Direct linearisation of discrete and continuous integrable systems: The KP hierarchy and its reductions

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The candidate confirms that the work submitted is his own, except where work which has formed part of jointly-authored publications has been included. The contribution of the candidate and the other author to this work has been explicitly indicated below. The candidate confirms that appropriate credit has been given within the thesis where reference has been made to the work of others.

Chapters 3, 4, 5 and 6 are based on the publications [**FN17a**], [**FN18**], [**FN17b**] and [**Fu18**], respectively. The contribution of the candidate to the results in the jointly authored publications [**FN17a**], [**FN18**] and [**FN17b**] was to provide the proofs and carry out all explicit computations; the concepts were developed in discussion with his co-author.

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Abstract

The thesis is concerned with the direct linearisation of discrete and continuous integrable systems, which aims to establish a unified framework to study integrable discrete and continuous nonlinear equations, and to reveal the underlying structure behind them.

The idea of the direct linearisation approach is to connect a nonlinear equation with a linear integral equation. By introducing an infinite-dimensional matrix structure to the linear integral equation, we are able to study various nonlinear equations in the same class and their interlinks simultaneously, as well as the associated integrability properties. Meanwhile, the linear integral equation also provides a general class of solutions to those nonlinear equations, in which the well-known soliton-type solutions to those nonlinear equations can be recovered very easily.

In the thesis, we consider discrete and continuous integrable equations associated with scalar linear integral equations. The framework is illustrated by three-dimensional models including the discrete and continuous Kadomtsev–Petviashvili-type equations as well as the discrete-time two-dimensional Toda lattice, and their dimensional reductions which result in a huge class of two-dimensional discrete and continuous integrable systems.

Abbreviations

1D	one-dimensional
2D	two-dimensional
2DTL	two-dimensional Toda lattice
3D	three-dimensional
4D	four-dimensional
BSQ	Boussinesq
BT	Bäcklund transform
DAGTE	discrete analogue of a generalised Toda equation
DL	direct linearisation
DT	Darboux transform
FG	Fordy–Gibbons
GD	Gel'fand–Dikii
HS	Hirota–Satsuma
HM	Hirota–Miwa
IST	inverse scattering method
KdV	Korteweg–de Vries
KK	Kaup–Kupershmidt
KP	Kadomtsev–Petviashvili
MDC	multi-dimensional consistency
SK	Sawada–Kotera

Contents

CHAPTER 1

Introduction

1. Integrability

In classical mechanics, the 'complete integrability' of a system is guaranteed by the existence of a sufficient number of functionally independent first integrals. The modern theory of integrable systems is a generalisation of the theory of finite-dimensional integrable systems. To be more precise, the notion is generalised from 'finite' to 'infinite'. A typical example is the famous Korteweg–de Vries (KdV) equation[1]

$$u_t = \frac{1}{4}u_{xxx} + \frac{3}{2}u_x^2, \tag{1.1}$$

which first appeared in the studies of wave theory. It was proven in [**Gar71**] (see also [**Miu76**]) that the KdV equation has an infinite number of commuting first integrals. We also note that such a property is equivalent to the existence of an infinite number of commuting symmetries, or the bi-Hamiltonian property, see e.g. [**Mag78**, **FF81**].

The integrability of the KdV equation can also be understood from many other viewpoints. In fact, it was Gardner, Greene, Kruskal and Miura [**GGKM67**] who first proved that the KdV equation is integrable in sense that the initial value problem of the KdV equation was solved by using a method called the inverse scattering transform (IST). In other words, in the IST scheme the integrability amounts to the existence of the N-soliton solution (where N is arbitrary). The key point in the IST approach is that the KdV equation arises as the compatibility condition of a pair of linear equations [**MGK68**] (which is now often known as Lax pair, see [**Lax68**])

$$L\phi = k^2\phi \quad \text{and} \quad \phi_t = A\phi, \tag{1.2}$$

where L and A are differential operators given by

$$L = \partial_x^2 + 2u_x \quad \text{and} \quad A = \partial_x^3 + 3u_x\partial_x + \frac{3}{2}u_{xx},$$

namely $L_t = [A, L]$, where $[A, L] \doteq AL - LA$ is the standard commutator. Therefore, the existence of a (non-fake) Lax pair[2] in a sense also guarantees the integrability of a nonlinear equation. Since then, the IST method was generalised and applied to other nonlinear equations, and simultaneously many new integrable systems were found, see e.g. [**AKNS73**, **AKNS74**, **ZS74**, **ZS79**], and also the monograph [**AC91**].

[1] For later convenience, here we use the potential form of the KdV equation. Its nonpotential form can be obtained by introducing the change of variable $\mathfrak{u} = u_x$, resulting in the well-known form $\mathfrak{u}_t = \frac{1}{4}\mathfrak{u}_{xxx} + 3\mathfrak{u}\mathfrak{u}_x$. Throughout the thesis, all soliton equations (except those modified equations in chapter 4) are derived in their potential form.

[2] There exist cases that a nonlinear equation cannot be solved by the IST method though it does have a Lax pair, see [**CN91**].

Other methods were also developed during the same period. Below we list a few famous ones. One is the so-called Bäcklund transform (BT) method, see [**WE73**]. We take the KdV equation as an example. The BT for equation (1.1) is given by

$$(u+\tilde{u})_x = 2p(\tilde{u}-u) - (\tilde{u}-u)^2, \tag{1.3}$$

where p is a parameter. It was proven that if u is a solution to the KdV equation, then $\tilde{u}$ solved from (1.3) provides a new solution (depending on the Bäcklund parameter p) to (1.1). Thus, one can start from a seed solution (for instance $u = 0$), and solve the Riccati equation for variable $\tilde{u}$, namely (1.3), with regard to different parameters by iteration, and end up with a solution involving an arbitrary number of free parameters.

The BT was carried out on the nonlinear level, cf. (1.3). A related method is the Darboux transform (DT) method, where the key step is to construct a transform on the associated linear system (1.2). Suppose that u is a solution to the KdV equation, and the spatial part of the Lax pair (1.2) has a solution f for given $k = p$. The DT

$$\tilde{\phi} = \phi_x - (\ln f)_x\phi, \tag{1.4a}$$

$$\tilde{u} = u + (\ln f)_x \tag{1.4b}$$

keeps both the linear system (1.2) and the nonlinear equation (1.1) invariant. That is to say, the DT generates a new solution $\tilde{u}$ from u, which now depends on the parameter p. By iteration, one can also construct solutions containing more and more parameters. The solution obtained from an N-fold DT is the same as the one arising from an N-fold BT if the same seed solution is selected.

Another useful method is Hirota's direct method, see e.g. [**Hir04**]. The idea of such a method is bilinearising a nonlinear equation. For instance, considering the bilinear transform $u = (\ln\tau)_x$, we can write down the bilinear form the KdV equation (1.1)

$$(D_x^4 - 4D_tD_x)\tau\cdot\tau = 0, \tag{1.5}$$

where D_x and D_t are Hirota's bilinear operators, and we refer to appendix A for the definition of Hirota's operator. Equation (1.5) can be solved by perturbation method, and its solution coincides with the result from BT or DT. The benefit of Hirota's method is that one can avoid the linear system and solve a nonlinear equation directly.

2. The KP hierarchy

2.1. Bilinearisation of the KP equation. Among integrable systems, one of the most important models is the Kadomtsev–Petviashvili (KP) equation

$$u_t = \frac{1}{4}u_{xxx} + \frac{3}{2}u_x^2 + \frac{3}{4}\partial_x^{-1}u_{yy}. \tag{1.6}$$

The KP equation is a (2+1)-dimensional integrable system, in the sense of having multi-soliton solution. Concretely, by introducing the bilinear transform $u = (\ln\tau)_x$, the KP equation (1.6) can be written as

$$(D_x^4 + 3D_y^2 - 4D_tD_x)\tau\cdot\tau = 0. \tag{1.7}$$

Then, the Hirota method shows that the N-soliton solution to the bilinear KP equation takes the form of (see e.g. [**Hir04**])

$$\tau = \sum_{J \subset I} \left(\prod_{i \in J} c_i \right) \left(\prod_{i,j \in J, i<j} A_{i,j} \right) \exp \left(\sum_{i \in J} \xi_i \right),$$

where $I = \{1, 2, \cdots, N\}$, J is any subset of I, c_i are arbitrary constants, and

$$\xi_i = (k_i + k_i')x + (k_i^2 - k_i'^2)y + (k_i^3 + k_i'^3)t, \quad A_{i,i'} = \frac{(k_i - k_j)(k_i' + k_j')}{(k_i - k_j')(k_i' + k_j)}.$$

We also note that such a solution can also be expressed by a determinant, such as a Wronskian or a Grammian [**Hir04**].

2.2. The Sato approach. The mathematics behind the KP equation was studied by Sato and Mori. In their papers [**SM80**, **SM81**, **Sat81**, **Sat83**], the concept of the KP hierarchy was first proposed, and it was shown that the structure of KP was completely governed by a single tau function solving the bilinear KP equations.

The KP hierarchy in the Sato scheme is associated with a pseudo-differential operator of order 1 as follows:

$$L = \partial + \mathfrak{u}_2 \partial^{-1} + \mathfrak{u}_3 \partial^{-2} + \cdots,$$

where $\mathfrak{u}_j = \mathfrak{u}_j(\mathbf{x})$ are functions of infinitely many independent variables $\mathbf{x} = (x_1, x_2, \cdots)$, and the operators ∂ and ∂^{-1} denote differentiation and integration with respect to the argument x_1, respectively, and therefore ∂^{-j} denotes the jth-composition of ∂^{-1}.

Considering the eigenvalue problem of this pseudo-differential operator given by

$$L\mathfrak{w} = k\mathfrak{w}, \tag{1.8}$$

and assuming that it has a formal solution

$$\mathfrak{w} = Me^{\xi(\mathbf{x},k)}, \quad \text{where} \quad M = 1 + \sum_{j=1}^{\infty} \mathfrak{w}_j \partial^{-j} \quad \text{and} \quad \xi(\mathbf{x}, k) = \sum_{j=1}^{\infty} k^j x_j,$$

one can deduce that the pseudo-differential operator L and the differential operator M obeys the following:

$$L = M \circ \partial \circ M^{-1}.$$

This relation actually implies that all the $\mathfrak{u}_j$ can be expressed by $\mathfrak{w}_j$.

If we further consider the linear system of equations

$$\frac{\partial \mathfrak{w}}{\partial x_j} = B_j \mathfrak{w}, \quad \text{where} \quad B_j = (L^j)_+, \tag{1.9}$$

and $(L_j)_+$ denotes the differential part of L_j, the compatibility between (1.8) and (1.9) gives rise to the Lax equations

$$\frac{\partial L}{\partial x_j} = [B_j, L]. \tag{1.10}$$

The above is an infinite set of equations for $\mathfrak{u}_2, \mathfrak{u}_3, \cdots$ and is referred to as the KP hierarchy. Meanwhile, the compatibility condition of the linear equations in (1.9), i.e.

$$\frac{\partial^2 \mathfrak{w}}{\partial x_i \partial x_j} = \frac{\partial^2 \mathfrak{w}}{\partial x_j \partial x_i},$$

also provides the Zakharov–Shabat representation of the KP hierarchy

$$\frac{\partial B_i}{\partial x_j} - \frac{\partial B_j}{\partial x_i} + [B_i, B_j] = 0.$$

The KP hierarchy (1.10) can also be reformulated to a hierarchy of scalar equations for $\mathfrak{u} = \mathfrak{u}_2$ and the simplest nontrivial one among them is the KP equation[3]

$$\mathfrak{u}_t = \frac{1}{4}\mathfrak{u}_{xxx} + 3\mathfrak{u}\mathfrak{u}_x + \frac{3}{4}\partial_x^{-1}\mathfrak{u}_{yy}.$$

where we have set $x = x_1$, $y = x_2$ and $t = x_3$.

The most important observation is that the wave function $\mathfrak{w}$ can actually be expressed by a single tau function via the equation

$$\mathfrak{w}(\mathbf{x}, k) = \frac{\tau\left(\mathbf{x} - \varepsilon(k^{-1})\right)}{\tau(\mathbf{x})} e^{\xi(\mathbf{x},k)}, \quad \text{where} \quad \varepsilon(k^{-1}) = \left(\frac{1}{k}, \frac{1}{2k^2}, \cdots\right).$$

Thus, all the $\mathfrak{w}_j$ are expressions of the tau function, and consequently $\mathfrak{u}_j$ are expressed by the tau function as well. For example,

$$\mathfrak{w}_1 = -\frac{\partial \tau}{\partial x_1}\Big/\tau, \quad \mathfrak{w}_2 = \frac{1}{2}\left(\frac{\partial^2 \tau}{\partial x_1^2} - \frac{\partial \tau}{\partial x_2}\right)\Big/\tau, \quad \cdots,$$
$$\mathfrak{u}_2 = \frac{\partial^2}{\partial x_1^2}\ln\tau, \quad \mathfrak{u}_3 = \frac{1}{2}\left(\frac{\partial^3}{\partial x_1^3} - \frac{\partial^2}{\partial x_1 \partial x_2}\right)\ln\tau, \quad \cdots.$$

The relation between $\mathbf{u}_2$ and τ actually plays the role of transferring the nonlinear KP hierarchy to its bilinear version. For instance, by this transform we immediately from (1.6) derive

$$(D_1^4 + 3D_2^2 - 4D_1D_3)\tau \cdot \tau = 0,$$

where D_j are the abbreviations of D_{x_j}, which is the first nontrivial member in the bilinear KP hierarchy.

2.3. Vertex operator, bilinear identity and classification of soliton equations. The Sato theory also motivated the Kyoto School to consider the theory of soliton equations from the aspect of the representation theory. It was observed that the tau function of the KP hierarchy (see e.g. [**MJD00**]) is generated by a vertex operator

$$X(k, k') = \exp\left(\sum_{j=1}^{\infty}(k^j - (-k')^j)x_j\right)\exp\left(-\sum_{j=1}^{\infty}\frac{1}{j}(k^{-j} - (-k')^{-j})\frac{\partial}{\partial x_j}\right), \tag{1.11}$$

and in this case the tau function for the N-soliton solution is given by

$$\tau = e^{c_1 X(k_1, k_1')} \cdots e^{c_N X(k_N, k_N')} 1, \tag{1.12}$$

[3]This is actually the nonpotential KP equation. We can retrieve its potential version, namely (1.6), by the transform $\mathfrak{u} = u_x$.

where $c_1, \cdots c_N$ are constants. In fact, it was proven that the tau function given above obeys a bilinear identity as follows:

$$\operatorname*{res}_{k=\infty}\left[e^{\xi(\mathbf{x},k)-\xi(\mathbf{x}',k)}\tau\left(\mathbf{x}-\varepsilon(k^{-1})\right)\tau\left(\mathbf{x}'+\varepsilon(k^{-1})\right)\right]=0. \tag{1.13}$$

By expanding the bilinear identity (1.13), one can obtain an infinite number of bilinear equations in the KP hierarchy, which are simultaneously solved by the same tau function given by (1.12). Below we list the first few bilinear KP equations:

$$(D_1^4-4D_1D_3+3D_2^2)\tau\cdot\tau=0, \tag{1.14a}$$

$$(D_1^3D_2+2D_3D_2-3D_1D_4)\tau\cdot\tau=0, \tag{1.14b}$$

$$(D_1^6-20D_1^3D_3-80D_3^2+144D_1D_5-45D_1^2D_2^2)\tau\cdot\tau=0, \tag{1.14c}$$

$$(D_1^6+4D_1^3D_3-32D_3^2-9D_1^2D_2^2+36D_2D_4)\tau\cdot\tau=0, \tag{1.14d}$$

in which the first equation is nothing but (1.7). Here we must point out that although on the nonlinear level every member in the KP hierarchy is a (2+1)-dimensional equation, the bilinear KP equations, instead, may involve higher dimensions (only the first bilinear equation can be written as a (2+1)-dimensional model). The vertex operator (1.11) is the infinitesimal generator of the BT of the bilinear KP hierarchy; more precisely, it plays the role of noncommuting symmetries (i.e. $X(k_i,k_i')$ and $X(k_j,k_j')$ do not commute in general) of the bilinear KP hierarchy.

More importantly, according to the Kyoto School, the vertex operator is deep down a representation of the infinite-dimensional algebra $\mathfrak{gl}(\infty)$ (also referred to as A_∞) on the polynomial space $\mathbb{C}[x_1,x_2,\cdots]$. This provides a way to understand soliton equations from the aspect of the group theory, and leads to a classification of integrable equations. Date, Jimbo, Kashiwara and Miwa generalised the KP hierarchy to its various modifications and considered their reductions, see e.g. [**DJKM83**, **JM83**]. In each case, a vertex operator is given, leading to a class of soliton equations. Examples include various KP-type hierarchies and a large class of (1+1)-dimensional soliton hierarchies. For example, the KdV hierarchy arises as the 2-periodic reduction of the KP hierarchy; concretely, by taking $k'=k$ in the vertex operator (1.11), we obtain the constraint $\frac{\partial}{\partial x_{2j}}\tau=0$, and as a result, a hierarchy of bilinear KdV equations including (1.5) arise from the bilinear KP hierarchy immediately. The same bilinear transform $u=(\ln\tau)_x$ brings us back to the nonlinear KdV hierarchy; e.g. (1.5) provides the KdV equation (1.1). The classification problem of (1+1)-dimensional soliton equations from the viewpoint of Kac–Moody algebras was also studied by Drinfel'd and Sokolov [**DS85**], and by Segal and Wilson [**SW85**], independently.

3. Discrete KP equation

3.1. Integrable discretisation. Discrete integrable systems have played an increasingly prominent role in both mathematics and physics during the past decades, cf. e.g. [**HJN16**]. The theory of discrete integrable systems has made important contributions to other areas of classical and modern mathematics, such as algebraic geometry, discrete geometry, Lie algebras, cluster algebras, orthogonal polynomials, quantum theory, special functions, and random matrix theory.

Discrete equations in a sense play the role of master equations in integrable systems theory. They encode entire hierarchies of corresponding continuous equations and can be understood as the Bianchi permutability property and Bäcklund transforms of both continuous and discrete equations. Furthermore, they have considerable significance in their own right, in view of the rich algebraic structure behind them.

Below we take the KP equation as an example, explaining how an integrable discrete equation is derived, as the superposition principle of a nonlinear differential equation. The KP equation (1.6) has its BT taking the following form:

$$\partial_y(\tilde{u}-u)=\partial_x^2(\tilde{u}+u)+\partial_x(\tilde{u}-u)^2-2p\partial_x(\tilde{u}-u), \tag{1.15a}$$

$$\begin{aligned}&\left(\partial_t-\frac{1}{4}\partial_x^3\right)(\tilde{u}-u)-\frac{3}{4}\partial_x\partial_y(\tilde{u}+u)\\&\quad=\frac{3}{2}\left[(\partial_x\tilde{u})^2-(\partial_x u)^2\right]+\frac{3}{4}\partial_y(\tilde{u}-u)^2-\frac{3}{2}p\partial_y(\tilde{u}-u),\end{aligned} \tag{1.15b}$$

where p is the Bäcklund parameter. In fact, differentiating the second equation in (1.15b) gives us

$$\begin{aligned}&\partial_x\left(\partial_t-\frac{1}{4}\partial_x^3\right)(\tilde{u}-u)-\frac{3}{4}\partial_y\left[\partial_x^2(\tilde{u}+u)-\partial_x(\tilde{u}-u)^2\right]\\&\quad=\frac{3}{2}\partial_x\left[(\partial_x\tilde{u})^2-(\partial_x u)^2\right]+\frac{3}{4}\partial_x\partial_y(\tilde{u}-u)^2-\frac{3}{2}p\partial_x\partial_y(\tilde{u}-u).\end{aligned}$$

Replacing $\partial_x^2(\tilde{u}+u)$ with the help of (1.15a), we obtain that

$$\left[\partial_x\left(\partial_t-\frac{1}{4}\partial_x^3\right)\tilde{u}-\frac{3}{4}\partial_y^2\tilde{u}-\frac{3}{2}\partial_x(\partial_x\tilde{u})^2\right]-\left[\partial_x\left(\partial_t-\frac{1}{4}\partial_x^3\right)u-\frac{3}{4}\partial_y^2u-\frac{3}{2}\partial_x(\partial_x u)^2\right]=0,$$

which implies that if u obeys the KP equation, $\tilde{u}$ is also a solution to (1.6). In concrete application, it is not necessary to solve (1.15b). For given u, we just need to solve $\tilde{u}$ from (1.15a) (which is actually a linearisable Burgers equation), and then substitute the $\tilde{u}$ into the KP equation (1.6) to determine the t-dependence.

The BT provides a way to construct richer and richer solutions by iteration starting from a seed solution. For instance, $u=0$ is a seed solution to the KP equation (1.6). However, once u becomes more and more complex, sometimes it would be rather difficult to solve (1.15a). An effective way to sort this issue is to make use of the superposition formula. In the case of the KP equation, we can consider two BTs (both taking the form of (1.15a), but with regard to two different Bäcklund parameters p and q), namely

$$\begin{aligned}\partial_y(\tilde{u}-u)&=\partial_x^2(\tilde{u}+u)-2p\partial_x(\tilde{u}-u)-\partial_x(\tilde{u}-u)^2,\\\partial_y(\hat{u}-u)&=\partial_x^2(\hat{u}+u)-2q\partial_x(\hat{u}-u)-\partial_x(\hat{u}-u)^2.\end{aligned}$$

The above formulae give the BTs from u to $\tilde{u}$ and from u to $\hat{u}$, respectively. In order to derive the superposition formula, we also need the BTs from $\hat{u}$ to $\tilde{\hat{u}}$ and from $\tilde{u}$ to $\hat{\tilde{u}}$, respectively, which are given as follows:

$$\begin{aligned}\partial_y(\tilde{\hat{u}}-\hat{u})&=\partial_x^2(\tilde{\hat{u}}+\hat{u})-2p\partial_x(\tilde{\hat{u}}-\hat{u})-\partial_x(\tilde{\hat{u}}-\hat{u})^2,\\\partial_y(\hat{\tilde{u}}-\tilde{u})&=\partial_x^2(\hat{\tilde{u}}+\tilde{u})-2q\partial_x(\hat{\tilde{u}}-\tilde{u})-\partial_x(\hat{\tilde{u}}-\tilde{u})^2.\end{aligned}$$

Now if we impose $\hat{\tilde{u}} = \tilde{\hat{u}}$ and eliminate the derivatives with respect to y from the above equations, we have

$$\partial_x(\hat{u} - \tilde{u}) = (p - q + \hat{u} - \tilde{u})(u - \tilde{u} - \hat{u} + \hat{\tilde{u}}). \tag{1.16}$$

This is the superposition formula for the KP equation. The superposition formula tells us that if u, $\tilde{u}$ and $\hat{u}$ are given, a more complex solution $\hat{\tilde{u}}$ (or $\tilde{\hat{u}}$) can be solved from an algebraic-differential relation, i.e. (1.16).

We can further consider three superposition formulae with regard to $(\tilde{\cdot}, p; \hat{\cdot}, q)$, $(\hat{\cdot}, q; \bar{\cdot}, r)$ and $(\bar{\cdot}, r; \tilde{\cdot}, p)$. By eliminating the derivative with respect to x, a purely algebraic relation can be constructed, which takes the form of

$$(p - \tilde{u})(q - r + \tilde{\bar{u}} - \hat{\tilde{u}}) + (q - \hat{u})(r - p + \hat{\tilde{u}} - \bar{\hat{u}}) + (r - \bar{u})(p - q - \bar{\hat{u}} - \tilde{\bar{u}}) = 0. \tag{1.17}$$

A lattice can now be built up from the BTs. We introduce a new notation $u_{n,m,h}(x, y, t)$ which is defined though $u = u_{n,m,h}(x, y, t)$ and obeys the following:

$$\begin{aligned} &\tilde{u} = u_{n+1,m,h}(x, y, t), \quad \hat{u} = u_{n,m+1,h}(x, y, t), \quad \bar{u} = u_{n,m,h+1}(x, y, t), \\ &\hat{\tilde{u}} = u_{n+1,m+1,h}(x, y, t), \quad \bar{\hat{u}} = u_{n,m+1,h+1}(x, y, t), \quad \tilde{\bar{u}} = u_{n+1,m,h+1}(x, y, t). \end{aligned}$$

This allows us to transfer the BTs to the dynamics on discrete variables. From this viewpoint, the BT (1.15a) is a semi-discrete equation involving two continuous variables x and y and one discrete variable n; the superposition formula (1.16) is a semi-discrete equation involving one continuous variable x and two discrete variable n, m; and equation (1.17) is a fully discrete equation of n, m and h. These are actually semi and full discretisations of the KP equation. Equation (1.17) was originally proposed in [**NCWQ84**, **NCW85**].

On the other hand, the fully discrete KP equation (1.17) can be thought of as a master model. By a suitable continuum limit step by step, one can recover (1.16), (1.15a), and finally (1.6). In order to perform the continuum limit, one needs to understand the solution structure of the discrete and continuous KP equations first. For this reason, we avoid discussing such a topic here in the introduction chapter, but in later chapters, we will explain both discrete and continuous equations in a single framework.

3.2. The Hirota–Miwa equation. The discrete KP equation (1.17) also has a bilinear form. In 1981, Hirota [**Hir81**] proposed a so-called discrete analogue of a generalised Toda equation (DAGTE)

$$[Z_1 \exp(D_1) + Z_2 \exp(D_2) + Z_3 \exp(D_3)]\, \tau \cdot \tau = 0,$$

where Z_1, Z_2 and Z_3 are arbitrary parameters. By following the definition of Hirota's derivative (see Appendix A), DAGTE can alternatively be written as ($D_1 = \delta D_x$, $D_2 = \epsilon D_y$, $D_3 = \kappa D_t$)

$$\begin{aligned} &Z_1 \tau(x + \delta, y, t)\tau(x - \delta, y, t) \\ &\qquad + Z_2 \tau(x, y + \epsilon, t)\tau(x, y - \epsilon, t) + Z_3 \tau(x, y, t + \kappa)\tau(x, y, t + \kappa) = 0, \end{aligned}$$

or equivalently ($x = n'\delta$, $y = m'\epsilon$, $t = h'\kappa$),

$$Z_1 \tau_{n'+1,m',h'} \tau_{n'-1,m',h'} + Z_2 \tau_{n',m'+1,h'} \tau_{n',m'-1,h'} + Z_3 \tau_{n',m',h'+1} \tau_{n',m',h'-1} = 0.$$

Introducing the change of variables $n = \frac{1}{2}(-n' + m' + h')$, $m = \frac{1}{2}(n' - m' + h')$ and $h = \frac{1}{2}(n' + m' - h')$ and following the notations in the discrete KP equation (1.17), we have

$$Z_1 \tilde{\tau} \bar{\hat{\tau}} + Z_2 \hat{\tau} \bar{\tilde{\tau}} + Z_3 \tau \hat{\tilde{\tau}} = 0.$$

Hirota's DAGTE is integrable for any arbitrary Z_1, Z_2 and Z_3. However, in order to study its soliton solution and also compare this with the bilinear KP equation (1.7), a more restrictive parametrisation, i.e. the sum of the coefficients equal to zero, is essential. In this case, the equation reads

$$(p-q)\bar{\tau}\hat{\tilde{\tau}} + (q-r)\tilde{\tau}\bar{\hat{\tau}} + (r-p)\hat{\tau}\bar{\tilde{\tau}} = 0. \tag{1.18}$$

Such a parametrisation was given by Miwa [**Miw82**], and nowadays equation (1.18) is often referred to as the Hirota–Miwa (HM) equation. The HM equation has very rich structure, and by reduction a lot of two-dimensional (2D) discrete integrable models can be obtained from it. The continuum limit of this equation generates all the continuous bilinear equations arising from the bilinear identity for the KP equation. The HM equation is connected with the discrete KP equation (1.17) via the transform

$$p + q + \hat{u} - \tilde{u} = (p-q)\frac{\tau\hat{\tilde{\tau}}}{\tilde{\tau}\hat{\tau}}$$

and its $(\hat{\cdot}, \bar{\cdot})$ and $(\bar{\cdot}, \tilde{\cdot})$ counterparts.

3.3. Multi-dimensional consistency. A key feature of the integrability of nonlinear systems is the phenomenon of multi-dimensional consistency (MDC) – the property that a nonlinear equation can be consistently extended to a family of equations by introducing an arbitrary number of discrete independent variables with their corresponding lattice parameters or continuous independent variables, cf. [**DS97**, **NW01**]. The MDC property was later employed to classify scalar affine-linear quadrilateral equations [**ABS03**] and octahedral equations [**ABS12**] by Adler, Bobenko and Suris.

We take the HM equation as an example, to explain the MDC property. The HM equation is a three-dimensional (3D) equation for tilde-, hat- and bar-directions associated with lattice parameters p, q and r, respectively. We introduce a fourth direction called dot-direction associated with its corresponding lattice parameter s, and embed the HM equation into a four-dimensional (4D) space. Thus, there exist four copies of the HM equation as follows:

$$(p-q)\bar{\tau}\boxed{\hat{\tilde{\tau}}} + (q-r)\tilde{\tau}\boxed{\bar{\hat{\tau}}} + (r-p)\hat{\tau}\boxed{\bar{\tilde{\tau}}} = 0, \tag{1.19a}$$

$$(p-q)\dot{\tau}\boxed{\hat{\tilde{\tau}}} + (q-s)\tilde{\tau}\dot{\hat{\tau}} + (s-p)\hat{\tau}\dot{\tilde{\tau}} = 0, \tag{1.19b}$$

$$(p-s)\bar{\tau}\dot{\tilde{\tau}} + (s-r)\tilde{\tau}\dot{\bar{\tau}} + (r-p)\dot{\tau}\boxed{\bar{\tilde{\tau}}} = 0, \tag{1.19c}$$

$$(s-q)\bar{\tau}\dot{\hat{\tau}} + (q-r)\dot{\tau}\boxed{\bar{\hat{\tau}}} + (r-s)\hat{\tau}\dot{\bar{\tau}} = 0. \tag{1.19d}$$

The four equations do not form a closed-form lattice. To show the consistency among these four equations, the following four equations are needed:

$$(p-q)\dot{\bar{\tau}}\boxed{\dot{\hat{\tilde{\tau}}}}+(q-r)\dot{\tilde{\tau}}\dot{\bar{\hat{\tau}}}+(r-p)\dot{\hat{\tau}}\dot{\bar{\tilde{\tau}}}=0, \tag{1.19e}$$

$$(p-q)\dot{\bar{\tau}}\boxed{\bar{\hat{\tilde{\tau}}}}+(q-s)\bar{\tilde{\tau}}\dot{\bar{\hat{\tau}}}+(s-p)\bar{\hat{\tau}}\dot{\bar{\tilde{\tau}}}=0, \tag{1.19f}$$

$$(p-s)\bar{\hat{\tau}}\boxed{\dot{\hat{\tilde{\tau}}}}+(s-r)\hat{\tilde{\tau}}\dot{\bar{\hat{\tau}}}+(r-p)\dot{\hat{\tau}}\boxed{\bar{\hat{\tilde{\tau}}}}=0, \tag{1.19g}$$

$$(s-q)\bar{\tilde{\tau}}\boxed{\dot{\hat{\tilde{\tau}}}}+(q-r)\dot{\tilde{\tau}}\boxed{\bar{\hat{\tilde{\tau}}}}+(r-s)\hat{\tilde{\tau}}\dot{\bar{\tilde{\tau}}}=0. \tag{1.19h}$$

In the 4D space, the eight equations totally involve 14 points $\tilde{\tau}$, $\hat{\tau}$, $\bar{\tau}$, $\dot{\tau}$, $\hat{\tilde{\tau}}$, $\bar{\tilde{\tau}}$, $\dot{\tilde{\tau}}$, $\bar{\hat{\tau}}$, $\dot{\hat{\tau}}$, $\dot{\bar{\tau}}$, $\bar{\hat{\tilde{\tau}}}$, $\dot{\hat{\tilde{\tau}}}$, $\dot{\bar{\tilde{\tau}}}$ and $\dot{\bar{\hat{\tau}}}$. For the 4D consistency of the HM equation, we need nine of these points being independent. Without loss of generality, we suppose that

$$\tilde{\tau},\quad \hat{\tau},\quad \bar{\tau},\quad \dot{\tau},\quad \dot{\tilde{\tau}},\quad \dot{\hat{\tau}},\quad \dot{\bar{\tau}},\quad \dot{\bar{\tilde{\tau}}}\quad \text{and}\quad \dot{\bar{\hat{\tau}}}$$

are given as independent initial data.

Under this assumption, the remaining the five points, namely $\hat{\tilde{\tau}}$, $\bar{\hat{\tau}}$, $\bar{\tilde{\tau}}$, $\bar{\hat{\tilde{\tau}}}$ and $\dot{\hat{\tilde{\tau}}}$, are to be determined from an overdetermined system composed of the above eight equations, i.e. equations (1.19a)–(1.19h). The consistency in the 4D space requires that they must be uniquely solved. To be more precise, we need $\hat{\tilde{\tau}}$, $\bar{\hat{\tau}}$ and $\bar{\tilde{\tau}}$ being uniquely determined from (1.19a)–(1.19d), and then $\bar{\hat{\tilde{\tau}}}$ and $\dot{\hat{\tilde{\tau}}}$ being consistently solved from (1.19e)–(1.19h).

In practice, the 4D consistency of the HM equation is proven by the following three steps:

(1) show that $\hat{\tilde{\tau}}$, $\bar{\tilde{\tau}}$ and $\bar{\hat{\tau}}$ solved from (1.19b), (1.19c) and (1.19d), respectively, satisfy (1.19a);
(2) substitute $\bar{\hat{\tilde{\tau}}}$ solved from (1.19f) into (1.19g) and (1.19h) and show that $\dot{\hat{\tilde{\tau}}}$ respectively solved from (1.19g) and (1.19h) are compatible.
(3) substitute $\dot{\hat{\tilde{\tau}}}$ solved from (1.19e) into (1.19g) and (1.19h) and show that $\bar{\hat{\tilde{\tau}}}$ respectively solved from (1.19g) and (1.19h) are compatible.

Since steps (2) and (3) are rather similar, below we only prove the first two steps.

Proof of step (1). Equations (1.19b), (1.19c) and (1.19d) can be written as

$$\begin{aligned}
(p-q)\frac{\hat{\tilde{\tau}}}{\tilde{\tau}\hat{\tau}}+(q-s)\frac{\dot{\hat{\tau}}}{\hat{\tau}\dot{\tau}}+(s-p)\frac{\dot{\tilde{\tau}}}{\tilde{\tau}\dot{\tau}}&=0,\\
(p-s)\frac{\dot{\tilde{\tau}}}{\tilde{\tau}\dot{\tau}}+(s-r)\frac{\dot{\bar{\tau}}}{\bar{\tau}\dot{\tau}}+(r-p)\frac{\bar{\tilde{\tau}}}{\tilde{\tau}\bar{\tau}}&=0,\\
(s-q)\frac{\dot{\hat{\tau}}}{\hat{\tau}\dot{\tau}}+(q-r)\frac{\bar{\hat{\tau}}}{\hat{\tau}\bar{\tau}}+(r-s)\frac{\dot{\bar{\tau}}}{\bar{\tau}\dot{\tau}}&=0.
\end{aligned}$$

Adding the three equations up, one obtains

$$(p-q)\frac{\hat{\tilde{\tau}}}{\tilde{\tau}\hat{\tau}}+(r-p)\frac{\bar{\tilde{\tau}}}{\tilde{\tau}\bar{\tau}}+(q-r)\frac{\bar{\hat{\tau}}}{\hat{\tau}\bar{\tau}}=0,$$

which is nothing but (1.19a), namely $\hat{\tilde{\tau}}$, $\bar{\tilde{\tau}}$ and $\bar{\hat{\tau}}$ solved from equations (1.19b), (1.19c) and (1.19d) are compatible with (1.19a). □

PROOF OF STEP (2). Equations (1.19b), (1.19c) and (1.19d) can alternatively be written in the following form:

$$
\begin{aligned}
(p-q)\frac{\hat{\tilde{\tau}}}{\dot{\tilde{\tau}}\dot{\hat{\tau}}}+(q-s)\frac{\tilde{\tau}}{\dot{\tau}\dot{\tilde{\tau}}}+(s-p)\frac{\hat{\tau}}{\dot{\tau}\dot{\hat{\tau}}}&=0,\\
(p-s)\frac{\bar{\tau}}{\dot{\tau}\dot{\bar{\tau}}}+(s-r)\frac{\tilde{\tau}}{\dot{\tau}\dot{\tilde{\tau}}}+(r-p)\frac{\bar{\tilde{\tau}}}{\dot{\tilde{\tau}}\dot{\bar{\tau}}}&=0,\\
(s-q)\frac{\bar{\tau}}{\dot{\tau}\dot{\bar{\tau}}}+(q-r)\frac{\bar{\hat{\tau}}}{\dot{\hat{\tau}}\dot{\bar{\tau}}}+(r-s)\frac{\hat{\tau}}{\dot{\tau}\dot{\hat{\tau}}}&=0.
\end{aligned}
$$

A suitable linear combination of these equations results in

$$
(p-q)(r-s)\frac{\hat{\tilde{\tau}}}{\dot{\tilde{\tau}}\dot{\hat{\tau}}}+(q-s)(r-p)\frac{\bar{\tilde{\tau}}}{\dot{\tilde{\tau}}\dot{\bar{\tau}}}+(p-s)(q-r)\frac{\bar{\hat{\tau}}}{\dot{\hat{\tau}}\dot{\bar{\tau}}}=0,
$$

or equivalently,

$$
(p-q)(r-s)\hat{\tilde{\tau}}\dot{\bar{\tau}}+(q-s)(r-p)\bar{\tilde{\tau}}\dot{\hat{\tau}}+(p-s)(q-r)\bar{\hat{\tau}}\dot{\tilde{\tau}}=0. \tag{1.20}
$$

We can reformulate (1.19f) as

$$
(p-q)\frac{\dot{\bar{\tau}}}{\bar{\tilde{\tau}}\bar{\hat{\tau}}}+(q-s)\frac{\dot{\bar{\hat{\tau}}}}{\bar{\hat{\tau}}\bar{\hat{\tilde{\tau}}}}+(s-p)\frac{\dot{\bar{\tilde{\tau}}}}{\bar{\tilde{\tau}}\bar{\hat{\tilde{\tau}}}}=0. \tag{1.21}
$$

Meanwhile, equations (1.19g) and (1.19h) have their respective equivalent form as follows:

$$
\begin{aligned}
(p-s)\frac{\dot{\hat{\tilde{\tau}}}}{\hat{\tilde{\tau}}\bar{\hat{\tilde{\tau}}}}+(s-r)\frac{\dot{\bar{\hat{\tau}}}}{\bar{\hat{\tau}}\bar{\hat{\tilde{\tau}}}}+(r-p)\frac{\dot{\hat{\tau}}}{\hat{\tilde{\tau}}\bar{\hat{\tau}}}&=0,\\
(s-q)\frac{\dot{\hat{\tilde{\tau}}}}{\hat{\tilde{\tau}}\bar{\hat{\tilde{\tau}}}}+(q-r)\frac{\dot{\tilde{\tau}}}{\hat{\tilde{\tau}}\dot{\tilde{\tau}}}+(r-s)\frac{\dot{\bar{\tilde{\tau}}}}{\bar{\tilde{\tau}}\bar{\hat{\tilde{\tau}}}}&=0.
\end{aligned}
$$

Both equations give the same $\dot{\hat{\tilde{\tau}}}$ if and only if

$$
(q-s)(s-r)\frac{\dot{\bar{\hat{\tau}}}}{\bar{\hat{\tau}}\bar{\hat{\tilde{\tau}}}}+(q-s)(r-p)\frac{\dot{\hat{\tau}}}{\hat{\tilde{\tau}}\bar{\hat{\tau}}}+(p-s)(q-r)\frac{\dot{\tilde{\tau}}}{\hat{\tilde{\tau}}\dot{\tilde{\tau}}}+(p-s)(r-s)\frac{\dot{\bar{\tilde{\tau}}}}{\bar{\tilde{\tau}}\bar{\hat{\tilde{\tau}}}}=0.
$$

Combining this equation with (1.21) and eliminating $\dot{\bar{\hat{\tau}}}$ and $\dot{\bar{\tilde{\tau}}}$, we end up with

$$
(q-s)(r-p)\frac{\dot{\hat{\tau}}}{\hat{\tilde{\tau}}\bar{\hat{\tau}}}+(p-s)(q-r)\frac{\dot{\tilde{\tau}}}{\hat{\tilde{\tau}}\dot{\tilde{\tau}}}+(p-q)(r-s)\frac{\dot{\bar{\tau}}}{\bar{\tilde{\tau}}\bar{\hat{\tau}}}=0,
$$

which is exactly the same as the proven equation (1.20). Hence, we have proven that $\dot{\hat{\tilde{\tau}}}$ arising from (1.19g) and (1.19h) coincide. □

The example of the HM equation is actually a very particular one, namely all the consistent lattice equations take the same form with regard to different lattice variables and parameters. We refer to this as "covariant". There are, however, "non-covariant" cases, namely the equations in the hierarchy take very different forms, though they are still compatible with each other.

We would also like to note that the MDC property is the analogue of the existence of commuting symmetries in the theory of continuous integrable systems. Actually, these covariant or non-covariant difference equations can be thought of as discrete symmetries for each other.

4. Direct linearisation approach

4.1. The KdV and KP equations. The direct linearisation (DL) approach was originally proposed by Fokas and Ablowitz [**FA81**] for solving the KdV equation, as a generalisation of the IST method. The idea of the DL is to relate a nonlinear integrable equation to a linear integral equation, and as a result, the solution space of the nonlinear equation can be constructed from the integral equation. The benefit of the DL method is that it provides considerable flexibility for a large class of solutions, compared with the traditional methods.

The associated linear integral equation for the KdV equation takes the form of

$$\phi(x,t;k) + \int_\Gamma \frac{\rho(x,t;k)}{k+l}\phi(x,t;l)\mathrm{d}\lambda(l) = \rho(x,t;k), \tag{1.22}$$

where $\phi(x,t;k)$ is the wave function of the independent variables x and t and the spectral parameter k, Γ and $\mathrm{d}\lambda(k)$ are an appropriate measure and a contour, respectively, and $\rho(x,t;k) = \exp(2kx + 2k^3t)$ is the plane wave factor. It was proven in [**FA81**] that the potential u defined by

$$u(x,t) = \int_\Gamma \phi(x,t;k)\mathrm{d}\lambda(k)$$

solves the KdV equation (1.1). Such a solution is very general since any solution of the linear integral equation (1.22) can provide a solution to the KdV equation. For instance, one can take a particular measure $\mathrm{d}\lambda(k)$ containing a finite number of singularities, and construct the soliton solutions to the KdV equation. Alternatively, a special measure connected with the similarity solution to the KdV equation is also allowed, which leads to the reduction from the KdV equation to the famous Painlevé II equation in the DL framework.

After then, the DL approach was generalised and was applied to other models [**FA83**, **SAF84**]. In [**FA83**], a more general integral equation in the form of

$$\phi(x,y,t;k) + \iint_D \frac{\rho(x,y,t;k)\sigma(x,y,t;l')}{k+l'}\phi(x,y,t;l)\mathrm{d}\zeta(l,l') = \rho(x,y,t;k) \tag{1.23}$$

was proposed for the KP equation (1.6), in which $\rho(x,y,t;k) = \exp(kx + k^2y + k^3t)$ and $\sigma(x,y,t;k') = \exp(k'x - k'^2y + k'^3t)$, and D and $\mathrm{d}\zeta(k,k')$ are now a domain and a measure on two spectral parameters k and k', respectively. The direct linearising solution to the KP equation is expressed by

$$u(x,y,t) = \iint_D \phi(x,y,t;k)\sigma(x,y,t;k')\mathrm{d}\zeta(k,k').$$

Different from the integral equation for KdV, the key point here is the separation of k and k' in the measure $\mathrm{d}\zeta(k,k')$ on the integration domain D, which is a reflection of an underlying nonlocal Riemann–Hilbert (or $\bar\partial$-) problem leading to 3D integrable hierarchies[4]

[4]The term 'nonlocal Riemann–Hilbert problem' was introduced by Zakharov and Manakov, see e.g. [**ZM85**]; in other words, a double integral is necessary in the linear integral equation for 3D integrable equations.

4.2. Integrable discrete equations. The DL framework was developed to study discrete integrable systems by Nijhoff, et al., see e.g. [**NQC83**, **QNCvdL84**, **NC95**, **NCWQ84**, **NCW85**, **NPCQ92**]. The main idea to construct an integrable discretisation of a nonlinear partial differential equation is replacing the plane wave factors by discrete ones.

In fact, the discrete KP equation (1.17) was originally constructed in the DL scheme. If we introduce the discrete plane wave factors

$$\rho(n,m,h;k) = (p+k)^n(q+k)^m(r+k)^h$$

and

$$\sigma(n,m,h;k') = (p-k')^{-n}(q-k')^{-m}(r-k')^{-h},$$

the potential

$$u(n,m,h) = \iint_D \phi(n,m,h;k)\sigma(n,m,h;k')\mathrm{d}\zeta(k,k').$$

was shown to solve the discrete KP equation (1.17).

We also note that introducing discrete plane wave factors in the DL scheme amounts to considering the BT for a continuous equation. In this sense, a fully discrete equation arising from the DL scheme should naturally be understood as the superposition formula for corresponding continuous equation, as was explained previously.

4.3. Motivation of the thesis. The DL framework brings us the following benefits: i) Solution to a nonlinear equation is determined by a linear integral equation, which allows a more general solution structure (which we refer to as the direct linearising solution), compared with those obtained by the traditional methods, such as the IST method, the bilinear method, etc.; ii) Both discrete and continuous variables are on the same footing in the scheme, which implies that in the DL we can treat both discrete and continuous equations equally; iii) The DL framework reveals the underlying structure of nonlinear equations (such as the unmodified, modified, bilinear forms) in the same class simultaneously, which makes it possible for us to study discrete and continuous integrable systems by class. More details will be explained in chapter 2, and examples will be given in later chapters.

The thesis aims to study integrable discrete and continuous equations associated with scalar linear integral equations within the DL framework. In order to realise this, our starting point is a linear integral equation allowing certain flexibility, which is represented by infinite-dimensional matrices and vectors. The master models we consider in this framework are the discrete and continuous KP (or also referred to as AKP) equations, as well as the discrete-time two-dimensional Toda lattice (2DTL). By performing dimensional reductions on these models, we establish the DL scheme for a huge class of integrable discrete and continuous equations.

5. Organisation of the thesis

The thesis is organised as follows.

Chapter 2 reviews the general theory of the DL approach. In the first part, we give an introduction to infinite-dimensional matrices and vectors, including the definitions of these notions, and some examples for their operations. In the second part, we discuss the infinite-dimensional matrix representation of a general linear integral equation, and introduce necessary quantities in the DL method.

In chapter 3, the DL framework of the three master scalar integrable discrete equations, namely the discrete AKP, BKP and CKP equations. By considering three different Cauchy kernels and their respective plane wave factors, we algebraically construct the discrete AKP, BKP and CKP equations, as well as their Lax pairs and exact solutions. In particular, a new parametrisation of the discrete CKP equation, which is corresponding to the soliton space of the discrete CKP, is proposed.

In chapter 4, a systematic framework is presented for the construction of hierarchies of continuous soliton equations. This is realised by considering scalar linear integral equations and their representations in terms of infinite-dimensional matrices, which give rise to all (2+1)- and (1+1)-dimensional soliton hierarchies associated with scalar differential spectral problems. The integrability characteristics for the obtained soliton hierarchies, including Miura-type transforms, tau functions, Lax pairs as well as soliton solutions, are also derived within this framework.

In chapter 5, the reduction by restricting the spectral parameters k and k' on a generic algebraic curve of degree $\mathcal{N}$ is performed for the discrete AKP, BKP and CKP equations, respectively. A variety of two-dimensional discrete integrable systems possessing a more general solution structure arise from the reduction, and in each case a unified formula for generic positive integer $\mathcal{N} \geq 2$ is given to express the corresponding reduced integrable lattice equations. The obtained extended two-dimensional lattice models give rise to many important integrable partial difference equations as special degenerations. Some new integrable lattice models such as the discrete Sawada–Kotera (SK), Kaup–Kupershmidt (KK) and Hirota–Satsuma (HS) equations in extended form are given as examples within the framework.

In chapter 6, the discrete-time 2DTL of A_∞-type is studied within the DL framework, which allows us to deal with several nonlinear equations in this class simultaneously and to construct more general solutions of these equations. The periodic reductions of this model are also considered, giving rise to the discrete-time 2DTLs of $A_{\mathcal{N}-1}^{(1)}$-type for $\mathcal{N} \geq 2$ (which amount to the negative flows of members in the discrete Gel'fand–Dikii (GD) hierarchy) and their integrability properties.

The last chapter, namely chapter 7, is dedicated to a conclusion of the whole thesis, as well as some remaining problems which will be considered in the near future.

CHAPTER 2

Direct linearisation

1. Overview

The DL was originated by Fokas and Ablowitz [**FA81**] as a formalisation of the Riemann–Hilbert method, to study the KdV equation, and its solution structure (including soliton solution as well as similarity solution related to Painlevé II). The DL scheme for the KdV equation is based on a linear integral equation involving a single integral with respect to a spectral variable. The linear integral equation was later extended to an integral equation associated with a double integral by the same authors [**FA83**], leading to the KP equation, i.e. a (2+1)-dimensional equation.

The DL was generalised by Nijhoff, Quispel, Capel, et al., to solve the nonlinear Schrödinger-type equations [**NvdLQ+82**, **NQCvdL83**], and the Boussinesq (BSQ)-type equations [**NPCQ92**]. The key point in such a generalisation is that those authors introduced an infinite-dimensional matrix structure in the DL scheme, which brings more flexibility and allows dealing with Miura-related equations (i.e. equations which are connected by differential/difference Miura transforms) in the same class simultaneously.

A further development of the DL method is the study of integrable discretisation of nonlinear equations. It was done by considering discrete plane wave factors in the linear integral equation, resulting in a huge number of integrable discrete models, such as the discrete KdV equation [**NQC83**, **NC95**], the discrete KP equation [**NCWQ84**, **NCW85**, **DN91**], as well as the discrete BSQ equation [**NPCQ92**, **Nij97**, **ZZN12**].

This chapter is a preliminary chapter for the whole thesis. In the first half of the chapter, we give an introduction to infinite-dimensional matrices and vectors. The general theory of the DL method is given in the second half, where we introduce all the necessary quantities.

2. Infinite-dimensional matrices and vectors

2.1. Finite-dimensional matrices and vectors. We give a very brief review of finite-dimensional matrices and vectors. A finite-dimensional matrix $\mathbf{A}$ of size $M \times N$ is an array taking the form of

$$\mathbf{A}_{M\times N} = \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,N} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,N} \\ \vdots & \vdots & \cdots & \vdots \\ A_{M,1} & A_{M,2} & \cdots & A_{M,N} \end{pmatrix},$$

where $A_{i,j}$ (in a certain field) for $i = 1, 2, \cdots, M$ and $j = 1, 2, \cdots, N$ are the entries of the matrix. A suitable way to understand such a matrix is to think of it as the following expression:

$$\mathbf{A}_{M\times N} = \sum_{i=1}^{M}\sum_{j=1}^{N} A_{i,j}\mathbf{E}_{M\times N}^{(i,j)}, \tag{2.1}$$

in which $\mathbf{E}_{M\times N}^{(i,j)}$ is an $M \times N$ matrix having its (m,n)-entry

$$(\mathbf{E}_{M\times N}^{(i,j)})_{m,n} = \delta_{m,i}\delta_{n,j}$$

with the usual Kronecker delta function, namely $\delta_{\cdot,\cdot}$ satisfies

$$\delta_{i,j} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \qquad \forall i,j \in \mathbb{Z}.$$

This implies that an arbitrary matrix $\mathbf{A}_{M\times N}$ taking the form of (2.1) can be generated by $\mathbf{E}_{M\times N}^{(i,j)}$.

The operations of finite-dimensional matrices are natural consequences of some properties of $\mathbf{E}_{M,N}^{(i,j)}$. The addition of two finite-dimensional matrices and the scalar multiplication of a finite-dimensional matrix by a real/complex number are obvious; while the multiplication of two finite-dimensional matrices of sizes $M_1 \times N_1$ and $M_2 \times N_2$ only works when $N_1 = M_2$. In fact, the fundamental relation we need in matrix multiplication is

$$\mathbf{E}_{M_1\times N_1}^{(i_1,j_1)}\mathbf{E}_{N_1\times N_2}^{(i_2,j_2)} = \delta_{j_1,i_2}\mathbf{E}_{M_1\times N_2}^{(i_1,j_2)}, \tag{2.2}$$

which results in the rule of the multiplication of two arbitrary matrices (if their sizes match) taking the form of (2.1).

An N-component column vector $\mathbf{a}_N = (a_1, a_2, \cdots, a_N)^{\mathrm{T}}$ is defined by

$$\mathbf{a}_N = \sum_{i=1}^{N} a_i \mathbf{e}_N^{(i)},$$

where $\mathbf{e}_N^{(i)}$ is an N-component column vector having its nth-component $(\mathbf{e}_N^{(i)})_n = \delta_{n,i}$, and a_i are the corresponding components. Similarly, an N-component row vector ${}^t\mathbf{a}_N$ can also be defined in this way – one can easily observe this by taking the transpose[1] of $\mathbf{a}_N$ and in this case ${}^t\mathbf{a}_N$ is expressed by ${}^t\mathbf{e}_N^{(i)}$ which satisfy $({}^t\mathbf{e}_N^{(i)})_n = \delta_{n,i}$.

The addition and the scalar multiplication for finite-dimensional vectors are also simple, namely the operations are acted on the corresponding components. The multiplication of a column vector and a row vector requires the essential relations

$$\mathbf{e}_M^{(i)}{}^t\mathbf{e}_N^{(j)} = \mathbf{E}_{M\times N}^{(i,j)} \quad \text{and} \quad {}^t\mathbf{e}_N^{(j)}\mathbf{e}_N^{(i)} = \delta_{i,j}, \tag{2.3}$$

which guarantee the multiplication of an arbitrary column vector and an arbitrary row vector.

Furthermore, from (2.3) we can derive the following fundamental relations:

$$\mathbf{E}_{M\times N}^{(i,j)}\mathbf{e}_N^{(i')} = \delta_{i',j}\mathbf{e}_M^{(i)}, \quad {}^t\mathbf{e}_M^{(i')}\mathbf{E}_{M\times N}^{(i,j)} = \delta_{i,i'}{}^t\mathbf{e}_N^{(j)}, \tag{2.4}$$

which govern the multiplication of a finite-dimensional matrix and a column/row vector.

[1]Due to a convention in the DL approach, in the thesis we also accept the notation ${}^t(\cdot)$ for the transpose, which means exactly the same as $(\cdot)^{\mathrm{T}}$.

In fact, elements in $\{\mathbf{E}^{(i,j)}_{N\times M}|i = 1,2,\cdots,M, j = 1,2,\cdots N\}$ constitute a basis for the linear space of $M \times N$ matrices, and elements in $\{\mathbf{e}^{(i)}_N|i = 1,2,\cdots,N\}$ (or $\{{}^t\mathbf{e}^{(i)}_N|i = 1,2,\cdots,N\}$) constitute a basis for the space of N-component vectors. One remark is that such bases are not unique, and here we choose the simplest ones for convenience. The important thing is understanding finite-dimensional matrices and vectors in such a way can easily help us to define infinite-dimensional matrices and vectors.

2.2. Infinite-dimensional matrices. In the finite-dimensional case, there are a finite number of indices labelling entries in a matrix/vector. In order to define an infinite-dimensional matrix, we need a "centred" matrix/vector together with the so-called index-raising operators to label all the entries. This involves the use of the objects (generators) $\mathbf{\Lambda}$, ${}^t\mathbf{\Lambda}$ and $\mathbf{O}$ in an associative algebra (with unit) $\mathcal{A}$ over a field $\mathcal{F}$, obeying the relation

$$\mathbf{O}\cdot {}^t\mathbf{\Lambda}^{-j}\cdot\mathbf{\Lambda}^{-i}\cdot\mathbf{O} = \delta_{j,i}\mathbf{O}, \quad i,j\in\mathbb{Z}, \tag{2.5}$$

where the powers ${}^t\mathbf{\Lambda}^i$ and $\mathbf{\Lambda}^j$ are the ith and jth compositions of ${}^t\mathbf{\Lambda}$ and $\mathbf{\Lambda}$ (and their inverse) respectively. In general, $\mathbf{\Lambda}$, ${}^t\mathbf{\Lambda}$ and $\mathbf{O}$ do not commute, and we require that $\mathbf{\Lambda}$ and ${}^t\mathbf{\Lambda}$ act as each other's transpose. From (2.5) it is also easy to see that $\mathbf{O}$ is a projector satisfying $\mathbf{O}^2 = \mathbf{O}$ by setting $i = j = 0$.

Following the idea in the case of finite-dimensional matrices, we define

$$\mathbf{E}^{(i,j)} = \mathbf{\Lambda}^{-i}\cdot\mathbf{O}\cdot{}^t\mathbf{\Lambda}^{-j}, \quad \forall i,j\in\mathbb{Z}, \tag{2.6}$$

which implies $\mathbf{O} = \mathbf{E}^{(0,0)}$.

A direct calculation shows that

$$\begin{aligned}\mathbf{E}^{(i_1,j_1)}\cdot\mathbf{E}^{(i_2,j_2)} &= \mathbf{\Lambda}^{-i_1}\cdot\mathbf{O}\cdot{}^t\mathbf{\Lambda}^{-j_1}\cdot\mathbf{\Lambda}^{-i_2}\cdot\mathbf{O}\cdot{}^t\mathbf{\Lambda}^{-j_2}\\ &= \delta_{j_1,i_2}\mathbf{\Lambda}^{-i_1}\cdot\mathbf{O}\cdot{}^t\mathbf{\Lambda}^{-j_2} = \delta_{j_1,i_2}\mathbf{E}^{(i_1,j_2)},\end{aligned} \tag{2.7}$$

where we have used the rule (2.5) for the second equality. The above relation is the analogue of (2.2) which will later govern the multiplication of two infinite-dimensional matrices.

Definition 2.1. *An, in general, infinite-dimensional matrix* $\mathbf{U}$ *is defined as*

$$\mathbf{U} = \sum_{i,j\in\mathbb{Z}} U_{i,j}\mathbf{E}^{(i,j)}, \tag{2.8}$$

where the coefficients $U_{i,j}$ *are the* (i,j)*-entries in the infinite-dimensional matrix, taking values in the field* $\mathcal{F}$.

From the definition, we can observe that members in $\{\mathbf{E}^{(i,j)}|i,j\in\mathbb{Z}\}$ form a basis of the linear space of infinite-dimensional matrices. The transpose operation on $\mathbf{U}$ is defined by

$${}^t\mathbf{U} = \sum_{i,j\in\mathbb{Z}} U_{j,i}\mathbf{E}^{(i,j)}.$$

In particular, we can easily prove that ${}^t\mathbf{O} = \mathbf{O}$ and ${}^t\mathbf{E}^{(i,j)} = \mathbf{E}^{(j,i)}$, following the definition (note that $\mathbf{O}$ and $\mathbf{E}^{(i,j)}$ have their respective (i',j')-entries $\delta_{i',0}\delta_{j',0}$ and $\delta_{i',i}\delta_{j',j}$).

