Well-Ordering Principles and Π^1_1-Comprehension + Bar Induction

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Abstract

This thesis proves that the statement "Every set $\mathfrak{X}$ is contained in a countable-coded ω-model of Π^1_1-CA + Bar Induction" is equivalent to the statement, "For all sets $\mathfrak{X}$, if $\mathfrak{X}$ is well-ordered, then the construction $\mathrm{OT}(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}})$ is well-ordered." Here $\mathrm{OT}(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}})$ stands for the Veblen hierarchy up to Ω_ω relativized through the addition of epsilon numbers $\mathfrak{E}_{\mathfrak{X}}$ above Ω_ω.

To my parents.

"A proof is a proof, and when you have a good proof it's because it's proven"

-Jean Chrétien

Acknowledgements

Thanks, first and foremost, to my advisor Michael Rathjen. It has been my immense pleasure to sit at the feet of the master and learn the ways of proof theory. Secondly, thanks to my parents for their unending support during my past decade of academia. I couldn't have walked this road without them. Finally, thanks to Samantha Penner, for everything outside of the Ivory Tower.

Contents

Chapter 1

Introduction

1.1 An Overview of Well-Ordering Principles

1.1.1 The Well-Ordering Principels WOP(f)

This thesis is part of an ongoing investigation into **well-ordering principles**; statements of the form

$$\mathbf{WOP}(f) : \forall \mathfrak{X}[\mathrm{WO}(\mathfrak{X}) \rightarrow \mathrm{WO}(f(\mathfrak{X}))].$$

Here, f is a proof-theoretic function which maps ordinals to ordinals, and $\mathrm{WO}(\mathfrak{X})$ stands for "$\mathfrak{X}$ is a well-ordering." There are now several examples in the literature, proving an equivalence between certain well-ordering principles and the various theories of reverse mathematics, modulo a weak base theory such as $\mathbf{RCA}_0$. The first such result is due to Girard [4].

Theorem 1.1.1 *The following are equivalent over* $\boldsymbol{RCA}_0$ *:*

1. $\boldsymbol{ACA}_0$.
2. $\boldsymbol{WOP}(2^{\mathfrak{X}})$

More recently, Marcone and Montalbán proved the following two results using the methods of computability theory: [5]

Theorem 1.1.2 *The following are equivalent over* $\boldsymbol{RCA}_0$:

1. $\boldsymbol{ACA}_0^+$.
2. $\boldsymbol{WOP}(\varepsilon_{\mathfrak{X}})$

Here, $\boldsymbol{ACA}_0^+$ is the system $\boldsymbol{ACA}_0$ plus the axiom:

$$\forall X \exists Y [Y_0 = X \wedge \forall n (Y)_{n+1} = TJ((Y)_n)]$$

where $TJ(U)$ is the Turing jump of U, as laid out in [11].

Theorem 1.1.3 *The following are equivalent over* $\boldsymbol{RCA}_0$:

1. $\boldsymbol{ATR}_0$.
2. $\boldsymbol{WOP}(\varphi\mathfrak{X}0)$

This latter result was based off of unpublished work by Friedman.

Based off of preprint drafts of this work, new demonstrations for these two theorems were found using the techniques of proof theory. Theorem 1.1.2 was proven by Afshari and Rathjen [1], while Theorem 1.1.3 was proven by Rathjen and Weiermann [10].

1.1.2 ω-models

In the development of these proof-theoretical techniques, Rathjen observed a similar equivalence between well-ordering proofs and statements of the form

Every set $\mathfrak{X}$ is contained in a countable-coded ω-model of T

where T is one of the systems of reverse mathematics [7].

Definition 1.1.4 *Let T be a theory in the language $\mathcal{L}_2$ of second-order arithmetic. A* ***countable-coded** ω**-model** of T is a set $W \subseteq \mathbb{N}$ which encodes the $\mathcal{L}_2$-model*

$$\mathbb{M} = (\mathbb{N}, \mathcal{S}, +, \cdot, 0, 1, <)$$

with $\mathcal{S} = \{(W)_n | n \in \mathbb{N}\}$ such that $\mathbb{M} \models T$. Here, $(W)_n = \{m | \langle n, m\rangle \in W\}$.

Note that an ω-model can be encoded in $\mathbf{RCA}_0$ [11].

Rathjen reformulated the results of Marcone and Montalbán as follows [7].

Theorem 1.1.5 *The following are equivalent over* $\boldsymbol{RCA}_0$:

1. $\boldsymbol{WOP}(\varepsilon_{\mathfrak{X}})$.
2. *Every set is contained in a countable-coded ω-model of* $\boldsymbol{ACA}_0$.

Theorem 1.1.6 *The following are equivalent over* $\boldsymbol{RCA}_0$:

1. $\boldsymbol{WOP}(\varphi\mathfrak{X}0)$.
2. *Every set is contained in a countable-coded ω-model of* Δ^1_1-$\boldsymbol{CA}_0$.

In the same paper, he proved the following result.

Theorem 1.1.7 *The following are equivalent over* $\boldsymbol{RCA}_0$:

1. $\boldsymbol{WOP}(\Gamma_{\mathfrak{X}})$.
2. *Every set is contained in a countable-coded ω-model of* $\boldsymbol{ATR}_0$.

In a separate paper, Rathjen and Vizcaíno proved a related result [8].

Theorem 1.1.8 *The following are equivalent over* $\boldsymbol{RCA}_0$:

1. ***WOP***$(\vartheta_{\mathfrak{X}})$.
2. *Every set is contained in a countable-coded* ω*-model of* $\boldsymbol{RCA}_0$+***Bar Induction***.

In adopting the ω-model approach, there seems to be a sharper parallel between the proof-theoretic functions found in the well-ordering principles and the systems being modelled. The case of $\mathbf{ACA}_0$ and $\varepsilon_{\mathfrak{X}}$, for example, closely resembles Gentzen's original ordinal bound for Peano Arithmetic [3]. In this thesis, we shall extend this parallel by proving the following result:

Theorem 1.1.9 *The following are equivalent over* $\boldsymbol{RCA}_0$:

1. ***WOP***$(\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}}))$.
2. *Every set is contained in a countable-coded* ω*-model of* Π^1_1***-CA + BI***.

1.1.3 The Theory Π^1_1-CA + BI

We shall now introduce the system Π^1_1**-CA + BI**. We formulate this system in language of second-order arithmetic, $\mathcal{L}_2$. Second order variables shall be denoted by capital letters, $U, V, W \dots$ while first order variables shall be denote by lower case letters $a, b, c \dots$. The language also contains the constant symbol 0, a symbol for every primitive recursive function, and the relations $=$ and $\in$ denoting first-sort equality and set membership respectively. The language also includes the standard logical connectives, $\wedge, \vee, \rightarrow, \neg$, as well as first-order quantifiers $\forall x, \exists y$ and second-order quantifiers $\forall X, \exists Y$.

Definition 1.1.10 *The system **ACA**$_0$ contains all the axioms of elementary number theory, defining $0,'$ (successorship), the equations defining primitive recursive functions, the induction axiom*

$$\forall X[0 \in X \wedge \forall x(x \in X \rightarrow x' \in X) \rightarrow \forall x(x \in X)]$$

*and the **arithmetical comprehension** schema*

$$\exists X \forall y[y \in X \leftrightarrow F(y)]$$

where $F(a)$ is an arithmetical formula (i.e. it contains no set quantifiers) and X is free in $F(a)$.

Definition 1.1.11 *Π^1_1-**CA** is a system which includes all the axioms of **ACA**$_0$ but has the Π^1_1-**comprehension** schema*

$$\exists X \forall y[y \in X \leftrightarrow F(y)]$$

where $F(a)$ is Π^1_1-formula (i.e. $F(a)$ is equivalent to some formula $\forall \overrightarrow{Y} G(a, \overrightarrow{Y})$, where $G(,a, \overrightarrow{U})$ is arithmetic, $\overrightarrow{Y} = \{Y_0, Y_1, \ldots Y_k\}$ for some finite k, and $\forall \overrightarrow{Y}$ is shorthand for $\forall Y_0 \forall Y_1 \ldots \forall Y_k$.).

Suppose $\prec$ is a two-place relation symbol, and $F(a)$ is an arbitrary $\mathcal{L}_2$-formula. We define:

$$Prog(\prec, F) := \forall x[\forall y(y \prec x \rightarrow F(y)) \rightarrow F(x)] \text{ (progressiveness)}$$

$$TI(\prec, F) := Prog(\prec, F) \rightarrow \forall x F(x) \text{ (transfinite induction)}$$

$$WF(\prec) := \forall X TI(\prec, X) \text{ (well-foundedness)}$$

Definition 1.1.12 ***Bar Induction** (denoted **BI** for short) is the axiom schema consisting of all formula with the form*

$$WF(\prec) \rightarrow TI(\prec, F)$$

where $\prec$ is an arithmetical relation and $F(a)$ is an arbitrary $\mathcal{L}_2$-formula.

Definition 1.1.13 *Π^1_1-**CA + BI** is the system Π^1_1-**CA** plus the Bar Induction schema.*

1.1.4 An outline of this thesis

In Chapter 2, we construct the relativized ordinal representation system, $\mathrm{OT}(\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}}))$, which consists of the Veblen hierarchy up to Ω_ω augmented by a set of epsilon numbers, $\mathfrak{E}_{\mathfrak{X}}$, above Ω_ω. We then carry out a well-ordering proof for that system predicated on the existence of an ω-model of Π^1_1**-CA + BI** that contains the set $\mathfrak{X}$. Chapter 3 lays preliminary groundwork for proving the existence of an ω-model. It introduces the concept of deduction chains, and the deduction tree D_Q, relative to an arbitrary set Q, and observe that if this tree is ill-founded then there is an infinite branch which yields an ω-model of Π^1_1**-CA + BI**. The second half of the chapter pertains to majorization and fundamental functions of ordinal terms in $\mathrm{OT}(\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}}))$. This is an essential, if technical, component of tracking ordinal heights during the next chapter. Chapter 4 is a proof that the deduction tree D_Q cannot be well-founded and thus an ω-model must exist which contains the set Q. We do this by embedding the deduction tree into a ramified sequent calculus, which yields a proof of the empty sequent. By leveraging cut-elimination we show that such a proof is impossible, and thus D_Q cannot be well-founded.

Chapter 2

A Well-Ordering Proof for $\mathbf{OT}(\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}}))$

In this chapter we shall construct our ordinal representation system $\mathrm{OT}(\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}})$. The first two parts of this chapter take a set-theoretic approach, working within **ZFC**. We conclude section 2.2 with an equivalent formal term structure that can be encoded in $\mathbf{RCA}_0$. The final part of this chapter then presents a well-ordering proof for $\mathrm{OT}(\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}}))$ from the axiom

"Every set is contained in a countable-coded ω-model of $\Pi^1_1 - \mathbf{CA}_0$.

This chapter closely follows the construction of the Veblen hierarchy as presented in [6].

2.1 The functions φ_α

Due to the complexity of $\mathrm{OT}(\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}}))$, we shall construct it over two sections. This section lays the groundwork, detailing the properties of the φ_α functions. These φ_α functions are not sufficient to create a primitive recursive representation system on their own, however, as there are strongly critical cases where $\varphi_\alpha 0 = \alpha$. In the next section we shall circumvent this difficulty by introducing the ψ_k functions, which will give these strongly critical cases a normal form representation.

Definition 2.1.1 *The* ***additive principle ordinals*** *are those ordinals $\alpha > 0$ such that $(\forall\eta < \alpha)\eta + \alpha = \eta$. They are enumerated by the function $\alpha \mapsto \omega^\alpha$. Note that this function is strictly increasing ($\alpha < \beta \implies \omega^\alpha < \omega^\beta$), and that $\omega^\lambda = sup\{\omega^\eta | \eta < \lambda\}$, where λ is a limit ordinal.*

An ordinal α is called an ε-number if $\alpha = \omega^\alpha$.

Definition 2.1.2 *The class of* ***critical ordinals of level*** *α, denoted $Cr(\alpha)$, is inductively defined as follows:*

1. *$Cr(0)$ is the class of additive principle ordinals.*
2. *φ_α is the function enumerating $Cr(\alpha)$.*
3. *$Cr(\alpha + 1) = \{\rho | \varphi_\alpha(\rho) = \rho\}$.*
4. *$Cr(\lambda) = \bigcap\{Cr(\xi) | \xi < \alpha\}$.*

Observe that each $Cr(\alpha)$ is an unbounded class of ordinals, and that φ_α is a strictly increasing function with $\varphi_\alpha\lambda = sup\{\varphi_\alpha\eta | \eta < \lambda\}$. Henceforth, we will write $\varphi\alpha\beta$ to denote $\varphi_\alpha\beta$.

Moreover, observe that $Cr(1)$ is the class of ε-numbers.

Lemma 2.1.3 (See [6] Lemma 9.3) *Suppose $\alpha = \varphi\gamma\delta$ and $\beta = \varphi\xi\eta$. Then $\alpha = \beta$ if and only if one of the following holds:*

1. *$\gamma < \xi$ and $\delta = \varphi\xi\eta$.*

2. $\gamma = \xi$ *and* $\delta = \eta$.

3. $\gamma > \xi$ *and* $\eta = \varphi\gamma\delta$.

Proof

For the first case, suppose $\gamma < \xi$. We know that $\beta \in Cr(\xi)$, and hence

$$\varphi\gamma(\varphi\xi\eta) = \varphi\xi\eta = \beta.$$

Thus $\alpha = \varphi\gamma\delta = \beta$ if and only if $\delta = \beta$. The second case is trivial, and the third case follows from the same argument as the first.

□

Lemma 2.1.4 (See [6] Lemma 9.3) *Suppose* $\alpha = \varphi\gamma\delta$ *and* $\beta = \varphi\xi\eta$. *Then* $\alpha < \beta$ *if and only if one of the following holds:*

1. $\gamma < \xi$ *and* $\delta < \varphi\xi\eta$.

2. $\gamma = \xi$ *and* $\delta < \eta$.

3. $\gamma > \xi$ *and* $\varphi\gamma\delta < \eta$.

Proof

For the first case, suppose $\gamma < \xi$. Since φ_γ is strictly increasing, we have

$$\alpha = \varphi\gamma\delta < \varphi\gamma(\varphi_\xi\eta) = \beta$$

if and only if $\delta < \varphi\xi\eta$.

The second case follows immediately from the face that φ_γ is strictly increasing.

For the third case, assume $\xi < \gamma$. Then $\alpha = \varphi\gamma\delta = \varphi\xi(\varphi\gamma\delta)$. Since φ_ξ is strictly increasing, it follows that

$$\alpha = \varphi_\xi(\varphi\gamma\delta) < \varphi_\xi\eta = \beta \iff \varphi\gamma\delta < \eta.$$

□

Lemma 2.1.5 (See [6] Lemma 9.4) $\varphi\alpha 0 < \varphi\beta 0 \iff \alpha < \beta$.

Proof

We know that $0 < \varphi\beta 0$ since all additive principle ordinals are non-zero. Hence, by (1) the preceding lemma, if $\alpha < \beta$ then $\varphi\alpha 0 < \varphi\beta 0$ and vice versa.

□

Lemma 2.1.6 (See [6] Lemma 9.4) $\alpha, \beta \leq \varphi\alpha\beta$.

Proof

We start by proving $\alpha \leq \varphi\alpha 0$, and by extension $\alpha \leq \varphi\alpha\beta$ since by lemma 2.1.4 (2) $\varphi\alpha 0 \leq \varphi\alpha\beta$. We proceed by transfinite induction on α.

In the base case, we know $0 < \varphi 00$. Now suppose $\alpha = \gamma + 1$, and we have $\gamma \leq \varphi\gamma 0$. Then we know that

$$\gamma + 1 \leq (\varphi\gamma 0) + 1 < \varphi(\gamma + 1)0$$

since $\varphi(\beta + 1)0$ is an additive principle number and hence cannot be a successor ordinal.

Now, suppose that $\alpha = \lambda$, a limit ordinal, and for all $\xi < \lambda$ we have

$$\xi \leq \varphi\xi 0 < \varphi\xi(\varphi\lambda 0) = \varphi\lambda 0.$$

It follows that $\lambda \leq \varphi\lambda 0$, and hence by transfinite induction we have $\alpha \leq \varphi\alpha 0$.

We now shall prove $\beta \leq \varphi\alpha\beta$ using transfinite induction on β. We have already proven the base case. Thus, suppose $\beta = \gamma + 1$ and that $\gamma \leq \varphi\alpha\gamma$. Then we know that

$$\gamma + 1 \leq (\varphi\alpha\gamma) + 1 < \varphi\alpha(\gamma + 1)$$

since $\varphi\alpha(\gamma + 1)$ is an additive principle number.

Finally, suppose $\beta = \lambda$, a limit ordinal, and that we have $\xi \leq \varphi\alpha\gamma$ for all $\xi < \lambda$. Then

$$\xi \leq \varphi\alpha\xi < \varphi\alpha(\varphi\alpha\lambda) = \varphi\alpha\lambda.$$

Consequently, we have $\lambda \leq \varphi\alpha\lambda$, and thus by transfinite induction we get $\beta \leq \varphi\alpha\beta$. $\square$

Lemma 2.1.7 (See [6] Lemma 9.5) *For every* $\rho \in Cr(0)$ *there exist unique ordinals* β, γ *such that* $\gamma < \rho$ *and* $\rho = \varphi\beta\gamma$.

Proof

Suppose $\alpha \in Cr(0)$. Then $0 < \alpha$ and there is γ such that $\gamma \leq \alpha = \varphi 0\gamma$. If $\gamma < \alpha$, then we are done. Otherwise, there is a least β such that $\alpha < \varphi\beta\alpha$. In other words, for all $\beta_0 < \beta$ we know that α is a fixed point of $\varphi\beta_0$. Hence, there is γ such that $\gamma \neq \alpha = \varphi\beta\gamma$, and by Lemma 2.1.6 we know $\gamma < \alpha$.

To show uniqueness, suppose $\alpha = \varphi\beta_0\gamma_0 = \varphi\beta_1\gamma_1$ and $\gamma_0, \gamma_1 < \alpha$. We shall assume for a contradiction that $\beta_0 \neq \beta_1$.

If $\beta_0 < \beta_1$, then $\gamma_0 < \alpha = \varphi\beta_1\gamma_1$, and thus by 2.1.4 (1) $\varphi\beta_0\gamma_0 < \varphi\beta_1\gamma_1$, contradicting our hypothesis. Similarly, if $\beta_1 < \beta_0$, then $\gamma_1 < \alpha = \varphi\beta_0\gamma_0$, and we find by 2.1.4 (3) that

$\varphi\beta_1\gamma_1 < \varphi\beta_0\gamma_0$, which is, again, a contradiction. Thus, $\beta_0 = \beta_1$. But then, by 2.1.4 (2) we know that $\gamma_0 = \gamma_1$, hence uniqueness.

□

Definition 2.1.8 *1. $\alpha =_{nf} \varphi\beta\gamma :\Longleftrightarrow \alpha = \varphi\beta\gamma$ and $\beta, \gamma < \alpha$.*

2. $\alpha =_{nf} \beta + \gamma :\Longleftrightarrow \alpha = \beta + \gamma$, with $\beta \in Cr(0)$, and $\gamma = \gamma_1 + \ldots + \gamma_n$ with $\gamma_i \in Cr(0)$ for all $i \leq n$ and $\beta \geq \gamma_1 \geq \ldots \geq \gamma_n$.

These normal forms are unique, due to the preceding lemma. It is important to note that this definition alone does not account for all critical ordinals. In particular, ordinals of the form $\varphi\alpha 0$ do not yet have an ordinal form representation.

Definition 2.1.9 *We define the class of* ***strongly critical ordinals*** *as*

$$SC := \{\alpha | \varphi\alpha 0 = \alpha\}.$$

2.2 OT($\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}})$) and the Functions ψ_k

In this section, we shall construct the full ordinal representation system OT($\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}})$). At the end of this section, give an equivalent presentation of OT($\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}})$) as a primitive recursive term system, which can be encoded into the system

$$\mathbf{RCA}_0 + \forall X \exists Y (X \in Y \wedge Y \text{ is an } \omega\text{-model of } \Pi^1_1\text{-}\mathbf{CA}_0 + \mathbf{BI}).$$

In this chapter, the notations $[\alpha, \beta]$ and (α, β) will stand for the inclusive and exclusive intervals from α to β respectively. We also establish the following conventions:

$$A < \alpha := (\forall \eta \in A)(\eta < \alpha)$$

$$\alpha < A := (\exists \eta \in A)(\alpha < \eta.)$$

The notations $A \leq \alpha$ and $\alpha \leq A$ are as above, replacing $<$ with $\leq$.

Definition 2.2.1 *Let* $\Omega_0 = 0$. *For* $0 < k \leq \omega$, *let* $\Omega_k = \aleph_k$. *It is worth noting that for regular cardinals* $\lambda > \omega$, *if* $\beta < \lambda$, *then* $\varphi\beta\lambda = \lambda$.

Fix a countable-coded set X *with a well-ordering* $<_X$. *For all* $u \in X$, $\{\mathfrak{E}_u\}_{u \in X}$ *enumerate the first* X ε*-numbers above* Ω_ω. *Thus, the following hold for all* $u, v \in X$*:*

1. $\Omega_\omega < \mathfrak{E}_u$
2. *If* $u <_X v$ *then* $\mathfrak{E}_u < \mathfrak{E}_v$
3. $\varphi 0 \mathfrak{E}_u = \mathfrak{E}_u$.

Definition 2.2.2 *For* $k < \omega$ *the sets* $C_k(\alpha)$ *and the ordinals* $\psi_k\alpha$ *are defined by recursion on* α, *with* $C_k(\alpha)$ *constructed inductively as follows:*

1. $\Omega_m \in C_k(\alpha)$ for all $m \leq \omega$.
2. $\mathfrak{E}_u \in C_k(\alpha)$ for all $u \in X$.
3. $[0, \Omega_k] \subseteq C_k(\alpha)$.
4. If $\xi, \eta \in C_k(\alpha)$, then $\xi + \eta \in C_k(\alpha)$.
5. If $\eta \in C_k(\alpha)$ then $\omega^\eta := \varphi 0 \eta \in C_k(\alpha)$.
6. If $\xi, \eta \in C_k(\alpha) \cap \Omega_\omega$, then $\varphi\xi\eta \in C_k(\alpha)$.
7. If $\xi < \alpha$ and $\xi \in C_k(\alpha)$, then $\psi_n\xi \in C_k(\alpha)$ for all $n < \omega$.
8. $\psi_k\alpha = \min\{\eta | \eta \notin C_k(\alpha)\}$.

Note that in (6), the φ function is defined purely over ordinals less than Ω_ω. This is because the φ function enumerates epsilon numbers, but epsilon numbers above Ω_ω are enumerated via $\mathfrak{E}_u$ for $u \in X$.

Definition 2.2.3 *1. If $\Omega_k \leq \alpha < \Omega_{k+1}$, then $S\alpha = \Omega_k$. We call $S\alpha$ the **level** of α. Similarly, let $\alpha^+ = \Omega_{k+1}$.*

Lemma 2.2.4 (See [6] Lemma 10.3) *1. If $\alpha \leq \beta$ then $C_k(\alpha) \subseteq C_k(\beta)$.*

2. $\Omega_k < \psi_k\alpha < \Omega_{k+1}$.

3. $\psi_k\alpha \in SC$

4. $\psi_k\alpha \neq \Omega_j$ or $\mathfrak{E}_u$ for any $j \leq \omega$ or $u \in X$.

5. $\psi_k\alpha = C_k(\alpha) \cap \Omega_{k+1}$.

Proof

(1) is proven by transfinite induction on α. The cases where α has the form $\xi + \eta$ or $\varphi\xi\eta$ reduce to the induction hypothesis. The critical case is when $\alpha = \psi_k(\gamma)$. If $\psi_k(\gamma) \in C_k(\alpha)$, then we have $\gamma < \alpha \leq \beta$. So $\psi_k(\gamma) \in C_k(\beta)$.

For (2), observe that since $[0, \Omega_k] \subseteq C_k(\alpha)$, clearly $\Omega_k < \psi_k\alpha$.

To show $\psi_k\alpha < \Omega_{k+1}$, we begin by constructing the sets $Cr^i_k(\alpha)$, as follows:

(i) $Cr^0_k(\alpha) = [0, \Omega_k] \cup \{\Omega_j\}_{j \leq \omega} \cup \{\mathfrak{E}_u\}_{u \in X}$.

(ii) Suppose $\xi, \eta \in Cr^i_k(\alpha)$. Then $\xi + \eta \in Cr^{i+1}_k(\alpha)$ and $\varphi\xi\eta \in Cr^{i+1}_k(\alpha)$.

(iii) If $\xi \in Cr^i_k(\alpha)$ and $\xi < \alpha$ then $\psi_k\xi \in Cr^{i+1}_k(\alpha)$.

Clearly, $\bigcup_{i<\omega} Cr^i_k(\alpha) = C_k(\alpha)$, and $|\Omega_k| = |Cr^0_k(\alpha)|$.

Suppose, then, that we have $|\Omega_k| = |Cr^i_k(\alpha)|$. To generate $Cr^{i+1}_k(\alpha)$, we take the closure of $Cr^i_k(\alpha)$ under a single iteration of the φ, $+$ and ψ_m functions. Hence $|Cr^{i+1}_k(\alpha)| = |Cr^i_k(\alpha)|$, and thus $|\bigcup_{i<\omega} Cr^i_k(\alpha)| = |Cr_k(\alpha)| < \Omega_{k+1}$ Therefore, by definition, $\psi_k\alpha < \Omega_{k+1}$.

To prove (3), first we observe that $\psi_k(\alpha)$ is an additive principle number. Otherwise, $\psi_k(\alpha)$ would be the sum of two ordinals in $C_k(\alpha)$, and thus we would have $\psi_k(\alpha) \in C_k(\alpha)$. Hence $\psi_k\alpha = \varphi\xi\gamma$, with $\xi \leq \psi_k\alpha$ and $\gamma < \psi_k\alpha$. Clearly we cannot have both $\xi, \gamma < \psi_k\alpha$ or we would have $\psi_k\alpha \in C_k(\alpha)$. Since $\gamma < \psi_k\alpha$, we must have $\xi = \psi_k\alpha$, and since $\psi_k\alpha \leq \varphi(\psi_k\alpha)0$, it follows that $\psi_k\alpha = \varphi(\psi_k\alpha)0$. Hence, $\alpha \in SC$.

(4) is immediately obvious from the construction of $C_k(\alpha)$.

(5) follows from (2) and the definition of $\psi_k(\alpha)$.

□

Lemma 2.2.5 (See [6] Lemma 10.4) *Let $\alpha \in C_k(\alpha)$ and $\beta \in C_l(\beta)$.*

1. $\psi_k\alpha = \psi_l\beta$ *if and only if* $k = l$ *and* $\alpha = \beta$.

2. $\psi_k\alpha < \psi_l\beta$ *if and only if* $k < l$ *or* $k = l \wedge \alpha < \beta$.

Proof

1. By part (2) of the preceding theorem, obviously $\psi_k\alpha \neq \psi_l\beta$ if $k \neq l$. Suppose there are α, β such that $\psi_k\alpha = \psi_k\beta$. Without loss of generality, suppose $\alpha < \beta$. Then by part (1) of the preceding lemma $\alpha \in C_k(\beta)$, and thus $\psi_k(\alpha) \in C_k(\beta)$. Thus $\psi_k\alpha \neq \psi_k\beta$.

2. This is a direct corollary of the result we just proved, combined with part (5) of the preceding lemma.

□

Lemma 2.2.6 (See [2] Lemma 2.7) *If* $\alpha < \beta$ *and there is no* $\delta \in C_k(\alpha)$ *such that* $\alpha \leq \delta < \beta$, *then* $\gamma \in C_k(\beta)$ *implies* $\gamma \in C_k(\alpha)$.

Proof

This is proved by induction on the construction of $\gamma \in C_k(\beta)$.

1. If $\gamma = 0$ or Ω_m or $\mathfrak{E}_u$, for any $m < \omega$, $u \in X$, then $\gamma \in C_k(\alpha)$ by definition.
2. If γ has the normal form $\phi\gamma_0\gamma_1$ or $\gamma_0+\gamma_1$, then $\gamma_0, \gamma_1 \in C_k(\beta)$ and thus by induction hypothesis, $\gamma_0, \gamma_1 \in C_k(\alpha)$. Thus, it follows that $\gamma \in C_k(\alpha)$.
3. Suppose $\gamma = \psi_k(\gamma_0)$. Then $\gamma_0 \in C_k(\beta)$ and $\gamma_0 < \beta$. By induction hypothesis, it follows that $\gamma_0 \in C_k(\alpha)$. By assumption, it is not the case that $\alpha \leq \gamma_0 < \beta$, so $\gamma_0 < \alpha$. Therefore, $\psi_k(\gamma_0) \in C_k(\alpha)$.

