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The Positive Maximum Principle on Lie Groups and Symmetric Spaces

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To my parents.

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Abstract

In this thesis we will use harmonic analysis to get new results in probability on Lie groups and symmetric spaces. We will establish necessary and sufficient conditions for the existence of a square integrable K-bi-invariant density of a K-bi-invariant measure. We will show that there is a topological isomorphism between K-bi-invariant smooth functions and a subspace of the Sugiura space of rapidly decreasing functions. Furthermore, we will extend Courrège's classical results to Lie groups and symmetric spaces, this consists of characterizing all linear operators on the space of smooth functions with compact support, that satisfy the positive maximum principle, as Lévy-type operators. We will specify some conditions under which such operators map to the Banach space of continuous functions vanishing at infinity, this allows us to study Feller semigroups and their generator in this context. We will show that on compact Lie groups all linear operators satisfying the positive maximum principle can be represented as pseudo-differential operators and on compact symmetric spaces they have analogous representations called spherical pseudo-differential operators.

Contents

Introduction

In this thesis we apply harmonic analysis to get new results in probability on Lie groups and symmetric spaces. On Euclidean spaces Paul Lévy was among the first to use Fourier analysis to study Markov processes. A special Markov process, named after Paul Lévy, called a Lévy process will be of particular interest to us. This is a stochastic process with stationary and independent increments and càdlàg paths. It is well-known that Lévy processes can be constructed from continuous convolution semigroups of probability measures. On Euclidean spaces we have the famous Lévy-Khintchine formula that characterizes convolution semigroups of measures via their characteristic functions.

On more general spaces, Perrin [50] first studied Brownian and Poisson processes on the rotation group $SO(3)$. Then in the early 50's, Itô [33] extended the notion of Brownian processes to general Lie groups and Yosida [60] defined such processes on Riemannian homogeneous spaces. In 1956, Hunt [32] gave an analogous result to the Lévy-Khintchine formula on Lie groups, indeed he provided a closed formula for the infinitesimal generator of a convolution semigroup of measures, therefore providing a characterization for Lévy processes in Lie groups. Since then, stochastic processes on Lie groups and more generally on locally compact groups have been extensively studied. More recently, Applebaum and Kunita [7] showed that the solution of a stochastic differential equation driven by a Brownian motion and a Poisson measure is a Lévy process and provided a characterization of Lévy processes on Lie groups. Heyer [26] offers a systematic and comprehensive introduction to convolution semigroups on locally compact groups in his monograph; on Lie groups see also Liao [45] and Applebaum [6].

On (non-compact) symmetric spaces, in 1964 Gangolli [22] gave a generalized Lévy-Khintchine formula, that classified spherical infinitely divisible laws. This spherical version of the Lévy-Khintchine formula was later obtained for convolution semigroups using Hunt's formula by Applebaum [1] and Liao and Wang [48].

We will also be interested in pseudo-differential operators and their relationship to stochastic processes. Pseudo-differential operators are generalisations of linear partial differential operators; they are an important tool to study elliptic operators, for instance they arose in the proof of the Atiyah-Singer index theorem [9] and in the study of boundary problems in the work of K.O. Friedrichs and P. Lax [43], [21]. The development of the theory around pseudo-differential operators is largely due to J. Kohn, L. Nirenberg [40] and Hörmander [29], who introduced symbol analysis.

On $\mathbb{R}^d$, the Feller semigroup and infinitesimal generator of a convolution semigroup of measures can be represented as pseudo-differential operators; important work in this area was carried out by Jacob, see his monographs [34] [35]. We will be interested more generally in the relationship between pseudo-differential operators and Feller processes. On compact Lie groups a theory of pseudo-differential operators was developed by Ruzhansky and Turunen [51] and on Heisenberg groups see Fischer and Ruzhansky [18]. Applebaum [4] [5] approached the topic by studying Lie group-valued Lévy processes and Markov processes with a slightly different notion of pseudo-differential operators on Lie groups.

In this thesis, we will approach pseudo-differential operators via the positive maximum principle. We are motivated by the work of Courrège [13], who classified all linear operators on Euclidean spaces that satisfy the positive maximum principle. He showed that these operators can be expressed as a sum of a second order elliptic operator and an integral term with a kernel. Furthermore, he also showed that such operators can be represented as pseudo-differential operators. He then extended his results to characterize Feller processes on manifolds, using local coordinates, see [14] (and also Bony et al. [10]). More recently, others were also inspired by Courrège's work in their application of his results to Feller processes on Euclidean spaces, see Jacob [34] [35], Schilling and Böttcher et al. [11].

Structure of the thesis

In chapter 1, we will introduce the most important notions and results on analysis in Lie groups. We will mostly refer the reader to Applebaum [6], Faraut [17] and Folland [20],[19].

In chapter 2, we will start our investigation on symmetric spaces and K-bi-invariant functions. Our main reference is Wolf's [59] work on positive definite functions, spherical functions and Gelfand pairs; we will provide completed and detailed development of his results, where he only outlines the proof, and where we could not find a complete presentation elsewhere.

In chapter 3, we will introduce spherical transforms of probability measures and functions. We will establish the relationship between spherical transforms and Fourier transform using the spherical Peter-Weyl theorem.

Chapter 4 contains regularity results related to Fourier transforms on compact

Lie groups from Applebaum [6], which we will then generalize to compact symmetric spaces with the help of spherical transforms. Using these regularity results we will establish necessary and sufficient conditions for the existence of a square integrable density of a K-bi-invariant measure on a compact Gelfand pair (G, K). This was published in Applebaum and Le Ngan [8] along with some further results on the existence of continuous densities for a convolution semigroup of measures, and a trace formula for a K-bi-invariant convolution semigroup. We will end the chapter, by using the regularity results in the K-bi-invariant case to generalize Sugiura's theorem [55], that is we will show that there is a topological isomorphism between K-bi-invariant smooth functions and a subspace of the Sugiura space of rapidly decreasing functions, for which see Applebaum [6].

In chapter 5, we will introduce a global theory of distributions on Lie groups based on the work of Ehrenpreis [15]. We will establish some useful properties of distributions of order 0 and order 2 on Lie groups.

In chapter 6, we will extend Courrège's results to Lie groups, but we will not adopt his approach and will instead follow Hoh's work [28], who simplifies the problem by studying appropriate linear functionals that satisfy the positive maximum principle. We will show that such functionals are distributions of order 2 and therefore have a closed form representation. For the pseudo-differential operators part, we will concentrate on compact Lie groups and we will show that all linear operators satisfying the positive maximum principle can be represented as pseudo-differential operators.

In the last chapter, we will first extend Courrège's theorem to symmetric spaces. Then we will introduce the notion of spherical pseudo-differential operators, and we will show that operators satisfying the positive maximum principle on compact symmetric spaces can be represented as spherical pseudo-differential operators.

Chapter 1

Preliminaries

1.1 Function spaces

Let X be a locally compact Hausdorff space and F denote $\mathbb{R}$ or $\mathbb{C}$, we are going to define function spaces on X. $B_b(X) = B_b(X, F)$ is the linear space of all bounded Borel measurable functions from X to F, it is a Banach space under the supremum norm $\|f\|_\infty := \sup_{x \in X} |f(x)|$ for all $f \in B_b(X)$. $C(X)$ is the space of F-valued continuous functions on X, $C_b(X)$ is the space of bounded continuous functions on X. A function f on X is said to *vanish at infinity* if given any $\varepsilon > 0$ there exists a compact set $H \subseteq X$ such that $|f(x)| < \varepsilon$ when $x \in H^C$. We will denote the space of continuous functions on X which vanish at infinity by $C_0(X)$. The *support* of a function f on X is the closure of the set $\{x \in X : f(x) \neq 0\}$ and it is denoted by $\text{supp}(f)$, the linear space of all continuous functions on X with compact support is denoted by $C_c(X)$. The spaces $C_b(X)$ and $C_0(X)$ are all Banach spaces under the supremum norm $\| \cdot \|_\infty$. Furthermore, $C_c(X)$ is dense in $C_0(X)$. In the first part of the thesis we will consider $F = \mathbb{C}$. From chapter 5 onwards, we will consider real valued functions.

1.2 Topological groups

This section provides an overview of some of the concepts of topological group theory which we use in later sections, for reference see [6] and [38] Chapter 1.

1.2.1 Representation theory

Definition. G is called a *topological group* if:
- G is a topological space,
- G is a group,

- the mapping $(g, h) \mapsto gh^{-1}$ from $G \times G$ to G is continuous.

From now on, all groups will be assumed to be topological groups, in particular we will only be looking at topological groups that are Hausdorff spaces.

If G is a topological group, then for all $g \in G$ we define *left translation* l_g, *right translation* r_g and *conjugation* c_g by

$$l_g h = gh, r_g h = hg \text{ and } c_g(h) = ghg^{-1}$$

for all $h \in G$. All three are automorphisms of G with inverses

$$l_g^{-1} = l_{g^{-1}}, r_g^{-1} = r_{g^{-1}} \text{ and } c_g^{-1} = c_{g^{-1}}, \text{ respectively.}$$

Definition. A *representation* (π, V) of a group G is a topological vector space V over $\mathbb{C}$ and a strongly continuous group homomorphism $\pi : G \to GL(V)$, where $GL(V)$ is the general linear group on V defined by

$$GL(V) := \{A : V \to V; A \text{ is a topological isomorphism}\},$$

and by strongly continuous we mean that for all $v \in V$, $g \mapsto \pi(g)v$ is continuous from G to V.

Remark 1.2.1. Later, when we consider the particular case where V is a Banach space, the representation π will be called a *Banach representation*.

Definition. Given a representation (π, V) of G, a closed subspace $W \subseteq V$ is a *π-invariant subspace* if $\pi(g)W \subseteq W$ for all $g \in G$. The restriction of π to W is itself a representation and is called a *subrepresentation*. On the quotient space V/W, π induces another representation called the *quotient representation* $\rho : G \to GL(V/W)$ given by $\rho(g)(v + W) = \pi(g)v + W$ for all $g \in G$ and $v \in V$. A representation (π, V) is called *irreducible* if it has no non-trivial sub-representations.

Definition. Let (π_1, V_1) and (π_2, V_2) be two representations of G. Then π_1 and π_2 are called *equivalent representations* if there exists a continuous linear isomorphism $T : V_1 \to V_2$ such that for all $g \in G$, $T\pi_1(g) = \pi_2(g)T$. Then T is called an *intertwining operator*.

A representation π of G on a complex separable Hilbert space V is said to be *unitary* if $\pi(g)$ is a unitary operator for every $g \in G$. If V_1 and V_2 are Hilbert spaces, with unitary representations (π_1, V_1) and (π_2, V_2) then these representations are said to be *unitary equivalent* if there exists a unitary intertwining operator between them.

Definition. From now on, we denote by $\widehat{G}$ the *unitary dual* of G which is the set of all equivalence classes of irreducible unitary representations, with respect to the equivalence relation of unitary isomorphism. If a representation $\pi \in \widehat{G}$ is finite dimensional we denote its dimension by d_π.

Example 1.2.2. The irreducible unitary representations of $G = U(1)$ are all one-dimensional and are given by $\rho_k(\lambda)(z) := \lambda^k z$ for all $\lambda \in U(1)$, $k \in \mathbb{Z}$, $z \in \mathbb{C}$. The unitary dual is therefore the dual group $\widehat{G} = \mathbb{Z}$.

1.2.2 Haar measure

Let $\mathcal{B}(X)$ denote the Borel σ-algebra of X. Recall that a Borel measure μ is called *outer regular* if for all $A \in \mathcal{B}(X)$,

$$\mu(A) = \inf\{\mu(O); A \subseteq O, O \text{ open in } X\},$$

and μ is called *inner regular* if

$$\mu(A) = \sup\{\mu(C); C \subseteq A, C \text{ compact in } X\}.$$

The measure μ is called *regular* if it is both inner regular and outer regular, such that all compact sets have finite measures, i.e. $\mu(C) < \infty$ for all compact $C \subseteq X$.

Definition. Let G be a locally compact group. A measure m_L on $(G, \mathcal{B}(G))$ is called a *left Haar measure* if

- m_L is a regular Borel measure,
- m_L is *left-invariant*: for all $A \in \mathcal{B}(G)$ and $g \in G$, $m_L(A) = m_L(gA)$.

We can define similarly a *right Haar measure*, m_R.

Definition. Given a left Haar measure m_L on the locally compact group G, for any complex valued Borel-measurable function f on G, we define for all $1 \leq p < \infty$

$$\|f\|_p := \left(\int_G |f(x)|^p m_L(dx)\right)^{1/p} \in [0, \infty],$$

and

$$\|f\|_\infty := \inf\{K > 0; |f(x)| < K \text{ a.e. with respect to } m_L\} \in [0, \infty].$$

Then for all $1 \leq p \leq \infty$ we denote by $L^p(G, \mathcal{B}(G), m_L)$ the space of all equivalence classes of complex-valued Borel-measurable functions that are equal m_L-almost everywhere, defined by

$$L^p(G, \mathcal{B}(G), m_L) := \{f : G \to \mathbb{C} \text{ Borel measurable}; \|f\|_p < \infty\}.$$

For all $1 \le p \le \infty$, the space $L^p(G, \mathcal{B}(G), m_L)$ is a Banach space with respect to the norm $\|\cdot\|_p$. In particular, $L^2(G, \mathcal{B}(G), m_L)$ is a Hilbert space with inner product $\langle f, h\rangle = \int_G f(x)\overline{h(x)}\, m_L(dx)$.

Theorem 1.2.3. *Let G be a locally compact Hausdorff group, then left and right Haar measures exist and are unique up to a positive multiplicative constant.*

Proof. See Theorem 11.8-11.9, p.344 in [19]. □

1.2.3 The modular function

Let m_L be a left Haar measure on G, then for any fixed $g \in G$ we can obtain another left Haar measure by right translation

$$m_L^g(A) := m_L(Ag),$$

for all $A \in \mathcal{B}(G)$. By the uniqueness of the Haar measure from Theorem 1.2.3, there exists a constant $\Delta(g) > 0$ such that

$$m_L^g(A) = \Delta(g) m_L(A), \qquad \text{for all } A \in \mathcal{B}(G)$$

Definition. The function $g \mapsto \Delta(g)$ from G to $(0, \infty)$ is called the *modular function.* The group G is called *unimodular* if $\Delta \equiv 1$.

Theorem 1.2.4. *The modular function is a continuous homomorphism from G to* $(0, \infty)$.

Proof. See Theorem 1.2.2, p.7 in [6]. □

Theorem 1.2.5. *Suppose G is locally compact.*

(i) The measure defined by $\widetilde{m}_R(A) = \int_A \Delta(x^{-1}) m_L(dx)$ *for all* $A \in \mathcal{B}(G)$ *is a right Haar measure on G.*

(ii) For all $f \in C_c(G)$*, we have*

$$\int_G f(x^{-1}) m_L(dx) = \int_G f(x)\Delta(x^{-1}) m_L(dx).$$

Proof. See Theorem 1.2.3, p.8 in [6]. □

Proposition 1.2.6. *If G is compact, then G is unimodular. Thus, any left Haar measure on G is a right Haar measure (and vice versa). In other words, any Haar measure on G is bi-invariant.*

Proof. Since the modular function is a continuous homomorphism and G is compact, $\Delta(G)$ is a compact subgroup of $(0, \infty)$; but $\{1\}$ is the only compact subgroup of $(0, \infty)$. The bi-invariance follows from Theorem 1.2.5 (i), since $\widetilde{m}_R = m_L$ on G. □

Remark 1.2.7. If G is a compact Hausdorff group, then every Haar measure m on G is finite. Indeed, by definition the Haar measure m is regular, so for all compact subsets $C \subseteq G$, $m(C) < \infty$. In particular, we have $m(G) < \infty$.

Definition. If G is compact, any Haar measure m (left or right) is unique up to multiplication with a non-negative constant, therefore we can define a unique *normalized Haar measure* on G by

$$\mu_G(A) = \frac{m(A)}{m(G)}, \quad \text{for all } A \in \mathcal{B}(G).$$

In the following, instead of writing $\int f(g)\ \mu_G(dg)$, we use the simpler notation $\int f(g)\ dg$ and we only specify μ_G in case of ambiguity.

1.2.4 Schur orthogonality and Peter-Weyl theorem

Let us state some of the most relevant results of harmonic analysis on compact topological groups; proofs can be found in the literature such as Chapter 6 of Faraut [17], Chapter 2 of Applebaum [6], Chapter 5 of Folland [20] and Chapter 4 of Knapp [38]. In this section we will assume that G is a compact Hausdorff group, and we use the simplified notation $L^2(G) := L^2(G, B(G), \mu_G)$.

Theorem 1.2.8. *An irreducible representation π of a compact group G on a complex Hilbert space V_π is finite dimensional.*

Proof. See [41] or p.28 in [6]. □

From now on we will write $d_\pi = \dim(V_\pi)$.

Remark 1.2.9. Note that every representation of a compact group on a Hilbert space is unitary under an appropriate inner product, see Proposition 2.2.1, p.27 in [6].

Definition. Let G be a compact group and π an irreducible representation of G acting on V_π. We define a closed subspace $\mathcal{M}_\pi$ of $L^2(G)$,

$$\mathcal{M}_\pi := \text{Span}\{g \mapsto \langle u, \pi(g)v\rangle; u, v \in V_\pi\}.$$

We will denote by $\mathcal{E}(G)$ the linear span of $\{\psi \in \mathcal{M}_\pi,\, \pi \in \widehat{G}\}$. For any representation $\pi \in \widehat{G}$ on V_π, let $e_1, e_2, \ldots, e_{d_\pi}$ be an orthogonal basis of V_π.

Theorem 1.2.10. *If π_1 and π_2 are distinct elements of $\widehat{G}$, then $\mathcal{M}_{\pi_1}$ and $\mathcal{M}_{\pi_2}$ are orthogonal.*

Proof. See Theorem 2.2.2, p.31 in [6] or Corollary 4.16, p.243 [38]. □

Theorem 1.2.11 (Schur Orthogonality Relation)**.** *Let $\pi_1, \pi_2 \in \widehat{G}$, then for all $\phi_i, \psi_i \in V_{\pi_i}, i = 1, 2,$*

$$\int_G \langle \pi_1(g)\phi_1, \psi_1\rangle \overline{\langle \pi_2(g)\phi_2, \psi_2\rangle} dg = \begin{cases} 0 & \text{if } \pi_1 \neq \pi_2 \\ \frac{1}{d_\pi}\langle \phi_1, \phi_2\rangle \overline{\langle \psi_1, \psi_2\rangle} & \text{if } \pi_1 = \pi_2 \end{cases}$$

Proof. See [6] p.31. □

We will now state the Peter-Weyl theorems, which will serve as the main tools for harmonic analysis on groups. The first theorem states that $L^2(G)$ is Hilbert direct sum of $\mathcal{M}_\pi$ for $\pi \in \widehat{G}$.

Theorem 1.2.12 (Peter-Weyl 1)**.** *The space $\mathcal{E}(G)$ is dense in $L^2(G)$.*

Theorem 1.2.13 (Peter-Weyl 2)**.** *We define the functions $\pi_{ij} : G \to \mathbb{C}$ by*

$$\pi_{ij}(g) := \langle \pi(g)e_i, e_j\rangle \qquad \text{for all } g \in G \text{ and } i, j = 1, 2, \ldots, d_\pi.$$

Then the set $\left\{d_\pi^{1/2}\pi_{ij}; 1 \le i, j, \le d_\pi, \pi \in \widehat{G}\right\}$ is a complete orthonormal basis for $L^2(G)$.

Theorem 1.2.14 (Peter-Weyl 3)**.** *The space $\mathcal{E}(G)$ is dense in $C(G)$.*

Proof. For proof of these theorems see p.33, in [6] or p.245, in [38]. □

1.3 Lie groups and Lie algebras

Definition. A *Lie group* G is a group that is also a smooth manifold, such that the mapping $(a, b) \mapsto ab^{-1}$ is smooth.

Definition. An *action* of a Lie group G on a manifold M is a mapping sending each $g \in G$ to a diffeomorphism $\rho(g)$ on M such that

i) $\rho(e) = \mathrm{id}_M$, that is $\rho(e)m = m$ for all $m \in M$,
ii) $\rho(gh) = \rho(g)\rho(h)$, for all $g, h \in G$.

We say that the action is *smooth* if the map $(g, m) \mapsto \rho(g)m$ is smooth from $G \times M$ to M, in the usual sense on manifolds, see [58] p.6. We will often denote the group action of G at a point $m \in M$ by $g.m$ for all $g \in G$.

Example 1.3.1. Many examples of Lie groups are matrix groups:

1. General linear group $GL(n, \mathbb{R})$
2. Special linear group $SL(n, \mathbb{R})$
3. Orthogonal group and special orthogonal group $O(n, \mathbb{R})$, $SO(n, \mathbb{R})$
4. Unitary group and special unitary group $U(n, \mathbb{R})$, $SU(n, \mathbb{R})$

Since G is a smooth manifold, we can use all its properties from differential geometry. Let us recall in particular, that we can equip the space of smooth vector fields on G with a *Lie bracket*. Given two smooth vector fields X and Y we can introduce a third smooth vector field $[[X, Y]]$, by

$$[[X, Y]](f) = X(Y(f)) - Y(X(f)), \qquad \text{for all } f \in C^\infty(G).$$

Definition. A *Lie algebra* L is a vector space over the field $\mathbb{F}(= \mathbb{R}$ or $\mathbb{C})$ equipped with a bilinear mapping $[\cdot, \cdot] : L \times L \to L$ satisfying: for all $X, Y, Z \in L$,

(i) $[X, Y] = -[X, Y]$ (anti-commutative)
(ii) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (Jacobi identity)

From (i) it follows that $[X, X] = 0$ for all $X \in L$. The bilinear map $[\cdot, \cdot]$ will be called the *Lie bracket of the the Lie algebra L*.

Definition. Let L, L' be two Lie algebras, then a *homomorphism of Lie algebras* is a linear map $f : L \to L'$ that preserves the Lie bracket,

$$f([X, Y]_L) = [f(X), f(Y)]_{L'}, \qquad \text{for all } X, Y \in L.$$

Definition. Let G be a Lie groups. A vector field $\xi : G \to TG$ on a Lie group is called *left-invariant*, if $T_e l_g(\xi(e)) = \xi(g)$, for all $g \in G$. The space of left-invariant vector fields on a Lie group G equipped with the Lie bracket is a Lie algebra and is called the *Lie algebra of the Lie group G*. We will denote it by $\mathfrak{g}$.

Let us briefly recall some definitions on smooth manifolds, that we will regularly apply to Lie groups.

Definition. Let M, N be two smooth manifolds and $F : M \to N$ a smooth mapping. The *differential* of F at $p \in M$ is a linear map $T_pF : T_pM \to T_{F(p)}N$ defined in the following way, for all $\varphi \in C^\infty(N)$ and $X \in T_pM$,

$$T_pF(X)(\varphi) = X(\varphi \circ F)(p).$$

It is easy to check that for all $X \in T_pM$, the map $T_pF(X) : C^\infty(N) \to \mathbb{R}$ is a derivation at $F(p)$, i.e. for all $f, g \in C^\infty(N)$

$$T_pF(X)(fg) = f(F(p))\, T_pF(X)(g) + g(F(p))\, T_pF(X)(f).$$

Theorem 1.3.2 (Chain rule for manifolds)**.** *Let M, N, L be smooth manifolds and $F : M \to N$, $H : N \to L$ smooth maps. Then the map $(H \circ F) : M \to L$ is also smooth and satisfies*

$$T_n(H \circ F) = T_{F(n)}H \circ T_nF, \qquad \textit{for all } n \in N,$$

where $T_n(H \circ F) : T_mM \to T_{(H\circ F)(m)}L$.

Proof. See [56], Theorem 8.5, p.79. □

Definition. Let X be a vector field on a manifold M and $p \in M$. A smooth curve $\gamma_X^p : (-\varepsilon, \varepsilon) \to M$, where $\varepsilon > 0$, is called an *integral curve* of X going through p if

- $\gamma_X^p(0) = p$,
- $(\gamma_X^p)'(t) = X(\gamma_X^p(t))$, for all $t \in (-\varepsilon, \varepsilon)$.

Note that on a smooth manifold M for any smooth $\phi : \mathbb{R} \to M$ by $\phi'(t)$ we mean $\phi'(t) = T_t\phi(\frac{d}{dt})$, where $T_t\phi : T_t\mathbb{R} \simeq \mathbb{R} \to T_{\phi(t)}M$ is the differential of ϕ as defined previously.

Definition. Let G be a Lie group, we call $\gamma : \mathbb{R} \to G$ a *one-parameter subgroup* of G if it is a continuous homomorphism from $\mathbb{R}$ to G.

Remark 1.3.3. An integral curve of a left-invariant vector field is a one parameter subgroup of G. Equivalently, any one-parameter subgroup of G is the integral curve of a left-invariant vector field, see Proposition 1.4, p.92 and Corollary 1.5, p.93 in [24].

Definition. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. We define the *exponential map* $\exp : \mathfrak{g} \to G$ as $\exp(X) := \gamma_X(1)$ for all $X \in \mathfrak{g}$.

Then by the chain rule, for all $t \in \mathbb{R}$ we get $\exp(tX) = \gamma_{tX}(1) = \gamma_X(t)$ and given that the integral curves γ_X for all $X \in \mathfrak{g}$ are homomorphisms from $\mathbb{R}$ to G, we obtain

$$\exp((s+t)X) = \exp(sX)\exp(tX), \qquad \text{for all } s, t \in \mathbb{R} \text{ and } X \in \mathfrak{g}.$$

Hence,

$$\exp(tX)^{-1} = \exp(-tX), \qquad \text{for all } t \in \mathbb{R} \text{ and } X \in \mathfrak{g}.$$

For a detailed proof see Theorem 3.7, p.27 in [37].

The exponential map is a diffeomorphic map from a neighbourhood V of the origin $\mathfrak{g}$ to a neighbourhood U of e in G. For a basis $\{X_1, X_2, \dots, X_d\}$ of $\mathfrak{g}$, there exist corresponding mappings $x_i : U \to \mathbb{R}$, $1 \le i \le d$ called *canonical coordinates* such that for all $1 \le i \le d$,

$$x_i\left(\exp\left(\sum_{j=1}^{d} a_j X_j\right)\right) = a_i,$$

when $\sum_{j=1}^{d} a_j X_j \in V$. See [6] p.14 and [45], p.11 for reference.

Remark 1.3.4. The space of left-invariant vector fields is isomorphic to the tangent space of G at e, T_eG. We can therefore identify $\mathfrak{g}$ with T_eG, see Theorem 2.27, p.15 in [37].

Let $X \in \mathfrak{g}$, we will denote the corresponding left-invariant vector field's value at $g \in G$ by $X(g) := T_e l_g X$. Then by the chain rule, for all $g \in G$ and $f \in C^\infty(G)$,

$$\begin{aligned} \frac{d}{dt} f(g\exp(tX))\bigg|_{t=0} &= T_o\left[f(g\exp(\cdot X))\right]\left(\frac{d}{dt}\right) = T_g f \circ T_e l_g \circ T_o \exp(\cdot X)\left(\frac{d}{dt}\right) \\ &= (T_g f \circ T_e l_g)(X(\gamma_X(0))) = (T_g f \circ T_e l_g)X(e) \\ &= T_g f X(g) = X f(g) \qquad (1.1) \end{aligned}$$

If V is a vector space, $GL(V)$ equipped with the commutator $[\alpha, \beta] = \alpha \circ \beta - \beta \circ \alpha$ is a Lie algebra, which we will denote by $\mathfrak{gl}(V)$. If V is finite dimensional, $GL(V)$ is a Lie group and $\mathfrak{gl}(V)$ is its Lie algebra, see Proposition 8.48, p.198 in [44].

Definition. Let $\mathcal{L}$ be a finite dimensional Lie algebra. A *Lie algebra representation* of $\mathcal{L}$ is a Lie algebra homomorphism from $\mathcal{L}$ to the Lie algebra $\mathfrak{gl}(V)$, where V is a vector space.

We will only be interested in finite dimensional representations of Lie algebras.

Definition. Let $\pi : G \to GL(V)$ be a finite-dimensional representation of a Lie group G on a vector space V. The differential of π at e, $d\pi := T_e\pi : \mathfrak{g} \to \mathfrak{gl}(V)$, is called the *derived representation of* $\mathfrak{g}$ and it is a Lie algebra representation of $\mathfrak{g}$.

The following diagram commutes

$$\begin{array}{ccc} G & \xrightarrow{\pi} & GL(V) \\ \uparrow{\scriptstyle \exp} & & {\scriptstyle \exp}\uparrow \\ \mathfrak{g} & \xrightarrow{d\pi} & \mathfrak{gl}(v) \end{array} \tag{1.2}$$

That is,

$$\pi(\exp(X)) = \exp(d\pi(X)), \quad \text{for all } X \in \mathfrak{g},$$

see Proposition 20.8, p.519 in [44]. Thus, one can calculate that for all $v \in V$,

$$d\pi(X)v = \frac{d}{dt}\pi(\exp(tX))v\Big|_{t=0}.$$

Let us look at a particular case. For any fixed $g \in G$, the *conjugate map* $c_g(x) := gxg^{-1}$, for $x \in G$ is an automorphism of G. Its differential at e is $\mathrm{Ad}(g) := T_e c_g : \mathfrak{g} \to \mathfrak{g}$. For all $g \in G$, $\mathrm{Ad}(g)$ is in $GL(\mathfrak{g})$, the mapping $\mathrm{Ad} : G \to GL(\mathfrak{g})$ is a representation of G on $\mathfrak{g}$ and it is called the *adjoint representation* of G on $\mathfrak{g}$. Indeed, since $c_{gh} = c_g \circ c_h$ for all $g, h \in G$, it then follows that $\mathrm{Ad}(gh) = \mathrm{Ad}(g) \circ \mathrm{Ad}(h)$.

The derived representation of Ad is denoted by $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ and it satisfies $\mathrm{ad}(X)(Y) = [\mathrm{X, Y}]$ for all $X, Y \in \mathfrak{g}$. As previously, the following diagram commutes

$$\begin{array}{ccc} G & \xrightarrow{\mathrm{Ad}} & GL(\mathfrak{g}) \\ \uparrow{\scriptstyle \exp} & & {\scriptstyle \exp}\uparrow \\ \mathfrak{g} & \xrightarrow{\mathrm{ad}} & \mathfrak{gl}(\mathfrak{g}) \end{array} \tag{1.3}$$

That is for all $X \in \mathfrak{g}$,

$$\mathrm{Ad}(\exp(X)) = \exp(\mathrm{ad}(X)).$$

One can also calculate that, for all $g \in G$ and $X \in \mathfrak{g}$

$$g \exp(X) g^{-1} = \exp(\mathrm{Ad}(g)X), \tag{1.4}$$

see p.127-128 in [25].

For $p \in \mathbb{N}$, we will denote by $C^p(G)$, the linear space of p times continuously differentiable functions on G, the subspace of functions with compact support is denoted $C^p_c(G)$ and the subspace of functions that vanish at infinity is denoted $C^p_0(G)$. The linear space of infinitely differentiable functions on G is $C^\infty(G) = \bigcap_{p\in\mathbb{N}} C^p(G)$, the subspace of smooth functions of compact support is denoted by $C^\infty_c(G) = C^\infty(G) \cap C_c(G)$. For all $g \in G$, $f \in C_0(G)$ we define $L_g f = f \circ l_g$, $R_g f = f \circ r_g$.

Theorem 1.3.5. *Let G be a connected Lie group. For any $p \in \mathbb{N}$ and f a real valued function on G, we have the following equivalence*

- *The mapping f is in $C^p(G,\mathbb{R})$.*
- *For all $X_1, X_2, ..., X_p \in \mathfrak{g}$, the mapping $g \mapsto X_1X_2 \cdots X_p f(g)$ is well-defined and continuous from G to $\mathbb{R}$.*

Proof. See [6] Theorem 1.3.5 p.21, as well as [55] p.42. □

Lemma 1.3.6. *For any $X, Y \in \mathfrak{g}$, $g, k \in G$ and $f \in C^2(G)$ we have*

$$\begin{aligned} X(f \circ c_k)(g) &= [Ad(k)X]f(c_k(g)), \\ XY(f \circ c_k)(g) &= [Ad(k)X]\,[Ad(k)Y]f(c_k(g)). \end{aligned}$$

Proof. This follows directly from equations (1.1) and (1.4). For all $f \in C^2(G)$, $X, Y \in \mathfrak{g}$ and $g, k \in G$

$$\begin{aligned} X(f \circ c_k)(g) &= \frac{d}{dt}(f \circ c_k)(g \exp(tX))\Big|_{t=0} = \frac{d}{dt} f(kgk^{-1}k \exp(tX)k^{-1})\Big|_{t=0} \\ &= \frac{d}{dt} f(kgk^{-1} \exp(\mathrm{Ad}(k)tX)) = \mathrm{Ad}(k)Xf(kgk^{-1}) \\ &= \mathrm{Ad}(k)Xf(c_k(g)). \end{aligned}$$

For the second derivative, with a similar method we have

$$\begin{aligned} XY(f \circ c_k)(g) &= \frac{d}{ds}\frac{d}{dt}(f \circ c_k)(g \exp(tY) \exp(sX))\Big|_{\substack{t=0\\ s=0}} \\ &= \frac{d}{ds}\frac{d}{dt} f(kgk^{-1}k \exp(tY)k^{-1}k \exp(sX)k^{-1})\Big|_{\substack{t=0\\ s=0}} \\ &= \frac{d}{ds}\frac{d}{dt} f(kgk^{-1} \exp(\mathrm{Ad}(k)tY) \exp(\mathrm{Ad}(k)sX))\Big|_{\substack{t=0\\ s=0}} \\ &= [\mathrm{Ad}(k)X][\mathrm{Ad}(k)Y]f(c_k(g)). \end{aligned}$$

□

1.3.1 Universal enveloping algebra of a Lie algebra

Definition. Let $\mathcal{L}$ be a Lie algebra. A unital associative algebra $\mathcal{U}(\mathcal{L})$ is called the *universal enveloping algebra* of $\mathcal{L}$ if there is a Lie algebra homomorphism $h : \mathcal{L} \to \mathcal{U}(\mathcal{L})$ such that for any Lie algebra homomorphism $f : \mathcal{L} \to A$ into a unital associative algebra A there is a unique associative algebra homomorphism $F : \mathcal{U}(\mathcal{L}) \to A$ such that $f = F \circ h$.

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{h} & \mathcal{U}(\mathcal{L}) \\ & \searrow^{f} & \downarrow F \\ & & A \end{array}$$

The universal enveloping algebra can be constructed as $\mathcal{U}(\mathcal{L}) := \mathcal{T}(\mathcal{L})/\mathcal{I}_{\mathcal{L}}$, where $\mathcal{T}(\mathcal{L})$ is the tensor algebra of $\mathcal{L}$,

$$\mathcal{T}(\mathcal{L}) = \bigoplus_{d=0}^{\infty} \mathcal{L}^{\otimes d},$$

where $\mathcal{L}^{\otimes d} = \mathcal{L} \otimes \cdots \otimes \mathcal{L}$ is the tensor product of $\mathcal{L}$ with itselt d times. The ideal $\mathcal{I}_{\mathcal{L}}$ is generated by the elements $a \otimes b - b \otimes a - [a, b] \in \mathcal{T}(\mathcal{L})$ for all $a, b \in \mathcal{L}$.

In particular, when $\mathcal{L} = \mathfrak{g}$ we can define the universal enveloping algebra of $\mathfrak{g}$ by $\mathcal{U}(\mathfrak{g}) = \mathcal{T}(\mathfrak{g})/\mathcal{I}_{\mathfrak{g}}$.

Theorem 1.3.7 (Poincaré-Birkhoff-Witt)**.** *Let $(x_i)_{i\in I}$ a basis of $\mathcal{L}$, with I being totally ordered. Then the family of monomials $h(x_{i_1})h(x_{i_2})\dots h(x_{i_n})$, where $i_1 \leq \cdots \leq i_n$, $n \geq 0$, is a basis of the universal enveloping algebra $\mathcal{U}(\mathcal{L})$ and $h : \mathcal{L} \to \mathcal{U}(\mathcal{L})$ is the Lie algebra homomorphism from the definition $\mathcal{U}(\mathcal{L})$.*

Proof. See Corollary C, p.92 [31]. □

Remark 1.3.8. In particular, for $\mathcal{U}(\mathfrak{g})$ this means for the basis $\{X_1, \dots, X_d\}$ of $\mathfrak{g}$, then the family $X_{i_1}^{j_1}, X_{i_2}^{j_2}, \dots, X_{i_n}^{j_n}$, where $1 \leq i_1 < i_2 < \cdots < i_n \leq d$ and $j_1, j_2, \dots, j_n \in \mathbb{N}$, is a basis of $\mathcal{U}(\mathfrak{g})$.

Let us now introduce the element of $\mathcal{U}(\mathfrak{g})$ that we will use the most. First define the following.

Definition. A Riemannian metric on a manifold M is a family of inner products $g_m : T_mM \times T_mM \to \mathbb{R}$ such that $m \mapsto g_m(X(m), Y(m))$ is smooth for any two smooth vector fields X and Y.

We can always equip a Lie group G with a Riemannian metric ρ, see the construction in [6] p.11. Then for a fixed basis $\{X_1, X_2, \dots, X_d\}$ of $\mathfrak{g}$ we define $\rho_{ij} := \rho(X_i, X_j)$

for all $1 \leq i, j \leq d$. The matrix (ρ_{ij}) is positive definite, so it is non-singular. Let us denote its inverse by (ρ_{ij}^{-1}). The *Laplace-Beltrami operator* on G equipped with ρ is $\Delta \in \mathcal{U}(\mathfrak{g})$ given by

$$\Delta = \sum_{i,j=1}^{d} \rho_{ij}^{-1} X_i X_j. \tag{1.5}$$

If G is compact and $X_1, X_2, \ldots, X_d$ is an orthonormal basis for $\mathfrak{g}$ with respect to the metric ρ, so that $\rho_{ij} = \delta_{ij}$ for all $1 \leq i, j \leq d$, then

$$\Delta = \sum_{i=1}^{d} X_i^2.$$

Proposition 1.3.9. *The Laplace-Beltrami operator Δ is independent of the choice of basis in $\mathfrak{g}$.*

Proof. See [6] Proposition 1.3.1 p.18. □

1.3.2 Weights

In this section we will suppose that the Lie group G is compact. We will start by providing a brief summary on weights, for references see [6] Chapter 2.5 and [31]. A *maximal torus* $\mathcal{T}$ in G is a maximal commutative subgroup of G. The dimension of $\mathcal{T}$ is called the *rank* of G. Note that any two maximal tori are conjugate, see [38] Corollary 4.35, p.255. Let t be the Lie algebra of $\mathcal{T}$, then t is a maximal abelian subalgebra of $\mathfrak{g}$, see [38] Proposition 4.30, p.252. Let π be a unitary representation of G, then the matrices $\{d\pi(X); X \in t\}$ are mutually commuting and therefore simultaneously diagonalizable, that is there exists a a non-singular matrix Q such that

$$Q d\pi(X) Q^{-1} = \operatorname{diag}(i\lambda_1(X). \ldots, i\lambda_{d_\pi}(X)), \qquad \text{for all } X \in t,$$

where each λ_k, $k = 1, \ldots, d_\pi$, is a linear functional, and we call these the *weights* of π. In particular, let us consider the adjoint representation of G acting on the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$. It is possible to equip $\mathfrak{g}_{\mathbb{C}}$ with an Ad-invariant inner product $(\cdot, \cdot)$, we will denote the corresponding norm on $\mathfrak{g}_{\mathbb{C}}$ by $|\cdot|$. Then the weights of the adjoint representation of G acting on $\mathfrak{g}_{\mathbb{C}}$ equipped with $(\cdot, \cdot)$ are called the *roots* of G. Let us denote by $\mathcal{P}$ the set of all roots of G. We can associate signs to each root the following way: let us fix $v \in t$ such that $\mathcal{P} \cap \{\eta(v) \in t^*; \eta(v) = 0\} = \emptyset$, where t^* is the dual space of t. Then we can split $\mathcal{P}$ into the set of positive roots $\mathcal{P}_+ := \{\alpha \in \mathcal{P}; \alpha(v) > 0\}$ and the set of negative roots $\mathcal{P}_- := \{\alpha \in \mathcal{P}; \alpha(v) < 0\}$ such that $\mathcal{P} = \mathcal{P}_+ \cup \mathcal{P}_-$. There exist a subset of positive roots $\mathcal{Q} \subset \mathcal{P}_+$ that forms a basis for t^* and every root $\alpha \in \mathcal{P}$ is a linear combination of the elements of $\mathcal{Q}$. The elements of $\mathcal{Q}$ are called *simple* or *fundamental roots* of G. It can be shown that for any fixed irreducible representation π of G, there exists a particular weight λ_π called the *highest weight* such that every weight of π is of the form

$$\mu_\pi = \lambda_\pi - \sum_{\alpha \in \mathcal{Q}} n_\alpha \alpha,$$

the terms n_α are non-negative integers, at least one of which is non-zero, see proof in [6] p.48. There is a one-to-one correspondence between $\widehat{G}$ and the space of highest weights D of irreducible representations, [54] Chapter 7, Theorem 1. $\widehat{G}$ is identified with D by this bijection. The norm $|\cdot|$ on D will be the norm derived from the inner product $(\cdot, \cdot)$.