Suppose that $\mathbf{U}$ and $\mathbf{V}$ are two infinite-dimensional matrices having entries $U_{i,j}$ and $V_{i,j}$, and $\mathfrak{p}$ is an element taken from the field $\mathcal{F}$. We can show that operations of these infinite-dimensional matrices such as addition, multiplication of two infinite-dimensional matrices as well as scalar multiplication of an infinite-dimensional matrix by $\mathfrak{p} \in \mathcal{F}$ obey the following rules:

$$\begin{aligned}
\mathbf{U} + \mathbf{V} &= \sum_{i,j\in\mathbb{Z}} (U_{i,j} + V_{i,j})\mathbf{E}^{(i,j)}, \\
\mathfrak{p}\,\mathbf{U} &= \sum_{i,j\in\mathbb{Z}} (\mathfrak{p}\,U_{i,j})\mathbf{E}^{(i,j)}, \\
\mathbf{U}\cdot\mathbf{V} &= \sum_{i,j\in\mathbb{Z}} \left(\sum_{i'\in\mathbb{Z}} U_{i,i'}V_{i',j}\right)\mathbf{E}^{(i,j)}.
\end{aligned}$$

The first two equations are proven directly as they follow from the definition (2.8). The proof of the third one requires (2.7). In fact,

$$\begin{aligned}
\mathbf{U}\cdot\mathbf{V} &= \sum_{i_1,j_1\in\mathbb{Z}}\sum_{i_2,j_2\in\mathbb{Z}} U_{i_1,j_1}V_{i_2,j_2}\mathbf{E}^{(i_1,j_1)}\cdot\mathbf{E}^{(i_2,j_2)} \\
&= \sum_{i_1,j_1\in\mathbb{Z}}\sum_{i_2,j_2\in\mathbb{Z}} U_{i_1,j_1}V_{i_2,j_2}\delta_{j_1,i_2}\mathbf{E}^{(i_1,j_2)} = \sum_{i_1,j_2\in\mathbb{Z}}\left(\sum_{i_2\in\mathbb{Z}} U_{i_1,i_2}V_{i_2,j_2}\right)\mathbf{E}^{(i_1,j_2)}.
\end{aligned}$$

We point out that the definition (2.8) also covers finite-dimensional matrices by restricting the number of non-zero coefficients to a finite number, i.e. $U_{i,j} = 0$ for $i \neq 1, 2, \cdots, M$ or $j \neq 1, 2, \cdots, N$ for some given positive integers M and N, resulting in an $M \times N$ finite-dimensional matrix given by $\mathbf{U} = \sum_{i=1,j=1}^{M,N} U_{i,j}\mathbf{E}^{(i,j)}_{M\times N}$.

Remark 2.2. *An alternative way to understand the objects $\mathbf{O}$, $\mathbf{\Lambda}$ and ${}^t\mathbf{\Lambda}$ is considering them as infinite-dimensional matrices of size $\infty\times\infty$ having their respective (i,j)-entry*

$$(\mathbf{O})_{i,j} = \delta_{i,0}\delta_{0,j}, \quad (\mathbf{\Lambda})_{i,j} = \delta_{i+1,j}, \quad \textit{and} \quad ({}^t\mathbf{\Lambda})_{i,j} = \delta_{i,j+1}.$$

Such a realisation is actually compatible with the statement above, see subsection 2.4. The infinite-dimensional matrices $\mathbf{\Lambda}$ and ${}^t\mathbf{\Lambda}$ play the role of the index-raising operators; to be more precise, $\mathbf{\Lambda}$ (resp. ${}^t\mathbf{\Lambda}$) raises the row (resp. column) index of an infinite-dimensional matrix by left (resp. right) multiplication.

2.3. Infinite-dimensional vectors. Following the same idea, we also introduce infinite-dimensional vectors as follows. Suppose that $\mathbf{o}$ and ${}^t\mathbf{o}$ are two objects obeying the relations

$${}^t\mathbf{o}\cdot{}^t\mathbf{\Lambda}^{-j}\cdot\mathbf{\Lambda}^{-i}\cdot\mathbf{o} = \delta_{j,i} \quad \text{and} \quad \mathbf{o}\cdot{}^t\mathbf{o} = \mathbf{O}, \tag{2.9}$$

where $i,j\in\mathbb{Z}$ and $\mathbf{o}$, ${}^t\mathbf{o}$, $\mathbf{\Lambda}$ and ${}^t\mathbf{\Lambda}$ do not commute with each other in general.

The objects $\mathbf{o}$ and ${}^t\mathbf{o}$ together with the index-raising operators $\mathbf{\Lambda}$ and ${}^t\mathbf{\Lambda}$ are the ingredients to construct infinite-dimensional vectors. We define

$$\mathbf{e}^{(i)} = \mathbf{\Lambda}^{-i}\cdot\mathbf{o}, \quad {}^t\mathbf{e}^{(i)} = {}^t\mathbf{o}\cdot{}^t\mathbf{\Lambda}^{-i}, \quad \forall i\in\mathbb{Z}$$

which play the roles of bases of an arbitrary column vector and an arbitrary row vector. Meanwhile, with the help of (2.9), we can show that

$$\mathbf{e}^{(i)} \cdot {}^t\mathbf{e}^{(j)} = \mathbf{\Lambda}^{-i} \cdot \mathbf{o} \cdot {}^t\mathbf{o} \cdot {}^t\mathbf{\Lambda}^{-j} = \mathbf{\Lambda}^{-i} \cdot \mathbf{O} \cdot {}^t\mathbf{\Lambda}^{-j} = \mathbf{E}^{(i,j)} \tag{2.10a}$$

as well as

$${}^t\mathbf{e}^{(j)} \cdot \mathbf{e}^{(i)} = {}^t\mathbf{o} \cdot {}^t\mathbf{\Lambda}^{-j} \cdot \mathbf{\Lambda}^{-i} \cdot \mathbf{o} = \delta_{j,i}. \tag{2.10b}$$

The two relations are the analogues of the ones given in (2.3).

Definition 2.3. *An infinite-dimensional column vector* $\mathbf{u}$ *and its transpose* ${}^t\mathbf{u}$ *(i.e. an infinite-dimensional row vector) are defined as*

$$\mathbf{u} = \sum_{i\in\mathbb{Z}} u_i \mathbf{e}^{(i)} \quad \textit{and} \quad {}^t\mathbf{u} = \sum_{i\in\mathbb{Z}} u_i {}^t\mathbf{e}^{(i)}, \tag{2.11}$$

respectively, where u_i *(as the corresponding components) are elements in the same field* $\mathcal{F}$. *Elements in the sets* $\{\mathbf{e}^{(i)}|i \in \mathbb{Z}\}$ *and* $\{{}^t\mathbf{e}^{(i)}|i \in \mathbb{Z}\}$ *form the respective bases for an arbitrary infinite-dimensional vector and its transpose, respectively.*

For two arbitrary infinite-dimensional vectors $\mathbf{u} = \sum_{i\in\mathbb{Z}} u_i \mathbf{e}^{(i)}$ and $\mathbf{v} = \sum_{j\in\mathbb{Z}} v_j \mathbf{e}^{(j)}$ and their transpose, where u_i, v_j are elements from the field $\mathcal{F}$, and for arbitrary $\mathfrak{p}$ also from $\mathcal{F}$, we have the basic operations as follows:

$$\begin{aligned}
\mathbf{u} + \mathbf{v} &= \sum_{i\in\mathbb{Z}} (u_i + v_i)\mathbf{e}^{(i)}, \quad {}^t\mathbf{u} + {}^t\mathbf{v} = \sum_{i\in\mathbb{Z}} (u_i + v_i){}^t\mathbf{e}^{(i)} \\
\mathfrak{p}\,\mathbf{u} &= \sum_{i\in\mathbb{Z}} (\mathfrak{p}\,u_i)\mathbf{e}^{(i)}, \quad \mathfrak{p}\,{}^t\mathbf{u} = \sum_{i\in\mathbb{Z}} (\mathfrak{p}\,u_i){}^t\mathbf{e}^{(i)}, \\
\mathbf{u} \cdot {}^t\mathbf{v} &= \sum_{i,j\in\mathbb{Z}} u_i v_j \mathbf{E}^{(i,j)}, \quad {}^t\mathbf{v} \cdot \mathbf{u} = \sum_{i\in\mathbb{Z}} u_i v_i.
\end{aligned}$$

The derivations of the first four relations are trivial as they can be observed from the definition of infinite-dimensional vectors immediately; while the last two equations follow from (2.10) and we give their derivations below:

$$\begin{aligned}
\mathbf{u} \cdot {}^t\mathbf{v} &= \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} u_i v_j \mathbf{e}^{(i)} \cdot {}^t\mathbf{e}^{(j)} = \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} u_i v_j \mathbf{E}^{(i,j)}, \\
{}^t\mathbf{v} \cdot \mathbf{u} &= \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} u_i v_j {}^t\mathbf{e}^{(j)} \cdot \mathbf{e}^{(i)} = \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} u_i v_j \delta_{i,j} = \sum_{i\in\mathbb{Z}} u_i v_i.
\end{aligned}$$

In other words, $\mathbf{u} \cdot {}^t\mathbf{v}$ is an infinite-dimensional matrix and ${}^t\mathbf{v} \cdot \mathbf{u}$ is a scalar quantity, which obey the same rules (except that the summation is over $\mathbb{Z}$) as those for finite-dimensional vectors.

The case of N-component (column and row) vectors is obtained by the restriction $u_i = 0$ for $i \neq 1, 2, \cdots, N$, leading to $\mathbf{u} = \sum_{i=1}^N u_i \mathbf{e}_N^{(i)}$ and ${}^t\mathbf{u} = \sum_{i=1}^N u_i {}^t\mathbf{e}_N^{(i)}$, namely a finite-dimensional vector can be considered as a degeneration of an infinite-dimensional vector.

Making use of (2.10), we can also show that $\mathbf{E}^{(i,j)}$, $\mathbf{e}^{(i)}$ and ${}^t\mathbf{e}^{(i)}$ satisfy the following relations:

$$\mathbf{E}^{(i,j)} \cdot \mathbf{e}^{(i')} = \delta_{i',j}\mathbf{e}^{(i)}, \quad {}^t\mathbf{e}^{(j')} \cdot \mathbf{E}^{(i,j)} = \delta_{i,j'}{}^t\mathbf{e}^{(j)}. \tag{2.12}$$

The relation (2.12) allows the multiplication of an infinite-dimensional matrix and an infinite-dimensional vector. For instance, for arbitrary $\mathbf{U}$ taking the form of (2.8) and $\mathbf{u}$ and ${}^t\mathbf{u}$ given by (2.11), we have

$$\begin{aligned}
\mathbf{U} \cdot \mathbf{u} &= \sum_{i,j\in\mathbb{Z}} \sum_{i'\in\mathbb{Z}} U_{i,j} u_{i'} \mathbf{E}^{(i,j)} \cdot \mathbf{e}^{(i')} = \sum_{i\in\mathbb{Z}} \left(\sum_{j\in\mathbb{Z}} U_{i,j} u_j \right) \mathbf{e}^{(i)}, \\
{}^t\mathbf{u} \cdot \mathbf{U} &= \sum_{i'\in\mathbb{Z}} \sum_{i,j\in\mathbb{Z}} u_{i'} U_{i,j} {}^t\mathbf{e}^{(i')} \cdot \mathbf{E}^{(i,j)} = \sum_{j\in\mathbb{Z}} \left(\sum_{i\in\mathbb{Z}} u_i U_{i,j} \right) {}^t\mathbf{e}^{(j)},
\end{aligned}$$

namely $\mathbf{U} \cdot \mathbf{u}$ is an infinite-dimensional column vector and ${}^t\mathbf{u} \cdot \mathbf{U}$ is an infinite-dimensional row vector.

2.4. Visualisations. The above subsection gives the definitions of infinite matrices and vectors in an algebraic way. In fact, these abstract notations have their respective visualisation. In this subsection, we give visualisations of the projector $\mathbf{O}$ (an "centred" matrix), the index-raising operators $\mathbf{\Lambda}$ and ${}^t\mathbf{\Lambda}$, as well as the "centred" vectors $\mathbf{o}$ and ${}^t\mathbf{o}$.

The projector $\mathbf{O}$ is a particular infinite-dimensional matrix when we take $U_{i,j} = \delta_{i,0}\delta_{j,0}$ in (2.8), namely $\mathbf{O}$ is an "centred" matrix has only nonzero value in the $(0,0)$-entry. Similarly, the unit infinite-dimensional matrix[2] $\mathbf{I}$ is a particular case of (2.8) when $U_{i,j} = \delta_{i,j}$. Both $\mathbf{O}$ and $\mathbf{I}$ have their respective visualisation as follows:

$$\mathbf{O} = \begin{pmatrix} \ddots & & & & \\ & 0 & & & \\ & & \boxed{1} & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix}, \quad \mathbf{I} = \begin{pmatrix} \ddots & & & & \\ & 1 & & & \\ & & \boxed{1} & & \\ & & & 1 & \\ & & & & \ddots \end{pmatrix},$$

where the boxes denote the "centres" (i.e. the $(0,0)$-entries) of the two matrices.

The index-raising operators $\mathbf{\Lambda}$ and ${}^t\mathbf{\Lambda}$ themselves are not infinite-dimensional matrices; however, we can visualise them by considering their operations on the unit infinite-dimensional matrix $\mathbf{I}$, namely $\mathbf{\Lambda} \cdot \mathbf{I}$ and $\mathbf{I} \cdot {}^t\mathbf{\Lambda}$. Observing that the index-raising operators acting on $\mathbf{E}^{(i,j)}$ gives rise to relations as follows:

$$\begin{aligned}
\mathbf{\Lambda} \cdot \mathbf{E}^{(i,j)} &= \mathbf{\Lambda}^{-i+1} \cdot \mathbf{O} \cdot {}^t\mathbf{\Lambda}^{-j} = \mathbf{E}^{(i-1,j)}, \\
\mathbf{E}^{(i,j)} \cdot {}^t\mathbf{\Lambda} &= \mathbf{\Lambda}^{-i} \cdot \mathbf{O} \cdot {}^t\mathbf{\Lambda}^{-j+1} = \mathbf{E}^{(i,j-1)},
\end{aligned}$$

[2]Sometimes we also use 1 (instead of $\mathbf{I}$) to denote a unit infinite-dimensional matrix.

according to the definition of $\mathbf{E}^{(i,j)}$, we consider $\mathbf{\Lambda}\cdot\mathbf{I}$ and $\mathbf{I}\cdot{}^t\mathbf{\Lambda}$, respectively, and obtain the following expressions:

$$\mathbf{\Lambda}\cdot\mathbf{I}=\sum_{i,j\in\mathbb{Z}}\delta_{i,j}\mathbf{E}^{(i-1,j)}=\sum_{i,j\in\mathbb{Z}}\delta_{i+1,j}\mathbf{E}^{(i,j)},$$
$$\mathbf{I}\cdot{}^t\mathbf{\Lambda}=\sum_{i,j\in\mathbb{Z}}\delta_{i,j}\mathbf{E}^{(i,j-1)}=\sum_{i,j\in\mathbb{Z}}\delta_{i,j+1}\mathbf{E}^{(i,j)}.$$

This implies that $\mathbf{\Lambda}\cdot\mathbf{I}$ and $\mathbf{I}\cdot{}^t\mathbf{\Lambda}$ have their respective (i,j)-entries $\delta_{i+1,j}$ and $\delta_{i,j+1}$. Therefore, the visualizations of them are given by

$$\mathbf{\Lambda}\cdot\mathbf{I}=\begin{pmatrix}\ddots & \ddots & & & \\ & 0 & 1 & & \\ & & \boxed{0} & 1 & \\ & & & 0 & \ddots \\ & & & & \ddots\end{pmatrix}\quad\text{and}\quad\mathbf{I}\cdot{}^t\mathbf{\Lambda}=\begin{pmatrix}\ddots & & & & \\ \ddots & 0 & & & \\ & 1 & \boxed{0} & & \\ & & 1 & 0 & \\ & & & \ddots & \ddots\end{pmatrix},$$

respectively.

Similarly, $\mathbf{o}$ and ${}^t\mathbf{o}$ are degenerations of $\mathbf{u}$ and ${}^t\mathbf{u}$ defined in (2.11), respectively, when $u_i=\delta_{i,0}$. Thus, we can conclude that $\mathbf{o}$ and ${}^t\mathbf{o}$ are the two "centred" vectors, and their visualisations are

$$\mathbf{o}=\Big(\cdots,0,\boxed{1},0,\cdots\Big)^{\mathrm{T}},\quad {}^t\mathbf{o}=\Big(\cdots,0,\boxed{1},0,\cdots\Big),$$

respectively.

The matrix $\mathbf{E}^{(i,j)}$ and the vectors $\mathbf{e}^{(i)}$ and ${}^t\mathbf{e}^{(i)}$ can also be visualised in the same way. By observing their corresponding coefficients, we can conclude that $\mathbf{E}^{(i,j)}$ is an infinite-dimensional matrix having the (i,j)-entry 1 and other entries zero; $\mathbf{e}^{(i)}$ is an infinite-dimensional column vector having the ith-component 1 and other components zero, and ${}^t\mathbf{e}^{(i)}$ is its transpose (therefore a row vector).

2.5. Some useful operations. In this subsection, we explain some operations on infinite-dimensional matrices and vectors, and give some concrete examples which would be used in the DL framework.

For an infinite-dimensional matrix $\mathbf{U}$, its (i,j)-entry $(\mathbf{U})_{i,j}$ is defined as the coefficient of $\mathbf{E}^{(i,j)}$ in the expansion (2.8). Similarly, the ith-components of $\mathbf{u}$ and ${}^t\mathbf{u}$ are defined as the coefficients of $\mathbf{e}^{(i)}$ and ${}^t\mathbf{e}^{(i)}$ according to (2.11). We also point out that the action $(\,\cdot\,)_{i,j}$ can all be expressed by $(\,\cdot\,)_{0,0}$ through the relation

$$(\mathbf{U})_{i,j}=U_{i,j}=(\mathbf{\Lambda}^i\cdot\mathbf{U}\cdot{}^t\mathbf{\Lambda}^j)_{0,0}.$$

Similarly, for an infinite-dimensional vector $\mathbf{u}$ and its transpose ${}^t\mathbf{u}$ as defined in (2.11), we have

$$(\mathbf{\Lambda}^i\cdot\mathbf{u})_0=u_i,\quad({}^t\mathbf{u}\cdot{}^t\mathbf{\Lambda}^j)_0=u_j.$$

Therefore, we only need the action $(\,\cdot\,)_{0,0}$ for convenience in the future. Sometimes, there also exist operations involving the index-raising operators $\mathbf{\Lambda}$ and ${}^t\mathbf{\Lambda}$ in their rational form acting on $\mathbf{U}$, or $\mathbf{u}$ (as well as ${}^t\mathbf{u}$). For instance[3],

$$\left(\frac{1}{1-\mathbf{\Lambda}}\cdot\mathbf{U}\right)_{0,0}=\left(\sum_{i=0}^{\infty}\mathbf{\Lambda}^i\cdot\mathbf{U}\right)_{0,0}=\sum_{i=0}^{\infty}(\mathbf{\Lambda}^i\cdot\mathbf{U})_{0,0}=\sum_{i=0}^{\infty}U_{i,0}.$$

Next, we give some examples of the action $(\,\cdot\,)$ on multiplications of infinite-dimensional matrices and vectors, especially for the case when the projector $\mathbf{O}$ is involved.

EXAMPLE 2.4. *Suppose that* $\mathbf{U}=\sum_{i,j\in\mathbb{Z}}U_{i,j}\mathbf{E}^{(i,j)}$ *and* $\mathbf{V}=\sum_{i,j\in\mathbb{Z}}V_{i,j}\mathbf{E}^{(i,j)}$. *We have*

$$(\mathbf{\Lambda}^{i_1}\cdot\mathbf{U}\cdot{}^t\mathbf{\Lambda}^{j_1}\cdot\mathbf{O}\cdot\mathbf{\Lambda}^{i_2}\cdot\mathbf{V}\cdot{}^t\mathbf{\Lambda}^{j_2})_{0,0}=U_{i_1,j_1}V_{i_2,j_2}.$$

PROOF. This relation can be proven directly. Notice $\mathbf{O}=\mathbf{O}^2$ (a particular case of (2.5)). We can prove that

$$\begin{aligned}&\mathbf{\Lambda}^{i_1}\cdot\mathbf{E}^{(i,j)}\cdot{}^t\mathbf{\Lambda}^{j_1}\cdot\mathbf{O}\cdot\mathbf{\Lambda}^{i_2}\cdot\mathbf{E}^{(i',j')}\cdot{}^t\mathbf{\Lambda}^{j_2}\\&=\mathbf{\Lambda}^{i_1-i}\cdot\mathbf{O}\cdot{}^t\mathbf{\Lambda}^{j_1-j}\cdot\mathbf{O}\cdot\mathbf{O}\cdot\mathbf{\Lambda}^{i_2-i'}\cdot\mathbf{O}\cdot{}^t\mathbf{\Lambda}^{j_2-j'}\\&=\delta_{j_1-j,0}\delta_{0,i_2-i'}\mathbf{\Lambda}^{i_1-i}\cdot\mathbf{O}\cdot{}^t\mathbf{\Lambda}^{j_2-j'}=\delta_{j_1-j,0}\delta_{0,i_2-i'}\mathbf{E}^{(i-i_1,j'-j_2)}.\end{aligned}$$

where the relation (2.5) is used. Then, it follows from the above relation that

$$\begin{aligned}&\mathbf{\Lambda}^{i_1}\cdot\mathbf{U}\cdot{}^t\mathbf{\Lambda}^{j_1}\cdot\mathbf{O}\cdot\mathbf{\Lambda}^{i_2}\cdot\mathbf{V}\cdot{}^t\mathbf{\Lambda}^{j_2}\\&=\sum_{i,j,i',j'\in\mathbb{Z}}U_{i,j}V_{i',j'}\mathbf{\Lambda}^{i_1}\cdot\mathbf{E}^{(i,j)}\cdot{}^t\mathbf{\Lambda}^{j_1}\cdot\mathbf{O}\cdot\mathbf{\Lambda}^{i_2}\cdot\mathbf{E}^{(i',j')}\cdot{}^t\mathbf{\Lambda}^{j_2}\\&=\sum_{i,j,i',j'\in\mathbb{Z}}U_{i,j}V_{i',j'}\delta_{j_1-j,0}\delta_{0,i_2-i'}\mathbf{E}^{(i-i_1,j'-j_2)}\\&=\sum_{i,j'\in\mathbb{Z}}U_{i,j_1}V_{i_2,j'}\mathbf{E}^{(i-i_1,j'-j_2)}=\sum_{i,j'\in\mathbb{Z}}U_{i+i_1,j_1}V_{i_2,j'+j_2}\mathbf{E}^{(i,j')},\end{aligned}$$

which implies that its $(0,0)$-entry is $U_{i_1,j_1}V_{i_2,j_2}$. □

From this example, we can see that $\mathbf{O}$ plays a particular role in the calculation. We explain this in the following remark:

REMARK 2.5. *The projector* $\mathbf{O}$ *separates the multiplication of infinite-dimensional matrices; more precisely, the right hand side of the above equation can be expressed as* $(\mathbf{\Lambda}^{i_1}\cdot\mathbf{U}\cdot{}^t\mathbf{\Lambda}^{j_1})_{0,0}(\mathbf{\Lambda}^{i_2}\cdot\mathbf{U}\cdot{}^t\mathbf{\Lambda}^{j_2})_{0,0}$.

We also have a similar relation for the multiplication of an infinite-dimensional matrix $\mathbf{U}$ and an infinite-dimensional column vector $\mathbf{u}$ when $\mathbf{O}$ is involved.

EXAMPLE 2.6. *Suppose that* $\mathbf{U}=\sum_{i,j\in\mathbb{Z}}U_{i,j}\mathbf{E}^{(i,j)}$ *and* $\mathbf{u}=\sum_i u_i\mathbf{e}^{(i)}$, *the following relation holds:*

$$(\mathbf{\Lambda}^{i_1}\cdot\mathbf{U}\cdot{}^t\mathbf{\Lambda}^{j_1}\cdot\mathbf{O}\cdot\mathbf{\Lambda}^{i_2}\cdot\mathbf{u})_0=U_{i_1,j_1}u_{i_2},$$

which still tells us that the projector $\mathbf{O}$ *separates the multiplication, namely the right hand side can be written as* $(\mathbf{\Lambda}^{i_1}\cdot\mathbf{U}\cdot\mathbf{\Lambda}^{j_1})_{0,0}(\mathbf{\Lambda}^{i_2}\cdot\mathbf{u})_0$.

[3]Here 1 denotes the identity.

PROOF. The proof of the relation is quite similar to the previous proof. The key point is that one needs to use $\mathbf{O} = \mathbf{O}^2 = \mathbf{O}\cdot\mathbf{o}\cdot{}^t\mathbf{o}$ as well as the rules given in (2.8) and (2.9). $\square$

Furthermore, we introduce the following two important infinite-dimensional vectors which will be quite important in the DL framework as follows:

$$\mathbf{c}_k = \sum_{i\in\mathbb{Z}} k^i\,\mathbf{e}^{(i)} = (\cdots, k^{-1}, \boxed{1}, k, \cdots)^{\mathrm{T}},$$
$${}^t\mathbf{c}_{k'} = \sum_{i\in\mathbb{Z}} k'^i\,{}^t\mathbf{e}^{(i)} = (\cdots, k'^{-1}, \boxed{1}, k', \cdots).$$

It is easy to verify that the vectors $\mathbf{c}_k$ and ${}^t\mathbf{c}_{k'}$ obviously satisfy

$$\mathbf{\Lambda}\cdot\mathbf{c}_k = k\mathbf{c}_k, \quad {}^t\mathbf{c}_{k'}\cdot{}^t\mathbf{\Lambda} = k'\,{}^t\mathbf{c}_{k'} \quad \text{and} \quad (\mathbf{c}_k)_0 = 1, \quad ({}^t\mathbf{c}_{k'})_0 = 1, \tag{2.13}$$

namely $\mathbf{c}_k$ and ${}^t\mathbf{c}_{k'}$ are eigenvectors of the index-raising operators $\mathbf{\Lambda}$ and ${}^t\mathbf{\Lambda}$, respectively, with their corresponding eigenvalues k and k'.

EXAMPLE 2.7. *The infinite-dimensional vectors $\mathbf{c}_k$ and ${}^t\mathbf{c}_{k'}$ obey the following relations:*

$$\mathbf{\Lambda}^i\cdot\mathbf{c}_k\cdot{}^t\mathbf{c}_{k'}\cdot{}^t\mathbf{\Lambda}^j = k^ik'^j\mathbf{c}_k\cdot{}^t\mathbf{c}_{k'}, \quad {}^t\mathbf{c}_{k'}\cdot{}^t\mathbf{\Lambda}^j\cdot\mathbf{O}\cdot\mathbf{\Lambda}^i\cdot\mathbf{c}_k = k^ik'^j.$$

PROOF. The first relation follows from (2.13) directly. For the second one, by direct calculation we have

$$\begin{aligned}
{}^t\mathbf{c}_{k'}\cdot{}^t\mathbf{\Lambda}^j\cdot\mathbf{O}\cdot\mathbf{\Lambda}^i\cdot\mathbf{c}_k &= \sum_{i',j'\in\mathbb{Z}} k^{i'}k'^{j'}\,{}^t\mathbf{e}^{(j')}\cdot{}^t\mathbf{\Lambda}^j\cdot\mathbf{O}\cdot\mathbf{\Lambda}^i\cdot\mathbf{e}^{(i')}\\
&= \sum_{i',j'\in\mathbb{Z}} k^{i'}k'^{j'}\,{}^t\mathbf{o}\cdot{}^t\mathbf{\Lambda}^{j-j'}\cdot\mathbf{o}\cdot{}^t\mathbf{o}\cdot\mathbf{\Lambda}^{i-i'}\cdot\mathbf{o}\\
&= \sum_{i',j'\in\mathbb{Z}} k^{i'}k'^{j'}\delta_{j-j',0}\delta_{0,i-i'} = k^ik'^j,
\end{aligned}$$

where we use $\mathbf{O} = \mathbf{o}\cdot{}^t\mathbf{o}$ the second equality and (2.9) for the third equality. $\square$

Based on the second relation, there is one more example which is needed in the later chapters.

EXAMPLE 2.8. *For the infinite-dimensional matrix $\mathbf{\Omega} = -\sum_{i=0}^{\infty}(-{}^t\mathbf{\Lambda})^{-i-1}\cdot\mathbf{O}\cdot\mathbf{\Lambda}^i$, we have*

$${}^t\mathbf{c}_{k'}\cdot\mathbf{\Omega}\cdot\mathbf{c}_k = \frac{1}{k+k'} \doteq \Omega_{k,k'}.$$

Thus, $\mathbf{\Omega}$ is an infinite-dimensional matrix representation of $\Omega_{k,k'}$.

PROOF. Direct calculation shows that

$${}^t\mathbf{c}'_k\cdot\mathbf{\Omega}\cdot\mathbf{c}_k = -\sum_{i=0}^{\infty}(-k')^{-i-1}k^i = \frac{1}{k'}\sum_{i=0}^{\infty}\left(-\frac{k}{k'}\right)^i = \frac{1}{k'}\frac{1}{1+\frac{k}{k'}} = \frac{1}{k+k'}.$$

The equality is proven. $\square$

REMARK 2.9. *The infinite-dimensional matrix $\mathbf{\Omega}$ obeys the relation*

$$\mathbf{\Omega}\cdot\mathbf{\Lambda} + {}^t\mathbf{\Lambda}\cdot\mathbf{\Omega} = \mathbf{O}.$$

2.6. Trace and determinant. In the approach we also need the notion of trace and determinant of an infinite-dimensional matrix, but only in the case when they involve the projector $\mathbf{O}$.

The trace of an infinite-dimensional matrix $\mathbf{U}$ taking the form of (2.8) is defined as

$$\operatorname{tr}\mathbf{U} = \sum_{i\in\mathbb{Z}} U_{i,i}. \tag{2.14}$$

Unlike the finite case, this definition may raise an issue about divergence due to the infinite summation. However, in our approach we only consider the case when $\mathbf{O}$ is involved, which avoids the divergence issue.

PROPOSITION 2.10. *The following identity for the trace holds:*

$$\operatorname{tr}(\mathbf{O}\cdot\mathbf{U}) = \operatorname{tr}(\mathbf{U}\cdot\mathbf{O}) = U_{0,0} = (\mathbf{U})_{0,0},$$

which allows us to transfer the trace of an infinite-dimensional matrix to a scalar quantity.

PROOF. We only prove $\operatorname{tr}(\mathbf{O}\cdot\mathbf{U}) = U_{0,0}$, and the proof of the other one is similar. A particular case (when $\mathbf{E}^{(i_1,j_1)} = \mathbf{E}^{(0,0)} = \mathbf{O}$) of (2.7) implies that $\mathbf{O}\cdot\mathbf{E}^{(i,j)} = \delta_{0,i}\cdot\mathbf{E}^{(0,j)}$. Thus, we have

$$\mathbf{O}\cdot\mathbf{U} = \sum_{i,j\in\mathbb{Z}} U_{i,j}\mathbf{O}\cdot\mathbf{E}^{(i,j)} = \sum_{i,j\in\mathbb{Z}} U_{i,j}\delta_{0,i}\mathbf{E}^{(0,j)} = \sum_{j\in\mathbb{Z}} U_{0,j}\mathbf{E}^{(0,j)},$$

and consequently the (i,j)-entry of $\mathbf{U}$ is expressed by $\delta_{0,i}U_{0,j}$. According to the definition (2.14), we have $\operatorname{tr}(\mathbf{O}\cdot\mathbf{U}) = \sum_{i\in\mathbb{Z}}\delta_{0,i}U_{0,i} = U_{0,0}$. □

PROPOSITION 2.11. *The trace has the properties*

$$\operatorname{tr}(\mathbf{U}+\mathbf{V}) = \operatorname{tr}\mathbf{U} + \operatorname{tr}\mathbf{V} \quad \textit{and} \quad \operatorname{tr}(\mathbf{U}\cdot\mathbf{V}) = \operatorname{tr}(\mathbf{V}\cdot\mathbf{U}).$$

where $\mathbf{U}$ *and* $\mathbf{V}$ *are two arbitrary infinite-dimensional matrices involving* $\mathbf{O}$.

These properties are exactly the same as the ones in the finite-dimensional matrix theory. The proof is similar, namely it is proven by using the definition. We give a concrete example (which will appear in a later chapter) as follows:

EXAMPLE 2.12. *Suppose that* $\mathbf{U}$ *is an arbitrary infinite-dimensional matrix,* $\mathbf{\Lambda}$ *and* ${}^t\mathbf{\Lambda}$ *are the index-raising operators, and* $\mathbf{O}$ *is the projector. We have the following identity:*

$$\operatorname{tr}((\mathbf{O}\cdot\mathbf{\Lambda} - {}^t\mathbf{\Lambda}\cdot\mathbf{O})\cdot\mathbf{U}) = (\mathbf{\Lambda}\cdot\mathbf{U} - \mathbf{U}\cdot{}^t\mathbf{\Lambda})_{0,0} = U_{1,0} - U_{0,1}.$$

PROOF. This is a consequence of the above two propositions. By using the linearity of the trace, we can prove

$$\operatorname{tr}((\mathbf{O}\cdot\mathbf{\Lambda} - {}^t\mathbf{\Lambda}\cdot\mathbf{O})\cdot\mathbf{U}) = \operatorname{tr}(\mathbf{O}\cdot\mathbf{\Lambda}\cdot\mathbf{U}) - \operatorname{tr}({}^t\mathbf{\Lambda}\cdot\mathbf{O}\cdot\mathbf{U}).$$

Next, using the cyclic permutation, the right hand side can be written as

$$\operatorname{tr}(\mathbf{O}\cdot(\mathbf{\Lambda}\cdot\mathbf{U})) - \operatorname{tr}(\mathbf{O}\cdot(\mathbf{U}\cdot{}^t\mathbf{\Lambda})).$$

The identity is proven in virtue of proposition 2.10. □

The determinant of a finite-dimensional non-zero square matrix $\mathbf{A}$ obeys the following important identity:

$$\ln(\det \mathbf{A}) = \operatorname{tr}(\ln \mathbf{A}),$$

which provides us with a way to generalise the notion to the infinite-dimensional case.

Definition 2.13. *The determinant of an infinite-dimensional matrix* $\mathbf{U}$ *is defined as*

$$\det \mathbf{U} = \exp\left(\operatorname{tr}(\ln \mathbf{U})\right).$$

The right hand side should be understand as a series expansion. In our framework, we only deal with a particular class of determinants taking the form of $\det(1+*)$, where $\mathbf{O}$ is involved in $*$. Under such an assumption, a determinant can be evaluated in terms of a scalar quantity; thus, we avoid the issue of divergence in the expansion.

In applications, we need the well-known Weinstein–Aronszajn formula to evaluate the determinant of an infinite-dimensional matrix. As examples, we list the rank 1 and rank 2 cases.

Example 2.14. *The determinant of infinite-dimensional matrices obeys the following identities:*

$$\det(1+\mathbf{U}\cdot\mathbf{O}\cdot\mathbf{V}) = 1 + (\mathbf{V}\cdot\mathbf{U})_{0,0},$$

$$\det(1+\mathbf{U}\cdot(\mathbf{O}\cdot\mathbf{\Lambda} - {}^t\mathbf{\Lambda}\cdot\mathbf{O})\cdot\mathbf{V}) = \det\begin{pmatrix} 1+(\mathbf{\Lambda}\cdot\mathbf{V}\cdot\mathbf{U})_{0,0} & -(\mathbf{\Lambda}\cdot\mathbf{V}\cdot{}^t\mathbf{\Lambda})_{0,0} \\ (\mathbf{V}\cdot\mathbf{U})_{0,0} & 1-(\mathbf{V}\cdot\mathbf{U}\cdot{}^t\mathbf{\Lambda})_{0,0}\end{pmatrix}.$$

The left hand sides of the above equalities should be understood as

$$\det(1+(\mathbf{U}\cdot\mathbf{o})\cdot({}^t\mathbf{o}\cdot\mathbf{V})) \quad \text{and} \quad \det\left(1+(\mathbf{U}\cdot\mathbf{o}, -\mathbf{U}\cdot{}^t\mathbf{\Lambda}\cdot\mathbf{o})\cdot\begin{pmatrix}{}^t\mathbf{o}\cdot\mathbf{\Lambda}\cdot\mathbf{V} \\ {}^t\mathbf{o}\cdot\mathbf{V}\end{pmatrix}\right),$$

respectively, where $\mathbf{U}\cdot\mathbf{o}$ and $\mathbf{U}\cdot{}^t\mathbf{\Lambda}\cdot\mathbf{o}$ are two infinite-dimensional column vectors, and ${}^t\mathbf{o}\cdot\mathbf{V}$ and ${}^t\mathbf{o}\cdot\mathbf{\Lambda}\cdot\mathbf{V}$ are two infinite-dimensional row vectors. Therefore, the above equalities are nothing but the infinite-dimensional matrix analogues of the rank 1 and rank 2 cases of the identity

$$\det(1+\mathbf{A}\mathbf{B}) = \det(1+\mathbf{B}\mathbf{A}),$$

where $\mathbf{A}$ and $\mathbf{B}$ are finite-dimensional matrices of sizes $M\times N$ and $N\times M$, respectively.

3. General theory of the direct linearisation

3.1. Notations. Before we discuss the DL framework, we introduce some notations in terms of continuous and discrete dynamics. We consider two infinite sets of variables

$$\mathbf{x} = (\cdots, x_{-2}, x_{-1}, x_1, x_2, \cdots) \quad \text{and} \quad \mathbf{n} = (\cdots, n_{-2}, n_{-1}, n_1, n_2, \cdots),$$

in which x_j are the continuous flow variables, and n_j are the discrete flow variables associated with their corresponding lattice parameters p_j. We also note that the suffices $j>0$ denote the positive flows and similarly $j<0$ denote negative flows. Consider a smooth function $f = f_n \doteq f(\mathbf{x};\mathbf{n};n)$, where n is an extra discrete variable associated with its corresponding lattice parameter zero. We introduce the notation ∂_j denoting the partial derivative with respect to the continuous variable x_j, namely

$$\partial_j f(\mathbf{x},\mathbf{n},n) = f_{x_j}(\mathbf{x},\mathbf{n},n) \doteq \frac{\partial}{\partial x_j} f(\mathbf{x};\mathbf{n};n).$$

We also introduce the discrete forward and backward shift operators T_{p_j} and $\mathrm{T}_{p_j}^{-1}$ in terms of n_j defined by

$$\mathrm{T}_{p_j} f(\cdots,n_j,\cdots) = f(\cdots,n_j+1,\cdots), \quad \mathrm{T}_{p_j}^{-1} f(\cdots,n_j+1,\cdots) = f(\cdots,n_j-1,\cdots),$$

and also the discrete forward and backward shift operators T_0 and T_0^{-1} for the discrete variable n defined by

$$\mathrm{T}_0 f(\mathbf{x},\mathbf{n},n) = f(\mathbf{x},\mathbf{n},n+1) \quad \text{and} \quad \mathrm{T}_0^{-1} f(\mathbf{x},\mathbf{n},n) = f(\mathbf{x},\mathbf{n},n-1).$$

Conventionally, we adopt the following short-hand notations:

$$\begin{aligned}
&\tilde{f} \doteq \mathrm{T}_{p_1} f, \quad \hat{f} \doteq \mathrm{T}_{p_2} f, \quad \bar{f} \doteq \mathrm{T}_{p_3} f, \quad \check{f} \doteq \mathrm{T}_{p_{-1}} f, \quad \dot{f} \doteq \mathrm{T}_0 f, \\
&\underset{\sim}{f} \doteq \mathrm{T}_{p_1}^{-1} f, \quad \underset{\wedge}{f} \doteq \mathrm{T}_{p_2}^{-1} f, \quad \underline{f} \doteq \mathrm{T}_{p_3}^{-1} f, \quad \underset{\vee}{f} \doteq \mathrm{T}_{p_{-1}}, \quad \underset{\cdot}{f} \doteq \mathrm{T}_0^{-1} f.
\end{aligned}$$

Besides the dot/underdot sign, sometimes we also use suffices to denote the discrete shifts with respect to n, namely

$$f_{n+1} \doteq \mathrm{T}_0 f_n, \quad f_{n-1} \doteq \mathrm{T}_0^{-1} f_n.$$

Combinations of the above notations (which will be used everywhere in the rest of the thesis) denote the compositions of their corresponding discrete shift operations.

3.2. Linear integral equation. The idea of the direct linearising method is to solve a nonlinear integrable equation by considering its associated linear integral equation, namely a solution to the linear integral equation can help to express a solution to the nonlinear equation by nonlinearisation. Reversely, considering the structure of a linear integral equation also provides a way to construct its related nonlinear equation. For the most general structure of the DL framework, we consider a linear integral equation taking the form of

$$\mathbf{u}_k + \iint_D \mathrm{d}\zeta(l,l')\rho_k \Omega_{k,l'} \sigma_{l'} \mathbf{u}_l = \rho_k \mathbf{c}_k, \tag{2.15}$$

where the wave function $\mathbf{u}_k$ is an infinite column vector having its ith-component $u_k^{(i)}$ as a smooth function of the continuous independent variables $\mathbf{x}$, the discrete independent variables $\mathbf{n}$ (and also their associated lattice parameters p_j), the extra discrete variable n, as well as the spectral parameter k; the Cauchy kernel $\Omega_{k,l'}$ is an algebraic expression of the spectral parameters k and l', independent of the independent variables $\mathbf{x}$, $\mathbf{n}$, n and lattice parameters $\{p_j|j \in \mathbb{Z}\}$; the plane wave functions ρ_k and $\sigma_{l'}$ are expressions of the discrete and continuous independent variables (i.e. flow variables) $\mathbf{x}$, $\mathbf{n}$, n as well as the lattice parameters $\{p_j|j \in \mathbb{Z}\}$, also depending on the spectral parameters k and l', respectively; the infinite column vector $\mathbf{c}_k$ takes the form of $\mathbf{c}_k = (\cdots,k^{-1},1,k,\cdots)^{\mathrm{T}}$; the measure $\mathrm{d}\zeta(l,l')$ depending on the spectral variables l and l', and the integration domain D can be determined later for particular classes of solutions.

REMARK 2.15. *Although the wave function* $\mathbf{u}_k$ *in the linear integral equation* (2.15) *is a multi-component (i.e. infinite-component) vector, it is still suitable to think of it as a scalar equation since each component* $u_k^{(i)}$ *solves an independent linear integral equation. The reason why we introduce an infinite number of components is because this brings a possibility to study all the gauge equivalent integrable equations in the same class simultaneously, see e.g. [**NCWQ84, NCW85**]. There is also a matrix analogue (real multi-component) of such a linear integral equation, see e.g. [**CWN86, NCW85**], but we do not discuss it in the thesis.*

REMARK 2.16. *The existence of two spectral variables* l *and* l' *in the measure* $\mathrm{d}\zeta(l,l')$ *on the domain* D *is crucial as they govern the general solution for 3D integrable hierarchies (one can see this in the soliton solution for the KP equation, cf. [**Hir04**]). Such a structure is also a reflection of an underlying nonlocal Riemann–Hilbert (or* $\bar{\partial}$*-) problem. In fact, the core step in a nonlocal Riemann–Hilbert problem is solving a linear integral equation, which is a special reduction of* (2.15)*, see chapter 5 of the monograph by Ablowitz and Clarkson [**AC91**].*

REMARK 2.17. *When the measure collapses, namely* l *and* l' *become algebraically dependent, the double integral turns out to be a single integral, which results in 2D integrable equations. This is how a dimensional reduction of a 3D equation is performed.*

REMARK 2.18. *The plane wave factors* ρ_k *and* $\sigma_{l'}$ *describe the dispersion of a nonlinear integrable equation, namely its linear structure. The Cauchy kernel, instead, reflects the nonlinear structure of the nonlinear equation; for instance, it produces the phase factor in a two-soliton solution, which is a characteristic helping to understand the nonlinear interaction between two solitons.*

3.3. Infinite-dimensional matrix representation. We have explained the linear integral equation in the DL. The next step is to give the infinite-dimensional matrix representation of the integral equation and the related quantities. Before we deal with this, we need to introduce some essential infinite-dimensional matrices, which reflect the structure of the linear integral equation.

We introduce an infinite-dimensional matrix $\mathbf{\Omega}$ defined by the following relation:

$$\Omega_{k,k'} = {}^t\mathbf{c}_{k'} \cdot \mathbf{\Omega} \cdot \mathbf{c}_k. \tag{2.16}$$

This relation implies that $\mathbf{\Omega}$ is the infinite-dimensional matrix representation of the Cauchy kernel $\Omega_{k,k'}$. We also define an infinite-dimensional matrix $\mathbf{C}$ as follows:

$$\mathbf{C} = \iint_D \mathrm{d}\zeta(k,k')\rho_k \mathbf{c}_k \cdot {}^t\mathbf{c}_{k'}\sigma_{k'}. \tag{2.17}$$

The core part of this infinite-dimensional matrix is $\rho_k\sigma_{k'}$, namely the effective dispersion in the linear integral equation. Thus, $\mathbf{C}$ is the infinite-dimensional matrix representation of the plane wave factor of the linear integral equation. Moreover, we define an infinite-dimensional matrix $\mathbf{U}$ as follows:

$$\mathbf{U} = \iint_D \mathrm{d}\zeta(k,k')\mathbf{u}_k \cdot {}^t\mathbf{c}_{k'}\sigma_{k'}. \tag{2.18}$$

The infinite-dimensional matrix $\mathbf{U}$ is a non-linearisation of the wave function – the wave function $\mathbf{u}_k$ solves the linear integral equation and we complement it with the plane wave factor $\sigma_{k'}$.

From the above definitions, we can conclude that the main idea to construct the infinite-dimensional matrix representations of the certain quantities in the linear integral equation is to get the infinite-dimensional vectors $\mathbf{c}_k$ and ${}^t\mathbf{c}_{k'}$ involved. Actually, such a treatment brings us all the powers of the spectral variables k and k' in the structure, which will later allow us deal with a class of Miura-related equations simultaneously.

We can now express the linear integral equation by the above infinite-dimensional matrices.

PROPOSITION 2.19. *The linear integral* (2.15) *has the infinite-dimensional matrix representation*

$$\mathbf{u}_k = (1 - \mathbf{U} \cdot \mathbf{\Omega}) \cdot \rho_k \mathbf{c}_k. \tag{2.19}$$

PROOF. According to the definition of $\mathbf{\Omega}$, we can replace the Cauchy kernel and rewrite the linear integral equation (2.15) as

$$\begin{aligned}\rho_k \mathbf{c}_k &= \mathbf{u}_k + \iint_D \mathrm{d}\zeta(l,l') \rho_k \sigma_{l'} \mathbf{u}_l \cdot {}^t\mathbf{c}_{l'} \cdot \mathbf{\Omega} \cdot \mathbf{c}_k \\ &= \mathbf{u}_k + \left(\iint_D \mathrm{d}\zeta(l,l') \mathbf{u}_l \cdot {}^t\mathbf{c}_{l'} \sigma_{l'} \right) \cdot \mathbf{\Omega} \cdot \rho_k \mathbf{c}_k,\end{aligned}$$

which implies that $\mathbf{u}_k = (1 - \mathbf{U} \cdot \mathbf{\Omega}) \cdot \rho_k \mathbf{c}_k$. □

COROLLARY 2.20. *As a consequence of* (2.19)*, the infinite-dimensional matrix* $\mathbf{U}$ *satisfies the following relation:*

$$\mathbf{U} = (1 - \mathbf{U} \cdot \mathbf{\Omega}) \cdot \mathbf{C}, \quad \text{or alternatively} \quad \mathbf{U} = \mathbf{C} \cdot (1 + \mathbf{\Omega} \cdot \mathbf{C})^{-1}. \tag{2.20}$$

PROOF. This follows from (2.19) directly. One can act $\iint_D \mathrm{d}\zeta(k,k') \cdot {}^t\mathbf{c}_{k'} \sigma_{k'}$ on (2.19), which gives rise to the above relation. □

In addition, we also need the tau function in the framework.

DEFINITION 2.21. *The tau function in the DL framework is defined as*

$$\tau = \det(1 + \mathbf{\Omega} \cdot \mathbf{C}). \tag{2.21}$$

The tau function contains the main structure of the linear integral equation, namely, the Cauchy kernel (in $\mathbf{\Omega}$), the plane wave factors and the measure (in $\mathbf{C}$). Thus, it can be thought of as a fundamental quantity in a sense.

REMARK 2.22. *Alternatively, one can also define the function by* $\tau = \det(1 + \mathbf{C} \cdot \mathbf{\Omega})$*. This is because*

$$\det(1 + \mathbf{C} \cdot \mathbf{\Omega}) = \exp\{\mathrm{tr}[\ln(1 + \mathbf{C} \cdot \mathbf{\Omega})]\} = \exp\{\mathrm{tr}[\ln(1 + \mathbf{\Omega} \cdot \mathbf{C})]\} = \det(1 + \mathbf{\Omega} \cdot \mathbf{C}),$$

which is exactly the same as the definition of the tau function (2.21)*.*

3.4. Algebraic construction of closed-form equations. The infinite-dimensional matrix representation helps to construct closed-form equations for a given class. The DL framework provides us with several perspectives to understand a class of equations.

For a given linear integral equation, namely for fixed plane wave factors ρ_k and $\sigma_{k'}$ (i.e. the linear structure), and fixed Cauchy kernel $\Omega_{k,k'}$ and measure $\mathrm{d}\zeta(k,k')$ (i.e. the nonlinear structure), the direct linearisation helps to algebraically construct a class of closed-form equations from the following perspectives: i) The components of $\mathbf{u}_k$ solve linear equations (i.e. Lax pairs); ii) The tau function τ solves the bilinear equations in Hirota's form; iii) The entries of $\mathbf{U}$ solve the nonlinear equations in the class; iv) The quantities given in (2.15), (2.21) and (2.18) provide the direct linearising solutions to the corresponding closed-form equations.

Remark 2.23. *The nonlinear variable* $\mathbf{U}$*, the multilinear variable* τ *and the linear variable* $\mathbf{u}_k$ *are not independent. A suitable way to connect these quantities is expressing (the entries of)* $\mathbf{U}$ *and (the components) of* $\mathbf{u}_k$ *by* τ*.*

The idea of constructing closed-form equations is the following: We consider the (discrete and continuous) dynamics of the quantities $\mathbf{U}$, τ and $\mathbf{u}_k$, and express the dynamics by some algebraic expressions of the infinite-dimensional matrix $\mathbf{U}$. Then some identities in terms of the infinite-dimensional matrix $\mathbf{U}$ naturally give rise to the closed-form equations. We will explain such a procedure in more details by examples in the later chapters.

3.5. Explicit solutions. Explicit solutions to a certain discrete or continuous integrable equation can be constructed from the DL framework very easily. Since all the equations are constructed based on $\mathbf{U}$, τ or $\mathbf{u}_k$, these quantities (involving a double integral) play the roles of the most "general" solutions to the corresponding discrete and continuous equations, which we refer to as the direct linearising solution.

The direct linearising solution is general in the sense of involving a continuous spectrum $\mathrm{d}\zeta(k,k')$. The soliton-type solution is a particular case of the direct linearising solution. One can take a particular measure involving a finite number of singularities (i.e. discrete spectrum)

$$\mathrm{d}\zeta(k,k') = \sum_{i=1}^{\mathcal{N}} \sum_{j=1}^{\mathcal{N}'} \mathrm{d}k\mathrm{d}k' \delta(k-k_i)\delta(k'-k'_j)$$

for arbitrary integers $\mathcal{N}$ and $\mathcal{N}'$. As a consequence, the linear integral equation turns out to be a linear system which is solvable, leading to soliton-type solution. The soliton solution obtained in such a way is normally expressed by a Cauchy matrix, which algebraically reveals the soliton phenomenon (i.e. soliton interaction) in physics. We comment that the rational/semi-rational solution is limit of soliton-type solution.

In addition, we also note that Painlevé-type solutions can also be obtained from the DL framework. However, the analysis is slightly more complicated and we do not discuss such a class of solutions in the thesis. One may refer to the paper [**FA81**] for the continuous case and [**NRGO01**] for the discrete case.

3.6. Multi-dimensional consistency. The MDC property is a consequence of the DL framework. As has been explained, the structure of the linear integral equation (2.15) guarantees the framework generates 3D integrable equations. Selecting any three independent variables from $(\mathbf{n}, \mathbf{x}, n)$ provides a nonlinear integrable equation. Thus, a hierarchy of discrete and continuous equations can be constructed from the DL framework, and all the equations are solved by a single potential variable (including an infinite number of independent flows). This is the MDC property as we discussed in the introduction chapter.

More precisely, all the independent variables are on the same footing. If the independent variables rely on the spectral parameters in a covariant way (with regard to their corresponding lattice parameters), we obtain a hierarchy of covariant consistent equations; for instance, the HM equation. However, selecting non-covariant independent variables in the plane wave factor generates a hierarchy of integrable equations taking different forms. In the sense of the common solution, these equations are should still be considered being multi-dimensionally consistent.

We take the MDC of the discrete modified KdV and sine–Gordon equations as a simple example, explaining the above statement (without giving concrete derivation of the two equations). These two equations share the same nonlinear structure, and the only difference between them is the choice of flow variables. The effective plane wave factor in this class is given by

$$\prod_{j=1}^{\infty}\left(\frac{p_j+k}{p_j-k}\right)^{n_j}\prod_{j=-\infty}^{-1}\left(\frac{p_j+k^{-1}}{p_j-k^{-1}}\right)^{n_j}.$$

The variables n_j for $j > 0$ and $j < 0$ describe the discrete positive and negative flows, respectively. The discrete modified KdV equation arises a closed-form equation describing the dynamics with respect to two positive flows; while selecting one positive flow and one negative flow gives us the discrete sine–Gordon equation. For instance, (n_1, n_2) and (n_1, n_{-1}) provide the discrete mKdV and sine–Gordon equations

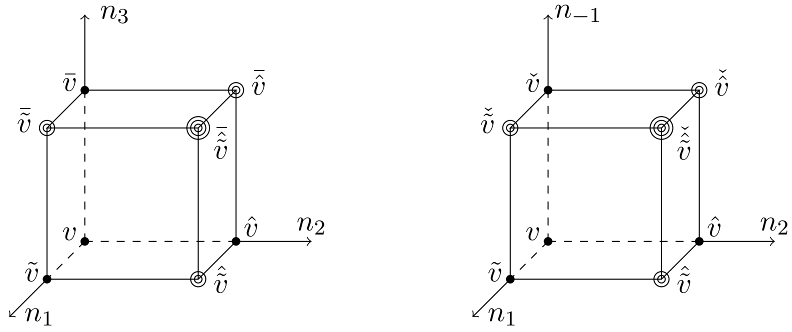
$$p_1(v\tilde{v}-\hat{v}\hat{\tilde{v}}) = p_2(v\hat{v}-\tilde{v}\hat{\tilde{v}}), \quad p_1p_{-1}(v\check{\tilde{v}}-\tilde{v}\check{v}) = v\tilde{v}\check{v}\check{\tilde{v}}-1,$$

respectively. Both equations have the same solution structure, since all the flow variables are included in the above effective plane wave factor. In other words, the potential v in both equations is the same. The discrete mKdV equation has covariant flow dynamics, namely n_1 and n_2 appear equally in the plane wave factor. Thus, to verify its 3D consistency, we can introduce the third variable n_3 and build up a closed cube, in which all the six equations on the cube take the form of the discrete modified equation, namely

$$\begin{gathered}
p_1(v\tilde{v}-\hat{v}\hat{\tilde{v}}) = p_2(v\hat{v}-\tilde{v}\hat{\tilde{v}}), \quad p_1(v\tilde{v}-\bar{v}\bar{\tilde{v}}) = p_3(v\bar{v}-\tilde{v}\bar{\tilde{v}}), \quad p_2(v\hat{v}-\bar{v}\bar{\hat{v}}) = p_3(v\bar{v}-\hat{v}\bar{\hat{v}}), \\
p_1(\bar{v}\bar{\tilde{v}}-\bar{\hat{v}}\bar{\hat{\tilde{v}}}) = p_2(\bar{v}\bar{\hat{v}}-\bar{\tilde{v}}\bar{\hat{\tilde{v}}}), \quad p_1(\hat{v}\hat{\tilde{v}}-\bar{\hat{v}}\bar{\hat{\tilde{v}}}) = p_3(\hat{v}\bar{\hat{v}}-\hat{\tilde{v}}\bar{\hat{\tilde{v}}}), \quad p_2(\tilde{v}\hat{\tilde{v}}-\bar{\tilde{v}}\bar{\hat{\tilde{v}}}) = p_3(\tilde{v}\bar{\tilde{v}}-\hat{\tilde{v}}\bar{\hat{\tilde{v}}}),
\end{gathered}$$

see Figure 1a. However, the discrete sine–Gordon equation has non-covariant flow variables n_1 and n_{-1} in it. If we consider a cube with regard to the flows n_1, n_2 and n_{-1}, it must have the discrete modified KdV equation on the bottom and top faces, namely

$$p_1(v\tilde{v}-\hat{v}\hat{\tilde{v}}) = p_2(v\hat{v}-\tilde{v}\hat{\tilde{v}}), \quad p_1(\check{v}\check{\tilde{v}}-\check{\hat{v}}\check{\hat{\tilde{v}}}) = p_2(\check{v}\check{\hat{v}}-\check{\tilde{v}}\check{\hat{\tilde{v}}}),$$



(A) MDC of the discrete modified KdV (B) MDC of the discrete sine–Gordon

FIGURE 1. Consistency around the cube

and the discrete sine–Gordon equation on the four side faces, i.e.

$$p_1p_{-1}(v\check{\tilde{v}} - \tilde{v}\check{v}) = v\tilde{v}\check{v}\check{\tilde{v}} - 1, \quad p_2p_{-1}(v\check{\hat{v}} - \hat{v}\check{v}) = v\hat{v}\check{v}\check{\hat{v}} - 1,$$
$$p_1p_{-1}(\hat{v}\check{\hat{\tilde{v}}} - \hat{\tilde{v}}\check{\hat{v}}) = \hat{v}\hat{\tilde{v}}\check{\hat{v}}\check{\hat{\tilde{v}}} - 1, \quad p_2p_{-1}(\tilde{v}\check{\hat{\tilde{v}}} - \hat{\tilde{v}}\check{\tilde{v}}) = \tilde{v}\hat{\tilde{v}}\check{\tilde{v}}\check{\hat{\tilde{v}}} - 1,$$

in order to make the cube closed, see Figure 1b.

