this is well-defined up to isomorphism class in T, i.e. to pairs $[G, X], [H, Y]$ in $\mathcal{T}$ we have a well-defined collection $\{[K_O, Z_O]\}_{O \in \mathcal{O}}$ in $\mathcal{T}$.

Theorem 3.19. With notation as above, the multiplication operation in $A(S;\mathfrak{M}_S)$ is given by

$$\gamma_{[G,X]} \cdot \gamma_{[H,Y]} = \sum_{O \in \mathcal{O}} \gamma_{[K_O,Z_O]}.$$

Thus we claim that the number of orbits is equal to the number of double cosets obtained from embeddings of $[G,X]$ and $[H,Y]$ into S, and moreover that the pairs $[K_O, Z_O]$ arising from the above construction are precisely the pairs arising from our original definition of the multiplication.

Before establishing the correspondence we introduce some notation and make some preliminary observations. Fix embeddings $\pi_1 : X \to \mathbb{N}$, $\pi_2 : Y \to \mathbb{N}$ such that $x_i \mapsto i$, for $1 \leq i \leq n$, and $y_j \mapsto j$ for $1 \leq j \leq m$. The pairs $[G,X]$ and $[H,Y]$ in $\mathcal{T}$ then give rise to subgroups $U = \tilde{\pi}_1(G) \times \mathrm{Sym}(\mathbf{n}')$ and $V = \tilde{\pi}_2(H) \times \mathrm{Sym}(\mathbf{m}')$ of S belonging to $\mathfrak{M}_S$.

Note that for $\sigma, \tau \in S$, we have $\sigma V = \tau V$ if $\sigma(j) = \tau(j)$ for $1 \leq j \leq m$. We then have $U\sigma V = U\tau V$ if $\sigma(j) = \tau(j)$ whenever $j \in \mathbf{m}$ and one of $\sigma(j), \tau(j)$ is in $\mathbf{n}$.

If $\sigma \in S$ is such that $\sigma \in \mathrm{Sym}(A) \times \mathrm{Sym}(B)$ for disjoint sets $A, B \subset \mathbb{N}$, then we can define elements $\sigma_A \in \mathrm{Sym}(A)$ and $\sigma_B \in \mathrm{Sym}(B)$ by putting $\sigma_A(i) = \sigma(i)$ for each $i \in A$, and $\sigma_B(i) = \sigma(i)$ for each $i \in B$. Then $\sigma = \sigma_A \sigma_B$. Similarly, for a subgroup $K \subset \mathrm{Sym}(A) \times \mathrm{Sym}(B)$, write K_A for the subgroup $\{\sigma_A \mid \sigma \in K\}$ of $\mathrm{Sym}(A)$.

Lemma 3.20. We have a bijection between the set $U\backslash S/V$ of (U,V)-double cosets of S, and the set $\mathcal{O}$ of orbits of the action of $G \times H$ on the set $\Delta_{X,Y}$ of partial injections $X \to Y$.

Proof. For a partial injection $f : X \to Y$, write O_f for the unique orbit to which f belongs. Given a double coset $U\sigma V$, define a partial injection $f_\sigma : X \to Y$ by putting $f(x_i) = y_j$ whenever $\sigma(j) = i$ with $j \in \mathbf{m}$ and $i \in \mathbf{n}$. We define a map $s : U\backslash S/V \to \mathcal{O}$ by putting $s(U\sigma V) = O_{f_\sigma}$.

In order to make sure that this map is well-defined, we need to check that if $\sigma' \in S$ is such that $U\sigma V = U\sigma' V$, then f_σ and $f_{\sigma'}$ are in the same orbit of $\Delta_{X,Y}$. Now if $U\sigma V = U\sigma' V$ then we have $u\sigma' = \sigma v$ for some $u \in U$ and $v \in V$. We can write $u = \tilde{\pi}_1(g)\tau_u$ and $v = \tilde{\pi}_2(h)\tau_v$, where $g \in G, h \in H$, and $\tau_1 \in \mathrm{Sym}(\mathbf{n}'), \tau_2 \in \mathrm{Sym}(\mathbf{m}')$. Suppose $(x_i, y_j) \in f_\sigma$, i.e. we have $j \in \mathbf{n}$ and $i \in \mathbf{m}$ with $\sigma(j) = i$. Then

$$\begin{aligned}
(u\sigma') \cdot (\tilde{\pi}_2(h^{-1}) \cdot j) &= (\sigma v) \cdot (\tilde{\pi}_2(h^{-1}) \cdot j) \\
&= (\sigma \tilde{\pi}_2(h)\tau_2) \cdot (\tilde{\pi}_2(h^{-1}) \cdot j) \\
&= (\sigma \tilde{\pi}_2(h)) \cdot (\tilde{\pi}_2(h^{-1}) \cdot j) \\
&= \sigma \cdot j \\
&= i
\end{aligned}$$

and so $\sigma' \cdot (\tilde{\pi}_2(h^{-1}) \cdot j) = u^{-1} \cdot i = \tilde{\pi}_1(g^{-1}) \cdot i$. It follows then that for $x_i \in X$ and $y_j \in Y$ we have $(g^{-1}x_i, h^{-1}y_j) \in f_{\sigma'}$ if and only if $(x_i, y_j) \in f_\sigma$, and hence $(g, h) \cdot f_{\sigma'} = f_\sigma$.

We define a map $t : \mathcal{O} \to U\backslash S/V$ as follows. Given $O \in \mathcal{O}$ and $f \in O$, choose any $\sigma \in S$ satisfying $\sigma(j) = i$ whenever $(x_i, y_j) \in f$, and $\sigma(j) \notin \mathbf{n}$ whenever $j \in \mathbf{m}$ is such that $j \notin \operatorname{im} f$. If $\tau \in S$ is another element with this property, we have $\sigma(j) = \tau(j)$ whenever $j \in \mathbf{m}$ and one of $\sigma(j), \tau(j)$ is in $\mathbf{n}$, and hence $U\sigma V = U\tau V$. It then makes sense to define $t(O) = U\sigma V$.

Running the above argument backwards shows that this does not depend on our choice of $f \in O$, and so the map is indeed well-defined. Moreover it is immediate that the maps s and t are mutually inverse, so we have a bijection between $\mathcal{O}$ and $U\backslash S/V$. □

Lemma 3.21. Let $U\sigma V$ be a (U, V)-double coset of S, let $O = s(U\sigma V)$, and let $K_O = \operatorname{Stab}_{G\times H} f_\sigma$. Consider the element $(U, \sigma V)$ of the S-set $S/U \times S/V$ and put

$$L = \operatorname{Stab}_S((U, \sigma V)) = \tilde{\pi}_1(G) \times \operatorname{Sym}(\mathbf{n}) \cap \sigma(\tilde{\pi}_2(H) \times \operatorname{Sym}(\mathbf{m}))\sigma^{-1}$$

and $K = L_{\mathbf{n}\cup\sigma(\mathbf{m})}$. Then the groups K and K_O are isomorphic.