□

Lemma 2.2.7 (See [2] Lemma 2.8) *If* $\beta = \min\{\xi|\alpha \leq \xi \in C_k(\alpha)\}$ *then* $C_k(\alpha) = C_k(\beta)$, *and thus* $\psi_k(\alpha) = \psi_k(\beta)$ *with* $\beta \in C_k(\beta)$.

Proof

Since $\alpha \leq \beta$, clearly $C_k(\alpha) \subseteq C_k(\beta)$ by part (1) of Lemma 2.2.4. By the preceding Lemma, it follows that $C_k(\beta) \subseteq C_k(\alpha)$. We have $\beta \in C_k(\beta)$ by our initial assumption.

□

Definition 2.2.8 $\alpha =_{nf} \psi_k\beta :\iff (\alpha = \psi_k\beta \wedge \beta \in C_k(\beta))$.

Definition 2.2.9 *The set of ordinal terms OT*$(\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{x}}}))$ *and the complexity* $\mathrm{G}\alpha < \omega$ *for* $\alpha \in$ *OT*$(\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{x}}}))$ *are defined inductively as follows:*

1. $0, \Omega_k, \mathfrak{E}_u \in$ *OT*$(\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{x}}}))$ *and* $\mathrm{G}0 = \mathrm{G}\Omega_k = \mathrm{G}\mathfrak{E}_u = 0$ *for* $k \leq \omega$.
2. *If* $\alpha =_{nf} \alpha_0 + \alpha_1 \wedge \alpha_0, \alpha_1 \in$ *OT*$(\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{x}}}))$ *then* $\alpha \in$ *OT*$(\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{x}}}))$ *and* $\mathrm{G}\alpha = \max\{\mathrm{G}\alpha_0\mathrm{G}\alpha_1\} + 1$.
3. *If* $\alpha =_{nf} \varphi\beta\gamma \wedge \beta, \gamma \in$ *OT*$(\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{x}}}))$ *then* $\alpha \in$ *OT*$(\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{x}}}))$ *and* $\mathrm{G}\alpha = \max\{\mathrm{G}\beta, \mathrm{G}\gamma\} + 1$.
4. *If* $\alpha =_{nf} \psi_k\beta \wedge \beta \in$ *OT*$(\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{x}}}))$ *then* $\alpha \in$ *OT*$(\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{x}}}))$ *and* $\mathrm{G}\alpha = \mathrm{G}\beta + 1$.

Using Lemma 2.1.7 and 2.2.7, we can see that every ordinal term $\alpha \in \mathrm{OT}(\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{x}}}))$ has a unique normal form, and thus $\mathrm{G}(\alpha)$ is well-defined.

Definition 2.2.10 *The set of ordinals* $K_k\alpha$ *for* $\alpha \in$ *OT*$(\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{x}}}))$ *are defined inductively as follows:*

1. $K_k0 = \emptyset$.
2. *If* $\alpha =_{nf} \alpha_1 + \ldots + \alpha_m$ *then* $K_k\alpha = \bigcup\{K_k\alpha_j | 1 \leq j \leq n\}$.
3. *If* $\alpha =_{nf} \varphi\beta\gamma$ *then* $K_k\alpha = K_k\beta \cup K_k\alpha$.

4. *If* $\alpha =_{nf} \psi_l\beta$ *then,*

$$K_k\alpha = \begin{cases} \emptyset & \textit{if } l < k \\ \{\beta\} \cup K_k\beta & \textit{if } k \leq l. \end{cases}$$

Lemma 2.2.11 (See [6] Lemma 10.9) *If* $\alpha \in$ *OT*$(\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}}))$ *then* $\alpha \in C_k(\beta)$ *if and only if* $K_k\alpha < \beta$.

Proof

Proved by induction on the construction of α in $C_m(\beta)$. The critical case is when $\alpha =_{nf} \psi_m\alpha_0$ and $k \leq m$.

First, suppose $\psi_m\alpha_0 \in C_k(\beta)$. Then $\alpha_0 \in C_k(\beta)$ and $\alpha_0 < \beta$. By induction hypothesis, $\{\alpha_0\} \cup K_k\alpha_0 < \beta$.

Now suppose $K_k\psi_m\alpha_0 < \beta$. Then $\{\alpha_0\} \cup K_k\alpha_0 < \beta$. By induction hypothesis $K_k\alpha_0 < \beta$ implies $\alpha_0 \in C_k(\beta)$ and since $\alpha_0 < \beta$ we obtain $\psi_k\alpha_0 \in C_k(\beta)$. $\square$

Definition 2.2.12 *We define the* $e(\alpha)$ *inductively as follows:*

1. $e(0) = 0$
2. $e(\Omega_k) = \{\Omega_k\}$
3. $e(\mathfrak{E}_u) = 1$
4. $e(\alpha) = 0$ *where* $\alpha =_{nf} \alpha_0 + \alpha_1$.
5. $e(\alpha) = \{\beta\}$ *where* $\alpha =_{nf} \varphi\beta\gamma$.
6. $e(\alpha) = \{\alpha\}$ *where* $\alpha =_{nf} \psi_k\beta$.

Lemma 2.2.13 (See [6] Lemma 10.10) *If* $\alpha = \varphi\beta\gamma$ *then* $\alpha =_{nf} \varphi\beta\gamma$ *if and only if* $e(\gamma) \leq \beta \wedge (\beta \notin SC \vee \gamma > 0)$.

Proof

Assume that $\alpha =_{nf} \varphi\beta\gamma$. Then $\gamma, \beta < \alpha$ by definition. Moreover, if $\beta \in SC$ and $\gamma = 0$ then $\beta =_{nf} \psi_k\beta_0 = \varphi\beta 0 = \alpha$, violating normal form. Hence $\beta \notin SC \vee \gamma \neq 0$. Proceed by induction on Gγ.

For the base case, if $\gamma = 0$, then the assertion is trivial. If $\gamma = \mathfrak{E}_u$, then $0 < \{1\} = e(\mathfrak{E}_u)$ so α cannot have the normal form $\phi 0\mathfrak{E}_u$. Now suppose $\gamma = \Omega_k \in \{\Omega_k\} = e(\gamma)$, for $k > 0$. Then $\Omega_k > \omega$. Thus if $\beta < e(\Omega_k)$, then $\varphi\beta\gamma = \gamma$, violating the normal form of α. Hence $e(\gamma) \leq \beta$.

Now suppose the assertion holds for G$\gamma_0 = n$, and G$\gamma = n + 1$.

If $\gamma =_{nf} \gamma_0 + \gamma_1$, then the assertion follows immediately by induction hypothesis.

If $\gamma =_{nf} \varphi\gamma_0\gamma_1$, then $\gamma_0, \gamma_1 < \gamma < \varphi\beta\gamma$, and $e(\gamma) = \{\gamma_0\}$. Suppose, for a contradiction, that $\beta < \gamma$. Since $\varphi\gamma_0\gamma_1 < \varphi\beta\gamma$, by Lemma 2.1.4, $\varphi\gamma_0\gamma_1 < \gamma$, which is a contradiction. Thus, $e(\gamma) = \{\gamma_0\} \leq \beta$.

If $\gamma =_{nf} \psi_k\gamma_0$, then $e(\gamma) = \{\gamma\}$ and by Lemma 2.1.5 $\gamma = \varphi\gamma 0 < \varphi\beta 0$, as needed.

For the opposite direction, we shall just consider the two critical cases of the induction step. Suppose $\alpha = \beta\gamma$, and $e(\gamma) \leq \beta$.

If $\gamma =_{nf} \varphi\gamma_0\gamma_1$ then $e(\gamma) = \{\gamma_0\} \leq \beta$ by assumption, and $\gamma_0, \gamma_1 < \gamma$ due to normal form. If $\gamma_0 < \beta$, then since $\gamma_1 < \gamma \leq \varphi\beta\gamma$, by Lemma 2.1.4 we have $\gamma < \varphi\beta\gamma$. Likewise, if $\gamma_0 = \beta$, then since $\gamma_1 < \gamma$, we have $\gamma < \varphi\beta\gamma$.

If $\gamma =_{nf} \psi_k\gamma_0$, then $e(\gamma) = \{\gamma\}$ and $\gamma \leq \beta$ by assumption. But then $\varphi\gamma 0 \leq \beta$, so $\gamma = \varphi\gamma 0 \leq \varphi\beta 0 < \varphi\beta\gamma$, as needed.

□

In order to ensure our proof can be carried out in $\mathbf{RCA}_0 + \forall X \exists Y(X \in Y \wedge Y$ is an ω-model of $\Pi^1_1\mathbf{CA}_0 + \mathbf{BI})$, we must present a primitive recursive term structure with an ordering relation equivalent to $\mathrm{OT}(\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}}))$.

Definition 2.2.14 *(Compare [8] Definition 2.9) Let $\mathfrak{X} = (X, <_X)$ be the well-ordering of the set $X \subseteq \mathbb{N}$ by the relation $<_X$. We shall recursively define a binary relational structure*

$$\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}}) = (|\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}})|, <),$$

together with a collection of functions

$$K^{\mathfrak{X}}_n : |\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}})| \mapsto \{\textit{finite subsets of } |\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}})|\},$$

for $n < \omega$, and an additional function,

$$e^{\mathfrak{X}} : |\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}})| \mapsto |\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}})|$$

such that

1. *$\Omega_m \in |\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}})|$, for all $m \leq \omega$, with $\Omega_0 = 0$. For all $m, n \leq \omega$ and $k \in \mathbb{N}$ we have $e^{\mathfrak{X}}(\Omega_m) = \{\Omega_m\}$, $K^{\mathfrak{X}}_k \Omega_m = \emptyset$, and if $m < n$ then $\Omega_m < \Omega_n$.*

2. *If $\alpha \in |\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}})|$ and $\alpha \neq 0$ then $0 < \alpha$.*

3. *For all $u \in \mathfrak{X}$ there is $\mathfrak{E}_u \in |\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}})|$, where $\Omega_\omega < \mathfrak{E}_u$, $e^{\mathfrak{X}}(\mathfrak{E}_u) = \{\mathfrak{E}_u\}$ and for all $n \in \mathbb{N}$, $K^{\mathfrak{X}}_n \mathfrak{E}_u = 0$. If $u, v \in \mathfrak{X}$ and $u <_X v$ then $\mathfrak{E}_u < \mathfrak{E}_v$.*

4. *If $\beta, \gamma \in |\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}})|$ with $e^{\mathfrak{X}}(\gamma) \leq \beta$ and ($\gamma \neq 0 \vee \beta$ does not have the form $\psi^{\mathfrak{X}}_m \alpha$), then $\varphi^{\mathfrak{X}} \beta\gamma \in |\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}})|$ with $e^{\mathfrak{X}}(\varphi^{\mathfrak{X}}\beta\gamma) = \{\beta\}$, and $K^{\mathfrak{X}}_n \varphi^{\mathfrak{X}}\beta\gamma = K^{\mathfrak{X}}_n \beta \cup K^{\mathfrak{X}}_n \gamma$.*

5. *Suppose $\alpha = \varphi^{\mathfrak{X}}\alpha_0\alpha_1 \in |\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}})|$ and $\beta = \varphi^{\mathfrak{X}}\beta_0\beta_1 \in |\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}})|$. Then $\alpha < \beta$*

if and only if

$$\alpha_0 < \beta_0 \textit{ and } \alpha_1 < \varphi^{\mathfrak{x}}\beta_0\beta_1 \textit{ or}$$

$$\alpha_0 = \beta_0 \textit{ and } \alpha_1 < \beta_1 \textit{ or}$$

$$\beta_0 < \alpha_0 \textit{ and } \varphi^{\mathfrak{x}}\alpha_1\alpha_0 < \beta_1.$$

6. $\varphi^{\mathfrak{x}}\beta\gamma \in |\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{x}}})|$ *and* α *has the form* $\Omega_k, \mathfrak{E}_u$, *or* $\psi^{\mathfrak{x}}_m\gamma$ *then* $\alpha < \varphi^{\mathfrak{x}}\beta\gamma$ *if* $\alpha < \beta$. *Otherwise* $\beta < \alpha$.

7. *If* $\alpha_1, \ldots, \alpha_k \in |\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{x}}})|$ *with* $\alpha_1 \geq \ldots \geq \alpha_k$ *with* $k \geq 2$, *then* $\omega^{\alpha_1} + \ldots + \omega^{\alpha_k} \in |\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{x}}})|$, *with* $e^{\mathfrak{x}}(\omega^{\alpha_1} + \ldots + \omega^{\alpha_k}) = \bigcup_{i\leq k} e^{\mathfrak{x}}(\alpha_i)$, *and* $K^{\mathfrak{x}}_n = \bigcup_{i\leq k} K^{\mathfrak{x}}_n\alpha_i$.

8. *If* $\alpha = \omega^{\alpha_1} + \ldots + \omega^{\alpha_k} \in |\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{x}}})|$ *and* β *has the form* $\Omega_n, \mathfrak{E}_u, \psi^{\mathfrak{x}}_k\gamma$, *or* $\varphi^{\mathfrak{x}}\gamma\eta$ *then*

$$\textit{if } \beta \leq \alpha_1 \textit{ then } \beta < \alpha \textit{ or}$$

$$\textit{if } \alpha_1 < \beta \textit{ then } \alpha < \beta.$$

9. *Suppose* $\alpha = \alpha = \omega^{\alpha_1} + \ldots + \omega^{\alpha_k} \in |\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{x}}})|$ *and* $\beta = \omega^{\beta_1} + \ldots + \omega^{\beta_k} \in |\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{x}}})|$. *Then* $\alpha < \beta$ *if and only if*

$$(m < n) \wedge \forall i \leq m(\alpha_i = \beta_1) \textit{ or}$$

$$\exists i \leq \min\{m, n\}\forall j < i[(\alpha_j = \beta_j) \wedge (\alpha_i < \beta_i)]$$

10. *If* $\alpha \in |\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{x}}})|$ *and* $K^{\mathfrak{x}}_m\alpha < \alpha$ *then* $\psi^{\mathfrak{x}}_m\alpha \in |\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{x}}})|$, *and* $e^{\mathfrak{x}}(\psi^{\mathfrak{x}}_n\alpha) = \{\psi^{\mathfrak{x}}_n\alpha\}$. *If* $m < n$ *then* $K^{\mathfrak{x}}_n\psi^{\mathfrak{x}}_m = \emptyset$. *If* $n \leq m$ *then* $K^{\mathfrak{x}}_n\psi^{\mathfrak{x}}_m = \{\alpha\} \cup K^{\mathfrak{x}}\alpha$.

11. *If* $\psi^{\mathfrak{x}}_m\alpha \in |\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{x}}})|$ *and* $m < n$ *then* $\psi^{\mathfrak{x}}_m\alpha < \Omega_n$, *and if* $n \leq m$ *then* $\Omega_n < \psi^{\mathfrak{x}}_m\alpha$.

12. *If* $\psi^{\mathfrak{x}}_m\alpha, \psi^{\mathfrak{x}}_n\beta \in |\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{x}}})|$ *and* $m < n$ *then* $\psi^{\mathfrak{x}}_m\alpha < \psi^{\mathfrak{x}}_n\beta$.

13. *If* $\psi^{\mathfrak{x}}_m\alpha, \psi^{\mathfrak{x}}_m\beta \in |\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{x}}})|$, *and* $\alpha < \beta$ *then* $\psi^{\mathfrak{x}}_m\alpha < \psi^{\mathfrak{x}}_m\beta$.

Recall that in our standard notation system $\omega^\alpha := \varphi 0\alpha$ and thus we are not introducing any problematic new symbols into the system. The term system $|\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}})|$ makes use of Cantor normal form, rather than the normal form we have established in this chapter, but this presents no great complications either. The only difference is that we will occasionally see an instance of ω^{α_i}, where α_i is an epsilon number, in which case we may recover the standard normal form by noting $\omega^{\alpha_i} = \alpha_i$.

Lemma 2.2.15 *1. The set* $|\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}})|$ *together with the binary relation* $<$ *and functions* $K_n^{\mathfrak{X}}$ *and* $e^{\mathfrak{X}}$ *is primitive recursive in* $\mathfrak{X}$.

2. $<$ *is a total linear ordering on* $|\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}})|$.

Proof

While the proof is not difficult, it would take up a great deal of space without adding any great insight. Instead, we simply note the parallels between the functions e and K_n and their respective counterparts, $e^{\mathfrak{X}}$ and $K_n^{\mathfrak{X}}$, with the behaviours shown in Lemmas 2.2.11 and 2.2.13

□

2.3 Distinguished Sets and Well-Ordering

In this section, we shall show OT$(\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}}))$ is well-ordered using the method of distinguished sets. We work in the background theory of

$$\mathbf{RCA}_0 + \forall X \exists Y (X \in Y \wedge Y \text{ is an } \omega\text{-model of } \Pi^1_1\text{-}\mathbf{CA}_0 + \mathbf{BI}).$$

In particular, we shall assume $\mathfrak{Y}$ is a countable-coded ω-model of Π^1_1-**CA**$_0$ + **BI**, with $X \in \mathfrak{Y}$.

Distinguished sets serve as a benchmark for provable well-ordering, which we shall ultimately leverage to find well-ordering up to Ω_ω. From there, we shall use the well-ordering of X within our model $\mathfrak{Y}$ to prove well-ordering up to $\mathfrak{E}_u$ for all $u \in X$.

Definition 2.3.1 *The **level** k strongly critical subterms of $\alpha \in$ OT$(\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}}))$ are inductively defined as follows:*

1. $SC_k(0) = SC_k(\mathfrak{E}_u) = \emptyset$ *for all* $u \in X$.
2. *If* $i < k$ *then* $SC_k(\Omega_i) = \{\Omega_i\}$. *Otherwise,* $SC_k(\Omega_i) = \emptyset$.
3. $SC_k(\alpha) = \{\alpha\}$ *if* $\alpha \in SC \cap \Omega_{k+1}$.
4. $SC_k(\alpha) = SC_k(\alpha_1) \cup SC_k(\alpha_2)$ *if* $\alpha =_{nf} \alpha_1 + \alpha_2$.
5. $SC_k(\alpha) = SC_k(\beta) \cup SC_k(\gamma)$ *if* $\alpha =_{nf} \varphi\beta\gamma$.
6. $SC_k(\alpha) = SC_k(\beta)$ *if* $\alpha =_{nf} \psi_m\beta$, *and* $\Omega_{k+1} \leq \alpha$, *for any* $m < \omega$.

Definition 2.3.2 *Let* $U \subseteq$ OT$(\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}}))$ *and* $F(a)$ *be an* L_2*-formula.*

1. $U \cap \alpha := \{\eta \in U | \eta < \alpha\}$.
2. $U \cap \alpha \subseteq F \iff (\forall \eta \in U \cap \alpha) F(\eta)$.
3. *Prg*$(U, F) \iff \forall \eta \in U[U \cap \eta \subseteq F \rightarrow F(\eta)]$.
4. $W[U] := \{\eta \in U | \forall Y[\mathit{Prg}(U, Y) \rightarrow U \cap \eta \subseteq Y]\}$
5. $M^U_k := \{\eta < \Omega_{k+1} | (\forall j, \Omega_j \in U \cap \Omega_k) SC_j(\eta) \subseteq U\}$.
6. $W^U_k := W[M^U_k]$.

Suppose $\alpha \in$ *OT*$(\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}}))$*, and* $S\alpha = \Omega_k$ *and* $\alpha^+ = \Omega_{k+1}$*. We establish the following conventions.*

$$W^U_{S\alpha} = W^U_k$$
$$M^U_{S\alpha} = M^U_k$$
$$W^U_{\alpha^+} = W^U_{k+1}$$
$$M^U_{\alpha^+} = M^U_{k+1}$$

Note that M^U_k is a set by arithmetical comprehension, while $W[U]$ (and therefore W^U_k) is a set by Π^1_1-CA$_0$.

Lemma 2.3.3 (See [6] Lemma 11.4) *1. Prg*$(U, S) \to W[U] \subseteq S$*.*

2. Prg$(U, W[U])$*.*

3. $[U \subseteq V \wedge$ *Prg*$(U, S)] \to$ *Prg*$(V, \{\eta | \eta \in U \to \eta \in S\})$*.*

4. Prg$(W[U], S) \to W[U] \subseteq S$*.*

5. $W[W[U]] = W[U]$*.*

6. $W[U \cap \alpha] \subseteq W[U]$ *for any* $\alpha \in$ *OT*$(\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}}))$*.*

7. $U \cap \Omega_k = V \cap \Omega_k \to (M^U_k = M^V_k \wedge W^U_k = W^V_k)$*.*

8. $\alpha \in W^U_k \leftrightarrow (\alpha \in M^U_k \wedge M^U_k \cap \alpha \subseteq W^U_k)$

Proof

(1) follows immediately from the definitions.

(2) Let $\alpha \in U$, and and suppose $U \cap \alpha \subseteq W[U]$. By (1) we have $W[U] \subseteq S$ for all S satisfying Prg(U, S). It follows that $U \cap \alpha \subseteq S$, and thus $\alpha \in S$, by definition of $W[U]$. Thus, Prg$(U, W[U])$.

(3) Assume $U \subseteq V$ and Prg(U, S), and let $\alpha \in V$, with $V \cap \alpha \subseteq \{\eta | \eta \in U \to \eta \in S\}$. Since $U \subseteq V$, we have

$$U \cap \alpha = U \cap (V \cap \alpha) \subseteq S,$$

so by Prg(U, S) we have $\alpha \in U \to \alpha \in S$, and therefore $\alpha \in \{\eta | \eta \in U \to \eta \in S\}$.

(4) Assume Prg$(W[U], S)$. By (3) we we have Prg$(U, \{\eta | \eta \in W[U] \to \eta \in S\})$. By (1) we get $W[U] \subseteq \{\eta | \eta \in W[U] \to \eta \in S\}$, and thus $W[U] \subseteq S$.

(5) $W[W[U]] \subseteq W[U]$ holds by definition. By (2) we know that Prg$(W[U], W[W[U]])$, and thus by (4) we get $W[U] \subseteq W[W[U]]$. So $W[U] = W[W[U]]$

(6) Let $\eta \in W[U \cap \alpha]$. Thus, we have $\eta \in U \cap \alpha$, and $\forall Y[\text{Prg}(U \cap \alpha, Y) \to (U \cap \alpha) \cap \eta \subseteq Y]$. Since $\eta < \alpha$, it follows that,

$$\forall Y[\text{Prg}(U, Y) \to U \cap \eta \subseteq Y].$$

So by definition, $\eta \in W[U]$.

(7) Let $U \cap \Omega_k = V \cap \Omega_k$. Then $M_k^U = \{\eta < \Omega_{k+1} | (\forall \Omega_j \in U \cap \Omega_k) SC_j(\eta) \in U\} = M_k^V$, and thus by definition $W_k^U = W_k^U$.

(8) By definition of W_k^U, we know that if $\alpha \in W_k^U$, then:

$$\text{Prg}(M_k^U, W_k^U) \to M_k^U \cap \alpha \subseteq W_k^U$$

and by (2), we deduce:

$$M_k^U \cap \alpha \subseteq W_k^U.$$

We obtain $\alpha \in M_k^U$ because $W_k^U \subseteq M_k^U$.

Similarly, suppose $\alpha \in M_k^U$ and $M_k^U \cap \alpha \subseteq W_k^U$. Then it is trivial to conclude:

$$\text{Prg}(M_k^U, W_k^U) \to M_k^U \cap \alpha \subseteq W_k^U.$$

By the definition of W^U_k, we may conclude:

$$\forall Y \text{Prg}(M^U_k, Y) \to M^U_k \cap \alpha \subseteq Y,$$

and thus $\alpha \in W^U_k$. □

Definition 2.3.4 *1. We say that $U \subseteq$ OT$(\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{x}}}))$ is a **distinguished set** if*

(a) $(\forall \alpha \in U) S\alpha \in U$ *and*

(b) $\forall i < \omega, \Omega_i \in U \to (U \cap \Omega_{i+1}) = W^U_i$.

We shall use $Ds(U)$ to denote that U is a distinguished set,

2. $\mathfrak{W} := \{\eta | \exists X[Ds(X) \wedge \eta \in X]\}$.

*We observe that $\mathfrak{W}$ is a Σ^1_2 statement, and thus is not provably a set in $\Pi^1_1 -$ **CA**$_0$.*

Henceforth, the variables Q and P will be used to represent distinguished sets.

Lemma 2.3.5 (See [6] Lemma 11.6) *1. $Q \subseteq W[Q]$, and thus $Q = W[Q]$.*

2. $Prg(Q, V) \to Q \subseteq V$.

Proof

(1) Suppose $\alpha \in Q$. Since Q is distinguished, we know that $S\alpha \in Q$, and thus

$$Q \cap \alpha^+ = W^Q_{S\alpha} = W[W^Q_{S\alpha}] = W[Q \cap \alpha^+] \subseteq W[Q].$$

(2) We know that Prg$(Q, V) \to W[Q] \subseteq V$. By part (1), it follows that $Q \subseteq V$. □

Note that if $<_Q$ is the restriction of $<$ to the distinguished set Q, then the preceding lemma gives $WO(<_Q)$.

Lemma 2.3.6 (See [6] Lemma 11.7) *1.* $(n \leq m \wedge \beta \in SC_m(\alpha)) \rightarrow SC_n(\beta) \subseteq SC_n(\alpha)$

2. $\alpha \in Q \wedge \Omega_k \in Q \rightarrow SC_k(\alpha) \subseteq Q$.

3. $\Omega_k \leq Q \rightarrow \Omega_k \in Q$.

Proof

(1) We proceed by induction on $G\alpha$. If $\alpha \in \{0, \Omega_h, \mathfrak{E}_u\}$ then $SC_m(\alpha) = \emptyset$ and the proposition holds vacuously. Now, suppose the proposition holds for $G\gamma \leq k$, and $G\alpha = k+1$. Suppose $n < m$ and $\beta \in SC_m(\alpha)$. The critical case is when $\alpha = \psi_j\gamma$ for $\Omega_j \leq \gamma$. Then $SC_m\alpha = \{\alpha\}$, and thus $\beta = \alpha$. So $SC_n(\beta) = SC_n(\alpha)$. The other cases follow by the induction hypothesis.

(2) Suppose $\alpha \in Q$ and $\Omega_k \in Q$. We have two cases. First, suppose $\Omega_k < S\alpha$. Since Q is distinguished, we know $S\alpha \in Q$ and thus $\alpha \in Q \cap \alpha^+ = W^Q_{S\alpha} \subseteq M^Q_{S\alpha}$. Moreover, we have $\Omega_k \in Q \cap S\alpha$, and thus $SC_k(\alpha) \subseteq Q$, by the definition of $M^Q_{S\alpha}$.

Now suppose that $\Omega_k \geq S\alpha$. Since Q is distinguished, we know $\alpha \in Q \cap \Omega_{k+1} = W^Q_k \subseteq M^Q_k$. By definition of M^Q_k, if $j < k$ then $SC_j(\alpha) \subseteq Q$, and by part (1), we have $(\forall \beta \in SC_k(\alpha))SC_j(\beta) \subseteq Q$. Thus, $SC_k(\alpha) \subseteq (\{\alpha\} \cup (M^Q_k \cap \alpha))$, and since $\alpha \in W^Q_k$, by Lemma 2.3.3 (6) we get:

$$SC_k(\alpha) \subseteq (\{\alpha\} \cup (M^Q_k \cap \alpha)) \subseteq W^Q_k.$$

(3) Suppose $\Omega_k \leq Q$. Then there is some $\alpha \in Q$ such that $\Omega_k \leq S\alpha$. Since Q is distinguished, if $\Omega_k = S\alpha$, then $\Omega_k \in Q$. Now suppose $\Omega_k < Q$. We know that $\Omega_k \in M^Q_{S\alpha}$ since $SC_j(\Omega_k) = \emptyset \subseteq Q$ for any j. Moreover, since $\alpha \in W^Q_{S\alpha}$, by Lemma 2.3.3 part 6 we know that $M^Q_{S\alpha} \cap \alpha^+ \subseteq W^Q_{S\alpha}$, and thus $\Omega_k \in W^Q_{S\alpha} = Q \cap \alpha^+$. $\square$

Lemma 2.3.7 (See [6] Lemma 11.8) *For all $k < \omega$, $Q \cap \Omega_{k+1} \subseteq W_k^Q$.*

Proof

Suppose $\alpha \in Q \cap \Omega_{k+1}$. Since Q is distinguished, $\alpha \in W^Q_{S\alpha}$, and thus by Lemma 2.3.3 $M^Q_{S\alpha} \cap \alpha \subseteq W^Q_{S\alpha}$. By Lemma 2.3.6 (8) we know that for all $j \leq k$, $SC_j(\alpha) \subseteq Q$, and thus $\alpha \in M_k^Q \cap \alpha^+$. Using Lemma 2.3.3 Prg$(M_k^Q \cap \alpha^+, U)$ implies:

$$\text{Prg}(M^Q_{S\alpha}, \{\eta | \eta \in M_k^Q \cap \alpha^+ \to \eta \in U\}),$$

and thus, by Lemma 2.3.3 (8)

$$M_k^Q \cap \alpha \subseteq M^Q_{S\alpha} \cap \alpha \subseteq W^Q_{S\alpha} \subseteq \{\eta | \eta \in M_k^Q \cap \alpha^+ \to \eta \in U\}.$$

We may therefore conclude that $M_k^Q \cap \alpha^+ \cap \alpha \subseteq U$, which proves that $\alpha \in W[M_k^Q \cap \alpha] \subseteq W_k^Q$. □

Theorem 2.3.8 (See [6] Lemma 11.9) $M_k^Q \cap \Omega_k \subseteq Q \to \Omega_k \in W_k^Q \wedge Ds(W_k^Q)$.