CHAPTER 3

Discrete KP-type equations

1. Overview

3D integrable discrete equations are often considered as the most general models in discrete integrable systems theory – 2D and 1D discrete integrable systems can normally be obtained from dimensional reductions of 3D equations. In fact, the algebraic/solution structures behind 3D discrete equations are much richer and sometimes they provide us with insights into the study of discrete integrable systems, while on the 2D/1D level a lot of key information collapses and the integrability sometimes cannot be easily determined. Under the assumption of the MDC, there are three important scalar models of KP-type in the 3D theory, namely the discrete AKP, BKP and CKP equations, which are the discretisations of the famous continuous (potential) AKP, BKP and CKP equations:

$$4u_{xt} - 3u_{yy} - (u_{xxx} + 6u_x^2)_x = 0, \tag{AKP}$$

$$9u_{xt} - 5u_{yy} + (-5u_{xxy} - 15u_x u_y + u_{xxxxx} + 15u_x u_{xxx} + 15u_x^3)_x = 0, \tag{BKP}$$

$$9u_{xt} - 5u_{yy} + (-5u_{xxy} - 15u_x u_y + u_{xxxxx} + 15u_x u_{xxx} + 15u_x^3 + \tfrac{45}{4}u_{xx}^2)_x = 0. \tag{CKP}$$

Here the letters ‘A’, ‘B’ and ‘C’ refer to the different types of infinite-dimensional Lie algebras which are associated with their respective hierarchies following the work of the Kyoto School, cf. a review paper [**JM83**] and references therein for their original research papers. The discrete AKP equation was first given by Hirota [**Hir81**] in the discrete bilinear form and therefore it is also known as the Hirota equation (also referred to as the HM (Hirota–Miwa) equation due to Miwa’s reparametrisation [**Miw82**] for its soliton solution). The discrete AKP equation is related to other nonlinear forms which can be referred to as the discrete KP equation [**NCWQ84**, **NCW85**], the discrete modified KP (mKP) equation [**DJM82**, **NCW85**] as well as the discrete Schwarzian KP (SKP) equation – a discrete equation in the form of a multi-ratio (see [**DN91**]). The discrete BKP equation was derived by Miwa [**Miw82**] (therefore also referred to as the Miwa equation) as a four-term bilinear equation and its nonlinear form in terms of multi-ratios was later given by Nimmo and Schief [**NS97**] (see also [**NS99**]). The discrete CKP equation was obtained from the star-triangle transform in the Ising model by Kashaev [**Kas96**] based on the idea of the local Yang–Baxter equations [**MN89**]. It was named CKP by Schief [**Sch03**] who revealed that Kashaev’s discrete model is the superposition property of the continuous CKP equation.

We note that the MDC property of the HM equation and the Miwa equation, i.e. the discrete AKP and BKP equations, can be proven by direct computation, however, the MDC property of the discrete CKP equation (as an equation in multi-quadratic form) is highly nontrivial and it was confirmed in [**TW09**]. Alternatively, Atkinson established the

MDC property of the discrete CKP equation by using discriminant factorisation [**Atk12**]. The reductions of the discrete KP-type equations give rise to a large number of lower-dimensional integrable models (cf. e.g. [**HJN16**] and references therein). In particular, the ultradiscretisations of the reduced 2D integrable discrete models as well as Yang–Baxter maps can be obtained from them [**KNW09**, **KNW10**].

The discrete AKP, BKP and CKP equations have been considered from the perspective of the underlying geometry in several papers, cf. Konopelchenko and Schief [**KS02a**, **KS02b**, **Sch03**], Doliwa [**Dol07**, **Dol10b**, **Dol10a**] and also Bobenko and Schief [**BS15**, **BS17**]. We propose a unified framework for the solution structure of these equations. The latter will comprise the structure of soliton-type solutions and those related to nonlocal Riemann–Hilbert problems. As a particular by-product, we obtain novel soliton solutions to the discrete CKP equation. The approach we adopt is the DL method, which was proven very effective in establishing solution structures of many integrable equations and their interrelations.

In the DL approach, one needs to select suitable Cauchy kernel, measure and plane wave factors to study a class of equations. The kernel and the plane wave factors must obey a closure relation in the so-called direct linearsing transform theory, which guarantees the existence of a general class of solutions to the resulting nonlinear equations, see [**FN17a**]. In this chapter, we consider the simplest kernel, and the plane wave factors which are linear or fractionally linear with respect to the lattice parameters as well as the spectral variables. The choice of the measure, differently, is related to the associated algebra of a class of nonlinear equations. Here besides a general measure (for AKP), we only consider antisymmetric (for BKP) and symmetric (for CKP) measures.

The chapter is organised as follows: Sections 2, 3 and 4 are contributed to the discrete AKP, BKP and CKP equations which are corresponding to the three particular cases under the general framework of the DL. In Section 5, soliton solutions to the discrete AKP, BKP and CKP equations are given from the scheme.

2. Discrete AKP equation

2.1. Cauchy kernel and plane wave factors. In the discrete AKP family, we select the Cauchy kernel and the measure as follows:

$$\Omega_{k,k'} = \frac{1}{k+k'}, \quad \mathrm{d}\zeta(k,k') \quad \text{being arbitrary,} \tag{3.1}$$

and they together govern the nonlinear behaviour of the resulting integrable discrete equations. As a remark, we point out that the Cauchy kernel is not unique. For instance, one can also choose the kernel being $\Omega_{k,k'} = \frac{k}{k+k'}$ or $\Omega_{k,k'} = \frac{k'}{k+k'}$. Such choices do not change the nonlinear structure, and the crucial point is that singularities (k and k' in the denominator) must be involved in the kernel. We also comment that in the discrete AKP family it is necessary to have the measure $\mathrm{d}\zeta(k,k')$ being arbitrary. Any restriction on the measure amounts to considering a sub-solution space of the discrete AKP family, resulting in reductions of integrable equations.

We choose plane wave factors ρ_k and $\sigma_{k'}$ in the discrete AKP family as follows:

$$\rho_k = \prod_{j=1}^{\infty} (p_j + k)^{n_j}, \quad \sigma_{k'} = \prod_{j=1}^{\infty} (p_j - k')^{-n_j},$$

In the plane wave factors, n_j are the discrete independent variables, and k and k' are the two spectral parameters. In concrete analysis, we often consider the multiplication of the two plane wave factors, namely

$$\rho_k \sigma_{k'} = \prod_{j=1}^{\infty} \left(\frac{p_j + k}{p_j - k'} \right)^{n_j}, \tag{3.2}$$

which is called the effective plane wave factor and describes the linear structure (i.e. the dispersions) of nonlinear integrable equations in the discrete AKP family.

Based on the above information, we are going to construct all the possible nonlinear equations in the discrete AKP family. The idea of constructing closed-form equations in the DL framework is transferring discrete shifts to algebraic relations of the entries of the infinite-dimensional matrix $\mathbf{U}$.

2.2. Infinite-dimensional matrix structure. The Cauchy kernel given in (3.1) obeys the relation

$$\Omega_{k,k'} k + k' \Omega_{k,k'} = 1.$$

Thus, considering the infinite-dimensional matrix representation of such a relation with the help of (2.16), we obtain its infinite-dimensional matrix analogue

$$\mathbf{\Omega} \cdot \mathbf{\Lambda} + {}^t\mathbf{\Lambda} \cdot \mathbf{\Omega} = \mathbf{O}. \tag{3.3}$$

In fact multiplying (3.3) by ${}^t\mathbf{c}_{k'}$ from the left and multiplying it by $\mathbf{c}_k$ from the right simultaneously, we immediately recover the previous relation for $\Omega_{k,k'}$. Based on (3.3), we can easily derive the following relations of $\mathbf{\Omega}$ with respect to the time evolutions.

PROPOSITION 3.1. *The infinite-dimensional matrix* $\mathbf{\Omega}$ *obeys the following time evolutions:*

$$\mathbf{\Omega} \cdot (p_j + \mathbf{\Lambda}) - (p_j - {}^t\mathbf{\Lambda}) \cdot \mathbf{\Omega} = \mathbf{O}. \tag{3.4}$$

We now consider the dynamics of $\mathbf{C}$. Notice that $\mathbf{C}$ is defined by (2.17), in which the plane wave factors for the discrete AKP family are given by (3.2). We can replace discrete shift operations on $\mathbf{C}$ by actions of the index-raising operators $\mathbf{\Lambda}$ and ${}^t\mathbf{\Lambda}$.

PROPOSITION 3.2. *The infinite-dimensional matrix* $\mathbf{C}$ *satisfies the following the discrete dynamical evolutions:*

$$\mathrm{T}_{p_j} \mathbf{C} \cdot (p_j - {}^t\mathbf{\Lambda}) = (p_j + \mathbf{\Lambda}) \cdot \mathbf{C}. \tag{3.5}$$

PROOF. We act T_{p_j} on $\mathbf{C}$ and obtain

$$\begin{aligned} \mathrm{T}_{p_j} \mathbf{C} &= \iint_D \mathrm{d}\zeta(k,k') \frac{p_j + k}{p_j - k'} \rho_k \mathbf{c}_k {}^t\mathbf{c}_{k'} \sigma_{k'} \\ &= \iint_D \mathrm{d}\zeta(k,k') \rho_k (p_j + \mathbf{\Lambda}) \cdot \mathbf{c}_k {}^t\mathbf{c}_{k'} \cdot (p_j - {}^t\mathbf{\Lambda})^{-1} \sigma_{k'} = (p + \mathbf{\Lambda}) \cdot \mathbf{C} \cdot (p - {}^t\mathbf{\Lambda})^{-1}, \end{aligned}$$

with the help of the property of $\mathbf{c}_k$ and ${}^t\mathbf{c}_{k'}$, which is exactly the first relation. $\square$

The relations given in propositions 3.1 and 3.2 are the fundamental infinite-dimensional matrix relations in the AKP family. They describe the linear and nonlinear structures in the infinite-dimensional matrix formalism, respectively.

2.3. Discrete nonlinear equations and Miura transforms. In this class, by choosing different potential variables, we show that there exist different forms of the discrete KP, including the discrete (unmodified) KP equation, the discrete modified KP equation, the discrete Schwarzian KP equation, as well as the famous HM equation.

As we can observe from the structure of the effective plane wave factor (3.2) that all the discrete variables n_j are on the same footing, without losing generality we only consider closed-form equations in terms of the discrete variables n_1, n_2 and n_3. The equations in terms of other independent variables take the same form with regard to the lattice parameters.

The discrete KP equations are derived from the dynamical relations of the infinite-dimensional matrix $\mathbf{U}$.

PROPOSITION 3.3. *The infinite-dimensional matrix* $\mathbf{U}$ *in the discrete AKP family obeys the dynamical evolutions*

$$\mathrm{T}_{p_j}\mathbf{U}\cdot(p_j - {}^t\mathbf{\Lambda}) = (p_j + \mathbf{\Lambda})\cdot\mathbf{U} - \mathrm{T}_{p_j}\mathbf{U}\cdot\mathbf{O}\cdot\mathbf{U}, \tag{3.6}$$

where j is any arbitrary positive integer.

PROOF. Acting T_{p_j} on $\mathbf{U}$ in (2.20) and right-multiplying it by $(p_j - {}^t\mathbf{\Lambda})$ gives rise to

$$\begin{aligned}\mathrm{T}_{p_j}\mathbf{U}\cdot(p_j - {}^t\mathbf{\Lambda}) &= (1 - \mathrm{T}_{p_j}\mathbf{U}\cdot\mathbf{\Omega})\cdot\mathrm{T}_{p_j}\mathbf{C}\cdot(p_j - {}^t\mathbf{\Lambda}) \\ &= (1 - \mathrm{T}_{p_j}\mathbf{U}\cdot\mathbf{\Omega})\cdot(p_j + \mathbf{\Lambda})\cdot\mathbf{C}\end{aligned}$$

in which the dynamical evolution of $\mathbf{C}$ is used. By using the dynamical relation for $\mathbf{\Omega}$, the right hand side can be written as

$$\begin{aligned}&(p_j + \mathbf{\Lambda})\cdot\mathbf{C} - \mathrm{T}_{p_j}\mathbf{U}\cdot\mathbf{\Omega}\cdot(p_j + \mathbf{\Lambda})\cdot\mathbf{C} \\ &= (p_j + \mathbf{\Lambda})\cdot\mathbf{C} - \mathrm{T}_{p_j}\mathbf{U}\cdot(\mathbf{O} + (p_j - {}^t\mathbf{\Lambda})\cdot\mathbf{\Omega})\cdot\mathbf{C} \\ &= (p_j + \mathbf{\Lambda})\cdot\mathbf{C} - \mathrm{T}_{p_j}\mathbf{U}\cdot\mathbf{O}\cdot\mathbf{C} - \mathrm{T}_{p_j}\mathbf{U}\cdot(p_j - {}^t\mathbf{\Lambda})\cdot\mathbf{\Omega}\cdot\mathbf{C}.\end{aligned}$$

Thus, we derive

$$(\mathrm{T}_{p_j}\mathbf{U})\cdot(p_j - {}^t\mathbf{\Lambda})\cdot(1 + \mathbf{\Omega}\cdot\mathbf{C}) = (p_j + \mathbf{\Lambda})\cdot\mathbf{C} - \mathrm{T}_{p_j}\mathbf{U}\cdot\mathbf{O}\cdot\mathbf{C},$$

and consequently the first relation if we multiply the above equation by $(1+\mathbf{\Omega}\cdot\mathbf{C})^{-1}$ from the right. $\square$

In order to construct the possible integrable lattice equations. We first of all introduce the following variables:

$$u = U_{0,0}, \quad v = 1 - U_{0,-1}, \quad w = 1 - U_{-1,0}, \quad z = U_{-1,-1} - \sum_{j\in\mathbb{Z}^+}\frac{n_j}{p_j}.$$

We consider the dynamical relations for the infinite-dimensional matrix $\mathbf{U}$ given in proposition 3.3 and list only the ones with respect to n_1, n_2 and n_3 below:

$$\tilde{\mathbf{U}} \cdot (p_1 - {}^t\mathbf{\Lambda}) = (p_1 + \mathbf{\Lambda}) \cdot \mathbf{U} - \tilde{\mathbf{U}} \cdot \mathbf{O} \cdot \mathbf{U}, \tag{3.7a}$$

$$\hat{\mathbf{U}} \cdot (p_2 - {}^t\mathbf{\Lambda}) = (p_2 + \mathbf{\Lambda}) \cdot \mathbf{U} - \hat{\mathbf{U}} \cdot \mathbf{O} \cdot \mathbf{U}, \tag{3.7b}$$

$$\bar{\mathbf{U}} \cdot (p_3 - {}^t\mathbf{\Lambda}) = (p_3 + \mathbf{\Lambda}) \cdot \mathbf{U} - \bar{\mathbf{U}} \cdot \mathbf{O} \cdot \mathbf{U}, \tag{3.7c}$$

which amount to (by taking the (i,j)-entry)

$$p_1\tilde{U}_{i,j} - \tilde{U}_{i,j+1} = p_1 U_{i,j} - U_{i+1,j} - \tilde{U}_{i,0}U_{0,j}, \tag{3.8a}$$

$$p_2\hat{U}_{i,j} - \hat{U}_{i,j+1} = p_2 U_{i,j} - U_{i+1,j} - \hat{U}_{i,0}U_{0,j}, \tag{3.8b}$$

$$p_3\bar{U}_{i,j} - \bar{U}_{i,j+1} = p_3 U_{i,j} - U_{i+1,j} - \bar{U}_{i,0}U_{0,j}. \tag{3.8c}$$

We now set $i = j = 0$ in (3.8), and obtain

$$p_1(\tilde{u} - u) + \tilde{u}u = \tilde{U}_{0,1} + U_{1,0},$$
$$p_2(\hat{u} - u) + \hat{u}u = \hat{U}_{0,1} + U_{1,0},$$
$$p_3(\bar{u} - u) + \bar{u}u = \bar{U}_{0,1} + U_{1,0}.$$

By eliminating $U_{1,0}$ and $U_{0,1}$, a closed-form equation is derived, which takes the form of

$$(p_1 - \tilde{u})(p_2 - p_3 + \tilde{\bar{u}} - \hat{\tilde{u}}) + (p_2 - \hat{u})(p_3 - p_1 + \hat{\tilde{u}} - \bar{\hat{u}}) + (p_3 - \bar{u})(p_1 - p_2 - \bar{\hat{u}} - \tilde{\bar{u}}) = 0.$$

This is the discrete (unmodified) KP equation.

Next, we take $i = 0, j = -1$ in (3.8) and obtain

$$p_1(v - \tilde{v}) = \tilde{u}v - U_{1,-1}, \quad p_2(v - \hat{v}) = \hat{u}v - U_{1,-1}, \quad p_3(v - \bar{v}) = \bar{u}v - U_{1,-1}.$$

We can now eliminate $U_{1,-1}$ and obtain the following relations between u and v:

$$p_1 - p_2 + \hat{u} - \tilde{u} = \frac{p_1\tilde{v} - p_2\hat{v}}{v}, \tag{3.9a}$$

$$p_2 - p_3 + \bar{u} - \hat{u} = \frac{p_2\hat{v} - p_3\bar{v}}{v}, \tag{3.9b}$$

$$p_3 - p_1 + \tilde{u} - \bar{u} = \frac{p_3\bar{v} - p_1\tilde{v}}{v}. \tag{3.9c}$$

The above relation is the Miura transform between the discrete KP equation and the discrete modified KP equation. We can further eliminate u in the Miura transforms and derive a closed-form scalar equation for v:

$$p_1\left(\frac{\hat{\tilde{v}}}{\hat{v}} - \frac{\tilde{\bar{v}}}{\bar{v}}\right) + p_2\left(\frac{\bar{\hat{v}}}{\bar{v}} - \frac{\hat{\tilde{v}}}{\tilde{v}}\right) + p_3\left(\frac{\tilde{\bar{v}}}{\tilde{v}} - \frac{\bar{\hat{v}}}{\hat{v}}\right) = 0,$$

which is the discrete modified KP equation.

We also consider the $i = j = -1$ case of (3.8). In this case, the equations in (3.8) turn out to be

$$p_1(z - \tilde{z}) = v\tilde{w}, \quad p_2(z - \hat{z}) = v\hat{w}, \quad p_3(z - \bar{z}) = v\bar{w}.$$

Making use of the identity $(v\tilde{w})\hat{}(v\hat{w})\bar{}(v\bar{w})\tilde{} = (v\tilde{w})\bar{}(v\hat{w})\tilde{}(v\bar{w})\hat{}$, we obtain a closed-form equation for z, namely the discrete Schwarzian KP equation

$$\frac{(\hat{\tilde{z}} - \hat{z})(\bar{\hat{z}} - \bar{z})(\tilde{\bar{z}} - \tilde{z})}{(\hat{\tilde{z}} - \tilde{z})(\bar{\hat{z}} - \hat{z})(\tilde{\bar{z}} - \bar{z})} = 1.$$

We can also obtain the Miura transform between the modified KP equation and the discrete Schwarzian KP equation as follows:

$$\frac{\hat{v}}{\tilde{v}} = \frac{p_1}{p_2}\frac{\hat{z} - \hat{\tilde{z}}}{\tilde{z} - \hat{\tilde{z}}}, \qquad \frac{\bar{v}}{\tilde{v}} = \frac{p_2}{p_3}\frac{\bar{z} - \bar{\hat{z}}}{\hat{z} - \bar{\hat{z}}}, \qquad \frac{\tilde{v}}{\bar{v}} = \frac{p_3}{p_1}\frac{\tilde{z} - \tilde{\bar{z}}}{\bar{z} - \tilde{\bar{z}}}. \tag{3.10a}$$

Due to the covariance of the discrete variables n_j in the effective plane wave factor (5.12), the discrete KP, modified KP and Schwarzian KP equations are all multidimensionally consistent. We conclude the results in this subsection in the following theorem.

THEOREM 3.4. *In the discrete KP family, there exist multidimensionally consistent equations including the discrete KP equation, the discrete modified KP equation and the discrete Schwarzian KP equation as follows:*

$$\begin{aligned}
&(p_1 - \tilde{u})(p_2 - p_3 + \tilde{\bar{u}} - \hat{\tilde{u}}) \\
&\quad + (p_2 - \hat{u})(p_3 - p_1 + \hat{\tilde{u}} - \bar{\hat{u}}) + (p_3 - \bar{u})(p_1 - p_2 - \bar{\hat{u}} - \tilde{\bar{u}}) = 0,
\end{aligned} \tag{3.11a}$$

$$p_1\left(\frac{\hat{\tilde{v}}}{\hat{v}} - \frac{\tilde{\bar{v}}}{\bar{v}}\right) + p_2\left(\frac{\bar{\hat{v}}}{\bar{v}} - \frac{\hat{\tilde{v}}}{\tilde{v}}\right) + p_3\left(\frac{\tilde{\bar{v}}}{\tilde{v}} - \frac{\bar{\hat{v}}}{\hat{v}}\right) = 0, \tag{3.11b}$$

$$\frac{(\hat{\tilde{z}} - \hat{z})(\bar{\hat{z}} - \bar{z})(\tilde{\bar{z}} - \tilde{z})}{(\hat{\tilde{z}} - \tilde{z})(\bar{\hat{z}} - \hat{z})(\tilde{\bar{z}} - \bar{z})} = 1. \tag{3.11c}$$

2.4. The HM equation. Next, we give another equation in the discrete KP family, which can be thought of as the fundamental model in the KP family. For convenience, we introduce a new variable

$$V_a = 1 - \left(\mathbf{U} \cdot \frac{1}{a + {}^t\mathbf{\Lambda}}\right)_{0,0}.$$

We consider the dynamical relations of the tau function.

PROPOSITION 3.5. *The tau function in the discrete KP class obeys*

$$\frac{\mathrm{T}_{p_j}\tau}{\tau} = V_{-p_j},$$

PROOF. This is proven directly. Direct calculation shows that

$$\begin{aligned}
\mathrm{T}_{p_j}\tau &= \mathrm{T}_{p_j}\det(1 + \mathbf{\Omega}\cdot\mathbf{C}) = \det(1 + \mathbf{\Omega}\cdot\mathrm{T}_{p_j}\mathbf{C}) \\
&= \det\left(1 + \mathbf{\Omega}\cdot(p_j + \mathbf{\Lambda})\cdot\mathbf{C}\cdot(p_j - {}^t\mathbf{\Lambda})^{-1}\right) \\
&= \det\left(1 + (\mathbf{O} + (p_j - {}^t\mathbf{\Lambda})\cdot\mathbf{\Omega})\cdot\mathbf{C}\cdot(p_j - {}^t\mathbf{\Lambda})^{-1}\right) \\
&= \det\left(1 + \mathbf{\Omega}\cdot\mathbf{C} + (p_j - {}^t\mathbf{\Lambda})^{-1}\cdot\mathbf{O}.\mathbf{C}\right) \\
&= \det(1 + \mathbf{\Omega}\cdot\mathbf{C})\det\left(1 + (p_j - {}^t\mathbf{\Lambda})^{-1}\mathbf{O}\cdot\mathbf{C}(1 + \mathbf{\Omega}\cdot\mathbf{C})^{-1}\right) \\
&= \tau\left(1 + (p_j - {}^t\mathbf{\Lambda})^{-1}\mathbf{O}\cdot\mathbf{U}\right)_{0,0},
\end{aligned}$$

where we used the rank 1 Weinstein–Aronszajn formula. ☐

Now we move onto the derivation of the HM equation. Multiplying the first equation (3.7) by $\frac{1}{a+{}^t\boldsymbol{\Lambda}}$ from the right and taking its $(0,0)$-entry, we derive

$$(p_1+a)(1-\tilde{V}_a)=p_1(1-V_a)+\left(\boldsymbol{\Lambda}\cdot\mathbf{U}\cdot\frac{1}{a+{}^t\boldsymbol{\Lambda}}\right)_{0,0}+\tilde{u}V_a.$$

Similarly, we also have

$$(p_2+a)(1-\hat{V}_a)=p_2(1-V_a)+\left(\boldsymbol{\Lambda}\cdot\mathbf{U}\cdot\frac{1}{a+{}^t\boldsymbol{\Lambda}}\right)+\hat{u}V_a$$

from the second equation in (3.7). By subtraction and setting $a=-p_1$, we obtain

$$p_1-p_2+\hat{u}+\tilde{u}=(p_1-p_2)\frac{\hat{V}_{-p}}{V_{-p}}.$$

Replacing V_{-p_1} by the expression of the tau function therefore gives rise to the bilinear transform between u and τ given by

$$p_1-p_2+\hat{u}-\tilde{u}=(p_1-p_2)\frac{\tau\hat{\tilde{\tau}}}{\hat{\tau}\tilde{\tau}}. \tag{3.12a}$$

Similarly, the relations with regard to the other two lattice directions can also be derived, namely

$$p_2-p_3+\bar{u}-\hat{u}=(p_2-p_3)\frac{\tau\bar{\hat{\tau}}}{\bar{\tau}\hat{\tau}}, \tag{3.12b}$$

$$p_3-p_1+\tilde{u}-\bar{u}=(p_3-p_1)\frac{\tau\tilde{\bar{\tau}}}{\tilde{\tau}\bar{\tau}}. \tag{3.12c}$$

Now we can eliminating u in (3.12) by adding the three equations up. As a result, a scalar closed-form equation for the tau function arises.

Theorem 3.6. *The tau function obeys a multidimensionally consistent hierarchy of bilinear discrete KP equations including*

$$(p_1-p_2)\bar{\tau}\hat{\tilde{\tau}}+(p_2-p_3)\tilde{\tau}\bar{\hat{\tau}}+(p_3-p_1)\hat{\tau}\tilde{\bar{\tau}}=0, \tag{3.13}$$

namely the HM equation.

2.5. Lax triplet. Finally we consider the Lax pair for the discrete equations in the above two subsections. The relations we need for the discrete KP equation are the following:

Proposition 3.7. *The infinite-dimensional vector* $\mathbf{u}_k$ *obeys the dynamical evolutions with respect to* n_j *as follows:*

$$\mathrm{T}_{p_j}\mathbf{u}_k=(p_j+\boldsymbol{\Lambda})\cdot\mathbf{u}_k-(\mathrm{T}_{p_j}\mathbf{U})\cdot\mathbf{O}\cdot\mathbf{u}_k, \tag{3.14}$$

where $j=1,2,\cdots$.

PROOF. Shifting the vector $\mathbf{u}_k$ with respect to n_j, we have

$$\begin{aligned}
\mathrm{T}_{p_j}\mathbf{u}_k &= (1-\mathrm{T}_{p_j}\mathbf{U}\cdot\mathbf{\Omega})\cdot\mathrm{T}_{p_j}\rho_k\mathbf{c}_k = (1-\mathrm{T}_{p_j}\mathbf{U}\cdot\mathbf{\Omega})\cdot(p_j+\mathbf{\Lambda})\cdot\rho_k\mathbf{c}_k\\
&= (p_j+\mathbf{\Lambda})\cdot\rho_k\mathbf{c}_k - \mathrm{T}_{p_j}\mathbf{U}\cdot\mathbf{\Omega}\cdot(p_j+\mathbf{\Lambda})\cdot\rho_k\mathbf{c}_k\\
&= (p_j+\mathbf{\Lambda})\cdot\rho_k\mathbf{c}_k - \mathrm{T}_{p_j}\mathbf{U}\cdot\big(\mathbf{O}+(p_j-{}^t\mathbf{\Lambda})\cdot\mathbf{\Omega}\big)\cdot\rho_k\mathbf{c}_k\\
&= (p_j+\mathbf{\Lambda})\cdot\rho_k\mathbf{c}_k - \mathrm{T}_{p_j}\mathbf{U}\cdot\mathbf{O}\cdot\rho_k\mathbf{c}_k + \big((p_j+\mathbf{\Lambda})\cdot\mathbf{U}-(\mathrm{T}_{p_j}\mathbf{U})\cdot\mathbf{O}\cdot\mathbf{U}\big)\cdot\mathbf{\Omega}\cdot\rho_k\mathbf{c}_k\\
&= (p_j+\mathbf{\Lambda})\cdot\mathbf{u}_k - (\mathrm{T}_{p_j}\mathbf{U})\cdot\mathbf{O}\cdot\mathbf{u}_k.
\end{aligned}$$

□

Without losing generality, we consider equations with regard to the lattice parameters p_1, p_2 and p_3 as follows:

$$\tilde{\mathbf{u}}_k = (p_1+\mathbf{\Lambda})\cdot\mathbf{u}_k - \tilde{\mathbf{U}}\cdot\mathbf{O}\cdot\mathbf{u}_k, \tag{3.15a}$$

$$\hat{\mathbf{u}}_k = (p_2+\mathbf{\Lambda})\cdot\mathbf{u}_k - \hat{\mathbf{U}}\cdot\mathbf{O}\cdot\mathbf{u}_k, \tag{3.15b}$$

$$\bar{\mathbf{u}}_k = (p_3+\mathbf{\Lambda})\cdot\mathbf{u}_k - \bar{\mathbf{U}}\cdot\mathbf{O}\cdot\mathbf{u}_k. \tag{3.15c}$$

We define the wave function $\phi = u_k^{(0)}$. By subtraction and taking the 0th component, we obtain the Lax triplet of the discrete KP equation (3.13).

THEOREM 3.8. *The discrete KP equation* (3.13) *has its Lax triplet representation as follows:*

$$\tilde{\phi}-\hat{\phi} = (p_1-p_2+\hat{u}-\tilde{u})\phi, \tag{3.16a}$$

$$\hat{\phi}-\bar{\phi} = (p_2-p_3+\bar{u}-\hat{u})\phi, \tag{3.16b}$$

$$\bar{\phi}-\tilde{\phi} = (p_3-p_1+\tilde{u}-\bar{u})\phi. \tag{3.16c}$$

The Lax triplets of other equations in this family can be obtained by replacing u by the corresponding dependent variable, via (3.9), (3.10) *and* (3.12).

Furthermore, we can eliminate u in the Lax triplet, and this results in a closed-form equation for ϕ:

$$\frac{\hat{\tilde{\phi}}-\bar{\tilde{\phi}}}{\hat{\phi}} + \frac{\bar{\hat{\phi}}-\tilde{\hat{\phi}}}{\bar{\phi}} + \frac{\tilde{\bar{\phi}}-\hat{\bar{\phi}}}{\tilde{\phi}} = 0.$$

This equation equation is the discrete modified KP equation (compare (3.11b)), which implies the wave function solves the modified equation.

2.6. Nonpotential discrete KP equation. In the discrete KP, mKP equations u and v are both potential variables, we would like to note that there also exists a non-potential form of the discrete KP equation. The discrete KP equation can be written in an alternative form which reads

$$\frac{(q-r+\bar{u}-\hat{u})\tilde{}}{q-r+\bar{u}-\hat{u}} = \frac{(r-p+\tilde{u}-\bar{u})\hat{}}{r-p+\tilde{u}-\bar{u}} = \frac{(p-q+\hat{u}-\tilde{u})\bar{}}{p-q+\hat{u}-\tilde{u}}. \tag{3.17}$$

Owing to the above form of the discrete KP equation, it is very natural for us to introduce the non-potential variables

$$P = q - r + \bar{u} - \hat{u}, \quad Q = r - p + \tilde{u} - \bar{u}, \quad R = p - q + \hat{u} - \tilde{u}, \tag{3.18}$$

and consequently a coupled system of P, Q and R can be derived from (3.17), i.e.

$$\frac{\tilde{P}}{P} = \frac{\hat{Q}}{Q} = \frac{\bar{R}}{R}, \quad P + Q + R = 0, \quad \tilde{P} + \hat{Q} + \bar{R} = 0, \tag{3.19}$$

where the first equation is Equation (3.17) itself and the other two follow from the definitions of P, Q and R directly. The system (3.19) can be treated as the discrete non-potential KP equation and it was proposed by Nimmo in [**Nim06**]. Now for a scalar form, we first eliminate R and obtain $\tilde{P}/P = \hat{Q}/Q, \bar{P} + \bar{Q} = \tilde{P} + \hat{Q}$, and then one can have the expression of P in terms of Q and consequently the expression of R in terms of Q:

$$P = \frac{Q\tilde{Q}(\bar{Q}(\hat{\tilde{Q}} - \tilde{\bar{Q}}) + \bar{\hat{Q}}(\bar{Q} - \hat{Q}))}{\hat{Q}(\tilde{Q}\bar{\hat{Q}} - \bar{Q}\hat{\tilde{Q}})}, \quad R = \frac{Q\bar{Q}(\tilde{Q}(\tilde{\bar{Q}} - \bar{\hat{Q}}) + \hat{\tilde{Q}}(\hat{Q} - \bar{Q}))}{\hat{Q}(\tilde{Q}\bar{\hat{Q}} - \bar{Q}\hat{\tilde{Q}})}.$$

Now if we express everything in Q, the system turns out to be a 10-point scalar discrete equation in the form of

$$\begin{aligned} &\tilde{Q}\tilde{\tilde{Q}}\hat{\tilde{Q}}\bar{\hat{\tilde{Q}}}\bar{\hat{Q}} - \tilde{\tilde{Q}}\hat{\tilde{Q}}\bar{\hat{\tilde{Q}}}\hat{Q}\bar{\hat{Q}} + \tilde{\tilde{Q}}\hat{\tilde{Q}}\bar{\hat{\tilde{Q}}}\bar{\hat{Q}}\bar{Q} - \tilde{Q}\tilde{\tilde{Q}}\hat{\tilde{\tilde{Q}}}\bar{\hat{Q}}\tilde{\bar{Q}} - \tilde{Q}\tilde{\tilde{Q}}\bar{\hat{\tilde{Q}}}\bar{\hat{Q}}\tilde{\bar{Q}} + \hat{\tilde{\tilde{Q}}}\hat{\tilde{Q}}\hat{Q}\bar{\hat{Q}}\tilde{\bar{Q}} \\ &+ \tilde{\tilde{Q}}\hat{\tilde{\tilde{Q}}}\hat{\tilde{Q}}\bar{Q}\tilde{\bar{Q}} - \hat{\tilde{\tilde{Q}}}\hat{\tilde{Q}}^2\bar{Q}\tilde{\bar{Q}} - \hat{\tilde{\tilde{Q}}}\hat{\tilde{Q}}\bar{\hat{Q}}\bar{Q}\tilde{\bar{Q}} + \hat{\tilde{\tilde{Q}}}\hat{\tilde{Q}}\bar{Q}\tilde{\bar{Q}}^2 + \tilde{Q}\tilde{\tilde{Q}}\bar{\hat{Q}}\tilde{\bar{Q}}\tilde{\tilde{\bar{Q}}} - \tilde{\tilde{Q}}\hat{\tilde{Q}}\bar{Q}\tilde{\bar{Q}}\tilde{\tilde{\bar{Q}}} = 0. \end{aligned} \tag{3.20}$$

It is this quintic equation which we refer to as the non-potential discrete KP equation and this scalar form, to the best of the author's knowledge, has not been given elsewhere. The Lax triplet of the equation can be obtained from (3.16) directly if we consider (3.18) and the expressions of P and R in terms of Q given above. We note that a different nonlinear form of the non-potential discrete KP equation was given in [**GRP$^+$07**] in which the equation is quartic but has 14 terms (compared to our quintic equation having 12 terms).

Before we move onto the discrete BKP and CKP equations, we would like to note that in the KP class there exist several nonlinear integrable equations which are connected with each other through Miura-type transforms. In fact they are the same equation in different forms because the core part of the scheme, namely the form of the kernel and the plane wave factors, has been fixed at very beginning. All these equations share the same bilinear structure, namely, the HM equation. In order to understand and compare equations in different classes, we need a uniformising quantity to characterise the algebraic structure behind the 3D discrete equations. The tau function could be a very strong candidate as it only contains the key information of an integrable system, namely, the Cauchy kernel and the plane wave factors. Therefore, below in this chapter we only mean the equation of the tau function, i.e. the HM equation (3.13), by the discrete AKP equation. While the other forms of the discrete AKP equation, namely the equations in u, v, and z can all be expressed by the tau function. In the next sections, we will mainly be concentrating on the tau function to understand the structure of the discrete BKP and CKP equations. Furthermore, we also note that the τ-equation does not necessarily need to be bilinear although the terminology was from the bilinear theory.

3. Discrete BKP equation

3.1. Infinite-dimensional matrix structure. For the discrete BKP equation, we choose the Cauchy kernel

$$\Omega_{k,k'} = \frac{1}{2}\frac{k-k'}{k+k'}. \tag{3.21}$$

which is antisymmetric in terms of the spectral variables k and k', namely $\Omega_{k',k} = -\Omega_{k,k'}$. This describes the nonlinear structure of the discrete BKP equation. Meanwhile, we also require that in this case the measure in (2.15), (2.20) and (2.17) obeys the antisymmetry property

$$\iint_D \cdot \mathrm{d}\zeta(k',k) = -\iint_D \cdot \mathrm{d}\zeta(k,k').$$

Compared with the discrete AKP equation, we can see that in the discrete BKP equation, the space of the spectral variables is smaller.

We also select the plane wave factors ρ_k and $\sigma_{k'}$ as follows:

$$\rho_k = \left(\frac{p_1+k}{p_1-k}\right)^{n_1}\left(\frac{p_2+k}{p_2-k}\right)^{n_2}\left(\frac{p_3+k}{p_3-k}\right)^{n_3},$$
$$\sigma_{k'} = \left(\frac{p_1+k'}{p_1-k'}\right)^{n_1}\left(\frac{p_2+k'}{p_2-k'}\right)^{n_2}\left(\frac{p_3+k'}{p_3-k'}\right)^{n_3},$$

which govern the linear structure of the discrete BKP equation. And thus, the effective plane wave factor is given by

$$\rho_k\sigma_{k'} = \left(\frac{p_1+k}{p_1-k}\frac{p_1+k'}{p_1-k'}\right)^{n_1}\left(\frac{p_2+k}{p_2-k}\frac{p_2+k'}{p_2-k'}\right)^{n_2}\left(\frac{p_3+k}{p_3-k}\frac{p_3+k'}{p_3-k'}\right)^{n_3}. \tag{3.22}$$

We can now write down the corresponding infinite-dimensional matrix relations for $\mathbf{\Omega}$ and $\mathbf{C}$ based on the Cauchy kernel and the plane wave factors. The operator $\mathbf{O}$ satisfies

$$\mathbf{\Omega}\cdot\mathbf{\Lambda} + {}^{t}\mathbf{\Lambda}\cdot\mathbf{\Omega} = \frac{1}{2}(\mathbf{\Omega}\cdot\mathbf{\Lambda} - {}^{t}\mathbf{\Lambda}\cdot\mathbf{O}), \tag{3.23}$$

which follows from the definition (2.16) and (3.21). Equation (3.23) immediately leads to the following property of $\mathbf{\Omega}$.

PROPOSITION 3.9. *The operator $\mathbf{\Omega}$ in the discrete BKP equation obeys the dynamical relations*

$$\mathbf{\Omega}\cdot\frac{p_j+\mathbf{\Lambda}}{p_j-\mathbf{\Lambda}} - \frac{p_j-{}^{t}\mathbf{\Lambda}}{p_j+{}^{t}\mathbf{\Lambda}}\cdot\mathbf{\Omega} = p_j\frac{1}{p_j+{}^{t}\mathbf{\Lambda}}\cdot(\mathbf{O}\cdot\mathbf{\Lambda} - {}^{t}\mathbf{\Lambda}\cdot\mathbf{O})\cdot\frac{1}{p_j-\mathbf{\Lambda}}. \tag{3.24}$$

and the antisymmetry property ${}^{t}\mathbf{\Omega} = -\mathbf{\Omega}$.

The effective plane wave factor (3.22) results in the infinite-dimensional matrix representation as follows.

PROPOSITION 3.10. *The infinite-dimensional matrix $\mathbf{C}$ obeys the dynamical relations*

$$\mathrm{T}_{p_j}\mathbf{C}\cdot\frac{p_j-{}^{t}\mathbf{\Lambda}}{p_j+{}^{t}\mathbf{\Lambda}} = \frac{p_j+\mathbf{\Lambda}}{p_j-\mathbf{\Lambda}}\cdot\mathbf{C}, \tag{3.25}$$

and also the antisymmetry property ${}^{t}\mathbf{C} = -\mathbf{C}$.

PROOF. The proof of the dynamical relation is similar to that of proposition 3.2 and we skip it. Notice that $\sigma_{k'} = \rho_{k'}$. Direct calculation shows that

$$ {}^t\mathbf{C} = \iint_D \mathrm{d}\zeta(k,k')\rho_{k'}\mathbf{c}_{k'}{}^t\mathbf{c}_k\rho_k = -\iint_D \mathrm{d}\zeta(k',k)\rho_{k'}\mathbf{c}_{k'}{}^t\mathbf{c}_k\rho_k = \mathbf{C}, $$

where we have used $\iint_D \cdot\mathrm{d}\zeta(k',k) = -\iint_D \cdot\mathrm{d}\zeta(k,k')$. □

3.2. The Miwa equation. We can now construct a closed-form equation of the tau function. To derive this equation, we need to build up the dynamical relations for the infinite-dimensional matrix $\mathbf{U}$.

PROPOSITION 3.11. *The infinite-dimensional matrix* $\mathbf{U}$ *satisfies*

$$ \mathrm{T}_{p_j}\mathbf{U}\cdot\frac{p_j - {}^t\mathbf{\Lambda}}{p_j + {}^t\mathbf{\Lambda}} = \frac{p_j + \mathbf{\Lambda}}{p_j - \mathbf{\Lambda}}\cdot\mathbf{U} - p_j\mathrm{T}_{p_j}\mathbf{U}\cdot\frac{1}{p_j + {}^t\mathbf{\Lambda}}\cdot(\mathbf{O}\cdot\mathbf{\Lambda} - {}^t\mathbf{\Lambda}\cdot\mathbf{O})\cdot\frac{1}{p_j - \mathbf{\Lambda}}\cdot\mathbf{U}, \quad (3.26) $$

and the antisymmetry property ${}^t\mathbf{U} = -\mathbf{U}$.

PROOF. The proof of the dynamical relations is similar to proposition 3.3 and we skip it. Below we give the proof of the antisymmetry property. We can observe from (2.20) that $\mathbf{U} = (\mathbf{C}^{-1} + \mathbf{\Omega})^{-1}$. Therefore, $\mathbf{U}$ being antisymmetric is a consequence of the antisymmetry property of $\mathbf{\Omega}$ and $\mathbf{C}$. □

For future convenience, we here introduce some new variables as follows:

$$ V_a = 1 - \left(\mathbf{U}\cdot\frac{a}{a - {}^t\mathbf{\Lambda}}\right)_{0,0} = 1 + \left(\frac{a}{a-\mathbf{\Lambda}}\cdot\mathbf{U}\right)_{0,0} = W_a, $$
$$ S_{a,b} = \left(\frac{a}{a-\mathbf{\Lambda}}\cdot\mathbf{U}\cdot\frac{b}{b - {}^t\mathbf{\Lambda}}\right)_{0,0} = -S_{b,a}, $$

where $V_a = W_a$ and $S_{a,b} = -S_{b,a}$ due to the antisymmetry property of $\mathbf{U}$.

Focusing on these new variables and considering $[(3.26)]_{0,0}$ and $[\frac{a}{a-\mathbf{\Lambda}}\cdot(3.26)]_{0,0}$ and also $[\frac{a}{a-\mathbf{\Lambda}}\cdot(3.26)\cdot\frac{b}{b-{}^t\mathbf{\Lambda}}]_{0,0}$, we derive the following relations as a consequence:

$$ V_{p_1}\tilde{V}_{-p_1} = 1, \quad (3.27a) $$
$$ 1 + 2V_{p_1}\tilde{S}_{-a,-p_1} = \frac{a-p_1}{a+p_1}(V_{p_1} - V_{-a}) + V_{p_1}\tilde{V}_{-a} \quad (3.27b) $$

as well as the dynamical relation for the variable $S_{a,b}$:

$$ \begin{aligned} &\frac{b+p_1}{b-p_1}\tilde{S}_{-a,-b} + \frac{p_1-a}{p_1+a}S_{-a,-b} - \frac{2b}{b-p_1}\tilde{S}_{-a,-p_1} + \frac{2a}{p_1+a}S_{p_1,-b} \\ &\quad + (1 - V_{-b})\tilde{S}_{-a,-p_1} - (1 - \tilde{V}_{-a} + 2\tilde{S}_{-a,-b})S_{p_1,-b} = 0, \end{aligned} \quad (3.27c) $$

together with the $(\hat{\cdot}, p_2)$ and $(\bar{\cdot}, p_3)$ counterparts.

In the discrete BKP case, we define the tau function by

$$ \tau^2 = \det(1 + \mathbf{\Omega}\cdot\mathbf{C}), $$

this is because both $\mathbf{\Omega}$ and $\mathbf{C}$ are antisymmetric, resulting in $\det(1+\mathbf{\Omega}\cdot\mathbf{C})$ being a perfect square, and such a definition will bring us some simplicity. Considering the dynamical evolution of the tau function, we have

$$
\begin{aligned}
\tilde{\tau}^2 &= \det(1+\mathbf{\Omega}\cdot\tilde{\mathbf{C}}) = \det\left(1+\mathbf{\Omega}\cdot\mathbf{C}+p_1\frac{1}{p_1-{}^t\mathbf{\Lambda}}\cdot(\mathbf{O}\cdot\mathbf{\Lambda}-{}^t\mathbf{\Lambda}\cdot\mathbf{O})\cdot\frac{1}{p_1-\mathbf{\Lambda}}\cdot\mathbf{C}\right)\\
&= \tau^2\det\begin{pmatrix} 1+p_1(\frac{\mathbf{\Lambda}}{p_1-\mathbf{\Lambda}}\cdot\mathbf{U}\cdot\frac{1}{p_1-{}^t\mathbf{\Lambda}})_{0,0} & -p_1(\frac{\mathbf{\Lambda}}{p_1-\mathbf{\Lambda}}\cdot\mathbf{U}\cdot\frac{{}^t\mathbf{\Lambda}}{p_1-{}^t\mathbf{\Lambda}})_{0,0}\\ p_1(\frac{1}{p_1-\mathbf{\Lambda}}\cdot\mathbf{U}\cdot\frac{1}{p_1-{}^t\mathbf{\Lambda}})_{0,0} & 1-p_1(\frac{1}{p_1-\mathbf{\Lambda}}\cdot\mathbf{U}\cdot\frac{{}^t\mathbf{\Lambda}}{p_1-{}^t\mathbf{\Lambda}})_{0,0}\end{pmatrix},
\end{aligned}
$$

in which we have used the rank 2 Weinstein–Aronszajn formula. The similar relations also hold for $(\hat{\cdot}, p_2)$ and $(\bar{\cdot}, p_3)$ as well as the undershifts $(\underset{\sim}{\cdot}, -p_1)$, $(\underset{\wedge}{\cdot}, -p_2)$ and $(\underline{\cdot}, -p_3)$. Taking (3.27a) into consideration, without losing generality, we have the dynamical relations for the tau function

$$
V_{p_1}=\frac{\tilde{\tau}}{\tau},\quad V_{-p_1}=\frac{\underset{\sim}{\tau}}{\tau},\quad V_{p_2}=\frac{\hat{\tau}}{\tau},\quad V_{-p_2}=\frac{\underset{\wedge}{\tau}}{\tau},\quad V_{p_3}=\frac{\bar{\tau}}{\tau},\quad V_{-p_3}=\frac{\underline{\tau}}{\tau}. \tag{3.28}
$$

Now if we set $a=p_2$ in (3.27b), the V-variables can be replaced by the tau function using (3.28), and therefore we can express $S_{-p_2,-p_1}$ by τ in the following formula:

$$
S_{-p_2,-p_1}=\frac{1}{2}\left[\frac{\underset{\wedge}{\tau}-\underset{\sim}{\tau}}{\tau}+\frac{p_2-p_1}{p_2+p_1}\left(1-\frac{\underset{\substack{\wedge\\ \sim}}{\tau}}{\tau}\right)\right]. \tag{3.29}
$$

Similar relations in terms of the other lattice parameters and discrete shifts can be obtained in a similar way. Finally, setting $a=p_2$ and $b=p_3$ in (3.27c) and replacing everything by the tau function, we end up with a closed-form 3D lattice equation of the tau function.

THEOREM 3.12. *The tau function in the discrete BKP case obeys the bilinear equation*

$$
\begin{aligned}
&(p_1-p_2)(p_2-p_3)(p_3-p_1)\tau\bar{\hat{\tilde{\tau}}}+(p_1+p_2)(p_1+p_3)(p_2-p_3)\tilde{\tau}\bar{\hat{\tau}}\\
&\quad+(p_3+p_1)(p_3+p_2)(p_1-p_2)\bar{\tau}\hat{\tilde{\tau}}+(p_2+p_3)(p_2+p_1)(p_3-p_1)\hat{\tau}\tilde{\bar{\tau}}=0,
\end{aligned} \tag{3.30}
$$

which is often referred to as the Miwa equation.

The difference between the Miwa equation and the HM equation (3.13) is that the former has an extra term (the fourth term).

REMARK 3.13. *Although equation* (3.30) *has an extra term compared with the HM equation, the solution space of the discrete BKP equation is actually smaller. This is because the measure in the discrete BKP equation must obey the antisymmetry property, namely the spectral variables are restricted in the BKP case.*

3.3. Lax triplet. Next, we consider the linear problem of the discrete BKP equation.

PROPOSITION 3.14. *The wave function* (2.19) *in the discrete BKP case obeys*

$$
\mathrm{T}_{p_j}\mathbf{u}_k=\frac{p_j+\mathbf{\Lambda}}{p_j-\mathbf{\Lambda}}\cdot\mathbf{u}_k-p_j\mathrm{T}_{p_j}\mathbf{U}\cdot\frac{1}{p_j+{}^t\mathbf{\Lambda}}\cdot(\mathbf{O}\cdot\mathbf{\Lambda}-{}^t\mathbf{\Lambda}\cdot\mathbf{O})\cdot\frac{1}{p_j-\mathbf{\Lambda}}\cdot\mathbf{u}_k. \tag{3.31}
$$

PROOF. The proof is similar to proposition, and thus, we omit it. □

By introducing $\phi=(\mathbf{u}_k)_0, \psi_a=(\frac{a}{a-\mathbf{\Lambda}}\cdot\mathbf{u}_k)_0$, we therefore from (4.21) have

$$
\tilde{\phi}=\tilde{V}_{-p_1}(\psi_{p_1}-\phi), \tag{3.32a}
$$

$$
\tilde{\psi}_{-a}=\frac{p_1-a}{p_1+a}(\psi_{-a}-\psi_{p_1})+(\tilde{V}_{-a}-2\tilde{S}_{-a,-p_1})\psi_{p_1}+\tilde{S}_{-a,-p_1}\phi \tag{3.32b}
$$

together with the similar equations associated with $(\hat{\cdot}, p_2)$ and $(\bar{\cdot}, p_3)$. Now we eliminate ψ by setting $a = p_2$, and as a result, we can derive the Lax triplet of the discrete BKP equation.

THEOREM 3.15. *The discrete BKP equation has its Lax triplet*

$$\hat{\tilde{\phi}} - \phi = \left(\frac{p_1 + p_2}{p_1 - p_2}\right) \frac{\hat{\tau}\tilde{\tau}}{\tau\hat{\tilde{\tau}}}(\hat{\phi} - \tilde{\phi}), \tag{3.33a}$$

$$\bar{\hat{\phi}} - \phi = \left(\frac{p_2 + p_3}{p_2 - p_3}\right) \frac{\bar{\tau}\hat{\tau}}{\tau\bar{\hat{\tau}}}(\bar{\phi} - \hat{\phi}), \tag{3.33b}$$

$$\bar{\tilde{\phi}} - \phi = \left(\frac{p_3 + p_1}{p_3 - p_1}\right) \frac{\tilde{\tau}\bar{\tau}}{\tau\bar{\tilde{\tau}}}(\hat{\phi} - \tilde{\phi}). \tag{3.33c}$$

A nonlinear form of the BKP equation can be obtained if we eliminate the tau function in the Lax triplet and it takes the form of

$$\frac{(\bar{\hat{\phi}} - \bar{\tilde{\phi}})(\hat{\tilde{\phi}} - \phi)}{(\bar{\tilde{\phi}} - \hat{\tilde{\phi}})(\phi - \bar{\hat{\phi}})} = \frac{(\tilde{\phi} - \hat{\phi})(\bar{\phi} - \bar{\hat{\tilde{\phi}}})}{(\hat{\phi} - \bar{\phi})(\bar{\hat{\tilde{\phi}}} - \tilde{\phi})},$$

where two cross-ratios are involved in the explicit form.

4. Discrete CKP equation

4.1. Infinite-dimensional matrix structure. In the discrete CKP case, we consider the Cauchy kernel

$$\Omega_{k,k'} = \frac{1}{k + k'}, \tag{3.34}$$

which is the same as the one for the discrete AKP equation. However, we impose an additional constraint

$$\iint_D \cdot \mathrm{d}\zeta(k', k) = \iint_D \cdot \mathrm{d}\zeta(k, k')$$

on the measure in the linear integral equation (2.15), which implies the Cauchy kernel is symmetric with regard to the two spectral parameters k and k'. The plane wave factors ρ_k and $\sigma_{k'}$ are exact the same as those in the discrete BKP equation, namely

$$\rho_k = \left(\frac{p_1 + k}{p_1 - k}\right)^{n_1} \left(\frac{p_2 + k}{p_2 - k}\right)^{n_2} \left(\frac{p_3 + k}{p_3 - k}\right)^{n_3},$$
$$\sigma_{k'} = \left(\frac{p_1 + k'}{p_1 - k'}\right)^{n_1} \left(\frac{p_2 + k'}{p_2 - k'}\right)^{n_2} \left(\frac{p_3 + k'}{p_3 - k'}\right)^{n_3},$$

which result in the effective plane wave factor taking the form of

$$\rho_k \sigma_{k'} = \left(\frac{p_1 + k}{p_1 - k}\frac{p_1 + k'}{p_1 - k'}\right)^{n_1} \left(\frac{p_2 + k}{p_2 - k}\frac{p_2 + k'}{p_2 - k'}\right)^{n_2} \left(\frac{p_3 + k}{p_3 - k}\frac{p_3 + k'}{p_3 - k'}\right)^{n_3}. \tag{3.35}$$

The $\mathbf{\Omega}$ therefore has the following property.

PROPOSITION 3.16. *The operator $\mathbf{\Omega}$ in the discrete CKP case satisfies*

$$\mathbf{\Omega} \cdot \frac{p_j + \mathbf{\Lambda}}{p_j - \mathbf{\Lambda}} - \frac{p_j - {}^t\mathbf{\Lambda}}{p_j + {}^t\mathbf{\Lambda}} \cdot \mathbf{\Omega} = 2p_j \frac{1}{p_j + {}^t\mathbf{\Lambda}} \cdot \mathbf{O} \cdot \frac{1}{p_j - \mathbf{\Lambda}}, \tag{3.36}$$

as well as the symmetry property ${}^t\mathbf{\Omega} = \mathbf{\Omega}$.

PROOF. The form of the kernel (3.34) implies that $\mathbf{\Omega}\cdot\mathbf{\Lambda}+{}^t\mathbf{\Lambda}\cdot\mathbf{\Omega}=\mathbf{O}$, and this leads to (3.36) directly. The proof of the symmetry property is similar to that of the antisymmetry property in the BKP case, cf. the proof of proposition 3.9, and thus, we do not repeat it. □

Since the effective plane wave factor for the discrete CKP equation is the same as (3.22), $\mathbf{C}$ satisfies the same dynamical relation. However, due to a different measure structure, it is instead symmetric.

PROPOSITION 3.17. *The infinite-dimensional matrix* $\mathbf{C}$ *obeys the dynamical relations*

$$\mathrm{T}_{p_j}\mathbf{C}\cdot\frac{p_j-{}^t\mathbf{\Lambda}}{p_j+{}^t\mathbf{\Lambda}}=\frac{p_j+\mathbf{\Lambda}}{p_j-\mathbf{\Lambda}}\cdot\mathbf{C} \tag{3.37}$$

with respect to p_j*, as well as the symmetry property* ${}^t\mathbf{C}=\mathbf{C}$.

PROOF. We omit the proof as is similar to that of proposition 3.10. □

4.2. The hyperdeterminant equation. In this subsection, we derive the discrete CKP equation based on the $\mathbf{\Omega}$ and $\mathbf{C}$ given above. The properties of $\mathbf{\Omega}$ and $\mathbf{C}$ can help us to derive the dynamical and algebraic relations of the infinite-dimensional matrix $\mathbf{U}$.

PROPOSITION 3.18. *The infinite-dimensional matrix* $\mathbf{U}$ *satisfies the dynamical relations with regard to* p_j

$$\mathrm{T}_{p_j}\mathbf{U}\cdot\frac{p_j-{}^t\mathbf{\Lambda}}{p_j+{}^t\mathbf{\Lambda}}=\frac{p_j+\mathbf{\Lambda}}{p_j-\mathbf{\Lambda}}\cdot\mathbf{U}-2p_j\mathrm{T}_{p_j}\mathbf{U}\cdot\frac{1}{p_j+{}^t\mathbf{\Lambda}}\cdot\mathbf{O}\cdot\frac{1}{p_j-\mathbf{\Lambda}}\cdot\mathbf{U}, \tag{3.38}$$

as well as the symmetry property ${}^t\mathbf{U}=\mathbf{U}$.

PROOF. The dynamical part is proven by following the same procedure of the proof of proposition 3.3. The symmetry property of $\mathbf{U}$ is a result of $\mathbf{\Omega}$ and $\mathbf{C}$ both being symmetric. □

In the discrete CKP case, we introduce the following new variables:

$$V_a=1-\left(\mathbf{U}\cdot\frac{1}{a+{}^t\mathbf{\Lambda}}\right)_{0,0}=1-\left(\frac{1}{a+\mathbf{\Lambda}}\cdot\mathbf{U}\right)_{0,0}=W_a,$$
$$S_{a,b}=\left(\frac{1}{a+\mathbf{\Lambda}}\cdot\mathbf{U}\cdot\frac{1}{b+{}^t\mathbf{\Lambda}}\right)_{0,0}=S_{b,a},$$

where $V_a=W_a$ and $S_{a,b}=S_{b,a}$ hold because of the symmetry property of $\mathbf{U}$. Now the dynamical evolution of $\mathbf{U}$ (3.38) can give rise to the evolutions for the introduced new variables as follows if we consider $[\frac{1}{a+\mathbf{\Lambda}}\cdot(3.38)]_{0,0}$ and $[\frac{1}{a+\mathbf{\Lambda}}\cdot(3.38)\cdot\frac{1}{b+{}^t\mathbf{\Lambda}}]_{0,0}$:

$$\tilde{V}_a+\frac{p_1-a}{p_1+a}V_a=2p_1\left(\frac{1}{p_1+a}-\tilde{S}_{a,p_1}\right)V_{-p_1}, \tag{3.39a}$$

$$[1-(p_1+a)\tilde{S}_{a,p_1}][1+(p_1-b)S_{-p_1,b}]+\frac{(p_1+a)(p_1+b)}{2p_1}\tilde{S}_{a,b}-\frac{(p_1-a)(p_1-b)}{2p_1}S_{a,b}=1 \tag{3.39b}$$

as well as the analogues in terms of $(\hat{\cdot},p_2)$ and $(\bar{\cdot},p_3)$. The tau function in this class is defined as (2.21). Since we know the dynamical evolutions of $\mathbf{\Omega}$ and $\mathbf{C}$ from (3.36) and

(3.37), respectively, with the help of them, we can prove the dynamical evolution of the tau function

$$
\begin{aligned}
\tilde{\tau} = & \det\left(1 + \boldsymbol{\Omega}\cdot\mathbf{C} + 2p_1\frac{1}{p_1 - {}^t\boldsymbol{\Lambda}}\cdot\mathbf{O}\cdot\frac{1}{p_1 - \boldsymbol{\Lambda}}\cdot\mathbf{C}\right) \\
& = \tau\left[1 + 2p_1\left(\frac{1}{p_1 - \boldsymbol{\Lambda}}\cdot\mathbf{U}\cdot\frac{1}{p_1 - {}^t\boldsymbol{\Lambda}}\right)_{0,0}\right],
\end{aligned}
$$

where the computation is very similar to the case in the discrete AKP equation. Similarly, we can also derive the evolution of the tau function in terms of the undershifts. Therefore the expressions for the tau function are given by

$$
\frac{\tilde{\tau}}{\tau} = 1 + 2p_1 S_{-p_1,-p_1}, \quad \frac{\underset{\sim}{\tau}}{\tau} = 1 - 2p_1 S_{p_1,p_1} \tag{3.40}
$$

together with the counterparts of the shifts $\hat{\cdot}$ and $\bar{\cdot}$ associated with their corresponding lattice parameters. Furthermore, if we set $a = p_1$ in (3.39a) and make use of the above relations, the dynamical relations of V-variables in terms of τ can easily be obtained, which are

$$
\frac{\tilde{V}_{p_1}}{V_{-p_1}} = \frac{\tau}{\tilde{\tau}}, \quad \frac{\underset{\sim}{V}_{-p_1}}{V_{p_1}} = \frac{\tau}{\underset{\sim}{\tau}}, \tag{3.41}
$$

and the similar relations also hold via the replacements $(\tilde{\cdot}, p_1) \leftrightarrow (\hat{\cdot}, p_2) \leftrightarrow (\bar{\cdot}, p_3)$. In order to find a closed-form equation in the tau function, we first set $a = p_2$ and $b = -p_1$ in (3.39b) respectively and this gives us

$$
\frac{1 - (p_1 + b)\tilde{S}_{p_1,b}}{1 + (p_1 - b)S_{-p_1,b}} = \frac{\tau}{\tilde{\tau}}, \quad \frac{1 + (p_1 - a)S_{a,-p_1}}{1 - (p_1 + a)\tilde{S}_{a,p_1}} = \frac{\tilde{\tau}}{\tau}, \quad (\tilde{\cdot}, p_1) \leftrightarrow (\hat{\cdot}, p_2) \leftrightarrow (\bar{\cdot}, p_3).
$$

Now if we set $a = b = p_2$ in (3.39b) and make use of the above obtained relations, an expression of the S-variable in terms of the tau function can be obtained as follow:

$$
[1 - (p_1 + p_2)S_{p_1,p_2}]^2 = \frac{(p_1 + p_2)^2\underset{\sim}{\tau}\underset{\wedge}{\tau} - (p_1 - p_2)^2\tau\underset{\sim}{\underset{\wedge}{\tau}}}{4p_1p_2\tau^2}. \tag{3.42}
$$

The expressions for S_{p_2,p_3} and S_{p_3,p_1} by the tau function as well as the S-quantity associated with $-p_1, -p_2, -p_3$ can be derived in a similar way. Equation (3.42) is the key relation in deriving a closed-form 3D lattice equation. In fact, if we set $a = p_2$ and $b = p_3$ in (3.39b) and express all the variables in the equation by the tau function, it turns out to be the discrete CKP equation.