Proof. Given $(g, h) \in K_O$ and $(x_i, y_j) \in f_\sigma$, we have $(g{\cdot}x_i, h{\cdot}y_j) = (x_{\tilde{\pi}_1(g)(i)}, y_{\tilde{\pi}_2(h)(j)}) \in f_\sigma$, and so the set

$$\mathbf{n}_X = \{i \in \mathbf{n} \mid (x_i, y_j) \in f_\sigma \text{ for some } y_j \in Y\}$$

is invariant under the natural action of $\tilde{\pi}_1(g)$. It follows then that

$$\tilde{\pi}_1(g) \in \operatorname{Sym}(\mathbf{n}_X) \times \operatorname{Sym}(\mathbf{n} - \mathbf{n}_X)$$

and hence we have a well-defined element $\tilde{\pi}_1(g)_{\mathbf{n}-\mathbf{n}_X}$. Similarly we define

$$\mathbf{m}_Y = \{j \in \mathbf{m} \mid (x_i, y_j) \in f_\sigma \text{ for some } x_i \in X\}$$

and note that

$$\sigma\tilde{\pi}_2(h)\sigma^{-1} \in \operatorname{Sym}(\sigma(\mathbf{m}_Y)) \times \operatorname{Sym}(\sigma(\mathbf{m} - \mathbf{m}_Y)).$$

We then define a map $\phi : K_O \to K$ by

$$\phi(g, h) = \tilde{\pi}_1(g)_{\mathbf{n}-\mathbf{n}_X}\sigma\tilde{\pi}_2(h)\sigma^{-1}.$$

Certainly $\phi(g, h) \in \operatorname{Sym}(\mathbf{n} \cup \sigma(\mathbf{m}))$, so in order to show that $\phi(g, h) \in K$ it is sufficient to show that it stabilises $(U, \sigma V)$. This will follow from showing that we have

$$\phi(g,h) \in \mathrm{Sym}(\mathbf{n}) \times \mathrm{Sym}(\sigma(\mathbf{m}) - \mathbf{n})$$

with $\phi(g,h)_{\mathbf{n}} = \tilde{\pi}_1(g)$, and

$$\phi(g,h) \in \mathrm{Sym}(\sigma(\mathbf{m})) \times \mathrm{Sym}(\mathbf{n} - \sigma(\mathbf{m}))$$

with $\phi(g,h)_{\sigma(\mathbf{m})} = \sigma\tilde{\pi}_2(g)\sigma^{-1}$. This latter equation is clear from the definition of $\phi(g,h)$, so it only remains to show the former.

Now if $i \in \mathbf{n} - \mathbf{n}_X$, then certainly $\phi(g,h)(i) = \tilde{\pi}_1(g)(i)$. If $i \in \mathbf{n}_X$, with say $(x_i, y_j) \in f_\sigma$, then we have that $\sigma(j) = i$. Suppose $g \cdot x_i = x_k$. Then we have $(x_k, y_l) \in f_\sigma$ with $h \cdot y_j = y_l$. It follows then that $\sigma(l) = j$. Now

$$\begin{aligned}\sigma\tilde{\pi}_2(h)\sigma^{-1}(i) &= \sigma\tilde{\pi}_2(h)(j)\\ &= \sigma(l)\\ &= k\\ &= \tilde{\pi}_1(g)(i)\end{aligned}$$

as required, and so $\phi(g,h)_{\mathbf{n}} = \tilde{\pi}_1(g)$.

Conversely, given some element $\tau \in K$, it is clear that τ must then be invariant on $\mathbf{n}$ and $\sigma(\mathbf{m})$. Reversing our argument above, we can then pull back $\tau_{\mathbf{n}}$ along $\tilde{\pi}_1$ to an element of G and $\tau_{\sigma(\mathbf{m})}$ along $\tilde{\pi}_2$ to an element of H, and this defines a map $K \to K_O$ which is the inverse of ϕ.

Finally, since

$$\begin{aligned}\phi(g,h) \cdot (g',h') &= \tilde{\pi}_1(g)_{\mathbf{n}-\mathbf{n}_X}\sigma\tilde{\pi}_2(h)\sigma^{-1}\tilde{\pi}_1(g')_{\mathbf{n}-\mathbf{n}_X}\sigma\tilde{\pi}_2(h')\sigma^{-1}\\ &= \tilde{\pi}_1(g)_{\mathbf{n}-\mathbf{n}_X}\tilde{\pi}_1(g')_{\mathbf{n}-\mathbf{n}_X}\sigma\tilde{\pi}_2(h)\pi_2(h')\sigma^{-1}\\ &= \tilde{\pi}_1(gg')_{\mathbf{n}-\mathbf{n}_X}\sigma\tilde{\pi}_2(hh')\sigma^{-1}\\ &= \phi(gg',hh'),\end{aligned}$$

the map ϕ is indeed a homomorphism of groups, and so we have $K \simeq K_O$. □

Lemma 3.22. Let K_O act on $Z_O = (X \sqcup Y) - \mathrm{im}(f)$, and K on $Z = \mathbf{n} \cup \sigma(\mathbf{m})$. Then (K_O, Z_O) and (K, Z) are isomorphic.

Proof. Let $\psi : Z_O \to Z$ be defined by putting $x_i \mapsto i$ for $1 \le i \le n$, and $y_j \mapsto \sigma(j)$ for all $j \in \mathbf{m}$ where y_j is not in the image of f. Let $\phi : K_O \to K$ be as defined in the proof of Lemma 3.21. Suppose $x_i \in X$. Then

$$\begin{aligned}\psi((g,h)\cdot x_i) &= \psi(x_{\tilde{\pi}_1(g)(i)})\\ &= \tilde{\pi}_1(g)(i)\\ &= \phi((g,h))\psi(x_i).\end{aligned}$$

Similarly, let $y_i \in Y - \mathrm{im}(f)$. Then

$$\begin{aligned}\psi((g,h)\cdot y_i) &= \psi(y_{\tilde{\pi}_2(h)(i)})\\ &= \sigma\tilde{\pi}_2(h)(i)\\ &= \sigma\tilde{\pi}_2(h)\sigma^{-1}\sigma(i)\\ &= \phi((g,h))\psi(y_i).\end{aligned}$$

The homomorphism $K_O \to \mathrm{Sym}(Z)$ induced by ψ is then the isomorphism ϕ as defined in Lemma 3.21, and so the pairs (K_O, Z_O) and (K, Z) are isomorphic. □

This completes the proof of Theorem 3.19. Theorem 3.18 now follows as an easy corollary:

Proof of Theorem 3.18. With notation as in Lemma 3.17, consider $\gamma_{[G,X]} \in A_n$ and $\gamma_{[H,Y]} \in A_m$. There is a unique partial injection $f : X \to Y$ satisfying $|(X \sqcup Y) - \mathrm{im}(f)| = |X| + |Y|$ given by the empty set. We have a corresponding orbit O with $(K_O, Z_O) = (G \times H, X \sqcup Y)$. Then in $\mathrm{gr}\, A(S, \mathfrak{M}_S)$ we have

$$(\gamma_{[G,X]} + A_{n-1}) \cdot (\gamma_{[H,Y]} + A_{m-1}) = \gamma_{[G\times H, X\sqcup Y]} + A_{n+m-1},$$

and so the isomorphism of abelian groups $\mathrm{gr}\, A(S, \mathfrak{M}_S) \to \mathcal{A}$ defined by putting $\gamma_{[G,X]} + A_{n-1} \mapsto \zeta_{[G,X]}$ is then an isomorphism of rings. □

3.3.2 Mackey system of Young subgroups

Recall that for a composition λ of n, we have a corresponding Young subgroup S_λ of S_n. Let $\mathfrak{Y}$ be the collection of all subgroups of the form $M_\lambda = S_\lambda \times \mathrm{Sym}(\mathbf{n}')$ together with their conjugates (and for λ the empty composition put $M_\lambda = S$). It is clear that $\mathfrak{Y}$ is a Mackey system for S, and that the set of conjugacy classes $\mathrm{ccs}(S, \mathfrak{Y})$ is in 1-1 bijection with the set of partitions. We have a Burnside ring $A(S, \mathfrak{Y})$ and a natural inclusion of rings $A(S, \mathfrak{Y}) \subset A(S, \mathfrak{M}_S)$. Write γ_λ for the element $\gamma_{[S_\lambda, \mathbf{n}]}$ of $A(S, \mathfrak{Y})$. Then for another partition μ, we have coefficients $c^\nu_{\lambda,\mu}$ defined by

$$\gamma_\lambda \cdot \gamma_\mu = \sum c^\nu_{\lambda,\mu} \gamma_\nu.$$

In order to describe the coefficients $c^\nu_{\lambda,\mu}$, we examine how the multiplication rule described above behaves in the subring $A(S, \mathfrak{Y})$.

Let $\lambda = (\lambda_1, \ldots, \lambda_p)$ be a partition of n and $\mu = (\mu_1, \ldots, \mu_q)$ a partition m. A numbering of μ by λ is a tuple $(t_{i,j})$, $0 \le i \le p$, $0 \le j \le q$, such that $t_{0,0} = 0$,

$$\sum_{0 \le j \le q} t_{i,j} = \lambda_i$$

for each $1 \le i \le p$, and

$$\sum_{0 \le i \le p} t_{i,j} = \mu_j$$

for each $1 \le j \le q$.

We can also describe numberings in terms of young tableau. Consider the young tableau associated to μ, and partially number the boxes using the numbers $1, \dots, p$ such that each i appears no more than λ_i times. Call this a tableau numbering of μ by λ. For $1 \le i \le p$ and $1 \le j \le q$, let $s_{i,j}$ denote the number of boxes numbered i in row j of the tableau. For $1 \le j \le q$, let $s_{0,j}$ be the number of boxes in row j left unnumbered; for $1 \le i \le p$, define $s_{i,0}$ by

$$\lambda_i - \sum_{1 \le j \le q} s_{i,j},$$

and put $s_{0,0} = 0$. It is clear that the tuple $S = (s_{i,j})$ then describes a numbering of μ by λ, and that any other tableau numbering of μ by λ which can be obtained by swapping boxes within the same row gives rise to the same tuple.

Define the join $J(\lambda, \mu, T)$ of λ and μ with respect to the numbering T to be the partition ν associated to T viewed as a composition.

Proposition 3.23. The coefficient $c^{\nu}_{\lambda,\mu}$ is equal to the number of numberings T of μ by λ such that $J(\lambda, \mu, T) = \nu$.

Proof. Let $f : \mathbf{n} \to \mathbf{m}$ be a partial injection. We can associate a tableau numbering of μ by λ to f by putting the number i in the jth box of the tableau if $f(l) = j$ and $\sum_{k=1}^{i-1} \lambda_k < l \le \sum_{k=1}^{i} \lambda_k$. The orbit of f under the action of $S_\lambda \times S_\mu$ then corresponds to all all tableau formed by swapping boxes within the same row; so to an orbit O we can associate a numbering T_O of μ by λ, and to each numbering we can associate a unique orbit. Let $\tau = J(\lambda, \mu, T_O)$. An element of $S_\lambda \times S_\mu$ then stabilises f precisely when it belongs to the subgroup S_τ. □

3.3.3 Refining the ghost ring

For sets A, B, write B^A for the set of functions $A \to B$. Then

$$B^A \simeq \prod_{a \in A} B,$$

where a function $f : A \to B$ corresponds to a tuple $(f(a))_{a \in A} \in \prod_{a \in A} B$.

Recall that for a finite group G, we have an embedding

$$A(G) \hookrightarrow \mathrm{Gh}(\mathrm{ccs}(G)) = \prod_{(H) \in \mathrm{ccs}(G)} \mathbb{Z} \simeq \mathbb{Z}^{\mathrm{ccs}(G)},$$

and we can regard each $a \in A$ as a function $\mathrm{ccs}(G) \to \mathbb{Z}$ where $a((H)) = \pi_{(H)}(a)$.

Both $\mathbb{Z}^{\mathrm{ccs}(G)}$ and $A(G)$ have $\mathbb{Z}$-module rank equal to $|\,\mathrm{ccs}(G)|$, and $A(G)$ has finite index in $\mathbb{Z}^{\mathrm{ccs}(G)}$. Similarly, for G a group and $\mathfrak{M}$ a Mackey system for G, we have an embedding

$$A(G,\mathfrak{M}) \hookrightarrow \mathbb{Z}^{\mathrm{ccs}(G,\mathfrak{M})}.$$

In the case of the restricted infinite symmetric group S, we have seen that $A(S,\mathfrak{M}_S)$ has countable $\mathbb{Z}$-module rank. However, $\mathbb{Z}^{\mathrm{ccs}(S,\mathfrak{M}_S)}$ does not, and so it seems reasonable to ask if there is a more suitable choice of ghost ring.

For $(H),(K)$ in $\mathrm{ccs}(S,\mathfrak{M}_S)$, write $(H) \leq (K)$ if H is subconjugate to K, and recall that by the proof of Proposition 3.13 that this defines a partial order on $\mathrm{ccs}(S,\mathfrak{M}_S)$. For $(K) \in \mathrm{ccs}(S,\mathfrak{M}_S)$, define

$$W((K)) = \{(H) \in \mathrm{ccs}(S,\mathfrak{M}_S) \mid (H) \leq (K)\}$$

and let $\mathcal{J}$ be the topology on $\mathrm{ccs}(S,\mathfrak{M}_S)$ generated by requiring that $W((K))$ is both open and closed, for each $(K) \in \mathrm{ccs}(S,\mathfrak{M}_S)$[2]. Let $C(S) \subset \mathbb{Z}^{\mathrm{ccs}(S,\mathfrak{M}_S)}$ be the set of continuous functions $\mathrm{ccs}(S,\mathfrak{M}_S) \to \mathbb{Z}$ with respect to the discrete topology on $\mathbb{Z}$.