1.3.3 Casimir spectrum

Let us list some known results which we will use later and can be all found in [6], Chapter 2.

Theorem 1.3.10. *For each $\pi \in \widehat{G}$, there exists a constant $\kappa_\pi \geq 0$ such that*

$$\Delta \pi_{i,j} = -\kappa_\pi \pi_{i,j}, \qquad \text{for all } i, j = 1, \ldots, d_\pi$$

we call $\{\kappa_\pi, \pi \in \widehat{G}\}$ the Casimir spectrum for G.

Theorem 1.3.11. *For each $\lambda \in D$,*

$$\kappa_\lambda = |\lambda - \rho|^2 - |\rho|^2,$$

where ρ is the half-sum of positive roots.

Corollary 1.3.12. *For all $\lambda \in D$,*

$$|\lambda|^2 \leq \kappa_\lambda \leq C(1 + |\lambda|^2),$$

where $C = \max\{2, |\rho|^2\}$.

Chapter 2

Spherical functions and K-bi-invariant functions

In this section we are interested in complex-valued functions that are constant on the double coset of a locally compact group, G. The notation is based on [59].

2.1 The convolution product

Let us introduce in this section the convolution product and some of its widely known properties. All the propositions which are discussed below can be found in [17]; however here we will provide more detailed proofs.

Definition. Let G be a locally compact group and $f, h \in L^1(G, \mathcal{B}(G), m_L)$. We define the *convolution product* of f and h by:

$$(f * h)(g) = \int_G f(x)h(x^{-1}g)dx, \quad \text{for all } g \in G.$$

The following proposition proves that the convolution product is well-defined from $L^1 \times L^1$ to L^1.

Proposition 2.1.1. *The convolution operation is well-defined and satisfies* $\|f*h\|_{L^1(G)} \leq \|f\|_{L^1(G)}\|h\|_{L^1(G)}$, *for all* $f, h \in L^1(G, \mathcal{B}(G), m_L)$. *Furthermore, it is associative on* $L^1(G, \mathcal{B}(G), m_L)$. *Thus,* $L^1(G, \mathcal{B}(G), m_L)$ *is a Banach algebra under convolution.*

Proof. Let us apply Fubini's theorem and a change of variable to see that for all

$f, h \in L^1(G, \mathcal{B}(G), m_L)$,

$$\begin{aligned}
\|f * h\|_{L^1} &= \int_G |(f*h)(g)|\, dg = \int_G \left| \int_G f(x)h(x^{-1}g)dx \right| dg \\
&\leq \int_G \int_G \left| f(x)h(x^{-1}g) \right| dx\, dg \\
&= \int_G \left(\int_G \left| h(x^{-1}g) \right| dg \right) |f(x)|\, dx \\
&= \int_G \left(\int_G |h(g')|\, dg' \right) |f(x)|\, dx \\
&= \int_G |h(g')|\, dg' \int_G |f(x)|\, dx = \|f\|_{L^1} \|h\|_{L^1}.
\end{aligned}$$

This implies that the convolution product is well defined from $L^1(G, \mathcal{B}(G), m_L) \times L^1(G, \mathcal{B}(G), m_L)$ to $L^1(G, \mathcal{B}(G), m_L)$.

To prove associativity, let $f, h, l \in L^1(G, \mathcal{B}(G), m_L)$ and $g \in G$, then by Fubini's theorem and a change of variable we have

$$\begin{aligned}
((f*h)*l)(g) &= \int_G (f*h)(x) l(x^{-1}g)dx \\
&= \int_G \int_G f(y)h(y^{-1}x)l(x^{-1}g)dy\, dx \\
&= \int_G f(y) \int_G h(x)l(x^{-1}y^{-1}g)dx\, dy \\
&= \int_G f(y)(h*l)(y^{-1}g)dy \\
&= (f*(h*l))(g).
\end{aligned}$$

□

Proposition 2.1.2. *The space $L^1(G, \mathcal{B}(G), m_L)$ can be equipped with an involution $f \mapsto f^*$ where*

$$f^*(g) = \overline{f(g^{-1})}\Delta(g^{-1}), \quad \text{for all } g \in G \text{ and } f \in L^1(G, \mathcal{B}(G), m_L).$$

Proof. For all $f \in L^1(G, \mathcal{B}(G), m_L)$, the equality $f^{**} = f$ follows by using the fact that Δ is a homomorphism, see Theorem 1.2.4. In fact, for all $g \in G$

$$f^{**}(g) = \overline{f^*(g^{-1})}\Delta(g^{-1}) = f(g)\Delta(g)\Delta(g^{-1}) = f(g).$$

Also for all $f, h \in L^1(G, \mathcal{B}(G), m_L)$ and $g \in G$, by applying a change of variable, the left-invariance of the Haar measure and the fact that the modular function Δ is a homomorphism, we have

$$\begin{aligned}(f*h)^*(g) &= \overline{(f*h)}(g^{-1})\Delta(g^{-1}) = \int_G \overline{f(x)}\ \overline{h(x^{-1}g^{-1})}\Delta(g^{-1})dx \\ &= \int_G \overline{f(g^{-1}x)}\ \overline{h(x^{-1})}\Delta(g^{-1})dx \\ &= \int_G \overline{f(g^{-1}x)}\Delta(g^{-1}x)\ \overline{h(x^{-1})}\Delta(x^{-1})dx \\ &= (h^* * f^*)(g).\end{aligned}$$

□

Remark 2.1.3. On $L^1(G, \mathcal{B}(G), m_L)$, the convolution product is commutative if and only if G is abelian, see Proposition 3.6.3 p.49 in [59].

2.2 The space of K-bi-invariant functions

Definition. Let G be a locally compact group and K a closed subgroup. A function f on G is said to be K-*left-invariant* if $f(kg) = f(g)$, K-*right-invariant* if $f(gk) = f(g)$ for all $g \in G$ and $k \in K$ and K-*bi-invariant* if it is both K-left-invariant and K-right-invariant.

Definition. For a locally compact group G and a compact closed K, we define the two *coset spaces* $K\backslash G := \{Kg; g \in G\}$, $G/K := \{gK; g \in G\}$ and the *double coset space* $K\backslash G/K := \{KgK; g \in G\}$. We denote by $C(K\backslash G)$, $C(G/K)$ and $C(K\backslash G/K)$ the closed subspaces of $C(G)$ which consists respectively of all K-*left-invariant*, K-*right-invariant* and K-*bi-invariant* functions. That is for all $k \in K$,

- $C(K\backslash G) := \{f \in C(G) : f(kg) = f(g) \text{ for all } k \in K \text{ and } g \in G\}$
- $C(G/K) := \{f \in C(G) : f(gk) = f(g) \text{ for all } k \in K \text{ and } g \in G\}$
- $C(K\backslash G/K) := C(K\backslash G) \cap C(G/K)$

These spaces are equipped with the corresponding induced topology. In the same way, we can define $C_b(K\backslash G/K)$, $C_c^\infty(K\backslash G/K)$, $C^p(K\backslash G/K)$ and $L^p(K\backslash G/K)$ for $p \geq 1$. We proceed similarly for the K-left-invariant and the K-right-invariant case.

Remark 2.2.1. There is a one-to-one correspondence between the space of K-bi-invariant continuous functions on G and the space of continuous functions on the double coset space. We denote both by $C(K\backslash G/K)$. Similarly for K-left- and K-right-invariant functions.

From now on we will assume that K is compact. The following operator will be extensively used throughout the thesis.

Proposition 2.2.2. *The linear mapping $\mathcal{Q}^K : f \mapsto f^{\#}$ from $C_b(G)$ to $C_b(K\backslash G/K)$, defined by*

$$f^{\#}(g) = \int_K \int_K f(k_1 g k_2) dk_1 dk_2, \quad \text{for all } g \in G.$$

is surjective and idempotent.

Proof. For all $f \in C_b(G)$, let us prove that $f^{\#}$ is in $C_b(K\backslash G/K)$; $f^{\#}$ is continuous on G by applying dominated convergence. To see that $f^{\#}$ is also K bi-invariant we use Proposition 1.2.5 and by applying a change of variable, we get for all g in G and $r_1, r_2 \in K$,

$$\begin{aligned} f^{\#}(r_1 g r_2) &= \int_K \int_K f(k_1 r_1 g r_2 k_2)\, dk_1\, dk_2 \\ &= \int_K \int_K f(l_1 g l_2)\, d(l_1 r_1^{-1})\, d(r_2^{-1} l_2) \\ &= \int_K \int_K f(l_1 g l_2)\, dl_1 dl_2 = f^{\#}(g). \end{aligned}$$

It then follows that $\mathcal{Q}^K$ is idempotent,

$$f^{\sharp\sharp}(g) = \int_K \int_K f^{\#}(k_1 g k_2)\, dk_1\, dk_2 = \int_K \int_K f^{\#}(g)\, dk_1\, dk_2 = f^{\sharp}(g)$$

Finally, $\mathcal{Q}^K$ is surjective, since for all $f \in C_b(K\backslash G/K) \subset C_b(G)$ we have $\mathcal{Q}^K f = f$. □

Corollary 2.2.3. *The mapping $\mathcal{Q}^K$ is an orthogonal projection from $L^2(G)$ to $L^2(K\backslash G/K)$.*

Proof. First we will prove that for all $f \in L^2(G)$, $f^{\sharp}$ is in $L^2(K\backslash G/K)$. By the Cauchy-Schwarz inequality, Fubini's theorem, Proposition 1.2.6, we get

$$\begin{aligned} \int_G |f^{\sharp}(g)|^2 dg &= \int_G \left| \int_K \int_K f(kgk')\, dk\, dk' \right|^2 dg \\ &\leq \int_G \int_K \int_K |f(kgk')|^2\, dk\, dk'\, dg \\ &= \int_K \int_K \int_G |f(kgk')|^2\, dg\, dk\, dk' \\ &= \int_K \int_K \int_G |f(g)|^2\, dg\, dk\, dk' = \|f\|_{L^2} < \infty. \end{aligned}$$

K-bi-invariance of $f^\sharp$ and the idempotence of $\mathcal{Q}^K : L^2(G) \to L^2(K/G\backslash K)$ can be proved the same way as in the continuous case in Proposition 2.2.2. Let us now show that $\mathcal{Q}^K$ is self-adjoint. For all $f, h \in L^2(G)$, we have

$$\begin{aligned}
\langle \mathcal{Q}^K f, h\rangle &= \int_G \mathcal{Q}^K f(g)\overline{h(g)}dg \\
&= \int_G \int_K \int_K f(kgk')\overline{h(g)}dk\, dk'\, dg \\
&= \int_K \int_K \int_G f(g)\overline{h(k'gk)}\, dg\, dk'\, dk \\
&= \int_G f(g) \int_K \int_K \overline{h(k'gk)}\, dk\, dk'\, dg \\
&= \int_G f(g)\overline{\mathcal{Q}^K h(g)}\, dg = \langle f, \mathcal{Q}^K h\rangle.
\end{aligned}$$

Thus, $\mathcal{Q}^K$ is an orthogonal projection from $L^2(G)$ to $L^2(K\backslash G/K)$. □

Example 2.2.4. We will identify the space of continuous K-bi-invariant functions $C(K\backslash G/K)$ with the space of K-left-invariant functions on the homogeneous space G/K. Let us consider the case where $G = SO(3)$ and $K = SO(2)$, then $G/K \simeq S^2$. Then for any function $f \in C(SO(3))$, its corresponding image by $\mathcal{Q}^K$ is the function $f^\sharp$ in $C(S^2)$, that is constant on the orbits.

2.3 Spherical measures and spherical functions

We are going to introduce spherical functions in this section; all proofs can be found in [59].

Definition. Let G be locally compact group, and K a closed subgroup. We call a measure ϑ on $(G, \mathcal{B}(G))$ a *spherical measure* for (G, K) if

i) ϑ is a non-zero Radon measure, i.e. it is an inner regular and locally finite Borel measure,
ii) ϑ is K-bi-invariant. i.e.: for all measurable $E \subseteq \mathcal{B}(G)$ and $k_1, k_2 \in K$, ϑ satisfies $\vartheta(k_1 E k_2) = \vartheta(E)$,
iii) the mapping $f \mapsto \vartheta(f)$ from the algebra $C_c(K\backslash G/K)$ with convolution operation to $\mathbb{C}$ is an algebra homomorphism, where

$$\vartheta(f) := \int_G f(g) d\vartheta(g).$$

Remark 2.3.1. Note that the term *spherical measure* is used here as per [59] and has a more restrictive meaning than that used in [1]. For the latter notion we will later introduce the term K-bi-invariant measure.

Theorem 2.3.2. *A spherical measure ϑ for (G, K) is absolutely continuous w.r.t the normalized Haar measure on G. In fact, there is a function $\omega \in C(K\backslash G/K)$ with $\omega(e) = 1$ such that for all $f \in C_c(K\backslash G/K)$*

$$\int_G f(x) d\vartheta(x) = \int_G f(x)\omega(x^{-1})\, dx.$$

That is, $x \mapsto \omega(x^{-1})$ is the Radon-Nikodym derivative of ϑ with respect to the Haar measure on G.

Proof. This is based on Theorem 8.2.4, p.157, in [59], here we will fill in some missing steps. Fix a function $h \in C_c(K\backslash G/K)$, such that $\vartheta(h) \neq 0$. By using the definition of a spherical measure and Fubini's theorem, we have for all $f \in C_c(K\backslash G/K)$

$$\begin{aligned}
\vartheta(f) &= \frac{1}{\vartheta(h)}\vartheta(f * h) = \frac{1}{\vartheta(h)}\int_G (f * h)(x) d\vartheta(x) \\
&= \frac{1}{\vartheta(h)}\int_G \left(\int_G f(y)h(y^{-1}x)dy\right) d\vartheta(x) \\
&= \frac{1}{\vartheta(h)}\int_G f(y)\left(\int_G h(y^{-1}x)d\vartheta(x)\right) dy \\
&= \int_G f(y)\omega(y^{-1})dy,
\end{aligned}$$

where $\omega(y) := \dfrac{1}{\vartheta(h)}\displaystyle\int_G h(yx)d\vartheta(x)$. To prove that ω is K-bi-invariant we use the K-bi-invariance of h and of the spherical measure ϑ. For all $k_1, k_2 \in K$ and $y \in G$, we have

$$\begin{aligned}
\omega(k_1yk_2) &= \frac{1}{\vartheta(h)}\int_G h(k_1yk_2x)d\vartheta(x) \\
&= \frac{1}{\vartheta(h)}\int_G h(k_1^{-1}k_1yk_2xk_1)d\vartheta(x) \\
&= \frac{1}{\vartheta(h)}\int_G h(yx)d\vartheta(x) = \omega(y)
\end{aligned}$$

Also,

$$\omega(e) = \frac{1}{\vartheta(h)}\int_G h(x)d\vartheta(x) = \frac{1}{\vartheta(h)}\vartheta(h) = 1.$$

To prove that ω is continuous, we use dominated convergence and the fact that h is continuous and has compact support. $\square$

Definition. A continuous function $\omega : G \to \mathbb{C}$ is a *spherical function* if
$\vartheta(f) := \displaystyle\int_G f(g)\omega(g^{-1})d\mu(g)$ is a spherical measure.

Theorem 2.3.3. *Let us consider a function $\omega : G \to \mathbb{C}$. The following statements are equivalent*

(i) *ω is a spherical function for (G, K)*
(ii) *ω is a continuous K bi-invariant function with $\omega(e) = 1$, and every function $f \in C_c(K\backslash G/K)$ is an eigenvector of the convolution operator T_ω with corresponding eigenvalue $\lambda_f \in \mathbb{C}$, i.e. $T_\omega f := f * \omega = \lambda_f \omega$.*
(iii) *ω is a continuous nonzero function such that for all $g_1, g_2 \in G$, we have*

$$\omega(g_1)\omega(g_2) = \int_K \omega(g_1 k g_2) dk.$$

Proof. See p.157 in [59]. □

In the next proposition the map $\lambda : C_c(K/G\backslash K) \to \mathbb{C}$ maps each $f \in C_c(K/G\backslash K)$ to the eigenvalue λ_f of the operator T_ω as given in Theorem 2.3.3.

Proposition 2.3.4. *The map $\lambda : C_c(K/G\backslash K) \to \mathbb{C}$ is an associative algebra homomorphism.*

Proof. Given the characterisation (ii) of Theorem 2.3.3, it follows that for any spherical function $\omega : G \to \mathbb{C}$, the corresponding spherical measure ϑ_ω satisfies for all $f \in C_c(K/G\backslash K)$

$$\vartheta_\omega(f) = (f * \omega)(e) = \lambda(f)\omega(e).$$

By definition $f \mapsto \vartheta_\omega(f)$ is a homomorphism from $C_c(K/G\backslash K)$ to $\mathbb{C}$ and $\omega(e) = 1$, so we get for all $f_1, f_2 \in C_c(K/G\backslash K)$

$$\vartheta_\omega(f_1 * f_2) = \vartheta_\omega(f_1)\vartheta_\omega(f_2).$$

That is

$$\lambda(f_1 * f_2) = \lambda(f_1)\lambda(f_2).$$

The remaining conditions for λ to be an algebra homomorphism follow from the linearity of the integral $\vartheta_\omega(f)$. □

2.4 Positive definite functions

Let us recall the definition and properties of positive definite functions based on Chapter 8.4 in [59]. In this section G will be a locally compact group.

Definition. A function $\phi : G \to \mathbb{C}$ is *positive definite* if for all $n \in \mathbb{N}$, $g_1, g_2, ..., g_n \in G$ and $c_1, c_2, ...c_n \in \mathbb{C}$, we have the following inequality

$$\sum_{i,j=1}^{n} \overline{c_i} c_j \phi(g_i^{-1} g_j) \geq 0.$$

Proposition 2.4.1. *Let $\phi : G \to \mathbb{C}$ be a positive definite function, then ϕ satisfies:*

(i) $\phi(e) \geq 0$,
(ii) $\phi(g^{-1}) = \overline{\phi(g)}$, for all $g \in G$,
(iii) $|\phi(g)| \leq \phi(e)$, for all $g \in G$.

Proof. This is a detailed version of the proof outlined in [59], Proposition 8.4.2. p165.

(i) Choose $n = 1$, $g_1 = e$ and $c_1 = 1$ and apply the definition of positive definiteness.
(ii) For $n = 2$, choose $g_1 = g$ and $g_2 = e$, then by positive definiteness we have for all $c_1, c_2 \in \mathbb{C}$,

$$\overline{c_1} c_2 \phi(g^{-1}) + |c_1|^2 \phi(e) + |c_2|^2 \phi(e) + \overline{c_2} c_1 \phi(g) \geq 0. \tag{2.1}$$

From (i) we saw that $\phi(e)$ and therefore $(|c_1| + |c_2|)\phi(e)$ are real numbers.
Thus, $\overline{c_1} c_2 \phi(g^{-1}) + \overline{c_2} c_1 \phi(g)$ has to be a real number as well. For simplicity, let us look at the real number $c\phi(g^{-1}) + \overline{c}\phi(g)$, where $c = \overline{c_1} c_2 \in \mathbb{C}$. Since it has no imaginary part, we have

$$\overline{c\phi(g^{-1}) + \overline{c}\phi(g)} = c\phi(g^{-1}) + \overline{c}\phi(g).$$

In particular, for $c = 1$ we obtain

$$\overline{\phi(g^{-1})} + \overline{\phi(g)} = \phi(g^{-1}) + \phi(g). \tag{2.2}$$

For $c = i$, we get

$$-i\overline{\phi(g^{-1})} + i\overline{\phi(g)} = i\phi(g^{-1}) - i\phi(g).$$

Therefore,

$$-\overline{\phi(g^{-1})} + \overline{\phi(g)} = \phi(g^{-1}) - \phi(g). \tag{2.3}$$

Adding (2.2) and (2.3), we get $\overline{\phi(g)} = \phi(g^{-1})$.

iii) The function ϕ is positive definite, therefore for all $n \in \mathbb{N}$ and $g_1, g_2, ..., g_n \in G$, the matrix with (i, j)th entry $\phi(g_i^{-1} g_j)$ is positive definite. Hence, the determinant of this matrix is non-negative, in particular when n=2, we have for all $g_1, g_2 \in G$,

$$\phi(g_1^{-1}g_1)\phi(g_2^{-1}g_2) - \phi(g_1^{-1}g_2)\phi(g_2^{-1}g_1) \geq 0.$$

Fix $g_2 = e$, then for any $g = g_1 \in G$ using (ii) we get

$$\phi(e)^2 - |\phi(g)|^2 \geq 0.$$

Thus, $\phi(e) \geq |\phi(g)|$. □

Definition. If π is a representation of G on a vector space V, then a vector $u \in V$ is called *cyclic* for π if $V = \overline{\mathrm{Span}\{\pi(g)u : g \in G\}}$. If K is a subgroup of G and $\pi(k)u = u$ for all $k \in K$ then u is said to be K*-fixed.*

Proposition 2.4.2. *Let π be a unitary representation of G on a Hilbert space H, and $u \in H$ a cyclic unit vector. Then $g \mapsto \phi(g) := \langle u, \pi(g)u \rangle$ is positive definite.*

Proof. Any representation π is a homomorphism, also $\pi(g)$ is a unitary operator for all $g \in G$. Therefore, for all $g_1, g_2, ..., g_n \in G$, $c_1, c_2, ..., c_n \in \mathbb{C}$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n}\overline{c_i}c_j\phi(g_i^{-1}g_j) &= \sum_{i=1}^{n}\sum_{j=1}^{n}\overline{c_i}c_j\langle u, \pi(g_i)^{-1}\pi(g_j)u\rangle \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\overline{c_i}c_j\langle \pi(g_i)u, \pi(g_j)u\rangle \\ &= \left\langle \sum_{i=1}^{n} c_i\pi(g_i)u, \sum_{j=1}^{n} c_j\pi(g_j)u \right\rangle \\ &= \left\| \sum_{i=1}^{n} c_i\pi(g_i)u \right\|^2 \geq 0.\end{aligned}$$

Hence, ϕ is positive definite. □

Example 2.4.3. Let $G = S^1$ act on $L^2(G)$ via the regular representation, $\pi(\alpha)f(\cdot) = f(\cdot + \alpha)$ for $e^{i\alpha} \in S^1$. The subspace V generated by $\sin(x)$ and $\cos(x)$, is G-invariant. Then $f(x) := \frac{1}{\sqrt{\pi}}\cos(x)$ is a cyclic unit vector of V. The map $\phi : \vartheta \mapsto \int f(x)\overline{f(x+\vartheta)}dx$ is positive definite.

To prove the converse of the previous proposition, we will need the following lemma.

Lemma 2.4.4. *Let π be a representation of G on a complex vector space V, equipped with a G-invariant non-negative definite Hermitian form $b : V \times V \to \mathbb{C}$ that satisfies*

$$|b(v,w)|^2 \leq |b(v,v)| \cdot |b(w,w)|, \qquad \text{for all } v, w, \in V.$$

We denote $\eta := \{\varepsilon \in V : b(\varepsilon, \varepsilon) = 0\}$, which is a subspace of V. Then b induces an inner product $\langle \cdot, \cdot \rangle$ on V/η, by

$$\langle [v], [w] \rangle := b(v,w), \qquad \text{for all } [v], [w] \in V/\eta,$$

And π induces a unitary representation ρ of G on V/η by

$$\rho(g)[v] := [\pi(g)(v)], \qquad \text{for all } g \in G, [v] \in V/\eta.$$

Proof. First, note that $\langle [v], [w] \rangle$ is well-defined for all $[v], [w] \in V/\eta$, i.e. it is independent of the choice of element in $[v]$ and $[w]$. Indeed, first observe that for all $v \in V$ and $\varepsilon \in \eta$ we have

$$|b(v,\varepsilon)|^2 \leq |b(v,v)||b(\varepsilon,\varepsilon)| = 0.$$

Thus, for all $v, w \in V$ and $\varepsilon, \varepsilon' \in \eta$ we have

$$\begin{aligned} \langle [v+\varepsilon], [w+\varepsilon'] \rangle &= b(v,w) + b(v,\varepsilon') + b(\varepsilon,w) + b(\varepsilon,\varepsilon') \\ &= b(v,w) = \langle [v], [w] \rangle \end{aligned}$$

Moreover, the bilinear form $\langle \cdot, \cdot \rangle$ is positive definite, if $[v] \in V/\eta$ such that $\langle [v], [v] \rangle = 0$, then $b(v,v) = 0$ and $v \in \eta$, thus $[v] = [0]$.

Let us now prove that the mapping $\rho : G \to \mathrm{Aut}(V/\eta)$ is also well-defined. By assumption, b is G invariant, so we have the following: for all $\varepsilon \in \eta$ and $g \in G$, $b(\pi(g)(\varepsilon), \pi(g)(\varepsilon)) = b(\varepsilon,\varepsilon) = 0$. Thus, $\pi(g)(\eta) \subseteq \eta$ for all $g \in G$. We then get

$$\rho(g)[v+\varepsilon] = [\pi(g)(v) + \pi(g)(\varepsilon)] = [\pi(g)(v)] = \rho(g)[v]$$

for all $v \in V$ and $\varepsilon \in \eta$. From the properties of π, it is easy to see that ρ is a representation of G on V/η. Furthermore, ρ is unitary on the complex Hilbert space V/η by G-invariance of b, i.e. for all $g \in G$, $v, w \in V$,

$$\begin{aligned} \langle \rho(g)[v], \rho(g)[w] \rangle &= \langle [\pi(g)(v)], [\pi(g)(w)] \rangle = b(\pi(g)(v), \pi(g)(w)) \\ &= b(v,w) = \langle [v], [w] \rangle. \end{aligned}$$

This shows $\rho(g)$ is an isometry, then unitarity follows as ρ is a representation. □

Theorem 2.4.5. *Let $\phi : G \to \mathbb{C}$ be a continuous positive definite function, with $\phi(e) = 1$. Then there exist a unitary representation π of G in a complex Hilbert space H_ϕ and a cyclic unit vector $u_\phi \in H_\phi$ such that*

$$\phi(g) = \langle u_\phi, \pi(g)u_\phi\rangle, \quad \textit{for all } g \in G. \tag{2.4}$$

Proof. Here we provide a very detailed proof based on the sketch in [59], Proposition 8.4.6., p.166.

We will construct H_ϕ with the help of Lemma 2.4.4. For any $g \in G$, we will use the usual notation of $\delta_g : G \to \{0, 1\}$ for the Dirac function taking the value 1 at g and 0 elsewhere; then we denote the complex vector space $V := \mathrm{Span}_{\mathbb{C}} \{\delta_g : g \in G\}$. Let $b : V \times V \to \mathbb{C}$ be the Hermitian form defined first on the basis vectors δ_g by

$$b(\delta_h, \delta_g) := \phi(g^{-1}h), \qquad \text{for all } h, g, \in G.$$

Then extend b to $V \times V$ complex linearly in the first variable, and complex antilinearly in the second. The Hermitian form b is non-negative definite, since for any $f = \sum_{i=1}^{n} \lambda_i \delta_{g_i} \in V$ where $n \in \mathbb{N}$, $g_i \in G$ and $\lambda_i \in \mathbb{C}$ for $i = 1, \ldots n$, the form b satisfies

$$b(f, f) = \sum_{i,j=1}^{n} \overline{\lambda_i}\lambda_j b(\delta_{g_j}, \delta_{g_i}) = \sum_{i,j=1}^{n} \overline{\lambda_i}\lambda_j \phi(g_i^{-1}g_j) \geq 0,$$

because ϕ is positive definite.

Let $\widetilde{L} : G \to \mathrm{Aut}(V)$ be the left regular representation of G given by $(\widetilde{L}(g)f)(x) = f(g^{-1}x)$, for all $g, x \in G$. First note that $\widetilde{L}(g)\delta_h = \delta_{g^{-1}h}$ for all $g, h \in G$. Then b is invariant under the action of $\widetilde{L}$, since for all $g \in G$ and $f_1 = \sum_{j=1}^{n} \lambda_j \delta_{g_j}$, $f_2 = \sum_{i=1}^{m} \beta_i \delta_{g_i} \in V$, we have

$$\begin{aligned} b(\widetilde{L}(g)f_1, \widetilde{L}(g)f_2) &= \sum_{j=1}^{n}\sum_{i=1}^{m} \overline{\beta_i}\lambda_j b(\delta_{g^{-1}g_j}, \delta_{g^{-1}g_i}) \\ &= \sum_{j=1}^{n}\sum_{i=1}^{m} \overline{\beta_i}\lambda_j \phi(g_j^{-1}gg^{-1}g_i) \\ &= \sum_{j=1}^{n}\sum_{i=1}^{m} \overline{\beta_i}\lambda_j \phi(g_j^{-1}g_i) \\ &= b(f_1, f_2) \end{aligned}$$

We denote $\eta_\phi := \{f \in V : b(f,f) = 0\}$, which is a subspace of V. By Lemma 2.4.4, b induces an inner product $\langle \cdot, \cdot \rangle$ on V/η_ϕ and $\widetilde{L}$ induces a unitary representation $\pi : G \to \mathrm{Aut}(V/\eta_\phi)$. Let H_ϕ be the Hilbert space completion of V/η_ϕ with respect to this inner product. Since V_ϕ/η_ϕ is dense in H_ϕ, if π is continuous, then it extends uniquely to a continuous representation denoted by the same letter $\pi : G \to \mathrm{Aut}(H_\phi)$ which is also unitary.
Let us now prove that π is indeed continuous, i.e. that for all $[f] \in V/\eta_\phi$ we have $\pi(g)[f] \to [f]$ as $g \to e$. Since $\langle \cdot, \cdot \rangle$ is induced by b, it suffices to show that for all $f \in V$, we have $b(\widetilde{L}(g)f - f, \widetilde{L}(g)f - f) \to 0$ as $g \to e$.

Fix $f = \sum_{i=1}^{n} c_i \delta_{g_i} \in V$, for some $n \in \mathbb{N}$, $c_i \in \mathbb{C}$, $g_i \in G$ for $i = 1, \ldots n$. Since

$$b(\widetilde{L}(g)f - f, \widetilde{L}(g)f - f) = 2b(f,f) - 2\,\mathrm{Re}\, b(\widetilde{L}(g)f, f),$$

it suffices to show that $b(\widetilde{L}(g)f, f) \to b(f,f)$ as $g \to e$. We have

$$b(\widetilde{L}(g)f, f) = \sum_{i=1}^{n} \overline{c_i} c_j b(\delta_{g^{-1}g_j}, \delta_{g_i}) = \sum_{i=1}^{n} \overline{c_i} c_j \phi(g_i^{-1} g^{-1} g_j).$$

The map $g \mapsto g_i^{-1} g^{-1} g_j$ is continuous from G to G and ϕ is continuous from G to $\mathbb{C}$, therefore $g \mapsto \phi(g_i^{-1} g^{-1} g_j)$ is continuous from G to $\mathbb{C}$. Hence, by linearity when $g \to e$ the right hand side converges to

$$\sum_{i=1}^{n} \overline{c_i} c_j \phi(g_i^{-1} g_j) = b(f,f),$$

implying that π is continuous.
Finally, $u_\phi := [\delta_e]$ is a cyclic unit vector. Indeed, since $\pi(g)[\delta_h] = [\delta_{gh}]$ for all $g, h \in G$,

$$H_\phi = \overline{\mathrm{Span}_{\mathbb{C}}\{[\delta_g] : g \in G\}} = \overline{\mathrm{Span}_{\mathbb{C}}\{\pi(g)u_\phi : g \in G\}}.$$

And by construction, we also have for all $g \in G$,

$$\langle u_\phi, \pi(g)u_\phi \rangle = \langle [\delta_e], [\delta_g] \rangle = b(\delta_e, \delta_g) = \phi(g).$$

□

2.5 Positive definite spherical functions

Definition. Let G be a locally compact group and K a closed subgroup. A function $\phi : G \to \mathbb{C}$ is called a *positive definite spherical function* for (G, K), if it is a positive definite function on G and a spherical function for (G, K).

2.5.1 Positive definite spherical functions and unitary representations

We proved in Theorem 2.4.5 that any positive definite function uniquely defines a cyclic vector and a unitary representation. In addition, for a positive definite spherical function, the unitary representation will be irreducible and the cyclic vector will be K-fixed, we will provide a detailed proof of this result following [59] p.167.

Definition. Let G be locally compact and K a compact subgroup of G. A unitary representation π of G on a Hilbert space V is called *spherical* if there exists a nonzero $u \in V$ that is fixed under all $k \in K$, i.e. such that $\pi(k)u = u$ for all $k \in K$. We shall denote the set of equivalence classes of unitary spherical representations by $\widehat{G}_K$,

$$\widehat{G}_K := \{\pi \in \widehat{G} : \pi \text{ is spherical}\}.$$

Then $\widehat{G}_K$ is in one-to-one correspondence with a subset of the highest weights D of the irreducible representations, that we will call restricted highest weights and we will denote this subset by D_S.

Remark 2.5.1. In the following whenever we say spherical representation, we mean spherical unitary representation.

Only for this section we will define bounded representations of a Lie group and of a Banach algebra, as we will use these notions in the proof of Theorem 2.5.3.

Definition. If π is a representation of G on a Banach space B, then π is called *bounded* if there exists a constant $M > 0$ such that the operator norm $\|\pi(g)\|_{\mathcal{L}(B)} \leq M$ for all $g \in G$.

In particular, any unitary and so spherical representation of G is a bounded representation.

Definition. *A representation of a Banach algebra* $\mathcal{A}$ on a Banach space B is an algebra homomorphism $\pi : \mathcal{A} \to \mathrm{End}(B)$, where $\mathrm{End}(B)$ is the space of all endomorphism on B. It is called *bounded* if there is a constant $M > 0$ such that $\|\pi(a)\|_{\mathcal{L}(B)} \leq M\|a\|_{\mathcal{A}}$ for all $a \in \mathcal{A}$.

In particular, if π is a bounded representation of a locally compact topological group G on a Banach space B, then we can define a map $\dot{\pi} : L^1(G) \to \mathrm{End}(B)$ by

$$\dot{\pi}(f) : b \mapsto \int_G f(x)\pi(x)b \, dm_L(x), \qquad \text{for all } b \in B, f \in L^1(G).$$

One can easily verify that $\dot{\pi}$ is a bounded representation of the Banach algebra $L^1(G)$ with respect to convolution.

We can now recall the following useful theorem from [59], Chapter 4 .

Theorem 2.5.2. *Let π be a bounded representation of G on B Banach space, and $B' \subset B$ be a closed subset. B' is G-invariant (under π) if and only if B' is $L^1(G)$-invariant (under $\dot{\pi}$).*

Theorem 2.5.3. *Let ϕ be a continuous positive definite spherical function for (G, K), such that $\phi(e) = 1$. Let π be the corresponding unitary representation acting on the Hilbert space H_ϕ and $u \in H_\phi$ the cyclic unit vector such that $\phi(g) = \langle u, \pi(g)u\rangle$. Then the pair (π, u) satisfy the following*

i) *π is irreducible,*
ii) *u is K-fixed, i.e. $\pi(k)u = u$ for all $k \in K$,*
iii) *the space of K-fixed vectors $V_\pi^K \subseteq V_\pi$ is one-dimensional and V_π^K is spanned by u.*

Proof. For a different approach, see Lemma 6.2.3 in [57]. Here, we will follow the steps outlined in [59] Theorem 8.4.8., p.167, however, our proof will be more detailed and we will fill out all missing arguments. Let us first deal with (ii) and (iii), then we will end with the proof of (i).

(ii) The function ϕ is spherical, therefore K-bi-invariant. In particular $\phi(g) = \phi(k^{-1}g)$ for all $g \in G$ and $k \in K$, that is

$$\langle u, \pi(g)u\rangle = \langle u, \pi(k^{-1})\pi(g)u\rangle = \langle \pi(k)u, \pi(g)u\rangle.$$

Thus, by linearity of the inner product we get $\langle \pi(k)u - u, \pi(g)u\rangle = 0$. By linearity again, for all $k \in K$, $(\pi(k)u - u)$ is orthogonal to the space $\mathrm{Span}\{\pi(g)u; g \in G\}$, which is dense in H_ϕ since u is a cyclic vector. This means $\pi(k)u = u$, for all $k \in K$.

For (iii), we will prove that the orthogonal complement of $\overline{\mathrm{Span}\{u\}}$ in V_π^K is $\{0\}$. We first claim that $v \perp \dot{\pi}(f)u$ for all $f \in C_c(G)$, i.e. that

$$\langle \dot{\pi}(f)u, v\rangle = 0 \text{ for all } f \in C_c(G). \tag{2.5}$$

For all $g \in G$ and $f \in C_c(K\backslash G/K)$, we have

$$\begin{aligned}
\langle \dot{\pi}(f)u, \pi(g)u \rangle &= \left\langle \int_G f(x)\pi(x)u \, dx, \pi(g)u \right\rangle \\
&= \int_G f(x)\langle \pi(x)u, \pi(g)u \rangle \, dx \\
&= \int_G f(x)\langle u, \pi(x^{-1})\pi(g)u \rangle \, dx \\
&= \int_G f(x)\phi(x^{-1}g) \, dx \\
&= (f * \phi)(g) = \lambda_f \phi(g) = \lambda_f \langle u, \pi(g)u \rangle,
\end{aligned}$$

where λ_f is the eigenvalue from Theorem 2.3.3. Since u is cyclic, this implies $\dot{\pi}(f)u = \lambda_f u$.