Proof

Since $SC_l(\Omega_k) = \emptyset$ for all $l < k$ we know $\Omega_k \in M_k^Q$. Moreover, we have $M_k^Q \cap \Omega_k \subseteq Q \cap \Omega_{k+1} \subseteq W_k^Q$, and thus by Lemma 2.3.3 we have $\Omega_k \in W_k^Q$.

Next, we shall prove that W_k^Q is a distinguished set.

(a) We begin by showing that if $\alpha \in W_k^Q$, then $S\alpha \in W_k^Q$. If $\alpha < \Omega_k$, then this is immediate from the fact that Q is distinguished, and $W_k^Q = Q \cap \Omega_{k+1}$. Otherwise $\Omega_k \leq \alpha \leq \Omega_{k+1}$, so $S\alpha = \Omega_k$.

(b) Next, we must show that if $\Omega_n \in W_k^Q$ then $W_k^Q \cap \Omega_{n+1} = W_n^{W_k^Q}$. Let $\Omega_n \in W_k^Q$.If $k < n$, then clearly $\Omega_n \notin W_k^Q$, so $n \leq k$. Thus, by Lemma 2.3.3 we have

$$W_k^Q \cap \Omega_n = Q \cap \Omega_{k+1} \cap \Omega_n = Q \cap \Omega_n = W_n^Q.$$

So W_k^Q is distinguished $\square$

The above theorem is significant, because it allows us to find non-empty distinguished sets. Even more importantly, we gain the following corollary:

Lemma 2.3.9 (See [6] Lemma 11.10) $Prg(P \cup Q, U) \to P \cup Q \subseteq U$

Proof

Assume $\text{Prg}(P \cup Q, U)$. Then for any α, $P \cap \alpha \subseteq P \cup Q$, we have

$$(P \cap \alpha \subseteq U) \wedge (Q \cap \alpha \subseteq U) \wedge (\alpha \in P \to \alpha \in U).$$

Moreover, by Lemma 2.3.3

$$(*) P \cap \alpha \subseteq U \to \text{Prg}(P \cup Q, \{\eta | \eta \in P \cap \alpha \to \eta \in U\})$$

which simplifies to

$$(**) P \cap \alpha \subseteq U \to \text{Prg}(Q, \{\eta | \eta < \alpha \to \eta \in U\}).$$

Since Q is distinguished, it follows by Lemma 2.3.5 and $(**)$ that

$$P \cap \alpha \subseteq U \to Q \cap \alpha \subseteq U$$

and therefore by $(*)$

$$(P \cap \alpha \subseteq U) \wedge (\alpha \in P) \to \alpha \in U,$$

i.e. $\text{Prg}(P, U)$. Again applying Lemma 2.3.5 we find that $P \subseteq U$. A similar argument yields that $Q \subseteq U$, and thus $P \cup Q \subseteq U$. $\square$

Lemma 2.3.10 (See [6] Lemma 11.11) $\Omega_k \in P \cup Q \wedge \Omega_k \leq P \wedge \Omega_k \leq Q \to P \cap \Omega_{k+1} = Q \cap \Omega_{k+1}$

Proof

Lemma 2.3.9 shows that we may perform induction over $P \cup Q$. For the base case, we note that if $0 \in P$, and $0 \leq Q$ then Q is nonempty and $P \cap 0 = Q \cap 0 = \emptyset$. The argument then proceeds much the same way as the induction step, below.

Thus, suppose $\Omega_k \in P$ and $\Omega_k \leq Q$. By induction hypothesis $P \cap \Omega_k = Q \cap \Omega_k$, and thus by Lemma 2.3.3 $\Omega_k \in W_k^P = W_k^Q \subseteq M_k^Q$. So by Lemma 2.3.6 we find $\Omega_k \in Q$, and since Q is distinguished, $P \cap \Omega_{k+1} = W_k^P = W_k^Q = Q \cap \Omega_{k+1}$. The same argument works for the opposite case, when $\Omega_k \in Q$ and $\Omega_k \leq P$. □

Theorem 2.3.11 (See [6] Lemma 11.12) $\alpha \in Q \rightarrow Q \cap \alpha^+ = \mathfrak{W} \cap \alpha^+$

Proof

Suppose $\alpha \in Q$. Obviously, $Q \cap \alpha^+ \subseteq \mathfrak{W} \cap \alpha^+$. Suppose, then, that $\eta \in \mathfrak{W} \cap \alpha^+$. Then there is some distinguished set P such that $\eta \in P \cap \alpha^+$. So $S\eta \in P \cup Q$, with $S\eta \leq \eta \in P$ and $S\eta \leq \alpha \in Q$. So by the preceding lemma, $\eta \in Q \cap \alpha^+$. □

We shall now examine the closure properties of distinguished sets, and by extension the closure of $\mathfrak{W}$.

Theorem 2.3.12 (See [6] Lemma 11.13) *1.* $\alpha, \beta \in Q \rightarrow \alpha + \beta \in Q$.

2. $\alpha, \beta \in \mathfrak{W} \rightarrow \alpha + \beta \in \mathfrak{W}$.

Proof

Suppose $\alpha, \beta \in Q$. If $S\alpha < S\beta$ then $\alpha + \beta = \beta \in Q$. So assume that $S\beta \leq S\alpha$. Thus, we have $\alpha, \beta \in W_{S\alpha}^Q$. Now let

$$U := \{\xi | \alpha + \xi \subseteq W_{S\alpha}^Q\}.$$

By definition $M^Q_{S\alpha}$ is closed under addition. Thus,

$$\eta \in M^Q_{S\alpha} \wedge M^Q_{S\alpha} \cap \eta \subseteq U \to \alpha + \eta \in M^Q_{S\alpha} \wedge M^Q_{S\alpha} \cap (\alpha + \eta) \subseteq W^Q_{S\alpha}.$$

So, applying Lemma 2.3.3 yields

$$\eta \in M^Q_{S\alpha} \wedge M^Q_{S\alpha} \cap \eta \subseteq U \to \alpha + \eta \in W^Q_{S\alpha}.$$

We may therefore conclude Prg$(M^Q_{S\alpha}, U)$, and thus $W^Q_{S\alpha} \subseteq U$ by Lemma 2.3.3, and thus $\alpha + \beta \in Q \cap \alpha^+$.

(2) follows immediately from (1). $\Box$

Lemma 2.3.13 (See [6] Lemma 11.14) *Let* $\mathfrak{F}(\alpha, \beta)$ *be the formula*

$$\alpha, \beta \in Q \wedge (\forall \xi \in Q \cap \alpha)(\forall \eta \in Q)(\varphi\xi\eta \in Q) \wedge (\forall \eta \in Q \cap \beta)(\varphi\alpha\eta \in Q).$$

The following statements are true:

1. $\mathfrak{F}(\alpha, \beta) \wedge \delta = \max\{\alpha, \beta\} \wedge \gamma \in M^Q_{S\delta} \cap \varphi\alpha\beta \to \gamma \in Q.$
2. $\mathfrak{F}(\alpha, \beta) \to \varphi\alpha\beta \in Q.$

Proof

For part (1) we proceed by induction on $G\gamma$. From $\mathfrak{F}(\alpha, \beta)$ we may deduce $\alpha, \beta \in Q \cap \delta^+ = W^Q_{S\delta}$. The assertion holds trivially, if $\gamma \in \{\Omega_k | \Omega_k < \delta^+\}$. If $\gamma =_{nf} \gamma_1 + \ldots \gamma_n$ then for $i \leq n, \gamma_i \in M^Q_{S\delta}$, and by induction hypothesis, $\gamma_i \in Q$. Hence $\gamma \in Q$. If $\gamma = \psi_k \eta$, then $\gamma \in SC$, and $\gamma \leq \alpha \vee \gamma \leq \beta$. Since $\alpha, \beta \in W^Q_{S\delta}$, we may conclude by Lemma 2.3.3 that

$$M^Q_{S\delta} \cap \gamma \subseteq M^Q_{S\delta} \cap \delta \subseteq W^Q_{S\delta},$$

and thus $\gamma \in W^Q_{S\delta} \subseteq Q$.

Finally, we consider when $\gamma =_{nf} \varphi\xi\eta$. By definition, $\xi, \eta \in M^Q_{S\delta}$ and hence by induction hypothesis $\xi, \eta \in Q$, If $\xi \leq \alpha$ we have $\gamma \in Q$ by $\mathfrak{F}(\alpha, \beta)$. If $\alpha < \xi$ then we must have $\gamma < \beta$ or else we would have $\varphi\alpha\beta < \gamma$ contrary to our hypothesis. Once again, we find

$$M^Q_{S\delta} \cap \gamma \subseteq M^Q_{S\delta} \cap \beta \subseteq W^Q_{S\delta},$$

and thus $\gamma \in W^Q_{S\delta} \subseteq Q$.

(2) Let $\delta = \max\{\alpha, \beta\}$. From part (i) we may deduce

$$\mathfrak{F}(\alpha, \beta) \to M^Q_{S\delta} \cap \varphi\alpha\beta \subseteq W^Q_{S\delta}.$$

By Lemma 2.3.6 we also find that

$$\mathfrak{F}(\alpha, \beta) \to \varphi\alpha\beta \in M^Q_{S\delta}.$$

These two statements, combined with Lemma 2.3.3 yield

$$\mathfrak{F}(\alpha, \beta) \to M^Q_{S\delta} \cap \varphi\alpha\beta \subseteq W^Q_{S\delta} = Q \cap \delta^+.$$

Hence $\mathfrak{F}(\alpha, \beta) \to \varphi\alpha\beta \in Q$. $\square$

Theorem 2.3.14 (See [6] Lemma 11.15) *1.* $\alpha, \beta \in Q \to \varphi\alpha\beta \in Q$.

2. $\alpha, \beta \in \mathfrak{W} \to \varphi\alpha\beta \in \mathfrak{W}$.

Proof

We shall prove this result in stages. First, let $\alpha \in Q$ and $V := \{\eta | \varphi\alpha\eta \in Q\}$. Now, assume $(\forall\xi \in Q \cap \alpha)(\forall\eta \in Q)(\varphi\xi\eta \in Q)$ and $Q \cap \gamma \subseteq V$. Then by the preceding lemma, $\varphi\alpha\gamma \in V$, and thus we have $\text{Prg}(Q, V)$. Since Q is distinguished, this means $Q \subseteq V$. In other words, we have:

$$(\forall\xi \in Q \cap \alpha)(\forall\eta \in Q)(\varphi\xi\eta \in Q) \to Q \subseteq V$$

Now, let $U = \{\xi | (\forall \eta \in Q)\varphi\xi\eta \in Q\}$. From the statement above, we may then deduce:

$$Q \cap \alpha \subseteq U \to \alpha \in U$$

i.e. Prg(Q, U), or $Q \subseteq U$. Thus Q, and by extension $\mathfrak{W}$ is closed under the φ function. □

Corollary 2.3.15 (See [6] Corollary 11.16) *1.* $S\alpha \leq \Omega_k \wedge \Omega_k \in Q \wedge SC_k(\alpha) \subseteq Q \to \alpha \in Q$.

2. $S\alpha \leq \Omega_k \wedge \Omega_k \in \mathfrak{W} \wedge SC_k(\alpha) \subseteq Q \to \alpha \in \mathfrak{W}$.

Proof

This follows immediately from Lemmas 2.3.13 and 2.3.15 above. □

Lemma 2.3.16 (See [6] Lemma 11.17) *1.* $\beta \in Q \wedge \alpha \in M^Q_{S\beta} \cap \beta \to \alpha \in Q$

2. $\beta \in \mathfrak{W} \wedge \alpha \in M^Q_{S\beta} \cap \beta \to \alpha \in \mathfrak{W}$

Proof

(1) If $\beta \in Q$ then $\beta \in Q \cap \beta^+ = W^Q_{S\beta}$. By Lemma 2.3.3 we have $M^Q_{S\beta} \cap \beta \subseteq W^Q_{S\beta}$ and hence $\alpha \in W^Q_{S\beta}$. (2), of course, follows immediately from (1). □

Definition 2.3.17 $\mathfrak{B}^Q_k := \{\alpha | (\forall \Omega_i \in Q \cap \Omega_k)[K_i\alpha < \alpha \to \psi_i\alpha \in Q]$

Lemma 2.3.18 (See [6] Lemma 11.19) *Assume* $\alpha \in M^Q_k, M^Q_k \cap \alpha \subseteq \mathfrak{B}^Q_k, \Omega_n \in Q \cap \Omega_k, K_n\alpha < \alpha$ *and* $\gamma \in M^Q_n \cap \psi_n\alpha$. *Then* $\gamma \in Q$.

Proof

Proceed by induction on $G\gamma$. If $\gamma \leq \Omega_n$ then $\gamma \in Q$ by Lemma 2.3.17. Suppose, then that $\Omega_n < \gamma$.

If $\gamma =_{nf} \gamma_1 + \ldots \gamma_m$ then by the induction hypothesis, for $i \leq m$ we have $\gamma_i \in Q$, and thus $\gamma \in Q$ by closure.

If $\gamma =_{nf} \varphi\xi\eta$ then $\xi, \eta \in Q$ by the induction hypothesis, and thus $\varphi\xi\eta \in Q$ by closure.

Finally, if $\gamma =_{nf} \psi_n\eta$ then we know $\eta < \alpha$, since ψ_k is a strictly increasing function. Since $n < k$ and $\gamma \in M_n^Q$, by Lemma 2.3.6 we know that

$$(\forall\Omega_t \in Q \cap \Omega_n)(\forall\beta \in SC_n(\eta))SC_t(\beta) \subseteq SC_t(\eta) \subseteq Q$$

By the construction of ψ_n, we know that $SC_n(\eta) < \psi_n\eta < \psi_n\alpha$ and thus we find that $SC_n(\eta) \subseteq M_n^Q \cap \psi_n\alpha$. By induction hypothesis, then, $SC_n(\eta) \subseteq Q$. In other words, we have:

$$(\forall t \leq n)[\Omega_t \in Q \cap \Omega_k \rightarrow SC_t(\eta) \subseteq Q]$$

From here, we may carry out a secondary induction to show that

$$(\forall\Omega_t \in Q \cap \Omega_k)(SC_t(\eta) \subseteq Q).$$

Suppose, then, that $\Omega_t \in Q \cap \Omega_k$. We have already proven the result for $t \leq n$. Suppose, then, that $n < t$. By our secondary induction hypothesis, we have $(\forall\Omega_{t'} \in Q \cap \Omega_t)SC_{t'}(\eta) \subseteq Q$. It follows that $SC_t(\eta) \subseteq M_t^Q$. We have $n < t$ and $K_n\alpha < \alpha$ by assumption. Since $\gamma =_{nf} \psi_n\eta$, we know that $K_n\eta < \eta$. From $n < t$ we may therefore deduce that $K_t\eta < \eta$ and $K_t\alpha < \alpha$. Thus, we have $SC_t(\eta) < \psi_t\eta < \psi_t\alpha$, and thus $SC_t(\eta) \subseteq M_t^Q \cap \psi_t\alpha$. Thus, by our primary induction hypothesis we may conclude that $SC_t(\eta) \subseteq Q$. Thus $\eta \in M_k^Q \cap \psi_k\alpha$, and since $\eta \in \mathfrak{B}_k^Q$ we have $\psi_k\eta \in Q$.
$\square$

Lemma 2.3.19 (See [6] Lemma 11.20) *Prg*$(M_k^Q, \mathfrak{B}_k^Q)$.

Proof

Suppose $\alpha \in M_u^Q$ and $M_u^Q \cap \alpha \subseteq \mathfrak{B}_u^Q$. We wish to show $\alpha \in \mathfrak{B}_u^Q$. So assume that $\Omega_v \in Q \cap \Omega_u$ and $K_v\alpha < \alpha$. By Lemma 2.3.19 we know that $M_b^Q \cap \psi_v\alpha \subseteq Q \cap \Omega_{v+1}$. For $\Omega_t \in Q \cap \Omega_v$, we have $SC_t(\psi_v\alpha) = SC_t(\alpha)$, and since $\alpha, \Omega_v \in Q$ by Lemma 2.3.6 we have $\psi_k\alpha \in W_v^Q \subseteq Q$, i.e. $\alpha \in \mathfrak{B}_u^Q$, and hence Prg$(M_u^Q, \mathfrak{B}_u^Q)$. □

Lemma 2.3.20 (See [6] Lemma 11.21) *1.* $\alpha, \Omega_k \in Q \wedge K_k\alpha < \alpha \rightarrow \psi_k\alpha \in Q$.

2. $\alpha, \Omega_k \in \mathfrak{W} \wedge K_k\alpha < \alpha \rightarrow \psi_k\alpha \in \mathfrak{W}$.

Proof

(1) Let $\delta = \max\{S\alpha, S\Omega_k\}$. Since we know Prg$(M_k^Q, \mathfrak{B}_k^Q)$ we have $W_k^Q \subseteq \mathfrak{B}_k^Q$. Since Q is distinguished, we have $\delta \in Q$ and thus $Q \cap \delta^+ = W_\delta^Q \subseteq \mathfrak{B}^\delta$. In particular, this yields $\alpha \in \mathfrak{B}_k^Q$ and thus $\psi_k\alpha \in Q$.

(2) of course, follows immediately from (1). □

Lemma 2.3.21 (See [6] Lemma 11.22) *Suppose* $U \subseteq \mathbb{N}$. *Then* $(\forall j \in U)Ds(Q_j) \rightarrow Ds(\cup\{Q_j | j \in U\})$.

Proof

Suppose $Ds(Q_j)$ holds for all $j \in U$. Then by arithmetical comprehension

$$Z := \cup\{Q_j | j \in U\}$$

is a set. If $\alpha \in Z$, then there is some $j \in U$ such that $\alpha \in Q_j$. Since Q_j is distinguished, $S\alpha \in Q_j \subseteq Z$.

Now, suppose that $\Omega_k \in Z$. Then $\Omega_k \in Q_i$ for some $i \in U$. By Theorem 2.3.12 we have

$$\mathfrak{W} \cap \Omega_{k+1} = Q_i \cap \Omega_{k+1} \subseteq Z \cap \Omega_{k+1} \subseteq \mathfrak{W} \cap \Omega_{k+1}$$

So $Z \cap \Omega_{k+1} = Q_i \cap \Omega_{k+1}$. By Lemma 2.3.3 we observe that $W_k^Z = W_k^{Q_i} = Z \cap \Omega_{k+1}$. $\square$

Lemma 2.3.22 *For all $n < \omega, \Omega_n$ there is a distinguished set Q such that $\Omega_n \in Q$. Thus, $Q_n \in \mathfrak{W}$. Moreover, $Ds(\mathfrak{W} \cap \Omega_\omega)$.*

Proof

Suppose $Q_0 = \emptyset$ and $Q_{n+1} = W_n^{Q_n}$. We will show that for all $n < \omega, \Omega_n \in Q_{n+1}$ and $Ds(Q_n)$. Hence, $\Omega_n \in \mathfrak{W}$.

Further, we claim that $M_n^{Q_n} = W_n^{Q_n}$ for all n.

For the base case, observe that $Ds(\emptyset)$ holds vacuously. Since $SC_0(0) = \emptyset$, and $M_0^{\emptyset} \cap 0 \subseteq \emptyset$, by Lemma 2.3.8 we have $0 \in W_0^{Q_0} = Q_1$.

To show $M_0^{\emptyset} = W_0^{\emptyset}$, suppose $\alpha \in M_0^{\emptyset}$. By induction on G$\alpha$ we shall prove that $M_0^{\emptyset} \cap \alpha \subseteq W_0^{\emptyset}$.

If G$\alpha = 0$, then $\alpha = 0$. Suppose for G$\gamma <$ Gα, we have $M_0^{\emptyset} \cap \gamma \subseteq W_0^{\emptyset}$.. If $\beta \in M_0^{\emptyset}$ and $\beta < \alpha$ then:

1. if $\alpha =_{nf} \alpha_0 + \alpha_1$, we have $\beta < \alpha_0$, or $\beta < \alpha_1$, so by induction hypothesis $\beta \in W_0^{\emptyset}$.
2. if$\alpha =_{nf} \varphi\alpha_0\alpha_1$, then by a secondary induction on Gβ, combined with Lemma 2.1.4 gives $\beta \in W_0^{\emptyset}$.
3. If $\alpha =_{nf} \psi_0\alpha_0$, then $\alpha \notin M_0^{\emptyset}$, since $SC_0(\psi_0\alpha) = \{\alpha\} \neq \emptyset$.

Hence, we have $\alpha \in M_0^{\emptyset}$ and $M_0^{\emptyset} \cap \alpha \subseteq W_0^{\emptyset}$, so by Lemma 2.3.3 (8), we have $\alpha \in W_0^{\emptyset} = Q_1$.

For the induction step, suppose that $\Omega_n \in Q_{n+1}$ and $Ds(Q_{n+1})$, with $M_n^{Q_n} = W_n^{Q_n}$. Since $Q_{n+1} = W_n^{Q_n}$ is distinguished, we know

$$M_{n+1}^{Q_{n+1}} \cap \Omega_{n+1} = \{\eta < \Omega_{n+1} | (\forall j, \Omega_j \in Q_n \cap \Omega_{n+1}) SC_j(\eta) \subseteq Q_n\} = M_n^{Q_n}$$

and by induction hypothesis, $M_n^{Q_n} = W_n^{Q_n}$. Thus, we may apply theorem 2.3.8 to find $\Omega_n \in Q_{n+2}$ and $Ds(Q_{n+2})$.

To show $M_{n+2}^{Q_{n+2}} = W_{n+2}^{Q_{n+2}}$, we again proceed by induction on Gα. The proof proceeds much the same as in the base case, though we must now consider the case where $\alpha =_{nf} \psi_k(\alpha_0)$, with $k < n+2$. Then $SC_{k+1}(\psi_k(\alpha_0)) = \{\psi_k(\alpha_0)\}$, and by definition of $M_{n+2}^{Q_{n+2}}$ this means $\psi_k(\alpha_0) \in Q_{n+1} \cap \Omega_{n+2}$. By induction hypothesis, Q_{n+1} is distinguished with $\Omega_{n+2} \in Q_{n+1}$, and thus $Q_{n+1} \cap \Omega_{n+2} = W_{n+2}^{Q_{n+2}}$, as needed.

To see that $\mathfrak{W} \cap \Omega_\omega$ is a distinguished set, we observe that by Lemma 2.3.11, $Q_{n+1} = Q_n \cap \Omega_{n+1} = \mathfrak{W} \cap \Omega_{n+1}$, and hence $Ds(\mathfrak{W} \cap \Omega_{n+1})$ for all $n < \omega$. By Lemma 2.3.21, then, $\cup\{\mathfrak{W} \cap \Omega_{j+1} | j < \omega\} = Ds(\mathfrak{W} \cap \Omega_{n+1})$, so $Ds(\mathfrak{W} \cap \Omega_{n+1})$.

□

Recall that $\mathfrak{Y}$ is a countable-coded ω-model of $\Pi_1^1 - \mathbf{CA} + \mathbf{BI}$. We say that U is **$\mathfrak{Y}$-definable** if $U = \{n \in \mathbb{N} | \mathfrak{Y} \models Y(n)\}$ for some formula $Y(x)$ of second-order arithmetic with parameters from $\mathfrak{Y}$.

Definition 2.3.23 *1.* $\mathfrak{M} := \{\alpha \mid \forall n < \omega\ SC_n(\alpha) \in \mathfrak{W}\}$.

2. $\alpha <_{\mathfrak{M}} \beta\ \ :\Leftrightarrow\ \ \alpha, \beta \in \mathfrak{M} \wedge \alpha < \beta$.

3. $COLL := \{\alpha \in \mathfrak{M} \mid \forall n < \omega\, (K_n\alpha < \alpha) \to \psi_n\alpha \in \mathfrak{W}\}$.

4. $Prg_{\mathfrak{M}}(U) = (\forall \alpha \in \mathfrak{M})[(\forall \beta <_{\mathfrak{M}} \alpha\ \beta \in U) \to \alpha \in U]$.

Lemma 2.3.24 $\mathfrak{W} \cap \Omega_\omega = \mathfrak{M} \cap \Omega_\omega$. *(Compare [8] Lemma 3.3)*

Proof

Let $\alpha \in \mathfrak{W} \cap \Omega_\omega$. By Lemma 2.3.22 there exists a distinguished set Q such that $\Omega_n, \alpha \in Q$. Then $SC_n(\alpha) \subseteq Q \subseteq \mathfrak{W}$ by Lemma 2.3.6 (2).

Now let $\alpha \in \mathfrak{M} \cap \Omega_\omega$. Choose n such that $\alpha < \Omega_n$. Then $SC_n(\alpha) \in \mathfrak{W}$, so that by Theorems 2.3.12 and 2.3.14, we get $\alpha \in \mathfrak{W}$. □

Lemma 2.3.25 *(Compare [8] Lemma 3.4) Let U be definable in our ω-model $\mathfrak{Y}$. Then we have*

$$\forall \alpha \in \mathfrak{W} \cap \Omega_\omega \, [(\forall \beta \in \mathfrak{W} \cap \alpha \;\; \beta \in U) \to \alpha \in U] \to \mathfrak{W} \cap \Omega_\omega \subseteq U.$$

Proof

We have

$$\forall \alpha \in Q \cap \Omega_\omega \, [(\forall \beta \in Q \cap \alpha \;\; \beta \in U) \to \alpha \in U] \to Q \cap \Omega_\omega \subseteq U \qquad (2.1)$$

for every distinguished set in $\mathfrak{Y}$, using Bar Induction inside that model. (2.1) yields the the desired assertion. □

Lemma 2.3.26 *(Compare [8] Lemma 3.6) If U is $\mathfrak{Y}$-definable then $Prg_{\mathfrak{M}}(U) \to \Omega_\omega, \Omega_\omega + 1 \in U$.*

Proof

Suppose $Prg_{\mathfrak{M}}(U)$. By Lemma 2.3.26 we get $\mathfrak{W} \cap \Omega_\omega \subseteq U$. As $\Omega_\omega, \Omega_\omega + 1 \in \mathfrak{M}$, $Prg_{\mathfrak{M}}(U)$ yields $\Omega_\omega, \Omega_\omega + 1 \in U$. □

Lemma 2.3.27 $Prg_{\mathfrak{M}}(COLL)$.

Proof

By induction on Gγ we first show that whenever (i) $\alpha \in \mathfrak{M}$, (ii) $\forall\beta <_{\mathfrak{M}} \alpha \ \beta \in COLL$, (iii) $n < \omega \ \wedge \ K_n\alpha < \alpha$, and (iv) $DS(Q) \ \wedge \ \Omega_n \in Q \ \wedge \ \gamma \in M_n^Q \cap \psi_n\alpha$ then $\gamma \in Q$.

So assume (i)-(iv). If $\gamma < \Omega_n$ or $\gamma =_{nf} \alpha_1 + \ldots + \alpha_m$ or $\gamma =_{nf} \varphi\alpha_0\alpha_1$ then this follows from the induction hypothesis using Theorems 2.3.12 and 2.3.14. Also if $\gamma = \Omega_n$ we have $\gamma \in Q$. Thus it remains to consider the case when $\gamma =_{nf} \psi_n\eta$ for some $\eta < \alpha$. Since $SC_n(\eta) < \psi_n\eta < \psi_n\alpha$ and the elements of $SC_n(\eta)$ are shorter that γ with respect to G and belong to $M_n^Q \cap \psi_n\alpha$, the induction hypothesis yields $SC_n(\eta) \subseteq Q$. To show that $\eta \in \mathfrak{M}$ we also have to verify that $SC_k(\eta) \subseteq \mathfrak{W}$ for $n < k < \omega$. To this end we employ a subsidiary induction on k. The subsidiary induction hypothesis yields that for all $n \leq k' < k$ one has $SC_{k'}(\eta) \subseteq \mathfrak{W}$. Thus $SC_k(\eta) \subseteq M_k^P$ for any distinguished set P with $\Omega_k \in P$. From $K_n\eta < \eta$ and $K_n\alpha < \alpha$ we can also deduce that $K_k\eta < \eta$, $K_k\alpha < \alpha$ and $SC_k(\eta) < \psi_k\eta < \psi_k\alpha$. Therefore we have $SC_k(\eta) \subseteq M_k^P \cap \psi_k\alpha$ and consequently, by applying the main induction hypothesis, $SC_k(\eta) \in P$. This completes the subsidiary induction proof. As a result, $SC_i(\eta) \subseteq \mathfrak{W}$ holds for all i, whence $\eta \in \mathfrak{M}$ so that by means of (ii) we obtain $\eta \in COLL$, and hence $\gamma = \psi_n\eta \in Q$.