Proposition 3.24. The embedding $\pi : A(S,\mathfrak{M}_S) \hookrightarrow \mathbb{Z}^{\mathrm{ccs}(S,\mathfrak{M}_S)}$ has image contained in $C(S)$.

We first give some preliminary definitions and observations. Let $K = G\times \mathrm{Sym}(A)$ for some cofinite set A and some $G \subset \mathrm{Sym}(A')$. The action of K partitions the set A' into a set of orbits $\mathcal{O}_K$. We define the signature of K to be the tuple $b = (b_1, \ldots, b_m)$ recording the size of these orbits in non-increasing order. We define the signature of a conjugacy class of subgroups (K) to be the signature of any $K' \in (K)$ and note that this is well-defined. For a tuple $b = (b_1, \ldots, b_m)$, let $\sum_{i=1}^{m} b_i$ be the length of b.

Lemma 3.25. For each $m \in \mathbb{N}$ there are only finitely many $(K) \in \mathrm{ccs}(S,\mathfrak{M}_S)$ such that the length of the signature of (K) is $\leq m$.

Proof. Suppose $(K) \in \mathrm{ccs}(S,\mathfrak{M}_S)$ has signature of length m. Then there is some $\hat{K}$ conjugate to K with $\hat{K} \subset \mathrm{Sym}(\mathbf{m}) \times \mathrm{Sym}(\mathbf{m}')$. It follows then that the number of elements of $\mathrm{ccs}(S,\mathfrak{M}_S)$ with signature of length m is bounded above by the number of subgroups of $\mathrm{Sym}(\mathbf{m})$, and hence the number of elements with signature of length $\leq m$ is finite. □

With K as above and $\sigma \in S$, note that each element of σK must map each orbit $O \in \mathcal{O}_K$ into $\sigma(O) = \{\sigma(a) \mid a \in O\}$. It follows that for $J \in \mathfrak{M}_S$, a necessary

[2]For G a compact Lie group, $\mathfrak{C}$ the set of conjugacy classes of closed subgroups of G with finite Weyl group, and $(K) \in \mathfrak{C}$, let $\mathfrak{W}((K)) = \{(H) \in \mathfrak{C} \mid (H) \leq (K)\}$. Then the topology on $\mathfrak{C}$ generated by requiring that $\mathfrak{W}((K))$ be both open and closed for each $(K) \in \mathfrak{C}$ is precisely the topology on $\mathfrak{C}$ induced by the Lie group structure of G. A Burnside ring $A(G)$ can be defined for G, and an embedding $A(G) \hookrightarrow \mathbb{Z}^{\mathfrak{C}}$. With respect to the induced topology on $\mathfrak{C}$ and the discrete topology on $\mathbb{Z}$, this is an embedding into the set of continuous functions $\mathfrak{C} \to \mathbb{Z}$. See [11] for details.

condition for J to fix the coset σK is for J to be invariant on the set $\sigma(O)$ for each $O \in \mathcal{O}_K$, i.e. each $\sigma(O)$ must be a union of orbits in $\mathcal{O}_J$. For an orbit $Q \in \mathcal{O}_J$, we say that Q is involved in σK if $Q \subset \sigma(O)$ for some $O \in \mathcal{O}_K$.

Suppose $J = H \times \mathrm{Sym}(B)$ for some cofinite set B and some $H \subset \mathrm{Sym}(B')$. For $Q \in \mathcal{O}_J$ and $h \in H$, we have $h \in \mathrm{Sym}(B' \cap Q') \times \mathrm{Sym}(Q) \times \mathrm{Sym}(B)$ and so can consider the element $h_{B' \cap Q'}$. We define the expansion of J by Q to be the subgroup $J' = H' \times \mathrm{Sym}(B \cup Q)$ in $\mathfrak{M}_S$ where

$$H' = \{h_{B' \cap Q'} \mid h \in H\}.$$

Note that $(J) \leq (J')$, and that if Q is not involved in σK, then J fixes σK if and only if J' does.

Proof of Proposition 3.24. We wish to show that for each $a \in A(S, \mathfrak{M}_S)$ the map $\mathrm{ccs}(S, \mathfrak{M}_S) \to \mathbb{Z}$ given by $(H) \mapsto \pi_{(H)}(a)$ is continuous. Since each a can be written as a sum $a = \sum_{(K) \in \mathrm{ccs}(S, \mathfrak{M}_S)} a_{(K)}[S/K]$ where $a_{(K)} \in \mathbb{Z}$ for each $(K) \in \mathrm{ccs}(S, \mathfrak{M}_S)$ and where only finitely many of the terms $a_{(K)}$ are non-zero, it is sufficient to show that for each K the map

$$\phi^{(K)} : (H) \mapsto |G/K^H|$$

is continuous. In order to show this, it is sufficient to show that the pre-image X_n of any $\{n\} \in \mathbb{Z}$ is open. We will do this for $n > 0$ by showing that each $Y_n = (\phi^{(K)})^{-1}([n, \infty])$ is both closed and open, and then observing that $X_n = Y_n \cap (C(S) - Y_{n+1})$. For $n = 0$ we note that $X_0 = C(S) - Y_1$, and for $n < 0$ we note that X_n is the empty set.

Suppose $n > 0$. We claim that there are finitely many maximal elements in Y_n with respect to the subconjugacy partial order. We will show this by establishing a bound on the length of the signature of a maximal element, and then applying Lemma 3.25. Let (K) have signature $(b_1, \ldots, b_q)$.

Suppose (J) is maximal, and some orbit Q of J is not involved in any σK in $(G/K)^J$. Then for J' the expansion of J by Q, we have $(J') \in Y_n$, and $(J) \leq (J')$, contradicting maximality of (J). So each orbit of J must be involved in some σK in $(G/K)^J$.

Let (J) have signature $(c_1, \ldots, c_r)$. Any orbit of J of cardinality greater than b_1 clearly can't be involved in any σK, so we have $c_i \leq b_1$ for each $1 \leq i \leq r$. It remains to establish a bound on r.

Each orbit of K can be a union of at most b_1 orbits of J, and so each σK fixed by J can have at most $b_1 q$ orbits of J involved in it. Since each orbit of J must be involved in some σK fixed by J, we have $\phi^{(K)}((J)) \geq \frac{r}{b_1 q}$. If $r > b_1 q n$, J can not be maximal, as expanding J by any orbit would give a J' with $\phi^{(K)}((J')) \geq n$. So we must have $r \leq b_1 q n$.

