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Polynomial Functors and W-Types for Groupoids

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Abstract

This thesis contributes to the semantics of Martin-Löf type theory and the theory of polynomial functors. We do so by investigating polynomial functors on the category of groupoids and their initial algebras, known as W-types. We consider several versions of polynomial functors: both simple and dependent, associated to either split, cloven or general fibrations. Our main results show the existence of W-types and their pullback stability in a variety of situations. These results are obtained working constructively, i.e. avoiding the use of excluded middle, the axiom of choice, power set axiom, ordinal iteration. We also extend the theory of natural models, by defining a version of them for Martin-Löf type theories where η-equality holds up to propositional, and not definitional equality.

Mojim staršem in sestri.
Hvala, ker mi vedno stojite ob strani.

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Contents

Introduction

The groupoid model of type theory

Dependent type theories, like Martin-Löf type theories [**37, 41**] or the Calculus of Constructions [**16**], are complex formal systems, the study of which can be performed by either syntactic methods, as employed, for example, in standard normalisation proofs [**38**] or semantic techniques, as employed, for example, to obtain realizability models [**15**]. Traditionally, semantic techniques have been particularly important to give mathematical insight into deduction rules and to establish relative consistency and independence results.

Among all the semantic models of Martin-Löf type theory considered so far, it is difficult to overstate the importance of the groupoid model, discovered by Martin Hofmann and Thomas Streicher in the '90s [**27, 28**]. In this model, types are interpreted as groupoids (i.e. categories in which every morphism has an inverse), while dependent types are interpreted as split fibrations (i.e. functors satisfying an appropriate version of the path-lifting property defining fibrations of topological spaces)[1]. First of all, the model provided a long-awaited independence result, establishing that the so-called principle of Uniqueness of Identity Proofs (UIP) cannot be proved in Martin-Löf type theory. Secondly, it provided a precursor to the homotopy-theoretic models (such as the simplicial model [**31**] and the cubical model [**13**]) that have led to the development of Homotopy Type Theory in the last decade [**47**].

The groupoid model is an example of a model given by a category with display maps [**29, 46**], in which dependent types are interpreted using a distinguished class of morphisms (called display maps). This is to be contrasted with models given by locally cartesian closed categories [**43**], in which dependent types are interpreted as arbitrary morphisms. This is forced upon us since the category of groupoids is not locally cartesian closed (as we briefly review in Chapter 1), but it is also essential to be able to make UIP fail. This approach however requires additional work, since one needs to show that the display maps under consideration (i.e. the split fibrations between groupoids) satisfy enough

[1]Strictly speaking, in [**27, 28**], the dependent types are interpreted using the equivalent notion of a functor from a small groupoid to the category of small groupoids.

closure properties to give an interpretation of the type constructors of Martin-Löf type theory. In particular, Hofmann and Streicher proved results showing that the groupoid model supports the interpretation of identity types (Id-types), dependent sums (Σ-types), dependent products (Π-types), and a type universe. Furthermore, they considered some inductive types, like the type of natural numbers and types of lists.

One of the goals of this thesis is to extend this work by showing that the groupoid model supports also the interpretation of well-ordering types (W-types) [**37**] and general tree types (dependent W-types) [**40, 42**]. Both of these are very important forms of inductive types, encompassing (up to equivalence) many well-known inductive types, including the type of natural numbers and list types mentioned above [**8, 18**]. Informally speaking, W-types provide a type-theoretic counterpart of free algebras for signatures with operations with arities that are not necessarily finite. More specifically, for a type A, whose elements are to be thought of as operations, and a dependent type $B(x)$, for $x \colon A$, where we think of the 'cardinality' of $B(a)$ as the arity of the operation $a \colon A$, we can construct a new type $(Wx \colon A)B(x)$, whose elements are the terms freely generated by this signature. Alternatively, one can think of these terms as wellfounded trees. In this view, the dependent type B is considered as the 'branching data' of the trees. Historically, W-types have been very important, for example to interpret constructive set theories into type theories [**2**]. Showing that the groupoid model supports W-types, as we shall do in this thesis, is intended to fill a gap in our understanding of the semantics of Martin-Löf type theory.

Polynomial functors on groupoids

In order to achieve our goal, we will exploit the category-theoretic understanding of W-types as initial algebras for a particular class of functors, known as polynomial functors, originally due to Moerdijk and Palmgren [**39**]. For a morphism

$$f \colon B \to A$$

in a locally cartesian closed category $\mathcal{E}$, the associated polynomial functor $P_f \colon \mathcal{E} \to \mathcal{E}$ can be written in the internal language of $\mathcal{E}$ [**43**] as

$$P_f(X) = \sum_{a \in A} X^{B(a)}$$

where $B(a) = f^{-1}(a)$, for $a \in A$, which motivates the name 'polynomial'. Initial algebras (which can be thought of as least fixpoints) of such functors are categorical counterparts of W-types. More generally, one can consider diagrams of the form

$$I \longleftarrow B \xrightarrow{\ f\ } A \longrightarrow J$$

and consider the associated (dependent) polynomial functor $P_f \colon \mathcal{E}_{/I} \to \mathcal{E}_{/J}$ between slice categories of $\mathcal{E}$. When $I = J$, this becomes an endofunctor and its initial algebras are counterparts of the dependent W-types [**1, 23**].

Since the work of Moerdijk and Palmgren, the theory of polynomial functors has been developed significantly [**24, 49**] and found applications also outside mathematical logic [**48, 34**]. However, much of the work on polynomial functors done to date has focused on polynomial functors on locally cartesian closed categories and hence cannot be applied to the category of groupoids. An important exception is represented by the work of Weber [**49**], which focused on polynomial functors on categories with pullbacks and can therefore be applied to the category of groupoids. However, while Weber was able to generalize part of the theory, not all the results on polynomial functors in [**23**] or [**24**] have a counterpart in his setting. For this, we shall consider polynomials in groupoids

(*) $$\mathbb{I} \longleftarrow \mathbb{B} \xrightarrow{F} \mathbb{A} \longrightarrow \mathbb{J}$$

in which the functor is a fibration, and their special case:

(**) $$1 \longleftarrow \mathbb{B} \xrightarrow{F} \mathbb{A} \longrightarrow 1$$

Indeed, fibrations in groupoids are exponentiable, thus allowing us to make the definition of a polynomial functor work. Furthermore, Weber does not consider questions of the existence of W-types.

We will improve on the existing theory by extending Weber's results in the particular case of the category of groupoids, exploiting the possibility of manipulating groupoids directly. In particular, we shall give a diagrammatic characterization of general natural transformations, not just cartesian ones, between polynomial functors. For this, we cannot apply the results in [**24**], since the proof of this fact given therein is not only developed in the context of locally cartesian closed, but is also incomplete (as it relies on global elements)[2]. We will also give more direct proofs of various results on polynomial functors, such as their closure under composition, adapting to the groupoid case the ideas in [**24**] rather than following the abstract approach of [**49**].

Let us also mention that there are motivations for studying polynomial functors on groupoids independent of the groupoid model of type theory. Finitary (discrete) polynomial functors in groupoids were also considered by Kock in [**34**]. He applied his work to trees of Feynman graphs, intersecting with the work of Baez and Dolan on stuff types [**9**]. Further, groupoids in general play a role in combinatorics [**9, 34, 50, 51**].

[2]A proof of the fact in locally cartesian categories can be obtained via the results in [**33**].

W-types in groupoids

Our main results establish the existence of initial algebras for various kinds of polynomial functors on groupoids. These results can be organized according to two parameters. The first is whether the polynomial functor is 'dependent' or 'simple', i.e. whether it is determined by a map as in (*) or by a diagram as in (**), respectively. The second is the hypothesis on the fibration $F\colon \mathbb{B} \to \mathbb{A}$, namely whether it is split, cloven or arbitrary. We can give an overview of the results in the thesis on W-types and of what remains to be done in table 1.

TABLE 1. W-types for various kinds of polynomial functors

	F split	F cloven	F general
Simple $1 \leftarrow \mathbb{B} \xrightarrow{F} \mathbb{A} \rightarrow 1$	Chapter 3	Chapter 6	Chapter 6
Simple in slice $\mathbb{A} \xrightarrow{F} \mathbb{B}$, $\mathbb{A} \rightarrow \mathbb{I}$, $\mathbb{B} \rightarrow \mathbb{I}$	Chapter 4		
Dependent $\mathbb{I} \leftarrow \mathbb{B} \xrightarrow{F} \mathbb{A} \rightarrow \mathbb{I}$	Chapter 5		

We prove these results working in a constructive metatheory, without assuming the law of excluded middle, the axiom of choice, fully impredicative principles (like the Power Set axiom), or the use of iteration along an ordinal. We essentially work in a constructive set theory allowing ourselves the possibility of defining W-types of sets [**3**].

This is similar in spirit to the work of Moerdijk and Palmgren [**39**] where they work in a suitably-defined 'predicative topos', equipped with W-types, and give a construction of W-types in categories of internal presheaves and sheaves. In fact, we shall make use of their work using W-types in categories of graphs, which are special cases of presheaf categories. This is in contrast with [**1**], in which W-types are obtained by transfinite iteration on ordinals, following the well-known approach [**4**].

In general the proofs in this thesis, concerning the existence of W-types, share a similar structure. We consider an inductively built set (or graph), and remove some of the elements, until we obtain a set (or a graph) that admits the structure of a groupoid and of a P-algebra. Then, we show initiality using additional facts about algebras for polynomial functors, for example, existence of the smallest subalgebra of a given algebra. One of the difficulties was avoiding the Power Set axiom, which we managed to accomplish using an inductive definition of the smallest subalgebra.

It should be mentioned that our results do not seem to follow from the work of van den Berg and Moerdijk in [**10**], where they constructed W-types in simplicial sets. There, they considered polynomial functors associated to Kan fibrations between Kan complexes and used ordinal iteration to define the appropriate initial algebras, and then showed that these are again Kan complexes. While the category of groupoids can be embedded in the category of simplicial sets via the nerve functor and the image under the nerve functor of a fibration is a Kan fibration, it is not clear whether applying the construction of van den Berg and Moerdijk to the image of a fibration of groupoids produces the nerve of a groupoid. Furthermore, as mentioned above, we are interested here in constructing W-types working a constructive metatheory.

Other related research includes the work of Emmenegger [**20**], where the author constructs W-types in the category of type-theoretic setoids, and of Dybjer and Moenclaey [**19**], who give a construction of finitary 1- and 2-higher inductive types in the groupoid model. We should also mention that Sozeau and Tabareau have begun a formalisation of the groupoid model within the Coq system [**44**], but without considering W-types yet. It would be natural to extend their work by formalising the constructions in this thesis, a topic that we leave for future work.

Natural models of type theory

The thesis also makes a contribution to the categorical semantics of type theory in general. In particular, we focus on the notion of a natural model of dependent type theory introduced by Awodey in [**7**], which also appeared in unpublished work of Fiore [**21**]. Natural models are essentially an alternative presentation of Dybjer's categories with families [**17**], formulated using only category-theoretic structures and universal properties. In this work, the key notion is that of a representable natural transformation, i.e. a transformation between presheaves whose pullbacks along morphisms with representable domain have representable domain.

In Awodey's work, natural models can be shown to model correctly a type theory with Id-types, unit type, Σ-types and Π-types. But, crucially, the approach taken forces the model to validate the so-called η-rules for unit type, Σ-types and Π-types as judgemental equalities. Here, we modify Awodey's approach and introduce a variant of his notion of a natural model that allows us to validate η-rules as propositional, rather than definitional, equalities. One motivation for this work derives from ongoing research by Nicola Gambino and Simon Henry, building on [**26**], on variants of the simplicial model of type theory. In their work, Π-types are interpreted as cofibrant replacements of dependent products, and therefore expected to validate the η-rule only propositionally.

Main contributions

In summary, the main contributions of this thesis are the following.

(1) We extend Weber's work on the theory of polynomials to include vertical morphisms of polynomials (Proposition 2.2.1). This result is specific to the category of groupoids and the proof of the statement linking vertical natural transformations and polynomial morphisms we produce differs from the incorrect one in [**24**].
(2) We show that the category of J-relative F-algebras is isomorphic to category of $F \circ K$-algebras, as soon as we have an adjoint pair $J \dashv K$ (Proposition 2.6.7).
(3) We produce a construction of W-types for split (Theorem 3.2.4), cloven (Theorem 6.1.4) and general fibrations (Theorem 6.2.16). The work is done in a constructive fashion, where we avoid the use of usual ordinal iteration, preferring to work in a constructive meta-theory without powerset, with inductive sets.
(4) We construct dependent W-types for split fibrations (Theorem 5.1.5).
(5) We show pullback stability for W-types in slices (Theorem 4.3.4) and the dependent W-types (Theorem 5.2.3).
(6) We produce a refinement of natural models, where η-equality is propositionally valid (Theorem 7.2.8).

Outline of the thesis

Chapter 1: This chapter reviews some background useful for the rest of the thesis. We recall the basic definitions of 2-categories, 2-functors, 2-natural transformations and 2-adjoints. Next, we discuss various forms of fibrations, recall the definition of cloven and split fibrations. We introduce the notion of generalized natural transformations and use them to define a 2-adjoint for cartesian product in slices of $\mathbf{Cat}$ for Conduché fibrations. This is then transferred to the 2-category of groupoids. Finally, we recall the construction of dependent products from the exponential objects.

Chapter 2: We recall the notion of polynomials and polynomial functors. We adapt the theory of polynomial functors to the category of groupoids. This chapter also recalls the concept of an endofunctor algebra and morphisms (both strict and pseudo). We give an inductive definition of smallest subalgebras for polynomial functors, avoiding the need for power set. We show that initiality implies 2-initiality for polynomial functors in groupoids. Further, we show that strict 2-initiality implies homotopy initiality. We conclude with a discussion of J-relative algebras and how they relate to endofunctor algebras.

Chapter 3: We construct W-types for split fibrations. This construction is then performed on an example. We also provide an alternative construction using W-types for presheaves given by Moerdijk and Palmgren.

Chapter 4: Using the previous chapter (Chapter 3), we construct W-types for split fibrations in slices. We show pullback stability.

Chapter 5: We construct dependent W-types for split fibrations. Again, we show pullback stability.

Chapter 6: We give a construction of W-types for cloven fibrations. Next we define triangle graphs, and give a construction of W-types for general fibrations.

Chapter 7: We recall the notion of natural models, and refine the established definition in order to model type theory with propositional η-equalities.

CHAPTER 1

Preliminaries

In this chapter we review some basic facts about 2-categories and in particular the 2-category of small categories, $\mathbf{Cat}$, and 2-category of groupoids, $\mathbf{Gpd}$.

We recall the various notions of fibrations and how they relate. We then show that despite the fact that $\mathbf{Cat}$ is not locally cartesian closed, we have exponentials in the slice categories when the exponent is a Conduché fibration. This is done by explicit construction, with a view towards our development.

The chapter ends by reviewing the construction of dependent products from exponentials, which will allow us to give an explicit definition of polynomial functors.

1.1. 2-categories

A 2-*category* is a category enriched over the category $\mathbf{Cat}$. That is, the hom-objects are categories and the composition morphisms are functors. We will unfold this definition to fix notation, and expose some basic facts of the theory of 2-categories. The interested reader is directed to consult the literature, in particular [**12**], [**32**] and [**35**].

Definition 1.1.1. A 2-category $\mathcal{C}$ consists of:

- A collection of objects, $\mathcal{C}_0$. We denote its members by $X, Y, \ldots$.
- For each X, Y in $\mathcal{C}_0$, a category $\mathcal{C}(X, Y)$. We denote the objects of $\mathcal{C}(X, Y)$ by $f : X \to Y$. Morphisms in $\mathcal{C}(X, Y)$ are denoted by $\alpha : f \Rightarrow g$.
- Composition functors $\circ_{X,Y,Z} : \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \to \mathcal{C}(X, Y)$, for $X, Y, Z \in \mathcal{C}_0$.
- Unit functors $\mathrm{id}_X : 1 \to \mathcal{C}(X, X)$, for $X \in \mathcal{C}_0$

This data is subject to additional conditions, namely that the composition functor is associative and that the unit functors are indeed unital for composition.

Elements of $\mathcal{C}_0$ are usually called 0-cells, and objects and morphisms of $\mathcal{C}(X, Y)$ are called 1- and 2-cells, respectively. Suppose we have $f, g, h : X \to Y$ and 2-cells $\alpha : f \Rightarrow g$ and $\beta : g \Rightarrow h$, we denote the vertical composition (that is composition in $\mathcal{C}(X, Y)$) as $\beta \cdot \alpha : f \Rightarrow h$.

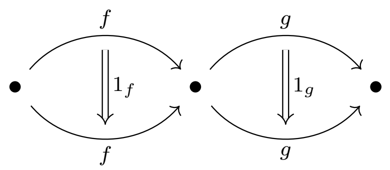
There is also horizontal composition. Suppose $f, g : X \to Y$, $h, i : Y \to Z$, $\alpha : f \Rightarrow g$ and $\beta : h \Rightarrow i$. Since $\circ_{X,Y,Z} : \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$ is a functor we have that $\beta \circ \alpha : h \circ f \Rightarrow i \circ g$.

Since $\mathcal{C}(X, Y)$ is a category, every 1-cell f of it comes equipped with an identity, which we denote by $1_f : f \Rightarrow f$.

We define the operation of *whiskering* as the horizontal composition of a 2-cell with the appropriate identity 2-cell. For $f : X \to Y$, $g, g' : Y \to Z$ and $\varphi : g \Rightarrow g'$, we have $\varphi \circ f : g \circ f \Rightarrow g' \circ f$, defined by

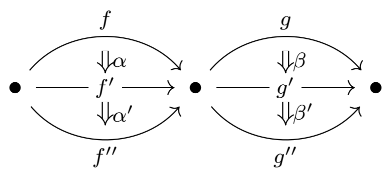
$$\varphi \circ f =_{\text{def}} \varphi \circ 1_f.$$

Functoriality of composition further forces additional equalities to hold, e.g.



$$1_g \circ 1_f = 1_{f \circ g}$$

and also the following (sometimes called the interchange law):



$$(\beta' \cdot \beta) \circ (\alpha' \cdot \alpha) = (\beta' \circ \alpha') \cdot (\beta \circ \alpha)$$

These imply:

$$\begin{aligned} (\alpha \cdot \beta) \circ 1_f &= (\alpha \circ 1_f) \cdot (\beta \circ 1_f) \\ 1_g \circ (\alpha \cdot \beta) &= (1_g \circ \alpha) \cdot (1_g \circ \beta) \\ \beta \circ \alpha &= (1_{g'} \circ \alpha) \cdot (\beta \circ 1_f) \\ &= (\beta \circ 1_{f'}) \cdot (1_g \circ \alpha) \end{aligned}$$

The functor $\mathrm{id}_X : 1 \to \mathcal{C}(X, X)$ provides us with the identity 1-cell for X. We will denote it with id_X and $1_{\mathrm{id}_X} = 1_X$, the identity 2-cell. One of the unital laws forces $f \circ \mathrm{id}_X = f$ and $\alpha \circ 1_X = \alpha$. The same holds when composing id_X on the other side.

Example 1.1.2. The prototypical example of a 2-category is $\mathbf{Cat}$, the category of (small) categories, with functors as 1-cells and natural transformations as 2-cells.

Example 1.1.3. A category is a *groupoid* if every morphism is an isomorphism. The 2-category of groupoids, $\mathbf{Gpd}$, is the full sub-2-category of $\mathbf{Cat}$, where the objects are groupoids.

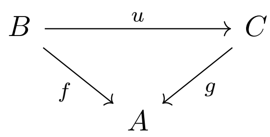
We briefly recall come constructions that will be used later.

Let $\mathcal{C}$ be a 2-category. We then have $\mathcal{C}^{\mathrm{op}}$, where we invert the directions of 1-cells.

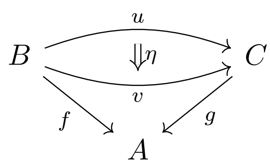
Let $\mathcal{C}$, $\mathcal{D}$ be two 2-categories. We denote with $\mathcal{C} \times \mathcal{D}$ the product 2-category. Its objects are pairs (X, X') for $X \in \mathcal{C}_0$ and $X' \in \mathcal{D}_0$. The morphism categories $\mathcal{C} \times \mathcal{D}((X, X'), (Y, Y'))$ are $\mathcal{C}(X, Y) \times \mathcal{D}(X', Y')$ and the composition and unit functors are products as well.

Let $\mathcal{C}$ be a 2-category and $A \in \mathcal{C}_0$, we define a *strict slice* category, $\mathcal{C}_{/A}$:

- Objects are morphisms $f : B \to A$
- Morphisms between two objects $f : B \to A$ and $g : C \to A$ are morphisms $u : B \to C$, making the following diagram commute:



- The 2-cells are 2-cells in $\mathcal{C}$, such that:



  That is, $\eta : u \Rightarrow v$, such that $g \circ \eta = \mathrm{id}_f$.
- All ways of composing 1 and 2-cells are inherited from $\mathcal{C}$

2-functors, 2-natural transformations.

Definition 1.1.4. A 2-*functor* $F : \mathcal{C} \to \mathcal{D}$ consists of a $\mathcal{D}$ object $F(A)$ for any $A \in \mathcal{C}_0$ and functor $F_{A,B} : \mathcal{C}(A, B) \to \mathcal{D}(FA, FB)$, for any pair $A, B \in \mathcal{C}_0$. The functor must additionally respect the enrichment structure:

$$F_{A,A}(\mathrm{id}_A) = \mathrm{id}_{FA}$$

$$F_{A,C}(g \circ f) = F_{B,C}(g) \circ F_{A,B}(f)$$

$$F_{A,C}(\beta \circ \alpha) = F_{B,C}(\beta) \circ F_{A,B}(\alpha)$$

For any 2-category $\mathcal{C}$, we write $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ for the identity 2-functor. Similarly, we write $\hom_{\mathcal{C}}(-, -) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Cat}$ for the $\hom$ 2-functor.

Definition 1.1.5. Given two 2-functors $F, G : \mathcal{C} \to \mathcal{D}$, we can define a *(strict)* 2-*natural transformation* $\eta : F \Rightarrow G$, as a collection of 1-cells $\eta_A : FA \to GA$ in $\mathcal{D}$ for $A \in \mathcal{C}_0$, such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C}(A,B) & \xrightarrow{F_{A,B}} & \mathcal{D}(FA,FB) \\ {\scriptstyle G_{A,B}}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{D}(FA,\eta_B)} \\ \mathcal{D}(GA,GB) & \xrightarrow[\mathcal{D}(\eta_A,GB)]{} & \mathcal{D}(FA,GB) \end{array}$$

This implies the usual naturality condition (for 1-cells),

$$\begin{array}{ccc} FA & \xrightarrow{\eta_A} & GA \\ {\scriptstyle F_{A,B}(f)}\big\downarrow & & \big\downarrow{\scriptstyle G_{A,B}(f)} \\ FB & \xrightarrow[\eta_B]{} & GB \end{array}$$

$$\eta_B \circ F_{A,B}(f) = G_{A,B}(f) \circ \eta_A$$

and, additionally, for a 2-cell α:

$$FA \underset{Fg}{\overset{Ff}{\overset{\curvearrowright}{\underset{\curvearrowright}{\Downarrow F(\alpha)}}}} FB \xrightarrow{\eta_B} GB \;=\; FA \xrightarrow{\eta_A} GA \underset{Gg}{\overset{Gf}{\overset{\curvearrowright}{\underset{\curvearrowright}{\Downarrow G(\alpha)}}}} GB$$

$$1_{\eta_B} \circ F_{A,B}(\alpha) = G_{A,B}(\alpha) \circ 1_{\eta_A}$$

The identity 2-natural transformation on a 2-functor F is given by $(1_F)_A = \mathrm{id}_{FA}$. The whiskering operation on 2-natural transformations is formally defined identically as in the usual case.

Definition 1.1.6. If $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ are 2-functors, then we say L is a *(strict)* 2-*adjoint* to R, if either of the two equivalent conditions hold:

- there exists a strict 2-natural isomorphism $\mathcal{D}(L(-), -) \Rightarrow \mathcal{C}(-, R(-))$
- there exists a pair of strict 2-natural transformations $\eta : 1_{\mathcal{C}} \Rightarrow RL$ and $\epsilon : LR \Rightarrow 1_D$, such that, the triangle laws are satisfied:

$$\begin{array}{ccc} L & \xrightarrow{L\eta} & LRL \\ & {\scriptstyle \mathrm{id}_L}\searrow & \big\downarrow{\scriptstyle \epsilon L} \\ & & L \end{array} \qquad \begin{array}{ccc} R & \xrightarrow{\eta R} & RLR \\ & {\scriptstyle \mathrm{id}_R}\searrow & \big\downarrow{\scriptstyle R\epsilon} \\ & & R \end{array}$$

1.2. Fibrations in Cat

In topology and homotopy theory the notion of fibration is a continuous map satisfying some lifting conditions. This idea can be presented in other settings. In this section we recall the various notions of fibration between categories.

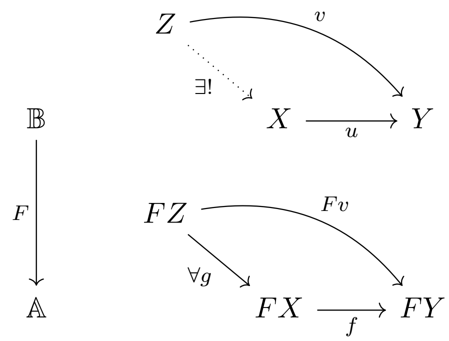
Definition 1.2.1. Let $F : \mathbb{B} \to \mathbb{A}$ be a functor. A morphism $u : X \to Y$ is an F-*cartesian morphism* over f, if for any arrow $v : Z \to Y$ in $\mathbb{B}$ and any $g : FZ \to FX$, such that the following commutes in $\mathbb{A}$:

$$\begin{array}{ccc} FZ & & \\ {\scriptstyle g}\downarrow & \searrow^{Fv} & \\ FX & \xrightarrow[f=Fu]{} & FY \end{array}$$

there exists a unique $w : Z \to X$ in $\mathbb{B}$ over g, making the following diagram commute in $\mathbb{B}$:

$$\begin{array}{ccc} Z & & \\ {\scriptstyle \exists! w}\downarrow & \searrow^{v} & \\ X & \xrightarrow[u]{} & Y \end{array}$$

Put together we can visualize the situation in the following diagram:



A morphism $u : X \to Y$ is F-*opcartesian* over f, if it is cartesian for $F^{\mathrm{op}} : \mathbb{B}^{\mathrm{op}} \to \mathbb{A}^{\mathrm{op}}$.

Definition 1.2.2. We say $F : \mathbb{B} \to \mathbb{A}$ is a *Grothendieck fibration* if for any $f : A \to FY$ in $\mathbb{A}$, there is an F-cartesian morphism u with codomain Y over f.

Definition 1.2.3. Let $F : \mathbb{B} \to \mathbb{A}$ and $F' : \mathbb{B}' \to \mathbb{A}$ be two fibrations. We say that a functor in $\mathbf{Cat}_{/\mathbb{A}}$ between F and F' is *cartesian* if it maps F-cartesian arrows in $\mathbb{B}$ to F'-cartesian arrows in $\mathbb{B}'$.

Definition 1.2.4. A fibration $F : \mathbb{B} \to \mathbb{A}$ is *cloven*, if it comes equipped with a choice of cartesian lifts. That is, for any $f : A \to B$ and X over B, an object f^*X and a cartesian morphism $\overline{f}_X : f^*X \to X$ over f:

$$\begin{array}{ccc} \mathbb{B} & & f^*X \xrightarrow{\overline{f}_X} X \\ \downarrow{\scriptstyle F} & & \\ \mathbb{A} & & A \xrightarrow{f} B = FX \end{array}$$

Remark 1.2.5. Similarly, we say that an opfibration is cloven, if it comes equipped with a choice of cartesian morphisms $\underline{f}_X : X \to f_! X$, for any $f : A \to B$ and X over A.

Let $F : \mathbb{B} \to \mathbb{A}$ be a functor. For any $A \in \mathbb{A}$, we denote with $\mathbb{B}_A$ the fibre over A, as defined by the following pullback:

$$\begin{array}{ccc} \mathbb{B}_A & \longrightarrow & \mathbb{B} \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle F} \\ 1 & \xrightarrow[A]{} & \mathbb{A} \end{array}$$

If F is cloven, then given a morphism $f : A \to B$ in $\mathbb{A}$, the cleavage data gives rise to a functor $f^* : \mathbb{B}_B \to \mathbb{B}_A$. f^* maps objects $X \in \mathbb{B}_B$ to f^*X and morphisms $u : X \to Y$, to the unique morphism making the following diagram commute:

$$\begin{array}{ccc} f^*X & \xrightarrow{\overline{f}_X} & X \\ {\scriptstyle f^*u}\big\downarrow & & \big\downarrow{\scriptstyle u} \\ f^*Y & \xrightarrow[\overline{f}_Y]{} & Y \end{array}$$

For any automorphism $f : A \to A$, the cleavage provides a natural transformation

$$\overline{f} : f^* \Rightarrow \mathrm{id}_{\mathcal{B}_A} \tag{1.1}$$

(or $\underline{f} : \mathrm{id}_{\mathcal{B}_A} \Rightarrow f_!$, for the case of opfibrations).

Suppose $f : A \to B$, $g : B \to C$ and X over C, we find ourselves with the following picture:

$$\begin{array}{ccccccc} & & (g \circ f)^*X & & \xrightarrow{\overline{(g\circ f)}_X} & & \\ \mathbb{B} & & f^*(g^*X) & \xrightarrow{\overline{f}_{g^*X}} & g^*X & \xrightarrow{\overline{g}_X} & X \\ {\scriptstyle F}\big\downarrow & & & & & & \\ \mathbb{A} & & A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

Since $\overline{f}$, $\overline{g}$, and, $\overline{g \circ f}$ are cartesian (since the composition of cartesian morphisms is again a cartesian morphism), there is a unique morphism $(g \circ f)^*X \to f^*(g^*X)$ making the diagram commute and the same holds for the other direction. This gives us a (unique) natural isomorphism $(g \circ f)^* \cong f^* \circ g^*$, which we will call *intermediating morphism*.

Definition 1.2.6. A cloven fibration $F : \mathbb{B} \to \mathbb{A}$ is called *split*, if for all $a \in \mathbb{A}$ and all $b \in \mathbb{B}$ over a:

$$\overline{\mathrm{id}_a}_b = \mathrm{id}_b$$

Further, for all $f : a \to a'$, $g : a' \to a''$ and $b \in \mathbb{B}$ over a:

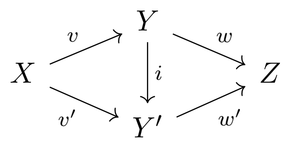
$$\overline{g}_{f^*b} \circ \overline{f}_b = \overline{g \circ f}_b$$

Definition 1.2.7. A functor $F : \mathbb{B} \to \mathbb{A}$ is an *isofibration*, if for any $Y \in \mathbb{B}$ and any isomorphism $f : A \to FY$, there exists an isomorphism $u : X \to Y$, such that $Fu = f$.

In [**14**] Conduché classified the functors in $\mathbf{Cat}$ that are exponentiable in slices.

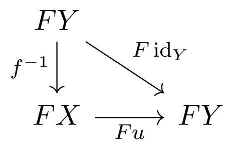
Definition 1.2.8. A functor $F : \mathbb{B} \to \mathbb{A}$ is *Conduché fibration*, if for any morphism $u : X \to Z$ and any factorization of Fu, $FX \xrightarrow{f} B \xrightarrow{g} FZ$, there is:

- a factorization $X \xrightarrow{v} Y \xrightarrow{w} Z$ of u, with $Fv = f$ and $Fw = g$, and,
- any two such factorizations are connected via a zig-zag of morphisms i over the identity id_B and make the triangles in the following diagram commute:

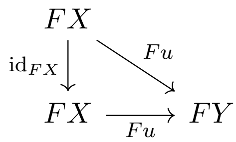


Lemma 1.2.9. *Any Grothendieck fibration in* $\mathbf{Cat}$ *is an isofibration.*

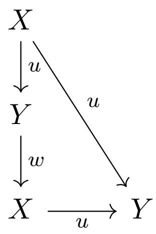
Proof. We will show that a cartesian arrow over an isomorphism is again an isomorphism. Let $F : \mathbb{B} \to \mathbb{A}$ be a Grothendieck fibration and let $f : A \to FY$ be an isomorphism in $\mathcal{A}$. We have a cartesian $u : X \to Y$ over f. This cartesian property gives us a section of u, which we will call $w : Y \to X$, since the following diagram commutes:



That is, $u \circ w = \mathrm{id}_Y$. Uniqueness of lifts will guarantee that $w \circ u = \mathrm{id}_X$, observe:



The unique lift of id_{FX} is id_X, but we also have that the following commutes (and $F(w \circ u) = \mathrm{id}_{FY}$):



∎

Lemma 1.2.10. *Any Grothendieck fibration in* $\mathbf{Cat}$ *is a Conduché fibration.*

Proof. Let $F : \mathbb{B} \to \mathbb{A}$ be a Grothendieck fibration, and suppose that we have a factorisation of Fu, $FX \xrightarrow{f} B \xrightarrow{g} FZ$. The first step is to construct a factorisation of u. Since, F is a fibration, we have a cartesian $w : Y \to Z$, over g. This means that $B = FY$, which in turn allows us to obtain a cartesian lift of f, say $q : X' \to Y$ over it. We have two arrows $Fu : FX \to FZ$ and $F(w \circ q) : FX' \to FZ$, with $w \circ q$ cartesian (since the composition of cartesian arrows is cartesian), that are mapped to the same morphism by F. As such we have a unique arrow over id_{FX}, say $q' : X \to X'$. Thus the factorisation of u in the domain is $q \circ q'$ followed by w.

Suppose now, we have another factorisation, $X \xrightarrow{\widetilde{q}} \widetilde{Y} \xrightarrow{\widetilde{w}} Z$. We first obtain a unique arrow over id_B, $l : \widetilde{Y} \to Y$, such that $w \circ l = \widetilde{w}$. This will be our connecting morphism. In order to see that, consider the unique arrow over id_{FX} obtained by the lifting problem assigned to $F(l \circ \widetilde{q})$ and Fq. Note that this same morphism also links u and $w \circ q$, so by uniqueness we have $q' = v$, and as such l is the linking morphism between the two factorisations. ∎

1.3. Exponentiability of fibrations

This part recalls exponentiability of functors in strict slices of $\mathbf{Cat}$.

Definition 1.3.1. A category $\mathcal{C}$ is locally cartesian closed, if it has a terminal object and for all $X \in \mathcal{C}$, the slice category $\mathcal{C}_{/X}$ is cartesian closed.

Remark 1.3.2. There are several other equivalent ways of defining the property of being locally cartesian closed. Another common definition, is to demand existence of a terminal object and left and right adjoint to the pullback functor $\Delta_f : \mathcal{C}_{/A} \to \mathcal{C}_{/B}$ for all morphisms $f : B \to A$ in $\mathcal{C}$. This will be reviewed in the last section of this chapter.

The first thing to note is that $\mathbf{Cat}$ is not locally cartesian closed (adapted from [**30**, page 48])

Lemma 1.3.3. $\mathbf{Cat}$ *is not locally cartesian closed.*

Proof. Consider $[n] = \{0, 1, \dots, n-1\}$, a groupoid with a unique arrow between each pair of objects and let $f : [2] \to [3]$, such that $f(0 \to 1) = 0 \to 1$ and $g : [2] \to [3]$, with $g(0 \to 1) = 1 \to 2$. Then $f + h : [2] + [2] \to [3]$ is a regular epi. Take $i : [2] \to [3]$, with $i(0 \to 1) = 0 \to 2$. The pullback of $f + h$ along i is no longer an epi, which should be the case if $\mathbf{Cat}$ is locally cartesian closed. (The pullback is $j_0 + j_1 : [1] + [1] \to [2]$, with j_i mapping 0 to i.). $\square$

Although, $\mathbf{Cat}$ is not locally cartesian closed, we have existence of exponentials in certain cases.

Definition 1.3.4. Let $F : \mathbb{B} \to \mathbb{A}$ be a Conduché fibration, $G : \mathbb{C} \to \mathbb{A}$ a functor, $f : A \to A'$ a morphism in $\mathcal{A}$, $T : \mathbb{B}_A \to \mathbb{C}$, $T' : \mathbb{B}_{A'} \to \mathbb{C}$, such that $G \circ T = F\mid_{\mathbb{B}_A}$ and $G \circ T' = F\mid_{\mathbb{B}_{A'}}$. A *generalized natural transformation* $\eta : T \rightsquigarrow T'$ over $f : A \to A'$, is a collection of arrows, $\eta_u : TX \to T'Y$ in $\mathbb{C}$, for all $u : X \to Y$ over f, such that whenever

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X' & \xrightarrow[v]{} & Y' \end{array}$$

commutes in $\mathbb{B}$ (with i being over id_A, j over $\mathrm{id}_{A'}$), we have that the following diagram commutes, as well:

$$\begin{array}{ccc} TX & \xrightarrow{\eta_u} & T'Y \\ {\scriptstyle Ti}\downarrow & & \downarrow{\scriptstyle T'j} \\ TX' & \xrightarrow[\eta_v]{} & T'Y \end{array}$$

This definition is similar to the definitions in [**25**] and to the modules viewpoint provided in [**45**]. The definition we provide is stated in more basic terms.

Note that the above definition says that for $\eta : T \rightsquigarrow T'$, and u over $f : A \to B$, i over id_A, j over id_B

$$T'j \circ \eta_u = \eta_{j \circ u}$$

$$\eta_u \circ T'i = \eta_{u \circ i}$$

Remark 1.3.5. Given a choice of a cleavage, we can see that any generalized natural transformation $\eta : T \rightsquigarrow T'$, defines a natural transformation $\epsilon : T \circ f^* \Rightarrow T'$. We need a collection of morphisms $\epsilon_X : Tf^*X \to T'X$. Let

$$\epsilon_X = \eta_{\overline{f}_X}$$

We need the following to commute:

$$\begin{array}{ccc} Tf^*X & \xrightarrow{\epsilon_X} & T'X \\ {\scriptstyle Tf^*u}\downarrow & & \downarrow{\scriptstyle T'f} \\ Tf^*Y & \xrightarrow[\epsilon_Y]{} & TY \end{array}$$

By the definition of generalized natural transformation, this commutes if the next diagram does:

$$\begin{array}{ccc} f^*X & \xrightarrow{\overline{f}_X} & X \\ {\scriptstyle f^*u}\downarrow & & \downarrow{\scriptstyle f} \\ f^*Y & \xrightarrow[\overline{f}_Y]{} & Y \end{array}$$

But this commutes, since it is exactly the definition of f^*.

Remark 1.3.6. We have a notion of whiskering for generalized natural transformations. Suppose $\eta : T \rightsquigarrow T'$, and $G : \mathbb{C} \to \mathbb{D}$, we define $G\eta : GT \rightsquigarrow GT'$ by letting

$$(G\eta)_u =_{\text{def}} G\eta_u$$

The required squares commute because of the functoriality of G.

Proposition 1.3.7. *Given T, $T' : \mathbb{B}_A \to \mathbb{C}$, generalized natural transformations $T \rightsquigarrow T'$ defined over id_A are in one-to-one correspondence to natural transformations $T \to T'$.*

Proof. ($\Rightarrow$) Given $\eta : T \rightsquigarrow T'$, then we define $\epsilon_X = \eta_{\mathrm{id}_X}$ (since id_X is over id_A). We need the following to commute for $u : X \to Y$:

$$\begin{array}{ccc} TX & \xrightarrow{\epsilon_X} & T'X \\ {\scriptstyle Tu}\downarrow & & \downarrow{\scriptstyle T'u} \\ TY & \xrightarrow[\epsilon_Y]{} & T'y \end{array}$$

This commutes, since the following does:

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u} \\ Y & \xrightarrow[\mathrm{id}_Y]{} & Y \end{array}$$

($\Leftarrow$) Given $\epsilon : T \Rightarrow T'$, define $\eta_u = \epsilon_Y \circ T(u)$. Then suppose:

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ W & \xrightarrow[v]{} & Z \end{array}$$

The desired property (that of, η being a generalized natural transformation) is verified by:

$$\begin{array}{ccccc} TX & \xrightarrow{Tu} & FY & \xrightarrow{\epsilon_Y} & T'y \\ {\scriptstyle Ti}\downarrow & & \downarrow{\scriptstyle Tj} & & \downarrow{\scriptstyle Tj} \\ TW & \xrightarrow[Tv]{} & TZ & \xrightarrow[\epsilon_Z]{} & T'z \end{array}$$

The left square commutes by functoriality of Y, the right commutes because of naturality of ϵ.