To verify $Prg_{\mathfrak{M}}(COLL)$, let $\alpha \in \mathfrak{M}$ and suppose that $\forall\beta <_{\mathfrak{M}} \alpha \ \beta \in COLL$. Suppose $K_n\alpha < \alpha$. Pick a distinguished set Q with $\Omega_n \in Q$. Then, by the first part of the proof, $M_n^Q \cap \psi_n\alpha \subseteq Q$. Since also $\psi_n\alpha \in M_n^Q$ we obtain $\psi_n\alpha \in Q$ and hence $\psi_n\alpha \in \mathfrak{W}$ as desired. $\Box$

Definition 2.3.28 *Let* $U \subset$ *OT*$(\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}}))$. *We define the* ***Gentzen Jump*** U^j *as follows:*

$$U^j = \{\gamma | \forall\delta \in \mathfrak{M}[\mathfrak{M} \cap \delta \subseteq U \rightarrow \mathfrak{M} \cap (\delta + \omega^\gamma) \subseteq U]\}$$

Lemma 2.3.29 *(Compare [8] Lemma 3.9) Let* U *be* $\mathfrak{Y}$*-definable. Then*

1. $\gamma \in U^j \rightarrow \mathfrak{M} \cap \omega^\gamma \subseteq U$.

2. $Prg_{\mathfrak{M}}(U) \to Prg_{\mathfrak{M}}(U^j)$.

Proof

(1) follows from the definition, with $\delta = 0$.

To show (2), suppose (a) $Prg_{\mathfrak{M}}(U)$, (b) $\gamma \in \mathfrak{M} \wedge \mathfrak{M} \cap \gamma \subseteq U^j$, and (c) $\mathfrak{M} \cap \delta \subseteq \mathfrak{M}$. We must show that $\mathfrak{M} \cap (\delta + \omega^\gamma) \subseteq U$. Let $\eta \in \mathfrak{M} \cap (\delta + \omega^\gamma)$.

If $\eta < \delta$, then $\eta \in U$ by (c). If $\eta = \delta$, then $\eta \in U$ by (a) and (c). If $\delta < \eta < \omega^\gamma$, then we have $\eta =_{nf} \delta + \omega^{\gamma_1} \ldots + \omega^{\gamma_n}$, for some $\gamma > \gamma_1 \leq \ldots \leq \gamma_n$. Since $\eta \in \mathfrak{M}$, it follows that $\gamma_i \in \mathfrak{M} \cap \gamma$. Now use (b) and (c) to obtain $\mathfrak{M} \cap (\delta + \omega^{\gamma_1}) \subseteq U$. By iterating this process, it follows that

$$\eta = \delta + \omega^{\gamma_1} + \ldots + \omega^{\gamma_n} \in U$$

So $\mathfrak{M} \cap (\delta + \omega^\gamma) \subseteq U$. It follows that $\gamma \in U^j$, and thus $Prg_{\mathfrak{M}}(U) \to Prg_{\mathfrak{M}}(U^j)$. □

Corollary 2.3.30 *(Compare [8] Lemma 3.10) Let $\mathfrak{I}(\delta)$ be the statement $Prg_{\mathfrak{M}}(U) \to \delta \in \mathfrak{M} \wedge \mathfrak{M} \cap \delta \subseteq U$ for all $\mathfrak{Y}$-definable sets U. Assume $\mathfrak{I}(\delta)$, and let $\delta_0 = \delta$ and $\delta_{n+1} = \omega^{\delta_n}$. Then $\mathfrak{I}(\delta_n)$ holds for all n.*

Proof

Proceed by induction on n. For $n = 0$, this is our starting assumption. Now suppose $\mathfrak{I}(\delta_n)$ holds. Assume $Prg_{\mathfrak{M}}(U)$. By the preceding lemma, we obtain $Prg_{\mathfrak{M}}(U^j)$, and hence $\delta_n \in U^j$ and $\mathfrak{M} \cap \delta_n \subseteq U^j$. Clearly, $\mathfrak{M} \cap 0 \subseteq U$. So $\mathfrak{M} \cap (0 + \omega^{\delta_n}) \subseteq U$, i.e. $\mathfrak{M} \cap \delta_{n+1} \subseteq U$. Since $Prg_{\mathfrak{M}}(U^j)$ entails $\delta \in \mathfrak{M}$, we also have $\delta_{n+1} \in \mathfrak{M}$. Thus, we have $\delta_{n+1} \in \mathfrak{M} \wedge \mathfrak{M} \cap \delta_{n+1} \subseteq U$ as desired.

□

Let $\omega_0(\alpha) := \alpha$ and $\omega_{n+1}(\alpha) = \omega^{\omega_n(\alpha)}$.

Lemma 2.3.31 *(Compare [8] Lemma 3.11)* $\mathfrak{I}(\mathfrak{E}_u)$ *for all* $u \in X$.

Since our background theory assumes that X is contained in an ω-model, and X is well-ordered, we can use transfinite induction over $<_X$.

We begin by observing that we have $\mathfrak{I}(\mathfrak{M} \cap \Omega_\omega + 1)$, by Lemma 2.3.26. Let u_0 be the $<_X$ least element of X. We have $\mathfrak{E}_{u_0} \in \mathfrak{M}$, and for all $\eta < \mathfrak{E}_{u_0}$ there exists n such that $\eta < \omega_n(\Omega_\omega + 1)$. By the preceding corollary, then, we have $Prg_{\mathfrak{M}}(U) \to \mathfrak{M} \cap \mathfrak{E}_{u_0} \subseteq U$, for all $\mathfrak{Y}$-definable sets U.

Now suppose that $u \in X$ is not the $<_X$-least element and for all $v <_X u$ we have $\mathfrak{I}(\mathfrak{E}_v)$. Since, for every $\eta < \mathfrak{E}_u$ there exists $v <_X u$ and $n < \omega$ such that $\eta < \omega_n(\mathfrak{E}_v + 1)$, the inductive assumption, together with the preceding corollary yields

$$Prg_{\mathfrak{M}}(U) \to \mathfrak{M} \cap \mathfrak{E}_u \subseteq U.$$

$\mathfrak{E}_u \in \mathfrak{M}$ is trivial.

Theorem 2.3.32 *(Compare [8] Lemma 3.12) For all* α, $\mathfrak{I}(\alpha)$.

Proof

Proceed by induction on $G\alpha$. Obviously, we have $\mathfrak{I}(0)$ and $\mathfrak{I}(\Omega_n)$ for all $n < \omega$. by Lemma 2.3.26 we also have $\mathfrak{I}(\Omega_\omega)$.

By the preceding lemma, we have $\mathfrak{I}(\mathfrak{E}_u)$ for all $u \in X$.

Suppose $\alpha =_{nf} \omega^{\alpha_1} + \ldots + \omega^{\alpha_n}$. Inductively we have $\mathfrak{I}(\alpha_i)$. Assume $Prg_{\mathfrak{M}}(U)$. Then $Prg_{\mathfrak{M}}(U^j)$ by Lemma 2.3.27. Hence $\alpha_i \cap \mathfrak{M} \subseteq U^j$. Using the definition of U^j repeatedly we conclude that $\alpha \cap \mathfrak{M} \subseteq U$. Moreover, $\alpha \in \mathfrak{M}$ since $\alpha_1, \ldots, \alpha_n \in \mathfrak{M}$.

Suppose $\alpha =_{nf} \varphi\xi\gamma$ with $\xi > 0$. Then $\alpha < \Omega_\omega$. Inductively, we have $\mathfrak{I}(\xi)$ and $\mathfrak{I}(\gamma)$, and thus $\xi, \gamma \in \mathfrak{W}$, whence $\gamma \in \mathfrak{W}$. Since $Prg_{\mathfrak{M}}(U) \to \mathfrak{W} \cap \Omega_\omega \subseteq U$ holds, we get $Prg_{\mathfrak{M}}(U) \to \alpha \in U$. Hence $\mathfrak{I}(\alpha)$.

Suppose $\alpha =_{nf} \psi_n\eta$. Inductively we have $\mathfrak{I}(\eta)$, especially $\eta \in COLL$ by Lemma 2.3.27. Thus $\alpha \in \mathfrak{W}$, which entails $\mathfrak{I}(\alpha)$.

□

Corollary 2.3.33 *(Compare [8] Lemma 3.13) OT*$(\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{x}}}))$ *is a well-ordering.*

Proof

By the preceding theorem, the proof is complete. □

Chapter 3

Prelude to ω-models: Deduction Chains and Majorization

3.1 The Deduction Tree D_Q

3.1.1 Deduction Chains

We wish to prove that if $WOP(\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}}))$ holds, then $\mathfrak{X}$ exists in a countable-coded ω-model of Π^1_1**CA**$_0$ + **BI**. We will prove this using the method of deduction chains. If $Q \subseteq \mathbb{N}$, then a deduction chain for Q is a series of sequents, beginning with the empty sequent. Each step introduces an axiom of Π^1_1**CA**$_0$ + **BI**, and decomposes one of the formulae from the preceding step into subformulae. These deduction chains can then be collected into a deduction tree D_Q. If D_Q is not well-founded, then it provides us with an ω-model $\mathbb{P}$ of Π^1_1**CA**$_0$ + **BI**.

In Chapter 4, we shall embed D_Q into a sequent calculus and leverage cut elimination to prove that D_Q cannot be well-founded.

Definition 3.1.1 *Henceforth, we will use the following conventions:*

1. *We enumerate the free set variables of $\mathcal{L}_2$ using, $U_0, U_1, U_2, \ldots$. If t is a closed $\mathcal{L}_2$ term, then $\bar{t}$ is the numerical value of t.*

2. *A **sequent** is a finite set of $\mathcal{L}_2$ sentences.*

3. *A **literal** is a an atomic sentence or negated atomic sentence, i.e. having the form $R(t_1, t_2, \ldots, t_n)$ or $\neg R(t_1, t_2, \ldots, t_n)$ where R is a predicate, and $t_1, t_2, \ldots t_n$ are closed terms.*

4. *A sequent $\Delta \Rightarrow \Gamma$is **axiomatic**, if:*

   (a) *Γ contains a true literal or Δ contains a false literal.*

   (b) *The formula $s \in U$ is in Γ and the fomula $t \in U$ is in Δ for some set variable U, and closed terms s, t such that $\bar{s} = \bar{t}$. We shall also consider sequents where either $\{t \in U, \neg s \in U\} \subseteq \Gamma$ or $\{t \in U, \neg s \in U\} \subseteq \Delta$ to be axiomatic.*

5. *A sequent is **reducible** if it is not axiomatic, and contains a formula which is not a literal.*

Next, we fix a set $Q \subseteq \mathbb{N}$. Ultimately, our deduction chains will provide an ω-model of $\Pi^1_1\mathbf{CA}_0 + \mathbf{BI}$ containing Q.

Definition 3.1.2

$$\bar{Q}(n) = \begin{cases} \bar{n} \in U_0 & \textit{if } n \in Q \\ \bar{n} \notin U_0 & \textit{otherwise.} \end{cases}$$

Definition 3.1.3 *Let $A_0, A_1, A_2, \ldots$ enumerate the (universal closures of) all instances of $\Pi^1_1 - CA$ and BI. Further, let us assume that A_i is an instance of $\Pi^1_1 - CA$ when i is even, and BI when i is odd.*

Definition 3.1.4 *Suppose $Q \subseteq \mathbb{N}$. A Q-deduction chain is a finite string of sequents,*

$$\Delta_0 \Rightarrow \Gamma_0, \Delta_1 \Rightarrow \Gamma_1, \ldots, \Delta_k \Rightarrow \Gamma_k$$

constructed as follows:

1. *$\Delta_0 \Rightarrow \Gamma_0$ is the sequent $\bar{Q}(0), A_0 \Rightarrow$*

2. *If $i < k$ then $\Delta_i \Rightarrow \Gamma_i$ is not axiomatic.*

3. *If $i < k$ and $\Delta_i \Rightarrow \Gamma_i$ is not reducible, then $\Delta_{i+1} \Rightarrow \Gamma_{i+1}$ is the sequent $\Delta_i, \bar{Q}(i + 1), A_{i+1} \Rightarrow \Gamma_i$,*

4. *If $\Delta_i \Rightarrow \Gamma_i$ is reducible, and $i < k$, then at least one of Δ_i or Γ_i contains a formula E that is not a literal. We call E the **redex**.*

   Suppose $i < k$, $\Delta_i \Rightarrow \Gamma_i$ is reducible, and $\Gamma_i = \Gamma_i', E, \Gamma_i''$ where Γ_i' contains only literals. We obtain $\Delta_{i+1} \Rightarrow \Gamma_{i+1}$ as follows:

   (a) *If $E \equiv \neg E_0$ then $\Delta_{i+1} \Rightarrow \Gamma_{i+1}$ has the form*

   $$\Delta_i, E_0, \bar{Q}(i + 1), A_{i+1} \Rightarrow \Gamma_i', \Gamma_i''.$$

   (b) *If $E \equiv E_0 \wedge E_1$ then $\Delta_{i+1} \Rightarrow \Gamma_{i+1}$ has the form*

   $$\Delta_i, \bar{Q}(i + 1), A_{i+1} \Rightarrow \Gamma_i', E_j, \Gamma_i''$$

   where $j \in \{1, 2, \}$.

   (c) *If $E \equiv E_0 \vee E_1$ then $\Delta_{i+1} \Rightarrow \Gamma_{i+1}$ has the form*

   $$\Delta_i, \bar{Q}(i + 1), A_{i+1} \Rightarrow \Gamma_i', E_0, E_1, \Gamma_i''.$$

   (d) *If $E \equiv \forall x E_0(x)$ then $\Delta_{i+1} \Rightarrow \Gamma_{i+1}$ has the form*

   $$\Delta_i, \bar{Q}(i + 1), A_{i+1} \Rightarrow \Gamma_i', E_0(\bar{t}), \Gamma_i''$$

   for an arbitrary $t \in \mathbb{N}$.

(e) *If* $E \equiv \exists x E_0(x)$ *then* $\Delta_{i+1} \Rightarrow \Gamma_{i+1}$ *has the form*

$$\Delta_i, \bar{Q}(i+1), A_{i+1} \Rightarrow \Gamma'_i, E_0(\bar{j}), E, \Gamma''_i$$

where j *is the least number such that* $E_0(\bar{j}) \notin \Gamma_0, \ldots, \Gamma_i$, *and* $\neg E_0(\bar{j}) \notin \Delta_0, \ldots \Delta_i$.

(f) *If* $E \equiv \forall X E_0(X)$ *then* $\Delta_{i+1} \Rightarrow \Gamma_{i+1}$ *has the form*

$$\Delta_i, \bar{Q}(i+1), A_{i+1} \Rightarrow \Gamma'_i, E_0(U_j), \Gamma''_i$$

for an arbitrary $j \in \mathbb{N}$.

(g) *If* $E \equiv \exists X E_0(X)$ *then* $\Delta_{i+1} \Rightarrow \Gamma_{i+1}$ *has the form*

$$\Delta_i, \bar{Q}(i+1), A_{i+1} \Rightarrow \Gamma'_i, E_0(U_j), E\Gamma''_i$$

where U_j *is the first set variable such that* $E_0(U_j) \notin \Gamma_0, \Gamma_1, \ldots, \Gamma_i$ *and* $\neg E_0(U_j) \notin \Theta_0, \Theta_1, \ldots, \Theta_i$.

5. *Now, suppose* $i < k$, $\Delta_i \Rightarrow \Gamma_i$ *is reducible, and* $\Delta_i = \Delta'_i, E, \Delta''_i$ *where* Δ'_i *contains only literals. We obtain* $\Delta_{i+1} \Rightarrow \Gamma_{i+1}$ *as follows:*

(a) *If* $E \equiv \neg E_0$ *then* $\Delta_{i+1} \Rightarrow \Gamma_{i+1}$ *has the form*

$$\Delta'_i, \Delta''_i, \bar{Q}(i+1), A_{i+1} \Rightarrow E_0, \Gamma_i$$

(b) *If* $E \equiv E_0 \vee E_1$ *then* $\Delta_{i+1} \Rightarrow \Gamma_{i+1}$ *has the form*

$$\Delta'_i, E_j, \Delta''_i, \bar{Q}(i+1), A_{i+1} \Rightarrow \Gamma_i$$

where $j \in \{1, 2, \}$.

(c) *If* $E \equiv E_0 \wedge E_1$ *then* $\Delta_{i+1} \Rightarrow \Gamma_{i+1}$ *has the form*

$$\Delta'_i, E_0, E_1, \Delta''_i, \bar{Q}(i+1), A_{i+1} \Rightarrow \Gamma_i$$

(d) If $E \equiv \exists x E_0(x)$ then $\Delta_{i+1} \Rightarrow \Gamma_{i+1}$ has the form

$$\Delta_i', E_0(\bar{t}), \Delta_i'', \bar{Q}(i+1), A_{i+1} \Rightarrow \Gamma_i$$

for an arbitrary $t \in \mathbb{N}$.

(e) If $E \equiv \forall x E_0(x)$ then $\Delta_{i+1} \Rightarrow \Gamma_{i+1}$ has the form

$$\Delta_i', E_0(\bar{j}), \Delta_i''\bar{Q}(i+1), A_{i+1} \Rightarrow \Gamma_i$$

where j is the least number such that $E_0(\bar{j}) \notin \Delta_0, \ldots, \Delta_i$ and $\neg E_0(\bar{j}) \notin \Gamma_0, \ldots, \Gamma_i$.

(f) If $E \equiv \exists X E_0(X)$ then $\Delta_{i+1} \Rightarrow \Gamma_{i+1}$ has the form

$$\Delta_i', E_0(U_j), \Delta_i'', \bar{Q}(i+1), A_{i+1} \Rightarrow \Gamma_i$$

for any $j \in \mathbb{N}$.

(g) If $E \equiv \forall X E_0(X)$ then $\Delta_{i+1} \Rightarrow \Gamma_{i+1}$ has the form

$$\Delta_i, E_0(U_j), \Delta_i''\bar{Q}(i+1), A_{i+1} \Rightarrow \Gamma_i$$

where U_j is the first set variable such that $E_0(U_j)$ does not appear in $\Delta_0, \ldots, \Delta_i$ and $\neg E_0(U_j)$ does not appear in $\Gamma_0, \ldots, \Gamma_i$.

Definition 3.1.5 *Let D_Q be the set of all Q-deduction chains. Then we call D_Q the* ***deduction tree*** *for Q.*

Claim: If D_Q is ill-founded, then there is a countable-coded ω-model $\mathcal{M}$ of $\Pi^1_1 - \mathbf{CA}_0 + \mathbf{BI}$, such that $Q \in \mathcal{M}$. This is provable in $\mathbf{RCA}_0$.

If D_Q is ill-founded, then it has an infinite path $\mathbb{P}$. We define the sets, M_i, as follows:

$$M_i = \{k | (\bar{k} \notin U_i) \in \mathbb{P}\}.$$

Here, we use the following shorthand. Let F be a formula, and $\Delta_i \Rightarrow \Gamma_i$ be a sequent appearing in $\mathbb{P}$. If $F \in \Delta_i$, then $\neg F \in \mathbb{P}$. If $F \in \Gamma_i$ then $F \in \mathbb{P}$.

We now create the $\mathcal{L}_2$ structure $\mathcal{M} = (\mathbb{N}, \{M_i | i \in \mathbb{N}\}, \in, +, \cdot, 0, 1, <)$. Under the assignment $U_i \mapsto M_i$, we have:

$$\text{If } F \in \mathbb{P}, \text{ then } \mathcal{M} \models \neg F,$$

and thus, $\mathcal{M}$ is an ω-model of Π^1_1**CA**$_0$ + **BI**. To see this, consider the following lemma.

Lemma 3.1.6 *Let Q be a subset of the naturals, and suppose that the corresponding deduction tree, D_Q, is ill-founded. Then D_Q has an infinite path $\mathbb{P}$ with the following properties:*

1. *All literals $E \in \mathbb{P}$ are false.*
2. *$\mathbb{P}$ does not contain both $s \in U_i$ and $t \notin U_i$, where s and t are constant terms such that $\bar{s} = \bar{t}$.*
3. *If $\mathbb{P}$ contains $E_0 \vee E_1$ then $\mathbb{P}$ contains E_0 and E_1.*
4. *If $\mathbb{P}$ contains $E_0 \wedge E_1$ then $\mathbb{P}$ contains E_0 or E_1.*
5. *If $\mathbb{P}$ contains $\exists x E(x)$ then $\mathbb{P}$ contains $E(\bar{n})$ for all $n \in \mathbb{N}$.*
6. *If $\mathbb{P}$ contains $\forall x E(x)$ then $\mathbb{P}$ contains $E(\bar{n})$ for some n.*
7. *If $\mathbb{P}$ contains $\exists X E(X)$ then $\mathbb{P}$ contains $E(U_m)$ for all $m \in \mathbb{N}$.*
8. *If $\mathbb{P}$ contains $\forall x E(x)$ then $\mathbb{P}$ contains $E(U_m)$ for some m.*
9. *$\mathbb{P}$ contains $\neg A_i$ for all i.*

Proof

(1) holds because if $\mathbb{P}$ contained a true literal, then it would contain an axiomatic sequent, and since all deduction chains end at an axiomatic sequent, $\mathbb{P}$ would be finite.

Likewise, (2) must also be true in order to prevent the occurrence of an axiomatic sequent. Note that there is no deduction chain rule which results in the elimination of an atomic formula. Hence, if $\Delta_k \Rightarrow \Gamma_k$ is a sequent appearing in $\mathbb{P}$, and $(s \in U_i) \in \Gamma_k$, then $(s \in U_i) \in \Gamma_{k+i}$ for sequents $\Delta_{k+i} \Rightarrow \Gamma_{k+i}$ appearing in $\mathbb{P}$. Thus, suppose $(s \in U_i) \in \mathbb{P}$ and $(t \notin U_i) \in \mathbb{P}$ with $\bar{t} = \bar{s}$. Then there must be some $\Delta_k \Rightarrow \Gamma_k$ appearing in $\mathbb{P}$ such that $(s \in U_i) \in \Gamma_k$ and either $(t \notin U_i) \in \Gamma_k$, or $(t \in U_i) \in \Delta_k$, which is an axiomatic sequent, producing a contradiction as before.

Conditions (3) through (8) are shown via induction on i, where $\Delta_i \Rightarrow \Gamma_i$ are sequents appearing in $\mathbb{P}$. $\mathbb{P}_n$ shall denote a finite segment of $\mathbb{P}$, containing the sequents $\Delta_0 \Rightarrow \Gamma_0, \ldots, \Delta_n \Rightarrow \Gamma_n$. The base case holds vacuously.

Suppose $(E_0 \vee E_1) \in \mathbb{P}_n$, but $E_0 \notin \mathbb{P}_n$ or $E_1 \notin \mathbb{P}_n$. Then $(E_0 \vee E_1)$ appears in $\mathbb{P}_n$ but is never the redex at any point. Hence the last sequent of $\mathbb{P}_n$. must be either $(i)\Delta_n \Rightarrow \Gamma_n, (E_0 \vee E_1)$, or $(ii)\neg(E_0 \vee E_1)\Delta_n \Rightarrow \Gamma_n$. Since D_Q is the union of all Q-deduction chains, in the case of (i) there is a deduction chain whose next sequent is $\bar{Q}(n+1), A_{n+1}\Delta_n \Rightarrow \Gamma_n, E_0, E_1$. Call this new segment $\mathbb{P}_{n+1}$. Then $E_0, E_1 \in \mathbb{P}_{n+1}$. The case of (ii) is similar, but requires an intermediary step to eliminate the $\neg$ first.

The case of $(E_0 \wedge E_1)$ follows a very similar process, but we must choose between two deduction chains - one containing E_0, the other containing E_1.

If $\forall x E(x) \in \mathbb{P}_{\ltimes}$ but for all m, $E(\bar{m}) \notin \mathbb{P}_n$, then, as before, the final sequent of $\mathbb{P}_n$ must be either $(i)\Delta_n \Rightarrow \Gamma_n, \forall x E(x)$, or $(ii)\neg\forall x E(x), \Delta_n \Rightarrow \Gamma_n$. Without loss of generality, we shall assume (i) is the case. Then for each $m \in \mathbb{N}$, there is a deduction chain in D_Q such that $\bar{Q}(n+1), A_{n+1}, \Delta_n \Rightarrow \Gamma_n, E(\bar{m})$ is the next sequent in the deduction chain. We may pick any of these chains, and call the new segment $\mathbb{P}_{n+1}$, with $E(\bar{m}) \in \mathbb{P}$. The same processes applies for $\forall X E(X)$.

If $\exists x E(x) \in \mathbb{P}_n$ but $E(\bar{m}) \notin \mathbb{P}_n$ for some m. Without loss of generality, assume this is the least such m. Then the final sequent of $\mathbb{P}_n$ is either $(i)\Delta_n \Rightarrow \Gamma_n, \exists x E(x)$, or $(ii)\neg\exists x E(x), \Delta_n \Rightarrow \Gamma_n$, since $\exists x E(x)$ is retained even if it is the redux formula. Again, without loss of generality we shall assume (i) is the case. Then there is a deduction chain in D_Q such that the next sequent is $\bar{Q}(n+1), A_{n+1}, \Delta_n \Rightarrow \Gamma_n, E(\bar{m}), \exists x E(x)$. We may, of course, iterate this process to find larger values of m if desired. The case for $\exists X E(X)$ is identical.

Thus, $\mathbb{P} = \bigcup_{n\in\mathbb{N}} \mathbb{P}_n$, and satisfies all of conditions (3) through (8).

Finally, we know $\neg A_i \in \mathbb{P}$ for all $i \in \mathbb{N}$, since the axioms are introduced on the left at every step of the deduction chains.

□

It is, of course, clause (9) that guarantees that $\mathcal{M}$ is an ω-model of $\Pi^1_1 - \mathbf{CA}_0 + \mathbf{BI}$, since $\mathcal{M} \models A_i$ for all i, where A_i are the axioms of $\Pi^1_1 - \mathbf{CA}_0 + \mathbf{BI}$.

3.2 Majorization and Fundamental Functions

In order to properly carry out the ordinal analysis of $\Pi^1_1\mathbf{CA}_0 + \mathbf{BI}$, we require a majorization relation and fundamental functions. These are necessary for the formulation of the Ω_{k+1}-rules in T^*_Q, which is our sequent calculus analogue. In particular, majorization and fundamental functions ensure that we can use $\psi_k(\alpha)$ to indicate proof heights, while ensuring that $\alpha \in C_k(\alpha)$. For more details see section 1.3 of [2].

In what follows, we shall use ω^γ as shorthand for $\varphi 0\gamma$, we shall use ω^α_n as shorthand for

$\omega^{\cdot^{\cdot^{\cdot^{\omega^\alpha}}}}$ iterated n times. We shall also let $\varphi_0\gamma = \gamma$ and $\varphi_{n+1}\gamma = \varphi(\varphi_n\gamma)1$.

3.2.1 Majorization

Definition 3.2.1 *1.* $\alpha \lhd_\tau \beta$ *if* $\alpha < \beta$ *and for all* δ, k, η *we have:*

$$(\alpha \leq \delta \leq min\{\beta, \eta\}) \wedge (\delta, \tau \in C_k(\eta)) \implies \alpha \in C_k(\eta)$$

2. $\alpha \lhd \beta$ *(*α *is* ***majorized*** *by* β*) if* $\alpha \lhd_0 \beta$

3. $\alpha \unlhd \beta$ *if either* $\alpha \lhd \beta$ *or* $\alpha = \beta$

The following basic properties are immediate consequences of the definition.

Lemma 3.2.2 (Basic properties) *[See [2], Lemma 4.1]*

1. *If* $\alpha \lhd \beta$ *then* $\alpha \lhd_\tau \beta$.
2. *If* $\alpha < \beta$ *then* $\alpha \lhd_\alpha \beta$.
3. *If* $\alpha < \beta < \gamma$ *and* $\alpha \lhd_\tau \gamma$ *then* $\alpha \lhd_\tau \beta$.
4. *If* $\alpha < \varphi 10$ *and* $\alpha < \beta$*, then* $\alpha \lhd \beta$.
5. *If* $0 < \beta < \varphi 10$ *then* $\alpha \lhd \alpha + \beta$.
6. *If* $\alpha < \beta < \Omega_1$ *then* $\alpha \lhd \beta$.
7. *If* $\alpha \lhd \beta$ *then* $\alpha + 1 \unlhd \beta$.
8. *If* $\Omega_i < \Omega_j$ *then* $\Omega_i \lhd \Omega_j$.
9. *If* $\mathfrak{E}_u < \mathfrak{E}_v$ *then* $\mathfrak{E}_u \lhd \mathfrak{E}_v$

10. *For all $i \leq \omega$ and all $u \in \mathfrak{X}, \Omega_i \lhd \mathfrak{E}_u$.*

Proof

(1) This is trivial.