Thus the length of the signature of a maximal element is bounded by $b_1^2 q n$, and so there are finitely many maximal elements.

Since $(H_1) \leq (H_2)$ implies $\phi^{(K)}(H_1) \geq \phi^{(K)}(H_2)$, we have that

$$Y_n = W(J_1) \cup \ldots \cup W(J_p)$$

for $J_1, \ldots, J_p$ a complete set of maximal elements in Y_n, and hence Y_n is both open and closed. □

Chapter 4

Further B-rings and B'-rings

4.1 Rational representation rings

Let G be a finite group. The isomorphism classes of finite-dimensional rational representations (i.e. linear representations over $\mathbb{Q}$) of G form a commutative semi-ring with unit, with addition given by direct sum and multiplication given by tensoring over $\mathbb{Q}$. The rational representation ring $RQ(G)$ is the Grothendieck ring associated to this semi-ring. For ϕ a finite-dimensional rational representation of G, write $[\phi]$ for the isomorphism class of ϕ in $RQ(G)$. We will first recall some basic facts about rational representation rings; for proofs and further details see [7] (Chapter V).

Let $\mathrm{cccs}(G)$ denote the set of conjugacy classes of cyclic subgroups of G, and let $\phi_1, \ldots, \phi_r$ be a complete irredundant set of irreducible finite-dimensional rational representations. Then $r = |\,\mathrm{cccs}(G)|$, and as an abelian group $RQ(G)$ is free on the elements $[\phi_1], \ldots, [\phi_r]$. For a vector space V and a linear operator $A : V \to V$, write $\mathrm{tr}\, A$ for the trace of A.

For each $g \in G$, we have a well-defined homomorphism of rings

$$\pi_g : RQ(G) \to \mathbb{Q},$$

where for a rational representation ϕ we put

$$\pi_g([\phi]) = \mathrm{tr}\, \phi(g)$$

and extend linearly. For a complex linear representation ϕ and $g \in G$, $\mathrm{tr}\, \phi(g)$ is an algebraic integer, so π_g has image in $\mathbb{Z}$. For $g, g' \in G$, $\pi_g = \pi_{g'}$ if and only if $\langle g \rangle$ is conjugate to $\langle g' \rangle$ in G, where $\langle g \rangle$ denotes the cyclic subgroup of G generated by g. For a cyclic subgroup $H = \langle g \rangle$, write π_H for the map π_g, and for $r \in RQ(G)$ write $r(H)$ for $\pi_H(r)$.

Lemma 4.1. The maps π_H combine to give an embedding

$$\pi : RQ(G) \to \prod_{(H) \in \mathrm{cccs}(G)} \mathbb{Z},$$

and this embedding defines a B-ring.

Proof. The only thing we need to check is that for H, J non-conjugate cyclic subgroups, we can find $r \in RQ(G)$ with $r(H) \neq 0$ and $r(J) = 0$. Given a G-set X, we have a rational representation φ_X defined on the vector space $\mathbb{Q}X$ with basis $\{e_x, x \in X\}$, where $\varphi_X(g)(e_x) = e_{g \cdot x}$. Then $\pi_H([\varphi_X])$ is just $|X^H|$. So our result follows from Lemma 1.11. □

As in the case of the Burnside ring, for $H, J \subset G$ non-conjugate cyclic subgroups we write $d(H, J)$ for $d_{RQ(G)}((H), (J))$. For $P \in \operatorname{Spec} \mathbb{Z}$ we write $q(H, P)$ for $q_{RQ(G)}((H), P)$.

For a rational prime p, each $g \in G$ can be written uniquely as $g = g_u g_r$ with g_u having order divisible by p and g_r having order coprime to p. Similarly, for $H \subset G$ a cyclic subgroup, we have a decomposition $H = H_u \times H_r$, where H_u is a cyclic subgroup of elements having p-power order and H_r a cyclic subgroup of elements having order prime to p. Then $O^p(H) = H_r$.

Proposition 4.2. Let $H, J \subset G$ be non-conjugate cyclic subgroups and p a rational prime. Then $p \mid d(H, J)$ if and only if $O^p(H)$ is conjugate to $O^p(J)$.

Proof. If $O^p(H)$ is not conjugate to $O^p(J)$, then by Lemma 1.12 and Proposition 1.9 there is a G-set X with $p \nmid |X^H| - |X^J|$. Letting φ_X be the corresponding permutation representation, we then have $p \nmid \pi_H([\varphi_X]) - \pi_J([\varphi_X])$, and so $p \nmid d(H, J)$.

The converse follows immediately from a more general result of Serre:

Lemma 4.3 ([21] Section 10.3, Lemma 7)**.** Let A be the subring of $\mathbb{C}$ generated by the $|G|$th roots of unity, let $R(G)$ be the ring of complex characters, and let χ be an element of $A \otimes_{\mathbb{Z}} R(G)$ which takes rational integer values. Then

$$\chi(g) \equiv \chi(g_r) \mod p.$$

□

Corollary 4.4. Let $H, J \subset G$ be non-conjugate cyclic subgroups and p a rational prime. Then $p \mid d_{RQ(G)}(H, J)$ if and only if $p \mid d_{A(G)}(H, J)$.

Proof. By Proposition 4.2, p divides $d_{RQ(G)}(H, J)$ if and only if $O^p(H)$ is conjugate to $O^p(J)$, and by Proposition 1.12 this is the case if and only if $p \mid d_{A(G)}(H, J)$. □

Since $RQ(G)$ is a B-ring, we have that each prime ideal is of the form $q(H, (p))$, for H a cyclic subgroup and p a rational prime or zero. Proposition 4.2 together with Proposition 1.9 then gives the following description of $\operatorname{Spec} RQ(G)$.

Corollary 4.5. For H, J cyclic subgroups of G and p a rational prime, $q(H, (p)) = q(J, (p))$ if and only if p is a rational prime and $O^p(H)$ is conjugate to $O^p(J)$.

Write $\mathbb{Z}_H$ for the $RQ(G)$-module $\mathbb{Z}_{(H)}$. Proposition 4.2 together with Proposition 1.7 then gives the following description of the degree 1 cohomology.

Corollary 4.6. Let H, J be cyclic subgroups of G. Then $\mathrm{Ext}^1_{RQ(G)}(\mathbb{Z}_H, \mathbb{Z}_J)$ is non-zero if and only if $(H) \neq (J)$ and $(O^p(H)) = (O^p(J))$ for some p, in which case $\mathrm{Ext}^1_{RQ(G)}(\mathbb{Z}_H, \mathbb{Z}_J)$ has a unique p-power summand.

Write $\overline{RQ(G)}$ for $RQ(G) \otimes \mathbb{F}_p$. We then have the following corollaries from the results of Chapter 1.5 together with Proposition 4.2.