Note that the map $\mathcal{Q}^K : C_c(G) \to C_c(K\backslash G/K)$ in surjective, see [20], p.61-2. Any function $f \in C_c(G)$ decomposes as $f = \mathcal{Q}^K f + (Id - \mathcal{Q}^K)f$, where from Proposition 2.2.2 we know that $f_1 := \mathcal{Q}^K f \in C_c(K\backslash G/K)$ and $f_2 := (Id - \mathcal{Q}^K f) \in \mathrm{Ker}(\mathcal{Q}^K)$. Therefore, for all $v \in V_\pi^K$ with $v \perp u$,

$$\begin{aligned}
\langle \dot{\pi}(f)u, v \rangle &= \langle \dot{\pi}(f_1)u, v \rangle + \langle \dot{\pi}(f_2)u, v \rangle \\
&= \langle \lambda_{f_1} u, v \rangle + \langle \dot{\pi}(f_2)u, v \rangle \\
&= 0 + \int_G f_2(x)\langle \pi(x)u, v \rangle \, dx.
\end{aligned}$$

The Haar measure on K is normalized, u and v are both K-fixed and $f_2^\sharp = 0$ for all $f \in C_c(G)$. It then follows by Fubini's theorem that for all $f \in C_c(G)$,

$$\begin{aligned}
\langle \dot{\pi}(f)u, v \rangle &= \int_K \int_K \int_G f_2(x)\langle \pi(x)u, v \rangle \, dx \, dk_1 dk_2 \\
&= \int_G \int_K \int_K f_2(x)\langle \pi(x)\pi(k_2)u, \pi(k_1^{-1})v \rangle \, dk_1 dk_2 \, dx \\
&= \int_G f_2^{\#}(x)\langle \pi(x)u, v \rangle \, dx = 0.
\end{aligned}$$

By a density argument, this implies

$$\langle \dot{\pi}(f)u, v \rangle = 0 \text{ for all } f \in L^1(G). \tag{2.6}$$

Let us define

$$S := \mathrm{Span}\{\dot{\pi}(f)u; f \in L^1(G)\},$$

which is a subspace of V_π. It is clear that S is $L^1(G)$-invariant (under $\dot{\pi}$). Thus, by Theorem 2.5.2, the space S is G-invariant (under π). Note that $u \in S$, so by G-invariance

$$\mathrm{Span}\{\pi(g)u : g \in G\} \subseteq S = \mathrm{Span}\{\dot{\pi}(f)u : f \in L^1(G)\}.$$

Thus, from (2.6) it follows that $\langle \pi(g)u, v\rangle = 0$ for all $g \in G$. But u is a cyclic vector, so this implies $\langle w, v\rangle = 0$ for all $w \in V_\pi$, i.e. $v = 0$.

(i) Suppose to the contrary that there exists a subspace $W \subset V_\pi$ that is G-invariant (under π). Then $W^\perp$ is also G-invariant. Indeed for all $g \in G$, $w_1 \in W$ and $w_2 \in W^\perp$, we have

$$\langle w_1, \pi(g^{-1})w_2\rangle = \langle \pi(g)w_1, w_2\rangle = \langle w_1, w_2\rangle = 0$$

Let $P : V_\pi \to W$ be the orthogonal projection on W. We claim that P intertwines with π. For all $v \in V_\pi$, there exist $w_1 \in W$ and $w_2 \in W^\perp$ such that $v = w_1 + w_2$. So for $g \in G$ and $v \in V_\pi$,

$$P\pi(g)v = P(\pi(g)w_1 + \pi(g)w_2) = P\left(\pi(g)w_1\right) = \pi(g)w_1 = \pi(g)P(v).$$

In particular, for all $k \in K$ we have, $\pi(k)P(u) = P(\pi(k)u) = P(u)$.

Thus, $P(V_\pi^K) \subset V_\pi^K$ and so $P(u) = \lambda u$ for some $\lambda \in \mathbb{C}$. Now applying the projection twice, we get:

$$\lambda u = P(u) = P^2(u) = \lambda^2 u.$$

This means λ is either 0 or 1. In the case where $\lambda = 0$, we have $Pu = \lambda u = 0$, so $P\pi(g)u = \lambda\pi(g)u = 0$ for all $g \in G$. It then follows that $\pi(g)u \subset W^\perp$, but u is cyclic for π, so $V_\pi = W^\perp$ and $W = \{0\}$. The case where $\lambda = 1$, we have $Pu = u$, so $P\pi(g)u = \pi(g)u \in W$ for all $g \in G$. Since u is cyclic, this implies $V_\pi = W$.
Thus, we have proved that π is irreducible. □

Theorem 2.5.4. *Let π be an irreducible unitary representation of G such that the space of K-fixed vectors is spanned by a unit vector u. Then, the function $\phi(\cdot) := \langle u, \pi(\cdot)u\rangle$ is positive definite and spherical for (G, K).*

Proof. We have already dealt with the positive definiteness in Proposition 2.4.2, so we only need to prove that ϕ is a spherical function, for which we will use the characterisation (ii) in Theorem 2.3.3. The representation π is continuous, so ϕ is also continuous. The vector u is K-fixed, so it then follows that ϕ is K-bi-invariant, i.e. for all $g \in G$ and $k_1, k_2 \in K$

$$\phi(k_1 g k_2) = \langle u, \pi(k_1 g k_2)u\rangle = \langle \pi(k_1^{-1})u, \pi(g)\pi(k_2)u\rangle = \langle u, \pi(g)u\rangle = \phi(g).$$

Now, for any $f \in C_c(K\backslash G/K)$, the vector $\dot{\pi}(f)u$ is K-fixed, since for all $k \in K$,

$$\pi(k)\dot{\pi}(f)u = \int_G f(x)\pi(kx)u\, dx = \int_G f(k^{-1}x)\pi(x)u\, dx = \dot{\pi}(f)u.$$

The space of K-fixed vectors V_π^K is spanned by u, thus there exists $\lambda_f \in \mathbb{C}$ such that $\dot{\pi}(f)u = \lambda_f u$. Let us now compute the convolution $f * \phi$, for all $g \in G$

$$\begin{aligned}(f * \phi)(g) &= \int_G f(x)\phi(x^{-1}g)dx = \int_G f(x)\langle u, \pi(x^{-1}g)u\rangle dx \\ &= \int_G f(x)\langle \pi(x)u, \pi(g)u\rangle dx = \left\langle \int_G f(x)\pi(x)udx, \pi(g)u \right\rangle \\ &= \langle \dot{\pi}(f)u, \pi(g)u\rangle = \lambda_f \langle u, \pi(g)u\rangle = \lambda_f \phi(g).\end{aligned}$$

We have proved that ϕ is a spherical function. □

2.6 Gelfand pairs

Definition. Let G be a locally compact group and K a compact subgroup. Then (G, K) is called a *Gelfand pair* if the convolution algebra $L^1(K\backslash G/K)$ is commutative.

Lemma 2.6.1. *If (G, K) is a Gelfand pair, then G is unimodular.*

Proof. See p.75 in [57] or p.154 in [59]. □

Proposition 2.6.2. *If $f \in C_c(K\backslash G/K)$ and $h \in C_c(G)$, then $(h * f)^\# = h^\# * f$ and $(f * h)^\# = f * h^\#$.*

Proof. By applying Fubini's theorem and the fact that the Haar measure is unimodular on G if (G, K) is a Gelfand pair, for all $g \in G$,

$$\begin{aligned}(h^\# * f)(g) &= \int_G \int_K \int_K h(k_1 y k_2) dk_1 dk_2\, f(y^{-1}g)dy \\ &= \int_K \int_K \int_G h(\tilde{y}) f(k_2 \tilde{y}^{-1} k_1 g) d\tilde{y}\, dk_1 dk_2 \\ &= \int_K \int_K \int_G h(\tilde{y}) f(k_2^{-1} k_2 \tilde{y}^{-1} k_1 g k_2) d\tilde{y} dk_1 dk_2 \\ &= \int_K \int_K \int_G h(\tilde{y}) f(\tilde{y}^{-1} k_1 g k_2) d\tilde{y} dk_1 dk_2 \\ &= (h * f)^\#(g).\end{aligned}$$

Also,

$$\begin{aligned}(f * h^{\#})(g) &= \int_G f(y) \int_K \int_K h(k_1 y^{-1} g k_2) dk_1 dk_2 \, dy \\ &= \int_K \int_K \int_G f(k_1^{-1} \tilde{y} k_2) h(\tilde{y}^{-1} k_1 g k_2) d\tilde{y} \, dk_1 dk_2 \\ &= \int_K \int_K \int_G f(\tilde{y}) h(\tilde{y}^{-1} k_1 g k_2) d\tilde{y} \, dk_1 dk_2 \\ &= (f * h)^{\#}(g).\end{aligned}$$

□

Proposition 2.6.3. *The convolution algebra $C_c(K\backslash G/K)$ is commutative if and only if for every irreducible unitary representation π of G on a Hilbert space H the subspace H_K of K-fixed vectors is at most one-dimensional.*

Proof. See [57] Prop. 6.3.1 on p.82. □

2.6.1 Compact Gelfand pairs

Let us suppose that G is compact and that K is a closed subgroup, such that (G, K) is a Gelfand pair.

Theorem 2.6.4. *Let G be a compact group and K a closed subgroup such that (G, K) is a Gelfand pair, then every continuous (G, K)-spherical function ϕ is positive definite.*

Proof. This proof is a modified version of the one in [59], p.204. First note that $\phi \in L^2_K(G)$ as it is K-bi-invariant and continuous on the compact group G. According to the Spherical Peter-Weyl Theorem 2.7.3, $L^2_K(G) = \bigoplus_{\pi \in \widehat{G}_K} \mathcal{M}_{V_\pi^K}$, where $\mathcal{M}_{V_\pi^K}$ is spanned by the function $\phi_\pi = \langle u_\pi, \pi(\cdot) u_\pi \rangle$. By Theorem 2.5.4 we know that ϕ_π is positive definite and spherical. Hence, ϕ has an L^2 decomposition $\phi = \sum_{\pi \in \widehat{G}_K} c_\pi \phi_\pi$. For all $\pi \in \widehat{G}_K$ the functions ϕ_π and ϕ are spherical, thus we can use the characterisation given in Theorem 2.3.3 (ii). So for all $f \in C_c(K\backslash G/K)$

$$\begin{aligned} f * \phi_\pi &= \lambda_\pi(f) \phi_\pi, \quad \text{for all } \pi \in \widehat{G}_K \\ f * \phi &= \lambda(f) \phi, \end{aligned}$$

where $\lambda(f), \lambda_\pi(f) \in \mathbb{C}$. In particular $(f * \phi_\pi)(e) = \lambda_\pi(f) \langle u_\pi, \pi(e) u_\pi \rangle = \lambda_\pi(f)$, and by linearity, $(f*\phi)(e) = \sum_{\pi\in\widehat{G}_K} c_\pi\lambda_\pi(f) = \lambda(f)$. Note that, by Proposition 2.3.4, the mappings $\lambda_\pi : C(K\backslash G/K) \to \mathbb{C}$ and $\lambda : C(K\backslash G/K) \to \mathbb{C}$ are algebra homomorphisms.

Also, for all $\pi \in \widehat{G}_K$ we have

$$\begin{aligned}\lambda_\pi(\phi_\pi) = (\phi_\pi * \phi_\pi)(e) &= \int_G \phi_\pi(g)\phi_\pi(g^{-1})dg \\ &= \int_G \phi_\pi(g)\overline{\phi_\pi(g)}dg = \int_G |\phi_\pi(g)|^2 dg > 0.\end{aligned}$$

For $\pi, \pi' \in \widehat{G}_K$ such that $\pi \neq \pi'$, by Schur's orthogonality relation we have

$$\phi_\pi * \phi_{\pi'} = \int_G \langle u_\pi, \pi(g)u_\pi\rangle\langle \pi'(g)u_{\pi'}, u_{\pi'}\rangle dg = 0.$$

Thus $\lambda(\phi_\pi * \phi_{\pi'}) = 0$. On the other hand, if there are two distinct non zero c_{π_1} and c_{π_2}, then we have

$$\lambda(\phi_{\pi_1} * \phi_{\pi_2}) = \lambda(\phi_{\pi_1})\lambda(\phi_{\pi_2}) = c_{\pi_1}\lambda_{\pi_1}(\phi_{\pi_1})c_{\pi_2}\lambda_{\pi_2}(\phi_{\pi_2}) \neq 0$$

This is a contradiction, so there can only be one nonzero c_π. Thus $\phi = c_\pi\phi_\pi$ and it is positive definite. □

Definition. The *character* $\chi_\pi : G \to \mathbb{C}$ of a finite dimensional representation π of the group G on V, is given by $\chi_\pi(g) := \mathrm{tr}(\pi(g))$, where tr denotes the trace of a matrix.

We will now show that in the particular case of compact Gelfand pairs, a spherical function can be expressed in terms of the character of the representation.

Proposition 2.6.5. *Let G be a compact group and K a closed subgroup, such that (G, K) is a Gelfand pair. Let ϕ be a spherical function for (G, K), π_ϕ the corresponding irreducible representation and u_ϕ the K-fixed unit vector such that $\phi(g) = \langle u_\phi, \pi_\phi(g)u_\phi\rangle$ for all $g \in G$. If χ_{π_ϕ} is the character of π_ϕ, then for all $g \in G$,*

$$\phi(g) = \int_K \overline{\chi_{\pi_\phi}(gk)}dk.$$

Proof. This is a well-known result, see [59] Proposition 9.10.2. For completeness, we will provide a proof that is self-contained in the context of this thesis. Since ϕ is a spherical function, from Theorem 2.6.4 we saw that it is necessarily positive definite; then by applying Theorem 2.5.3 we know that V_π^K is spanned by u_ϕ.

The projection $P_\pi : V_\pi \to V_\pi^K$ given by (2.7) satisfies $P_\pi(e_1) = e_1$ and $P_\pi(e_i) = 0$ for all $i \neq 1$, where $e_1 = u_\phi$. Thus, we have

$$\mathrm{tr}(P_\pi\ \pi(g)\ P_\pi) = \overline{\phi(g)}, \qquad \text{for all } g \in G.$$

By idempotence, we also have $\mathrm{tr}(P_\pi \pi(g) P_\pi) = \mathrm{tr}(\pi(g)(P_\pi)^2) = \mathrm{tr}(\pi(g)P_\pi)$. This gives us, for all $g \in G$,

$$\overline{\phi(g)} = \mathrm{tr}(\pi(g)P_\pi) = \mathrm{tr}\left(\int_K \pi(g)\pi(k)\ dk\right) = \int_K \mathrm{tr}(\pi(gk))\ dk = \int_K \chi_{\pi_\phi}(gk)dk.$$

□

2.7 Spherical functions on compact groups

In this section G is a compact Lie group and K a closed subgroup, such that (G, K) is a Gelfand pair. We will provide a spherical version of the original Peter-Weyl Theorem for functions on $K\backslash G/K$. We will denote by $L^2_K(G) = L^2(K\backslash G/K) \subset L^2(G)$ the subspace of K-bi-invariant square integrable functions on G.

Lemma 2.7.1. *Let G be a compact group and (π, V_π) an irreducible representation. If V_π decomposes into $V_\pi = V_\pi^1 \oplus V_\pi^2$, then the space of matrix elements decomposes into $\mathcal{M}_\pi = \mathcal{M}_\pi^1 \oplus \mathcal{M}_\pi^2$ where*

$$\mathcal{M}_\pi^i = Span\{g \mapsto \langle u, \pi(g)v\rangle : u, v \in V_\pi^i\}, \quad i = 1, 2.$$

Proof. Denote by $P : V_\pi \to V_\pi^1$, the orthogonal projection from V_π to V_π^1. We will prove that the mapping $\mathcal{E}_P : \langle u, \pi(\cdot)v\rangle \mapsto \langle Pu, \pi(\cdot)Pv\rangle$ is also an orthogonal projection from $\mathcal{M}_\pi$ to $\mathcal{M}_\pi^1$. By linearity of the inner product and of π, $\mathcal{E}_P$ is also linear. Moreover, $\mathcal{E}_P$ satisfies $\mathcal{E}_P = \mathcal{E}_P^2$ as P is a projection, that is for all $g \in G$ and $u, v \in V_\pi$

$$\mathcal{E}_P^2\left(\langle u, \pi(g)v\rangle\right) = \left\langle P^2u, \pi(g)P^2v\right\rangle = \langle Pu, \pi(g)Pv\rangle = \mathcal{E}_P(\langle u, \pi(g)v\rangle).$$

To see that $\mathcal{E}_P$ is an orthogonal projection, we will prove that $\mathcal{E}_P$ is self-adjoint. It is sufficient to consider $f, h \in \mathcal{M}_\pi$ with $f(g) = \langle u_1, \pi(g)v_1\rangle$ and $h(g) = \langle u_2, \pi(g)v_2\rangle$ where $u_1, u_2, v_1, v_2 \in V_\pi$. Then by Schur's orthogonality relation Theorem 1.2.11, and the fact that P is self-adjoint, we have

$$\langle \mathcal{E}_P\ f, h\rangle_{L^2(G)} = \frac{1}{d_\pi}\langle v_2, Pv_1\rangle\overline{\langle u_2, Pu_1\rangle} = \frac{1}{d_\pi}\langle Pv_2, v_1\rangle\overline{\langle Pu_2, u_1\rangle} = \langle f, \mathcal{E}_P\ h\rangle_{L^2(G)}.$$

Thus, $\mathcal{M}_\pi$ can be decomposed into $\mathcal{M}_\pi = \mathcal{M}_\pi^1 \oplus \mathcal{M}_\pi^2$. □

From now on we will mostly be interested in the particular decomposition $V_\pi = V_\pi^K \oplus (V_\pi^K)^\perp$, where $\pi \in \widehat{G}$. We have a corollary to the previous lemma,

Corollary 2.7.2. *Let G be a compact group and K a compact subgroup. Fix a representation $\pi \in \widehat{G}$. We define a map $P : V_\pi \to V_\pi^K$ by*

$$Pv := \int_K \pi(k)v\; dk, \qquad \textit{for all } v \in V_\pi. \tag{2.7}$$

Then P is an orthogonal projection and for all $g \in G$ and $u, v \in V_\pi$,

$$\mathcal{E}_P\langle u, \pi(\cdot)v\rangle)(g) = \mathcal{Q}^K\langle u, \pi(\cdot)v\rangle)(g)$$

Proof. The mapping P is linear and idempotent. Indeed, for all $v \in V_\pi$, by a change of variable, $k = k'^{-1}k$ and by using the fact that the Haar measure is invariant and is normalized on K we get

$$\begin{aligned} P^2v &= \int_K \pi(k') \int_K \pi(k)v\; dk\; dk' = \int_K\int_K \pi(k'k)v\; dk\; dk' \\ &= \int_K\int_K \pi(k)vdkdk' = \int_K \pi(k)vdk = Pv. \end{aligned}$$

By Theorem 1.2.5 (ii) and unimodularity of the Haar measure we have for all $u, v \in V$,

$$\begin{aligned} \langle Pu, v\rangle &= \left\langle \int_K \pi(k)u\; dk, v \right\rangle = \int_K \langle \pi(k)u, v\rangle\; dk = \int_K \langle u, \pi(k^{-1})v\rangle \\ &= \left\langle u, \int_K \pi(k^{-1})v\; dk \right\rangle = \left\langle u, \int_K \pi(k)v\; dk \right\rangle = \langle u, Pv\rangle \end{aligned}$$

Hence, P is self-adjoint. This proves that P is an orthogonal projection from V_π to V_π^K. From Lemma 2.7.1 we know that $\mathcal{E}_P$ is also an orthogonal projection from $\mathcal{M}_\pi$ to $\mathcal{M}_{V_\pi^K}$ and we have for all $g \in G$ and $u, v \in V_\pi$,

$$\begin{aligned} \mathcal{E}_P\,\langle u, \pi(g)v\rangle :=& \left\langle \int_K \pi(k)u\; dk,\; \pi(g)\int_K \pi(k)v\; dk \right\rangle \\ =& \int_K\int_K \langle u, \pi(kgk')v\rangle dkdk' \\ =& \langle u, \pi(\cdot)v\rangle^\sharp(g) \\ =& \mathcal{Q}^K(\langle u, \pi(\cdot)v\rangle)(g). \end{aligned}$$

□

From now on, for any representation $\pi \in \widehat{G}_K$, we denote by $\phi_\pi := \langle u_\pi, \pi(\cdot)u_\pi \rangle$ the spherical function associated to the K-fixed unit vector $u_\pi \in V_\pi^K$.

Theorem 2.7.3 (Spherical Peter-Weyl 1). *The family $\{\sqrt{d_\pi}\phi_\pi, \pi \in \widehat{G}_K\}$ is an orthonormal basis for $L^2_K(G)$.*

Proof. This result is well-known, see [59] Prop 9.10.4 p.205 and [25] Theorem 3.5 p.533. However, the proof given here is original as far as the author is aware.
Let us consider the decomposition $V_\pi = V_\pi^K \oplus (V_\pi^K)^\perp$, then by Proposition 2.7.1 we have $\mathcal{M}_\pi = \mathcal{M}_{V_\pi^K} \oplus \mathcal{M}_{(V_\pi^K)^\perp}$. The original Peter-Weyl Theorem 1.2.12 states that $L^2(G) = \bigoplus_{\pi \in \widehat{G}} \mathcal{M}_\pi$, so by linearity and continuity of the projection $\mathcal{Q}^K : L^2(G) \to L^2_K(G)$, we have

$$L^2_K(G) = \mathcal{Q}^K(L^2(G)) = \bigoplus_{\pi \in \widehat{G}} \mathcal{Q}^K \mathcal{M}_\pi.$$

To simplify the notation write $\mathcal{M}_\pi^K := \mathcal{Q}^K \mathcal{M}_\pi$. We claim that $\mathcal{M}_\pi^K = \mathcal{M}_{V_\pi^K}$, which will imply $L^2_K(G) = \bigoplus_{\pi \in \widehat{G}_K} \mathcal{M}_{V_\pi^K}$. Note that here, the direct sum is over $\widehat{G}_K$ since for any $\pi \notin \widehat{G}_K$, we have $V_\pi = \{0\}$ and therefore $\mathcal{M}_{V_\pi^K} = \{0\}$.

First, take any $f = \sum_{i=1}^{N} \alpha_{ij} \langle u_i, \pi(\cdot)v_j \rangle \in \mathcal{M}_{V_\pi^K}$, for some $u_i, v_j \in V_\pi^K$, $\alpha_{ij} \in \mathbb{R}$, and $i, j = 1, \ldots N$. We will show that f is K-bi-invariant. For all $k_1, k_2 \in K$ and $g \in G$, each u_i and v_j are K-fixed, we have

$$\begin{aligned} f(k_1 g k_2) &= \sum_{i=1}^{N} \alpha_{ij} \left\langle \pi(k_1^{-1})u_i, \pi(g)\pi(k_2)v_j \right\rangle \\ &= \sum_{i=1}^{N} \alpha_{ij} \left\langle u_i, \pi(g)v_j \right\rangle = f(g). \end{aligned}$$

Thus, $f \in \mathcal{M}_\pi^K$ and so $\mathcal{M}_{V_\pi^K} \subseteq \mathcal{M}_\pi^K$. On the other hand, any element of $\mathcal{M}_\pi^K$ can be written as $\mathcal{Q}^K f$ where $f = \sum_{ij=1}^{N} \alpha_{ij} \langle \psi_i, \pi(\cdot)\varphi_j \rangle \in \mathcal{M}_\pi$ for some $\psi_i, \varphi_j \in V_\pi$ and $\alpha_{ij} \in \mathbb{R}$ with $i, j = 1, \ldots, N$. Thus, by Corollary 2.7.2, we get

$$\begin{aligned} \mathcal{Q}^K f(g) &= \sum_{ij=1}^{N} \alpha_{ij} \mathcal{Q}^K \left\langle \psi_i, \pi(g)\varphi_j \right\rangle = \sum_{ij=1}^{N} \alpha_{ij} \mathcal{E}_{P_\pi} \left\langle \psi_i, \pi(g)\varphi_j \right\rangle \\ &= \sum_{ij=1}^{N} \alpha_{ij} \left\langle P_\pi \psi_i, \pi(g) P_\pi \varphi_j \right\rangle, \end{aligned}$$

where P_π is defined in (2.7). For any $\pi \in \widehat{G}_K$, since P_π is a projection onto V_π^K each of $P_\pi\psi_i, P_\pi\varphi_j$ are elements of V_π^K. Thus, $\mathcal{Q}^K f \in \mathcal{M}_{V_\pi^K}$ and we have the inclusion $\mathcal{M}_\pi^K \subseteq \mathcal{M}_{V_\pi^K}$. It then follows that $\mathcal{M}_\pi^K = \mathcal{M}_{V_\pi^K}$ for all $\pi \in \widehat{G}^K$ and
$L_K^2(G) = \bigoplus_{\pi\in\widehat{G}_K} \mathcal{M}_{V_\pi^K}$.

Furthermore, if $\pi \in \widehat{G}_K$, by Proposition 2.6.3 there exists a K-fixed unit vector u_π which spans V_π^K, and $\mathcal{M}_\pi^K = \mathrm{Span}\{g \mapsto \langle u_\pi, \pi(g)u_\pi\rangle\}$. We can now conclude that $\{\sqrt{d_\pi}\langle u_\pi, \pi(g)u_\pi\rangle; \pi \in \widehat{G}_K\}$ is an orthonormal basis of $L_K^2(G)$. □

Let us denote by $\mathcal{E}_K(G)$ the span of all functions in $\mathcal{M}_\pi^K$, $\pi \in \widehat{G}_K$, i.e.

$$\mathcal{E}_K(G) := \mathrm{Span}\{\phi_\pi; \pi \in \widehat{G}_K\}.$$

Theorem 2.7.4 (Spherical Peter-Weyl 2)**.** *$\mathcal{E}_K(G)$ is dense in $L_K^2(G)$.*

Proof. This follows directly from Theorem 2.7.3. □

Theorem 2.7.5 (Spherical Peter-Weyl 3)**.** *$\mathcal{E}_K(G)$ is dense in $C(K\backslash G/K)$.*

Proof. Let $f \in C(K\backslash G/K) \subset C(G)$. By Theorem 1.2.14, for all $\varepsilon > 0$ there exists $h \in \mathcal{E}(G)$ such that $\|f - h\|_\infty \leq \varepsilon$. Let us recall from Proposition 2.2.2 that the mapping $\mathcal{Q}^K : C(G) \to C(K\backslash G/K)$ is surjective and idempotent. Thus, for all $f \in C(K\backslash G/K)$, we have $\mathcal{Q}^K f = f$ and

$$\begin{aligned}
\|f - \mathcal{Q}^K h\|_\infty = \|\mathcal{Q}^K(f-h)\|_\infty &= \sup_{g\in G}\left|\int_K\int_K (f-h)(k_1gk_2)dk_1dk_2\right| \\
&\leq \int_K\int_K \sup_{g\in G}\left|(f-h)(k_1gk_2)\right| dk_1dk_2 \\
&= \int_K\int_K \|R_{k_2}L_{k_1}(f-h)\|_\infty \, dk_1dk_2 \\
&= \int_K\int_K \|(f-h)\|_\infty \, dk_1dk_2 \\
&= \|(f-h)\|_\infty \leq \varepsilon. \qquad (2.8)
\end{aligned}$$

Consequently, we have

$$\|f - \mathcal{Q}^K h\|_\infty \leq \varepsilon.$$

We need to show that for $h \in \mathcal{E}(G)$, $\mathcal{Q}^K h$ is in $\mathcal{E}_K(G)$. By definition of $\mathcal{E}(G)$, for all $h \in \mathcal{E}(G)$ there exists a finite subset $S \in \widehat{G}$, $\alpha_\pi \in \mathbb{C}$ and $h_\pi(\cdot) = \langle\psi_\pi, \pi(\cdot)\varphi_\pi\rangle \in \mathcal{M}_\pi$ for each $\pi \in S$, such that $h = \sum_{\pi\in S}\alpha_\pi h_\pi$.

By linearity of $\mathcal{Q}^K$ and by Corollary 2.7.2, we have for all $g \in G$,

$$\mathcal{Q}^K h(g) = \sum_{\pi \in S} \alpha_\pi \mathcal{Q}^K h_\pi = \sum_{\pi \in S} \alpha_\pi \mathcal{E}_{P_\pi} h_\pi \in \mathcal{E}_K(G),$$

Thus, we have proved that $\mathcal{Q}^K h \in \mathcal{E}_K(G)$ and we can conclude that $\mathcal{E}_K(G)$ is dense in $C(K \backslash G/K)$. $\square$

2.8 Homogeneous spaces and symmetric spaces

Definitions and known results in this section are based on [12], [24] and [58].

Definition. A group action of a topological group G on a topological space M is called *transitive* if for all $p, q \in M$, there exist $g \in G$ such that $g.p = q$.
The *isotropy subgroup* (or stabilizer subgroup) of G at a point $p \in M$ is the subgroup of G that fixes p, defined as

$$\mathrm{Stab}_p := \{g \in G; g.p = p\}.$$

Remark 2.8.1. *For all $x \in M$, $Stab_x$ is indeed a subgroup of G, since $\forall g, h \in Stab_x$*

$$(hg).x = h.(g.x) = h.x = x$$

Definition. A *homogeneous space* M is a manifold with a transitive action of a locally compact group G.

Further on, we will mostly be interested in the case where M is a smooth manifold and G is a Lie group, so the following theorem will be useful for us.

Theorem 2.8.2. *Let $\rho : G \times M \to M$, $\rho(g, x) = g.x$ be a transitive action of a Lie group G on a manifold M. Fix a point $x \in M$ and denote $K := Stab_x$, then the map $\alpha : G/K \to M$ defined by $\alpha(gK) = \rho(g, x)$ is a diffeomorphism.*

Proof. See [58], Theorem 3.62, p.123 $\square$

We will now provide a short summary on symmetric spaces, which are a special case of homogeneous spaces.

Definition. Let M be a Riemannian manifold, $p \in M$. A curve $\gamma : I \to M$, for an interval $I \subset \mathbb{R}$, is called a *geodesic* if the family of tangent vectors γ' is parallel with respect to γ, in the sense defined in Helgason [24], p.28. A geodesic is called *maximal* if it has largest possible domain. For each point $p \in M$ and tangent vector

$v \in T_pM$, we denote by $\xi_v^p : \mathbb{R} \to M$ the maximal geodesic through p tangent to v and $\xi_v^p(0) = p$ and $(\xi_v^p)'(0) = v$. We define the *exponential map* of the manifold M at p, $\mathrm{Exp}_p : T_pM \to M$ by $\mathrm{Exp}_p(v) = \xi_v^p(1)$. There exists $\varepsilon > 0$ such that Exp_p is a diffeomorphism from $\{X \in T_pM : |X| < \varepsilon\}$ to its image. Furthermore, on a sufficiently small open ball $B_r(p)$, $r > 0$, we define the map $s_p : B_r(p) \to B_r(p)$ by $s_p(\mathrm{Exp}_p(tv)) = \mathrm{Exp}_p(-tv)$ is called a *local geodesic symmetry* at p.

We call a (connected) Riemannian manifold M a *locally symmetric space* if at each $p \in M$, there exists an open ball $B_r(p)$, such that the corresponding symmetry s_p is an isometry. Furthermore, M is called a *(globally) symmetric space* if at each $p \in M$, s_p extends to a global isometry $s_p : M \to M$. That is, s_p is an involutive isometry, i.e. $s_p^2 = id$, and p is an isolated fixed point of s_p, i.e. there is a neighbourhood V of p where p is the only fixed point of s_p.

If M is a symmetric space, we will denote by $I(M)$ the group of all Riemannian isometries of M onto itself, it has a natural Lie group structure and $I(M)$ acts transitively on M, see [24], Lemma 3.2, p.170. Let K be the compact subgroup of $I(M)$ that leaves some point $p_0 \in M$ fixed, then M and $I(M)/K$ are diffeomorphic, see [24] Theorem 3.3, p.173. Thus, a symmetric space is a homogeneous space.

Proposition 2.8.3. *If $M = I(M)/K$ is a symmetric space, then $(I(M), K)$ is a Gelfand pair.*

Proof. See [59] Corollary 8.1.4, p.154. □

Definition. Let G be a connected Lie group and K a closed subgroup. If s is an involutive automorphism s of G, we denote by G_s the fixed point set of s and by $(G_s)^0$ the connected component of G_s containing the identity element. We say that (G, K) is a *symmetric pair* if there exist an involutive analytic automorphism s of G such that $(G_s)^0 \subset K \subset G_s$. Furthermore, if $\mathrm{Ad}_G(K)$ is a compact subgroup of $GL(\mathfrak{g})$, we call (G, K) a *Riemannian symmetric pair.*

Proposition 2.8.4. *If (G, K) is a Riemannian symmetric pair, then G/K is a symmetric space.*

Proof. See [49] Theorem 1.3, p.73 or [24] Proposition 3.4, p.174. □

In this thesis we will be particularly interested in compact symmetric spaces.

Chapter 3

Fourier and spherical transforms

3.1 The Fourier transform

Definition. The *Hilbert-Schmidt inner product* on the matrix algebra $M_n(\mathbb{C})$ is defined by $\langle A, B\rangle_{HS} := \operatorname{tr}(AB^*)$ for $A, B \in M_n(\mathbb{C})$. The corresponding norm is $\|A\|_{HS}^2 = \langle A, A\rangle_{HS}$ for all $A \in M_n(\mathbb{C})$.

Remark 3.1.1. It is useful to note here that by the Cauchy-Schwarz inequality, for all $A \in M_n(\mathbb{C})$ and $k = 1, \ldots, n$,

$$|\langle Ae_k, e_k\rangle|^2 \leq \|Ae_k\|^2 \leq \sum_{i=1}^{n} \|Ae_i\|^2 = \|A\|_{HS}^2. \tag{3.1}$$

i.e. $|\langle Ae_k, e_k\rangle| \leq \|A\|_{HS}$, for all $A \in M_n(\mathbb{C})$ and $k = 1, \ldots, n$.

Let G be a compact group, then we define the set $\mathcal{M}(\widehat{G}) := \bigcup_{\pi\in\widehat{G}} M_{d_\pi}(\mathbb{C})$.
A mapping $F : \widehat{G} \mapsto \mathcal{M}(\widehat{G})$ is called *compatible* if $F(\pi) \in M_{d_\pi}(\mathbb{C})$ for each $\pi \in \widehat{G}$. We denote by $\mathcal{L}(\widehat{G})$ the linear space of compatible mappings where addition and scalar multiplication are defined pointwise, and by $\mathcal{H}_2(\widehat{G})$ the subspace of $\mathcal{L}(\widehat{G})$ which satisfies $|||F|||_2^2 := \sum_{\pi\in\widehat{G}_K} d_\pi \|F(\pi)\|_{HS}^2 < \infty$, for all $F \in \mathcal{H}_2(\widehat{G})$. The space $\mathcal{H}_2(\widehat{G})$ is a complex Hilbert space with inner product

$$\langle\langle F, G\rangle\rangle = \sum_{\pi\in\widehat{G}} d_\pi \langle F(\pi), G(\pi)\rangle_{HS}.$$

Let us now define the Fourier transform of a function on a group.

Definition. Let G be a compact group and $f \in L^1(G)$. For each $\pi \in \widehat{G}$, we introduce the *(non-commutative) Fourier transform* $\mathcal{F} : f \mapsto \widehat{f}$ where

$$\widehat{f}(\pi) := \int_G \pi(g^{-1}) f(g) dg.$$

This is a matrix valued integral, where each coefficient is

$$\widehat{f}(\pi)_{i,j} := \int_G \pi_{i,j}(g^{-1}) f(g) dg, \quad \text{for all} 1 \leq i, j \leq d_\pi.$$

When we restrict the domain of $\mathcal{F}$ to $L^2(G)$, we get two important results, these being the Fourier expansion and the Plancherel formula. We will only state them here, for the proof see [6] Theorem 2.3.1 p.36.

Theorem 3.1.2 (Fourier expansion)**.** *Let G be a compact group, then*

$$f = \sum_{\pi \in \widehat{G}} d_\pi tr(\widehat{f}(\pi)\pi(\cdot)), \qquad \text{for all } f \in L^2(G).$$

Theorem 3.1.3 (Parseval-Plancherel identity)**.** *The operator $\mathcal{F}$ is an isometry from $L^2(G)$ to $\mathcal{H}_2(\widehat{G})$ and for all $f, h \in L^2(G)$,*

$$\int_G f(g)\overline{h(g)} dg = \sum_{\pi \in \widehat{G}} d_\pi \left\langle \widehat{f}(\pi), \widehat{h}(\pi) \right\rangle_{HS}.$$

In particular,

$$\int_G |f(g)|^2 dg = \sum_{\pi \in \widehat{G}} d_\pi \left\| \widehat{f}(\pi) \right\|_{HS}^2.$$

We can also define the Fourier transform of a probability measure on $(G, \mathcal{B}(G))$. Let $\mathcal{P}(G)$ denote the set of all Borel probability measures on a compact group G.

Definition. For all $\pi \in \widehat{G}$ and $\mu \in \mathcal{P}(G)$ we define the *Fourier transform of the measure* μ at π as the matrix-valued integral

$$\widehat{\mu}(\pi) := \int_G \pi(g^{-1}) \mu(dg),$$

which is a bounded linear operator on V_π. When G is compact, it is a matrix valued integral.

3.2 The spherical transform

Let G be a locally compact group, and K a compact subgroup, such that (G, K) is a Gelfand pair. $S(G, K)$ denotes the set of all spherical functions for (G, K).

Definition. The *spherical transform of a function* for (G, K) is the map $\mathcal{F}^S : f \mapsto \widehat{f}^S$ from $C_c(K\backslash G/K)$ to a space of functions on $S(G, K)$, where $\widehat{f}^S$ is defined by

$$\widehat{f}^S(\phi) = \int_G f(g)\phi(g)dg,$$

and ϕ is a spherical function in $S(G, K)$.

Definition. Let μ be a K-bi-invariant probability measure in $\mathcal{P}(G)$. We denote the set of all such measures by $\mathcal{P}_K(G)$. Then the *spherical transform of the measure* μ is the complex valued mapping

$$\widehat{\mu}^S(\phi) = \int_G \phi(g)\mu(dg),$$

for any spherical function $\phi \in S(G, K)$.

From now on let G be compact, so all continuous spherical functions are positive definite. For all $\pi \in \widehat{G}$ let us fix a basis $e_1, \ldots, e_{d_\pi}$ of V_π. In the special case where π is a spherical representation, i.e. $\pi \in \widehat{G}_K$, by definition there exists a non-zero unit K-fixed vector $u_\pi \in V_\pi$. Following Theorem 2.5.3, we know that V_π^K is spanned by the vector u_π and we will set $e_1 = u_\pi$. We have the following characterisation of K-bi-invariant measures, with the help of their Fourier transform. We will denote by P_π the map defined in (2.7) for $\pi \in \widehat{G}$.

Theorem 3.2.1. *Let μ be a probability measure on a compact group G. Then the following statements are equivalent:*

a) The measure μ is K-bi-invariant.
b) $P_\pi \, \widehat{\mu}(\pi) \, P_\pi = \widehat{\mu}(\pi)$, for all $\pi \in \widehat{G}$.
c) $\widehat{\mu}(\pi)_{ij} = 0$ for all $\pi \notin \widehat{G}_K$, or $\pi \in \widehat{G}_K$ and $(i, j) \neq (1, 1)$.