(2) Obviously, when $\tau = \alpha$ we get $\alpha \in C_k(\eta) \to \alpha \in C_k(\eta)$, so if $\alpha < \beta$ then $\alpha \lhd_\alpha \beta$.

(3) Suppose $\alpha < \gamma < \beta$ and $\alpha \lhd_\tau \gamma$. Then, we have for all δ, τ $(\alpha \leq \delta \leq min\{\gamma, \eta\}) \wedge (\delta, \tau \in C_k(\eta)) \implies \alpha \in C_k(\eta)$. Since $\gamma < \beta$, if we assume that $(\alpha \leq \delta \leq \min\{\beta, \eta\}) \wedge (\delta \in C_k(\eta))$, then this is merely a stronger version of the same condition, and hence $\alpha \in C_k(\eta)$ as desired.

(4) Note that if $\alpha < \beta < \varphi 10$, then α and β are constructed solely using $0, +$, and ω^x. As all $C_k(\eta)$ are closed under $+$ and ω, it follows that $\alpha \in C_k(\eta)$ for any η, and thus $\alpha \lhd \beta$.

(5) Suppose $\beta < \varphi 10$. We wish to show that if $\alpha \in C_k(\eta)$ then $\alpha + \beta \in C_k(\eta)$. But, as noted in the proof for part (4), since $\beta < \varphi 10$ it follows that $\beta \in C_k(\eta)$, and thus by closure under $+$, we have $\alpha + \beta \in C_k(\eta)$.

(6) We prove this by induction on the construction of α. If $\alpha = 0$, then $0 \lhd \beta$ follows from (4). Otherwise, suppose the statement holds up to α, and that $\beta \in C_k(\eta)$. If $\alpha =_{nf} \omega^{\alpha_0} + \alpha_1$ or $\alpha =_{nf} \varphi \alpha_0 \alpha_1 + \alpha_2$ then by induction hypothesis we get $\alpha_0, \alpha_1, \alpha_2 \lhd \beta$, and hence $\alpha \in C_k(\eta)$ by closure. Thus, the critical case is when $\alpha = \psi_0(\alpha_0)$, with $\alpha_0 \in C_0(\alpha_0)$. Note that by Lemma 2.2.4 (5), $\psi_0(\alpha_0) = C_0(\alpha_0) \cap \Omega_1$. So either $\alpha_0 < \psi_0(\alpha_0) < \beta$ (in which case the argument follows similarly to the previous cases) or $\Omega_{k+1} \leq \alpha$ for some k. Thus, let us consider the latter case.

Then $\beta = \psi_0(\beta_0)$ with $\beta_0 \in C_0(\beta_0)$, and $\alpha_0 < \beta_0$. Hence, if $\beta \in C_k(\eta)$, then $\alpha_0 < \beta_0 < \eta$. Thus $\psi_0(\alpha_0) < \psi_0(\eta)$. By Lemma 2.2.4 (5), we may restate this as $\alpha \in (C_0(\eta) \cap \Omega_1) \subseteq C_k(\eta)$. Hence $\alpha \lhd \beta$.

(7) Assume $\alpha \lhd \beta$. Then, $\alpha < \beta$, so either $\alpha + 1 = \beta$ in which case we are done, or $\alpha + 1 < \beta$. We then observe that if $\alpha \in C_k(\eta)$, then $\alpha + 1 \in C_k(\eta)$ by closure under addition.

(8) Observe that $\Omega_i \in C_k(\eta)$ for all k and all η. Hence if $\Omega_i < \Omega_j$ then $\Omega_i \lhd \Omega_j$.

(9) & (10) Since $\mathfrak{E}_u \in C_k(\eta)$ for all u, k, and η, these results follow from a similar argument to (8).

□

Lemma 3.2.3 (See [2], Lemma 4.2) $\alpha \lhd_\tau \beta$ *and* $\beta \lhd_\tau \gamma \implies \alpha \lhd_\tau \gamma$

Proof

Clearly, we have $\alpha < \gamma$. Suppose $\alpha \leq \delta \leq min\{\gamma, \eta\}$, and $\delta, \tau \in C_k(\eta)$. If $\delta \leq \beta$, then by $\alpha \lhd_\tau \beta$ we have $\alpha \in C_k(\eta)$. If $\beta < \delta$, then by $\beta \lhd_\tau \gamma$ we have $\beta < min\{\gamma, \eta\}$, which means $\beta \in C_k(\eta)$. By $\alpha \lhd_\tau \beta$, it again follows that $\alpha \in C_k(\eta)$. Hence, $\alpha \lhd_\tau \gamma$. □

Lemma 3.2.4 (See [2], Lemma 4.3) *If* $\alpha \lhd_\tau \beta$ *and* $\beta < \omega^{\gamma+1}$*, then* $\omega^\gamma + \alpha \lhd_\tau \omega^\gamma + \beta$.

Proof

By assumption, we have $\omega^\gamma + \alpha \leq \omega^\gamma + \beta$. Suppose $\omega^\gamma + \alpha \leq \delta \leq min\{\omega^\gamma + \beta, \eta\}$ and $\delta, \tau \in C_k(\eta)$. By normal form, we know that $\delta = \omega^\gamma + \delta_0$, where $\alpha \leq \delta_0 \leq min\{\beta, \eta\}$. By definition of $C_k(\eta)$, it follows that $\gamma, \delta_0 \in C_k(\eta)$, and by $\alpha \lhd_\tau \beta$ we have $\alpha \in C_k(\eta)$. It thus follows that $\omega^\gamma + \alpha \in C_k(\eta)$. □

Corollary 3.2.5 (See [2] Corollary 4.3) $(\omega^\alpha) \cdot n \lhd (\omega^\alpha) \cdot (n + 1)$

This follows easily by induction on n from the previous lemma, with the base case being $0 \lhd \omega^\alpha$, using Lemma 3.2.2, part 4.

Lemma 3.2.6 *If $\alpha \lhd_\tau \beta$ then $\omega^\alpha \cdot n \lhd_\tau \omega^\beta$.*

Proof

Clearly, we have $\omega^\alpha \cdot n < \omega^\beta$. When $n = 0$, this follows directly from lemma 3.2.2 (4). Otherwise, assume $\omega^\alpha \cdot n \leq \delta \leq min\{\omega^\beta, \eta\}$ with $\delta, \tau \in C_k(\eta)$. By normal form, we know $\delta = \omega^{\delta_1} + \delta_2$, where $\alpha \leq \delta_1 \leq min\{\beta, \eta\}$, and $\delta_1 \in C_k(\eta)$. Hence, by $\alpha \lhd_\tau \beta$, we know $\alpha \in C_k(\eta)$, and thus $\omega^\alpha \cdot n \in C_k(\eta)$

□

Lemma 3.2.7 *Let us fix an ordinal η. Then:*

1. *if $\alpha \lhd_\tau \beta$ and $\beta < \varphi\eta(\gamma + 1)$, then $(\varphi\eta\gamma) + \alpha \lhd_\tau (\varphi\eta\gamma) + \beta$.*
2. *for all α we have $(\varphi\eta\alpha) \cdot n \lhd (\varphi\eta\alpha) \cdot (n + 1)$.*
3. *if $\alpha \lhd_\tau \beta$ then $(\varphi\eta\alpha) \cdot n \lhd \varphi\eta\beta$.*

Proof

(1) Suppose $\alpha \lhd_\tau \beta$ and $\beta < \varphi\eta(\gamma + 1)$. Since $\alpha < \beta < \varphi\eta(\gamma + 1)$, we know that $(\varphi\eta\gamma) + \alpha$ and $(\varphi\eta\gamma) + \beta$ are in normal form. Now, suppose we have δ, k, ξ such that $(\varphi\eta\gamma) + \alpha \leq \delta \leq \min\{(\varphi\eta\gamma) + \beta, \xi\}$ and $\delta, \tau \in C_k(\xi)$. Using normal form, we know that $\delta = (\varphi\eta\gamma) + \delta_0$, with $(\varphi\eta\gamma), \delta_0 \in C_k(\xi)$ and $\alpha \leq \delta_0 \leq \min\{\beta, \xi\}$. Thus, we may apply $\alpha \lhd_\tau \beta$ to obtain $\alpha \in C_k(\xi)$. Combined with $(\varphi\eta\gamma) \in C_k(\xi)$ this gives us $(\varphi\eta\gamma) + \alpha \in C_k(\xi)$ as desired.

(2) The base case, where $0 \lhd \varphi\eta\alpha$ follows from 3.2.2 (2). Otherwise, we know that $(\varphi\eta\alpha) \cdot n < (\varphi\eta\alpha) \cdot n(n+1)$. Suppose that we have δ, k, ξ such that $(\varphi\eta\alpha) \cdot n \leq \delta \leq \min\{(\varphi\eta\alpha) \cdot n(n+1), \xi\}$. Then $\delta =_{nf} \varphi\eta\alpha + \delta_0$ with $\varphi\eta\alpha \in C_k(\xi)$. Thus, by closure under addition, $(\varphi\eta\alpha) \cdot n \in C_k(\xi)$.

(3) In the case where $n = 0$, this follows from 3.2.2 (2). Otherwise, suppose $\alpha \lhd_\tau \beta$. Then obviously $(\varphi\eta\alpha) \cdot n \lhd_\tau \varphi\eta\beta$. Now assume we have δ, k, ξ such that $(\varphi\eta\alpha) \cdot n \leq \delta \leq \min\{\varphi\eta\beta, \xi\}$, and $\delta, \tau \in C_k(\xi)$. Then $\delta = (\varphi\eta\delta_0) + \delta_1$ with $\eta, \delta_0, \delta_1 \in C_k(\xi)$ and $\alpha \leq \delta_0 \leq \min\{\beta, \xi\}$. Using $\alpha \lhd_\tau$ we have $\alpha \in C_k(\xi)$ and combined with $\eta \in C_k(\xi)$ and closure under addition we get $(\varphi\eta\alpha) \in C_k(\xi)$.

□

Lemma 3.2.8 (See [2] Lemma 4.6) *If $\alpha \lhd_\tau \beta$, with $\tau \in C_k(\alpha)$ and $\beta \in C_k(\beta)$ then $\alpha \in C_k(\alpha)$ and $\psi_k(\alpha) \lhd_\tau \psi_k(\beta)$.*

Proof

First, we shall show that $\alpha \in C_k(\alpha)$. By Lemma 2.2.7 we know that for $\gamma = \min\{\xi | \alpha \leq \xi \in C_k(\alpha)\}$, we have $\gamma \in C_k(\gamma) = C_k(\alpha)$. By assumption, we also have $\tau \in C_k(\gamma)$. Since $\beta \in C_k(\beta)$ we have $\alpha \leq \gamma = \min\{\beta, \gamma\}$. Thus, by definition of $\alpha \lhd_\tau \beta$, we have $\alpha \in C_k(\gamma) = C_k(\alpha)$.

Since $\alpha \in C_k(\alpha)$ and $\beta \in C_k(\beta)$, with $\alpha < \beta$, it follows that $\psi_k\alpha < \psi_k\beta$. Now, suppose $\psi_k\alpha \leq \delta \leq \min\{\psi_k\beta, \eta\}$ and $\delta, \tau \in C_m(\eta)$. We shall prove $\psi_k\alpha \in C_m(\eta)$ by induction on the construction of δ. Since $\psi_k\alpha \leq \delta \leq \psi_k\beta$, we know $\Omega_k \leq \delta < \Omega_{k+1}$. Thus, we must consider the following cases:

1. If $\delta < \Omega_m$ then $\psi_k\alpha < \Omega_m$, and thus $\psi_k\alpha \in C_m(\eta)$.

2. If $\Omega_m < \delta =_{nf} \varphi\delta_0, \delta_1$, then $\delta_0, \delta_1 \in C_m(\eta)$. Recall that $\psi_k\alpha = \varphi\psi_k(\alpha)0$. It follows, therefore, that either $\psi_k\alpha \leq \delta_0$ or $\psi_k\alpha < \delta_1$. Either way, owing to the induction hypothesis, $\psi_k\alpha \in C_m(\eta)$.

3. If $\Omega_m < \delta =_{nf} \delta_0 + \delta_1$ for $\delta_0, \delta_1 \in C_m(\eta)$, then $\psi_k\alpha \leq \delta_0$, and thus $\psi_k\alpha \in C_m(\eta)$.

4. If $\Omega_m < \delta =_{nf} \psi_m\delta_0$, then $\delta_0 \in C_m(\eta)$ and $\delta_0 < \eta$. Now, if $k < m$ then $\psi_k\alpha < \Omega_m$ and thus $\psi_k\alpha \in C_m(\eta)$. Otherwise, $k = m$. Then $\alpha < \delta_0 < \eta$. Furthermore, since $\psi_k\delta_0 < \psi_k\beta$, we have $\delta_0 < \beta$. Since $\alpha \lhd_\tau \beta$ we have $\alpha \in C_m(\eta)$, and since $\alpha < \eta$ we have $\psi_k\alpha \in C_m(\eta)$.

$\square$

Corollary 3.2.9 (See [2] Corollary 4.6) $\alpha = \alpha_0 + 1 \in C_k(\alpha)$ *implies* $\alpha_0 \in C_k(\alpha)$ *and* $\psi_k(\alpha_0) \lhd \psi_k(\alpha)$.

Proof

The proof follows immediately from the preceding lemma with $\tau = 0$. We need only note that $\alpha_0 \lhd \alpha$, via Lemma 3.2.2 (5), since $1 < \varphi 10$.

$\square$

3.2.2 Fundamental Functions

Definition 3.2.10 *A function* $f : dom(f) \to OT(\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{x}}}))$ *with the domain* $dom(f) \subseteq OT(\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{x}}}))$ *is a* ***fundamental function*** *if the following hold:*

1. *If $\beta \in$ dom(f) and $\alpha < \beta$, then $\alpha \in$ dom(f) and $f(\alpha) \lhd_\alpha f(\beta)$.*

2. *If $\beta \in$ dom(f) and $f(0) \leq \delta < f(\beta)$ then there is an $\alpha \in$ dom(f) such that $f(\alpha) \leq \delta < f(\alpha+1)$ and $f(\alpha) \lhd f(\alpha+1)$.*

3. *If $\alpha \in$ dom(f) and $f(\alpha) \in C_k(\eta)$ then $\alpha \in C_k(\eta)$.*

Lemma 3.2.11 (See [2] Lemma 5.1) *If f is a fundamental function and $\alpha \in$ dom(f), then $\alpha \leq f(\alpha)$.*

Proof

Suppose for a contradiction that α is the least ordinal in dom(f) such that $f(\alpha) < \alpha$. Then, by property (1) of fundamental functions, $f(\alpha) \in$ dom(f) and $f(f(\alpha)) \lhd_{f(\alpha)} f(\alpha)$. So in particular, $f(f(\alpha)) < f(\alpha)$, but this contradicts our initial assumption. Thus, $\alpha \leq f(\alpha)$ for all $\alpha \in$ dom(f).

□

Definition 3.2.12 *Let Id_β be the identity function with dom$(Id_\beta) = \{\alpha \in OT(\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}}))|\alpha \leq \beta\}$ and $Id_\beta(\alpha) = \alpha$ for all $\alpha \in$ dom(Id_β).*

Lemma 3.2.13 (See [2] Lemma 5.2) *Id_β is a fundamental function.*

Proof

This is obvious from the definition of a fundamental function. □

Definition 3.2.14 *Let f be a fundamental function.*

1. *$\omega^\gamma + f$ is the function with dom$(\omega^\gamma + f) = \{\alpha \in$ dom$(f) | f(\alpha) < \omega^{\gamma+1}\}$ and $(\omega^\gamma + f)(\alpha) = \omega^\gamma + f(\alpha)$ for all $\alpha \in$ dom$(\omega^\gamma + f)$.*

2. *ω^f is the function with dom(ω^f) = dom(f) and $(\omega^f)(\alpha) = \omega^{f(\alpha)}$ for all $\alpha \in$ dom(ω^f).*

3. *$\varphi\gamma f$ is the function with dom$(\varphi\gamma f)$ = dom(f), where $(\varphi\gamma f)(\alpha) = \varphi\gamma(f\alpha)$.*

4. *Let $\psi_k f$ be the function with dom$(\psi_k f) = \{\alpha \in$ dom$(f) | \alpha < \Omega_{k+1}, f(\alpha) \in C_k(f(\alpha))\}$ and $(\psi_k f)(\alpha) = \psi_k(f(\alpha))$ for all $\alpha \in$ dom$(\psi_k f)$.*

Lemma 3.2.15 (See [2] Lemma 5.3) *If f is a fundamental function, then so are*

1. *$\omega^\gamma + f$*

2. *ω^f*

3. *$\varphi\gamma f$*

4. *$\psi_k f$*

Proof

We shall briefly sketch the proofs for the first three functions. The proof for ψ_k is considerably more involved, and will be handled in full.

1. Property (1) follows by virtue of Lemma 3.2.4. For property (2), we observe that if $\omega^\gamma + f(0) \leq \delta < \omega^\gamma + f(\beta)$, then $\delta = \omega^\gamma + \delta_0$, with $f(0) \leq \delta < f(\beta)$, and hence α can be found such that $f(\alpha) \leq \delta_0 < f(\alpha + 1)$. To show $\omega^\gamma + f(\alpha) \lhd \omega^\gamma + f(\alpha + 1)$, note that if $\delta \in C_k(\eta)$, then $\omega^\gamma, \delta_0 \in C_k(\eta)$ by closure, and from $f(\alpha) \lhd f(\alpha + 1)$, combined with $f(\alpha) \leq \delta_0 < f(\alpha + 1)$, we get $f(\alpha) \in C_k(\eta)$.

Hence, by closure, $\omega^\gamma + f(\alpha) \in C_k(\eta)$, as needed. Property (3) is shown by first noting that if $\omega^\gamma + f(\alpha) \in C_k(\eta)$ then $f(\alpha) \in C_k(\eta)$. $\alpha \in C_k(\eta)$ then follows because f is fundamental.

2. Property (1) holds due to Lemma 3.2.6, where $n = 1$. For property (2), we note that if $\omega^{f(0)} \leq \delta < \omega^{f(\beta)}$ then $\delta = \omega^{\delta_0} + \delta_1$, with $f(0) \leq \delta_0 < f(\beta)$. Since f is fundamental, we can find α such that $f(\alpha) \leq \delta_0 < f(\alpha+1)$, and thus $\omega^{f(\alpha)} \leq \delta < \omega^{f(\alpha+1)}$. Supposing $\delta \in C_k(\eta)$, we find that $\delta_0 \in C_k(\eta)$ by closure. Since $f(\alpha) \lhd f(\alpha+1)$, this gives us $f(\alpha) \in C_k(\eta)$ and, by closure, $\omega^{f(\alpha)} \in C_k(\eta)$. Hence, $\omega^{f(\alpha)} \lhd \omega^{f(\alpha+1)}$. Property (3) is proven by noting that if $\omega^{f(\alpha)} \in C_k(\eta)$ then $f(\alpha) \in C_k(\eta)$. $\alpha \in C_k(\eta)$ then follows because f is fundamental.

3. Property (1) holds due to Lemma 3.2.7, where $n = 1$. For property (2), we note that if $\varphi\gamma f(0) \leq \delta < \varphi\gamma f(\beta)$ then $\delta = (\varphi\gamma\delta_0) + \delta_1$, with $f(0) \leq \delta_0 < f(\beta)$. Since f is fundamental, we can find α such that $f(\alpha) \leq \delta_0 < f(\alpha+1)$, and thus$(\varphi\gamma f(\alpha)) \leq \delta_0 < (\varphi\gamma f(\alpha+1))$. If $\delta_0 \in C_k(\eta)$, then since $f(\alpha) \lhd f(\alpha+1)$, we get $f(\alpha) \in C_k(\eta)$, and hence, by closure, $\varphi\gamma f(\alpha) \in C_k(\eta)$. Thus $\varphi\gamma f(\alpha) \lhd \varphi\gamma f(\alpha+1)$. Property (3) follows immediately via the closure of $C_n(\eta)$, where $\varphi\gamma f(\alpha) \in C_n(\eta)$, since f is fundamental.

4. We now consider the ψ_k function.

   (a) Suppose $\beta \in \text{dom}(\psi_k f)$ and $\alpha < \beta$. We must show $\alpha \in \text{dom}(\psi_k f)$ and $\psi_k f(\alpha) \lhd_\alpha \psi_k f(\beta)$.

   Since f is a fundamental function and $\alpha < \beta$ we have $\alpha \in \text{dom}(f)$ and $f(\alpha) \lhd_\alpha f(\beta)$. We also have $f(\beta) \in C_k(f(\beta))$ and $\beta < \Omega_{k+1}$ by the definition of $\text{dom}(\psi_k f)$.

   By Lemma 2.2.7 there is $\gamma = \min\{\xi | f(\alpha) \leq \xi \in C_k(f(\alpha))\}$ such that $C_k(\gamma) = C_k(f(\alpha))$. Since $f(\beta) \in C_k(f(\beta))$, it follows that $f(\alpha) \leq \gamma \leq f(\beta)$.

If $\gamma = f(\beta)$, then $f(\beta) \in C_k(f(\beta)) = C_k(f(\alpha))$. By property (3) of fundamental functions, this implies $\beta \in C_k(f(\alpha))$. Since $\alpha < \beta < \Omega_{k+1}$, and $\beta \in C_k(f(\alpha))$ it follows that $\alpha \in C_k(f(\alpha))$.

If $\gamma < f(\beta)$, then there exists $\tau < \beta$ such that $f(\tau) \leq \gamma < f(\tau + 1)$, $\alpha \leq \tau$ and $f(\tau) \lhd f(\tau + 1)$. Since $\gamma \in C_k(\gamma)$, it follows that $f(\tau) \in C_k(\gamma)$. By property (3) of fundamental functions this means $\tau \in C_k(\gamma)$ and since $\alpha \leq \tau$ we get $\alpha \in C_k(\tau) = C_k(f(\alpha))$.

Thus, we have $\alpha \in C_k(f(\alpha))$. Combining this with the fact that $f(\alpha) \lhd_\alpha f(\beta)$, it follows that $f(\alpha) \in C_k(f(\alpha))$. Hence $\alpha \in \text{dom}(\psi_k f)$ and $\psi_k f(\alpha) \lhd \psi_k f(\beta)$ by Lemma (3.2.8).

(b) Suppose $\beta \in \text{dom}(\psi_k f)$ and $\psi_k f(0) \leq \delta < \psi_k f(\beta)$. We shall prove there exists $\alpha < \beta$ such that $f(\alpha) \leq \delta < f(\alpha + 1)$, using induction on the construction of δ.

For the base case, if $\delta = 0, \Omega_m$, or $\mathfrak{E}_u$ then δ cannot be between $\psi_k f(0)$ and $\psi_k f(\beta)$, so the statement holds vacuously.

If $\delta =_{nf} \delta_0 + \delta_1$, then by induction hypothesis, there is α such that $\psi_k f(\alpha) \leq \delta_0 < \psi_k f(\alpha + 1)$, and thus $\psi_k f(\alpha) \leq \delta_0 + \delta_1 < \psi_k f(\alpha + 1)$.

If $\delta =_{nf} \varphi\delta_0\delta_1$. Choose the greater of δ_0, δ_1. We shall assume $\delta_0 > \delta_1$ in this case, though the opposite argument proceeds much the same way. By our induction hypothesis, we find $\psi_k f(\alpha) \leq \delta_0 < \psi_k(\alpha + 1)$. Then by Lemma 2.1.4 we see that $\psi_k f(\alpha) \leq \varphi\delta_0\delta_1 \leq \varphi\delta_0\delta_0 < \psi_k(\alpha + 1)$, as required.

(c) Suppose $\beta \in \text{dom}(\psi_k f)$. If $k < m$ then obviously $\beta \in C_m(\eta)$ since $\beta < \Omega_{k+1}$. If $k = m$ and $\psi_k f(\beta) \in C_m(f(\eta))$ then $f(\beta) \in C_m(\eta)$ by definition, and since f is fundamental, it follows that $\beta \in C_m(\eta)$.

□

The following Lemma is central to the proof of the collapsing theorem in Chapter 4, which sets an upper bound on the height of proofs.

Lemma 3.2.16 (See [2] Lemma 5.4) *If f is a fundamental function with $\alpha, \Omega_{k+1} \in$ dom(f), $\alpha < \beta = \psi_k(f(\alpha))$, and $f(\alpha) \lhd f(\Omega_{k+1})$, then $f(\beta) \lhd f(\Omega_{k+1})$.*

Proof

Since $\beta = \psi_k f(\alpha) < \Omega_{k+1}$, by property (3) of fundamental functions we obtain $f(\beta) \lhd_\beta f(\Omega_{k+1})$. Now suppose that $f(\beta) \leq \delta \leq \min\{\Omega_{k+1}, \eta\}$ and $\delta \in C_t(\eta)$, for some t. We will prove that $\beta \in C_t(\eta)$.

Since $\alpha < \beta$, we get $f(\alpha) < f(\beta) < \eta$. Moreover, $f(\alpha) \leq \delta \leq \min\{f(\Omega_{k+1}), \eta\}$, so by $f(\alpha) \lhd f(\Omega_{k+1})$, we have $f(\alpha) \in C_t(\eta)$.

Since $f(\alpha) \in C_t(\eta)$ and $f(\alpha) < \eta$, it follows that $\beta = \psi_k f(\alpha) \in C_t(\eta)$, as desired. Applying $f(\beta) \lhd_\beta f(\Omega_{k+1})$, we get $f(\beta) \in C_t(\eta)$. Thus $f(\beta) \lhd f(\Omega_{k+1})$.

$\square$

Readers may note that the above proof is somewhat simpler than that presented in [2]. This is because we are using only a fragment of the full ordinal representation system presented in that book. In the full system, there is a function $\Omega_x : \tau \mapsto \Omega_\tau$, where τ is itself an ordinal term. When dealing with larger τ, one has to be careful, and ensure that $\tau \in C_t(\eta)$ before applying the corresponding ψ_τ function.

Corollary 3.2.17 (See [2] Corollary 5.4) *If f is a fundmental function with $\Omega_{k+1} \in$ dom(f) then $f(\psi_k f(0)) \lhd f(\Omega_{k+1})$.*

Proof

This is a direct application of the preceding lemma, with $\alpha = 0$, observing that

$f(0) \lhd f(\Omega_{k+1})$. $\square$

Lemma 3.2.18 (See [2] Lemma 5.5) *Let f be a fundamental function, where $\Omega_{k+1} \in$ dom$(f), j \leq k, f(\Omega_{k+1}) \in C_j(f(\Omega_{k+1}))$, and let (β_n) be the sequence where $\beta_0 = 0$ and $\beta_{n+1} = \psi_j(f(\beta_n))$. Let g be the function with dom$(g) = \{\alpha|\alpha \leq \omega\}$, where $g(n) = \psi_j(f(\beta_n))$, and $g(\omega) = \psi_j(f(\Omega_{k+1}))$. Then g is a fundamental function.*

Proof

To prove the first property of fundamental functions, we begin by noting that if $\gamma \in$ dom$(g) = \{\alpha|\alpha \leq \omega\}$, then for all $\alpha < \gamma, \alpha \in$ dom(g).

Next, we shall prove

$$(1)\ \beta_n < \beta_{n+1} \text{ and } f(\beta_n) \lhd f(\Omega_{k+1})$$

using induction on n.

When $n = 0$ we have $0 < \psi_k(f(0))$ and by part 4 of Lemma 3.2.2 $0 \lhd f(\Omega_{k+1})$.

For the induction step, assume $\beta_n < \beta_{n+1}$ and $f(\beta_n) \lhd f(\Omega_{k+1})$. Note that $f(\Omega_{k+1}) \in C_j(f(\Omega_{k+1})) \subseteq C_k(f(\Omega_{k+1}))$. Combining this with $f(\beta_n) \lhd f(\Omega_{k+1})$, we apply Lemma 3.2.8 to obtain $f(\beta_n) \in C_k(f(\beta_n))$. Since f is fundamental, and $\beta_n < \beta_{n+1}$ we get $f(\beta_n) < f(\beta_{n+1})$. Hence $\psi_j(f(\beta_n)) = \beta_{n+1} \in C_j(f(\beta_{n+1}))$, and thus

$$\beta_{n+1} = \psi_j(f(\beta_n)) < \psi_j(f(\beta_{n+1})) = \beta_{n+2}.$$

Since $\beta_n < \beta_{n+1}$ and $f(\beta_n) \lhd f(\Omega_{k+1})$, then by Lemma 3.2.16 we get $f(\psi_k(f(\beta_n))) = f(\beta_{n+1}) \lhd f(\Omega_{k+2})$, which completes the inductive proof of (1).