Given $\eta : T \rightsquigarrow T'$, mapping it to a natural transformation and back, we obtain $\eta'_u = \eta_{\mathrm{id}_Y} \circ Tu$, but we know that $\eta_{\mathrm{id}_Y} \circ Tu = \eta_u$. Similarly, given $\epsilon : T \Rightarrow T'$, we obtain $\epsilon'_X = \epsilon_X \circ T\,\mathrm{id}_X$, which equals ϵ_X. □

Proposition 1.3.8. *Let* $(f : A \to B, \eta) : T^1 \rightsquigarrow T^2$ *and* $(g : B \to C, \epsilon) : T^2 \rightsquigarrow T^3$. *We can define a generalized natural transformation* $(g \circ f, \epsilon \circ \eta) : T^1 \rightsquigarrow T^3$*, called the* composite *of* (f, η) *and* (g, ϵ). *Furthermore, this operation is associative and unital.*

Proof. Given (f, η) and (g, ϵ) as in the proposition statement, we first define the composition $\epsilon \circ \eta : T^1 \rightsquigarrow T^3$ over $g \circ f$. Let u over $(g \circ f)$. Since F is a Grothendieck fibration, we have that it satisfies the Conduché criterion. Then $g \circ f$ is a factorization of Fu in $\mathcal{A}$, we have $w \circ v$, a factorization of u in $\mathcal{B}$. Set $(\epsilon \circ \eta)_u = \epsilon_w \circ \eta_v$. This assignment is *essentially* unique, by the second condition of the Conduché criteria. Suppose we had a different factorization $w' \circ v'$. We then have a zig-zag of morphisms connecting the two factorisations. Suppose for the moment, that the zig-zag is of length 1, that is, $i : Y \to Y'$ over id_B, connecting the two factorization:

$$\begin{array}{ccccc} X & \xrightarrow{v} & Y & \xrightarrow{w} & Z \\ {\scriptstyle \mathrm{id}_X}\| & & \downarrow{\scriptstyle i} & & \|{\scriptstyle \mathrm{id}_Z} \\ X & \xrightarrow[v']{} & Y' & \xrightarrow[w']{} & Z \end{array}$$

By the definition of generalized natural transformations, the following commutes:

$$\begin{array}{ccccc} T^1X & \xrightarrow{\eta_v} & T^2Y & \xrightarrow{\epsilon_w} & T^3Z \\ {\scriptstyle \mathrm{id}_{T^1X}}\| & & \downarrow{\scriptstyle T^2 i} & & \|{\scriptstyle \mathrm{id}_{T^3Z}} \\ T^1X & \xrightarrow[\eta_{v'}]{} & T^2Y' & \xrightarrow[\epsilon_w]{} & T^3Z \end{array}$$

Thus, $\epsilon_w \circ \eta_v = \epsilon_{w'} \circ \eta_v$.

In case the zig-zag of intermediary morphisms was longer, we can see that we only need to repeat the argument for each morphism along the path.

To see that this really defines a generalized natural transformation, suppose

$$\begin{array}{ccc} X & \xrightarrow{u} & Z \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X' & \xrightarrow[u']{} & Z' \end{array}$$

Again, by Conduché criterion, we have $w \circ v = u$ and $w' \circ v' = u'$. Further, the factorizations $(j \circ w) \circ v = w \circ (v' \circ i)$ are connected by a zig-zag of morphisms. As before, let us suppose the path of intermediary morphisms is of length 1. This allows us to propose the following diagram (where l is the previously mentioned intermediary morphism):

$$\begin{array}{ccccc} X & \xrightarrow{v} & Y & \xrightarrow{j \circ w} & Z' \\ {\scriptstyle \mathrm{id}_X}\Big\| & & \downarrow{\scriptstyle l} & & \Big\|{\scriptstyle \mathrm{id}'_Z} \\ X & \xrightarrow[v' \circ i]{} & Y' & \xrightarrow[w']{} & Z' \end{array}$$

Since both ϵ and η are generalized natural transformations, the following commutes:

$$\begin{array}{ccccc} T_1X & \xrightarrow{\eta_v} & T_2Y & \xrightarrow{\epsilon_{j \circ w}} & T_3Z' \\ {\scriptstyle T_1\,\mathrm{id}_X}\Big\| & & \downarrow{\scriptstyle T_2 l} & & \Big\|{\scriptstyle T_3\,\mathrm{id}'_Z} \\ T_1X & \xrightarrow[\eta_{v' \circ i}]{} & T_2Y' & \xrightarrow[\epsilon_{w'}]{} & T_3Z' \end{array}$$

So, $\epsilon_{j \circ w} \circ \eta_v = \epsilon_{w'} \circ \eta_{v' \circ i}$, but from the discussion before this means that $T_3 j \circ \epsilon_w \circ \eta_v = \epsilon_{w'} \circ \eta_{v'} \circ T_1 i$. As in the previous argument, if the zig-zag of the intermediary morphisms is longer, we simply repeat the argument along each step.

Suppose $\eta : T_1 \rightsquigarrow T_2$, $\epsilon : T_2 \rightsquigarrow T_3$ and $\chi : T_3 \rightsquigarrow T_4$ (over f, g, and h, respectively). Take u over $(h \circ g \circ f)$. Observe $((\chi \circ \epsilon) \circ \eta)_u = (\chi \circ \epsilon)_w \circ \eta_v = (\chi_s \circ \epsilon_t) \circ \eta_v$ and $(\chi \circ (\epsilon \circ \eta))_u = \chi_{s'} \circ (\epsilon_{t'} \circ \eta_{v'})$. Shifting things around a bit, we get $(\chi_{s'} \circ \epsilon_{t'}) \circ \eta_{v'} = (\chi \circ \epsilon)_{s' \circ t'} \circ \eta_{v'}$. We showed before that composition of natural transformation isn't sensitive to any particular factorization, as such $(\chi \circ \epsilon)_{s' \circ t'} \circ \eta_{v'} = (\chi \circ \epsilon)_w \circ \eta_v$. Thus composition of generalized natural transformations is associative.

Finally, let $(f : A \to B, \eta) : T^1 \rightsquigarrow T^2$, we define the identity arrow $(\mathrm{id}_a, \mathrm{id}_{T^1}) : T^1 \rightsquigarrow T^1$ as

$$(\mathrm{id}_{T^1})_u = T^1 u$$

Then, we have that $(f, \eta) \circ (\mathrm{id}_a, \mathrm{id}_{T^1}) = (f, \eta \circ \mathrm{id}_{T^1})$. Let $u : X \to Y$, be over f. By definition of composition, we have:

$$\begin{aligned} (\eta \circ \mathrm{id}_{T^1})_u &= \eta_u \circ (\mathrm{id}_{T^1})_{\mathrm{id}_X} \\ &= \eta_u \circ T^1(\mathrm{id}_X) \\ &= \eta_u \end{aligned}$$

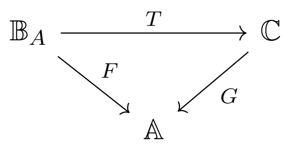
Composing with the identity on the left side is analogous. □

It is a well known result that the product functor $(-) \times F : \mathbf{Cat}_{/\mathbb{A}} \to \mathbf{Cat}_{/\mathbb{A}}$ has a right adjoint in $\mathbf{Cat}_{/\mathbb{A}}$ if and only if F is a Conduché fibration. We give the proof of one of the implications, by explicitly constructing the exponential object using generalized natural transformations.

Theorem 1.3.9 ([**14**])**.** *Let $F : \mathbb{B} \to \mathbb{A}$. If F is a Conduché fibration, then $(-) \times \mathbb{B} : \mathbf{Cat}_{/\mathbb{A}} \to \mathbf{Cat}_{/\mathbb{A}}$ has a right 2-adjoint.*

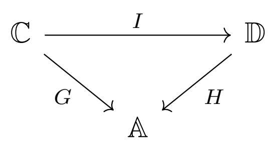
Proof. Let $F : \mathbb{B} \to \mathbb{A}$ be a Conduché fibration and $G : \mathbb{C} \to \mathbb{A}$. We define $\mathbb{C}^{\mathbb{B}} \xrightarrow{\pi} \mathbb{A}$, the *exponential object*, as follows:

- objects are pairs $(A, T : \mathbb{B}_A \to \mathbb{C})$, such that $G \circ T = F$:



- morphisms between (A, T) and (A', T') are pairs (f, η), where $f : A \to A'$ in $\mathbb{A}$ and $\eta : T \rightsquigarrow T'$ defined over f, with composition defined as above,
- $\mathbb{C}^{\mathbb{B}} \xrightarrow{\pi} \mathbb{A}$, projects on the first component.

$(-)^{\mathbb{B}}$ in fact defines a 2-functor $\mathbf{Cat}_{/\mathbb{A}} \to \mathbf{Cat}_{/\mathbb{A}}$. Suppose a map $I : \mathbb{C} \to \mathbb{D}$:



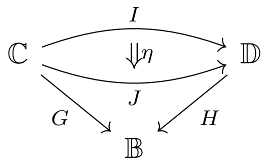
Then $I^{\mathbb{B}}$ maps (A, T) to $(A, I \circ T)$ and morphism (f, η) to $(f, I \circ \eta)$. See that

$$\begin{aligned}(I \circ \mathrm{id}_T)_u &= ITu \\ &= (\mathrm{id}_{I \circ T})_u\end{aligned}$$

Further,

$$\begin{aligned}(I \circ \epsilon)_w \circ (I \circ \eta)_v &= I\epsilon_w \circ I\eta_v \\ &= I(\epsilon_w \circ \eta_v) \\ &= I \circ (\epsilon \circ \eta)_u\end{aligned}$$

As such, $I^{\mathbb{B}}$ is well defined. Further observe that $J^{\mathbb{B}} \circ I^{\mathbb{B}} = (I \circ J)^{\mathbb{B}}$. Now, suppose $\eta : I \Rightarrow J$:



$(\eta^{\mathbb{B}})_{(A,T)}$ is set to be $(\mathrm{id}_A, (\eta \circ T)')$, where $(\eta \circ T)'$ is the generalized natural transformation assigned to natural transformation $\eta \circ T$, that is $(\eta_{Tu} \circ FTu)_u$. All of the 2-functoriality conditions are straightforward verifications (as above).

For $G : \mathbb{C} \to \mathbb{A}$ and $H : \mathbb{D} \to \mathbb{A}$ we exhibit an isomorphism of categories:

$$\Phi_{G,H} : \mathbf{Cat}_{/\mathbb{A}}(\mathbb{C} \times \mathbb{B}, \mathbb{D}) \to \mathbf{Cat}_{/\mathbb{A}}(\mathbb{C}, \mathbb{D}^{\mathbb{B}})$$

which will be 2-natural in G and H. We define $\Phi_{G,H}$ to act as follows:

- Given $I : \mathbb{C} \times \mathbb{B} \to \mathbb{D}$, we construct a functor $\Phi_I : \mathbb{C} \to \mathbb{D}^{\mathbb{B}}$:
$$X \mapsto (GX, I(X, -))$$
$$f : X \to Y \mapsto (Gf, (I(f, g))_g)$$
The functoriality of the above amounts a bunch of trivial verifications (similar to the one done before).
- Given $\eta : I \Rightarrow J$, we construct a natural transformation $\eta^{\Phi} : \Phi_I \Rightarrow \Phi_J$, by setting:
$$(\eta^{\Phi})_X = (\mathrm{id}_{GX}, (\eta_{(X,Z} \circ I(\mathrm{id}_X, u))_u)$$
Where $u : Y \to Z \in F^{-1}(\mathrm{id}_{GX})$. Again, the required checks are skipped as they are very similar to the ones already performed.

Φ is bijective on objects. Given a $T : \mathbb{C} \to \mathbb{D}^{\mathbb{B}}$, its preimage is $\widehat{T} : \mathbb{C} \times \mathbb{B} \to \mathbb{D}$:

$$(X, Y) \mapsto (\pi_2 TX)Y$$
$$(u, v) \mapsto (\pi_2 Tu)_v$$

It is also bijective on morphisms. Given $\eta : T \Rightarrow T'$, its preimage is $\widehat{\eta} : \widehat{T} \Rightarrow \widehat{T'}$, where $\widehat{\eta}_{X,Y} = (\pi_2 \eta_X)_{\mathrm{id}_Y}$ (it can be shown that $\widehat{eta}$ is natural). Thus, Φ is an isomorphism of categories.

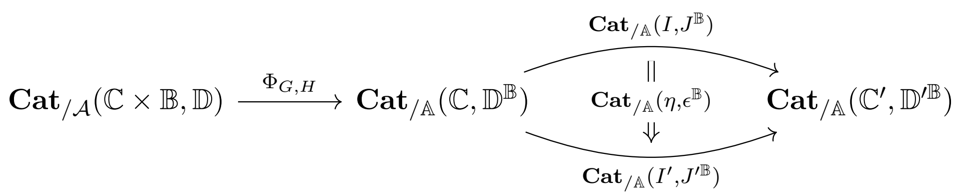
We show that $\Phi_{-,-}$ is natural. Let $G' = G \circ I : \mathbb{C}' \to \mathbb{A}$ and $H' = H \circ J : \mathbb{D}' \to \mathbb{A}$. Then, for $T : \mathbb{C} \times \mathbb{B} \to \mathbb{D}$, we get the following by simply unfolding the definitions:

$$\begin{aligned}
(\mathbf{Cat}_{/\mathbb{A}}(I, J^{\mathbb{B}}) \circ \Phi_{G,H}(T))(X) &= (J^{\mathbb{B}} \circ \Phi_T \circ I)(X) \\
&= (GIX, JT(IX, -)) \\
&= (G'X, JT(IX, -)) \\
(\mathbf{Cat}_{/\mathbb{A}}(I, J^{\mathbb{B}}) \circ \Phi_{G,H}(T))(u) &= (G'u, JT(Iu, -)) \\
(\Phi_{G',H'} \circ \mathbf{Cat}_{/\mathbb{A}}(I \times \mathbb{B}, J)(T))(X) &= \Phi_{J \circ T \circ I \times \mathbb{B}}(X) \\
&= (G'X, JT(IX, -)) \\
(\Phi_{G',H'} \circ \mathbf{Cat}_{/\mathbb{A}}(I \times \mathbb{B}, J)(T))(u) &= (G'u, JT(Iu, -))
\end{aligned}$$

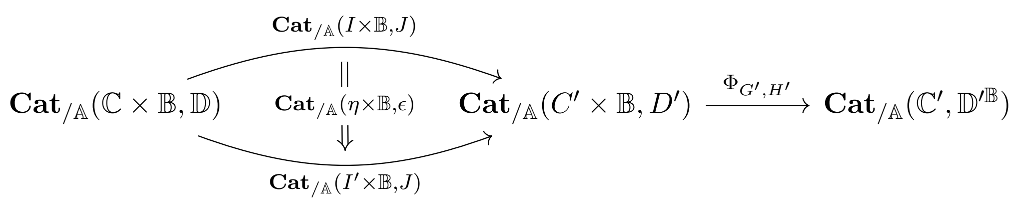
Further, for $\eta : T \Rightarrow T' : \mathbb{C} \times \mathbb{B} \to \mathbb{D}$):

$$\begin{aligned}
(\mathbf{Cat}_{/\mathbb{A}}(I, J^{\mathbb{B}}) \circ \Phi_{G,H}(\eta))_X &= J^{\mathbb{B}}(\eta^{\Phi})_{IX} \\
&= (\mathrm{id}_{G'X}, (J(\eta_{(IX,-)} \circ T(\mathrm{id}_{IX}, -)))) \\
(\Phi_{G',H'} \circ \mathbf{Cat}_{/\mathbb{A}}(I \times \mathbb{B}, J)(\eta))_X &= (\mathrm{id}_{G'X}, (J \circ \eta \circ I \times \mathbb{B})_{(X,-)} \circ (J \circ T \circ I \times \mathbb{B})(\mathrm{id}_X, -)) \\
&= (\mathrm{id}_{G'X}, J(\eta_{(IX,-)} \circ T(\mathrm{id}_{IX}, -)))
\end{aligned}$$

In order to prove 2-naturality, further assume that we have $\eta : I \Rightarrow I' : \mathbb{C}' \to \mathbb{C}$ and $\epsilon : J \Rightarrow J' : \mathbb{D} \to \mathbb{D}'$. We would like that:



equals to:



We can see that this is indeed the case (unfolding the natural transformation defined by the first diagram):

$$\begin{aligned}((\mathbf{Cat}_{/\mathbb{A}}(\eta,\epsilon^{\mathbb{B}})\circ\Phi_{G,H})_T)_X &= (\epsilon^{\mathbb{B}}\circ\Phi_T\circ\eta)_X\\ &= J'^{\mathbb{B}}(\Phi_T\eta_X)\circ\epsilon^{\mathbb{B}}_{\Phi_{TIX}}\\ &= (G\eta_X, J'(T(\eta_X,-))\circ(\mathrm{id}_{G'X},(\epsilon\circ T(\mathrm{id}_{IX},-))')\end{aligned}$$

Let $u : Y \to Z$ over $\mathrm{id}_{G'X}$, then $\epsilon_{T(IX,Z)} \circ JT(\mathrm{id}_{IX}, u) = J'T(\mathrm{id}_{IX}, u) \circ \epsilon_{T(IX,Y)}$. Also keep in mind that $G\eta_X = \mathrm{id}_{G'X}$.

$$(G\eta_X, J'(T(\eta_X,-))\circ(\mathrm{id}_{G'X},(\epsilon\circ T(\mathrm{id}_{IX},-))') = (\mathrm{id}_{G'X}, J'(T(\eta_X,-))\circ\epsilon)$$

Unfolding the natural transformation, given by the second diagram:

$$\begin{aligned}((\Phi_{G',H'}\circ\mathbf{Cat}_{/\mathbb{A}}(\eta\times\mathbb{B},\epsilon))_T)_X &= (\Phi_{G',H'}(\eta\times\mathbb{B}\circ T\circ\epsilon))_X\\ &= (\mathrm{id}_{G'X}, ((\eta\times\mathbb{B}\circ T\circ\epsilon)_{(X,Z)}\circ JT(\mathrm{id}_{IX},u))_u)\\ &= (\mathrm{id}_{G'X}, (J'(T(\eta_X,\mathrm{id}_Z))\circ\epsilon_{T(IX,Z)}\circ JT(\mathrm{id}_{IX},u))_u)\\ &= (\mathrm{id}_{G'X}, J'(T(\eta_X,-))\circ\epsilon)\end{aligned}$$

So, Φ as defined, is a 2-natural isomorphism. □

Let $F : \mathbb{B} \to \mathbb{A}$ be a cloven opfibration and $G : \mathbb{C} \to \mathbb{A}$ be a functor. Using the additional data from the cleavage we can define a simpler version of exponential object $\mathbb{C}^{\mathbb{B}}$. The objects remain the same as in the general case. The morphisms between (a_1, F_1) and (a_2, F_2) are now defined to be pairs $(u : a_1 \to a_2, \eta : F_1 \Rightarrow F_2 \circ \alpha_!)$, with composition defined to be:

$$(u^2,\eta^2)\circ(u^1,\eta^1) =_{\mathrm{def}} (u^2\circ u^1, (1_{F_3}\circ\Phi_{u^1,u^2})\cdot(\eta^2\circ 1_{u^1_!})\cdot\eta^1)$$

where $(u^i, \eta^i) : (a_i, F_i) \to (a_{i+1}, F_{i+1})$. The identities are of the form $(\mathrm{id}_a, (F(\underline{\mathrm{id}_{a_b}}))_b)$.

Lemma 1.3.10. $\mathbb{C}^{\mathbb{B}}$ *with composition, as defined above, is a category.*

Proof. We begin, by showing that the composition is associative. Suppose we have $(u^i, \eta^i) : (a_i, F_i) \to (a_{i+1}, F_{i+1})$ for $i = 1, 2, 3$, then using the cleavage and associated properties of cartesian morphism and intermediating morphisms:

$$
\begin{aligned}
&(u^3,\eta^3)\circ((u^2,\eta^2)\circ(u^1,\eta^1)) = \\
&= (u^3,\eta^3)\circ(u^2\circ u^1, (1_{F_3}\circ\Phi_{u^1,u^2})\cdot(\eta^2\circ 1_{u^1_!})\cdot\eta^1) \\
&= (u^3\circ u^2\circ u^1, (1_{F_4}\circ\Phi_{u^2\circ u^1,u^3})\cdot(\eta_3\circ 1_{(u^2\circ u^1)_!})\cdot(1_{F_3}\circ\Phi_{u^1,u^2})\cdot(\eta^2\circ 1_{u^1_!})\cdot\eta^1) \\
&= (u^3\circ u^2\circ u^1, (1_{F_4}\circ\Phi_{u^2\circ u^1,u^3})\cdot(\eta_3\circ\Phi_{u^1,u^2})\cdot(\eta^2\circ 1_{u^1_!})\cdot\eta^1)
\end{aligned}
$$

Composing the other way, we see:

$$
\begin{aligned}
&((u^3,\eta^3)\circ(u^2,\eta^2))\circ(u^1,\eta^1) = \\
&= (u^3\circ u^2, (1_{F_4}\circ\Phi_{u^2,u^3})\cdot(\eta^3\circ 1_{u^2_!})\cdot\eta^2)\circ(u^1,\eta^1) \\
&= (u^3\circ u^2\circ u^1, (1_{F_4}\circ\Phi_{u^1,u^3\circ u^2})\cdot(((1_{F_4}\circ\Phi_{u^2,u^3})\cdot(\eta^3\circ 1_{u^2_!})\cdot\eta^2)\circ 1_{u^1_!})\cdot\eta^1) \\
&= (u^3\circ u^2\circ u^1, (1_{F_4}\circ\Phi_{u^1,u^3\circ u^2})\cdot(1_{F_4}\circ\Phi_{u^2,u^3}\circ 1_{u^1_!})\cdot(\eta^3\circ 1_{u^2_!}\circ 1_{u^1_!})\cdot(\eta^2\circ 1_{u^1_!})\cdot\eta^1) \\
&= (u^3\circ u^2\circ u^1, (1_{F_4}\circ(\Phi_{u^1,u^3\circ u^2}\cdot(\Phi_{u^2,u^3}\circ 1_{u^1_!})))\cdot(\eta^3\circ 1_{u^2_!}\circ 1_{u^1_!})\cdot(\eta^2\circ 1_{u^1_!})\cdot\eta^1) \\
&= (u^3\circ u^2\circ u^1, (1_{F_4}\circ(\Phi_{u^2\circ u^1,u^3}\cdot(1_{u^3_!}\circ\Phi_{u^1,u^2})))\cdot(\eta^3\circ 1_{u^2_!}\circ 1_{u^1_!})\cdot(\eta^2\circ 1_{u^1_!})\cdot\eta^1) \\
&= (u^3\circ u^2\circ u^1, (1_{F_4}\circ\Phi_{u^2\circ u^1,u^3})\cdot(\eta^3\circ\Phi_{u^1,u^2})\cdot(\eta^2\circ 1_{u^1_!})\cdot\eta^1)
\end{aligned}
$$

Composing with an identity:

$$
\begin{aligned}
(u,\eta)\circ(\mathrm{id}_{a_1},\mathrm{id}_{(a_1,F_1)}) &= (u,(1_{F_2}\cdot\Phi_{\mathrm{id}_{a_1},u})\cdot(\eta\cdot 1_{\mathrm{id}_{a_1!}})\cdot\mathrm{id}_{(a_1,F_1)}) \\
((1_{F_2}\cdot\Phi_{\mathrm{id}_{a_1},u})\cdot(\eta\cdot 1_{\mathrm{id}_{a_1!}})\cdot\mathrm{id}_{(a_1,F_1)})_b &= \mathrm{id}_{F_2u_!b}\circ F_2(\Phi_{\mathrm{id}_{a_1},u})_b\circ\eta_{\mathrm{id}_{a_1!}b}\circ F_1(\mathrm{id}_{\mathrm{id}_{a_1!}b})\circ F_1(\underline{\mathrm{id}_{a_1}}_b) \\
&= F_2(\Phi_{\mathrm{id}_{a_1},u})_b\circ\eta_{\mathrm{id}_{a_1!}b}\circ F_1(\underline{(}\,\mathrm{id}_{a_1})_b) \\
&= F_2((\Phi_{\mathrm{id}_{a_1},u})_b\circ u_!(\underline{\mathrm{id}_{a_1}}_b))\circ\eta_b \\
&= \eta_b \\
(\mathrm{id}_{a_2},\mathrm{id}_{(a_2,F_2)})\circ(u,\eta) &= (u,(1_{F_2}\cdot\Phi_{u,\mathrm{id}_{a_2}})\cdot(\mathrm{id}_{(a_2,F_2)}\cdot 1_{u_!})\cdot\eta)
\end{aligned}
$$

Note that the following hold

$$
\begin{aligned}
(1_{F_2}\circ\Phi_{u,\mathrm{id}_{a_2}})_b\circ(\mathrm{id}_{(a_2,F_2)}\circ 1_{u_!})_b &= F_2((\Phi_{u,\mathrm{id}_{a_2}})_b\circ\underline{\mathrm{id}_{a_2}}_{u_!e}) \\
&= \mathrm{id}_{F_2u_!e}
\end{aligned}
$$

As such we can conclude:

$$
(1_{F_2}\cdot\Phi_{u,\mathrm{id}_{a_2}})\cdot(\mathrm{id}_{(a_2,F_2)}\cdot 1_{u_!})\cdot\eta)_b = \eta_b
$$

□

Proposition 1.3.11. *The exponential object defined for fibrations equipped with a cleavage (as in Lemma 1.3.10) is isomorphic to the exponential object for general fibrations (where we forget the additional cleavage structure).*

Proof. To see this, we construct a mapping in both directions:

$$(u, \eta : F_1 \rightsquigarrow F_2) \overset{\Psi}{\longmapsto} (u, (\eta_{\underline{u}_b})_b)$$

$$(u, \eta : F_1 \Rightarrow F_2 u_!) \overset{\widehat{\Psi}}{\longmapsto} (u, (F_2\Phi_v \circ \eta_{b_1})_{v: b_1 \to b_2 \in p^{-1}(u)})$$

Where Φ_v is the inverse of the unique arrow defined by the cartesian arrow $\underline{u}_{b_1}$ in the following diagram:

$$\begin{array}{ccc} & & b_2 \\ & \overset{v}{\nearrow} & \\ b_1 & \xrightarrow[\underline{u}_{b_1}]{} & u_! b_1 \end{array}$$

A long string of calculations ensures that these really are functors. Suppose $(u, \eta : F_1 \rightsquigarrow F_2)$, then $\widehat{\eta}$ as defined by Ψ is a natural transformation $F_1 \Rightarrow F_2 u_!$, since this diagram commutes:

$$\begin{array}{ccc} b & \xrightarrow{\underline{u}_b} & u_! e \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle u_! f} \\ b' & \xrightarrow{\underline{u}_{b'}} & u_! e' \end{array}$$

As $\eta : F_1 \rightsquigarrow F_2$ is a generalized natural equivalence, the following diagram also commutes:

$$\begin{array}{ccc} F_1 b & \xrightarrow{\eta_{\underline{u}_b}} & F_2 u_! b \\ \downarrow{\scriptstyle F_1 f} & & \downarrow{\scriptstyle F_2 u_! f} \\ F_1 b' & \xrightarrow{\eta_{\underline{u}_{b'}}} & F_2 u_! b' \end{array}$$

Notice that the following diagram commutes:

$$\begin{array}{ccccc} b & \xrightarrow{\underline{u^1}_b} & u^1_! b & \xrightarrow{\underline{u^2}_{u^1_! b}} & u^2_! u^1_! b \\ \| & & & & \downarrow{\scriptstyle (\Phi_{u^1, u^2})_b} \\ b & & \xrightarrow{\underline{u^2 \circ u^1}_b} & & (u^2 \circ u^1)_! b \end{array}$$

We know that $(\eta^2 \circ \eta^1)_{\underline{u^2}_{u^1_! b} \circ \underline{u^1}_b} = \eta^2_{\underline{u^2}_{u^1_! b}} \circ \eta^1_{\underline{u^1}_b}$, hence the following rectangle commutes:

$$\begin{array}{ccccc}
F_1 b & \xrightarrow{\eta^1_{\underline{u^1}_b}} & F_2 u^1_! b & \xrightarrow{\eta^2_{\underline{u^2}_{u^1_! b}}} & F_3 u^2_! u^1_! b \\
\Vert & & & & \big\downarrow{\scriptstyle F_3((\Phi_{u^1,u^2})_b)} \\
F_1 b & & \xrightarrow{\quad (\eta^2 \circ \eta^1)_{\underline{u^2 \circ u^1}_b} \quad} & & F_3(u^2 \circ u^1)_! b
\end{array}$$

This means that Ψ respects composition. Lastly, identities: $\Psi(\mathrm{id}_{(a,F)}) = (\mathrm{id}_a, (F(\underline{u}_b))_b)$, which is the identity in $\widehat{q}^p$.

To check that $\widehat{\Psi}$ is a well defined functor, suppose that we have a commuting square of the form:

$$\begin{array}{ccc}
b_1 & \xrightarrow{v} & b_2 \\
\big\downarrow{\scriptstyle \delta_1} & & \big\downarrow{\scriptstyle \delta_2} \\
b'_1 & \xrightarrow{v'} & b'_2
\end{array}$$

where v, v' are over u and δ_i are in $\mathbb{B}_{a_i}$.

Since $\underline{u}_{b_1}$, $\underline{u}_{b'_1}$ are cartesian, we have that the right square in the following diagram commutes:

$$\begin{array}{ccccc}
 & & \overset{v}{\frown} & & \\
b_1 & \xrightarrow{\underline{u}_{b_1}} & u_! b_1 & \xrightarrow{\Phi_v} & b_2 \\
\big\downarrow{\scriptstyle \delta_1} & & \big\downarrow{\scriptstyle u_! \delta_1} & & \big\downarrow{\scriptstyle \delta_2} \\
b'_1 & \xrightarrow{\underline{u}_{b'_1}} & u_! b'_1 & \xrightarrow{\Phi_{v'}} & b'_2 \\
 & & \underset{v'}{\smile} & &
\end{array}$$

This in turn implies that the following diagram commutes:

$$\begin{array}{ccccc}
F_1 b_1 & \xrightarrow{\eta_{b_1}} & F_2 u_! b_1 & \xrightarrow{F_2 \Phi_v} & F_2 b_2 \\
\big\downarrow{\scriptstyle F_1 \delta_1} & & \big\downarrow{\scriptstyle F_1 u_! \delta_1} & & \big\downarrow{\scriptstyle F_2 \delta_2} \\
F_1 b'_1 & \xrightarrow{\eta_{v'}} & F_2 u_! b'_1 & \xrightarrow{F_2 \Phi_{v'}} & F_2 b'_2
\end{array}$$

So $\widehat{\Psi}$ maps natural transformations to generalized natural transformations.

Suppose we have (u^i, η^i), for $i = 1, 2$. We can see that $\Phi_w \circ u^2_!(\Phi_v) = \Phi_{w \circ v} \circ \Phi_{u_1,u_2}$, for any factorization $w \circ v = l$ over $u_2 \circ u_1$. This implies that:

$$\begin{aligned}
F_3(\Phi_w) \circ \eta^2_v \circ F_2(\Phi_v) \circ \eta^1_v &= F_3(\Phi_w \circ u^2_! \Phi_v) \circ \eta^2_{u^1_! v} \circ \eta^1_v \\
&= F_3(\Phi_l) \circ F_3(\Phi_{u_1,u_2}) \circ \eta^2_{u^1_! b_1} \circ \eta^1_{b_1}
\end{aligned}$$

Thus, $\widehat{\Psi}$ respects composition. Again, $\widehat{\Psi}(\mathrm{id}_{(a,F)}) = (\mathrm{id}_a, (F(\Phi_v \circ \underline{\mathrm{id}_a}_b))_v)$, but this is by definition equal to $(\mathrm{id}_a, (F(v))_v)$. Hence, $\widehat{\Psi}$ is a functor.

Since $\Phi_{\underline{u}_b} = id_{u_! b}$, we have that $\Psi \circ \widehat{\Psi}((u, \eta)) = (u, \eta)$

On the other hand the following square commutes:

$$\begin{array}{ccc} b_1 & \xrightarrow{\underline{u}_{b_1}} & u_! b_1 \\ \| & & \downarrow{\scriptstyle \Phi_v} \\ b_1 & \xrightarrow{v} & b_2 \end{array}$$

for any v over u. This means that for any generalized natural transformation η over u:

$$\eta_v = F_2(\Phi_v) \circ \eta_{\underline{u}_{b_1}}$$

Hence, $\widehat{\Psi} \circ \Psi((u, \eta)) = (u, \eta)$. □

Let $F : \mathbb{B} \to \mathbb{A}$ be a split opfibration, and $G : \mathbb{C} \to \mathbb{D}$ a functor. Then the above construction for cloven fibrations reduces even further, since the intermediating isomorphism is the identity:

Corollary 1.3.12. *The exponential object, $\mathbb{C}^{\mathbb{B}}$ for a split fibration is defined as follows: the objects remain the same as in the general case and the morphisms between (a_1, F_1) and (a_2, F_2) are now defined to be pairs $(u : a_1 \to a_2, \varphi : F_1 \Rightarrow F_2 \circ u_!)$. Composition is defined to be:*

$$(v, \eta) \circ (u, \varphi) =_{\text{def}} (v \circ u, (\eta^2 \circ u_!) \cdot \varphi)$$

The identities are of the form $(\mathrm{id}_a, (F(\underline{\mathrm{id}_b}_b))_b)$.

1.4. Fibrations in $\mathbf{Gpd}$

We recall some basic facts about fibrations in the context of $\mathbf{Gpd}$. Note that any natural transformation $\eta : F \Rightarrow G$ in $\mathbf{Gpd}$, is necessarily a natural isomorphism (thus we can see $\mathbf{Gpd}$ as enriched over itself). $\mathbf{Gpd}$ inherits various constructions from $\mathbf{Cat}$: terminal object, products, equalizers, exponential object, pullbacks, pushouts.

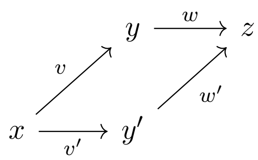
Lemma 1.4.1. *Any isofibration $F : \mathbb{B} \to \mathbb{A}$ in $\mathbf{Gpd}$ is Conduché fibration.*

Proof. Suppose there is a factorization of Fu in $\mathbb{A}$:

$$\begin{array}{ccccc} & & a & & \\ & {\scriptstyle f}\nearrow & & \searrow{\scriptstyle g} & \\ px & & \xrightarrow{\quad Fu \quad} & & pz \end{array}$$

Then, since F is an isofibration we have $w : x \to y$ over g. Set v to be $w^{-1} \circ u$ (and note $pv = g^{-1} \circ pu = f$). Thus we have obtained a factorization $w \circ v = w \circ w^{-1} \circ u$ of u in $\mathbb{B}$.

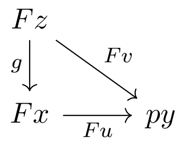
Suppose now, that there exist two such factorizations:



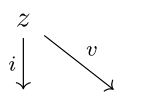
We can define $i : y \to y'$, by setting it to $v' \circ v^{-1}$. This links the two factorizations, and further $F(v' \circ v^{-1}) = F(v') \circ F(v)^{-1} = \mathrm{id}_b$ □

Lemma 1.4.2. *Any isofibration $F : \mathbb{B} \to \mathbb{A}$ in $\mathbf{Gpd}$ is a Grothendieck fibration.*

Proof. This proof boils down to the fact that any isomorphism over an isomorphism is cartesian, which we will now show. Let $f : a \to Fy$ and since f is iso, we have $u : x \to y$ over it. Assume $v : z \to y$ and $g : Fz \to Fy$, such that:



Notice that $g = (Fu)^{-1} \circ Fv$, so let us set $i = u^{-1} \circ v$, which implies $Fi = g$. Further, this makes the following diagram commute:



Further, assume $i' : z \to x$, such that $F(i') = g$ and making the above commute. Then $u \circ i' = u \circ i$, and since u is iso, $i' = i$. Thus u is cartesian. □

Lemma 1.4.3. *Any Conduché functor $F : \mathbb{B} \to \mathbb{A}$ in $\mathbf{Gpd}$ is an isofibration*

Proof. Let F be Conduché and consider $f : x \to Fy$. f gives us a factorization of $F\,\mathrm{id}_y$, namely $f \circ f^{-1}$, and by the Conduché property we have a factorization of id_y in $\mathbb{B}$, $v \circ u$, with u over f. Since $\mathbb{B}$ is a groupoid, u is an isomorphism. □

Since all notions coincide when domain and codomain are groupoids, we will refer to them collectively as just fibrations.

We define $\mathrm{Fib}(\mathbb{I})$ to be the subcategory of $\mathbf{Gpd}_{/\mathbb{I}}$ where the objects are fibrations and morphisms are cartesian functors.

Proposition 1.4.4. *Any map in $\mathbf{Gpd}_{/\mathbb{A}}$ between two fibrations is cartesian.*

Proof. Let $F : \mathbb{B} \to \mathbb{A}$ and $F' : \mathbb{B}' \to \mathbb{A}$, be two fibrations and $G : \mathbb{B} \to \mathbb{B}'$ be a morphism in $\mathbf{Gpd}_{/\mathbb{A}}$. Let $u : b \to b'$ in $\mathbb{B}$ be a F-cartesian map. We claim that Gu is a F'-cartesian map.

Suppose we have $v : c \to Gb'$ in $\mathbb{B}'$ and $h : F'c \to F'Gb$ in $\mathbb{A}$ such that the following diagram commutes in $\mathbb{A}$:

$$\begin{array}{ccc} F'c & & \\ {\scriptstyle h}\downarrow & \searrow^{F'v} & \\ F'Gb & \xrightarrow[F'Gu]{} & F'Gb' \end{array}$$

Since all of these maps are isomorphisms, we can set $q = (Gu)^{-1} \circ v$, obtaining a map in $\mathbb{A}'$ over h, making the required triangle commute. Since these are all isomorphisms, it is the unique such map. $\square$

Corollary 1.4.5. $\mathrm{Fib}(\mathbb{I})$ *is a full sub-category* $\mathbf{Gpd}_{/\mathbb{I}}$. $\square$

We can define the cobase change in terms of base change $u_! = (u^{-1})^*$, so the same holds for it. This isomorphism will be denoted by $\Phi_{u,v} : v_! \circ u_! \Rightarrow (v \circ u)_!$ and $\underline{u}_b = \overline{u^{-1}}_b$. Uniqueness of $\Phi_{u,v}$ gives us the following:

$$\Phi_{u,w \circ v} \cdot (\Phi_{v,w} \circ 1_{u_!}) = \Phi_{v \circ u, w} \cdot (1_{w_!} \circ \Phi_{u,v})$$

Exponentiability. Given a fibration $F : \mathbb{B} \to \mathbb{A}$, and a functor $G : \mathbb{C} \to \mathbb{A}$ in $\mathbf{Gpd}$, we consider the exponential object $\mathbb{C}^{\mathbb{B}}$ in $\mathbf{Cat}_{/\mathbb{A}}$.

Proposition 1.4.6. *The category* $\mathbb{C}^{\mathbb{B}}$*, as defined in Theorem 1.3.9 is a groupoid.*

Proof. We produce the inverse of a generalized natural transformation. Let $\eta : T \rightsquigarrow T'$ (over f) be one such. We begin by setting $(\eta^{-1})_u = \eta^{-1}_{u^{-1}}$, defined over f^{-1}. Let v over id_a. Observe:

$$\begin{array}{ccccccc} b & \xrightarrow{v} & b' & \quad & Tb & \xrightarrow{(\eta^{-1}\circ\eta)_v} & Fb' \\ {\scriptstyle v}\downarrow & & \| & & {\scriptstyle Fv}\downarrow & & \| \\ b' & = & b' & & Fb' & \xrightarrow[(\eta^{-1}\circ\eta)_{\mathrm{id}_{b'}}]{} & Fb' \end{array}$$

Note that $\mathrm{id}_{b'}$ factorizes as $u^{-1} \circ u$ for any $u : b' \to x$ in $\mathbb{B}$. Then $(\eta^{-1} \circ \eta)_{\mathrm{id}_{b'}} = \eta_u^{-1} \circ \eta_u = \mathrm{id}_{b'}$, and hence, $(\eta^{-1} \circ \eta)_v = Tv$. Composition on the other side is similar. $\square$

Corollary 1.4.7. *Let* $F : \mathbb{B} \to \mathbb{A}$*. If* F *is a fibration, then* $(-) \times \mathbb{B} : \mathbf{Gpd}_{/\mathbb{A}} \to \mathbf{Cat}_{/\mathbb{A}}$ *has a right* 2*-adjoint.* $\square$

1.5. Dependent products

Given a functor $\mathbb{B} \to \mathbb{A}$, we define the pullback functor $\Delta_F : \mathbf{Cat}_{/\mathbb{A}} \to \mathbf{Cat}_{/\mathbb{B}}$ as follows:

- For $G : \mathbb{C} \to \mathbb{A}$, $\Delta_F \mathbb{C}$ is the category with:
  - objects are pairs (b, c), where $c \in \mathbb{C}$ and $b \in \mathbb{B}$, such that $Fb = Gc$
  - given two (b, c) and $(b', c') \in \Delta_F \mathbb{C}$, morphisms between them are pairs (u, v), where $u : b \to b'$ and $v : c \to c'$, such that $Fu = Gv$
  - composition is inherited from $\mathbb{C}$ and $\mathbb{B}$

  $\Delta_F \mathbb{C}$ is seen as an object in $\mathbf{Cat}_{/\mathbb{B}}$, by projecting on the first component.
- Given $s : G \to G'$ in $\mathbf{Cat}_{/\mathbb{A}}$, the map $\Delta_F s : \Delta_F \mathbb{C} \to \Delta_F \mathbb{C}'$, simply applies s to the second component:

$$\Delta_F s(x, y) = (x, sy)$$

- Given a natural transformation $\varphi : s \Rightarrow s'$ in $\mathbf{Cat}_{/\mathbb{A}}$, we define $\Delta_F \varphi$ as:

$$(\Delta_F \varphi)_{(b,c)} = (\mathrm{id}_b, \varphi_c)$$

Proposition 1.5.1. *Let $F : \mathbb{B} \to \mathbb{A}$. If F is a Conduché fibration, then the pullback functor $\Delta_F : \mathbf{Cat}_{/\mathbb{B}} \to \mathbf{Cat}_{/\mathbb{A}}$ has a right adjoint.*

Proof. We adapt the proof in [**6**]. Suppose $F : \mathbb{B} \to \mathbb{A}$ is a Conduché fibration, and let s be the canonical isomorphism $1 \times \mathbb{B} \cong \mathbb{B}$. Using the adjunction from before, we obtain $\widehat{s} : 1 \to \mathbb{B}^{\mathbb{B}}$. Unfolding the definition we see that $\widehat{s}$, work in the following way: :

$$\begin{aligned} a &\mapsto (a, \iota_a) \\ u &\mapsto (u, (v)_v) \end{aligned}$$

(we denote with ι_b the inclusion functor $\mathbb{B}_a \hookrightarrow \mathbb{B}$).