Proof. This result has been published in [8]. First show that (a) implies (b). Let $\pi \in \widehat{G}$ and $\psi, \phi \in V_\pi$. Given that μ is K-bi-invariant and the Haar measure is normalized, by a change of variable we have

$$\begin{aligned}\langle \widehat{\mu}(\pi)\psi, \phi\rangle &= \int_K \int_K \int_G \langle \pi(kg^{-1}k')\psi, \phi\rangle \; d\mu(g) \; dkdk' \\ &= \int_G \left\langle \int_K \pi(k)dk \; \pi(g^{-1}) \int_K \pi(k')dk'\psi, \phi \right\rangle \\ &= \langle P_\pi \widehat{\mu}(\pi) P_\pi \psi, \phi\rangle\end{aligned}$$

We can then conclude the result. To show that (b) implies (c), it is sufficient to use that for all $\pi \in \widehat{G}$, $P_\pi : V_\pi \to V_\pi^K$ is an orthogonal projection (as seen in Corollary 2.7.2). When π is not spherical, i.e. $\pi \notin \widehat{G}_K$, the space of K-fixed vectors is $V_\pi^K = \{0\}$, so $P_\pi v = 0$ for all $v \in V_\pi$. Thus, for $\pi \notin \widehat{G}_K$ and $i, j = 1, \dots, d_\pi$,

$$\widehat{\mu}(\pi)_{ij} = \langle \widehat{\mu}(\pi)e_i, e_j\rangle = \langle P_\pi \widehat{\mu}(\pi) P_\pi e_i, e_j\rangle = \langle 0, e_j\rangle = 0$$

When $\pi \in \widehat{G}_K$, then V_π^K is spanned by e_1 and $e_2, \dots, e_{d_\pi} \in (V_\pi^K)^\perp$, thus if $i \neq 1$ then $P_\pi e_i = 0$. Thus, for all $i, j \neq (1,1)$ we have

$$\widehat{\mu}(\pi) = \langle P_\pi \widehat{\mu}(\pi) P_\pi e_i, e_j\rangle = 0.$$

Let us show that (c) implies (a). We have

$$\int_G \langle \pi(g)e_i, e_j\rangle \mu(dg) = 0, \text{ for all } \pi \notin \widehat{G}_K, \text{ or } \pi \in \widehat{G} \text{ and } (i,j) \neq (1,1). \tag{3.2}$$

Take any $f \in \mathcal{M}_\pi$ for $\pi \in \widehat{G}$, then f is of the form

$$f(g) = \sum_{i,j=1}^{d_\pi} \alpha_{ij} \langle \pi(g)e_i, e_j\rangle, \qquad \text{for all } g \in G$$

We can split this sum into two parts,

$$f(g) = \alpha_{11}\langle \pi(g)e_1, e_1\rangle + \sum_{(i,j)\neq(1,1)}^{d_\pi} \alpha_{ij}\langle \pi(g)e_i, e_j\rangle, \qquad \text{for all } g \in G,$$

where the first term is an element of $\mathcal{M}_\pi^K$ and the second term is an element of $(\mathcal{M}_\pi^K)^\perp$. So, given (3.2) and Corollary 2.7.2 we get for all $g \in G$,

$$\int_G f(g)\mu(dg) = \int_G \alpha_{11}\langle \pi(g)e_1, e_1\rangle \; d\mu(g) = \int_G \mathcal{Q}^K f(g) \; d\mu(g).$$

By Theorem 1.2.14, we can extend this result to all $f \in C_c(G)$. Thus, for all $f \in C_c(G)$

we have

$$\int_G f(g)d\mu(g) = \int_G f(g)\ \mu(k_1\ dg\ k_2), \qquad \text{for all } g \in G, k_1, k_2 \in K.$$

It then follows by uniqueness of the measure from the Riesz representation theorem that μ is a K-bi-invariant measure.

□

We deduce from Theorem 3.2.1 that if μ is a K-bi-invariant probability measure then for all $\pi \in \widehat{G}_K$,

$$\widehat{\mu}^S(\phi_\pi) = \widehat{\mu}(\pi)_{11}.$$

Whereas, if $\pi \notin \widehat{G}_K$, then $\widehat{\mu}^S(\phi_\pi) = 0$.

Corollary 3.2.2. *Let $f \in C(G)$, then the following statements are equivalent*

a) The function f is K-bi-invariant
b) $P_\pi \widehat{f}(\pi) P_\pi = \widehat{f}(\pi)$ for all $\pi \in \widehat{G}$
c) $\widehat{f}(\pi)_{ij} = 0$ for all $\pi \notin \widehat{G}_K$, or $\pi \in \widehat{G}_K$ and $(i, j) \neq (1, 1)$.

Proof. For all $f \in C(K\backslash G/K)$, let us decompose $f = f^+ - f^-$, where we use the usual notation $f^+(g) := \max(f(g), 0)$ and $f^-(g) = -\min(f(g), 0)$ for all $g \in G$.

We can then define the positive finite Borel measures $\mu^+(A) = \int_A f^+(g)dg$ and $\mu^-(A) = \int_A f^-(g)dg$ for all $A \in \mathcal{B}(G)$, after normalizing them over G we will get two probability measures. We observe that $\widehat{\mu^\pm}(\pi) = \widehat{f^\pm}(\pi)$ for all $\pi \in \widehat{G}$. Thus, we can conclude by applying Theorem 3.2.1 and by using $\widehat{f}(\pi) = \widehat{f}^+(\pi) - \widehat{f}^-(\pi)$, for all $\pi \in \widehat{G}$. □

As a consequence of this last result, if $f \in C(K\backslash G/K)$ then for all $\pi \in \widehat{G}_K$, the matrix $\widehat{f}(\pi)$ has all non zero coefficients except for

$$\widehat{f}(\pi)_{11} = \widehat{f}^S(\phi_\pi) \tag{3.3}$$

When $\pi \notin \widehat{G}$, $\widehat{f}(\pi)$ is the zero matrix, and $\widehat{f}^S(\phi_\pi) = 0$. Let $A = (a_{ij}) \in M_{n\times n}(\mathbb{C})$ we define $R_n \in M_{n\times n}(\mathbb{C})$ by

$$R_n := \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \dots & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix} \tag{3.4}$$

Consider the matrix

$$R_nAR_n = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & \dots & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix}$$

In Corollary 3.2.2, we have seen that if $f \in C(K\backslash G/K)$ then

$$\widehat{f}(\pi) = R_{d_\pi}\widehat{f}(\pi)R_{d_\pi} \qquad \text{for all } \pi \in \widehat{G}. \tag{3.5}$$

In particular, when $\pi \notin \widehat{G}_K$, we just have $\widehat{f}(\pi) = 0$.

Note that Corollary 3.2.2 also holds for $f \in L^2(G)$, by using similar arguments to the proof given for Theorem 3.2.1. Thus, the equations (3.3) and (3.5) also hold for all $f \in L^2(K\backslash G/K)$.

We can now obtain a spherical version of the Parseval-Plancherel formula.

Theorem 3.2.3 (Spherical Parseval-Plancherel)**.** *Let G be a compact group and K a closed subgroup. Then for all $f, h \in L^2(K\backslash G/K)$,*

$$\int_G f(g)\overline{h(g)}dg = \sum_{\pi\in\widehat{G}_K} d_\pi \widehat{f}^S(\phi_\pi)\overline{\widehat{h}^S(\phi_\pi)}.$$

So, in particular

$$\int_G |f(g)|^2 dg = \sum_{\pi\in\widehat{G}} d_\pi |\widehat{f}^S(\phi_\pi)|^2.$$

Proof. This is a known result, see [59] Theorem 9.5.1, p.193, that we include for completeness. Our proof here is different from that in [59] and is self-contained in the context of this thesis. The proof follows directly from the original Parseval-Plancherel formula and Corollary 3.2.2. For all $f, h \in L^2(K\backslash G/K)$,

$$\begin{aligned}\int_G f(g)\overline{h(g)}dg &= \sum_{\pi\in\widehat{G}} d_\pi \left\langle \widehat{f}(\pi), \widehat{h}(\pi)\right\rangle_{HS} \\ &= \sum_{\pi\in\widehat{G}_K} d_\pi \widehat{f}(\pi)_{11}\overline{\widehat{h}(\pi)_{11}} \\ &= \sum_{\pi\in\widehat{G}_K} d_\pi \widehat{f}^S(\phi_\pi)\overline{\widehat{h}^S(\phi_\pi)},\end{aligned}$$

and similarly,

$$\begin{aligned}\int_G |f(g)|^2 dg &= \sum_{\pi\in\widehat{G}} d_\pi \|\widehat{f}(\pi)\|_{HS}^2 \\ &= \sum_{\pi\in\widehat{G}_K} d_\pi |\widehat{f}(\pi)_{11}|^2 \\ &= \sum_{\pi\in\widehat{G}_K} d_\pi |\widehat{f}^S(\phi_\pi)|^2.\end{aligned}$$

□

Theorem 3.2.4. *[Spherical Fourier expansion] For all $f \in L^2(K\backslash G/K)$,*

$$f = \sum_{\pi\in\widehat{G}_K} d_\pi \widehat{f}^S(\phi_\pi)\phi_\pi$$

Proof. This is also a known result, see [59] Theorem 9.4.1, p.191, that we include for completeness. Here we will provide our own short proof using the original Fourier expansion and (3.5), so for all $g \in G$ and $f \in L^2(K\backslash G/K)$

$$\begin{aligned}f(g) &= \sum_{\pi\in\widehat{G}} d_\pi \mathrm{tr}\left(\widehat{f}(\pi)\pi(g)\right) \\ &= \sum_{\pi\in\widehat{G}} d_\pi \mathrm{tr}\left(R_\pi\widehat{f}(\pi)R_\pi\pi(g)\right) \\ &= \sum_{\pi\in\widehat{G}} d_\pi \mathrm{tr}\left(\widehat{f}(\pi)R_\pi\pi(g)R_\pi\right) \\ &= \sum_{\pi\in\widehat{G}} d_\pi \mathrm{tr}\left((R_\pi\widehat{f}(\pi)R_\pi)(R_\pi\pi(g)R_\pi)\right) \\ &= \sum_{\pi\in\widehat{G}_K} d_\pi \widehat{f}^S(\phi_\pi)\phi_\pi(g).\end{aligned}$$

This is exactly what we wanted. □

Chapter 4

Application of spherical transforms: densities of K-bi-invariant measures

4.1 Regularity results and estimates

In this chapter G will be a compact Lie group.

Proposition 4.1.1. *There exists a constant $C \geq 0$ such that for any irreducible representation π of G with corresponding highest weight λ, we have*

$$d_\lambda \leq C|\lambda|^m$$

where $m = \#\mathcal{P}_+$ is the number of positive roots of G.

See Corollary 2.5.2 in [6].

Theorem 4.1.2. *Let π be an irreducible representation of G and λ the corresponding highest weight, then*

$$\|d\pi_\lambda(X)\|_{HS}^2 \leq C|\lambda|^{m+2}|X|^2, \qquad \textit{for all } X \in \mathfrak{g}.$$

Also, for all $X, Y \in \mathfrak{g}$

$$\|d\pi_\lambda(X)d\pi_\lambda(Y)\|_{HS}^2 \leq C|\lambda|^{2m+4}|X|^2|Y|^2,$$

for some constant $C \geq 0$.

Proof. For the proof of the first part see Theorem 3.4.1 in [6]. For the second inequality, we use the equivalence of the Hilbert-Schmidt norm $\|\cdot\|_{HS}$ and the operator norm

$\|\cdot\|_{op}$. For all $X, Y \in \mathfrak{g}$ there exist some constants $K_1, K_2 \geq 0$ such that

$$\begin{aligned}\|d\pi_\lambda(X)d\pi_\lambda(Y)\|_{HS} &\leq K_1\|d\pi(X)d\pi(Y)\|_{op}\\ &\leq K_1\|d\pi(X)\|_{op}\|d\pi(Y)\|_{op}\\ &\leq K_1K_2\|d\pi(X)\|_{HS}\|d\pi(Y)\|_{HS}\\ &\leq C|\lambda|^{m+2}|X||Y|\end{aligned}$$

for some constant $C \geq 0$. □

Denote $D_0 = D\backslash\{0\}$. Let us introduce the *Sugiura zeta function* ζ for all $s \in \mathbb{C}$,

$$\zeta(s) := \sum_{\lambda\in D_0} \frac{1}{|\lambda|^{2s}} \in \mathbb{C} \cup \{\infty\}.$$

Then we have the following results from [55], p.37-38, and [6] p.75.

Theorem 4.1.3. *Suppose G has rank r. Then the series $\sum_{\lambda\in D_0} \frac{1}{|\lambda|^{2s}}$ converges absolutely if $\Re(s) > \frac{r}{2}$.*

Theorem 4.1.4. *Let G be a compact connected Lie group and $f \in C(G)$. Then the Fourier series of f,*

$$\sum_{\lambda\in D} d_\lambda tr(\widehat{f}(\lambda)\pi_\lambda)$$

converges absolutely and uniformly on G if one of the following conditions are satisfied

i) $f \in C^{2p}(G, \mathbb{C})$ where $p \in \mathbb{N}$ and $4p > d$.

ii) $\left\|\widehat{f}(\lambda)\right\|_{HS} = O(|\lambda|^s)$ as $|\lambda| \to \infty$ with $s > r + \frac{3m}{2}$

Proof. See [6], Theorem 3.3.1, p.75. □

It then follows that every smooth function of $C^\infty(G)$ has a uniformly convergent Fourier series.

4.2 Existence and square-integrability of K-bi-invariant densities

If a probability measure $\mu \in \mathcal{P}(G)$ is absolutely continuous with respect to the Haar measure m on G, then the Radon-Nikodym theorem ensures that there exists a function $f \in L^1(G)$ such that $\mu(A) = \int_A f(g)\ m(dg)$ for all $A \in \mathcal{B}(G)$. The Radon-

Nikodym derivative f is called the *density* of the measure μ (w.r.t to the Haar measure m).

Let us recall the main result of existence and square-integrability of densities on compact Lie groups from Applebaum [6] and [2].

Theorem 4.2.1. *If G is a compact Lie group, then a measure $\mu \in \mathcal{P}(G)$ has an L^2-density f_μ if and only if $\sum_{\pi\in\widehat{G}} d_\pi \|\widehat{\mu}(\pi)\|_{HS}^2 < \infty$. In this case we have the L^2-Fourier expansion*

$$f_\mu = \sum_{\pi\in\widehat{G}} d_\pi tr(\widehat{\mu}(\pi)\pi(\cdot))$$

Proof. See Theorem 4.5.1 p.101 in [6] or [2]. □

For K-bi-invariant measures, we will obtain an analogous result using spherical transforms. Let us start with a preliminary result.

Lemma 4.2.2. *Let G be a compact Lie group and K a closed subgroup. Suppose that $\mu \in \mathcal{P}(K\backslash G/K)$ has a probability density function f on G. Then f is K-bi-invariant almost everywhere.*

Proof. First note that G is compact, therefore it is unimodular, by Proposition 1.2.6. Thus, by a change of variable we have for all $k_1, k_2 \in K$, we have

$$\begin{aligned}\int_G h(g)f(k_1gk_2)\, dg &= \int_G h(k_1^{-1}gk_1^{-1})f(g)dg = \int_G h(k_1^{-1}gk_2^{-1})\mu(dg)\\ &= \int_G h(g)\mu(dg) = \int_G h(g)f(g)dg\end{aligned}$$

By the Riesz representation theorem we can then conclude that f is K-bi-invariant. □

Theorem 4.2.3. *Let G be a compact group and K a compact subgroup such that (G, K) is a Gelfand pair. If μ is a K-bi-invariant probability measure on G, then μ has a L^2-density f_μ if and only if*

$$\sum_{\pi\in\widehat{G}_K} d_\pi |\widehat{\mu}^S(\pi)|^2 < \infty. \tag{4.1}$$

And in this case, the density has the L^2-Fourier expansion

$$f_\mu = \sum_{\pi\in\widehat{G}_K} d_\pi \widehat{\mu}^S(\pi)\, \phi_\pi(\cdot). \tag{4.2}$$

Proof. This result can be proved using Theorem 3.2.1 and Theorem 4.2.1, see [8] Theorem 3.3. Here, will use another approach based on the general compact group case in [6] Theorem 4.5.1 p.101, so this proof has the advantage of being self-contained. For necessity, suppose that μ is absolutely continuous w.r.t to the Haar measure and that the density f_μ of the measure μ is square-integrable. i.e. $f_\mu \in L^2(G)$. Then, by Lemma 4.2.2, the density function f_μ is K-bi-invariant almost everywhere w.r.t the normalized Haar measure, i.e. $f_\mu \in L^2(K\backslash G/K)$. The spherical transform of f_μ is

$$\widehat{f_\mu}^S(\phi_\pi) = \int_G f_\mu(g)\phi_\pi(g)dg = \widehat{\mu}^S(\pi), \qquad \text{for all } \pi \in \widehat{G}_K$$

By the spherical Parseval-Plancherel identity,

$$\sum_{\pi\in\widehat{G}_K} d_\pi \left|\widehat{\mu}^S(\pi)\right|^2 = \sum_{\pi\in\widehat{G}_K} d_\pi \left|\widehat{f}^S(\phi_\pi)\right|^2 = \sum_{\pi\in\widehat{G}_K} d_\pi \left|\langle f_\mu, \phi_\pi\rangle\right|^2 = \|f\|^2_{L^2_K} < \infty$$

For sufficiency, suppose the inequality (4.1) holds and let f_μ be the L^2-limit of the following series

$$f_\mu := \sum_{\pi\in\widehat{G}_K} d_\pi \, \widehat{\mu}^S(\pi)\phi_\pi(\cdot). \tag{4.3}$$

Note that by K-bi-invariance of the spherical functions, f_μ is also K-bi-invariant. So by applying the spherical Parseval-Plancherel identity we get

$$\begin{aligned}
\|f_\mu\|^2_{L^2_K(G)} &= \sum_{\pi\in\widehat{G}_K} d_\pi \left|\widehat{f_\mu}^S(\phi_\pi)\right|^2 \\
&= \sum_{\pi\in\widehat{G}_K} d_\pi \left|\int_G f_\mu(g)\phi_\pi(g)\, dg\right|^2 \\
&= \sum_{\pi\in\widehat{G}_K} d_\pi \left|\widehat{\mu}^S(\pi)\right|^2 < \infty.
\end{aligned}$$

For any $h \in \mathcal{E}_K(G)$ there is a finite subset S of $\widehat{G}_K$ such that

$$h = \sum_{\pi\in S} d_\pi \langle h, \phi_\pi\rangle \phi_\pi.$$

Thus, we have by dominated converges and orthogonality of the basis $\{\phi_\pi : \pi \in \widehat{G}_K\}$

in $L^2_K(G)$,

$$\begin{aligned}
\int_G h(g)\overline{f_\mu(g)}dg &= \int_G \sum_{\pi\in S} d_\pi \langle h, \phi_\pi\rangle \phi_\pi(g) \sum_{\pi'\in\widehat{G}_K} d_{\pi'} \overline{\widehat{\mu}^S(\pi')}\ \overline{\phi_{\pi'}(g)}dg \\
&= \sum_{\pi\in S} d_\pi \langle h, \phi_\pi\rangle \sum_{\pi'\in\widehat{G}_K} d_{\pi'} \overline{\widehat{\mu}_s(\pi')} \int_G \phi_\pi(g)\overline{\phi_{\pi'}(g)}dg \\
&= \sum_{\pi\in S} d_\pi \langle h, \phi_\pi\rangle \overline{\widehat{\mu}_s(\pi)} \\
&= \sum_{\pi\in S} d_\pi \langle h, \phi_\pi\rangle \int_G \phi_\pi(\sigma)\mu(d\sigma) \\
&= \int_G \sum_{\pi\in S} d_\pi \langle h, \phi_\pi\rangle \phi_\pi(\sigma)\mu(d\sigma) \\
&= \int_G h(\sigma)\mu(d\sigma).
\end{aligned}$$

By density of $\mathcal{E}_K(G)$ in $C(K\backslash G/K)$, we can use dominated convergence to extend the previous result to all functions $h \in C(K\backslash G/K)$. Thus, we have

$$\int_G h(g)\overline{f_\mu(g)}dg = \int_G h(g)\mu(dg), \quad \text{for all } h \in C(K\backslash G/K).$$

From the Riesz representation theorem, f_μ is real-valued and it is the density of the measure μ. Furthermore, by uniqueness of the measure in the Riesz representation theorem, the K-bi-invariance of the function f_μ implies that μ is K-bi-invariant. □

Corollary 4.2.4. *Let G be a compact group and K a compact subgroup such that (G, K) is a Gelfand pair and let $\mu \in \mathcal{P}(K\backslash G/K)$. If the series $\sum_{\pi\in\widehat{G}_K} d_\pi\ \widehat{\mu}^S(\pi)\phi_\pi(\cdot)$ converges uniformly on G, then μ has a continuous density f_μ.*

Proof. This follows the proof of the result on a general compact Lie group, given in [6], Proposition 4.5.1, p102.

Define the function $f_\mu(g) := \sum_{\pi\in\widehat{G}_K} d_\pi \widehat{\mu}^S(\pi)\phi_\pi(g)$ for all $g \in G$. Then $f_\mu \in C(K\backslash G/K)$ as it is the uniform limit of continuous K-bi-invariant functions. Furthermore, G is compact so we also have $f_\mu \in L^2_K(K\backslash G/K)$, that is
$\|f_\mu\|^2_{L^2_K} = \sum_{\pi\in\widehat{G}_K} d_\pi |\widehat{\mu}^S(\pi)|^2 < \infty$. We can use Theorem 4.2.3 to conclude that f_μ is the density of μ. □

4.3 Regularity of K-bi-invariant functions and spherical Fourier series

In this section we suppose that G is a compact Lie group and K is a closed subgroup such that (G, K) is a Gelfand pair. As we are parametrizing $\widehat{G}$ by the highest weights D, let us simplify the notation of the Fourier transform by writing $\widehat{f}(\lambda) := \widehat{f}(\pi_\lambda)$ for all $\lambda \in D$ and $f \in C(G)$. We will first state some known regularity results that can all be found in [6], Chapter 3.3, we will then proceed to obtain their spherical versions.

Theorem 4.3.1. *For $f \in C(G)$ the Fourier series $\sum_{\lambda\in D} d_\lambda tr(\widehat{f}(\lambda)\pi_\lambda)$ converges absolutely and uniformly to f if*

$$\|\widehat{f}(\lambda)\|_{HS} = O(|\lambda|^{-s}),$$

as $|\lambda| \to \infty$ with $s > r + \frac{3m}{2}$.

Let us denote by D_0 the set of highest weights excluding $\{0\}$, i.e.: $D_0 = D - \{0\}$.

Theorem 4.3.2. *We have the following regularity results on the Fourier transform*

i) If $f \in C^{2p}(G)$ where $p \in \mathbb{N}$, then $\|\widehat{f}(\lambda)\|_{HS} = o(|\lambda|^{-2p})$ as $|\lambda| \to \infty$, $\lambda \in D$.
ii) If $f \in C(G)$, then $f \in C^\infty(G)$ if and only if $\|\widehat{f}(\lambda)\|_{HS} = o(|\lambda|^{-p})$ for all $p \in \mathbb{N}$ as $|\lambda| \to \infty$.

In the case where the function f is K-bi-invariant we apply Theorem 3.2.2 and Corollary 3.2.2 to obtain the spherical versions of the previous regularity results.

Theorem 4.3.3. *For $f \in C(K\backslash G/K)$ the spherical series $\sum_{\lambda\in D_S} d_\lambda \widehat{f}^S(\lambda)\, \phi_\lambda$ converges absolutely and uniformly to f if*

$$|\widehat{f}^S(\lambda)| = O(|\lambda|^{-s}),$$

as $|\lambda| \to \infty$ with $\lambda \in D_S$ and $s > r + \frac{3m}{2}$.

Proof. The proof follows directly from Corollary 3.2.2. Recall that for any $\lambda \in D_S$, $\widehat{f}(\lambda)$ is a matrix with only one non zero coefficient $\widehat{f}(\lambda)_{11} = \widehat{f}^S(\lambda)$, thus

$$\|\widehat{f}(\lambda)\|_{HS} = \|R_{d_\lambda}\widehat{f}(\lambda)R_{d_\lambda}\|_{HS} = |\widehat{f}(\lambda)_{11}| = |\widehat{f}^S(\lambda)|, \tag{4.4}$$

$$\mathrm{tr}(\widehat{f}\,(\lambda)\pi_\lambda) = \mathrm{tr}\left(R_{d_\lambda}\widehat{f}(\lambda)R_{d_\lambda}\right) = \widehat{f}(\lambda)_{11}(\pi_\lambda)_{11} = \widehat{f}^S(\lambda)\phi_\lambda, \tag{4.5}$$

where $R_\lambda \in M_{d_\lambda}(\mathbb{C})$ is defined in (3.4). So, by Corollary 3.2.2 and (4.5) we have

$$\sum_{\lambda\in D_S} d_\lambda \widehat{f}^S(\lambda)\,\phi_\lambda = \sum_{\lambda\in D} d_\lambda \mathrm{tr}(\widehat{f}(\lambda)\pi_\lambda). \tag{4.6}$$

We can now apply the regularity result from Theorem 4.3.1 to conclude that the series (4.6) converges absolutely and uniformly to f if $|\widehat{f}^S(\lambda)| = \|\widehat{f}(\lambda)\|_{HS} = O(|\lambda|^{-s})$ $|\lambda| \to \infty$ with $\lambda \in D_S$ and $s > r + \frac{3m}{2}$. □

Theorem 4.3.4. *If $f \in C^{2p}(K\backslash G/K)$ where $p \in \mathbb{N}$, then*

$$|\widehat{f}^S(\lambda)| = o(|\lambda|^{-2p})$$

as $|\lambda| \to \infty$, $\lambda \in D_S$.

Proof. For all $f \in C^{2p}(K\backslash G/K)$ we can apply Theorem 4.3.2, (i) and using the same argument from (4.4), we have $|\widehat{f}^S(\lambda)| = \|\widehat{f}(\lambda)\|_{HS} = o(|\lambda|^{-2p})$ as $|\lambda| \to \infty$, $\lambda \in D_S$. □

Theorem 4.3.5. *If $f \in C(K\backslash G/K)$, then $f \in C^\infty(K\backslash G/K)$ if and only if $|\widehat{f}^S(\lambda)| = o(|\lambda|^{-p})$ for all $p \in \mathbb{N}$, as $|\lambda| \to \infty$ and $\lambda \in D_S$.*

Proof. This follows from Theorem 4.3.2, (ii) and Corollary 3.2.2 and equation 4.4. □

We will now define a class of rapidly decreasing functions on D following the work of [55].

Definition. The *Sugiura space of rapidly decreasing function* is the set $\mathcal{S}(D)$ of compatible matrix-valued functions F on D such that for all $p \in \mathbb{N}$,

$$\lim_{|\lambda|\to\infty} |\lambda|^p \|F(\lambda)\|_{HS} = 0. \tag{4.7}$$

For all $\lambda, \in D$, $F(\lambda)$ is an element of $M_{d_\lambda \times d_\lambda}(\mathbb{C})$.

Definition. The *spherical Sugiura space of rapidly decreasing functions*, denoted by $\mathcal{S}(D_S)$ is the set of functions $F : D_S \to \mathbb{C}$ such that for all $p \in \mathbb{N}$,

$$\lim_{|\lambda|\to\infty} |\lambda|^p |F(\lambda)| = 0. \tag{4.8}$$

Let us denote by $\mathcal{S}_0(D_S)$ the subspace of $\mathcal{S}(D)$ such that

$$\mathcal{S}_0(D_S) = \left\{ F \in \mathcal{S}(D) : R_{d_\lambda} F(\lambda) R_{d_\lambda} = \begin{cases} F(\lambda), & \text{if } \lambda \in D_S \\ 0, & \text{if } \lambda \notin D_S \end{cases} \right\},$$

where R_{d_λ} is defined in (3.4). So a function $F \in \mathcal{S}_0(D_S)$, if $F \in \mathcal{S}(D)$ such that $F(\lambda)_{11} \in \mathbb{C}$ and $F(\lambda)_{ij} = 0$ for $i, j \neq (1,1), \lambda \in D_S$, and $F(\lambda) = 0$ for $\lambda \notin D_S$. Thus, $\mathcal{S}_0(D_S)$ is a set of complex-valued matrices with only one non-zero coefficient at $(1,1)$.

Proposition 4.3.6. *There is a one-to-one correspondence between $\mathcal{S}(D_S)$ and $\mathcal{S}_0(D_S)$.*

Proof. Let us denote by $h : \mathcal{S}(D_S) \to \mathcal{S}_0(D_S)$ the mapping that sends each function $a \in \mathcal{S}(D_S)$ to a matrix valued function $A \in S_0(D_S)$ such that for all $\lambda \in D_s$, $A(\lambda) = a(\lambda)R_{d_\lambda}$.

First, check that A is indeed an element of $\mathcal{S}_0(D)$; for all $\lambda \in D_S$, we have by definition $\|A(\lambda)\|_{HS} = |a(\lambda)|\|R_{d_\lambda}\|_{HS} = |a(\lambda)|$ and a satisfying (4.8) implies that A satisfies (4.7). Furthermore, h is injective, since for all $a, b \in \mathcal{S}(D_S)$, $h(a) = h(b)$ implies that $a(\lambda) = [h(a)(\lambda)]_{11} = [h(b)(\lambda)]_{11} = b(\lambda)$. The map h is also surjective, since for all $A \in \mathcal{S}_0(D_S)$, we have $A_{11} \in \mathcal{S}(D_S)$ and $h(A_{11})(\lambda) = A_{11}(\lambda)R_{d_\pi} = A$. □

Consequently, there is an injection $j : \mathcal{S}(D_S) \to \mathcal{S}(D)$ corresponding to the map h. Let us also denote by i the injection $i : f \mapsto f$ from $C^\infty(K\backslash G/K)$ to $C^\infty(G)$.

Theorem 4.3.7. *The following diagram commutes.*

$$\begin{array}{ccc} C^\infty(G) & \xrightarrow{\mathcal{F}} & \mathcal{S}(D) \\ \uparrow i & & \uparrow j \\ C^\infty(K\backslash G/K) & \xrightarrow{\mathcal{F}^S} & \mathcal{S}(D_S) \end{array} \tag{4.9}$$

Proof. First note that $\mathcal{F}$ maps from $C^\infty(G)$ to $\mathcal{S}(D)$. Indeed, using Theorem 4.3.2, (ii) we have for all $p \geq 1$

$$\lim_{|\lambda|\to\infty} |\lambda|^p \|\widehat{f}(\lambda)\|_{HS} = 0,$$

which means by definition that $\widehat{f} \in \mathcal{S}(D)$.

Similarly, $\mathcal{F}^S$ maps from $C^\infty(K\backslash G/K)$ to $\mathcal{S}(D_S)$. Using Theorem 4.3.4, we have for all $p \geq 1$

$$\lim_{|\lambda|\to\infty} |\lambda|^p |\widehat{f}^S(\lambda)| = 0,$$

thus, by definition $\widehat{f}^S \in \mathcal{S}(D_S)$. We want to show that $(\mathcal{F} \circ i)f = (j \circ \mathcal{F}^S)f$, for all $f \in C^\infty(K\backslash G/K)$. From Corollary 3.2.2 we know that for all $\lambda \in D_S$, the Fourier transform $\mathcal{F}f(\lambda)$ is a $d_\lambda \times d_\lambda$ matrix that has one non-zero entry at $(1,1)$ equal to the spherical transform $\mathcal{F}^S f(\lambda)$. That is, for all $f \in C^\infty(K\backslash G/K)$ and $\lambda \in D_S$,

$$R_{d_\lambda}\left(\mathcal{F}f(\lambda)\right)R_{d_\lambda} = \mathcal{F}f(\lambda) = \left((\mathcal{F}^S f)(\lambda)\right)R_{d_\lambda}. \tag{4.10}$$

Also, $j : \mathcal{S}(D_S) \to \mathcal{S}(D)$ transforms $\mathcal{F}^S f(\lambda)$ to a $d_\lambda \times d_\lambda$ matrix, whose coefficients are all zero with the exception of the $(1,1)$th entry which is equal to $\mathcal{F}^S f(\lambda)$ itself. That is, for all $f \in C^\infty(K\backslash G/K)$ and $\lambda \in D_S$, we have

$$j\left(\mathcal{F}^S f(\lambda)\right) = (\mathcal{F}^S f(\lambda))R_{d_\lambda} \tag{4.11}$$

Thus, from equations (4.10) and (4.11) we conclude that

$$[(\mathcal{F} \circ i)f](\lambda) = (\mathcal{F}^S f(\lambda))R_{d_\lambda} = [(j \circ \mathcal{F}^S)f](\lambda), \qquad \text{for all } \lambda \in D_S.$$

Finally, when $\lambda \notin D_S$, both sides of the last identity are equal to 0. So, we can conclude that the diagram (4.9) commutes. □

Theorem 4.3.8. *The Fourier transform $\mathcal{F} : f \mapsto \widehat{f}$ is a topological isomorphism of $C^\infty(G)$ onto $\mathcal{S}(D)$.*

Proof. See [55] Theorem 4, p.44. □

Theorem 4.3.9. *The spherical transformation $\mathcal{F}^S : f \mapsto \widehat{f}^S$ is a linear isomorphism between $C^\infty(K\backslash G/K)$ and $\mathcal{S}(D_S)$.*

Proof. We are following the proof of [6] Theorem 3.4.3 p.78 and of [55] Theorem 4 p.44. The mapping $\mathcal{F}^S$ is linear by definition of the spherical transform. To see that $\mathcal{F}^S$ is an injective map, let $f_1, f_2 \in C^\infty(K\backslash G/K)$ such that $\mathcal{F}^S f_1 = \mathcal{F}^S f_2$, that is such that $\widehat{f_1}^S(\lambda) = \widehat{f_2}^S(\lambda)$ for all $\lambda \in D_S$. From Theorem 4.3.5 we know that $\widehat{f_1}^S(\lambda), \widehat{f_2}^S(\lambda) \in \mathcal{S}(D_S)$, and in particular $|\widehat{f_i}^S(\lambda)| = O(|\lambda|^{-s})$ for $i = 1, 2$ with $s > r + \frac{3m}{2}$. Now by applying Corollary 4.3.3 the following two series $\sum_{\lambda \in D_S} d_\lambda \widehat{f_1}^S(\lambda)\phi_\lambda$ and $\sum_{\lambda \in D_S} d_\lambda \widehat{f_2}^S(\lambda)\phi_\lambda$ converge absolutely and uniformly to f_1 and f_2 respectively. Hence, by uniqueness of the limit and the fact that $\sum_{\lambda \in D_S} d_\lambda \widehat{f_1}^S(\lambda)\phi_\lambda = \sum_{\lambda \in D_S} d_\lambda \widehat{f_2}^S(\lambda)\phi_\lambda$ we conclude that $f_1 = f_2$. The mapping $\mathcal{F}^S$ is therefore injective.

To prove that $\mathcal{F}^S$ is surjective, let us consider a function $F \in \mathcal{S}(D_S)$. We seek to find a function f_F in $C^\infty(K\backslash G/K)$ such that $\mathcal{F}(f_F) = F$. Moreover, by Theorem 4.3.5, for any $\varepsilon > 0$, there exists $\lambda_0 \in D_S$ such that for all $|\lambda| \geq |\lambda_0|$ and $p \in \mathbb{N}$, $|F(\lambda)| \leq \varepsilon |\lambda|^{-p}$. Thus, we have

$$\begin{aligned}\sum_{\lambda \in D_S, |\lambda| \geq |\lambda_0|} d_\lambda |F(\lambda)\phi_\lambda| &\leq \varepsilon \sum_{\lambda \in D_S, |\lambda| \geq |\lambda_0|} d_\lambda |\lambda|^{-p} \\ &\leq C\varepsilon \sum_{\lambda \in D_S, |\lambda| \geq |\lambda_0|} |\lambda|^{-p+m}\end{aligned}$$

The last inequality follows from Proposition 4.1.1. The series on the right-hand side converges for any $p \in \mathbb{N}$ that satisfies $p > r + m$. Therefore, the series $\sum_{\lambda \in D_S} d_\lambda F(\lambda)\ \phi_\lambda$ converges absolutely and uniformly and we can define the function $f_F(g) = \sum_{\lambda \in D_S} d_\lambda F(\lambda)\ \phi_\lambda(g)$ for all $g \in G$. By uniqueness of the spherical series expansion, we have $F = \widehat{f_F}^S = \mathcal{F}^S(f_F)$. It follows from Theorem 4.3.5 that f_F is a smooth function.

Also, f is a K-bi-invariant function, since the spherical functions ϕ_λ, $\lambda \in D_S$ are all K-bi-invariant: for all $k_1, k_2 \in K$,

$$f(k_1 g k_2) = \sum_{\lambda \in D_S} d_\lambda F(\lambda)\ \phi_\lambda(k_1 g k_2) = \sum_{\lambda \in D_S} d_\lambda F(\lambda)\ \phi_\lambda(g) = f(g).$$

This allows to conclude our proof that $\mathcal{F}^S$ is a linear isomorphism between $C^\infty(K\backslash G/K)$ and $\mathcal{S}(D_S)$. □

Let us introduce the topologies of $C^\infty(G)$ and $\mathcal{S}(D)$. The space of complex valued smooth functions on G, $C^\infty(G)$ is topologized by the family of seminorms defined by

$$\{p_U(f) = \|Uf\|_\infty : U \in \mathcal{U}(g)\}, \tag{4.12}$$

where $\mathcal{U}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$, see [55].

The topology of the vector space $\mathcal{S}(D)$ is generated by the family of seminorms

$$\{q_r(F) = \max_{\lambda \in D} |\lambda|^r \|F(\lambda)\|_{HS} : r > 0\}.$$

We will now introduce the topologies of $C^\infty(K\backslash G/K)$ and $\mathcal{S}(D_S)$. Similarly to $C^\infty(G)$, the topology of $C^\infty(K\backslash G/K)$ is the subspace topology induced by the same family of seminorms (4.12). The topology of $\mathcal{S}(D_S)$ is generated by the family of seminorms $\{\widetilde{q_r}(F) := \max_{\lambda \in D_S} |\lambda|^r |F(\lambda)|\}$, with $r > 0$.

Theorem 4.3.10. *The spherical transformation $\mathcal{F}^S : f \mapsto \widehat{f}^S$ is a topological isomorphism between $C^\infty(K\backslash G/K)$ and $\mathcal{S}(D_S)$.*

Proof. For $\mathcal{F}^S$ to be a topological isomorphism it is sufficient to prove that $\mathcal{F}^S$ and $(\mathcal{F}^S)^{-1}$ are continuous since we have already proven that $\mathcal{F}^S$ is a linear isomorphism.

We shall first prove that $\{\widetilde{q_r}(\widehat{f}^S(\lambda)), r > 0\}$ and $\{q_r(\widehat{f}(\lambda)), r > 0\}$ are the same on

$C^\infty(K\backslash G/K)$. Indeed for all $f \in C^\infty(K\backslash G/K)$ by Theorem 3.2.2 we have

$$\begin{aligned}\widetilde{q}_r(\mathcal{F}^S f) &= \max_{\lambda\in D_S} |\lambda|^r \left|\widehat{f}^S(\lambda)\right| \\ &= \max_{\lambda\in D} |\lambda|^r \|\widehat{f}(\lambda)\|_{HS} \\ &= q_r(\mathcal{F}f)\end{aligned}$$

We know from Theorem 4.3.8 that $\mathcal{F}$ is continuous from $C^\infty(G)$ to $\mathcal{S}(D)$, it then follows, that there exist $N \in \mathbb{N}$, $U_1, U_2, ..., U_N \in \mathcal{U}(\mathfrak{g})$ and a constant $M_r > 0$ such that

$$\widetilde{q}_r(\mathcal{F}^S f) = q_r(\mathcal{F}f) \leq M_r \sum_{i=1}^{N} p_{U_i}(f)$$

This proves that $\mathcal{F}^S : f \mapsto \widehat{f}^S$ is a continuous mapping from $C^\infty(K\backslash G/K)$ to $\mathcal{S}(D_S)$.