Next, using $f(\beta_n) \lhd f(\Omega_{k+1})$ and $f(\Omega_{k+1}) \in C_j(f(\Omega_{k+1}))$ we apply Lemma 3.2.8 to get $f(\beta_n) \in C_j(f(\beta_n))$. Combined with $f(\beta_n) < f(\beta_{n+1})$, this yields

$$(2)\ g(n) = \psi_j(f(\beta_n)) < \psi_j(f(\beta_{n+1})) = g(n+1).$$

Applying 3.2.8 once more yields

$$(3)\ g(n) = \psi_j(f(\beta_n)) \lhd \psi_j(f(\Omega_{k+1})) = g(\omega).$$

Hence, if $\gamma \in \mathrm{dom}(g)$ and $\alpha < \gamma$, then $g(\alpha) \lhd \psi_j(f(\Omega_{k+1}))$. By part 3 of Lemma 3.2.2, $g(\alpha) \lhd g(\gamma)$, and thus $g(\alpha) \lhd_\alpha g(\gamma)$. Thus, we have proven the first property of fundamental functions for g.

For the second property, we begin by proving

$$(4)\ \text{if } \gamma < \psi_k(f(\Omega_{k+1})),\text{. then there is } n \text{ such that } \beta_n \leq \gamma < \beta_{n+1}.$$

using induction on the construction of γ.

For the base case, if $\gamma < \beta$ then $\beta_0 = 0 \leq \gamma < \beta_1$.

Otherwise, assume $\beta_1 = \psi_k(f(0)) \leq \gamma < \psi_k(f(\Omega_{k+1}))$. If $\gamma =_{nf} \omega^{\gamma_0} < \gamma_1$, then the assertion follows immediately from the induction hypothesis. If $\gamma =_{nf} \varphi\gamma_0\gamma_1$ then $\gamma_0, \gamma_1 < \varphi\gamma_0\gamma_1$. We take the greater of γ_0 and γ_1. We shall assume γ_0 in this case, but the proof is similar for γ_1. Then by induction hypothesis, we can find $\beta_n \leq \gamma_0 < \beta_{n+1}$. Using lemma 2.1.4, we have $\beta_n \leq \varphi\gamma_0\gamma_1 < \varphi\gamma_0\gamma_0 < \beta_{n+1}$.

This leaves the case where $\gamma = \psi_k(\gamma_0)$, with $\gamma_0 \in C_k(\gamma_0)$, with $f(0) \leq \gamma_0 < f(\Omega_{k+1})$. Since f is fundamental, we may find $\alpha < \Omega_{k+1}$ such that $f(\alpha) \leq \gamma_0 < f(\alpha+1)$ and $f(\alpha) \lhd f(\alpha+1)$. Applying part 3 of Lemma 3.2.2 yields $f(\alpha) \lhd \gamma_0 <$ and since $\gamma_0 \in C_k(\gamma_0)$, we may apply Lemma 3.2.8 to get $f(\alpha) \in C_k(f(\alpha)) \subseteq C_k(\gamma_0)$. Since f is fundamental, $\alpha \in C_k(\gamma_0)$ and since $\alpha < \Omega_{k+1}$ we know $\alpha < \psi_k(\gamma_0)$. Applying our induction hypothesis, we get $\beta_n \leq \alpha < \alpha + 1 < \beta_{n+1}$. Hence

$$f(\beta_n) \leq f(\alpha) \leq \gamma_0 < f(\alpha+1) < f(\beta_{n+1})$$

and thus $\beta_{n+1} = \psi_k(f(\beta_n)) \leq \psi_k(\gamma_0) = \gamma < \psi_k(f(\beta_{n+1})) = \beta_{n+1}$. This completes the proof of (4).

Next, by induction on the construction of δ we shall prove

$$(5) \text{ if } g(0) \leq \delta < g(\omega), \text{ then there is } n \text{ such that } g(n) \leq \delta < g(n+1).$$

If $\delta =_{nf} \omega^{\delta_0} + \delta_1$ or $\delta =_{nf} \varphi\delta_0\delta_1$, then this follows from the induction hypothesis, much like in the proof of (4).

Otherwise, $\delta = \psi_j(\delta)$ with $\delta_0 \in C_j(\delta_0)$ and $f(0) \leq \delta_0 < f(\Omega_{k+1})$. Since f is fundamental, we may find $\alpha < \Omega_{k+1}$ such that $f(\alpha) \leq \delta_0 < f(\alpha+1)$, with $f(\alpha) \lhd f(\alpha+1)$. Much like the proof of (4), this yields, $f(\alpha) \in C_j(\delta_0)$ and hence $\alpha \in C_j(\delta_0) \subseteq C_j(f(\Omega_{k+1}))$. This, in turn, gives us $\alpha < \psi_k(f(\Omega_{k+1}))$ so we may apply (4), to find $\beta_n \leq \alpha < \alpha + 1 < \beta_{n+1}$. Thus,

$$f(\beta_n) \leq f(\alpha) \leq \delta_0 < f(\alpha+1) < f(\beta_{n+1})$$

which leads to

$$g(n) = \psi_j(f(\beta_n)) \leq \psi_j(\delta_0) < \psi_j(f(\beta_{n+1})) = g(n+1).$$

And since $g(n) \lhd g(n+1)$ follows from (3) by use of part 3 of Lemma 3.2.2, this completes the proof of property 2.

For the third property of fundamental functions note that if $\gamma \leq \omega$, then $\gamma \in C_h(\eta)$ for all h and all η. $\Box$

Chapter 4

Existence of ω-models for $\Pi^1_1 - \mathbf{CA}_0 + \mathbf{BI}$

4.1 The Infinitary Calculus T^*_Q

The calculus T^* appears in [9], and was subsequently adapted in [8] to prove a well-ordering principle result for ACA$_0$+ Bar Induction. By extending the Ω-rule, we can adapt this system for the ordinal analysis of $\Pi^1_1 - \mathbf{CA}_0 + \mathbf{BI}$. We shall also fix a set $Q \subseteq \mathbb{N}$. Hence, for every set Q (and thus for every deduction chain D_Q) there is a corresponding calculus T^*_Q. The language $\mathcal{L}^Q_2$ is the usual language of second-order arithmetic augmented by the unary predicate $\overline{Q}$. We shall use x' to denote the successor of x. As usual, numerical variables shall be denoted via lowercase letters x, y, z, etc. Likewise, set variables shall be denoted via capital letters, U, V, W, etc.

In the first section, we shall prove some basic properties of T^*_Q, ultimately showing that the axioms of $\Pi^1_1 - \mathbf{CA}_0 + \mathbf{BI}$ are provable in T^*_Q. The next section proves the cut elimination and collapsing theorems for T^*_Q. The last section embeds D_Q into T^*_Q, and then leverages cut elimination to show that D_Q cannot be well-founded (and hence there is an ω-model of $\Pi^1_1 - \mathbf{CA}_0 + \mathbf{BI}$ containing Q.).

Definition 4.1.1 *Let A, B be formulas in L^Q_2. Then*

1. *$SV(A)$ is the set of all free set variables that appear within the scope of a set quantifier. For example, if A is the formula $x \in U \wedge \forall V(y \in V \wedge y \in W)$, then $SV(A) = \{W\}$.*

2. *the **length** $l(A)$ is defined as follows:*

   (a) *If A is atomic then $l(A) = 0$.*

   (b) $l(A * B) = \max\{l(A), l(B)\} + 1$ *where* $* \in \{\wedge, \vee, \rightarrow\}$

   (c) $l(\forall X A(X)) = l(\exists X A(X)) = l(A(U^0)) + 1$.

   (d) $l(\forall x A(x)) = l(\exists x A(x)) = l(A(0)) + 1$.

Definition 4.1.2 *Let Σ, Γ be sets of $\mathcal{L}_2^Q$ formulas. Then we call $\Sigma \Rightarrow \Gamma$ a **sequent**.*

Definition 4.1.3 (Weak Formulas) *The set of **weak formulas** is inductively defined as follows:*

1. *All atomic formulas are weak.*

2. *If A and B are weak formulas, then $A \wedge B, A \vee B, A \rightarrow B, \neg A, \forall x A$, and $\exists x A$ are weak formulas.*

3. *$\forall X A(X)$ and $\exists X A(X)$ are weak formulas if $A(U)$ is a weak formula and $U \notin SV(A(U))$.*

It should be noted that the set of Π^1_1-formulas is a subset of the weak formulas. We shall shall prove that comprehension over weak formulae is in fact equivalent to Π^1_1 comprehension. To do so, we require some further definitions.

Definition 4.1.4 (Weak and Strong set quantifiers) *We define **weak** and **strong set quantifiers** inductively as follows:*

1. *Atomic formulas contain no set quantifiers.*

2. *A set quantifier in $(A \wedge B), (A \vee B), (A \to B)$, or $\neg A$ is a weak quantifier if the corresponding quantifier in A or B is weak. Likewise, the quantifier is strong if the corresponding quantifier in A or B is strong.*

3. *A set quantifier in $\forall x A(x)$ or $\exists x A(x)$ is weak if the corresponding quantifier in $A(0)$ is weak. Likewise, the quantifier is strong if the corresponding quantifier in $A(0)$ is strong. Note that $A(0)$ is merely canonical reference point. The precise term substituted for x does not affect whether a given set quantifier is strong or weak.*

4. *The set quantifier $\forall X$ in $\forall X A(X)$ is a weak quantifier if $\forall X A(X)$ is a weak formula. Otherwise, it is a strong quantifier. Other set quantifiers appearing in $\forall X A(X)$ are weak if the corresponding quantifier in $A(U)$ is weak, and strong if the corresponding quantifier in $A(U)$ is strong.*

   Any formula containing a strong quantifier is a ***strong formula****.*

Definition 4.1.5 (Formulas in T^*_Q) *The formulas of the system T^*_Q are generated from $\mathcal{L}^Q_2$ formulas through the following procedure:*

1. *Every free number variable is replaced by a closed term.*

2. *Every free set variable U is replaced by U^n where $n \in \mathbb{N}$.*

3. *Every strong predicate quantifier $\forall X$ and $\exists X$ is replaced by $\forall^\omega X$ and $\exists^\omega X$ respectively.*

   The formula obtained by eliminating the superscripts n and ω shall be called the ***corresponding formula*** *in $\mathcal{L}^Q_2$.*

Definition 4.1.6 *The* ***grade*** *$gr(A)$ of a T^*_Q formula A is defined inductively as follows:*

1. $gr(A) = 0$ *if A is an atomic formula, or has the form $\forall X F(X)$ or $\exists X F(X)$, where F is a weak formula.*

2. $gr(\neg A) = gr(\forall x A) = (\exists x A) = gr(A) + 1.$

3. $gr(A * B) = \max\{gr(A), gr(B)\} + 1$ *where* $* \in \{\wedge, \vee, \rightarrow\}$

4. $gr(\forall^\omega X A(X)) = gr(\exists^\omega X A(X)) = gr(A(U^0)) + 1.$

In order to stratify weak formulas, we shall also define the notion of **stage**. Since we shall be using a two-sided sequent calculus in T^*_Q, we require dual notions of stage.

Definition 4.1.7 *The **stage right**, $stR(A)$, of a formula A is defined inductively as follows:*

1. $stR(A) = stR(\neg A) = 0$ *if A is a atomic formula with no set variables.*

2. $stR(t \in U^n) = stR(\neg t \in U^n) = n.$

3. $stR(A * B) = \max\{stR(A), stR(B)\}$, *if* $* \in \{\wedge, \vee\}$.

4. $stR(\neg(A * B)) = stR(\neg A * \neg B)$ *if* $* \in \{\wedge, \vee\}$.

5. $stR(A \rightarrow B) = \max\{stR(\neg A), stR(B)\}.$

6. $stR(\neg(A \rightarrow B)) = stR(A \wedge \neg B).$

7. $stR(\forall x A(x)) = stR(\exists x A(x)) = stR(A(0)).$

8. $stR(\neg \forall x A(x)) = stR(\neg \exists x A(x)) = stR(\neg A(0)).$

9. $stR(\exists X A(X)) = stR(\neg \forall X A(X)) = stR(A(U^0)) + 1.$

10. $stR(\forall X A(X)) = stR(\neg \exists X A(X)) = stR(A(U^0)).$

11. $stR(\forall^\omega XA(X)) = stR(\neg\forall^\omega XA(X)) = stR(\exists^\omega XA(X)) = stR(\neg\exists^\omega XA(X)) = \omega$.

12. $stR(\neg\neg A) = stR(A)$.

*The **stage left**, $stL(A)$, of a formula A is defined in the same way, except for*

$$(8) stL(\forall XA(X)) = stL(\neg\exists XA(X)) = stL(A(U^0)) + 1.$$

and

$$(9) stL(\exists XA(X)) = stL(\neg\forall XA(X)) = stR(A(U^0))$$

Thus, all weak formulas have finite stages, and all strong formulas have stage ω. We may occasionally say $stL(\Gamma) \leq n$ or $stR(\Gamma) \leq m$, where Γ is a set of formulas. In such a case, we mean that for every formula $A \in \Gamma$ we have $stL(A) \leq n$ or $stR(A) \leq m$.

This dual notion of stage is necessary, since A appearing in the antecedent of a sequent is equivalent to $\neg A$ appearing in the succedent and vice versa. These notions of stage track the alternations of quantifiers as we ascend the arithmetical heirarchy, with $stR(A)$ tracking Σ^1_n-formulas, and $stL(A)$ tracking Π^1_n-formulas. From time to time, we may also use notations such as $stL(\Sigma) \leq n$, where Σ is a set of formulas, to denote that for every formula $A \in \Sigma$, $stL(A) \leq n$. The same notations hold for $stR(A)$.

In what follows, we shall use $*_1$ to designate a placeholder for an arbitrary term.

As we shall see, comprehension over weak formulas is equivalent to Π^1_1-comprehension. To show this, we first require the following lemma:

Lemma 4.1.8 *Let $F(a)$ be a weak formula. Then there exists a formula $G(a, U_0, \ldots U_k) \in \Pi^1_1 \cup \Sigma^1_1$ containing at most one second-order quantifier, such that $F(a) \equiv G(a, A_0(a, *_1), \ldots A_k(a, *_1))$ where $A_i(a, b)$ are weak formulas of length less* *than $F(a)$, and $G(a, A_0(a, *_1), \ldots, A_k(a, *_1))$ is obtained by replacing every expression $b \in U_0$ with the formula $A_0(a, b)$, then replacing every instance of $b \in U_1$ with $A_0(a, b)$, etc., and $SV(G(a, A_0(a, *_1), \ldots A_k(a, *_1))) = SV(F(a))$.*

Proof

Proceed by induction on the length (not the grade) of $F(a)$.

1. If $F(a)$ is a atomic formula, then $F(a)$ is Π^1_1 by definition, and hence $F(a) = G(a)$.

2. If $F(a)$ has the form $\forall X F_0(a, X)$ then by induction hypothesis, we have $F_0(a, V) \equiv G_0(a, A_1(a, *_1) \ldots, A_k(a, *_1), U)$ where $G_0(a, U_1, \ldots U_k, U) \in \Pi^1_1 \cup \Sigma^1_1$ with at most one set quantifier, and $SV(G(a, A_0(a, *_1), \ldots A_k(a, *_1))) = SV(F(a, V))$. If $G_0(a, U_1, \ldots U_k, U)$ does not contain a set quantifier, then this is trivial. Otherwise, suppose $QY\ B_0(a, b, U_1, \ldots U_k)$ is the largest subformula of $G_0(a, U_1, \ldots U_k, U)$ bounded by the quantifier QY. Then $QY\ B_0(a, b, U_1, \ldots U_k)$ is a weak formula. Moreover, $QY\ B_0(a, b, A_1(a, *_1), \ldots A_k(a, *_1))$ is weak, since $A_1, \ldots A_k$ are weak. Thus, we define $A_{k+1}(a, b) := QY\ B_0(a, b, A_1(a, *_1), \ldots A_k(a, *_1))$, and replace all inbstances of $QY\ B_0(a, b, U_1, \ldots U_k)$ with the formula $b \in U_k$ to obtain $G_0'(a, U_1, \ldots, U_k, U_{k+1}, V)$. We may iterate this process, replacing the next-largest subformula at each step, until no second-order quantifiers remain, with each subformula $QY B_i$ being substituted with a formula A_{k+i} for all $i \leq m$, where m is the number of iterations needed to remove all second-order quantifiers. Then

$$\forall X F_0(a, X) \equiv \forall X G_0'(a, A_1(a, *_1), \ldots, A_{k+1}(a, *_1), \ldots, A_{k+m}(a, *_1), X),$$

where

$$\forall X G_0'(a, A_1(a, *_1), \ldots, A_{k+1}(a, *_1), \ldots, A_{k+m}(a, *_1), X) \in \Pi^1_1 \cup \Sigma^1_1$$

with only a single set quantifier, and

$$SV(F(a)) = SV(\forall X G_0'(a, A_1(a, *_1), \dots, A_{k+1}(a, *_1), \dots, A_{k+m}(a, *_1), X)).$$

3. If $F(a) \equiv \exists X F_0(X, a)$ is proven similarly to the universal case.

4. If $F(a) \equiv \neg F_0(a)$, then by induction hypothesis we have $F_0(a) \equiv G(a, A_1(a, *_1) \dots A_k(a, *_1))$ and hence $F(a) \equiv \neg G(a, A_1(a, *_1) \dots A_k(a, *_1))$.

5. If $F(a) \equiv F_0(a) \vee F_1(a)$ then let $A_0(a, *_1) = F_0(a)$ and $A_1(a, *_1) = F_1(a)$. $G(a, U_1, U_2) = a \in U_1 \vee a \in U_2$. Likewise, if $F(a) \equiv A_1(a) \wedge A_2(a)$ then $G(a, U_1, U_2) = a \in U_1 \wedge a \in U_2$, and $A_0(a, *_1) = F_0(a)$ and $A_1(a, *_1) = F_1(a)$.

6. If $F(a) \equiv F_0(a) \rightarrow F_1(a)$ then $A_0(a, *_1) = F_0(a)$ and $A_1(a, *_1) = F_1(a)$. $G(a, U_1, U_2) = (\neg a \in U_1) \vee a \in U_2$.

7. If $F(a) \equiv \forall x F_0(a, x)$. Then let $G(a, U_1) = \forall z z \in U_1$ and $F_0(a, *_1) = A_1(a, *_1)$.

8. If $F(a) \equiv \exists x F_0(a, x)$. Then let $G(a, U_1) = \exists z, z \in U_1$ and $F_0(a, *_1) = A_1(a, *_1)$.

□

Definition 4.1.9 (Axioms of T_Q^*) *Let Σ, Γ be sets of T_Q^* formulas. The following are axioms of T_Q^*.*

1. *If A is a true atomic formula, then $\Sigma \Rightarrow \Gamma, A$ is an axiom.*

2. *If A is a false atomic formula, then $A, \Sigma \Rightarrow \Gamma$ is an axiom.*

3. *If $n \in Q$ and t is a closed term with value n, then $\Sigma \Rightarrow \Gamma, \overline{Q}(t)$ is an axiom.*

4. *If $n \notin Q$ and t is a closed term with value n, then $\overline{Q}(t), \Sigma \Rightarrow \Gamma$ is an axiom.*

5. *If $A(s_1, \ldots, s_n)$ is a weak formula of grade 0 and s_i and t_i are equivalent terms for $1 \leq i \leq n$, then $A(s_1, \ldots, s_n), \Sigma \Rightarrow \Gamma, A(t_1, \ldots, t_n)$*

Lemma 4.1.10 *Comprehension over weak formulas is equivalent to comprehension over Π^1_1 formulas over $\boldsymbol{RCA}_0$*

Proof

Again, let $F(a)$ be a weak formula and proceed by induction on the length of the formula. We shall show that $\{z|F(z)\}$ is a set.

If $F(a)$ is atomic, then it is arithmetic, and hence $\{z|F(z)\}$ is a set by Π^1_1 comprehension.

Otherwise, by the preceding lemma we know that there is a formula $G(a, U_1, \ldots, U_k) \in \Pi^1_1 \cup \Sigma^1_1$ and weak formulas $A_i(a, *_1)$ of length less than $F(a)$ such that $F(a) \equiv G(a, A_1(a, *_1), \ldots, A_k(a, *_1))$. Inductively, we have the sets $V_i = \{z|A_i(z, *_1)\}$.

Then

$$\{z|F(z)\} \equiv \{z|G(z, \langle z, *_1\rangle \in V_1, \ldots, \langle z, *_1\rangle \in V_k)\},$$

where $< a, *_1 >$ is a coding of the pair $(a, *_1)$. And since $G \in \Pi^1_1 \cup \Sigma^1_1$, we find that $\{z|G(z, \langle z, *_1\rangle \in V_1, \ldots, \langle z, *_1\rangle \in V_k)\}$ is a set, and thus $\{z|F(z)\}$ is a set.

$\square$

4.1.1 Inference Rules of T^*_Q

Let $\Gamma, \Theta, \Sigma, \Xi$ be sets of T^*_Q formulas, and let t be a closed term.The sequent calculus T^*_Q has the following first-order rules of inference:

- $\neg$L $\dfrac{\Gamma \Rightarrow \Theta, A}{\neg A \Gamma \Rightarrow \Theta}$
- $\neg$R $\dfrac{A, \Gamma \Rightarrow \Theta}{\Gamma \Rightarrow \Theta, \neg A}$
- $\wedge$L $\dfrac{A, \Gamma \Rightarrow \Theta}{A \wedge B, \Gamma \Rightarrow \Theta}, \dfrac{B, \Gamma \Rightarrow \Theta}{A \wedge B, \Gamma \Rightarrow \Theta}$
- $\wedge$R $\dfrac{\Gamma \Rightarrow \Theta, A \qquad \Sigma \Rightarrow \Xi, B}{\Gamma, \Sigma \Rightarrow \Theta, \Xi A \wedge B}$
- $\vee$L $\dfrac{A, \Gamma \Rightarrow \Theta \qquad B, \Sigma \Rightarrow \Xi}{A \vee B, \Gamma, \Sigma \Rightarrow \Theta, \Xi}$
- $\vee$R $\dfrac{\Gamma \Rightarrow \Theta, A}{\Gamma \Rightarrow \Theta, A \vee B}, \dfrac{\Gamma \Rightarrow \Theta, B}{\Gamma \Rightarrow \Theta, A \vee B}$
- $\rightarrow$L $\dfrac{\Gamma \Rightarrow \Theta, A \qquad B, \Sigma \Rightarrow \Xi}{A \rightarrow B, \Gamma, \Sigma \Rightarrow \Theta, \Xi}$
- $\rightarrow$R $\dfrac{\Gamma, A \Rightarrow B, \Theta}{\Gamma \Rightarrow \Theta, A \rightarrow B}$
- $\forall_1$L $\dfrac{A(t), \Gamma \Rightarrow \Theta}{\forall x A(x), \Gamma \Rightarrow \Theta}$
- $\exists_1$R $\dfrac{\Gamma \Rightarrow \Theta, A(t)}{\Gamma \Rightarrow \Theta, \exists x A(x)}$

$$\omega_1\text{L}\ \frac{A(0), \Gamma \Rightarrow \Theta \qquad A(1), \Gamma \Rightarrow \Theta \qquad \ldots}{\exists x A(x), \Gamma \Rightarrow \Theta}$$

$$\omega_1\text{R}\ \frac{\Gamma \Rightarrow \Theta, A(0) \qquad \Gamma \Rightarrow \Theta, A(1) \qquad \ldots}{\Gamma \Rightarrow \Theta, \forall x A(x)}$$

Additionally, there are the following second-order rules of inference.

Suppose $stR(\forall X A(X)) = stL(\exists X A(X)) = n < \omega$, and U^i does not occur in the conclusion of the inference for any $i < \omega$. Then:

- $\forall_2 R_n$ $\dfrac{\Gamma \Rightarrow \Theta, A(U^n)}{\Gamma \Rightarrow \Theta, \forall X A(X)}$
- $\exists_2 L_n$ $\dfrac{A(U^n), \Gamma \Rightarrow \Theta}{\exists X A(x), \Gamma \Rightarrow \Theta}$

Now, suppose U does not occur in the conclusion of the inference. Then:

- ω_2L $\dfrac{A(U^0), \Gamma \Rightarrow \Theta \qquad A(U^1), \Gamma \Rightarrow \Theta \qquad \ldots}{\exists^\omega X A(X), \Gamma \Rightarrow \Theta}$
- $\forall_2$L $\dfrac{A(U^n), \Gamma \Rightarrow \Theta}{\forall^\omega X A(X), \Gamma \Rightarrow \Theta}$ for any $n < \omega$.

- ω_2R $\dfrac{\Gamma \Rightarrow \Theta, A(U_0) \qquad \Gamma \Rightarrow \Theta, A(U^1) \qquad \ldots}{\Gamma \Rightarrow \Theta, \forall^\omega X A(X)}$

- $\exists_2$R $\dfrac{\Gamma \Rightarrow \Theta, A(U^n)}{\Gamma \Rightarrow \Theta, \exists^\omega X A(X)}$ for any $n < \omega$.

Collectively, we call these first- and second-order rules the **principal inferences** of T^*_Q.

We also have the **cut rule**

Cut $\dfrac{\Gamma \Rightarrow \Theta, A \qquad A, \Sigma \Rightarrow \Xi}{\Gamma, \Sigma \Rightarrow \Theta, \Xi}$

We say that $l(A)$ is the **grade** of the cut.

Definition 4.1.11 *Let $\Sigma \Rightarrow \Gamma$ be a sequent, $\gamma \in$ OT$(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}})$and $\rho < \omega$. We define the relation $T^*_Q \mid\!\frac{\gamma}{\rho}\, \Sigma \Rightarrow \Gamma$ inductively as follows:*

1. *If $\Sigma \Rightarrow \Gamma$ is an axiom of T^*_Q then $T^*_Q \mid\!\frac{\gamma}{\rho}\, \Sigma \Rightarrow \Gamma$ for all γ, ρ.*

2. *If $T^*_Q \mid\!\frac{\beta}{\rho}\, \Sigma_i \Rightarrow \Gamma_i$ with $\beta \lhd \gamma$ for every premise of a principal inference, or a cut of grade $\rho_0 < \rho$, then $T^*_Q \mid\!\frac{\gamma}{\rho}\, \Sigma_i \Rightarrow \Gamma_i$ holds for the conclusion $\Sigma \Rightarrow \Gamma$ of that inference.*

3. *(Ω_{n+1}R-rule) Let f be a fundamental function with $\Omega_{n+1} \in$ dom(f). $T^*_Q \mid\!\frac{\gamma}{\rho}\, \Sigma \Rightarrow \Gamma$ holds if the following are satisfied:*

   (a) $f(\Omega_{n+1}) \trianglelefteq \gamma$

   (b) $T^*_Q \mid\!\frac{f(0)}{\rho}\, \Sigma \Rightarrow \Gamma, \forall X F(X)$, *where* $stR(\forall X F(X)) \leq n$.

   (c) $T^*_Q \mid\!\frac{\alpha}{0}\, \Xi \Rightarrow \Theta, \forall X F(X)$, *implies* $T^*_Q \mid\!\frac{f(\alpha)}{\rho}\, \Xi, \Sigma \Rightarrow \Theta, \Gamma$ *for every* $\alpha < \Omega_{n+1}$ *and every set of weak formulas* Ξ, Θ *where* $stR(\Xi) + 1, stR(\Theta) \leq n$.

4. *(Ω_{n+1}L-rule) Let f be a fundamental function with $\Omega_{n+1} \in$ dom(f). $T^*_Q \mid\!\frac{\gamma}{\rho}\, \Sigma \Rightarrow \Gamma$ holds if the following are satisfied:*

(a) $f(\Omega_{n+1}) \trianglelefteq \gamma$

(b) $T^*_Q \left|\frac{f(0)}{\rho}\right. \exists X F(X), \Sigma \Rightarrow \Gamma$, *where* $stL(\exists X F(X)) \leq n$.

(c) $T^*_Q \left|\frac{\alpha}{0}\right. \exists X F(X), \Xi \Rightarrow \Theta$, *implies* $T^*_Q \left|\frac{f(\alpha)}{\rho}\right. \Xi, \Sigma \Rightarrow \Theta, \Gamma$ *for every* $\alpha < \Omega_{n+1}$ *and every set of weak formulas* Ξ, Θ *where* $stL(\Xi), stL(\Theta) + 1 \leq n$.

*We call γ the **height**, and ρ the **cut rank** of the proof.*

It should be noted that $T^*_Q \left|\frac{\alpha}{0}\right. \Xi \Rightarrow \Theta, \forall X F(X)$ and $T^*_Q \left|\frac{\alpha}{0}\right. \exists X F(X), \Xi \Rightarrow \Theta$ occur negatively in part (c) of the Ω_{n+1}R- and Ω_{n+1}L-rules respectively. If an Ω_{k+1}-rule with $n \leq k$ was required to derive the this negative occurrence, this would undermine the inductive definition of derivability in T^*_Q. However, since we know $\alpha < \Omega_{n+1}$ and $\Omega_{n+1} \leq f(\Omega_{n+1}) \trianglelefteq \gamma$, we know that any application of an Ω_k-rule must have $k \leq n$. Thus, derivability has an iterative inductive definition. The primary induction occurs over γ, and defines the basic derivability predicate, while a secondary induction on k defines derivability with the use of Ω_{n+1} rules with $n < k$.