Theorem 3.19. *The tau function obeys a 3D discrete equation*

$$
\begin{aligned}
&[(p_1 - p_2)^2(p_2 - p_3)^2(p_3 - p_1)^2\tau\bar{\hat{\tilde{\tau}}} + (p_1 + p_2)^2(p_1 + p_3)^2(p_2 - p_3)^2\tilde{\tau}\bar{\hat{\tau}} \\
&\quad - (p_2 + p_3)^2(p_2 + p_1)^2(p_3 - p_1)^2\hat{\tau}\bar{\tilde{\tau}} - (p_3 + p_1)^2(p_3 + p_2)^2(p_1 - p_2)^2\bar{\tau}\hat{\tilde{\tau}}]^2 \\
&\quad - 4(p_1^2 - p_2^2)^2(p_1^2 - p_3^2)^2[(p_2 + p_3)^2\hat{\tilde{\tau}}\bar{\tilde{\tau}} - (p_2 - p_3)^2\tilde{\tau}\bar{\hat{\tilde{\tau}}}][(p_2 + p_3)^2\hat{\tau}\bar{\tau} - (p_2 - p_3)^2\tau\bar{\hat{\tau}}] = 0,
\end{aligned} \tag{3.43}
$$

which is referred to as the discrete CKP equation.

This is a new parametrisation for the discrete CKP equation in contrast to the existing form in [**Kas96**,**Sch03**]. This version of the discrete CKP equation, i.e. (3.43), can provide soliton solutions as the lattice parameters p_1, p_2, p_3 are introduced. Furthermore, we note that the left hand side of the equation takes the form of Cayley's $2\times2\times2$ hyperdeterminant.

4.3. Lax triplet. Now we consider the linear problem of the discrete CKP equation. Following (2.19) and propositions 3.16 and 3.17, we obtain the dynamical evolutions of $\mathbf{u}_k$ with regard to the lattice parameters p_j.

PROPOSITION 3.20. *The wave function $\mathbf{u}_k$ in the CKP equation obeys dynamical relations*

$$\mathrm{T}_{p_j}\mathbf{u}_k = \frac{p_j + \mathbf{\Lambda}}{p_j - \mathbf{\Lambda}} \cdot \mathbf{u}_k - 2p_j \mathrm{T}_{p_j}\mathbf{U} \cdot \frac{1}{p_j + {}^t\mathbf{\Lambda}} \cdot \mathbf{O} \cdot \frac{1}{p_j - \mathbf{\Lambda}} \cdot \mathbf{u}_k. \tag{3.44}$$

By taking $[(3.44)]_{0,0}$ and $[\frac{1}{a+\mathbf{\Lambda}} \cdot (3.44)]_{0,0}$ respectively, we obtain the following:

$$\psi_{-p_1} = \frac{1}{2p_1\tilde{V}_{p_1}}(\tilde{\phi} + \phi), \quad \tilde{\psi}_a = \frac{p_1 - a}{p_1 + a}\psi_a - \frac{2p_1}{p_1 + a}[1 - (p_1 + a)\tilde{S}_{a,p_1}]\psi_{-p_1}$$

together with their $(\hat{\cdot}, p_2)$ and $(\bar{\cdot}, p_3)$ counterparts, in which $\phi = (\mathbf{u}_k)_0, \psi_a = (\frac{1}{a+\mathbf{\Lambda}} \cdot \mathbf{u}_k)_0$. Now if we set $a = -p_2$ and eliminate the ψ-variables in the linear problem, we can obtain the Lax triplet of the discrete CKP equation, which is

$$\hat{\tilde{\phi}} + \tilde{\phi} = \frac{p_1 + p_2}{p_1 - p_2}\frac{\hat{\tilde{V}}_{p_2}}{\hat{V}_{p_2}}(\phi + \hat{\phi}) + \frac{2p_2}{p_1 - p_2}\frac{\hat{\tilde{V}}_{p_2}}{\tilde{V}_{p_1}}[1 - (p_1 - p_2)\tilde{S}_{p_1,-p_2}](\phi + \tilde{\phi}) \tag{3.45}$$

together with its counterparts associated with the other two directions, where the V- and S-variables can be expressed in terms of the tau function via Equations (3.41) and (3.42), respectively.

5. Soliton solution structure

We now discuss how explicit soliton solutions for the discrete AKP, BKP and CKP equations arising from the DL constructed in the previous sections. Actually, these soliton solutions structure are constructed by restricting the double integral (in the integral equation) on a domain which contains a finite number of singular points. In other words, a measure that can introduce a finite number of singularities is required. As we noted previously, a 3D integrable lattice equation may have different forms but the solution structure hidden behind them is always the same. For convenience, we only study soliton solution to the τ-equations, i.e. the HM (AKP) equation (3.13), the Miwa (BKP) equation (3.30) and the Kashaev (CKP) equation (3.43). Soliton solutions to the other forms can easily be recovered via the associated discrete differential transforms.

In the soliton reduction, we need the following finite matrices: an $N \times N'$ full-rank constant matrix $\mathbf{A}$ with entries $A_{i,j}$, and a generalised Cauchy matrix

$$\mathbf{M} = (M_{j,i})_{N'\times N}, \quad M_{j,i} = \sigma_{k'_j}\Omega_{j,i}\rho_{k_i}, \tag{3.46}$$

where ρ_{k_i} and $\sigma_{k'_j}$ are the corresponding plane wave factors and $\Omega_{j,i} = \Omega_{k_i,k'_j}$ is the Cauchy kernel. Next, we impose the following condition on the measure $\mathrm{d}\zeta(k,k')$:

$$\mathrm{d}\zeta(k,k') = \sum_{i=1}^{N}\sum_{j=1}^{N'} A_{i,j}\delta(k - k_i)\delta(k' - k'_j)\mathrm{d}k\mathrm{d}k'. \tag{3.47}$$

In fact, k_i and k'_j are the singular points in the domain D for soliton solutions. The condition of the measure (3.47) immediately gives rise to the degeneration of $\mathbf{C}$ and it takes the form of

$$\mathbf{C} = \sum_{i=1}^{N}\sum_{j=1}^{N'} A_{i,j}\rho_{k_i}\mathbf{c}_{k_i}{}^t\mathbf{c}_{k'_j}\sigma_{k'_j}. \tag{3.48}$$

The degeneration helps us to reduce the nonlocality (i.e. the double integral) in the problem and brings us a finite summation which leads to soliton solutions of the 3D lattice equations in the form of a finite matrix. We conclude it as the following important statement:

$$\det(1+\mathbf{\Omega}\cdot\mathbf{C}) = \det(\mathbf{I}_{N'\times N'} + \mathbf{M}_{N'\times N}\mathbf{A}_{N\times N'}) = \det(\mathbf{I}_{N\times N} + \mathbf{A}_{N\times N'}\mathbf{M}_{N'\times N}). \tag{3.49}$$

Either equality in Equation (3.49) provides us with a general formula for soliton solutions to the 3D lattice equations. In practice, the restriction on the measure $\mathrm{d}\zeta(k,k')$ may require that the constant matrix $\mathbf{A}$ satisfies certain conditions. As a result, soliton solutions can be constructed. In the following, we follow from this idea and give the soliton solutions to the discrete AKP, BKP and CKP equations one by one.

5.1. Discrete AKP equation. The plane wave factors and the Cauchy kernel as follows:

$$\begin{aligned}
\rho_{k_i} &= (p+k_i)^{n_1}(q+k_i)^{n_2}(r+k_i)^{n_3},\\
\sigma_{k'_j} &= (p-k'_j)^{-n_1}(q-k'_i)^{-n_2}(r-k'_j)^{-n_3},\\
\Omega_{j,i} &= \frac{1}{k_i+k'_j}.
\end{aligned}$$

As there is no restriction on the measure $\mathrm{d}\zeta(k,k')$ in the discrete AKP case, the matrix $\mathbf{A}$ can be arbitrary (but non-degenerate). Therefore, the (N,N')-soliton solution to the HM equation (3.13) is

$$\tau = \det(\mathbf{I}+\mathbf{AM}), \quad M_{j,i} = \frac{\rho_{k_i}\sigma_{k'_j}}{k_i+k'_j}, \quad i=1,2,\cdots,N, \quad j=1,2,\cdots,N'. \tag{3.50}$$

5.2. Discrete BKP equation. The plane wave factors and the Cauchy kernel for the discrete BKP equation are given by

$$\rho_{k_i} = \left(\frac{p_1+k_i}{p_1-k_i}\right)^{n_1}\left(\frac{p_2+k_i}{p_2-k_i}\right)^{n_2}\left(\frac{p_3+k_i}{p_3-k_i}\right)^{n_3}, \quad \sigma_{k'_j} = \rho_{k'_j}, \quad \Omega_{j,i} = \frac{1}{2}\frac{k_i-k'_j}{k_i+k'_j},$$

In addition, the antisymmetry condition $\mathrm{d}\zeta(k,k') = -\mathrm{d}\zeta(k',k)$ on the measure leads to $A_{i,j} = -A_{j,i}$ in the matrix $\mathbf{A}$, i.e.

$$\mathrm{d}\zeta(k,k') = \sum_{i,j=1}^{2N} A_{i,j}\delta(k-k_i)\delta(k'-k'_j)\mathrm{d}k\mathrm{d}k', \quad A_{i,j} = -A_{j,i}. \tag{3.51}$$

As a result, the N-soliton solution to the Miwa equation (3.30) is determined by

$$\tau^2 = \det(\mathbf{I} + \mathbf{AM}), \quad M_{j,i} = \rho_{k_i} \frac{1}{2} \frac{k_i - k'_j}{k_i + k'_j} \sigma_{k'_j}, \quad A_{i,j} = -A_{j,i}, \quad i,j = 1,2,\cdots,2N. \tag{3.52}$$

As the matrices $\mathbf{A}$ and $\mathbf{M}$ are both antisymmetric it can be shown that the determinant $\det(\mathbf{I} + \mathbf{AM})$ must be a perfect square. In other words, the tau function itself can be expressed by a Pfaffian.

5.3. Discrete CKP equation. The plane wave factors and the Cauchy kernel in the discrete CKP equation take the form of

$$\rho_{k_i} = \left(\frac{p_1 + k_i}{p_1 - k_i}\right)^{n_1} \left(\frac{p_2 + k_i}{p_2 - k_i}\right)^{n_2} \left(\frac{p_3 + k_i}{p_3 - k_i}\right)^{n_3}, \quad \sigma_{k'_j} = \rho_{k'_j}, \quad \Omega_{j,i} = \frac{1}{k_i + k'_j}.$$

Due to the symmetry property of the measure $\mathrm{d}\zeta(k,k') = \mathrm{d}\zeta(k',k)$, one has to set $N' = N$ and impose the symmetry condition on $\mathbf{A}$, and therefore

$$\mathrm{d}\zeta(k,k') = \sum_{i,j=1}^{N} A_{i,j} \delta(k - k_i)\delta(k' - k'_j)\mathrm{d}k\mathrm{d}k', \quad A_{i,j} = A_{j,i}. \tag{3.53}$$

Clearly the N-soliton solution to the Kashaev equation (3.43) can then be written in the form of

$$\tau = \det(\mathbf{I} + \mathbf{AM}), \quad M_{j,i} = \frac{\rho_{k_i}\sigma_{k'_j}}{k_i + k'_j}, \quad A_{i,j} = A_{j,i}, \quad i,j = 1,2,\cdots,N. \tag{3.54}$$

6. Concluding remarks

By suitably choosing the Cauchy kernel and the plane wave factors in (2.15), the DL scheme was established for the discrete AKP, BKP and CKP equations. In each class, we not only provide possible nonlinear equations and their associated linear problems, but also construct their soliton solutions.

In the AKP class, we obtained three nonlinear equations (expressed by u, v and z, respectively) and one bilinear equation (i.e. the HM equation). However, in either the BKP class or the CKP class only the equation of the tau function has been constructed, though there still exists a nonlinear discrete BKP equation expressed by the wave function. It would be an interesting question to see whether we can find more nonlinear equations, such as equations expressed by the unmodified variable u, the modified variable v and the Schwarzian variable z, in the BKP and CKP classes.

We only give an implicit Lax pair for the discrete CKP equation, i.e. we cannot replace the V- and S-variables in (3.45) by the tau function explicitly. In fact, according to Kashaev [**Kas96**], the discrete CKP arises in the theory of the local Yang–Baxter equations, namely a 3D generalisation of the theory of 2D Lax equations which was invented by Maillet and Nijhoff [**MN89**]. This implies that the discrete CKP equation could arise as the compatibility of a '3D Lax equation', rather than a classical Lax pair.

CHAPTER 4

Continuous KP-type hierarchies and their reductions

1. Overview

Integrable nonlinear partial differential equations (PDEs) arise in a variety of areas in modern mathematics and physics. Over the past few decades, there have been many methods developed for the construction and solutions for such equations, including the IST method, Riemann–Hilbert approach, finite-gap integration, Hirota's bilinear method and methods based on representation theory, cf. the monographs [**AC91**, **NMPZ84**, **Hir04**, **Kac94**]. Among those models, the KP hierarchy is often considered to be the most fundamental one and the most popular construction of this hierarchy is Sato's approach based on pseudo-differential operators [**SM81**]. This is related to the observation that the KP hierarchy is closely related to infinite-dimensional Grassmannians [**Sat81**]. This idea was further developed by Date, Jimbo, Kashiwara and Miwa who classified soliton equations by using transformation groups associated with infinite-dimensional Lie algebras based on the so-called fundamental bilinear identity leading to the hierarchy of equations in Hirota's form in terms of the tau function (cf. e.g. [**JM83**, **MJD00**] and references therein for the original research papers by the Kyoto School). A classification of soliton solutions for the KP hierarchy was recently given by Kodama (see monograph [**Kod17**]).

The pseudo-differential operator approach may have some disadvantages. One disadvantage is that it singles out a preferred independent variable from all the flow variables in the hierarchy whereas the latter can be considered all to be on the same footing. The other disadvantage is that it can only be discretised to obtain semi-discrete equations but no fully-discrete equations have been obtained yet from this approach. In contrast, we will promote the DL approach for treating the KP hierarchy in which no preselected independent variable is required to set up the constitutive relations, and which also allows in a natural way to a full discretisation.

In chapter 3 the DL was established for the three families of the discrete KP-type equations, namely the discrete AKP, BKP and CKP equations, extending the earlier results on discrete KP equations of A-type [**NCWQ84**]. Based on the new insights provided by the previous chapter, in the current chapter we revisit the continuous hierarchies for the AKP, BKP and CKP equations as well as their dimensional reductions. The resulting (1+1)-dimensional hierarchies include the following examples: the KdV, BSQ, generalised HS, HS, SK and KK, bidirectional SK (bSK) and bidirectional KK (bKK), and Ito hierarchies. The DL framework is presented in the language of infinite matrices and this treatment provides the following: i) different nonlinear forms in the each class together with the Miura-type transforms; ii) the multilinear form in terms of the tau function for each class; iii) the Lax pairs of both nonlinear equations and the multilinear equation for each class; iv) the soliton solutions for both nonlinear and multilinear equations. In spite of the undeniable virtues of other approaches, we believe that the DL framework provides the most comprehensive treatment of all these results.

The chapter is organised as follows: section 2 is contributed to the three important (2+1)-dimensional soliton hierarchies, namely, the AKP, BKP and CKP hierarchies. The dimensional reductions of the higher-dimensional models are discussed in section 3 which includes a number of (1+1)-dimensional integrable models. Finally in section 4, soliton solutions to all the soliton hierarchies will be given as particular cases as the general DL framework.

2. (2+1)-dimensional soliton hierarchies

In this section, some particular $\mathbf{C}$ and $\mathbf{\Omega}$ are given and the DL scheme is established. The resulting models include the AKP, BKP and CKP hierarchies. The three hierarchies play the role of the master integrable systems in our framework and in section 3 we will see that they generate a number of (1+1)-dimensional soliton hierarchies by suitable dimensional reductions.

2.1. The AKP hierarchy. The AKP hierarchy, normally known as the KP hierarchy, is associated with the infinite-dimensional Lie algebra A_∞, namely $\mathfrak{gl}(\infty)$. In this class, we consider a particular infinite matrix $\mathbf{C}$ given by

$$\mathbf{C} = \iint_D \mathrm{d}\zeta(l,l')\rho_l \mathbf{c}_l {}^t\mathbf{c}_{l'}\sigma_{l'}, \quad \rho_k = \exp\left(\sum_{j=1}^{\infty} k^j x_j\right), \quad \sigma_{k'} = \exp\left(-\sum_{j=1}^{\infty}(-k')^j x_j\right), \tag{4.1}$$

where ρ_k and $\sigma_{k'}$ are the plane wave factors. Differentiating $\mathbf{C}$ with respect to x_j and notice the form of ρ_k and $\sigma_{k'}$, one can obtain the dynamical evolution of $\mathbf{C}$:

$$\partial_j \mathbf{C} = \mathbf{\Lambda}^j \cdot \mathbf{C} - \mathbf{C} \cdot (-{}^t\mathbf{\Lambda})^j, \quad j \in \mathbb{Z}^+, \tag{4.2}$$

where $\partial_j \doteq \partial_{x_j}$. The operator $\mathbf{\Omega}$, namely, the infinite matrix version of the Cauchy kernel in this case is defined by the relation

$$\mathbf{\Omega} \cdot \mathbf{\Lambda} + {}^t\mathbf{\Lambda} \cdot \mathbf{\Omega} = \mathbf{O}. \tag{4.3}$$

In fact, the Cauchy kernel can be recovered from $\mathbf{\Omega}$ and is given by $\Omega_{k,l'} = \frac{1}{k+l'}$. This generalises the previous Fokas–Ablowitz result, cf. equation (1.23). In fact, the generalisation was made in the following two aspects: i) higher-order flow variables are added in the plane wave factors ρ_k and $\sigma_{k'}$, leading to higher-order equations in the KP hierarchy; ii) infinite-dimensional vectors $\mathbf{c}_k$ and ${}^t\mathbf{c}_{k'}$ are introduced, which allows to deal with modifications of the KP hierarchy (more precisely, the modified and Schwarzian KP hierarchies) simultaneously.

One can now consider the dynamical evolution of $\mathbf{U}$ defined as (2.20). In fact, making use of Equations (4.2) and (4.3), we end up with the relation for $\mathbf{U}$ as follow:

$$\partial_j \mathbf{U} = \mathbf{\Lambda}^j \cdot \mathbf{U} - \mathbf{U} \cdot (-{}^t\mathbf{\Lambda})^j - \mathbf{U} \cdot \mathbf{O}_j \cdot \mathbf{U}, \quad j \in \mathbb{Z}^+, \tag{4.4}$$

where $\mathbf{O}_j \doteq \sum_{i=0}^{j-1}(-{}^t\mathbf{\Lambda})^i \cdot \mathbf{O} \cdot \mathbf{\Lambda}^{j-1-i}$. The equations in (4.4) are all compatible as they are simultaneously solved by a common $\mathbf{U}$ defined by (2.18). Equations (4.4) can be considered as the AKP hierarchy expressed by the infinite matrix $\mathbf{U}$.

Following the definition of $\mathbf{O}_j$, one can easily prove the recurrence relation

$$\mathbf{O}_{i+j} = \mathbf{O}_i \cdot \mathbf{\Lambda}^j + (-{}^t\mathbf{\Lambda})^i \cdot \mathbf{O}_j. \tag{4.5}$$

Making use of the recurrence relation of $\mathbf{O}_j$ (4.5), one can find the recurrence relation for the dynamical relation of $\mathbf{U}$ (4.4) via some straightforward computation and they are given by:

$$(\partial_{i+j} + \partial_i\partial_j)\mathbf{U} = (\mathbf{\Lambda}^i - \mathbf{U}\cdot\mathbf{O}_i)\cdot(\partial_j\mathbf{U}) + (\mathbf{\Lambda}^j - \mathbf{U}\cdot\mathbf{O}_j)\cdot(\partial_i\mathbf{U}),$$
$$(\partial_{i+j} - \partial_i\partial_j)\mathbf{U} = (\partial_j\mathbf{U})\cdot((-{}^t\mathbf{\Lambda})^i + \mathbf{O}_i\cdot\mathbf{U}) + (\partial_i\mathbf{U})\cdot((-{}^t\mathbf{\Lambda})^j + \mathbf{O}_j\cdot\mathbf{U}).$$

The importance of the above relations is that in the first one only $\mathbf{\Lambda}$ is involved and in the second one only ${}^t\mathbf{\Lambda}$ is involved. This observation provides us with a possibility to reduce the degree of $\mathbf{\Lambda}$ and ${}^t\mathbf{\Lambda}$ in the dynamical relation of $\mathbf{U}$, i.e. (4.4). In fact, taking $i = j = 1$, one can have

$$\mathbf{\Lambda}\cdot\mathbf{U} = \frac{1}{2}\partial_1^{-1}(\partial_2\mathbf{U} + \partial_1^2\mathbf{U} + 2\mathbf{U}\cdot\mathbf{O}\cdot(\partial_1\mathbf{U})), \tag{4.6a}$$
$$\mathbf{U}\cdot(-{}^t\mathbf{\Lambda}) = \frac{1}{2}\partial_1^{-1}(\partial_2\mathbf{U} - \partial_1^2\mathbf{U} - 2(\partial_1\mathbf{U})\cdot\mathbf{O}\cdot\mathbf{U}). \tag{4.6b}$$

With the help of Equation (4.6), one can eliminate $\mathbf{\Lambda}$ and ${}^t\mathbf{\Lambda}$ in (4.4) and obtain a differential relation for only $\mathbf{U}$ and $\mathbf{O}$. Consider the entry $U_{0,0} = (\mathbf{U})_{0,0} \doteq u$ in the infinite matrix $\mathbf{U}$, the KP (i.e. AKP) hierarchy can be derived. The first nontrivial equation (when $j = 3$), namely, the KP equation, is given by

$$u_{x_3x_1} = \left(\frac{1}{4}u_{x_1x_1x_1} + \frac{3}{2}u_{x_1}^2\right)_{x_1} + \frac{3}{4}u_{x_2x_2}. \tag{4.7}$$

We note that a slightly different derivation of the KP equation (4.7) from the DL framework can be found in [**Wal01**]. Other nonlinear forms in the AKP class can also be derived if one defines $v = [\ln(1 - U_{0,-1})]_{x_1}$, $w = -U_{1,-1}/(1 - U_{0,-1})$ and $z = U_{-1,-1} - x_1$. In fact, (4.4) leads to some Miura-type transforms between these variables:

$$2u_{x_1} = -v_{x_1} - v^2 + \partial_{x_1}^{-1}v_{x_2}, \quad v = \frac{1}{2}\frac{z_{x_1x_1} + z_{x_2}}{z_{x_1}}. \tag{4.8}$$

We omit the derivation of these transforms and one can verify these identities by substituting the derivatives with the entries in the infinite matrix $\mathbf{U}$ by using (4.4). By these transforms, one can from (4.7) derive

$$v_{x_3x_1} = \left(\frac{1}{4}v_{x_1x_1x_1} - \frac{3}{2}v^2v_{x_1}\right)_{x_1} + \frac{3}{2}v_{x_1x_1}\partial_{x_1}^{-1}v_{x_2} + \frac{3}{2}v_{x_1}v_{x_2} + \frac{3}{4}v_{x_2x_2}, \tag{4.9}$$

$$\left(\frac{z_{x_3}}{z_{x_1}}\right)_{x_1} = \frac{1}{4}\{z, x_1\}_{x_1} + \frac{3}{4}\frac{z_{x_2}}{z_{x_1}}\left(\frac{z_{x_2}}{z_{x_1}}\right)_{x_1} + \frac{3}{4}\left(\frac{z_{x_2}}{z_{x_1}}\right)_{x_2}, \quad \{z, x_1\} \doteq \frac{z_{x_1x_1x_1}}{z_{x_1}} - \frac{3}{2}\frac{z_{x_1x_1}^2}{z_{x_1}^2}. \tag{4.10}$$

The two equations are referred to as the modified KP (mKP) equation and the Schwarzian KP (SKP) equation respectively. $\{z, x_1\}$ defined above is the Schwarzian derivative of z

with respect to x_1 and it is Möbius invariant, namely it is invariant under a fractionally linear transform and therefore it is clear to see that the SKP equation has a Möbius symmetry (cf. [**Wei83**]). In addition, one can also find $w_{x_1} = \frac{1}{2}(v_{x_1} - v^2 + \partial_{x_1}^{-1} v_{x_2})$, which implies that the variable w also obeys the KP equation (4.7). One remark here is that the AKP class can be obtained from different $\mathbf{\Omega}$. For instance, one can replace $\mathbf{O}$ in (4.3) by $\frac{1}{2}(\mathbf{O} \cdot \mathbf{\Lambda} - {}^t\mathbf{\Lambda} \cdot \mathbf{O})$, then $U_{0,0}$ will also give us a slightly different form (a weak form) of the mKP equation.

From the above result we have seen that the AKP hierarchy has a lot of nonlinear forms. Therefore we need a quantity that can describe the evolution of the AKP hierarchy in a unified way. The tau function defined by $\tau = \det(1 + \mathbf{\Omega} \cdot \mathbf{C})$ can actually be a very good candidate as it contains the information of $\mathbf{\Omega}$ and $\mathbf{C}$ that is given at the beginning of our scheme. This reminds us of considering the dynamical evolution of the tau function. In fact some simple calculation using the rank 1 Weinstein–Aronszajn formula shows that

$$\partial_j \ln \tau = \partial_j \ln(\det(1 + \mathbf{\Omega} \cdot \mathbf{C})) = \operatorname{tr}(\mathbf{O}_j \cdot \mathbf{U}), \tag{4.11}$$

where in the derivation the identity $\ln(\det \mathbf{W}) = \operatorname{tr}(\ln \mathbf{W})$ for an arbitrary matrix $\mathbf{W}$ is used. When $j = 1$, it gives us the multilinear transform $u = (\ln \tau)_{x_1}$ and as a result the equation (4.7) turns out to be the bilinear equation

$$(D_1^4 - 4D_1D_3 + 3D_2^2)\tau \cdot \tau = 0. \tag{4.12}$$

The higher-order equations for the tau function can be obtained from the nonlinear AKP hierarchy in u via the same transform. But we note that the obtained τ-equations may no longer be bilinear, instead they will be multilinear. It is this set of multilinear equations that governs the algebraic structure behind the AKP class.

In the DL framework, the fundamental object is the infinite matrix $\mathbf{U}$ which generates the whole KP hierarchy with the help of (4.4) and (4.6); more precisely, eliminating $\mathbf{\Lambda}$ and ${}^t\mathbf{\Lambda}$ in these relations and considering the $(0, 0)$-entry gives us the whole KP hierarchy expressed by the nonlinear variable u. The bilinear transform $u = (\ln \tau)_{x_1}$ then brings us a hierarchy of multilinear equations expressed by the same tau function. These multilinear equations are deep down equivalent to the bilinear KP equations in the Sato scheme, see e.g. [**JM83**]. The difference is that in the bilinear framework more and more independent variables must be involved in higher-order bilinear equations, but in our approach each multilinear equation only depends on three dynamical variables, namely x_1, x_2 and x_j.

Now we consider the linear problem of the AKP hierarchy. The differentiation of (2.19) together with Equations (4.4) and (4.3) bring us the dynamical evolution for $\mathbf{u}_k$ in the AKP class given by

$$\partial_j \mathbf{u}_k = \mathbf{\Lambda}^j \cdot \mathbf{u}_k - \mathbf{U} \cdot \mathbf{O}_j \cdot \mathbf{u}_k. \tag{4.13}$$

Like how we deal with the nonlinear variable $\mathbf{U}$, a similar derivation provides us with the following important relations for $\mathbf{u}_k$:

$$(\partial_{i+j} - \partial_i \partial_j)\mathbf{u}_k = (\partial_j \mathbf{U}) \cdot \mathbf{O}_i \cdot \mathbf{u}_k + (\partial_i \mathbf{U}) \cdot \mathbf{O}_j \cdot \mathbf{u}_k, \quad j \in \mathbb{Z}^+.$$

By taking $i = j = 1$ and setting $\phi = u_k^{(0)}$ in the above relation, the linear problem can be obtained and it is as the following:

$$\phi_{x_2} = (\partial_1^2 + 2u_{x_1})\phi. \tag{4.14a}$$

The linear equation (4.14a) in ϕ governs the linear structure of the whole AKP hierarchy, namely, it is the spatial part of the Lax pairs for all the members in the AKP hierarchy. The temporal evolutions of the hierarchy can be derived by considering the corresponding flows separately. In practice, one can have from (4.13)

$$\mathbf{\Lambda} \cdot \mathbf{u}_k = \partial_1 \mathbf{u}_k + \mathbf{U} \cdot \mathbf{O} \cdot \mathbf{u}_k$$

by taking $j = 1$. This relation together with (4.6) can help to reduce the order of $\mathbf{\Lambda}$ and ${}^t\mathbf{\Lambda}$ in $\partial_j \mathbf{u}_k$ following from (4.13) and therefore the temporal evolutions can be derived. For instance, if we consider the x_3-flow in (4.13), we have the temporal evolution

$$\phi_{x_3} = \left[\partial_1^3 + 3u_{x_1}\partial_1 + \frac{3}{2}(u_{x_1x_1} + u_{x_2})\right]\phi. \tag{4.14b}$$

The compatibility condition of the spatial part (4.14a) and the temporal evolution (4.14b), namely, $\phi_{x_2x_3} = \phi_{x_3x_2}$, gives us the KP equation (4.7). The Lax pairs for the other nonlinear and multilinear forms can be calculated by replacing u in (4.14) by the other variables via the Miura-type transforms (4.8) and the multilinear transform. The temporal evolution of the higher-order equations in the hierarchy can be derived in a similar way.

2.2. The BKP hierarchy. The BKP hierarchy is associated with the algebra B_∞ which is a sub-algebra of A_∞, therefore the BKP hierarchy can be understood as a sub-hierarchy of AKP (in the sense of having a more restrictive measure in its solution, see below). In this case, we take a particular infinite matrix $\mathbf{C}$ as follow:

$$\mathbf{C} = \iint_D \mathrm{d}\zeta(l,l')\rho_l \mathbf{c}_l {}^t\mathbf{c}_{l'}\rho_{l'}, \quad \rho_k = \exp\left(\sum_{j=0}^{\infty} k^{2j+1}x_{2j+1}\right), \quad \mathrm{d}\zeta(l',l) = -\mathrm{d}\zeta(l,l'). \tag{4.15}$$

From the above formula one can see the difference between BKP and AKP is that in the BKP hierarchy only odd independent variables are involved and besides the measure in the integral has the antisymmetry property. One can easily find that $\mathbf{C}$ satisfies the dynamical relation

$$\partial_{2j+1}\mathbf{C} = \mathbf{\Lambda}^{2j+1} \cdot \mathbf{C} + \mathbf{C} \cdot {}^t\mathbf{\Lambda}^{2j+1}, \quad j = 0, 1, 2, \cdots \tag{4.16}$$

as well as the antisymmetry property ${}^t\mathbf{C} = -\mathbf{C}$. Now we require that the infinite matrix version $\mathbf{\Omega}$ of the kernel obeys the algebraic relation

$$\mathbf{\Omega} \cdot \mathbf{\Lambda} + {}^t\mathbf{\Lambda} \cdot \mathbf{\Omega} = \frac{1}{2}(\mathbf{O} \cdot \mathbf{\Lambda} - {}^t\mathbf{\Lambda} \cdot \mathbf{O}). \tag{4.17}$$

Actually we have pointed out in the previous section that replacing the right hand side of (4.3) by $\frac{1}{2}(\mathbf{O} \cdot \mathbf{\Lambda} - {}^t\mathbf{\Lambda} \cdot \mathbf{O})$ also gives us the AKP hierarchy in a slightly different form. So Equation (4.17) is just another representation of (4.3). In other words, the infinite matrix relation for $\mathbf{\Omega}$ is preserved from AKP to BKP, but here we impose in addition the antisymmetry property on the measure as one can see in (4.15). We would also like to note that this actually provides us with the Cauchy kernel $\Omega_{k,l'} = \frac{1}{2}\frac{k-l'}{k+l'}$ in the linear integral equation. Using Equations (4.16) and (4.17), and differentiating the general structure of $\mathbf{U}$, i.e. (2.20), one immediately obtains the dynamical evolution of $\mathbf{U}$ with respect to the independent variable x_j given by

$$\partial_{2j+1}\mathbf{U} = \mathbf{\Lambda}^{2j+1}\cdot\mathbf{U} + \mathbf{U}\cdot{}^t\mathbf{\Lambda}^{2j+1} - \frac{1}{2}\mathbf{U}\cdot(\mathbf{O}_{2j+1}\cdot\mathbf{\Lambda} - {}^t\mathbf{\Lambda}\cdot\mathbf{O}_{2j+1})\cdot\mathbf{U}, \quad j = 0,1,2,\cdots. \tag{4.18}$$

In addition, it can be proven that the infinite matrix $\mathbf{U}$ obeys the antisymmetry property ${}^t\mathbf{U} = -\mathbf{U}$. This can be derived from the antisymmetry property of $\mathbf{C}$ and $\mathbf{\Omega}$ (the antisymmetry of $\mathbf{\Omega}$ is obvious since the kernel is antisymmetric in k and l'). We can now refer to (4.18) together with the antisymmetry property of $\mathbf{U}$ as the BKP hierarchy in infinite matrix form because the structure of BKP (i.e. the infinite matrix $\mathbf{C}$ and the operator $\mathbf{\Omega}$) has been contained in the two relations.

In order to get a closed-form scalar equation, we set $u = U_{1,0} = -U_{0,1}$. Like AKP, consider x_1-, x_3- and x_{2j+1}-flows in the equation (4.18) and eliminating all the other variables apart from u, the nonlinear form of the x_{2j+1}-flow of the BKP hierarchy can be obtained. Among them the first nontrivial equation is the x_5-flow, which is the BKP equation, i.e.

$$9u_{x_5x_1} - 5u_{x_3x_3} + (-5u_{x_1x_1x_3} - 15u_{x_1}u_{x_3} + u_{x_1x_1x_1x_1x_1} + 15u_{x_1}u_{x_1x_1x_1} + 15u_{x_1}^3)_{x_1} = 0. \tag{4.19}$$

Other nonlinear forms (e.g. the modified BKP equation, etc.) in the BKP class also exist like AKP, but multi-component forms must be involved. We omit them here because we would prefer to be focusing on scalar nonlinear forms.

The tau function in the BKP class is defined as $\tau^2 = \det(1 + \mathbf{\Omega}\cdot\mathbf{C})$. The reason why we define τ^2 here instead of τ is because the determinant must be in the form of a perfect square due to the antisymmetry property of $\mathbf{\Omega}$ and $\mathbf{C}$ and this treatment will lead to the equations in τ written in a more elegant form. Some direct computation yields the dynamical evolution of $\partial_{2j+1}\ln\tau^2$ and the simplest one gives rise to the transform $(\ln\tau^2)_{x_1} = 2(\ln\tau)_{x_1} = u$, which transfers the nonlinear BKP hierarchy to its multilinear form. The first nontrivial multilinear equation in the hierarchy can be obtained from (4.19), which is given by

$$(D_1^6 - 5D_1^3D_3 - 5D_3^2 + 9D_1D_5)\tau\cdot\tau = 0, \tag{4.20}$$

where D_i is Hirota's operator. This is the bilinear form of the nonlinear BKP equation (4.19).

Now we consider the linear problem for the BKP hierarchy. In fact, differentiating (2.19) with respect to x_{2j+1}, one has the dynamical evolution

$$\partial_{2j+1}\mathbf{u}_k = \mathbf{\Lambda}^{2j+1}\cdot\mathbf{u}_k - \frac{1}{2}\mathbf{U}\cdot(\mathbf{O}_{2j+1}\cdot\mathbf{\Lambda} - {}^t\mathbf{\Lambda}\cdot\mathbf{O}_{2j+1})\cdot\mathbf{u}_k, \quad j = 0,1,2,\cdots, \tag{4.21}$$

where (4.18) and (4.21) are used. One can refer to this relation as the linear problem for the BKP hierarchy in infinite matrix form. Similarly to the linear problem for AKP, we denote $\phi = u_k^{(0)}$ and after eliminating the other components $u_k^{(i)}$ for $i \neq 0$ in Equation

(4.21), one can get the linear problems for the whole BKP hierarchy. For instance the spectral problem and the temporal evolution for the BKP equation (4.19) are given by

$$\phi_{x_3} = (\partial_1^3 + 3u_{x_1}\partial_1)\phi, \tag{4.22a}$$

$$\phi_{x_5} = \left[\partial_1^5 + 5u_{x_1}\partial_1^3 + 5u_{x_1x_1}\partial_1^2 + \left(\frac{10}{3}u_{x_1x_1x_1} + 5u_{x_1}^2 + \frac{5}{3}u_{x_3}\right)\partial_1\right]\phi. \tag{4.22b}$$

The linear problem for (4.20) can be obtained using the transform $u = 2(\ln\tau)_{x_1}$. We note that we made use of the dynamical evolution and the antisymmetry property of the infinite matrix of $\mathbf{U}$ in the derivation. The temporal parts for the other members in the hierarchy can be calculated in the same way by considering the corresponding flows in (4.21).

2.3. The CKP hierarchy. For the KP hierarchy of C-type (associated with the infinite-dimensional Lie algebra C_∞), we define the infinite matrix $\mathbf{C}$ as

$$\mathbf{C} = \iint_D \mathrm{d}\zeta(l,l')\rho_l\mathbf{c}_l{}^t\mathbf{c}_{l'}\rho_{l'}, \quad \rho_k = \exp\left(\sum_{j=0}^{\infty} k^{2j+1}x_{2j+1}\right), \quad \mathrm{d}\zeta(l',l) = \mathrm{d}\zeta(l,l'). \tag{4.23}$$

From the definition one can see that the only difference between the BKP and CKP classes is that the measure is now symmetric in the CKP case. The symmetric measure immediately provides the property $\mathbf{C} = {}^t\mathbf{C}$. Since there are only odd flows in the plane wave factors in the $\mathbf{C}$, the dynamical evolutions of the infinite matrix $\mathbf{C}$ only involve the odd flows and they are given by

$$\partial_{2j+1}\mathbf{C} = \mathbf{\Lambda}^{2j+1}\cdot\mathbf{C} + \mathbf{C}\cdot{}^t\mathbf{\Lambda}^{2j+1}, \tag{4.24}$$

which is exactly the same as that in the BKP class. We now require that the $\mathbf{\Omega}$ in the CKP class obeys the algebraic relation

$$\mathbf{\Omega}\cdot\mathbf{\Lambda} + {}^t\mathbf{\Lambda}\cdot\mathbf{\Omega} = \mathbf{O}. \tag{4.25}$$

This relation for the operator $\mathbf{\Omega}$ is the same as (4.3), and this tells us that the Cauchy kernel in this case is $\Omega_{k,l'} = \frac{1}{k+l'}$. Making use of the symmetry property of $\mathbf{C}$ and $\mathbf{\Omega}$ and following the definition of $\mathbf{U}$ (2.20), we have the symmetry property ${}^t\mathbf{U} = \mathbf{U}$. One can now consider the dynamical evolution of $\mathbf{U}$ with the help of (4.24) and (4.25). By differentiating $\mathbf{U}$ with respect to the independent variables x_{2j+1}, we have

$$\partial_{2j+1}\mathbf{U} = \mathbf{\Lambda}^{2j+1}\cdot\mathbf{U} + \mathbf{U}\cdot{}^t\mathbf{\Lambda}^{2j+1} - \mathbf{U}\cdot\mathbf{O}_{2j+1}\cdot\mathbf{U}, \quad j = 0,1,2,\cdots. \tag{4.26}$$

The dynamical relation (4.26), together with the symmetry property of $\mathbf{U}$, can be thought of as the CKP hierarchy in infinite matrix form.

We now look for scalar closed-form equations from the infinite matrix structure by choosing particular entries in $\mathbf{U}$. In fact, one can take $u = U_{0,0}$ and select x_1, x_3 and x_{2j+1} as the independent variables for the x_{2j+1}-flow of the CKP hierarchy in (4.26). As a result the CKP hierarchy can be found and the first nontrivial equation is the CKP equation

$$
\begin{aligned}
&9u_{x_5x_1} - 5u_{x_3x_3} \\
&\quad + \left(-5u_{x_1x_1x_3} - 15u_{x_1}u_{x_3} + u_{x_1x_1x_1x_1x_1} + 15u_{x_1}u_{x_1x_1x_1} + 15u_{x_1}^3 + \frac{45}{4}u_{x_1x_1}^2\right)_{x_1} = 0.
\end{aligned}
\tag{4.27}
$$

By selecting other entries in $\mathbf{U}$, one can also obtain other nonlinear forms from the infinite matrix structure nevertheless the price one has to pay is that they may be in multi-component form and thus we omit them here. A unified structure describing the CKP class should be the form in the tau function. In this case, it is defined by $\tau = \det(1+\mathbf{\Omega}\cdot\mathbf{C})$ like that in AKP and therefore it obeys the dynamical evolution $\partial_{2j+1}(\ln\tau) = \operatorname{tr}(\mathbf{O}_j\cdot\mathbf{U})$ in which $\mathbf{U}$ is symmetric. The first one of them gives us the transform $u = (\ln\tau)_{x_1}$ and therefore one obtains the multilinear form of the CKP equation (4.27) given by

$$
\begin{aligned}
&4\tau^3\tau_{x_1x_1x_1x_1x_1x_1} + 5\tau^2\tau_{x_1x_1x_1}^2 - 24\tau^2\tau_{x_1}\tau_{x_1x_1x_1x_1x_1} - 30\tau\tau_{x_1}\tau_{x_1x_1}\tau_{x_1x_1x_1} \\
&+ 45\tau_{x_1}^2\tau_{x_1x_1}^2 + 60\tau\tau_{x_1}\tau_{x_1x_1x_1x_1} - 60\tau_{x_1}^3\tau_{x_1x_1x_1} + 60\tau^2\tau_{x_1}\tau_{x_1x_1x_3} - 60\tau\tau_{x_1}^2\tau_{x_1x_3} \\
&+ 60\tau_{x_1}^3\tau_{x_3} - 60\tau\tau_{x_1}\tau_{x_1x_1}\tau_{x_3} + 20\tau^2\tau_{x_1x_1x_1}\tau_{x_3} - 20\tau^3\tau_{x_1x_1x_1x_3} + 20\tau^2\tau_{x_3}^2 - 20\tau^3\tau_{x_3x_3} \\
&+ 36\tau^3\tau_{x_1x_5} - 36\tau^2\tau_{x_1}\tau_{x_5} = 0.
\end{aligned}
\tag{4.28}
$$

This quadrilinear form is analogous to the result for the discrete CKP equation (see chapter 3) which is in the form of Cayley's $2\times2\times2$ hyperdeterminant. In other words, Equation (4.28) can be understood as the continuous analogue of a hyperdeterminant. The multilinear transform also brings the higher-order equations in u in the hierarchy to the corresponding multilinear forms in the tau function.

Similarly to how we derive (4.26), one can from (4.26) and (4.25) obtain the dynamical evolution of $\mathbf{u}_k$ as follow:

$$
\partial_{2j+1}\mathbf{u}_k = \mathbf{\Lambda}^{2j+1}\mathbf{u}_k - \mathbf{U}\cdot\mathbf{O}_{2j+1}\cdot\mathbf{u}_k, \quad j = 0,1,2,\cdots. \tag{4.29}
$$

One can refer to these relations together with $\mathbf{U} = {}^t\mathbf{U}$ as the linear problems for the CKP hierarchy in infinite matrix form. In fact, if we fix x_3 and x_1 and set $\phi = u_k^{(0)}$, the spatial part of the linear problem is derived by getting rid the other components in $\mathbf{u}_k$, and it is given by

$$
\phi_{x_3} = \left(\partial_1^3 + 3u_{x_1}\partial_1 + \frac{3}{2}u_{x_1x_1}\right)\phi. \tag{4.30a}
$$

This is the spatial part for the whole CKP hierarchy. The temporal part can also be calculated from (4.29) as well. For the CKP equation (4.28), namely the time variable is fixed at x_5, we have the temporal evolution

$$
\begin{aligned}
\phi_{x_5} = \Bigg[&\partial_1^5 + 5u_{x_1}\partial_1^3 + \frac{15}{2}u_{x_1x_1}\partial_1^2 \\
&+ \left(\frac{35}{6}u_{x_1x_1x_1} + 5u_{x_1}^2 + \frac{5}{3}u_{x_3}\right)\partial_1 + \left(\frac{5}{3}u_{x_1x_1x_1x_1} + 5u_{x_1}u_{x_1x_1} + \frac{5}{6}u_{x_1x_3}\right)\Bigg]\phi.
\end{aligned}
\tag{4.30b}
$$

The temporal evolutions for the higher-order equations in the hierarchy can be obtained similarly but the formulae of them become more and more complex.

3. (1+1)-dimensional soliton hierarchies

Dimensional reductions of higher-dimensional hierarchies can always be thought of a powerful tool to obtain (1+1)-dimensional integrable hierarchies. In general, following Sato's scheme, such reductions are normally done by imposing certain conditions on pseudo-differential operators and consequently the Lax operators for the corresponding lower-dimensional hierarchies arise, cf. e.g. the Kyoto School [**MJD00**], Konopelchenko and Strampp [**KS91**, **KS92**], Cheng and Li [**CL91**], and also Loris and Willox [**LW99**, **Lor99**], etc. Among these dimensional reductions, some of them give rise to integrable systems associated with matrix or nonlocal scalar Lax structure which we think will arise from the view point of the DL starting from a matrix linear integral equation. We do not discuss these systems in the current chapter and only consider the dimensional reductions leading to scalar differential spectral problems.

Dimensional reductions can also be realised within the DL framework. As we can see in the previous section, the integral equation associated with (2+1)-dimensional soliton hierarchies is a nonlocal Riemann–Hilbert problem, namely, there must be a double integral in it. When the double integral collapses, the integral equation turns out to be a local Riemann–Hilbert problem which is associated with (1+1)-dimensional soliton models. This method was recently used in [**ZZN12**] in order to construct an extended discrete BSQ equation and under such reductions more general solutions can be found for the obtained reduced soliton hierarchies. In this section, we generalise this method generically to the AKP, BKP and CKP hierarchies and as a result we obtain a huge class of (1+1)-dimensional soliton hierarchies.

3.1. Dimensional reductions of the KP-type hierarchies.

3.1.1. *Reductions of the AKP hierarchy.* We take the measure in a particular form as

$$\mathrm{d}\zeta(l,l') = \sum_{j=1}^{\mathcal{N}-1} \mathrm{d}\lambda_j(l)\mathrm{d}l'\delta(l'+\omega^j l), \tag{4.31}$$

where ω is an $\mathcal{N}$th root of unity, namely, $\omega^{\mathcal{N}} = 1$. The linear integral equation for the AKP hierarchy with a double integral then turns out to be an integral equation with only a single integral as follows:

$$\mathbf{u}_k + \sum_{j=1}^{\mathcal{N}-1} \int_{\Gamma_j} \mathrm{d}\lambda_j(l) \frac{\rho_k \sigma_{-\omega^j l}}{k-\omega^j l} \mathbf{u}_l = \rho_k \mathbf{c}_k, \tag{4.32a}$$

$$\rho_k = \exp\left(\sum_{j=1}^{\infty} k^j x_j\right), \quad \sigma_{k'} = \exp\left(-\sum_{j=1}^{\infty} (-k')^j x_j\right), \tag{4.32b}$$

where Γ_j is a certain contour (as degeneration of the domain D) for the integral and $\mathrm{d}\lambda_j(l)$ is the associated measure. This integral equation is the one for the $\mathcal{N}$th member in the GD hierarchy. In fact, such a reduction imposed on the AKP hierarchy also gives reduced forms of the infinite matrix $\mathbf{C}$ and the infinite matrix $\mathbf{U}$ respectively:

$$\mathbf{C} = \sum_{j=1}^{\mathcal{N}-1} \int_{\Gamma_j} \mathrm{d}\lambda_j(l) \rho_l \mathbf{c}_l {}^t\mathbf{c}_{-\omega^j l} \sigma_{-\omega^j l}, \quad \mathbf{U} = \sum_{j=1}^{\mathcal{N}-1} \int_{\Gamma_j} \mathrm{d}\lambda_j(l) \mathbf{u}_l {}^t\mathbf{c}_{-\omega^j l} \sigma_{-\omega^j l}. \tag{4.33}$$

Nevertheless, one can easily verify the representation of $\mathbf{U}$ in terms of the infinite matrices (2.20) will be invariant, i.e. we still have $\mathbf{U} = (1 - \mathbf{U} \cdot \mathbf{\Omega}) \cdot \mathbf{U}$, and so is that of $\mathbf{u}_k$ (2.19), namely, $\mathbf{u}_k = (1 - \mathbf{U} \cdot \mathbf{\Omega}) \cdot \mathbf{u}_k$. The same thing will also happen in the dimensional reductions of the BKP and CKP hierarchies.

Consider the property of the $\mathcal{N}$th root of unity, we can obviously see from the structure of the $\mathbf{C}$, i.e. Equation (4.33), that $\mathbf{C}$ satisfies not only the dynamical relation (4.2) but also the following algebraic relation:

$$\partial_j \mathbf{C} = \mathbf{\Lambda}^j \cdot \mathbf{C} - (-{}^t\mathbf{\Lambda})^j \cdot \mathbf{C} = 0, \quad j = 0 \mod \mathcal{N}. \tag{4.34}$$

In other words, from the view point of the DL framework, some dynamical relations degenerate to algebraic relations. This observation immediately leads to a similar property for the nonlinear variable $\mathbf{U}$ which is given by

$$\partial_j \mathbf{U} = \mathbf{\Lambda}^j \cdot \mathbf{U} - \mathbf{U} \cdot (-{}^t\mathbf{\Lambda})^j - \mathbf{U} \cdot \mathbf{O}_j \cdot \mathbf{U} = 0, \quad j = 0 \mod \mathcal{N}. \tag{4.35}$$

In fact, this algebraic relation can be proven by differentiating (2.20) with respect to x_j and making use of (4.3) (which is invariant under the reduction), (4.2) and (4.34). Equations (4.4) together with (4.35) can be considered as the hierarchy in GD of rank $\mathcal{N}$ in infinite matrix form. Furthermore, for the multilinear variable τ one can easily derive the constraints $\partial_j \tau = 0$ for $j = 0 \mod \mathcal{N}$ due to (4.33) and the variable $\mathbf{u}_k$ for the linear problem now obeys the algebraic relation

$$\partial_j \mathbf{u}_k = \mathbf{\Lambda}^j \cdot \mathbf{u}_k - \mathbf{U} \cdot \mathbf{O}_j \cdot \mathbf{u}_k = k^j \mathbf{u}_k, \quad j = 0 \mod \mathcal{N}, \tag{4.36}$$

which together with (4.13) will provide the Lax pairs for the reduced (1+1)-dimensional hierarchies.

3.1.2. *Reductions of the BKP hierarchy.* In the BKP class, we introduce the reduction of the measure in a slightly different way, which is given by

$$\mathrm{d}\zeta(l, l') = \sum_{j=1}^{\mathcal{N}-1} \mathrm{d}\lambda_j(l) \mathrm{d}l' \delta(l' + \omega^j l) - \sum_{j=1}^{\mathcal{N}-1} \mathrm{d}\lambda_j(l') \mathrm{d}l \delta(l + \omega^j l'). \tag{4.37}$$

The reason why we take this more complex form compared to (4.31) is because in the BKP hierarchy the measure $\mathrm{d}\zeta(l, l')$ is antisymmetric and only the dimensional reduction (4.37) can preserve the property. Like the reductions of AKP, we have that in the BKP hierarchy the reduced integral equation is given by

$$\mathbf{u}_k + \sum_{j=1}^{\mathcal{N}-1} \int_{\Gamma_j} \mathrm{d}\lambda_j(l) \frac{1}{2} \rho_k \frac{k + \omega^j l}{k - \omega^j l} \rho_{-\omega^j l} \mathbf{u}_l - \sum_{j=1}^{\mathcal{N}-1} \int_{\Gamma_j} \mathrm{d}\lambda_j(l') \frac{1}{2} \rho_k \frac{k - l'}{k + l'} \rho_{l'} \mathbf{u}_{-\omega^j l} = \rho_k \mathbf{c}_k. \tag{4.38}$$

And the reduced infinite matrix $\mathbf{C}$ becomes

$$\mathbf{C} = \sum_{j=1}^{\mathcal{N}-1} \int_{\Gamma_j} \mathrm{d}\lambda_j(l) \rho_l \mathbf{c}_l {}^t\mathbf{c}_{-\omega^j l} \rho_{-\omega^j l} - \sum_{j=1}^{\mathcal{N}-1} \int_{\Gamma_j} \mathrm{d}\lambda_j(l') \rho_{-\omega^j l'} \mathbf{c}_{-\omega^j l'} {}^t\mathbf{c}_{l'} \rho_{l'}, \tag{4.39}$$

where $\rho_k = \exp(\sum_{j=0}^{\infty} k^{2j+1} x_{2j+1})$ and obviously it still obeys ${}^t\mathbf{C} = -\mathbf{C}$. Simultaneously, for the nonlinear variable $\mathbf{U}$ we have

$$\mathbf{U} = \sum_{j=1}^{\mathcal{N}-1} \int_{\Gamma_j} \mathrm{d}\lambda_j(l) \mathbf{u}_l {}^t\mathbf{c}_{-\omega^j l} \rho_{-\omega^j l} - \sum_{j=1}^{\mathcal{N}-1} \int_{\Gamma_j} \mathrm{d}\lambda_j(l') \mathbf{u}_{-\omega^j l'} {}^t\mathbf{c}_{l'} \rho_{l'}, \tag{4.40}$$

which also obeys the antisymmetry property ${}^t\mathbf{U} = -\mathbf{U}$ following from that of $\mathbf{\Omega}$ defined in (4.17) and that of the reduced $\mathbf{C}$.

Some straightforward computation shows that the infinite matrix $\mathbf{C}$ in the $\mathcal{N}$-reduction of the BKP hierarchy obeys

$$\mathbf{\Lambda}^j \cdot \mathbf{C} - (-{}^t\mathbf{\Lambda})^j \cdot \mathbf{C} = 0, \quad j = 0 \mod \mathcal{N}, \tag{4.41}$$

and this together with (2.20) and (4.17) give rise to the algebraic relation for the nonlinear variable $\mathbf{U}$ which is given by

$$\mathbf{\Lambda}^j \cdot \mathbf{U} - \mathbf{U} \cdot (-{}^t\mathbf{\Lambda})^j - \frac{1}{2}\mathbf{U} \cdot (\mathbf{O}_j \cdot \mathbf{\Lambda} - {}^t\mathbf{\Lambda} \cdot \mathbf{O}_j) \cdot \mathbf{U} = 0, \quad j = 0 \mod \mathcal{N}. \tag{4.42}$$

This relation obviously implies that $\partial_{2j+1}\mathbf{U} = 0$ when $2j + 1 = 0 \mod \mathcal{N}$ in (4.18) and consequently we have also $\partial_{2j+1}\tau = 0$. Similarly one can also from (2.19) prove that the reduction (4.37) gives us

$$k^j \mathbf{u}_k = \mathbf{\Lambda}^j \cdot \mathbf{u}_k - \frac{1}{2}\mathbf{U} \cdot (\mathbf{O}_j \cdot \mathbf{\Lambda} - {}^t\mathbf{\Lambda} \cdot \mathbf{O}_j) \cdot \mathbf{u}_k, \quad j = 0 \mod \mathcal{N}, \tag{4.43}$$

and particularly when $2j+1 = 0 \mod \mathcal{N}$ this algebraic relation implies $\partial_{2j+1}\mathbf{u}_k = k^{2j+1}\mathbf{u}_k$ if one follows from (4.21). This relation together with (4.21) will later give us the Lax pairs for the (1+1)-dimensional hierarchies arising from the reductions of the BKP class.