Then, we define the dependent product functor $\Pi_F : \mathbf{Cat}_{/\mathbb{B}} \to \mathbf{Cat}_{/\mathbb{A}}$ as the pullback in the following diagram for $G : \mathbb{C} \to \mathbb{B}$:

$$\begin{array}{ccc} \Pi_F G & \longrightarrow & (F \circ G)^F \\ \big\downarrow & \lrcorner & \big\downarrow G^F \\ \mathbb{B} & \longrightarrow & F^F \end{array}$$

Since representable functors $\mathbf{Cat}_{/\mathbb{A}}(\mathbb{Q} \xrightarrow{H} \mathbb{A}, -)$ preserves limits (in particular conical limits), we get the following pullback diagram (in $\mathbf{Cat}$):

$$\begin{array}{ccc} \mathbf{Cat}_{/\mathbb{A}}(\mathbb{Q} \xrightarrow{H} \mathbb{A}, \Pi_F G) & \longrightarrow & \mathbf{Cat}_{/\mathbb{A}}(\mathbb{Q} \xrightarrow{H} \mathbb{A}, (F \circ G)^F) \\ \big\downarrow & \lrcorner & \big\downarrow \\ 1 & \longrightarrow & \mathbf{Cat}_{/\mathbb{A}}(\mathbb{Q} \xrightarrow{H} \mathbb{A}, F^F) \end{array}$$

Applying the natural isomorphism from above:

$$\begin{array}{ccc} \mathbf{Cat}_{/\mathbb{A}}(\mathbb{Q} \xrightarrow{H} \mathbb{A}, \Pi_F G) & \longrightarrow & \mathbf{Cat}_{/\mathbb{A}}(H \times F, (F \circ G)) \\ \downarrow & \lrcorner & \downarrow \\ 1 & \longrightarrow & \mathbf{Cat}_{/\mathbb{A}}(H \times F, F) \end{array}$$

However, $\mathbf{Cat}_{/\mathbb{B}}(\Delta_F H, G)$ is also a pullback of this diagram. Thus we obtain an isomorphism $\mathbf{Cat}_{/\mathbb{B}}(\Delta_F H, G) \cong \mathbf{Cat}_{/\mathbb{A}}(H, \Pi_F G)$ (which is natural). □

Remark 1.5.2. The same proof can be performed in $\mathbf{Gpd}$.

We can spell out the concrete definition of the strict dependent product functor. Given a Conduché fibration $F : \mathbb{B} \to \mathbb{A}$, we define a strict dependent product functor $\Pi_F : \mathbf{Cat}_{/\mathbb{B}} \to \mathbf{Cat}_{/\mathbb{A}}$ by:

- for $G : \mathbb{C} \to \mathbb{B}$, $\Pi_F G$ is a category with objects $(a, T : \mathbb{B}_a \to \mathbb{C})$, where $G \circ T = \iota_a$:

$$\begin{array}{ccc} \mathbb{B}_a & \xrightarrow{T} & \mathbb{C} \\ & \searrow^{\iota_a} \quad \swarrow_{G} & \\ & \mathbb{B} & \end{array}$$

  and morphisms $(f, \varphi : T \rightsquigarrow T')$ such that $G \cdot \varphi = (v)_v$,
- for $s : G \to H$:

$$\Pi_F(s)(a, T) = (a, s \circ T)$$

$$\Pi_F(s)(f, \varphi) = (f, (s(\varphi_v))_v)$$

- for $\eta : s \Rightarrow t$, $(\Pi_F \eta)_{(a,T)} = (\mathrm{id}_a, (\eta \cdot T)')$.

This definition can be simplified when F is a cloven (or split) fibration. We show this in Section 2.1, when giving an explicit description of polynomial functors.

Suppose that the following diagram is a pullback in $\mathbf{Cat}$

$$\begin{array}{ccc} \mathbb{B} & \xrightarrow{f} & \mathbb{A} \\ {\scriptstyle u}\downarrow & \lrcorner & \downarrow{\scriptstyle v} \\ \mathbb{D} & \xrightarrow[g]{} & \mathbb{C} \end{array}$$

The *Beck-Chevalley* (BC) condition states that:

$$\Sigma_f \Delta_u \cong \Delta_v \Sigma_g$$

$$\Pi_f \Delta_u \cong \Delta_v \Pi_g$$

Proposition 1.5.3. *The Beck-Chevalley condition holds if g in the above diagram is a fibration.*

Proof. Suppose we have a diagram like the one above. The first BC condition holds due to uniqueness of pullbacks. Let us focus on the second one. Since g is a fibration, and the square is a pullback, f is one as well. Let $w : Y \to A$, $q : Z \to D$ and consider an arrow $w \to \Delta_v \Pi_g q$. Then:

$$\begin{array}{ccc} h & \longrightarrow & \Delta_v \Pi_g q \\ \hline \Delta_g \Sigma_v h & \longrightarrow & q \\ \hline \Sigma_u \Delta_f h & \longrightarrow & q \\ \hline h & \longrightarrow & \Pi_f \Delta_u q \end{array}$$

The Yoneda lemma now guarantees that $\Pi_f \Delta_u \cong \Delta_v \Pi_g$ □

CHAPTER 2

Polynomial Functors in Gpd

In this chapter we recall the definition of polynomials and polynomial functors, adapt the theory of polynomial functors to the category of groupoids (following the work of [**24, 49**]), and exhibit some examples.

Next, we define endofunctor algebras, show some facts about the category of algebras and define W-types, along with some examples of them. This is relaxed to obtain the definition of homotopy initial algebras, before showing that strict initiality implies homotopy initiality.

Finally, we define J-relative algebras for a functor $F : \mathcal{C} \to \mathcal{D}$ in the sense of [**5**]. We show that in some cases, J-relative algebras can be seen as usual endofunctor algebras where we modify the functor F.

2.1. Polynomial Functors

Definition 2.1.1. A *polynomial* in $\mathbf{Gpd}$ is a diagram of the form:

(*)
$$\begin{array}{ccccc} & \mathbb{B} & \xrightarrow{F} & \mathbb{A} & \\ {\scriptstyle S}\swarrow & & & & \searrow{\scriptstyle R} \\ \mathbb{I} & & & & \mathbb{J} \end{array}$$

where F and R are fibrations. We say that a polynomial is cloven (split) if F and R are cloven (split) fibrations. Note that all groupoids are fibrant, that is, for $\mathbb{A}$ a groupoid, the unique map $\mathbb{A} \to 1$ is a fibration.

To a polynomial as in (*) we assign a polynomial functor, P_F, which is defined as the composition of:

$$\mathbf{Gpd}_{/\mathbb{I}} \xrightarrow{\Delta_S} \mathbf{Gpd}_{/Bg} \xrightarrow{\Pi_F} \mathbf{Gpd}_{/Ag} \xrightarrow{\Sigma_R} \mathbf{Gpd}_{/\mathbb{J}}$$

The exponentiability of fibrations is used to obtain Π_F. In general a functor is called a polynomial functor, if it is 2-naturally isomorphic to the polynomial functor assigned to a polynomial.

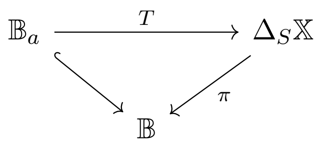
Unfolding the definition above, we can provide an explicit description of P_F.

- Let $G : \mathbb{X} \to \mathbb{I}$, then $P_F(\mathbb{X})$ is an object in $\mathbf{Gpd}_{/\mathbb{J}}$:

$$P_F\mathbb{X} \xrightarrow{\pi} \mathbb{A} \xrightarrow{R} \mathbb{J}$$

  where:

  - the objects of $P_F\mathbb{X}$ are pairs (a, T), where $a \in \mathbb{A}$ and $T : \mathbb{B}_a \to \Delta_S\mathbb{X}$, such that:



    Using type theoretic notation, we would write $T : \Pi_{b:\mathbb{B}_a}\mathbb{X}_{Sb}$, that is every $b \in \mathbb{B}_a$ maps to the G-fibre above Sb.
  - for $(a, T), (a', T') \in P_F\mathbb{X}$, a morphism $(f, \varphi) : (a, T) \to (a', T')$, consists of $f : a \to a'$ in $\mathbb{A}$ and a generalized natural transformation $\varphi : T \rightsquigarrow T'$, such that $(\pi \cdot \varphi)_v = v$ for $v : b \to b' \in Bg$ over f. That is, φ is the identity on the first component
  - given two morphisms (f, φ) and (f', φ'), their composition is:

$$(f' \circ f, \varphi' \circ \varphi)$$

- for $s : \mathbb{X} \to \mathbb{X}'$ in $\mathbf{Gpd}_{/\mathbb{I}}$, we get $P_Fs : P_F\mathbb{X} \to P_F\mathbb{X}'$, which is defined as follows:

$$(P_Fs)(a, T) = (a, s \circ T)$$
$$(P_Fs)(f, \varphi) = (f, s \cdot \varphi)$$

- for a natural transformation $\psi : s \Rightarrow s'$, we get a natural transformation $P_F\psi : P_Fs \Rightarrow P_Fs'$:

$$(P_F\psi)_{(a,T)} = (\mathrm{id}_a, (\psi_{Tb'} \circ sTu)_{u:b\to b'})$$

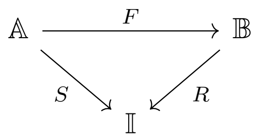
We now adapt this definition for the case where $F : \mathbb{B} \to \mathbb{A}$ is a cloven fibration. If $G : \mathbb{X} \to \mathbb{I}$, the groupoid $P_F(\mathbb{X})$ is defined by:

- objects are pairs (a, T) as before,
- given $(a, T), (a', T') \in P_F\mathbb{X}$, the morphisms are pairs $(f, \varphi) : (a, T) \to (a', T')$, where $f : a \to a'$ and $\varphi : T \Rightarrow T'f_!$, where $(G\eta)_b = \underline{f}_b$ for $b \in \mathbb{B}_a$,
- given two morphisms (f, φ), (f', φ') the composition is defined by:

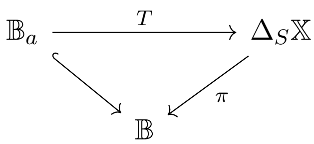
$$(f', \varphi') \circ (f, \varphi) = (f' \circ f, (1_{T''} \circ \Phi_{f,f'}) \cdot (\varphi' \circ 1_{f_!}) \cdot \varphi)$$

The same holds if F is split, but the composition simplifies since the intermediating isomorphism, $\Phi_{f,f'}$, is the identity.

Given a diagram as in (*). Suppose F is split and further assume $\mathbb{I} = \mathbb{J}$ and $R \circ F = S$:



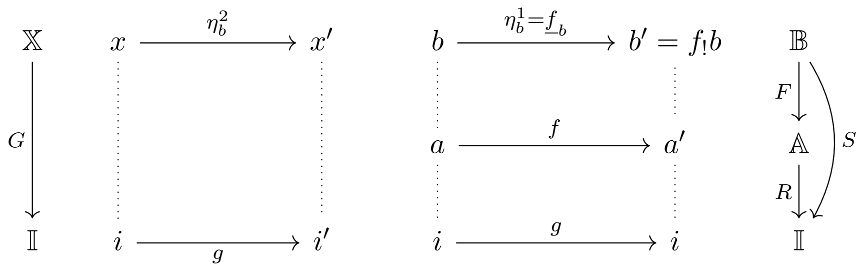
Let $G : \mathbb{X} \to \mathbb{I}$ be an object of $\mathbf{Gpd}_{/\mathbb{I}}$. We focus on the objects of $P_F\mathbb{X}$ first. Since $T : \mathbb{B}_a \to \Delta_S\mathbb{X}$, needs to make the following triangle commute:



we see that T is identity on $\mathbb{B}_a$. Further since its codomain is a pullback, it has to be the case that $G\pi_1 Tb = S\pi_2 Tb$, but $S\pi_2 Tb = RF\pi_2 Tb = Ra$. This means, that the above functor encodes the same data as a functor of the form $T : \mathbb{B}_a \to \mathbb{X}_i$, where $i = Ra$. We prefer to represent the above set as:

$$(P\mathbb{X})_0 = \{(i, a, T) \mid i \in \mathbb{I}, a \in \mathbb{A}_i, T : \mathbb{B}_a \to \mathbb{X}_i\}$$

Next, let us turn our attention to morphisms. Let $(f, \varphi) : (a, T) \to (a', T')$, that is, $\varphi_b = (\varphi_b^1 : b \to b', \varphi_b^2 : x \to x')$. Observe that $\eta_b^2 : x \to x'$, has to be over $Rf = g$, since $G\eta_b^2 = S\eta_b^1$:

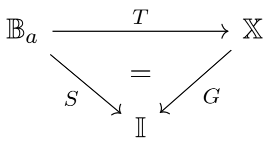


We can then represent the morphisms in $P_F\mathbb{X}$ as:

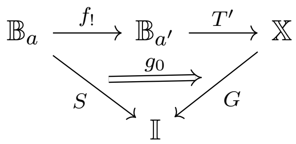
$$(g : i \to i', f : a \to a', \varphi : T \Rightarrow T' \circ u_!)$$

where f is over g and components of φ are over g as well.

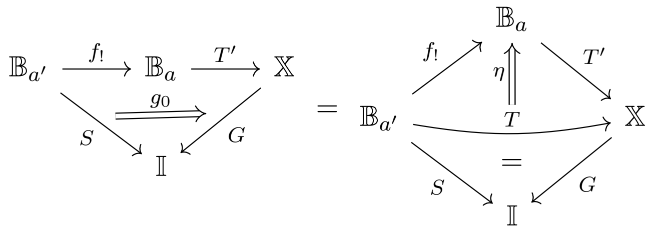
Alternatively, for the reader familiar with lax slices, we can present the above condition as a 2-cell in the lax slice $\mathbf{Gpd}/\!\!/_{\mathbb{I}}$ between:



and



where $(g_0)_b =_{\mathrm{def}} g$. That is the following pasting condition holds:



Let $F : \mathbb{B} \to \mathbb{A}$ be a fibration. Further simplifications can be made in the case when $\mathbb{I} = \mathbb{J} = 1$. Then the objects of $P_F\mathbb{X}$ are pairs (a, T), where $a \in \mathbb{A}$ and $T : \mathbb{B}_a \to \mathbb{X}$. We can see that this is the same as just applying the dependent product functor.

Examples of polynomial functors.

Example 2.1.2. Let A be a discrete groupoid (i.e. a set) and set $\mathbb{A}$ to be $\mathbb{Z}_2 + A$, where $\mathbb{Z}_2$ is a groupoid with one object, $\bullet$, and one non-trivial, involutive arrow τ. Additionally, let $\mathbb{B}$ be **J**, the walking isomorphism groupoid, that is, a groupoid with two objects 0, 1, and two non-trivial arrows $0 \to 1$, and $1 \to 0$, that are inverses of each other.

We define $F : \mathbb{B} \to \mathbb{A}$, which maps objects of $\mathbb{B}$ to $\bullet$ and the non-trivial arrows 01, and 10 to τ. Obviously this is a split fibration. For $\mathbb{X}$ a groupoid, we see that $P_F\mathbb{X}$ consists of two types of objects:

$$(a, 0 \to \mathbb{X})$$

for any $a \in A$, and:

$$(\bullet, 1 + 1 \to \mathbb{X})$$

The only interesting morphisms appear over the objects marked with $\bullet$:

- $(\mathrm{id}_\bullet, \varphi) : (\bullet, T) \to (\bullet, T')$, where

$$\varphi_0 : T0 \to T'0$$
$$\varphi_1 : T1 \to T'1$$

- $(\tau, \varphi) : (\bullet, T) \to (\bullet, T')$, where

$$\varphi_0 : T0 \to T'1$$
$$\varphi_1 : T1 \to T'0$$

Since, the domain of T, and of T' is discrete, there are no additional conditions.

The examples given in this section have discrete fibers, but this is not necessary. For example, if we were to add an additional isomorphism in the first example, this would amount to adding a requirement that the two branches at every point must be isomorphic.

We can generalize the above example:

Example 2.1.3. Let X be a set and consider G a subgroup of the symmetric group $S(X)$, then for $A \in \mathbf{Set}$, we define $\mathbb{A}$ to be $G + A$. Define $\mathbb{B}$ as follows:

$$\mathbb{B}_0 = X$$
$$\mathbb{B}(x, x') = \{\pi \in G \mid \pi x = x'\}$$

We set f to essentially act as a projection, mapping all objects of $\mathbb{B}$ to $\bullet$ and morphisms $x \xrightarrow{\pi} x'$ to $\bullet \xrightarrow{\pi} \bullet$. This is again a split fibration.

Using groupoids we can construct the symmetric list monad:

Example 2.1.4. Let $\mathbb{A}$ be $1 + \Sigma_{n\in\mathbb{N}} S_n$ and $\mathbb{B}$ be $\Sigma_{n\in\mathbb{N}}[n]$, where $[n]_0 = \{0, \ldots, n-1\}$ and $[n](i, j) = \{\pi \in S_n \mid \pi i = j\}$. $F : \mathbb{B} \to \mathbb{A}$ acts as a projection.

Then $P_F\mathbb{X}$ is the groupoid of lists consisting of objects of $\mathbb{X}$. Between lists that are equal modulo some permutation we have isomorphisms.

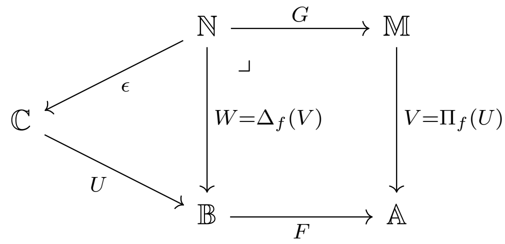
If we take inspiration from the previous example and limit ourselves to some subgroups of S_n, we can obtain cyclic lists, etc.

This polynomial monad is sometimes denoted by $\mathbf{S}$, and its $\mathbf{Cat}$ monad algebras correspond to symmetric (strict) monoidal categories [**11**].

Example 2.1.5. In [**48**] Weber gives a construction assigning to each *symmetric collection* T over I a polynomial $\mathbf{Cat}_{/I} \to \mathbf{Cat}_{/I}$ and further a cartesian morphism into $\mathbf{S}$. Given an operad structure on a collection, he further gives a monad structure to a polynomial functor assigned to the collection. An inspection of his construction shows that the polynomial obtained is actually a polynomial in groupoids.

Composition of polynomial functors. The next result establishes the type-theoretic counterpart to the so-called axiom of choice.

Proposition 2.1.6 (Weber [**49**])**.** *Let* $F : \mathbb{B} \to \mathbb{A}$ *be an object in* $\mathrm{Fib}(\mathbb{A})$ *and further let* $U : \mathbb{C} \to \mathbb{B}$*. Consider the diagram:*



Then $\Pi_F \Sigma_U \cong \Sigma_V \Pi_G \Delta_\epsilon$

Proof. Weber shows ([**49**]) that (P, Q, R) is a distributivity pullback around (F, U) if and only if $\delta_{P,Q,R} : \Sigma_R \Pi_Q \Delta_P \to \Pi_F \Sigma_U$ is an isomorphism. Further, he shows that the above is a distributive pullback (even more, the initial such). □

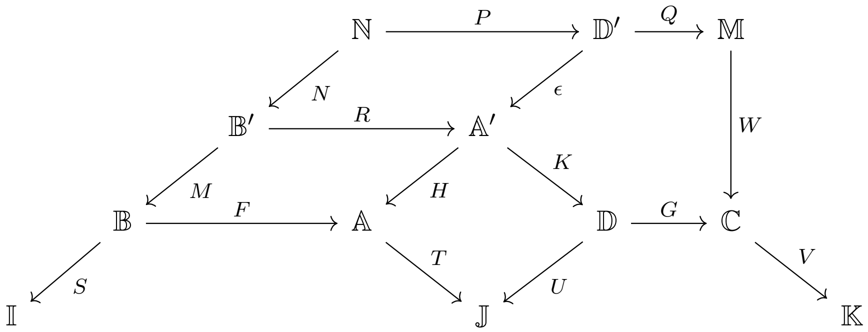
We can compose polynomials. This composition is defined in [**24**] and further extended to categories with pullbacks in [**49**]. We recall their construction. Suppose we have two polynomials, F:

$$\mathbb{I} \xleftarrow{S} \mathbb{B} \xrightarrow{F} \mathbb{A} \xrightarrow{T} \mathbb{J}$$

and G:

$$\mathbb{J} \xleftarrow{U} \mathbb{D} \xrightarrow{G} \mathbb{C} \xrightarrow{V} \mathbb{K}$$

We define the operation of substitution $G \circ F$ to be:



Where the squares are pullbacks, and the pentagon is the distributivity diagram from before. Since F was a fibration, both R and P are as well, further since G was one, Q is one. Similarly, T being a fibration makes K one, as well. Pushforward of a fibration is a fibration, hence W is a fibration. Composing two fibrations gives a fibration and we obtain a properly defined polynomial.

Thanks to Beck-Chevalley (Proposition 1.5.3), distributivity and pseudo-functoriality of the Σ, Π and Δ functors, we can produce the following calculations:

$$\begin{aligned} P_G \circ P_F &= \Sigma_V \Pi_G \Delta_U \Sigma_T \Pi_F \Delta_S \\ &\cong \Sigma_V \Pi_G \Sigma_K \Delta_H \Pi_F \Delta_S \\ &\cong \Sigma_V \Sigma_W \Pi_Q \Delta_\epsilon \Delta_H \Pi_F \Delta_S \\ &\cong \Sigma_V \Sigma_W \Pi_Q \Pi_P \Delta_N \Delta_M \Delta_S \\ &\cong \Sigma_{(VW)} \Pi_{(QP)} \Delta_{(SMN)} \\ &= P_{G \circ F} \end{aligned}$$

We will see later in this chapter that this operation is associative and unital up to coherence and gives rise to a bicategory, exactly as in [**24, 49**].

2.2. Morphisms of Polynomial Functors

We essentially follow [**24**]. The proof of Proposition 2.8 therein contains an error, which we fix in the case of $\mathbf{Gpd}$[1]. In their proof the state that one can without loss of generality consider the case where $A = 1$, which was later pointed out to not be the case. We instead manually construct the required morphism and show that the assignment is unique by looking at specific objects in $\mathbf{Gpd}_{/\mathbb{I}}$, which fully determine the action of the vertical natural transformation.

Recall that a natural transformation $\varphi : F \Rightarrow G : \mathbb{B} \to \mathbb{A}$ is *cartesian*, if the naturality square is a pullback. That is, for all $f : X \to Y \in \mathbb{B}$:

$$\begin{array}{ccc} FX & \xrightarrow{\varphi_X} & GX \\ {\scriptstyle Ff}\downarrow & \lrcorner & \downarrow{\scriptstyle Gf} \\ FY & \xrightarrow[\varphi_Y]{} & GY \end{array}$$

is a pullback.

Define $\mathrm{PolyFun}(\mathbf{Gpd}_{/\mathbb{I}}, \mathbf{Gpd}_{/\mathbb{J}})$ to be the category of polynomial functors and 2-natural transformations between them.

Let $\mathcal{F} : \mathrm{PolyFun}(\mathbf{Gpd}_{/\mathbb{I}}, \mathbf{Gpd}_{/\mathbb{J}}) \to \mathrm{Fib}_{/\mathbb{J}}$ (or $\mathrm{SFib}_{/\mathbb{J}}$), that acts in the following way:

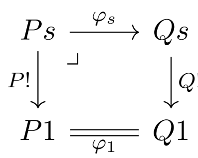
$$\mathcal{F}(P) = P(1)$$

$$\mathcal{F}(\varphi) = \varphi_1$$

[1]As mentioned in the introduction, a correction can be made in case of locally closed cartesian categories using [**33**]

Then $\mathcal{F}$ is a Grothendieck fibration, where the cartesian arrows are exactly cartesian natural transformations and vertical arrows are those with $\varphi_1 = \mathrm{id}$.

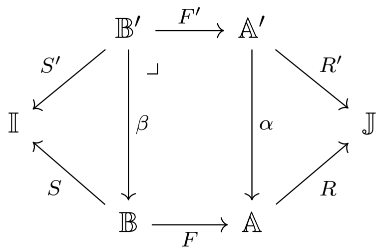
Let $\varphi : P \Rightarrow Q$, be vertical and cartesian. Then φ is an isomorphism. Let s be an object in the domain of P. Since the domain also contains a terminal object 1, we have an arrow $! : s \to 1$. The following diagram is a pullback:



Since φ_s is a pulllback of the identity, it is an isomorphism.

Let us now consider the diagram of the following shape:

(2.1)



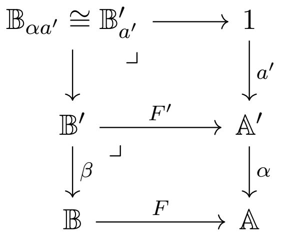
As shown in [**24**], such a diagram induces a cartesian natural transformation $\varphi : P_{F'} \Rightarrow P_F$, defined as the following composite:

$$\begin{aligned}
\Sigma_{R'}\Pi_{F'}\Delta_{S'} &\cong \Sigma_{R'}\Pi_{F'}\Delta_\beta\Delta_S && \text{(by } S' = S\beta\text{)}\\
&\cong \Sigma_{R'}\Delta_\alpha\Pi_F\Delta_S && \text{(by Beck-Chevalley)}\\
&= \Sigma_R\Sigma_\alpha\Delta_\alpha\Pi_F\Delta_S && \text{(by } R' = R\alpha\text{)}\\
&\Rightarrow \Sigma_R\Pi_F\Delta_S && \text{(compose with the counit)}
\end{aligned}$$

The composition cartesian is because individual components are. Explicitly at the point $\mathbb{X}$:

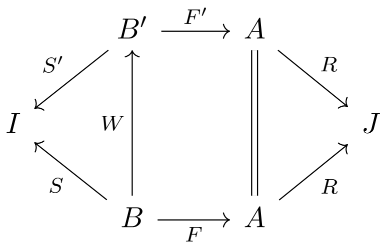
$$\varphi_{\mathbb{X}}(a', T) = (\alpha a', T \circ \beta_{a'})$$
$$\varphi_{\mathbb{X}}(f', \psi) = (\alpha f', (\psi_{\beta_{f'} v})_v)$$

Where $\beta_{a'} : \mathbb{B}_{\alpha a'} \to \mathbb{B}'_{a'}$ (and $\beta_{f'} : \mathbb{B}_{\alpha f'} \to \mathbb{B}'_{f'}$) are isomorphisms determined by the pullback square:



Turning our attention to diagrams of the form:

(2.2)



We can see, that diagrams of this shape give us a vertical natural transformation $\varphi : P_{F'} \Rightarrow P_F$. For $\mathbb{X}$, we have that $\varphi_{\mathbb{X}}$ is as follows:

$$\varphi_{\mathbb{X}}(a, T) = (a, \lambda x : \mathbb{B}_a.(x, TWx))$$
$$\varphi_{\mathbb{X}}(g, \psi) = (g, \lambda u : F^{-1}(g).(u, \psi_{Wu}))$$

Technically Tx (and ψ_u) are elements of $\Delta_{S'}\mathbb{X}$, however for ease of reading we omit writing the projection.

We can see that φ is natural. Suppose $h : \mathbb{X} \to \mathbb{Y}$ in $\mathbf{Gpd}_{/\mathbb{I}}$, :

$$(\varphi_{\mathbb{X}} \circ P_{F'}(h))(a, T) = (a, \lambda x.(x, hTWx))$$
$$(P_F(h) \circ \varphi_{\mathbb{Y}})(a, T) = (a, \lambda x.(x, hTWx))$$

and similarly for morphisms.

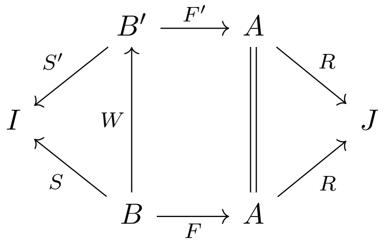
Now suppose $\alpha : h \Rightarrow h'$, we can see that φ is in fact 2-natural:

$$\varphi_{\mathbb{Y}}(P_{F'}\alpha)_{a,T} = (\mathrm{id}_a, \lambda(u : b \to b').\alpha_{TWb'} \circ TWu)$$
$$(P_F\alpha)_{\varphi_{\mathbb{X}}(a,T)} = (\mathrm{id}_a, \lambda(u : b \to b').\alpha_{TWb'} \circ TWu)$$

We see that $\varphi_1 = \mathrm{id}_A$, so it is vertical for $\mathcal{F}$.

As noted before, the proof of the next statement is quite different from the one in [**24**].

Proposition 2.2.1. *Every vertical* 2*-natural transformation* $\varphi : P_{F'} \Rightarrow P_F$ *can be assigned a commuting diagram of the shape (2.2):*



Proof. Let $\varphi : P_{F'} \Rightarrow P_F$ be a vertical natural transformation. The first thing to notice is that for any such, the following commutes (for any $\mathbb{X} \in \mathbf{Gpd}_{/\mathbb{I}}$):

$$\begin{array}{ccc} P_{F'}\mathbb{X} & \xrightarrow{\varphi_{\mathbb{X}}} & P_F\mathbb{X} \\ \downarrow & & \downarrow \\ \mathbb{A} \cong P_{F'}1 & = & P_F 1 \cong \mathbb{A} \end{array}$$

Meaning that for any $(a, T) \in P_{F'}\mathbb{X}$, we have that $\varphi_{\mathbb{X}}(a, T) = (a, \widehat{T})$ (and similarly for morphisms). That is, φ is the identity on the first component.

If $a \in \mathbb{A}$, we can see $\mathbb{B}'_a$ over $\mathbb{I}$, and define $\delta_a : \mathbb{B}'_a \to \mathbb{B}' \times_{\mathbb{I}} \mathbb{B}'_a$ as the diagonal functor. Notice that (a, δ_a) is an object of $P_{F'}\mathbb{B}'_a$. Applying $\varphi_{Bg'_a}$ to it we obtain an element of $P_F\mathbb{B}'_a$ of type $(a, \widehat{\delta_a} : \mathbb{B}_a \to \mathbb{B} \times_{\mathbb{I}} \mathbb{B}'_a)$.

We define $W : \mathbb{B} \to \mathbb{B}'$ on objects first:

$$W(b : \mathbb{B}'_a) = \pi_2 \widehat{\delta_a}(b)$$

We will omit writing the projection in the next part.

Since for $b \in \mathbb{B}_a$, $\widehat{\delta_a}$ maps it to an element of the form $(b, b') \in \mathbb{B} \times_{\mathbb{I}} \mathbb{B}'_a$:

$$Sb = S'b'$$

Further since $\mathbb{B}'_a$ is over $a \in \mathbb{A}$, we have:

$$Fb = F'b'$$

Since φ is natural, the following commutes:

$$\begin{array}{ccc} P_{F'}\mathbb{B}'_a & \xrightarrow{P_{F'}\iota} & P_{F'}\mathbb{B}' \\ {\scriptstyle \varphi_{\mathbb{B}'_a}}\downarrow & & \downarrow{\scriptstyle \varphi_{\mathbb{B}'}} \\ P_F\mathbb{B}'_a & \xrightarrow[P_F\iota]{} & P_F\mathbb{B}' \end{array}$$

Meaning that $(a, \delta_a : \mathbb{B}'_a \to \mathbb{B}' \times_{\mathbb{I}} \mathbb{B}')$ gets mapped (via $\varphi_{\mathbb{B}'}$) to $(a, \mathbb{B}_a \xrightarrow{\widehat{\delta_a}} \mathbb{B} \times_{\mathbb{I}} \mathbb{B}'_a \xrightarrow{\iota} \mathbb{B} \times \mathbb{B}')$. This will allow us to define W on morphisms as well.

Let $g : a \to a' \in \mathbb{A}$, and define $(g, \delta_g) : (a, \delta_a) \to (a', \delta_{a'})$, by setting:

$$\delta_g = \lambda u.(u, u)$$

Then for $u : b \to b'$ over $g : a \to a'$, we set:

$$W(u) = \widehat{\delta}_g(u)$$

We now proceed to show that W is a functor.

Given id_b over id_a, then δ_{id_a} is actually the identity natural generalized natural transformation. Since $\varphi_{\mathbb{B}'}$ is a functor, we have that $\varphi_{\mathbb{B}'}(\mathrm{id}_a, \delta_{\mathrm{id}_a}) = (\mathrm{id}_a, \widehat{\delta_{\mathrm{id}_a}}) = \mathrm{id}_{(a, \widehat{\delta_a})}$, and $W(\mathrm{id}_b) = \mathrm{id}_{Wb}$.

Suppose $u : b \to b'$, $v : b' \to b''$ over g, g' respectively. Then first notice that:

$$\delta'_g \circ \delta_g = \delta_{g' \circ g}$$

Since $\varphi_{\mathbb{B}'}$ is a functor:

$$\widehat{\delta_{g'}} \circ \widehat{\delta} g = \widehat{\delta_{g'} \circ \delta_g} = \widehat{\delta_{g' \circ g}}$$

Then:

$$\begin{aligned} W(v) \circ W(u) &= \widehat{\delta_{g'}}(v) \circ \widehat{\delta} g(u) \\ &= (\widehat{\delta_{g'}} \circ \widehat{\delta} g)(v \circ u) \\ &= \widehat{\delta_{g' \circ g}}(v \circ u) \\ &= W(v \circ u) \end{aligned}$$

Define $\mathbb{B}'_g$ to be the pullback in the following square:

$$\begin{array}{ccc} \mathbb{B}'_g & \longrightarrow & \mathbf{J} \\ \downarrow & & \downarrow g \\ \mathbb{B}' & \xrightarrow{F'} & \mathbb{A} \end{array}$$

Then (a, δ_a) is also an object of $P_{F'}\mathbb{B}'_g$ and $(\mathrm{id}_a, \delta_g)$ is a morphism $(a, \delta_a) \to (a', \delta_{a'})$ in $P_{F'}\mathbb{B}'_g$, so we get that:

$$SWu = S'Wu$$

$$FWu = F'Wu$$

as before.

We now wish to show that φ is fully determined by W. Let $\mathbb{X} \in \mathbf{Gpd}_{/\mathbb{I}}$, and suppose $(a, T : \mathbb{B}'_a \to \Delta_{s'}\mathbb{X}) \in P_{F'}\mathbb{X}$. Note that T can be seen as a map in $\mathbf{Gpd}_{/\mathbb{I}}$, then we get the following:

$$\begin{array}{ccc} P_{F'}\mathbb{B}'_a & \xrightarrow{P_{F'}T} & P_{F'}\mathbb{X} \\ {\scriptstyle \varphi_{\mathbb{B}'_a}}\downarrow & & \downarrow{\scriptstyle \varphi_{\mathbb{X}}} \\ P_F\mathbb{B}'_a & \xrightarrow[P_FT]{} & P_F\mathbb{X} \end{array}$$

Further we have that $(P_{F'}T)(a, \delta_a) = (a, T)$, so:

$$\varphi_{\mathbb{X}}(a, T) = (a, \Delta_S T \circ \widehat{\delta_a})$$

That is:

$$\widehat{T}(x) = (x, TWx)$$

A similar trick can be performed with morphisms. Given $(g, \psi) : (a, T) \to (a', T')$ in $P_{F'}\mathbb{X}$. We can see that ψ defines a functor of the type

$$\psi : \mathbb{B}'_g \to \Delta_{s'}\mathbb{X}$$

ψ acts as T over $\mathbb{B}'_a \hookrightarrow \mathbb{B}'_g$ (and T' over $\mathbb{B}'_{a'}$ respectively), and like ψ (and ψ^{-1}) for u over g. Further we have that the following commutes:

$$\begin{array}{ccc} P_{F'}\mathbb{B}'_g & \xrightarrow{P_{F'}\psi} & P_{F'}\mathbb{X} \\ {\scriptstyle \varphi_{\mathbb{B}'_g}}\downarrow & & \downarrow{\scriptstyle \varphi_{\mathbb{X}}} \\ P_F\mathbb{B}'_g & \xrightarrow[P_F\psi]{} & P_F\mathbb{X} \end{array}$$

As before, we get that $(P_{F'}\psi)(g, \delta_g) = (g, \psi)$ and tracing the above diagram, we can conclude:

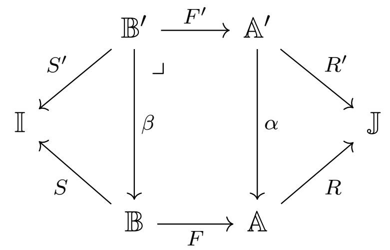
$$\varphi_{\mathbb{X}}(g, \psi) = (P_F\psi)(g, \widehat{\delta_g})$$

Expanding this we get:

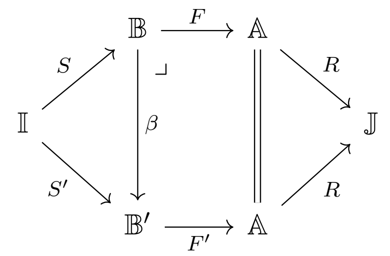
$$\widehat{\psi}_u = \psi_{Wu}$$

Hence we can see that $\varphi : P_{F'} \Rightarrow P_F$ is fully determined by $W : \mathbb{B} \to \mathbb{B}'$. $\square$

Proposition 2.2.2. *Let $\varphi : P_{F'} \Rightarrow P_F$ be a cartesian 2-natural transformation. Then it is uniquely represented by a diagram of the shape (2.1):*



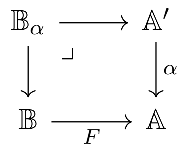
Proof. To start with, suppose φ is both vertical and cartesian. From the previous proposition, we get $\beta : \mathbb{B} \to \mathbb{B}'$. Note that we have already shown that φ must be an isomorphism, so we can also obtain β' for φ^{-1}. Uniqueness gives us that β and β' must be inverses of each other. This gives us the diagram of the required form:



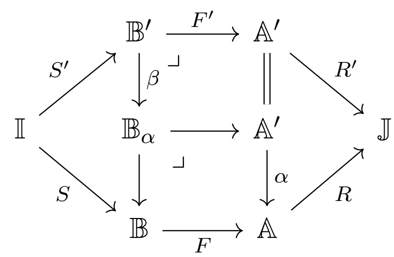
Now, we relax the constraint and allow φ to only be cartesian. We get

$$\alpha : \mathbb{A}' \cong P_{F'}1 \xrightarrow{\varphi_1} P_F 1 \cong \mathbb{A}$$

Consider the following pullback:



The above defines a new polynomial, one that is also above $\mathbb{A}'$ for the Grothendieck fibration $\mathcal{F}$, which we will call P_{F_α}. We now have a cartesian arrow $\varphi : P_{F'} \Rightarrow P_F$ and another cartesian arrow $\psi : P_{F_\alpha} \Rightarrow P_F$. This means that there exists a cartesian and vertical arrow $\xi : P_{F'} \Rightarrow P_{F_\alpha}$, such that $\varphi = \psi \circ \xi$. Applying the previous consideration we obtain:



□

Consider now a diagram of the shape:

$$
\begin{array}{ccccccc}
I & \xleftarrow{U} & D & \xrightarrow{G} & C & \xrightarrow{V} & J \\
\Vert & & \uparrow & & \Vert & & \Vert \\
\Vert & & B' & \longrightarrow & C & & \Vert \\
\Vert & & \downarrow & \lrcorner & \downarrow & & \Vert \\
I & \xleftarrow[S]{} & B & \xrightarrow[F]{} & A & \xrightarrow[T]{} & J
\end{array}
\tag{2.3}
$$

Diagrams of this shape will be called *morphisms* between polynomials G and F. Applying what we know, we get a 2-natural transformation $P_G \Rightarrow P_F$. We can also show that the converse holds:

Proposition 2.2.3. *Every* 2*-natural transformation between polynomial functors is represented in an essentially unique way by a diagram above (2.3):*

$$
\begin{array}{ccccccc}
I & \xleftarrow{U} & D & \xrightarrow{G} & C & \xrightarrow{V} & J \\
\Vert & & \uparrow & & \Vert & & \Vert \\
\Vert & & B' & \longrightarrow & C & & \Vert \\
\Vert & & \downarrow & \lrcorner & \downarrow & & \Vert \\
I & \xleftarrow[S]{} & B & \xrightarrow[F]{} & A & \xrightarrow[T]{} & J
\end{array}
$$

Proof. Given a natural transformation $\varphi : P_G \Rightarrow P_F$, we can factor it as a cartesian natural transformation followed by a vertical one (thanks to $\mathcal{F}$ being a fibration). Applying the two propositions we just showed, we obtain a diagram of the desired shape. □

We define a new category $\mathrm{Poly}(\mathbb{I}, \mathbb{J})$ where the objects are polynomials and morphisms are diagrams as described above (2.3). Given two such diagrams, stacked on top of each other, we see that we have a cartesian followed by a vertical natural transformation. Looking at the natural transformations assigned to them, we use the fibration property of $\mathcal{F}$ to transform it into vertical followed by cartesian, and take the diagram assigned to the newly obtained natural transformation. The last step is to simply compose the squares.