We shall now prove that the mapping $(\mathcal{F}^S)^{-1} : \mathcal{F}^S f \mapsto f$ is continuous from $\mathcal{S}(D_S)$ to $C^\infty(K\backslash G/K)$.

Let $f \in C^\infty(K\backslash G/K)$, by Theorem 4.3.8 we know that the mapping $\mathcal{F}^{-1} : \mathcal{F}f \mapsto f$ is continuous from $\mathcal{S}(D)$ to $C(G)$, that is for all $U \in \mathcal{U}(\mathfrak{g})$ there exist a finite subset $A_U \subset \mathbb{N}$ and a constant C_U such that

$$\begin{aligned}\|p_U f\|_\infty &\leq C_U \sum_{r\in A_U} q_r(\mathcal{F}f) \\ &= C_U \sum_{r\in A_U} \max_{\lambda\in D} |\lambda|^r \ \|\mathcal{F}(f)(\lambda)\|_{HS}. \end{aligned} \tag{4.13}$$

By applying Theorem 4.3.7 and equation (4.13) we get

$$\begin{aligned}\|p_U f\|_\infty &\leq C_U \sum_{r\in A_U} \max_{\lambda\in D_S} |\lambda|^r \ |\widehat{f}^S(\lambda)| \\ &= C_U \sum_{r\in A_U} \max_{\lambda\in D_S} |\lambda|^r \ |\mathcal{F}^S f(\lambda)| \\ &= C_U \sum_{r\in A_U} \widetilde{q}_r(\mathcal{F}^S f)\end{aligned}$$

We can now conclude that $\mathcal{F}^S$ is a topological isomorphism from $C(K\backslash G/K)$ to $\mathcal{S}(D_S)$. □

Chapter 5

Distributions on Lie groups

Let G be a Lie group and let $\{X_1, X_2, \cdots, X_d\}$ be a basis of $\mathfrak{g}$. Following the work of Ehrenpreis [15] we will adopt a global theory of distributions on Lie groups. From now on, we will only be interested in real-valued functions on G.

Definition. A linear functional $T : C_c^\infty(G) \to \mathbb{R}$ is called a *distribution* on G if for any compact set $H \subseteq G$, there exist two constants $C_H > 0$, $k \in \mathbb{N}$ such that for all $f \in C_c^\infty(H)$,

$$|Tf| \le C_H \left(\sup_H |f| + \sum_{i=1}^{d} \sup_H |X_i f| + \cdots + \sum_{i_1, i_2, \ldots, i_k = 1}^{d} \sup_H |X_{i_1} X_{i_2} \cdots X_{i_k} f| \right). \quad (5.1)$$

The linear space of all distributions in G will be denoted by $\mathcal{D}'(G)$. If the same constant k can be used for all compact set H, then T is said to be of *order* k. Note that the vector fields in general do not commute, therefore there are more distinct terms on the right hand side (rhs) than in the corresponding definition of the distribution on $\mathbb{R}^d$.

It will be helpful for us to use a similar notation to that on $\mathbb{R}^d$. First we introduce the norm $\|f\|_{\infty,H} := \sup_{g \in H} |f(g)|$ for all $f \in C_c^\infty(G)$ with support H. Then, we will write $C_H \sum_{|\alpha| \le k} \|X^\alpha f\|_{\infty,H}$ for the rhs of (5.1), with the understanding that for any two elements X_{i_1}, X_{i_2} of the basis of $\mathfrak{g}$, the terms $\|X_{i_1} X_{i_2} f\|_{\infty,H}$ and $\|X_{i_2} X_{i_1} f\|_{\infty,H}$ are not equal in general, with an obvious extension to higher order terms. We will regularly use the notation $\langle T, f \rangle = Tf$. A distribution T is said to be *identically zero* on a neighbourhood Ω of a point $g \in G$ if for any test function $f \in C_c^\infty(G)$ with $\mathrm{supp}(f) \subseteq \Omega$, we have $Tf = 0$. Then the support of a distribution T, denoted $\mathrm{supp}(T)$, is the complement of the set of points $g \in G$ such that T is identically zero in a neighbourhood of g.

Definition. Let $T : C_c^\infty(G) \to \mathbb{R}$ be a distribution and $h \in C^\infty(G)$, then we obtain another distribution $hT : C_c^\infty(G) \to \mathbb{R}$ given by

$$hT(\phi) = T(h\phi), \qquad \text{for all } \phi \in C_c^\infty(G).$$

Remark 5.0.1. Let μ be a Radon measure on G, then the linear functional defined by $\langle T, f\rangle := \int_G f(g)\mu(dg)$ for $f \in C_c(G)$, satisfies

$$|\langle T, f\rangle| \leq C_H \sup_{g\in H} |f(g)|, \qquad \text{for all } f \in C_c^\infty(H),$$

with $C_H = \mu(H)$. Thus, T is a a distribution of order 0, and we can identify T with μ. Moreover, T is a positive distribution, that is for all $f \geq 0$, $\langle T, f\rangle \geq 0$

Proposition 5.0.2. *A positive distribution is of order 0.*

Proof. The proof is the same as in the $\mathbb{R}^d$ case see [30], Theorem 2.17, p.38 or [27], Proposition 2.3, p.270. Let $H \subseteq G$ be a compact subset and $\phi \in C_c^\infty(G)$ take values in $[0, 1]$ such that $\phi = 1$ on H. For all $f \in C_c^\infty(H)$, we have

$$|f(g)| \leq |f(g)\phi(g)| \leq \sup_{\sigma\in H} |f(\sigma)|\phi(g), \text{ for all } g \in G. \tag{5.2}$$

Let T be a positive distribution, then $T\phi \geq 0$ and by linearity from the inequalities (5.2) we have

$$-\sup_{\sigma\in H} |f(\sigma)| \cdot T\phi \leq Tf \leq \sup_{\sigma\in H} |f(\sigma)| \cdot T\phi.$$

That is,

$$|Tf| \leq \sup_{\sigma\in H} |f(\sigma)| \cdot T\phi.$$

So T is of order 0 with $C_H = T\phi$. □

Corollary 5.0.3. *If T is a distribution of order 0, then it uniquely extends to a linear functional on $C_c(G)$. Furthermore, if T is positive, then so is its extension and it uniquely defines a Radon measure on $(G, \mathcal{B}(G))$.*

Proof. We know that $C_c^\infty(G)$ is dense in $C_0(G)$ w.r.t. the supremum norm, therefore every function in $C_c(G)$ can be uniformly approximated by a sequence in $C_c^\infty(G)$. For any $f \in C_c(G)$, let $(f_n)_{n\in\mathbb{N}} \in C_c^\infty(G)$ be a sequence converging to f. We define an extension of T to $C_c(G)$ by

$$\widetilde{T}f = \lim_{n\to\infty} Tf_n.$$

Note that $\widetilde{T}$ is uniquely defined, since if there are two sequences $(f_n)_{n\in\mathbb{N}} \in C_c^\infty(G)$ and $(f'_n)_{n\in\mathbb{N}} \in C_c^\infty(G)$ converging to the same function $f \in C_c(G)$, then $\lim_{n\to\infty} T(f_n - f'_n) = 0$. Let T be a positive distribution, then for all positive function $f \in C_c(G) \geq 0$ there exists a sequence of positive functions $(f_n)_{n\in N} \in C_c^\infty(G)$, $f_n \geq 0$ such that $\lim_{n\to\infty} f_n = f$. Then $\widetilde{T}f = \lim_{n\to\infty} Tf_n \geq 0$.

We can apply the Riesz representation theorem to obtain a unique Radon measure μ on G, that is

$$\widetilde{T}f = \int_G f(g)\mu(dg), \text{ for all } f \in C_c(G).$$

□

Theorem 5.0.4. *Let P be a distribution on G of order k with support $\{e\}$, then P has the form*

$$Pf = \sum_{|\alpha|\leq k} \frac{1}{|\alpha|!} a_\alpha X^\alpha f(e), \qquad \textit{for all } f \in C_c^\infty(G),$$

where $a_\alpha \in \mathbb{R}$ for all $|\alpha| \leq k$.

Proof. The distribution P will not vanish on any open canonical coordinate neighbourhood of e. Let us consider such a neighbourhood $U \subset G$ of e. Then there exists a diffeomorphism $\phi : U \to \widetilde{U}$, where $\widetilde{U} \subset \mathbb{R}^d$ is a neighbourhood of 0 and the map is given by $\phi(g) = (x_1(g), \dots, x_d(g))$ with $\phi(e) = 0$. Then there is a linear isomorphism $J_\phi : C_c^\infty(U) \to C_c^\infty(\widetilde{U})$ given by

$$J_\phi f = f \circ \phi^{-1} =: \widetilde{f}.$$

Also, for all $X_i \in T_eG$, $i = 1, \dots, d$, we have

$$\widetilde{X_i} := J_\phi \; X_i \; J_\phi^{-1} \in T_0(\mathbb{R}^d).$$

So given the basis $\{\partial_1, \dots, \partial_d\}$ of the tangent space $T_0(\mathbb{R}^d)$, the vector field $\widetilde{X_i}$ at $x \in \widetilde{U}$ can be decomposed as

$$\widetilde{X_i}(x) = \sum_{i=1}^d a_{ij}(x)\partial_i,$$

where for all $i, j = 1 \dots, d$, the functions $a_{ij} \in C^\infty(\widetilde{U})$ satisfy $a_{ij}(0) = \delta_{ij}$. This means that at 0 we have $\widetilde{X}_i = \partial_i$ (compare with [32] and [45] p.11).

We define the linear functional $\widetilde{P} : C_c^\infty(\widetilde{U}) \to \mathbb{R}$ by

$$\widetilde{P}\widetilde{f} := Pf = P\left(\widetilde{f} \circ \phi\right), \qquad \text{for all } \widetilde{f} \in C_c^\infty(\widetilde{U}).$$

Then we have,

$$
\begin{aligned}
|\widetilde{P}\widetilde{f}| = |Pf| &\le C \sum_{|\alpha|\le k} \|X^\alpha f\|_{\infty,G} \\
&= C \sum_{|\alpha|\le k} \|J_\phi X^\alpha f\|_{\infty,\mathbb{R}^d} \\
&= C \sum_{|\alpha|\le k} \|J_\phi J_\phi^{-1} \widetilde{X}^\alpha J_\phi f\|_{\infty,\mathbb{R}^d} \\
&= C \sum_{|\alpha|\le k} \|\widetilde{X}^\alpha \widetilde{f}\|_{\infty,\mathbb{R}^d}. \qquad (5.3)
\end{aligned}
$$

For any two vector fields $\widetilde{X}_{i_1}, \widetilde{X}_{i_2}$, where $i_1, i_2 = 1, \dots, d$ and for all $\widetilde{f} \in C_c^\infty(\widetilde{U})$ with support $H \subset \widetilde{U}$, we have at $x \in \widetilde{U}$

$$
\begin{aligned}
\widetilde{X}_{i_1}\widetilde{X}_{i_2}\widetilde{f}(x) &= \left(\sum_{j=1}^d a_{i_1,j}(x)\partial_j\right)\left(\sum_{k=1}^d a_{i_2,k}(x)\partial_k\right)\widetilde{f}(x) \\
&= \sum_{j=1}^d\sum_{k=1}^d a_{i_1,j}(x)a_{i_2,k}(x)\partial_j\partial_k\widetilde{f}(x) + \sum_{j=1}^d\sum_{k=1}^d a_{i_1,j}(x)(\partial_j a_{i_2,k})(x)\partial_k\widetilde{f}(x).
\end{aligned}
$$

So,

$$
\|\widetilde{X}_{i_1}\widetilde{X}_{i_2}\widetilde{f}\|_{\infty,\mathbb{R}^d} \le C_H \left(\sum_{j,k=1}^d \sup_{x\in H}|\partial_j\partial_k\widetilde{f}(x)| + \sum_{j=1}^d \sup_{x\in H}|\partial_j\widetilde{f}(x)|\right),
$$

where $C_H = \max_{j,k=1,\dots,d}\sup_{x\in H}|a_{i_1,j}(x)a_{i_2,k}(x)| + \max_{j,k=1,\dots,d}\sup_{x\in H}|a_{i_1,j}(x)(\partial_k a_{i_2,k})(x)|$. We iterate these steps for any family of indices $i_1, \dots, i_k = 1, \dots, d$, and the vector fields $\widetilde{X}_{i_1}\widetilde{X}_{i_2}\cdots\widetilde{X}_{i_k}$. By construction, we have

$$
\widetilde{X}_{i_1}\widetilde{X}_{i_2}\cdots\widetilde{X}_{i_k}\widetilde{f}(x) = \prod_{l=1}^k\left(\sum_{j_l=1}^d a_{i_l,j_l}(x)\partial_{j_l}\right)\widetilde{f}(x).
$$

Then as previously, we get

$$
\begin{aligned}
&\left\|\widetilde{X}_{i_1}\widetilde{X}_{i_2}\cdots\widetilde{X}_{i_k}\widetilde{f}\right\|_{\infty,\mathbb{R}^d} \\
&\le C_H\left(\sum_{j_1=1}^d \|\partial_{j_1}\widetilde{f}\|_{\infty,\mathbb{R}^d} + \sum_{j_1,j_2=1}^d \|\partial_{j_1}\partial_{j_2}\widetilde{f}\|_{\infty,\mathbb{R}^d} + \cdots + \sum_{j_1,j_2,\dots,j_k=1}^d \|\partial_{j_1}\partial_{j_2}\cdots\partial_{j_k}\widetilde{f}\|_{\infty,\mathbb{R}^d}\right). \qquad (5.4)
\end{aligned}
$$

C_H in the last line is a constant that depends on the functions

$$\{a_{i_l,j_l} \in C^\infty(\widetilde{U}) : l = 1, \ldots, k \text{ and } i_l, j_l = 1, \ldots, d\}.$$

Thus, from (5.3) and (5.4) we get

$$|\widetilde{P}\widetilde{f}| \leq C \sum_{|\alpha| \leq k} \|\partial^\alpha \widetilde{f}\|_{\infty, \mathbb{R}^d}, \qquad \text{for all } f \in C_c^\infty(\widetilde{U}).$$

So $\widetilde{P}$ extends to a distribution on $\mathbb{R}^d$ with support $\{0\}$. We can then use the original result on $\mathbb{R}^d$, see [30] p.46 Theorem 2.3.4., to conclude that for all $\widetilde{f} \in C_c^\infty(\widetilde{U})$

$$\widetilde{P}\widetilde{f} = \sum_{|\alpha| \leq k} a_\alpha \partial^\alpha \widetilde{f}(0),$$

where $a_\alpha \in \mathbb{R}$ for all $|\alpha| \leq k$. Thus, there exists $a'_\alpha \in \mathbb{R}$ for all $|\alpha| \leq k$, such that

$$\begin{aligned}
Pf = \widetilde{P}\widetilde{f} &= \sum_{|\alpha| \leq k} a_\alpha \partial^\alpha \widetilde{f}(0) = \sum_{|\alpha| \leq k} a'_\alpha \widetilde{X}^\alpha \widetilde{f}(0) \\
&= \sum_{|\alpha| \leq k} a'_\alpha (J_\phi X^\alpha J_\phi^{-1})(J_\phi f)(0) \\
&= \sum_{|\alpha| \leq k} a'_\alpha J_\phi (X^\alpha f)(0) \\
&= \sum_{|\alpha| \leq k} a'_\alpha X^\alpha f(e), \qquad \text{for all } f \in C_c^\infty(U).
\end{aligned}$$

Note that the coefficient a'_α are due to the non-commutativity of the vector fields, as when we apply the definition (5.1) of a distribution on G we need to distinguish all permutations of subsets of $X_1, \ldots, X_d$. □

Remark 5.0.5. In the particular case where P is a distribution of order 2 on G with support at $\{e\}$, P is of the form

$$Pf = -cf(e) + \sum_{i=1}^{d} b_i X_i f(e) + \sum_{i,j=1}^{d} \frac{1}{2} a_{ij} X_i X_j f(e), \qquad \text{for all } f \in C_c^\infty(G), \tag{5.5}$$

where for all $i, j = 1, \ldots, d$, we have $c, b_i, a_{ij} \in \mathbb{R}$. Furthermore, the commutativity $\partial_i \partial_j f = \partial_j \partial_i f$ implies $X_i X_j f(e) = X_j X_i f(e)$ so $a_{ij} = a_{ji}$, for all $i, j = 1, \ldots, d$.

Chapter 6

The positive maximum principle on Lie groups and a generalized Courrège's Theorem

6.1 Linear operators satisfying the positive maximum principle

The main goal of this section is to prove a generalization of the Courrège theorem [13] to Lie groups using the methodology of Hoh [28] on Euclidean spaces. For this we need to extend the definitions and properties of the functionals and operators satisfying the positive maximum principle on a Lie group G. Let us denote by $\mathrm{Fun}(G, \mathbb{R})$ the space of real-valued functions on G.

Definition. Let E be a subspace of $\mathrm{Fun}(G, \mathbb{R})$. A linear operator $A : E \to \mathrm{Fun}(G, \mathbb{R})$ is said to satisfy the *positive maximum principle* ("PMP") if for all $\varphi \in E$ such that $\varphi(g) = \sup_{\sigma \in G} \varphi(\sigma) \geq 0$ for some $g \in G$, then $A\varphi(g) \leq 0$.

In particular, we will be looking at the cases where E is $C_c^\infty(G)$, $C_c^\infty(G/K)$, $C_c^\infty(K\backslash G)$ or $C_c^\infty(K\backslash G/K)$.

Definition. A linear functional $T : C_c^\infty(G) \to \mathbb{R}$ is called *almost positive* if for all $\varphi \in C_c^\infty(G)$ such that $\varphi \geq 0$ and $\varphi(e) = 0$, then $T\varphi \geq 0$.
We say that T satisfies the *positive maximum principle* if for all $\varphi \in C_c^\infty(G)$ such that $\varphi(e) = \sup_{x \in G} \varphi(x) \geq 0$, then $T\varphi \leq 0$.

Remark 6.1.1. Suppose that $T : C_c^\infty(G) \to \mathbb{R}$ is an almost positive linear functional. Then for any two functions $f, h \in C_c^\infty(G)$ such that $f(e) = h(e) = 0$ with $|f(g)| \leq h(g)$

for all $g \in G$, we have $|T(f)| \leq T(h)$. To see why this is so, observe that since $f - h \leq 0$ with $(f-h)(e) = 0$ and $-h - f \leq 0$ with $(-h-f)(e) = 0$, we can use the almost positivity of T on both inequalities. Then by linearity of T we get respectively $T(f) \leq T(h)$ and $-T(h) \leq T(f)$, so that $|T(f)| \leq T(h)$.

Proposition 6.1.2. *If $T : C_c^\infty(G) \to \mathbb{R}$ satisfies the positive maximum principle then it is almost positive.*

Proof. Let $\varphi \in C_c^\infty(G)$ be such that $\varphi \geq 0$ and $\varphi(e) = 0$. So the function $-\varphi \in C_c^\infty(G)$ attains its non-negative supremum at e. By the PMP, $T(-\varphi) \leq 0$ and the linearity of T gives $T\varphi \geq 0$. □

Let us now explore the relationship between operators and functionals satisfying the positive maximum principle. For any linear operator $A : C_c^\infty(G) \to \mathrm{Fun}(G, \mathbb{R})$, we define a family of linear functionals $A_g : C_c^\infty(G) \to \mathbb{R}$, $g \in G$ by

$$A_g\varphi := (L_g A L_{g^{-1}})\varphi(e) = A(L_{g^{-1}}\varphi)(g). \tag{6.1}$$

Lemma 6.1.3. *A linear operator $A : C_c^\infty(G) \to$ Fun$(G, \mathbb{R})$ satisfies the positive maximum principle if and only if for all $g \in G$ the functional A_g satisfies the positive maximum principle.*

Proof. For necessity, let $\varphi \in C_c^\infty(G)$ be a test function with $\varphi(e) = \sup_{\sigma \in G} \varphi(\sigma) \geq 0$. Then for any $g \in G$, the translated function $(L_{g^{-1}}\varphi) \in C_c^\infty(G)$ satisfies $(L_{g^{-1}}\varphi)(g) = \sup_{\sigma \in G}(L_{g^{-1}}\varphi)(\sigma) \geq 0$. Now, since A satisfies the PMP we get

$$A_g\varphi = A(L_{g^{-1}}\varphi)(g) \leq 0.$$

For sufficiency, let $\varphi \in C_c^\infty(G)$ be such that $\varphi(g) = \sup_{\sigma \in G} \varphi(\sigma)$ for some $g \in G$. Then the translated function $L_g\varphi \in C_c^\infty(G)$ satisfies $(L_g\varphi)(e) = \sup_{\sigma \in G} L_g\varphi(\sigma)$. The functional A_g satisfies the PMP, thus we have

$$A\varphi(g) = (L_g A L_{g^{-1}} L_g)\varphi(e) = A_g(L_g\varphi) \leq 0.$$

□

Let us recall that the exponential map is a diffeomorphism from a neighbourhood V of the origin in $\mathfrak{g}$ to a neighbourhood U of e in G. For the given basis $(X_1, X_2, \ldots, X_d)$ of $\mathfrak{g}$, there exists a family of smooth functions $x_i : U \to \mathbb{R},\ i = 1, \ldots, d$ such that

$$g = \exp\left(\sum_{i=1}^{d} x_i(g) X_i\right) \qquad \text{for all } g \in U \tag{6.2}$$

and for all $i = 1, \ldots, d$,

$$x_i\left(\exp\left(\sum_{k=1}^{d} a_k X_k\right)\right) = a_i$$

if $\sum_{k=1}^{d} a_k X_k \in V$, see [47]. We call the functions x_i, *canonical coordinate functions*, they satisfy $x_i(e) = 0$ and $X_i x_k(e) = \delta_{ik}$ for all $i, k = 1, \ldots, d$. From now on, for all $i = 1, \ldots, d$ the function x_i will be extended to G so that $x_i \in C_c^\infty(G)$.

Theorem 6.1.4. *[Taylor's Theorem] Let $f \in C_0^2(G), g \in G$, then for all $l \in U$ there exists $l' \in U$ such that*

$$f(gl) = f(g) + \sum_{i=1}^{d} x_i(l) X_i f(g) + \frac{1}{2} \sum_{i,j=1} x_i(l) x_j(l) X_i X_j f(gl')$$

Proof. See [6] Theorem 5.3.1, p.127. □

Remark 6.1.5. As a direct consequence of the theorem, we also have the following inequality: for all $f \in C_0^2(G), g \in G, l \in U$ we have

$$\begin{aligned} \left| f(gl) - f(g) - \sum_{i=1}^{d} x_i(l) X_i f(g) \right| &\leq \frac{1}{2} \sum_{i,j=1}^{d} |x_i(l) x_j(l)| \sup_{v \in G} |X_i X_j f(v)| \\ &\leq \frac{1}{2} d \max_{i,j=1,\ldots,d} \|X_i X_j f\|_\infty \sum_{i=1}^{d} x_i^2(l) \end{aligned} \tag{6.3}$$

by the Cauchy-Schwarz inequality.

Also, by a change of variable we can reformulate Taylor's theorem:

Corollary 6.1.6. *Let $f \in C_0^2(G)$, $g \in G$, then for all $h \in l_g U$ there exists $h' \in l_g U$ such that*

$$f(h) = f(g) + \sum_{i=1}^{d} (L_{g^{-1}} x_i)(h) X_i f(g) + \frac{1}{2} \sum_{i,j=1}^{d} (L_{g^{-1}} x_i)(h)(L_{g^{-1}} x_j)(h) X_i X_j f(h').$$

Proof. This follows directly from Theorem 6.1.4, by putting $h = gl$ for all $g \in G$ and $l \in U$. So $x_i(l) = (L_{g^{-1}} x_i)(h)$ for all $i = 1, \ldots, d$ and $h' = gl'$. □

Note that here we translated the local coordinate functions, so we get

$$X_j(L_{g^{-1}} x_i)(g) = \delta_{ji}(e) \qquad \text{for all } g \in G \text{ and } i, j = 1 \ldots, d$$

Lemma 6.1.7. *Any function $\varphi \in C_c^\infty(G)$ with local maximum $\varphi(\sigma) = \sup_{g \in V} \varphi(g)$ for some $\sigma \in G$ on a neighbourhood V of σ, satisfies $X\varphi(\sigma) = 0$ for all $X \in \mathfrak{g}$.*

Proof. For any $X \in \mathfrak{g}$, let us introduce the function $t \mapsto \varphi(\sigma \exp(tX))$ from $\mathbb{R}$ to $\mathbb{R}$. This function is smooth and attains a local maximum at $t = 0$, therefore we have

$$\frac{d}{dt}\varphi(\sigma \exp(tX))\bigg|_{t=0} = X\varphi(\sigma) = 0, \qquad \text{for all } X \in \mathfrak{g}. \tag{6.4}$$

□

We will now mostly follow the approach adopted in Hoh [28], that we will generalize to G. Let us first recall some properties of Hausdorff spaces, that we will use in the following theorem.

Proposition 6.1.8. *If X is a Hausdorff space then any finite number of points $x_1, \ldots, x_n \in X$ can be separated by pairwise disjoint open sets.*

Proof. Let us denote by Ω the topology on X and by S the set of finite points $S = \{x_1, \ldots, x_n\} \subset X$. From the definition of a Hausdorff space, we know that for any pair $x_i, x_j \in S$, such that $x_i \neq x_j$, there exist two open sets $U_{ij}, U_{ji} \in \Omega$ such that $x_i \in U_{ij}$, $x_j \in U_{ji}$ and $U_{ij} \cap U_{ji} = \emptyset$.

Let us now fix a point $x_i \in S$, then we can define another neighbourhood of x_i by $U_i := \bigcap_{j=1}^{n} U_{ij}$, which is a finite intersection of elements in Ω, therefore $U_i \in \Omega$. Furthermore, for all $x_i, x_k \in S$ such that $x_i \neq x_k$, we have

$$(U_i \cap U_k) \subseteq (U_{ik} \cap U_{ki}) = \emptyset.$$

Thus, we have proved that there exists a family of pairwise disjoint open sets $U_1, \ldots, U_n \in \Omega$ containing $x_1, \ldots, x_n$ respectively. □

Theorem 6.1.9. *Let $T : C_c^\infty(G) \to \mathbb{R}$ be an almost positive functional, then T is a distribution of order 2, that is for any compact subset $H \subset G$ there is a constant $C_H \geq 0$ such that*

$$|T\varphi| \leq C_H \left(\sup_{x \in H} |\varphi(x)| + \sum_{i=1}^{d} \sup_{x \in H} |X_i\varphi(x)| + \sum_{i,j=1}^{d} \sup_{x \in H} |X_iX_j\varphi(x)| \right)$$

for all test functions $\varphi \in C_c^\infty(G)$ with support in H.

Proof. Let $\varphi \in C_c^\infty(G)$ with $\mathrm{supp}(\varphi) = H$ where $H \subset G$ is a compact set. Define the constant

$$M := \sup_{g \in H} |\varphi(g)| + \sum_{i=1}^{d} \sup_{g \in H} |X_i\varphi(g)| + \sum_{i,j=1}^{d} \sup_{g \in H} |X_iX_j\varphi(g)|.$$

The set H is covered by $\bigcup_{k\in H} l_k U$ and since it is compact, there is a finite subcover $\{U_1,\dots U_N\}$, where $U_i := l_{k_i}U$ for some $k_i \in H$, $i = 1,\dots,N$. Furthermore, since G is a Hausdorff space, we can separate the points $k_1, k_2, \dots, k_N$ by a family of pairwise disjoint open sets $W_1,\dots,W_N \subset G$ such that $k_i \in W_i$ for all $i = 1,\dots,N$, see Proposition 6.1.8.

In the following we will use Taylor's theorem at each k_l, $l = 1,\dots,N$, with the translated local coordinates. First, denote the constant

$$C_1 := \max_{\substack{r=1,\dots,d\\ l=1,\dots,N}} \left\{ \sup_{g\in G} \left|(L_{k_l^{-1}}x_r)(g)\right| + \sum_{i=1}^{d} \sup_{g\in G} \left|X_i(L_{k_l^{-1}}x_r)(g)\right| + \sum_{i,j=1}^{d} \sup_{g\in G} \left|X_iX_j(L_{k_l^{-1}}x_r)(g)\right| \right\}.$$

We will now define a function $\widetilde{\varphi} \in C_c^\infty(G)$ which will serve as test function for the argument below.

$$\widetilde{\varphi}(g) := \varphi(g) - \sum_{l=1}^{N} \varphi(k_l)\varepsilon_l(g) - \sum_{l=1}^{N}\sum_{r=1}^{d} L_{k_l^{-1}}x_r(g)\ \varepsilon_l(g)\ X_r\varphi(k_l), \qquad \text{for all } g \in G, \tag{6.5}$$

where for all $l = 1,\dots,N$ the function $\varepsilon_l \in C_c^\infty(G)$ takes values in $[0,1]$ with support in W_l and is equal to 1 in a neighbourhood of k_l. Let us denote the constant

$$C_2 := \max_{l=1,\dots,N} \left\{ \sup_{g\in G} |\varepsilon_l(g)| + \sum_{i=1}^{d} \sup_{g\in G} |X_i\varepsilon_l(g)| + \sum_{i,j=1}^{d} \sup_{g\in G} |X_iX_j\varepsilon_l(g)| \right\}.$$

Then from (6.5) for all $g \in G$ and $i, j = 1,\dots,d$, we have

$$\begin{aligned}
|X_iX_j\widetilde{\varphi}(g)| \le & |X_iX_j\varphi(g)| + \sum_{l=1}^{N} |\varphi(k_l)||X_iX_j\varepsilon_l(g)| \\
& + \sum_{l=1}^{N}\sum_{r=1}^{d} |X_r\varphi(k_l)| \left| X_iX_j \left(\varepsilon_l(g)(L_{k_l^{-1}}x_r)(g)\right)\right| \\
\le & M + C_2M + M\sum_{l=1}^{N}\sum_{r=1}^{d} \left|\varepsilon_l(g)X_iX_j(L_{k_l^{-1}}x_r)(g)\right| \\
& + M\sum_{l=1}^{N}\sum_{r=1}^{d} \Bigg(\left|(L_{k_l^{-1}}x_r)(g)X_iX_j\varepsilon_l(g)\right| + |X_j\varepsilon_l(g)||X_i(L_{k_l^{-1}}x_r)(g)| \\
& \qquad + |X_i\varepsilon_l(g)||X_j(L_{k_l^{-1}}x_r)(g)| \Bigg) \\
\le & M + C_2M + 4MC_1C_2 = CM
\end{aligned}$$

where $C \geq 0$ is a constant depending on $\varepsilon_l, l = 1\ldots, N$ and the local coordinate functions. In particular, at $k_m \in U_m$ for all $m = 1, \ldots, N$, we have

$$\begin{aligned}\widetilde{\varphi}(k_m) &= \varphi(k_m) - \sum_{l=1}^{N} \varphi(k_l)\varepsilon_l(k_m) - \sum_{l=1}^{N}\sum_{r=1}^{d}(L_{k_l^{-1}}x_r)(k_m)\varepsilon_l(k_m)X_r\varphi(k_l) \\ &= \varphi(k_m) - \varphi(k_m)\varepsilon_m(k_m) - \sum_{r=1}^{d}(L_{k_m^{-1}}x_r)(k_m)X_r\varphi(k_m) = 0. \qquad (6.6)\end{aligned}$$

Next we will calculate $X_i\widetilde{\varphi}(k_m)$, for all $i = 1, \ldots d$. For this, first note that ε_m equals 1 in a neighbourhood of k_m, so by using Lemma 6.1.7 for the function $t \mapsto \varepsilon_m(k_m \exp(tX))$ for all $X \in \mathfrak{g}$, we have $X\varepsilon_m(k_m) = \frac{d}{dt}\varepsilon_m(k_m \exp(tX)) = 0$, for all $X \in \mathfrak{g}$. Thus,

$$\begin{aligned}X_i\widetilde{\varphi}(k_m) &= X_i\varphi(k_m) - \sum_{l=1}^{N} \varphi(k_l)\ X_i\varepsilon_l(k_m) - \sum_{l=1}^{N}\sum_{k=1}^{d} X_i\left((L_{k_l^{-1}}x_r)(k_m)\varepsilon_l(k_m)\right)\ X_k\varphi(k_l) \\ &= X_i\varphi(k_m) - \varphi(k_m)\ X_i\varepsilon_m(k_m) - \sum_{k=1}^{d} X_i(L_{k_m^{-1}}x_r)(k_m)X_k\varphi(k_m) \\ &= X_i\varphi(k_m) - \sum_{k=1}^{d}\delta_{ik}X_k\varphi(k_m) = 0. \qquad (6.7)\end{aligned}$$

For any $g \in H$, there exists at least one $m = 1, \ldots, N$ such that $g \in U_m$. Thus, by equations (6.6) and (6.7), we have $\widetilde{\varphi}(g) = \widetilde{\varphi}(g) - \widetilde{\varphi}(k_m) - \sum_{i=1}^{d}(L_{k_m^{-1}}x_r)(g)\ X_i\widetilde{\varphi}(k_m)$. Using this with the corollary of Taylor's theorem 6.1.6 and the Cauchy-Schwarz inequality we have for all $g \in G$

$$\begin{aligned}|\widetilde{\varphi}(g)| &= \left|\widetilde{\varphi}(g) - \widetilde{\varphi}(k_m) - \sum_{i=1}^{d}(L_{k_m^{-1}}x_r)(g)\ X_i\widetilde{\varphi}(k_m)\right| \\ &\leq \frac{1}{2}\sum_{i,j=1}\left|(L_{k_m^{-1}}x_i)(g)(L_{k_m^{-1}}x_j)(g)\right|\ \sup_{v\in G}|X_iX_j\widetilde{\varphi}(v)| \\ &\leq \frac{1}{2}dCM\sum_{i=1}^{d}\left|(L_{k_m^{-1}}x_i)(g)\right|^2\end{aligned}$$

As a consequence, we have

$$|\widetilde{\varphi}(g)| \leq \frac{1}{2}dCM\sum_{m=1}^{N}\sum_{i=1}^{d}\left|(L_{k_m^{-1}}x_i)(g)\right|^2, \qquad \text{for all } g \in H. \qquad (6.8)$$

Since $\widetilde{\varphi}(g) = 0$ when $g \notin H$, we can extend the inequality (6.8) to all $g \in G$. For simplicity, let us introduce the function $h \in C_c^\infty(G)$ defined by

$$h(g) = \sum_{m=1}^{N} \sum_{i=1}^{d} \left|(L_{k_m^{-1}} x_i)(g)\right|^2 \text{ for all } g \in G.$$

Observe that $\widetilde{\varphi}(e) = h(e) = 0$ and $|\widetilde{\varphi}(g)| \leq \frac{1}{2} dCMh(g)$ for all $g \in G$. Thus, since T is almost positive, by the argument of Remark 6.1.1 we have

$$|T(\widetilde{\varphi})| \leq \frac{1}{2} dCM\, T(h).$$

So, by linearity of T and the definition of $\widetilde{\varphi}$ in (6.5) we have

$$\begin{aligned} |T(\varphi)| &\leq |T(\widetilde{\varphi})| + \sum_{l=1}^{N} |\varphi(k_l)||T(\varepsilon_l)| + \sum_{l=1}^{N} \sum_{k=1}^{d} \left|T(\varepsilon_l(L_{k_{l-1}} x_r))\right| |X_k \varphi(k_l)| \\ &\leq \frac{1}{2} dCM\, T(h) + M \sum_{l=1}^{N} |T(\varepsilon_l)| + M \sum_{l=1}^{N} \sum_{k=1}^{d} \left|T(\varepsilon_l x_r^{U_l})\right|. \end{aligned}$$

Since the cover $(U_1, U_2, \ldots, U_N)$ depends on the compact set H, so do all the local coordinates and the functions ε. We can then conclude that there is a constant $C_H > 0$ that depends on H such that

$$|T(\varphi)| \leq C_H M.$$

□

Definition. We say that a Borel measure μ on $G^* = G \backslash \{e\}$ is a *Lévy measure* if for every canonical coordinate neighbourhood U of e we have

$$\int_{U^*} \left(\sum_{i=1}^{d} x_i^2(g) \right) \mu(dg) < \infty, \text{ and } \mu(U^C) < \infty.$$

Note that this definition is a slight variation from those found in the literature: p.38 in [47] or p.128 in [6].

Theorem 6.1.10. *Let $T : C_c^\infty(G) \to \mathbb{R}$ be a linear functional satisfying the positive maximum principle. Then there exists a unique family of constants $a_{ij}, b_i, c \in \mathbb{R}$, $i, j = 1, \ldots d$ and a unique Radon measure μ on $G^* = G \backslash \{e\}$ such that*

i) $(a_{ij})_{i,j=1,\ldots,d}$ *is a symmetric non-negative definite matrix*
ii) $c \geq 0$
iii) μ *is a Lévy measure*

and T is of the form

$$T\varphi = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}X_iX_j\varphi(e) + \sum_{i=1}^{d} b_iX_i\varphi(e) - c\varphi(e)$$
$$+\int_{G^*}\left(\varphi(g) - \varphi(e) - \sum_{i=1}^{d} x_i(g)X_i\varphi(e)\right)\mu(dg), \tag{6.9}$$

for all $\varphi \in C_c^\infty(G)$.

Proof. Let V be a compact neighbourhood of e such that $\overline{U} \subseteq V$, and let $\phi_1, \phi_2 \in C_c^\infty(G)$ be two functions taking values in $[0,1]$ with supports respectively in V and in U^C, such that $\phi_1(g) = 1$ for all $g \in U$, and $\phi_2(g) = 1$ for all $g \in V^C$.

We introduce the function $\xi := \sum_{i=1}^{d} x_i^2(\cdot)\cdot\phi_1 + \phi_2$ from G to $\mathbb{R}$ and note that $\xi(e) = 0$. Then $\xi \cdot T$ is a positive distribution. Indeed, for all $\varphi \in C_c^\infty(G)$ such that $\varphi \geq 0$, we also have $\xi\cdot\varphi \geq 0$ with $\xi(e)\varphi(e) = 0$. So by almost positivity of T we get

$$\langle \xi\cdot T, \varphi\rangle = \langle T, \xi\cdot\varphi\rangle \geq 0.$$

Since $\xi \cdot T$ is a positive distribution, it is of order 0 by Proposition 5.0.2. Thus, by Corollary 5.0.3 there exists a unique regular Borel measure ν on G such that $\nu = \xi\cdot T$, i.e. $\langle \xi\cdot T, f\rangle = \int_G f(g)\nu(dg)$ for all $f \in C_c^\infty(G)$. We define another Borel measure $\mu = \frac{1}{\xi}\cdot\nu|_{G^*}$ on G^*, so that we have

$$\langle T, f\rangle = \langle \xi\cdot T, \frac{1}{\xi}f\rangle = \int_{G^*} f(g)\frac{1}{\xi(g)}\nu(dg) = \int_{G^*} f(g)\mu(dg), \qquad \text{for all } f \in C_c^\infty(G^*). \tag{6.10}$$

The set V is a compact neighbourhood of e and we define $V^* = V/\{e\}$, so by regularity of measure ν we have

$$\int_{U^*}\left(\sum_{i=1}^{d} x_i^2(g)\right)\mu(dg) = \int_{V^*}\xi(g)\mu(dg) \leq \nu(V) < \infty \tag{6.11}$$

Let $\alpha, \beta \in C_c^\infty(G)$ be two functions taking values in $[0,1]$, such that $\mathrm{supp}(\alpha) \subset U$ and $\mathrm{supp}(\beta) \subset U^C$ with $\alpha(e) = 1$, so

$$\alpha(e) + \beta(e) = 1 + 0 = \sup_{g\in G}(\alpha+\beta)(g).$$

By the positive maximum principle, $T(\alpha+\beta) \leq 0$, so by linearity of T, we get

$T\beta \le -T\alpha$, that is

$$\langle T, \beta\rangle = \int_{G^*} \beta(g)\mu(dg) \le -\langle T, \alpha\rangle$$

Taking the supremum over all possible β it then follows that

$$\mu(U^C) \le -\langle T, \alpha\rangle < \infty \tag{6.12}$$

So the measure μ is indeed a Lévy measure.