3.1.3. *Reductions of the CKP hierarchy.* While in the CKP class, we take a particular measure in the form of

$$\mathrm{d}\zeta(l, l') = \sum_{j=1}^{\mathcal{N}-1} \mathrm{d}\lambda_j(l) \mathrm{d}l' \delta(l' + \omega^j l) + \sum_{j=1}^{\mathcal{N}-1} \mathrm{d}\lambda_j(l') \mathrm{d}l \delta(l + \omega^j l'). \tag{4.44}$$

This reduction on the measure has been symmetrised, namely, the reduced measure preserves the symmetry property, and it in turn implies that we now have the integral equation for the reduced hierarchies from the CKP class as

$$\mathbf{u}_k + \sum_{j=1}^{\mathcal{N}-1} \int_{\Gamma_j} \mathrm{d}\lambda_j(l) \frac{\rho_k \rho_{-\omega^j l}}{k - \omega^j l} \mathbf{u}_l + \sum_{j=1}^{\mathcal{N}-1} \int_{\Gamma_j} \mathrm{d}\lambda_j(l') \frac{\rho_k \rho_{l'}}{k + l'} \mathbf{u}_{-\omega^j l} = \rho_k \mathbf{c}_k, \tag{4.45}$$

where ρ_k is exactly the same as the one in the reductions of BKP, namely

$$\rho_k = \exp\left(\sum_{j=0}^{\infty} k^{2j+1} x_{2j+1}\right).$$

The infinite matrix $\mathbf{C}$ in this case is now expressed by

$$\mathbf{C} = \sum_{j=1}^{\mathcal{N}-1} \int_{\Gamma_j} \mathrm{d}\lambda_j(l) \rho_l \mathbf{c}_l {}^t\mathbf{c}_{-\omega^j l} \rho_{-\omega^j l} + \sum_{j=1}^{\mathcal{N}-1} \int_{\Gamma_j} \mathrm{d}\lambda_j(l') \rho_{-\omega^j l'} \mathbf{c}_{-\omega^j l'} {}^t\mathbf{c}_{l'} \rho_{l'}, \tag{4.46}$$

and it obeys the symmetry property ${}^t\mathbf{C} = \mathbf{C}$. Furthermore the nonlinear variable $\mathbf{U}$ now turns out to be

$$\mathbf{U} = \sum_{j=1}^{\mathcal{N}-1} \int_{\Gamma_j} \mathrm{d}\lambda_j(l) \mathbf{u}_l {}^t\mathbf{c}_{-\omega^j l} \rho_{-\omega^j l} + \sum_{j=1}^{\mathcal{N}-1} \int_{\Gamma_j} \mathrm{d}\lambda_j(l') \mathbf{u}_{-\omega^j l'} {}^t\mathbf{c}_{l'} \rho_{l'}, \tag{4.47}$$

and due to the symmetry property of $\mathbf{C}$ and $\mathbf{\Omega}$ in the reductions, from (2.20) one still has ${}^t\mathbf{U} = \mathbf{U}$.

Noticing that $\omega^{\mathcal{N}} = 1$, one can easily prove that the infinite matrix $\mathbf{C}$ obeys the algebraic relation

$$\mathbf{\Lambda}^j \cdot \mathbf{C} - (-{}^t\mathbf{\Lambda})^j \cdot \mathbf{C} = 0, \quad j = 0 \mod \mathcal{N}, \tag{4.48}$$

which is exactly the same as the one in the reductions of BKP, i.e. (4.41), and consequently one can now follow from (2.20) and obtain the important algebraic relation for the nonlinear dynamical variable $\mathbf{U}$ as follow:

$$\mathbf{\Lambda}^j \cdot \mathbf{U} - \mathbf{U} \cdot (-{}^t\mathbf{\Lambda})^j - \mathbf{U} \cdot \mathbf{O}_j \cdot \mathbf{U} = 0, \quad j = 0 \mod \mathcal{N}. \tag{4.49}$$

In the particular case when $2j+1 = 0 \mod \mathcal{N}$, this above algebraic relation yields $\partial_{2j+1}\mathbf{U} = 0$. While from (4.48) we can prove $\partial_{2j+1}\mathbf{C} = 0$ and consequently $\partial_{2j+1}\tau = 0$ in the same case.

On the linear level, taking (4.49) into consideration, one can find from (2.19) that the linear variable $\mathbf{u}_k$ satisfies the algebraic relation

$$k^j \mathbf{u}_k = \mathbf{\Lambda}^j \cdot \mathbf{u}_k - \mathbf{U} \cdot \mathbf{O}_j \cdot \mathbf{u}_k, \quad j = 0 \mod \mathcal{N}, \tag{4.50}$$

which implies $\partial_{2j+1}\mathbf{u}_k = k^{2j+1}\mathbf{u}_k$ when $2j+1 = 0 \mod \mathcal{N}$ and this relation is the ingredient for Lax pairs of the reduced lower-dimensional equations.

Table 1 shows the first few examples of the dimensional reductions of the KP-type equations. One remark is that the 2-reduction of the BKP hierarchy leads to triviality due to the antisymmetry property of the measure $\mathrm{d}\zeta(l,l')$. In the following subsections, we give the DL scheme for the hierarchies of all the equations listed in the table as examples according to the generic scheme for the dimensional reductions given in this subsection.

3.2. The KdV hierarchy. The KdV hierarchy is from the 2-reduction of the AKP hierarchy. The general scheme for KdV is given by the dynamical relation and the algebraic relation

$$\partial_{2j+1}\mathbf{U} = \mathbf{\Lambda}^{2j+1} \cdot \mathbf{U} + \mathbf{U} \cdot {}^t\mathbf{\Lambda}^{2j+1} - \mathbf{U} \cdot \mathbf{O}_{2j+1} \cdot \mathbf{U}, \quad j = 0, 1, 2, \cdots, \tag{4.51a}$$

$$\mathbf{\Lambda}^2 \cdot \mathbf{U} - \mathbf{U} \cdot {}^t\mathbf{\Lambda}^2 - \mathbf{U} \cdot \mathbf{O}_2 \cdot \mathbf{U} = 0. \tag{4.51b}$$

These relations constitute the infinite matrix version of the KdV hierarchy. In order to obtain some closed-form equations, namely some scalar equations, from the scheme, we take $u = U_{0,0}$, $v = [\ln(1 - U_{0,-1})]_{x_1}$ and $z = U_{-1,-1} - x_1$ and these variables give rise

Dimensional reductions	(1+1)-dimensional soliton hierarchies
2-reduction of AKP	KdV
3-reduction of AKP	BSQ
4-reduction of AKP	generalised HS [**SH82**]
3-reduction of BKP/CKP	SK [**SK74**] and KK [**Kau80**]
5-reduction of BKP/CKP	bidirectional SK and bidirectional KK [**DP01**]
2-reduction of CKP	KdV
4-reduction of CKP	HS [**HS81**, **SH82**, **DS85**]
4-reduction of BKP	KdV
6-reduction of BKP	Ito [**Ito80**]

TABLE 1. Some examples for the dimensional reductions of AKP, BKP and CKP

to the KdV, modified KdV (mKdV) and Schwarzian KdV (SKdV) hierarchies. The first nontrivial equation (when $j=1$ in (4.51a)) in the KdV hierarchy is the KdV equation:

$$u_{x_3}=\frac{1}{4}u_{x_1x_1x_1}+\frac{3}{2}u_{x_1}^2. \tag{4.52}$$

From (4.51a) and (4.51b) one can also find the connection between u and v as well as that between v and z, i.e. the Miura-type transforms among the nonlinear variables given by

$$u_{x_1}=-\frac{1}{2}(v_{x_1}+v^2),\quad v=\frac{1}{2}\frac{z_{x_1x_1}}{z_{x_1}}. \tag{4.53}$$

The above transforms help us to derive other nonlinear forms in the KdV class from (4.52) and we find

$$v_{x_3}=\frac{1}{4}v_{x_1x_1x_1}-\frac{3}{2}v^2v_{x_1}, \tag{4.54}$$

$$\frac{z_{x_3}}{z_{x_1}}=\frac{1}{4}\{z,x\}\doteq\frac{1}{4}\frac{z_{x_1x_1x_1}}{z_{x_1}}-\frac{3}{8}\frac{z_{x_1x_1}^2}{z_{x_1}^2}. \tag{4.55}$$

The equation in v is the mKdV equation and the equation in z is the SKdV equation. Actually, since the 2-reduction of AKP is equivalent to $\partial_2\mathbf{U}=0$ as we have pointed out in subsection 3.1, the above results can very easily be obtained from the results in the AKP class discussed in subsection 2.1. The higher-order equations in the mKdV and SKdV hierarchies can be calculated via the same Miura-type transforms.

The tau function of the KdV class is the same as the one for the AKP class with an additional constraint $\partial_{2j}\tau=0$. Therefore the same transform $u=(\ln\tau)_{x_1}$ substituted into (4.52) gives us the bilinear KdV equation

$$(D_1^4-4D_1D_3)\tau\cdot\tau=0, \tag{4.56}$$

which can also be obtained from (4.12) directly by reduction.

The infinite matrix version of the linear problem of the KdV hierarchy can be obtained from the reduction of that for the AKP hierarchy and it is given by

$$k^2\mathbf{u}_k=\mathbf{\Lambda}^2\cdot\mathbf{u}_k-\mathbf{U}\cdot\mathbf{O}_2\cdot\mathbf{u}_k,\quad \partial_{2j+1}\mathbf{u}_k=\mathbf{\Lambda}^{2j+1}\cdot\mathbf{u}_k-\mathbf{U}\cdot\mathbf{O}_{2j+1}\cdot\mathbf{u}_k,\quad j=0,1,2,\cdots. \tag{4.57}$$

By taking $\phi = (\mathbf{u}_k)_0$, one can obtain the Lax pair for the KdV hierarchy and the one for the KdV equation (4.52) is

$$L^{\mathrm{KdV}}\phi = k^2\phi, \quad L^{\mathrm{KdV}} = \partial_1^2 + 2u_{x_1}, \tag{4.58a}$$

$$\phi_{x_3} = \left(\partial_1^3 + 3u_{x_1}\partial_1 + \frac{3}{2}u_{x_1x_1}\right)\phi. \tag{4.58b}$$

Equation (4.58a) is the spectral problem for the whole KdV hierarchy. The Lax pair for the other nonlinear forms (4.54) and (4.55) and the bilinear form (4.56) can be written down if one replaces u by the other variables according the transforms given in this subsection.

The 2-reduction of the CKP hierarchy is exactly the same as the scheme from that of AKP, therefore the same results can be obtained. In fact, the additional constraint ${}^t\mathbf{U} = \mathbf{U}$ from CKP (which AKP does not have) only appears in the KdV class ($\mathcal{N} = 2$) as a special case, and it is equivalent to the algebraic relation (4.51b).

The 4-reduction of the BKP hierarchy is slightly different and the infinite matrix structure of $\mathbf{U}$ is given by

$$\partial_{2j+1}\mathbf{U} = \mathbf{\Lambda}^{2j+1}\cdot\mathbf{U} + \mathbf{U}\cdot{}^t\mathbf{\Lambda}^{2j+1} - \frac{1}{2}\mathbf{U}\cdot(\mathbf{O}_{2j+1}\cdot\mathbf{\Lambda} - {}^t\mathbf{\Lambda}\cdot\mathbf{O}_{2j+1})\cdot\mathbf{U}, \quad j = 0,1,2,\cdots,$$

$$\mathbf{\Lambda}^4\cdot\mathbf{U} - \mathbf{U}\cdot{}^t\mathbf{\Lambda}^4 - \frac{1}{2}\mathbf{U}\cdot(\mathbf{O}_4\cdot\mathbf{\Lambda} - {}^t\mathbf{\Lambda}\cdot\mathbf{O}_4)\cdot\mathbf{U} = 0, \quad {}^t\mathbf{U} = -\mathbf{U}.$$

Choosing the entry $u = U_{1,0}$ in the infinite matrix one can derive the KdV hierarchy and the first nontrivial equation is $u_{x_3} + \frac{1}{2}u_{x_1x_1x_1} + \frac{3}{2}u_{x_1}^2 = 0$ which is the same as (4.52) up to some scaling on the independent variables and the dependent variable. The bilinear transform $u = 2(\ln\tau)_{x_1}$ according to BKP gives us the bilinear form $(D_1^4 + 2D_1D_3)\tau\cdot\tau = 0$ obviously. The infinite matrix structure for the linear variable $\mathbf{u}_k$ in this case obeys

$$k^4\mathbf{u}_k = \mathbf{\Lambda}^4\cdot\mathbf{u}_k - \frac{1}{2}\mathbf{U}\cdot(\mathbf{O}_4\cdot\mathbf{\Lambda} - {}^t\mathbf{\Lambda}\cdot\mathbf{O}_4)\cdot\mathbf{u}_k,$$

$$\partial_{2j+1}\mathbf{u}_k = \mathbf{\Lambda}^{2j+1}\cdot\mathbf{U} + \mathbf{U}\cdot{}^t\mathbf{\Lambda}^{2j+1} - \frac{1}{2}\mathbf{U}\cdot(\mathbf{O}_{2j+1}\cdot\mathbf{\Lambda} - {}^t\mathbf{\Lambda}\cdot\mathbf{O}_{2j+1})\cdot\mathbf{U},$$

where $j = 0,1,2,\cdots$, and it leads to a 4th-order Lax pair for KdV if we take $\phi = u_k^{(0)}$:

$$L^{\mathrm{KdV}}\phi = k^4\phi, \quad \phi_{x_3} = (\partial_1^3 + 3u_{x_1}\partial_1)\phi,$$

where $L^{\mathrm{KdV}} = \partial_1^4 + 4u_{x_1}\partial_1^2 + 2u_{x_1x_1}\partial_1$.

3.3. The BSQ hierarchy. In this section we consider the 3-reduction of AKP. From the general structure in subsection 3.1, when $\mathcal{N} = 3$ we have the dynamical relation for $\mathbf{U}$ together with the associated algebraic relation as follows:

$$\partial_j\mathbf{U} = \mathbf{\Lambda}^j\cdot\mathbf{U} - \mathbf{U}\cdot(-{}^t\mathbf{\Lambda})^j - \mathbf{U}\cdot\mathbf{O}_j\cdot\mathbf{U}, \quad j \neq 0 \mod 3, \quad j \in \mathbb{Z}^+, \tag{4.59a}$$

$$\mathbf{\Lambda}^3\cdot\mathbf{U} + \mathbf{U}\cdot{}^t\mathbf{\Lambda}^3 - \mathbf{U}\cdot\mathbf{O}_3\cdot\mathbf{U} = 0. \tag{4.59b}$$

The relations constitute the infinite matrix representation of the BSQ hierarchy. In fact, these relations are equivalent to the dynamical relations for the AKP hierarchy (4.4) subject to a constraint $\partial_j\mathbf{U} = 0$ for $j = 0 \mod 3$. If one introduces the nonlinear variables $u = U_{0,0}$, $v = [\ln(1 - U_{0,-1})]_{x_1}$ and $z = U_{-1,-1} - x_1$, from the (4.59a) and (4.59b) one can obtain the BSQ, modified BSQ (mBSQ) and Schwarzian BSQ (SBSQ) hierarchies respectively. The first nontrivial equation in each hierarchies are

$$\left(\frac{1}{4}u_{x_1x_1x_1}+\frac{3}{2}u_{x_1}^2\right)_{x_1}+\frac{3}{4}u_{x_2x_2}=0, \tag{4.60}$$

$$\left(\frac{1}{4}v_{x_1x_1x_1}-\frac{3}{2}v^2v_{x_1}\right)_{x_1}+\frac{3}{2}v_{x_1x_1}\partial_{x_1}^{-1}v_{x_2}+\frac{3}{2}v_{x_1}v_{x_2}+\frac{3}{4}v_{x_2x_2}=0, \tag{4.61}$$

$$\frac{1}{4}\{z,x_1\}_{x_1}+\frac{3}{4}\frac{z_{x_2}}{z_{x_1}}\left(\frac{z_{x_2}}{z_{x_1}}\right)_{x_1}+\frac{3}{4}\left(\frac{z_{x_2}}{z_{x_1}}\right)_{x_2}=0. \tag{4.62}$$

These equations are usually referred to as the BSQ, mBSQ and SBSQ equations respectively. Furthermore, it is obvious to see that the Miura transforms for AKP (4.8) are invariant under the 3-reduction and therefore Equations (4.60), (4.61) and (4.62) are still connected by the following Miura-type transforms

$$u_{x_1}=-\frac{1}{2}(v_{x_1}+v^2-\partial_{x_1}^{-1}v_{x_2}),\quad v=\frac{1}{2}\frac{z_{x_1x_1}+z_{x_2}}{z_{x_1}}. \tag{4.63}$$

One can now consider the bilinear form of the BSQ equation. In fact, we have already known in subsection 3.1 that in the BSQ class $\tau_{x_3}=0$ and hence from the bilinear KP equation (4.12) one immediately obtains the bilinear form

$$(D_1^4+3D_2^2)\tau\cdot\tau=0. \tag{4.64}$$

Alternatively it can also be obtained by replacing u in (4.60) by $(\ln\tau)_{x_1}$ which holds for AKP and its dimensional reductions. The multilinear form for the other members in the BSQ hierarchy can be calculated in the same way.

The linear problem for the BSQ hierarchy in infinite matrix structure is given by

$$k^3\mathbf{u}_k=\mathbf{\Lambda}^3\cdot\mathbf{u}_k-\mathbf{U}\cdot\mathbf{O}_3\cdot\mathbf{u}_k,\quad \partial_j\mathbf{u}_k=\mathbf{\Lambda}^j\cdot\mathbf{u}_k-\mathbf{U}\cdot\mathbf{O}_j\cdot\mathbf{u}_k,\quad j\neq 0 \mod 3,\quad j\in\mathbb{Z}^+. \tag{4.65}$$

Eliminating the index-raising operator $\mathbf{\Lambda}$ and setting $\phi=u_k^{(0)}$, we can derive the Lax pair for the BSQ hierarchy. For instance the Lax pair for the x_2-flow, i.e. the BSQ equation, is in the form of

$$L^{\mathrm{BSQ}}\phi=k^3\phi,\quad L^{\mathrm{BSQ}}=\partial_1^3+3u_{x_1}\partial_1+\frac{3}{2}(u_{x_1x_1}+u_{x_2}), \tag{4.66a}$$

$$\phi_{x_2}=(\partial_1^2+2u_{x_1})\phi. \tag{4.66b}$$

The bilinear transform and the Miura-type transforms substituted into the formulae yields the Lax pairs for the modified and Schwarzian hierarchies.

3.4. The generalised HS hierarchy. We now consider $\mathcal{N}=4$ in the dimensional reductions of the AKP class. In fact, the (1+1)-dimensional integrable hierarchy in this case can longer be written in scalar form. In other words, we will obtain a hierarchy of multicomponent systems. Later one can see that the first nontrivial system is a 3-component generalisation of the famous HS hierarchy (a coupled KdV hierarchy) and here we refer to it as the generalised HS (gHS) hierarchy. According to the general framework, we can write down the infinite matrix formula of $\mathbf{U}$, namely, the dynamical relation and the algebraic relation for $\mathbf{U}$ given by

$$\partial_j \mathbf{U} = \mathbf{\Lambda}^j \cdot \mathbf{U} - \mathbf{U} \cdot (-{}^t\mathbf{\Lambda})^j - \mathbf{U} \cdot \mathbf{O}_j \cdot \mathbf{U}, \quad j \neq 0 \mod 4, \quad j \in \mathbb{Z}^+, \tag{4.67a}$$

$$\mathbf{\Lambda}^4 \cdot \mathbf{U} - \mathbf{U} \cdot {}^t\mathbf{\Lambda}^4 - \mathbf{U} \cdot \mathbf{O}_4 \cdot \mathbf{U} = 0, \tag{4.67b}$$

which can be understood as the gHS hierarchy in infinite matrix form. Now we introduce the following nonlinear variables which are some combinations of the entries in the infinite matrix $\mathbf{U}$:

$$u = U_{0,0}, \quad v = [\ln(1 - U_{0,-1})]_{x_1}, \quad w = -\frac{U_{1,-1}}{1 - U_{0,-1}},$$

$$r = U_{1,0} - U_{0,1}, \quad s = U_{1,0} + \frac{U_{0,0}U_{1,-1} - U_{2,-1}}{1 - U_{0,-1}},$$

$$p = U_{3,0} + U_{0,3} - U_{2,1} - U_{1,2} + 2(U_{0,0}U_{1,1} - U_{1,0}U_{0,1}),$$

$$q = U_{3,0} - U_{0,3} + U_{2,1} - U_{1,2} + 2\frac{U_{1,-1}(U_{2,0} - U_{0,2}) - U_{2,-1}(U_{1,0} - U_{0,1})}{1 - U_{0,-1}}.$$

Taking u, p and r as the dependent variables of a system, one can from the infinite matrix structure, i.e. (4.67a) and (4.67b), find the gHS hierarchy. The first nontrivial equation is the x_3-flow, namely, the gHS equation

$$u_{x_3} = \frac{1}{4}u_{x_1x_1x_1} + \frac{3}{2}u_{x_1}^2 + \frac{3}{4}(p - r^2), \tag{4.68a}$$

$$p_{x_3} = -\frac{1}{2}p_{x_1x_1x_1} - 3u_{x_1}p_{x_1}, \tag{4.68b}$$

$$r_{x_3} = -\frac{1}{2}r_{x_1x_1x_1} - 3u_{x_1}r_{x_1}. \tag{4.68c}$$

This equation was first given in [**SH82**] as the 4-reduction on the pseudo-differential operator in the KP hierarchy in order to derive the HS equation. In subsection 3.5 we will see that this is very natural from the view point of the DL framework. The infinite matrix structure also provides us with a Miura transform as follow:

$$u_{x_1} = -\frac{1}{2}(v_{x_1} + v^2 - s + r), \quad p = -s_{x_1x_1} - s^2 - 2vs_{x_1} + 2sr. \tag{4.69}$$

This transform brings us from the gHS hierarchy to its modification. For instance, substituting the transform into the gHS equation gives us the generalised modified HS (gmHS) equation

$$v_{x_3} = \frac{1}{4}v_{x_1x_1x_1} - \frac{3}{2}v^2v_{x_1} + \frac{3}{2}(vs - vr)_{x_1} + \frac{3}{4}(s + r)_{x_1x_1}, \tag{4.70a}$$

$$s_{x_3} = -\frac{1}{2}s_{x_1x_1x_1} - \frac{3}{2}ss_{x_1} - \frac{3}{2}(v_{x_1} - v^2)s_{x_1} + \frac{3}{2}rs_{x_1}, \tag{4.70b}$$

$$r_{x_3} = -\frac{1}{2}r_{x_1x_1x_1} + \frac{3}{2}rr_{x_1} + \frac{3}{2}(v_{x_1} + v^2)r_{x_1} - \frac{3}{2}sr_{x_1}. \tag{4.70c}$$

Furthermore, one can find another Miura transform in the form of

$$w_{x_1} = \frac{1}{2}(v_{x_1} - v^2 + s - r), \quad q = r_{x_1x_1} - r^2 - 2vr_{x_1} + 2sr,$$

and this transform connects the gmHS hierarchy with a hierarchy in terms of the variables w, q and s. The hierarchy in w, q and s is exactly the same as the gHS hierarchy. For example the variables obeys the gHS equation (4.68) if one considers the x_3-flow. At the moment we do not need this hierarchy as w, q and s do not bring us a new nonlinear form for the gHS hierarchy but later in subsection 3.5 we will see that the gHS hierarchies in (u, p, r) and (w, q, s) will lead to two different nonlinear forms in the C-type reduction.

Since the gHS is the 4-reduction of AKP, the independent variables apart from x_{4j} for $j \in \mathbb{Z}^+$ still exist in the hierarchy. Notice this and consider (4.11), one can identify that $u = (\ln\tau)_{x_1}$, $r = (\ln\tau)_{x_2}$ and $p = (\ln\tau)_{x_2x_2} + (\ln\tau)^2_{x_2}$. The multilinear transforms help us to reformulate the gHS hierarchy and to get its multilinear form. For instance, we can from (4.68) derive

$$(D_1^4 - 4D_1D_3 + 3D_2^2)\tau\cdot\tau = 0, \tag{4.71a}$$
$$(D_1^3D_2 + 2D_2D_3)\tau\cdot\tau = 0, \tag{4.71b}$$
$$(D_1^3D_2 + 2D_2D_3)\tau\cdot\tau_{x_2} = 0. \tag{4.71c}$$

The system of bilinear equations is a system with independent variables to x_1, x_2 and x_3. Since there is no x_2 in the nonlinear form, here x_2 can be thought of as an auxiliary variable. If fact, the bilinear form is nothing but the bilinear KP equation in addition to two constraints.

The linear problem for the gHS class in infinite matrix structure turns out to be

$$k^4\mathbf{u}_k = \mathbf{\Lambda}^4\cdot\mathbf{u}_k - \mathbf{U}\cdot\mathbf{O}_4\cdot\mathbf{u}_k, \quad \partial_j\mathbf{u}_k = \mathbf{\Lambda}^j\cdot\mathbf{u}_k - \mathbf{U}\cdot\mathbf{O}_j\cdot\mathbf{u}_k, \quad j\neq 0 \mod 4, \quad j\in\mathbb{Z}^+. \tag{4.72}$$

By considering the x_j-flows, it gives us the Lax pair for the gHS hierarchy if one sets $\phi = u_k^{(0)}$. Let us take the x_3-flow as an example. The first equation in (4.72) gives us the spectral problem and the second one when $j = 3$ gives the corresponding temporal part for (4.68):

$$L^{\mathrm{gHS}}\phi = k^4\phi, \tag{4.73a}$$
$$\phi_{x_3} = \left[\partial_1^3 + 3u_{x_1}\partial_1 + \frac{3}{2}(u_{x_1x_1} + r_{x_1})\right]\phi, \tag{4.73b}$$

where the differential operator is given by

$$L^{\mathrm{gHS}} = \partial_1^4 + 4u_{x_1}\partial_1^2 + (4u_{x_1x_1} + 2r_{x_1})\partial_1 + 2u_{x_1x_1x_1} + 4u_{x_1}^2 + p + r_{x_1x_1} - r^2.$$

The Lax pairs for the hierarchies in other forms such as the modified hierarchy and the bilinear hierarchy can be calculated using the transforms given above.

3.5. The HS hierarchy. Now we consider the HS hierarchy as the 4-reduction of the CKP class. The infinite matrix structure in this class is

$$\partial_{2j+1}\mathbf{U} = \mathbf{\Lambda}^{2j+1}\cdot\mathbf{U} + \mathbf{U}\cdot{}^t\mathbf{\Lambda}^{2j+1} - \mathbf{U}\cdot\mathbf{O}_{2j+1}\cdot\mathbf{U}, \quad j = 0, 1, 2\cdots, \tag{4.74a}$$
$$\mathbf{\Lambda}^4\cdot\mathbf{U} - \mathbf{U}\cdot{}^t\mathbf{\Lambda}^4 - \mathbf{U}\cdot\mathbf{O}_4\cdot\mathbf{U} = 0, \quad {}^t\mathbf{U} = \mathbf{U}. \tag{4.74b}$$

If one compares it with the structure in the gHS class, it is obvious to see that the only difference between HS and gHS is that in the HS class one only considers odd flows and in addition the infinite matrix $\mathbf{U}$ satisfies the symmetry property. Therefore one can define the nonlinear variables exactly the same as those in gHS, nevertheless some of the variables in gHS now become zero due to the symmetry property. Concretely, we have $r = q = 0$. The nonlinear variables in the class are now given by

$$u = U_{0,0}, \quad v = [\ln(1 - U_{0,-1})]_{x_1}, \quad w = -\frac{U_{1,-1}}{1 - U_{0,-1}}, \quad z = U_{-1,-1} - x_1,$$

$$p = 2U_{3,0} - 2U_{2,1} + 2(U_{0,0}U_{1,1} - U_{1,0}^2), \quad s = U_{1,0} + \frac{U_{0,0}U_{1,-1} - U_{2,-1}}{1 - U_{0,-1}},$$

The hierarchy based on u and p can be obtained from the gHS hierarchy in (u, p, r) by setting $r = 0$ directly. The first nontrivial nonlinear equation in the hierarchy is

$$u_{x_3} = \frac{1}{4}u_{x_1x_1x_1} + \frac{3}{2}u_{x_1}^2 + \frac{3}{4}p, \tag{4.75a}$$

$$p_{x_3} = -\frac{1}{2}p_{x_1x_1x_1} - 3u_{x_1}p_{x_1}, \tag{4.75b}$$

which was first given by Satsuma and Hirota as an equivalent form of the HS equation (cf. [**SH82**]). While setting $q = 0$ in the gHS hierarchy expressed by (w, q, s), one can easily find the original form of the HS hierarchy and the HS equation is

$$w_{x_3} = \frac{1}{4}w_{x_1x_1x_1} + \frac{3}{2}w_{x_1}^2 - \frac{3}{4}s^2, \tag{4.76a}$$

$$s_{x_3} = -\frac{1}{2}s_{x_1x_1x_1} - 3w_{x_1}s_{x_1}. \tag{4.76b}$$

So from the view point of the DL, it is very clear that the HS equation is the 4-reduction of the CKP equation and the treatment that taking one dependent variable to be zero in a 3-component HS equation (the gHS equation) given in [**SH82**] naturally follows from the symmetry property of the infinite matrix $\mathbf{U}$. The modified hierarchy can be calculated in a similar way (taking $r = 0$) in the gmHS hierarchy. The first nontrivial equation is the modified HS (mHS) equation

$$v_{x_3} = \frac{1}{4}v_{x_1x_1x_1} - \frac{3}{2}v^2v_{x_1} + \frac{3}{2}(vs)_{x_1} + \frac{3}{4}s_{x_1x_1}, \tag{4.77a}$$

$$s_{x_3} = -\frac{1}{2}s_{x_1x_1x_1} - \frac{3}{2}ss_{x_1} - \frac{3}{2}(v_{x_1} - v^2)s_{x_1}, \tag{4.77b}$$

which was first derived in [**JM83**] with some misprints in the first equation and the correct form was later given in [**WGHZ99**] following from the Miura transform for (4.75). In fact, the Miura-type transforms in our framework can be obtained very easily from (4.69) by imposing the symmetry constraint of $\mathbf{U}$:

$$u_{x_1} = -\frac{1}{2}(v_{x_1} + v^2 - s), \quad p = -s_{x_1x_1} - s^2 - 2vs_{x_1}, \tag{4.78a}$$

$$w_{x_1} = \frac{1}{2}(v_{x_1} - v^2 + s), \quad v = \frac{1}{2}\frac{z_{x_1x_1}}{z_{x_1}}. \tag{4.78b}$$

The first two transforms is the Miura transform between (4.75) and (4.77) which coincides with the result in [**WGHZ99**]. The transform between w and (v, s) reveals the clear link between the mHS equation and the HS equation, which was probably not given before, to the best of the author's knowledge. We note that there is also a "semi-modification" of the HS equation by considering the factorisation of a 4th-order Lax operator (cf. [**BEF95**]).

The transform between v and z gives rise to the Schwarzian form of the HS class and it in multicomponent form is written as

$$\frac{z_{x_3}}{z_{x_1}} = \frac{1}{4}\{z, x_1\} + \frac{3}{2}s, \tag{4.79a}$$

$$s_{x_3} = -\frac{1}{2}s_{x_1x_1x_1} - \frac{3}{2}ss_{x_1} - \frac{3}{4}s_{x_1}\{z, x_1\}. \tag{4.79b}$$

We refer to this equation as the Schwarzian HS (SHS) equation. Eliminating s in the system, we obtain a scalar equation where only z in involved:

$$\begin{aligned} 2\left(\frac{z_{x_3}}{z_{x_1}} - \frac{1}{4}\{z, x_1\}\right)_{x_3} + \left(\frac{z_{x_3}}{z_{x_1}} - \frac{1}{4}\{z, x_1\}\right)_{x_1x_1x_1} \\ + \left(2\frac{z_{x_3}}{z_{x_1}} + \{z, x_1\}\right)\left(\frac{z_{x_3}}{z_{x_1}} - \frac{1}{4}\{z, x_1\}\right)_{x_1} = 0. \end{aligned} \tag{4.80}$$

This scalar equation was first proposed in [**Wei85**] as an example which is Möbius invariant. Equation (4.75) also gives us a scalar form in u if we eliminate p. The obtained scalar equation is a higher-order equation with respect to x_3 given by

$$8u_{x_3x_3} + 2u_{x_1x_1x_1x_3} = (u_{x_1x_1x_1x_1x_1} + 18u_{x_1}u_{x_1x_1x_1} + 9u_{x_1x_1}^2 + 24u_{x_1}^3)_{x_1}, \tag{4.81}$$

which is referred to as the bidirectional HS (bHS) equation and the explicit form was first written down in [**TH03**].

The equation (4.81) makes it possible for us to find the multilinear form of the HS class. In fact, in the CKP class we still have the transform $u = (\ln\tau)_{x_1}$ and it gives us a quadrilinear equation as follow:

$$\begin{aligned} &\tau^3\tau_{x_1x_1x_1x_1x_1x_1} - \tau^2\tau_{x_1x_1x_1}^2 + 12\tau\tau_{x_1}^2\tau_{x_1x_1x_1x_1} - 12\tau_{x_1}^3\tau_{x_1x_1x_1} \\ &+ 3\tau^2\tau_{x_1x_1}\tau_{x_1x_1x_1x_1} + 9\tau_{x_1}^2\tau_{x_1x_1}^2 - 6\tau\tau_{x_1}\tau_{x_1x_1}\tau_{x_1x_1x_1} - 6\tau^2\tau_{x_1}\tau_{x_1x_1x_1x_1x_1} \\ &+ 6\tau^2\tau_{x_1x_1}\tau_{x_1x_3} + 6\tau^2\tau_{x_1}\tau_{x_1x_1x_3} - 12\tau\tau_{x_1}\tau_{x_1x_1}\tau_{x_3} + 12\tau_{x_1}^3\tau_{x_3} - 12\tau\tau_{x_1}^2\tau_{x_1x_3} \\ &+ 2\tau^2\tau_{x_1x_1x_1}\tau_{x_3} - 2\tau^3\tau_{x_1x_1x_1x_3} + 8\tau^2\tau_{x_3}^2 - 8\tau^3\tau_{x_3x_3} = 0, \end{aligned} \tag{4.82}$$

The formal linear problem for the HS hierarchy in infinite matrix satisfies the following dynamical and algebraic relations:

$$k^4\mathbf{u}_k = \mathbf{\Lambda}^4 \cdot \mathbf{u}_k - \mathbf{U} \cdot \mathbf{O}_4 \cdot \mathbf{u}_k, \quad \partial_{2j+1}\mathbf{u}_k = \mathbf{\Lambda}^{2j+1} \cdot \mathbf{u}_k - \mathbf{U} \cdot \mathbf{O}_{2j+1} \cdot \mathbf{u}_k, \quad j = 0, 1, 2, \cdots. \tag{4.83}$$

The relations look the same as (4.72) but in fact $\mathbf{U}$ now obeys some additional constraints as we have pointed out in (4.74). The wave function $\phi = u_k^{(0)}$ as a component in the infinite vector $\mathbf{u}_k$ gives the Lax pairs for the whole hierarchy. The explicit form for the one of Equation (4.75) as an example is given as follow:

$$L^{\mathrm{HS}}\phi = k^4\phi, \quad L^{\mathrm{HS}} = \partial_1^4 + 4u_{x_1}\partial_1^2 + 4u_{x_1x_1}\partial_1 + \frac{5}{3}u_{x_1x_1x_1} + 2u_{x_1}^2 + \frac{4}{3}u_{x_3}, \tag{4.84a}$$

$$\phi_{x_3} = \left(\partial_1^3 + 3u_{x_1}\partial_1 + \frac{3}{2}u_{x_1x_1}\right)\phi. \tag{4.84b}$$

The Lax pairs for the other nonlinear forms and the multilinear form can be calculated simply by the listed differential transforms. We note that the Lax pair for (4.76) was first given in [**DF82**].

3.6. The SK and KK hierarchies. The $(2j+1)$-reduction of the BKP and CKP classes result in the same infinite-dimensional Lie algebra and therefore the obtained (1+1)-dimensional integrable hierarchies are the same. In this subsection we consider the 3-reductions of BKP and CKP hierarchies. We start with the 3-reduction of BKP. The dynamical relation (4.18) together with (4.42) for $\mathcal{N}=3$ constitute the infinite matrix structure of the SK hierarchy, i.e.

$$\partial_{2j+1}\mathbf{U} = \mathbf{\Lambda}^{2j+1}\cdot\mathbf{U} + \mathbf{U}\cdot {}^t\mathbf{\Lambda}^{2j+1} - \frac{1}{2}\mathbf{U}\cdot(\mathbf{O}_{2j+1}\cdot\mathbf{\Lambda} - {}^t\mathbf{\Lambda}\cdot\mathbf{O}_{2j+1})\cdot\mathbf{U}, \tag{4.85a}$$

$$\mathbf{\Lambda}^3\cdot\mathbf{U} + \mathbf{U}\cdot {}^t\mathbf{\Lambda}^3 - \frac{1}{2}\mathbf{U}\cdot(\mathbf{O}_3\cdot\mathbf{\Lambda} - {}^t\mathbf{\Lambda}\cdot\mathbf{O}_3)\cdot\mathbf{U} = 0, \quad {}^t\mathbf{U} = -\mathbf{U} \tag{4.85b}$$

for $2j+1\neq 0 \mod 3$ where $j=0,1,2,\cdots$. These relations constitute the infinite matrix version of the SK hierarchy and they are equivalent to the infinite matrix structure of the BKP hierarchy in addition to $\partial_3\mathbf{U}=0$. One can take $u=U_{1,0}=-U_{0,1}$ and from the BKP hierarchy the SK hierarchy can immediately be derived. The first nontrivial equation in the hierarchy is a 5th-order equation

$$9u_{x_5} + u_{x_1x_1x_1x_1x_1} + 15u_{x_1}u_{x_1x_1x_1} + 15u_{x_1}^3 = 0, \tag{4.86}$$

which is the SK equation. The bilinear transform in the BKP class shows that $u=2(\ln\tau)_{x_1}$ and consequently one can have the bilinear form of the SK hierarchy. Consider the x_5-flow, i.e. Equation (4.86), we obtain a bilinear equation

$$(D_1^6 + 9D_1D_5)\tau\cdot\tau = 0, \tag{4.87}$$

which can also be derived directly from the bilinear BKP equation because it has been shown in subsection 3.1 that the 3-reduction also implies that $\tau_{x_3}=0$ in the BKP hierarchy. The linear problem following from BKP under the 3-reduction constraint turns out to be

$$k^3\mathbf{u}_k = \mathbf{\Lambda}^3\cdot\mathbf{u}_k - \frac{1}{2}\mathbf{U}\cdot(\mathbf{O}_3\cdot\mathbf{\Lambda} - {}^t\mathbf{\Lambda}\cdot\mathbf{O}_3)\cdot\mathbf{u}_k, \tag{4.88a}$$

$$\partial_{2j+1}\mathbf{u}_k = \mathbf{\Lambda}^{2j+1}\cdot\mathbf{u}_k - \frac{1}{2}\mathbf{U}\cdot(\mathbf{O}_{2j+1}\cdot\mathbf{\Lambda} - {}^t\mathbf{\Lambda}\cdot\mathbf{O}_{2j+1})\cdot\mathbf{u}_k \tag{4.88b}$$

in infinite matrix structure, where $2j+1\neq 0 \mod 3$, $j=0,1,2,\cdots$. The wave function $\phi = u_k^{(0)}$ gives rise to the Lax pairs for the SK hierarchy and the first nontrivial one is

$$L^{\mathrm{SK}}\phi = k^3\phi, \quad L^{\mathrm{SK}} = \partial_1^3 + 3u_{x_1}\partial_1, \tag{4.89a}$$

$$\phi_{x_5} = \left[\partial_1^5 + 5u_{x_1}\partial_1^3 + 5u_{x_1x_1}\partial_1^2 + \left(\frac{10}{3}u_{x_1x_1x_1} + 5u_{x_1}^2\right)\partial_1\right]\phi. \tag{4.89b}$$

We omit looking for other nonlinear forms here as the 3-reduction of CKP gives the same (1+1)-dimensional soliton hierarchies and it is more convenient to find the other forms there.

Now we consider the 3-reduction of the CKP hierarchy. The infinite matrix structure (namely the dynamical and algebraic relations for $\mathbf{U}$) is as follow:

$$\partial_{2j+1}\mathbf{U} = \mathbf{\Lambda}^{2j+1}\cdot\mathbf{U} + \mathbf{U}\cdot{}^{t}\mathbf{\Lambda}^{2j+1} - \mathbf{U}\cdot\mathbf{O}_{2j+1}\cdot\mathbf{U}, \tag{4.90a}$$

$$\mathbf{\Lambda}^{3}\cdot\mathbf{U} + \mathbf{U}\cdot{}^{t}\mathbf{\Lambda}^{3} - \mathbf{U}\cdot\mathbf{O}_{3}\cdot\mathbf{U} = 0, \quad {}^{t}\mathbf{U} = \mathbf{U}, \tag{4.90b}$$

for $2j+1 \neq 0 \mod 3$ where $j = 0,1,2,\cdots$, which is the infinite matrix version of the hierarchy in the KK and SK class. One can now combine the entries in the infinite matrix $\mathbf{U}$ and introduce the following nonlinear variables:

$$u = U_{0,0}, \quad v = [\ln(1-U_{0,-1})]_{x_1}, \quad w = -\frac{U_{1,-1}}{1-U_{0,-1}}, \quad z = U_{-1,-1} - x_1.$$

From Equations (4.90a) and (4.90b) one can first of all find a hierarchy based on the variable u, which is the KK hierarchy. As an example in the hierarchy we consider the x_5-flow which is the KK equation

$$9u_{x_5} + u_{x_1x_1x_1x_1x_1} + 15u_{x_1}u_{x_1x_1x_1} + 15u_{x_1}^3 + \frac{45}{4}u_{x_1x_1}^2 = 0. \tag{4.91}$$

Next, one can find that the infinite matrix dynamical and algebraic relations give rise to the following Miura-type transforms:

$$u_{x_1} = -\frac{2}{3}v_{x_1} - \frac{1}{3}v^2, \quad w_{x_1} = \frac{1}{3}v_{x_1} - \frac{1}{3}v^2, \quad v = \frac{1}{2}\frac{z_{x_1x_1}}{z_{x_1}}. \tag{4.92}$$

The first two transforms are the remarkable nonlinear transforms which were proposed by Fordy and Gibbons, cf. [**FG80c**, **FG80a**]. Making use of them, one can derive the Fordy–Gibbons (FG) and SK hierarchies from KK respectively. The first nontrivial members in the respective hierarchy are given by

$$9v_{x_5} + v_{x_1x_1x_1x_1x_1} - 20vv_{x_1}v_{x_1x_1} - 20v^2v_{x_1x_1x_1} - 5v_{x_1x_1}^2 - 5v_{x_1}^3 + 5v^4v_{x_1} = 0, \tag{4.93}$$

$$9w_{x_5} + w_{x_1x_1x_1x_1x_1} + 15w_{x_1}w_{x_1x_1x_1} + 15w_{x_1}^3 = 0. \tag{4.94}$$

We note that the equation in v is normally referred to as the FG equation and the equation in w is exactly the same as the SK equation (4.86). The FG equation can be thought of as the modification for both of the SK and KK equations. The Miura-type transform between v and z in (4.92) provides us with

$$9\frac{z_{x_5}}{z_{x_1}} + \{z,x_1\}_{x_1x_1} + \frac{1}{4}\{z,x_1\}^2 = 0. \tag{4.95}$$

This equation is obviously Möbius invariant. Since there are SK and KK equations as the unmodified equations in the class we refer to this equation as the Schwarzian FG equation and it was proposed originally in [**Wei84**].

The tau function defined in the CKP class obeys that $u = (\ln\tau)_{x_1}$ and this transform substituted into the nonlinear form (4.91) gives the multilinear form for the SK and KK class, i.e.

$$\begin{aligned}&4\tau^3\tau_{x_1x_1x_1x_1x_1x_1} + 5\tau^2\tau_{x_1x_1x_1}^2 - 24\tau^2\tau_{x_1}\tau_{x_1x_1x_1x_1x_1} - 30\tau\tau_{x_1}\tau_{x_1x_1}\tau_{x_1x_1x_1} + 45\tau_{x_1}^2\tau_{x_1x_1}^2\\ &+ 60\tau\tau_{x_1}\tau_{x_1x_1x_1x_1} - 60\tau_{x_1}^3\tau_{x_1x_1x_1} + 36\tau^3\tau_{x_1x_5} - 36\tau^2\tau_{x_1}\tau_{x_5} = 0.\end{aligned} \tag{4.96}$$

The multilinear forms for the higher-order equations in the hierarchy can be derived under the same transform.

Finally, we consider the linear problem in this class. According to the reduction of the CKP hierarchy. We have the infinite matrix structure

$$k^3\mathbf{u}_k = \mathbf{\Lambda}^3 \cdot \mathbf{u}_k - \mathbf{U} \cdot \mathbf{O}_3 \cdot \mathbf{u}_k, \tag{4.97a}$$

$$\partial_{2j+1}\mathbf{u}_k = \mathbf{\Lambda}^{2j+1} \cdot \mathbf{U} + \mathbf{U} \cdot {}^t\mathbf{\Lambda}^{2j+1} - \mathbf{U} \cdot \mathbf{O}_{2j+1} \cdot \mathbf{U}, \tag{4.97b}$$

where $2j+1 \neq 0 \mod 3$ for $j = 0, 1, 2, \cdots$. By taking $\phi = u_k^{(0)}$, we can eliminate the other components in (4.97) and find a closed-form relation in ϕ, which is the Lax pair for the x_{2j+1}-flow in the hierarchy. For instance, when $j = 2$, we get the Lax pair for the KK equation (4.91), which is given by

$$L^{\mathrm{KK}}\phi = k^3\phi, \quad L^{\mathrm{KK}} = \partial_1^3 + 3u_{x_1}\partial_1 + \frac{3}{2}u_{x_1x_1}, \tag{4.98a}$$

and

$$\begin{aligned}\phi_{x_5} = &\Bigg[\partial_1^5 + 5u_{x_1}\partial_1^3 + \frac{15}{2}u_{x_1x_1}\partial_1^2 \\ &+ \left(\frac{35}{6}u_{x_1x_1x_1} + 5u_{x_1}^2\right)\partial_1 + \left(\frac{5}{3}u_{x_1x_1x_1x_1} + 5u_{x_1}u_{x_1x_1}\right)\Bigg]\phi.\end{aligned} \tag{4.98b}$$

The Miura-type transforms therefore lead us to the Lax pairs for the other nonlinear forms (4.93), (4.94) and (4.95) and the multilinear transform gives us the Lax pair for the multilinear form (4.96).

3.7. Other higher-rank hierarchies. Other higher-rank (1+1)-dimensional soliton hierarchies can be obtained in a similar way and in those cases more and more nonlinear forms can be found in each class. In this subsection we only give some of them as examples.

We first consider the 5-reduction of the BKP hierarchy. The infinite matrix structure is given by

$$\partial_{2j+1}\mathbf{U} = \mathbf{\Lambda}^{2j+1} \cdot \mathbf{U} + \mathbf{U} \cdot {}^t\mathbf{\Lambda}^{2j+1} - \frac{1}{2}\mathbf{U} \cdot (\mathbf{O}_{2j+1} \cdot \mathbf{\Lambda} - {}^t\mathbf{\Lambda} \cdot \mathbf{O}_{2j+1}) \cdot \mathbf{U}, \tag{4.99a}$$

$$\mathbf{\Lambda}^5 \cdot \mathbf{U} + \mathbf{U} \cdot {}^t\mathbf{\Lambda}^5 - \frac{1}{2}\mathbf{U} \cdot (\mathbf{O}_5 \cdot \mathbf{\Lambda} - {}^t\mathbf{\Lambda} \cdot \mathbf{O}_5) \cdot \mathbf{U} = 0, \quad {}^t\mathbf{U} = -\mathbf{U} \tag{4.99b}$$

for $2j+1 \neq 0 \mod 5$ where $j = 0, 1, 2, \cdots$. Equations (4.99a) together with (4.99b) constitute the bSK hierarchy in infinite matrix. The infinite matrix structure is equivalent to (4.18) in addition to $\partial_{2j+1}\mathbf{U} = 0$ for $2j+1 = 0 \mod 5$. Like the BKP class, we take the nonlinear variable $u = U_{1,0}$, one can find the nonlinear form of the bSK hierarchy. For example, when $j = 1$ in (4.99a) we find the x_3-flow, which is the bSK equation (cf. [**DP01**])

$$-5u_{x_3x_3} + (-5u_{x_1x_1x_3} - 15u_{x_1}u_{x_3} + u_{x_1x_1x_1x_1x_1} + 15u_{x_1}u_{x_1x_1x_1} + 15u_{x_1}^3)_{x_1} = 0. \tag{4.100}$$

The bilinear transforms that follows from the BKP hierarchy is $u = 2(\ln\tau)_{x_1}$ and it gives us the bilinear form of the bSK equation:

$$(D_1^6 - 5D_1^3D_3 - 5D_3^2)\tau \cdot \tau = 0. \tag{4.101}$$

Equation (4.101) was actually given earlier in [**Ram81**] in Painlevé test for bilinear formalism and therefore this bilinear equation is also referred to as the Ramani equation. The higher-order equations in the hierarchy can be obtained by using the same transform. The 5-reduction of the linear problem of the BKP hierarchy gives rise the following relations for the infinite vector $\mathbf{u}_k$:

$$k^5\mathbf{u}_k = \mathbf{\Lambda}^5 \cdot \mathbf{u}_k - \frac{1}{2}\mathbf{U}\cdot(\mathbf{O}_5\cdot\mathbf{\Lambda} - {}^t\mathbf{\Lambda}\cdot\mathbf{O}_5)\cdot\mathbf{u}_k, \tag{4.102a}$$

$$\partial_{2j+1}\mathbf{u}_k = \mathbf{\Lambda}^{2j+1}\cdot\mathbf{U} + \mathbf{U}\cdot{}^t\mathbf{\Lambda}^{2j+1} - \frac{1}{2}\mathbf{U}\cdot(\mathbf{O}_{2j+1}\cdot\mathbf{\Lambda} - {}^t\mathbf{\Lambda}\cdot\mathbf{O}_{2j+1})\cdot\mathbf{U}, \tag{4.102b}$$

where $2j+1 \neq 0 \mod 5$ for $j = 0, 1, 2, \cdots$. The dynamical relations for $\mathbf{u}_k$ yield the Lax pairs for the whole bSK hierarchy if one fixes the variable $\phi = u_k^{(0)}$. The simplest nontrivial one is the linear problem of the bSK equation (4.100) whose explicit form is

$$L^{\mathrm{bSK}}\phi = k^5\phi, \tag{4.103a}$$

$$\phi_{x_3} = (\partial_1^3 + 3u_{x_1}\partial_1)\phi, \tag{4.103b}$$

where the Lax operator is given by

$$L^{\mathrm{bSK}} = \partial_1^5 + 5u_{x_1}\partial_1^3 + 5u_{x_1x_1}\partial_1^2 + \left(\frac{10}{3}u_{x_1x_1x_1} + 5u_{x_1}^2 + \frac{5}{3}u_{x_3}\right)\partial_1.$$

Likewise, the 5-reduction of the CKP hierarchy has the infinite matrix structure in $\mathbf{U}$ as follow:

$$\partial_{2j+1}\mathbf{U} = \mathbf{\Lambda}^{2j+1}\cdot\mathbf{U} + \mathbf{U}\cdot{}^t\mathbf{\Lambda}^{2j+1} - \mathbf{U}\cdot\mathbf{O}_{2j+1}\cdot\mathbf{U}, \tag{4.104a}$$

$$\mathbf{\Lambda}^5\cdot\mathbf{U} + \mathbf{U}\cdot{}^t\mathbf{\Lambda}^5 - \mathbf{U}\cdot\mathbf{O}_5\cdot\mathbf{U} = 0, \quad {}^t\mathbf{U} = \mathbf{U} \tag{4.104b}$$

for $2j+1 \neq 0 \mod 5$ where $j = 0, 1, 2, \cdots$. From the structure if one chooses $u = U_{0,0}$, a closed-form hierarchy can be derived. The x_3-flow in the hierarchy gives us the bKK equation

$$\begin{aligned} &-5u_{x_3x_3} \\ &\quad+ \left(-5u_{x_1x_1x_3} - 15u_{x_1}u_{x_3} + u_{x_1x_1x_1x_1x_1} + 15u_{x_1}u_{x_1x_1x_1} + 15u_{x_1}^3 + \frac{45}{4}u_{x_1x_1}^2\right)_{x_1} = 0, \end{aligned} \tag{4.105}$$

which was first introduced in [**DP01**].

The multilinear transform in the CKP hierarchy is given by $u = (\ln\tau)_{x_1}$ and therefore a quadrilinear form of Equation (4.105) can be obtained:

$$\begin{aligned} &4\tau^3\tau_{x_1x_1x_1x_1x_1x_1} + 5\tau^2\tau_{x_1x_1x_1}^2 - 24\tau^2\tau_{x_1}\tau_{x_1x_1x_1x_1x_1} - 30\tau\tau_{x_1}\tau_{x_1x_1}\tau_{x_1x_1x_1} + 45\tau_{x_1}^2\tau_{x_1x_1}^2 \\ &+ 60\tau\tau_{x_1}\tau_{x_1x_1x_1x_1} - 60\tau_{x_1}^3\tau_{x_1x_1x_1} + 60\tau^2\tau_{x_1}\tau_{x_1x_1x_3} - 60\tau\tau_{x_1}^2\tau_{x_1x_3} + 60\tau_{x_1}^3\tau_{x_3} \\ &- 60\tau\tau_{x_1}\tau_{x_1x_1}\tau_{x_3} + 20\tau^2\tau_{x_1x_1x_1}\tau_{x_3} - 20\tau^3\tau_{x_1x_1x_1x_3} + 20\tau^2\tau_{x_3}^2 - 20\tau^3\tau_{x_3x_3} = 0. \end{aligned} \tag{4.106}$$

The dynamical and algebraic relations for the infinite vector $\mathbf{u}_k$ in this case are given by

$$k^5\mathbf{u}_k = \mathbf{\Lambda}^5\cdot\mathbf{u}_k - \mathbf{U}\cdot\mathbf{O}_5\cdot\mathbf{u}_k, \quad \partial_{2j+1}\mathbf{u}_k = \mathbf{\Lambda}^{2j+1}\cdot\mathbf{U} + \mathbf{U}\cdot{}^t\mathbf{\Lambda}^{2j+1} - \mathbf{U}\cdot\mathbf{O}_{2j+1}\cdot\mathbf{U} \tag{4.107}$$

for $2j+1 \neq 0 \mod 5$ where $j = 0, 1, 2, \cdots$. Taking $\phi = u_k^{(0)}$, one can calculate the Lax pair for the bKK hierarchy. For the x_3-flow, namely, the bKK equation (4.105), one has the Lax pair

$$L^{\mathrm{bKK}}\phi = k^5\phi, \tag{4.108a}$$

$$\phi_{x_3} = \left(\partial_1^3 + 3u_{x_1}\partial_1 + \frac{3}{2}u_{x_1x_1}\right)\phi, \tag{4.108b}$$

where the differential operator is given as

$$\begin{aligned} L^{\mathrm{bKK}} &= \partial_1^5 + 5u_{x_1}\partial_1^3 + \frac{15}{2}u_{x_1x_1}\partial_1^2 \\ &\quad + \left(\frac{35}{6}u_{x_1x_1x_1} + 5u_{x_1}^2 + \frac{5}{3}u_{x_3}\right)\partial_1 + 5u_{x_1}u_{x_1x_1} + \frac{5}{3}u_{x_1x_1x_1x_1} + \frac{5}{6}u_{x_1x_3}. \end{aligned}$$

Finally we consider the 6-reduction of the BKP hierarchy. We list the dynamical and algebraic relations as follows:

$$\partial_{2j+1}\mathbf{U} = \mathbf{\Lambda}^{2j+1}\cdot\mathbf{U} + \mathbf{U}\cdot{}^t\mathbf{\Lambda}^{2j+1} - \frac{1}{2}\mathbf{U}\cdot(\mathbf{O}_{2j+1}\cdot\mathbf{\Lambda} - {}^t\mathbf{\Lambda}\cdot\mathbf{O}_{2j+1})\cdot\mathbf{U}, \quad j = 0, 1, 2, \cdots, \tag{4.109}$$

$$\mathbf{\Lambda}^6\cdot\mathbf{U} - \mathbf{U}\cdot{}^t\mathbf{\Lambda}^6 - \frac{1}{2}\mathbf{U}\cdot(\mathbf{O}_6\cdot\mathbf{\Lambda} - {}^t\mathbf{\Lambda}\cdot\mathbf{O}_6)\cdot\mathbf{U} = 0, \quad {}^t\mathbf{U} = -\mathbf{U}. \tag{4.110}$$

The first nontrivial equation from the hierarchy based on $U_{1,0} \doteq u$ is the Ito equation:

$$u_{x_3x_3} + 2(u_{x_1x_1x_3} + 3u_{x_1}u_{x_3})_{x_1} = 0, \tag{4.111}$$

and the transform $u = 2(\ln\tau)_{x_1}$ gives us the bilinear form of the Ito equation:

$$(D_3^2 + 2D_1^3D_3)\tau\cdot\tau = 0. \tag{4.112}$$

Similarly, from the reduction of BKP one can also obtain the dynamical and algebraic relations for the infinite vector $\mathbf{u}_k$ in the Ito class:

$$k^6\mathbf{u}_k = \mathbf{\Lambda}^6\cdot\mathbf{u}_k - \frac{1}{2}\mathbf{U}\cdot(\mathbf{O}_6\cdot\mathbf{\Lambda} - {}^t\mathbf{\Lambda}\cdot\mathbf{O}_6)\cdot\mathbf{u}_k, \tag{4.113a}$$

$$\partial_{2j+1}\mathbf{u}_k = \mathbf{\Lambda}^{2j+1}\cdot\mathbf{U} + \mathbf{U}\cdot{}^t\mathbf{\Lambda}^{2j+1} - \frac{1}{2}\mathbf{U}\cdot(\mathbf{O}_{2j+1}\cdot\mathbf{\Lambda} - {}^t\mathbf{\Lambda}\cdot\mathbf{O}_{2j+1})\cdot\mathbf{U}, \quad j = 0, 1, 2, \cdots. \tag{4.113b}$$

The component $u_k^{(0)} \doteq \phi$ gives us the Lax pair for the Ito equation (the Lax pair for the other members in the hierarchy can be calculated in a similar way):

$$L^{\mathrm{Ito}}\phi = k^6\phi, \tag{4.114a}$$

$$\phi_{x_3} = (\partial_1^3 + 3u_{x_1}\partial_1)\phi, \tag{4.114b}$$

where the spectral problem is associated with a 6th-order differential operator given by

$$\begin{aligned} L^{\mathrm{Ito}} &= \partial_1^6 + 6u_{x_1}\partial_1^4 + 9u_{x_1x_1}\partial_1^3 \\ &\quad + (9u_{x_1x_1x_1} + 9u_{x_1}^2 + 2u_{x_3})\partial_1^2 + (3u_{x_1x_1x_1x_1} + 9u_{x_1}u_{x_1x_1} + u_{x_1x_3})\partial_1. \end{aligned}$$

One comment here is that in the original paper [**Ito80**] a Lax pair for Equation (4.111) was derived from the bilinear Bäcklund transform of the equation (cf. (4.11) and (4.12) in [**Ito80**]) and it is effectively a 4th-order Lax pair having two spectral parameters. While from the DL framework the Ito equation arises as the 6-reduction of BKP and it has a 6th-order Lax operator.

4. Soliton solutions

Solutions for the soliton equations arising from the DL framework can be obtained naturally. In fact, the $\mathbf{U}$ defined in (2.20) involving an integral gives us general solutions for (2+1)-dimensional integrable hierarchies. The reductions of the infinite matrix $\mathbf{U}$, namely the $\mathbf{U}$ defined in Equations (4.33), (4.40) and (4.47), provide solutions to the respective (1+1)-dimensional hierarchies. By choosing a specific measure and an integration domain, one then has special classes of solutions for these hierarchies. In this section we only consider soliton-type solutions to the hierarchies that arise from our framework.

4.1. Solitons for the (2+1)-dimensional hierarchies. To construct soliton solutions, we introduce a measure containing a finite number of distinct singularities. In this subsection we consider solutions for the (2+1)-dimensional hierarchies.