Proposition 2.2.4. *For* $\mathbb{I}$ *and* $\mathbb{J}$*, the functor:*

$$\mathrm{Ext} : \mathrm{Poly}(\mathbb{I}, \mathbb{J}) \to \mathrm{PolyFun}(\mathbf{Gpd}_{/\mathbb{I}}, \mathbf{Gpd}_{/\mathbb{J}})$$

is an equvalence of categories

Proof. This is a consequence of the previous proposition. □

Theorem 2.2.5. *There exists a bicategory* $\mathrm{Poly}_{\mathbf{Gpd}}$*, called the bicategory of polynomials in groupoids, having small groupoids as objects, polynomials as* 1*-cells and polynomial* *morphisms as* 2*-cells, such that the functors*

$$\mathrm{Ext} : \mathrm{Poly}(\mathbb{I}, \mathbb{J}) \to \mathrm{PolyFun}(\mathbf{Gpd}_{/\mathbb{I}}, \mathbf{Gpd}_{/\mathbb{J}})$$

extend to a biequivalence:

$$\mathrm{Ext} : \mathrm{Poly}_{\mathbf{Gpd}} \to \mathrm{PolyFun}$$

Proof. See the construction of $\mathrm{Poly}_{\mathcal{E}}$ in section 2.16 of [**24**]. □

2.3. Algebras for endofunctors

The main topic of this thesis is discussing the so-called W-types for groupoids. We begin by reviewing some definitions and facts about endofunctors on a category and then move to 2-categorical aspects. Let us first recall what endofunctor algebras are:

Definition 2.3.1. Let $\mathbb{C}$ be a category and $F : \mathbb{C} \to \mathbb{C}$ an endofunctor. An F-algebra is a pair $(X, \sup_X)$, where $X \in \mathbb{C}$ and $\sup_X : FX \to X$.

We collect F-algebras into a category $F\text{-alg}_s$, where:

- the objects are F-algebras, that is pairs $(X, \sup_X : FX \to X)$,
- the morphisms between $(X, \sup_X)$ and $(Y, \sup_Y)$ are morphisms $f : X \to Y$ of $\mathbb{C}$, such that:

$$f \circ \sup_X = Ff \circ \sup_Y$$

$$\begin{array}{ccc} FX & \xrightarrow{Ff} & FY \\ {\scriptstyle \sup_X}\downarrow & & \downarrow{\scriptstyle \sup_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

- composition and identities are inherited from $\mathbb{C}$

The following is a well-established lemma in the theory of endofunctor algebras (cf. [**11**]):

Lemma 2.3.2. *The forgetful functor* $U : F\text{-alg}_s \to \mathbb{C}$ *creates limits.* □

Definition 2.3.3. An initial algebra for F is an initial object in $F\text{-alg}_s$.

One of the most important theorems about endofunctor algebras is Lambek's Lemma [**36**], stating that if $(X, \sup_X)$ is initial algebra for F, then X is isomorphic to FX via $\sup_X$.

2-categories of algebras. Let $F : \mathcal{C} \to \mathcal{C}$, now be a 2-endofunctor. We collect F-algebras into a 2-category $F\text{-alg}_s$, where:

- the objects and morphisms are the same as in the 1-categorical version,

- 2-cells between two morphisms f and g, are those 2-cells of $\mathcal{C}$, $\alpha : f \Rightarrow g$, such that:

$$1_{\sup_Y} \circ F\alpha = \alpha \circ 1_{\sup_X}$$

$$PX \underset{Pg}{\overset{Pf}{\underset{\Downarrow P\alpha}{\rightrightarrows}}} PY \xrightarrow{\sup_Y} Y \quad = \quad PX \xrightarrow[\sup_X]{} X \underset{g}{\overset{f}{\underset{\Downarrow \alpha}{\rightrightarrows}}} Y$$

- composition and identities are inherited from $\mathcal{C}$

We can extend the previous proposition about limits to 2-limits as well.

Lemma 2.3.4. *F-alg_s has all strict 2-limits that $\mathcal{C}$ has..* □

Definition 2.3.5. An algebra $(W, \sup_W)$ is *strictly 2-initial*, if the hom-category $\hom(W, X)$ (for any other algebra) is isomorphic to the terminal category 1. That is, for any other algebra $(X, \sup_X)$, there exists a unique algebra morphism $f : W \to X$ and the only 2-cell $\alpha : f \Rightarrow f$ is the identity.

We can relax the definition of F-alg_s to define the category of algebras and pseudomorphisms, which we will denote with F-alg:

- objects are pairs $(X, \sup_X : FX \to X)$,
- morphisms are pairs $(f, \bar{f}) : (X, \sup_X) \to (Y, \sup_Y)$, consisting of a map $f : X \to Y$ and a 2-cell $\bar{f} : \sup_Y \circ Ff \Rightarrow f \circ \sup_X$:

$$\begin{array}{ccc} FX & \xrightarrow{Ff} & FY \\ {\scriptstyle \sup_X}\downarrow & \Downarrow \bar{f} & \downarrow{\scriptstyle \sup_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

- 2-cells between $(f, \bar{f}) \Rightarrow (g, \bar{g})$ are 2-cells $\phi : f \to g$, satisfying the following equation:

$$\begin{array}{ccc} FX & \underset{Fg}{\overset{Ff}{\underset{\Downarrow F\phi}{\rightrightarrows}}} & FY \\ {\scriptstyle \sup_X}\downarrow & \Downarrow \bar{g} & \downarrow{\scriptstyle \sup_Y} \\ X & \xrightarrow[g]{} & Y \end{array} \quad = \quad \begin{array}{ccc} PX & \xrightarrow{Ff} & PY \\ {\scriptstyle \sup_X}\downarrow & \Downarrow \bar{f} & \downarrow{\scriptstyle \sup_Y} \\ X & \underset{g}{\overset{f}{\underset{\Downarrow \phi}{\rightrightarrows}}} & Y \end{array}$$

Remark 2.3.6. Traditionally algebra pseudomaps means that the 2-cells are isomorphisms. In our case this is automatic, since any natural transformation in **Gpd** is an isomorphism.

We propose the following definition, for what it means to be a homotopy initial algebra (this is inspired by type-theoretic notions in [**8**]):

Definition 2.3.7. An algebra $(W, \sup_W)$ is *homotopy initial*, if the hom-category $F\text{-alg}(W, X)$ (for any other algebra X) is equivalent to the terminal category 1. That is, for any other algebra $(X, \sup_X)$, there exists a pseudomorphism $f : W \to X$ and, for any other pseudomorphism g, there exists a unique algebra 2-cell $\alpha : f \Rightarrow g$.

2.4. Algebras for polynomial endofunctors

We now specialize the definitions from the previous section to the case where the endofunctor is $P_F : \mathbf{Gpd}_{/\mathbb{I}} \to \mathbf{Gpd}_{/\mathbb{I}}$, a polynomial functor.

A subalgebra of $\mathbb{X}$ is an algebra U, with an inclusion map $f : U \hookrightarrow \mathbb{X}$, which is an algebra map. It is known that every algebra has a unique smallest subalgebra. In general, this fact relies on powerset. For polynomial functors we can give an inductive construction of the smallest subalgebra. So let P_F, be the polynomial endofunctor associated to

$$\mathbb{I} \xleftarrow{S} \mathbb{B} \xrightarrow{F} \mathbb{A} \xrightarrow{R} \mathbb{I}$$

and consider an algebra $(\mathbb{X} \to \mathbb{I}, \sup_{\mathbb{X}} : P_F\mathbb{X} \to \mathbb{X})$. We will show that $\mathbb{X}$ has a unique smallest subalgebra. To that end, we inductively define a graph, which we then show can be made into a groupoid over $\mathbb{I}$ and further equipped with an algebra structure, that makes it into a subalgebra.

Let G, be the smallest graph X, such that the following holds

(1) If $(a, T) \in P_F\mathbb{X}$, and for every $b \in \mathbb{B}_a$ $Tb \in X_0$ and for every $u : b \to b' \in \mathbb{B}_a$ $Tu \in X(Tb, Tb')$, we have that $\sup_{\mathbb{X}}(a, T) \in X_0$

(2) If $(f, \varphi) : (a, T) \to (a', T') \in P_F\mathbb{X}$, $\sup_{\mathbb{X}}(a, T) \in X_0$, $\sup_{\mathbb{X}}(a', T') \in X_0$, and for every $u : b \to b'$ over f, $\varphi(u) \in X(Tb, T'b')$, then $\sup_{\mathbb{X}}(f, \varphi) \in X(\sup_{\mathbb{X}}(a, T), \sup_{\mathbb{X}}(a', T'))$

(3) If $u : x \to x'$ and $v : x' \to x''$ are two edges in X, then their $\mathbb{X}$-composition $v \circ u \in X$.

Remark 2.4.1. We would like to provide some justification as to why the above inductive definition is valid. As we will see in Chapter 3, we can construct an inductive set, and then equip it with a graph structure. Alternatively, we can consider the inductive definitions in presheaf categories, as described in [**39**].

To illustrate how a set satisfying these inductive constraints is built, we suppose that we are operating with only discrete groupoids. We define

$$\overline{B} = \{((a, T), b, Tb) \mid (a, T) \in P_F\mathbb{X}, b \in \mathbb{B}_a\}$$

Consider the polynomial:

$$\mathbb{X} \xleftarrow{\pi_3} \overline{B} \xrightarrow{\pi_1} P_F\mathbb{X} \xrightarrow{\text{sup}} \mathbb{X}$$

Then the initial algebra for the above polynomial in $\mathbf{Set}$, satisfies the first constraint in the inductive definition. Finally, one considers a quotient of this initial algebra.

Lemma 2.4.2. *G admits the structure of a groupoid.*

Proof. By definition of G, we have the operation of composition, inherited from $\mathbb{X}$ (and is as such associative). Suppose $x \in G$, then there exists $(a, T) \in P_F(\mathbb{X})$ such that $\sup_{\mathbb{X}}(a, T) = x$. By the definition of G, we have that $Tu \in G$ for all $u : b \to b'$ in $\mathbb{B}_a$. Note that the identity arrow for (a, T) in $P_F\mathbb{X}$ is defined to be $(\text{id}_a, (Tu)_u)$, hence $\sup_{\mathbb{X}}(\text{id}, (Tu)_u)$ is in G. Further $\sup_{\mathbb{X}}$ is functor, hence id_x is in G.

We would like to show that X is a groupoid. If $h : x \to x' \in X$, we will show that there is an inverse by induction. We have two cases corresponding to (2) and (3).

In the first case we have that there exist $(f, \varphi) \in P_F\mathbb{X}$ such that $u = \sup_{\mathbb{X}}(f, \varphi)$ and $\varphi(u) \in X$. By induction, we have that $\varphi(u)^{-1}$ are edges in X. Since the inverse of a generalized natural transformation is defined as $\varphi^{-1}(u) = \varphi(u^{-1})^{-1}$, we have that $\sup_{\mathbb{X}}(f^{-1}, \varphi^{-1}) \in X$. Since $\sup_{\mathbb{X}}$ is a functor, $h^{-1} \in X$.

The other case gives us a pair of morphisms v, u such that $h = v \circ u$. By induction, we have that v^{-1} and u^{-1} are in X. Then we have that $u^{-1} \circ v^{-1} \in X$ and h again has an inverse. □

Proposition 2.4.3. *Every P_F-algebra has a unique smallest subalgebra.*

Proof. Let $(\mathbb{X} \to \mathbb{I}, \sup_{\mathbb{X}} : P_F\mathbb{X} \to \mathbb{X})$, be a P_F-algebra. By Lemma 2.4.2 we have a subgroupoid of $\mathbb{X}$. We can equip G with an algebra structure – the restriction of $\sup_{\mathbb{X}}$ to G. This makes G into a subalgebra.

Let Y be another subalgebra of $\mathbb{X}$. Then underlying graph of Y satisfies the conditions in the definition of G. Suppose $(a, T) \in P_F\mathbb{X}$. If Tb and Tu are members of Y, we have that $(a, T) \in P_FY$. Since $\sup_Y$ must be a restriction of $\sup_{\mathbb{X}}$ we have that $\sup_Y(a, T) = \sup_{\mathbb{X}}(a, T) \in Y$. Similar reasoning applies to morphisms. Since G is the smallest graph satisfying these constraints we get an inclusion map $G \hookrightarrow Y$, which is also an algebra morphism. □

Proposition 2.4.4. *The structure map of the smallest subalgebra of $\mathbb{X}$ is epimorphism.*

Proof. We know that a functor $F : \mathcal{C} \to \mathcal{D}$ is epimorphism, if it is surjective on objects and the closure under composition of graph $\text{im}(F)$ equals $\mathcal{D}$.

Let G be the smallest subalgebra of $\mathbb{X}$. We see that $\sup_G$ is surjective on objects, since if $g \in G$, we have $(a, T) \in P_F(\mathbb{X})$, such that $\sup_{\mathbb{X}}(a, T) = g$. Further, any morphism in G is a composition of morphisms given by $\sup_G$. □

2.5. W-types

Definition 2.5.1. A *W-type* is the initial object in the category of algebras for a polynomial functor.

For the case of polynomial functors for groupoids, we can show that the seemingly weaker formulation of initiality implies 2-initiality as well. We begin by considering the interval category, **I**:

$$0 \longrightarrow 1 \ .$$

One of the basic results for groupoids states that 2-cells in $\mathbf{Gpd}(Y, X)$ are in 1-to-1 correspondence to 1-cells in $\mathbf{Gpd}(\mathbb{Y}, \mathbb{X}^{\mathbf{I}})$. Given $\alpha : f \Rightarrow g$, we produce $\overline{\alpha} : \mathbb{Y} \to \mathbb{X}^{\mathbf{I}}$, by setting:

$$\overline{\alpha}(y) = \alpha_y : f(y) \to g(y)$$
$$\overline{\alpha}(u) = (fu, gu)$$

Likewise, given a morphism $T : \mathbb{Y} \to \mathbb{X}^{\mathbf{I}}$, we obtain a pair of functors (∂_0, ∂_1 are the domain/codomain fibrations):

$$T^0(y) = \partial_0(Ty) \qquad\qquad T^1(y) = \partial_1(Ty)$$
$$T^1(u) = \pi_0(Tu) \qquad\qquad T^1(u) = \pi_1(Tu)$$

and a natural transformation $\varphi^T : T^0 \Rightarrow T^1$ by letting, for $y \in \mathbb{Y}$, $\varphi^T_y : T^0 y \to T^1 y$ to be:

$$\varphi^T_y = T(y)$$

Lemma 2.5.2. *Let $(\mathbb{X}, \sup_{\mathbb{X}})$ be a P_F-algebra. Then $\mathbb{X}^{\mathbf{I}}$ can be equipped with an algebra structure as well.*

Proof. Let $\mathbb{X}$ be an algebra as in the statement. Define $\sup_{\mathbb{X}^{\mathbf{I}}} : P_F \mathbb{X}^I \to \mathbb{X}^I$:

$$\sup_{\mathbb{X}^{\mathbf{I}}}(a, T : \mathcal{B}_a \to \mathbb{X}^{\mathbf{I}}) = \sup_{\mathbb{X}}(\mathrm{id}_a, \varphi^T)$$
$$\sup_{\mathbb{X}^{\mathbf{I}}}(f, \alpha) = (\sup_{\mathbb{X}}(f, \alpha^0), \sup_{\mathbb{X}}(f, \alpha^1))$$

A series of trivial calculations show that $\sup_{\mathbb{X}^{\mathbf{I}}}$ is a functor. □

Lemma 2.5.3. *Given two algebra morphisms $f, g : (\mathbb{X}, \sup_{\mathbb{X}}) \to (\mathbb{Y}, \sup_{\mathbb{Y}})$ and an algebra 2-cell $\alpha : f \Rightarrow g$ in P_F-alg_s, the associated morphism $\overline{\alpha} : (\mathbb{X}, \sup_{\mathbb{X}}) \to (\mathbb{Y}^I, \sup_{\mathbb{Y}^I})$ is an algebra morphism.*

Proof. If $(a, T) \in P_F X$, then:

$$\overline{\alpha}(\sup_{\mathbb{X}}(a,T)) = \alpha_{\sup_{\mathbb{X}}(a,T)} : f(\sup_{\mathbb{X}}(a,T)) \to g(\sup_{\mathbb{X}}(a,T))$$
$$\sup_{\mathbb{Y}^I}(a, \overline{\alpha} \circ T) = \sup_{\mathbb{Y}}(\mathrm{id}_a, \alpha \cdot T) : \sup_{\mathbb{Y}}(a, f \circ T) \to \sup_{\mathbb{Y}}(a, g \circ T)$$

Note that since f and g are algebra morphisms and α is an algebra 2-cell:

$$\alpha_{\sup_{\mathbb{X}}(a,T)} = \sup_{\mathbb{Y}}(\mathrm{id}_a, \alpha \cdot T)$$

Let (u, φ) be a morphism in $P_F\mathbb{X}$:

$$\begin{aligned}\overline{\alpha}(\sup_{\mathbb{X}}(u,\varphi)) &= (f \sup_{\mathbb{X}}(u,\varphi), g \sup_{\mathbb{X}}(u,\varphi)) \\ &= (\sup_{\mathbb{Y}}(u, f \cdot \varphi), \sup_{\mathbb{Y}}(u, g \cdot \varphi)) \\ &= \sup_{\mathbb{Y}^I}(u, \overline{\alpha} \cdot \varphi)\end{aligned}$$

□

Theorem 2.5.4. *If $(\mathbb{W}, \sup_{\mathbb{W}})$ is initial, then it is strictly 2-initial.*

Proof. We claim that for every P_F algebra $(\mathbb{X}, \sup_{\mathbb{X}})$ there is an isomorphism of categories:

$$P_F\text{-}\mathrm{alg}_s((\mathbb{W}, \sup_{\mathbb{W}}), (\mathbb{X}, \sup_{\mathbb{X}})) \cong 1$$

By initiality of $\mathbb{W}$, we know there exists a unique algebra morphism $f : \mathbb{W} \to \mathbb{X}$. So it only remains to show that the only algebra 2-cell $\alpha : f \Rightarrow f$ is the identity. We have already shown that any algebra 2-cell:

$$\mathbb{W} \underset{f}{\overset{f}{\rightrightarrows}} \mathbb{X} \quad (\Downarrow\alpha)$$

corresponds to an algebra morphism

$$f^{\alpha} : \mathbb{W} \to \mathbb{X}^I$$

But, by initiality of $\mathbb{W}$, there exists a unique algebra morphism $\mathbb{W} \to \mathbb{X}^I$, and we know that f^{id} exists. So we must have $f^{\mathrm{id}} = f^{\alpha}$ and hence $\alpha = \mathrm{id}_f$. □

Let P be a polynomial endofunctor assigned to a polynomial:

$$\begin{array}{ccccc} & \mathbb{B} & \xrightarrow{F} & \mathbb{A} & \\ {}^{S}\swarrow & & & & \searrow^{R} \\ \mathbb{I} & & & & \mathbb{I} \end{array}$$

Proposition 2.5.5. *The following is equivalent.*

(1) $(\mathbb{W}, \sup_{\mathbb{W}})$ *is homotopy initial for* P.

(2) For any split fibration $p : \mathbb{E} \to \mathbb{W}$*, which is a strict algebra morphism, there exists a section* $s : \mathbb{W} \to \mathbb{E}$*, with an algebra pseudomap structure,* $\bar{s}$*. Further, for any other section with algebra pseudomap structure* $(g, \bar{g})$*, there exists a unique algebra* 2*-cell* $\alpha : f \to g$.

Proof.

$(1) \Rightarrow (2)$ Suppose $(\mathbb{W}, \sup_{\mathbb{W}})$ is a homotopy-initial algebra. Further suppose we have a split fibration $p : \mathbb{E} \to \mathbb{W}$, which is a strict algebra map. By homotopy initiality we get an algebra pseudomorphism $(s, \bar{s}) : \mathbb{W} \to \mathbb{E}$. We can compose this map with p, and thus obtain a map $(p \circ s, p \cdot \bar{s}) : \mathbb{W} \to \mathbb{W}$. Again, by homotopy initiality we get an algebra 2-cell $\theta : 1_{\mathbb{W}} \Rightarrow p \circ s$.

We now define $s' : \mathbb{W} \to \mathbb{E}$, which will be the section of p. Since p is a split fibration we have:

$$\begin{array}{ccc} (\theta^*_w)(sw) & \xrightarrow{\underline{\theta_w}_{sw}} & sw \\ & & \downarrow p \\ w & \xrightarrow{\theta_w} & (ps)(w) \end{array}$$

Let $u : w \to w'$. The interesting diagram now is:

$$\begin{array}{ccc} e' & \xrightarrow{\underline{u}_{\theta^*_{w'},sw}} & \theta'^*_w(sw') \\ & & \downarrow p \\ w & \xrightarrow{u} & w' = p(\theta^*_{w'}(sw')) \end{array}$$

Since θ is a natural transformation, the following commutes:

$$\begin{array}{ccc} w & \xrightarrow{\theta_w} & psw \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle psu} \\ w' & \xrightarrow[\theta_{w'}]{} & psw' \end{array}$$

and additionally, p is split. Hence $e' = \theta^*_w(sw)$. Set s' as follows:

$$s'(w) = (\theta^*_w)(sw)$$

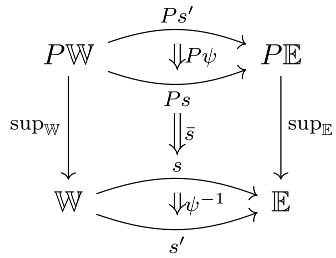
$$s'(u) = \underline{u}_{\theta^*_{w'}(sw')}$$

This definition is functorial, since p is split. We thus obtain a section of p.

Define $\psi : s' \Rightarrow s$, a natural transformation, by setting:

$$\psi_w : (\theta_w^*)(sw) \to sw$$
$$\psi_w = \underline{\theta_w}_{sw}$$

Note that, ψ is not only natural, but also a 1-cell in $\mathbf{Gpd}_{/\mathbb{I}}$. Using it, we define an algebra pseudomap structure for s':



Suppose now, that we have another section with pseudomap structure $(t, \bar{t}) : \mathbb{W} \to \mathbb{E}$. Since W is homotopy initial, we get a unique 2-cell $s' \Rightarrow$.

$(2) \Rightarrow (1)$ Let $(\mathbb{X}, \sup_{\mathbb{X}})$ be an algebra, by Lemma 2.3.4, $\mathbb{W} \times \mathbb{X}$ is an algebra and π_i are strict algebra morphisms. Further, they are split. By (2) we get a section, with a pseudomap structure $(s, \bar{s}) : \mathbb{W} \to \mathbb{W} \times \mathbb{X}$.

We now have an algebra pseudomap $(\pi_2 \circ s, \pi_2 \cdot \bar{s}) : \mathbb{W} \to \mathbb{X}$. Suppose $(g, \bar{g}) : \mathbb{W} \to \mathbb{X}$ is another such. We can use it to produce another section to π_1, $(\langle \mathrm{id}_{\mathbb{W}}, g\rangle, \langle 1, \bar{g}\rangle) : \mathbb{W} \to \mathbb{W} \times \mathbb{X}$. By our assumption we get an algebra 2-cell $\varphi : s \Rightarrow \langle \mathrm{id}_{\mathbb{W}}, g\rangle$, which in turn gives a 2-cell $\pi_2 \cdot s : \pi_2 s \Rightarrow g$.

Suppose $\psi : \pi_2 s \Rightarrow g$ is another algebra 2-cell. Since $\pi_1 s = \mathrm{id}_{\mathbb{W}}$, we have that $\langle \mathrm{id}_{\mathbb{W}}, \psi\rangle$ is a 2-cell between two sections. By uniqueness, $\langle \mathrm{id}_{\mathbb{W}}, \psi\rangle = \phi$ and $\psi = \pi_2\phi$, so $\pi_2\phi$ is unique.

□

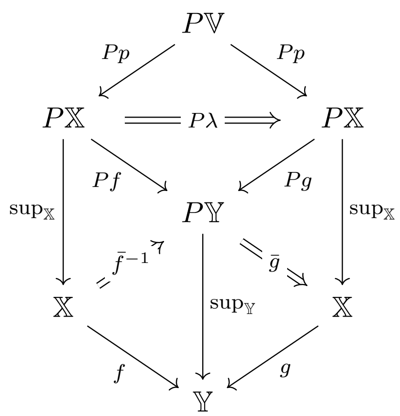
Blackwell, Kelly, Power [**11**] produced several results regarding 2-limits in F-alg, where F is a 2-monad. We adapt them for the case of P-alg, where P is a polynomial 2-functor:

Proposition 2.5.6. *Let $f, g : \mathbb{X} \to \mathbb{Y}$ be a pair of parallel algebra pseudomorphisms. The inserter of these two arrows $(\mathbb{V}, p : \mathbb{V} \to \mathbb{X}, \lambda : fp \Rightarrow gp)$, computed in $\mathbf{Gpd}_{/\mathbb{I}}$, is again an algebra. Further, p is a strict algebra morphism, and λ is an algebra 2-cell.*

Proof. Let $(i : \mathbb{X} \to \mathbb{I}, \sup_{\mathbb{X}})$ and $(j : \mathbb{Y} \to \mathbb{I}, \sup_{\mathbb{Y}})$ be two algebras and f, g be algebra pseudo morphisms as in the above statement of the proposition. We compute the inserter in $\mathbf{Gpd}_{/\mathbb{I}}$, that is:

$$\begin{aligned}
\mathbb{V} &= \{(x, h : fx \to gx) \mid j(h) = \mathrm{id}_{ix}\} \\
\mathbb{V}((x,h),(x',h')) &= \{k \in \mathbb{X}(x,x') \mid fk \circ h' = h \circ gk\} \\
p(x,h) &= x \\
p(k) &= k \\
\lambda_{x,h} &= h
\end{aligned}$$

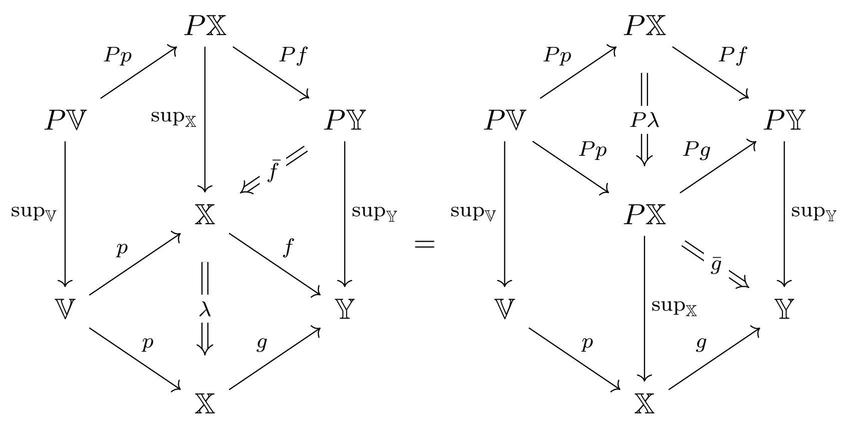
To make $P\mathbb{V}$ into an algebra we will use the universal property of the inserter in $\mathbf{Gpd}_{/\mathbb{I}}$. To that end, take $P\mathbb{V}, \sup_X \circ Pp$. We need to produce a natural transformation $\psi : f \circ \sup_X \circ Pp \Rightarrow g \circ \sup_{\mathbb{X}} \circ Pp$. Consider the following 2-cell:



That is, out ψ is defined to be:

$$f \circ \sup_{\mathbb{X}} \circ Pp \xrightarrow{\bar{f}^{-1} \cdot Pp} \sup_{\mathbb{Y}} \circ Pf \circ Pp \xrightarrow{\sup_{\mathbb{Y}} \cdot P\lambda} \sup_{\mathbb{Y}} \circ Pg \circ Pp \xrightarrow{\bar{g} \cdot Pp} g \circ \sup_{\mathbb{X}} \circ Pp$$

By the universal property of inserters in $\mathbf{Gpd}_{/\mathbb{I}}$, we have a unique morphism, which we denote by $\sup_{\mathbb{V}} : P\mathbb{V} \to \mathbb{V}$, such that $p \circ \sup_{\mathbb{V}} = \sup_{\mathbb{X}} \circ Pp$, and $\lambda \cdot \sup_{\mathbb{V}} = \psi$. Hence, $(\mathbb{V}, \sup_{\mathbb{V}})$ is an algebra, and p is a strict morphism. All of these conditions ensure that we have the following equality:



And hence, λ is an algebra 2-cell. □

Proposition 2.5.7. *Let $\alpha, \beta : f \Rightarrow g$ be a pair of parallel algebra 2-cells. The equifier of these two arrows $(\mathbb{E}, e : \mathbb{E} \to \mathbb{X})$, computed in $\mathbf{Gpd}_{/\mathbb{I}}$, is again an algebra, and e is a strict algebra morphism.*

Proof. Let $\alpha, \beta : f \Rightarrow g : \mathbb{X} \to \mathbb{Y}$, be a pair of parallel algebra 2-cells. Consider $\mathbb{E}$, the full subcategory of $\mathbb{X}$, such that

$$\mathbb{E}_0 = \{x \in \mathbb{X}_0 \mid \alpha_e = \beta_e\}$$

This is an equifier of α and β in $\mathbf{Gpd}_{/\mathbb{I}}$. Let $e : \mathbb{E} \to \mathbb{X}$ be the inclusion map. We would like that α and β whiskered by $\sup_{\mathbb{X}} \circ Pe$ are equal. We can produce the following equations:

$$(\alpha \cdot \sup_{\mathbb{X}} Pe) \circ (\bar{f} \cdot Pe) = (\bar{g} \cdot Pe) \circ (\sup_{\mathbb{Y}} \cdot P\alpha \cdot Pe)$$

Since $\alpha e = \beta e$, we have:

$$= (\bar{g} \cdot Pe) \circ (\sup_{\mathbb{Y}} \cdot P\beta \cdot Pe)$$

And, again, since β is a 2-cell:

$$= (\beta \cdot \sup_{\mathbb{X}} Pe) \circ (\bar{f} \cdot Pe)$$

Because $\bar{f} \cdot Pe$ is an isomorphism, we obtain:

$$(\alpha \cdot \sup_{\mathbb{X}} Pe) = (\beta \cdot \sup_{\mathbb{X}} Pe)$$

By the universal property of equifiers, we get a map $\sup_{\mathbb{E}} : P\mathbb{E} \to \mathbb{E}$, such that $\sup_{\mathbb{X}} \circ Pe = e \circ \sup_{\mathbb{E}}$, that is, e is a strict algebra morphism. $\square$

Theorem 2.5.8. *The strict initial algebra $\mathbb{W}$ for a polynomial functor P is homotopy-initial as well.*

Proof. Let $\mathbb{W}$ be the strict initial algebra and let $(\mathbb{X}, \sup_{\mathbb{X}})$ be another algebra. We get a strict algebra morphism $\mathbb{W} \to \mathbb{X}$. Suppose now, that we have two algebra pseudo morphisms $f, g : \mathbb{W} \to \mathbb{X}$.

Applying the previous proposition, we get an inserter $\mathbb{V} \xrightarrow{p} \mathbb{W}$ and an algebra 2-cell $\lambda : fp \to gp$. Since $\mathbb{V}$ is again an algebra, we get a map $\mathbb{W} \xrightarrow{u} \mathbb{V}$. This map, must be a section to p, since $\mathrm{id}_{\mathbb{W}}$ is the unique strict algebra map $\mathbb{W} \to \mathbb{W}$. Hence $\lambda \cdot u$ is an algebra 2-cell $f \to g$.

Let $\alpha : f \Rightarrow g$, be another such 2-cell and consider the equifier of α and $\lambda \cdot u$. As we saw in the proof of the previous proposition, we define it as a full subcategory of $\mathbb{W}$, where the objects are:

$$E_0 = \{w \in \mathbb{W}_0 \mid \alpha_w = (\lambda \cdot u)_w\}$$

with the equifier map, being a simple inclusion. We know that as $\mathbb{W}$ has no non-trivial algebras, it must be the case that $\mathbb{E} = \mathbb{W}$ and hence, $\alpha = (\lambda \cdot u)$. $\square$

2.6. J-relative algebras

In [**5**] and [**22**] the authors explore the concept of relative monads. We adapt their work to define a simpler notion of J-relative F-algebras.

Definition 2.6.1. Let $F : \mathbb{C} \to \mathbb{D}$ and $J : \mathbb{C} \to \mathbb{D}$ be two functors. Then a *J-relative F-algebra* is a pair (X, χ), where $X \in \mathbb{D}$ and χ is a natural transformation of type $\mathbb{D}(J-, X) \Rightarrow \mathbb{D}(F-, X)$.

Given (X, χ) and (Y, ψ), we say that $f : X \to Y$ is a J-relative F-algebra morphism, if for all $Z \in \mathbb{C}$ and all $h : JZ \to X$, the following commutes:

$$\begin{array}{ccc} FZ & \xrightarrow{\chi(h)} & X \\ & \searrow^{\psi(f \circ h)} & \downarrow f \\ & & Y \end{array}$$

We arrange J-relative algebras into a category $F\text{-alg}^J$, where the objects are algebras, the morphisms are algebra morphisms and composition is inherited from $\mathbb{D}$.

Remark 2.6.2. If F is an endofunctor and J is the identity functor, we recover the usual definition of F-algebras.

Consider the polynomial functor P, assigned to the following polynomial:

$$\begin{array}{ccccc} & \mathbb{B} & \xrightarrow{F} & \mathbb{A} & \\ \swarrow^{S} & & & & \searrow^{R} \\ \mathbb{I} & & & & \mathbb{J} \end{array}$$

Further, suppose we have a morphism $\sigma : \mathbb{I} \to \mathbb{J}$. Let $P' : \mathbf{Gpd}_{/\mathbb{J}} \to \mathbf{Gpd}_{/\mathbb{J}}$, be defined as the composite:

$$\mathbf{Gpd}_{/\mathbb{J}} \xrightarrow{\Delta_\sigma} \mathbf{Gpd}_{/\mathbb{I}} \xrightarrow{P} \mathbf{Gpd}_{/\mathbb{I}}$$

which is again a polynomial functor.

We will explore Σ_σ-relative P-algebras, in terms of more familiar P'-algebras. We will construct a mapping taking $(\mathbb{X}, \chi : \mathbf{Gpd}_{/\mathbb{J}}(\Sigma_\sigma -, \mathbb{X}) \Rightarrow \mathbf{Gpd}_{/\mathbb{J}}(P-, \mathbb{X}))$ a Σ_σ-relative P-algebra to $\Phi(\mathbb{X}, \chi)$, a P'-algebra. Since $\Sigma_\sigma \dashv \Delta_\sigma$, we have a counit $\epsilon : \Sigma_\sigma\Delta_\sigma \Rightarrow 1$, in particular an arrow of the type $\epsilon_\mathbb{X} : \Sigma_\sigma\Delta_\sigma\mathbb{X} \to \mathbb{X}$. Then $\chi_{\Delta_\sigma\mathbb{X}} : \mathbf{Gpd}_{/\mathbb{J}}(\Sigma_\sigma\Delta_\sigma\mathbb{X}, \mathbb{X}) \to \mathbf{Gpd}_{/\mathbb{J}}(P\Delta_\sigma\mathbb{X}, \mathbb{X}))$ provides us with a P'-algebra structure morphism:

$$\chi(\epsilon_\mathbb{X}) : P\Delta_\sigma\mathbb{X} \to \mathbb{X}$$

That is, we define $\Phi(\mathbb{X}, \chi)$ as:

$$\Phi(\mathbb{X}, \chi) = (\mathbb{X}, \chi(\epsilon_\mathbb{X}))$$

Proposition 2.6.3. *The mapping* Φ *extends to a functor* $\Phi : P\text{-alg}^{\Sigma_\sigma} \to P'\text{-alg}_s$.

Proof. We will show that the Σ_σ-relative algebra morphism $f : (\mathbb{X}, \chi) \to (\mathbb{Y}, \psi)$, is also a P'-algebra morphism $\Phi(\mathbb{X}, \chi) \to \Phi(\mathbb{Y}, \psi)$. Since f is a Σ_σ-relative morphism, we know the following:

$$\begin{aligned} f \circ \sup_\mathbb{X} &= f \circ \xi(\epsilon_\mathbb{X}) \\ &= \psi(f \circ \epsilon_\mathbb{X}) \end{aligned}$$

Further, we have that ψ is natural, so the following commutes:

$$\begin{array}{ccc} \mathbf{Gpd}_{/\mathbb{J}}(\Sigma_\sigma\Delta_\sigma Y, Y) & \xrightarrow{\psi} & \mathbf{Gpd}_{/\mathbb{J}}(P_F\Delta_\sigma Y, Y) \\ \downarrow{\scriptstyle -\circ\Sigma_\sigma\Delta_\sigma f} & & \downarrow{\scriptstyle -\circ P_F\Delta_\sigma f} \\ \mathbf{Gpd}_{/\mathbb{J}}(\Sigma_\sigma\Delta_\sigma X, Y) & \xrightarrow{\psi} & \mathbf{Gpd}_{/\mathbb{J}}(P_F\Delta_\sigma X, Y) \end{array}$$

That is, we have the following:

$$\begin{aligned}\sup_{\mathbb{Y}} \circ P_F\Delta_\sigma f &= \psi(\epsilon_{\mathbb{Y}}) \circ P_F\Delta_\sigma f \\ &= \psi(\epsilon_{\mathbb{Y}} \circ \Sigma_\sigma\Delta_\sigma f) \\ &= \psi(f \circ \epsilon_{\mathbb{X}}) \qquad\qquad \text{since } \epsilon \text{ is a natural transformation}\end{aligned}$$

Since the morphisms remain unchanged and composition in both categories is inherited from $\mathbf{Gpd}_{/\mathbb{J}}$ we get a functor $P\text{-alg}^{\Sigma_\sigma} \to P'\text{-alg}_s$ □

Further, we can construct a mapping Φ^{-1} taking a P'-algebra to a Σ_σ-relative P-algebra. Let $(\mathbb{X}, \sup_{\mathbb{X}} : P'\mathbb{X} \to \mathbb{X})$ be a P'-algebra. We have a natural isomorphism ψ : $\mathbf{Gpd}_{/\mathbb{J}}(\Sigma_\sigma -, -) \Rightarrow \mathbf{Gpd}_{/\mathbb{I}}(-, \Delta_\sigma -)$, using it we can define $\chi : \mathbf{Gpd}_{/\mathbb{J}}(\Sigma_\sigma -, \mathbb{X}) \Rightarrow \mathbf{Gpd}_{/\mathbb{J}}(P-, \mathbb{X}))$:

$$\chi_{\mathbb{Y}}(u) = \sup_{\mathbb{X}} \circ P(\psi_{\mathbb{Y},\mathbb{X}}(u))$$

We now set Φ^{-1} to be:

$$\Phi^{-1}(\mathbb{X}, \sup_{\mathbb{X}}) = (\mathbb{X}, \chi)$$

Proposition 2.6.4. *The mapping Φ^{-1} extends to a functor $\Phi : P'\text{-alg}_s \to P\text{-alg}^{\Sigma_\sigma}$.*

Proof. Given a P'-algebra morphism $f : (\mathbb{X}, \sup_{\mathbb{X}}) \to (\mathbb{Y}, \sup_{\mathbb{Y}})$, we can see that f is also a Σ_σ-relative P-algebra morphism $\Phi^{-1}(\mathbb{X}, \sup_{\mathbb{X}}) \to (\mathbb{Y}, \sup_{\mathbb{Y}})$. Let $\Phi^{-1}(\mathbb{X}, \sup_{\mathbb{X}}) = (\mathbb{X}, \chi)$ and $\Phi^{-1}(\mathbb{Y}, \sup_{\mathbb{Y}}) = (\mathbb{Y}, \psi)$. If $\mathbb{Z} \in \mathbf{Gpd}_{/\mathbb{I}}$ and $h : \Sigma_\sigma\mathbb{Z} \to \mathbb{X}$, then:

$$\begin{aligned}f \circ \chi(h) &= f \circ \sup_{\mathbb{X}} \circ P(\psi_{\mathbb{Z},\mathbb{X}}(h)) \\ &= \sup_{\mathbb{Y}} \circ P\Delta_\sigma f \circ P(\psi_{\mathbb{Z},\mathbb{X}}(h)) \\ &= \sup_{\mathbb{Y}} \circ P(\psi_{\mathbb{Z},\mathbb{Y}}(f \circ h)) \\ &= \psi(f \circ h)\end{aligned}$$

Thus, Φ^{-1} is a functor, as morphisms remain unchanged and composition in both domain and codomain is inherited from $\mathbf{Gpd}_{/\mathbb{J}}$. □

Proposition 2.6.5. *The functor $\Phi : P\text{-alg}^{\Sigma_\sigma} \to P'\text{-alg}_s$ is an isomorphism.*

Proof. We have shown that we have two functors going back and forth in between the two categories. It now suffices to show that these two are inverses of each other on objects (since they are identical on morphisms).