We introduce a linear functional $S : C_c^\infty(G) \to \mathbb{R}$, by

$$S\varphi := \int_{G^*} \left[\varphi(g) - \varphi(e) - \sum_{i=1}^{d} x_i(g) X_i\varphi(e)\right] \mu(dg), \qquad \text{for all } \varphi \in C_c^\infty(G).$$

The integral is finite by a second order Taylor's expansion and properties of the measure μ. By Lemma 6.1.7 for all $\varphi \in C_c^\infty(G)$ such that $\varphi(e) = \sup_{g\in G} \varphi(g) \ge 0$ we have $X_i\varphi(e) = 0$, therefore

$$S\varphi = \int_{G^*} (\varphi(g) - \varphi(e))\ \mu(dg) \le 0.$$

That is, S satisfies the positive maximum principle. By Proposition 6.1.2 and Theorem 6.1.9 both T and S are distributions of order 2, thus so is their difference $P := T - S$. If $\varphi \in C_c^\infty(G)$ with $e \notin \mathrm{supp}(\varphi)$, then

$$S\varphi = \int_{G^*} \varphi(g)\mu(dg) = T\varphi.$$

That is $P\varphi = 0$ for all such $\varphi \in C_c^\infty(G)$, therefore $\mathrm{supp}(P) \subseteq \{e\}$. Thus, using Theorem 5.0.4, P is of the form

$$P\varphi = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij} X_i X_j \varphi(e) + \sum_{i=1}^{d} b_i X_i\varphi(e) - c\varphi(e).$$

We will now prove that the constant c is positive. Let $(\varphi_k)_{k\in\mathbb{N}}$ be a sequence of non-negative, monotone increasing functions in $C_c^\infty(G)$ such that $\varphi_k = 1$ in a neighbourhood of e which is pointwise convergent to $\mathbb{1}_G$, then by the monotone convergence theorem

$$S\varphi_k = \int_{G^*} (\varphi_k(g) - 1)\ \mu(dg) \to 0, \qquad \text{as } k \to \infty.$$

So

$$T\varphi_k = P\varphi_k + S\varphi_k = -c\varphi_k(e) + S\varphi_k \to -c, \qquad \text{as } k \to \infty.$$

Note that for all $k \in \mathbb{N}$, $\varphi_k(e) = \sup_{g\in G} \varphi_k(g) \geq 0$, and since T satisfies the positive maximum principle we have $T\varphi_k \leq 0$. Thus, $c \geq 0$.

From Remark 5.0.5, we already know that (a_{ij}) is symmetric. We will now prove that (a_{ij}) is also positive definite. Let $(\varepsilon_k)_{k\in\mathbb{N}}$ be a sequence of monotone decreasing functions in $C_c^\infty(G)$, taking values in $[0,1]$ such that $\varepsilon_k = 1$ in a neighbourhood of e with $V_k \subset V_{k'}$ when $k > k'$ and $\bigcap_{k\in\mathbb{N}} V_k = \{e\}$. Let us also denote by $f_\xi \in C_c^\infty(G)$ the function defined by $f_\xi := \frac{1}{2}\sum_{i,j=1}^{d} \xi_i\xi_j x_i(\cdot)x_j(\cdot)$, where $\xi = (\xi_1,\ldots,\xi_d) \in \mathbb{R}^d$. Note that for all $1 \leq k, l \leq d$, we have

$$\begin{aligned} X_lX_kf_\xi &= \frac{1}{2}\sum_{i,j=1}^{d} \xi_i\xi_j X_lX_k(x_ix_j) \\ &= \frac{1}{2}\sum_{i,j=1}^{d} \xi_i\xi_j(\delta_{ki}\delta_{lj} + \delta_{kj}\delta_{li}) \\ &= \frac{1}{2}(\xi_k\xi_l + \xi_l\xi_k) = \xi_l\xi_k. \end{aligned}$$

Furthermore,

$$\begin{aligned} T(\varepsilon_k \cdot f_\xi) &= P(\varepsilon_k \cdot f_\xi) + S(\varepsilon_k \cdot f_\xi) \\ &= \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j + \int_{G^*} \varepsilon_k(g)f_\xi(g)\,\mu(dg). \end{aligned}$$

Thus, by dominated convergence $T(\varepsilon_k \cdot f) \to \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j$ as k goes to infinity. Also, since T is almost positive, $T(\varepsilon_k \cdot f) \geq 0$, so $\sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j \geq 0$ and (a_{ij}) is positive definite.

It is clear that (a_{ij}) and c are uniquely defined for all T. Moreover, ν was uniquely defined and therefore μ is also uniquely defined. This allows as to calculate $\sum_{i=1}^{d} b_iX_i\varphi(e)$ for any $\varphi \in C_c^\infty(G)$, so the vector $b = (b_1,\ldots,b_d) \in \mathbb{R}^d$ is also uniquely defined. □

We will now establish the converse of Theorem 6.1.10.

Theorem 6.1.11. *Every linear functional on $C_c^\infty(G)$ of the form (6.9) satisfies the positive maximum principle.*

Proof. By Lemma 6.1.7, for any function $\varphi \in C_c^\infty(G)$ such that $\varphi(e) = \sup_{g\in G} \varphi(g) \geq 0$, we have $X\varphi(e) = 0$ for all $X \in \mathfrak{g}$. Thus, in (6.9) both the first order differential part and the integral part satisfy the positive maximum principle. So now we only have to deal with the second order differential part.

Fix $h \in U$ and let us consider the function $H : t \mapsto \varphi\left(\exp\left(t\sum_{i=1}^d x_i(h)X_i\right)\right)$ from $\mathbb{R}$ to G. Let P_2 be the second order Taylor polynomial of H around 0,

$$P_2(t) = \varphi(e) + t\sum_{i=1}^d x_i(h)X_i\varphi(e) + \frac{t^2}{2}\sum_{i,j=1}^d x_i(h)x_j(h)X_iX_j\varphi(e).$$

Then $H(t) - P_2(t) = o(t^2)$ as $t \to 0$. That is

$$\varphi\left(\exp\left(t\sum_{i=1}^d x_i(h)X_i\right)\right) - \varphi(e) - t\sum_{i=1}^d x_i(h)X_i\varphi(e) - \frac{t^2}{2}\sum_{i,j=1}^d x_i(h)x_j(h)X_iX_j\varphi(e) = o(t^2).$$

Thus, by Lemma 6.1.7 and the fact that $\varphi(e) = \sup_{g\in G} \varphi(g)$ we have

$$o(t^2) + \frac{t^2}{2}\sum_{i,j=1}^d x_i(h)x_j(h)X_iX_j\varphi(e) = \varphi\left(\exp\left(t\sum_{i=1}^d x_i(h)X_i\right)\right) - \varphi(e) \leq 0.$$

Hence, dividing by t^2 and taking the limit $t \to 0$ we get

$$\sum_{i,j=1}^d x_i(h)x_j(h)X_iX_j\varphi(e) \leq 0.$$

The map $h \mapsto (x_1(h), \ldots, x_d(h))$ is a diffeomorphism from U to an open neighbourhood $\widetilde{U}$ of $0 \in \mathbb{R}^d$. Let us fix an open ball $B_r(0) \subset \widetilde{U}$ of radius $r > 0$. Then given any $\lambda \in \mathbb{R}^d$, there exists $c(\lambda) > 0$ such that $c(\lambda)\lambda \in B_r(0)$, i.e. for all $i = 1, \ldots, d$, $c(\lambda)\lambda_i = x_i(h)$ for some $h \in U$. Then we have

$$\sum_{i,j=1}^d \lambda_i\lambda_j X_iX_j f(e) = \frac{1}{c(\lambda)^2}\sum_{i,j=1}^d x_i(h)x_j(h)X_iX_j f(e) \leq 0$$

We conclude that any linear functional on $C_c^\infty(G)$ of the form (6.9) satisfies the positive maximum principle. □

We are now in a position to generalize Courrège's theorem, see Theorem 3.4 in [13], from Euclidean spaces to Lie groups. For a detailed proof on Euclidean spaces, see also Jacob and Schilling [36] and Jacob [34], p.360.

Definition. A *Lévy kernel* is a family of measures $\{\mu(g,\cdot), g \in G\}$ on $(G, \mathcal{B}(G))$ such that for all $g \in G$, $\mu(g,\cdot)$ is a Lévy measure.

Theorem 6.1.12. *A linear operator $A : C_c^\infty(G) \to B(G)$ satisfies the positive maximum principle if and only if for all $g \in G$ there exist*

- *a unique real symmetric non negative definite matrix $(a_{ij}(g))$*
- *a unique vector $b(g) \in \mathbb{R}^d$*
- *a unique constant $c(g) \geq 0$*
- *a unique Lévy kernel*

such that, for all $\varphi \in C_c^\infty(G)$ and $g \in G$,

$$\begin{aligned} A\varphi(g) =& \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(g)X_iX_j\varphi(g) - \sum_{i=1}^{d} b_i(g)X_i\varphi(g) - c(g)\varphi(g) \\ &+ \int_{G^*}\left[\varphi(g\tau) - \varphi(g) - \sum_{i=1}^{d} x_i(\tau)X_i\varphi(g)\right]\mu(g,d\tau). \end{aligned} \tag{6.13}$$

Proof. From Lemma 6.1.3, we know that A satisfies the PMP if and only if for all $g \in G$ the linear functional A_g as defined in (6.1) satisfies the PMP. Thus, for each $g \in G$, the functional A_g has the form (6.9),

$$\begin{aligned} A_g\varphi =& \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(g)X_iX_j\varphi(e) - \sum_{i=1}^{d} b_i(g)X_i\varphi(e) - c(g)\varphi(e) \\ &+ \int_{G^*}\left(\varphi(\tau) - \varphi(e)\sum_{i,j=1}^{d} x_i(\tau)X_i\varphi(e)\right)\mu(g,d\tau). \end{aligned}$$

Then in particular for the function $L_g\varphi \in C_c^\infty(G)$, we get

$$\begin{aligned} A\varphi(g) = A_g(L_g\varphi) =& \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(g)X_iX_j\varphi(g) - \sum_{i=1}^{d} b_i(g)X_i\varphi(g) - c(g)\varphi(g) \\ &+ \int_{G^*}\left(\varphi(g\tau) - \varphi(g) - \sum_{i=1}^{d} x_i(\tau)X_i\varphi(g)\right)\mu(g,d\tau). \end{aligned}$$

□

We will be particularly interested in the case where A is the generator of a Feller semigroup on $C_0(G)$. So in the following, we will establish sufficient conditions for

$A : C_c^\infty(G) \to C_0(G)$. For the most recent results on Euclidean spaces see Kühn and Schilling [42].

First, let us introduce some simplifying notations. Define the map $H : C_c^\infty(G) \to C(G \times G)$ by

$$Hf(g,\tau) := f(g\tau) - f(g) - \sum_{i=1}^{d} x_i(\tau) X_i f(g), \quad \text{for all } f \in C_c^\infty(G), \tau, g \in G. \tag{6.14}$$

Theorem 6.1.13. *Let A be a linear operator on $C_c^\infty(G)$ satisfying the positive maximum principle such that*

i) The functions $a_{ij}(\cdot), b_i(\cdot), c(\cdot)$, for all $i, j = 1, \dots, d$ are continuous on G.
ii) If $(g_n)_{n\in\mathbb{N}}$ is a sequence such that $g_n \to g$ as $n \to \infty$ for some $g \in G$, then

$$\lim_{n\to\infty} \int_{U^*} \sum_{i=1}^{d} x_i^2(\tau) \left|\mu(g, d\tau) - \mu(g_n, d\tau)\right| = 0$$

and

$$\lim_{n\to\infty} \int_{U^C} \left|\mu(g, d\tau) - \mu(g_n, d\tau)\right| = 0,$$

Then the operator A maps $C_c^\infty(G)$ to $C(G)$.

Proof. A satisfies the positive maximum principle, therefore it is of the form (6.13). When a_{ij}, b_i, c are continuous functions on G, the differential part in (6.13) will be clearly continuous. To deal with the integral part, we will split it into two.

$$\int_{G^*} \left[f(g\tau) - f(g) - \sum_{i=1}^{d} x_i(\tau) X_i f(g)\right] \mu(g, d\tau) = \int_{U^C} Hf(g,\tau)\mu(g, d\tau) + \int_{U^*} Hf(g,\tau)\mu(g,\tau), \tag{6.15}$$

for all $f \in C_c^\infty(G)$ and $g \in G$.

First, let us prove the continuity of the integral over U^C, for this we will use both the uniform continuity of the function f and assumption (ii). As $g_n \to g$ when $n \to \infty$, we have

$$\begin{aligned}
&\left|\int_{U^C} Hf(g,\tau)\mu(g,\tau) - \int_{U^C} Hf(g_n,\tau)\, \mu(g_n,\tau)\right| \leq \\
&\left|\int_{U^C} \left[f(g\tau) - f(g_n\tau) - f(g) + f(g_n) - \sum_{i=1}^{d} x_i(\tau) \left(X_i f(g) - X_i f(g_n)\right)\right] \mu(g, d\tau)\right| \\
&+ \left|\int_{U^C} Hf(g_n,\tau) \left[\mu(g_n.d\tau) - \mu(g, d\tau)\right]\right|
\end{aligned} \tag{6.16}$$

By smoothness of the function f, when $g_n \to g$ we have $f(g_n) \to f(g)$, $f(g_n\tau) \to f(g\tau)$ and $X_i f(g_n) - X_i f(g) \to 0$, for all $\tau \in U^C$ and $i = 1, \ldots, d$. Thus, by dominated convergence, the first integral on the right hand side of (6.16) converges to 0, when $g_n \to g$. As for the last term in (6.16), we have an upper bound

$$\left| \int_{U^C} \left[f(g_n\tau) - f(g_n) - \sum_{i=1}^{d} x_i(\tau) X_i f(g_n) \right] [\mu(g_n.d\tau) - \mu(g, d\tau)] \right| \\ \leq K_f \int_{U^C} |\mu(g_n, d\tau) - \mu(g, \tau)| \tag{6.17}$$

where $K_f = 2 \sup_{g \in G} |f(g)| + \sum_{i=1}^{d} \sup_{g \in G} |x_i(g)||X_i f(g)|$. Using (ii), the right hand side converges to 0 as $g_n \to g$. We will now deal with the continuity of the integral over U^*. For all $g \in G$, we have

$$\left| \int_{U^*} Hf(g, \tau)\, \mu(g, d\tau) - \int_{U^*} Hf(g_n, \tau)\, \mu(g_n, d\tau) \right| \\ \leq \left| \int_{U^*} [Hf(g, \tau) - Hf(g_n, \tau)]\, \mu(g, d\tau) \right| + \left| \int_{U^*} Hf(g_n, \tau) [\mu(g_n, d\tau) - \mu(g, d\tau)] \right| \tag{6.18}$$

Again, by smoothness of $f \in C_c^\infty(G)$, when $g_n \to g$ we have $Hf(g_n, \tau) \to Hf(g, \tau)$, for all $\tau \in U$. Also, by Taylor's formula, for all $\tau \in U$ there exist $\tau', \tau'' \in U$ such that

$$|Hf(g, \tau) - Hf(g_n, \tau)| = \left| \frac{1}{2} \sum_{i,j=1}^{d} x_i(\tau) x_j(\tau) [X_i X_j f(g\tau') - X_i X_j f(g_n \tau'')] \right|$$

Let us denote $M'' := \max_{i,j=1,\ldots,d} \sup_{g \in G} |X_i X_j f(g)|$. Then from the previous equation and by the Cauchy-Schwarz inequality, we get

$$|Hf(g, \tau) - Hf(g_n, \tau)| \leq M'' d \sum_{i=1}^{d} x_i^2(\tau)$$

The right hand side is integrable over U^* with respect to the measure $\mu(g, \cdot)$, therefore by dominated convergence $\int_{U^*} [Hf(g, \tau) - Hf(g_n, \tau)]\, \mu(g, d\tau)$ goes to 0 as $g_n \to g$. For the last integral in (6.18), similarly as before, we use Taylor's formula and the

Cauchy-Schwarz inequality which gives

$$|Hf(g_n,\tau)| \leq \frac{d}{2}M'' \sum_{i=1}^{d} x_i^2(\tau), \qquad \text{for all } g_n \in G, \tau \in U^*. \tag{6.19}$$

From the assumption (ii) it then follows that the sequence $\int_{U^*} Hf(g_n,\tau)[\mu(g_n,\tau) - \mu(g,\tau)]$ converges to 0 as $g_n \to g$. Hence the right hand side in (6.18) converges to zero. We then conclude that Af is a continuous function on G. □

Theorem 6.1.14. *Let A be a linear operator on $C_c^\infty(G)$ satisfying the positive maximum principle such that*

iii)

$$\lim_{\sigma\to\infty} \int_{U^*} \left(\sum_{i=1}^{d} x_i^2(\tau) \right) \mu(\sigma, d\tau) = 0$$

and

$$\lim_{\sigma\to\infty} \mu(\sigma, U^C) = 0,$$

Then Af vanishes at infinity for all $f \in C_c^\infty(G)$.

Proof. Note that the differential part of Af in (6.13) has compact support for all $f \in C_c^\infty(G)$, so when $\sigma \to \infty$ the differential part of $Af(\sigma)$ will vanish for any $f \in C_c^\infty(G)$. As for the integral part in (6.13), we will split it into two again

$$\int_{G^*} Hf(\sigma,\tau)\mu(\sigma,d\tau) = \int_{U^*} Hf(\sigma,\tau)\mu(\sigma,d\tau) + \int_{U^C} Hf(\sigma,\tau)\mu(\sigma,d\tau), \qquad \text{for all } \sigma \in G.$$

Thus, using the inequality (6.19), for all $\sigma \in G$

$$\begin{aligned} \left| \int_{U^*} Hf(\sigma,\tau)\mu(\sigma,d\tau) \right| &\leq \int_{U^*} |Hf(\sigma,\tau)|\, \mu(\sigma,d\tau) \\ &\leq \frac{d}{2}M'' \int_{U^*} \left(\sum_{i=1}^{d} x_i^2(\tau) \right) \mu(\sigma,d\tau). \end{aligned}$$

Therefore, using the condition iii), given any $\varepsilon > 0$, there is a compact set W_1 such that for all $\sigma \in W_1^C$,

$$\left| \int_{U^*} Hf(\sigma,\tau)\mu(\sigma,d\tau) \right| < \varepsilon/2.$$

For the second integral,

$$\left| \int_{U^C} Hf(\sigma,\tau)\mu(\sigma,d\tau) \right| \leq CM \cdot \mu(\sigma, U^C),$$

where C and M are as defined in (6.17). Using condition iii), there is a compact set W_2 such that for all $\sigma \in W_2^C$,

$$\left| \int_{U^C} Hf(\sigma,\tau)\mu(\sigma,d\tau) \right| < \varepsilon/2$$

Since $W_1 \cup W_2$ is compact, by summation we get

$$\left| \int_{G^*} Hf(\sigma,\tau)\mu(\sigma,d\tau) \right| < \varepsilon \qquad \text{for all } \sigma \in (W_1 \cup W_2)^C.$$

We have proved that the integral part of Af also vanishes at infinity. □

Corollary 6.1.15. *Let A be a linear operator on $C_c^\infty(G)$ satisfying the positive maximum principle such that conditions i), ii) from Theorem 6.1.13 and condition iii) from Theorem 6.1.14 are also satisfied. Then A maps $C_c^\infty(G)$ to $C_0(G)$.*

Proof. This follows directly from Theorem 6.1.13 and Theorem 6.1.14. □

6.2 The positive maximum principle and killed Hunt's formula

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space where the σ-algebra $\mathcal{F}$ is equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$. We will consider a Markov process $(Y_t)_{t\geq 0}$ with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$, taking values in G. We will denote the transition probabilities of $(Y_t)_{t\geq 0}$ by $p_t(\sigma, A) := P(Y_t \in A | Y_0 = \sigma)$ for all $A \in \mathcal{B}(G)$ and $\sigma \in G$. We then have a one-parameter contraction semigroup of operators on $B_b(G)$ given for all $f \in B_b(G)$, $\sigma \in G$ by

$$T_t f(\sigma) = \int_G f(\tau) p_t(\sigma, d\tau). \tag{6.20}$$

The family of operators $(T_t)_{t\geq 0}$ is called a *Feller semigroup* if

i) $T_t(C_0(G)) \subseteq C_0(G)$ for all $t \geq 0$,

ii) $\lim_{t\to 0} \|T_t f - f\|_\infty = 0$ for all $f \in C_0(G)$.

In this case we say that $(Y_t)_{t\geq 0}$ is a Feller process. We will denote by A the infinitesimal generator of the Feller semigroup $(T_t)_{t\geq 0}$. The following lemma is well-known, see [34], p.332. We will include the proof for completeness.

Lemma 6.2.1. *Let $(T_t)_{t\geq 0}$ be a Feller semigroup with generator A such that $C_c^\infty(G) \subseteq Dom(A)$, then A satisfies the positive maximum principle.*

Proof. Let $f \in C_c^\infty(G)$ such that $f(g_0) = \sup_{\sigma \in G} f(\sigma) \geq 0$ for some $g_0 \in G$. We have

$$Af(g_0) = \lim_{t \to 0} \frac{(T_t f - f)(g_0)}{t} = \lim_{t \to 0} \frac{1}{t} \int_G [f(\tau) - f(g_0)]\, p_t(g_0, d\tau) \leq 0.$$

This proves that A satisfies the PMP. □

The generator A of a Feller process satisfies the PMP so by Theorem 6.1.12, A can be characterized by a quadruple $((a_{ij}), b, c, \mu)$ and we say that this quadruple is associated to A. In particular, we will explore the relationship between the transition probabilities of the Feller process and the Lévy kernel associated to its generator. This generalizes a well-known result about convolution semigroups in $\mathbb{R}^d$, see [53], Corollary 8.9, p.45, which has recently been extended to the current context, in the case where $G = \mathbb{R}^d$, in [42] Theorem 3.2.

Proposition 6.2.2. *Let $(p_t)_{t \geq 0}$ be the transition probabilities associated to a Feller process, with Feller semigroup $(T_t)_{t \geq 0}$ and generator A. Assume that $C_c^\infty(G) \subseteq Dom(A)$, then for all $g \in G$ and $f \in C_c^\infty(G)$ vanishing on a neighbourhood of g,*

$$\lim_{t \to 0} \frac{1}{t} \int_G f(\tau) p_t(g, d\tau) = \int_{G^*} f(g\tau) \mu(g, d\tau),$$

where μ is the Lévy kernel associated to A.

Proof. By definition, we have for all $f \in C_c^\infty(G)$, $g \in G$

$$Af(g) = \lim_{t \to 0} \frac{1}{t}(T_t f(g) - f(g)) = \lim_{t \to 0} \frac{1}{t} \int_G (f(\tau) - f(g)) p_t(g, d\tau) \tag{6.21}$$

Given that the generator A satisfies the PMP, by Lemma 6.1.3 for all $g \in G$ the distribution A_g also satisfies the PMP, where A_g is defined in (6.1). From (6.10) we have shown that for all $g \in G$ and $f \in C_c^\infty(G)$, $A_g f = \int_{G^*} f(\tau)\, \mu(g, d\tau)$ where $\mu(g, \cdot)$ is a Lévy measure. Thus, for all $g \in G$ and $f \in C_c^\infty(G)$ we have

$$Af(g) = A_g(L_g f) = \int_{G^*} f(g\tau)\, \mu(g, d\tau). \tag{6.22}$$

In particular when $f \in C_c^\infty(G)$ vanishes on a neighbourhood of $g \in G$, from (6.21) and (6.22) we get

$$\lim_{t \to 0} \frac{1}{t} \int_G f(\tau) p_t(g, d\tau) = \int_{G^*} f(g\tau)\, \mu(g, d\tau).$$

□

Definition. A family of probability measures $(\mu_t)_{t\geq 0}$ on $(G, \mathcal{B}(G))$ is called a *convolution semigroup* of probability measures if

i) $\mu_{s+t} = \mu_s * \mu_t$, for all $s, t \geq 0$
ii) $\mu_0 = \delta_e$,
iii) $\lim_{t\to 0} \mu_t = \mu_0$. (weak convergence)

Let $(\mu_t)_{t\geq 0}$ be a convolution semigroup of probability measures on G. It then follows that μ_t is infinitely divisible for all $t \geq 0$, see [6] p.123.. These semigroup of measures arise as the laws of Lévy processes on G, see p.10 in [45]. A Lévy process is a Feller process and the associated $C_0(G)$-semigroup, $(T_t)_{t\geq 0}$ defined on $C_0(G)$, is called a *Hunt semigroup* and is defined by

$$T_t f(g) = \int_G f(g\tau)\mu_t(d\tau), \qquad \text{for all } t \geq 0, f \in C_0(G) \text{ and } g \in G. \tag{6.23}$$

If we compare this to (6.20), we see that

$$p_t(\sigma, B) = \mu_t(\sigma^{-1}B)$$

for all $\sigma \in G$ and $B \in \mathcal{B}(G)$. The infinitesimal generator A of a Hunt semigroup is called the *Hunt generator* . From the definition of the Hunt semigroup it follows that $L_\sigma T_t = T_t L_\sigma$ for all $\sigma \in G$ and $t \geq 0$. Furthermore, we know that if $f \in \mathrm{Dom}(A)$ then $L_g f \in \mathrm{Dom}(A)$ for all $g \in G$. Therefore, $L_\sigma A f = A L_\sigma f$, for all $f \in \mathrm{Dom}(A)$, see Lemma 5.3.2 in [6] and [32]. It is well known that $C_c^\infty(G) \subseteq \mathrm{Dom}(A)$, see [47]. Thus,

$$L_\sigma A f = A L_\sigma f, \qquad \text{for all } f \in C_c^\infty(G).$$

The following proposition is the classical result that the Hunt semigroup is precisely the left-invariant Feller semigroup.

Proposition 6.2.3. *Let $(T_t)_{t\geq 0}$ be a Feller semigroup in $C_0(G)$ such that $L_g T_t = T_t L_g$ for all $t \geq 0$, $g \in G$ and $p_0(e, \cdot) = \delta_e(\cdot)$. Then $(T_t)_{t\geq 0}$ is the Hunt semigroup associated with a convolution semigroup of measures.*

Proof. We follow the same steps as in the Euclidean case, see [3], Theorem 3.3.1, p.161-2. Let $(T_t)_{t\geq 0}$ be the Feller semigroup of a Feller process Y with transition probabilities $(p_t)_{t\geq 0}$. By definition (6.20), we have for all $f \in C_0(G)$, $g, \sigma \in G$,$t \geq 0$,

$$L_g(T_t f)(\sigma) = \int_G f(\tau) p_t(g\sigma, d\tau)$$
$$T_t(L_g f)(\sigma) = \int_G f(g\tau)\, p_t(\sigma, d\tau) = \int_G f(g\tau)\, p_t(\sigma, g^{-1}d\tau).$$

The semigroup $(T_t)_{t\geq 0}$ is invariant w.r.t. to left translation so, by the Riesz representation theorem we get

$$p_t(g\sigma, B) = p_t(\sigma, g^{-1}B), \qquad \text{for all } g, \sigma \in G, B \in \mathcal{B}(G), t \geq 0.$$

For all $t \geq 0$, define $\mu_t := p_t(e, \cdot)$, so we have $\mu_0 = p_0(e, \cdot) = \delta_e(\cdot)$. Furthermore, for all $B \in \mathcal{B}(G)$, $t \geq 0$ and $g \in G$,

$$p_t(g, B) = p_t(e, g^{-1}B) = \mu_t(g^{-1}B).$$

The using the Chapman-Kolmogorov equations for all $t, s \geq 0$, $B \in \mathcal{B}(G)$

$$\mu_{s+t}(B) = p_{s+t}(e, B) = \int_G p_t(\tau, B)p_s(e, d\tau) = \int_G \mu_t(\tau^{-1}B)\mu_s(d\tau),$$

so $(\mu_t)_{t\geq 0}$ is a convolution semigroup of measures. Vague continuity follows from the fact that $(T_t)_{t\geq 0}$ is a Feller semigroup. From [26], p.25 Theorem 1.1.19 we know that in the case of a family of probability measures on locally compact spaces, vague continuity is equivalent to weak continuity. □

We will now recall the well-known Hunt's theorem [32], for this first we define the space

$$C_0^{(2)}(G) := \{f \in C_0(G); X_i f \in C_0(G) \text{ and } X_j X_k f \in C_0(G) \text{ for all } 1 \geq i, j, k \geq d\}.$$

Then, we have $C_c^\infty(G) \subseteq C_0^{(2)}(G)$ and $C_0^{(2)}(G)$ is dense in $C_0(G)$.

Theorem 6.2.4 (Hunt's Theorem). *If A is the generator of a semigroup of operators associated to a convolution semigroup of measures $(\mu_t)_{t\geq 0}$, then*

1. $C_0^{(2)}(G) \subseteq Dom(A)$
2. *for all $f \in C_0^{(2)}(G)$, $g \in G$,*

$$\begin{aligned} Af(g) =& \frac{1}{2}\sum_{i,j=1}^{d} a_{ij} X_i X_j f(g) + \sum_{i=1}^{d} b_i X_i f(g) \\ &+ \int_G \left(f(g\tau) - f(g) - \sum_{i=1}^{d} x_i(\tau) X_i f(g) \right) \mu(d\tau) \end{aligned} \tag{6.24}$$

where (a_{ij}) is a non-negative definite, symmetric matrix, $(b_1, \ldots, b_d) \in \mathbb{R}^d$ and μ is a Lévy measure on $(G, \mathcal{B}(G))$. Conversely, any linear operator represented as (6.24), is the restriction to $C_0^2(G)$ of the Hunt generator corresponding to a unique convolution semigroup of probability measures.

Proof. See [32], [26] Theorem 4.2.4 p.262 or [6] Theorem 5.3.3, p.132. □

Definition. Let $(\mu_t)_{t\geq 0}$ be a convolution semigroup of probability measures with Hunt semigroup $(T_t)_{t\geq 0}$. For a fix $c > 0$, we will consider a family of measures $(\widetilde{\mu}_t)_{t\geq 0}$ on $(G, \mathcal{B}(G))$ given by

$$\widetilde{\mu}_t = e^{-ct}\mu_t, \quad \text{for all } t \geq 0.$$

Thus, we have $\widetilde{\mu}_t(G) = e^{-ct} < 1$ for all $t \geq 0$. Then $(\widetilde{\mu}_t)_{t\geq 0}$ is a *convolution semigroup of sub-probability measures*, that is, it is a family of measures with total mass not exceeding 1 and satisfying the conditions i)-iii), that we listed above for convolution semigroups of measures. Furthermore, we will define a family of operators $(S_t)_{t\geq 0}$ on $C_0(G)$ by

$$S_t = e^{-ct}T_t \quad \text{for all } t \geq 0,$$

then $(S_t)_{t\geq 0}$ is a C_0-contraction semigroup and we have for all $f \in C_0(G)$, $g \in G$, $t \geq 0$,

$$S_t f(g) = e^{-ct}T_t f(g) = \int_G f(g\tau)e^{-ct}\mu_t(d\tau) = \int_G f(g\tau)\widetilde{\mu}_t(d\tau).$$

Let us denote the infinitesimal generator of $(S_t)_{t\geq 0}$ by $\widetilde{A}$, then for all $f \in C_0^2(G)$

$$\widetilde{A}f = \frac{d}{dt}S_t f\bigg|_{t=0} = \frac{d}{dt}(e^{-ct}T_t)f\bigg|_{t=0} = -cf + Af$$

$(S_t)_{t\geq 0}$ cannot be considered as a Feller semigroup in the classical sense, but is can be associated to a so called *killed Lévy process* on G, see [3] p.405. We will call the operator $\widetilde{A}$, a *killed Hunt generator*.

Following the proof of Lemma 6.2.1 it is easy to see that $\widetilde{A}$ satisfies the positive maximum principle. Conversely we have the following result.

Theorem 6.2.5. *If $A : C_c^\infty(G) \to C(G)$ satisfies the positive maximum principle and is such that $L_g Af = AL_g f$ for all $f \in C_c^\infty(G), g \in G$ then*

$$\begin{aligned} Af(g) =& \frac{1}{2}\sum_{i,j=1}^d a_{ij}X_iX_j f(g) + \sum_{i=1}^d b_i X_i f(g) - cf(g) \\ &+ \int_G \left(f(g\tau) - f(g) - \sum_{i=1}^d x_i(\tau)X_i f(g) \right) \mu(d\tau), \end{aligned} \tag{6.25}$$

where $\{a_{ij}\}$ is a non-negative definite, symmetric matrix, $(b_1, \ldots, b_d) \in \mathbb{R}^d, c \geq 0$ and μ is a Lévy measure on $(G, \mathcal{B}(G))$. Furthermore A extends to the killed Hunt generator associated to a unique convolution semigroup of sub–probability measures on G.

Proof. The operator A satisfies the PMP so by Theorem 6.1.12 it is of the from (6.13). Furthermore, A is invariant under left translation on $C_c^\infty(G)$. Thus,

$$\begin{aligned} Af(g) &= AL_g f(e) = L_g Af(e) \\ &= \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(e)X_iX_jf(g) + \sum_{i=1}^{d} b_i(e)X_if(g) - c(e)f(g) \\ &+ \int_{G^*}\left[f(g\tau) - f(g) - \sum_{i=1}^{d} x_i(\tau)X_if(g)\right]\mu(e,d\tau). \end{aligned}$$

We then define $a_{ij} = a_{ij}(e)$, $b_i = b_i(e)$, $c = c(e)$ for all $i,j = 1,\dots,d$, and $\mu(\cdot) = \mu(e,\cdot)$. Next write (6.25) as $Af = -cf + Bf$. It is clear that A and B may be extended to linear operators A_1 and B_1 on $C_0^{(2)}(G)$, so that $A_1 f = -cf + B_1 f$ for all $f \in C_0^{(2)}(G)$. Then by Hunt's theorem, B_1 extends to the Hunt generator associated to a unique convolution semigroup $(\mu_t)_{t\geq 0}$, and then the required convolution semigroup of sub–probability measures is given, as above, by defining $\widetilde{\mu}_t = e^{-ct}\mu_t$ for each $t \geq 0$. Moreover by Theorem 5.3.4 on p.137 of [3], $C_c^\infty(G)$ is a core for the Hunt generator, and so for the killed Hunt generator, from which we see that the action of A on $C_c^\infty(G)$ uniquely determines $(\widetilde{\mu}_t, t \geq 0)$. □

6.3 The positive maximum principle and pseudo-differential operators

Pseudo-differential operators can be defined on manifolds using local coordinates. On compact Lie groups, there is a global approach developed in [51], see also [52] and [18]. Later [5] provided an extended definition. Here we give a "working definition" in the spirit of [51].

In this section we will assume that the conditions from Corollary 6.1.15 are satisfied, so that A maps $C_c^\infty(G)$ to $C_0(G)$. Any linear operator A on $C_0(G)$ that satisfies the PMP is dissipative, see [16] Lemma 2.1, p.165. Furthermore, any dissipative linear operator on a Banach space is closeable, see [16] Lemma 2.2, p.16., therefore A has a closed extension $\overline{A}$ with $C_c^\infty(G) \subseteq \mathrm{Dom}(\overline{A})$. We will also assume that G is a compact Lie group and we will rely on the Peter-Weyl theorem in this section.

Let us denote $\mathcal{R}(\widehat{G}) = \bigcup_{\pi\in\widehat{G}} \mathcal{L}(V_\pi)$ and for all $\pi \in \widehat{G}$, I_π denotes the identity matrix acting in V. If $\lambda \in D$ we will equivalenty denote $I_\lambda = I_{\pi_\lambda}$.

Definition. A linear operator $A : C^\infty(G) \to C(G)$ is called a *pseudo-differential operator* if there exists a mapping $\sigma_A : G \times \widehat{G} \to \mathcal{R}(\widehat{G})$, such that $\sigma_A(g,\pi) \in \mathcal{L}(V_\pi)$

for all $g \in G, \pi \in \widehat{G}$ and

$$\begin{aligned} \sigma_A(g,\pi) &= \pi(g^{-1})A\pi(g), \qquad \text{for all } g \in G, \pi \in \widehat{G}, \\ Af(g) &= \sum_{\pi \in \widehat{G}} d_\pi \mathrm{tr}\left(\sigma_A(g,\pi)\widehat{f}(\pi)\pi(g)\right), \qquad \text{for all } f \in C^\infty(G). \end{aligned} \tag{6.26}$$

In this case, σ_A is called the *symbol* of the operator A.

In particular, let $A : C^\infty(G) \to C_0(G)$ be a linear operator satisfying the positive maximum principle, then A is of the form (6.13). We want to prove that such operator A is a pseudo-differential operator and calculate its symbol. First note that by definition of the derived representation we have

$$X\pi(g) = \frac{d}{dt}\pi(g\exp(tX)) = \pi(g)\frac{d}{dt}\pi(\exp(tX)) = \pi(g)d\pi(X), \tag{6.27}$$

for all $\pi \in \widehat{G}, X \in \mathfrak{g}$ and $g \in G$.

Let us define the matrix $A\pi := [A\pi_{kl}]_{k,l=1\ldots,d_\pi}$, then by applying the formula (6.13) for each function π_{ij}, $i,j = 1,\ldots,d_\pi$ we get for all $g \in G$,

$$\begin{aligned} A\pi(g) =& \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(g)\pi(g)d\pi(X_i)d\pi(X_j) - c(g)\pi(g) + \sum_{i=1}^{d} b_i(g)\pi(g)d\pi(X_i) \\ &+ \int_{G^*} \pi(g)\left\{\pi(\tau) - I_\pi - \sum_{i=1}^{d} x_i(\tau)d\pi(X_i)\right\}\mu(g,d\tau) \end{aligned}$$

Let us simplify the notation by introducing a function $J_A : G\times\widehat{G} \to \mathcal{M}_{d_\pi\times d_\pi} = \mathcal{L}(V_\pi)$, which takes the form

$$\begin{aligned} J_A(g,\pi) :=& \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(g)d\pi(X_i)d\pi(X_j) + \sum_{i=1}^{d} b_i(g)d\pi(X_i) - c(g)I_\pi \\ &+ \int_{G^*} \left\{\pi(\tau) - I_\pi - \sum_{i=1}^{d} x_i(\tau)d\pi(X_i)\right\}\mu(g,d\tau), \end{aligned} \tag{6.28}$$

for all $\pi \in \widehat{G}$ and $g \in G$. Then we have

$$A\pi(g) = \pi(g)J_A(g,\pi), \qquad \text{for all } g \in G, \pi \in \widehat{G}. \tag{6.29}$$

For simplicity for all $\lambda \in D$ we will denote $J_A(g,\lambda) := J_A(g,\pi_\lambda)$.