Solitons for the AKP hierarchy. For the AKP hierarchy, one can take a particular measure as

$$\mathrm{d}\zeta(l,l')=\sum_{i=1}^{N}\sum_{j=1}^{N'}A_{i,j}\delta(l-k_i)\delta(l'-k'_j)\mathrm{d}l\mathrm{d}l', \tag{4.115}$$

from which one can easily see that now singularities k_i and k'_j are introduced into the measure. This now turns out to be a $\overline{\partial}$ problem (cf. [**AC91**]) and the infinite matrix $\mathbf{U}$ defined in (2.20) and the linear integral equation (2.15) can be reformulated as

$$\mathbf{U}=\sum_{i=1}^{N}\sum_{j=1}^{N'}A_{i,j}\mathbf{u}_{k_i}{}^{t}\mathbf{c}_{k'_j}\sigma_{k'_j},\quad \mathbf{u}_k+\sum_{i=1}^{N}\sum_{j=1}^{N'}A_{i,j}\frac{\rho_k\sigma_{k'_j}}{k+k'_j}\mathbf{u}_{k_i}=\rho_k\mathbf{c}_k.$$

If one now takes k to be k_i in the above relation, it becomes a set of linear equations for the infinite vector $\mathbf{u}_{k_i}$ and therefore the explicit expression of $\mathbf{U}$ is obtained, in other words, the (N,N')-soliton solution for the AKP hierarchy is constructed. In practice, we introduce the generalised Cauchy matrix $\mathbf{M}$ defined as

$$\mathbf{M}=(M_{j,i})_{N'\times N},\quad M_{j,i}=\frac{\rho_{k_i}\sigma_{k'_j}}{k_i+k'_j}, \tag{4.116}$$

$$\rho_k=\exp\left(\sum_{j=1}^{\infty}k^jx_j\right),\quad \sigma_{k'}=\exp\left(-\sum_{j=1}^{\infty}(-k')^jx_j\right), \tag{4.117}$$

and $\mathbf{A}=(A_{i,j})_{N\times N'}$ is an arbitrary matrix, and consequently the entries in the infinite matrix $\mathbf{U}$ can therefore be expressed by

$$U_{i,j}=\mathbf{r}^{\mathrm{T}}\mathbf{K}^i(1+\mathbf{A}\mathbf{M})^{-1}\mathbf{A}\mathbf{K}'^j\mathbf{s}, \tag{4.118}$$

where the vectors $\mathbf{r}$, $\mathbf{s}$ and the matrices $\mathbf{K}$, $\mathbf{K}'$ are given by

$$\mathbf{r} = (\rho_{k_1}, \cdots, \rho_{k_N})^{\mathrm{T}}, \quad \mathbf{s} = (\sigma_{k'_1}, \cdots, \sigma_{k'_{N'}})^{\mathrm{T}},$$
$$\mathbf{K} = \mathrm{diag}(k_1, \cdots, k_N), \quad \mathbf{K}' = \mathrm{diag}(k'_1, \cdots, k'_{N'}).$$

Similarly one can consider the tau function defined by $\tau = \det(1 + \mathbf{\Omega} \cdot \mathbf{C})$ together with (4.115), and this gives us the explicit formula for the tau function taking the form

$$\tau = \det(1 + \mathbf{A}\mathbf{M}), \tag{4.119}$$

which solves the corresponding multilinear equations. We note that this expression is corresponding to the Grammian representation of the KP soliton, which is equivalent to the Wronskian representation.

Solitons for the BKP hierarchy. In the BKP hierarchy, we take a particular measure

$$\mathrm{d}\zeta(l, l') = \sum_{i,j=1}^{2N} A_{i,j}\delta(l - k_i)\delta(l' - k'_j)\mathrm{d}l\mathrm{d}l', \quad A_{i,j} = -A_{j,i}. \tag{4.120}$$

The reason why $A_{i,j}$ is antisymmetric is that this treatment preserves the antisymmetry property of the measure in the BKP hierarchy in infinite matrix form. After some similar computation one can find the N-soliton solution for the BKP hierarchy is expressed by

$$U_{i,j} = \mathbf{r}^{\mathrm{T}}\mathbf{K}^i(1 + \mathbf{A}\mathbf{M})^{-1}\mathbf{A}\mathbf{K}'^j\mathbf{r}', \tag{4.121}$$

where the generalised Cauchy matrix in this case is given by

$$\mathbf{M} = (M_{j,i})_{2N\times 2N}, \quad M_{j,i} = \frac{1}{2}\rho_{k_i}\frac{k_i - k'_j}{k_i + k'_j}\rho_{k'_j}, \quad \rho_k = \exp\left(\sum_{j=1}^{\infty} k^{2j+1}x_{2j+1}\right), \tag{4.122}$$

and $\mathbf{r}$, $\mathbf{s}$, $\mathbf{K}$ and $\mathbf{K}'$ are defined by

$$\mathbf{r} = (\rho_{k_1}, \cdots, \rho_{k_{2N}})^{\mathrm{T}}, \quad \mathbf{r}' = (\rho_{k'_1}, \cdots, \rho_{k'_{2N}})^{\mathrm{T}},$$
$$\mathbf{K} = \mathrm{diag}(k_1, \cdots, k_{2N}), \quad \mathbf{K}' = \mathrm{diag}(k'_1, \cdots, k'_{2N'}),$$

and $\mathbf{A} = (A_{i,j})_{2N\times 2N}$ is a skew-symmetric matrix. The solution to the multilinear forms can be expressed by τ and following its definition in subsection 2.2, we have

$$\tau^2 = \det(1 + \mathbf{A}\mathbf{M}). \tag{4.123}$$

The tau function itself in the BKP hierarchy can be expressed by a Pfaffian apparently because $\mathbf{A}$ and $\mathbf{M}$ are both skew-symmetric matrices.

Solitons for the CKP hierarchy. Likewise we take the symmetric measure in the CKP hierarchy as follow:

$$\mathrm{d}\zeta(l, l') = \sum_{i,j=1}^{N} A_{i,j}\delta(l - k_i)\delta(l' - k'_j)\mathrm{d}l\mathrm{d}l', \quad A_{i,j} = A_{j,i}. \tag{4.124}$$

Similarly to the BKP hierarchy, the reason why we require $A_{i,j}$ symmetric is that this condition can preserve the symmetry condition of the measure in the CKP hierarchy. We then define $\mathbf{r}$, $\mathbf{r}'$ as

$$\mathbf{r} = (\rho_{k_1}, \cdots, \rho_{k_N})^{\mathrm{T}}, \quad \mathbf{r}' = (\rho_{k'_1}, \cdots, \rho_{k'_N})^{\mathrm{T}},$$

and let $\mathbf{K}$ and $\mathbf{K}'$ be exactly the same as those in the AKP hierarchy. Suppose $\mathbf{A} = (A_{i,j})_{N\times N}$ is a symmetric matrix, and introducing the generalised Cauchy matrix

$$\mathbf{M} = (M_{j,i})_{N\times N}, \quad M_{j,i} = \frac{\rho_{k_i}\rho_{k'_j}}{k_i + k'_j}, \quad \rho_k = \exp\left(\sum_{j=0}^{\infty} k^{2j+1} x_{2j+1}\right), \tag{4.125}$$

we have the N-soliton solution for the nonlinear form of the CKP hierarchy and it is determined by the expression for the entry $U_{i,j}$ as follow:

$$U_{i,j} = \mathbf{r}^{\mathrm{T}}\mathbf{K}^i(1+\mathbf{AM})^{-1}\mathbf{A}\mathbf{K}'^j\mathbf{r}'. \tag{4.126}$$

Similarly, the tau function takes the form

$$\tau = \det(1+\mathbf{AM}), \tag{4.127}$$

providing the N-soliton solution to the multilinear form of the CKP hierarchy.

4.2. Solitons for the (1+1)-dimensional hierarchies.

4.2.1. *Solitons for the reductions of AKP.* For the $\mathcal{N}$-reduction of the AKP hierarchy we already have the general expressions for the infinite matrix $\mathbf{U}$, namely (4.33) and the the reduced integral equation (4.32). One can restrict the measures $\lambda_j(l)$ to be a particular form involving singularities as

$$\mathrm{d}\lambda_j(l) = \sum_{j'=1}^{N_j} A_{j,j'}\delta(l - k_{j,j'})\mathrm{d}l, \tag{4.128}$$

and as a result the infinite matrix $\mathbf{U}$ and the linear integral equation (4.32) under the particular measure can be written as

$$\mathbf{u}_k + \sum_{j=1}^{\mathcal{N}-1}\sum_{j'=1}^{N_j} A_{j,j'}\frac{\rho_k\sigma_{-\omega^j k_{j,j'}}}{k-\omega^j k_{j,j'}}\mathbf{u}_{k_{j,j'}} = \rho_k\mathbf{c}_k, \quad \mathbf{U} = \sum_{j=1}^{\mathcal{N}-1}\sum_{j'=1}^{N_j} A_{j,j'}\mathbf{u}_{k_{j,j'}}{}^{t}\mathbf{c}_{-\omega^j k_{j,j'}}\sigma_{-\omega^j k_{j,j'}}.$$

Here $\mathcal{N}$ denotes the $\mathcal{N}$-reduction, namely $\omega^{\mathcal{N}} = 1$, while N_j denotes the number of solitons in solution. So are the notations for the reduced hierarchy of BKP and CKP. Taking $k = k_{i,i'}$ in the above relations, one can obtain the expression for $\mathbf{U}$ by solving $\mathbf{u}_{k_{i,i'}}$ in the first equation given above. The expression of the entries in $\mathbf{U}$ is

$$U_{i,j} = \mathbf{r}^{\mathrm{T}}\mathbf{K}^i(1+\mathbf{AM})^{-1}\mathbf{A}\mathbf{K}'^j\mathbf{s}, \tag{4.129}$$

in which the generalised Cauchy matrix is defined by

$$\mathbf{M} = (M_{(j,j'),(i,i')})_{j,i=1,\cdots,\mathcal{N}-1,j'=1,\cdots,N_j,i'=1,\cdots,N_i}, \quad M_{(j,j'),(i,i')} = \frac{\rho_{k_{i,i'}}\sigma_{-\omega^j k_{j,j'}}}{k_{i,i'} - \omega^j k_{j,j'}} \tag{4.130}$$

with ρ_k and σ_k defined as those in (4.116). The matrix $\mathbf{M}$ should be understood as a $(\mathcal{N}-1)\times(\mathcal{N}-1)$ block matrix in which the (j,i)-entry is a rectangular matrix of size $N_j \times N_i$. In other words, the indices (j,i) denote the blocks and the indices (j',i') denote the entries in each block. While the vectors $\mathbf{r}$ and $\mathbf{s}$, and the matrices $\mathbf{K}$, $\mathbf{K}'$ and $\mathbf{A}$ are given by

$$
\begin{aligned}
\mathbf{r} &= (\rho_{k_{1,1}}, \cdots, \rho_{k_{1,N_1}}; \cdots; \rho_{k_{j,1}}, \cdots, \rho_{k_{j,N_j}}; \cdots; \rho_{k_{\mathcal{N}-1,1}}, \cdots, \rho_{k_{\mathcal{N}-1,N_{\mathcal{N}-1}}})^{\mathrm{T}}, \\
\mathbf{s} &= (\sigma_{-\omega k_{1,1}}, \cdots, \sigma_{-\omega k_{1,N_1}}; \cdots; \sigma_{-\omega^j k_{j,1}}, \cdots, \sigma_{-\omega^j k_{j,N_j}}; \cdots; \\
&\qquad \sigma_{-\omega^{\mathcal{N}-1} k_{\mathcal{N}-1,1}}, \cdots, \sigma_{-\omega^{\mathcal{N}-1} k_{\mathcal{N}-1,N_{\mathcal{N}-1}}})^{\mathrm{T}}, \\
\mathbf{K} &= \mathrm{diag}(k_{1,1}, \cdots, k_{1,N_1}; \cdots; k_{j,1}, \cdots, k_{j,N_j}; \cdots; k_{\mathcal{N}-1,1}, \cdots, k_{\mathcal{N}-1,N_{\mathcal{N}-1}}), \\
\mathbf{K}' &= \mathrm{diag}(-\omega k_{1,1}, \cdots, -\omega k_{1,N_1}; \cdots; -\omega^j k_{j,1}, \cdots, -\omega^j k_{j,N_j}; \cdots; \\
&\qquad -\omega^{\mathcal{N}-1} k_{\mathcal{N}-1,1}, \cdots, -\omega^{\mathcal{N}-1} k_{\mathcal{N}-1,N_{\mathcal{N}-1}}), \\
\mathbf{A} &= \mathrm{diag}(A_{1,1}, \cdots, A_{1,N_1}; \cdots; A_{j,1}, \cdots, A_{j,N_j}; \cdots; A_{\mathcal{N}-1,1}, \cdots, A_{\mathcal{N}-1,N_{\mathcal{N}-1}}).
\end{aligned}
$$

The tau function takes the form

$$\tau = \det(1 + \mathbf{AM}) \tag{4.131}$$

with $\mathbf{M}$ defined as (4.130).

The formulae for $U_{i,j}$ and τ govern the soliton solutions for the equations arising as the reductions of the AKP hierarchy. For instance, the cases $\mathcal{N} = 2, 3, 4$ are corresponding to the solitons of the KdV, BSQ and gHS hierarchies, respectively.

4.2.2. *Solitons for the reductions of BKP.* One can take the particular measures $\lambda_j(l)$ like (4.128), and as a result the infinite matrix $\mathbf{U}$ and the linear integral equation (4.38) under the particular measures can be written as

$$
\begin{aligned}
&\mathbf{u}_k + \sum_{j=1}^{\mathcal{N}-1} \sum_{j'=1}^{N_j} A_{j,j'} \frac{1}{2} \rho_k \frac{k + \omega^j k_{j,j'}}{k - \omega^j k_{j,j'}} \rho_{-\omega^j k_{j,j'}} \mathbf{u}_{k_{j,j'}} - \sum_{j=1}^{\mathcal{N}-1} \sum_{j'=1}^{N_j} A_{j,j'} \frac{1}{2} \rho_k \frac{k - k_{j,j'}}{k + k_{j,j'}} \rho_{k_{j,j'}} \mathbf{u}_{-\omega^j k_{j,j'}} = \rho_k \mathbf{c}_k, \\
&\mathbf{U} = \sum_{j=1}^{\mathcal{N}-1} \sum_{j'=1}^{N_j} A_{j,j'} \mathbf{u}_{k_{j,j'}} {}^t\mathbf{c}_{-\omega^j k_{j,j'}} \rho_{-\omega^j k_{j,j'}} - \sum_{j=1}^{\mathcal{N}-1} \sum_{j'=1}^{N_j} A_{j,j'} \mathbf{u}_{-\omega^j k_{j,j'}} {}^t\mathbf{c}_{k_{j,j'}} \rho_{k_{j,j'}}.
\end{aligned}
$$

Similarly we can calculate the expressions of the entries in $\mathbf{U}$ and obtain:

$$
\begin{aligned}
U_{i,j} = &\begin{pmatrix} \mathbf{r} \\ \mathbf{r}' \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \mathbf{K} & 0 \\ 0 & \mathbf{K}' \end{pmatrix}^{i} \left(1 + \begin{pmatrix} \mathbf{A} & 0 \\ 0 & -\mathbf{A} \end{pmatrix} \begin{pmatrix} \mathbf{M} & 0 \\ 0 & \mathbf{M}' \end{pmatrix} \right)^{-1} \\
&\times \begin{pmatrix} \mathbf{A} & 0 \\ 0 & -\mathbf{A} \end{pmatrix} \begin{pmatrix} \mathbf{K}' & 0 \\ 0 & \mathbf{K} \end{pmatrix}^{j} \begin{pmatrix} \mathbf{r}' \\ \mathbf{r} \end{pmatrix},
\end{aligned}
\tag{4.132}
$$

in which the generalised Cauchy matrix is a block matrix defined by

$$\mathbf{M} = (M_{(j,j'),(i,i')})_{j,i=1,\cdots,\mathcal{N}-1, j'=1,\cdots,N_j, i'=1,\cdots,N_i}, \quad M_{(j,j'),(i,i')} = \frac{1}{2} \rho_{k_{i,i'}} \frac{k_{i,i'} + \omega^j k_{j,j'}}{k_{i,i'} - \omega^j k_{j,j'}} \rho_{-\omega^j k_{j,j'}}, \tag{4.133a}$$

$$\mathbf{M}' = (M'_{(j,j'),(i,i')})_{j,i=1,\cdots,\mathcal{N}-1, j'=1,\cdots,N_j, i'=1,\cdots,N_i}, \quad M'_{(j,j'),(i,i')} = \frac{1}{2} \rho_{-\omega^i k_{i,i'}} \frac{-\omega^i k_{i,i'} - k_{j,j'}}{-\omega^i k_{i,i'} + k_{j,j'}} \rho_{k_{j,j'}}, \tag{4.133b}$$

where ρ_k is given in (4.122), and the vectors $\mathbf{r}$ and $\mathbf{r}'$, and the matrices $\mathbf{K}$, $\mathbf{K}'$ and $\mathbf{A}$ are given by

$$\begin{aligned}
\mathbf{r} &= (\rho_{k_{1,1}},\cdots,\rho_{k_{1,N_1}};\cdots;\rho_{k_{j,1}},\cdots,\rho_{k_{j,N_j}};\cdots;\rho_{k_{\mathcal{N}-1,1}},\cdots,\rho_{k_{\mathcal{N}-1,N_{\mathcal{N}-1}}})^{\mathrm{T}},\\
\mathbf{r}' &= (\rho_{-\omega k_{1,1}},\cdots,\rho_{-\omega k_{1,N_1}};\cdots;\rho_{-\omega^j k_{j,1}},\cdots,\rho_{-\omega^j k_{j,N_j}};\cdots;\\
&\qquad \rho_{-\omega^{\mathcal{N}-1}k_{N,1}},\cdots,\rho_{-\omega^{\mathcal{N}-1}k_{\mathcal{N}-1,N_{\mathcal{N}-1}}})^{\mathrm{T}},\\
\mathbf{K} &= \mathrm{diag}(k_{1,1},\cdots,k_{1,N_1};\cdots;k_{j,1},\cdots,k_{j,N_j};\cdots;k_{\mathcal{N}-1,1},\cdots,k_{\mathcal{N}-1,N_{\mathcal{N}-1}}),\\
\mathbf{K}' &= \mathrm{diag}(-\omega k_{1,1},\cdots,-\omega k_{1,N_1};\cdots;-\omega^j k_{j,1},\cdots,-\omega^j k_{j,N_j};\cdots;\\
&\qquad -\omega^{\mathcal{N}-1}k_{\mathcal{N}-1,1},\cdots,-\omega^{\mathcal{N}-1}k_{\mathcal{N}-1,N_{\mathcal{N}-1}}),\\
\mathbf{A} &= \mathrm{diag}(A_{1,1},\cdots,A_{1,N_1};\cdots;A_{j,1},\cdots,A_{j,N_j};\cdots;A_{\mathcal{N}-1,1},\cdots,A_{\mathcal{N}-1,N_{\mathcal{N}-1}}).
\end{aligned}$$

The tau function is determined by

$$\tau^2 = \det\left[1+\begin{pmatrix}\mathbf{A} & 0\\ 0 & -\mathbf{A}\end{pmatrix}\begin{pmatrix}\mathbf{M} & 0\\ 0 & \mathbf{M}'\end{pmatrix}\right], \tag{4.134}$$

which solves the multilinear forms of the reduced hierarchies from BKP.

Following the above general formulae for $U_{i,j}$ and τ, we can see that the particular case when $\mathcal{N}=3$ gives us the solitons solutions to (4.86) and (4.87), and the case when $\mathcal{N}=6$ gives us the soliton solutions to (4.111) and (4.112).

4.2.3. *Solitons for the reductions of CKP.* Likewise we impose (4.128) on the reduced integral equation (4.45) and the reduced infinite matrix $\mathbf{U}$ (4.47) in the CKP hierarchy. This gives us the following relations:

$$\begin{aligned}
&\mathbf{u}_k+\sum_{j=1}^{\mathcal{N}-1}\sum_{j'=1}^{N_j}A_{j,j'}\frac{\rho_k\rho_{-\omega^j k_{j,j'}}}{k-\omega^j k_{j,j'}}\mathbf{u}_{k_{j,j'}}+\sum_{j=1}^{\mathcal{N}-1}\sum_{j'=1}^{N_j}A_{j,j'}\frac{\rho_k\rho_{k_{j,j'}}}{k+k_{j,j'}}\mathbf{u}_{-\omega^j k_{j,j'}}=\rho_k\mathbf{c}_k,\\
&\mathbf{U}=\sum_{j=1}^{\mathcal{N}-1}\sum_{j'=1}^{N_j}A_{j,j'}\mathbf{u}_{k_{j,j'}}{}^t\mathbf{c}_{-\omega^j k_{j,j'}}\rho_{-\omega^j k_{j,j'}}+\sum_{j=1}^{\mathcal{N}-1}\sum_{j'=1}^{N_j}A_{j,j'}\mathbf{u}_{-\omega^j k_{j,j'}}{}^t\mathbf{c}_{k_{j,j'}}\rho_{k_{j,j'}}.
\end{aligned}$$

If one introduces the following block generalised Cauchy matrices:

$$\mathbf{M}=(M_{(j,j'),(i,i')})_{j,i=1,\cdots,\mathcal{N}-1,j'=1,\cdots,N_j,i'=1,\cdots,N_i},\quad M_{(j,j'),(i,i')}=\frac{\rho_{k_{i,i'}}\rho_{-\omega^j k_{j,j'}}}{k_{i,i'}-\omega^j k_{j,j'}}, \tag{4.135a}$$

$$\mathbf{M}'=(M_{(j,j'),(i,i')})_{j,i=1,\cdots,\mathcal{N}-1,j'=1,\cdots,N_j,i'=1,\cdots,N_i},\quad M'_{(j,j'),(i,i')}=\frac{\rho_{-\omega^i k_{i,i'}}\rho_{k_{j,j'}}}{-\omega^i k_{i,i'}+k_{j,j'}}, \tag{4.135b}$$

the entries $U_{i,j}$ in the infinite matrix $\mathbf{U}$ in this class can be written as

$$\begin{aligned}
U_{i,j}=&\begin{pmatrix}\mathbf{r}\\ \mathbf{r}'\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}\mathbf{K} & 0\\ 0 & \mathbf{K}'\end{pmatrix}^{i}\left(1+\begin{pmatrix}\mathbf{A} & 0\\ 0 & \mathbf{A}\end{pmatrix}\begin{pmatrix}\mathbf{M} & 0\\ 0 & \mathbf{M}'\end{pmatrix}\right)^{-1}\\
&\times\begin{pmatrix}\mathbf{A} & 0\\ 0 & \mathbf{A}\end{pmatrix}\begin{pmatrix}\mathbf{K}' & 0\\ 0 & \mathbf{K}\end{pmatrix}^{j}\begin{pmatrix}\mathbf{r}'\\ \mathbf{r}\end{pmatrix},
\end{aligned}\tag{4.136}$$

where $\mathbf{r}$, $\mathbf{r}'$, $\mathbf{K}$ and $\mathbf{K}'$ and $\mathbf{A}$ are the same as those given in the soliton solutions for the reduced hierarchies from BKP. The tau function given by

$$\tau = \det\left[1 + \begin{pmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{A} \end{pmatrix}\begin{pmatrix} \mathbf{M} & 0 \\ 0 & \mathbf{M}' \end{pmatrix}\right] \tag{4.137}$$

provides soliton solutions to the hierarchies in the multilinear form.

In this case, the expressions of the $U_{i,j}$ and the tau function provide soliton solutions for the $\mathcal{N}$-reduced hierarchies from CKP. For example, the $\mathcal{N} = 3$ case comprises the solitons for the equations in the SK and KK family, and the $\mathcal{N} = 4$ case provides those for equations in the HS family.

5. Concluding remarks

We presented a unified framework to understand (2+1)- and (1+1)-dimensional soliton hierarchies associated with scalar linear integral equations. The framework provides many of the requisite integrability characteristics such as the solution structure (including explicit solutions such as soliton solutions), the associated linear problem (Lax pair), Miura-type transforms among different nonlinear forms and the multilinear form for the tau function for each soliton hierarchy. All well-known soliton hierarchies associated with a scalar differential spectral problem are covered.

We obtained all the (1+1)-dimensional soliton hierarchies from reductions of (2+1)-dimensional models by imposing certain conditions on the measures and integration domains in the corresponding linear integral equations. As a by-product, we attained a richer solution structure for the higher-rank soliton equations where block Cauchy matrices are involved. Furthermore, our approach has some advantage over the "standard approach" to the KP hierarchy which exploits pseudo-differential operators in that in the DL framework it is not necessary to single out a particular flow variable (usually called $x \doteq x_1$) in order to set up the framework.

Other integrability characteristics such as recursion operators and Hamiltonian structures were not considered in the chapter. Nevertheless from the view point of the DL, they can be obtained by considering squared eigenfunctions, cf. the procedure given in [**CWN86**]. However, the integrability of the equations in the hierarchies that we obtained is self-evident from the fact that they possess infinite families of explicit solutions from our framework.

While, the starting point in the current chapter is the DL scheme for the KP hierarchy associated with the A_∞ algebra. The other examples such as BKP and CKP are the hierarchies associated with the sub-algebras B_∞ and C_∞ which are contained in the AKP case. And so are the reduced (1+1)-dimensional hierarchies arising from the AKP, BKP and CKP hierarchies. There also exist (3+1)-dimensional soliton equations beyond this case. A recent result [**JN14**] showed that the KP hierarchy has an elliptic extension and the model also leads to elliptic extensions of the "sub-hierarchies" (e.g. the elliptic KdV hierarchy [**NP03**]). However, the algebras hidden behind these models are not yet clear.

There also exists the so-called DKP hierarchy, named after the infinite-dimensional algebra D_∞. However, the DKP hierarchy is a sub-case of the two-component AKP hierarchy, whose corresponding linear structure is beyond the integral equation (2.15). We will report the relevant results elsewhere in the future.

CHAPTER 5

Reductions of the discrete KP-type equations

1. Overview

Studying 3D integrable lattice models might be a very proper and effective path to understand the theory of integrable systems – the higher dimension brings richer structure and the integrability structure of 2D discrete systems is hidden in that of 3D lattice equations. Among 3D discrete integrable systems, there are three typical lattice equations of the KP-type, namely the discrete AKP [**Hir81**], BKP [**Miw82**] and CKP [**Kas96**, **Sch03**] equations. The discrete KP-type equations are shown to possess the MDC property [**ABS12**, **HJN16**, **TW09**, **Atk12**], and this can also be understood under the DL framework for both discrete and continuous KP hierarchies, see chapters 3 and 4. The continuous KP hierarchies play the role of master models in the continuous theory since they reduce to all the soliton hierarchies associated with scalar differential spectral problems, see e.g. [**JM83**] and chapter 4. Therefore it is an interesting question to understand how reductions work for the discrete KP-type equations and consequently to establish the corresponding discrete theory.

Reductions of the discrete KP-type equations apart from the discrete CKP equation have been considered. One approach is to perform reductions on the tau functions in their bilinear form, namely simple constraints can be imposed on the tau functions describing the bilinear 3D lattice equations and as a result many 2D discrete integrable equations such as the discrete KdV, BSQ, SK equations are obtained (see e.g. [**DJM83**, **TH96**, **HZ13**]). The idea behind such a procedure is actually a reduction on the two spectral parameters k and k', namely they are related through an algebraic curve. However, in those works, some fixed lattice parameters are involved in the algebraic curve, i.e. the preselected discrete variables and their associated lattice parameters are treated specifically compared with the others. This implies that the key point that all the independent variables must be on the same footing is broken, and consequently solution structures of the reduced 2D lattice models become more restrictive. Instead, a trilinear discrete BSQ equation as an example was derived from the DL approach recently [**ZZN12**], where the associated algebraic curve is a generic one (independent of lattice parameters). The trilinear discrete BSQ equation possesses the most general solution structure since no restriction on the lattice parameters is required. This observation motivates us to reconsider reductions of the discrete KP-type equations generically.

In chapter 3, the DL framework was established for the discrete AKP, BKP and CKP equations and their integrability. This unified framework enables us to study reductions of the discrete KP-type equations from this view point in the current chapter. The starting point is a generic algebraic curve taking the form of

$$C(t,k) \doteq P(t) - P(k) = 0, \quad \text{where} \quad P(t) = \sum_{j=1}^{\mathcal{N}} \alpha_j t^j \quad \text{for} \quad \mathcal{N} = 2, 3, \cdots . \tag{5.1}$$

Without losing generality, we take $\alpha_{\mathcal{N}} = 1$, namely the polynomial $P(k)$ is monic. Since $P(t)$ is a polynomial of degree $\mathcal{N}$, $P(t) - P(k)$ can be theoretically factorised as $\prod_{j=1}^{\mathcal{N}}(t - \omega_j(k))$ where $\omega_j(k)$ depend on the parameters α_j. This implies that Equation (5.1) as an algebraic equation of t has $\mathcal{N}$ roots $t = \omega_j(k)$. It is worth pointing out that the curve degenerates when $\alpha_j = 0$ for $j = 1, 2, \cdots, \mathcal{N} - 1$ and in this case we have the roots $t = \omega^j k$ for $j = 1, 2, \cdots$, where ω is one of the primitive $\mathcal{N}$th roots of unity defined by $\omega = \exp(2\pi \mathrm{i}/\mathcal{N})$ with i the imaginary unit. In order to avoid the ambiguity in the future, we require that $\omega_j(k)|_{\alpha_1=\cdots=\alpha_{\mathcal{N}-1}=0} = \omega^j k$ for $j = 1, 2, \cdots, \mathcal{N}$. Hence the generic curve (5.1) has $\mathcal{N} - 1$ primitive roots $t = \omega_j(k)$ for $j = 1, 2, \cdots, \mathcal{N} - 1$ (because more parameters α_j are introduced in the curve when the degree $\mathcal{N}$ increases) and the non-primitive root $\omega_{\mathcal{N}}(k) = k$ as it solves (5.1) for any $\mathcal{N}$.

The reduction is performed by restricting the spectral parameters k and k' on the algebraic curve $C(-k', k) = 0$, namely by taking any of the primitive roots $k' = -\omega_j(k)$ with $j = 1, 2, \cdots, \mathcal{N} - 1$ (or their combination as we will see later). As a result, 2D integrable lattice models in their extended form in terms of the extra parameters α_j arise from such a reduction where the extra parameters α_j, $j = 1, 2, \cdots, \mathcal{N} - 1$ are introduced into the discrete systems. The degeneration $\alpha_j = 0$ gives rise to $k' = -\omega^j k$, which is corresponding to the unextended discrete systems. One can observe that no lattice parameters are involved in such a reduction and therefore the 2D lattice models possess the most general solution structure. For each KP-type equation, a unified expression (coupled system form) for the reduced 2D integrable lattice equations in extended form is given for generic positive integer $\mathcal{N} \geq 2$. Examples include the discrete KdV and BSQ equations, a new discretisation of the SK equation, and some 2D discrete integrable systems which have not been investigated in the literature before, such as the discrete KK and HS equations.

The chapter is organised as follows. In Section 2, we briefly review the general structure of discrete and continuous integrable hierarchies, and explain the MDC property of integrable lattice equations, from the view point of the DL. In Section 3, a comparison between the old and new reductions is made, illustrated by the discrete BSQ and KdV equations. The main result, namely a general reduction scheme, is proposed in Section 4, and examples including the discrete BSQ, SK, KK and HS equations in extended form together with their general solution structure are also given in the section.

2. 3D integrable discrete equations

2.1. Discrete and continuous KP-type hierarchies. The three typical scalar 3D integrable models, namely the discrete and continuous AKP, BKP and CKP, can be recovered by choosing the following Cauchy kernels, plane wave factors as well as measures. Since this chapter is contributed to reductions of the discrete KP-type equations, we do not list the continuous equations here, but the continuous structures will still be given for comparison.

2.1.1. *The AKP hierarchy.* For the discrete and continuous AKP hierarchies, we choose the Cauchy kernel and the plane wave factors as follows:

$$
\begin{aligned}
&\Omega_{k,k'} = \frac{1}{k+k'}, \\
&\rho_k = \prod_{\gamma=1}^{\infty} (p_\gamma + k)^{n_\gamma} \exp\left(\sum_{\gamma=1}^{\infty} k^\gamma x_\gamma\right), \\
&\sigma_{k'} = \prod_{\gamma=1}^{\infty} (p_\gamma - k')^{-n_\gamma} \exp\left(-\sum_{\gamma=1}^{\infty} (-k')^\gamma x_\gamma\right),
\end{aligned}
\tag{5.2}
$$

and in this case, there is no particular requirement for the measure $\mathrm{d}\zeta(k,k')$. The tau function for soliton solution given by

$$
\tau = \det(1+\mathbf{AM}), \quad (\mathbf{M})_{j,i} = M_{j,i} = \frac{\rho_{k_i}\sigma_{k'_j}}{k_i + k'_j}, \quad i = 1,2,\cdots,N, \quad j = 1,2,\cdots,N', \tag{5.3}
$$

where $\mathbf{A}$ with entries $A_{i,j}$ is an arbitrary $N \times N'$ non-degenerate matrix, solves simultaneously the discrete, semi-discrete and continuous AKP hierarchies, see chapters 3 and 4. The tau function satisfies an infinite number of copies of the same 3D discrete equation with regard to lattice directions and parameters – this is the MDC property (which will be explained later) and the whole family of discrete equations should be understood as the discrete AKP hierarchy. Without loss of generality, we write down the discrete equation involving the lattice directions n_1, n_2 and n_3 and it takes the form of

$$
(p_1 - p_2)\bar{\tau}\hat{\tilde{\tau}} + (p_2 - p_3)\tilde{\tau}\bar{\hat{\tau}} + (p_3 - p_1)\hat{\tau}\bar{\tilde{\tau}} = 0, \tag{5.4}
$$

namely the HM equation, see [**Hir81**, **Miw82**].

2.1.2. *The BKP hierarchy.* For the discrete BKP hierarchy, we choose the Cauchy kernel and the plane wave factors and impose the antisymmetry property on the measure as follows:

$$
\begin{aligned}
&\Omega_{k,k'} = \frac{1}{2}\frac{k-k'}{k+k'}, \\
&\rho_k = \prod_{\gamma=1}^{\infty} \left(\frac{p_\gamma + k}{p_\gamma - k}\right)^{n_\gamma} \exp\left(\sum_{\gamma=0}^{\infty} k^{2\gamma+1} x_{2\gamma+1}\right), \quad \sigma_{k'} = \rho_{k'}, \\
&\iint_D \cdot \mathrm{d}\zeta(k,k') = -\iint_D \cdot \mathrm{d}\zeta(k',k).
\end{aligned}
\tag{5.5}
$$

Then the DL framework reveals that the tau function for soliton solution given by

$$
\tau^2 = \det(1+\mathbf{AM}), \quad (\mathbf{M})_{j,i} = M_{j,i} = \rho_{k_i}\frac{1}{2}\frac{k_i - k'_j}{k_i + k'_j}\sigma_{k'_j}, \quad A_{i,j} = -A_{j,i} \tag{5.6}
$$

for $i,j = 1,2,\cdots,2N$, solves the discrete, semi-discrete as well as continuous BKP hierarchies (see chapter 3 for the discrete one and chapter 4 for the continuous one). The discrete equation describing the dynamical evolutions with respect to the lattice directions

n_1, n_2 and n_3 takes the form of

$$(p_1-p_2)(p_2-p_3)(p_3-p_1)\tau\bar{\hat{\tilde{\tau}}}+(p_1+p_2)(p_1+p_3)(p_2-p_3)\tilde{\tau}\bar{\hat{\tau}}$$
$$+(p_3+p_1)(p_3+p_2)(p_1-p_2)\bar{\tau}\hat{\tilde{\tau}}+(p_2+p_3)(p_2+p_1)(p_3-p_1)\hat{\tau}\tilde{\bar{\tau}}=0, \tag{5.7}$$

which is usually referred to as the Miwa equation [**Miw82**]. In addition, the tau function can be expressed by a Pfaffian due to the antisymmetry of the matrices $\mathbf{A}$ and $\mathbf{M}$.

2.1.3. *The CKP hierarchy.* The following Cauchy kernel, plane wave factors and measure are chosen for the discrete and continuous CKP hierarchies:

$$\Omega_{k,k'}=\frac{1}{k+k'},$$
$$\rho_k=\prod_{\gamma=1}^{\infty}\left(\frac{p_\gamma+k}{p_\gamma-k}\right)^{n_\gamma}\exp\left(\sum_{\gamma=0}^{\infty}k^{2\gamma+1}x_{2\gamma+1}\right),\quad \sigma_{k'}=\rho_{k'}, \tag{5.8}$$
$$\iint_D\cdot\mathrm{d}\zeta(k,k')=\iint_D\cdot\mathrm{d}\zeta(k',k).$$

The tau function for soliton solution in this case is given by

$$\tau=\det(1+\mathbf{AM}),\quad (\mathbf{M})_{j,i}=M_{j,i}=\frac{\rho_{k_i}\sigma_{k'_j}}{k_i+k'_j},\quad A_{i,j}=A_{j,i},\quad i,j=1,2,\cdots,N, \tag{5.9}$$

and it was proven in chapters 3 and 4 that the tau function solves the discrete, semi-discrete and continuous CKP hierarchies. An example involving the lattice variables n_1, n_2 and n_3 takes the form of Cayley's $2\times2\times2$ hyperdeterminant as follows:

$$[(p_1-p_2)^2(p_2-p_3)^2(p_3-p_1)^2\tau\bar{\hat{\tilde{\tau}}}+(p_1+p_2)^2(p_1+p_3)^2(p_2-p_3)^2\tilde{\tau}\bar{\hat{\tau}}$$
$$-(p_2+p_3)^2(p_2+p_1)^2(p_3-p_1)^2\hat{\tau}\tilde{\bar{\tau}}-(p_3+p_1)^2(p_3+p_2)^2(p_1-p_2)^2\bar{\tau}\hat{\tilde{\tau}}]^2$$
$$-4(p_1^2-p_2^2)^2(p_1^2-p_3^2)^2[(p_2+p_3)^2\hat{\tilde{\tau}}\tilde{\bar{\tau}}-(p_2-p_3)^2\tilde{\tau}\bar{\hat{\tilde{\tau}}}][(p_2+p_3)^2\hat{\tau}\bar{\tau}-(p_2-p_3)^2\tau\bar{\hat{\tau}}]=0. \tag{5.10}$$

This is the discrete CKP equation and it serves the superposition property of the continuous CKP equation. Compared with the discrete AKP and BKP equations, this model is no longer in Hirota's bilinear form and instead it takes multiquadratic and quadrilinear form.

2.2. Reduction on the measure. The 3D lattice models, namely the discrete AKP, BKP and CKP equations, are all associated with a nonlocal Riemann–Hilbert problem where the measure depends on two spectral parameters and a double integral is involved. The key point in the reduction problem is to reduce the double integral in (2.15) to a single one connected to a local Riemann–Hilbert problem. The reduction can be performed on the measure $\mathrm{d}\zeta(k,k')$, see e.g. [**ZZN12**]. In the cases of the discrete and continuous AKP, BKP and CKP hierarchies, we consider the following reductions, respectively:

$$\mathrm{d}\zeta(k,k') = \sum_{j=1}^{\mathcal{N}-1} \mathrm{d}\lambda_j(k)\mathrm{d}k'\delta(k'+\omega_j(k)), \tag{5.11a}$$

$$\mathrm{d}\zeta(k,k') = \sum_{j=1}^{\mathcal{N}-1} \Big[\mathrm{d}\lambda_j(k)\mathrm{d}k'\delta(k'+\omega_j(k)) - \mathrm{d}\lambda_j(k')\mathrm{d}k\delta(k+\omega_j(k'))\Big], \tag{5.11b}$$

$$\mathrm{d}\zeta(k,k') = \sum_{j=1}^{\mathcal{N}-1} \Big[\mathrm{d}\lambda_j(k)\mathrm{d}k'\delta(k'+\omega_j(k)) + \mathrm{d}\lambda_j(k')\mathrm{d}k\delta(k+\omega_j(k'))\Big]. \tag{5.11c}$$

The reason why the reductions for BKP and CKP are more complex is because the measures in these cases obey antisymmetry and symmetry properties, respectively, cf. (5.5) and (5.8), and such properties must be preserved for the compatibility between the 3D equations and the reduced 2D equations.

Such a procedure, i.e. reduction on the measure, is effectively restricting spectral parameters acting as free parameters in solutions to a hierarchy of 3D integrable equations on an algebraic curve, or more concretely, from (5.3), (5.6) and (5.9); in other words, imposing a constraint on the measure is deep down restricting the solution space of a 3D hierarchy, and thus the problem turns to be looking for the corresponding reduced hierarchy of 2D equations describing the dynamics of the reduced solution space. This can be realised via translating the algebraic curve of the spectral parameters to its corresponding symmetry constraint on the tau function, and then the reduced 2D hierarchy can be obtained by applying the symmetry constraint to the 3D hierarchy.

3. Discrete BSQ and KdV equations

The reduction problem of a 3D lattice integrable equation amounts to imposing a constraint on the spectral parameters k and k'. In this section, we illustrate how a generic algebraic curve for the spectral parameters affects the reduction result by concrete examples including the discrete BSQ and KdV equations.

There are two existing discrete BSQ equations, namely a bilinear one and a trilinear one. By performing the reduction associated with the generic algebraic curve, the connection between the two equations is clearly revealed. Following this idea, the reduction to the discrete KdV equation is also explained in the same way.

3.1. Discrete BSQ equation: bilinear versus trilinear. The discrete BSQ equation arises as the so-called 3-reduction of the discrete AKP equation. In order to understand the structure of the discrete BSQ equation, we start with considering the effective plane wave factor in the tau function of the discrete AKP equation, i.e.

$$\rho_k\sigma_{k'} = \prod_{\gamma=1}^{\infty}\left(\frac{p_\gamma+k}{p_\gamma-k'}\right)^{n_\gamma}\exp\left(\sum_{\gamma=1}^{\infty}(k^\gamma-(-k')^\gamma)x_\gamma\right), \tag{5.12}$$

and also the Cauchy kernel $\Omega_{k,k'} = 1/(k+k')$. Two different reductions can be imposed on the discrete AKP equation, i.e. the HM equation, and they result in the bilinear and trilinear discrete BSQ equations, respectively.

3.1.1. *Bilinear discrete BSQ equation.* One can impose an algebraic relation on the spectral parameters k and k' as follows:

$$R(p_1, p_2, p_3) = \left(\frac{p_1 + k}{p_1 - k'}\right)\left(\frac{p_2 + k}{p_2 - k'}\right)\left(\frac{p_3 + k}{p_3 - k'}\right) = 1, \tag{5.13}$$

which implies

$$(\rho_k \sigma_{k'})^{\bar{\hat{\tilde{}}}} = 1, \quad \text{and consequently} \quad \bar{\hat{\tilde{\mathbf{C}}}} = 1,$$

according to (2.17). By observing the definition of the tau function, i.e. (2.21), we can deduce that such a reduction leads to a symmetry constraint $\bar{\hat{\tilde{\tau}}} = \tau$. The following bilinear 2D equation arises from the HM equation (5.4) as a consequence of imposing the symmetry constraint:

$$(p_1 - p_2)\hat{\tilde{\tau}}\underset{\hat{\sim}}{\tau} + (p_2 - p_3)\tilde{\tau}\underset{\sim}{\tau} + (p_3 - p_1)\hat{\tau}\underset{\wedge}{\tau} = 0, \tag{5.14}$$

which is the bilinear discrete BSQ equation given by Date, Jimbo and Miwa, see [**DJM83**].

We would like to argue that such a reduction is not sufficient to describe the BSQ structure. This is because the continuous BSQ equation $(D_1^4 + 3D_2^2)\tau \cdot \tau = 0$ arises from the continuous AKP equation $(D_1^4 - 4D_1D_3 + 3D_2^2)\tau \cdot \tau = 0$ by imposing the symmetry constraint $\partial_{x_3}\tau = 0$, where x_3 is the third flow variable in the continuous AKP hierarchy (see (5.12)) and D is Hirota's bilinear operator. Such a reduction procedure is deep down governed by the algebraic constraint $k^3 = (-k')^3$, cf. (5.2) and (5.3), which implies that (5.13) does not completely reflect the algebraic structure of BSQ.

For the sake of the consistency between the discrete and continuous BSQ equations, some extra conditions are necessary, namely p_1, p_2 and p_3 must be connected in order to reduce (5.13) to $k^3 = (-k')^3$. By expanding (5.13), one can find that the lattice parameters must be restricted by the relations $p_2 = \omega p_1$ and $p_3 = \omega^2 p_1$, where ω is a primitive cube root of unity given by $\omega = \exp(2\pi\mathfrak{i}/3)$, cf. [**Hie15**] (this was also implied in the continuum limit of the equation in [**DJM83**]). Under this assumption the equation (5.14) turns out to be

$$\hat{\tilde{\tau}}\underset{\hat{\sim}}{\tau} + \omega\tilde{\tau}\underset{\sim}{\tau} + \omega^2\hat{\tau}\underset{\wedge}{\tau} = 0, \tag{5.15}$$

which is a discretisation of the bilinear BSQ equation. This reduction also brings us the reduced effective plane wave factors for the equation given by

$$\rho_k \sigma_{-\omega^j k} = \prod_{\gamma=2}^{3}\left(\frac{\omega^{\gamma-1}p_1 + k}{\omega^{\gamma-1}p_1 + \omega^j k}\right)^{n_\gamma} \prod_{\gamma\neq 2,3,\gamma\in\mathbb{Z}^+}\left(\frac{p_\gamma + k}{p_\gamma + \omega^j k}\right)^{n_\gamma} \exp\left(\sum_{\gamma=1}^{\infty}(k^\gamma - (\omega^j k)^\gamma)x_\gamma\right) \tag{5.16}$$

for $j = 1, 2$. This amounts to a constraint on the lattice parameters and hence truncates the dynamics. In other words, (5.15) is a truncated equation in the discrete hierarchy. Furthermore, due to the dependence of p_2 on p_1, the semi-discrete BSQ equation probably cannot be recovered from the continuum limit of (5.15) – the limit immediately leads to the fully continuous one since there is effectively only one lattice parameter p_1 in the equation.

Equations (5.14) and (5.15) are the ones describing the dynamical evolutions with respect to n_1 and n_2 and both take bilinear forms. Similarly, the equations for lattice directions (n_2, n_3) or (n_1, n_3) also take the same forms because the restrictions on p_1, p_2 and p_3 imply that the lattice variables n_1, n_2 and n_3 are treated particularly, cf. (5.13). A natural question is that what an equation in the 2D discrete hierarchy would be if at least one of the discrete variables is selected from $\{n_\gamma | \gamma \geq 4\}$ (for instance one may consider the equation describing the dynamical evolutions with respect to n_4 and n_5). The answer is that such equations are in trilinear form and the general structure is explained below.

3.1.2. *Trilinear discrete BSQ equation.* The reduction (5.13) implies that the spectral variables k and k' are restricted on an algebraic curve

$$(-k')^3 - k^3 + (p_1 + p_2 + p_3)((-k')^2 - k^2) + (p_1p_2 + p_2p_3 + p_3p_1)((-k') - k) = 0.$$

However, specific lattice parameters p_1, p_2 and p_3 are involved in this algebraic curve. To perform the most general reduction, we need to consider the reduction based on a generic cubic curve

$$C(-k', k) = (-k')^3 - k^3 + \alpha_2((-k')^2 - k^2) + \alpha_1((-k') - k) = 0, \tag{5.17}$$

where α_1 and α_2 are constants independent of the lattice parameters, and denote the three roots by $k' = -\omega_j(k)$ for $j = 1, 2, 3$, in which $\omega_1(k)$ and $\omega_2(k)$ are the two primitive roots depending on the parameters α_1 and α_2, and $\omega_3(k) = k$. In the case of $\alpha_1 = \alpha_2 = 0$, the generic cubic curve degenerates to the reduction to the BSQ equation $k^3 = (-k')^3$ and its solutions are given by $k' = -\omega^j k$. In other words, we have $\omega_j(k)|_{\alpha_1=\alpha_2=0} = \omega^j k$. As we have pointed out at the end of Section 2, imposing the constraint on the spectral parameters is equivalent to considering a subspace of the general soliton solution space (i.e. (5.3)) of the AKP hierarchy, and thus we expect to find a 2D discrete hierarchy describing the subspace.

The reduction based on the cubic curve provides us with the effective plane wave factors as follows:

$$\rho_k \sigma_{-\omega_j(k)} = \prod_{\gamma=1}^{\infty} \left(\frac{p_\gamma + k}{p_\gamma + \omega_j(k)} \right)^{n_\gamma} \exp\left(\sum_{\gamma=1}^{\infty} (k^\gamma - (\omega_j(k))^\gamma) x_\gamma \right), \quad j = 1, 2. \tag{5.18}$$

The direct linearisation approach shows in [**ZZN12**] that such a reduction leads to a hierarchy of extended discrete BSQ equations in trilinear form and the one associated with n_1 and n_2 takes the form of

$$\begin{aligned} &(p_1 - p_2)^2 \hat{\tilde{\tau}} \tau \underset{\sim}{\underset{\wedge}{\tau}} - (3p_2^2 - 2\alpha_2 p_2 + \alpha_1) \tilde{\tau} \tau \underset{\sim}{\tau} - (3p_1^2 - 2\alpha_2 p_1 + \alpha_1) \hat{\tau} \tau \underset{\wedge}{\tau} \\ &\quad + [(p_1^2 + p_1p_2 + p_2^2) - \alpha_2(p_1 + p_2) + \alpha_1](\underset{\sim}{\hat{\tau}} \tilde{\tau} \underset{\wedge}{\tau} + \underset{\wedge}{\tilde{\tau}} \hat{\tau} \underset{\sim}{\tau}) = 0. \end{aligned} \tag{5.19}$$

For the unextended BSQ equation, we can consider the degeneration $\alpha_1 = \alpha_2 = 0$ and it gives the effective plane wave factors

$$\rho_k \sigma_{-\omega^j k} = \prod_{\gamma=1}^{\infty} \left(\frac{p_\gamma + k}{p_\gamma + \omega^j k} \right)^{n_\gamma} \exp\left(\sum_{\gamma=1}^{\infty} (k^\gamma - (\omega^j k)^\gamma) x_\gamma \right), \quad j = 1, 2, \tag{5.20}$$

for the unextended fully discrete BSQ hierarchy. The equation in the hierarchy for directions (n_1, n_2) takes the trilinear form of

$$(p_1-p_2)^2\hat{\tilde{\tau}}\tau\underset{\hat{\tilde{}}}{\tau} - 3p_2^2\tilde{\tau}\tau\underset{\tilde{}}{\tau} - 3p_1^2\hat{\tau}\tau\underset{\hat{}}{\tau} + (p_1^2+p_1p_2+p_2^2)(\underset{\tilde{}}{\hat{\tau}}\tilde{\tau}\underset{\hat{}}{\tau} + \underset{\hat{}}{\tilde{\tau}}\hat{\tau}\underset{\tilde{}}{\tau}) = 0. \tag{5.21}$$

We note that the nonlinear forms of this trilinear equation (5.21) were even given earlier from the view point of the DL in [**NPCQ92, Nij99**]. From the structure of the plane wave factors we observe that no restriction is imposed on the lattice parameters $\{p_\gamma\}$ and therefore compared to the bilinear BSQ equations, the trilinear ones possess more general structure. In fact, the bilinear equations can be recovered from the trilinear equations. One can take $\alpha_1 = p_1p_2 + p_2p_3 + p_3p_1$ and $\alpha_2 = p_1 + p_2 + p_3$, then the extended BSQ equation (5.19) turns out to be Equation (5.14) immediately. This is because the cubic curve (5.17) under such a degeneration becomes (5.13). And the equation (5.15) can be recovered from the unextended BSQ equation (5.21) by taking $p_2 = \omega p_1$ – this can be obtained if one compares their plane wave factors (5.20) and (5.16).

The statement can be made as follows: The extended trilinear BSQ equation (5.19) possesses the most general solution structure in the BSQ class. Equation (5.21) as an equation without restrictions on the lattice parameters should be referred to as the discrete BSQ equation since it is consistent with the continuous BSQ hierarchy in the sense of the curve relation $k^3 = (-k')^3$. While Equations (5.14) and (5.15) taking bilinear forms are degenerations of the trilinear equations when extra conditions are imposed on the curve parameters α_1 and α_2 as well as the lattice parameters p_2 and p_3. Therefore one may think of them as truncations of the extended discrete BSQ equation (5.19) and the discrete BSQ equation (5.21) respectively.

Furthermore, we would like to make a few comments on the MDC property of the above discrete BSQ equations. The MDC property is preserved in dimensional reductions of multi-dimensionally consistent 3D equations. Thus the various BSQ equations, namely (5.14), (5.15), (5.19) and (5.21), all possess the MDC property since the common solutions for the discrete hierarchies follow from their respective reductions on that of the discrete AKP hierarchy. In the cases of the non-truncated discrete BSQ equations (5.19) and (5.21), all the discrete BSQ equations in their respective hierarchies take the same form, which is guaranteed by their respective reduced plane wave factors (5.18) and (5.20). While in the truncated cases (5.14) and (5.15), the equations in each discrete hierarchy may look differently. Taking (5.14) as an example, the equation involving the lattice variables n_1 and n_2 is bilinear, i.e. (5.14) itself; however, the equation describing the dynamical evolutions with respect to n_4 and n_5 is trilinear – we can start from the extended discrete BSQ equation for (n_4, n_5) which takes the same form of (5.19) but involves lattice parameters p_4, p_5 and extra parameters α_1, α_2; then the same degeneration from (5.19) to (5.14), namely $\alpha_1 = p_1p_2 + p_2p_3 + p_3p_1$ and $\alpha_2 = p_1 + p_2 + p_3$, gives rise to a trilinear equation containing parameters p_4, p_5 and p_1, p_2, p_3.

3.2. Discrete KdV equation. A similar comparison can also be made for the discrete KdV class (although it is not as obvious as the case in the discrete BSQ class). One can introduce the 2-reduction on the spectral parameters k and k' as follows:

$$R(p_2, p_3) = \left(\frac{p_2 + k}{p_2 - k'}\right)\left(\frac{p_3 + k}{p_3 - k'}\right) = 1. \tag{5.22}$$

This is equivalent to the constraint on the tau function $\bar{\hat{\tau}} = \tau$. Making use of the constraint a reduced discrete equation can immediately be derived from the HM equation (5.4), i.e.

$$(p_1 - p_2)\hat{\tilde{\tau}}\underset{\hat{}}{\tau} + (p_2 - p_3)\tau\tilde{\tau} + (p_3 - p_1)\underset{\hat{}}{\tilde{\tau}}\hat{\tau} = 0, \tag{5.23}$$

which is the bilinear discrete KdV equation given in [**DJM83**].

Similarly to the case in the discrete BSQ equation, in order to match the discrete KdV equation to its continuous counterpart, one has to take $p_3 = -p_2$ and therefore the reduction condition becomes the relation $k^2 = (-k')^2$, which reduces the continuous KP hierarchy to the continuous KdV hierarchy. This gives us the effective plane wave factor that takes the form of

$$\rho_k\sigma_k = \left(\frac{-p_2 + k}{-p_2 - k}\right)^{n_3} \prod_{\gamma\neq 3, \gamma\in\mathbb{Z}^+} \left(\frac{p_\gamma + k}{p_\gamma - k}\right)^{n_\gamma} \exp\left(\sum_{\gamma=1}^{\infty}(k^\gamma - (-k)^\gamma)x_\gamma\right). \tag{5.24}$$

Simultaneously this restriction of the lattice parameter p_3 on Equation (5.25) gives rise to the 6-point bilinear discrete KdV equation

$$(p_1 + p_2)\underset{\hat{}}{\tilde{\tau}}\hat{\tau} - (p_1 - p_2)\underset{\hat{}}{\tau}\hat{\tilde{\tau}} = 2p_2\tau\tilde{\tau}. \tag{5.25}$$

In the above reduction one can still see that some lattice parameters are restricted and we argue that it leads to some truncated solution structure.

We now consider the reduction associated with a generic quadratic curve of spectral parameters which is independent of any lattice parameters as follows:

$$C(-k', k) = (-k')^2 - k^2 + \alpha_1((-k') - k) = 0. \tag{5.26}$$

The curve has two roots $k' = -\omega_1(k) = k + \alpha_1$ and $k' = -\omega_2(k) = -k$. Applying such a reduction to (5.12) provides us with the effective plane wave factor:

$$\rho_k\sigma_{-\omega_1(k)} = \prod_{\gamma=1}^{\infty}\left(\frac{p_\gamma + k}{p_\gamma + \omega_1(k)}\right)^{n_\gamma} \exp\left(\sum_{\gamma=1}^{\infty}(k^\gamma - (\omega_1(k))^\gamma)x_\gamma\right), \tag{5.27}$$

where we only choose the primitive root $k' = -\omega_1(k) = k + \alpha_1$. Then the tau function defined in (5.3) under this reduction solves the following two compatible equations:

$$(p_1 + p_2 - \alpha_1)\underset{\tilde{}}{\hat{\tau}}\tilde{\tau} + (p_1 - p_2)\underset{\tilde{}}{\tau}\hat{\tilde{\tau}} = (2p_1 - \alpha_1)\tau\hat{\tau}, \tag{5.28a}$$

$$(p_1 + p_2 - \alpha_1)\underset{\hat{}}{\tilde{\tau}}\hat{\tau} - (p_1 - p_2)\underset{\hat{}}{\tau}\hat{\tilde{\tau}} = (2p_2 - \alpha_1)\tau\tilde{\tau}, \tag{5.28b}$$

in which either can be thought of as the discrete analogue of the bilinear KdV equation. The derivation of the bilinear discrete KdV equations will be given in the next section, and here we note that the equations are actually embedded in the extended discrete BSQ equation (5.19) because the generic cubic algebraic curve (5.17) contains the curve (5.26). Equation (5.28) is the analogue of the trilinear equation (5.19) on the KdV level.

By taking $\alpha_1 = 0$ in (5.28), we obtain two compatible equations as follows:

$$(p_1 + p_2)\underset{\sim}{\hat{\tau}}\tilde{\tau} + (p_1 - p_2)\underset{\sim}{\tau}\hat{\tilde{\tau}} = 2p_1\tau\hat{\tau}, \quad (p_1 + p_2)\underset{\wedge}{\tilde{\tau}}\hat{\tau} - (p_1 - p_2)\underset{\wedge}{\tau}\hat{\tilde{\tau}} = 2p_2\tau\tilde{\tau}. \tag{5.29}$$

Considering the compatibility of the two equations, we obtain a third lattice equation taking the form of

$$(p_1 - p_2)^2\hat{\tilde{\tau}}\underset{\hat{\sim}}{\tau} - (p_1 + p_2)^2\underset{\wedge}{\tilde{\tau}}\underset{\sim}{\hat{\tau}} + 4p_1p_2\tau^2 = 0, \tag{5.30}$$

which is known as the discrete-time Toda equation. Since the curve (5.26) degenerates when $\alpha_1 = 0$, namely $k' = -\omega_j(k) = (-1)^{j-1}k$ for $j = 1, 2$, the equations in (5.29) and Equation (5.30) are all governed by the effective plane wave factor

$$\rho_k\sigma_k = \prod_{\gamma=1}^{\infty}\left(\frac{p_\gamma + k}{p_\gamma - k}\right)^{n_\gamma}\exp\left(\sum_{\gamma=1}^{\infty}(k^\gamma - (-k)^\gamma)x_\gamma\right), \tag{5.31}$$

where we only choose the primitive root $k' = k$ in order to avoid triviality. This in turn implies that any single equation in (5.29) and (5.30) can be considered as the bilinear discrete KdV equation – they are just different bilinear forms and the tau functions are the same.