Let $(\mathbb{X}, \chi)$ be a Σ_σ-relative algebra. Then, as before we get $\Phi(\mathbb{X}, \chi) = (\mathbb{X}, \chi(\epsilon_\mathbb{X}))$, a P'-algebra. Going back around, we get $\Phi^{-1}(\Phi(\mathbb{X}, \chi)) = (\mathbb{X}, \widehat{\chi})$, then:

$$\begin{aligned}
\widehat{\chi}_\mathbb{X}(u) &= \chi(\epsilon_\mathbb{X}) \circ P(\psi_{\mathbb{X},\mathbb{Y}} u) \\
&= \chi(\epsilon_\mathbb{X} \circ \Sigma_\sigma(\psi_{\mathbb{Y},\mathbb{X}}(u))) \\
&= \chi(\epsilon_\mathbb{X} \circ \Sigma_\sigma(\Delta_\sigma u \circ \eta_\mathbb{Y}) \\
&= \chi(u)
\end{aligned}$$

To show that the mappings are inverse in the other direction as well assume $(\mathbb{X}, \sup_\mathbb{X})$ is a P'-algebra. Then we get $\Phi^{-1}(\mathbb{X}, \sup_\mathbb{X}) = (\mathbb{X}, \chi)$ a Σ_σ-relative algebra. Applying Φ to it we get $\Phi(\Phi^{-1}(\mathbb{X}, \sup_\mathbb{X})) = (\mathbb{X}, \chi(\epsilon_\mathbb{X}))$. We can show that $\chi(\epsilon_\mathbb{X}) = \sup_\mathbb{X}$:

$$\begin{aligned}
\chi(\epsilon_\mathbb{X}) &= \sup_\mathbb{X} \circ P(\psi_{\Delta_\sigma \mathbb{X}, \mathbb{X}}(\epsilon_\mathbb{X})) \\
&= \sup_\mathbb{X} \circ P(\mathrm{id}_{\Delta_\sigma \mathbb{X}}) \\
&= \sup_\mathbb{X}
\end{aligned}$$

□

Corollary 2.6.6. *The initial P'-algebra is also the initial Σ_σ-relative P-algebra.*

The above can be generalized to any pair of adjoint functors $J \dashv K$.

Proposition 2.6.7. *Given a functor $F : \mathcal{C} \to \mathcal{D}$, and a pair of adjoint functors $J : \mathcal{C} \to \mathcal{D}$ and $K : \mathcal{D} \to \mathcal{C}$, the category of J-relative F-algebras, $F\text{-alg}^J$, is isomorphic to the category $(F \circ K)\text{-alg}_s$.*

Proof. The proofs of the previous propositions use no particular properties of either polynomials or groupoids. □

Corollary 2.6.8. *Suppose we have the following polynomial:*

$$\mathbb{I} \xleftarrow{S} \mathbb{B} \xrightarrow{F} \mathbb{A} \xrightarrow{R} \mathbb{J}$$

Let P be the polynomial functor assigned to it. Further suppose we have a fibration *$\sigma : \mathbb{J} \to \mathbb{I}$. Then:*

$$(P \circ \Pi_\sigma)\text{-alg}_s \cong P\text{-alg}^{\Delta_\sigma}$$

Note that $P \circ \Pi_\sigma$ is again a polynomial functor (thanks to the Beck-Chevalley condition), of type $\mathbf{Gpd}_{/\mathbb{J}} \to \mathbf{Gpd}_{/\mathbb{J}}$.

CHAPTER 3

W-types for split fibrations

The goal of this chapter is to give a construction of W-types for a simple polynomial functor on groupoids, associated to a polynomial of the form:

$$1 \longleftarrow \mathbb{B} \xrightarrow{F} \mathbb{A} \longrightarrow 1$$

where F is a split fibration. The main result is the construction of such a groupoid and showing that the algebra structure defined on it is initial (Theorem 3.2.3). The particular steps of the construction are illustrated by an example.

We do our work in a constructive fashion, as opposed to the classical fixpoint construction, which requires the use of ordinals.

In the last section (Section 3.4), we show an alternative construction using the work of Moerdijk and Palmgren.

The construction and techniques presented in this chapter will be used throughout the rest of the thesis, in particular when showing the existence of W-types for polynomials defined in slices and dependent polynomials.

3.1. Construction of W-types

Background. Let $F : \mathbb{B} \to \mathbb{A}$ be a split fibration of groupoids. This will remain fixed for the rest of the section. If $f : a \to a'$ in $\mathbb{A}$, we have a collection of arrows of the form $\underline{f}_b : b \to f_! b$, indexed by $b \in \mathbb{B}_a$, obtained from the splitting data:

$$\begin{array}{ccc} \mathbb{B} & & b \xrightarrow{\underline{f}_b} f_! b \\ \big\downarrow F & & \vdots \qquad\quad \vdots \\ \mathbb{A} & & a \xrightarrow{\ f\ } a' \end{array}$$

We call this set $\mathcal{F}_f$:

$$\mathcal{F}_f =_{\text{def}} \left\{ \underline{f}_b : b \to f_! b \mid b \in \mathbb{B}_a \right\} \tag{3.1}$$

We also need graphs. Here, we use graph to denote what is usually called a directed graph in category theory, i.e. a reflexive directed multigraph. Correspondingly, when we when we say subgraphs, we mean reflexive directed submultigraphs. We need to establish some notation. For a graph (X_0, X_1), where X_0 is the set of vertices, X_1 is the set of edges, we write σ, τ, ρ, for the source, target and reflexivity maps:

$$X_0 \underset{\tau}{\overset{\sigma}{\leftleftarrows}} \xrightarrow{\rho} X_1 \ .$$

When we talk about edges between two vertices, we will use categorical notation and $f : a \to a'$ to denote an edge f with $\sigma(f) = a$ and $\tau(f) = a'$.

Construction of W. Consider the polynomial functor $P_F : \mathbf{Gpd} \to \mathbf{Gpd}$ defined by the composition

$$\mathbf{Gpd} \xrightarrow{\Delta_{\mathbb{B}}} \mathbf{Gpd}_{/\mathbb{B}} \xrightarrow{\Pi_F} \mathbf{Gpd}_{/\mathbb{A}} \xrightarrow{\Sigma_{\mathbb{A}}} \mathbf{Gpd}$$

For $\mathbb{X} \in \mathbf{Gpd}$, we have:

- The objects of $P_F\mathbb{X}$ consist of pairs (a, T), where $a \in \mathbb{A}$ and T is a functor of the form $T : \mathbb{B}_a \to \mathbb{X}$,
- For $(a, T), (a', T') \in P_F\mathbb{X}$, the morphisms consist of pairs $(f, \varphi) : (a, T) \to (a', T')$, where $f : a \to a'$ and $\varphi : T \Rightarrow T' f_!$.

We will construct an initial algebra for P_F in the following steps.

(1) Define sets $\overline{A}$, $\overline{B}$ and a function:

$$\overline{F} : \overline{B} \to \overline{A}$$

Then we take the initial algebra $W^{(s)}$ of the polynomial functor $P_{\overline{F}} : \mathbf{Set} \to \mathbf{Set}$ and define a graph structure on it.

(2) Define a subgraph $\widetilde{W}$ of $W^{(s)}$ and equip it with a groupoid structure.

(3) Using the groupoid structure of $\widetilde{W}$, define a sub-groupoid W. We will then show that W admits a P_F-algebra structure and that it is the initial algebra.

Step 1. Now, following the above plan, we begin by defining the two sets:

$$\overline{A} =_{\text{def}} \mathbb{A}_0 \sqcup \mathbb{A}_1$$

that is, $\overline{A}$ is the set of objects and arrows of $\mathbb{A}$. Next, for $a \in \mathbb{A}$ define $\overline{B}_a$:

$$\overline{B}_a =_{\text{def}} \{b \in \mathbb{B}_0 \mid Fb = a\} \cup \{u \in \mathbb{B}_1 \mid Fu = \text{id}_a\}$$

i.e. it is the set of objects over a and vertical morphism. For $f : a \to a'$ in $\mathbb{A}$, we define the set $\overline{B}_f$

$$\overline{B}_f =_{\text{def}} \overline{B}_a \sqcup \overline{B}_{a'} \sqcup \mathcal{F}_f.$$

This is the disjoint union of objects over a and a', the vertical morphisms over both, and the morphisms over f obtained from the splitting data of 3.1. Using these, we define $\overline{B}$:

$$\overline{B} =_{\text{def}} \left(\coprod_{a \in \mathbb{A}_0} \overline{B}_a \right) \sqcup \left(\coprod_{f \in \mathbb{A}_1} \overline{B}_f \right)$$

Finally we define a function $\overline{F} : \overline{B} \to \overline{A}$, that maps all elements of $\overline{B}_a$ to a, and elements of $\overline{B}_f$ to f.

Let $W^{(s)}$ be the initial algebra of $P_{\overline{F}} : \mathbf{Set} \to \mathbf{Set}$. This is the collection of elements of the form $\sup(x, T)$, where x is in $\overline{A}$ and T is a function $\overline{B}_x \to W^{(s)}$. The algebra structure map $\sup : P_{\overline{F}} W^{(s)} \to W^{(s)}$, maps a pair $(x, T : \overline{B}_x \to W^{(s)})$ to $\sup(a, T)$.

Lemma 3.1.1. *$W^{(s)}$ admits the structure of a graph.*

Proof. We wish to consider $W^{(s)}$ as a reflexive graph. First we define the set of vertices, $(W^{(s)})_0$, which consists of the elements of the form $\sup(a, T)$, for $a \in \mathbb{A}$ and $T : \overline{B}_a \to W^{(s)}$. The edges, $(W^{(s)})_1$, are the elements of the form $\sup(f, \varphi)$, where $f : a \to a'$ in $\mathbb{A}$ and $\varphi : \overline{B}_f \to W^{(s)}$. We will now define graph structure maps. Take an element of the edges set, $\sup(f : a \to a', \varphi : \overline{B}_f \to W^{(s)})$. The source and target maps are defined as follows:

$$\sigma(\sup(f, \varphi)) =_{\text{def}} \sup(a, \varphi|_{\overline{B}_a})$$
$$\tau(\sup(f, \varphi)) =_{\text{def}} \sup(a', \varphi|_{\overline{B}_{a'}})$$

Since $\overline{B}_a \subset \overline{B}_f$, when we restrict φ to it, we obtain a function of the form $\overline{B}_a \to W^{(s)}$, and thus an element of $(W^{(s)})_0$. The same holds for the target map.

Given a vertex $\sup(a, T : \overline{B}_a \to W^{(s)})$, we define the reflexive arrow $\sup(a, T) \to \sup(a, T)$, by constructing a map $\rho^* T : \overline{B}_{\text{id}_a} \to W^{(s)}$. $\overline{B}_{\text{id}_a}$ is defined to be $\overline{B}_a \cup \overline{B}_a \cup \mathcal{F}_{\text{id}_a}$, and we can see that all of its components are either equal to $\overline{B}_a$, or a subset of it, so we set:

$$(\rho^* T)(x) =_{\text{def}} T(x)$$

Given the above, we define the reflexivity structure map of the graph, that takes a vertex and returns an edge, as:

$$\rho(\sup(a, T)) =_{\text{def}} \sup(\text{id}_a, (\rho^* T))$$

Both target and source maps are sections of this. We will denote $\rho(\sup(a, T))$ by $\text{id}_{\sup(a,T)}$.

□

Step 2. Note that for $x \in \overline{A}$ (either a vertex or an edge), $\overline{B}_x$ can be equipped with a reflexive graph structure as well. We consider the elements that are objects in $\mathbb{B}$, to be vertices, and the elements that are morphisms in $\mathbb{B}$, are considered to be the edges. Reflexive edges are the identity morphisms.

Definition 3.1.2. Let $x \in \overline{A}$ and $\sup(x, T) \in W^{(s)}$. We say that $\sup(x, T)$ is a *hereditary graph morphism (hg-morphism)*, if:

- $T : \overline{B}_x \to W^{(s)}$ is a graph morphism, and,
- for all $x \in \overline{B}_x$, $T(x)$ is a hereditary graph morphism.

Denote by $\widetilde{W}$, the subset of $W^{(s)}$, consisting of hereditary graph morphisms.

Lemma 3.1.3. *$\widetilde{W}$ is a subgraph of $W^{(s)}$.*

Proof. We need to show that the structure maps of $W^{(s)}$ preserve the property of being an hg-morphism. Let $\sup(f, \varphi)$ be an edge that is an hg-morphism. The actions of σ and τ restrict the domain of φ to a subgraph. Therefore, if the original arrow was an hg-morphism, so is the resulting arrow.

If $\sup(a, T)$ is a vertex and an hg-morphism, then the edge $\sup(\mathrm{id}_a, \rho^*T)$ is an hg-morphism as well:

- T is a graph morphism of type $\overline{B}_a \to W^{(s)}$. If we unfold the definition of $\overline{B}_{\mathrm{id}_a} = \overline{B}_a \sqcup \overline{B}_a \sqcup \mathcal{F}_{\mathrm{id}_a}$, we see that ρ^*T is a graph morphism for the $\overline{B}_a$ components. An arrow of the form $\underline{\mathrm{id}_{a_b}} : b \to b$, is mapped to id_{Tb} by ρ^*T, since T is a graph morphism, which is an arrow of the appropriate source and target. Hence, ρ^*T is a graph morphism.
- For all $x \in \overline{B}_{\mathrm{id}_a}$, $\rho^*T(x) = T(x)$ and we know that Tx is hg-morphism.

□

Proposition 3.1.4. *$\widetilde{W}$ is isomorphic, as a graph, to the smallest graph X, such that:*

- *if $a \in \mathbb{A}$ and $T : \mathbb{B}_a \to X$ is a graph morphism, then (a, T), is a vertex in X,*
- *if $a, a' \in \mathbb{A}$, $f : a \to a'$, $(a, T), (a', T') \in X_0$, and φ is a collection of arrows in X of the form:*

$$\varphi = \left(\varphi_b : Tb \to T' f_! b \mid b \in \mathbb{B}_a\right)$$

then (f, φ) is an edge from (a, T) to (a', T').

Proof. Let X be such a graph. Then we define a map $H : \widetilde{W} \to X$ by recursion on the elements of $\widetilde{W}$:

$$H(\sup(a,T)) =_{\text{def}} (a, HT)$$

Since $\varphi(\underline{f}_b) : Tb \to T' f_! b$, H again just acts on components:

$$H(\sup(f, \varphi : \overline{B}_f \to W^{(s)})) =_{\text{def}} (f, (H(\varphi(\underline{f}_b))|b \in \mathbb{B}_a))$$

By unfolding the relevant definitions one can easily show H is a graph morphism. Next, we define the inverse for H, $H^{-1} : X \to \widetilde{W}$:

$$H^{-1}((a,T)) = \sup(a, H^{-1} \circ T)$$
$$H^{-1}((f,\varphi)) = \sup(f, \overline{\varphi}) \qquad \text{for } (f,\varphi) : \sup(a,T) \to \sup(a',T')$$

where:

$$\overline{\varphi}(x) = \begin{cases} H^{-1} \circ T(x) & \text{if } x \in B_a \\ H^{-1} \circ T'(x) & \text{if } x \in B_{a'} \\ H^{-1}\varphi_b & \text{if } x = \underline{f}_b \text{ for } \underline{f}_b \in \mathcal{F}_f \end{cases}$$

By induction, one can prove that H^{-1} is a graph morphism, and also that H and H^{-1} are inverses to each other. □

We will denote the inductively defined graph by $\widetilde{W}'$, which will be used further on in this chapter.

By recursion we can define the operation of composition on the edges of $\widetilde{W}$. Let $\sup(f,\varphi) : \sup(a,T) \to \sup(a',T')$ and $\sup(f',\varphi') : \sup(a',T') \to \sup(a'',T'')$. We define $\sup(f',\varphi') \circ \sup(f,\varphi) : \sup(a,T) \to \sup(a'',T'')$, by letting:

$$\sup(f',\varphi') \circ \sup(f,\varphi) =_{\text{def}} \sup(f' \circ f, (\varphi' \circ \varphi))$$

where:

$$(\varphi' \circ \varphi)\big|_{\overline{B}_a} =_{\text{def}} \varphi|_{\overline{B}_a}$$
$$(\varphi' \circ \varphi)\big|_{\overline{B}_{a''}} =_{\text{def}} \varphi'|_{\overline{B}_{a''}}$$
$$(\varphi' \circ \varphi)(\underline{f' \circ f}_b) =_{\text{def}} \varphi'(\underline{f'}_{f_! b}) \circ \varphi(\underline{f}_b)$$

The last line of the definition of $\varphi' \circ \varphi$ is well defined, since F is a split fibration. In particular for any $\underline{f' \circ f}_b$, we have $\underline{f}_b : b \to f_! b$ and $\underline{f'}_{f_! b} : f_! b \to f'_! f_! b = (f' \circ f)_! b$, such that $\underline{f' \circ f}_b = \underline{f'}_{f_! b} \circ \underline{f}_b$. Further, since φ and φ' are hg-morphisms, the edges match up.

Lemma 3.1.5. *$\widetilde{W}$ admits the structure of a category.*

Proof. We will show that the operation of composition as defined above is:

- well defined,
- associative, and
- unital with respect to reflexivity arrows.

As such, $\widetilde{W}$ is a category.

In order for the operation of composition to be well defined, we need to check that it preserves the hereditary graph morphism property. Let $\sup(f,\varphi) : \sup(a,T) \to \sup(a',T')$ and $\sup(f',\varphi') : \sup(a',T') \to \sup(a'',T'')$ be two edges in $\widetilde{W}$. We proceed by induction, and suppose that $(\varphi' \circ \varphi)(x)$ is an hg-morphism for all $x \in \overline{B}_{f'\circ f}$. Then what remains to be shown is that $\varphi' \circ \varphi$ is a graph morphism. It is clear that $\varphi' \circ \varphi$ is a graph morphism when restricted to $\overline{B}_a$ and $\overline{B}_{a'}$ since both of components were. Now, take an arrow from $\mathcal{F}_{f'\circ f}$, say $\underline{f' \circ f}_b$. By the above definition, $\varphi' \circ \varphi(\underline{f' \circ f}_b)$ is assigned to be $\varphi'(\underline{f'}_{f_!b}) \circ \varphi(\underline{f}_b)$, that is, an arrow $Tb \to T''(f' \circ f)_!b$.

To show that $\circ$ is associative, let $\sup(f,\varphi) : \sup(a,T) \to \sup(a',T)'$, $\sup(f',\varphi') : \sup(a',T') \to \sup(a'',T'')$ and $\sup(f'',\varphi'') : \sup(a,T'') \to \sup(a''',T''')$. Then, by induction assume:

$$\varphi''(\underline{f''}_{(f'\circ f)_!b}) \circ (\varphi'(\underline{f'}_{f_!b}) \circ \varphi(\underline{f}_b)) = (\varphi''(\underline{f''}_{(f'\circ f)_!b}) \circ \varphi'(\underline{f'}_{f_!b})) \circ \varphi(\underline{f}_b)$$

Thanks to the fact that F is a split fibration we have:

$$\begin{aligned}
(\varphi'' \circ (\varphi' \circ \varphi))(\underline{f'' \circ f' \circ f}_b) &= \varphi''(\underline{f''}_{(f'\circ f)_!b}) \circ (\varphi' \circ \varphi)(\underline{f' \circ f}_b)) \\
&= \varphi''(\underline{f''}_{(f'\circ f)_!b}) \circ (\varphi'(\underline{f'}_{f_!b}) \circ \varphi(\underline{f}_b)) \\
((\varphi'' \circ \varphi') \circ \varphi)(\underline{f'' \circ f' \circ f}_b) &= (\varphi'' \circ \varphi')(\underline{f'' \circ f'}_{f_!b}) \circ \varphi(\underline{f}_b) \\
&= (\varphi''(\underline{f''}_{(f'\circ f)_!b}) \circ \varphi'(\underline{f'}_{f_!b})) \circ \varphi(\underline{f}_b)
\end{aligned}$$

Again, let $\sup(f,\varphi) : \sup(a,T) \to \sup(a',\varphi')$ and by induction assume $\varphi(\underline{f}_b) \circ \mathrm{id}_{Tb} = \varphi(\underline{f}_b)$ for all b. By unrolling the definition of composition we have that $\sup(f,\varphi) \circ \mathrm{id}_{\sup(a,T)} = \sup(f,(\varphi \circ \rho^*T))$. Then, since F is split:

$$\begin{aligned}
(\varphi \circ r^*T)(\underline{f}_b) &= \varphi(\underline{f}_b) \circ \rho^*T(\underline{\mathrm{id}_a}_b) \\
&= \varphi(\underline{f}_b) \circ \rho^*T(\mathrm{id}_b) = \varphi(\underline{f}_b) \circ \mathrm{id}_{Tb}
\end{aligned}$$

Similar steps allow us to conclude that $\mathrm{id}_{\sup(a',T')}$ will be a right identity for $\sup(f,\varphi)$.

□

Proposition 3.1.6. *$\widetilde{W}$ admits the structure of a groupoid.*

Proof. We also define inverses for arrows. If $\sup(f,\varphi)$ is an arrow, we define its inverse recursively by setting:

$$(\sup(f,\varphi))^{-1} = \sup(f^{-1},\varphi^{-1})$$
$$\varphi^{-1}(\underline{f^{-1}}_b) = (\varphi(\underline{f}_{f_!^{-1}b}))^{-1}$$

Let $\sup(f,\varphi) : \sup(a,T) \to \sup(a',T')$ and assume, by induction, that $\varphi(\underline{f}_b)^{-1} \circ \varphi(\underline{f}_b) = \mathrm{id}_{Tb}$, then, by the fact that F is split:

$$\begin{aligned}(\varphi^{-1}\circ\varphi)(\underline{\mathrm{id}_a}_b) &= \varphi^{-1}(\underline{f^{-1}}_{f_!b}) \circ \varphi(\underline{f}_b)\\ &= (\varphi(\underline{f}_{f_!^{-1}f_!b}))^{-1} \circ \varphi(\underline{f}_b)\\ &= \varphi(\underline{f}_b)^{-1} \circ \varphi(\underline{f}_b) = \mathrm{id}_{Tb}\end{aligned}$$

So, $(\sup(f,\varphi))^{-1} \circ \sup(f,\varphi) = \mathrm{id}_{\sup(a,T)}$, and similarly $\sup(f,\varphi) \circ (\sup(f,\varphi))^{-1} = \mathrm{id}_{\sup(a',T')}$. □

Similarly, we can define a composition operation on $\widetilde{W}'$. Take (f,φ) and (f',φ'), then define $(f',\varphi') \circ (f,\varphi)$ using recursion:

$$(f',\varphi') \circ (f,\varphi) = (f' \circ f, \varphi' \circ \varphi)$$

where $(\varphi' \circ \varphi)_b = \varphi'_{f_!b} \circ \varphi_b$.

Step 2. To obtain a groupoid, which will be an initial algebra for P_F, we define additional predicates on the objects and morphisms of $\widetilde{W}$.

Definition 3.1.7.

- For an object $\sup(a,T)$ of $\widetilde{W}$, we say that it is *functorial*, if:
  - for $u : b \to b'$ and $v : b' \to b''$ in $\overline{B}_a$:

$$T(v \circ u) = Tv \circ Tu$$

  - for $b \in \overline{B}_a$, Tb is functorial and
  - for $u : b \to b'$ in $\overline{B}_a$, Tu is natural.
- For a morphism $\sup(f,\varphi) : \sup(a,T) \to \sup(a',T')$, we say that it is *natural* if,
  - for $u : b \to b'$ in $\overline{B}_a$, we have that:

$$T'f_!u \circ \varphi(\underline{f}_b) = \varphi(\underline{f}_{b'}) \circ Tu$$

$$\begin{array}{ccc} Tb & \xrightarrow{\varphi(\underline{f}_b)} & T'f_!b \\ {\scriptstyle Tu}\downarrow & & \downarrow{\scriptstyle T'f_!u} \\ Tb' & \xrightarrow[\varphi(\underline{f}_{b'})]{} & T'f_!b' \end{array}$$

and

– for $\underline{f}_b : b \to f_!b$ in $\overline{B}_f$, $\varphi(\underline{f}_b)$ is natural

Let W be the subgraph of $\widetilde{W}$ consisting of functorial vertices and natural edges.

Proposition 3.1.8. *W admits the structure of a subgroupoid of $\widetilde{W}$.*

Proof. Let us begin by showing that composition is well-defined, that is, it preserves naturality. This is clear, by the following reasoning:

$$\begin{aligned} T''(f' \circ f)_!u \circ (\varphi' \circ \varphi)(\underline{f' \circ f}_b) &= T''f'_!(f_!u) \circ \varphi'(\underline{f'}_{f_!b}) \circ \varphi(\underline{f}_b) \\ &= \varphi'(\underline{f'}_{f_!b'}) \circ T'f_!u \circ \varphi(\underline{f}_b) \\ &= \varphi'(\underline{f'}_{f_!b'}) \circ \varphi(\underline{f}_{b'}) \circ Tu \\ &= (\varphi' \circ \varphi)(\underline{f' \circ f}_{b'}) \circ Tu \end{aligned}$$

The identity $\mathrm{id}_{\sup(a,T)}$ is natural:

$$\begin{aligned} T(\mathrm{id}_a)_!u \circ (r^*T)(\underline{\mathrm{id}_a}_b) &= Tu \circ T\,\mathrm{id}_b \\ &= Tu = T\,\mathrm{id}_{b'} \circ Tu \\ &= (r^*T)(\underline{id_a}_b) \circ Tu \end{aligned}$$

If $\sup(f,\varphi)$ is natural, then we have:

$$T'f \circ \varphi(\underline{f}_{f_!^{-1}b}) = \varphi(\underline{f}_{f_!^{-1}b'}) \circ Tf_!^{-1}u$$

If we multply by inverses, we get:

$$\begin{aligned} \varphi(\underline{f}_{f_!^{-1}b'})^{-1} \circ T'u &= Tf_!^{-1}u \circ \varphi(\underline{f}_{f_!^{-1}b})^{-1} \\ \varphi^{-1}(\underline{f^{-1}}_{b'}) \circ T'u &= Tf_!^{-1}u \circ \varphi(\underline{f_b^{-1}}) \end{aligned}$$

Hence, inverses of natural arrows are natural. □

We can transfer the properties of functoriality and naturality to $\widetilde{W}'$. Functoriality remains the same, but naturality nominally changes:

- for a morphism $(f,\varphi) : (a,T) \to (a',T')$ in $\widetilde{W}'$, we say that it is *natural* if,
  - for $b \in \mathbb{B}_a$, φ_b is natural, and

– for $u : b \to b'$ in $\overline{B}_a$, we have that:

$$T' f_! u \circ \varphi_b = \varphi_{b'} \circ Tu$$

Again, inductive reasoning allows us to conclude that H and H' preserve these properties.

Define W' as the subgroupoid $\widetilde{W'}$ consisting of functorial objects and natural morphisms. This new groupoid is isomorphic to W.

3.2. The initial algebra structure of W

We will show that W is the initial algebra for $P_F : \mathbf{Gpd} \to \mathbf{Gpd}$.

Proposition 3.2.1. *W admits the structure of a P_F-algebra.*

Proof. We will define $\sup_W : P_F W \to W$ using $\sup : P_{\overline{F}} W^{(s)} \to W^{(s)}$:

$$\begin{aligned} &\sup_W(a, T) = \sup(a, T) \\ &\sup_W(f, \varphi) = \sup(f, \bar{\varphi}) \qquad \text{where } f : a \to a', \varphi : T \Rightarrow Tf_! \end{aligned}$$

and $\bar{\varphi}$ is defined to be:

$$\bar{\varphi} = \begin{cases} T(x) & \text{if } x \in \overline{B}_a \\ T'(x) & \text{if } x \in \overline{B}_{a'} \\ \varphi_b & \text{if } x = \underline{f}_b \in \mathcal{F}_f \end{cases}$$

If (a, T) is an object of $P_F W$, it appears also as an element of $P_{\overline{F}} W^{(s)}$. Further, T_b are all functorial, Tu are all natural and T is a functor, $\sup(a, T)$ is a functorial object, and so is an object in W.

If (f, φ) is a morphism in $P_F W$, we get that $\bar{\varphi}(\underline{f}_b)$ is natural and $T' f_! u \circ \varphi_b = T' f_! u \circ \bar{\varphi}(\underline{f}_b) = \bar{\varphi}(\underline{f}_{b'}) \circ Tu = \varphi_{b'} \circ Tu$. $\sup(f, \bar{\varphi})$ is thus a natural arrow and a morphism in W.

Simply unfolding the definition of $\sup_W$, identities, and composition shows that $\sup_W$ is a functor. □

Proposition 3.2.2. *Every subalgebra of W is equal to W.*

Proof. Take $U \hookrightarrow W$, the smallest subalgebra of W (which exists by Proposition 2.4.3). Let $\sup_W(x, H)$ be an element of W (either an object or a morphism) and suppose by induction that for any element $y \in \overline{B}_x$, Hx is in U. Thus H can be seen as a map $\overline{B}_x \to U$

and an element of $P_F U$. Since the inclusion is an algebra morphism $\sup_W(x, H) \in U$. This means that it is a bijection and $U = W$. □

The proof of the next statement (initiality of W) adapts the argument in [**30**, Section A2.5].

Theorem 3.2.3. *W is a initial algebra for the polynomial functor $P_F : \mathbf{Gpd} \to \mathbf{Gpd}$.*

Proof. Let X be a P_F-algebra and let $(P, Q) : G \hookrightarrow W \times X$ be the smallest subalgebra of $W \times X$. Assume, by induction, that for any $\sup(x, T)$, for any $y \in \widehat{B}_x$, and $g, g' \in G$, if $Pg = Pg' = Ty$ then $g = g'$. That is, we would like to show that P is injective.

Suppose now that we have two g, g' in G such that $Pg = Pg' = \sup_W(a, T) \in W$. Since $\sup_G$ is surjective on objects, we have a collection of objects in $P_F G$ that map to either g or g'. For any two $(a, H), (a', H') \in \sup_G^{-1}(g) \cup \sup_G^{-1}(g')$, $\sup_W(a, PH) = \sup_W(a, PH') = \sup_W(a, T)$, by virtue of P being an algebra morphism. In turn, this means $PHy = PH'y = Ty$, and by our induction hypothesis $Hy = H'y$, which by extensionality means, $g = g'$.

Let $\sup_W(f, \varphi)$ be a morphism, and suppose we have $g, g' \in G$, such that $Pg = Pg' = \sup(f, \varphi)$. Since $\mathrm{im}(\sup_G) = G$, we have in $P_F G$ a sequence of morphisms $(f^n, \varphi^n), \ldots, (f^1, \varphi^1)$, such that, $\sup_G(f^n, \varphi^n) \circ \cdots \circ \sup_G(f^1, \varphi^1) = g$ (and the same for g').

As it turns out we can show that in this case, we actually have that the sequence of morphisms is already composable in $P_F G$. Suppose $(f, \varphi) : (a^1, T^1) \to (a^2, T^2)$, $(f', \varphi') : (a^3, T^3) \to (a^4, T^4)$, such that $\sup_G(a^2, T^2) = \sup_G(a^3, T^3)$. That is, $\sup_G(f^1, \eta^1)$ and $\sup_G(f^2, \eta^2)$ are composable in G. Since P is an algebra morphism:

$$\begin{aligned} Pu &= P(\sup_G(f', \varphi') \circ \sup_G(f, \varphi)) \\ &= \sup_W(f, P \cdot \varphi') \circ \sup_W(f, P \cdot \varphi) \end{aligned}$$

Since $\sup_W$ is a bijection, we see that $a_2 = a_3$ and $PT^2 = PT^3$, by the previous argument we get $T^2 = T^3$. Thus morphisms are composable in $P_F G$.

Due to this, we only need to consider preimages of the form (f, φ) for g and g'. Let $(f, \varphi), (f', \varphi') \in \sup_G^{-1}(g) \cup \sup_G^{-1}(g')$. As before, we get $P\varphi_b = P\varphi'_b = \varphi(\underline{f}_b)$ and we see that $\varphi = \varphi'$ by our induction hypothesis.

P is injective and G is a subalgebra of W. By the previous proposition P must be a bijection. This gives us a morphism $W \to W \times X \to X$.

Suppose now we had two morphisms $f, g : W \to X$. Then by taking an equalizer of them, we would obtain a subalgebra of W, but this subalgebra is equal to W by Proposition 3.2.2, which implies that $f = g$. □

We can conclude that W is 2-initial.

Theorem 3.2.4. *W is strictly 2-initial.*

Proof. This is an application of Theorem 2.5.4 and Theorem 3.2.3. □

3.3. Examples of the construction

In this section we will work out our construction on an example, justifying the steps made.

Consider the data from Example 2.1.2, which we quickly recall here. Given a discrete groupoid A, we define a split fibration $F : J \to \mathbb{Z}_2 + A$, which maps objects of J to $\bullet \in \mathbb{Z}_2$ and the two maps $0 \to 1$ and $1 \to 0$ to τ (the involutive arrow of $\mathbb{Z}_2$). To fit with the style of the presentation, we will refer to the domain as $\mathbb{B}$ (and the codomain as $\mathbb{A}$).

In the first step of the construction we defined sets $\overline{A}$, $\overline{B}$ and a set function $\overline{F} : \overline{B} \to \overline{A}$. In this case:

$$\begin{aligned}
\overline{A} &= A \sqcup \{\bullet, \mathrm{id}_\bullet, \tau\} \\
\overline{B}_\bullet &= \{0, 1, \mathrm{id}_0, \mathrm{id}_1\} \\
\overline{B}_a &= \emptyset \\
\overline{B}_\tau &= \overline{B}_\bullet \sqcup \overline{B}_\bullet \sqcup \{0 \to 1, 1 \to 0\} \\
\overline{B}_{\mathrm{id}_\bullet} &= \overline{B}_\bullet \sqcup \overline{B}_\bullet \sqcup \{\mathrm{id}_0, \mathrm{id}_1\} \\
\overline{B}_{\mathrm{id}_a} &= \emptyset \\
\overline{B} &= \overline{B}_\bullet \sqcup \overline{B}_\tau \sqcup \overline{B}_{\mathrm{id}_\bullet} \sqcup (\bigsqcup_{a \in A} \overline{B}_a \sqcup \overline{B}_{\mathrm{id}_a}) \\
F(x : B_y) &= y
\end{aligned}$$

The set $W^{(s)}$ associated to the function $\overline{F}$ is then equipped with a graph structure. For example given a function $t : \overline{B}_\bullet \to W^{(s)}$, we have it as a vertex of the form $\sup(\bullet, t)$. To it we have the associated reflexive arrow $\sup(\mathrm{id}_\bullet, t')$, where t' is essentially the same as t.

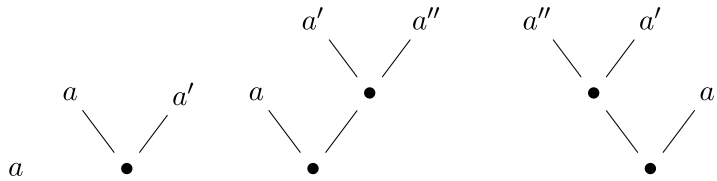
In the next step we consider the above sets as graphs, for example $\overline{B}_\bullet$ would be visualized as:

$$\mathrm{id}_0 \circlearrowleft 0 \qquad 1 \circlearrowright \mathrm{id}_1$$

We can quickly see that $W^{(s)}$ contains elements that are not useful for our purposes. For example, consider the function $T : \overline{B}_\bullet \to W^{(s)}$:

$$T(x) = \sup(a, \emptyset \to W^{(s)})$$

Then $\sup(\bullet, T)$ is a vertex in our graph, but T is not a graph morphism (in fact it sends arrows in $\overline{B}_\bullet$ to vertices). We defined the property of *hereditary graph morphism*, in order to get rid of elements like this. We then obtain a groupoid $\widetilde{W}$. In this particular case, $\widetilde{W}$ is the initial algebra. This is due to the fact that the individual fibers of F over objects $x \in \mathbb{A}$ are discrete groupoids. We can visualize W as a set of binary well-founded trees with leaves labeled in A. For example:



The morphisms marked with τ provide us with isomorphisms that swap branches at a particular level. This means that the last two trees in the above drawing are isomorphic. Further, if A is not just a set, but some general groupoid, we preserve isomorphisms on the leaves.

In order to show the usefulness of the last step, we need to consider a fibration F, whose fibers are not discrete. Let $\mathbb{Z}$ be the groupoid with one object associated to the additive group on integers. We now define the following fibration:

$$F : \mathbb{Z} \to 1 + \mathbb{Z}$$

which maps $\mathbb{Z}$ to 1. $\widetilde{W}$ associated to this fibration contains elements which behave in unexpected ways. We will denote the elements in the codomain with $\widehat{\cdot}$ to distinguish the two instances of $\mathbb{Z}$.

Consider the following. We have an object in $\widetilde{W}$ of the following form $\sup(\widehat{\bullet}, \emptyset \xrightarrow{!} \widetilde{W})$ and for every $\widehat{n} \in \mathbb{Z}$, we have a morphism:

$$\sup(\widehat{n}, \emptyset \xrightarrow{!} \widetilde{W}) : \sup(\widehat{\bullet}, !) \to \sup(\widehat{\bullet}, !)$$

Composing two morphisms of this form looks like:

$$\sup(\widehat{n}, !) \circ \sup(\widehat{m}, !) = \sup(\widehat{n + m}, !)$$

Now, define the following function $T : \overline{B}_1 \to \widetilde{W}$:

$$T(\bullet) =_{\text{def}} \sup(\widehat{\bullet}, !)$$

$$T(n) =_{\text{def}} \begin{cases} \sup(\widehat{0}, !) & \text{if n = 0} \\ \sup(\widehat{1}, !) & \text{otherwise} \end{cases}$$

While T is a graph morphism, we can quickly see that it isn't a functor, since for $n, m \neq 0$:

$$T(n+m) = \sup(1, !)$$
$$T(n) \circ T(m) = \sup(1, !) \circ \sup(1, !) = \sup(2, !)$$

Hence $\sup(1, T)$ is a member of $\widetilde{W}$, but $(1, T)$ does not appear in $P_F\widetilde{W}$ and $\sup$ cannot be a bijection.

To show why the naturality condition is necessary, let $n \neq m$ and define:

$$T(\bullet \xrightarrow{k} \bullet) =_{\text{def}} \sup(\widehat{\bullet}, !) \xrightarrow{n \cdot k} \sup(\widehat{\bullet}, !)$$
$$T'(\bullet \xrightarrow{k} \bullet) =_{\text{def}} \sup(\widehat{\bullet}, !) \xrightarrow{m \cdot k} \sup(\widehat{\bullet}, !)$$

$\sup(1, T)$ and $\sup(1, T')$ are functorial. We will exhibit an arrow $\sup(\mathrm{id}_1, \varphi) : \sup(1, T) \to \sup(1, T')$ which is a member of $\widetilde{W}$. $\overline{B}_{\mathrm{id}_1}$ is in this case $\overline{B}_1 \sqcup \overline{B}_1 \sqcup \{\mathrm{id}_\bullet\}$, let $l \in \mathbb{Z}$ and define $\varphi : \overline{B}_{\mathrm{id}_1} \to \widetilde{W}$, to be:

$$\varphi|_{\overline{B}_1} = T \qquad \text{the first copy of } \overline{B}_1$$
$$\varphi|_{\overline{B}_1} = T' \qquad \text{the other copy of } \overline{B}_1$$
$$\varphi(\mathrm{id}_\bullet) = \sup(l, !)$$

This is a graph morphism, so it is in $\widetilde{W}$. In order for $(\mathrm{id}_1, \overline{\varphi})$ to appear in $P\widetilde{W}$, the following needs to commute (for all $k \in \mathbb{Z}$):

$$\begin{array}{ccc} T\bullet & \xrightarrow{l} & T'\bullet \\ \big\downarrow{\scriptstyle n \cdot k} & & \big\downarrow{\scriptstyle m \cdot k} \\ T\bullet & \xrightarrow{l} & T'\bullet \end{array}$$

but this is only true in the case $n = m$. Thus $(\mathrm{id}_1, \overline{\varphi})$, does not appear in $P\widetilde{W}$.

In this case, it is only after we remove non-functorial objects and non-natural arrows, that we obtain an initial algebra.