We should also note the stage restrictions on the $\Omega + n + 1$ rules. Since the Ω_{n+1}R-rules have an active formula of the form $\forall X F(X)$ we must track the usage of universal quantifiers, hence we use stR. We require $stR(\Xi) + 1 \leq n$ since a formula A on the left side of a sequent can be moved to the right side via a $\neg$R inference, and $stR(\neg A) = stR(A) + 1$. The dual case naturally holds for the Ω_{n+1}L-rules.

Lemma 4.1.12 (Weakening And Inversion) *[See [8] Lemma 5.14]*

1. Weakening: *If* $T^*_Q \left|\frac{\alpha}{\delta}\right. \Gamma \Rightarrow \Sigma$ *and* $\Gamma \subseteq \Delta, \Sigma \subseteq \Theta$, *with* $\alpha \trianglelefteq \beta$ *and* $\delta \leq \rho$ *then* $T^*_Q \left|\frac{\beta}{\rho}\right. \Delta \Rightarrow \Theta$.
2. *If* $T^*_Q \left|\frac{\alpha}{\rho}\right. \Gamma \Rightarrow \Sigma, A \wedge B$ *then* $T^*_Q \left|\frac{\alpha}{\rho}\right. \Gamma \Rightarrow \Sigma, A$ *and* $T^*_Q \left|\frac{\alpha}{\rho}\right. \Gamma \Rightarrow \Sigma, B$.
3. *If* $T^*_Q \left|\frac{\alpha}{\rho}\right. A \vee B, \Gamma \Rightarrow \Sigma$ *then* $T^*_Q \left|\frac{\alpha}{\rho}\right. A, \Gamma \Rightarrow \Sigma$ *and* $T^*_Q \left|\frac{\alpha}{\rho}\right. B, \Gamma \Rightarrow \Sigma$.

4. *If* $T^*_Q \mathrel{|\!\frac{\alpha}{\rho}} \Gamma \Rightarrow \Sigma, A \vee B$ *then* $T^*_Q \mathrel{|\!\frac{\alpha}{\rho}} \Gamma \Rightarrow \Sigma, A, B$.

5. *If* $T^*_Q \mathrel{|\!\frac{\alpha}{\rho}} A \wedge B, \Gamma \Rightarrow \Sigma$ *then* $T^*_Q \mathrel{|\!\frac{\alpha}{\rho}} A, B, \Gamma \Rightarrow \Sigma$.

6. *If* $T^*_Q \mathrel{|\!\frac{\alpha}{\rho}} A \to B, \Gamma \Rightarrow \Sigma$ *then* $T^*_Q \mathrel{|\!\frac{\alpha}{\rho}} B, \Gamma \Rightarrow \Sigma$ *and* $T^*_Q \mathrel{|\!\frac{\alpha}{\rho}} \Gamma \Rightarrow A, \Sigma$

7. *If* $T^*_Q \mathrel{|\!\frac{\alpha}{\rho}} \Gamma \Rightarrow \Sigma, A \to B$ *then* $T^*_Q \mathrel{|\!\frac{\alpha}{\rho}} \Gamma, A \Rightarrow \Sigma, B$

8. *If* s *and* t *are terms and* $s = t$ *and* $T^*_Q \mathrel{|\!\frac{\alpha}{\rho}} \Gamma \Rightarrow \Sigma, F(s)$ *then* $T^*_Q \mathrel{|\!\frac{\alpha}{\rho}} \Gamma \Rightarrow \Sigma, F(t)$. *Likewise, if* $T^*_Q \mathrel{|\!\frac{\alpha}{\rho}} F(s), \Gamma \Rightarrow \Sigma$ *then* $T^*_Q \mathrel{|\!\frac{\alpha}{\rho}} F(t), \Gamma \Rightarrow \Sigma$.

9. *If* $T^*_Q \mathrel{|\!\frac{\alpha}{\rho}} \Gamma \Rightarrow \Sigma, \forall x F(x)$ *then* $T^*_Q \mathrel{|\!\frac{\alpha}{\rho}} \Gamma \Rightarrow \Sigma, F(s)$ *for all terms* s.

10. *If* $T^*_Q \mathrel{|\!\frac{\alpha}{\rho}} \exists x F(x), \Gamma \Rightarrow \Sigma$ *then* $T^*_Q \mathrel{|\!\frac{\alpha}{\rho}} F(s), \Gamma \Rightarrow \Sigma$ *for all terms* s.

11. *If* $T^*_Q \mathrel{|\!\frac{\alpha}{\rho}} \Gamma \Rightarrow \Sigma, \forall^\omega X F(X)$ *then* $T^*_Q \mathrel{|\!\frac{\alpha}{\rho}} \Gamma \Rightarrow \Sigma, F(U^n)$ *for all* n.

12. *If* $T^*_Q \mathrel{|\!\frac{\alpha}{\rho}} \exists^\omega X F(X), \Gamma \Rightarrow \Sigma$ *then* $T^*_Q \mathrel{|\!\frac{\alpha}{\rho}} F(U^n), \Gamma \Rightarrow \Sigma$ *for all* n.

Proof

(1) is the standard weakening principle, and proceeds via induction on α. When $\alpha = 0$, $\Gamma \Rightarrow \Sigma$ is an axiom and the proof is trivial. Otherwise, suppose $T^*_Q \mathrel{|\!\frac{\alpha}{\delta}} \Gamma \Rightarrow \Sigma$ and $\Gamma \subseteq \Delta, \Sigma \subseteq \Theta$, with $\alpha \trianglelefteq \beta$ and $\delta \leq \rho$. If the last inference was any non-Ω_{k+1} rule, then we have $T^*_Q \mathrel{|\!\frac{\alpha_0}{\delta}} \Gamma_i \Rightarrow \Sigma_i$, where $\alpha_0 \triangleleft \alpha$, and $\{\Gamma_i \Rightarrow \Sigma_i\}_{i<\omega}$ are the premises of the inference. Applying the induction hypothesis yields

$$T^*_Q \mathrel{|\!\frac{\alpha}{\rho}} \Delta, \Gamma_i \Rightarrow \Theta, \Sigma_i,$$

since $\alpha_0 \triangleleft \alpha$, and reapplying the inference gives

$$T^*_Q \mathrel{|\!\frac{\beta}{\rho}} \Delta \Rightarrow \Sigma_i.$$

Otherwise, if the last inference was an Ω_{k+1} rule, there is a fundamental function f such that $f(\Omega_{n+1}) \trianglelefteq \alpha$. We note that $\Omega_{n+1} \leq f(\Omega_{n+1}) \leq \beta$, so by Lemma 3.2.2 (3), we have $f(\Omega_{n+1}) \trianglelefteq \beta$. Hence, we may apply our induction hypothesis to the premises of the inference, then reapply the Ω_{n+1}-rule to get

$$T^*_Q \left|\frac{\beta}{\rho}\right. \Delta \Rightarrow \Sigma_i.$$

The rest are standard inversion principles used throughout proof theory, also proved by induction on the length of the formula. □

Lemma 4.1.13 (See [8] Lemma 5.15) $T^*_Q \left|\frac{2\cdot\alpha}{0}\right. \Gamma, A(s_1, \ldots, s_n) \Rightarrow \Sigma, A(t_1, \ldots, t_n)$ *when* $gr(A(s_1, \ldots, s_n)) \trianglelefteq \alpha$ *and* $s_i = t_i$ *for all* $i \leq n$.

Proof

We prove this by induction. Note that when $gr(A(s_1, \ldots, s_n)) = 0$ the sequent in question is an axiom. From there, we observe that in each induction step we require the application of a left-side rule and a right-side rule, which adds 2 to the height of the proof as needed. Since the grade is always finite, Lemma 3.2.2 (4) guarantees that $gr(A(s_1, \ldots, s_n)) \trianglelefteq \alpha$.

□

Lemma 4.1.14 (See [8] Lemma 5.16)

1. $T^*_Q \left|\frac{2m+1}{0}\right. \overline{0} \in U^n, \forall x(x \in U^n \to x' \in U^n) \Rightarrow \overline{m} \in U^n$ *for all* $m, n \in \mathbb{N}$.
2. $T^*_Q \left|\frac{\omega+4}{0}\right. \forall X[(\overline{0} \in X \wedge \forall x(x \in X \to x' \in X)) \to \forall x(x \in X)]$.

Proof

(1) Proceed by induction on m. We have $T^*_Q \mid\frac{1}{0}\ \overline{0} \in U^n, \forall x(x \in U^n \to x' \in U^n) \Rightarrow \overline{0} \in U^n$ as an axiom.

Now, suppose we have $T^*_Q \mid\frac{2m+1}{0}\ \overline{0} \in U^n, \forall x(x \in U^n \to x' \in U^n) \Rightarrow \overline{m} \in U^n$. Then we have:

$$\dfrac{\dfrac{\overline{0} \in U^n, \forall x(x \in U^n \to x' \in U^n) \Rightarrow \overline{m} \in U^n \qquad \overline{m+1} \in U^n \Rightarrow \overline{m+1} \in U^n}{\overline{0} \in U^n, \forall x(x \in U^n \to x' \in U^n), (\overline{m} \in U^n \to \overline{m+1} \in U^n) \Rightarrow \overline{m+1} \in U^n}\ {\to}\mathrm{L}}{\overline{0} \in U^n, \forall x(x \in U^n \to x' \in U^n) \Rightarrow \overline{m+1} \in U^n}\ \forall_1\mathrm{L}$$

The two inferences get us a height of $2m+3$ as desired.

(2) By the previous result, using the ωR-rule we get

$$T^*_Q \mid\frac{\omega}{0}\ \overline{0} \in U^n, \forall x(x \in U^n \to x' \in U^n) \Rightarrow \forall x(x \in U^n).$$

Two applications of $\wedge$R yield

$$T^*_Q \mid\frac{\omega+2}{0}\ (\overline{0} \in U^n \wedge \forall x(x \in U^n \to x' \in U^n)) \Rightarrow \forall x(x \in U^n).$$

Using $\to$R we get

$$T^*_Q \mid\frac{\omega+3}{0} \Rightarrow (\overline{0} \in U^n \wedge \forall x(x \in U^n \to x' \in U^n)) \to \forall x(x \in U^n).$$

Finally, since our formula is arithmetic, we may use $\forall_2$R to obtain

$$T^*_Q \mid\frac{\omega+4}{0} \Rightarrow (\forall X[\overline{0} \in X \wedge \forall x(x \in X \to x' \in X)) \to \forall x(x \in X)].$$

□

Definition 4.1.15 *Let $F(U^n)$ and $A(a)$ be formulas such that no variable bound in $F(U^n)$ occurs bound in $A(a)$. Then $F(A)$ is the formula obtained by replacing every instance of $t \in U^n$ with $A(t)$. By ensuring F and A do not share bound variables, the result is, in fact, a well-formed formula.*

Lemma 4.1.16 (See [8] Lemma 5.18) *Suppose $\alpha \lhd \Omega_{n+1}$ and let*

$$\Gamma(U^0) = \{G_1(U^0), \ldots, G_{m_\Gamma}(U^0)\}$$

and

$$\Delta(U^0) = \{F_1(U^0), \ldots, F_{m_\Delta}(U^0)\}$$

be finite sets of weak formulas such that $stL(G_i(U^0)) \leq n$ for $i \leq m_\Gamma$ and $stR(F_i(U^0)) \leq n$ for $i \leq m_\Delta$. For an arbitrary formula $A(a)$,

$$\textit{if } T^*_Q \left|\frac{\alpha}{0}\right. \Gamma(U^0) \Rightarrow \Delta(U^0) \textit{ then } T^*_Q \left|\frac{\Omega_{n+1}+\alpha}{0}\right. \Gamma(A) \Rightarrow \Delta(A).$$

Proof

Note that $\Omega_{n+1} \in C_j(\eta)$ for all j and all η. Hence if $\alpha \lhd \Omega_{n+1}$ then $\alpha \in C_j(\eta)$. Hence $\alpha \lhd \Omega_{n+1} + \alpha$. This satisfies our conditions for the proof height.

We proceed by induction on α. If $\alpha = 0$ then $\Gamma(U^0) \Rightarrow \Delta(U^0)$ is an axiom. Then either U^0 occurs only in side-formulas, in which case $\Delta(A) \Rightarrow \Gamma(A)$ is still an axiom, or $t \in U^0, s \in U^0$ are the active formulas of the axiom. In this case by Lemma 4.1.13 we know $T^*_Q \left|\frac{\omega+\omega}{0}\right. \Gamma(A) \Rightarrow \Delta(A)$. Hence we may simply assign the proof a height of $\Omega_{n+1} + \alpha$, with Lemma 3.2.2 (4) ensuring that $\alpha \unlhd \Omega_{n+1} + \alpha$.

If $T^*_Q \left|\frac{\alpha}{0}\right. \Gamma(U^0) \Rightarrow \Delta(U^0)$ is the result of an inference, then most of the cases follow by induction hypothesis. Since the proof is cut free, the final inference cannot be a cut. Furthermore, since $\Gamma(U^0), \Delta(U^0)$ are weak formulas whose respective stages are less than $n+1$ the final inference cannot be an ω_2-rule, or an Ω_{h+1}-rule where $n+1 \leq h$.

The one remaining case is when the inference is an Ω_{h+1}-rule where $h \leq n$. We shall use Ω_{h+1}R, though the proof is essentially the same for the left-handed rule as well.

By assumption, we have a fundamental function f such that $f(\Omega_{h+1}) \unlhd \alpha$, and we know that

$$T^*_Q \left|\frac{f(0)}{0}\right. \Gamma(U^0) \Rightarrow \Delta(U^0), \forall X H(X) \text{ for } stR(\forall X H(X)) \leq h$$

and

$T^*_Q \left|\frac{\beta}{0}\right. \Sigma \Rightarrow \Theta, \forall X H(X)$ implies $T^*_Q \left|\frac{f(\beta)}{0}\right. \Sigma, \Gamma(U^0) \Rightarrow \Theta, \Delta(U^0)$ where $stL(\Sigma), stR(\Theta) \leq h$.

Using the induction hypothesis we find:

$$(1) T^*_Q \left|\frac{\Omega_{n+1}+f(0)}{0}\right. \Gamma(A) \Rightarrow \Delta(A), \forall X H(X) \text{ for } stR(\forall X H(X)) \leq h$$

and

$$(2) T^*_Q \left|\frac{\beta}{0}\right. \Sigma \Rightarrow \Theta, \forall X H(X) \text{ implies } T^*_Q \left|\frac{\Omega_{n+1}+f(\beta)}{0}\right. \Sigma, \Gamma(A) \Rightarrow \Theta, \Delta(A)$$

where $stL(\Sigma), stR(\Theta) \leq h$.

Observe that $\Omega_{n+1}+f$ is fundamental with $dom(\Omega_{n+1}+f) = \{\beta | \beta \in dom(f) \wedge \alpha < \Omega_h\}$. Moreover, $\Omega_{n+1} + f(\Omega_h) \trianglelefteq \Omega_{n+1} + \alpha$. This allows us to conclude

$$T^*_Q \left|\frac{\Omega_{n+1}+\alpha}{0}\right. \Gamma(A) \Rightarrow \Delta(A)$$

via the Ω_{n+1}R-rule as desired.

□

Lemma 4.1.17 (See [8] Lemma 5.19) *Suppose we have*

$$stR(\Delta) + 1, stR(\Gamma) \leq stR(\forall X F(X)) = n,$$

and let $\alpha < \Omega_{n+1}$.

If $T^*_Q \left|\frac{\alpha}{0}\right. \Gamma \Rightarrow \Delta, \forall X F(X)$ *then* $T^*_Q \left|\frac{\alpha}{0}\right. \Gamma \Rightarrow \Delta, F(U^n)$.

Likewise, suppose $stL(\Gamma) + 1, stL(\Delta) \leq stL(\exists X F(X)) = n$.

If $T^*_Q \left|\frac{\alpha}{0}\right. \exists X F(X), \Gamma \Rightarrow \Delta$ *then* $T^*_Q \left|\frac{\alpha}{0}\right. F(U^n), \Gamma \Rightarrow \Delta$.

Proof

We shall only concern ourself with the $\forall XF(X)$ case. The case for $\exists XF(X)$ proceeds in much the same fashion. We proceed by induction on α, observing that if $\alpha = 0$ then $\Gamma \Rightarrow \Delta, \forall XF(X)$ is an axiom, where $\forall XF(X)$ occurs as a side formula (since we require $stR(\Delta) < stR(\forall XF(X))$) and hence $\forall XF(X) \notin \Delta$. Thus, we may replace $\forall XF(X)$ with $F(U^n)$ without any difficulty.

Now, assume that $T^*_Q \left|\frac{\alpha}{0}\right. \Gamma \Rightarrow \Delta, \forall XF(X)$ follows via an inference rule. If the last inference was a $\forall_2 R_n$-rule, then the proof is trivial. We also observe that the proof is cut free, and since $\alpha < \Omega_{n+1}$, no use of Ω_{n+1}-rules may occur. If the final inference is an Ω_{k+1} rule for $k < n$ then crucially $n < stR(\forall XF(X))$, so $\forall XF(X)$ cannot be the primary formula of the inference, where the removal of a universal quantifier might invalidate the inference. In all other cases, the proof follows immediately from the induction hypothesis.

□

The following result and its corollaries establish the pivotal role of the Ω_{n+1}-rules in T^*_Q.

Lemma 4.1.18 (See [8] Lemma 5.20) *(1) $T^*_Q \left|\frac{(\Omega_{n+1}\cdot 2)}{0}\right. F(A) \Rightarrow \exists XF(X)$ for $\exists XF(X)$ a weak formula with $stR(\exists XF(X)) = n$ and $A(a)$ an arbitrary formula.*

(2) $T^*_Q \left|\frac{(\Omega_{n+1}\cdot 2)}{0}\right. \forall XF(X) \Rightarrow F(A)$ for $\forall XF(X)$ a weak formula with $stL(\exists XF(X)) = n$ and $A(a)$ an arbitrary formula.

Proof

By Lemma 3.2.15 (1), $f(\alpha) = \Omega_{n+1} + \alpha$ is a fundamental function (noting that $\omega^{\Omega}_{n+1} = \Omega_{n+1}$.)

Then $T^*_Q \left|\frac{f(0)}{0}\right. F(A), \exists XF(X) \Rightarrow \exists XF(X)$ by Lemma 4.1.13.

Suppose $\beta < \Omega_{n+1}$ and Σ, Θ are sets of weak formulas, such that $stR(\Sigma) + 1, stR(\Theta) \leq n$. If

$$T^*_Q \left|\frac{\beta}{0}\right. \exists X F(X), \Sigma \Rightarrow \Theta$$

then by Lemmas 4.1.16 and 4.1.17 we obtain

$$T^*_Q \left|\frac{f(\beta)}{0}\right. F(A), \Sigma \Rightarrow \Theta$$

and by Lemma 4.1.12

$$T^*_Q \left|\frac{f(\beta)}{0}\right. F(A), \Sigma \Rightarrow \Theta, \exists X F(X).$$

Thus, by the Ω_{n+1}L-rule, we obtain:

$$T^*_Q \left|\frac{(\Omega_{n+1}\cdot 2)}{0}\right. F(A) \Rightarrow \exists X F(X).$$

The case for (2) follows a similar argument using the Ω_{n+1}R-rule.

□

Corollary 4.1.19 (Provability of Weak Comprehension) *[See [8] Lemma 5.21]*

$T^*_Q \left|\frac{(\Omega_{n+1}\cdot 2)+1}{gr(B(0))+3}\right. \emptyset \Rightarrow \exists X \forall y (y \in X \leftrightarrow B(y))$ *for all weak formulas* $B(a)$ *such that* $st(B(a)) \leq n$.

Proof

By the previous lemma we obtain

$$(*) T^*_Q \left|\frac{(\Omega_{n+1}\cdot 2)}{0}\right. \forall y (B(y) \leftrightarrow B(y)) \Rightarrow \exists X \forall y (y \in X \leftrightarrow B(y)).$$

By lemma 4.1.13 we get $B(t) \Rightarrow B(t)$ for all terms t. From this we may derive $\Rightarrow \forall y (B(y) \leftrightarrow B(y))$ in a cut free proof of finite height.

(Note: $B(t) \Rightarrow B(t) \equiv \forall y ((B(y) \rightarrow B(y)) \wedge (B(y) \rightarrow B(y)))$, and therefore has $gr(B(0)) + 3$.)

Cutting this sequent with $(*)$ yields

$$T^*_Q \,\Big|\frac{(\Omega_{n+1}\cdot 2)+1}{gr(B(0)+3)}\, \emptyset \Rightarrow \exists X \forall y (y \in X \leftrightarrow B(y))$$

as needed. □

Corollary 4.1.20 (See [8] Lemma 5.22) *For all relations $\prec$ definable via weak formulas (allowing parameters), and for an arbitrary formula $A(a)$ we have*

$$T^*_Q \,\Big|\frac{\Omega_\omega+\omega}{0}\, \emptyset \Rightarrow \forall \overrightarrow{X} \forall \overrightarrow{x} (WF(\prec) \to TI(\prec, A(a))),$$

where $\forall \overrightarrow{X} \forall \overrightarrow{x}$ bind the free variables of $(WF(\prec) \to TI(\prec, A(a)))$.

Proof

By lemma 4.1.18 we have $T^*_Q \,\Big|\frac{\Omega_\omega}{0}\, (WF(\prec))' \Rightarrow (TI(\prec, A))'$ where $'$ denotes any assignment of variables to closed terms. We may then apply $\to$R followed by ω_1R and $\forall_2$R sufficiently many times to close off any free variables, giving us

$$T^*_Q \,\Big|\frac{\Omega_\omega+\omega}{0}\, \emptyset \Rightarrow \forall \overrightarrow{X} \forall \overrightarrow{x} (WF(\prec) \to TI(\prec, A(a))),$$

as desired. □

4.1.2 The Reduction Procedure for T^*_Q

Lemma 4.1.21 *If C is a true literal and $T^*_0 \,\Big|\frac{\delta}{\rho}\, \Gamma_0, C \Rightarrow \Gamma_1$, then $T^*_0 \,\Big|\frac{\delta}{\rho}\, \Gamma_0 \Rightarrow \Gamma_1$. Likewise, if C is a false literal and $T^*_0 \,\Big|\frac{\delta}{\rho}\, \Gamma_0 \Rightarrow \Gamma_1, C$, then $T^*_0 \,\Big|\frac{\delta}{\rho}\, \Gamma_0 \Rightarrow \Gamma_1$.*

Proof

The is proved inductively on δ. In the base case, it is obvious $\Gamma_0 \Rightarrow \Gamma_1$ must be an axiom. The inductive step follows immediately from the induction hypothesis, since no inference rule can introduce a literal to the proof.

□

Lemma 4.1.22 (See [8] Lemma 5.23) *Suppose $gr(C) = \rho$ and $\delta \trianglelefteq \alpha = \omega^{\alpha_0} + \ldots \omega^{\alpha_n}$ with $\omega^{\alpha_0} \geq \ldots \geq \omega^{\alpha_n} \geq \delta$.*

1. *If C is atomic, or has the form $\exists x F(x), \exists X F(X), \exists^\omega F(X), A \vee B$, where $T^*_Q \left|\frac{\alpha}{\rho}\right. \Delta_0, C \Rightarrow \Delta_1$ and $T^*_Q \left|\frac{\delta}{\rho}\right. \Gamma_0 \Rightarrow \Gamma_1, C$, then $T^*_Q \left|\frac{\alpha+\delta}{\rho}\right. \Delta_0, \Gamma_0 \Rightarrow \Delta_1, \Gamma_1$.*
2. *If C has the form $\forall x F(x), \forall X F(X), \forall^\omega F(X), A \wedge B$, where $T^*_Q \left|\frac{\alpha}{\rho}\right. \Gamma_0 \Rightarrow \Gamma_1, C$ and $T^*_Q \left|\frac{\delta}{\rho}\right. \Delta_0, C \Rightarrow \Delta_1$ then $T^*_Q \left|\frac{\alpha+\delta}{\rho}\right. \Delta_0, \Gamma_0 \Rightarrow \Delta_1, \Gamma_1$.*

Proof

We shall prove (1) here. The proof for (2) is the dual case of (1). We proceed by induction on δ.

1. Suppose $\delta = 0$. Then $\Gamma_0 \Rightarrow \Gamma_1, C$ is an axiom. Then we have three subcases.
   (a) $\Gamma_0 \Rightarrow \Gamma_1$ is an axiom. Then by Lemma 4.1.12 $T^*_Q \left|\frac{\alpha+\delta}{\rho}\right. \Delta_0, \Gamma_0 \Rightarrow \Delta_1, \Gamma_1$.
   (b) C is a true literal. Then by Lemma 4.1.21 we have $T^*_Q \left|\frac{\alpha}{\rho}\right. \Delta_0 \Rightarrow \Delta_1$ and by weakening, $T^*_Q \left|\frac{\alpha+\delta}{\rho}\right. \Delta_0, \Gamma_0 \Rightarrow \Delta_1, \Gamma_1$.
   (c) C has the form $A(s_1, \ldots, s_m)$ and $A(t_1, \ldots, t_m) \in \Gamma_0$ where s_i and t_i are equivalents terms. Then from $T^*_Q \left|\frac{\alpha}{\rho}\right. \Delta_0, A(s_1, \ldots, s_m) \Rightarrow \Delta_1$ we get

$$T^*_Q \left|\frac{\alpha}{\rho}\right. \Delta_0, A(t_1, \ldots, t_m) \Rightarrow \Delta_1$$

by Lemma 4.1.12. And since $A(t_1, \ldots, t_m) \in \Gamma_0$, by Lemma 4.1.13 we have $T^*_Q \left|\frac{\alpha}{\rho}\right. \Delta_0, \Gamma_0 \Rightarrow \Delta_1, \Gamma_1$.

(d) $C = \exists X F(X)$ with $C \in \Gamma_0$. Then by weakening, $T^*_Q \left|\frac{\alpha}{\rho}\right. \Delta_0, \Gamma_0 \Rightarrow \Delta_1, \Gamma_1$, with $\exists X F(X)$ being absorbed into Γ_0.

In what follows, we shall assume that in the proof of $T^*_Q \left|\frac{\delta}{\rho}\right. \Gamma_0 \Rightarrow \Gamma_1, C$ the final inference is one where C is the principle formula or where an Ω_{k+1} rule was used. Otherwise, the proof follows from applying the induction hypothesis to the premises of the inference, and then carrying out the inference once again.

2. If $C = A \vee B$ was obtained by a $\vee$R-rule, then we have $(i) T^*_Q \left|\frac{\delta_0}{\rho}\right. \Gamma_0 \Rightarrow \Gamma_1, A, B$ for some $\delta_0 \lhd \delta$, and $gr(A), gr(B) < \rho$. By inversion we have $(ii) T^*_Q \left|\frac{\alpha}{\rho}\right. \Delta_0, A \Rightarrow \Delta_1$ and $(iii) T^*_Q \left|\frac{\alpha}{\rho}\right. \Delta_0, B \Rightarrow \Delta_1$. Applying our induction hypothesis to (i) and (ii) gives $T^*_Q \left|\frac{\alpha+\delta_0}{\rho}\right. \Gamma_0, \Delta_0 \Rightarrow \Gamma_1, \Delta_1, B$, and a applying a cut with (iii) gives $T^*_Q \left|\frac{\alpha+\delta_0+1}{\rho}\right. \Gamma_0, \Delta_0 \Rightarrow \Gamma_1, \Delta_1$, with $\alpha + \delta_0 + 1 \unlhd \alpha + \delta$. A weakening concludes the proof of this case.

3. If $C = \exists x F(x)$ was obtained by a $\exists_1$R-rule then $T^*_Q \left|\frac{\delta_0}{\rho}\right. \Gamma_0 \Rightarrow \Gamma_1, F(t)$ for some $t \in \mathbb{N}$ and by inversion $T^*_Q \left|\frac{\alpha}{\rho}\right. \Delta_0, F(s) \Rightarrow \Delta_1$ for all $s \in \mathbb{N}$. Taking the case where $t = s$, we may apply our induction hypothesis to get $T^*_Q \left|\frac{\alpha+\delta_0}{\rho}\right. \Gamma_0, \Delta_0 \Rightarrow \Gamma_1, \Delta_1$.