Corollary 4.7. Suppose $p \nmid |G|$. Then $\overline{RQ(G)}$ is semisimple.

Proof. By the proof of Corollary 1.29 we have $p \nmid d_{A(G)}(H, J)$ for each pair H, J of non-conjugate subgroups. Then by Corollary 4.4, $p \nmid d_{RQ(G)}(H, J)$ for each pair H, J of non-conjugate cyclic subgroups, and by Corollary 1.26 $\overline{RQ(G)}$ is semisimple. □

Corollary 4.8. Let $H, J \subset G$ be cyclic subgroups such that $O^p(H)$ is not conjugate to $O^p(J)$ for any rational prime p. Then

$$\mathrm{Ext}^l_{RQ(G)}(\mathbb{Z}_H, \mathbb{Z}_J) = 0$$

for all $l \geq 0$.

Proof. By Proposition 4.2, $p \nmid d(H, J)$ for any rational prime p, and the result follows by Corollary 1.27. □

Corollary 4.9. Suppose $p \mid |G|$ and $p^2 \nmid |G|$. Let H be a non-trivial p-subgroup of G. For $J \subset G$ a cyclic subgroup and $l \geq 1$, write $M_{J,l}$ for the p-part of $\mathrm{Ext}^l_{RQ(G)}(\mathbb{Z}_J, \mathbb{Z}_J)$; for $K \subset G$ a cyclic subgroup not conjugate to J and $l \geq 1$, write $N_{J,K,l}$ for the p-part of $\mathrm{Ext}^l_{RQ(G)}(\mathbb{Z}_J, \mathbb{Z}_K)$. Then

1. $M_{J,l}$ is non-zero for some $l \geq 1$ if and only if $(J) \in \{(H), (e)\}$,
2. $N_{J,K,l}$ is non-zero for some $l \geq 1$ if and only if $\{(J), (K)\} = \{(H), (e)\}$,
3. if $(J) \in \{(H), (e)\}$ then

$$M_{J,l} \simeq \begin{cases} 0 & \text{if } l \text{ odd} \\ \mathbb{Z}/p\mathbb{Z} & \text{if } l \text{ even,} \end{cases}$$

4. if $\{(J), (K)\} = \{(H), (e)\}$, then

$$N_{J,K,l} \simeq \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{if } l \text{ odd} \\ 0 & \text{if } l \text{ even.} \end{cases}$$

Proof. Write e for the trivial subgroup of G. Since $p^2 \nmid |G|$, the subgroup H is a Sylow p-subgroup of G, and all non-trivial p-subgroups of G are conjugate to H. Then for non-conjugate cyclic subgroups J, J' in G, we have $p \mid d(J, J')$ if and only if $\{(J),(J')\} = \{(H),(e)\}$. Let $\mathcal{E}$ be the set of equivalence classes of the relation $\sim_p$ on $\mathrm{cccs}(G)$. Then $\{(e),(H)\}$ is the only class of cardinality 2, and all other classes have cardinality 1. The remainder of the proof follows that of Corollary 1.31. □

Corollary 4.10. Suppose $|G|$ is square-free. Then for all cyclic subgroups $H, J \subset G$, $\mathrm{Ext}^l_{RQ(G)}(\mathbb{Z}_H, \mathbb{Z}_J)$ is either zero for all $l \geq 1$, or periodic with period 2.

4.2 B'-rings

Another natural candidate for consideration within the framework of B-rings is the ordinary representation ring $R(G)$ of a finite group G, i.e. the Grothendieck ring associated to the semi-ring of isomorphism classes of finite-dimensional complex linear representations. However $R(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ is not necessarily a product of 1-dimensional $\mathbb{Q}$-algebras, and so by Proposition 1.3, $R(G)$ is not isomorphic to a B-ring. The obvious thing to do is relax this condition.

Definition 4.11. A B'-ring is a commutative torsion-free reduced ring which is of finite rank as a $\mathbb{Z}$-module.

This raises the question of the appropriate embedding for such a ring. For I a finite set and $\mathcal{A}_I = \{A_i\}_{i\in I}$ a multiset of subrings of rings of algebraic integers, define $\mathrm{Gh}'(I) = \prod_{i\in I} A_i$. For a subring $R \subset \mathrm{Gh}'(I)$ let π_i be the projection $R \to A_i$ corresponding to $i \in I$. For $r \in R$ write $r(i)$ for $\pi_i(r)$.

Proposition 4.12. A ring S is a B'-ring if and only if there exists a finite set I and a multiset $\mathcal{A}_I$ of subrings of rings of algebraic integers, together with an isomorphism $S \simeq R \subset \mathrm{Gh}'(I)$ such that for each distinct pair $i, j \in I$, there exists an $r \in R$ with $r(i) \neq 0$ and $r(j) = 0$, and such that for each $i \in I$, the associated projection map $\pi_i : R \to A_i$ is surjective.

Proof. Since $\mathbb{Q} \otimes_{\mathbb{Z}} S$ is a finite-dimensional commutative reduced $\mathbb{Q}$-algebra, by the Artin-Wedderburn theorem ([4] Theorem 1.3.5) we have an isomorphism

$$\theta : \mathbb{Q} \otimes S \to \prod_{i\in I} E_i,$$

for some finite set I and some multiset $\mathcal{E}_I = \{E_i\}_{i\in I}$ of finite field extensions of $\mathbb{Q}$. For each $i \in I$, let A'_i be the ring of algebraic integers of E_i. As in the proof of Proposition 1.3, we identify $R = \theta(1 \otimes S)$ as a subring of $\prod_{i\in I} A'_i$ isomorphic to S. For each $i \in I$ we have a projection map

$$\pi_i : R \to A'_i$$

onto the factor corresponding to $i \in I$. Let A_i be the subring $\pi_i(R) \subset A'_i$. We can then regard R as sitting inside $\mathrm{Gh}'(I) = \prod_{i\in I} A_i$. The remainder of the proof follows that of Proposition 1.3. □

For I a finite set, $\mathcal{A}_I = \{A_i\}_{i\in I}$ a multiset of subrings of rings algebraic integers, and $R \subset \mathrm{Gh}'(I)$ a B'-ring, define an R-module structure on A_i by

$$r \cdot n = r(i)n.$$

For the remainder of this chapter let I be some fixed finite set and $\mathcal{A}_I$ some multiset of subrings of rings of algebraic integers.

We restate results on B-rings which follow in this more general setting without any adjustment.

Lemma 4.13. Let $R \subset \mathrm{Gh}'(I)$ be a B'-ring. Then the modules $\mathrm{Ext}^l_R(A_i, A_j)$ are finite for any $l \geq 1$ and $i, j \in I$.