3.4. An alternative construction

Moerdijk and Palmgren constructed W-types in category of internal presheaves [**39**], working in a suitably-defined 'predicative topos'. Since graphs can be seen as presheaves, we will present an alternative construction, where we start by constructing an initial algebra for a polynomial functor on graphs. We then proceed similarly, by first defining a binary operation on edges and carving out particular elements, thus obtaining a groupoid. Finally, we show that what we obtain is an initial algebra for the polynomial functor on the groupoids we started with.

In this section, we follow Moerdijk and Palmgren's notation to facilitate comparison. Let $\mathbf{R}$ be the following category (with the identity arrows omitted):

$$0 \underset{\xrightarrow[t]{}}{\overset{\xrightarrow{s}}{\xleftarrow{\;\;r\;\;}}} 1$$

where s and t are sections of r. We can see that a reflexive graph can be represented as a presheaf $X : \mathbf{R}^{\mathrm{op}} \to \mathbf{Set}$ and graph morphisms as natural transformations between such presheaves. Let $\mathbf{RGraph} = [\mathbf{R}^{\mathrm{op}}, \mathbf{Set}]$ and $y : \mathbf{R} \to \mathbf{RGraph}$ be the Yoneda embedding. We will consider a split fibration $F : \mathbb{B} \to \mathbb{A}$ as a natural transformation $F : B \Rightarrow A$, between the presheaves (the underlying reflexive graphs of the groupoids $\mathbb{A}$ and $\mathbb{B}$).

Let $I \in \mathbf{R}$ and $x \in A(I)$. Then we define $B_{I,x}$ to be the following pullback in the presheaf category:

$$\begin{array}{ccc} B_{I,x} & \longrightarrow & B \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle F} \\ yI & \xrightarrow[x]{} & A \end{array}$$

Unfolding this definition:

- for $I = 0$ and $x = a$ where $a \in \mathbb{A}$, we obtain a graph with vertices $b \in \mathbb{B}_a$ and edges $u : b \to b'$, such that $Fb = a$ and $Fu = \mathrm{id}_a$,
- for $I = 1$ and $x = f : a \to a'$, we obtain a graph with vertices (s, b) and (t, b'), with $Fb = a$ and $Fb' = a'$. The edges are:
  - $(s \circ r, u) : (s, b) \to (s, b')$, with $u : b \to b'$, such that $Fu = \mathrm{id}_a$,
  - $(t \circ r, u) : (t, b) \to (t, b')$, such that $Fu = \mathrm{id}_{a'}$, and
  - $(\mathrm{id}_1, u) : (s, b) \to (t, b')$, such that $Fu = f$.

Remark 3.4.1. In the case of $B_{0,a}$, the graph inherits the composition from $\mathbb{B}$. For $B_{1,f}$, the composition is somewhat reminiscent of the collage of a profunctor:

- for two compatible edges $(s \circ r, v)$, $(s \circ r, u)$ (and analogously for arrows of type $t \circ r$):

$$(s \circ r, v) \circ (s \circ r, u) =_{\text{def}} (s \circ r, v \circ u)$$

,

- for compatible $(s \circ r, u)$, (id_1, v):

$$(\text{id}_1, v) \circ (s \circ r, u) =_{\text{def}} (\text{id}_1, v \circ u)$$

- for compatible (id_1, v), $(t \circ r, v)$:

$$(t \circ r, v) \circ (\text{id}_1, u) =_{\text{def}} (\text{id}_1, v \circ u)$$

Let $W^{(g)}$ be a W-type for F, as defined by the Moerdijk-Palmgren presheaf construction. Unfolding the definition, we obtain:

- The vertices of $W^{(g)}$ are of the form $\sup(a, T)$, where $a \in A(0)$ and T is a natural transformation of the form $B_{0,a} \Rightarrow W^{(g)}$
- The edges are of the form $\sup(f, \varphi)$, where $f \in A(1)$ and $\varphi : B_{1,f} \Rightarrow W^{(g)}$
- The source action does the following. let $\sup(f : a \to a', \varphi) \in W^{(g)}(1)$, then $\sup(f, \varphi) \cdot s = \sup(a, s^*\varphi)$, where:

$$(s^*\varphi)(b) = \varphi(s, b)$$
$$(s^*\varphi)(u) = \varphi(s \circ r, u)$$

  and analogously for the target action.
- The reflexivity action takes $\sup(a, T) \in W^{(g)}(0)$ and returns $\sup(a, T) \cdot r = \sup(\text{id}_a, r^*T)$. $r^*T : B_{1,\text{id}_a} \Rightarrow W^{(g)}$ ignores the first component of the argument, that is, it "acts" as T on all elements:

$$r^*T(x, y) = Ty$$

Given the above, we can define an operation of composition on edges. Let $\sup(f, \varphi) : \sup(a, T) \to \sup(a', T')$ and $\sup(f', \varphi') : \sup(a', T') \to \sup(a'', T'')$ be two arrows. We define $\sup(f', \varphi') \circ \sup(f, \varphi)$, to be

$$\sup(f', \varphi') \circ \sup(f, \varphi) =_{\text{def}} \sup(f' \circ f, \varphi' \circ \varphi)$$

where $\varphi' \circ \varphi$ is defined recursively in the following way:

$$\begin{aligned}(\varphi' \circ \varphi)(\mathrm{id}_1, u) &=_{\mathrm{def}} \varphi'(\mathrm{id}_1, \underline{u} \circ \underline{f'}_{f_! b}) \circ \varphi(\mathrm{id}_1, \underline{f}_b)\\ (\varphi' \circ \varphi)(s, b) &=_{\mathrm{def}} \varphi(s, b)\\ (\varphi' \circ \varphi)(t, b) &=_{\mathrm{def}} \varphi'(t, b)\\ (\varphi' \circ \varphi)(s \circ r, u) &=_{\mathrm{def}} \varphi(s \circ r, u)\\ (\varphi' \circ \varphi)(t \circ r, u) &=_{\mathrm{def}} \varphi'(t \circ r, u)\end{aligned}$$

Induction and a few calculations show that this edge satisfies the constraints (of being a natural transformation), and is in the $W^{\mathbf{RGraph}}$.

We have that composition is associative. We show this by induction. Let $\sup(f, \varphi) : \sup(a, T) \to \sup(a', T')$, $\sup(f', \varphi') : \sup(a', T') \to \sup(a'', T'')$ and $\sup(f'', \varphi'') : \sup(a'', T'') \to \sup(a''', T''')$. Looking at the definition of composition the only interesting case is of the form $(\mathrm{id}_1, u) \in B_{1, f'' \circ f' \circ f}$:

$$\begin{aligned}(\varphi'' \circ (\varphi' \circ \varphi))(\mathrm{id}_1, u) &= \varphi''(\mathrm{id}_1, \underline{u} \circ \underline{f''}_{f'_! f_! b}) \circ (\varphi' \circ \varphi)(\mathrm{id}_1, \underline{f' \circ f}_b)\\ &= \varphi''(\mathrm{id}_1, \underline{u} \circ \underline{f''}_{f'_! f_! b}) \circ (\varphi'(\mathrm{id}_1, \underline{f'}_{f_! b}) \circ \varphi(\mathrm{id}_1, \underline{f}_b))\\ ((\varphi'' \circ \varphi') \circ \varphi)(\mathrm{id}_1, u) &= (\varphi'' \circ \varphi')(\mathrm{id}_1, \underline{u} \circ \underline{f'' \circ f'}_{f_! b}) \circ \varphi(\mathrm{id}_1, \underline{f}_b)\\ &= (\varphi''(\mathrm{id}_1, \underline{u} \circ \underline{f''}_{f'_! f_! b}) \circ \varphi'(\mathrm{id}_1, \underline{f'}_{f_! b})) \circ \varphi(\mathrm{id}_1, \underline{f}_b)\end{aligned}$$

The fact that the arguments to φ, φ' and φ'' are the same in both cases, comes from the fact that we are dealing with split fibrations and cartesian morphisms.

The right identitiy for $\sup(a, T)$ is $\sup(\mathrm{id}_a, r^* T)$, that is, the arrow obtained from the reflexivity map. To show that this is a right identity we use induction and simply unfold the definition of composition. We denote this arrow by $\mathrm{id}_{\sup(a,T)}$.

In order to get an initial algebra for $P_F : \mathbf{Gpd} \to \mathbf{Gpd}$, we will define a hereditary predicate, similar to the one in Section 3.1. This predicate will allow us to equip a subgraph of $W^{(g)}$ with a groupoid structure.

Definition 3.4.2. We say that an element $\sup(x, H)$ is *functorial* if:

- for all composable u, v in B_x, $H(v \circ u) = Hv \circ Hu$
- for all objects b in B_x, Hb is functorial, and

Let $W^{\mathbf{R}}$ denote the subgraph of $W^{(g)}$ consisting of functorial vertices and edges.

Proposition 3.4.3. *$W^{\mathbf{R}}$ admits the structure of a groupoid.*

Proof. The first thing to note is that, composition preserves the property of functoriality. To show that, let $\sup(f,\varphi) : \sup(a,T) \to \sup(a',T')$, $\sup(f,\varphi) : \sup(a',T') \to \sup(a'',T'')$ be two composable functorial edges. First, $\sup(f',\varphi') \circ \sup(f,\varphi)$ satisfies the hereditary condition of functoriality, since both components of the composition do. The only interesting cases are where one of the composable arrows is marked with id_1. If $(s \circ r, u) : (s,b) \to (s,b')$, $(\mathrm{id}_1, v) : (s,b') \to (t,b'')$ over $f' \circ f$, then:

$$
\begin{aligned}
(\varphi' \circ \varphi)(\mathrm{id}_1, v \circ u) &= \varphi'(\mathrm{id}_1, \underline{v \circ u} \circ \underline{f'}_{f_!b}) \circ \varphi(\mathrm{id}_1, \underline{f}_b), \\
(\varphi' \circ \varphi)(\mathrm{id}_1, v) \circ (\varphi' \circ \varphi)(s \circ r, u) &= \varphi'(\mathrm{id}_1, \underline{v} \circ \underline{f'}_{f_!b'}) \circ \varphi(\mathrm{id}_1, \underline{f_{b'}}) \circ \varphi(s \circ r, u) \\
&= \varphi'(\mathrm{id}_1, \underline{v} \circ \underline{f'}_{f_!b'}) \circ \varphi(\mathrm{id}_1, f_! u \circ \underline{f}_b) \qquad \text{since } \varphi \text{ is functorial}
\end{aligned}
$$

Composition in $B_{1,x}$ gives us that $(\mathrm{id}_1, f_! u \circ \underline{f}_b) = (t \circ r, f_! u) \circ (\mathrm{id}_1, \underline{f}_b)$. Using this, we get the next line:

$$
\varphi'(\mathrm{id}_1, \underline{v} \circ \underline{f'}_{f_!b'}) \circ \varphi(\mathrm{id}_1, f_! u \circ \underline{f}_b) = \varphi'(\mathrm{id}_1, \underline{v} \circ \underline{f'}_{f_!b}) \circ \varphi(t \circ r, f_! u) \circ \varphi(\mathrm{id}_1, \underline{f}_b)
$$

Since φ and φ' are composable arrows, we have $\varphi'(s \circ r, f_! u) = \varphi(t \circ r, f_! u)$. Further φ' is functorial:

$$
\begin{aligned}
\varphi'(\mathrm{id}_1, \underline{v} \circ \underline{f'}_{f_!b}) \circ \varphi(t \circ r, f_! u) \circ \varphi(\mathrm{id}_1, \underline{f}_b) &= \varphi'(\mathrm{id}_1, \underline{v} \circ \underline{f'}_{f_!b'} \circ f_! u) \circ \varphi(\mathrm{id}_1, \underline{f}_b) \\
&= \varphi'(\mathrm{id}_1, \underline{v \circ u} \circ \underline{f'}_{f_!b}) \circ \varphi(\mathrm{id}_1, \underline{f}_b) \quad \text{since } F \text{ is split}
\end{aligned}
$$

In order to finish with functoriality one last case remains. Suppose that $(\mathrm{id}_1, u) : (s,b) \to (t,b')$ and $(t \circ r, v) : (t,b') \to (t,b'')$:

$$
\begin{aligned}
(\varphi' \circ \varphi)(t \circ r, v) \circ (\varphi' \circ \varphi)(\mathrm{id}_1, u) &= \varphi'(t \circ r, v) \circ \varphi'(\mathrm{id}_1, \underline{u} \circ \underline{f'}_{f_!b}) \circ \varphi(\mathrm{id}_1, \underline{f}_b) \\
&= \varphi'(\mathrm{id}_1, v \circ \underline{u} \circ \underline{f}_{f_!b}) \circ \varphi(\mathrm{id}_1, \underline{f}_b) \qquad \varphi' \text{ is functorial} \\
&= \varphi'(\mathrm{id}_1, \underline{v \circ u} \circ \underline{f'}_{f_!b}) \circ \varphi(\mathrm{id}_1, \underline{f}_b) \qquad \text{since } F \text{ is split}
\end{aligned}
$$

Next, we need to check that $\mathrm{id}_{\sup(a',T')}$ becomes the identity when composing with it on the left as well. Let $\sup(f,\varphi)$ be a functorial arrow, and let $(\mathrm{id}_1, u) : (s,b) \to (t,b')$:

$$
\begin{aligned}
(\mathrm{id}_{\sup(a',T')} \circ \varphi)(\mathrm{id}_1, u) &= T'(\underline{u} \circ \underline{\mathrm{id}_a}_{f_!b}) \circ \varphi(\mathrm{id}_1, \underline{f}_b) \\
&= \varphi(t \circ r, \underline{u}) \circ \varphi(\mathrm{id}_1, \underline{f}_b) \\
&= \varphi(\mathrm{id}_1, u)
\end{aligned}
$$

Using recursion, we define inverse arrows. Let $\sup(f, \varphi) : \sup(a, T) \to \sup(a', T')$, be a functorial arrow. Recursively define $\sup(f, \varphi)^{-1}$ as $\sup(f^{-1}, \varphi^{-1})$, where:

$$
\begin{aligned}
(\varphi^{-1})(\mathrm{id}_1, u : b \to b') &=_{\mathrm{def}} \varphi(s \circ r, \underline{u}) \circ (\varphi(\underline{f}_{f_!^{-1}b}))^{-1} \\
(\varphi^{-1})(s, b) &=_{\mathrm{def}} \varphi(t, b) \\
(\varphi^{-1})(t, b) &=_{\mathrm{def}} \varphi(s, b) \\
(\varphi^{-1})(s \circ r, u) &=_{\mathrm{def}} \varphi(t \circ r, u) \\
(\varphi^{-1})(t \circ r, u) &=_{\mathrm{def}} \varphi(s \circ r, u)
\end{aligned}
$$

This can be shown to be a natural transformation, so it is an element of $W^{(g)}$.

We first show that this is indeed an inverse for functorial arrows, by induction. Let $\sup(f, \varphi)$ be such an arrow. Then $\sup(f, \varphi)^{-1} \circ \sup(f, \varphi)$, is clearly equal to $\mathrm{id}_{\sup(a,T)}$ on all elements of B_{id_a} of the form (s, b), (t, b), $(s \circ r, u)$, $(t \circ r, u)$. Let's take a look at how $\varphi^{-1} \circ \varphi$ behaves on the elements of the form $(\mathrm{id}_1, u : b \to b')$:

$$
\begin{aligned}
(\varphi^{-1} \circ \varphi)(\mathrm{id}_1, u) &= \varphi^{-1}(\mathrm{id}_1, \underline{u} \circ \underline{f^{-1}}_{f_!b}) \circ \varphi(\mathrm{id}_1, \underline{f}_b) && \text{definition of composition} \\
&= \varphi(s \circ r, \underline{\underline{u} \circ \underline{f}^{-1}_{f_!b}}) \circ \varphi(\mathrm{id}_1, \underline{f}_{f_!^{-1}f_!b})^{-1} \circ \varphi(\mathrm{id}_1, \underline{f}_b) && \text{definition of } \varphi^{-1} \\
&= \varphi(s \circ r, u) \circ \varphi(\mathrm{id}_1, \underline{f}_b)^{-1} \circ \varphi(\mathrm{id}_1, \underline{f}_b) && \text{since } F \text{ is split} \\
&= \varphi(s \circ r, u) \\
&= \mathrm{id}_{\sup(a,T)}(\mathrm{id}_1, u)
\end{aligned}
$$

Composition with the inverse on the other side gives us:

$$
\begin{aligned}
(\varphi \circ \varphi^{-1})(\mathrm{id}_1, u) &= \varphi(\mathrm{id}_1, \underline{u} \circ \underline{f}_{f_!^{-1}b}) \circ \varphi^{-1}(\mathrm{id}_1, \underline{f^{-1}}_b) && \text{definition of composition} \\
&= \varphi(\mathrm{id}_1, \underline{u} \circ \underline{f}_{f_!^{-1}b}) \circ \varphi(s \circ r, \underline{\underline{f^{-1}}}_b) \circ \varphi(\mathrm{id}_1, \underline{f}_{f_!^{-1}b})^{-1} && \text{definition of } \varphi^{-1} \\
&= \varphi(\mathrm{id}_1, u \circ \underline{f}_{f_!^{-1}b}) \circ \varphi(s \circ r, \mathrm{id}_{f_!^{-1}b}) \circ \varphi(\mathrm{id}_1, \underline{f}_{f_!^{-1}b})^{-1} && \text{since } F \text{ is split} \\
&= \varphi(t \circ r, u) \circ \varphi(\mathrm{id}_1, \underline{f}_{f_!^{-1}b}) \circ \varphi(\mathrm{id}_1, \underline{f}_{f_!^{-1}b})^{-1} && \text{since } \varphi \text{ is functorial} \\
&= \varphi(t \circ r, u) && \text{by induction} \\
&= \mathrm{id}_{\sup(a',T')}(\mathrm{id}_1, u)
\end{aligned}
$$

We can check that the inverse as defined is functorial, when the original arrow is such. Again, the only interesting case is when one of the arrows is tagged with id_1. First, let

$(s \circ r, u) : (s, b) \to (s, b')$, $(\mathrm{id}_1, v) : (s, b') \to (t, b'')$:

$$\begin{aligned}
\varphi^{-1}(\mathrm{id}_1, v) \circ \varphi^{-1}(s \circ r, u) &= \varphi(s \circ r, \underline{v}) \circ \varphi(\mathrm{id}_1, \underline{f}_{f_!^{-1}b'})^{-1} \circ \varphi(t \circ r, u) && \text{definition of } \varphi^{-1} \\
&= \varphi(s \circ r, \underline{v}) \circ \varphi(\mathrm{id}_1, \underline{f}_{f_!^{-1}b'})^{-1} \circ \varphi(t \circ r, u^{-1})^{-1} && \varphi \text{ is functorial} \\
&= \varphi(s \circ r, \underline{v}) \circ (\varphi(\mathrm{id}_1, u^{-1} \circ \underline{f}_{f_!^{-1}b'}))^{-1} \\
&= \varphi(s \circ r, \underline{v}) \circ (\varphi(\mathrm{id}_1, \underline{f}_{f_!^{-1}b} \circ (f_!^{-1}u)^{-1}))^{-1} && \text{since } F \text{ is split} \\
&= \varphi(s \circ r, \underline{v}) \circ \varphi(s \circ r, (f_!^{-1}u)^{-1})^{-1} \circ \varphi(\mathrm{id}_1, \underline{f}_{f_!^{-1}b})^{-1} \\
&= \varphi(s \circ r, \underline{v} \circ f_!^{-1}u) \circ \varphi(\mathrm{id}_1, \underline{f}_{f_!^{-1}b})^{-1} \\
&= \varphi^{-1}(\mathrm{id}_1, v \circ u)
\end{aligned}$$

Now, let us consider $(t \circ r, v)$, (id_1, u):

$$\begin{aligned}
\varphi^{-1}(t \circ r, v) \circ \varphi^{-1}(\mathrm{id}_1, u) &= \varphi(s \circ r, v) \circ \varphi(\mathrm{id}_1, \underline{u}) \circ \varphi(\mathrm{id}_1, \underline{f}_{f_!^{-1}b})^{-1} \\
&= \varphi(s \circ r, v \circ \underline{u}) \circ \varphi(\mathrm{id}_1, \underline{f}_{f_!^{-1}b})^{-1} \\
&= \varphi^{-1}(\mathrm{id}_1, v \circ u)
\end{aligned}$$

□

Since we now have a groupoid structure on $W^{\mathbf{R}}$ it makes sense to compare it to W constructed in the previous chapter. As it turns out, these two objects are isomorphic, which allows us to conclude that we have another description of initial algebras for P_F

Proposition 3.4.4. *$W^{\mathbf{R}}$ is isomorphic to W*

Proof. We construct a pair of graph morphisms $(-)^* : W^{(g)} \to \widetilde{W}$ and $(-)_* : \widetilde{W} \to W^{(g)}$. These two graph morphisms will turn out to be groupoid isomorphisms, when restricted to $W^{\mathbf{R}}$ and W. The functions are defined recursively on the set of well founded trees :

$$
\begin{aligned}
(\sup(a,T))^* &=_{\text{def}} \sup(a,T^*) & T^*x &=_{\text{def}} (Tx)^* \\
(\sup(f,\varphi))^* &=_{\text{def}} \sup(f,\varphi^*) & \varphi^*(b:B_a) &=_{\text{def}} \varphi(s,b)^* \\
& & \varphi^*(u:B_a) &=_{\text{def}} \varphi(s\circ r,u)^* \\
& & \varphi^*(b:B_{a'}) &=_{\text{def}} \varphi(t,b)^* \\
& & \varphi^*(u:B_{a'}) &=_{\text{def}} \varphi(t\circ r,u)^* \\
& & \varphi^*(\underline{f}_b) &=_{\text{def}} \varphi(\mathrm{id}_1,\underline{f}_b)^* \\
(\sup(a,T))_* &=_{\text{def}} \sup(a,T_*) & T_*(x) &=_{\text{def}} (Tx)_* \\
(\sup(f,\varphi))_* &=_{\text{def}} \sup(f,\varphi_*) & \varphi_*(s,b) &=_{\text{def}} T(b)_* \\
& & \varphi_*(t,b) &=_{\text{def}} T'(b)_* \\
& & \varphi_*(s\circ r,u) &=_{\text{def}} T(u)_* \\
& & \varphi_*(t\circ r,u) &=_{\text{def}} T'(u)_* \\
& & \varphi_*(\mathrm{id}_1,u) &=_{\text{def}} T'(\underline{u})_*\circ\varphi(\underline{f}_b)_*
\end{aligned}
$$

These two functions obviously preserve the source and targets, and using induction can be shown to preserve the identity arrows as well.

Using induction we can also see that $(-)^*$ preserves composition:

$$
\begin{aligned}
(\varphi'^*\circ\varphi^*)(\underline{f'\circ f}_b) &= \varphi(\mathrm{id}_1,\underline{f'}_{f_!b})^*\circ\varphi(\mathrm{id}_1,\underline{f}_b)^* \\
(\varphi'\circ\varphi)^*(\underline{f'\circ f}_b) &= (\varphi(\mathrm{id}_1,\underline{f'}_{f_!b})\circ\varphi(\mathrm{id}_1,\underline{f}_b))^*
\end{aligned}
$$

Further, we see that $(-)^*$ maps functorial objects to functorial objects. To see that a functorial arrow $\sup(f,\varphi)$ maps to a natural arrow, consider the following:

$$
\begin{aligned}
\varphi^*(\underline{f}_{b'})\circ T^*u &= \varphi(\mathrm{id}_1,\underline{f}_b)^*\circ(T(u))^* \\
&= (\varphi(\mathrm{id}_1,\underline{f}_b)\circ\varphi(s\circ r,u))^* \\
&= (\varphi(t\circ r,f_!u)\circ\varphi(\mathrm{id}_1,\underline{f}_b)^* \\
&= T^*(f_!u)\circ\varphi^*(\underline{f}_b)
\end{aligned}
$$

$(-)_*$ preserves composition when restricted to W. In order to show that, we actually need to assume a stronger induction hypothesis, that is, $(-)_*$ preserves composition, maps functorial objects to functorial objects and natural arrows to functorial arrows.

Most of the checks are trivial and amount to simply unfolding the various definitions and applying the induction hypothesis. We spell the details of some of them here. We start by showing that natural arrows are mapped to functorial arrows. Let $\sup(f,\varphi) : \sup(a,T) \to \sup(a',T') \in W$ be a natural arrow, and let $(s \circ r, u) : (s,b) \to (s,b')$, $(\mathrm{id}_1, v) : (s,b') \to (t,b'')$ be arrows in $B_{1,f}$, then:

$$\begin{aligned}
\varphi_*(\mathrm{id}_1, v) \circ \varphi_*(s \circ r, u) &= T'(\underline{v})_* \circ \varphi(\underline{f}_{b'})_* \circ T(u)_* \\
&= T'(\underline{v})_* \circ (\varphi(\underline{f}_{b'}) \circ T(u))_* && \text{by induction} \\
&= T'(\underline{v})_* \circ (T'(f_! u) \circ \varphi(\underline{f}_b))_* && \varphi \text{ is natural} \\
&= (T'(\underline{v}) \circ T(f_! u))_* \circ \varphi(\underline{f}_b))_* && \text{by induction} \\
&= T'(\underline{v \circ u})_* \circ \varphi(\underline{f}_b) && \text{since } F \text{ is split} = \varphi_*(\mathrm{id}_1, v \circ u)
\end{aligned}$$

Now consider $(\mathrm{id}_1, u) : (s,b) \to (t,b')$, $(t \circ r, v) : (t,b') \to (t,b'')$ arrows in $B_{1,f}$:

$$\begin{aligned}
\varphi_*(t \circ r, v) \circ \varphi_*(\mathrm{id}_1, u) &= T'(v)_* \circ T'(\underline{u})_* \circ \varphi(\mathrm{id}_1, \underline{f}_b) \\
&= (T'(v) \circ T'(\underline{u}))_* \circ \varphi(\mathrm{id}_1, \underline{f}_b)_* && \text{by induction} \\
&= (T'(\underline{v \circ u}))_* \circ \varphi(\mathrm{id}_1, \underline{f}_b)_* && \text{since } T \text{ is functorial and } F \text{ is split} \\
&= \varphi_*(\mathrm{id}_1, v \circ u)
\end{aligned}$$

The main concern of the induction statement was to show that $(-)^*$ preserves composition. Let $\sup(f,\varphi) : \sup(a,T) \to \sup(a',T')$, $\sup(f',\varphi') : \sup(a',T') \to \sup(a'',T'')$ be two arrows in W and $(\mathrm{id}_1, u) : (s,b) \to (t,b')$ be an arrow in $B_{1,f' \circ f}$. Then:

$$\begin{aligned}
(\varphi'_* \circ \varphi_*)(\mathrm{id}_1, u) &= \varphi'_*(\mathrm{id}_1, \underline{u} \circ \underline{f'}_{f_! b}) \circ \varphi_*(\mathrm{id}_1, \underline{f}_b) && \text{definition of composition in } W^{\mathbf{R}} \\
&= T''(\underline{u})_* \circ \varphi(\underline{f}_b)_* \circ T(\mathrm{id}_{f_! b})_* \circ \varphi(\underline{f}_b)_* \\
&= T''(\underline{u})_* \circ \varphi(\underline{f}_b)_* \circ \varphi(\underline{f}_b)_* && \text{by induction} \\
(\varphi' \circ \varphi)_*(\mathrm{id}_1, u) &= T''(\underline{u})_* \circ (\varphi(\underline{f}_{f_! b}) \circ \varphi(\underline{f}_b))_* && \text{definition of composition in } W \text{ and of } (-)_* \\
&= T''(\underline{u})_* \circ \varphi(\underline{f}_{f_! b})_* \circ \varphi(\underline{f}_b)_* && \text{by induction}
\end{aligned}$$

Other checks are even more trivial.

We can show that these two groupoid morphisms are inverses, again by induction. We will only sketch out the details for the arrows. Let $\sup(f,\varphi) : \sup(a,T) \to \sup(a',T')$

be a functorial arrow in $W^{\mathbf{R}}$ and $(\mathrm{id}_1, u) : (s, b) \to (t, b)$ in $B_{1,f}$:

$$\begin{aligned}
(\varphi^*)_*(\mathrm{id}_1, u) &= T^*(\underline{u})_* \circ \varphi^*(\underline{f}_b)_* \\
&= ((T'(\underline{u}) \circ \varphi(\mathrm{id}_1, \underline{f}_b))^*)_* && \text{since composition is preserved} \\
&= (\varphi(\mathrm{id}_1, u)^*)_* && \text{since } \varphi \text{ matches } T \\
&= \varphi(\mathrm{id}_1, u) && \text{by induction.}
\end{aligned}$$

Similarly, let $\sup(f, \varphi) : \sup(a, T) \to \sup(a', T')$ be an arrow in W.

$$\begin{aligned}
(\varphi_*)^*(\underline{f}_b) &= \varphi_*(\mathrm{id}_1, \underline{f}_b)^* \\
&= ((T'(\mathrm{id}_{f_! b}) \circ \varphi(\underline{f}_b))_*)^* && \text{since composition is preserved and } F \text{ is split} \\
&= (\varphi(\underline{f}_b)_*)^* && \text{since } T \text{ is functorial} \\
&= \varphi(\underline{f}_b) && \text{by induction.}
\end{aligned}$$

□

CHAPTER 4

W-types for split fibrations in slices

In type theory W-types are types that are defined inductively in a well-founded manner. We start this chapter by looking at type-theoretic rules for W-types and comparing them to the categorical version of W-types. The results in the previous chapter do not yet provide a semantic counterpart to type-theoretic rules, as we only consider polynomials over 1, that is, the empty context.

As is the tradition in type theory, we begin with the formation rule:

$$\frac{\Gamma, x : A \vdash B(x) : \mathrm{type}}{\Gamma \vdash (Wx : A)B(x) : \mathrm{type}}$$

Categorically, we would interpret the rule in the following way: the premises say we have a commutative diagram of the form (where all morphisms are split fibrations):

$$\begin{array}{ccc} \mathbb{B} & \xrightarrow{F} & \mathbb{A} \\ & {\scriptstyle S}\searrow \quad \swarrow{\scriptstyle R} & \\ & \Gamma & \end{array}$$

The conclusion asserts that we have an split fibration over Γ:

$$W_F \to \Gamma$$

The introduction rule is:

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash t : B(a) \to (Wx : A)B(x)}{\Gamma \vdash \mathrm{sup}(a, t) : (Wx : A)B(x)}$$

Interpreting this in category theory means that W_F is an algebra for $P_F : \mathbf{Gpd}_{/\Gamma} \to \mathbf{Gpd}_{/\Gamma}$:

$$\begin{array}{ccc} P_F W_F & \xrightarrow{\mathrm{sup}} & W_F \\ & \searrow \quad \swarrow & \\ & \Gamma & \end{array}$$

The elimination rule for W-types corresponds to initiality.

In this chapter we construct initial algebras for polynomials functors assigned to morphism in slices, using the initial algebras from the previous chapter. Further we show that these are stable under pullback.

4.1. Polynomial functors in slices

Let $\mathbb{I} \in \mathbf{Gpd}$, to remain fixed throughout this section and the next. Further suppose $F : \mathbb{B} \to \mathbb{A}$ is a split fibration in $\mathbf{Gpd}_{/\mathbb{I}}$, that is, we have a commutative diagram:

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{\quad F \quad} & \mathbb{B} \\ & {}_{S}\searrow \quad \swarrow_{R} & \\ & \mathbb{I} & \end{array}$$

Note that we assume the triangle commutes strictly.

Define $P_F^{\mathbb{I}} : \mathbf{Gpd}_{/\mathbb{I}} \to \mathbf{Gpd}_{/\mathbb{I}}$ to be the composite:

$$\mathbf{Gpd}_{/\mathbb{I}} \xrightarrow{\Delta_S} \mathbf{Gpd}_{/\mathbb{B}} \xrightarrow{\Pi_F} \mathbf{Gpd}_{/\mathbb{A}} \xrightarrow{\Sigma_R} \mathbf{Gpd}_{/\mathbb{I}}$$

When both super- and subscripts are obvious, we will omit them. We will show that the polynomial functor $P : \mathbf{Gpd}_{/\mathbb{I}} \to \mathbf{Gpd}_{/\mathbb{I}}$ in the slice category has an initial algebra, and further, that this initial algebra is stable under pullback (in a precise sense specified in the section).

To start with, we will first begin by recalling the action of the polynomial functor P. Let $\mathbb{X} \xrightarrow{H} \mathbb{I}$ be an object of $\mathbf{Gpd}_{/\mathbb{I}}$. Then, the set of objects of $P(\mathbb{X})$ has elements of the form (i, a, T), where $i \in \mathbb{I}$, $a \in \mathbb{A}_i$ and $T : \mathbb{B}_a \to \mathbb{X}_i$:

$$P\mathbb{X} = \{(i, a, T) \mid i \in \mathbb{I}, a \in \mathbb{A}_i, T : \mathbb{B}_a \to \mathbb{X}_i\}$$

By $\mathbb{X}_i$ (and $\mathbb{A}_i$), we mean the pullback of $\mathbb{X} \xrightarrow{H} \mathbb{I}$ ($\mathbb{A} \xrightarrow{R} \mathbb{I}$) along $1 \xrightarrow{i} \mathbb{I}$.

We can represent the morphisms $(v, u, \eta) : (i, a, T) \to (i', a', T')$ in $P\mathbb{X}$:

$$(v : i \to i', u : a \to a', \eta : T \Rightarrow T' \circ u_! : \mathbb{B}_a \to \mathbb{X})$$

where u is over v and components of η are over v as well.

4.2. Construction of $\mathbf{W}_F^{\mathbb{I}}$

Let us observe what happens when P is applied twice. The objects of $P^2\mathbb{X}$ are

$$(i, a : \mathbb{A}_i, T : \mathbb{B}_a \to (P\mathbb{X})_i)$$

Given two objects (i, a, T), (i', a', T') a morphism in $(v, u, \eta) : (i, a, T) \to (i', a', T')$ in $P^2\mathbb{X}$ between them is

$$(v : i \to i', u : a \to a', \eta : T \Rightarrow T \circ u')$$

with u and η_b over v.

This gives the idea of considering the simple polynomial $P_F : \mathbf{Gpd} \to \mathbf{Gpd}$ defined as the composite $\Sigma_{\mathbb{A}}\Pi_F\Delta_{\mathbb{B}}$ and defining a hereditary predicate on W_F (the initial algebra for P_F), carving out a subgroupoid and thus obtaining an initial algebra for $P_F^{\mathbb{I}}$. A similar idea appears in [**39**] in the context of pretoposes with dependent products. First we define a map $\rho : W_F \to \mathbb{I}$, which is simply the composition $W_F \xrightarrow{\pi_1} \mathbb{A} \xrightarrow{R} \mathbb{I}$.

Definition 4.2.1. We say that $\sup(a, T) \in W_F$ is $\mathbb{I}$-*constant*, if

- for all $b \in \mathbb{B}_a$, $\rho(T(b)) = Ra$, and,
- for all $x \in \mathbb{B}$, $T(x)$ is $\mathbb{I}$-constant

And $\sup(f, \varphi) : \sup(a, T) \to \sup(a', T')$ is $\mathbb{I}$-*constant*, if:

- for all $b \in \mathbb{B}_a$, $\rho(\varphi(\underline{u}_b)) = Rf$, and,
- for all $x \in \mathbb{B}_f$, $\varphi(x)$ is $\mathbb{I}$-constant.

Let $W_F^{\mathbb{I}}$ be the subgraph of W_F, consisting of $\mathbb{I}$-constant vertices and arrows. As before, we will omit the super- and subscript, when they are obvious. As a matter of convenience we equip the objects and morphisms with indices from $\mathbb{I}$, that is we will write $\sup(i, a, T)$ for $i = ra$ (and similarly for morphisms).

Proposition 4.2.2. *$W_F^{\mathbb{I}}$ can be equipped with a groupoid structure, making it a subgroupoid of W_F.*

Proof.

(1) Suppose $\sup(u, v, \varphi) : \sup(i, a, T) \to \sup(i', a', T')$ and $\sup(u', v', \varphi') : \sup(i', a', T') \to \sup(i'', a'', T'')$ are $\mathbb{I}$-constant, then $\sup(u', v', \varphi') \circ \sup(u, v, \varphi)$ is $\mathbb{I}$-constant.

We know that $\sup(u', v', \varphi') \circ \sup(u, v, \varphi) = \sup(u' \circ u, v' \circ v, \varphi' \circ \varphi)$, where $(\varphi' \circ \varphi)(\underline{v' \circ v_b}) = \varphi'(\underline{v'}_{v_!b}) \circ \varphi(\underline{v}_b)$. Now since ρ is a functor:

$$\rho(\varphi'(\underline{v'}_{v_!b}) \circ \varphi(\underline{v}_b)) = \rho(\varphi'(\underline{v'}_{v_!b})) \circ \rho(\varphi(\underline{v}_b)) = u' \circ u$$

The second condition follows from the fact that the two components of the composition are $\mathbb{I}$-constant.

(2) $\mathrm{id}_{\sup(i,a,T)}$ is $\mathbb{I}$-constant, for an $\mathbb{I}$-constant $\sup(i, a, T)$

We need to check that $\rho(r^*T(\underline{id}_b)) = \mathrm{id}_i$. Since $r^*T(\underline{\mathrm{id}}_b) = T(\mathrm{id}_b) = \mathrm{id}_{T(b)}$ and T is $\mathbb{I}$-constant, we have that $\rho(T(b)) = i$ and hence $\rho(\mathrm{id}_{T(b)}) = \mathrm{id}_i$. Further since $(r^*T)(x) = T(x)$, we have that $(r^*T)(x)$ is $\mathbb{I}$-constant for all x.

(3) If $\sup(u, v, \varphi)$ is $\mathbb{I}$-constant, then $\sup(u, v, \varphi)^{-1}$ is also $\mathbb{I}$-constant

First $\sup(u, v, \varphi)^{-1} = \sup(u^{-1}, v^{-1}, \varphi^{-1})$, where $\varphi^{-1}(\underline{v^{-1}}_b) = \varphi(\underline{u}_{u_!^{-1}b})^{-1}$.

Then:

$$\rho(\varphi(\underline{u}_{u_!^{-1}b})^{-1}) = \rho(\varphi(\underline{u}_{u_!^{-1}b}))^{-1} = u^{-1}$$

Again, the second condition follows from the fact that $\sup(u, v, \varphi)$ was $\mathbb{I}$-constant.

$\square$

Proposition 4.2.3. *$W_F^{\mathbb{I}}$ admits the structure of a $P_F^{\mathbb{I}}$-algebra.*

Proof. Take $(i, a : \mathbb{A}_i, T : \mathbb{B}_a \to (W^{\mathbb{I}})_i) \in P^{\mathbb{I}}W^{\mathbb{I}}$. First, we notice that $T(x)$ are all $\mathbb{I}$-constant and $\rho(T(b)) = i$. Hence, there is a $\sup(i, a, T)$ in $W^{\mathbb{I}}$. A similar argument applies to morphisms. We denote this morphism by $\sup : PW \to W$. $\square$

Proposition 4.2.4. $\sup : P^{\mathbb{I}}W^{\mathbb{I}} \to W^{\mathbb{I}}$ *is an isomorphism.*

Proof. By induction, we show that $\sup$ is bijective. $\square$

Proposition 4.2.5. *Every subalgebra of W is equal to W.*

Proof. Let $G \hookrightarrow W$ be the smallest subalgebra (by Proposition 2.4.3). Take $\sup(i, a, T)$ and suppose, by induction, that $T(x)$ has a preimage in G, that is, $T(x) \in G$, for all $x \in \mathbb{B}_a$. Then $(i, a, T' : \mathbb{B}_a \to G_i) \in PG$, where $T'(x) = T(x)$, since $\sup(i, a, T)$ is $\mathbb{I}$-constant. Hence $\sup(i, a, T) \in G$. The same reasoning applies for arrows as well. $\square$

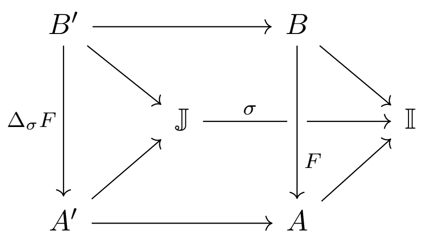
Note that the above proof is the essentially the same as in the case of W_F (Proposition 3.2.2), except for the extra consideration of $\mathbb{I}$-constancy.

Theorem 4.2.6. *$W^{\mathbb{I}}$ is initial for $P^{\mathbb{I}} : \mathbf{Gpd}_{/\mathbb{I}} \to \mathbf{Gpd}_{/\mathbb{I}}$.*

Proof. The content of the proof is essentially the same as in Theorem 3.2.3. $\square$

4.3. Pullback stability of W-types

Let $F : \mathbb{B} \to \mathbb{A}$ be a split fibration in $\mathbf{Gpd}_{/\mathbb{I}}$ as before, and additionally, consider a functor $\sigma : \mathbb{J} \to \mathbb{I}$. Consider the following diagram, obtained by pulling back along σ:



It is well known if F is split, that $\Delta_\sigma F$ is also split, so it makes sense asking how do (initial) algebras of $P^{\mathbb{J}}_{\Delta_\sigma F}$ and $P^{\mathbb{I}}_F$ relate. In this section we will show that $\Delta_\sigma W^{\mathbb{I}}_F$ is, in fact, the initial algebra for $P^{\mathbb{J}}_{\Delta_\sigma}$.