The second discrete KdV equation in (5.29) looks exactly the same as (5.25). But in fact, the reduced equation from a generic algebraic curve possesses more general structure as can be seen if one compares (5.31) and (5.24). More concretely, in the previous case, a restriction is made on the lattice parameters, i.e. $p_3 = -p_2$, this leads the equation involving the discrete variables n_2 and n_3 to its degeneration $\bar{\tilde{\tau}} = \tau$, namely the constraint itself. And this is also the reason why the first equation in (5.29) is missing in the former case. While in the latter case, all the discrete variables are on the same footing and all the discrete KdV equations in the hierarchy are covariant.

Furthermore, similar to the discrete BSQ case, if we take $\alpha_1 = p_2 + p_3$, the quadratic curve (5.26) turns out to be the reduction condition (5.22), in other words, the equation (5.23) is a truncated version of the extended discrete KdV equation (5.28). This also reflects on the equations themselves, namely we can from (5.28) recover the following bilinear discrete equations:

$$(p_1 - p_3)\underset{\sim}{\hat{\tau}}\tilde{\tau} + (p_1 - p_2)\underset{\sim}{\tau}\hat{\tilde{\tau}} = (2p_1 - p_2 - p_3)\tau\hat{\tau}, \tag{5.32a}$$

$$(p_1 - p_3)\underset{\wedge}{\tilde{\tau}}\hat{\tau} - (p_1 - p_2)\underset{\wedge}{\tau}\hat{\tilde{\tau}} = (p_2 - p_3)\tau\tilde{\tau}, \tag{5.32b}$$

in which the second equation is exactly the same as (5.23), while the first one was missing in [**DJM83**].

The MDC property of the discrete KdV equations, similarly to the discrete BSQ equations, is still preserved according to the general theory. One remark here is that the equations in (5.29) and (5.30) should be understood as members in the same discrete KdV hierarchy.

In the above analysis of the discrete BSQ and KdV equations, we considered the reduction associated with $\alpha_{\mathcal{N}} = 1$. The results show that the reduced 2D lattice equations have more general structure, and the bilinear discrete BSQ and KdV equations given in the literature are the truncations of the obtained 2D integrable lattice equations by taking particular values for the parameters α_i.

However, one can observe that α_1 and α_2 can be removed from Equations (5.28) and (5.19), respectively, after some simple transforms on the lattice parameters p and q. For instance, the transforms $p_1 \to p_1 + \alpha_1/2$ and $p_2 \to p_2 + \alpha_1/2$ immediately bring (5.28) to (5.29). Similar transforms can also eliminate α_2 in (5.19). This is because the term of the second highest degree, i.e. $\alpha_{\mathcal{N}-1}(t^{\mathcal{N}-1} - k^{\mathcal{N}-1})$, in the algebraic curve (5.1) can be absorbed by certain transforms on t and k; in other words, $\alpha_{\mathcal{N}-1}$ is redundant in the reductions. Thus, without losing generality, for dimensional reductions of the discrete KP equations, we only need to consider a generic algebraic curve of degree $\mathcal{N}$ in the form of

$$C_{\mathcal{N}}(t,k) \doteq P_{\mathcal{N}}(t) - P_{\mathcal{N}}(k) = 0, \tag{5.33}$$

where

$$P_{\mathcal{N}}(t) = t^{\mathcal{N}} + \sum_{j=1}^{\mathcal{N}-2} \alpha_j t^j, \quad \mathcal{N} = 2, 3, \cdots .$$

For convenience, we still denote the $\mathcal{N}$ roots to the algebraic equation by $t = \omega_j(k)$ for $j = 1, 2, \cdots, \mathcal{N}$ in which ω_j satisfies $\omega_j(k)|_{\alpha_1=\cdots=\alpha_{\mathcal{N}-2}=0} = \omega^j k$ with $\omega = \exp(2\pi \mathrm{i}/\mathcal{N})$.

Considering the algebraic curve $C_{\mathcal{N}}(-k', k) = 0$, we find that in the case of $\mathcal{N} = 2$, the curve takes the form of $C_2(-k,k) = (-k')^2 - k^2 = 0$. Thus there is no extended form for the discrete KdV equation, and every single equation in (5.29) and (5.30) can be thought of as its standard form. When $\mathcal{N} = 3$, the algebraic curve $C_3(-k', k) = (-k')^3 - k^3 + \alpha_1((-k') - k) = 0$ contains only one parameter α_1. Hence the extended BSQ equation can be obtained immediately by taking $\alpha_2 = 0$ in (5.19), which takes the form of

$$(p_1 - p_2)^2 \hat{\tilde{\tau}} \tau \underset{\hat{\tilde{}}}{\tau} - (3p_2^2 + \alpha_1)\tilde{\tau}\tau\underset{\tilde{}}{\tau} - (3p_1^2 + \alpha_1)\hat{\tau}\tau\underset{\hat{}}{\tau} + (p_1^2 + p_1 p_2 + p_2^2 + \alpha_1)(\underset{\tilde{}}{\hat{\tau}}\tilde{\tau}\underset{\hat{}}{\tau} + \underset{\hat{}}{\tilde{\tau}}\hat{\tau}\underset{\tilde{}}{\tau}) = 0. \tag{5.34}$$

The degeneration case $\alpha_1 = 0$ then gives us the standard trilinear BSQ equation (5.21).

4. General reduction scheme

In this section, we would like to consider the reduction associated with a generic algebraic curve involving extended parameters α_j for the discrete AKP, BKP and CKP equations. The aim is to give a unified expression of the reduced 2D lattice equations in each case. Examples include some new integrable lattice equations such as the discrete SK, KK and HS equations.

As we discussed in the previous section, $\alpha_{\mathcal{N}-1}$ is redundant in the reduction and it can be removed by some suitable transforms on the spectral parameters. Therefore, for reductions of the discrete AKP equation, we only consider constraint $C_{\mathcal{N}}(-k,k) = 0$ defined by (5.33), and it is corresponding to the reduction condition (5.11a). However, since the measure reductions for BKP and CKP, i.e. (5.11b) and (5.11c), must be anti-symmetrised and symmetrised, respectively, due to their original properties on the 3D level, cf. (5.6) and (5.8), in both cases the generic curve must take the following form:

$$C_{\mathcal{N}}(t,k) \doteq P_{\mathcal{N}}(t) - P_{\mathcal{N}}(k) = 0 \quad \text{with} \quad P_{\mathcal{N}}(t) = \begin{cases} t^{\mathcal{N}} + \sum_{i=1}^{j-1} \alpha_{2i} t^{2i}, & \mathcal{N} = 2j, \\ t^{\mathcal{N}} + \sum_{i=0}^{j-1} \alpha_{2i+1} t^{2i+1}, & \mathcal{N} = 2j+1 \end{cases} \tag{5.35}$$

for $j = 1, 2, \cdots$, and we still denote all the roots of the curve by $t = \omega_j(k)$. This curve immediately leads to the fact that

$$C_{\mathcal{N}}(-k', k) = 0 \quad \Longleftrightarrow \quad C_{\mathcal{N}}(-k, k') = 0, \tag{5.36}$$

because $P(t)$ in the curve (5.35) is either an odd degree polynomial or an even degree polynomial. The reduction $C(-k, k') = 0$ for the discrete AKP, BKP and CKP equations can be translated to a symmetry constraint and we give it in the following proposition:

Proposition 5.1. *The reduction condition* $C_{\mathcal{N}}(-k', k) = 0$ *determined by* (5.33) *(resp.* (5.35)*) for the discrete AKP (resp. BKP and CKP) equation(s) provides the symmetry constraint*

$$\left(\prod_{j=1}^{\mathcal{N}} \mathrm{T}_{-\omega_j(-p_\gamma)} \right) \tau \Big|_{C_{\mathcal{N}}(-k',k)=0} = \tau \quad \textit{for} \quad \gamma = 1, 2, \cdots. \tag{5.37}$$

Proof. As is mentioned in Subsection 3 the tau function in the framework of the DL is defined via $\det(1 + \mathbf{\Omega} \cdot \mathbf{C})$ and the dynamical variables only appear in the effective plane wave factor $\rho_k \sigma_{k'}$ in the infinite matrix $\mathbf{C}$, cf. (2.17). Thus, we only need to prove in each case that

$$\left(\prod_{j=1}^{\mathcal{N}} \mathrm{T}_{-\omega_j(-p_\gamma)} \right) \rho_k \sigma_{k'} \Big|_{C_{\mathcal{N}}(-k',k)=0} = \rho_k \sigma_{k'} \quad \text{for} \quad \gamma = 1, 2, \cdots. \tag{5.38}$$

In the case of AKP, namely the plane wave factors given by (5.3), we can calculate that

$$\begin{aligned} \left(\prod_{j=1}^{\mathcal{N}} \mathrm{T}_{-\omega_j(-p_\gamma)} \right) \rho_k \sigma_{k'} &= \prod_{j=1}^{\mathcal{N}} \left(\frac{-\omega_j(-p_\gamma) + k}{-\omega_j(-p_\gamma) - k'} \right) \rho_k \sigma_{k'} \\ &= \frac{P_{\mathcal{N}}(k) - P_{\mathcal{N}}(-p_\gamma)}{P_{\mathcal{N}}(-k') - P_{\mathcal{N}}(-p_\gamma)} \rho_k \sigma_{k'} = \rho_k \sigma_{k'}, \end{aligned}$$

where in the last step we used the relation $P_{\mathcal{N}}(k) = P_{\mathcal{N}}(-k')$, i.e. the constraint of the spectral variables $C_{\mathcal{N}}(-k', k) = 0$ governed by the curve (5.33). Likewise, in the cases of BKP and CKP, we can from the plane wave factors given by (5.5) and (5.8) obtain that

$$\begin{aligned} \left(\prod_{j=1}^{\mathcal{N}} \mathrm{T}_{-\omega_j(-p_\gamma)} \right) \rho_k \sigma_{k'} &= \prod_{j=1}^{\mathcal{N}} \left(\frac{-\omega_j(-p_\gamma) + k}{-\omega_j(-p_\gamma) - k} \right) \left(\frac{-\omega_j(-p_\gamma) + k'}{-\omega_j(-p_\gamma) - k'} \right) \rho_k \sigma_{k'} \\ &= \left(\frac{P_{\mathcal{N}}(k) - P_{\mathcal{N}}(-p_\gamma)}{P_{\mathcal{N}}(-k) - P_{\mathcal{N}}(-p_\gamma)} \right) \left(\frac{P_{\mathcal{N}}(k') - P_{\mathcal{N}}(-p_\gamma)}{P_{\mathcal{N}}(-k') - P_{\mathcal{N}}(-p_\gamma)} \right) \rho_k \sigma_{k'} = \rho_k \sigma_{k'}, \end{aligned}$$

where the last equality holds because of the constraints given in (5.36). Thus, in any case Equation (5.38) is proven and consequently the symmetry constraint (5.37) is obtained. □

This is the generic reduction on the bilinear/multilinear forms of the discrete KP equations and in the following we show that under such a reduction, coupled systems of 2D lattice equations can be obtained. In practice, we define some new variables $\sigma_i(p_\gamma)$ as

$$\sigma_i(p_\gamma) = \left(\prod_{j=1}^{i} \mathrm{T}_{-\omega_j(-p_\gamma)}\right)\tau, \quad i = 0, 1, 2, \cdots, \mathcal{N}-1. \tag{5.39}$$

Following from the reduction (5.37) and the definition (5.39), one can easily prove that the new variables obey the relations as follows:

$$\sigma_0(p_\gamma) = \tau, \quad \sigma_i(p_\gamma) = \mathrm{T}_{-\omega_i(-p_\gamma)}\sigma_{i-1}(p_\gamma), \quad \sigma_{\mathcal{N}-1}(p_\gamma) = \mathrm{T}_{p_\gamma}^{-1}\tau. \tag{5.40}$$

These variables appear in the reduced 2D lattice equations, namely they are the components in the obtained coupled systems of discrete equations.

We can now consider a generic 3D discrete integrable equation that takes the form of

$$\mathcal{F}_{p_1,p_2,p_3}(\tau, \tilde{\tau}, \hat{\tau}, \bar{\tau}, \hat{\tilde{\tau}}, \bar{\hat{\tau}}, \bar{\tilde{\tau}}, \bar{\hat{\tilde{\tau}}}) = \mathcal{F}_{p_1,p_2,p_3}(\tau, \tilde{\tau}, \hat{\tau}, \mathrm{T}_{p_3}\tau, \hat{\tilde{\tau}}, \mathrm{T}_{p_3}\hat{\tau}, \mathrm{T}_{p_3}\tilde{\tau}, \mathrm{T}_{p_3}\hat{\tilde{\tau}}) = 0, \tag{5.41}$$

Without losing generality, we take $p_3 = -\omega_i(-p_1)$ for $i = 1, 2, \cdots, \mathcal{N}-1$ respectively, and as a result we obtain a coupled system of $\mathcal{N}-1$ discrete equations as follows:

$$\mathcal{F}_{p_1,p_2,-\omega_i(-p_1)}(\sigma_{i-1}(p_1), \tilde{\sigma}_{i-1}(p_1), \hat{\sigma}_{i-1}(p_1), \sigma_i(p_1), \hat{\tilde{\sigma}}_{i-1}(p_1), \hat{\sigma}_i(p_1), \tilde{\sigma}_i(p_1), \hat{\tilde{\sigma}}_i(p_1)) = 0, \tag{5.42}$$

for $i = 1, 2, \cdots, \mathcal{N}-1$, where $\sigma_0(p_1) = \tau$ and $\sigma_{\mathcal{N}-1}(p_1) = \underset{\sim}{\tau}$ according to (5.40). This coupled system has effectively $\mathcal{N}-1$ dependent variables $\tau, \sigma_1(p_1), \cdots \sigma_{\mathcal{N}-2}(p_1)$. Likewise, one can also take $p_3 = -\omega_i(-p_2)$ for $i = 1, 2, \cdots, \mathcal{N}-1$ respectively and obtain a coupled system in the same form but in terms of the lattice parameter p_2.

4.1. Reductions of the discrete AKP equation. The discrete AKP equation, i.e. the HM equation, takes the form of (5.4). We can therefore write down its $\mathcal{N}$-reduction as follows:

$$(p_1 - p_2)\sigma_i\hat{\tilde{\sigma}}_{i-1} + (p_2 + \omega_i(-p_1))\tilde{\sigma}_{i-1}\hat{\sigma}_i - (\omega_i(-p_1) + p_1)\hat{\sigma}_{i-1}\tilde{\sigma}_i = 0, \quad i = 1, 2, \cdots, \mathcal{N}-1, \tag{5.43}$$

where we denote $\sigma_i = \sigma_i(p)$ and in addition $\sigma_0(p_1) = \tau$ and $\sigma_{\mathcal{N}-1}(p_1) = \underset{\sim}{\tau}$.

The coupled system is integrable (similarly for the reductions of BKP and CKP) in the sense of having N-soliton solution. The N-soliton solution of the coupled system (also of the other equations in the hierarchy) is given by $\tau = \det(1 + \mathbf{AM})$. Here the constant matrix $\mathbf{A}$ and the Cauchy matrix $\mathbf{M}$ are given by (cf. chapter 3 as the structure is the same as that in the continuous case, similarly for the results below for the reductions of the discrete BKP and CKP equations)

$$\mathbf{A} = \mathrm{diag}(A_{1,1}, \cdots, A_{1,N_1}; \cdots; A_{j,1}, \cdots, A_{j,N_j}; \cdots; A_{\mathcal{N}-1,1}, \cdots, A_{\mathcal{N}-1,N_{\mathcal{N}-1}}),$$

$$\mathbf{M} = (M_{(j,j'),(i,i')})_{j,i=1,\cdots,\mathcal{N}-1,j'=1,\cdots,N_j,i'=1,\cdots,N_i}, \quad M_{(j,j'),(i,i')} = \frac{\rho_{k_{i,i'}}\sigma_{-\omega_j(k_{j,j'})}}{k_{i,i'} - \omega_j(k_{j,j'})}, \tag{5.44}$$

where the plane wave factors take the form of

$$\rho_k = \prod_{\gamma=1}^{\infty}\left[(p_\gamma+k)^{n_\gamma}\prod_{j=1}^{\mathcal{N}-1}(-\omega_j(-p_\gamma)+k)^{n_\gamma^{(j)}}\right]\exp\left(\sum_{\gamma=1}^{\infty}k^\gamma x_\gamma\right),$$

$$\sigma_{k'} = \prod_{\gamma=1}^{\infty}\left[(p_\gamma-k')^{-n_\gamma}\prod_{j=1}^{\mathcal{N}-1}(-\omega_j(-p_\gamma)-k')^{-n_\gamma^{(j)}}\right]\exp\left(\sum_{\gamma=1}^{\infty}(-k')^\gamma x_\gamma\right).$$

As long as the structure of the tau function is given, the σ_i can be easily obtained by acting the discrete shifts on the tau function according to the definition (5.39). In the following, we give the nontrivial examples for $\mathcal{N}=2$ and $\mathcal{N}=3$.

Discrete KdV equation ($\mathcal{N}=2$). The case of $\mathcal{N}=2$ only gives the curve $C_2(-k,k) = (-k')^2-k^2=0$. In this case we take $p_3=-\omega_1(-p_1)=-p_1$ and obtain the first equation in (5.29). Alternatively, we can take $r=-\omega_1(-p_2)=-p_2$ and then general formula (5.43) gives rise to the second equation in (5.29). The two equations are compatible in the sense of the MDC property, i.e. the common tau function given above.

Discrete BSQ equation ($\mathcal{N}=3$). When $\mathcal{N}=3$, according to the general scheme, the variable σ_1 is needed and the coupled system form arises. We can then take $p_3=-\omega_1(-p_1)$ and $p_3=-\omega_2(-p_1)$ and obtain the coupled system for τ and $\sigma\doteq\sigma_1(p_1)$:

$$(p_1-p_2)\sigma\hat{\tilde{\tau}}+(p_2+\omega_1(-p_1))\tilde{\tau}\hat{\sigma}-(\omega_1(-p_1)+p_1)\hat{\tau}\tilde{\sigma}=0, \tag{5.45a}$$

$$(p_1-p_2)\underset{\sim}{\tau}\hat{\tilde{\sigma}}+(p_2+\omega_2(-p_1))\tilde{\sigma}\hat{\underset{\sim}{\tau}}-(\omega_2(-p_1)+p_1)\hat{\sigma}\tau=0. \tag{5.45b}$$

We refer to this system as the extended discrete BSQ equation. Alternatively, we can also derive a coupled system in a similar form for τ and $\sigma_1(p_2)$, which is compatible with (5.45).

We also note that eliminating σ in (5.45) gives us the trilinear extended discrete BSQ equation (5.34); the tau function for the BSQ equation solves the coupled system (5.45) and the trilinear equation (5.34) simultaneously. The degeneration $\alpha_1=0$ gives the unextended version of Equation (5.45) and it is corresponding to (5.21).

4.2. Reductions of the discrete BKP equation. The $\mathcal{N}$-reduction of the discrete BKP equation (5.7) following from the general scheme gives us the following coupled system (if one takes $p_3=-\omega_i(-p_1)$ for $i=1,2,\cdots,\mathcal{N}-1$ without loss of generality):

$$\begin{aligned}
&(p_1-p_2)(q+\omega_i(-p_1))(\omega_i(-p_1)+p_1)\sigma_{i-1}\hat{\tilde{\sigma}}_i\\
&\quad-(p_1+p_1)(p_1-\omega_i(-p_1))(p_2+\omega_i(-p_1))\tilde{\sigma}_{i-1}\hat{\sigma}_i\\
&\quad-(\omega_i(-p_1)-p_1)(\omega_i(-p_1)-p_2)(p_1-p_2)\sigma_i\hat{\tilde{\sigma}}_{i-1}\\
&\quad+(p_2-\omega_i(-p_1))(p_2+p_1)(\omega_i(-p_1)+p_1)\hat{\sigma}_{i-1}\tilde{\sigma}_i=0
\end{aligned} \tag{5.46}$$

for $i=1,2,\cdots,\mathcal{N}-1$, where we denote $\sigma_i\doteq\sigma_i(p_1)$ for convenience and $\sigma_0(p_1)=\tau$ and $\sigma_{\mathcal{N}-1}=\underset{\sim}{\tau}$. Alternatively, one can also take $p_3=-\omega_i(-p_2)$ for $i=1,2,\cdots,\mathcal{N}-1$ and obtain a similar coupled system associated with the lattice parameter p_2, which is compatible with (5.46) due to the MDC property.

The coupled system has the N-soliton solution τ determined by

$$\tau^2 = \det \left[1 + \begin{pmatrix} \mathbf{A} & 0 \\ 0 & -\mathbf{A} \end{pmatrix} \begin{pmatrix} \mathbf{M} & 0 \\ 0 & \mathbf{M}' \end{pmatrix} \right], \tag{5.47}$$

where the matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \mathrm{diag}(A_{1,1}, \cdots, A_{1,N_1}; \cdots; A_{j,1}, \cdots, A_{j,N_j}; \cdots; A_{\mathcal{N}-1,1}, \cdots, A_{\mathcal{N}-1,N_{\mathcal{N}-1}}),$$

and the Cauchy matrices $\mathbf{M}$ and $\mathbf{M}'$ take the form of

$$\mathbf{M} = (M_{(j,j'),(i,i')})_{j,i=1,\cdots,\mathcal{N}-1,j'=1,\cdots,N_j,i'=1,\cdots,N_i}, \ M_{(j,j'),(i,i')} = \rho_{k_{i,i'}} \frac{1}{2} \frac{k_{i,i'} + \omega_j(k_{j,j'})}{k_{i,i'} - \omega_j(k_{j,j'})} \rho_{-\omega_j(k_{j,j'})}, \tag{5.48a}$$

$$\mathbf{M}' = (M'_{(j,j'),(i,i')})_{j,i=1,\cdots,\mathcal{N}-1,j'=1,\cdots,N_j,i'=1,\cdots,N_i}, \ M'_{(j,j'),(i,i')} = \rho_{-\omega_i(k_{i,i'})} \frac{1}{2} \frac{\omega_i(k_{i,i'}) + k_{j,j'}}{\omega_i(k_{i,i'}) - k_{j,j'}} \rho_{k_{j,j'}}, \tag{5.48b}$$

respectively. Here the plane wave factor ρ_k is given by

$$\rho_k = \prod_{\gamma=1}^{\infty} \left[\left(\frac{p_\gamma + k}{p_\gamma - k} \right)^{n_\gamma} \prod_{j=1}^{\mathcal{N}-1} \left(\frac{-\omega_j(-p_\gamma) + k}{-\omega_j(-p_\gamma) - k} \right)^{n_\gamma^{(j)}} \right] \exp \left(\sum_{\gamma=0}^{\infty} k^{2\gamma+1} x_{2\gamma+1} \right).$$

Below we give an example for $\mathcal{N} = 3$, which is the discrete SK equation in extended form.

Discrete SK equation ($\mathcal{N} = 3$). The coupled system that describes the structure of the extended discrete SK equation according to the above scheme is given by

$$\begin{aligned} &(p_1 - p_2)(p_2 + \omega_1(-p_1))(\omega_1(-p_1) + p_1)\tau\hat{\tilde{\sigma}} \\ &\quad - (p_1 + p_2)(p_1 - \omega_1(-p_1))(p_2 + \omega_1(-p_1))\tilde{\tau}\hat{\sigma} \\ &\quad - (\omega_1(-p_1) - p_1)(\omega_1(-p_1) - p_2)(p_1 - p_2)\sigma\hat{\tilde{\tau}} \\ &\quad + (p_2 - \omega_1(-p_1))(p_2 + p_1)(\omega_1(-p_1) + p_1)\hat{\tau}\tilde{\sigma} = 0, \end{aligned} \tag{5.49a}$$

$$\begin{aligned} &(p_1 - p_2)(p_2 + \omega_2(-p_1))(\omega_2(-p_1) + p_1)\sigma\hat{\tau} \\ &\quad - (p_1 + p_2)(p_1 - \omega_2(-p_1))(p_2 + \omega_2(-p_1))\tilde{\sigma}\hat{\underset{\sim}{\tau}} \\ &\quad - (\omega_2(-p_1) - p_1)(\omega_2(-p_1) - p_2)(p_1 - p_2)\underset{\sim}{\tau}\hat{\tilde{\sigma}} \\ &\quad + (p_2 - \omega_2(-p_1))(p_2 + p_1)(\omega_2(-p_1) - p_1)\hat{\sigma}\tau = 0, \end{aligned} \tag{5.49b}$$

where we take $p_3 = -\omega_1(-p_1)$ and $p_3 = -\omega_2(-p_1)$ respectively in the discrete BKP equation and denote $\sigma \doteq \sigma_1(p_1)$. Similarly one can also derive a compatible coupled system for τ and $\sigma_1(p_2)$. Equation (5.49) is a new integrable difference equation. It is not yet clear whether or not one can eliminate the σ-function from the coupled systems and express the extended discrete SK equation by only the tau function as a scalar multilinear equation. The degeneration $\alpha_1 = \alpha_2 = 0$ is corresponding to the unextended version of the coupled system (5.49), which should be referred to as the discrete SK equation.

We also note that a different reduction $\hat{\hat{\tilde{\tau}}} = \tau$ was considered in [**HZ13**] (see also [**TH96**] for a slightly different reduction) for the discrete BKP equation (5.7) resulting in the following bilinear equation:

$$\begin{aligned}&(p_1-p_2)(p_2-p_3)(p_3-p_1)\tau^2+(p_1+p_2)(p_1+p_3)(p_2-p_3)\tilde{\tau}\underaccent{\tilde}{\tau}\\ &\quad+(p_3+p_1)(p_3+p_2)(p_1-p_2)\underaccent{\tilde}{\underaccent{\hat}{\tau}}\hat{\tilde{\tau}}+(p_2+p_3)(p_2+p_1)(p_3-p_1)\hat{\tau}\underaccent{\hat}{\tau}=0. \qquad (5.50)\end{aligned}$$

However, the algebraic curve for the spectral parameters k and k' behind such a reduction is

$$\left(\frac{p_1+k}{p_1-k}\right)\left(\frac{p_2+k}{p_2-k}\right)\left(\frac{p_3+k}{p_3-k}\right)\left(\frac{p_1+k'}{p_1-k'}\right)\left(\frac{p_2+k'}{p_2-k'}\right)\left(\frac{p_3+k'}{p_3-k'}\right)=1, \qquad (5.51)$$

which is effectively an algebraic curve biquadratic in k and k' and it does not match (5.1). This implies that the obtained bilinear equation does not possess the structure of the SK equation, but it is still an integrable equation having different nonlinear behaviour subject to the algebraic curve (5.51). Differently, in the BSQ case the algebraic curve associated with the reduction $\bar{\hat{\tilde{\tau}}}=\tau$ is a cubic curve, i.e. (5.13), and hence the reduced equation (5.14) can still be thought of a discretisation of the BSQ equation, although it is a truncation.

4.3. Reductions of the discrete CKP equation. Finally, we consider the $\mathcal{N}$-reduction of the discrete CKP equation. If one takes $r=-\omega_i(-p_1)$ for $i=1,2,\cdots,\mathcal{N}-1$ (or alternatively $r=-\omega_i(-p_2)$ is also allowed), the general reduction scheme in the case of the discrete CKP provides us with the following coupled system of lattice equations:

$$\begin{aligned}&\Big[(p_1-p_2)^2(p_2+\omega_i(-p_1))^2(\omega_i(-p_1)+p_1)^2\sigma_{i-1}\hat{\tilde{\sigma}}_i+(p_1+p_2)^2(p_1-\omega_i(-p_1))^2(p_2+\omega_i(-p_1))^2\tilde{\sigma}_{i-1}\hat{\sigma}_i\\ &\quad-(p_2-\omega_i(-p_1))^2(p_2+p_1)^2(\omega_i(-p_1)+p_1)^2\hat{\sigma}_{i-1}\tilde{\sigma}_i-(\omega_i(-p_1)-p_1)^2(\omega_i(-p_1)-p_2)^2(p_1-p_2)^2\sigma_i\hat{\tilde{\sigma}}_{i-1}\Big]^2\\ &\quad-4(p_1^2-p_2^2)^2(p_1^2-(\omega_i(-p_1))^2)^2\Big[(p_2-\omega_i(-p_1))^2\hat{\tilde{\sigma}}_{i-1}\tilde{\sigma}_i-(p_2+\omega_i(-p_1))^2\tilde{\sigma}_{i-1}\hat{\tilde{\sigma}}_i\Big]\\ &\quad\times\Big[(p_2-\omega_i(-p_1))^2\hat{\sigma}_{i-1}\sigma_i-(p_2+\omega_i(-p_1))^2\sigma_{i-1}\hat{\sigma}_i\Big]=0, \qquad (5.52)\end{aligned}$$

for $i=1,2,\cdots,\mathcal{N}-1$, where we still have from (5.40) that $\sigma_0(p_1)=\tau$ and $\sigma_{\mathcal{N}-1}(p_1)=\underaccent{\tilde}{\tau}$ and denote $\sigma_i\doteq\sigma_i(p_1)$. And again we note that the coupled system for τ and $\sigma_i(p_2)$ is compatible with (5.52).

The soliton-type solution to the coupled system (5.52) (also to its hierarchy) takes the form of

$$\tau=\det\left[1+\begin{pmatrix}\mathbf{A} & 0\\ 0 & \mathbf{A}\end{pmatrix}\begin{pmatrix}\mathbf{M} & 0\\ 0 & \mathbf{M}'\end{pmatrix}\right] \qquad (5.53)$$

with the same matrix $\mathbf{A}$ in Subsection 4.2 and the Cauchy matrices $\mathbf{M}$ and $\mathbf{M}'$ given by

$$\mathbf{M}=(M_{(j,j'),(i,i')})_{j,i=1,\cdots,\mathcal{N}-1,j'=1,\cdots,N_j,i'=1,\cdots,N_i}, \quad M_{(j,j'),(i,i')}=\frac{\rho_{k_{i,i'}}\rho_{-\omega_j(k_{j,j'})}}{k_{i,i'}-\omega_j(k_{j,j'})}, \qquad (5.54a)$$

$$\mathbf{M}'=(M_{(j,j'),(i,i')})_{j,i=1,\cdots,\mathcal{N}-1,j'=1,\cdots,N_j,i'=1,\cdots,N_i}, \quad M'_{(j,j'),(i,i')}=\frac{\rho_{-\omega_i(k_{i,i'})}\rho_{k_{j,j'}}}{-\omega_i(k_{i,i'})+k_{j,j'}}, \qquad (5.54b)$$

where ρ_k is also exactly the same as the one given in Subsection 4.2. The cases when $\mathcal{N}=3$ and $\mathcal{N}=4$ would be quite interesting in this class since they give rise to discretisations of the KK and HS equations in extended form and such results (even the unextended cases) have not yet be given elsewhere, to the best of the author's knowledge.

Discrete KK equation ($\mathcal{N}=3$). The discrete KK equation can be obtained as a two-component system for τ and $\sigma \doteq \sigma_1(p_1)$ if one takes $p_3=-\omega_1(-p_1)$ and $p_3=-\omega_2(-p_1)$ respectively in the discrete CKP equation. The extended discrete KK equation takes the form of

$$\begin{aligned}
&\Big[(p_1-p_2)^2(p_2+\omega_1(-p_1))^2(\omega_1(-p_1)+p_1)^2\tau\hat{\tilde{\sigma}}+(p_1+p_2)^2(p_1-\omega_1(-p_1))^2(p_2+\omega_1(-p_1))^2\tilde{\tau}\hat{\sigma}\\
&\quad-(p_2-\omega_1(-p_1))^2(p_2+p_1)^2(\omega_1(-p_1)+p_1)^2\hat{\tau}\tilde{\sigma}-(\omega_1(-p_1)-p_1)^2(\omega_1(-p_1)-p_2)^2(p_1-p_2)^2\sigma\hat{\tilde{\tau}}\Big]^2\\
&\quad-4(p_1^2-p_2^2)^2(p_1^2-(\omega_1(-p_1))^2)^2\Big[(p_2-\omega_1(-p_1))^2\hat{\tilde{\tau}}\tilde{\sigma}-(p_2+\omega_1(-p_1))^2\tilde{\tau}\hat{\tilde{\sigma}}\Big]\\
&\quad\times\Big[(p_2-\omega_1(-p_1))^2\hat{\tau}\sigma-(p_2+\omega_1(-p_1))^2\tau\hat{\sigma}\Big]=0, \qquad (5.55a)\\
&\Big[(p_1-p_2)^2(p_2+\omega_2(-p_1))^2(\omega_2(-p_1)+p_1)^2\sigma\hat{\tau}+(p_1+p_2)^2(p_1-\omega_2(-p_1))^2(p_2+\omega_2(-p_1))^2\tilde{\sigma}\underset{\sim}{\hat{\tau}}\\
&\quad-(p_2-\omega_2(-p_1))^2(p_2+p_1)^2(\omega_2(-p_1)+p_1)^2\hat{\sigma}\tau-(\omega_2(-p_1)-p_1)^2(\omega_2(-p_1)-p_2)^2(p_1-p_2)^2\underset{\sim}{\tau}\hat{\tilde{\sigma}}\Big]^2\\
&\quad-4(p_1^2-p_2^2)^2(p_1^2-(\omega_2(-p_1))^2)^2\Big[(p_2-\omega_2(-p_1))^2\hat{\tilde{\sigma}}\tau-(p_2+\omega_2(-p_1))^2\tilde{\sigma}\hat{\tau}\Big]\\
&\quad\times\Big[(p_2-\omega_2(-p_1))^2\hat{\sigma}\underset{\sim}{\tau}-(p-2+\omega_2(-p_1))^2\sigma\underset{\sim}{\hat{\tau}}\Big]=0, \qquad (5.55b)
\end{aligned}$$

which is a new integrable discrete equation. Alternatively, a compatible system for τ and $\sigma_1(p_2)$ can also be derived if one takes $r=-\omega_1(-p_2)$ and $p_3=-\omega_2(-p_2)$ respectively in the discrete CKP equation. The unextended KK equation is a particular case of (5.55) when $\alpha_1=\alpha_2=0$.

Similarly to the discrete SK equation, the reduction $\bar{\hat{\tilde{\tau}}}=\tau$ together with the discrete CKP equation (5.10) gives rise to a very simple scalar quadrilinear discrete equation

$$\begin{aligned}
&[(p_1-p_2)^2(p_2-p_3)^2(p_3-p_1)^2\tau^2+(p_1+p_2)^2(p_1+p_3)^2(p_2-p_3)^2\tilde{\tau}\underset{\sim}{\tau}\\
&\quad-(p_2+p_3)^2(p_2+p_1)^2(p_3-p_1)^2\hat{\tau}\underset{\wedge}{\tau}-(p_3+p_1)^2(p_3+p_2)^2(p_1-p_2)^2\underset{\substack{\wedge\\ \sim}}{\tau}\hat{\tilde{\tau}}]^2\\
&\quad-4(p_1^2-p_2^2)^2(p_1^2-p_3^2)^2[(p_2+p_3)^2\hat{\tilde{\tau}}\underset{\wedge}{\tau}-(p_2-p_3)^2\tilde{\tau}\tau][(p_2+p_3)^2\hat{\tau}\underset{\substack{\wedge\\ \sim}}{\tau}-(p_2-p_3)^2\tau\underset{\sim}{\tau}]=0.
\end{aligned} \qquad (5.56)$$

But its solution structure does not reflect the structure of the KK equation since the reduction is still associated with the biquadratic algebraic curve (5.51) rather than the algebraic curve $C_3(-k',k)=0$ determined by (5.35).

Discrete HS equation ($\mathcal{N}=4$). The discrete HS equation is a three-component system for τ, $\sigma_1(p_1)$ and $\sigma_2(p_1)$ (or consistently a three-component system for τ, $\sigma_1=\sigma_1(p_2)$ and $\sigma_2=\sigma_2(p_2)$ in terms of the lattice direction associated with the parameter q). In practice, one can take $p_3=-\omega_i(-p_1)$ for $i=1,2,3$ respectively in the discrete CKP without losing generality. We omit the explicit formulae here and one can refer to the case of $\mathcal{N}=4$ in (5.52) , which is the extended HS equation. And the HS equation is the case when $\alpha_1=\alpha_2=0$, i.e. the unextended case.

Finally, we would like to point out that the coupled systems we have listed in this section are the integrable discretisations of their corresponding continuous hierarchies in the sense that the respective continuum limits of the tau functions lead to the continuous ones given in chapter 4. The limit procedure on the discrete equations might be slightly subtle – one can calculate the continuum limits of the discrete equations first and then eliminate the variables σ_i.

5. Concluding remarks

Reductions of the discrete AKP, BKP and CKP equations are considered in a unified way. For lower-rank examples, the coupled systems could be reformulated into scalar equations which only involve the tau function, for instance, the discrete KdV and BSQ equations can be written as the bilinear equation (5.29) and the trilinear equation (5.21) respectively. However, this is highly nontrivial, especially in the coupled systems as the reductions of the discrete CKP equation due to the complexity of its quadrilinear form. We believe that the coupled system form is a proper way to express the 2D reduced discrete equations since a unified expression can be written down explicitly.

We only considered the reductions on the bilinear/multilinear form of the discrete AKP, BKP and CKP equations since the tau function is the best candidate to describe the solution structure (namely, the effective plane wave factor and the Cauchy kernel) of an integrable equation/hierarchy. An interesting question would be looking for the nonlinear forms of the obtained 2D lattice integrable equations. This can be done via the DL framework and successful examples include the discrete KdV equation [**HJN16**] and the discrete BSQ equation [**ZZN12**]. However, the nonlinear forms for the 2D lattice equations arising from the reductions of the discrete BKP and CKP equations are not yet clear. In fact, on the continuous level there are some remarkable transforms between the SK and KK equations [**FG80c**, **FG80a**] (see also chapter 4). Nevertheless, no such result has been found on the discrete level (even not in the semi-discrete case according to [**AP11**]). In addition, there exists a trilinear discrete Tzitzeica equation [**Sch99**] (see also [**Adl11**]). It is also not clear if this equation is related to the discrete SK equation (5.49).

Finally, it was pointed out in [**OST10**] that the famous pentagram map is deeply related to the discrete BSQ equation in the sense that the pentagram map leads to the continuous BSQ equation in continuum limit. Recently, it was shown by Hietarinta and Maruno [**Hie15**] that the pentagram map as a discrete system can be bilinearised and its bilinear form arises as a reduction of the HM equation. However, it was not clarified how the full lattice BSQ equation, i.e. the trilinear equation (5.21), is related to the pentagram map.

CHAPTER 6

Discrete-time two-dimensional Toda lattices

1. Overview

The two-dimensional Toda lattice (2DTL)

$$\partial_1\partial_{-1}\varphi_n = -e^{\varphi_{n+1}-\varphi_n} + e^{\varphi_n-\varphi_{n-1}}, \tag{6.1}$$

where φ_n is the potential as a function of two continuous time variables x_1 and x_{-1} and one discrete spatial variable n, and ∂_1 and ∂_{-1} denote the partial derivatives $\partial/\partial x_1$ and $\partial/\partial x_{-1}$, respectively, was originally proposed by Mikhailov [**Mik79**] in 1979, as an integrable generalisation of the famous one-dimensional Toda lattice [**Tod67**]. Equation (6.1) can alternatively be written in the form of (cf. e.g. Jimbo and Miwa's review paper [**JM83**])

$$\partial_1\partial_{-1}\theta_n = -\sum_{m\in\mathbb{Z}} a_{n,m}e^{-\theta_m}, \tag{6.2}$$

where $\theta_n = \varphi_{n-1} - \varphi_n$ and $a_{n,m}$ are the entries of the Cartan matrix corresponding to the infinite-dimensional algebra A_∞, namely

$$a_{n,m} = \begin{cases} 2, & n = m, \\ -1, & n = m \pm 1, \\ 0, & \text{otherwise,} \end{cases}$$

and thus, it is also reasonable to refer to this model as the 2DTL of A_∞-type. The 2DTL is mathematically remarkable because it is still integrable when the algebra A_∞ is replaced by various algebras, see Mikhailov, Olshanetsky and Perelomov [**MOP83**], Fordy and Gibbons [**FG80b**, **FG83**], Wilson [**Wil81**], and also the Kyoto School [**UT84**, **DJM83**, **JM83**]. Among these the two typical ones are the 2DTLs of $A_1^{(1)}$-type and $A_2^{(2)}$-type which can be written in scalar form, namely the well-known sinh–Gordon and Tzitzeica equations

$$\partial_1\partial_{-1}\varphi_0 = e^{2\varphi_0} + e^{-2\varphi_0} \quad \text{and} \quad \partial_1\partial_{-1}\varphi_0 = e^{2\varphi_0} - e^{-\varphi_0}. \tag{6.3}$$

Exact solutions to the 2DTLs of different types are also obtained by using various methods, see e.g. [**Mik81**,**HOS88**,**NW97**] for soliton solutions, and also [**BMW17**] for higher-rank solutions. For the algebraic structure of the 2DTL, we refer to Ueno and Takasaki [**UT84**] (see also [**Tak18**]).

An effective way to construct the discrete 2DTL is to discretise the bilinear 2DTL, attributed to Hirota [**Hir81**] and Miwa [**Miw82**]. The mechanism of such a discretisation was explained by Date, Jimbo and Miwa [**DJM82**] from the viewpoint of the theory of transformation groups, motivated by the idea in [**Miw82**]. Moreover, the q-discretisation of the 2DTL, which is closely related to quantum groups, was also obtained by the bilinear approach in [**KOS94**]. Another approach to discretise the 2DTL was introduced by Fordy and Gibbons [**FG80b**, **FG83**], where the discrete equation arises as the superposition formula of two Bäcklund transforms for the continuous-time 2DTL on the nonlinear level. A recent significant progress on the discrete-time 2DTLs was made in [**GHY12**, **Smi15**], where the integrability is guaranteed in sense that first integrals and symmetries are preserved in the discretisation.

Our motivation is to consider the discrete-time 2DTL within the DL framework. In the present chapter, three nonlinear discrete equations in the class of the discrete-time 2DTL of A_∞-type are constructed in the direct linearisation, including the bilinear and modified discrete-time 2DTL equations given by Date, Jimbo and Miwa in [**DJM82**], and also an unmodified equation which is a new parametrisation of an unnamed octahedron-type equation that appeared recently in [**FNR15**]. Their associated Lax pairs are also obtained in the scheme, which completes the incomplete part in [**DJM82**] (though the author firmly believes that those authors were able to construct the Lax pairs in their framework). The direct linearising solutions for these equations are natural consequences of the scheme, reducing to soliton solutions as a particular degenerate case. The periodic reductions are also considered in the framework, resulting in the corresponding 2D discrete integrable systems, namely the discrete-time 2DTLs of $A^{(1)}_{\mathcal{N}-1}$-type for $\mathcal{N} \geq 2$ (amounting to the negative flows of members in the discrete GD hierarchy), in which the $A^{(1)}_1$ class provides the well-known discrete sinh–Gordon (or sine–Gordon) equation and its gauge equivalent models. When $\mathcal{N} \geq 3$, we obtain the new discrete integrable systems very recently given by Fordy and Xenitidis [**FX17**]. The nonlinear forms, the bilinear formalism, the Lax matrices and the direct linearising solutions for these models are all discussed, from the perspective of the direct linearisation.

The chapter is organised as follows. Sections 2 and 3 are dedicated to the direct linearisation schemes of the discrete-time 2DTL equations of A_∞-type and $A^{(1)}_{\mathcal{N}-1}$-type, respectively, and their integrability properties.

2. Discrete-time two-dimensional Toda lattice of A_∞-type

2.1. Nonlinear and linear structures. To algebraically construct equations in the class of the discrete-time 2DTL of A_∞-type, we consider the following Cauchy kernel and measure:

$$\Omega_{k,k'} = \frac{1}{k+k'}, \quad \mathrm{d}\zeta(k,k') \quad \text{being arbitrary,} \tag{6.4}$$

as well as the plane wave factors

$$\rho_k = (p_1+k)^{n_1}(p_{-1}+k)^{n_{-1}}k^n \quad \text{and} \quad \sigma_{k'} = (p_1-k')^{-n_1}(p_{-1}-k'^{-1})^{-n_{-1}}(-k')^{-n}. \tag{6.5}$$

In concrete calculation, the two plane wave factors normally combine together (cf. the infinite matrix $\mathbf{C}$), and form the effective plane wave factor

$$\rho_k\sigma_{k'} = \left(\frac{p_1+k}{p_1-k'}\right)^{n_1}\left(\frac{p_{-1}+k^{-1}}{p_{-1}-k'^{-1}}\right)^{n_{-1}}\left(-\frac{k}{k'}\right)^{n}, \tag{6.6}$$

which completely governs the dynamics in the resulting nonlinear models.

2.2. Soliton solution. According to the general statement, equations (2.15), (2.21) and (2.18) provide us with the direct linearising solutions to the discrete-time 2DTL of A_∞-type. As an example, below we give the explicit formulae for soliton solutions. We take a particular measure involving a finite number of singularities, namely

$$\mathrm{d}\zeta(k,k') = \sum_{i=1}^{N}\sum_{j=1}^{N'} A_{i,j}\delta(k-k_i)\delta(k'-k'_j)\mathrm{d}k\mathrm{d}k', \tag{6.7}$$

where $A_{i,j}$ is the (i,j)-entry of a constant full-rank $N\times N'$ matrix $\mathbf{A}$, k_i and k'_j are the singularities, and the delta function here should be understood as $\delta(k-k_j) = \frac{1}{2\pi\mathrm{i}}\frac{1}{k-k_j}$ with i being the imaginary unit. Let the domain D contain all these singularities. This reduces the linear integral equation (2.15) to

$$\mathbf{u}_k + \sum_{i=1}^{N}\sum_{j=1}^{N'} A_{i,j}\frac{\rho_k\sigma_{k'_j}}{k+k'_j}\mathbf{u}_{k_i} = \rho_k\mathbf{c}_k, \tag{6.8}$$

according to the residue theorem. Now we introduce an $N'\times N$ generalised Cauchy matrix $\mathbf{M}$ with its (j,i)-entry

$$M_{j,i} = \frac{\rho_{k_i}\sigma_{k'_j}}{k_i+k'_j}, \quad j=1,2,\cdots,N', \quad i=1,2,\cdots,N,$$

where $\rho_k\sigma_{k'}$ is given by (6.6). Taking $k=k_i$ for $i=1,2,\cdots,N$ in the reduced equation (6.8), we obtain

$$(\mathbf{u}_{k_1},\mathbf{u}_{k_2},\cdots,\mathbf{u}_{k_N}) + (\mathbf{u}_{k_1},\mathbf{u}_{k_2},\cdots,\mathbf{u}_{k_N})\mathbf{A}\mathbf{M} = (\rho_{k_1}\mathbf{c}_{k_1},\rho_{k_2}\mathbf{c}_{k_2},\cdots,\rho_{k_N}\mathbf{c}_{k_N})$$

and thus, $\mathbf{u}_{k_i}$ can be expressed by

$$(\mathbf{u}_{k_1},\mathbf{u}_{k_2},\cdots,\mathbf{u}_{k_N}) = \mathbf{r}^{\mathrm{T}}\,\mathrm{diag}(\mathbf{c}_{k_1},\mathbf{c}_{k_2},\cdots,\mathbf{c}_{k_N})(1+\mathbf{A}\mathbf{M})^{-1},$$

where $\mathbf{r} = (\rho_{k_1},\rho_{k_2},\cdots,\rho_{k_N})^{\mathrm{T}}$. Meanwhile, considering (2.18) together with (6.7), we also have

$$\mathbf{U} = \sum_{i=1}^{N}\sum_{j=1}^{N'} A_{i,j}\mathbf{u}_{k_i}{}^{t}\mathbf{c}_{k'_j}\sigma_{k'_j} = (\mathbf{u}_{k_1},\mathbf{u}_{k_2},\cdots,\mathbf{u}_{k_N})\mathbf{A}\,\mathrm{diag}(\mathbf{c}_{k'_1},\mathbf{c}_{k'_2},\cdots,\mathbf{c}_{k'_{N'}})\mathbf{s}. \tag{6.9}$$

where $\mathbf{s} = (\sigma_{k'_1},\sigma_{k'_2},\cdots,\sigma_{k'_{N'}})^{\mathrm{T}}$. Therefore, substituting $\mathbf{u}_{k_i}$ in (6.9) by using the previous formula, we end up with the explicit expression of $\mathbf{U}$ as follows:

$$\mathbf{U} = \mathbf{r}^{\mathrm{T}}\,\mathrm{diag}(\mathbf{c}_{k_1},\mathbf{c}_{k_2},\cdots,\mathbf{c}_{k_N})(1+\mathbf{A}\mathbf{M})^{-1}\mathbf{A}\,\mathrm{diag}(\mathbf{c}_{k'_1},\mathbf{c}_{k'_2},\cdots,\mathbf{c}_{k'_{N'}})\mathbf{s}.$$

Furthermore, we can also consider the soliton expression for the tau function. According to the definition of the determinant of an infinite matrix, we have the following relation for the tau function:

$$\ln\tau = \ln\left(\det(1+\boldsymbol{\Omega}\cdot\mathbf{C})\right) = \mathrm{tr}\left(\ln(1+\boldsymbol{\Omega}\cdot\mathbf{C})\right).$$

Thus, by expansion the right hand side can be written as

$$\operatorname{tr}\left(\sum_{\gamma=1}^{\infty}(-1)^{\gamma+1}\frac{1}{\gamma}(\boldsymbol{\Omega}\cdot\mathbf{C})^{\gamma}\right)=\sum_{\gamma=1}^{\infty}(-1)^{\gamma+1}\frac{1}{\gamma}\operatorname{tr}(\boldsymbol{\Omega}\cdot\mathbf{C})^{\gamma}.$$

Considering the degeneration (6.7) and the definition of $\mathbf{C}$, i.e. (2.17), we can derive

$$\operatorname{tr}(\boldsymbol{\Omega}\cdot\mathbf{C})^{\gamma}=\operatorname{tr}\left(\boldsymbol{\Omega}\cdot\left(\sum_{i=1}^{N}\sum_{j=1}^{N'}A_{i,j}\rho_{k_i}\mathbf{c}_{k_i}{}^{t}\mathbf{c}_{k'_j}\sigma_{k'_j}\right)\right)^{\gamma}=\operatorname{tr}(\mathbf{AM})^{\gamma},$$

where the cyclic permutation of the trace is used in the last step. Thus, we end up with the relation

$$\begin{aligned}\ln\tau&=\sum_{\gamma=1}^{\infty}(-1)^{\gamma+1}\frac{1}{\gamma}\operatorname{tr}(\mathbf{AM})^{\gamma}=\operatorname{tr}\left(\sum_{\gamma=1}^{\infty}(-1)^{\gamma+1}\frac{1}{\gamma}(\mathbf{AM})^{\gamma}\right)\\&=\operatorname{tr}\left(\ln(1+\mathbf{AM})\right)=\ln\left(\det(1+\mathbf{AM})\right).\end{aligned}$$

We have derived the soliton formulae for the linear, nonlinear and bilinear variables. For convenience in the future, we give the formulae for the components $u_k^{(i)}$ in the wave function, the entries $U_{i,j}$ in the infinite matrix $\mathbf{U}$ as well as the tau function τ, and conclude all the results in the following theorem:

THEOREM 6.1. *For the Cauchy kernel, the measure and the effective plane wave factor given in* (6.4), (6.7) *and* (6.6), *the wave function, the tau function and the potential variable in the* (N,N')*-soliton form are determined by the following:*

$$\begin{aligned}&(u_{k_1}^{(i)},u_{k_2}^{(i)},\cdots,u_{k_N}^{(i)})=\mathbf{r}^{\mathrm{T}}\mathbf{K}^{i}(1+\mathbf{AM})^{-1},\\&\tau=\det(1+\mathbf{AM}),\\&U_{i,j}=\mathbf{r}^{\mathrm{T}}\mathbf{K}^{i}(1+\mathbf{AM})^{-1}\mathbf{A}\mathbf{K}'^{j}\mathbf{s},\end{aligned}$$

in which $\mathbf{K}=\operatorname{diag}(k_1,k_2,\cdots,k_N)$ *and* $\mathbf{K}'=\operatorname{diag}(k'_1,k'_2,\cdots,k'_{N'})$.

2.3. Discrete dynamics. In this subsection, we discuss the infinite matrix formalism in the direct linearisation based on the nonlinear and linear structures of the discrete-time 2DTL, i.e. (6.4) and (6.6), which will be used to construct closed-form equations in the next subsection.

Equation (6.4) implies that the Cauchy kernel obeys the relation $\Omega_{k,k'}k+k'\Omega_{k,k'}=1$. Recalling the definition of $\boldsymbol{\Omega}$, i.e. (2.16), we can deduce that

$$\boldsymbol{\Omega}\cdot\boldsymbol{\Lambda}+{}^{t}\boldsymbol{\Lambda}\cdot\boldsymbol{\Omega}=\mathbf{O},\tag{6.10a}$$

and consequently it can be generalised to

$$\begin{aligned}&\boldsymbol{\Omega}\cdot(p_1+\boldsymbol{\Lambda})-(p_1-{}^{t}\boldsymbol{\Lambda})\cdot\boldsymbol{\Omega}=\mathbf{O},\\&\boldsymbol{\Omega}\cdot(p_{-1}+\boldsymbol{\Lambda}^{-1})-(p_{-1}-{}^{t}\boldsymbol{\Lambda}^{-1})\cdot\boldsymbol{\Omega}={}^{t}\boldsymbol{\Lambda}^{-1}\cdot\mathbf{O}\cdot\boldsymbol{\Lambda}^{-1}.\end{aligned}\tag{6.10b}$$

The above relations form the nonlinear structure of the infinite matrix formalism. Below we give the linear structure. Observing that the effective plane wave factor (6.6) satisfies

$$\widetilde{(\rho_k\sigma_{k'})} = \left(\frac{p_1+k}{p_1-k'}\right)\rho_k\sigma_{k'}, \quad \check{(\rho_k\sigma_{k'})} = \left(\frac{p_{-1}+k^{-1}}{p_{-1}-k'^{-1}}\right)\rho_k\sigma_{k'}, \quad \dot{(\rho_k\sigma_{k'})} = \left(-\frac{k}{k'}\right)\rho_k\sigma_{k'},$$

and the property of $\mathbf{c}_k$ given in (2.13), we can derive from (2.17) the dynamical evolutions of $\mathbf{C}$ as follows:

$$\tilde{\mathbf{C}}\cdot(p_1 - {}^t\boldsymbol{\Lambda}) = (p_1+\boldsymbol{\Lambda})\cdot\mathbf{C}, \quad \check{\mathbf{C}}\cdot(p_{-1} - {}^t\boldsymbol{\Lambda}^{-1}) = (p_{-1}+\boldsymbol{\Lambda}^{-1})\cdot\mathbf{C}, \quad \dot{\mathbf{C}}\cdot(-{}^t\boldsymbol{\Lambda}) = \boldsymbol{\Lambda}\cdot\mathbf{C}. \tag{6.11}$$

Equations (6.10) and (6.11) are the fundamental relations which can help to build up the dynamics for key ingredients, namely $\mathbf{U}$, $\mathbf{u}_k$ and τ, in the scheme.

PROPOSITION 6.2. *The infinite matrix* $\mathbf{U}$ *obeys the following dynamical evolutions with respect to the lattice variables* n_1, n_{-1} *and* n*:*

$$\tilde{\mathbf{U}}\cdot(p_1 - {}^t\boldsymbol{\Lambda}) = (p_1+\boldsymbol{\Lambda})\cdot\mathbf{U} - \tilde{\mathbf{U}}\cdot\mathbf{O}\cdot\mathbf{U}, \tag{6.12a}$$

$$\check{\mathbf{U}}\cdot(p_{-1} - {}^t\boldsymbol{\Lambda}^{-1}) = (p_{-1}+\boldsymbol{\Lambda}^{-1})\cdot\mathbf{U} - \check{\mathbf{U}}\cdot{}^t\boldsymbol{\Lambda}^{-1}\cdot\mathbf{O}\cdot\boldsymbol{\Lambda}^{-1}\cdot\mathbf{U}, \tag{6.12b}$$

$$\dot{\mathbf{U}}\cdot(-{}^t\boldsymbol{\Lambda}) = \boldsymbol{\Lambda}\cdot\mathbf{U} - \dot{\mathbf{U}}\cdot\mathbf{O}\cdot\mathbf{U}, \tag{6.12c}$$

which are the fundamental relations for constructing nonlinear equations in the class of the 2DTL of A_∞*-type.*

PROOF. We only prove the first equation. According to the infinite matrix representation (2.20), we have

$$\tilde{\mathbf{U}}\cdot(p_1 - {}^t\boldsymbol{\Lambda}) = (1 - \tilde{\mathbf{U}}\cdot\boldsymbol{\Omega})\cdot\tilde{\mathbf{C}}\cdot(p_1 - {}^t\boldsymbol{\Lambda}).$$

In light of the first dynamical relation in (6.11), this relation can be written as

$$\tilde{\mathbf{U}}\cdot(p_1 - {}^t\boldsymbol{\Lambda}) = (1 - \tilde{\mathbf{U}}\cdot\boldsymbol{\Omega})\cdot(p_1+\boldsymbol{\Lambda})\cdot\mathbf{C} = (p_1+\boldsymbol{\Lambda})\cdot\mathbf{C} - \tilde{\mathbf{U}}\cdot\boldsymbol{\Omega}\cdot(p_1+\boldsymbol{\Lambda})\cdot\mathbf{C}.$$

Notice the second relation in (6.10). The above equation can further be reformulated as

$$\tilde{\mathbf{U}}\cdot(p_1 - {}^t\boldsymbol{\Lambda}) = (p_1+\boldsymbol{\Lambda})\cdot\mathbf{C} - \tilde{\mathbf{U}}\cdot\left[\mathbf{O} + (p_1 - {}^t\boldsymbol{\Lambda})\cdot\boldsymbol{\Omega}\right]\cdot\mathbf{C},$$

which amounts to

$$\tilde{\mathbf{U}}\cdot(p_1 - {}^t\boldsymbol{\Lambda})\cdot(1+\boldsymbol{\Omega}\cdot\mathbf{C}) = (p_1+\boldsymbol{\Lambda})\cdot\mathbf{C} - \tilde{\mathbf{U}}\cdot\mathbf{O}\cdot\mathbf{C}.$$

Multiplying this equation by $(1+\boldsymbol{\Omega}\cdot\mathbf{C})^{-1}$ from the right and recalling $\mathbf{U} = \mathbf{C}\cdot(1+\boldsymbol{\Omega}\cdot\mathbf{C})^{-1}$, we obtain the first equation in (6.12). The second and third equations are proven similarly. $\square$

Equations in (6.12) can further help to derive the dynamics of the wave function $\mathbf{u}_k$.

Proposition 6.3. *The wave function $\mathbf{u}_k$ satisfies the following dynamical evolutions with respect to the discrete flow variables n_1, n_{-1} and n:*

$$\tilde{\mathbf{u}}_k = (p_1 + \mathbf{\Lambda}) \cdot \mathbf{u}_k - \tilde{\mathbf{U}} \cdot \mathbf{O} \cdot \mathbf{u}_k, \tag{6.13a}$$

$$\check{\mathbf{u}}_k = (p_{-1} + \mathbf{\Lambda}^{-1}) \cdot \mathbf{u}_k - \check{\mathbf{U}} \cdot {}^t\mathbf{\Lambda}^{-1} \cdot \mathbf{O} \cdot \mathbf{\Lambda}^{-1} \cdot \mathbf{u}_k, \tag{6.13b}$$

$$\dot{\mathbf{u}}_k = \mathbf{\Lambda} \cdot \mathbf{u}_k - \dot{\mathbf{U}} \cdot \mathbf{O} \cdot \mathbf{u}_k, \tag{6.13c}$$

which will be used to construct the closed-form linear equations in the class of the 2DTL of A_∞-type.