Lemma 4.14. Let $R \subset \mathrm{Gh}'(I)$ be a B'-ring. Then

$$\mathrm{Hom}_R(A_i, A_j) \simeq \begin{cases} 0 & \text{if } i \neq j \\ A_i & \text{if } i = j. \end{cases}$$

Recall that for $R \subset \mathrm{Gh}(I)$ a B-ring, we defined positive integers $d(i,j)$ for distinct $i, j \in I$ such that

i. $\mathrm{Ext}^1_R(\mathbb{Z}_i, \mathbb{Z}_j) \simeq \mathbb{Z}_j/d(j,i)\mathbb{Z}_j$;

ii. $q(i,(p)) = q(j,(p))$ if and only if $p \mid d(i,j)$.

We can't define $d(i,j)$ as before, as we do not necessarily have an ordering on a given subring of a ring of algebraic integers.

Definition 4.15. Let $R \subset \mathrm{Gh}'(I)$ be a B'-ring, and for $j \in I$ let K_j be the kernel of the map $\pi_i : R \to A_j$. For $i \in I$ with $i \neq j$, define $d_R(i,j)$ to be the ideal $\pi_i(K_j)$ of A_j. When the ring R is clear, we will simply write $d(i,j)$.

While it no longer makes sense to ask if $d(i,j) = d(j,i)$, we do however have the following.

Lemma 4.16. $A_i/d(i,j)A_i \simeq A_j/d(j,i)A_j$.

Proof. Note that

$$d(i,j) = \pi_i(K_j) \simeq K_j/(K_j \cap K_i) \simeq (K_i + K_j)/K_i.$$

Since $R/K_i \simeq A_i$, we have

$$\begin{aligned} A_i/d(i,j) \simeq& (R/K_i)/((K_i + K_j)/K_i) \\ &\simeq R/(K_i + K_j) \\ &\simeq (R/K_j)/((K_i + K_j)/K_j) \\ &\simeq A_j/d(j,i). \end{aligned}$$

□

For A a subring of a ring of algebraic integers, E its field of fractions, and $J \subset A$ an ideal, we can define an A-submodule of E

$$J^{-1} = \{e \in E \mid eJ \subset A\}.$$

If A is a principal ideal domain, then $J = (j)$ for some $j \in A$, and $J^{-1} = (j^{-1})$. Then we have an isomorphism of A-modules $J^{-1}/A \simeq A/J$. We can state the equivalent of property i above.

Proposition 4.17. Let $R \subset \mathrm{Gh}'(I)$ be a B'-ring, $i, j \in I$. Then

$$\mathrm{Ext}^1_R(A_i, A_j) \simeq \begin{cases} d(j,i)^{-1}/A_j & \text{if } i \neq j \\ 0 & \text{if } i = j. \end{cases}$$

Proof. We first recall the proof of Proposition 1.7, most of which goes through in this more general setting. Putting $K_i = \ker \pi_i$, and letting ι denote the inclusion $K_i \to R$, we have

$$\mathrm{Ext}^1_R(A_i, A_j) \simeq \mathrm{Hom}_R(K_i, A_j)/\bar{\iota}(\mathrm{Hom}_R(R, A_j)),$$

where $\bar{\iota} = \mathrm{Hom}_R(\iota, A_j)$.

Suppose $i \neq j$, and recall that any homomorphism of R-modules $K_i \to A_j$ factors through $\pi_j|_{K_i} : K_i \to d(i,j)$. We can then rewrite the above as

$$\mathrm{Ext}^1_R(A_i, A_j) \simeq \mathrm{Hom}_{A_j}(d(i,j), A_j)/\overline{\sigma}(\mathrm{Hom}_{A_j}(A_j, A_j)),$$

where σ denotes the inclusion $d(i,j) \hookrightarrow A_j$ and $\overline{\sigma} = \mathrm{Hom}_{A_j}(\sigma, A_j)$.

Since $\mathrm{Hom}_{A_j}(d(i,j), A_j)/\overline{\sigma}(\mathrm{Hom}_{A_j}(A_j, A_j))$ is finite, it is torsion. Then for any homomorphism of A_j-modules $\tau : d(i,j) \to A_j$, there exists some $n \in \mathbb{Z}$ such that $n\tau$ is given by multiplication by some element a in A_j. Let E be the field of fractions of A_j. Then τ is given by multiplication by $a/n \in E$. It follows that each $e \in E$ with $e \cdot d(i,j) \subset A_j$ gives rise to a homomorphism of A_j-modules $d(i,j) \to A_j$, and all homomorphisms are of this form. So we get

$$\mathrm{Hom}_{A_j}(d(i,j), A_j) \simeq d(i,j)^{-1}$$

with $d(i,j)^{-1}$ as defined above. Moreover this isomorphism sends the submodule $\overline{\sigma}(\mathrm{Hom}_{A_j}(A_j, A_j))$ of $\mathrm{Hom}_{A_j}(d(j,i), A_j)$ to the submodule A_j of $d(j,i)^{-1}$, so we have

$$\mathrm{Ext}^1_R(A_i, A_j) \simeq d(j,i)^{-1}/A_j$$

as claimed.

If $i = j$, again any homomorphism $K_i \to A_i$ factors through $\pi_i(K_i) = 0$, and so

$\mathrm{Ext}^1(A_i, A_i) = 0.$ □

Let I be a finite set and $\mathcal{A}_I$ a multiset of subrings of rings of algebraic integers. The spectrum of $\mathrm{Gh}'(I)$ is given by

$$\mathrm{Spec}\,\mathrm{Gh}'(I) = \bigsqcup_{i \in I} \mathrm{Spec}\, A_i,$$

and as before, for $i \in I$ and $P \in \mathrm{Spec}\, A_i$ we write $Q(i, P)$ for the corresponding prime. For each $i \in I$, let e_i be the corresponding idempotent of $\mathrm{Gh}'(I)$. Let $R \subset \mathrm{Gh}'(I)$ be a B'-ring. Since the projection maps $\pi_i : R \to A_i$ are all surjective, for any $x \in \mathrm{Gh}'(I)$ we can find elements $x_i \in R$ such that

$$x = \sum_{i \in I} e_i x_i.$$

Now since each e_i is trivially integral over R, it follows that $\mathrm{Gh}'(I)$ is integral over R, and so we have a surjection

$$\mathrm{Spec}\,\mathrm{Gh}'(I) \to \mathrm{Spec}\, R$$

given by $Q(i, P) \mapsto Q(i, P) \cap R =: q_R(i, P)$. For $P \in \mathrm{Spec}\, A_i$ and J an ideal of A_i, write $P \mid J$ for $J \subset P$.

We can now state our equivalent of Proposition 1.9.

Lemma 4.18. Let $R \subset \mathrm{Gh}'(I)$ be a B'-ring and suppose $P \in \mathrm{Spec}\, A_i$ is such that $P \mid d(i, j)$. Then there exists $Q \in \mathrm{Spec}\, A_j$ with $q(i, P) = q(j, Q)$ and $Q \mid d(j, i)$.