Proposition 4.3.1. *There exists a natural isomorphism $\Delta_\sigma P_F \cong P_{\Delta_\sigma F}\Delta_\sigma$.*

Proof. We begin by noticing that the following squares are all pullbacks:

$$\begin{array}{ccc} \cdot & \xrightarrow{\Delta_\sigma F} & \cdot \\ {\scriptstyle \Delta_s\sigma}\downarrow & & \downarrow{\scriptstyle \Delta_r\sigma} \\ \cdot & \xrightarrow[F]{} & \cdot \end{array} \qquad \begin{array}{ccc} \cdot & \xrightarrow{\Delta_\sigma r} & \cdot \\ {\scriptstyle \Delta_r\sigma}\downarrow & & \downarrow{\scriptstyle \sigma} \\ \cdot & \xrightarrow[r]{} & \cdot \end{array}$$

By Beck-Chevalley (Proposition 1.5.3) we get the following (where the squares commute up to a natural isomorphism):

$$\begin{array}{ccccccc} \mathbf{Gpd}_{/\mathbb{I}} & \xrightarrow{\Delta_s} & \mathbf{Gpd}_{/\mathbb{B}} & \xrightarrow{\Pi_F} & \mathbf{Gpd}_{/\mathbb{A}} & \xrightarrow{\Sigma_r} & \mathbf{Gpd}_{/\mathbb{I}} \\ {\scriptstyle \Delta_\sigma}\downarrow & & \downarrow{\scriptstyle \Delta_{\Delta_s\sigma}} & & \downarrow{\scriptstyle \Delta_{\Delta_r\sigma}} & & \downarrow{\scriptstyle \Delta_\sigma} \\ \mathbf{Gpd}_{/\mathbb{J}} & \xrightarrow{\Delta_{\Delta_\sigma s}} & \mathbf{Gpd}_{/\Delta_\sigma\mathbb{B}} & \xrightarrow{\Pi_{\Delta_\sigma F}} & \mathbf{Gpd}_{/\Delta_\sigma\mathbb{A}} & \xrightarrow{\Sigma_{\Delta_\sigma r}} & \mathbf{Gpd}_{/\mathbb{J}} \end{array}$$

The composition of these natural isomorphisms give the desired natural isomorphism $\Delta_\sigma P_F \Rightarrow P_{\Delta_\sigma F}\Delta_\sigma$. ☐

Proposition 4.3.2. $\Delta_\sigma : \mathbf{Gpd}_{/\mathbb{I}} \to \mathbf{Gpd}_{/\mathbb{J}}$ *lifts to a functor* $P^{\mathbb{I}}_F\text{-alg}_s \to P^{\mathbb{J}}_{\Delta_\sigma F}\text{-alg}_s$*:*

$$\begin{array}{ccc} P^{\mathbb{I}}_F\text{-alg}_s & \xrightarrow{\Delta_\sigma} & P^{\mathbb{J}}_{\Delta_\sigma F}\text{-alg}_s \\ \downarrow & & \downarrow \\ \mathbf{Gpd}_{/\mathbb{I}} & \xrightarrow{\Delta_\sigma} & \mathbf{Gpd}_{/\mathbb{J}} \end{array}$$

Proof. If $\mathbb{X}$ is a $P^{\mathbb{I}}_F$-algebra, we begin by pulling it back:

$$\begin{array}{cccc} \Delta_\sigma P^{\mathbb{I}}_F\mathbb{X} & \longrightarrow & P^{\mathbb{I}}_F\mathbb{X} & \\ \downarrow & \searrow & \downarrow & \searrow \\ & \mathbb{J} & \longrightarrow & \mathbb{I} \\ \downarrow & \nearrow & \downarrow & \nearrow \\ \Delta_\sigma\mathbb{X} & \longrightarrow & \mathbb{X} & \end{array}$$

But by the previous proposition, $\Delta_\sigma P^{\mathbb{I}}_F\mathbb{X} \cong P^{\mathbb{J}}_{\Delta_\sigma F}\Delta_\sigma\mathbb{X}$. Composing it with $\Delta_\sigma \sup_{\mathbb{X}}$ we get an algebra structure for $\Delta_\sigma\mathbb{X}$:

$$P^{\mathbb{J}}_{\Delta_\sigma F}\Delta_\sigma\mathbb{X} \to \Delta_\sigma P^{\mathbb{I}}_F\mathbb{X} \to \Delta_\sigma\mathbb{X}$$

The naturality of $\Delta_\sigma P_F \cong P_{\Delta_\sigma F}\Delta_\sigma$ allows us to transfer algebra morphisms as well. ☐

Proposition 4.3.3. $\Delta_\sigma W^{\mathbb{I}}_F$ *admits the structure of a $P^{\mathbb{J}}_{\Delta_\sigma F}$-algebra, and further*

$$P^{\mathbb{J}}_{\Delta_\sigma F}\Delta_\sigma W^{\mathbb{I}}_F \cong \Delta_\sigma W^{\mathbb{I}}_F.$$

Proof. A pullback preserves isomorphisms, and given the previous previous proposition, we know that $\Delta_\sigma W_F^{\mathbb{I}}$ is a $P^{\mathbb{J}}_{\Delta_\sigma F}$-algebra. □

Theorem 4.3.4. *$W^{\mathbb{J}}_{\Delta_\sigma F}$ is isomorphic as a $P^{\mathbb{J}}_{\Delta_\sigma F}$-algebra to $\Delta_\sigma W_F^{\mathbb{I}}$.*

Proof. Let $H : W^{\mathbb{J}}_{\Delta_\sigma} \to \Delta_\sigma W_F^{\mathbb{I}}$, be the unique algebra morphism, given by the initiality of $W^{\mathbb{J}}_{\Delta_\sigma F}$. We propose the following induction statement, which will be used to construct the isomorphism (and similarly for the morphisms):

$$\forall(\sup^F(i,a,T) \in W_F^{\mathbb{I}}).\forall(j \in \mathbb{J}).(j,(\sup^F(i,a,T)) \in \Delta_\sigma W_F^{\mathbb{I}} \Rightarrow$$
$$\exists!\,\sup^{\Delta_\sigma F}(j,a,U) \in W^{\mathbb{J}}_{\Delta_\sigma F}.H(\sup^{\Delta_\sigma F}(j,a,U)) = (j,(\sup^F(i,a,T)).$$

Let $(j,\sup^F(i,a,T))$ be in $\Delta_\sigma W_F^{\mathbb{I}}$, then the first thing to notice is that for all $b \in \mathbb{B}_a$, and for all $u : b \to b' \in \mathbb{B}_a$, both $(j,T(b))$ and $(\mathrm{id}_j, T(u))$ are in $\Delta_\sigma W_F^{\mathbb{I}}$, since T is $\mathbb{I}$-constant.

Define $U : \mathbb{B}_a \to W^{\mathbb{J}}_{\Delta_\sigma}$ by setting:

$$U(b) =_{\mathrm{def}} \sup^{\Delta_\sigma F}(j,a',U^b)$$
$$U(u) =_{\mathrm{def}} \sup^{\Delta_\sigma F}(\mathrm{id}_j,v,U^u)$$

where U^x is the unique element of $W^{\mathbb{J}}_{\Delta_\sigma}$, assigned to $T(x)$. By uniqueness imposed in the induction statement, we get that U is a functor. Further it is $\mathbb{J}$-constant, by definition. Since H is an algebra morphism we have $\sup^{\Delta_\sigma F}(j,a,U)$, that maps to $(j,\sup^F(i,a,T))$.

Suppose U' is another such, and that $H(\sup^{\Delta_\sigma F}(j,a,U')) = (j,\sup^F(i,a,T))$. Then, we also have that $HU'x = (j,Tx)$. However our induction hypothesis claims that $U(x)$ is the only such, therefore $U'x = Ux$ and U is unique.

The argument for morphisms is similar, except that the uniqueness condition in the inductive hypothesis provides us with the naturality condition. □

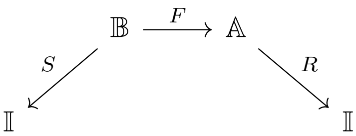
CHAPTER 5

Dependent W-types for split fibrations

Dependent polynomial functors as described in Section 2.1 and the initial algebras associated to them are supposed to model *general trees* as in [**40**]:

$$\frac{\begin{array}{c}\Gamma, i : I \vdash A(i) : \text{type} \\ \Gamma, i : I, a : A(i) \vdash B(i,a) : \text{type} \\ \Gamma, i : I, a : A(i), b : B(i,a) \vdash s(i,a,b) : I\end{array}}{\Gamma, i : I \vdash W(I,A,B)(s) : \text{type}}$$

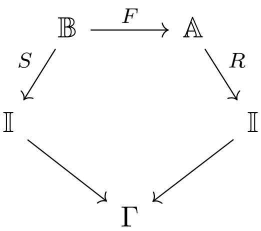
Interpreting the above rule categorically means that, given the following diagram (where F and R are split fibrations):



We have an split fibration over $\mathbb{I}$:

$$W_F \to \mathbb{I}$$

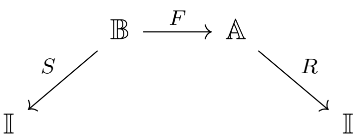
While it's true that $R \circ F$ does not necessarily equal S, we have that the following commutes:



In this chapter we will construct initial algebra for dependent polynomial functors and show that these are stable under pullback.

5.1. Construction of dependent W-types

Let $\mathbb{I} \in \mathbf{Gpd}$. This will remain fixed throughout this chapter. Further suppose, we are given the following polynomial:



where F is a split fibration. Note that we do not assume that $R \circ F = S$. We denote by P_F the polynomial functor assigned to the above polynomial:

$$\mathbf{Gpd}_{/\mathbb{I}} \xrightarrow{\Delta_S} \mathbf{Gpd}_{/\mathbb{B}} \xrightarrow{\Pi_F} \mathbf{Gpd}_{/\mathbb{A}} \xrightarrow{\Sigma_R} \mathbf{Gpd}_{/\mathbb{I}}$$

To investigate how a potential initial algebra of P_F will behave, assume we have one. That is, if $W_F \xrightarrow{\text{sup}} \mathbb{I}$ is the initial algebra of P_F, then the objects of $P_F W_F$ are triples (i, a, T), where $i \in \mathbb{I}$, $a \in \mathbb{A}_i$ and $T : \mathbb{B}_a \to \Delta_S W_F$, such that:

$$\begin{array}{ccc} \mathbb{B}_a & \xrightarrow{\quad T \quad} & \Delta_S W_F \\ & \searrow \quad \swarrow_{\pi} & \\ & \mathbb{B} & \end{array}$$

If we examine $T : \mathbb{B} \to \Delta_S W_F$ a bit closer we see that:

$$T_b = (b, w) \qquad\qquad \text{where } Sb = \pi w$$

Abusing type theoretic notation, this means:

$$T \in \prod_{b \in \mathbb{B}_a} W_{S(b)}$$

Let W be the initial algebra for $P : \mathbf{Gpd} \to \mathbf{Gpd}$, defined as the composition:

$$\mathbf{Gpd} \xrightarrow{\Delta_{\mathbb{B}}} \mathbf{Gpd}_{/\mathbb{B}} \xrightarrow{\Pi_F} \mathbf{Gpd}_{/\mathbb{A}} \xrightarrow{\Sigma_{\mathbb{A}}} \mathbf{Gpd}$$

Inspired by the previous chapter, we define a hereditary predicate. We define $\rho : W \to \mathbb{I}$ as the composite $\xrightarrow{\pi_1} \mathbb{A} \xrightarrow{R} \mathbb{I}$.

Definition 5.1.1. We say that $\sup(a, T) \in W$ is $\mathbb{I}$-*coherent* if:

- for all $b \in \mathbb{B}_a$, $\rho(T_b) = Sb$, and,
- for all $x \in \mathbb{B}_a$, T_x is $\mathbb{I}$-coherent

Further, $\sup(u, \varphi) : \sup(a, T) \to \sup(a', T')$ is $\mathbb{I}$-*coherent* if:

- for all $b \in \mathbb{B}_a$, $\rho(\varphi(\underline{u}_b)) = S\underline{u}_b$, and,
- for all $x \in \mathbb{B}_f$, $\varphi(x)$ is $\mathbb{I}$-coherent.

Let W_F be the subgraph of W consisting of $\mathbb{I}$-coherent vertices and arrows.

Proposition 5.1.2. *W_F can be equipped with a groupoid structure, that makes it a subgroupoid of W.*

Proof.

(1) Let $\sup(u,\varphi) : \sup(a,T) \to \sup(a',T')$ and $\sup(u',\varphi') : \sup(a',T') \to \sup(a'',T'')$ be composable and $\mathbb{I}$-coherent. Then the composition is again $\mathbb{I}$-coherent.

Since ρ is a functor:

$$\begin{aligned} \rho((\varphi' \circ \varphi)(\underline{u' \circ u_b})) &= \rho(\varphi'(\underline{u'}_{u_!b}) \circ \varphi(\underline{u}_b)) \\ &= \rho(\varphi'(\underline{u'}_{u_!b})) \circ \rho(\varphi(\underline{u}_b)) && \rho \text{ is a functor} \\ &= S(\underline{u'}_{u_!b}) \circ S(\underline{u}_b) && \varphi, \varphi' \text{ are } \mathbb{I}\text{-coherent} \\ &= S(\underline{u' \circ u_b}) \end{aligned}$$

(2) If $\sup(a,T)$ is $\mathbb{I}$-coherent, then the identity morphism $\mathrm{id}_{\sup(a,T)} = \sup(\mathrm{id}_a,\varphi)$ is also $\mathbb{I}$-coherent.

Take $b \in \mathbb{B}_a$. Note that $\rho(T_b) = Sb$, by assumption and hence $\rho(\varphi(\underline{\mathrm{id}}_b)) = \rho(\mathrm{id}_{T_b}) = S(\mathrm{id}_b)$. Thus the identity morphisms are $\mathbb{I}$-coherent.

(3) Suppose $\sup(u,\varphi)$ is $\mathbb{I}$-coherent. Then the inverse is as well:

$$\begin{aligned} \rho(\varphi^{-1}(\underline{u^{-1}}_b)) &= \rho(\varphi(\underline{u}_{u_!^{-1}b})^{-1}) \\ &= \rho(\varphi(\underline{u}_{u_!^{-1}b}))^{-1} && \text{since } \rho \text{ is a functor} \\ &= (S(\underline{u}_{u_!^{-1}b}))^{-1} && \text{since } \varphi \text{ is a } \mathbb{I}\text{-coherent} \\ &= S(\underline{u^{-1}}_b) \end{aligned}$$

□

Since W_F is a subgroupoid of W, we can see it over $\mathbb{I}$ via ρ:

$$W_F \hookrightarrow W \xrightarrow{\pi} \mathbb{A} \xrightarrow{R} \mathbb{I}$$

Proposition 5.1.3. *W_F admits a P_F-algebra structure.*

Proof. Let $(a,T) \in P_F W_F$. We wish to show that $\sup(a,T) \in W$ is $\mathbb{I}$-coherent and lies in W_F. Since $T : \mathbb{B}_a \to \Delta_s W_F$, we have that $\rho(T_b) = Sb$. Further for any $x \in \mathbb{B}_a$ is T_x is $\mathbb{I}$-coherent. The same reasoning applies to morphisms. □

Proposition 5.1.4. *Every subalgebra of W_F is equal to W_f.*

Proof. Let $G \hookrightarrow W_F$ be the smallest subalgebra (by Proposition 2.4.3). Take $\sup(a,T)$ and suppose, by induction, that $T(x)$ has a preimage in G, that is, $T(x) \in G$, for all $x \in \mathbb{B}_a$. Then $(a, T' : \mathbb{B}_a \to \Delta_s G) \in PG$, where $T'(x) = T(x)$, since $\sup(a,T)$ is $\mathbb{I}$-coherent. Hence $\sup(a,T) \in G$. The same reasoning applies for arrows as well. □

The above proposition allows us to conclude the following:

Theorem 5.1.5. *W_F is the initial algebra for P_F.*

Proof. This proof is similar to that of Theorem 3.2.3. □

Another way of defining W_F is as a particular equalizer, as in [**23**]. Let W and $W_{F\times\mathbb{I}}$ be the initial algebras for $F : \mathbb{B} \to \mathbb{A}$ and $F \times \mathrm{id}_{\mathbb{I}} : \mathbb{B} \times \mathbb{I} \to \mathbb{A} \times \mathbb{I}$. Let $\xi : W \to W_{F\times\mathbb{I}}$ be defined recursively as follows:

$$\xi(\sup(a, T)) = \sup(Ra, a, \xi \circ T)$$
$$\xi(\sup(u, \varphi)) = \sup(Ru, u, \xi \circ \varphi)$$

While defining ξ above, we should make sure that it is well defined, that is, $\xi \circ T$ and $\xi \circ \varphi$ need to be functors. We take $W_F^{\mathbf{Set}}$, as in the chapter where we first constructed the W-types for split fibrations and define ξ on those sets. Then we can show ξ is a reflexive graph morphism and further preserves the properties of functoriality and naturality.

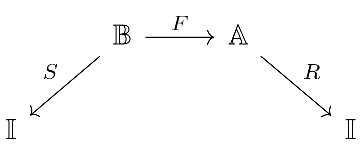
Further we define $\alpha : W_{F\times\mathbb{I}} \times \mathbb{B} \to W_{F\times\mathbb{I}}$, and $\psi :\to W_{F\times\mathbb{I}} \to\to W_{F\times\mathbb{I}}$:

$$\alpha(\sup(i, a, T), b) = \sup(Sb, a, (\lambda x : \mathbb{B}_a).\alpha(Tx, x))$$
$$\alpha(\sup(v, u, \varphi), h) = \sup(Sh, u, (\lambda \mathbb{B}_u).\alpha(\varphi x, x))$$
$$\psi(\sup(i, a, T)) = \sup(i, a, (\lambda x : \mathbb{B}_a).\alpha(Tx, x))$$
$$\psi(\sup(v, u, \varphi)) = \sup(v, u, (\lambda \mathbb{B}_u).\alpha(\varphi x, x))$$

Then $\xi' : W \to W_{f\times\mathbb{I}}$, where the other morphism appearing in the equalizer is defined to be $\psi \circ \xi$. Unrolling the definitions we see that the equalizer exactly satisfies the condition of being $\mathbb{I}$-coherent.

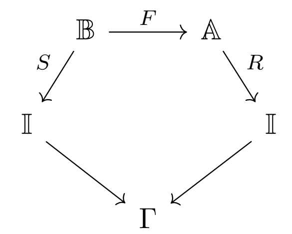
5.2. Pullback stability of dependent W-types

As before we assume the following data:

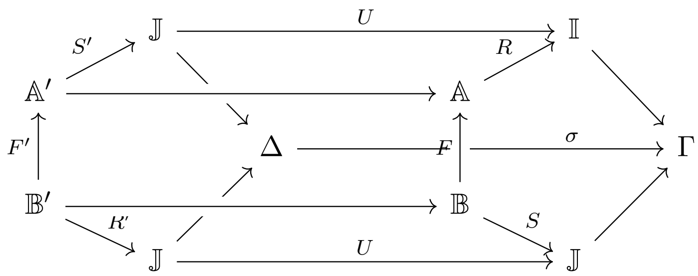


where F is a split fibration. As mentioned at the beginning, the following commutes:

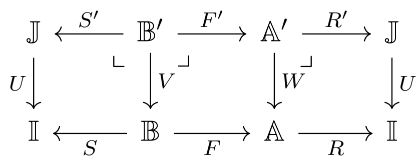
(∗)



Suppose now that we have an arrow $\sigma : \Delta \to \Gamma$. Then pulling back diagram $*$ along σ gives us:



While operating over Γ links up well with type theoretic interpretation of dependent W-types, we will work with a slightly more general description. We will assume that we have an arrow $u : \mathbb{J} \to \mathbb{I}$, and the following commutative diagram (where all the squares are pullbacks):



We will denote the polynomial $\mathbb{J} \xleftarrow{S'} \mathbb{B} \xrightarrow{F'} \mathbb{A} \xrightarrow{R'} \mathbb{J}$ by U^*F.

Proposition 5.2.1. *There is a natural isomorphism $\Delta_U P_F \cong P_{U^*F}\Delta_U$.*

Proof. This is simply a chain of Beck-Chevalley (cf. Proposition 1.5.3) isomorphisms:

$$
\begin{aligned}
\Delta_U \Sigma_R \Pi_F \Delta_S &\cong \Sigma_{R'} \Delta_V \Pi_F \Delta_S \\
&\cong \Sigma_{R'} \Pi_{F'} \Delta_W \Delta_V \\
&\cong \Sigma_{R'} \Pi_{F'} \Delta_{S'} \Delta_U \\
&= P_{U^*F} \Delta_U
\end{aligned}
$$

□

Corollary 5.2.2. $\Delta_U : \mathbf{Gpd}_{/\mathbb{I}} \to \mathbf{Gpd}_{/\mathbb{J}}$ *lifts to a functor* P_F-alg $\to P_{U^*F}$-alg. *Further, since pullbacks preserve isomorphisms, $\Delta_u W_F^{\mathbb{I}}$ is a fixpoint for P_{U^*F}.*

Let us prepare the terrain for the final theorem of this section. For $a' \in \mathbb{A}'$, we begin by noticing that $\mathbb{B}'_{a'}$ is isomorphic to $\mathbb{B}_{Wa'}$:

$$\begin{array}{ccc} \mathbb{B}'_{a'} & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow a' \\ \mathbb{B}' & \xrightarrow{F'} & \mathbb{A}' \\ V \downarrow & \lrcorner & \downarrow W \\ \mathbb{B} & \xrightarrow[F]{} & \mathbb{A} \end{array}$$

Observe that $\mathbb{B}_{Wa'}$ is the pullback of $W \circ a'$, that is, the outer square. We will denote this isomorphism by $\chi^{a'} : \mathbb{B}_{Wa} \to \mathbb{B}'_{a'}$. Note that the same observation holds for $\mathbb{B}_{u'} \cong \mathbb{B}_{Wu'}$.

Unfolding the definition of the algebra structure map $\sup_{\Delta_U W_F^{\mathbb{I}}} : P_{U^*F}\Delta_u W_F^{\mathbb{I}} \to \Delta_U W_F^{\mathbb{I}}$ obtained in Corollary 5.2.2, we get for (j, a', T), where $j \in \mathbb{J}$, $a' \in \mathbb{A}'_j$ and $T : \mathbb{B}_{a'} \to \Delta_{S'}\Delta_U W_F^{\mathbb{I}}$:

$$\sup_{\Delta_U W_F^{\mathbb{I}}}(j, a', T) = (j, \sup_{W_F^{\mathbb{I}}}(Uj, Wa', T' : \mathbb{B}_{Wa'} \to \Delta_U W_F^{\mathbb{I}}))$$

where T' is defined as the composite:

$$\mathbb{B}_{Wa'} \xrightarrow{\chi^{a'}} \mathbb{B}'_{a'} \xrightarrow{T} \Delta_{S'}\Delta_U W_F^{\mathbb{I}} \to \Delta_U W_F^{\mathbb{I}}$$

Suppose we have $(j, \sup(i, a, T)) \in \Delta_U W_F^{\mathbb{I}}$. Since the algebra structure map is an isomorphism, we have (j, a', T'), that maps to it. Further we have that:

$$T'_{(\chi^{a'})^{-1}b} = (j', T_b)$$

Theorem 5.2.3. *$\Delta_U W_F^{\mathbb{I}}$ is isomorphic to $W_{U^*F}^{\mathbb{J}}$ as a P_{U^*F} algebra.*

Proof. This proof is quite similar to the proof of the analogous statement in the previous chapter. However, there are some steps that require a bit more careful consideration. Let $H : W_{U^*F}^{\mathbb{J}} \to \Delta_U W_F^{\mathbb{I}}$ be the unique algebra morphism, given by initiality of $W_{U^*F}^{\mathbb{J}}$. By induction on $W_F^{\mathbb{I}}$ we assume the following induction hypothesis (and similarly for arrows):

$$\begin{aligned} &\forall \sup_{W_F^{\mathbb{I}}}(i, a, T) \in W_F^{\mathbb{I}}.\forall j \in \mathbb{J}.(j, \sup_{W_F^{\mathbb{I}}}(i, a, T)) \in \Delta_U W_F^{\mathbb{I}} \Rightarrow \\ &\qquad \exists!\sup_{W_{U^*F}^{\mathbb{J}}}(j, a', T') \in W_{U^*F}^{\mathbb{J}}.H(\sup_{W_{U^*F}^{\mathbb{J}}}(j, a', T')) = (j, \sup_{W_F^{\mathbb{I}}}(i, a, T)) \end{aligned}$$

Let $(j, \sup(i, a, T)) \in \Delta_U W_F^{\mathbb{I}}$. By previous considerations we have a series of $(j_b, T_b) \in \Delta_U W_F^{\mathbb{I}}$ and our induction hypothesis gives a unique $\sup_{W_{U^*F}^{\mathbb{J}}}(j_b, a'_b, T'^b)$ for each $b \in \mathbb{B}_a$, that maps to (j_b, T_b) via H.

Let a' be such that $Wa' = a$. We define $T' : \mathbb{B}'_{a'} \to W_{U^*F}^{\mathbb{J}}$, by setting $T(x) = T'^{(\chi^{a'})^{-1}x}$. Uniqueness of smaller trees guarantees that T is functorial.

Let $b' \in \mathbb{B}'_{a'}$. We know that $S'b' = S'((\chi^{a'})^{-1}b') = j_{(\chi^{a'})^{-1}b'}$, which allows us to conclude that T' is $\mathbb{J}$-coherent and hence (j, a', T') is in $P_{U^*F}W^{\mathbb{J}}_{U^*F}$. This maps to $(j, \sup(i, a, T))$ via H by construction.

Since all T'^x are unique, we get that T is unique as well.

The argument for morphisms is similar, except that uniqueness also provides us with the naturality condition.

This means that H is bijective and hence $\Delta_U W^{\mathbb{I}}_F$ and $W^{\mathbb{J}}_{U^*F}$ are isomorphic. □

CHAPTER 6

W-types for cloven and general fibrations

In this chapter we first construct W-types for simple polynomials, where the fibration is equipped with a cleavage, which is not necessarily split. We do this in a similar fashion to the case when the fibration is split (Chapter 3).

Next, we consider the case of a general fibration. In order to construct the W-type for this case we introduce the notion of triangle graphs, which is another presentation of 2-truncated simplicial sets. Since we have W-types for presheaves [**39**], we propose an inductively defined triangle graph, which we show admits the structure of a groupoid and is the initial algebra for the simple polynomial functor.

6.1. W-types for cloven fibrations

Let $F : \mathbb{B} \to \mathbb{A}$ be a cloven fibration, not necessarily split. Our goal is to construct a W-type for the polynomial associated to it, defined as the composition:

$$\mathbf{Gpd} \xrightarrow{\Delta_{\mathbb{B}}} \mathbf{Gpd}_{/\mathbb{B}} \xrightarrow{\Pi_F} \mathbf{Gpd}_{/\mathbb{A}} \xrightarrow{\Sigma_{\mathbb{A}}} \mathbf{Gpd}$$

(Section 2.1 describes the action of this functor explicitly). In order to do so, we make use of similar techniques as in chapter 3. We will omit the construction of the graph from a $\mathbf{Set}$-valued W-type, and instead define an inductive graph in the style of $\widetilde{W}'$. In the proofs in chapter 3 we use the fact that F is a split fibration, but in the present case we need to take additional care when defining the composition operation. Let X be the smallest inductive graph:

- if $a \in \mathbb{A}$ and $T : \mathbb{B}_a \to X$ is a graph morphism, then (a, T), is a vertex in X,
- if $a, a' \in \mathbb{A}$, $f : a \to a'$, $(a, T), (a', T') \in X_0$, and φ is a collection of arrows in X of the form:

  $$\varphi = \big(\varphi_b : Tb \to T' f_! b \mid b \in \mathbb{B}_a\big)$$

  then (f, φ) is an edge from (a, T) to (a', T').

We will denote this graph by $\widetilde{W}_F$.

Unfolding the definition of W-types for reflexive graphs, we can see what the reflexivity map does. To each vertex (a, T) we associate a reflexive arrow, which we write

$\mathrm{id}_{(a,T)} : (a,T) \to (a,T)$. The arrow is defined as

$$(\mathrm{id}_a, (T(\underline{\mathrm{id}_a}_b) : Tb \to T(\mathrm{id}_a)_! b \mid b \in \mathbb{B}_a))$$

We define a binary operation on the edges of $\widetilde{W_F}$, which will become associative after we remove some of the elements of the graph. The operation is defined recursively. Suppose we have $\sup(f,\varphi) : \sup(a,T) \to \sup(a',T')$ and $\sup(f',\varphi') : \sup(a',T') \to \sup(a'',T'')$. We assume that we have defined a composition operation on

$$(-) \circ (-) : \widetilde{W_F}(T'b, T'b') \times \widetilde{W_F}(T'b', T''b'') \to \widetilde{W_F}(Tb, T''b'')$$

Then we define:

$$(f',\varphi') \circ (f,\varphi) =_{\mathrm{def}} (f' \circ f, \varphi' \circ \varphi)$$
$$(\varphi \circ \varphi)_b =_{\mathrm{def}} (T''\Phi_{f,f'} \circ \varphi'_{f_! b}) \circ \varphi_b$$

The reflexivity arrows, will become identities for this operation, once we restrict the structure.

As said, the composition operation is not necessarily associative for all arrows. Notice however that the definition matches the one for the exponential object in the slices. Using this definition we can define a predicate that will allow us to obtain a subgraph which turns out to be a groupoid with this composition operation.

Definition 6.1.1. Let $\sup(a,T)$ be a vertex in $\widetilde{W_F}$. We say that $\sup(a,T)$ is *functorial* if:

- if $T : \mathbb{B}_a \to \widetilde{W_F}$ preserves the composition operation and identity arrows, that is:

$$T(v \circ u) = T(u) \circ T(v)$$
$$T(\mathrm{id}_b) = \mathrm{id}_{Tb}$$

- for all $b \in \mathbb{B}_a$, Tb is again functorial.

Further, let $\sup(f,\varphi) : \sup(a,T) \to \sup(a',T')$ be an edge in $\widetilde{W_F}$. We say that $\sup(f,\varphi)$ is *natural* if:

- for all $u : b \to b' \in \mathbb{B}_a$:

$$\varphi_{b'} \circ Tu = T' f_! u \circ \varphi_b$$

- for all $b \in \mathbb{B}_a$, we have that φ_b is natural.

Let W_F be the smallest subgraph of $\widetilde{W_F}$ consisting of functorial vertices and natural edges.

Proposition 6.1.2. *W_F admits the structure of a groupoid.*

Proof. Let $\sup(f^i, \varphi^i) : \sup(a^i, T^i) \to \sup(a^{i+1}, T^{i+1})$ for $1 \leq i \leq 3$. We would like to show that

$$(\varphi^3 \circ \varphi^2) \circ \varphi^1 = \varphi^3(\circ\varphi^2 \circ \varphi^1)$$

Unfolding the definition of composition, we get:

$$(\varphi^3(\circ\varphi^2 \circ \varphi^1))_b = (T^4(\Phi_{f^2 \circ f^1}) \circ \varphi^3_{(f^2 \circ f^1)_! b}) \circ ((T^3(\Phi_{f^1,f^2}) \circ \varphi^2_{f^1_! b}) \circ \varphi^1_b)$$
$$(\varphi^3 \circ (\varphi^2 \circ \varphi^1))_b = (T^4(\Phi_{f^1,f^3 \circ f^2}) \circ ((T^4(\Phi_{f^2,f^3}) \circ \varphi^3_{f^2_! f^1_! b}) \circ \varphi^2_{f^1_! b})) \circ \varphi^1_b$$

Suppose, by induction, that composition is associative for arrows of the form $W_F(Tb, T'b')$ (for any T, T', b, b'). We can then show that the next diagram commutes:

$$\begin{array}{ccccccccc}
T^1 b & \xrightarrow{\varphi^1_b} & T^2 f^1_! b & \xrightarrow{\varphi^2_{f^1_! e}} & T^3 f^2_! f^1_! b & \xrightarrow{\varphi^3_{f^2_! f^1_! b}} & T^4 f^3_! f^2_! f^1_! b & \xrightarrow{T^4(\Phi_{f^2,f^3})_b} & T^4(f^3 \circ f^2)_! f^1_! b \\
 & & & & \downarrow{\scriptstyle T^3(\Phi_{f^1,f^2})_e} & & \downarrow{\scriptstyle T^4 f^3_!(\Phi_{f^1,f^2})_b} & & \downarrow{\scriptstyle T^4(\Phi_{f^1,f^3 \circ f^2})_b} \\
 & & & & T^3(f^2 \circ f^1)_! b & \xrightarrow[\varphi^3_{(f^2 \circ f^1)_! b}]{} & T^4 f^3_!(f^2 \circ f^1)_! b & \xrightarrow[T^4(\Phi_{f^2 \circ f^1, f^3})_b]{} & T^4(f^3 \circ f^2 \circ f^1)_! b
\end{array}$$

The left-hand square commutes since we assumed φ^i are natural, and the right-hand square commutes since we assumed T^i to be functorial. The upper path is equal to $((\varphi^3 \circ \varphi^2) \circ \varphi^1)_b$ and the bottom path is equal to $(\varphi^3 \circ (\varphi^2 \circ \varphi^1))_b$. Hence composition in W_F is associative.

The proofs for the left and right unit laws and inverses are similar to the computations performed in the previous chapter when discussing the exponential object for cloven fibrations (cf. Lemma 1.3.10). □

Proposition 6.1.3. *W_F can be equipped with an algebra structure* $\sup : P_F W_F \to W_F$. *Further* $\sup$ *is an isomorphism.*

Proof. Applying P_F to W_F, as defined above, produces an isomorphic object. Given $(a, T) \in P_F W_F$, observe that $T : \mathbb{B}_a \to W_F$ satisfies the constraints of being a functorial graph morphism and hence $\sup(a, T)$ is a already present in W_F. Similarly for morphisms. Thus we define $\sup : P_F W_F \to W_F$:

$$(a, T) \mapsto \sup(b, t)$$
$$(f, \varphi) \mapsto \sup(f, \varphi)$$

Note that this has an immediate inverse. □

Theorem 6.1.4. $(W_F, \sup)$ *is the initial algebra for* P_F.

Proof. The content of this proof is very similar to the one for split fibrations (Theorem 3.2.3), so we omit the details. ☐

6.2. W-types for general fibrations

To construct the initial algebra for a general fibration, we first introduce the notion of a triangle graph. Let Δ_2 be the 2-truncated simplex category:

$$0 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; 1 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; 2$$

Similarly to the reflexive graphs before, we now consider *reflexive triangle graphs*, as presheaves $[\Delta_2^{\mathrm{op}}, \mathbf{Set}]$. Unfolding this, we have the following definition.

Definition 6.2.1. A *triangle graph G* consists of:

- a set of vertices G_0,
- for every $x, y \in G_0$, a set of edges $G_1(x, y)$,
- for every $f \in G_1(x, y)$, $g \in G_1(y, z)$ and $h \in G_1(x, z)$, a set of triangles $G_2(h, g, f)$.

A triangle graph is *reflexive*, if it comes equipped with functions:

$$r_0 : \Pi_{x:G_0} G_1(x, x)$$

$$r_{01} : \Pi_{x,y:G_0} \Pi_{f:G_1(x,y)} G_2(f, f, r_0 x)$$

$$r_{12} : \Pi_{x,y:G_0} \Pi_{f:G_1(x,y)} G_2(f, r_0 y, f)$$

Definition 6.2.2. A *triangle graph morphism* $\varphi : G \to H$ consists of:

- a function $\varphi_0 : G_0 \to H_0$,
- for every $x, y \in G_0$, a function $\varphi_{x,y} : G_1(x, y) \to H_1(\varphi x, \varphi y)$
- for every $f \in G_1(x, y)$, $g \in G_1(y, z)$ and $h \in G_1(x, z)$, a function $\varphi_{h,g,f}$: $G_2(h, g, f) \to G_2(\varphi h, \varphi g, \varphi f)$

A *morphism of reflexive triangle graphs* has the same data, but must additionally commute with the reflexivity functions, e.g.:

$$r_0^H \circ \varphi_0 = \varphi_{x,x} \circ r_0^G$$

$$\begin{array}{ccc} G_0 & \xrightarrow{\quad \varphi_0 \quad} & H_0 \\ \big\downarrow{\scriptstyle r_0^G} & & \big\downarrow{\scriptstyle r_0^H} \\ \Pi_x G_1(x, x) & \xrightarrow{\varphi} & \Pi_x H_1(x, x) \end{array}$$

Example 6.2.3. We can view any small category $\mathcal{C}$ as a reflexive triangle graph, by considering the objects as vertices, morphisms as edges, identities as the reflexive edges and a single triangle whenever $h = g \circ f$.

Suppose $F : \mathbb{B} \to \mathbb{A}$ is a fibration. In the case of F split or cloven we had the option of using the cleavage data to specify the composition operation. Unfortunately, we are unable to do so in general, as it requires axiom of choice. However, using triangle graphs, we can encode the composition directly into the graph.

Using [**39**] we can obtain W-types for presheaves. This allows us to consider W-types for triangle graphs.

Proposition 6.2.4. *The reflexive triangle graph W is the smallest reflexive triangle graph X, such that:*

(1) If $a \in \mathbb{A}_0$ and $T : \mathbb{B}_a \to X$ a reflexive triangle graph morphism, then $\sup(a, T) \in X_0$

(2) If $\sup(a, T)$*,* $\sup(a', T')$ *are vertices in X, $f : a \to a' \in \mathbb{A}(a, a')$ and further we have*

$$\varphi : \prod_{u:b \to b' \in F^{-1}(f)} X_1(Tb, T'b')$$

along with:

$$\epsilon : \prod_{\substack{u:b \to b' \in F^{-1}(f)\\ d:\widehat{b} \to b \in F^{-1}(\mathrm{id}_a)}} X_2(\varphi(u \circ d), \varphi(u), Td)$$

$$\overline{\epsilon} : \prod_{\substack{u:b \to b' \in F^{-1}(f)\\ d:b' \to \widehat{b'} \in F^{-1}(\mathrm{id}_{a'})}} X_2(\varphi(d \circ u), T'd, \varphi(u))$$

then $\sup(f, \varphi, \epsilon, \overline{\epsilon}) \in X_1(\sup(a, T), \sup(a', T'))$.

(3) If we have

$$\sup(f, \varphi, \epsilon, \overline{\epsilon}) \in X_1(\sup(a, T), \sup(a', T'))$$
$$\sup(f', \varphi', \epsilon', \overline{\epsilon'}) \in X_1(\sup(a', T'), \sup(a'', T''))$$
$$\sup(f' \circ f, \varphi'', \epsilon'', \overline{\epsilon''}) \in X_1(\sup(a, T), \sup(a'', T''))$$

and:

$$\xi : \prod_{\substack{u:b \to b' \in F^{-1}(f)\\ v:b' \to b'' \in F^{-1}(f')}} X_2(\varphi''(v \circ u), \varphi'(v), \varphi(u))$$

then $\sup(\xi) \in X_2(\sup(f' \circ f, \varphi'', \epsilon'', \overline{\epsilon''}), \sup(f', \varphi', \epsilon', \overline{\epsilon'}), \sup(f, \varphi, \epsilon, \overline{\epsilon}))$.

Remark 6.2.5. Unfolding the construction in [**39**], we can see what the reflexivity actions do:

(1) Let $\sup(a, T) \in W_0$, then its reflexive arrow is set to be:

$$r_0(\sup(a, T)) = \sup(\mathrm{id}_a, \varphi, \epsilon, \overline{\epsilon})$$

where

$$\varphi(u : b \to b') = Tu$$

$$\epsilon(u, d) = T \left(\begin{array}{ccc} \cdot & & \\ {\scriptstyle d}\downarrow & \searrow^{u \circ d} & \\ \cdot & \xrightarrow[u]{} & \cdot \end{array} \right)$$

$$\overline{\epsilon}(u, d) = T \left(\begin{array}{ccc} \cdot & \xrightarrow{u} & \cdot \\ & {}_{u \circ d}\searrow & \downarrow{\scriptstyle d} \\ & & \cdot \end{array} \right)$$

That is $r_0(\sup(a, T)) = \sup(\mathrm{id}_a, T, T, T)$.