4. If $C = \exists X F(X)$, then C cannot be the principle formula of an inference.

5. If $C = \exists^\omega X F(X)$ was obtained via a $\exists_2$R-rule, then $T^*_Q \left|\frac{\delta_0}{\rho}\right. \Gamma_0 \Rightarrow \Gamma_1, F(U^n)$ for all $n \in \mathbb{N}$ and by inversion $T^*_Q \left|\frac{\alpha}{\rho}\right. \Delta_0, F(U^m) \Rightarrow \Delta_1$ for some $m \in \mathbb{N}$. Taking the case where $n = m$, we may apply our induction hypothesis to get

$$T^*_Q \left|\frac{\alpha+\delta_0}{\rho}\right. \Gamma_0, \Delta_0 \Rightarrow \Gamma_1, \Delta_1.$$

6. If $T^*_Q \left|\frac{\delta_0}{\rho}\right. \Gamma_0 \Rightarrow \Gamma_1, C$ is obtained by a Ω_{k+1}R inference then we have a fundamental function f such that

(a) $f(\Omega_{k+1}) \trianglelefteq \delta$

(b) $T_Q^* \mid\frac{f(0)}{\rho} \Gamma_0 \Rightarrow \Gamma_1, C, \forall X F(X)$, with $stR(\forall X F(X)) \leq n$.

(c) $T_Q^* \mid\frac{\beta}{0} \Xi \Rightarrow \Theta, \forall X F(X)$ implies $T_Q^* \mid\frac{f(\beta)}{0} \Xi, \Gamma_0 \Rightarrow \Theta, \Gamma_1$, where $stR(\Xi) + 1, stR(\Theta) \leq n$, and $\beta < \Omega_{k+1}$.

Applying the induction hypothesis to (b) and (c) yields:

$$\text{(b*) } T_Q^* \mid\frac{\alpha+f(0)}{\rho} \Gamma_0, \Delta_0 \Rightarrow \Gamma_1, \Delta_1, \forall X F(X)$$

and

$$T_Q^* \mid\frac{\alpha+f(\beta)}{0} \Xi, \Gamma_0, \Delta_0 \Rightarrow \Theta, \Gamma_1, \Delta_1.$$

Since $f(\Omega_{k+1}) \trianglelefteq \delta \trianglelefteq \alpha + \delta$, we have $f(\Omega_{k+1}) \trianglelefteq \alpha + \delta$, and $\alpha + f$ is a fundamental function. Hence. we may apply Ω_{k+1}R once again to find $T_Q^* \mid\frac{\alpha+\delta}{\rho} \Gamma_0, \Delta_0 \Rightarrow \Gamma_1, \Delta_1$.

7. If $T_Q^* \mid\frac{\delta_0}{\rho} \Gamma_0 \Rightarrow \Gamma_1, C$ is obtained by a Ω_{k+1}L inference, then the proof is similar to the Ω_{k+1}R case.

□

Lemma 4.1.23 (Cut Elimination) *[See [8] Lemma 5.24]*

If $T_Q^* \mid\frac{\alpha}{\rho+1} \Gamma_0 \Rightarrow \Gamma_1$, *then* $T_Q^* \mid\frac{\omega^\alpha}{\rho} \Gamma_0 \Rightarrow \Gamma_1$.

Proof

Proceed by induction on α. We need only deal with the critical case, where the final inference of the proof is a cut of grade ρ.

Thus, suppose we have $T_Q^* \left|\frac{\alpha_0}{\rho+1}\right. \Gamma_0 \Rightarrow \Gamma_1, C$ and $T_Q^* \left|\frac{\alpha_0}{\rho+1}\right. C, \Gamma_0 \Rightarrow \Gamma_1$, where $\alpha_0 \lhd \alpha$ $gr(C) = \rho$. By our induction hypothesis, we get $T_Q^* \left|\frac{\omega^{\alpha_0}}{\rho}\right. \Gamma_0 \Rightarrow \Gamma_1, C$ and $T_Q^* \left|\frac{\omega^{\alpha_0}}{\rho}\right. C, \Gamma_0 \Rightarrow \Gamma_1$.

If C has the form $\exists x F(x), \exists X F(X), \exists^\omega X F(X), a \vee b$ then we apply Lemma 4.1.22 (1), and we are finished. Likewise, if C has the form $\forall x F(x), \forall X F(X), \forall^\omega X F(X), a \wedge b$ then we apply Lemma 4.1.22 (2). Since $\alpha_0 \lhd \alpha$, we may apply weakening to get $T_Q^* \left|\frac{\omega^{\alpha}}{\rho}\right. \Gamma_0 \Rightarrow \Gamma_1$.

If C has the form $A \rightarrow B$, then we may apply inversion to $T_Q^* \left|\frac{\omega^{\alpha_0}}{\rho}\right. \Gamma_0 \Rightarrow \Gamma_1, C$ which gives us

$$(i)\ T_Q^* \left|\frac{\omega^{\alpha_0}}{\rho}\right. \Gamma_0, A \Rightarrow \Gamma_1, B.$$

Applying inversion to $T_Q^* \left|\frac{\omega^{\alpha_0}}{\rho}\right. A \rightarrow B, \Gamma_0 \Rightarrow \Gamma_1$ gives us

$$(ii)\ T_Q^* \left|\frac{\omega^{\alpha_0}}{\rho}\right. \Gamma_0 \Rightarrow \Gamma_1, A \text{ and } (iii)\ T_Q^* \left|\frac{\omega^{\alpha_0}}{\rho}\right. \Gamma_0, B \Rightarrow \Gamma_1.$$

We may then apply the cut rule to (i) and (ii), followed by a cut with (iii) to get, $T_Q^* \left|\frac{\omega^{\alpha_0}+2}{\rho}\right. \Gamma_0, A \Rightarrow \Gamma_1$. Weakening one again gives us the desired result.

If C has the form $\neg A$ then we apply inversion to both derivations, followed by a cut, much like in the previous case.

□

Theorem 4.1.24 (Collapsing Theorem) *[See [8] Lemma 5.25]*

Suppose Γ_0, Γ_1 *are sets of weak formulas, such that* $stL(\Gamma_0), stR(\Gamma_1) \leq n$, *and let* $\alpha \in C_n(\alpha)$. *Then:*

$$\textit{If } T_Q^* \left|\frac{\alpha}{0}\right. \Gamma_0 \Rightarrow \Gamma_1 \textit{ then } T_Q^* \left|\frac{\psi_n \alpha}{0}\right. \Gamma_0 \Rightarrow \Gamma_1.$$

Proof

We proceed by induction on α. The base case is trivial.

For the induction step, assume the proposition holds up to α. If If $T^*_Q \left|\frac{\alpha}{0}\right. \Gamma_0 \Rightarrow \Gamma_1$ is derived by any inference other than an Ω_{k+1} rule, then we have some $\alpha_0 \lhd \alpha$ such that If $T^*_Q \left|\frac{\alpha_0}{0}\right. \Gamma^i_0 \Rightarrow \Gamma^i_1$, where $\Gamma^i_0 \Rightarrow \Gamma^i_1$ is the ith premise of the inference. By Corollary 3.2.8 we find $\alpha_0 \in C_n(\alpha_0)$ and $\psi_n(\alpha_0) \lhd \psi_n(\alpha)$. Hence, by weakening, we are finished.

We shall prove the cases for the Ω_{k+1}R-rules, but the left-sided cases proceed similarly.

If $T^*_Q \left|\frac{\alpha}{0}\right. \Gamma_0 \Rightarrow \Gamma_1$ is the result of an Ω_{k+1}R inference, and $k < n$. Then we have a fundamental function f such that $f(\Omega_{k+1}) \trianglelefteq \alpha$. Moreover $\psi_n(f)$ is a fundamental function with $\Omega_{k+1} \in \text{dom}(f)$, and $f(\Omega_{k+1}) \trianglelefteq \psi_n f(\Omega_{k+1})$. Thus, we may apply Ω_{k+1}R to get If $T^*_Q \left|\frac{\psi_n f(\Omega_{k+1})}{0}\right. \Gamma_0 \Rightarrow \Gamma_1$.

Since $f(\Omega_{k+1}) \trianglelefteq \alpha$, and $\alpha \in C_n(\alpha)$, we have $\psi_n(f(\Omega_{k+1})) \trianglelefteq \psi_n(\alpha)$ and once again, by weakening, we are finished.

Finally, suppose If $T^*_Q \left|\frac{\alpha}{0}\right. \Gamma_0 \Rightarrow \Gamma_1$ is the result of an Ω_{k+1}R inference, and $n \leq k$.

Then we have $f(\Omega_{k+1}) \trianglelefteq \alpha$, and

$$(1)\ T^*_Q \left|\frac{f(0)}{0}\right. \Gamma_0 \Rightarrow \Gamma_1, \forall X F(X).$$

and

$$(2)\ T^*_Q \left|\frac{\beta}{0}\right. \Xi \Rightarrow \Theta, \forall X F(X) \text{ implies } T^*_Q \left|\frac{f(\beta)}{0}\right. \Xi, \Gamma_0 \Rightarrow \Theta, \Gamma_1,$$

for all $\beta < \Omega_{k+1}$ and all sets of weak formulas Ξ, Θ such that $stR(\Xi) + 1, stR(\Theta) \leq n$.

Applying out induction hypothesis to (1), we get $T^*_Q \left|\frac{\psi_n(f(0))}{0}\right. \Gamma_0 \Rightarrow \Gamma_1, \forall X F(X)$. Hence, by (2), taking $\Xi = \Gamma_0$ and $\Theta = \Gamma_1$ we obtain

$$T^*_Q \left|\frac{f(\psi_n(f(0)))}{0}\right. \Gamma_0 \Rightarrow \Gamma_1,$$

with Corollary 3.2.17 proving that $f(\psi_n(f(0))) \lhd f(\Omega_k{+}1) \trianglelefteq \alpha$. We apply our induction hypothesis one more time, to get

$$T^*_Q \left|\frac{\psi_n(f(\psi_n(f(0))))}{0}\right. \Gamma_0 \Rightarrow \Gamma_1,$$

and since $f(\psi_n(f(0))) \lhd \alpha$, we have $\psi_n(f(\psi_n(f(0)))) \lhd \psi_n(\alpha)$. □

4.1.3 Embedding D_Q into T^*_Q

In this section, we shall show that if D_Q is a well-founded tree, then we may embed it into T^*_Q, and thereby obtain a proof of the empty sequent. We may then leverage cut elimination and the collapsing theorem in order to show that such a proof is impossible, and therefore D_Q is not well founded, thereby proving our central result. Suppose that $\mathfrak{X}$ is the Kleene-Brouwer ordering of D_Q. We shall use $D_Q \overset{\tau}{\vdash} \Gamma_0 \Rightarrow \Gamma_1$ to denote that the sequent $\Gamma_0 \Rightarrow \Gamma_1$ is attached to the node τ. Also, recall that in enumerating the axioms of $\Pi^1_1 - \mathbf{CA}_0 + \mathbf{BI}$ we specified that A_i is always an instance of $\Pi^1_1 - \mathbf{CA}_0$ when i is even, and an instance of Bar Induction when odd.

Recall that $\omega_n\alpha$ as shorthand to indicate $\omega^{\cdot^{\cdot^{\cdot^{\omega^\alpha}}}}$, iterated n times.

Definition 4.1.25 *A T^*_Q formula F^* is said to be an* ***interpretation*** *of a $\mathcal{L}^Q_2$-formula F, if F^* is the result of replacing every free set variable U in F with U^m for some $m < \omega$ and every strong predicate quantifier $\forall X$ with $\forall^\omega X$. If Γ is a set of formulas, we shall then $\Gamma^* := \{F^* | F \in \Gamma\}$ for some interpretation *.*

Theorem 4.1.26 (See [8] Lemma 5.26) $D_Q \overset{\tau}{\vdash} \Delta \Rightarrow \Gamma$ *implies* $\exists k < \omega, T^*_Q \overset{\mathfrak{E}_\tau + k}{\underset{0}{\vdash}} \Delta^* \Rightarrow \Gamma^*$.

Proof

We proceed by induction on τ, i.e. the Kleene-Brouwer ordering on D_Q. Recall that by Lemma 3.2.2 (9), that $\mathfrak{E}_u \lhd \mathfrak{E}_v$ for all $u < v$.

If τ is an end node, then $\Delta \Rightarrow \Gamma$ is axiomatic, and therefore $\Delta^* \Rightarrow \Gamma^*$ is axiomatic. Hence $T^*_Q \overset{\mathfrak{E}_\tau + k}{\underset{0}{\vdash}} \Delta^* \Rightarrow \Gamma^*$ follows by Lemma 4.1.12.

Now suppose τ is not an end node.

If τ is not reducible, then there is a node τ_0 immediately above τ such that $D_Q \stackrel{\tau_0}{\vdash} A_i, \bar{Q}(i)\Delta \Rightarrow \Gamma$. Applying our induction hypothesis, we have

$$T^*_Q \vdash^{\mathfrak{E}_{\tau_0}+k_0}_{0} A^*_i, \bar{Q}(i)^*\Delta^* \Rightarrow \Gamma^*.$$

We also have $T^*_Q \vdash^0_0 \bar{Q}(i)$ and, using Corollary 4.1.19 (for even i) and Corollary 4.1.20 (for odd i) we have $T^*_Q \vdash^{\Omega_\omega+\omega}_0 A_i$. Since $\Omega_\omega + \omega \lhd \mathfrak{E}_{\tau_0} + k_0$, by applying two cuts, we obtain

$$T^*_Q \vdash^{\mathfrak{E}_{\tau_0}+k_0+2}_{n} \Delta^* \Rightarrow \Gamma^*,$$

with $n \neq 0$ and by applying cut elimination we find $T^*_Q \vdash^{\omega_n(\mathfrak{E}_{\tau_0}+k_0+2)}_0 \Delta^* \Rightarrow \Gamma^*$. Recall that $\omega_n(\alpha)$ is shorthand for $\omega^{\cdot^{\cdot^{\cdot^{\omega^\alpha}}}}$. Since $\omega_n(\mathfrak{E}_{\tau_0} + k_0 + 2) \lhd \mathfrak{E}_\tau$, we may apply weakening to obtain $T^*_Q \vdash^{\mathfrak{E}_\tau+k}_0 \Delta^* \Rightarrow \Gamma^*$ as desired.

Now, suppose that $\Delta \Rightarrow \Gamma$ is reducible, and of the form $\Delta \Rightarrow \Gamma', E, \Gamma''$ where E is the redex, and Γ'_i contains only literals. Any case where $E \in \Delta$ has a dual case in Γ that proceeds by a similar process.

Suppose E has the form $\forall x F(x)$. Then for each m there is a node τ_m immediately above τ such that:

$$D_Q \stackrel{\tau_m}{\vdash} A_i, \bar{Q}(i), \Delta \Rightarrow \Gamma', F(m), \Gamma''.$$

Applying our induction hypothesis yields:

$$T^*_Q \vdash^{\mathfrak{E}_{\tau_m}+k_m}_{0} A_i, \bar{Q}(i), \Delta^* \Rightarrow \Gamma'^*, F(m), \Gamma''^*.$$

As above, we cut $\bar{Q}(i)$ and $A(i)$, yielding

$$T^*_Q \vdash^{\mathfrak{E}_{\tau_m}+k_m+2}_{n} \Delta^* \Rightarrow \Gamma'^*, F(m), \Gamma''^*$$

and Cut Elimination yields

$$T^*_Q \vdash^{\omega_n(\mathfrak{E}_{\tau_m}+k_m+2)}_{0} \Delta^* \Rightarrow \Gamma'^*, F(m), \Gamma''^*.$$

and thus

$$T^*_Q \left|\frac{\mathfrak{E}_\tau}{0}\right. \Delta^* \Rightarrow \Gamma'^*, F(m), \Gamma''^*.$$

Finally, we apply ω_1R to obtain

$$T^*_Q \left|\frac{\mathfrak{E}_\tau}{0}\right. \Delta^* \Rightarrow \Gamma'^*, \forall x F(x), \Gamma''^*$$

as desired.

The case for $\exists x F(x) \in \Delta$ proceeds by much the same procedure, and indeed the cases involving the remaining first-order quantifiers and logical connectives in general resemble finitary versions of the above case. The critical cases revolve around the secondary quantifiers.

If E has the form $\forall X F(X)$, then we know that there is τ_0 immediately above τ with $D_Q \left|\frac{\tau_0}{}\right. A_i, \bar{Q}(i), \Delta \Rightarrow \Gamma', F(U), \Gamma''$. Applying our induction hypothesis, and making the usual cuts we get:

$$T^*_Q \left|\frac{\mathfrak{E}_{\tau_0}+k+2}{0}\right. \Delta^* \Rightarrow \Gamma'^*, F^*(U^m), \Gamma''^*,$$

for every interpretation $*$ and hence for every $n \leq \omega$. If $\forall X F(X)$ is weak, then take $n = stR(\forall X F(X))$ and apply $\forall_2\mathrm{R}_m$, to get

$$T^*_Q \left|\frac{\mathfrak{E}_{\tau_0}+k+3}{n}\right. \Delta^* \Rightarrow \Gamma'^*, \forall X F^*(X), \Gamma''^*.$$

Otherwise, apply the $\omega_2 R$-rule to get

$$T^*_Q \left|\frac{\mathfrak{E}_{\tau_0}+k+3}{n}\right. \Delta^* \Rightarrow \Gamma'^*, \forall^\omega X F^*(X), \Gamma''^*.$$

We may then apply Cut Elmination and Lemma 4.1.12 as usual to get the appropriate cut rank and proof height. The case for $\exists X F(X) \in \Delta$ proceeds similarly.

Finally, if E has the form $\exists X F(X)$, then we have τ_0 above τ such that $D_Q \left|\frac{\tau_0}{}\right. \Delta \Rightarrow \Gamma', \exists X F(X), \Gamma''$. Applying the induction hypothesis and requisite cuts, we get:

$$T^*_Q \left|\frac{\mathfrak{E}_{\tau_0}+k+2}{n}\right. \Delta^* \Rightarrow \Gamma'^*, F^*(U^m), \Gamma''^*.$$

If $\exists X$ was a strong quantifier, then we apply $\exists_2$R, followed by cut elimination to get

$$T^*_Q \,\Big|\!\frac{\omega_n(\mathfrak{E}_{\tau_0}+k+3)}{0}\, \Delta^* \Rightarrow \Gamma'^*, \exists^\omega X F^*(X), \Gamma''^*.$$

Otherwise, invoking Lemma 4.1.18 we obtain $T^*_Q \,\Big|\!\frac{\Omega_{n+1}\cdot 2}{0}\, F(U^m) \Rightarrow \exists X F^*(X)$. Applying a cut yields

$$T^*_Q \,\Big|\!\frac{\mathfrak{E}_{\tau_0}+k+4}{n}\, \Delta^* \Rightarrow \Gamma'^*, \exists X F^*(X), \Gamma''^*.$$

We then apply cut elimination as usual, and then raise the ordinal using Lemma 4.1.12 to complete the proof. This case is analogous to when $\forall X F(X) \in \Delta$.

□

Corollary 4.1.27 (See [8] Lemma 5.27) *If D_Q is well-founded, then $T^*_Q \,\Big|\!\frac{\psi_0(\omega_n(\mathfrak{E}_{\tau_0}+k))}{0}$ $\emptyset \Rightarrow \emptyset$ for some $n, k < \omega$, and τ_0 the root node of D_Q.*

Proof

By the preceding lemma, we have

$$T^*_Q \,\Big|\!\frac{\mathfrak{E}_{\tau_0}+k}{0}\, \bar{Q}(0), A_0 \Rightarrow \emptyset.$$

Using Lemma 4.1.19 We also have

$$T^*_Q \,\Big|\!\frac{\Omega_{k+1}}{0}\, \emptyset \Rightarrow A_0.$$

and $T^*_Q \,\Big|\!\frac{0}{0}\, \emptyset \Rightarrow \bar{Q}(0)$ is axiomatic. Applying two cuts yields

$$T^*_Q \,\Big|\!\frac{\mathfrak{E}_{\tau_0}+k}{n}\, \emptyset \Rightarrow \emptyset.$$

We then invoke Cut Elimination and the Collapsing Theorem to get

$$T^*_Q \,\Big|\!\frac{\psi_0(\omega_n\mathfrak{E}_{\tau_0}+k)}{0}\, \emptyset \Rightarrow \emptyset.$$

□

Theorem 4.1.28 (See [8] Lemma 5.28) *D_Q is not well-founded.*

Proof

If D_Q were well-founded then we would have $T^*_Q \left|\frac{\psi_0(\omega_n(\mathfrak{E}_{\tau_0}+k))}{0}\right. \emptyset \Rightarrow \emptyset$ for some $n, k < \omega$, and τ_0 the root node of D_Q by the preceding lemma. However, if we carry out an induction on $\alpha \leq \Omega_\omega$ we can see that if $T^*_Q \left|\frac{\alpha}{0}\right. \Delta \Rightarrow \Gamma$ then either $\Gamma \neq \emptyset$ or $\Delta \neq \emptyset$ and hence there can be no proof of the empty sequent. $\square$

It remains only to show that this proof can be carried out in the base theory$\mathbf{RCA}_0 +$ $\mathbf{WOP}(\psi_0(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}}))$, which we shall abbreviate as **S** in the following argument. Note that through Theorem 1.1.1 that we may bootstrap up to $\mathbf{ACA}_0$ in **S**. Of particular concern is the notion of derivability in T^*_Q, since it appears to be defined through an iterated inductive definition, introducing new Ω_k-rules at each step. This is not available in our base theory, **S**. However, we will show that derivability can in fact be proven using a fixed-point argument, which is permissible in $\mathbf{ACA}_0$. Our argument is adapted from that found at the end of [8].

Given a set Q, we can prove there exists an ω-model $\mathcal{A}$, with $Q \in \mathcal{A}$ and $\mathcal{A} \models$ **BI** thanks to Theorem 1.1.8.

Now, suppose $\alpha \in \mathrm{OT}(\mathfrak{E}_{\Omega_{\omega+\mathfrak{X}}}), \rho < \omega$ and $\Theta \Rightarrow \Gamma$ is a sequent of T^*_Q. We first desire a derivability predicate D_0 such that $D_0(\alpha, \rho, \Theta \Rightarrow \Gamma)$ if and only if $\Theta \Rightarrow \Gamma$ is axiomatic, or $\Theta \Rightarrow \Gamma$ the result of a non-Ω_k inference in T^*_Q, with premises $(\Theta_i \Rightarrow \Gamma_i)_{i \in I}$, and $\beta_i \lhd \alpha$, such that for all $i \in I, D_0(\beta_i, \rho, \Theta_i \Rightarrow \Gamma_i)$. If this inference is a cut, we also require that the the cut rank is less than ρ.

We may view this as a fixed-point statement which, when combined with transfinite induction over $OT(\mathfrak{E}_{\Omega_{\omega+X}})$ implicitly defines derivability in T^*_Q, minus the Ω_k-rules.

Now, $\mathbf{ATR}_0$ proves $\Sigma^1_1 - \mathbf{AC}$, the axiom of choice for Σ^1_1 formulas (see Theorem V.8.3 in [11]). Moreover, $\mathbf{ACA}_0 + \Sigma^1_1 - \mathbf{AC}$ proves the Second Recursion Theorem: for every P-positive arithmetical formula $A(u, P)$ there exists a Σ^1_1 formula $F(U)$ such that $\forall x[F(x) \leftrightarrow A(x, F)]$ where $A(x, F)$ is obtained from $A(u, P)$ by replacing every instance of $P(t)$ with $F(t)$.

Thus, arguing in $\mathcal{A}$, we may find an ω-model $\mathcal{B}_0 \in A$ such that $X \in \mathcal{B}_0$ and $\mathcal{B}_0 \models \mathbf{ATR}_0$. By applying the Second Recursion Theorem we may define D_0 within $\mathcal{B}_0$ and therefore D_0 is a set in $\mathcal{A}$.

We may now apply Theorem 1.1.7 iteratively, to create a tower of ω-models, $\mathcal{B}_0 \in \mathcal{B}_1, \in \mathcal{B}_2 \ldots$. In each ω-model, $\mathcal{B}_i$ we define a corresponding derivability predicate, D_i, and require that $D_i \in \mathcal{B}_{i+1}$ for all $i \in \mathbb{N}$.

Each predicate D_{i+1} encapsulates derivability using Ω_k-rules for $k \leq i+1$. This is defined much the same as D_0, but we require that in the instance of an Ω_{i+1} inference that the negative occurrences must satisfy the D_i derivability predicate. With this done, we may apply **ATR** to collect these derivability predicates into a single predicate $D_\omega \in \mathcal{A}$.

Finally, we should note that the notion of derivability involves quantifying over the set of fundamental functions, which is not permitted in $\mathbf{ACA}_0$. However, the only fundamental functions used in our proof are primitive recursive. Hence, by restricting ourselves to primitive recursive fundamental functions, we may carry out the quantification in $\mathbf{ACA}_0$.

Chapter 5

Conclusion

5.1 Further Avenues of Investigation

This thesis can be regarded as an extension of the methods used in [8], which was the first paper on well-ordering principles to require the use of a Ω-rule, and consequently a fixed-point argument at the conclusion to ensure the proof could be carried out over a base theory of $\mathbf{RCA}_0$. However, this thesis and [8] still closely follow the example of existing proof-theoretic research. For example, [2] and [6] present ordinal analyses of systems up to Δ^1_2-**CA** + Bar Rule and Δ^1_2-**CA** + **BI** respectively, with similarly powerful ordinal representation systems. In [2] this is accomplished by a generalized Ω_α function, whereas [6] uses a collection of $\Phi\gamma\alpha$ functions, where Φ_0 enumerates the class $Kr(0)$ of uncountable cardinals, and Φ_γ enumerates the fixed points of Φ_{γ_0} for all $\gamma_0 < \gamma$, in much the same way as the φ functions enumerate fixed points of the additive principle numbers. It would be only natural to seek out similar well-ordering principles for these more powerful systems.

Of course, there is also the question of finding a well-ordering principle for Π^1_1-$\mathbf{CA}_0$. At the time of this writing, the author's thesis advisor is working on a paper that will prove:

Theorem 5.1.1 *The following are equivalent over* $\textbf{\textit{RCA}}_0$:

1. $\textbf{\textit{WOP}}(\Omega_\omega \cdot \mathfrak{X})$.

2. *Every set is contained in a countable-coded* ω*-model of* Π^1_1*-$\textbf{\textit{CA}}_0$*.

Here, $OT(\Omega_\omega \cdot \mathfrak{X})$ is constructed in a manner similar to $OT(\psi_0(\mathfrak{E}_{\Omega_{\Omega+\mathfrak{X}}}))$, except instead of having epsilon numbers, $\mathfrak{E}_u$ for all $u \in X$, we have the terms $\Omega_\omega \cdot u$. Consequently, we also lose closure under ω^γ above $OT(\Omega_\omega \cdot \mathfrak{X})$. Removing Bar Induction also removes the problem of strong formulas appearing in the deduction tree D_Q, which accounts for the reduced height needed to embed D_Q into T^*_Q.

5.2 The Utility of ω-Models

As stated in the introduction of this thesis, there are two ways to present well-ordering principle results. Compare Theorems 1.1.2 and 1.1.5, referenced from [5] and [7] respectively. In the case of Theorem 1.1.2, we have an equivalence between **WOP**$(\varepsilon_\mathfrak{X})$ and $\mathbf{ACA}_0^+$. In Theorem 1.1.5, we instead have an equivalence between **WOP**$(\varepsilon_\mathfrak{X})$ and the statement "Every set is contained in a countable-coded ω-model of $\mathbf{ACA}_0$." During the defence of this thesis, the examiners asked why one might prefer the latter presentation to the former.

There are several reasons. The first, is that it lines up more easily with established proof-theoretic results. In this thesis we adapted the ordinal analysis of Π^1_1**-CA + BI** found in [2] and [6], using it as a road map for the proof that "Every set is contained in a countable-coded ω-model of Π^1_1**-CA + BI**." Likewise, Theorems 1.1.5 and 1.1.7 should hold a certain ring of familiarity to those versed in proof-theory. Thus, moving forward we should look to the ω-model presentation as a guide for how to adapt existing proof theoretic research to the investigation of well-ordering principles.

Nevertheless, one might argue that it is more intuitive to work directly within a system, as with the presentation of Theorem 1.1.2. Certainly, if $\mathbf{WOP}(f\mathfrak{X})$ is equivalent over $\mathbf{RCA}_0$ to the statement "Every set is contained in a countable-coded ω-model of T," for some theory T, then we should be able to find a theory T' such that $\mathbf{WOP}(f\mathfrak{X})$ is directly equivalent to T'. However, T' may not necessarily be an intuitive system to work in.

Take Theorem 1.1.2, for example. $\mathbf{ACA}_0^+$ includes an axiom regarding the existence of Turing jumps. As the result was proved using computability theory, this makes a great deal of sense. However, non-computability theorists may find it more intuitive to work with ω-models of a the familiar system $\mathbf{ACA}_0$. Indeed, the fixed point argument at the end of Chapter 4 is a scenario where it seems advantageous to work in the ω-model presentation, as we can iteratively generate larger ω-models to suit our needs.

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