This follows from the slightly more general result:

Lemma 4.19. Suppose P is a proper ideal of A_i with $P \mid d(i, j)$. Then the set

$$Q = \{r(j) \in A_j \mid r \in R, r(i) \in P\}$$

is a proper ideal of A_j, with $Q \mid d(j, i)$.

Proof. It is clear that Q is an abelian group and that $d(j, i) \subset Q$. For $q \in Q$ and $a \in A_j$, we can choose $r, s \in R$ with $r(j) = q$, $r(i) \in P$, and $s(j) = a$. Then $rs \in R$ satisfies $(rs)(i) \in P$ and $(rs)(j) = qa$. So $aq \in Q$ and Q is an ideal of A_j.

Put $\hat{P} = \{r(i) \in A_i \mid r \in R, r(j) \in Q\}$. Certainly $P \subset \hat{P}$. We claim that $\hat{P} \subset P$ and hence $\hat{P} = P$. Consider $\hat{p} \in \hat{P}$, and let $r \in R$ be such that $r(i) = \hat{p}$ and $r(j) = q \in Q$. By the definition of Q, there then exists $r' \in R$ with $r(j) = q$ and $r'(i) = p$ for some $p \in P$. Put $t = r - r'$. Then $t(j) = 0$ and hence $t(i) \in d(i, j) \subset P$. But $t(i) = \hat{p} - p \in P$ and hence $\hat{p} \in P$. So $\hat{P} = P$. Now if $Q = A_j$ then it is clear from the definition that $\hat{P} = A_i$. But $\hat{P} = P$ is proper so Q is proper, as required. □

4.2.1 Character rings

Let G be a finite group. The isomorphism classes of finite-dimensional complex representations (i.e. linear representations over $\mathbb{C}$) of G form a commutative semi-ring with unit, with addition given by direct sum and multiplication given by tensoring over $\mathbb{C}$. The representation ring $R(G)$ is the Grothendieck ring associated to this semi-ring. For ϕ a finite-dimensional representation of G, write $[\phi]$ for the isomorphism class of ϕ in $R(G)$. We will first recall some basic facts about representation rings; for proofs and further details see Chapter V of [7].

Let $\mathrm{cc}(G)$ denote the set of conjugacy classes of G, and let $\phi_1, \ldots, \phi_r$ be a complete irredundant set of irreducible finite-dimensional complex representations. Then $r = |\,\mathrm{cc}(G)|$, and as an abelian group $R(G)$ is free on the elements $[\phi_1], \ldots, [\phi_r]$. For $g \in G$, write (g) for the conjugacy class of g.

For each $g \in G$, we have a well-defined homomorphism of rings

$$\pi_g : R(G) \to \mathbb{C}$$

where for a complex representation ϕ we put

$$\pi_g([\phi]) = \mathrm{tr}\,\phi(g)$$

and extend linearly. Let A be the ring of algebraic integers and recall that for a complex representation ϕ and $g \in G$, $\mathrm{tr}\,\phi(g)$ is an algebraic integer. For $g \in G$, we have a subring A_g of A given by all $\pi_g(r), r \in R(G)$. For $g, g' \in G$, we have $\pi_g = \pi_{g'}$ if and only if $(g) = (g')$.

It follows that the homomorphisms π_g allow us to embed $R(G)$ as a subring of $\prod_{(g)\in \mathrm{cc}(G)} A_g$. It is clear in the sense of Definition 4.11 that $R(G)$ is a B'-ring. However the embedding just defined is not in general a B'-ring embedding in the sense of Proposition 4.12.

Example 4.20. Let $G = C_3$, the cyclic group of order 3. Let ω be the algebraic integer $e^{2\pi i/3}$ and A the subring of $\mathbb{C}$ generated by ω. We can identify $R(G)$ as the subring of $\mathbb{Z} \times A \times A$ which as a $\mathbb{Z}$-module is spanned by the elements $(1, 1, 1)$, $(1, \omega, \omega^2)$, and $(1, \omega^2, \omega)$. Let t denote the element $(1, \omega, \omega^2)$ and note that $t^2 = (1, \omega^2, \omega)$.

Now the regular representation of C_3 corresponds to the element $(3, 0, 0)$ of $R(G)$, so we certainly have the separability condition verified for the first slot. However, we claim that there is no $x = (x_1, x_2, x_3) \in R(G)$ with $x_2 = 0$ and $x_3 \neq 0$. Since $R(G)$ is spanned by the elements $1_{R(G)}, t, t^2$, for any element $y \in R(G)$ we can find some polynomial f_y with integer coefficients such that $f_y(t) = y$. Suppose we have $f_x(t) = x$ with x as above. Then $x_2 = 0$ implies that $f_x(\omega) = 0$, in which case $x_3 = f_x(\omega^2) = 0$, since ω and ω^2 are both roots of the minimal polynomial for ω over $\mathbb{Z}$.

For each $(H) \in \mathrm{cccs}(G)$, fix some choice of $g \in G$ satisfying $(\langle g \rangle) = (H)$, and write π_H for π_g and A_H for A_g.

Proposition 4.21. The maps π_H combine to give an embedding

$$\pi : R(G) \to \prod_{(H) \in \mathrm{cccs}(G)} A_H,$$

and this embedding defines a B'-ring.

Proof. Any rational representation of G is automatically a complex representation, so our verification of the separability condition for the rational representation ring in the proof of Lemma 4.1 is also a verification of the separability condition for the claimed embedding above.

It remains to show that this is indeed an embedding. We know we have an embedding

$$\iota : R(G) \hookrightarrow \prod_{(g) \in \mathrm{cc}(G)} A_g,$$

and we can regard our map π as the map ι followed by the map

$$\alpha : \prod_{(g) \in \mathrm{cc}(G)} A_g \to \prod_{(H) \in \mathrm{cccs}(G)} A_H$$

given by dropping the additional slots.

Suppose then that $x \in R(G)$ is such that $\pi(x) = 0$. If $x \neq 0$, then $\iota(x) \neq 0$, and so x is only non-zero in the slots dropped by the map α. Since $\pi_g(x)$ is an algebraic integer, for each $(g) \in \mathrm{cc}(G)$ we can find some $f_g \in \mathbb{Z}[y]$ such that $f_g(\pi_g(x)) \in \mathbb{Z} - \{0\}$, and hence we can find some polynomial f and some $g \in G$ such that $f(x) \in \ker \pi$ is integer-valued and $\pi_g(f(x))$ is non-zero.

So we can assume that $x \in \ker \pi$ is integer-valued. But for any integer-valued virtual character y we have that $y(g) = y(h)$ whenever g and h generate conjugate cyclic subgroups ([14] Lemma 5.22). It follows then that $\pi_H(x) \neq 0$ for $(H) = (\langle g \rangle)$, a contradiction. So $\ker \pi = 0$ and π is an embedding. □

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