(2) Let $\sup(f, \varphi, \epsilon, \overline{\epsilon}) \in W_1(\sup(a, T), \sup(a', T'))$, then:

$$r_{01}(\sup(f, \varphi, \epsilon, \overline{\epsilon})) = \sup(\epsilon)$$
$$r_{12}(\sup(f, \varphi, \epsilon, \overline{\epsilon})) = \sup(\overline{\epsilon})$$

Proposition 6.2.6. *Let*

$$\sup(f, \varphi, \epsilon, \overline{\epsilon}) : \sup(a, T) \to \sup(a', T')$$
$$\sup(f', \varphi', \epsilon', \overline{\epsilon}') : \sup(a', T') \to \sup(a'', T'')$$

be two arrows in W. If

$$W_2(\sup(f' \circ f, \varphi'', \epsilon'', \overline{\epsilon''}), \sup(f', \varphi', \epsilon', \bar{\epsilon}'), \sup(f, \varphi, \epsilon, \overline{\epsilon}))$$

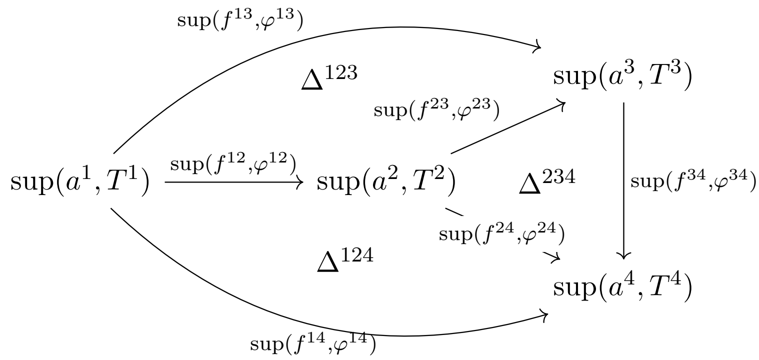
is inhabited, then it has a unique member.

Proof. We prove the above claim by induction, so suppose that for any $u : b \to b'$, $u' : b' \to b''$ over f, f', $W_2(\varphi''(u' \circ u), \varphi(u'), \varphi(u))$ has a unique member, if inhabited.

Suppose $\Delta, \Delta' \in W_2(\sup(f' \circ f, \varphi'', \epsilon'', \overline{\epsilon''}), \sup(f', \varphi', \epsilon', \bar{\epsilon}'), \sup(f, \varphi, \epsilon, \overline{\epsilon}))$, then $W_2(\varphi''(u' \circ u), \varphi(u'), \varphi(u))$ is inhabited, namely we have $\Delta(u', u)$ and $\Delta'(u', u)$. By our induction hypothesis, $\Delta(u', u) = \Delta'(u', u)$. Hence $\Delta = \Delta'$. □

In the light of this proposition, we will omit ϵ and $\overline{\epsilon}$, that is we will write $\sup(f, \varphi)$ instead of $\sup(f, \varphi, \epsilon, \overline{\epsilon})$.

Proposition 6.2.7. *Suppose we find ourselves in the following situation:*

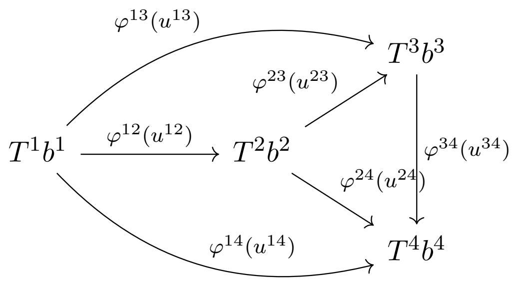


And further, we have the following:

$$\Delta^{123} : W_2(\sup(f^{13},\varphi^{13}), \sup(f^{23},\varphi^{23}), \sup(f^{12},\varphi^{12}))$$
$$\Delta^{124} : W_2(\sup(f^{14},\varphi^{14}), \sup(f^{24},\varphi^{24}), \sup(f^{12},\varphi^{12}))$$
$$\Delta^{234} : W_2(\sup(f^{24},\varphi^{24}), \sup(f^{34},\varphi^{34}), \sup(f^{23},\varphi^{23}))$$

Then there exists $\Delta^{134} : W_2(\sup(f^{14},\varphi^{14}), \sup(f^{34},\varphi^{34}), \sup(f^{13},\varphi^{13}))$.

Proof. Suppose, by induction, that the above holds for diagrams of the form:



where

$$u^{13} = u^{23} \circ u^{12}$$
$$u^{24} = u^{34} \circ u^{23}$$
$$u^{14} = u^{24} \circ u^{12}$$

with triangles $\Delta^{123}(u^{23}, u^{12}), \Delta^{124}(u^{24}, u^{12})$ and $\Delta^{234}(u^{34}, u^{23})$. This allows us to obtain a candidate

$$\Delta^{134}_{u^{34},u^{23},u^{12}} : W_2(\varphi^{14}(u^{34} \circ u^{23} \circ u^{12}), \varphi^{34}(u^{34}), \varphi^{13}(u^{23} \circ u^{12}))$$

for each u^{12}, u^{23}, and u^{34} over f^{12}, f^{23}, f^{34} respectively. Further this is the unique such, due to the previous proposition (Proposition 6.2.6).

Let u^{13} and u^{34} be over f^{13}, f^{34}. By the Conduché property, we have a factorisation of $u^{13} = u^{23} \circ u^{34}$, for some u^{23}, u^{34}. Suppose we have another factorisation v^{23}, v^{12}, but note that:

$$\begin{aligned} &W_2(\varphi^{14}(u^{34} \circ u^{23} \circ u^{12}), \varphi^{34}(u^{34}), \varphi^{13}(u^{23} \circ u^{12})) \\ =&W_2(\varphi^{14}(u^{34} \circ u^{13}), \varphi^{34}(u^{34}), \varphi^{13}(u^{13})) \\ =&W_2(\varphi^{14}(u^{34} \circ v^{23} \circ v^{12}), \varphi^{34}(u^{34}), \varphi^{13}(v^{23} \circ v^{12})) \end{aligned}$$

And by uniqueness of triangles (Proposition 6.2.6) we get:

$$\Delta^{134}_{u^{34},u^{23},u^{12}} = \Delta^{134}_{u^{34},v^{23},v^{12}}$$

Hence we can define $\Delta^{134}(u^{34}, u^{13}) = \Delta^{134}_{u^{34},u^{23},u^{12}}$, for any factorisation of u^{13}. □

Remark 6.2.8. We can extend the above to say that if we instead have the outer triangle (Δ^{134}), but are missing Δ^{124}, we can obtain it as well. The proof is analogous.

Remark 6.2.9. This proposition states that W satisfies the inner horn filling condition, if we see it as a simplicial set.

We now show that we have existence of composition.

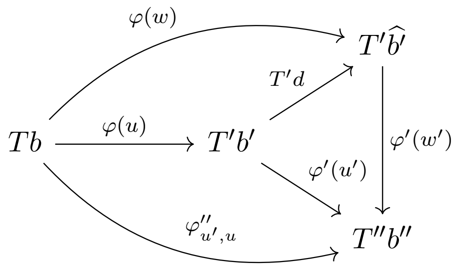
Proposition 6.2.10. *If we have two arrows of the form* $\sup(f, \varphi) : \sup(a, T) \to \sup(a', T')$ *and* $\sup(f', \varphi') : \sup(a', T') \to \sup(a'', T'')$*, then there exists a unique* φ''*, such that* $\sup(f' \circ f, \varphi'') : \sup(a, T) \to \sup(a'', T'')$ *and* $W_2(\sup(f' \circ f, \varphi''), \sup(f', \varphi), \sup(f, \varphi))$ *is inhabited.*

Proof. Assume by induction, that the statement holds for the diagrams of the form:

$$Tb \xrightarrow{\varphi(u)} T'b' \xrightarrow{\varphi(u')} T''b''$$

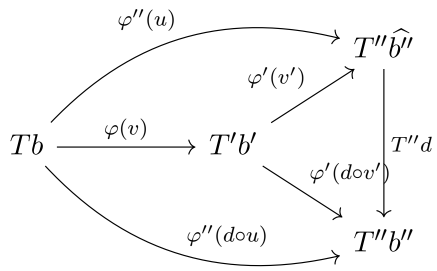
Let u be over $f' \circ f$. By the Conduché property, we have a factorisation $v = u' \circ u$, and the induction hypothesis then gives us a candidate $\varphi''_{u',u}$.

Given another factorisation $u = w' \circ w$, the Conduché property gives us a morphism $d : b' \to \widehat{b'}$ linking the two factorizations. Hence we find ourselves in the following situation:



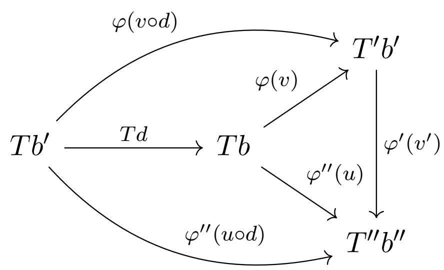
The upper triangle is $\overline{\epsilon}^{\varphi}(d, u)$, the rightmost triangle is $\epsilon^{\varphi}(w', d)$ and the lower comes from the inductive hypothesis. By Proposition 6.2.7, we have existence of the outer triangle. Since by induction hypothesis, there exists a unique arrow, such that $W_2(\varphi''_{w',w}, \varphi'(w'), \varphi(w))$ is inhabited, we get $\varphi''_{w',w} = \varphi''_{u',u}$. We set $\varphi''(u) = \varphi''_{v',v}$ for any factorisation $u = v' \circ v$.

In order for $\sup(f' \circ f, \varphi'')$ to be an arrow in W, we need to construct its ϵ, $\overline{\epsilon}$. Suppose $d : b'' \to \widehat{b''}$ and consider:



The upper and bottom triangles exist by our induction hypothesis and the right triangle is $\overline{\epsilon}^{\varphi'}(d, v')$. By Proposition 6.2.7, we have existence of the outer triangle, which we set to be our $\overline{\epsilon}(d, u)$.

Suppose now that we have $d : \widehat{b} \to b$ and consider:



The upper triangle is $\epsilon^{\varphi}(v, d)$, the right and the outer triangle are given by the induction hypothesis. We obtain the lower triangle by Remark 6.2.8, which we set to be our $\epsilon(u, d)$.

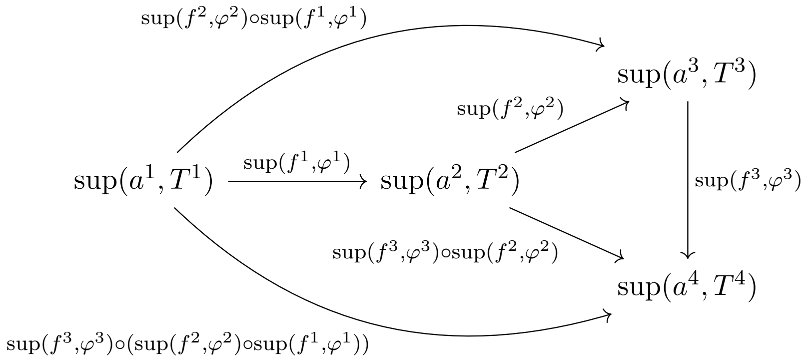
Suppose we have another $\sup(f' \circ f, \gamma) : \sup(a, T) \to \sup(a'', T'')$, such that $W_2(\sup(f' \circ f, \gamma), \sup(f', \varphi'), \sup(f, \varphi))$ is inhabited. Call that member Δ'. Let u over $f' \circ f$, and $u = v' \circ v$ for some v', v. We have then that $\Delta'(v', v) \in W_2(\gamma(u), \varphi'(u'), \varphi(u))$, but by our induction hypothesis $\varphi''_{v',v}$ is the unique such, which means $\gamma(u) = \varphi''(u)$ $\square$

Proposition 6.2.11. *W admits the structure of a category.*

Proof. Using Proposition 6.2.10 we can define the operation of composition. If $\sup(f, \varphi) : \sup(a, T) \to \sup(a', T')$ and $\sup(f', \varphi') : \sup(a', T') \to \sup(a'', T'')$, we define:

$$\sup(f', \varphi') \circ \sup(f, \varphi) = \sup(f' \circ f, \varphi'')$$

where φ'' is the unique arrow obtained from Proposition 6.2.10. Suppose now we have $\sup(a^i, T^i) : \sup(a_i, T_i) \to \sup(a_{i+1}, T_{i+1})$ for $i = 1, 2, 3$, then we find ourselves in the following situation:



The upper, right and outer triangles exist by our definition of composition. The existence of the lower triangle is given by Remark 6.2.8. But Proposition 6.2.10 says the the composite is unique, hence:

$$\sup(f^3, \varphi^3) \circ (\sup(f^2, \varphi^2) \circ \sup(f^1, \varphi^1)) = (\sup(f^3, \varphi^3) \circ \sup(f^2, \varphi^2)) \circ \sup(f^1, \varphi^1)$$

The identities are given by r_0. We write $\mathrm{id}_{\sup(a,T)} = r_0(\sup(a, T))$. Then we can see that for any $\sup(f, \varphi) : \sup(a, T) \to \sup(a', T')$, $\sup(f, \varphi) \circ \mathrm{id}_{\sup(a,T)} = \sup(f, \varphi)$, since we have

$$\epsilon(u, d) : W_2(\varphi(u \circ d), \varphi(u), \varphi(d))$$

and composition is unique. The same reasoning applies to the other identity, except we use $\bar{\epsilon}$. □

Proposition 6.2.12. *Let $\sup(a, T)$ be a vertex in W_0. Then $T : \mathbb{B}_a \to W$ is a functor.*

Proof. This follows from the fact that T is a reflexive graph morphism and the fact that W has unique triangles. That is, we have:

$$T(id_b) = \mathrm{id}_{Tb}$$

Further, $T^2_{v\circ u,v,u} : \mathbb{B}(v \circ u, v, u) \to W^2(T(v \circ u), Tv, Tu)$ and by Proposition 6.2.6 we now have that:

$$T(v \circ u) = Tv \circ Tu$$

□

Proposition 6.2.13. *The category W is a groupoid.*

Proof. Let $\sup(f, \varphi) : \sup(a, T) \to \sup(a', T')$ be a morphism. Suppose by induction that for all u over f, there exists $\varphi(u)^{-1}$. We can define $\sup(f^{-1}, \varphi^{-1}, \epsilon, \bar{\epsilon}) : \sup(a', T') \to \sup(a, T)$, which will be the inverse of $\sup(f, \varphi)$. Set $\varphi^{-1}(u) = (\varphi(u^{-1}))^{-1}$. To show that ϵ exists, we use the categorical structure. Note that if we have $f \circ g = h$, this actually means we have a triangle between h, g and f. Suppose $d : \widehat{b'} \to b'$ over id_a and $u : b' \to b$ over f^{-1}, and we would like to show that:

$$\varphi^{-1}(u \circ d) = \varphi^{-1}(u) \circ T'(d)$$

Unfolding the definition of φ^{-1}, this means that we would like to show:

$$\varphi(d^{-1} \circ u^{-1})^{-1} = \varphi(u^{-1})^{-1} \circ T'(d)$$

Since T' is a functor and d is iso, this is the same as showing:

$$\begin{aligned} \varphi(d^{-1} \circ u^{-1})^{-1} &= \varphi(u^{-1})^{-1} \circ T'(d^{-1})^{-1} \\ &= (T'(d^{-1}) \circ \varphi(u^{-1}))^{-1} \end{aligned}$$

This is necessarily true, since φ is a morphism in W.

Similarly for $\bar{\epsilon}$. To show that $\sup(f^{-1}, \varphi^{-1})$ is indeed the inverse, simply amounts to unfolding the definition of composition. □

Proposition 6.2.14. *Let $\sup(f, \varphi) : \sup(a, T) \to \sup(a', T')$ in W. Then φ is a generalized natural transformation $T \rightsquigarrow T'$.*

Proof. Suppose that, the following commutes in $\mathbb{B}$

$$\begin{array}{ccc} b & \xrightarrow{u} & b' \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d'} \\ \widehat{b} & \xrightarrow[v]{} & \widehat{b'} \end{array}$$

Where u and v are over f, and d, d' over $id_a, \mathrm{id}_{a'}$ respectively. We know that $\varphi(v) \circ T(d) = \varphi(v \circ d)$, but since the diagram just above commutes, we have $\varphi(v \circ d) = \varphi(d' \circ u) = T'(d') \circ \varphi(u)$. So the following diagram commutes in W:

$$\begin{array}{ccc} Tb & \xrightarrow{\varphi(u)} & T'b' \\ {\scriptstyle Td}\downarrow & & \downarrow{\scriptstyle T'd'} \\ T\widehat{b} & \xrightarrow[\varphi(v)]{} & T\widehat{b'} \end{array}$$

□

Proposition 6.2.15. *Let $F : \mathbb{B} \to \mathbb{A}$ be a fibration, and $P_F : \mathbf{Gpd} \to \mathbf{Gpd}$ be the polynomial functor associated to F. Then the map $P_F W \to W$ is an isomorphism.*

Proof. We can see that objects and morphisms in $P_F W$ match the requirements made in the inductive definition of W, hence they are present there already. □

Theorem 6.2.16. *W is a strictly 2-initial algebra for P_F.*

Proof. As in the case for split fibrations (Theorem 3.2.3), we first show that W is initial, by considering the smallest subalgebra $G \overset{(P,Q)}{\hookrightarrow} W \times \mathbb{X}$, for any other P_F algebra $(\mathbb{X}, \sup_{\mathbb{X}})$. By induction we show that P must be injective. Finally, Theorem 2.5.4 gives us 2-initiality.

The key difference with respect to the proof for Theorem 3.2.3, is that morphisms (f, φ) are not defined for just the cleavage data (since we do not necessarily have one), but for all u over f. Note however, that does not play an essential role in the proof. □

CHAPTER 7

Natural models and η-equality

In [**7**] Awodey defines the concept of natural models. This chapter briefly recalls the results from that paper, after which we propose some refinements to the original definitions in order to model types where η-equalities for Π and Σ are propositional and not definitional.

An example of why modeling type theory with propositional η-equality is interesting is given by homotopy type theory. In [**26**] Simon Henry constructs a weak model structure on simplicial sets. In ongoing joint work with Nicola Gambino, they give a construction where Π-types are interpreted as a cofibrant replacement of the right adjoint to the pullback, obtaining only propositional η-equality.

7.1. Review of natural models

Natural models of dependent type theory were first established by Awodey in [**7**]. In this section we briefly recall the definitions and results he obtained. We fix a category $\mathbb{C}$ and write $y : \mathbb{C} \to \widehat{\mathbb{C}}$ for the Yoneda embedding.

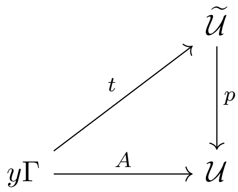
Definition 7.1.1. A natural transformation between two presheaves, $\mathcal{U}$ and $\widetilde{\mathcal{U}} \in \hat{\mathcal{C}}$, $p : \widetilde{\mathcal{U}} \to \mathcal{U}$ is *representable*, if for every $\Gamma \in \mathcal{C}$ and $A \in \mathcal{U}(\Gamma)$, we are given an object $\Gamma.A \in \mathcal{C}$, a map $p_A : \Gamma.A \to \Gamma$, and a $q_A \in \widetilde{\mathcal{U}}(\Gamma.A)$, such that the following square is a pullback:

$$\begin{array}{ccc} y\Gamma.A & \xrightarrow{q_A} & \widetilde{\mathcal{U}} \\ \downarrow{\scriptstyle yp_A} & \lrcorner & \downarrow{\scriptstyle p} \\ y\Gamma & \xrightarrow{A} & \mathcal{U} \end{array}$$

Definition 7.1.2. A small category $\mathcal{C}$, possesses a *natural model structure* if it comes equipped with two presheaves, $\mathcal{U}$ and $\widetilde{\mathcal{U}}$, and a natural transformation, $p : \widetilde{\mathcal{U}} \to \mathcal{U}$, which is representable.

One way of modeling type theory is via so-called *categories with families*, first defined by Peter Dybjer in [**17**]. Awodey observed that natural models immediately give a CwF structure (Proposition 1.2 in [**7**]).

We consider $\mathcal{C}$ to be a category of contexts, then given $p : \widetilde{\mathcal{U}} \to \mathcal{U}$, we have $\mathcal{U}(\Gamma)$ as the set of types and $\widetilde{\mathcal{U}}(\Gamma)$ as the set of terms, with p as the typing of those terms. Thus thanks to the Yoneda Lemma, given $A \in \mathcal{U}(\Gamma)$ and $t \in p_{\Gamma}^{-1}(A)$, we have the following typing diagram:

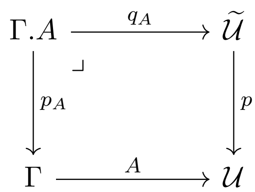


Naturality of p gives us substitution. Given $\sigma : \Delta \to \Gamma$, we have that

$$\Gamma \vdash A : \textsc{Types} \Rightarrow \Delta \vdash A\sigma : \textsc{Types}$$

$$\Gamma \vdash a : A \Rightarrow \Delta \vdash a\sigma : A\sigma$$

Representability matches the notion of context extension. Given a $\Gamma \in \mathcal{C}$ and a type $A \in \mathcal{U}\Gamma$, by the definition of representability, we get the following pullback:



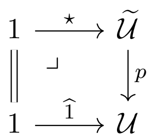
From now on we assume the reader to be familiar with locally cartesian closed categories, as we use the internal language of $\widehat{\mathcal{C}}$

Definition 7.1.3. A natural model p supports unit types, if we are given:

$$\widehat{1} : 1 \to \mathcal{U}$$

$$\star : 1 \to \widetilde{\mathcal{U}}$$

making the following diagram a pullback:



Definition 7.1.4. A natural model p supports dependent products, if we are given the following maps:

$$\hat{\lambda} : \sum_{A:\mathcal{U}} \widetilde{\mathcal{U}}^A \to \widetilde{\mathcal{U}},$$
$$\hat{\Pi} : \sum_{A:\mathcal{U}} \mathcal{U}^A \to \mathcal{U},$$

making the following diagram a pullback:

$$\begin{array}{ccc} \sum_{A:\mathcal{U}} \widetilde{\mathcal{U}}^A & \xrightarrow{\hat{\lambda}} & \widetilde{\mathcal{U}} \\ {\scriptstyle \sum_{A:\mathcal{U}} p^A}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ \sum_{A:\mathcal{U}} \mathcal{U}^A & \xrightarrow{\hat{\Pi}} & \mathcal{U} \end{array} \tag{7.1}$$

Definition 7.1.5. A natural model p supports dependent sums, as soon as we are given the following maps:

$$\text{pair} : \sum_{A:\mathcal{U}} \sum_{B:\mathcal{U}^A} \sum_{a:A} B(a) \to \widetilde{\mathcal{U}},$$
$$\hat{\Sigma} : \sum_{A:\mathcal{U}} \mathcal{U}^A \to \mathcal{U},$$

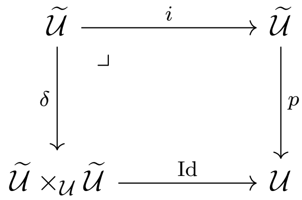
making the following diagram a pullback:

$$\begin{array}{ccc} \sum_{A:\mathcal{U}} \sum_{B:\mathcal{U}^A} \sum_{a:A} B(a) & \xrightarrow{\text{pair}} & \widetilde{\mathcal{U}} \\ {\scriptstyle \pi}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ \sum_{A:\mathcal{U}} \mathcal{U}^A & \xrightarrow{\hat{\Sigma}} & \mathcal{U} \end{array}$$

Definition 7.1.6. A natural model p supports extensional equality, if we are given the following maps:

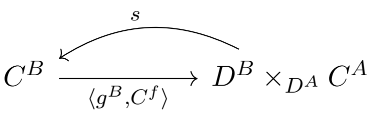
$$i : \widetilde{\mathcal{U}} \to \widetilde{\mathcal{U}}$$
$$\text{Id} : \widetilde{\mathcal{U}} \times_{\mathcal{U}} \widetilde{\mathcal{U}} \to \mathcal{U}$$

making the following diagram a pullback:

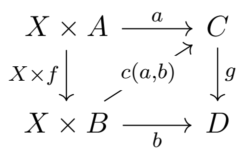


Natural models can also model intensional equality. However in order to do so, we first need to introduce the notion of left-lifting structure – i.e. a refined notion of left lifting property, where the filler is given in a functorial manner.

Definition 7.1.7. A left-lifting structure s for f with respect to g, is a section of the comparison map $\langle g^B, C^f\rangle : C^B \to D^B \times_{D^A} C^A$:



Equivalently, we could demand that we have a choice (natural in X) of diagonal fillers $c(a, b)$ in the following diagram:

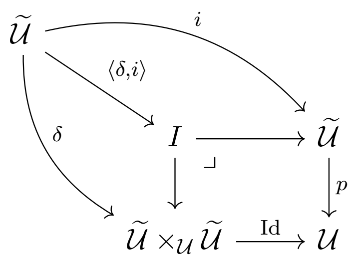


Definition 7.1.8. A natural model p models intensional equality, as soon as there exist the two following maps:

$$i : \widetilde{\mathcal{U}} \to \widetilde{\mathcal{U}}$$

$$\mathrm{Id} : \widetilde{\mathcal{U}} \times_{\mathcal{U}} \widetilde{\mathcal{U}} \to \mathcal{U}$$

making the following diagram commute:



We also require that the canonical map $\langle \delta, i\rangle$ has a left-lifting structure j with respect to p, when we consider them as maps over $\mathcal{U}$, $\langle \delta, i\rangle \pitchfork_j \mathcal{U}^* p$.

Type theory distinguishes two notions of equality, that is *definitional* and *propositional* equality. If we say two terms $t, t' : A$ are propositionally equal, we take it to mean, that

$\mathrm{Id}_A(t, t')$ is inhabited. These two notions do not necessarily coincide, unless we assume extensionality for Id types.

In the particular case of Σ and Π types, we have the following options:

$$\Sigma - \eta \frac{t : (\Sigma x : A)B(x)}{\langle \pi_1 t, \pi_2 t\rangle = t : (\Sigma x : A)B(x)} \qquad \frac{t : (\Sigma x : A)B(x)}{\eta_t : \mathrm{Id}(\langle \pi_1 t, \pi_2 t\rangle, t)} \text{ Prop } - \Sigma - \eta$$

$$\Pi - \eta \frac{t : (\Pi x : A)B(x)}{(\lambda x.tx) = t : (\Pi x : A)B(x)} \qquad \frac{t : (\Pi x : A)B(x)}{\eta_t : \mathrm{Id}((\lambda x.tx), t)} \text{ Prop } - \Pi - \eta$$

In what follows, we refine Definition 7.1.4 and Definition 7.1.5 to allow for Π and Σ types with propositional η-equality.

7.2. Refinement of Natural Models

We take inspiration from the definition of intensional Id-types (in [**7**]), where the author relaxes the constraint of having a pullback square, but instead demands that we posses a certain class of fillers.

We will model both Σ and Π types in a way that makes the β-rule (i.e. computational rule) valid judgmentally and the η-rule valid *propositionally*. To start with, we provide the rules for Σ-types with a split operator.

$$\Sigma\text{-intro} \frac{\Gamma \vdash A : \text{TYPES} \qquad \Gamma.x : A \vdash B(x) : \text{TYPES}}{\Gamma \vdash (\Sigma x : A)B(x) : \text{TYPES}}$$

$$\Sigma\text{-form} \frac{\Gamma \vdash a : A \qquad \Gamma.x : A \vdash b : B(x)}{\Gamma \vdash \langle a, b\rangle : (\Sigma x : A)B(x)}$$

$$\Sigma\text{-elim} \frac{\Gamma.z : (\Sigma x : A)B(x) \vdash E(z) : \mathcal{U} \qquad \Gamma.x : A, y : B(x) \vdash e(x,y) : E(\langle x, y\rangle) \qquad \Gamma \vdash u : (\Sigma x : A)B(x)}{\Gamma \vdash \mathrm{split}(e, u) : E(u)}$$

$$\Sigma\text{-comp} \frac{\Gamma \vdash \mathrm{split}(e, \langle u, v\rangle) : E(\langle u, v\rangle)}{\Gamma \vdash \mathrm{split}(e, \langle u, v\rangle) = e(u, v) : E(\langle u, v\rangle)}$$

TABLE 1. Rules for Σ with split

Given this version of Σ, we can define projections as:

$$\pi_1 t \equiv \mathrm{split}((x, y)x, t)$$

$$\pi_2 t \equiv \mathrm{split}((x, y)y, t)$$

Definition 7.2.1. We say that a natural model structure supports dependent sums with propositional η-equality, if we are given:

$$\text{pair} : \sum_{A:\mathcal{U}} \sum_{B:\mathcal{U}^A} \sum_{a:A} B(a) \to \widetilde{\mathcal{U}}$$
$$\hat{\Sigma} : \sum_{A:\mathcal{U}} \mathcal{U}^A \to \mathcal{U},$$

such that the following diagram commutes:

$$\begin{array}{ccc} \sum_{A:\mathcal{U}} \sum_{B:\mathcal{U}^A} \sum_{a:A} B(a) & \xrightarrow{\text{pair}} & \widetilde{\mathcal{U}} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p} \\ \sum_{A:\mathcal{U}} \mathcal{U}^A & \xrightarrow{\hat{\Sigma}} & \mathcal{U} \end{array} \tag{7.1}$$

and with a diagonal filler in the following diagram:

$$\begin{array}{ccc} \sum_{A:\mathcal{U}} \sum_{\mathcal{B}:\mathcal{U}^{\mathcal{A}}} \sum_{E:\mathcal{U}^{(\hat{\Sigma}A)B}} \sum_{e:\Pi_{x:A}\Pi_{y:B}E(\langle x,y\rangle)} \sum_{a:A} B(a) & \xrightarrow{\text{eval}} & \widetilde{\mathcal{U}} \\ \downarrow & \nearrow{\scriptstyle \text{split}} & \downarrow{\scriptstyle p} \\ \sum_{A:\mathcal{U}} \sum_{\mathcal{B}:\mathcal{U}^{\mathcal{A}}} \sum_{E:\mathcal{U}^{(\hat{\Sigma}A)B}} \sum_{e:\Pi_{x:A}\Pi_{y:B}E(\langle x,y\rangle)} (\hat{\Sigma}A)B & \xrightarrow{\text{eval}} & \mathcal{U} \end{array} \tag{7.2}$$

where the top eval works as follows: $\langle A, B, E, e, a, b\rangle \mapsto e(a, b)$, the bottom eval maps $\langle A, B, E, e, u\rangle \mapsto E(u)$ and the left vertical morphism maps $\langle A, B, E, e, a, b\rangle \mapsto \langle A, B, E, e, \langle a, b\rangle\rangle$.

Proposition 7.2.2. *A natural model with the structure in Definition 7.2.1 models the rules for Σ-types of Section 7.2.*

Proof.

Introduction Rule: We have two morphisms $A : \Gamma \to \mathcal{U}$, and $B : \Gamma.A \to \mathcal{U}$, allowing us to define $\langle A, B\rangle : \Gamma \to \sum_{A:\mathcal{U}} \mathcal{U}^A$. This morphism post-composed with $\hat{\Sigma}$ gets us the typing morphism required.

Formation Rule: We have $A : \Gamma \to \mathcal{U}$, $B : \Gamma.A \to \mathcal{U}$, $a : \Gamma \to \widetilde{\mathcal{U}}$, and $b : \Gamma.B \to \widetilde{\mathcal{U}}$, such that the necessary triangles commute. Then, it follows that we have $\langle A, B, a, b\rangle : \Gamma \to \sum_{A:\mathcal{U}} \sum_{B:\mathcal{U}^A} \sum_{a:A} B(a)$. Post-composed with pair we obtain a morphism, corresponding to the term $\langle a, b\rangle$ defined above of the required type.

Elimination Rule On the model side we have the following morphism (making certain typing triangles commute):

$$A : \Gamma \to \mathcal{U}$$
$$B : \Gamma.A \to \mathcal{U}$$
$$E : \Gamma.(\hat{\Sigma}A)B \to \mathcal{U}$$
$$e : \Gamma.A.B \to \widetilde{\mathcal{U}}$$
$$u : \Gamma \to \widetilde{\mathcal{U}}$$

Which implies that there is a tuple morphism:

$$\langle A, B, E, e, u\rangle : \Gamma \to \sum_{A:\mathcal{U}} \sum_{B:\mathcal{U}^A} \sum_{E:\mathcal{U}^{(\hat{\Sigma}A)B}} \sum_{e:\Pi_{x:A}\Pi_{y:B}E(\langle x,y\rangle)} (\hat{\Sigma}A)B.$$

We then interpret $\mathrm{split}(\mathrm{e}, \mathrm{u})$ as $\mathrm{split} \circ \langle A, B, E, e, u\rangle$:

$$\Gamma \xrightarrow{\langle A,B,E,e,u\rangle} \sum_{A:\mathcal{U}} \sum_{B:\mathcal{U}^A} \sum_{E:\mathcal{U}^{(\hat{\Sigma}A)B}} \sum_{e:\Pi_{x:A}\Pi_{y:B}E(\langle x,y\rangle)} (\hat{\Sigma}A)B \xrightarrow{\mathrm{split}} \widetilde{\mathcal{U}}$$

Computation Rule Since we have that split commutes with the evaluation maps in 7.2, the computation rule holds.

□

Remark 7.2.3. We can show that Definition 7.2.1 models propositional η-equality. Let

$$A : \Gamma \to \mathcal{U}$$
$$B : \Gamma.A \to \mathcal{U}$$

Then for any $a : A, b : B(a)$

$$\langle \pi_1\langle a, b\rangle, \pi_2\langle a, b\rangle\rangle = \langle a, b\rangle$$

from the computation rule. Thus

$$a : A, b : B(a) \vdash \mathrm{refl}(\langle a, b\rangle) : \mathrm{Id}(\langle a, b\rangle, \langle \pi_1\langle a, b\rangle, \pi_2\langle a, b\rangle\rangle).$$

Applying the Σ elimination rule, we obtain:

$$c : (\Sigma A)B \vdash \mathrm{split}((x)(y)\,\mathrm{refl}(\langle x, y\rangle), c) : \mathrm{Id}(c, \langle \pi_1 c, \pi_2 c\rangle).$$

Similar steps are taken in the model (where we assume the existence of Id types à la Awodey). First, observe that given $\Gamma.A.B$ we have a morphism $\langle a, b\rangle : \Gamma.A.B \to \widetilde{\mathcal{U}}$ (using the pair morphism from the definition before). Taking pair in the place of e in the above schema, we obtain $\mathrm{split}(\mathrm{pair}, \langle a, b\rangle)$. Further we see that the following diagram commutes:

$$\begin{array}{ccc} \Gamma.A.B & \xrightarrow{\mathrm{split}(\mathrm{pair},\langle a,b\rangle)} & \widetilde{\mathcal{U}} \\ \Big\downarrow{\scriptstyle\langle a,b\rangle} & & \Big\downarrow{\scriptstyle p} \\ \widetilde{\mathcal{U}} & \xrightarrow{\quad p \quad} & \mathcal{U} \end{array}$$

Thus we have $\mathrm{Id}(\mathrm{split}(\mathrm{pair}, \langle a, b\rangle), \langle a, b\rangle) : \Gamma.A.B \to \mathcal{U}$ and $\mathrm{refl}(\langle a, b\rangle) : \Gamma.A.B \to \widetilde{\mathcal{U}}$. Taking $\mathrm{refl}(\langle a, b\rangle)$ as the e in the above schema, and $\mathrm{Id}(\langle \pi_1 u, \pi_2 u\rangle, u)$ in place of E (and supposing the existence of $u : \Gamma \to \widetilde{\mathcal{U}}$), we get $\mathrm{split}(\mathrm{refl}(\langle x, y\rangle), u)$, of type $\mathrm{Id}(\langle \pi_1 u, \pi_2 u\rangle, u)$.

Next we define Π-types with funsplit operator.

$$\Pi\text{-intro}\ \frac{\Gamma \vdash A : \textsc{Types} \qquad \Gamma.x : A \vdash B(x) : \textsc{Types}}{\Gamma \vdash (\Pi x : A)B(x) : \textsc{Types}}$$

$$\Pi\text{-form}\ \frac{\Gamma.x : A \vdash b : B(x)}{\Gamma \vdash \lambda x.b : (\Pi x : A)B(x)}$$

$$\Pi\text{-elim}\ \frac{\Gamma \vdash f : (\Pi x : A)B(x) \qquad \Gamma.x : (\Pi x : A)B(x) \vdash C : \mathcal{U} \qquad \Gamma.x : A.y(x) : B(x) \vdash d(y) : C(\lambda x.y)}{\Gamma \vdash \mathrm{funsplit}(f, d) : C(f)}$$

$$\Pi\text{-comp}\ \frac{\Gamma.x : A \vdash b : B(x) \qquad \Gamma.x : (\Pi x : A)B(x) \vdash C : \mathcal{U} \qquad \Gamma.x : A.y(x) : B(x) \vdash d(y) : C(\lambda x.y)}{\Gamma \vdash \mathrm{funsplit}(\lambda x.b, d) = d(b) : C(\lambda x.b)}$$

TABLE 2. Rules for Π with funsplit

Definition 7.2.4. We say that a natural model structure supports dependent products with funsplit and propositional η-rule, if we are given:

$$\hat{\lambda} : \sum_{A:\mathcal{U}} \widetilde{\mathcal{U}}^A \to \widetilde{\mathcal{U}}$$
$$\hat{\Pi} : \sum_{A:\mathcal{U}} \mathcal{U}^A \to \mathcal{U},$$

such that the following diagram commutes:

$$\begin{array}{ccc} \sum_{A:\mathcal{U}} \widetilde{\mathcal{U}}^A & \xrightarrow{\hat{\lambda}} & \widetilde{\mathcal{U}} \\ \downarrow & & \downarrow{\scriptstyle p} \\ \sum_{A:\mathcal{U}} \mathcal{U}^A & \xrightarrow{\hat{\Pi}} & \mathcal{U} \end{array} \tag{7.3}$$

We demand existence of the diagonal filler in the following diagram:

$$\begin{array}{ccc} \sum_{A:\mathcal{U}} \sum_{B:\mathcal{U}^A} \sum_{C:\mathcal{U}^{(\hat{\Pi} A)B}} \sum_{b:\Pi_{x:A}B} (\Pi_{y:\Pi_{x:A}B} C(\hat{\lambda} y)) & \xrightarrow{\text{eval}} & \widetilde{\mathcal{U}} \\ \downarrow & \overset{\text{funsplit}}{\nearrow} & \downarrow{\scriptstyle p} \\ \sum_{A:\mathcal{U}} \sum_{B:\mathcal{U}^A} \sum_{C:\mathcal{U}^{(\hat{\Pi} A)B}} (\hat{\Pi} A)B & \xrightarrow{\text{eval}} & \mathcal{U} \end{array} \tag{7.4}$$

Proposition 7.2.5. *A natural model with the structure in Definition 7.2.4 models the rules for* Π*-types of table 2.*

Proof. The rules get transcribed in a similar way as they do in the case of Σ-types. □

Remark 7.2.6. Definition 7.2.1 models the propositional η-rule.

Since we have funsplit we can define apply in the following way:

$$\text{apply}(f, a) = \text{funsplit}(f, \lambda_x x(a))$$

Thanks to how funsplit is defined, regular β-equality holds for the above apply operator. Hence:

$$a : A, b : B \vdash \hat{\lambda}_x \,\text{apply}(\hat{\lambda}_a b, a) = \hat{\lambda}_x b$$

By Id-introduction:

$$a : A, b : B \vdash \text{refl}(\hat{\lambda}_x b) : \text{Id}(\hat{\lambda}_a \,\text{apply}(\hat{\lambda}_x b, a), \hat{\lambda}_x b)$$

Now applying, Π-elimination:

$$a : A, f : \Pi_A B \vdash \text{funsplit}(f, \lambda_b \,\text{refl}(\hat{\lambda}_x b(x))) : \text{Id}(\hat{\lambda}_a \,\text{apply}(f, a), f)$$

The proof is written in a type theoretic way, and a translation to diagrammatic form is tedious, but not difficult. The reader should pay attention to the fact that certain constructors have a hat (ex. $\hat{\lambda}$) above them, these constructors are in fact the constructors inside the model.

Remark 7.2.7. The apply version of the Π-types is the way the dependent functions are more commonly defined. This is the way the author of [**7**] chooses to model it. The application for Π types is given by composition in the category, rather than by additional structure. This makes it difficult to have a version of Π-types without the definitional η-rule.

Awodey proposes a weakening of Definition 7.1.4 (Corollary 2.5, of the same paper [**7**]), where the square in Diagram 7.1, is assumed to be a weak-pullback square and further we are given a section of the canonical map $\sum_{A:\mathcal{U}} \widetilde{\mathcal{U}}^A \to (\sum_{A:\mathcal{U}} \mathcal{U}^A) \times_{\mathcal{U}} \widetilde{\mathcal{U}}$. This additional structure gives the β-rule, but not the η-rule. In particular, assuming the η-rule as well, gives us that the square in Diagram 7.1 is a pullback.

Combining the definitions in this section with the unit and intensional equality types given by [**7**], we can summarize the results as follows:

Theorem 7.2.8. *Assume $\widehat{\mathcal{C}}$ has a natural model structure $p : \widetilde{\mathcal{U}} \to \mathcal{U}$ and additionally*

(1) models unit types as in Definition 7.1.3
(2) models Id *types as in Definition 7.1.8*
(3) models Σ *types as in Definition 7.2.1*
(4) models Π *types as in Definition 7.2.4*

Then $\widehat{\mathcal{C}}$ is a model of Martin-Löf type theory with 1, Σ, Π, Id, *with* Σ *and* Π *types satisfying propositional η-equality.*

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