Lemma 6.3.1. *There is a constant $C \geq 0$ such that for all $\lambda \in D$,*

$$\sup_{g\in G} \|J_A(g,\lambda)\|_{HS} \leq C(1+|\lambda|^{m+2}).$$

Proof. Let us start by establishing some preliminary steps, that are necessary for the proof. First, by definition of derived representations we have for all $\pi \in \widehat{G}$ and $\tau \in U$,

$$\pi(\tau) = \pi\left(\exp\left(\sum_{i=1}^{d} x_i(\tau)X_i\right)\right) = \exp\left(d\pi\left(\sum_{i=1}^{d} x_i(\tau)X_i\right)\right)$$
$$= \exp\left(\sum_{i=1}^{d} x_i(\tau)d\pi(X_i)\right) \tag{6.30}$$

We will also use the fact that by Theorem 4.1.2, for all $\lambda \in D$,

$$\|I_\lambda\|_{HS} = \sqrt{d_\lambda} \leq C|\lambda|^{m/2},$$

for some constant $C \geq 0$. In the following, C will denote a generic constant, which may vary from line to line.

Let us return to the proof, we will start with the integral part. We first split the integral in (6.28) into $\int_{G^*} = \int_{U^*} + \int_{U^C}$. By Taylor's series expansion, Theorem 4.1.2 and the Cauchy-Schwarz inequality, we get for all $\pi \in \widehat{G}$,

$$\left\|\int_{U^*}\left(\pi(\tau) - I_\pi - \sum_{i=1}^{d} x_i(\tau)d\pi(X_i)\right)\nu(g,d\tau)\right\|_{HS}$$
$$\leq \int_{U^*}\left\|\exp\left(\sum_{i=1}^{d} x_i(\tau)d\pi(X_i)\right) - I_\pi - \sum_{i=1}^{d} x_i(\tau)d\pi(X_i)\right\|_{HS}\nu(g,d\tau)$$
$$\leq \int_{U^*}\frac{1}{2}\sum_{i=1}^{d}|x_i(\tau)x_j(\tau)|\ \|d\pi(X_i)d\pi(X_j)\|_{HS}\,\nu(g,d\tau)$$
$$\leq Cd\max\{|X_1|,\ldots,|X_d|\}\left(\int_{U^*}\sum_{i=1}^{d} x_i(\tau)^2\nu(g,d\tau)\right)|\lambda|^{m+2}$$

So taking the supremum over G on both sides we get for all $\lambda \in D$

$$\sup_{g\in G}\left\|\int_{U^*}\left(\pi_\lambda(\tau) - I_\lambda - \sum_{i=1}^{d} x_i(\tau)d\pi_\lambda(X_i)\right)\nu(g,d\tau)\right\|_{HS} \leq C|\lambda|^{m+2}, \tag{6.31}$$

for some constant $C \geq 0$. For the second part of the integral, by Theorem 4.1.2 and

Proposition 4.1.1 we have for all $g \in G$,

$$\begin{aligned}
&\left\| \int_{U^C} \left(\pi_\lambda(\tau) - I_\lambda - \sum_{i=1}^{d} x_i(\tau) d\pi_\lambda(X_i) \right) \nu(g, d\tau) \right\|_{HS} \\
&\leq \int_{U^C} \left(\|\pi_\lambda(\tau)\|_{HS} + \|I_\lambda\|_{HS} + \sum_{i=1}^{d} |x_i(\tau)| \|d\pi_\lambda(X_i)\|_{HS} \right) \nu(g, d\tau) \\
&= (2d_\lambda^{1/2} + C \max_{i=1,\dots,N} \{\|d\pi_\lambda(X_i)\|_{HS}\})\ \nu(g, U^C) \\
&\leq C(|\lambda|^{\frac{m}{2}} + |\lambda|^{\frac{m+2}{2}})\ \nu(g, U^C).
\end{aligned}$$

So taking the supremum over G on both sides, we get

$$\sup_{g \in G} \left\| \int_{U^C} \left(\pi_\lambda(\tau) - I_\lambda - \sum_{i=1}^{d} x_i(\tau) d\pi_\lambda(X_i) \right) \nu(g, d\tau) \right\|_{HS} \leq C(|\lambda|^{\frac{m}{2}} + |\lambda|^{\frac{m+2}{2}}), \quad (6.32)$$

for some constant $C \geq 0$. Thus, going back to the definition of J_A in (6.28), by the inequalities (6.31) and (6.32) we get

$$\begin{aligned}
\sup_{g \in G} \|J(g, \lambda)\|_{HS} \leq & C|\lambda|^{m+2} \sup_{g \in G} \sum_{i,j=1}^{d} |a_{ij}(g)| + C|\lambda|^{\frac{m+2}{2}} \sup_{g \in G} \sum_{i=1}^{d} |b_i(g)| \\
& + C|\lambda|^{\frac{m}{2}} \sup_{g \in G} |c(g)| + C|\lambda|^{m+2} + C(|\lambda|^{\frac{m}{2}} + |\lambda|^{\frac{m+2}{2}}) \\
\leq & C(|\lambda|^{\frac{m}{2}} + |\lambda|^{\frac{m+2}{2}} + |\lambda|^{m+2})
\end{aligned}$$

This means that, when $|\lambda| < 1$, we have $\sup_{g \in G} \|J(g, \lambda)\|_{HS} \leq C$ and when $|\lambda| \geq 1$, we have $\sup_{g \in G} \|J(g, \lambda)\|_{HS} \leq C|\lambda|^{m+2}$. Thus,

$$\sup_{g \in G} \|J(g, \lambda)\|_{HS} \leq C(1 + |\lambda|^{m+2}), \qquad \text{for all } \lambda \in D$$

for some constant $C \geq 0$. □

Lemma 6.3.2. *For all $f \in C^\infty(G)$, the series*

$$\sum_{\lambda \in D} d_\lambda tr \left(J_A(g, \pi_\lambda) \widehat{f}(\lambda) \pi_\lambda(g) \right) \quad (6.33)$$

converges absolutely and uniformly for all $g \in G$.

Proof. First note that for all $\lambda \in D$ and $g \in G$

$$\begin{aligned} \|\widehat{f}(\lambda)\pi_\lambda(g)\|_{HS} &= \operatorname{tr}\left(\pi_\lambda(g)^*\widehat{f}(\lambda)^*\widehat{f}(\lambda)\pi_\lambda(g)\right) \\ &= \operatorname{tr}\left(\widehat{f}(\lambda)^*\widehat{f}(\lambda)\pi_\lambda(g)\pi_\lambda(g)^*\right) \\ &= \operatorname{tr}\left(\widehat{f}(\lambda)^*\widehat{f}(\lambda)\right) \\ &= \|\widehat{f}(\lambda)\|_{HS} \end{aligned} \tag{6.34}$$

Then, we use the fact that $\operatorname{tr}(AB) \leq \|A\|_{HS}\|B\|_{HS}$, so by (6.34), Lemma 6.3.1 and Proposition 4.1.1, we get for all $g \in G$ and $\lambda \in D$

$$\begin{aligned} d_\lambda \left|\operatorname{tr}\left(J_A(g,\lambda)\widehat{f}(\lambda)\pi_\lambda(g)\right)\right| &\leq d_\lambda \|J_A(g,\lambda)\|_{HS}\|\widehat{f}(\lambda)\pi_\lambda(g)\|_{HS} \\ &\leq C(1+|\lambda|^{m+2})|\lambda|^m\|\widehat{f}(\lambda)\|_{HS}, \end{aligned} \tag{6.35}$$

for some constant $C \geq 0$.

Now, recall from Theorem 4.3.8 that $\widehat{f} \in S(D)$ for all $f \in C_c^\infty(G)$, so

$$\lim_{|\lambda|\to\infty} |\lambda|^p \; \|\widehat{f}(\lambda)\|_{HS} = 0, \qquad \text{for all } p \in \mathbb{N}.$$

Hence, for $\varepsilon = 1$, there exists $\lambda_0 \in D_0$ such that for all $|\lambda| > |\lambda_0|$, we have $\|\widehat{f}(\lambda)\|_{HS} \leq \frac{1}{|\lambda|^p}$. So when we look at the tail behaviour of the series (6.33).

$$\sup_{g\in G} \sum_{|\lambda|>|\lambda_0|} d_\lambda \left|\operatorname{tr}\left(J_A(g,\lambda)\widehat{f}(\lambda)\pi_\lambda(g)\right)\right| \leq C \sum_{|\lambda|>|\lambda_0|} \frac{|\lambda|^m(1+|\lambda|^{m+2})}{|\lambda|^p} < \infty$$

Choose $p > 2m+2+r$, then the convergence of the Sugiura zeta function in Theorem 4.1.3 allows us to conclude that the right hand side is finite. Thus, the absolute and uniform convergence of (6.33) follows. □

Theorem 6.3.3. *Let $A : C^\infty(G) \to C(G)$ be a linear operator satisfying the positive maximum principle, then A is a pseudo differential operator with symbol J_A.*

Proof. From (6.29) it follows that when $f \in \mathcal{E}(G)$, we have

$$Af(g) = \sum_{\lambda \in D} d_\lambda \operatorname{tr}(J_A(g,\lambda)\widehat{f}(\lambda)\pi_\lambda(g)), \qquad \text{for all } g \in G \tag{6.36}$$

Now, take any $f \in C^\infty(G)$, then by Theorem 4.1.4 the Fourier series $\sum_{\lambda\in D} d_\lambda \operatorname{tr}(\widehat{f}(\lambda)\pi_\lambda(g))$ converges absolutely and uniformly to $f(g)$ for all $g \in G$. We will impose a norm ordering of the space of weights D, wherein if two weights have the same weight we choose an arbitrary ordering between them. Then the partial sums $f_N := \sum_{i=1}^{N} d_{\lambda_i} \mathrm{tr}(\widehat{f}(\lambda_i)\pi_{\lambda_i}(\cdot))$ for all $N \in \mathbb{N}$, are in $\mathcal{E}(G)$ and the sequence $(f_N)_{N\in\mathbb{N}}$ converges uniformly to f.

For all $\lambda \in D$ and $i, j = 1, \ldots, d_\lambda$, we have

$$\begin{aligned}
\widehat{f_N}(\lambda)_{ij} &= \int_G (\pi_\lambda)_{ij}(g^{-1}) f_N(g) dg \\
&= \int_G (\pi_\lambda)_{ij}(g^{-1}) \sum_{p=1}^{N} d_{\lambda_p} \mathrm{tr}\left(\widehat{f}(\lambda_p)\pi_{\lambda_p}(g)\right) dg \\
&= \int_G (\pi_\lambda)_{ij}(g^{-1}) \sum_{p=1}^{N} d_{\lambda_p} \sum_{k,l=1}^{d_{\lambda_p}} \widehat{f}(\lambda_p)_{kl}\ (\pi_\lambda)_{lk}(g)\ dg \\
&= \sum_{p=1}^{N} \sum_{k,l=1}^{d_{\lambda_p}} \widehat{f}(\lambda_p)_{kl}\ d_{\lambda_p} \int_G (\pi_\lambda)_{ij}(g^{-1})\ (\pi_{\lambda_p})_{lk}(g)\ dg \\
&= \begin{cases} \widehat{f}(\pi_\lambda)_{ij} & \lambda \in \lambda_1, \ldots, \lambda_N \\ 0 & \lambda \notin \lambda_1, \ldots, \lambda_N \end{cases}
\end{aligned}$$

The last line follows from the fact that $\{\sqrt{d_\lambda}(\pi_\lambda)_{ij} : \lambda \in D, i, j = 1, \ldots, d_\lambda\}$ is an orthonormal basis of $L^2(G)$. Thus, we have

$$\begin{aligned}
Af_N(g) &= \sum_{\lambda\in D} d_\lambda \mathrm{tr}\left(J_A(g, \pi_\lambda)\widehat{f_N}(\lambda)\pi_\lambda(g)\right) \\
&= \sum_{i=1}^{N} d_{\lambda_i} \mathrm{tr}\left(J_A(g, \pi_{\lambda_i})\widehat{f}(\lambda_i)\pi_{\lambda_i}(g)\right)
\end{aligned}$$

By Lemma 6.3.2, the sequence of partial sums Af_N converges uniformly. Furthermore, $\overline{A}$ is closed, therefore

$$Af(g) = \lim_{N\to\infty} Af_N(g) = \sum_{\lambda\in D} d_\lambda \mathrm{tr}(J_A(g, \pi_\lambda)\widehat{f}(\lambda)\pi_\lambda(g)).$$

□

Chapter 7

The positive maximum principle on symmetric spaces

7.1 K-bi-invariant linear operators

7.1.1 Adjoint representation on G/K

Let G be a Lie group and $K \subset G$ be a closed subgroup, we denote by M the homogeneous space $M = G/K$. We denote by $\natural$ the natural map that maps G to G/K by $\natural(g) = gK$ for all $g \in G$. Furthermore, for any $h \in G$, we denote by σ_h the action of left-translation on G/K, i.e. $\sigma_h(gK) := hgK$ for all $h \in G$ and $gK \in X$. Then, we have for all $h \in G$,

$$\natural \circ l_h = \sigma_h \circ \natural.$$

If $x = gK \in G/K$, we often write $\sigma_h(gK)$ as $h.x$ for all $h \in G$.

Definition. Let $\pi : G \to GL(V)$ be a representation of a Lie group G on a vector space V, and let $W \subset V$ a G-invariant subspace of V. Then there is an induced representation of the group G on the quotient space V/W, $\widetilde{\pi} : G \to GL(V/W)$, called the *quotient representation*, that is given by $\widetilde{\pi}(g)(vW) = \pi(g)(v)W$ for all $g \in G$ and $vW \in V/W$.

Given σ_k we can define an adjoint representation of the homogeneous space G/K the following way

Definition. The homomorphism $\mathrm{Ad}^{G/K} : K \to GL(T_oG/K)$ defined by

$$\mathrm{Ad}^{G/K}(k)(\widetilde{X}) = T_o\sigma_k(\widetilde{X}), \qquad \text{for all } \widetilde{X} \in T_o(G/K), k \in K.$$

is called the *isotropy representation* of the homogeneous space G/K.

Remark 7.1.1. The isotropy representation is a representation of the Lie subgroup K on $T_o(G/K)$. Indeed, we have $\mathrm{Ad}^{G/K}(e) = T_o\sigma_e = T_o(\mathrm{Id}_{G/K}) = \mathrm{Id}_{T_o(G/K)}$ and for all $k_1, k_2 \in K$, by the chain rule we have

$$\mathrm{Ad}^{G/K}(k_1)\mathrm{Ad}^{G/K}(k_2) = T_o\sigma_{k_1} \circ T_o\sigma_{k_2} = T_o(\sigma_{k_1}\sigma_{k_2}) = T_o(\sigma_{k_1k_2}) = \mathrm{Ad}^{G/K}(k_1k_2)$$

Suppose the subgroup $K \subset G$ is compact, then there exists an $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{g}$, i.e. an inner product $\langle \cdot, \cdot \rangle$ for which $\mathrm{Ad}(k)$ acts isometrically on $\mathfrak{g}$ for each $k \in K$. To see this, take any inner product $\langle \cdot, \cdot \rangle'$ on $\mathfrak{g}$ and define $\langle X, Y \rangle := \int_K \langle \mathrm{Ad}(k)X, \mathrm{Ad}(k)Y \rangle' dk$. We will denote by $\mathfrak{k}$ the Lie algebra of K, and by $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the $\mathrm{Ad}(K)$-invariant inner product, i.e. $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, and this is called the Cartan decomposition. Then $\mathfrak{p}$ is $\mathrm{Ad}(K)$-invariant, that is $\mathrm{Ad}(k)\mathfrak{p} \subseteq \mathfrak{p}$, for all $k \in K$. For the fixed basis $\{X_1, X_2, \ldots, X_d\}$ of $\mathfrak{g}$, we will assume that $\{X_1, X_2, \ldots, X_n\}$ is a basis for $\mathfrak{p}$ and $\{X_{n+1}, X_{n+2}, \ldots, X_d\}$ is a basis for $\mathfrak{k}$.

Lemma 7.1.2. *The isotropy representation can be identified with the quotient representation of the adjoint representation of the Lie subgroup K on the Lie algebra $\mathfrak{g}$.*

Proof. This is a known result mentioned in [23], Lemma 3.1, p.132. Here we will include our own proof. Since $T_eK \subset T_eG$ is $\mathrm{Ad}(K)$-invariant, we have the quotient representation $\widetilde{\mathrm{Ad}} : K \to GL(T_eG/T_eK)$ of the adjoint representation of K on T_eG given by

$$\widetilde{\mathrm{Ad}}(k)(Y + T_eK) = \mathrm{Ad}(k)(Y) + T_eK$$

for all $Y + T_eK \in T_eG/T_eK$ and $k \in K$. Also, we know that there is an isomorphism $T_o(G/K) \simeq T_eG/T_eK$, see Theorem 2.11 in [37]. Note that for all $k \in K$ and $g \in G$,

$$\sigma_k(gK) = kgK = (kgk^{-1})K = c_k(g)K$$

That is $\sigma_k \circ \natural = \natural \circ c_k$ on G, so $T_e(\sigma_k \circ \natural) = T_e(\natural \circ c_k)$ on T_eG. So by the chain rule we have $T_o\sigma_k \circ T_e\natural = T_e\natural \circ \mathrm{Ad}(k)$, that is

$$\mathrm{Ad}^{G/K}(k) \circ T_e\natural = T_e\natural \circ \mathrm{Ad}(k)$$

So we have the following commuting diagram for any $k \in K$

$$\begin{array}{ccccc}
T_eG & \xrightarrow{T_e\natural} & T_o(G/K) & \xleftrightarrow{\;i\;} & T_eG/T_eK \\
{\scriptstyle \mathrm{Ad}(k)}\downarrow & & {\scriptstyle \mathrm{Ad}^{G/K}(k)}\downarrow & \swarrow {\scriptstyle \widetilde{\mathrm{Ad}}(k)} & \\
T_eG & \xrightarrow{T_e\natural} & T_o(G/K) & &
\end{array}$$

where $i = T_e\natural|_{\mathfrak{p}}$ is the canonical isomorphism between T_eG/T_eK and $T_o(G/K)$. Indeed, we have for all $Y \in T_o(G/K)$, $i(Y) = Y' + \mathfrak{k}$, where $Y' \in T_eG$.

$$\begin{aligned}\widetilde{\mathrm{Ad}}(k)i(Y) &= \mathrm{Ad}(k)Y' + \mathfrak{k} \\ &= T_e\natural \circ \mathrm{Ad}(k)Y' \\ &= \mathrm{Ad}^{G/K}(k) \circ T_e\natural(Y') \\ &= \mathrm{Ad}^{G/K}(k)(Y' + \mathfrak{k}) \\ &= \mathrm{Ad}^{G/K}(k)i(Y)\end{aligned}$$

□

Differentiating the natural map at the origin we get for all $X \in \mathfrak{g}$

$$T_e\natural(X) = \frac{d}{dt}[\natural \circ \exp(\cdot X)(t)]\bigg|_{t=0} = \frac{d}{dt}(\exp(tX)K)\bigg|_{t=0} \tag{7.1}$$

So in particular we get $T_e\natural(\mathfrak{k}) = 0$, that is $\mathrm{Ker}(T_e\natural) = \mathfrak{k}$. But $T_e\natural$ is also surjective, so we get a canonical isomorphism $\mathfrak{p} = \mathfrak{g}/\mathfrak{k} \simeq T_o(G/K)$. So for all $i > n$, if $f \in C_c^\infty(G/K)$ then we have $X_i f = 0$.

From now on, for simplicity both the isotropy representation and the quotient representation of the adjoint representaion of the Lie group K, will be denoted by Ad.

7.1.2 Canonical local coordinate functions on G/K

For the basis $X_1, X_2, \ldots, X_n$ of $\mathfrak{p}$, we are going to define a family of canonical local coordinate functions $y_i : \widetilde{U} \to \mathbb{R}$, where $\widetilde{U}$ is a neighbourhood of o in G/K. This section is based on [45], p.40.

The map Ψ from $\mathbb{R}^n$ to G/K defined by

$$\Psi : y = (y_1, y_2, \ldots, y_n) \mapsto \natural\left(e^{\sum_{i=1}^n y_i X_i}\right)$$

is a diffeomorphism from a neighbourhood V in 0 of $\mathbb{R}^n$ to the neighbourhood $\Psi(V)$ of $o = eK$ in G/K. For any $x = \Psi(y) \in \Psi(V)$ we can write $\frac{\partial}{\partial y_i} f(x) = \frac{\partial}{\partial y_i} f \circ \Psi(y)$ for all $f \in C^1(G/K)$ and $1 \le i \le n$. So $\frac{\partial}{\partial y_i}$ can be considered as a vector field on $\Psi(V)$. From now on, for all $x \in \Psi(V) \subset G/K$ we denote by $y(x) \in \mathbb{R}^n$ the point for which $x = \Psi(y(x))$, so for each $i = 1, 2, \ldots, n$, $y_i(\cdot) = \Psi_i^{-1} : \Psi(V) \to \mathbb{R}$ is a smooth function that we can extend to G/K such that $y_i \in C_c^\infty(G/K)$ and we will call $y_1, y_2, \ldots, y_n$ canonical local coordinate functions on G/K.

Proposition 7.1.3. *We have for all $x \in G/K$ and for all $k \in K$*

$$k.x = k.\Psi(y(x)) = \natural\left(e^{\sum_{i=1}^{n} y_i(x)Ad(k)X_i}\right) \tag{7.2}$$

and the canonical local coordinate functions on G/K satisfy

$$\sum_{i=1}^{n} y_i(x)Ad(k)X_i = \sum_{i=1}^{n} y_i(k.x)X_i. \tag{7.3}$$

Proof. See [47], p.40.

□

Let us from now on order the canonical local coordinate functions $\{x_i; i = 1, \ldots, d\}$ on G such that they match with the local coordinate functions on G/K, i.e.

$$y_i \circ \natural = x_i, \qquad \text{for all } i = 1, \ldots, n. \tag{7.4}$$

So from (7.3) we get

$$\sum_{i=1}^{n} x_i(g)\mathrm{Ad}(k)X_i = \sum_{i=1}^{n} x_i(kg)X_i, \qquad \text{for all } g \in G \text{ and } k \in K. \tag{7.5}$$

7.1.3 Invariant differential operators

We will now introduce some notations for differential operators following [24] p.385. Let $f \in C(G)$ and α be a endomorphism of G, then we denote $f^\alpha := f \circ \alpha^{-1}$. For any mapping $A : C(G) \to C(G)$, we write $A^\alpha : f \mapsto (Af^{\alpha^{-1}})^\alpha$ on $C(G)$. A function f is said to be *invariant* under α if $f^\alpha = f$ and an operator A is said to be *invariant* under α if $A^\alpha = A$. Given two homeomorphisms α, β from G to G, we have

$$f^{\alpha\circ\beta} = f \circ (\alpha \circ \beta)^{-1} = (f \circ \beta^{-1}) \circ \alpha^{-1} = (f^\beta)^\alpha.$$

Thus,

$$A^{\alpha\circ\beta} f = [A(f^{(\alpha\circ\beta)^{-1}})]^{\alpha\circ\beta} = \left[\left[A\left(\left(f^{\alpha^{-1}}\right)^{\beta^{-1}}\right)\right]^{\beta}\right]^{\alpha} = \left(A^\beta(f^{\alpha^{-1}})\right)^\alpha = (A^\beta)^\alpha f. \tag{7.6}$$

In this section, we will be interested in the case where $\alpha = l_k$ and $\beta = r_k$ where $k \in K$. A linear transformation $D : C_c^\infty(G) \to C_c^\infty(G)$ is called a *differential operator* on G, if for at any point $p \in G$ and each local chart (φ, U) around p there exists a finite number of functions $a_\alpha \in C^\infty(U)$ such that for all $f \in C_c^\infty(G)$ with support in the neighbourhood U,

$$Df(q) = \sum_\alpha a_\alpha(q)\ (D^\alpha(f\circ\varphi^{-1}))(\varphi(q)), \qquad \text{if } q\in U,$$
$$Df(q) = 0, \qquad \text{if } q\notin U.$$

In particular, all vector fields are differential operators. Let us call a differential operator *left-invariant* if it is invariant under l_g for all $g\in G$. Then $\mathrm{D}(G)$ denotes the set of all G left-invariant differential operators on G. $\mathrm{D}_K(G)$ will denote the subspace of operators in $\mathrm{D}(G)$ that are also K-right invariant. For each $g\in G$, $\mathrm{Ad}(g):\mathfrak{g}\to\mathfrak{g}$ extends uniquely to an automorphism of $\mathrm{D}(G)$, we will denote this extension by $\mathrm{Ad}(g)$ as well, see [24] p.392.

It is easy to verify that if X is a left-invariant vector field on G, then it is a left-invariant differential operator on $C_c^\infty(G)$ in the above sense.

From Lemma 2.18 and equation (7.6) we also know that, for all $X\in\mathfrak{g}$, $f\in C_c^\infty(G)$, $g\in G$,

$$\mathrm{Ad}(g)Xf = X^{c_g}f = \left(X^{l_g}\right)^{r_{g^{-1}}} f.$$

So by G left-invariance of the differential operator $X : C_c^\infty(G)\to C_c^\infty(G)$ we have

$$[\mathrm{Ad}(g)X]f = \left(X^{l_g}\right)^{r_{g^{-1}}} f = X^{r_{g^{-1}}}f, \qquad \text{for all } f\in C_c^\infty(G), g\in G. \tag{7.7}$$

(See also [24], p.391-392)

Furthermore, in the particular case where $f\in C_c^\infty(G/K)$, $g\in G$, $k\in K$, $X_i\in\mathfrak{p}$

$$\begin{aligned} X_i f(gk) &= \frac{d}{dt} f(gk\exp(tX_i))\Big|_{t=0} = \frac{d}{dt} f(gk\exp(tX_i)k^{-1})\Big|_{t=0} \\ &= \frac{d}{dt} f(g\exp(t\mathrm{Ad}(k)X_i))\Big|_{t=0} = \mathrm{Ad}(k)X_i f(g). \end{aligned} \tag{7.8}$$

And similarly, we have

$$\begin{aligned} X_iX_j f(gk) &= \frac{d}{ds}\frac{d}{dt} f(gk\exp(tX_i)k^{-1}k\exp(sX_j)k)\Big|_{t=0,s=0} \\ &= \frac{d}{ds}\frac{d}{dt} f(g\exp(t\mathrm{Ad}(k)X_i)\exp(s\mathrm{Ad}(k)X_j))\Big|_{t=0,s=0} \\ &= [\mathrm{Ad}(k)X_i][\mathrm{Ad}(k)X_j]f(g) \end{aligned} \tag{7.9}$$

Note that from (7.7) and (7.8), it then follows that for $X_i\in\mathfrak{p}$, $f\in C_c^\infty(G/K)$,

$g \in G$ and $k \in K$,

$$X_i f(gk) = X_i^{r_{k^{-1}}} f(g) = X_i(f \circ r_k)(gk^{-1}) = X_i(f \circ r_k)(g). \tag{7.10}$$

That is, on G/K, for all $i = 1, \ldots, n$, the differential operator X_i is K-right-invariant.

For all $k \in K$, let $([\mathrm{Ad}(k)]_{ij})$ be the matrix associated to $\mathrm{Ad}(k) : \mathfrak{p} \to \mathfrak{p}$ in the basis $\{X_1, X_2, \ldots, X_n\}$ with respect to the $\mathrm{Ad}(K)$-invariant inner product $\langle \cdot, \cdot \rangle$. That is, for all $k \in K$, $[\mathrm{Ad}(k)]_{ij} = \langle \mathrm{Ad}(k)X_i, X_j \rangle$ and

$$\mathrm{Ad}(k)X_j = \sum_{i=1}^{n} [\mathrm{Ad}(k)]_{ji} X_i \qquad \text{for all } j = 1, \ldots, n. \tag{7.11}$$

Furthermore, since for all $k \in K$ the inner product $\langle \cdot, \cdot \rangle$ is $Ad(k)$-invariant, the matrix $([\mathrm{Ad}(k)]_{ij})$ is orthogonal. Similarly to [47] p.67, we will call a symmetric $n \times n$ real valued matrix (a_{ij}), *$Ad(K)$-invariant* if

$$a_{ij} = \sum_{p,q=1}^{n} a_{pq} [\mathrm{Ad}(k)]_{pi} [\mathrm{Ad}(k)]_{qj}, \qquad \text{for all } i, j = 1, \ldots, n \text{ and } k \in K.$$

In this case by (7.11), we have $\sum_{i,j=1}^{n} a_{ij} [\mathrm{Ad}(k)X_i][\mathrm{Ad}(k)X_j] = \sum_{i,j=1}^{n} a_{ij} X_i X_j$ for all $k \in K$, $i, j = 1, \ldots, n$.

Note that $X_1, \ldots, X_n$ may be regarded as vector fields on G/K and as G left-invariant differential operators on G/K.

7.2 The positive maximum principle and K-bi-invariance

Lemma 7.2.1. *Let $A : C_c^\infty(G) \to Fun(G)$ be a linear operator, then we have*

a) $A : C_c^\infty(G/K) \to Fun(G/K)$ if and only if $R_k Af = Af$, for all $f \in C_c^\infty(G/K)$ and $k \in K$. In this case, we say that A is K-right invariant

Similarly,

b) $A : C_c^\infty(K\backslash G) \to Fun(K\backslash G)$ if and only if $L_k Af = Af$, for all $f \in C_c^\infty(K\backslash G)$ and $k \in K$ In this case, we say that A is K-left invariant.

Finally, $A : C_c^\infty(K\backslash G/K) \to Fun(K\backslash G/K)$ if and only if both a) and b) are satisfied. In this case, A is called K-bi-invariant.

Proof. If a) is true then, clearly $Af \in \mathrm{Fun}(K\backslash G)$ for all $f \in C_c^\infty(K\backslash G)$. Conversely, for all $f \in C_c^\infty(K\backslash G)$ having $Af \in \mathrm{Fun}(K\backslash G)$ precisely means that $R_k Af = Af$ for all $k \in K$. The rest can be proved similarly. □

We will now establish the closed form for a K-left-invariant, K-right-invariant and K-bi-invariant linear operator satisfying the positive maximum principle. We will only prove the the first two cases, then the K-bi-invariant case follows.

Theorem 7.2.2. *Let $A : C_c^\infty(K\backslash G) \to Fun(K\backslash G)$ be a linear operator. Then A satisfies the positive maximum principle if and only if for all $g \in G$ there exist*

- *a unique real symmetric, non-negative definite matrix $(a_{ij}(g))$ such that $a_{ij}(kg) = a_{ij}(g)$, for all $k \in K$, $i, j = 1\ldots, d$,*
- *a unique vector $b(g) = (b_1(g), \ldots, b_d(g)) \in \mathbb{R}^d$ such that $b_i(kg) = b_i(g)$, for all $k \in K$, $i = 1, \ldots, d$,*
- *a unique $c(g) \geq 0$ such that $c(kg) = c(g)$, for all $k \in K$,*
- *a unique Lévy kernel μ such that $\mu(kg, A) = \mu(g, A)$, for all $A \in \mathcal{B}(G)$ and $k \in K$,*

such that A is of the form (6.13).

Proof. First, from Lemma 7.2.1, b), we have $Af(g) = L_k Af(g) = Af(kg)$ for all $f \in C_c^\infty(K\backslash G)$, $k \in K$ and $g \in G$. Furthermore, for all $X \in \mathfrak{g}$, $f \in C_c^\infty(K\backslash G)$, $g \in G$ and $k \in K$ we have $Xf(kg) = Xf(g)$. Thus, for all $f \in C_c^\infty(K\backslash G)$, $k \in K$ and $g \in G$,

$$\begin{aligned} Af(g) =& Af(kg) \\ =& \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(kg)X_iX_jf(kg) + \sum_{i=1}^{d} b_i(kg)X_if(kg) - c(kg)f(kg) \\ &+ \int_{G^*}\left[f(kg\tau) - f(kg) - \sum_{i=1}^{d} x_i(\tau)X_if(kg)\right]\mu(kg, d\tau) \\ =& \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(kg)X_iX_jf(g) + \sum_{i=1}^{d} b_i(kg)X_if(g) - c(kg)f(g) \\ &+ \int_{G^*}\left[f(g\tau) - f(g) - \sum_{i=1}^{d} x_i(\tau)X_if(g)\right]\mu(kg, d\tau) \end{aligned} \tag{7.12}$$

By uniqueness of the coefficient functions, we have $a_{ij}(kg) = a_{ij}(g)$, $b_i(kg) = b_i(g)$, $c(kg) = c(g)$, for all $g \in G$, $k \in K$, $i, j = 1, \ldots, d$ and uniqueness of the Lévy kernel gives

$$\mu(kg, B) = \mu(g, B), \qquad \text{for all } k \in K, g \in G, B \in \mathcal{B}(G).$$

Conversely, if A is linear operator on $C_c^\infty(K\backslash G)$ of the form (6.13) such that the coefficient functions and the Lévy kernel satisfy the above conditions, then from the calculations in (7.12) it is clear that $Af(kg) = Af(g)$ for all $f \in C_c^\infty(K\backslash G)$, $g \in G$ and $k \in K$. □

Theorem 7.2.3. *$A : C_c^\infty(G/K) \to Fun(G/K)$ is a linear operator satisfying the positive maximum principle if and only if for all $g \in G$ there exist*

- *a unique real symmetric, non-negative definite matrix $(a_{ij}(g))$, such that for all $k \in K$,*
$$(a_{ij}(g)) = [Ad(k)]^T (a_{ij}(gk))[Ad(k)]$$
- *a unique vector $b(g) \in \mathbb{R}^d$ such that for all $k \in K$,*
$$b(g) = [Ad(k)]^T \, b(gk)$$
- *a unique $c(g) \geq 0$ such that $c(gk) = c(g)$, for all $k \in K$,*
- *a unique Lévy kernel μ such that $\mu(gk, A) = \mu(g, kAk')$, for all $A \in \mathcal{B}(G)$ and $k, k' \in K$.*

such that A is of the form (6.13).

Proof. The operator A satisfies the PMP therefore it is of the form (6.13). First recall that after equation (7.1) we established that for all $i > n$ and $f \in C_c^\infty(G/K)$, we have $X_i f(g) = 0$. Thus, by the calculations in (7.8) and (7.9) and a change of variable, we have for all $k \in K$, $g \in G$ and $f \in C_c^\infty(G/K)$,

$$\begin{aligned}
Af(g) =& R_k Af(g) = Af(gk) \\
=& \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(gk) X_i X_j f(gk) + \sum_{i=1}^{d} b_i(gk) X_i f(gk) - c(gk) f(gk) \\
& + \int_{G^*} \left[f(gk\tau) - f(gk) - \sum_{i=1}^{d} x_i(\tau) X_i f(gk) \right] \mu(gk, d\tau) \\
=& \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(gk)[\mathrm{Ad}(k)X_i][\mathrm{Ad}(k)X_j] f(g) + \sum_{i=1}^{n} b_i(gk)[\mathrm{Ad}(k)X_i] f(g) - c(gk) f(g) \\
& + \int_{G^*} \left[f(g\tau) - f(g) - \sum_{i=1}^{n} x_i(k^{-1}\tau)[\mathrm{Ad}(k)X_i] f(g) \right] \mu(gk, k^{-1}d\tau) \qquad (7.13)
\end{aligned}$$

First note that by (7.5), we have

$$\sum_{i=1}^{n} x_i(k^{-1}\tau)[\mathrm{Ad}(k)X_i] = \sum_{i=1}^{n} x_i(\tau) X_i$$

Then using the decomposition (7.11), we get for all $f \in C_c^\infty(G/K)$,

$$\begin{aligned} Af(g) =& \frac{1}{2}\sum_{p,q=1}^{n}\sum_{i,j=1}^{n} a_{ij}(gk)[\mathrm{Ad}(k)]_{pi}[\mathrm{Ad}(k)]_{qj}X_pX_qf(g) \\ &+\sum_{p=1}^{n}\sum_{i=1}^{n} b_i(gk)[\mathrm{Ad}(k)]_{pi}X_pf(g) - c(g)f(g) \\ &+\int_{G^*}\left[f(g\tau)-f(g)-\sum_{i=1}^{n}x_i(\tau)X_if(g)\right]\mu(gk,k^{-1}d\tau). \end{aligned}$$

By uniqueness of the Lévy kernel, for all $g \in G$, $k \in K$ and $B \in \mathcal{B}(G)$

$$\mu(g,B) = \mu(gk,k^{-1}B). \tag{7.14}$$

Furthermore, since we are considering K-right-invariant functions, for all $k \in K$ the integral part is also equal to

$$\begin{aligned} &\int_{G^*}\Big[f(g\tau)-f(g)-\sum_{i=1}^{n}x_i(\tau)X_if(g)\Big]\mu(g,d\tau) \\ &\quad= \int_{G^*}\left[f(g\tau k)-f(g)-\sum_{i=1}^{n}x_i(\tau)X_if(g)\right]\mu(g,d\tau) \\ &\quad= \int_{G^*}\left[f(g\tau)-f(g)-\sum_{i=1}^{n}x_i(\tau k^{-1})X_if(g)\right]\mu(g,d\tau k^{-1}) \\ &\quad= \int_{G^*}\left[f(g\tau)-f(g)-\sum_{i=1}^{n}x_i(\tau)X_if(g)\right]\mu(g,d\tau k^{-1}) \end{aligned}$$

The last line follows from (7.4). Thus, from uniqueness of the Lévy kernel we have for all $k \in K$

$$\mu(g,Bk) = \mu(g,B). \tag{7.15}$$

Combining (7.14) and (7.15) we get for all $k, k' \in K$,

$$\mu(g,B) = \mu(gk,k^{-1}Bk')$$

By uniquenes of the coefficient functions, for all $g \in G$ and $k \in K$

$$\begin{aligned} c(g) &= c(gk) \\ a_{pq}(g) &= \sum_{i,j=1} a_{ij}(gk)[\mathrm{Ad}(k)]_{ip}[\mathrm{Ad}(k)]_{jq} \\ b_p(g) &= \sum_{i=1}^{n} b_i(gk)[\mathrm{Ad}(k)]_{ip} \end{aligned}$$

That is, $(a_{ij}(g)) = [\mathrm{Ad}(k)]^T(a_{ij}(gk))[\mathrm{Ad}(k)]$ and $b(g) = [\mathrm{Ad}(k)]^T\, b(gk)$.

Conversely if A is a linear operator on $C_c^\infty(G/K)$ of the form (6.13) such that the coefficient functions and the Lévy kernel satisfy the above conditions, from the calculations in (7.13) is is clear that $Af(gk) = Af(g)$ for all $f \in C_c^\infty(G/K)$, $g \in G$ and $k \in K$. □

Corollary 7.2.4. *A linear operator $A : C_c^\infty(K\backslash G/K) \to Fun(K\backslash G/K)$ satisfies the positive maximum principle if and only if for all $g \in G$ there exist*

- *a unique real symmetric matrix $(a_{ij}(g))$ such that for all $k, k' \in K$*

$$(a_{ij}(g)) = [Ad(k')]^T(a_{ij}(kgk'))[Ad(k')] \tag{7.16}$$

- *a unique vector $b(g) \in \mathbb{R}^n$ such that for all $k, k' \in K$*

$$b(g) = [Ad(k')]^T\, b(kgk') \tag{7.17}$$

- *a unique $c(g) \geq 0$ such that for all $k, k' \in K$, $c(g) = c(kgk')$,*
- *a unique Lévy kernel μ such that for all $k, k', k'' \in K$ and $B \in \mathcal{B}(G)$*

$$\mu(g, B) = \mu(kgk', (k')^{-1}Bk'') \tag{7.18}$$

such that A is of the form (6.13).