Proof. Once again, we only prove the first equation. Forward-shifting equation (2.19) with respect to n_1 by one unit gives us

$$\tilde{\mathbf{u}}_k = (1 - \tilde{\mathbf{U}} \cdot \mathbf{\Omega}) \cdot \tilde{\rho}_k \mathbf{c}_k = (1 - \tilde{\mathbf{U}} \cdot \mathbf{\Omega}) \cdot (p_1 + k)\rho_k \mathbf{c}_k = (1 - \tilde{\mathbf{U}} \cdot \mathbf{\Omega}) \cdot (p_1 + \mathbf{\Lambda}) \cdot \rho_k \mathbf{c}_k,$$

where the last step holds due to (2.13). In virtue of the second relation in (6.10), we further obtain

$$\begin{aligned}\tilde{\mathbf{u}}_k &= (p_1 + \mathbf{\Lambda}) \cdot \rho_k \mathbf{c}_k - \tilde{\mathbf{U}} \cdot \mathbf{\Omega} \cdot (p_1 + \mathbf{\Lambda}) \cdot \rho_k \mathbf{c}_k \\ &= (p_1 + \mathbf{\Lambda}) \cdot \rho_k \mathbf{c}_k - \tilde{\mathbf{U}} \cdot \left[\mathbf{O} + (p_1 - {}^t\mathbf{\Lambda}) \cdot \mathbf{\Omega}\right] \cdot \rho_k \mathbf{c}_k.\end{aligned}$$

With the help of (6.12a), this can be written as

$$\tilde{\mathbf{u}}_k = (p_1 + \mathbf{\Lambda}) \cdot \rho_k \mathbf{c}_k - \tilde{\mathbf{U}} \cdot \mathbf{O} \cdot \rho_k \mathbf{c}_k - \Big[(p_1 + \mathbf{\Lambda}) \cdot \mathbf{U} - \tilde{\mathbf{U}} \cdot \mathbf{O} \cdot \mathbf{U}\Big] \cdot \mathbf{\Omega} \cdot \rho_k \mathbf{c}_k,$$

which is nothing but (6.13a) since $\mathbf{u}_k = (1 - \mathbf{U} \cdot \mathbf{\Omega}) \cdot \rho_k \mathbf{c}_k$ (see (2.19)). ☐

In the framework, we also need to consider dynamical evolutions of the tau function with respect to these lattice variables. Below we only list two relations which will be used later.

Proposition 6.4. *The tau function obeys evolutions*

$$\frac{\dot{\tau}}{\tau} = 1 - U_{0,-1} \quad \textit{and} \quad \frac{\underset{\cdot}{\tau}}{\tau} = 1 - U_{-1,0} \tag{6.14}$$

with respect to the lattice direction n.

Proof. Notice that the tau function is defined by (2.21). We can calculate that

$$\begin{aligned}\dot{\tau} &= \det(1 + \mathbf{\Omega} \cdot \dot{\mathbf{C}}) = \det\left(1 + \mathbf{\Omega} \cdot \mathbf{\Lambda} \cdot \mathbf{C} \cdot (-{}^t\mathbf{\Lambda})^{-1}\right) \\ &= \det\left(1 + (\mathbf{O} - {}^t\mathbf{\Lambda} \cdot \mathbf{\Omega}) \cdot \mathbf{C} \cdot (-{}^t\mathbf{\Lambda})^{-1}\right),\end{aligned}$$

where the last equation holds because of (6.10). We can then further reformulate this equation as

$$\begin{aligned}\dot{\tau} &= \det\left(1 + \mathbf{\Omega} \cdot \mathbf{C} - {}^t\mathbf{\Lambda}^{-1} \cdot \mathbf{O} \cdot \mathbf{C}\right) = \det\left(1 + \mathbf{\Omega} \cdot \mathbf{C}\right) \det\left(1 - (1 + \mathbf{\Omega} \cdot \mathbf{C})^{-1} \cdot {}^t\mathbf{\Lambda}^{-1} \cdot \mathbf{O} \cdot \mathbf{C}\right) \\ &= \tau\left[1 - (\mathbf{C} \cdot (1 + \mathbf{\Omega} \cdot \mathbf{C})^{-1} \cdot {}^t\mathbf{\Lambda}^{-1})_{0,0}\right] = \tau\left[1 - (\mathbf{U} \cdot {}^t\mathbf{\Lambda}^{-1})_{0,0}\right] = \tau(1 - U_{0,-1}),\end{aligned}$$

where the rank 1 Weinstein–Aronszajn formula is used to evaluate the determinant in terms of a scalar quantity, see chapter 3. This is exactly the first identity for the tau function. The second one can be derived similarly and we skip the proof. ☐

2.4. Closed-form nonlinear equations and associated linear systems. In order to construct closed-form equations, we introduce the unmodified variable u, and the modified variables v and w as follows:

$$
\begin{aligned}
u &\doteq U_{0,0} = (\mathbf{U})_{0,0},\\
v &\doteq 1 - U_{0,-1} = 1 - (\mathbf{U}\cdot {}^t\mathbf{\Lambda}^{-1})_{0,0},\\
w &\doteq 1 - U_{-1,0} = 1 - (\mathbf{\Lambda}^{-1}\cdot \mathbf{U})_{0,0}.
\end{aligned}
$$

These variables and the tau function are connected with each other via certain difference transforms. We list all these relations in the proposition below.

PROPOSITION 6.5. *The unmodified variable u and the modified variable v are related via the following discrete Miura transform:*

$$
p_1 + \dot{u} - \tilde{u} = p_1\frac{\tilde{v}}{v}, \quad 1 + p_{-1}(u - \check{u}) = \frac{\check{v}}{\underset{\cdot}{v}}. \tag{6.15a}
$$

The bilinear transforms between the nonlinear variables u, v and the bilinear variable τ are given by

$$
p_1 + \dot{u} - \tilde{u} = p_1\frac{\tau\dot{\tilde{\tau}}}{\dot{\tau}\tilde{\tau}}, \quad 1 + p_{-1}(u - \check{u}) = \frac{\underset{\cdot}{\tau}\dot{\check{\tau}}}{\tau\check{\tau}} \quad \textit{and} \quad v = \frac{\dot{\tau}}{\tau}, \tag{6.15b}
$$

respectively. The two modified variables v and w satisfy a simple relation $w = 1/\underset{\cdot}{v}$.

PROOF. Recalling (6.14) and the definitions of v and w, we can easily observe that $v = \dot{\tau}/\tau$ and $w = \underset{\cdot}{\tau}/\tau$, which are the bilinear transforms for the two modified variables, leading to the relation $v\dot{w} = 1$ immediately,[1] i.e. $w = 1/\underset{\cdot}{v}$. Taking the $(0,-1)$-entry of (6.12a) and (6.12c), respectively, we obtain

$$
p_1(v - \tilde{v}) = U_{1,-1} + \tilde{u}v \quad \text{and} \quad U_{1,-1} + \dot{u}v = 0.
$$

Subtracting the two relations and eliminating $U_{1,-1}$ gives rise to the first half in the Miura transform (6.15a). Next, we take the $(0,0)$-entry of (6.12b), which results in

$$
1 + p_{-1}(u - \check{u}) = \check{v}w. \tag{6.16}
$$

This is exactly the second half of the Miura transform once the variable w is replaced by $1/\underset{\cdot}{v}$. The bilinear transform between u and τ can then be derived by substituting v in the Miura transform (6.15a) with the tau function via $v = \dot{\tau}/\tau$. ☐

With the help of these transforms, we are able to construct closed-form linear systems based on the eigenfunction $\phi \doteq (\mathbf{u}_k)_0 = u_k^{(0)}$.

[1] Such an identity can alternatively be derived from the infinite matrix relation (6.12c) by taking the $(-1,-1)$-entry, cf. [**NCW85**]. Here we can observe that this relation is a consequence of a trivial identity based on the tau function.

THEOREM 6.6. *In the class of the discrete-time 2DTL of A_∞-type, we have the following linear systems:*

$$\tilde{\phi} = (p_1 + \dot{u} - \tilde{u})\phi + \dot{\phi} = p_1 \frac{\tilde{v}}{v}\phi + \dot{\phi} = p_1 \frac{\tau\dot{\tilde{\tau}}}{\dot{\tau}\tilde{\tau}}\phi + \dot{\phi}, \tag{6.17a}$$

$$\check{\phi} = p_{-1}\phi + [1 + p_{-1}(u - \check{u})]\,\underset{\cdot}{\phi} = p_{-1}\phi + \frac{\check{v}}{\underset{\cdot}{v}}\underset{\cdot}{\phi} = p_{-1}\phi + \frac{\underset{\cdot}{\tau}\dot{\check{\tau}}}{\tau\check{\tau}}\underset{\cdot}{\phi}. \tag{6.17b}$$

which are the Lax pairs for their corresponding nonlinear integrable lattice equations of u, v and τ.

PROOF. We only prove the linear equations involving the unmodified variable u. The other equalities are natural consequences of this under the difference transforms given in (6.15a) and (6.15b). The relation (6.13c) can be expressed by

$$\mathbf{\Lambda} \cdot \mathbf{u}_k = \dot{\mathbf{u}}_k + \dot{\mathbf{U}} \cdot \mathbf{O} \cdot \mathbf{u}_k, \quad \text{or alternatively} \quad \mathbf{\Lambda}^{-1} \cdot \mathbf{u}_k = \underset{\cdot}{\mathbf{u}}_k - \mathbf{\Lambda}^{-1} \cdot \mathbf{U} \cdot \mathbf{O} \cdot \underset{\cdot}{\mathbf{u}}_k,$$

which helps us to eliminate $\mathbf{\Lambda} \cdot \mathbf{u}_k$ and $\mathbf{\Lambda}^{-1} \cdot \mathbf{u}_k$ in (6.13a) and (6.13b), respectively, and obtain the following relations involving only $\mathbf{u}_k$:

$$\tilde{\mathbf{u}}_k = (p_1 + \dot{\mathbf{U}} \cdot \mathbf{O} - \tilde{\mathbf{U}} \cdot \mathbf{O}) \cdot \mathbf{u}_k + \dot{\mathbf{u}}_k,$$
$$\check{\mathbf{u}}_k = p_{-1}\mathbf{u}_k + (1 - \check{\mathbf{U}} \cdot {}^t\mathbf{\Lambda}^{-1} \cdot \mathbf{O}) \cdot (1 - \mathbf{\Lambda}^{-1} \cdot \mathbf{U} \cdot \mathbf{O}) \cdot \underset{\cdot}{\mathbf{u}}_k$$

The 0th-component of the first equation immediately gives rise to the "tilde" equation, i.e. (6.17a). For the "check" equation in (6.17b), we consider the 0th-component of the second equation and obtain

$$\check{\phi} = p_{-1}\phi + \check{v}w\underset{\cdot}{\phi},$$

which then turns out to be the second equation (6.17b) once $\check{v}w$ is replaced by $1+p_{-1}(u-\check{u})$, with the help of the identity (6.16). □

The compatibility condition of the equations in each Lax pair listed in (6.17) gives us a 3D nonlinear integrable difference equation.

THEOREM 6.7. *The unmodified variable u, the unmodified variable v and the tau function τ solve the following 3D integrable discrete equations:*

$$\frac{p_1 + \dot{\check{u}} - \tilde{\check{u}}}{p_1 + u - \underset{\cdot}{\tilde{u}}} = \frac{1 + p_{-1}(\tilde{u} - \tilde{\check{u}})}{1 + p_{-1}(u - \check{u})}, \tag{6.18a}$$

$$p_1 p_{-1}\left(\frac{\tilde{\check{v}}}{\check{v}} - \frac{\tilde{v}}{v}\right) = -\frac{\dot{\check{v}}}{v} + \frac{\tilde{\check{v}}}{\underset{\cdot}{\tilde{v}}}, \tag{6.18b}$$

$$p_1 p_{-1}(\tau\tilde{\check{\tau}} - \tilde{\tau}\check{\tau}) = \tau\tilde{\check{\tau}} - \dot{\check{\tau}}\underset{\cdot}{\tilde{\tau}}, \tag{6.18c}$$

which we refer to as the unmodified, modified and bilinear discrete-time 2DTL equations of A_∞-type, respectively. The direct linearising solutions for these equations are governed by (2.18) *and* (2.21)*, respectively. The (N, N')-soliton solutions are given in theorem 6.1.*

PROOF. Equations (6.18a) and (6.18b) follow from the u and v parts of the linear equations (6.17a) and (6.17b). The compatibility condition $\tilde{\check{\phi}} = \check{\tilde{\phi}}$ in terms of τ gives a quartic equation

$$p_1 p_{-1}\left(\tau\check{\tilde{\tau}}\dot{\check{\tau}}\dot{\tilde{\tau}}-\tilde{\tau}\check{\tau}\dot{\tau}\dot{\check{\tilde{\tau}}}\right)=\tau\tilde{\tau}\check{\tilde{\tau}}\dot{\dot{\check{\tau}}}-\underset{\cdot}{\tilde{\tau}}\dot{\tau}\dot{\check{\tau}}\dot{\check{\tilde{\tau}}}.$$

This is a weak equation defined on 10 points and by discrete integration we obtain the bilinear equation (6.18c). $\square$

Remark 6.8. *Making use of the transform $v = 1/\underset{\cdot}{w}$, we can derive another modified equation expressed by w which is dual to* (6.18b)*, cf.* (6.14)*, and its Lax pair can be obtained from* (6.17) *under the same transform. These is also a closed-form equation based on the Schwarzian variable $z \doteq U_{-1,-1}$, which takes a dual form of the unmodified equation* (6.18a)*. This is because in the discrete-time 2DTL the effective plane wave factor* (6.6) *depends on k, k' and k^{-1}, k'^{-1} in a covariant way, leading to the fact that u and z are dual to each other.*

The bilinear equation (6.18c) and the modified equation (6.18b) were originally given by Date, Jimbo and Miwa [**DJM82**] within the framework of transformation groups for soliton equations.[2] Through a point transformation $\mathfrak{u} = u-n_1p_1-n_{-1}/p_{-1}$, the unmodified equation (6.18a) can be written as

$$\frac{(\mathfrak{u}-\check{\mathfrak{u}})(\dot{\check{\mathfrak{u}}}-\check{\tilde{\mathfrak{u}}})}{(\mathfrak{u}-\underset{\cdot}{\tilde{\mathfrak{u}}})(\tilde{\mathfrak{u}}-\check{\tilde{\mathfrak{u}}})}=1,$$

which appeared very recently in [**FNR15**] (after a recombination of all the discrete shifts). The parametrisation in (6.18a), however, describes its solution structure in a more natural way and allows to consider its continuum limit. Furthermore, by a transform $v = \exp\varphi$ we obtain from (6.18b)

$$p_1p_{-1}\left(\exp(\check{\tilde{\varphi}}-\check{\varphi})-\exp(\tilde{\varphi}-\varphi)\right)=-\exp(\dot{\check{\varphi}}-\varphi)+\exp(\check{\tilde{\varphi}}-\underset{\cdot}{\tilde{\varphi}}), \tag{6.19}$$

which is the discrete analogue of (6.1).

2.5. Continuum limits. We set up the continuum limit scheme for the discrete-time 2DTL of A_∞-type. The continuous independent variables are introduced as follows:

$$x_1=\frac{n_1}{p_1},\quad x_{-1}=\frac{n_{-1}}{p_{-1}},\quad n_1,n_{-1}\to\infty,\quad p_1,p_{-1}\to\infty.$$

To respect the tradition, we mark the discrete variable n explicitly as a suffix in the corresponding variables. The continuum limit scheme results in the maps between the discrete and continuous spaces, namely for $f = f_{n_1,n_{-1},n} = f_n(x_1, x_{-1})$ we have

$$\tilde{f}=f_n(x_1+1/p_1,x_{-1}),\quad \check{f}=f_n(x_1,x_{-1}+1/p_{-1}),\quad \dot{f}=f_{n+1}(x_1,x_{-1}).$$

as well as

$$\underset{\sim}{f}=f_n(x_1-1/p_1,x_{-1}),\quad \underset{\vee}{f}=f_n(x_1,x_{-1}-1/p_{-1}),\quad \underset{\cdot}{f}=f_{n-1}(x_1,x_{-1}).$$

where f can be any of the variables u, v and τ.

---

[2]A two-component extension of this modified equation was given in [**NCW85**].

In the continuum limit, the discrete bilinear equation (6.18c) turns out to be

$$\frac{1}{2}D_1D_{-1}\tau_n\cdot\tau_n=\tau_n^2-\tau_{n+1}\tau_{n-1},$$

where D_1 and D_{-1} are Hirota's bilinear operators (see appendix A for the definition) with respect to x_1 and x_{-1}. This is the bilinear 2DTL, see e.g. [**Hir81**,**DJM82**]. Similarly we also have the continuum limits of the unmodified and modified equations as follows:

$$\partial_1\ln(1-\partial_{-1}u_n)=u_{n+1}-2u_n+u_{n-1},\quad \partial_1\partial_{-1}\ln v_n=-\frac{v_{n+1}}{v_n}+\frac{v_n}{v_{n-1}}.$$

These equations are different nonlinear forms of the 2DTL (6.1), as we can still see the characteristic of the Cartan matrix corresponding to A_∞ in both equations. By transform $v_n=\exp\varphi_n$, the v_n equation becomes (6.1), which can alternatively be derived by taking the limit of (6.19). All these potentials are connected with each other via transforms

$$\varphi_n=\ln v_n=\ln\frac{\tau_{n+1}}{\tau_n}\quad\text{and}\quad u_{n+1}-u_n=\partial_1\ln v_n=\partial_1\ln\frac{\tau_{n+1}}{\tau_n},$$

which are actually the continuum limits of the bilinear transforms (6.15b) and Miura transforms (6.15a). The u_n equation can also be written in a slightly different form

$$\partial_1\partial_{-1}\ln(1-s_n)=s_{n+1}-2s_n+s_{n-1}$$

via transform $s_n=\partial_{-1}u_n$, which is the nonpotential form of the u_n equation and was the form considered in [**HIS88**].

The Lax pairs of the continuous equations can be recovered from the discrete ones in the same limit scheme. One can also take the limit only with respect to n_1 or n_{-1}, from which the semi-discrete equations will arise.

3. Discrete-time two-dimensional Toda lattices of $A^{(1)}_{\mathcal{N}-1}$-type

3.1. Periodic reductions. Performing the $\mathcal{N}$-periodic reduction (for integer $r\geq 2$) of the discrete-time 2DTL of A_∞-type is equivalent to considering sub-algebra $A^{(1)}_{\mathcal{N}-1}$, see e.g. [**JM83**]. In the direct linearisation framework, such a reduction can be realised by taking the measure

$$\mathrm{d}\zeta(k,k')=\sum_{j\in J}\mathrm{d}\lambda_j(k)\mathrm{d}k'\delta(k'+\omega^jk),\tag{6.20}$$

where $\mathrm{d}\lambda_j(k)$ are the measures only depending on the spectral variable k, and $\omega=\exp(2\pi\mathrm{i}/\mathcal{N})$ and $J=\{j|0<j<\mathcal{N}\text{ are integers coprime to }\mathcal{N}\}$, i.e. ω^j are all $\mathcal{N}$th primitive roots of unity. In other words, a constraint is imposed on the two spectral parameters, restricting k and k' on an algebraic curve $k^{\mathcal{N}}=(-k')^{\mathcal{N}}$, cf. chapter 4. As a consequence, the linear integral equation (2.15) associated with a double integral degenerates, and becomes one with only a single integral, namely

$$\mathbf{u}_k+\sum_{j\in J}\int_{\Gamma_j}\mathrm{d}\lambda_j(l)\rho_k\Omega_{k,-\omega^jl}\sigma_{-\omega^jl}\mathbf{u}_l=\rho_k\mathbf{c}_k,\tag{6.21}$$

where Γ_j are the corresponding contours. Meanwhile, reduction (6.20) also results in

$$\mathbf{C}=\sum_{j\in J}\int_{\Gamma_j}\mathrm{d}\lambda_j(k)\rho_k\mathbf{c}_k{}^t\mathbf{c}_{-\omega^j k}\sigma_{-\omega^j k}\quad\text{and}\quad \mathbf{U}=\sum_{j\in J}\int_{\Gamma_j}\mathrm{d}\lambda_j(k)\mathbf{u}_k{}^t\mathbf{c}_{-\omega^j k}\sigma_{-\omega^j k},\qquad(6.22)$$

where ρ_k and $\sigma_{k'}$ still take their respective forms given in (6.5), but one now has to keep in mind that the reduced effective plane wave factors become

$$\rho_k\sigma_{-\omega^j k}=\left(\frac{p_1+k}{p_1+\omega^j k}\right)^{n_1}\left(\frac{p_{-1}+k^{-1}}{p_{-1}+(\omega^j k)^{-1}}\right)^{n_{-1}}\left(\frac{1}{\omega^j}\right)^{n}.\qquad(6.23)$$

Thus, the variables $\phi=(\mathbf{u}_k)_0$, $u=(\mathbf{U})_{0,0}$, $v=1-(\mathbf{U})_{0,-1}$ and $\tau=\det(1+\boldsymbol{\Omega}\cdot\mathbf{C})$ governed by (6.21) and (6.22) play the roles of the direct linearising solutions to the corresponding linear and nonlinear equations. For explicit solitons, we can still take particular measures which bring a finite number of poles as we have done in subsection 2.2. We omit the derivation here since the structure of the direct linearising solution is already clear and the resulting soliton structure is very similar to that for the 3D case in theorem 6.1. The only comment here is that in this case a block Cauchy matrix structure will arise, cf. chapter 4, since all the primitive roots are involved. As the simplest example, one can take $k'_j=-\omega k_j$ (i.e. we only consider one primitive root) in theorem 6.1. This immediately provides us with the soliton solutions of the reduced equations.

3.2. Constraints on the linear and nonlinear variables. The reduction (6.20) provides not only the reduced direct linearising solutions, but also the corresponding constraints on the linear, bilinear and nonlinear variables in the 3D theory, leading to 2D equations. In fact, we can observe from (6.23) that the plane wave factors obeys $\rho_k(n+\mathcal{N})\sigma_{-\omega^j k}(n+\mathcal{N})=\rho_k(n)\sigma_{-\omega^j k}(n)$, since ω^j are the rth primitive roots of unity, which implies that the reduced infinite matrix $\mathbf{C}$ given in (6.22) satisfies $\mathbf{C}(n+\mathcal{N})=\mathbf{C}(n)$ as its dynamics rely on the effective plane wave factors $\rho_k\sigma_{-\omega^j k}$. Recalling that the dynamics of the infinite matrix $\mathbf{U}$ and the tau function τ explicitly depend on $\mathbf{C}$ and the dynamics of the wave function $\mathbf{u}_k$ are determined by $\mathbf{U}$ and ρ_k, cf. (2.20), (2.21) and (2.19), we can therefore obtain

$$\mathbf{U}(n+\mathcal{N})=\mathbf{U}(n),\quad \tau(n+\mathcal{N})=\tau(n)\quad\text{and}\quad \mathbf{u}_k(n+\mathcal{N})=k^{\mathcal{N}}\mathbf{u}_k(n).$$

For future convenience, in the discrete-time 2DTL of $A^{(1)}_{\mathcal{N}-1}$-type, we introduce a suffix n for each variable. For example, we mark $u=u_n$, $\dot{u}=u_{n+1}$ and $\underset{\cdot}{u}=u_{n-1}$ (and similar for v, τ and ϕ). Taking the corresponding components or entries for the above relations, we have the constraints for all the variables we need and conclude them in the following proposition.

Proposition 6.9. *The $\mathcal{N}$-periodic reduction results in the following constraints on the nonlinear, bilinear and linear variables:*

$$u_{n+\mathcal{N}}=u_n,\quad v_{n+\mathcal{N}}=v_n,\quad \tau_{n+\mathcal{N}}=\tau_n,\quad \phi_{n+\mathcal{N}}=k^{\mathcal{N}}\phi_n,\qquad(6.24)$$

which implies that we only need to consider u_i, v_i, τ_i and ϕ_i for $i=0,1,\cdots,\mathcal{N}-1$ in the $\mathcal{N}$-periodic reduction.

Furthermore, there are also additional relations arising from the reduction.

PROPOSITION 6.10. *The unmodified and modified variables* u_n *and* v_n *obey identities*

$$\prod_{i=0}^{\mathcal{N}-1}(p_1+u_{i+1}-\tilde{u}_i)=p_1^{\mathcal{N}},\quad \prod_{i=0}^{\mathcal{N}-1}[1+p_{-1}(u_i-\check{u}_i)]=1 \quad \textit{and} \quad \prod_{i=0}^{\mathcal{N}-1}v_i=1, \tag{6.25}$$

in the class of the discrete-time 2DTL of $A^{(1)}_{\mathcal{N}-1}$*-type.*

PROOF. These identities are easily proven as they all follow from (6.15b) and proposition 6.9. For instance,

$$\prod_{i=0}^{\mathcal{N}-1}(p_1+u_{i+1}-\tilde{u}_i)=p_1^{\mathcal{N}}\prod_{i=0}^{\mathcal{N}-1}\frac{\tau_i\tilde{\tau}_{i+1}}{\tau_{i+1}\tilde{\tau}_i}=p_1^{\mathcal{N}}\frac{\tau_0\tilde{\tau}_{\mathcal{N}}}{\tau_{\mathcal{N}}\tilde{\tau}_0}=p_1^{\mathcal{N}}$$

and

$$\prod_{i=0}^{\mathcal{N}-1}v_n=\prod_{i=0}^{\mathcal{N}-1}\frac{\tau_{i+1}}{\tau_i}=\frac{\tau_{\mathcal{N}}}{\tau_0}=1,$$

where $\tau_{\mathcal{N}}=\tau_0$ is used in the last step in each equation. The second identity is proven similarly. □

The identities in proposition 6.10 were also given in [**FX17**] and were referred to as discrete first integrals. Here we can see that they are consequences of the periodicity of the tau function in the reduction.

3.3. Reduced bilinear and nonlinear equations. We can now construct the bilinear and nonlinear equations in the class of the discrete-time 2DTL of $A^{(1)}_{\mathcal{N}-1}$-type, with the help of the constraints generated by the periodic reduction.

We first consider the bilinear equation, which is derived from (6.18c). By selecting $n=0,1,\cdots,\mathcal{N}-1$ and making use of $\tau_{n+\mathcal{N}}=\tau_n$, we obtain a coupled system of bilinear discrete equations

$$p_1p_{-1}(\tau_0\check{\tilde{\tau}}_0-\tilde{\tau}_0\check{\tau}_0)=\tau_0\check{\tilde{\tau}}_0-\check{\tau}_1\tilde{\tau}_{\mathcal{N}-1}, \tag{6.26a}$$

$$p_1p_{-1}(\tau_n\check{\tilde{\tau}}_n-\tilde{\tau}_n\check{\tau}_n)=\tau_n\check{\tilde{\tau}}_n-\check{\tau}_{n+1}\tilde{\tau}_{n-1},\quad n=1,2,\cdots,\mathcal{N}-2, \tag{6.26b}$$

$$p_1p_{-1}(\tau_{\mathcal{N}-1}\check{\tilde{\tau}}_{\mathcal{N}-1}-\tilde{\tau}_{\mathcal{N}-1}\check{\tau}_{\mathcal{N}-1})=\tau_{\mathcal{N}-1}\check{\tilde{\tau}}_{\mathcal{N}-1}-\check{\tau}_0\tilde{\tau}_{\mathcal{N}-2}, \tag{6.26c}$$

which we refer to as the bilinear discrete-time 2DTL of $A^{(1)}_{\mathcal{N}-1}$-type.

Similarly, the unmodified discrete-time 2DTL of $A^{(1)}_{\mathcal{N}-1}$-type is derived from (6.18a) with the help of $u_{n+\mathcal{N}}=u_n$ and also takes a coupled system form

$$\frac{p_1+\check{u}_1-\check{\tilde{u}}_0}{p_1+u_0-\tilde{u}_{\mathcal{N}-1}}=\frac{1+p_{-1}(\tilde{u}_0-\check{\tilde{u}}_0)}{1+p_{-1}(u_0-\check{u}_0)}, \tag{6.27a}$$

$$\frac{p_1+\check{u}_{n+1}-\check{\tilde{u}}_n}{p_1+u_n-\tilde{u}_{n-1}}=\frac{1+p_{-1}(\tilde{u}_n-\check{\tilde{u}}_n)}{1+p_{-1}(u_n-\check{u}_n)},\quad n=1,2,\cdots,\mathcal{N}-2, \tag{6.27b}$$

$$\frac{p_1+\check{u}_0-\check{\tilde{u}}_{\mathcal{N}-1}}{p_1+u_{\mathcal{N}-1}-\tilde{u}_{\mathcal{N}-2}}=\frac{1+p_{-1}(\tilde{u}_{\mathcal{N}-1}-\check{\tilde{u}}_{\mathcal{N}-1})}{1+p_{-1}(u_{\mathcal{N}-1}-\check{u}_{\mathcal{N}-1})}. \tag{6.27c}$$

This system should also be referred to as the negative flow[3] of the $\mathcal{N}$th member in the discrete unmodified GD hierarchy. Although the variable u obeys an additional identity (see proposition 6.10), in this case it seems not possible to eliminate one dependent variable in (6.27) and express the coupled system by $\mathcal{N}-1$ components. However, for the positive flow of a member in the discrete GD hierarchy, one can reduce the number of components by one, making use of such an identity, see [**FX17**].

The modified discrete-time 2DTL of $A^{(1)}_{\mathcal{N}-1}$-type can be written as the following coupled system of discrete equations:

$$p_1p_{-1}\left(\frac{\check{\tilde{v}}_0}{\check{v}_0}-\frac{\tilde{v}_0}{v_0}\right)=-\frac{\check{v}_1}{v_0}+\frac{\check{\tilde{v}}_0}{\tilde{v}_{\mathcal{N}-1}}, \tag{6.28a}$$

$$p_1p_{-1}\left(\frac{\check{\tilde{v}}_n}{\check{v}_n}-\frac{\tilde{v}_n}{v_n}\right)=-\frac{\check{v}_{n+1}}{v_n}+\frac{\check{\tilde{v}}_n}{\tilde{v}_{n-1}},\quad n=1,2,\cdots,\mathcal{N}-2, \tag{6.28b}$$

$$p_1p_{-1}\left(\frac{\check{\tilde{v}}_{\mathcal{N}-1}}{\check{v}_{\mathcal{N}-1}}-\frac{\tilde{v}_{\mathcal{N}-1}}{v_{\mathcal{N}-1}}\right)=-\frac{\check{v}_0}{v_{\mathcal{N}-1}}+\frac{\check{\tilde{v}}_{\mathcal{N}-1}}{\tilde{v}_{\mathcal{N}-2}}, \tag{6.28c}$$

which can also be referred to as the negative flow of the $\mathcal{N}$th member in the discrete modified GD hierarchy. In this case, it is possible to eliminate one of the components with the help of proposition 6.10. Without loss of generality, we can replace $v_{\mathcal{N}-1}$ by $v_0,\cdots,v_{\mathcal{N}-2}$ according to $v_{\mathcal{N}-1}=1/\prod_{i=0}^{\mathcal{N}-2}v_i$. Thus, the modified system can also be written as

$$p_1p_{-1}\left(\frac{\check{\tilde{v}}_0}{\check{v}_0}-\frac{\tilde{v}_0}{v_0}\right)=-\frac{\check{v}_1}{v_0}+\check{\tilde{v}}_0\prod_{i=0}^{\mathcal{N}-2}\tilde{v}_i,$$

$$p_1p_{-1}\left(\frac{\check{\tilde{v}}_n}{\check{v}_n}-\frac{\tilde{v}_n}{v_n}\right)=-\frac{\check{v}_{n+1}}{v_n}+\frac{\check{\tilde{v}}_n}{\tilde{v}_{n-1}},\quad n=1,2,\cdots,\mathcal{N}-3,$$

$$p_1p_{-1}\left(\frac{\check{\tilde{v}}_{\mathcal{N}-2}}{\check{v}_{\mathcal{N}-2}}-\frac{\tilde{v}_{\mathcal{N}-2}}{v_{\mathcal{N}-2}}\right)=-\frac{1}{v_{\mathcal{N}-2}\prod_{i=0}^{\mathcal{N}-2}\check{v}_i}+\frac{\check{\tilde{v}}_{\mathcal{N}-2}}{\tilde{v}_{\mathcal{N}-3}},$$

The transform $v_n=\exp\varphi_n$ gives us the exponential form of the discrete-time 2DTL of $A^{(1)}_{\mathcal{N}-1}$-type.

The constraints (6.24) simultaneously bring us the Miura and bilinear transforms

$$p_1+u_{n+1}-\tilde{u}_n=p_1\frac{\tilde{v}_n}{v_n}=p_1\frac{\tau_n\tilde{\tau}_{n+1}}{\tau_{n+1}\tilde{\tau}_n},\quad p_1+u_0-\tilde{u}_{\mathcal{N}-1}=p_1\frac{\tilde{v}_{\mathcal{N}-1}}{v_{\mathcal{N}-1}}=p_1\frac{\tau_{\mathcal{N}-1}\tilde{\tau}_0}{\tau_0\tilde{\tau}_{\mathcal{N}-1}} \tag{6.29a}$$

for $n=0,1,\cdots,r-2$, and

$$\begin{aligned}1+p_{-1}(u_0-\check{u}_0)&=\frac{\check{v}_0}{v_{\mathcal{N}-1}}=\frac{\tau_{\mathcal{N}-1}\check{\tau}_1}{\tau_0\check{\tau}_0},\\1+p_{-1}(u_n-\check{u}_n)&=\frac{\check{v}_n}{v_{n-1}}=\frac{\tau_{n-1}\check{\tau}_{n+1}}{\tau_n\check{\tau}_n},\\1+p_{-1}(u_{\mathcal{N}-1}-\check{u}_{\mathcal{N}-1})&=\frac{\check{v}_{\mathcal{N}-1}}{v_{\mathcal{N}-2}}=\frac{\tau_{\mathcal{N}-2}\check{\tau}_0}{\tau_{\mathcal{N}-1}\check{\tau}_{\mathcal{N}-1}}\end{aligned} \tag{6.29b}$$

[3] Compare (6.23) with the effective plane wave factor

$$\rho_k\sigma_{-\omega^jk}=\left(\frac{p_1+k}{p_1+\omega^jk}\right)^{n_1}\left(\frac{p_2+k}{p_2+\omega^jk}\right)^{n_2}$$

for the discrete GD hierarchy, see [**NPCQ92**].

for $n = 1, 2, \cdots, \mathcal{N} - 2$, which relate (6.27), (6.28) and (6.26) with each other.

The unmodified equation (6.27) and the modified equation (6.28) are new parametrisations of the integrable lattice equations corresponding to equivalence class $[(0, 1; \mathcal{N} - 1, 0)]$ given in [**FX17**], allowing exact solutions and continuum limits.

3.4. Lax matrices. The Lax pairs for the reduced equations (6.26), (6.27) and (6.28) can be constructed from (6.17) under the constraints given in proposition 6.9. We only write down the Lax pair for the unmodified equation in u_n, and those for the modified and bilinear equations can be obtained by using the Miura and bilinear transforms given in (6.29).

The "tilde" equation of the Lax pair is a consequence of (6.17a) and the constraints on u_n and ϕ_n in (6.24), taking the form of

$$\widetilde{\begin{pmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_{\mathcal{N}-2} \\ \phi_{\mathcal{N}-1} \end{pmatrix}} = \begin{pmatrix} p_1 + u_1 - \tilde{u}_0 & 1 & & & \\ & p_1 + u_2 - \tilde{u}_1 & 1 & & \\ & & \ddots & \ddots & \\ & & & p_1 + u_{\mathcal{N}-1} - \tilde{u}_{\mathcal{N}-2} & 1 \\ k^{\mathcal{N}} & & & & p_1 + u_0 - \tilde{u}_{\mathcal{N}-1} \end{pmatrix} \begin{pmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_{\mathcal{N}-2} \\ \phi_{\mathcal{N}-1} \end{pmatrix}. \tag{6.30a}$$

This linear equation is also compatible with the positive flows of members in the discrete GD hierarchy, and it is gauge equivalent to the one given in [**NPCQ92**]. The dynamics describing the negative flow is contained in the "check" part, which follows from (6.17b) and $\phi_{-1} = k^{-\mathcal{N}} \phi_{\mathcal{N}-1}$ in proposition 6.9 and is given by

$$\check{\begin{pmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_{\mathcal{N}-2} \\ \phi_{\mathcal{N}-1} \end{pmatrix}} = \begin{pmatrix} p_{-1} & & & & * \\ 1 + p_{-1}(u_1 - \check{u}_1) & p_{-1} & & & \\ & \ddots & \ddots & & \\ & & 1 + p_{-1}(u_{\mathcal{N}-2} - \check{u}_{\mathcal{N}-2}) & p_{-1} & \\ & & & 1 + p_{-1}(u_{\mathcal{N}-1} - \check{u}_{\mathcal{N}-1}) & p_{-1} \end{pmatrix} \begin{pmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_{\mathcal{N}-2} \\ \phi_{\mathcal{N}-1} \end{pmatrix}, \tag{6.30b}$$

where $* = k^{-\mathcal{N}}[1 + p_{-1}(u_0 - \check{u}_0)]$. The compatibility condition of the Lax pair gives rise to the coupled system (6.27).

An alternative way to construct such a linear problem was discussed within a framework of $\mathbb{Z}_N$ graded algebras. In fact, by introducing a gauge transform

$$\begin{pmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_{\mathcal{N}-2} \\ \psi_{\mathcal{N}-1} \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ & k^{-1} & & & \\ & & \ddots & & \\ & & & k^{-(\mathcal{N}-2)} & \\ & & & & k^{-(\mathcal{N}-1)} \end{pmatrix} \begin{pmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_{\mathcal{N}-2} \\ \phi_{\mathcal{N}-1} \end{pmatrix},$$

the linear equations in (6.30) become more or less the same as the ones given in [**FX17**]. To be more precise, the "tilde" part becomes exactly the same (up to a translation on the potential variables u_n), and the "check" part has the same matrix structure but in our case it depends on k^{-1} explicitly.

3.5. $A^{(1)}_1$ and $A^{(1)}_2$: Negative flows of discrete KdV and BSQ. We list two concrete examples explicitly, namely the discrete-time 2DTLs of $A^{(1)}_1$-type and $A^{(1)}_2$-type. The $A^{(1)}_1$ class is corresponding to the negative flows of the discrete KdV-type equations, including the discrete sine–Gordon equation which was discovered and rediscovered in the literature on numerous occasions. The $A^{(1)}_2$ class gives the negative flows of the discrete BSQ-type equations.

The $A^{(1)}_1$ class includes the negative flows of the discrete unmodified and modified KdV equations, and the bilinear discrete-time 2DTL of $A^{(1)}_1$ as follows:

$$
\begin{aligned}
&\frac{p_1+\check{u}_1-\check{\tilde{u}}_0}{p_1+u_0-\tilde{u}_1}=\frac{1+p_{-1}(\tilde{u}_0-\check{\tilde{u}}_0)}{1+p_{-1}(u_0-\check{u}_0)},\quad \frac{p_1+\check{u}_0-\check{\tilde{u}}_1}{p_1+u_1-\tilde{u}_0}=\frac{1+p_{-1}(\tilde{u}_1-\check{\tilde{u}}_1)}{1+p_{-1}(u_1-\check{u}_1)},\\
&p_1p_{-1}\left(\frac{\check{\tilde{v}}_0}{\check{v}_0}-\frac{\tilde{v}_0}{v_0}\right)=-\frac{1}{v_0\check{v}_0}+\tilde{v}_0\check{\tilde{v}}_0,\\
&p_1p_{-1}(\tau_0\check{\tilde{\tau}}_0-\tilde{\tau}_0\check{\tau}_0)=\tau_0\check{\tilde{\tau}}_0-\check{\tau}_1\tilde{\tau}_1,\quad p_1p_{-1}(\tau_1\check{\tilde{\tau}}_1-\tilde{\tau}_1\check{\tau}_1)=\tau_1\check{\tilde{\tau}}_1-\check{\tau}_0\tilde{\tau}_0.
\end{aligned}
$$

The transform $v_0=\exp\varphi_0$ then brings us

$$
p_1p_{-1}\left(\exp(\check{\tilde{\varphi}}_0-\check{\varphi}_0)-\exp(\tilde{\varphi}_0-\varphi_0)\right)=\exp(\tilde{\varphi}_0+\check{\tilde{\varphi}}_0)-\exp(-\varphi_0-\check{\varphi}_0),
$$

which is the discrete analogue of the continuous-time sinh–Gordon equation (i.e. the first equation in (6.3)). This equation can also be written in another form, namely the discrete sine–Gordon equation

$$
p_1p_{-1}\sin(\vartheta_0+\check{\tilde{\vartheta}}_0-\tilde{\vartheta}_0-\check{\vartheta}_0)=\sin(\vartheta_0+\tilde{\vartheta}_0+\check{\vartheta}_0+\check{\tilde{\vartheta}}_0),
$$

by transform $\varphi_0=2\mathrm{i}\vartheta_0$. One comment here is that the unmodified equation in u_0 and u_1 can also be written in a coupled system composed of the unmodified and modified variables u_0 and v_0, with the help of the Miura transform $p_1+u_1-\tilde{u}_0=p_1\tilde{v}_0/v_0$.

Now we consider the class of the discrete-time 2DTL of $A^{(1)}_2$-type, namely the 3-periodic reduction. The bilinear discrete-time 2DTL of $A^{(1)}_2$-type according to the general framework (6.26) is given by

$$
\begin{aligned}
p_1p_{-1}(\tau_0\check{\tilde{\tau}}_0-\tilde{\tau}_0\check{\tau}_0)&=\tau_0\check{\tilde{\tau}}_0-\check{\tau}_1\tilde{\tau}_2,\\
p_1p_{-1}(\tau_1\check{\tilde{\tau}}_1-\tilde{\tau}_1\check{\tau}_1)&=\tau_1\check{\tilde{\tau}}_1-\check{\tau}_2\tilde{\tau}_0,\\
p_1p_{-1}(\tau_2\check{\tilde{\tau}}_2-\tilde{\tau}_2\check{\tau}_2)&=\tau_2\check{\tilde{\tau}}_2-\check{\tau}_0\tilde{\tau}_1,
\end{aligned}
$$

which is also the negative flow of the trilinear discrete BSQ equation discussed in [**ZZN12**].

The unmodified equation in this class is a three-component system involving u_0, u_1 and u_2, which takes the form of

$$\begin{aligned}
\frac{p_1+\check{u}_1-\check{\tilde{u}}_0}{p_1+u_0-\tilde{u}_2}&=\frac{1+p_{-1}(\tilde{u}_0-\check{\tilde{u}}_0)}{1+p_{-1}(u_0-\check{u}_0)},\\
\frac{p_1+\check{u}_2-\check{\tilde{u}}_1}{p_1+u_1-\tilde{u}_0}&=\frac{1+p_{-1}(\tilde{u}_1-\check{\tilde{u}}_1)}{1+p_{-1}(u_1-\check{u}_1)},\\
\frac{p_1+\check{u}_0-\check{\tilde{u}}_2}{p_1+u_2-\tilde{u}_1}&=\frac{1+p_{-1}(\tilde{u}_2-\check{\tilde{u}}_2)}{1+p_{-1}(u_2-\check{u}_2)}.
\end{aligned}$$

This equation is also not decoupled and acts as the negative flow of the 9-point discrete unmodified BSQ equation proposed in [**NPCQ92**].

The modified equation is also a three-component system of discrete equations, composed of v_0, v_1 and v_2, according to (6.28). In this case, we can make use of the identity $v_2=1/(v_0v_1)$ (cf. equation (6.25)) and eliminate the component v_2. As a consequence, we have

$$p_1p_{-1}\left(\frac{\check{\tilde{v}}_0}{\check{v}_0}-\frac{\tilde{v}_0}{v_0}\right)=-\frac{\check{v}_1}{v_0}+\check{\tilde{v}}_0\tilde{v}_0\tilde{v}_1,\quad p_1p_{-1}\left(\frac{\check{\tilde{v}}_1}{\check{v}_1}-\frac{\tilde{v}_1}{v_1}\right)=-\frac{1}{\check{v}_0\check{v}_1v_1}+\frac{\check{\tilde{v}}_1}{\tilde{v}_0},$$

which is the negative flow of the 9-point discrete modified BSQ equation which was given in [**NPCQ92**]. The exponential form of the discrete-time 2DTL of $A_2^{(1)}$-type is derived by the transforms $v_0=\exp\varphi_0$ and $v_1=\exp\varphi_1$, taking the form of

$$\begin{aligned}
&p_1p_{-1}\left(\exp(\check{\tilde{\varphi}}_0-\check{\varphi}_0)-\exp(\tilde{\varphi}_0-\varphi_0)\right)=\exp(\check{\tilde{\varphi}}_0+\tilde{\varphi}_0+\tilde{\varphi}_1)-\exp(\check{\varphi}_1-\varphi_0),\\
&p_1p_{-1}\left(\exp(\check{\tilde{\varphi}}_1-\check{\varphi}_1)-\exp(\tilde{\varphi}_1-\varphi_1)\right)=\exp(\check{\tilde{\varphi}}_1-\tilde{\varphi}_0)-\exp(-\check{\varphi}_0-\check{\varphi}_1-\varphi_1),
\end{aligned}$$

which is the $A_2^{(1)}$-type analogue of (6.19).

The continuum limits of these discrete equations follow the same scheme in subsection 2.5, leading to the negative flows in the continuous KdV and BSQ hierarchies.

4. Concluding remarks

The direct linearisation scheme was established for the discrete-time 2DTL equations of A_∞-type and $A_{\mathcal{N}-1}^{(1)}$-type. For each algebra, a class of nonlinear (including bilinear) equations arise and their integrability is guaranteed in the sense of having Lax pairs and direct linearising solutions.

For convenience we mainly focused on equations expressed by the (potential) unmodified variable, the (potential) modified variable, as well as the tau function, because the resulting equations take relatively simple forms (i.e. scalar octahedron-type equations). There certainly exist alternative nonlinear forms, and sometimes a nonlinear equation could even take the form of a coupled system, which normally happens in a closed-form equation expressed by a nonpotential variable, see e.g. [**HIS88**, **KOS94**].

In fact, in the framework there is also another nonpotential form of the discrete-time 2DTL of A_∞-type. Introducing nonpotential variables

$$P\doteq p_{-1}^{-1}+u-\check{u}=p_{-1}^{-1}\frac{\check{v}}{\underset{\cdot}{v}}=p_{-1}^{-1}\frac{\underset{\cdot}{\tau}\check{\dot{\tau}}}{\tau\check{\tau}}\quad\text{and}\quad Q\doteq p_1+\dot{u}-\tilde{u}=p_1\frac{\tilde{v}}{v}=p_1\frac{\tau\dot{\tilde{\tau}}}{\dot{\tau}\tilde{\tau}},$$

we obtain the nonpotential equation taking the form of a coupled system of P and Q, namely

$$\check{Q}/\underset{\cdot}{Q} = \tilde{P}/P, \quad \check{Q} - Q = \tilde{P} - \dot{P}, \tag{6.31}$$

which follow from the unmodified equation (6.18a) and the modified equation (6.18b), respectively. Equation (6.31) is still a 6-point equation (if we count both components P and Q), similar to equations given in theorem 6.7. The Lax pair of this coupled system takes the form of

$$\tilde{\phi} = Q\phi + \dot{\phi}, \quad \check{\phi} = p_{-1}(\phi + P\underset{\cdot}{\phi})$$

which is derived from (6.17) by replacing u by the nonpotential variables. Following the idea about deriving the nonpotential discrete KP equation given in chapter 3, one can eliminate either P or Q in (6.31). As a consequence, a 10-point scalar equation in terms of only Q or P can be expected, which we leave as an exercise. This is not surprising as we have already seen such a lattice structure in the derivation of the bilinear equation (6.18c), see the proof of theorem 6.7.

The solution structure of the discrete-time 2DTL of A_∞-type is related to that of the discrete KP equation. To be more precise, the nonlinear structure (6.4) is exactly the same, and the linear structure (6.6) is a slight deformation of that in KP and it can be reconstructed from the plane wave factor of the discrete KP equation in a subtle way. However, such a deformation leads to discrete integrable systems having different lattice structures (though in this case they are still octahedron-type equations).

The unmodified and modified equations (6.27) and (6.28) are corresponding to some new equations equivalence class $[(0,1;\mathcal{N}-1,0)]$ in [**FX17**]. Thus, we identify that these new equations are the nonlinear equations in the class of the discrete-time 2DTL of $A_{\mathcal{N}-1}^{(1)}$-type, which arise as the dimensional reductions of (6.18a) and (6.18b). Equations in other equivalence classes in [**FX17**] are actually also associated with the algebra $A_{\mathcal{N}-1}^{(1)}$. To put it another way, these new equations share the same kernel and measure in the direct linearisation, and obey different (discrete) time evolutions.

The direct linearisation of the discrete-time 2DTLs of other types (such as B_∞ and C_∞ and their reductions) will be reported elsewhere. From the author's experience, the structures of these equations are very different. This is mainly because of the fractionally linear dependence on the spectral parameters in their dispersions. However, such an issue does not occur in the continuous theory, and various continuous-time 2DTLs of different types arise very naturally as reductions of the 2DTL of A_∞-type.

CHAPTER 7

Conclusions

1. Summary

We studied algebraic and solution structures of discrete and continuous integrable systems within the DL framework. Our starting point was to study a scalar linear integral equation with flexibility, as a consequence, a large number of integrable discrete and continuous nonlinear equations together with their solutions and other integrability characteristics were constructed (which include some new integrable equations such as the discrete SK equation (5.49) and the discrete KK equation (5.55), as well as some new parametrisations such as the discrete CKP equation (3.43) and the unmodified discrete 2DTL of A_∞-type (6.18a)). The DL scheme not only allowed constructing a more large class of solutions to the related nonlinear equations, in sense that the direct linearising solution involved an integral with respect to the spectral variable(s), but also made it possible to understand the underlying structure for the whole class of integrable nonlinear equations, once suitable Cauchy kernel and plane wave factors were selected.

The thesis included the DL framework of the following discrete and continuous models: i) the 3D discrete integrable systems, including the discrete AKP, BKP and CKP equations; ii) the continuous AKP, BKP and CKP hierarchies and their dimensional reductions; iii) the reductions of the discrete AKP, BKP and CKP equations; iv) the discrete-time 2DTL equations of A_∞- and $A^{(1)}_{\mathcal{N}-1}$-types.

In chapter 3, we revisited the DL of the discrete AKP equation, by reviewing the results in [**NCWQ84**, **NCW85**, **DN91**] and rediscovering the HM equation and the Lax pair of the discrete AKP equation from the DL approach. We also extended the early result obtained by Miwa [**Miw82**], namely the soliton solution of the discrete BKP equation (i.e. the Miwa equation) was generalised to the direct linearising solution. The most important contribution in this chapter was that we gave a new parametrisation of the discrete CKP equation. Compared with the results in [**Kas96**, **Sch03**], this new parametrisation helped us to construct the direct linearising solution (including soliton solution) for the discrete CKP equation.

Chapter 4 was a review of the continuous soliton theory from the aspect of the DL approach. The first half of the chapter was dedicated to the DL scheme of the continuous AKP, BKP and CKP equations. In the second half, we performed dimensional reductions of these KP-type equations, resulting in a variety of famous (1+1)-dimensional soliton equations, including the KdV, BSQ, SK, KK and HS equations etc. The main contribution of this chapter was that the direct linearising solutions to all those nonlinear equations were constructed.

In chapter 5, the dimensional reduction related a generic algebraic curve of degree $\mathcal{N}$ was performed on the discrete AKP, BKP and CKP equations in the chapter 3. This resulted in a class of new discrete integrable systems, such as the discrete SK, KK and HS equations. We also explained the connection between various discrete BSQ equations and showed that the discrete BSQ equation possessing the full structure is a trilinear one.

The DL scheme of the discrete-time 2DTL equations was considered in chapter 6. On the 3D level, namely the discrete-time 2DTL of A_∞-type, we recovered the old results obtained in [**DJM82**] and showed that a new 3D discrete equation which recently appeared in [**FNR15**] also belonged to this class. The periodic reduction of the discrete-time 2DTL of A_∞-type leaded to the discrete-time 2DTL of $A_{\mathcal{N}-1}^{(1)}$-type, corresponding to a class of new discrete equations which were proposed in [**FX17**] very recently. Our scheme provided a suitable parametrisation allowing exact solutions and continuum limits for these new equations.

2. Outlook

There are still a lot of remaining problems for the DL of integrable systems, particularly for integrable discrete models. Below we list some of the problems which we are interested in and will consider in the future.

The first interesting problem is the classification of discrete integrable systems. At the current stage, the DL scheme has been successfully established for the discrete AKP, BKP and CKP equations, as well as their reductions. However, the reductions of the discrete BKP and CKP equations have only been done on the bilinear level. It would be an interesting problem to understand these models from their nonlinear forms and Miura-type maps and linear forms (i.e. Lax pairs). A related question is to extend the result of the discrete-time 2DTLs of A_∞-type to the B_∞- and C_∞-types and their reductions. The next step towards the full classification is to establish the DL of the discrete KP equation associated with the infinite-dimensional algebra D_∞, and its reductions. This requires us to understand the discrete two-component AKP equation first. To realise this, we need to systematically study a matrix linear integral equation (which was originally considered in [**Nij88**]), and its related nonlinear equations.

The second problem is about symmetries of 3D discrete integrable systems. The DL has also been proven an effective method to study symmetries of integrable lattice equations, see [**NRGO01**]. We aim to extend this to the 3D case, and construct the master symmetries of the discrete KP-type equations.

The third question is to consider conservation laws of integrable discrete equations. In fact, there exists an adjoint linear integral equation dual to (2.15), and the product of the two wave functions forms the square eigenfunction generating infinitely many conservation laws and helping to construct recursion operators.

In the DL approach, the direct linearising solution to a nonlinear equation involves an integral with respect to the spectral variable, which will lead to various explicit solutions depending on the choice of the form of the measure. In the thesis, we only discussed the soliton case. An interesting problem would be considering other classes of solution within the DL framework. For instance, similarity solution is also covered by the DL approach, leading to reductions to the discrete Painlevé-type equations, see e.g. [**NRGO01**]. We are going to investigate various exact solutions for the 3D and 2D discrete integrable systems.

APPENDIX A

Hirota's bilinear operator

Definition A.1. *Given two smooth functions $f(x)$ and $g(x)$ of a single variable x, the Taylor expansion of $f(x+y)g(x-y)$ around $y=0$ is given by*

$$f(x+y)g(x-y) = \sum_{j=0}^{\infty} \frac{1}{j!}(D_x^j f \cdot g) y^j.$$

The operator $(f,g) \mapsto D_x^j f \cdot g$ is the Hirota derivative.

Following the definition, we can write down the definition of the Hirota operator in a more direct way, namely

$$D_x^j f \cdot g = \frac{\partial^j}{\partial y^j} f(x+y)g(x-y)\Big|_{y=0}.$$

Alternatively, the Hirota operator is also defined by

$$D_x^j f(x) \cdot g(x') = (\partial_x - \partial_{x'})^j f(x) g(x')\Big|_{x'=x}.$$

Remark A.2. *When $g \equiv 1$, the Hirota derivative degenerates and becomes the usual derivative, namely $D_x f \cdot 1 = \partial_x f$, according to the definition.*

The Hirota operator also has the following property.

Proposition A.3. *For smooth functions $f = f(x,t)$ and $g = g(x,t)$, we have*

$$D_x^i D_t^j f \cdot g = (-1)^{i+j} D_x^i D_t^j g \cdot f.$$

Proof. This follows from the definition. In fact, we have

$$(\partial_x - \partial_{x'})^i (\partial_t - \partial_{t'})^j f(x,t) g(x',t') = (-1)^{i+j} (\partial_{x'} - \partial_x)^i (\partial_{t'} - \partial_t)^j g(x',t') f(x,t).$$

The identity is proven if we set $x' = x$ and $t' = t$. □

Corollary A.4. *$D_x^i D_t^j f \cdot f \equiv 0$ if $i+j$ is an odd number.*

The Hirota operator is closely related to logarithm function. Below we list some useful identities which are used in this thesis. Suppose that $f = f(x,t)$ and $g = g(x,t)$ are two smooth functions and D_x and D_t are the Hirota operators with respect to x and t. The following identities hold:

$$
\begin{aligned}
2\,(\ln f)_{xx} &= \frac{D_x^2 f \cdot f}{f^2}, \\
2\,(\ln f)_{xxxx} &= \frac{D_x^4 f \cdot f}{f^2} - 3\left(\frac{D_x^2 f \cdot f}{f^2}\right)^2, \\
2\,(\ln f)_{xxxx} &= \frac{D_x^6 f \cdot f}{f^2} - 15\frac{D_x^4 f \cdot f}{f^2}\frac{D_x^2 f \cdot f}{f^2} + 30\left(\frac{D_x^2 f \cdot f}{f^2}\right)^3, \\
2\,(\ln f)_{xt} &= \frac{D_x D_t f \cdot f}{f^2}.
\end{aligned}
$$

These identities help us to transfer a nonlinear equation to its bilinear form, e.g. from the KP equation to the bilinear KP equation.

Reversely, in order to transfer a bilinear equation to its nonlinear form, we also need identities which can be used to express $D_x^j f \cdot g$ by logarithm functions. We give a few examples below.

$$
\begin{aligned}
\frac{D_x f \cdot g}{fg} &= \left(\ln \frac{f}{g}\right)_x, \\
\frac{D_x^2 f \cdot g}{fg} &= \left[\left(\ln \frac{f}{g}\right)_x\right]^2 + \left(\ln \frac{f}{g}\right)_{xx} + 2\,(\ln g)_{xx}, \\
\frac{D_x^3 f \cdot g}{fg} &= \left[\left(\ln \frac{f}{g}\right)_x\right]^3 + \left(\ln \frac{f}{g}\right)_{xxx} + 3\left(\ln \frac{f}{g}\right)_x \left[\left(\ln \frac{f}{g}\right)_{xx} + 2\,(\ln g)_{xx}\right], \\
\frac{D_x D_t f \cdot g}{fg} &= \left(\ln \frac{f}{g}\right)_x \left(\ln \frac{f}{g}\right)_t + \left(\ln \frac{f}{g}\right)_{xt} + 2\,(\ln g)_{xt}.
\end{aligned}
$$

The above identities involve two functions f and g. Sometimes we also need identities involving only a single function f. Examples are as follows:

$$
\begin{aligned}
\frac{D_x^2 f \cdot f}{f^2} &= 2\,(\ln f)_{xx}, \\
\frac{D_x^4 f \cdot f}{f^2} &= 2\,(\ln f)_{xxxx} + 12\,[(\ln f)_{xx}]^2, \\
\frac{D_x^6 f \cdot f}{f^2} &= 2\,(\ln f)_{xxxxxx} + 60(\ln f)_{xx}(\ln f)_{xxxx} + 120\,[(\ln f)_{xx}]^3, \\
\frac{D_x D_t f \cdot f}{f^2} &= 2\,(\ln f)_{xt}.
\end{aligned}
$$

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