Proof. This follows from combining Theorem 7.2.2 and Theorem 7.2.3. □

7.3 Feller semigroups and generators on symmetric spaces

Definition. A *spherical Feller C_0-semigroup* on $C_0(K\backslash G/K)$ is a Feller C_0-semigroup that satisfies

(SF1) $R_k T_t f = T_t f$, for all $k \in K, t \geq 0, f \in C_0(K\backslash G/K)$

(SF2) $L_k T_t f = T_t f$, for all $k \in K, t \geq 0, f \in C_0(K\backslash G/K)$.

The Feller process associated to a spherical Feller semigroup is called a *spherical Feller process.*

Proposition 7.3.1. *A Feller process Z is a spherical Feller process if and only if Z has transition probabilities $(p_t)_{t\geq 0}$ that satisfy*

$$p_t(\sigma k, B) = p_t(\sigma, B), \tag{7.19}$$
$$p_t(k\sigma, B) = p_t(\sigma, B), \tag{7.20}$$

for all $t \geq 0, B \in \mathcal{B}(G), \sigma \in G, k \in K$.

Proof. Let us start with necessity. For all $k \in K, t \geq 0, \sigma \in G, f \in C_0(K\backslash G/K)$, we have

$$R_k T_t f(\sigma) = T_t f(\sigma k) = \int_G f(\tau) p_t(\sigma k, d\tau),$$

Suppose we have (SF1), then by the Riesz representation theorem

$$p_t(\sigma k, B) = p_t(\sigma, B), \qquad \text{for all } \sigma \in G, k \in K, B \in \mathcal{B}(G), t \geq 0. \tag{7.21}$$

Similarly, supposing (SF2) implies

$$p_t(k\sigma, B) = p_t(\sigma, B), \qquad \text{for all } \sigma \in G, k \in K, B \in \mathcal{B}(G), t \geq 0. \tag{7.22}$$

For sufficiency, we have for all $k \in K, \sigma \in G, t \geq 0$

$$R_k T_t f(\sigma) = \int_G f(\tau) p_t(\sigma k, d\tau) = \int_G f(\tau) p_t(\sigma, d\tau) = T_t f(\sigma), \text{ for } f \in C_0(G/K)$$
$$L_k T_t f(\sigma) = \int_G f(\tau) p_t(k\sigma, d\tau) = \int_G f(\tau) p_t(\sigma, d\tau) = T_t f(k\sigma) \text{ for } f \in C_0(K\backslash G)$$

□

Lemma 7.3.2. *Let $\mathcal{L}$ be the generator of a spherical Feller process, then it satisfies*

$R_k \mathcal{L} f = \mathcal{L} f$, for all $k \in K, t \geq 0$ and $f \in D_{\mathcal{L}} \subset C_0(K\backslash G/K)$

$L_k \mathcal{L} f = \mathcal{L} f$, for all $k \in K, t \geq 0$, and $f \in D_{\mathcal{L}} \subset C_0(K\backslash G/K)$.

Proof. First note that for all $k \in K$, R_k is an isometry, so by the definition of spherical

Feller semigroups, we have for all $f \in D_{\mathcal{L}}$

$$\lim_{t\to 0}\left\|\frac{T_t f - f}{t} - \mathcal{L}f\right\|_\infty = \lim_{t\to 0}\left\|R_k\left(\frac{T_t f - f}{t} - \mathcal{L}f\right)\right\|_\infty = \lim_{t\to 0}\left\|\frac{T_t R_k f - f}{t} - R_k\mathcal{L}f\right\|_\infty$$
$$= \lim_{t\to 0}\left\|\frac{T_t f - f}{t} - R_k\mathcal{L}f\right\|_\infty$$

It follows that $R_k\mathcal{L}f = \mathcal{L}f$. The left K-invariance is proved similarly. □

Corollary 7.3.3. *Let $\mathcal{L}$ be the generator of a spherical Feller process with $C_c^\infty(K\backslash G/K) \subseteq D_{\mathcal{L}}$, then it has the form (6.13) with conditions as in Theorem 7.2.4.*

Proof. Following Lemma 7.3.2 the generator $\mathcal{L}$ is K-bi-invariant so by Lemma 7.2.1, we have $\mathcal{L} : C_c^\infty(K\backslash G/K) \to \mathrm{Fun}(K\backslash G/K)$. Furthermore, by Lemma 6.2.1, since $\mathcal{L}$ is the generator of a Feller process, it satisfies the PMP therefore it has the form (6.13). Then we can directly apply Theorem 7.2.4 to get the conditions on the coefficients and the Lévy kernel. □

Definition. A family of probability measures $(\mu_t)_{t\geq 0}$ on $(G, \mathcal{B}(G))$ is a *generalized convolution semigroup* of measures on G if

- $\mu_{s+t} = \mu_t * \mu_s$, for all $t, s \in \mathbb{R}$,
- $\mu_t \to \mu_0$ as $t \to 0$.

Note that the convolution semigroup defined in 6.2 is a generalized convolution semigroup with $\mu_0 = \delta_e$. We will call a generalized convolution semigroup K-left-invariant, K-right-invariant and K-bi-invariant if for all $t \geq 0$, the measure μ_t is respectively K-left-invariant, K-right-invariant and K-bi-invariant.

Note that it was proved in [46], Proposition 2, that a convolution semigroup is K-left-invariant if and only if it is K-right-invariant if and only if it is K-bi-invariant.

Lemma 7.3.4. *The Hunt semigroup $(T_t)_{t\geq 0}$ of a K-bi-invariant generalized convolution semigroup $(\mu_t)_{t\geq 0}$ is a spherical Feller semigroup.*

Proof. From (6.23), it follows that given any convolution semigroup of measures $(\mu_t)_{t\geq 0}$, the associated semigroup $(T_t)_{t\geq 0}$ is left-invariant, that is $L_g T_t f = T_t L_g f$ for all $g \in G$, $t \geq 0$, $f \in C_0(G)$. So in particular this implies that $(T_t)_{t\geq 0}$ satisfies (SF2). To prove (SF1), for all $t \geq 0$, $B \in \mathcal{B}(G)$, $k \in K$ and $\sigma \in G$ we have

$$R_k T_t f(g) = \int_G f(gk\tau)\mu_t(d\tau) = \int_G f(g\tau)\mu_t(k^{-1}d\tau) = \int_G f(g\tau)\mu_t(d\tau) = T_t f(g).$$

□

Let us recall the following useful result from the literature.

Lemma 7.3.5. *Let $(\mu_t)_{t\geq 0}$ be a convolution semigroup on G, then μ_0 is a Haar measure on a compact subgroup H of G. If $(\mu_t)_{t\geq 0}$ is K-right invariant, then $K \subset H$.*

Proof. For the first part of statement see [26], Theorem 1.2.10. The second part can be found in [46], Proposition 1, p.711. □

We assume from now on that $K = H$, for simplicity.

Theorem 7.3.6. *The following are equivalent*

i) $(\mu_t)_{t\geq 0}$ is a convolution semigroup of measures such that μ_t is K-bi-invariant for all $t \geq 0$,
ii) The Hunt semigroup $(T_t)_{t\geq 0}$ on $C_0(G/K)$ satisfies $R_k T_t f = T_t f$ for all $t \geq 0$, $k \in K$ and $f \in C_0(G/K)$,
iii) The generator A satisfies $R_k A f = Af$ for all $k \in K$ and $f \in D_A$.

Proof. i) implies ii), since for all $g \in G$, $k \in K$, $f \in C_0(G/K)$

$$\begin{aligned} R_k T_t f(g) = T_t f(gk) &= \int_G f(gkh)\mu_t(dh) \\ &= \int_G f(gh)\mu_t(k^{-1}dh) = \int_G f(gh)\mu_t(dh) \\ &= T_t f(g). \end{aligned} \tag{7.23}$$

For ii) implies i), suppose ii) is satisfied, that is $\int_G f(gh)\mu_t(kdh) = \int_G f(gh)\mu_t(dh)$ for all $t \geq 0$, $f \in C_0(G/K)$, $k \in K$ and $g \in G$. In particular, for $g = e$, we have for all $f \in C_0(G/K)$ and $k \in K$,

$$\int_G f(h)\mu_t(kdh) = \int_G f(h)\mu_t(dh)$$

So by the Riesz representation theorem we have $\mu_t(kB) = \mu_t(B)$ for all $t \geq 0$, $k \in K$ and $B \in \mathcal{B}(G)$.

The result follows, since the K-right invariance of the convolution semigroup is equivalent with its K-bi-invariance.

ii) implies iii) is by the same argument as in Lemma 7.3.2 □

Remark 7.3.7. By Lemma 7.3.2, the generator A of a spherical Hunt semigroup commutes with G-left and K-right translation on $C_c^\infty(G)$. In particular A maps $C_c^\infty(K\backslash G/K)$ to $\mathrm{Fun}(K\backslash G/K)$.

The characterisation of the generator of a K-bi-invariant convolution semigroup can be found in the work of [48] and [47] p.139. Here, we will proceed as previously using Courrège's theorem and a special case of Theorem 7.2.3.

Theorem 7.3.8. *Let $A : C_c^\infty(G/K) \to C_0(G/K)$ be the Hunt generator of a K-bi-invariant convolution semigroup of measures, then for all $f \in C_c^\infty(G/K)$ we have*

$$\begin{aligned} Af(g) =& \frac{1}{2}\sum_{i,j=1}^{n} a_{ij} X_i X_j f(g) + \sum_{i=1}^{n} b_i X_i f(g) \\ &+ \int_{G^*} \left[f(g\tau) - f(g) - \sum_{i=1}^{n} x_i(\tau) X_i f(g) \right] \mu(d\tau), \end{aligned} \tag{7.24}$$

where (a_{ij}) is an Ad(K)-invariant, non-negative definite, symmetric matrix, $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$ is a vector such that $\sum_{i=1}^{n} b_i X_i$ is an Ad(K)-invariant vector field in $\mathfrak{p}$ and μ is a K-bi-invariant Lévy measure on $(G, \mathcal{B}(G))$.

Proof. By Hunt's Theorem 6.2.4 we have the form of the operator A. For the additional conditions on the coefficients, we use the result of Theorem 7.2.4. We have $a_{ij} = a_{ij}(e)$ and $b_i = b_i(e)$ for all $i, j = 1, \ldots, n$ from Theorem 7.2.4, so for all $k \in K$

$$a_{ij} = [\mathrm{Ad}(k)]^T (a_{ij}) [\mathrm{Ad}(k)],$$

and

$$\sum_{i=1}^{n} b_i X_i = \sum_{i=1}^{n} \sum_{p=1}^{n} b_p [\mathrm{Ad}(k)]_{ip} X_i = \sum_{p=1}^{n} b_p \mathrm{Ad}(k) X_p \tag{7.25}$$

So the vector $\sum_{i=1}^{n} b_i X_i \in \mathfrak{p}$ is Ad(K)-invariant. Furthermore, the Lévy measure is $\mu(\cdot) = \mu(e, \cdot)$ on $(G, \mathcal{B}(G))$.

□

Let us look separately at the the second order differential part of A. The following results are due to Liao [47].

Proposition 7.3.9. *Let $(a_{ij})_{n\times n}$ and $X_0 \in \mathfrak{p}$ be Ad(K)-invariant, then $P := \sum_{i,j=1}^{n} a_{ij} X_i X_j + X_0$ is a differential operator in $D(G/K)$. Conversely, any second order linear differential operator T in $D(G/K)$ with $T1 = 0$ may be written in the form P for a unique pair of Ad(K)-invariant matrix (a_{ij}) and Ad(K)-invariant vector $X_0 \in \mathfrak{p}$.*

Proof. See [47], Proposition 3.3, p.77. □

Theorem 7.3.10. *Suppose that the representation $Ad: K \to GL(\mathfrak{p})$ acts irreducibly on $\mathfrak{p}$ and $dim(G/K) > 1$. Then any second order differential operator in $D(G/K)$ with $T1 = 0$ is of the form*

$$T = a\sum_{i=1}^{n} X_i^2 =: a\Delta_X$$

where Δ_X is the Laplace-Beltrami operator.

Proof. This result can be found in [47], Proposition 5.5 p.140. Here we will provide a simpler proof using Schur's Lemma. Following Proposition 7.3.9, any second order differential operator $T \in D(G/K)$ with $T1 = 0$ is given by

$$T = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}X_iX_j + X_0, \tag{7.26}$$

where (a_{ij}) is an $\mathrm{Ad}(K)$-invariant symmetric matrix and $X_0 \in \mathfrak{p}$ is also $\mathrm{Ad}(K)$-invariant. By $\mathrm{Ad}(K)$-invariance of X_0, the space spanned by X_0 is invariant under $\mathrm{Ad}(k)$ for all $k \in K$, but since $\mathrm{Ad}(\cdot)$ is irreducible and $\dim(G/K) > 1$ this space has to be $\{0\}$.

The matrix representation $[\mathrm{Ad}(k)] = ([\mathrm{Ad}(k)]_{ij})$ restricted to $\mathfrak{p}$ is in $O(n)$. Let $U : \mathfrak{p} \to \mathbb{R}^n$ be the unitary isomorphism that maps $\{X_1, X_2, \ldots, X_n\}$ to $\{e_1, e_2, \cdots, e_n\}$ which is the natural orthonormal basis of $\mathbb{R}^n$, so it preserves the inner product: $\langle UX, UY\rangle = \langle X, Y\rangle$ for all $X, Y \in \mathfrak{p}$.

Then for all $k \in K$, the matrix representation $[\mathrm{Ad}(k)]$ of $\mathrm{Ad}(k)$ is equal to

$$[\mathrm{Ad}(k)] = U\mathrm{Ad}(k)U^T,$$

and we have by the $\mathrm{Ad}(k)$-invariance of the inner product $\langle\cdot,\cdot\rangle_{\mathfrak{p}}$, for all $x, y \in \mathbb{R}^n$

$$\begin{aligned}
\langle[\mathrm{Ad}(k)]x, [\mathrm{Ad}(k)]y\rangle_{\mathbb{R}^n} &= \langle U\mathrm{Ad}(k)U^{-1}x, U\mathrm{Ad}(k)U^{-1}y\rangle_{\mathbb{R}^n} \\
&= \langle \mathrm{Ad}(k)U^{-1}x, \mathrm{Ad}(k)U^{-1}y\rangle_{\mathfrak{p}} \\
&= \langle U^{-1}x, U^{-1}y\rangle_{\mathfrak{p}} \\
&= \langle x, y\rangle_{\mathbb{R}^n}
\end{aligned}$$

We next show that $[\mathrm{Ad}(K)]$ acts irreducibly on $\mathbb{R}^n$. Let W be an $[\mathrm{Ad}(K)]$-invariant subspace of $\mathbb{R}^n$, then for all $k \in K$ and $x \in W$, we have $[\mathrm{Ad}(k)].x \in W$. That is $U\mathrm{Ad}(k)U^{-1}x \in W$, i.e. $\mathrm{Ad}(k)U^{-1}x \in U^{-1}W$ where $U^{-1}W$ is a closed subspace of $\mathfrak{p}$. But $\mathrm{Ad}(K)$ is irreducible on $\mathfrak{p}$ and $\dim(G/K) > 1$ so $U^{-1}W = \{0\}$ and by the unitarity of U we get $\{W\} = \{0\}$. By the $\mathrm{Ad}(K)$-invariance of $\{a_{ij}\}$, we have for all

$k \in K$

$$[\mathrm{Ad}(k)]^T \{a_{ij}\} [\mathrm{Ad}(k)] = \{a_{ij}\}$$

That is,

$$[\mathrm{Ad}(k)]\{a_{ij}\} = \{a_{ij}\}[\mathrm{Ad}(k)]$$

So by the famous Schur's lemma, see [39] Ch.V, Prop 5.1, there exists $a \in \mathbb{R}$, such that $\{a_{ij}\} = aI_{n\times n}$. Thus, any second order differential operator $T \in D(G/K)$ can be simplified from the general form (7.26) to

$$T = \frac{1}{2}\sum_{i,j=1}^{n} a\delta_{ij}X_iX_j = \frac{1}{2}\sum_{i=1}^{n} aX_iX_i = \frac{1}{2}a\Delta_X.$$

□

Remark 7.3.11. Note that in Proposition 7.3.9 and in Theorem 7.3.10, when the matrix (a_{ij}) is a non-negative definite matrix, then $a \geq 0$. Furthermore, when $(a_{ij})(\cdot)$ is continuous so is $a(\cdot)$ and when $(a_{ij})(\cdot)$ is K-bi-invariant so is $a(\cdot)$.

Corollary 7.3.12. *Suppose that the representation $Ad : K \to GL(\mathfrak{p})$ acts irreducibly on $\mathfrak{p}$ and $dim(G/K) > 1$. Let $A : C_c^\infty(K\backslash G/K) \to C_0(K\backslash G/K)$ be the Hunt generator of a K-bi-invariant convolution semigroup of measures, then for all $f \in C_c^\infty(K\backslash G/K)$ we have*

$$Af(g) = \frac{1}{2}a\Delta_X f(g) + \int_{G^*}\left[f(g\tau) - f(g) - \sum_{i=1}^{n} x_i(\tau)X_i f(g)\right]\mu(d\tau), \qquad (7.27)$$

where $a \geq 0$ is a unique constant and μ is a unique Lévy measure such that for all $k, k' \in K$ and $B \in \mathcal{B}(G)$,

$$\mu(kBk') = \mu(B).$$

Proof. A is the Hunt generator of a K-bi-invariant convolution semigroup, so by Theorem 7.3.8, A is of the form (7.24). By Theorem 7.3.10, we have seen that the second order differential operator part simplifies to $\frac{1}{2}a\Delta_X f(g)$ for a unique constant $a \geq 0$, so A is of the form (7.27). From Corollary 7.3.12, we also know that the Lévy measure satisfies

$$\mu(kB) = \mu(B), \text{ for all } k \in K \text{ and } B \in \mathcal{B}(G).$$

Furthermore, from (7.4) we have $x_i(gk) = x_i(g)$ for all $k \in K$ and $i = 1, \ldots, n$. So for all $f \in C_c^\infty(G/K)$ by a change of variable, we have for all $k \in K$

$$\begin{aligned}
Af(g) &= \frac{1}{2}a\Delta_X f(g) + \int_{G^*}\left[f(g\tau k) - f(g) - \sum_{i=1}^n x_i(\tau)X_i f(g)\right]\mu(d\tau),\\
&= \frac{1}{2}a\Delta_X f(g) + \int_{G^*}\left[f(g\tau) - f(g) - \sum_{i=1}^n x_i(\tau k^{-1})X_i f(g)\right]\mu(d\tau k^{-1})\\
&= \frac{1}{2}a\Delta_X f(g) + \int_{G^*}\left[f(g\tau) - f(g) - \sum_{i=1}^n x_i(\tau)X_i f(g)\right]\mu(d\tau k^{-1})
\end{aligned}$$

By uniqueness of the Lévy measure μ, this implies that for all $k \in K$ and $B \in \mathcal{B}(G)$

$$\mu(Bk) = \mu(B).$$

□

7.4 The positive maximum principle and pseudo-differential operators on symmetric spaces

From now on we will assume that, G is compact and K is a closed subgroup such that (G, K) is a Gelfand pair. We also assume that G/K is irreducible, that is the representation $\mathrm{Ad} : K \to GL(\mathfrak{p})$ acts irreducibly on $\mathfrak{p}$ and $\dim(G/K) > 1$.

Definition. A linear operator $A : C^\infty(K\backslash G/K) \to C(K\backslash G/K)$ is called a *spherical pseudo-differential operator* if there exits a map $\widetilde{\sigma_A} : G \times \widehat{G}_K \to \mathbb{R}$, called the *spherical symbol* of A such that for all $f \in C^\infty(K\backslash G/K)$, $g \in G$ and $\pi \in \widehat{G}_K$

$$\begin{aligned}
A\phi_\pi(g) &= \widetilde{\sigma_A}(g,\pi)\phi_\pi(g)\\
Af(g) &= \sum_{\pi\in\widehat{G}_K} d_\pi\, \widetilde{\sigma_A}(g,\pi)\widehat{f}^S(\phi_\pi)\phi_\pi(g)
\end{aligned}$$

Let $A : C^\infty(K\backslash G/K) \to C_0(K\backslash G/K)$ be a linear operator satisfying the positive maximum principle, then A is of the form (6.13) with conditions from Corollary 7.2.4. We want to prove that such operator A is a spherical pseudo-differential operator and calculate its spherical symbol. For this we need some additional conditions.

(A1) $\sup_{g\in G}\max_{i=1,\dots,d}\int_{G^*}|x_i(\tau)|\mu(g, d\tau) < \infty$.

(A2) for all $i, j = 1, \dots, n$ and any $B \in \mathcal{B}(G)$, the functions $a_{ij}(\cdot)$, $b_i(\cdot)$ and $\mu(\cdot, B)$ are K-bi-invariant.

From (A1) it follows that for all $f \in C_c^\infty(K\backslash G/K)$, $i = 1, \dots, n$ and $g \in G$, we have

$$\left| \int_{G^*} x_i(\tau) X_i f(g) \mu(g, \tau) \right| < \infty.$$

Condition (A2) and conditions (7.17), (7.16), (7.18) from Corollary 7.2.4 implies the following set of properties (K-Bi):

- for all $g \in G$, $b(g) = [\mathrm{Ad}(k)]^T b(g)$. Thus, by (7.25) the vector $\sum_{i=1}^{n} b_i(g) X_i \in \mathfrak{p}$ is $\mathrm{Ad}(K)$-invariant;
- for all $g \in G$ the matrix $(a_{ij}(g))$ is $\mathrm{Ad}(K)$-invariant;
- for all $g \in G$ the Lévy measure $\mu(g, \cdot)$ is K-bi-invariant, because $\mu(g, kBk') = \mu(gk^{-1}, Bk') = \mu(g, B)$, for all $B \in \mathcal{B}(G)$ and $k, k' \in K$.

We will also use the fact that from (3.1)

$$|\phi_\pi(\tau) - 1| \le \|\pi(\tau) - I_\pi\|_{HS}, \qquad \text{for all } \pi \in \widehat{G}_K \text{ and } \tau \in G. \tag{7.28}$$

Let $A : C^\infty(K\backslash G/K) \to C_0(K\backslash G/K)$ be a linear operator satisfying the PMP. By Corollary 7.2.4 and condition (A1), the operator A can be written for all $f \in C^\infty(K\backslash G/K)$, $g \in G$ as

$$\begin{aligned} Af(g) =& \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(g) X_i X_j f(g) + \sum_{i=1}^{n} b_i(g) X_i f(g) - \sum_{i=1}^{n} \left(\int_{G^*} x_i(\tau) \mu(g, d\tau) \right) X_i f(g) \\ & - c(g) f(g) + \int_{G^*} \left(f(g\tau) - f(g) \right) \mu(g, d\tau) \end{aligned} \tag{7.29}$$

Remark 7.4.1. Condition (A1) can be dropped if we write the integral part as a principal value.

Remark 7.4.2. When A is the generator of a convolution semigroup, we know that A is left-invariant, therefore it is completely determined by its action at e. Furthermore, in Gangolli's Lévy-Khintchine formula, [22], there is no first order differential term in the integral, see [47] Theorem 5.3, p.139.

Note that for all $\pi \in \widehat{G}_K$, by K-bi-invariance of ϕ_π and Lemma 1.3.6 we have for all $k \in K$ and $g \in G$

$$X\phi_\pi(g) = X(\phi_\pi \circ c_k)(g) = \left([\mathrm{Ad(k)} X] \phi_\pi \right) (kgk^{-1}).$$

Using this and the K-bi-invariance of the Lévy kernel with respect to the first variable, we have for all $g \in G$, $k \in K$ $i = 1, \ldots, n$ and $\pi \in \widehat{G}_K$

$$\begin{aligned}
\int_{G^*} x_i(\tau) X_i \phi_\pi(g)\, \mu(g, d\tau) &= \int_{G^*} x_i(\tau) X_i \phi_\pi(g)\, \mu(g, d\tau) \\
&= \int_{G^*} x_i(\tau) [\mathrm{Ad}(k) X_i] \phi_\pi(kgk^{-1})\, \mu(g, d\tau) \\
&= \int_{G^*} x_i(\tau) [\mathrm{Ad}(k) X_i] \phi_\pi(g)\, \mu(kgk^{-1}, d\tau) \\
&= \int_{G^*} x_i(\tau) [\mathrm{Ad}(k) X_i] \phi_\pi(g)\, \mu(g, d\tau) \\
&= \left[\int_K \int_{G^*} x_i(\tau) \mathrm{Ad}(k) \mu(g, d\tau)\, dk \right] X_i \phi_\pi(g)
\end{aligned}$$

All the above integrals are well-defined since we have the assumption (A1). The vector field $X_0 = \left[\int_K \int_{G^*} x_i(\tau) \mathrm{Ad}(k) \mu(g, d\tau)\, dk \right] X_i \in \mathfrak{p}$ is $\mathrm{Ad}(K)$-invariant, because the normalized Haar measure on K is unimodular. The space spanned by X_0 is invariant under $\mathrm{Ad}(k)$ for all $k \in K$, but since $\mathrm{Ad}(\cdot)$ is irreducible and $\dim(G/K) > 1$ this space has to be $\{0\}$. Therefore,

$$\int_{G^*} x_i(\tau) X_i \phi_\pi(g)\, \mu(g, d\tau) = 0 \qquad \text{for all } i = 1 \ldots, n, \pi \in \widehat{G}_K, g \in G \tag{7.30}$$

More generally, given the irreducibility of G/K, there is no non-zero $\mathrm{Ad}(K)$-invariant vector in $\mathfrak{p}$.

To investigate an interesting class of pseudo-differential operators in this context, we simply generalize the generator of a convolution semigroup of measures on a irreducible symmetric space, as studied by Gangolli [22], Applebaum [1] and Liao and Wang [48]. From now on we consider operators of the form

$$Af(g) = a(g) \Delta_X f(g) + \int_G (f(g\tau) - f(g))\, \mu(g, d\tau), \quad \text{for all } f \in C^\infty(K \backslash G/K).$$

We assume the first moment condition (A1); and that the matrix a and the Lévy measure μ satisfy the properties (K-Bi).

When applying A to the spherical function ϕ_π, $\pi \in \widehat{G}_K$, by (7.29) and (7.30) we get for all $g \in G$,

$$A\phi_\pi(g) = a(g) \Delta_X \phi_\pi(g) - c(g) \phi_\pi(g) + \int_{G^*} (\phi_\pi(g\tau) - \phi_\pi(g))\, \mu(g, d\tau). \tag{7.31}$$

We know that for all $\pi \in \widehat{G}_K$, ϕ_π is an eigenvector of Δ_X, so there exists a constant

β_π such that $\Delta_X \phi_\pi = \beta_\pi \phi_\pi$. Indeed, from Corollary 1.3.10 we know that it is exactly $\beta_\pi = -\kappa_\pi$, where $\{\kappa_\pi, \pi \in \widehat{G}\}$ is the Casimir spectrum.

We will now show that for all $\pi \in \widehat{G}_K$, the spherical functions ϕ_π is an eigenvector for the operator A.

Theorem 7.4.3. *For all* $\pi \in \widehat{G}_K$, $g \in G$

$$A\phi_\pi(g) = \widetilde{J_A}(g,\pi)\phi_\pi(g),$$

where

$$\widetilde{J_A}(g,\pi) = a(g)\beta_\pi - c(g) + \int_{G^*} (\phi_\pi(\tau) - 1)\, \mu(g, d\tau).$$

Proof. First, let us prove that the integral $\int_K \int_{G^*} (\phi_\pi(gk\tau) - \phi_\pi(g))\ \mu(g, d\tau)dk$ is well-defined and that we can interchange the integral signs. For this we will split the integral into $\int_K \int_{U^*} + \int_K \int_{U^C}$. For the second integral we are going to use the fact that $\mu(g,\cdot)$ is a Lévy measure for all $g \in G$ and by the inequality (7.28) we have

$$\begin{aligned}\left| \int_K \int_{U^C} (\phi_\pi(gk\tau) - \phi_\pi(g))\ \mu(g, d\tau)dk \right| &\leq \int_K \int_{U^*} \|\pi(gk\tau) - \pi(g)\|_{HS}\ \mu(g, d\tau)dk \\ &\leq C\mu(g, U^C) < \infty,\end{aligned}$$

for some constant $C \geq 0$. For the first integral we will use the characterization of spherical functions from Theorem 2.3.3, Taylor's expansion and the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}&\left| \int_{U^*} \int_K (\phi_\pi(gk\tau) - \phi_\pi(g))\ \mu(g, d\tau)dk \right| \\ &\quad \leq \int_{U^*} |\phi_\pi(g)||\phi_\pi(\tau) - 1|\ \mu(g, d\tau) \\ &\quad \leq \int_{U^*} |\langle (\pi(\tau) - I_\pi)u_\pi, u_\pi \rangle|\ \mu(g, d\tau) \\ &\quad = \int_{U^*} \sum_{i=1}^d |x_i(\tau)| \left| \left\langle d\pi(X_i) \exp\left(\theta \sum_{i=1}^d x_i(\tau) d\pi(X_i)\right) u_\pi, u_\pi \right\rangle \right| \mu(g, d\tau) \\ &\quad \leq \int_{U^*} \sum_{i=1}^d |x_i(\tau)| \|d\pi(X_i)u_\pi\|_{V_\pi} \mu(g, d\tau) \\ &\quad \leq \max_{i=1,\dots,d} \|d\pi(X_i)u_\pi\|_{V_\pi} \int_{U^*} \sum_{i=1}^d |x_i(\tau)| \mu(g, d\tau) < \infty,\end{aligned}$$

where $\theta \in [0,1]$. So by Fubini's theorem we have

$$\int_K \int_{G^*} (\phi_\pi(gk\tau) - \phi_\pi(g)) \ \mu(g, d\tau)dk = \int_{G^*} \int_K (\phi_\pi(gk\tau) - \phi_\pi(g)) \, dk \, \mu(g, d\tau)$$

Using the facts that the Lévy kernel is K-bi-invariant with respect to the second variable, the Haar measure on K is normalized and from the characterization of the spherical functions in Theorem 2.3.3, we can write (7.31) the following way, for all $\pi \in \widehat{G}_K$, $g \in G$

$$\begin{aligned} A\phi_\pi(g) &= a(g)\beta_\pi\phi_\pi(g) - c(g)\phi_\pi(g) + \int_K \int_{G^*} (\phi_\pi(g\tau) - \phi_\pi(g)) \ \mu(g, k^{-1}d\tau) \ dk \\ &= a(g)\beta_\pi\phi_\pi(g) - c(g)\phi_\pi(g) + \int_{G^*} \int_K (\phi_\pi(gk\tau) - \phi_\pi(g)) \, dk \, \mu(g, d\tau) \\ &= a(g)\beta_\pi\phi_\pi(g) - c(g)\phi_\pi(g) + \phi_\pi(g) \int_{G^*} (\phi_\pi(\tau) - 1) \ \mu(g, d\tau). \end{aligned}$$

Thus, we have for all $\pi \in \widehat{G}_K$ and $g \in G$

$$A\phi_\pi(g) = \left(a(g)\beta_\pi - c(g) + \int_{G^*} (\phi_\pi(\tau) - 1) \, \mu(g, d\tau) \right) \phi_\pi(g) \tag{7.32}$$

This means that for all $\pi \in \widehat{G}_K$, ϕ_π is an eigenvector for A and we have

$$A\phi_\pi(g) = \widetilde{J_A}(g, \pi)\phi_\pi(g) \qquad \text{for all } g \in G, \tag{7.33}$$

where

$$\widetilde{J_A}(g, \pi) = a(g)\beta_\pi - c(g) + \int_{G^*} (\phi_\pi(\tau) - 1) \, \mu(g, d\tau).$$

□

Note that the symbol $\widetilde{J_A}$ is K-bi-invariant with respect to its first variable. We will equivalently write $\widetilde{J_A}(g, \lambda) = \widetilde{J_A}(g, \pi_\lambda)$ for all $g \in G$ and $\lambda \in D_S$.

Remark 7.4.4. In the compact case, the class of operators as given by (7.32) is more general than that obtained for K-bi-invariant convolution semigroups of measures characterized by Gangolli [22] and more recently by Liao and Wang [48], if the condition (A1) is also imposed in that context. We conjecture that in our general case, the first moment condition (A1) can be dropped, however at the moment we don't know how to do this.

Lemma 7.4.5. *There is a constant $C \geq 0$ such that for all $\lambda \in D_S$*

$$\sup_{g \in G} |\widetilde{J_A}(g, \lambda)| \leq C(1 + |\lambda|^{\frac{m+2}{2}}).$$

Proof. We are going to base the proof on that of Lemma 6.3.1 on Lie groups. Similarly as before, C will denote a generic constant which may vary from line to line. First, let us note that from (3.1) for all $\lambda \in D_S$ and $g \in G$

$$\begin{aligned} \left| \int_{U^*} (\phi_\lambda(\tau) - 1)\, \mu(g, d\tau) \right| &= \left| \left\langle \int_{U^*} (\pi_\lambda(\tau) - \pi_\lambda(e))\, \mu(g, d\tau) u, u \right\rangle \right| \\ &\leq \left\| \int_{U^*} (\pi_\lambda(\tau) - I_{\pi_\lambda})\, \mu(g, d\tau) \right\|_{HS} \end{aligned} \tag{7.34}$$

Then we will use (6.30) and Taylor's expansion, so for all $\lambda \in D_S$ and $g \in G$

$$\begin{aligned} \left\| \int_{U^*} (\pi_\lambda(\tau) - I_{\pi_\lambda})\, \mu(g, d\tau) \right\|_{HS} &\leq \int_{U^*} \|\pi_\lambda(\tau) - I_{\pi_\lambda}\|_{HS}\, \mu(g, d\tau) \\ &= \int_{U^*} \left\| \exp\left(\sum_{i=1}^d x_i(\tau) d\pi_\lambda(X_i) \right) - I_{\pi_\lambda} \right\|_{HS} \mu(g, d\tau) \\ &\leq \int_{U^*} \sum_{i=1}^d |x_i(\tau)| \|d\pi_\lambda(X_j)\|_{HS}\, \mu(g, d\tau) \\ &\leq C \max\{|X_1|, \dots, |X_d|\} \left(\sum_{i=1}^d \int_{U^*} |x_i(\tau)| \mu(g, d\tau) \right) |\lambda|^{\frac{m+2}{2}} \end{aligned} \tag{7.35}$$

Combining (7.35) and (7.34) and taking the supremum over G, we get

$$\begin{aligned} \sup_{g \in G} \left| \int_{U^*} (\phi_\lambda(\tau) - 1)\, \mu(g, d\tau) \right| &\leq \sup_{g \in G} \left\| \int_{U^*} (\pi_\lambda(\tau) - I_{\pi_\lambda})\ \mu(g, d\tau) \right\|_{HS} \\ &\leq C |\lambda|^{\frac{m+2}{2}} \end{aligned} \tag{7.36}$$

For the rest of the integral we follow the steps from the proof of Lemma 6.3.1, we get

$$\sup_{g \in G} \left| \int_{U^C} (\phi_\lambda(\tau) - 1)\, \mu(g, d\tau) \right| \leq C |\lambda|^{\frac{m}{2}} \tag{7.37}$$

Combining the two estimates (7.36) and (7.37), then using Corollary 1.3.12 for β_π and given that a and c are bounded on G we get

$$\sup_{g \in G} |\widetilde{J}_A(g, \phi_\pi)| \leq C \sup_{g \in G} a(g)(1 + |\lambda|^2) + \sup_{g \in G} c(g) + C(|\lambda|^{\frac{m+2}{2}} + |\lambda|^{\frac{m}{2}})$$

So the result follows. □

Lemma 7.4.6. *For all $f \in C^{\infty}(K\backslash G/K)$, the series*

$$\sum_{\lambda \in D_S} d_\lambda \widetilde{J_A}(g,\lambda)\widehat{f}^S(\lambda)\phi_\lambda(g) \tag{7.38}$$

converges absolutely and uniformly for all $g \in G$.

Proof. Recall from Theorem 4.3.10 that $\widehat{f}^S \in S(D_S)$ for all $f \in C^{\infty}(K\backslash G/K)$ so

$$\lim_{|\lambda|\to\infty, \lambda \in D_S} |\lambda|^p \, |\widehat{f}^S(\lambda)| = 0, \qquad \text{for all } p \in \mathbb{N}.$$

We will now follow the proof of Lemma 6.3.2, there exists $\lambda_0 \in D_S/\{0\}$ for all $p \in \mathbb{N}$

$$\sup_{g\in G} \sum_{|\lambda|>|\lambda_0|} d_\lambda \left| \widetilde{J_A}(g,\lambda)\widehat{f}^S(\lambda)\phi_\lambda(g) \right| \le C \sum_{|\lambda|>|\lambda_0|} \frac{|\lambda|^m(1+|\lambda|^{\frac{m+2}{2}})}{|\lambda|^p} < \infty$$

Choose $p > \frac{3m+2}{2} + r$, and the result follows from the convergence of the Sugiura Zeta function in Theorem 4.1.3. □

Theorem 7.4.7. *Let $A : C^{\infty}(K\backslash G/K) \to C_0(K\backslash G/K)$ be a linear operator satisfying the positive maximum principle, (A1) and (A2). Then A is a spherical pseudo differential operator with symbol $\widetilde{J_A}$.*

Proof. From (7.33) it follows that when $f \in \mathcal{E}_K(G)$, we have

$$Af(g) = \sum_{\lambda \in D_S} d_\lambda \widetilde{J_A}(g,\lambda)\widehat{f}^S(\lambda)\phi_\lambda(g), \qquad \text{for all } g \in G \tag{7.39}$$

Now, take any $f \in C^{\infty}(K\backslash G/K)$, then by Theorem 3.2.4 the spherical Fourier series $\sum_{\lambda \in D_S} d_\lambda \widehat{f}^S(\phi_\lambda)\phi_\lambda(g)$ converges absolutely and uniformly to $f(g)$ for all $g \in G$. Then the partial sums $f_N := \sum_{i=1}^{N} d_{\lambda_i}\widehat{f}^S(\lambda_i)\phi_{\lambda_i}$ for all $N \in \mathbb{N}$, are in $\mathcal{E}_K(G)$ and the sequence $(f_N)_{N\in\mathbb{N}}$ converges uniformly to f. Following similar calculations as in Theorem 6.3.3, we have for all $N \in \mathbb{N}$, $g \in G$,

$$Af_N(g) = \sum_{i=1}^{N} d_{\lambda_i} \widetilde{J_A}(g,\lambda_i)\widehat{f}^S(\lambda_i)\phi_{\lambda_i}(g).$$

By Lemma 7.4.6, the sequence of partial sums Af_N converges uniformly. Furthermore,

$\overline{A}$ is closed, therefore

$$Af(g) = \lim_{N\to\infty} Af_N(g) = \sum_{\lambda\in D_S} d_\lambda \widetilde{J}_A(g,\lambda)\widehat{f}^S(\lambda)\phi_\lambda(g),$$

and the result follows. □

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